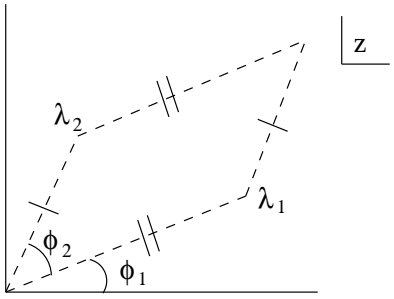
Chapter 7

One-loop Amplitudes

Following *"String Theory"* by J. Polchinski, Vol.I.
Notes written by students (work still in progress).
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All two dimensional surfaces can be classified by their genus (the number of handles on a closed oriented Riemann surface). Earlier we showed the tree amplitudes could be represented by a sphere(for closed strings) or a disk(for open strings). For higher order corrections, the amplitudes can be represented by surfaces with a larger genus number. For example, the one loop correction will be represented by a surface with genus number equal to one, ie a torus. As before we can represent incoming and outgoing particles as vertex operators(punctures). To begin, we will look at the torus with no punctures. Unlike the sphere two tori with no punctures are conformally inequivalent. We must find the moduli space for the torus itself.

Geometry



Let $\phi_2 > \phi_1$.
Define $\tau = \frac{\lambda_2}{\lambda_1}$ where $\mathrm{Im}\tau > 0$. τ will uniquely specify the conformal structure of the torus.

Define a lattice with coordinates z,$\bar{z}$.

$$\begin{aligned} z &\approx z + \lambda_1 \\ z &\approx z + \lambda_2 \\ &\cdot \\ &\cdot \\ z &\approx z + n\lambda_2 + m\lambda_2 \qquad n, m \in \mathbb{Z} \end{aligned}$$

Do a conformal transformation.

$$\begin{aligned} z &\rightarrow \zeta z \\ \lambda_1 &\rightarrow \frac{1}{\zeta}\lambda_1 \\ \lambda_2 &\rightarrow \frac{1}{\zeta}\lambda_2 \end{aligned}$$

τ depends on the ratio of the λ's and doesn't change. We can see that the conformal transformation doesn't change the conformal structure of the surface as it shouldn't. This also means that one can't make two conformally inequivalent surfaces equivalent through a conformal transformation. We must find a class of transformations that will make two surfaces conformally equivalent. This will be done later.

We have seen how z can transform, but λ can also transform. Transforming λ will look geometrically like the parallelogram is being stretched. We need to find how we can transform both λs such that the conformal structure is unchanged. Take a general matrix transformation where the four parameters (a,b,c,d) are integers.

$$\begin{pmatrix} \lambda_2' \\ \lambda_1' \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}$$

With the conditions $\mathrm{ad} - \mathrm{bc} = 1, \quad \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in \mathbb{Z}$.
This is just SL(2,$\mathbb{Z}$), which is a subgroup of SL(2,$\mathbb{C}$).

Below is the proof τ is invariant under the above transformation of λ.

$$\begin{aligned} z &\approx z + n'\lambda_1' + m'\lambda_2' \\ &= z + (m'\ n')\begin{pmatrix} \lambda_2' \\ \lambda_1' \end{pmatrix} \\ &= z + (m'\ n')\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \\ &= z + (m\ n)\begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \end{aligned}$$

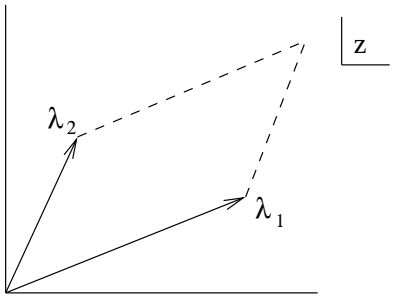
Define a parameter describing a torus

$$\tau = \frac{\lambda_1}{\lambda_2},$$

and the transformation between these $\tau \longrightarrow \tau'$ is

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Same torus described if under this transformation.

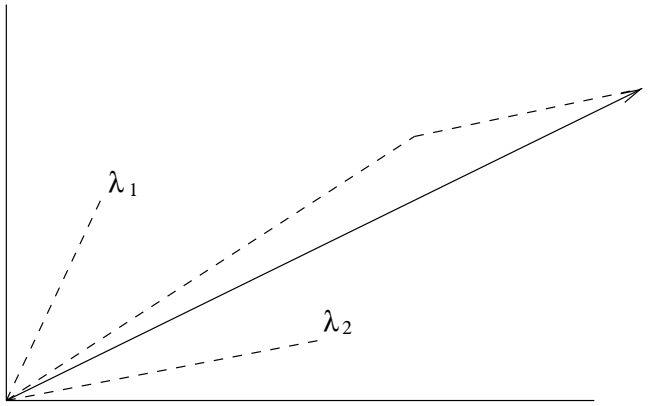


where a, b, c and d $\in \mathbb{Z}$. This SL(2,$\mathbb{Z}$) transformation, $\tau \to \tau'$, describes the same torus. There are two kinds of transformation that cover all the members in the group, i.e.

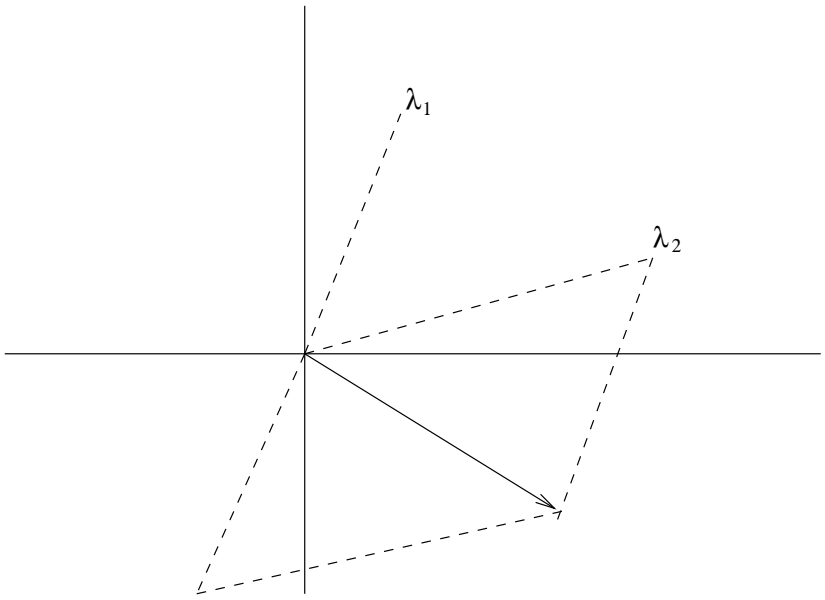
$$T \quad : \quad \tau \longrightarrow \tau + 1 \tag{7.1}$$

$$S \quad : \quad \tau \longrightarrow -\frac{1}{\tau} \tag{7.2}$$

If let $\lambda_2' = \lambda_2 + \lambda_1$ and $\lambda_1' = \lambda_1$, then $\lambda_1' + \lambda_2' = 2\lambda_1 + \lambda_2$ corresponding to $T : \tau \longrightarrow \tau + 1$ as the following picture.



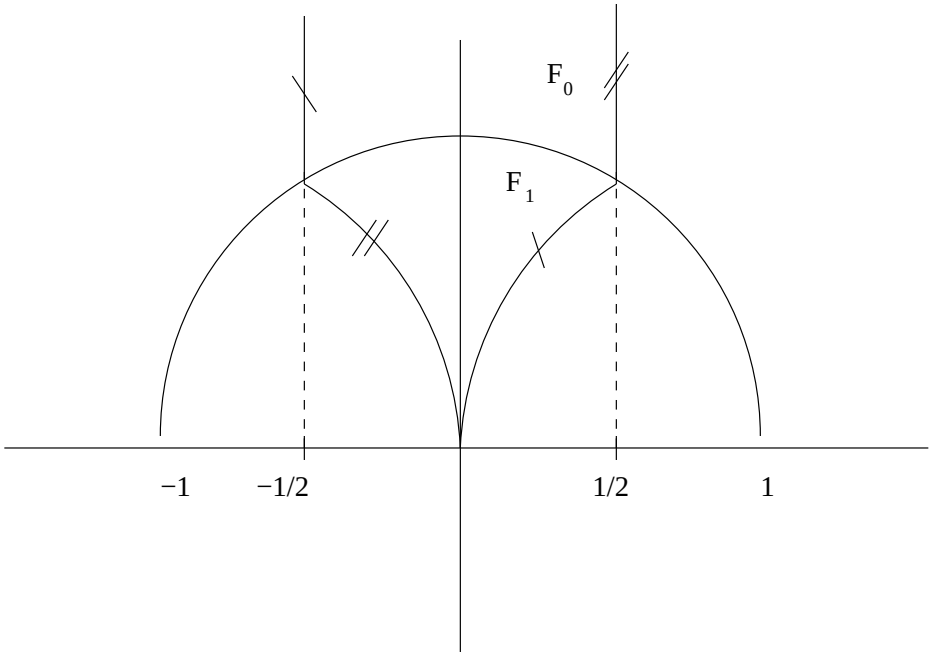
If let $\lambda_2' = \lambda_1$ and $\lambda_1' = -\lambda_2$, then $\lambda_1' + \lambda_2' = \lambda_1 - \lambda_2$ coresponding to $S : \tau \longrightarrow -\frac{1}{\tau}$ as the following picture.



