Chapter 8

Toroidal Compactification and T-Duality

Following *"String Theory"* by J. Polchinski, Vol.II.
Notes written by students (work still in progress).
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8.1 Compactification

Review of Electromagnetism

$$\phi, A \to A_\mu \text{ - potentials} \tag{8.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ Field-strength Tensor} \tag{8.2}$$

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu) \text{ Action} \tag{8.3}$$

$$\delta S = 0 \text{ -Variation of the Action} \tag{8.4}$$

The variation of the action yields Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = J^\nu \text{ with } J^\nu = (\rho, \vec{J}) \tag{8.5}$$

$$Q = \int d^3x\rho \text{ -Electric charge} \tag{8.6}$$

Conservation Laws:

$$\partial_\mu J^\mu = 0 \tag{8.7}$$

$$\frac{dQ}{dt} = 0 \tag{8.8}$$

Gauge Symmetry:

$$A_\mu \longrightarrow A_\mu - \partial_\mu\lambda, \quad \text{Conservation of current comes from gauge symmetry} \tag{8.9}$$

Matter: Scalar Field ϕ

$$p_\mu = (E, \vec{p}) \qquad p^\mu p_\mu = -E^2 + p^2 = -m^2 \tag{8.10}$$

$$p_\mu \to i\partial_\mu \quad \text{-to quantize, momentum goes to an operator} \tag{8.11}$$

$$\partial_\mu \partial^\mu \Phi = -m^2 \Phi \quad \text{-(Klein-Gordon equation)} \tag{8.12}$$

$$i\frac{\partial}{\partial t}\Phi = -\frac{1}{2m}\nabla^2 \Phi \quad \text{-(Schroedinger equation)} \tag{8.13}$$

Action Principle:

$$S = \frac{1}{2}\int d^4x(\partial_\mu \Phi^* \partial^\mu \Phi + m^2|\Phi|^2) \text{ -This action cannot possibly describe a charged field} \tag{8.14}$$

Conserved current:

$$j_\mu \sim \Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^* \tag{8.15}$$

$$\partial^\mu j_\mu \sim 0 \tag{8.16}$$

$$p_\mu \to p_\mu - qA_\mu \quad \text{(momentum for particle in an external field)} \tag{8.17}$$

$$\partial_\mu \to \partial_\mu + iqA_\mu = D_\mu \quad \text{(momentum going to an operator)} \tag{8.18}$$

$$S_{\text{matter}} = \frac{1}{2}\int d^4x(D_\mu \Phi^* D^\mu \Phi + m^2|\Phi|^2) \text{ New action, invariant under gauge transformation} \tag{8.19}$$

Term that's linear in A_μ:

$$-\frac{iq}{2}A^\mu \Phi^* \partial_\mu \Phi + (\partial_\mu \Phi^*)\frac{iq}{2}A^\mu \Phi \tag{8.20}$$

$$= A^\mu(-\frac{iq}{2}\Phi^* \partial_\mu \Phi + \frac{iq}{2}\Phi \partial_\mu \Phi^*) = A^\mu j_\mu \tag{8.21}$$

$$S = S_{\text{fields}} + S_{\text{matter}} \quad \text{-total action} \tag{8.22}$$

$$S_{\text{fields}} = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{8.23}$$

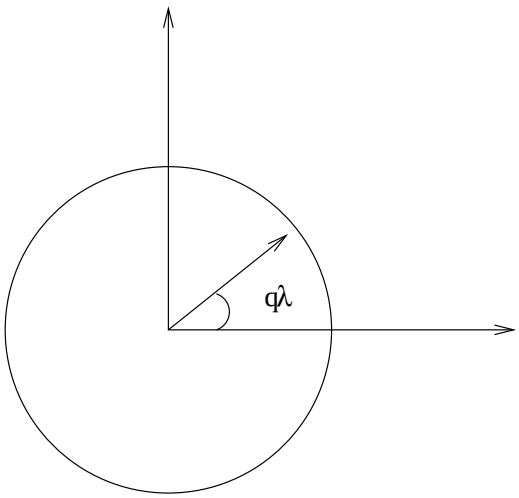
Is Gauge Symmetry in S_{matter}?