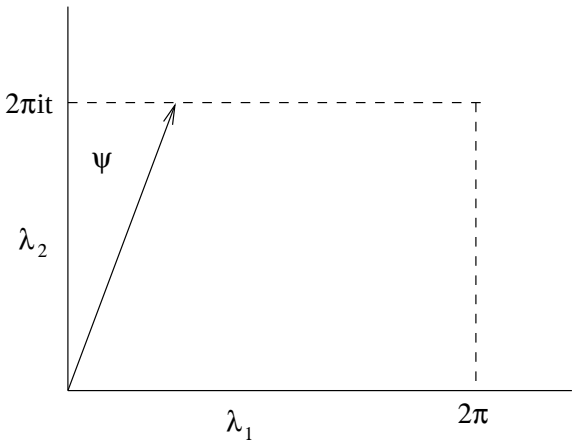
Next if consider a transformation

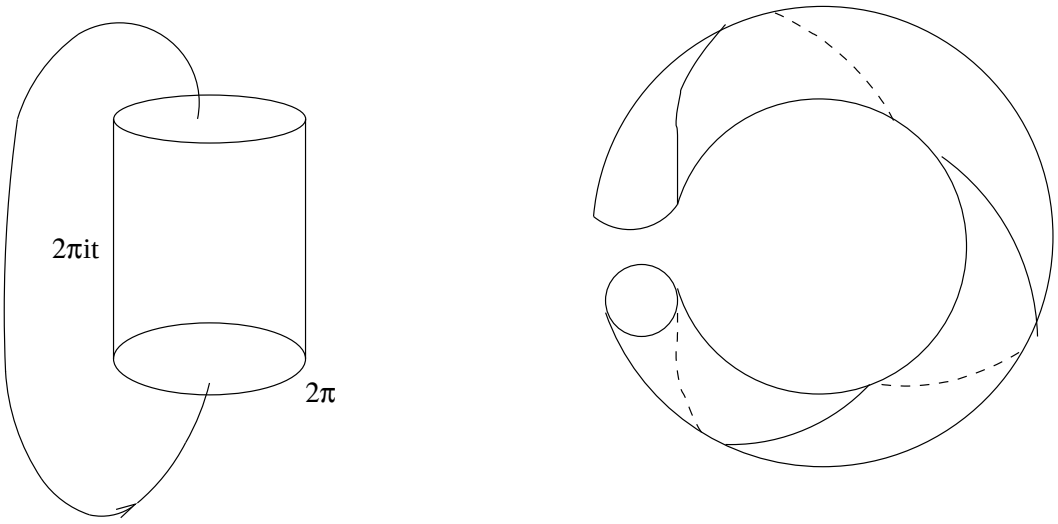
$$\begin{aligned} it' &= -\frac{1}{it} \text{ or} \\ t' &= \frac{1}{t} \end{aligned} \tag{7.3}$$

which maps points from (1,∞) to (0,1) as the picture below. The transformation excludes the zero region and removes infinity from the theory. If in this case $\tau = e^{i\theta}$, then τ will transform as $\tau = e^{i\theta} \longrightarrow -\frac{1}{e^{i\theta}} = e^{-i(\theta+\pi)}$, which is an unit circle transforming to an unit circle. To calculate the loop amplitude τ needed to be integrated over all transformations among torus and averaged over the volume of the group. It is conventional and convenient to integrate over area F_0 as shown below.

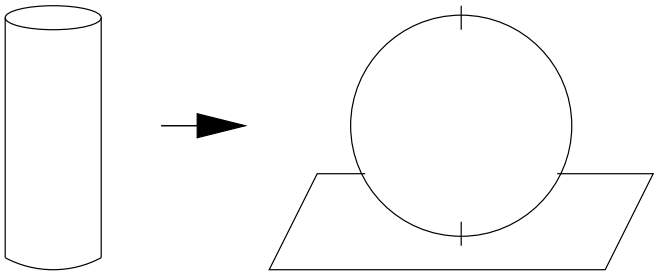


Example : Fix $\lambda_1 = 2\pi$ and $\lambda_2 = 2\pi\tau$ where $\tau = it, t \in \mathbb{R}$. From the torus picture, ψ appears when $\psi \in \mathbb{Z}$, making the torus twisiting with angle ψ.

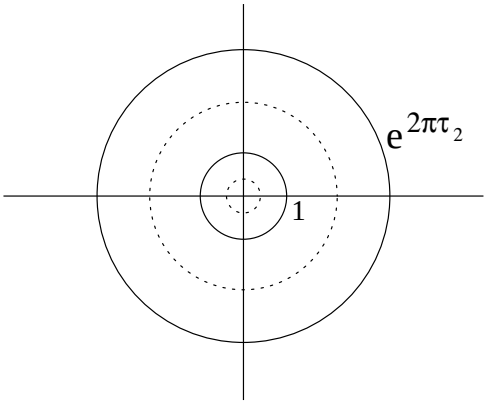




From $\tilde{z} = e^{iz}$, the transformation can be shown as



where $z \simeq z + m\lambda_1 + n\lambda_2$. That is if $z \longrightarrow z + 2\pi$, then $\tilde{z} \longrightarrow \tilde{z}$. From moving $\tilde{z}$ on an unit circle, $\tilde{z} \longrightarrow \tilde{z}e^{-2\pi i\tau}$, if $\tau = \tau_1 + i\tau_2$, then $|\tilde{z}| \approx |\tilde{z}|e^{2\pi\tau_2}$



From quantum mechanics, let $|n>$ be an intial state of a system at $\tau = 0$, when time has evolved for $2\pi\tau$, the state will evolve to $|n(\tau)> = e^{-iH(2\pi\tau)}|n>$, where the Hamiltonian for the left mover is $H = L_0 - \frac{D}{24}$. The

translation amplitude and $z(\tau)$ are

$$\begin{aligned} A_n(\tau) &= < n(\tau)|n(0) > \\ &= < n|e^{iH(2\pi\tau)}|n > \qquad (7.4) \\ z(\tau) &= \sum_n A_n(\tau). \qquad (7.5) \end{aligned}$$

If $\tau_1 = 0$ or $\tau' = i\tau_2$ then

$$\begin{aligned} A_n &= < n|e^{-2\pi\tau_2 H}|n >, \text{ where } H|n >= E_n|n > \\ Z(\tau) &= \sum_n e^{-2\pi\tau_2 E_n} \qquad (7.6) \end{aligned}$$

$Z(\tau)$ looks the same as the statistical mechanics partition function or in other words quantum mechanics is zero-temperature statistical mechanics, where $\tau \sim 1/T$. Therefore $Z(\tau) = \sum A_n(\tau)$ can be considered as a generalized partition function. $Z(\tau)$ can be writen as

$$\begin{aligned} Z(\tau) &= \sum_n e^{2\pi i\tau E_n} \\ &= \mathrm{Tr}(e^{2\pi i\tau(L_0 - \frac{D}{24})}), \quad \text{for left mover part} \qquad (7.7) \end{aligned}$$

The whole $Z(\tau)$ are

$$Z(\tau) = \mathrm{Tr}(e^{2\pi i\tau(L_0 - \frac{D}{24})}\ e^{-2\pi i\tilde{\tau}(\tilde{L}_0 - \frac{D}{24})}) \qquad (7.8)$$

where

$$L_0 + \tilde{L_0} = \frac{\alpha'}{2}p^2 + N + \tilde{N} \qquad (7.9)$$

generates translation in time and

$$L_0 - \tilde{L_0} = N - \tilde{N} \qquad (7.10)$$

generates translation in σ along the string with

$$\begin{aligned} N &= \sum_{n=1}^{\infty} \alpha_{\mu n}\alpha^{\mu}_{-n}, \qquad (7.11) \\ \tilde{N} &= \sum_{n=1}^{\infty} \tilde{\alpha}_{\mu n}\tilde{\alpha}^{\mu}_{-n}. \qquad (7.12) \end{aligned}$$

To simplify the notation let $q = e^{2\pi i\tau}$, Im $\tau > 0$, then $Z(\tau)$ is

$$\begin{aligned} Z(\tau) &= \mathrm{Tr}(q^{L_0 - \frac{D}{24}}\ \bar{q}^{\tilde{L}_0 - \frac{D}{24}}) \\ &= (q^{-\frac{D}{24}}\bar{q}^{-\frac{D}{24}})\ \mathrm{Tr}\ e^{-2\pi\tau_2(\frac{\alpha'}{2}p^2)}\ \mathrm{Tr}(q^N\bar{q}^{\tilde{N}}) \qquad (7.13) \end{aligned}$$

next use $N(\alpha_{-n})^l|0 >= nl\alpha_{-n}|0 >$, then

$$\mathrm{Tr}(q^N\bar{q}^{\tilde{N}}) = \prod_{\mu}\prod_{n}(\sum_{\ell=0}^{\infty} q^{n\ell_{\mu n}})(\sum_{\tilde{\ell}=0}^{\infty} \bar{q}^{\ n\tilde{\ell}_{\mu n}})$$

and also use

$$\sum_{l=0}^{\infty} q^{nl} = \frac{1}{1-q^n} \quad \text{and} \quad \sum_{l=0}^{\infty} q^{-nl} = \frac{1}{1-q^{-n}}$$

then

$$\begin{aligned} \mathrm{Tr}(q^N \tilde{q}^{\tilde{N}}) &= \left[\prod_n (1-q^n)^{-1} \prod_n (1-\bar{q}^{\ n})^{-1}\right]^D \\ &= \prod_{n=1}^{\infty} |1-q^n|^{-2D}. \end{aligned} \tag{7.14}$$

Suppose the space-time is a box with length L each side then $\vec{p} \sim \frac{2\pi\vec{n}}{L}$ and

$$L^D \sum \frac{1}{L^D} f(\vec{p}) \quad \longrightarrow \quad \int \frac{d^D p}{(2\pi)^D} f(\vec{p}), \text{ for large } L$$

and let $E = p_0 \longrightarrow i p_0$ to make the integral finite, i.e.

$$\begin{aligned} \mathrm{Tr}\ e^{-2\pi i \tau_2 (\frac{\alpha'}{2} p^2)} &\longrightarrow iL^D \int \frac{d^D p}{(2\pi)^D} f(\vec{p}) \\ &\longrightarrow i(L^D \int \frac{d^D p}{(2\pi)^D}\ e^{-2\pi i \tau_2 (\frac{\alpha'}{2} p^2)})^D \\ &= i(L \frac{1}{2\pi\sqrt{\alpha' \tau_2}})^D. \end{aligned} \tag{7.15}$$

The total $Z(\tau)$ is

$$Z(\tau) = i|q|^{-\frac{D}{12}} L^D (2\pi\sqrt{\alpha'\tau_2})^{-D} \prod_{n=1}^{\infty} (1-q^n)^{-2D} \tag{7.16}$$

$Z(\tau)$ is invarance under two kinds of tansformations i.e.

$$\tau \longrightarrow \tau + 1, \tag{7.17}$$

which is obviously invariant, $q = e^{2\pi i \tau} \longrightarrow q$ and

$$\tau \longrightarrow -\frac{1}{\tau} \tag{7.18}$$

which can be shown its invarance as following.

Scalar Partition Function

$$\begin{aligned} Z(\tau) &= \mathrm{Tr}[q^{L_0 - \frac{D}{24}} - \tilde{q}^{\tilde{L}_0 - \frac{D}{24}}] : \quad q = e^{2\pi i \tau} \\ &= iL^D \left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}} |q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)|^{-2} \right)^D \end{aligned} \tag{7.19}$$

Z(τ) is invariant under modular transformations ($\tau \to \tau + 1,\ \tau \to -\frac{1}{\tau}$).
Define the ϑ function:

$$\vartheta(\nu,\tau)=\sum_{n=0}^{\infty}e^{i\pi n^2\tau+2\pi in\nu}=\sum_{n=-\infty}^{\infty}q^{\frac{n^2}{2}}z^n=\vartheta_3(\nu|\tau)$$

The ϑ function obeys the Schroedinger equation.

$$\frac{i}{\pi}\frac{\partial^2\vartheta}{\partial\nu^2}+4\frac{\partial\vartheta}{\partial\tau}=0$$

What are the transformation properties of the ϑ function.

$$\vartheta(\nu+1,\tau)=\vartheta(\nu,\tau) \tag{7.20}$$

$$\begin{aligned}\vartheta(\nu+\tau,\tau) &= \sum_n q^{\frac{n^2}{2}}z^nq^n=q^{-\frac{1}{2}}\sum_n q^{\frac{1}{2}(n+1)^2}z^n \qquad \frac{1}{2}n^2+n=\frac{1}{2}(n+1)^2-\frac{1}{2}\\ &= q^{-\frac{1}{2}}\sum_n q^{\frac{n^2}{2}}z^{n-1}\\ &= q^{-\frac{1}{2}}z^{-1}\sum_n q^{\frac{n^2}{2}}z^n\\ &= q^{-\frac{1}{2}}z^{-1}\vartheta(\nu,\tau)\\ &= e^{-\pi i\tau-2\pi i\nu}\vartheta(\nu,\tau)\end{aligned} \tag{7.21}$$

These two conditions uniquely determine ϑ up to a multiplicative constant.
ϑ can also be represented as a product of qs.

$$\vartheta(\nu,\tau)=\prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-\frac{1}{2}})(1+z^{-1}q^{m-\frac{1}{2}}) \tag{7.22}$$

We need to show that this new representation of ϑ also meets the above transformations.

$$\vartheta(\nu+1|\tau) = \vartheta(\nu|\tau) \tag{7.23}$$

$$\begin{aligned}\vartheta(\nu+\tau,\tau) &= \frac{1+z^{-1}q^{-\frac{1}{2}}}{1+zq^{-\frac{1}{2}}}\vartheta(\nu,\tau)\\ &= q^{-\frac{1}{2}}z^{-1}\vartheta(\nu,\tau)\end{aligned} \tag{7.24}$$

Define a general ϑ with characteristics a,b:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu,\tau)=\sum_{n=-\infty}^{\infty}e^{\pi i(n+a)^2\tau+2\pi i(n+a)(\nu+b)}$$

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\nu,\tau)=\vartheta_3(\nu|\tau)$$

$$\begin{aligned}-\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(\nu,\tau) &= \vartheta_1(\nu|\tau)\\ &= i\sum_n(-i)^nq^{\frac{1}{2}(n-\frac{1}{2})^2}z^{n-\frac{1}{2}}\\ &= 2e^{\pi i\tau/4}\sin(\pi\nu)\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m)\end{aligned} \tag{7.25}$$

where ϑ has zeros at $z = q^{\overset{+}{-}m}$.

Modular Transformations:

In this section we look at the ϑ, η, and Z functional behavior under modular transformations. For a partition function to have any physical meaning, it must be modular invariant. The modular transformations are given by:
$T : \tau \to \tau + 1$
$S : \tau \to -\frac{1}{\tau}$

What happens to ϑ under these transformations?
T:

$$\begin{aligned}
\vartheta_3(\nu|\tau+1) &= \sum_n e^{i\pi n^2(\tau+1)+2\pi i n\nu} \\
&= \sum_n \mathrm{e}^{\mathrm{i}\pi \mathrm{n}^2\tau+2\pi \mathrm{i}\mathrm{n}\nu+(\pi \mathrm{i}\mathrm{n}-\pi \mathrm{i}\mathrm{n})} \\
&= \sum_n \mathrm{e}^{\mathrm{i}\pi \mathrm{n}^2\tau+2\pi \mathrm{i}\mathrm{n}(\nu+\frac{1}{2})+\pi \mathrm{i}\mathrm{n}(\mathrm{n}-1)} \\
&= \sum_n \mathrm{e}^{\pi \mathrm{i}\mathrm{n}(\mathrm{n}-1)}\mathrm{e}^{\mathrm{i}\pi \mathrm{n}^2\tau+2\pi \mathrm{i}\mathrm{n}(\nu+\frac{1}{2})} \\
&= \sum_n \mathrm{e}^{\text{even integer}}\mathrm{e}^{\mathrm{i}\pi \mathrm{n}^2\tau+2\pi \mathrm{i}\mathrm{n}(\nu+\frac{1}{2})} \\
&= \sum_n \mathrm{e}^{\mathrm{i}\pi \mathrm{n}^2\tau+2\pi \mathrm{i}\mathrm{n}(\nu+\frac{1}{2})} \\
&= \vartheta_3(\nu+\frac{1}{2}|\tau) \qquad (7.26)
\end{aligned}$$

S:

$$\begin{aligned}
\vartheta_3(\nu|\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} z^n \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{\frac{x^2}{2}} z^x \delta(x-n)\, dx \\
&= \int_{-\infty}^{\infty} dx\, q^{\frac{x^2}{2}} z^x \sum_{n=-\infty}^{\infty} \delta(x-n) \qquad (7.27)
\end{aligned}$$

Lemma: Show

$$\sum_n \delta(x-n) = \sum_{m=-\infty}^{\infty} e^{2\pi i m x} = f(x)$$

Proof:

$$x \in \mathbb{Z} \quad f(x) = \sum_{m=-\infty}^{\infty} 1 = \infty$$

What if x$\notin \mathbb{Z}$?

$$\begin{aligned} f(x) &= \sum_{m=0}^{\infty} e^{2\pi i x m} + \sum_{m=0}^{\infty} e^{-2\pi i x m} - 1 \\ &= \frac{1}{1-e^{2\pi i x}} + \frac{1}{1-e^{-2\pi i x}} - 1 \\ &= 2\mathrm{Re}\frac{1}{1-e^{-2\pi i x}} - 1 \\ &= 2\mathrm{Re}\frac{e^{-i\pi x}}{-2i\sin(\pi x)} - 1 \\ &= 1 - 1 = 0 \end{aligned} \tag{7.28}$$

$\therefore$ f(x)=0 unless x is $\mathbb{Z}$.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\ f(x) = \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx e^{2\pi i x m} = \sum_m \frac{e^{2\pi i x m}}{2\pi i m}|_{-\frac{1}{2}}^{\frac{1}{2}} = 1 \quad \text{for m} = 0.$$

$$\begin{aligned} \vartheta_3(\nu|\tau) &= \sum_{m=-\infty}^{\infty} \int_{m=-\infty}^{\infty} dx\ q^{x^2/2} z^x e^{2\pi i x m} \\ &= \sum_m \int dx\ e^{\pi i \tau x^2 + 2\pi i \nu x + 2\pi i x m} \\ &= \sum_m \int dx\ e^{\pi i \tau x^2 + 2\pi i x(\nu+m)} \\ &= \sum_m \int dx\ e^{\pi i \tau (x+\frac{\nu+m}{\tau})^2 - \pi i \frac{(\nu+m)^2}{\tau}} \\ &= \sum_m e^{-\pi i \frac{(\nu+m)^2}{\tau}} \int dx\ e^{\pi i \tau (x+\frac{\nu+m}{\tau})^2} \\ &= \sum_m e^{-\pi i \frac{(\nu+m)^2}{\tau}} \frac{1}{\sqrt{-i\tau}} \\ &= \frac{1}{\sqrt{-i\tau}} e^{\frac{-i\pi\nu^2}{\tau}} \sum_m e^{-i\pi m^2/\tau - 2\pi i \nu m/\tau} \\ &= \frac{1}{\sqrt{-i\tau}} e^{\frac{-i\pi\nu^2}{\tau}} \vartheta_3(\frac{\nu}{\tau}| -\frac{1}{\tau}) \end{aligned} \tag{7.29}$$

$$\vartheta_3(\frac{\nu}{\tau}| -\frac{1}{\tau}) = \sqrt{-i\tau} e^{\pi i \nu^2/\tau} \vartheta(\nu|\tau)$$

$$\vartheta_1(\nu|\tau+1) = e^{\pi i \tau/4} \vartheta_1(\nu|\tau)$$

$$\vartheta_1(\frac{\nu}{\tau}| -\frac{1}{\tau}) = -\sqrt{-i\tau} e^{\pi i \nu^2/\tau} \vartheta_1(\nu|\tau)$$

$$\vartheta_1(\nu|\tau) = 2e^{\pi i \tau/4} \sin(\pi\nu) \prod_{m=1}^{\infty} (1-q^m)(1-zq^m)(1-z^{-1}q^m)$$

$$\begin{aligned}\partial_\nu \vartheta_1(\nu|\tau)|_{\nu=0} &= 2\pi \mathrm{e}^{\pi \mathrm{i}\tau/4}\cos(\pi\nu)\prod_{\mathrm{m}=1}^{\infty}(1-\mathrm{q}^{\mathrm{m}})(1-\mathrm{zq}^{\mathrm{m}})(1-\mathrm{z}^{-1}\mathrm{q}^{\mathrm{m}}) \\ &\quad +2e^{\pi i\tau/4}\sin(\pi\nu)\partial_\nu\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m)|_{\nu=0} \qquad (7.30)\end{aligned}$$

for $\nu = 0 \;\; z = e^{2\pi i\nu} = 1$

$$\begin{aligned}\partial_\nu \vartheta_1(0|\tau) &= 2\pi e^{\pi i\tau/4}\left[\prod_{m=1}^{\infty}(1-q^m)\right]^3 \\ &= 2\pi\left(q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m)\right)^3 \\ &= 2\pi(\eta(\tau))^3 \quad \text{"Dedekind function"} \qquad (7.31)\end{aligned}$$

$$\eta(\tau) = \left(\frac{\partial_\nu \vartheta_1(0|\tau)}{2\pi}\right)^{1/3}$$