$$\Phi \to e^{iq\lambda}\Phi \text{ Gauge transformation} \tag{8.24}$$

$$\begin{aligned} D_\mu \Phi &\to (\partial_\mu + iqA_\mu - iq\partial_\mu \lambda)e^{iq\lambda}\Phi \\ &= iq\partial_\mu \lambda(e^{iq\lambda}\Phi) + e^{iq\lambda}\partial_\mu \Phi + (iqA_\mu - iq\partial_\mu \lambda)e^{iq\lambda}\Phi \\ &= e^{iq\lambda}(\partial_\mu + iqA_\mu)\Phi = e^{iq\lambda}D_\mu \Phi \to \end{aligned} \tag{8.25}$$

$$S_{\text{matter}} \text{ is invariant under gauge transformation} \tag{8.26}$$

$$|\phi| \to |\phi| \tag{8.27}$$



-Kaluza-Klein proposed that this circle is real instead of imaginary

5 Dimensions

$$x^m = (t, x, y, z, u) = (x^\mu, u) \tag{8.28}$$

Line element

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + 2G_{\mu u}dx^\mu du + G_{uu}du^2 \tag{8.29}$$

Assumption:

$$G_{MN}(x^\mu, u), \quad \partial_u G_{MN} = 0 \quad \text{-invariant with respect to u (Gauge Invariance)} \tag{8.30}$$

$$A_\mu = \frac{G_{\mu u}}{G_{uu}} \quad \text{-The vector potential is just part of the metric} \tag{8.31}$$

$$g_{\mu\nu} = G_{\mu\nu} - G_{uu}A_\mu A_\nu \quad \text{-Reparametrizing} \tag{8.32}$$

$$\begin{aligned} \therefore ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + G_{uu}A_\mu A_\nu dx^\mu dx^\nu + 2A_\mu G_{uu}dx^\mu du + G_{uu}du^2 \\ &= g_{\mu\nu}dx^\mu dx^\nu + G_{uu}(A_\mu A_\nu dx^\mu dx^\nu + 2A_\mu dx^\mu du + du^2) \\ &= g_{\mu\nu}dx^\mu dx^\nu + G_{uu}(du + A_\mu dx^\mu)^2 \end{aligned} \tag{8.33}$$

$$\begin{aligned} \text{Let: } u &\longrightarrow u + \lambda(x^\mu) \\ A_\mu &\longrightarrow A_\mu - \partial_\mu\lambda \end{aligned} \tag{8.34}$$

$$du + A_\mu dx^\mu \to du + d\lambda(x^\mu) + A_\mu dx^\mu - \partial_\mu\lambda dx^\mu \to du + A_\mu dx^\mu \quad (d\lambda(x^\mu) = \partial_\mu\lambda dx^\mu) \tag{8.35}$$

The line element is invariant under translations of u AND Gauge Invariance

Gauge invariance is required for Kaluza-Klein Theory

(8.36)

From Quantum Mechanics:

$$p_\mu \text{ conjugate to u} \tag{8.37}$$

$$p_\mu \to -i\partial_\mu \tag{8.38}$$

$$e^{iap_u} f(u) = f(u+a) \tag{8.39}$$

For the one curled dimension:

$$f(u+2\pi R) = f(u) \quad \therefore e^{i2\pi R p_u} = 1 \tag{8.40}$$

$$Rp_u = n \in \mathbb{Z} \tag{8.41}$$

$$p_u = \frac{n}{R}, \quad \text{momentum is quantized} \tag{8.42}$$

$$\text{Eigenfunctions: } e^{i\frac{n}{R}u} \tag{8.43}$$

Massless Scalar Field Φ :

$$\Phi(x^M), \quad m = 0 \tag{8.44}$$

$$\partial_M \partial^M \Phi = 0, \text{ Klein-Gordon equation in five dimensions} \tag{8.45}$$

$$p^M p_M = -m^2 = 0, \text{ only five dimensional beings see zero mass} \tag{8.46}$$

Expand Φ in complete set of eigenfunctions:

$$\Phi(x^M) = \sum_{n=-\infty}^{\infty} a_n(x^\mu) e^{i\frac{n}{R}u}, \quad a_n(x^\mu) \text{ are Kaluza-Klein modes} \tag{8.47}$$

Looking at one mode:

$$\partial_\mu \partial^\mu a_n - \frac{n^2}{R^2} a_n = 0 \tag{8.48}$$

$$m = \frac{|n|}{R}, \text{ n=0 is massless} \tag{8.49}$$

At early universe, energy $<< \frac{1}{R}$

Charges

Isolate effects of A_μ:

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.50}$$

$$\text{Set } G_{uu} = 1 \tag{8.51}$$

$$ds^2 = dx^\mu dx_\mu + (du + A_\mu dx^\mu)^2 \text{ -line element from before} \tag{8.52}$$

$$\text{Let } u \to u + \lambda(x^\mu) \tag{8.53}$$

$$a_n \to a_n e^{i\frac{n}{R}\lambda} \quad from \quad \Phi(x^\mu) \to \sum_{n=-\infty}^{\infty} a_n(x^\mu) e^{i\frac{n}{R}u} e^{i\frac{n}{R}\lambda} \tag{8.54}$$

$$\Phi \to e^{iq\lambda}\Phi \overset{implies}{\longrightarrow} \frac{n}{R} = q = \overset{+}{-} m \tag{8.55}$$

$$D^M D_M(a_n e^{i\frac{n}{R}u}) = 0 \quad \text{-Wave equation} \tag{8.56}$$

$$\text{Becomes: } (\partial_\mu + i\frac{n}{R}A_\mu)^2 a_n = \frac{n^2}{R^2}a_n \tag{8.57}$$

From Before:

$$D^\mu D_\mu \Phi = m^2 \Phi \tag{8.58}$$

$$D_\mu = \partial_\mu + iqA_\mu \tag{8.59}$$

$$\int d^5x\sqrt{-G}\, R^{(5)} \quad \text{Action for 5D gravity} \tag{8.60}$$

$$R^{(5)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.61}$$

$$\det G = -1 \tag{8.62}$$

If you add back the curvature:

$$ds^2 = g_{\mu\nu}dx^\mu dx_\mu + (du + A_\mu dx^\mu)^2 \tag{8.63}$$

$$R^{(5)} = R^{(4)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.64}$$

$$\det \text{ G} = \det \text{ g} \tag{8.65}$$

Closed Strings

$$S = \frac{1}{2\pi\alpha'}\int d^2z(\partial U\bar{\partial}U + \partial X^\mu \bar{\partial}X_\mu) \tag{8.66}$$

$$U = U + 2\pi R \qquad p = \frac{n}{R}, n \in \mathbb{Z} \tag{8.67}$$

Expansion for X^μ(Solution to the wave equation):

$$X^\mu = x^\mu + 2\pi\alpha' p\frac{\tau}{l} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m e^{-2\pi im(\sigma+\tau)/l} + \tilde{\alpha}_m e^{-2\pi im(\sigma-\tau)/l}) \tag{8.68}$$

$$\frac{\partial^2 X}{\partial\sigma^2} = \frac{\partial^2 X}{\partial\tau^2} \tag{8.69}$$

$$z = e^{2\pi im(\sigma+\tau)/l} \text{ and } \bar{z} = e^{-2\pi im(\sigma-\tau)/l} \tag{8.70}$$

$$z\bar{z} = e^{4\pi i\tau/l} \tag{8.71}$$

$$X^\mu = x^\mu - \frac{1}{2}i\ \alpha'\ p\ ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m \bar{z}^{-m}) \tag{8.72}$$

$$\partial X^\mu = -\frac{1}{2}i\alpha' p\frac{1}{z} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m z^{-m-1}(\text{ Holomorphic }) \tag{8.73}$$

$$\bar{\partial} X^\mu = -\frac{1}{2}i\alpha' p\frac{1}{\bar{z}} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde{\alpha}_m \bar{z}^{-m-1}(\text{ Antiholomorphic }) \tag{8.74}$$

$$p = \frac{1}{2\pi\alpha'}\oint (dz\partial X - d\bar{z}\bar{\partial}X) \quad \text{-Center of mass momentum} \tag{8.75}$$

Go around string:

$$\oint (dz\partial X + d\bar{z}\bar{\partial}X) = 0 \quad \text{-X is a single valued function} \tag{8.76}$$