Modular Properties of the Dedekind function:

$\tau \to \tau + 1$

$$\eta(\tau+1) = e^{\pi 1/12}\eta(\tau)$$

$$\begin{aligned}\frac{1}{\tau}\partial_\nu \vartheta_1(0|-\frac{1}{\tau}) &= -\sqrt{-i\tau}\,\partial_\nu \vartheta_1(0|\tau) \\ \partial_\nu \vartheta_1(0|-\frac{1}{\tau}) &= (-i\tau)^{3/2}\partial_\nu \vartheta_1(0|\tau) \qquad (7.32)\end{aligned}$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$$

So we see that ϑ and η are not invariant under modular transformations. We really only care about the partition function, which depends on η. Check and see how the partition function transforms under the modular transformations from knowing how η transforms.

$$Z(\tau) = iL^D\left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}}|\eta(\tau)|^{-2}\right)^D$$

$T : \tau \to \tau + 1$

$$Z(\tau) = Z(\tau+1) \qquad (7.33)$$

$S : \tau \to -\frac{1}{\tau}$

$$\begin{aligned} \eta(-\frac{1}{\tau}) &= (-i\tau)^{1/2}\eta(\tau) \\ |\eta(-\frac{1}{\tau})|^2 &= \frac{1}{|\tau|}|\eta(\tau)|^2 \end{aligned} \tag{7.34}$$

$$\frac{1}{\sqrt{\tau_2}} \overset{\tau\to-\frac{1}{\tau}}{\to} \tau \to \frac{-1}{\tau_1+i\tau_2} = -\frac{(\tau_1-i\tau_2)}{|\tau|^2} \Rightarrow \tau_2 \to \frac{\tau_2}{|\tau|^2}$$

$$Z(-\frac{1}{\tau}) = Z(\tau)$$

$\therefore$ $Z(\tau)$ is modular invariant!
From

$$\begin{aligned} Z(\tau) &= iL^D(\frac{1}{2\pi\sqrt{\alpha'\tau_2}}|\eta(\tau)|^{-2})^D \\ \eta(\tau) &= q^{\frac{1}{24}}\prod_{m=0}^{\infty}(1-q^m), \end{aligned}$$

$\eta(\tau)$ supposes to be raised to the power D-2 in $Z(\tau)$ equation, excluding time and longitudinal components. bc system is introduced for heping to obtain the factor D - 2.

$$L_0 = \sum n : b_{-n}c_n : -1 \tag{7.35}$$

where -1 comes from $\lambda(1-\lambda)/2$, when $\lambda = 2$. From chapter 2 the energy-momentum tensor transforms as

$$T \longrightarrow (\frac{\partial\tilde{z}}{\partial z})^2 T + \frac{c}{24} \tag{7.36}$$

where c is -26 in bc system, then shift L_0 by 13/12, i.e.

$$\begin{aligned} L_0 &\longrightarrow L_0 + \frac{13}{12} \\ L_0 &\longrightarrow \sum n : b_{-n}c_n : +\frac{1}{2} \end{aligned} \tag{7.37}$$

Z_{bc} can be obtained in the similar fashion as $Z(\tau)$, i.e.

$$Z_{bc} = q^{1/2}\ \bar{q}^{\ 1/2}\ \mathrm{Tr}\ q^{N_{bc}}\bar{q}^{\ \bar{N}_{bc}} \tag{7.38}$$

$$\begin{aligned} N_{bc} &= \sum_{n=-\infty}^{\infty} n : b_n c_{-n} : \\ &= b_{-1}c_1 + c_{-1}b_1 + (n=2) + (n=3) + \ldots\ , \text{ using } \{b_{-n}, c_n\} = 1 \\ (b_{-1}c_1)^2 &= b_{-1}c_1 b_{-1}c_1 = b_{-1}\{c_1, b_{-1}\}c_1 \\ &= b_{-1}c_1, \end{aligned} \tag{7.39}$$

$(b_{-1}c_1)^2$ can be considered as a projection operator with eigenvalue 0 and 1, which can be shown by defining

$$|0>=|\psi>,\ |\chi>=c_0|\psi>,\ |1>=b_{-1}|0>, \tag{7.40}$$

then

$$b_{-1}c_1|1>=b_{-1}c_1b_{-1}|0>=b_{-1}|0>=|1>\ . \tag{7.41}$$

If let $c_n^\dagger = c_{-n}$ and $b_n^\dagger = b_{-n}$, then $< \psi|\psi >=< \chi|\chi >= 0$. If also define $< \psi|c_0|\psi >=< \chi|\psi >\equiv 1$ and $< 1'| = - < 0|c_1$ from $|1 >= b_{-1}|0 >$, then

$$< 1'|c_0|1 >= - < 0|c_1 c_0 b_{-1}|0 >= 1. \tag{7.42}$$

The first two terms in N_{bc}(n = 1), $b_{-1}c_1 + c_{-1}b_1$, has 4 eigenvectors, $\{|0 >, |1 >, |1' >, |2 >\}$ corresponding to 3 eigenvalues,$\{0, 1, 2\}$, where $|1 >, |1' >$ are degenerated with an eigenvalue 1, therefore when inserting the identity operation the trace will become

$$\begin{aligned} \text{Tr } A &= \text{Tr} \sum_n |n >< n|A = \sum_n < n|A|n >, \text{ then} \\ \text{Tr } q^{b_{-1}c_1 + c_{-1}b_1} &= \text{Tr}(|0 >< 0'| + |1 >< 1'| + |1 >< 1'| + |2 >< 2'|) q^{b_{-1}c_1 + c_{-1}b_1} \end{aligned}$$

where $|0 >= |0' >$, $|2 >= |2' >$ and

$$\begin{aligned} \sum_{i=0}^{2} q^i &= q^0 - q^1 - q^1 + q^2 \\ &= (1-q)^2. \end{aligned}$$

The n = 2 terms and the n = 3 terms in $N_{bc} = b_{-1}c_1 + c_{-1}b_1 + (n = 2) + (n = 3) + ...$ can be calculated in the same fashion with the result $(1 - q^2)^2$ for n = 2 and $(1 - q^3)^2$ for n = 3 and so on ... , then

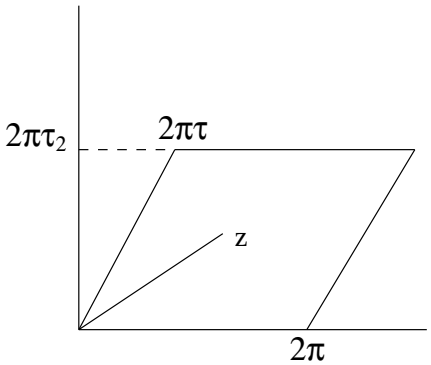
$$\text{Tr } q^{N_{bc}} = \prod_{m=1}^{\infty} (1 - q^m)^2 \tag{7.43}$$

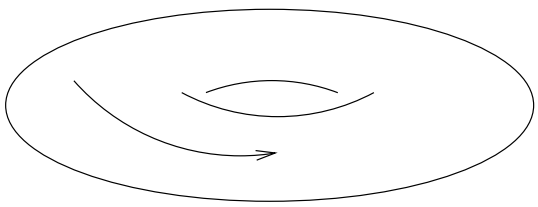
$$\begin{aligned} Z_{bc} &= q^{1/2} \bar{q}^{\,1/2} | \prod_{m=1}^{\infty} (1 - q^m)|^4 \\ &= |q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)|^4 \\ &= |\eta(\tau)|^4. \end{aligned} \tag{7.44}$$

Z_{bc} is invariant under the transformation $\tau \longrightarrow -1/\tau$, and $Z = ZZ_{bc}$ i.e.

$$\begin{aligned} Z &= ZZ_{bc} \\ &= iL^D (2\pi\sqrt{\alpha' \tau_2})^{-D} |\eta(\tau)|^{-2(D-2)} \end{aligned} \tag{7.45}$$

as expected that η raised to the power of $D - 2$. On a sphere the transformation undergoes inside a group $SL(2, \mathbb{C})$ and the translation transformation on torus can be considered as a 1-1 map on z-lattice plane.





On a torus the number of translations is equal to the area of a torus or in other words the volume of the group is equal the area of the lattice in z plane, $(2\pi)^2\tau_2$ and the physical quantites have to be averaged over all translations, i.e.

$$\frac{\int d\tau d\bar{\tau}}{2(2\pi)^2\tau_2}.$$

The factor 1/2 comes from the reflection effect, then $Z_{physical}$ is

$$Z_{physical} = iL^D \int_{F_0} \frac{d\tau d\bar{\tau}}{\tau_2} (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}. \tag{7.46}$$

The behavior of Z vacuum for small l or large E is

$$Z \sim \int_0 \frac{dl}{l^{d/2+1}} \longrightarrow \infty \tag{7.47}$$

which is ultraviolet diverge.