Compactify:

$$\oint (dz\partial U + d\bar{z}\bar{\partial}U) \neq 0 \tag{8.77}$$

$$U(\sigma + l) = U(\sigma) + 2\pi R w \tag{8.78}$$

In expansion, X, we are going to add a term linear in σ to satisfy wave equation:

$$2\pi R w\frac{\sigma}{l} \tag{8.79}$$

To write in $z, \bar{z}$:

$$\frac{z}{\bar{z}} = e^{4\pi i\sigma/l} \longrightarrow 2\pi R w\frac{\sigma}{l} \rightarrow -\frac{1}{2}iRwln\Big(\frac{z}{\bar{z}}\Big) \tag{8.80}$$

$$U = u - \frac{i}{2}\alpha'\frac{n}{R}ln|z|^2 - \frac{i}{2}Rwln\left(\frac{z}{\bar{z}}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m \bar{z}^{-m}) \tag{8.81}$$

$$\partial U = -\frac{i}{2}\alpha'\frac{n}{R}z^{-1} - \frac{i}{2}Rwz^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m z^{-m-1} \quad \text{(Holomorphic)} \tag{8.82}$$

$$\bar{\partial} U = -\frac{i}{2}\alpha'\frac{n}{R}\bar{z}^{-1} + \frac{i}{2}Rw\bar{z}^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\tilde{\alpha}_m \bar{z}^{-m-1} \quad \text{(Antiholomorphic)} \tag{8.83}$$

$$\partial U = -\frac{i\alpha'}{2}\left(\frac{n}{R} + \frac{Rw}{\alpha'}\right)z^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m z^{-m-1} \tag{8.84}$$

$$\bar{\partial} U = -\frac{i\alpha'}{2}\left(\frac{n}{R} - \frac{Rw}{\alpha'}\right)\bar{z}^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\tilde{\alpha}_m \bar{z}^{-m-1} \tag{8.85}$$

$$p_L = \frac{n}{R} + \frac{Rw}{\alpha'} \tag{8.86}$$

$$p_R = \frac{n}{R} - \frac{Rw}{\alpha'} \tag{8.87}$$

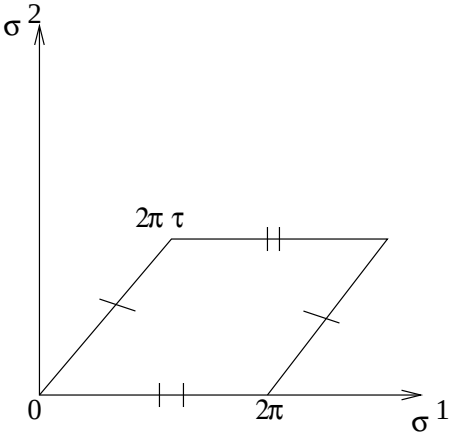
$$p = \frac{1}{2\pi\alpha'}\oint (dz\partial U - d\bar{z}\bar{\partial}U) = \frac{1}{2}(p_L + p_R)$$
$$\oint (dz\partial U + d\bar{z}\bar{\partial}U) = \frac{2\pi\alpha'}{2}(p_L - p_R) = 2\pi R w \tag{8.88}$$

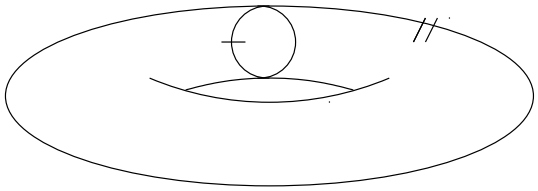
$$L_o = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \tag{8.89}$$

$$\tilde{L}_o = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n \tag{8.90}$$

8.2 Partition Function

$$Z = Tr\left(q^{L_o - 1/24}\bar{q}^{(L_o - 1/24)}\right) \text{ with } q = e^{2\pi i\tau},\ \ \tau = \tau_1 + i\tau_2,\ \ \ l = 2\pi \tag{8.91}$$





$$Z = Tr(q\bar{q})^{\frac{\alpha' p^2}{4} - \frac{1}{24}} \prod_{n=1}^{\infty} \left(\sum_{N_n=0}^{\infty} q^{nN_n} \sum_{\tilde{N}_n=0}^{\infty} \bar{q}^{n\tilde{N}_n} \right)$$
$$= \left[Tr(q\bar{q})^{\frac{\alpha' p^2}{4}}\right] (q\bar{q})^{-1/24} \prod_{n=1}^{\infty} (1-q^n)^{-1}(1-\bar{q}^n)^{-1} = |\eta(\tau)|^{-2}\left[Tr(q\bar{q})^{\frac{\alpha' p^2}{4}}\right] \tag{8.92}$$

$$V\int \frac{dp}{2\pi}(q\bar{q})^{\frac{\alpha' p^2}{2}} = V\int \frac{dp}{2\pi} e^{-\pi\alpha'\tau_2 p^2} = \frac{V}{2\pi}\frac{1}{\sqrt{\alpha'\tau_2}} \tag{8.93}$$

$$|q| = e^{-2\pi\tau_2} \qquad \tau = \tau_1 + i\tau_2 \tag{8.94}$$

$$\therefore Z = |\eta(\tau)|^{-2}\frac{V}{2\pi}\frac{1}{\sqrt{\alpha'\tau_2}} \tag{8.95}$$