Cosmological Constant

Earlier, we looked at a single particle, and found a bogus Hamiltonian.

$$\begin{aligned} E &= \sqrt{p^2 + m^2} \\ \chi &= \frac{1}{2}(p_\mu p^\mu + m^2) = 0 \end{aligned}$$

What information can we gain by evaluating the vacuum amplitude for the string? The vacuum amplitude defined earlier is given by,

$$Z = \int_0^\infty \frac{dl}{2\pi} Z(l). \tag{7.48}$$

where

$$Z(l) = L^D \int \frac{d^D p}{(2\pi)^D} e^{-\chi l} = iL^D \left(\frac{1}{\sqrt{2\pi l}}\right)^D e^{-m^2 l/2}$$

There is an ultraviolet divergence when $l = 0$. Cut off the integration region at ϵ and later let $\epsilon \to 0$. Apply a Wick rotation ($p_0 \to ip_0$) and integrate over p_0.

$$\begin{aligned} Z(l) &= i\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \exp[-\frac{l}{2}(p_0^2 + \vec{p}^2 + m^2)] \\ &\sim \frac{1}{\sqrt{l}} \exp[-\frac{l}{2}(\vec{p}^2 + m^2)] \end{aligned} \tag{7.49}$$

Now do the integral over l:

$$\begin{aligned} Z &= \int_\epsilon^\infty \frac{dl}{l} \frac{1}{\sqrt{l}} \exp[-\frac{l}{2}(\vec{p}^2 + m^2)] \\ &\sim \sqrt{\vec{p}^2 + m^2} \int_\epsilon^\infty \frac{dx}{x^{3/2}} \, \mathrm{e}^{-\mathrm{x}/2} \\ &\sim \sqrt{\vec{p}^2 + m^2} * \text{constant} \end{aligned} \tag{7.50}$$

$$\begin{aligned} Z(l) &\sim L^D \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} \sqrt{\vec{p}^2 + m^2} \\ &\sim L^D \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} E(\vec{p}) \end{aligned} \tag{7.51}$$

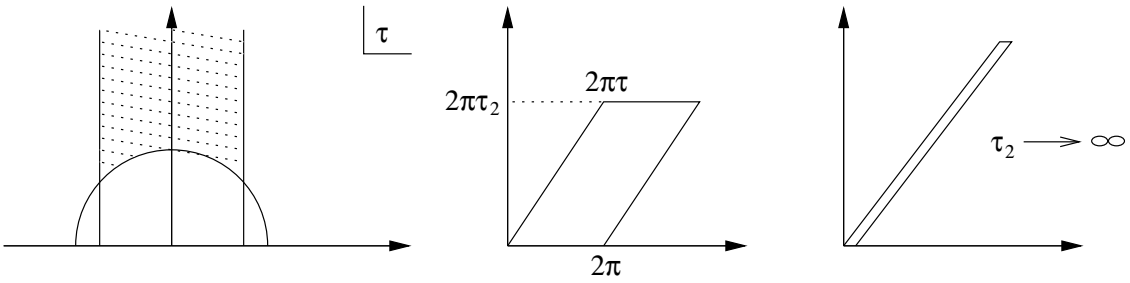
This is equivalent to integrating over all possible modes of the string. We can define the energy density as Λ.

$$\Lambda = \frac{Z}{L^D} \sim \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} E(\vec{p}) \qquad \text{"Cosmological Constant"} \tag{7.52}$$

This implies the cosmological constant is the sum of the vacuum energies from each of the states of the string.

How does the Cosmological constant play a role in one loop correction of the vacuum?

$$\Lambda \sim \int_{F_0} \frac{d\tau\, d\bar{\tau}}{\tau_2} (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)} \tag{7.53}$$



There will be no ultraviolet divergence, unlike the particle case. We see in the figure above the torus will get long as $\tau_2 \to \infty$. Look at the properties of the integral over τ_2 and see what we can learn.

$$|\eta(\tau)| = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$$

$$|q| = |\mathrm{e}^{2\pi \mathrm{i} \tau}| = \mathrm{e}^{-2\pi\tau_2} \to 0$$

Expand η for small q, and keep the dominate contribution.

$$\begin{aligned} |\eta(\tau)| &= |q|^{1/24}(1 - q + O(q^2) + ...) \\ |\eta(\tau)|^{-2} &= |q|^{-1/24}(1 + 2q + O(q^2) + ...) \end{aligned} \tag{7.54}$$

Ends up being an expression of the modes in terms of the mass. The expansion would look good if the tachyon term was not included.

$$|q|^{-1/12} = \mathrm{e}^{-\pi\tau_2/6}$$

For D=26:

$$|q|^{-(D-2)/12} = \mathrm{e}^{-4\pi\tau_2}$$

$$\Lambda \sim \int \frac{d\tau_2}{\tau_2} \tau_2^{-D/2} \mathrm{e}^{4\pi\tau_2} \tag{7.55}$$

We can relate this to equation (7.48) and find the mass of the particle.

$$\begin{aligned} l = 2\pi\alpha'\tau_2 : \qquad 4\pi\tau_2 &= -m^2 l/2 \\ &= -m^2\pi\alpha'\tau_2 \\ m^2 &= -\frac{4}{\alpha'} \qquad \text{"TACHYON!"} \end{aligned} \tag{7.56}$$

If we can find a theory that doesn't include the tachyon, then we are in business. If there is no tachyon we find that the cosmological constant will be finite. If superstring theory is introduced, the cosmological constant will vanish, and not match experimental results.

Thermodynamics

In this section we look at various thermodynamic properties of the partition function.

$$\tau_1 = 0 \qquad \tau = i\tau_2 \qquad q = e^{-2\pi i\tau_2}$$

In the limits above what can we find out about the partition function?

$$\begin{aligned} Z(\tau_2) &= |q|^{-D/12}\mathrm{Tr}\left(q^{L_0}\bar{q}^{\tilde{L}_0}\right) \\ &= \mathrm{e}^{-\pi \mathrm{D}\tau_2/6}\mathrm{Tr}\left(\mathrm{e}^{-2\pi\tau_2 \mathrm{H}}\right) : \qquad \mathrm{H} = \mathrm{L}_0 + \tilde{\mathrm{L}}_0 \\ &= \mathrm{e}^{-\pi \mathrm{D}\tau_2/6}\sum_{\mathrm{n}} \mathrm{e}^{-2\pi\tau_2 \mathrm{E_n}} \ : \qquad \frac{1}{\mathrm{T}} = 2\pi\tau_2 \end{aligned} \tag{7.57}$$

High-T limit: $\tau_2 \to 0 \Rightarrow q \to 1$

In this limit Z is modular invariant.

Small-T limit: $\tau_2 \to \infty \Rightarrow q \to q' = \mathrm{e}^{-2\pi/\tau_2} \to 0$

$$Z(\tau_2) = \sum_n \mathrm{e}^{-\mathrm{E_n}/\mathrm{T}} \simeq \mathrm{e}^{-\mathrm{E}_0/\mathrm{T}} + \sum_{\mathrm{n}\neq 0} \mathrm{e}^{-\mathrm{E_n}/\mathrm{T}} + ... \tag{7.58}$$

Only the $E_0 = 0$ will survive in the sum.

$$Z\left(\frac{1}{\tau_2}\right) = \mathrm{e}^{-\frac{\pi \mathrm{D}}{6\tau_2}}$$

$$\tau_2 \to 0 \ : \qquad Z(\tau_2) = Z\left(\frac{1}{\tau_2}\right) = \mathrm{e}^{-\frac{\pi \mathrm{D}}{6\tau_2}} \tag{7.59}$$

Entropy:

$$Z = \sum_E \rho(E)\mathrm{e}^{-\mathrm{E}/\mathrm{T}} \qquad \text{saddle point approximation}$$

S=lnρ

$$Z = \sum \mathrm{e}^{\mathrm{S}-\mathrm{E}/\mathrm{T}} \qquad \mathrm{d}(\mathrm{S} - \mathrm{E}/\mathrm{T}) = 0 \Rightarrow \mathrm{dS} = \frac{\mathrm{dE}}{\mathrm{T}} \tag{7.60}$$

Free energy:

$$F = -T \ln Z = -T\left(S - \frac{E}{T}\right) = E - TS$$

$$S = -\frac{\partial F}{\partial T}$$

$$Z(\tau_2) = \mathrm{e}^{-\pi \mathrm{D} \tau_2/6} = \mathrm{e}^{-\mathrm{D}/12\mathrm{T}} \tag{7.61}$$

$$\begin{aligned} Z(\tau)\mathrm{e}^{-\mathrm{D}/12\mathrm{T}} &= Z\left(\frac{(2\pi)^2}{T}\right)\mathrm{e}^{-\pi^2 \mathrm{DT}/12} \\ Z(\tau) &= \mathrm{e}^{\mathrm{D}/12\mathrm{T}}\mathrm{e}^{-\pi^2 \mathrm{DT}/12}\mathrm{Z}\left(\frac{(2\pi)^2}{\mathrm{T}}\right) \end{aligned} \tag{7.62}$$

Z is a slowly varying function in the saddle point approximation. When we go from statistical mechanics to thermodynamics $Z\left(\frac{1}{T}\right)$ is ignored.

$$\ln Z(T) = \frac{D}{12T} - \frac{\pi^2 DT}{12}$$

$$\begin{aligned} F &= -\frac{D}{12} + \frac{\pi^2 DT^2}{12} \\ &= \frac{D}{12}(\pi^2 T^2 - 1) \end{aligned} \tag{7.63}$$

$$S = -\frac{\partial F}{\partial T} = \frac{D\pi^2 T}{6}$$

$$E = F + TS = \frac{D}{12}(\pi^2 T^2 + 1) \qquad S \leq \pi E$$

as T$\rightarrow \infty$ $\quad E \sim T^2$ "Casimir energy in 2-D", where E for blackbody in 4-D is $E \sim T^4$.

$$\begin{aligned} \frac{D}{12}\pi^2 T^2 &= E - \frac{D}{12} \\ S^2 &= \left(\frac{D\pi^2}{6}\right)^2 T^2 \\ &= \frac{D^2}{26}\pi^4 \frac{12}{\pi^2 D}\left(E - \frac{D}{12}\right) \\ &= 4\frac{D\pi^2}{12}\left(E - \frac{D}{12}\right) \\ S &= 2\pi\sqrt{\frac{D}{12}\left(E - \frac{D}{12}\right)} \qquad \text{"Cardy"} \end{aligned} \tag{7.64}$$

or $S \leq \pi E$ which is the same as the black hole entropy, where Bekenstein equation is $S \leq S_B \sim E$. If equation (7.64) is generalized to 4-D, it gives the Friedman-Robertson-Walker universe equation for the Hubble constant.

From chapter2

$$\begin{aligned} T(z)T(0) &= \frac{D}{2z^4} + \frac{2}{z^2}T(z) + \frac{1}{z}\partial T(z) + ... \\ T_{bc}(z)T_{bc}(0) &= -\frac{26}{2z^4} + ... \end{aligned}$$

the term D/$2z^4$ is an anomaly breaking comformal symmetry however when combining with the second equation this term is canceled with -26/$2z^4$ for D = 26. From the action

$$S = \frac{1}{2\pi}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \bar{\partial}_b X_\mu \tag{7.65}$$

where γ_{ab} is a metric on the world sheet and γ is a determinent of Laplace operator on the world sheet or

$$f[\gamma] = \int \mathcal{D}X e^{-S[X]}, \text{ and} \tag{7.66}$$

$$\frac{1}{\det\nabla^2} \sim e^{-X\nabla^2 X}, \tag{7.67}$$

which depends on γ. From a gauge invariant equation

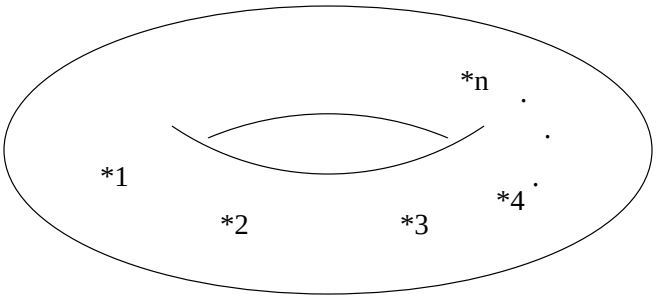
$$D = (i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi = 0 \tag{7.68}$$

$$\det(D) = \int [\mathcal{D}\psi] e^{-\int \bar{\psi}(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi} \tag{7.69}$$

If changing gauge $A_\mu \longrightarrow A_\mu + \partial_\mu\omega$, the determinent is not invariant.

Amplitude on a torus

From $V_i =: e^{ik_i\cdot X}(z_i) \times (c.c.) :$, the amplitude can be writen as



$$A(z_1, z_2, ..., z_n) = \text{Tr } V_1 \text{ .... } V_n q^{(L_0 - D/24)} \times (c.c.) \tag{7.70}$$

where

$$\begin{aligned} X_L &= x + \frac{\alpha'}{2}pz + i\sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}\alpha_n e^{-inz}, \text{ and} \\ [x, p] &= i. \end{aligned}$$

To simplify let $\alpha' = \frac{1}{2}$ and consider first only $e^{ik_i\cdot x/2}e^{ik_i\cdot pz_i/4}$ factors in the amplitude,

$$\int \frac{d^Dp}{(2\pi)^D} < p|q^{p^2/2}e^{ik_1\cdot x/2}e^{ik_1\cdot pz_1/4} \ ... \ e^{ik_m\cdot x/2}e^{ik_m\cdot pz_m/4}|p >$$

where $e^{ik_m\cdot x/2}|p>$ is shifted to $|p+k_m>$ also $e^{ik_{m-1}\cdot x/2}|p+k_m> \longrightarrow |p+k_m+k_{m-1}>$ and so on, then

$$< p|p+k_m+k_{m-1}+...+k_1 >= \delta^D(k_1+...+k_m) \tag{7.71}$$

and use $q = e^{i2\pi\tau}$, $I = \sum_E |E>< E|$ or Tr $A = \sum_E < E|A|E >$.

$\int \frac{d^Dp}{(2\pi)^D} < p|q^{p^2/2}e^{ik_1\cdot x/2}e^{ik_1\cdot pz_1/4} \ ... \ e^{ik_m\cdot x/2}e^{ik_m\cdot pz_m/4} \times (c.c.)|p >$

$$\begin{aligned} &\sim \int \frac{d^Dp}{(2\pi)^D} e^{i\pi\tau p^2} e^{i(k_1z_1+...+k_mz_m)\cdot p}\delta^D(k_1+...+k_m)\times(c.c.) \\ &\sim \int \frac{d^Dp}{(2\pi)^D} e^{-\pi\tau_2 p^2} e^{i\mathrm{Im}(k_1z_1+...+k_mz_m)\cdot p}\delta^D(k_1+...+k_m)\times(c.c.) \end{aligned}$$

Integrate over p by completing the square and shifting $p \longrightarrow p - \frac{1}{\pi\tau_2}\mathrm{Im}(k_1z_1+...+k_mz_m)$

$$\sim \frac{1}{(\pi\tau_2)^D} e^{\frac{1}{\pi\tau_2}(\mathrm{Im}(k_1z_1+...+k_mz_m))^2}\delta^D(k_1+...+k_m)\times(c.c.).$$

Use the conservation of momentum and combine the complex conjugate terms of x and p, then

$$\sim \prod_{i<j} e^{\frac{k_i\cdot k_j}{\pi\tau_2}(z_i-z_j)^2} \times \prod_{i<j} e^{-\frac{i}{4}k_i\cdot k_j(z_i-z_j)}\delta^D(k_1+...+k_m)\times(c.c.)$$

or $\int \frac{d^Dp}{(2\pi)^D} < p|q^{p^2/2}e^{ik_1\cdot x/2}e^{ik_1\cdot pz_1/4} \ ... \ e^{ik_m\cdot x/2}e^{ik_m\cdot pz_m/4} \times (c.c.)|p >$

$$= \prod_{i<j} |e^{-(i/4)k_i\cdot k_j(z_i-z_j)}|^2 \ e^{(k_i\cdot k_j/\pi\tau_2)\mathrm{Im}(z_i-z_j)^2}\delta^D(k_1+...+k_m).$$

Next consider α_n terms by letting

$$T_n^\mu = \mathrm{Tr} \prod_{i=1}^{N} e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz}}e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz}}q^{\alpha_{-n}\alpha_n} \quad , \quad n>0. \tag{7.72}$$

Let $\kappa \in \mathbb{C}$,

$$|\kappa >= e^{\frac{1}{n}\kappa\cdot\alpha_{-n}}|0>, \tag{7.73}$$

called $|\kappa>$ coherent states and the identity from $|\kappa>$ expansion is

$$I = \frac{1}{2\pi}\int d^2\kappa e^{-|\kappa|^2/n}|\kappa >< \kappa| \tag{7.74}$$

From $N = \alpha_{-n}\alpha_n$ and $N|\ell>=\ell|\ell>$, $\ell \in \mathbb{N}$, $|\ell>$ is defined as

$$\begin{aligned} |\ell> &= \frac{1}{\sqrt{\ell!}}\frac{(\alpha_{-n})^\ell)}{\sqrt{n}}|0>, \text{ then} \\ <\ell|\ell> &= \frac{1}{\ell!n^\ell} < 0|\alpha_n^\ell\alpha_{-n}^\ell|0>= 1 \\ <\ell|\kappa> &= \frac{1}{\ell!} < \ell|(\frac{\kappa}{n})^\ell\alpha_{-n}^\ell|0> \end{aligned} \tag{7.75}$$

$$
\begin{aligned}
&= (\frac{\kappa}{n})^\ell \frac{1}{\sqrt{\ell!}} \\
< \ell|I|\ell' > &= \frac{1}{n\pi}\frac{1}{\sqrt{(\ell!)^2}}\int d^2\kappa\ e^{-|\kappa|^2/n}(\frac{\bar\kappa}{n})^\ell(\frac{\kappa}{n})^{\ell'} \\
\ell \neq \ell',\ < \ell|I|\ell' > &= 0,\ \text{use polar coordinates to integrate} \\
\ell = \ell',\ < \ell|I|\ell > &= \frac{1}{n\pi\alpha'}\int d^2k e^{-|\kappa|^2/n}(\frac{|\kappa|}{n})^{2\ell} \\
&= 1,\ \text{ i.e.} \\
< \ell|I|\ell' > &= \delta_{\ell\ell'}
\end{aligned}
\tag{7.76}
$$

then substitude the identity expansion into T_n^μ

$$
T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n} < \kappa|\prod_{i=1}^n e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz_i}} e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz_i}}\ q^{\alpha_{-n}\alpha_n}|\kappa >, \tag{7.77}
$$

where $q^N|\kappa >= |q^n\kappa >$ and $q^{\alpha_{-n}\alpha_n}|\kappa >= |q^n\kappa >= e^{\frac{1}{n}q^n\kappa\alpha_{-n}}|0>$, i.e.

$$
\begin{aligned}
q^N\alpha_{-n}q^{-N} &= q^n q_{-n} \\
(q^N\alpha_{-n}q^{-N})^\ell &= q^N\alpha_{-n}^\ell q^{-N} = q^{n\ell}\alpha_{-n}^\ell \\
q^N e^{\frac{1}{n}\kappa\alpha_{-n}}q^{-N} &= e^{\frac{1}{n}q^n\kappa\alpha_{-n}},\ \text{ and use } e^Ae^B = e^Be^Ae^{[A,B]}.
\end{aligned}
$$

The T_n^μ will be

$$
T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n} < \kappa|\prod_{i=1}^m e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz_i}} e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz}} e^{\frac{1}{n}q^n\kappa\alpha_{-n}}\ |0> \tag{7.78}
$$

To calculate the expectation value, α_n have to be moved to the right and α_{-n} to the left. First let move the $\alpha_{\pm n}$ terms inside $\prod_{i=1}^m e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz_i}} e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz}}$

$\prod_{i=1}^{m-1} e^{-\frac{1}{n}k_{i+1}\cdot\alpha_n e^{-niz_i}} e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz}}$

$$
= \prod_{i<j} e^{-\frac{1}{n}k_i\cdot k_j e^{-ni(z_i-z_j)}}\ (\prod_{i=1}^m e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz_i}})\ (\prod_{i=1}^m e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz}}).
$$

Next move the term $e^{-\frac{1}{n}\kappa\alpha_n}$ from $<\kappa|$ to the right and $e^{\frac{1}{n}q^n\kappa\alpha_{-n}}$ to the left will obtain

$$
e^{-\frac{1}{n}|\kappa|^2q^n}\prod_i e^{\frac{1}{n}k_i\cdot\bar\kappa e^{niz_i}}\prod_i e^{-\frac{1}{n}k_i\cdot\kappa q^n e^{-niz_i}}
$$

.
Then T_n^μ is

$$
T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n}\ e^{-\frac{1}{n}|\kappa|^2q^n}(\prod_i e^{\frac{1}{n}k_i\cdot\bar\kappa e^{niz_i}})(\prod_i e^{-\frac{1}{n}k_i\cdot\kappa q^n e^{-niz_i}})\ \prod_{i<j} e^{-\frac{1}{n}k_i\cdot k_j e^{-ni(z_i-z_j)}} \tag{7.79}
$$

and use $\frac{1}{\pi}\int d^2\kappa e^{-c|\kappa|^2}\ e^{a\kappa+b\bar\kappa} = \frac{1}{c}e^{-ab/c}$, then

$$
T_n^\mu \sim \frac{1}{1-q^n}e^{(\sum k_i e^{-niz_i})(\sum k_i e^{niz_i})/n(1-q^n)}\ \prod_{i<j} e^{-\frac{1}{n}k_i\cdot k_j e^{-ni(z_i-z_j)}} \tag{7.80}
$$

From

$$\begin{aligned} A &\sim \text{Tr}\ (V_1...V_m\ q^{L_0-D/24}\bar{q}^{\tilde{L}_0-D/24}) \\ &\sim T_p\prod_{n,\mu}T_n^\mu \end{aligned}$$

where

$$T_p \sim (\frac{1}{2\pi\sqrt{\tau_2/2}})^D\delta^D(k_1+...+k_m)\prod_{i<j}e^{-\frac{k_i\cdot k_j}{2\pi\tau_2}(\text{Im}(z_i-z_j))^2}|e^{-\frac{1}{4}k_i\cdot k_j(z_i-z_j)}|^2 \tag{7.81}$$

$$T_n^\mu \sim (1-q^n)^{-1}\prod_{i<j}e^{-\frac{1}{n}k_ik_je^{ni(z_i-z_j)}}exp\{-\frac{(\sum_i k_ie^{niz_i})(\sum_j k_je^{-niz_j})q^n}{n(1-q^n)}\} \tag{7.82}$$

$$\prod_{\mu,n}T_n^\mu = (1-q^n)^{-D}\prod_{i<j}exp\{-k_i\cdot k_j\frac{e^{ni(z_i-z_j)}+q^ne^{-ni(z_i-z_j)}-2q^n}{n(1-q^n)}\}$$

using

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{x^n}{1-y^n}=\sum_{n=0}^{\infty}\ln(1-xy^n)\ ,\ \text{then}$$

$$\prod_{\mu,n}T_n^\mu = (1-q^n)^{-D}\prod_{i<j}(1-e^{i(z_i-z_j)})^{k_i\cdot k_j/2}\left[\frac{(1-q^ne^{i(z_i-z_j)})(1-q^ne^{-i(z_i-z_j)})}{(1-q^n)^2}\right]^{k_i\cdot k_j/2},$$

$$\begin{aligned} \text{then } T_p\prod_{n,\mu}T_n^\mu &\sim [\prod_{i<j}e^{-\frac{k_i\cdot k_j}{2\pi\tau_2}(\text{Im}(z_i-z_j))^2}|e^{-\frac{1}{4}k_i\cdot k_j(z_i-z_j)}|^2]\ [\prod_{i<j}(1-e^{i(z_i-z_j)})^{k_i\cdot k_j/2}] \\ &\sim \prod_{i<j}\left[\sin\frac{(z_i-z_j)}{2}\right]^{k_i\cdot k_j/2} \end{aligned}$$

However from

$$\vartheta_1(\nu|\tau) = 2e^{i\pi\tau/2}\sin(\pi\nu)\prod(1-q^n)^3\frac{(1-zq^n)(1-q^n/n)}{(1-q^n)} \tag{7.83}$$

$$\partial_\nu\vartheta_1(\nu|\tau)|_{\nu=0} = 2e^{\pi i\tau/4}\prod(1-q^n)^3 \tag{7.84}$$

where $z=e^{2\pi i\nu}=e^{i(z_i-z_j)}$ or $\nu=\frac{z_i-z_j}{2\pi}$, then

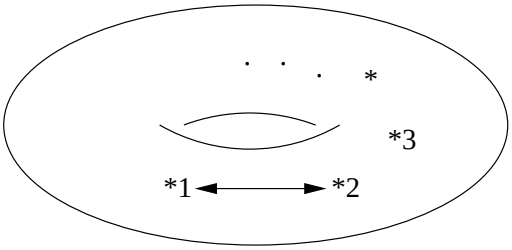
$$\begin{aligned} T_p\prod_{n,\mu}T_n^\mu &\sim \prod_{i<j}\frac{\vartheta_1(\frac{z_i-z_j}{2\pi}|\tau)}{\partial_\nu\vartheta_1(0|\tau)})^{\alpha' k_i\cdot k_j},\ \text{where} \\ \vartheta_1(\nu|\tau+1) &= e^{i\pi/4}\vartheta_1(\nu|\tau) \qquad (7.85) \\ \vartheta_1(\frac{\nu}{\tau}|-\frac{1}{\tau}) &= -i\sqrt{-i\tau}e^{\pi i\nu^2/\tau}\vartheta_1(\nu|\tau). \qquad (7.86) \end{aligned}$$

When $z_i\longrightarrow z_j$ or $z_i-z_j\longrightarrow 0$, then

$$\vartheta_1(\frac{z_i-z_j}{2\pi}|\tau) \sim \sin\pi\nu\sim z_i-z_j$$

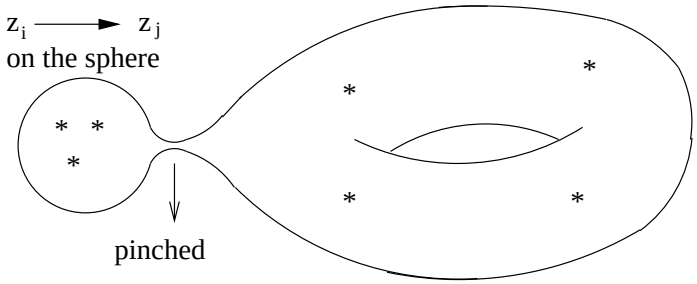
That is when two punctures on the torus surface get closer, the amplitude will be proportional to

$$\text{Amp} \sim \prod_{z_i\to z_j}(z_i-z_j)^{\alpha' k_i\cdot k_j}, \tag{7.87}$$

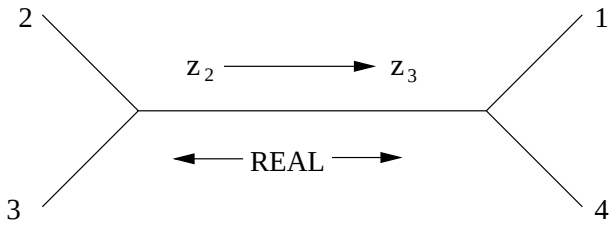


which is the same result as on the sphere because the topological surfaces around the neighborhood of *1 and *2 is looked almost the same from their points of view.

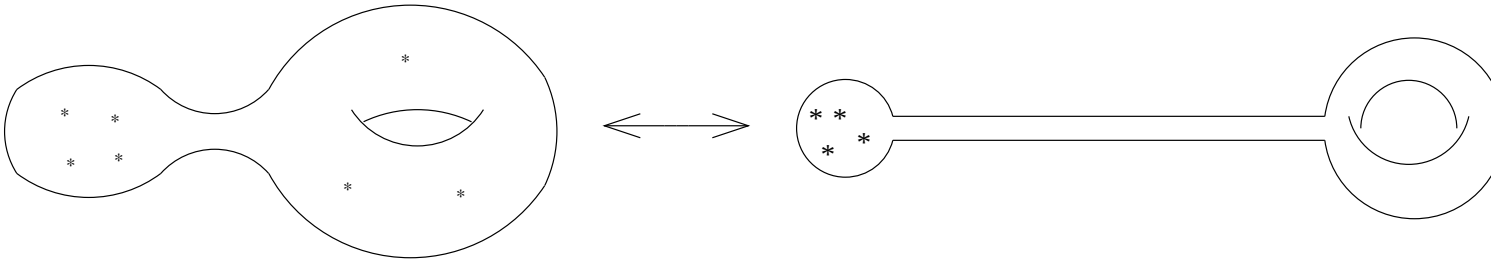
In particle physics, there are two kinds of divergence. One is ultraviolent divergence which occurs when two particles come closer in a small distance. The other is infrared divergence which occurs when two particles go far apart. In string theory there are no ultraviolet divergence because of the convergence of amplitude on a sphere. For $z_i \longrightarrow z_j$ on a torus, the torus can be equivalently topologically considered as a torus connected with a sphere containing all the $z_i \longrightarrow z_j$ points, making the amplitude converge.



On the sphere the amplitude is $A =< v_1(\infty)V_2(1)V_3(z)V_4(0) >$ or in field theory, the diagram for two point particles coming together is

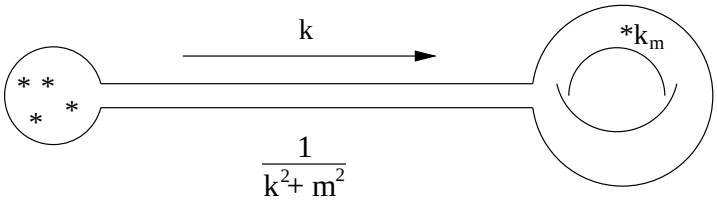


A torus can be also topologically transformed to

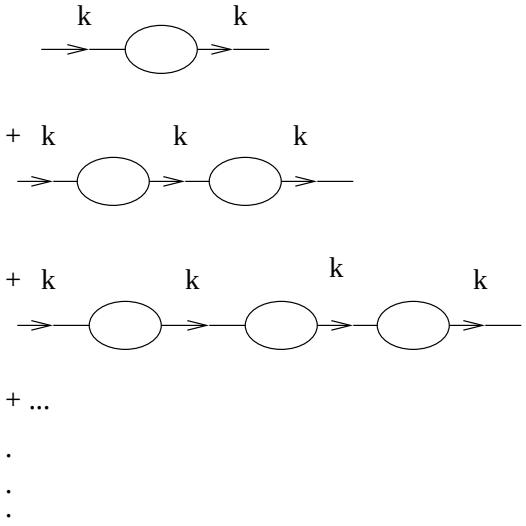


The long connection tube between the sphere and the torus can be considered as a long physical state happening after two incoming legs merge together in field theory. There are some other special cases for torus.

Special case I
All but one vertex comes together with a propagator $\frac{1}{k^2+m^2}$



From conservation of momentum $k = k_m$, if let $k^2 = -m^2$, then $\frac{1}{k^2+m^2}$ diverges. To correct the problem let $k^2 \neq m^2$ by adding new torus between the previous sphere and torus, causing the amplitde shifting by small amount δ for one loop, $\frac{\delta^2}{k^2+m^2}$ for two loops, $\frac{\delta^3}{(k^2+m^2)^2}$ for three loops and so on, then



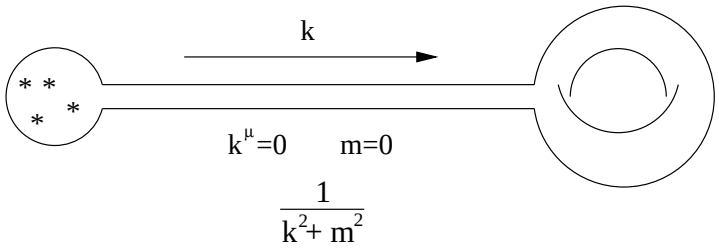
This can be summed as a geometric series.

$$\delta + \frac{\delta^2}{k^2+m^2} + \frac{\delta^3}{(k^2+m^2)^2} + ... \quad = \quad \delta \frac{1}{1-\frac{\delta}{k^2+m^2}}$$

$$= \quad \delta \frac{k^2 + m^2}{k^2 + m^2 - \delta} \tag{7.88}$$

This means that the poles have shifted by an amount δ, and $m^2 \longrightarrow m^2 - \delta$ is more appropriated to use for expanding the amplitude than just m^2.

Special case II
All vertices come together



From the conservation of momentum, $k^\mu = 0$, with no mass $m = 0$, the propagator is $\frac{1}{k^2+m^2}$ divergent. This can be corrected by inserting an identity operator, $\sum_E |E >< E|$, in $\vec{k}$. One term of $\sum |E >< E|$ will give a massless mode, i.e ∂X^μ term or

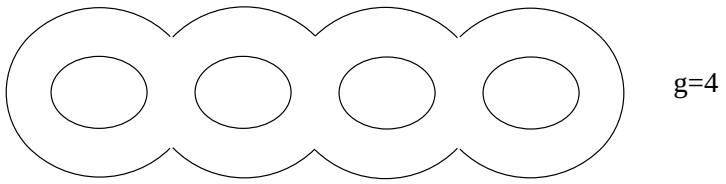
$$\int c\bar{\partial} X_\mu \partial X^\mu d^2 z,$$

where c is a scalar function. This term is added to correct the action impling that the flat space, Lorentz metric, is not a good choice for the expansion.

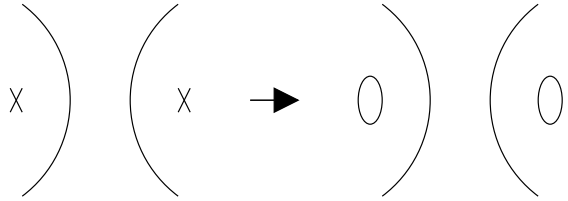
$$\begin{aligned} S \quad &\sim \quad \int \bar{\partial} X_\mu \partial X^\mu - \int c\bar{\partial} X_\mu \partial X^\mu \\ &= \quad \int (1-c)\bar{\partial} X_\mu \partial X^\mu, \text{ where} \\ \eta_{\mu\nu}\bar{\partial} X^\mu \partial X^\nu \quad &\longrightarrow \quad G_{\mu\nu}\bar{\partial} X_\mu \partial X^\mu \end{aligned}$$

where $G_{\mu\nu} = (1-c)\eta_{\mu\nu}$. $\eta_{\mu\nu}$ is not a good for zero approximation which is shown that the curve space must be taken to the consideration at the beginning when writing down the action.

Higher genus surfaces



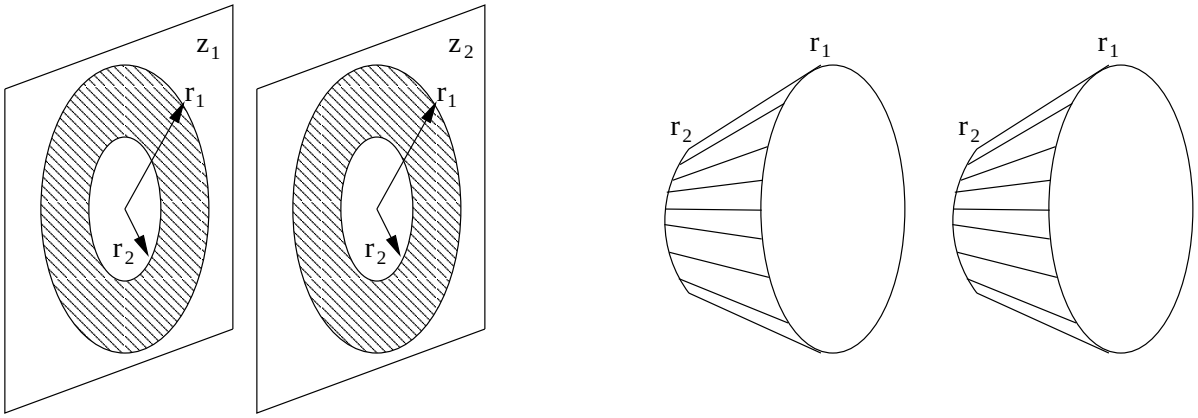
The process of gluing amplitudes together to make higher order corrections is known as plumbing. For example, if we have two Riemann surfaces which need to be glued we look near the vertex operator of each sheet and stretch the puncture into a small hole.

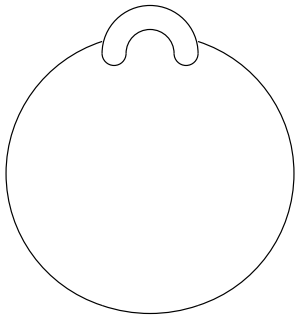


We form a single surface by identifying pairs of points in each patch such that

$$z_1 z_2 = R_1 R_2 \mathrm{e}^{\mathrm{i}\theta} = \mathrm{q}\,.$$

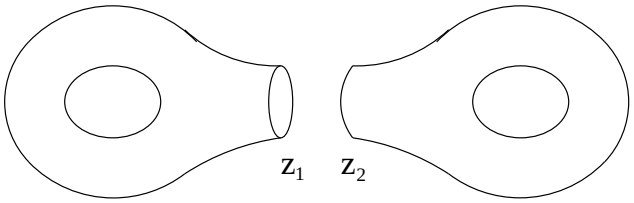
The next step is to stretch each hole into a small cone, then connect them together.





z_1 and z_2 on the sphere.

z_1 and z_2 on torus gives surface with g=2

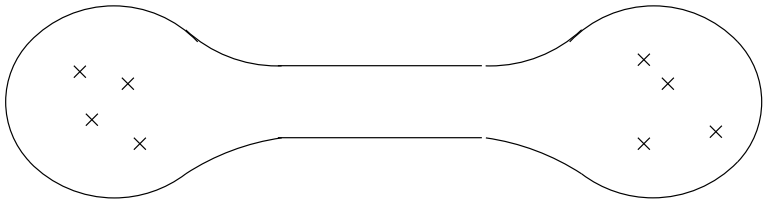


A sphere with a small handle is conformally equivalent to a very long intermediate leg.

Important limit: $q \to 0$, if R_1 and let $R_2 \to 0$, then the neighborhood around the puncture is equivalent to a cylinder or a handle.

Amplitudes:

a) Trivial case: Two spheres connected by a cylinder.



$$\begin{aligned} A_1 &= \langle 0|V_1...V_m|E\rangle \\ A_2 &= \langle E|V_{m+1}...V_n|0\rangle \end{aligned}$$

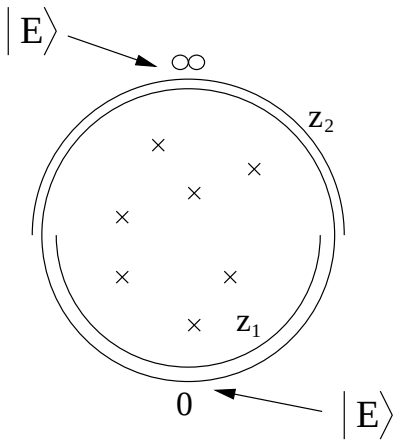
where

$$|E\rangle = V_E|0\rangle \quad \text{and} \quad \mathrm{I} = \sum_{\mathrm{E}} |\mathrm{E}\rangle\langle\mathrm{E}|$$

$$\sum_E A_1 A_2 = \langle 0|V_1...V_n|0\rangle \tag{7.89}$$

b) Torus:

$q = 1 \qquad z_1 z_2 = 1$



$$\begin{aligned} \langle 0|V_E(\infty)V_1V_2...V_mV_E(0)|0\rangle &= \sum\langle E|V_1V_2...V_m|E\rangle \\ &= Tr(V_1V_2...V_m) \end{aligned} \tag{7.90}$$

and use a transformation $z_2 \to qz_1$ to calculate the amplitude, where

$$\left(\frac{\partial z_2'}{\partial z_2}\right)^h \left(\frac{\partial \bar{z}_2'}{\partial \bar{z}_2}\right)^{\tilde{h}} = q^h \bar{q}^{\tilde{h}}$$

$$\begin{aligned} L_0|E\rangle &= h|E\rangle \\ \tilde{L}_0|E\rangle &= \tilde{h}|E\rangle \end{aligned}$$

$$\mathrm{Tr}\langle V_1...V_m q^h \bar{q}^{\tilde{h}}\rangle \tag{7.91}$$