The partition function is modular invariant:

$$\tau \to \tau + 1 \tag{8.96}$$

$$\tau \to -1/\tau \tag{8.97}$$

Looking at the wave equation:

$$\partial\bar{\partial}X = 0 \tag{8.98}$$

$$X = x - \frac{i\alpha'}{2} p\, ln|z|^2 + ... \tag{8.99}$$

$$\partial X = -\frac{i\alpha'}{2} p z^{-1} + ... \tag{8.100}$$

$$\bar{\partial}X = -\frac{i\alpha'}{2} p \bar{z}^{-1} + ... \tag{8.101}$$

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'}\int d^2z \partial X \bar{\partial} X = \frac{1}{2\pi\alpha'}\int d^2z \left(\frac{\alpha'}{2}\right)^2 p^2 |z|^{-2} \\ &= p^2 \frac{\alpha'}{8\pi} 8\pi^2 \tau_2 = \alpha' \tau_2 p^2 \pi \qquad \left[\frac{1}{2}\int \frac{d^2z}{|z|^2} = 2\pi 2\pi\tau_2 \text{ - V of torus}\right] \end{aligned} \tag{8.102}$$

In compactified case:

$$Tr(q\bar{q})^{\frac{\alpha' p^2}{4}} = Tr\left(q^{\frac{\alpha' p_L^2}{4}} \bar{q}^{\frac{\alpha' p_R^2}{4}}\right) = \sum_{n,w} q^{\frac{\alpha' p_L^2}{4}} \bar{q}^{\frac{\alpha' p_R^2}{4}} = \sum e^{2\pi i \tau_1 (\frac{\alpha'}{4} p_L^2 - \frac{\alpha'}{4} p_R^2)} e^{-2\pi\tau_2 (\frac{\alpha'}{4} p_L^2 + \frac{\alpha'}{4} p_R^2)} \tag{8.103}$$

$$= \sum e^{2\pi i \tau_1 \frac{\alpha'}{4} 4 \frac{n}{R} \frac{Rw}{\alpha'}} e^{-2\pi\tau_2 \frac{\alpha'}{4} 2\left(\frac{n^2}{R^2} + \left(\frac{Rw}{\alpha'}\right)^2\right)} \tag{8.104}$$

$$= \sum e^{2\pi i \tau_1 nw} e^{-\pi\tau_2 \alpha' \left(\frac{n^2}{R^2} + \left(\frac{Rw}{\alpha'}\right)^2\right)} \tag{8.105}$$

We have to show that this sum is modularly invariant

Poisson Resummation:

$$\sum_n e^{-\pi a n^2 + 2\pi i b n} = \frac{1}{\sqrt{a}} \sum_m e^{-\pi \frac{(m-b)^2}{a}} \tag{8.106}$$

$$\begin{aligned} \therefore Tr(q\bar{q})^{\frac{\alpha' p^2}{4}} &= \frac{R}{\sqrt{\tau_2 \alpha'}} \sum_{m,w} e^{\frac{-\pi R^2}{\alpha' \tau_2}(m - \tau_1 w)^2 - \pi\tau_2 \alpha' \frac{R^2 w^2}{\alpha'^2}} \\ &= \frac{R}{\sqrt{\tau_2 \alpha'}} \sum_{m,w} e^{\frac{-\pi R^2}{\alpha' \tau_2}[(m-\tau_1 w)^2 + \tau_2^2 w^2]} = \frac{R}{\sqrt{\tau_2 \alpha'}} \sum_{m,w} e^{\frac{-\pi R^2}{\alpha' \tau_2}|(m - \tau w)|^2} \end{aligned} \tag{8.107}$$

$$\begin{aligned} |m - \tau w|^2 &= (m - (\tau_1 + i\tau_2)w)(m - (\tau_1 - i\tau_2)w) \\ &= m^2 - mw(\tau_1 + i\tau_2) - mw(\tau_1 - i\tau_2) + (\tau_1 + i\tau_2)(\tau_1 - i\tau_2)w^2 \\ &= m^2 - 2mw\tau_1 + (\tau_1^2 + \tau_2^2)w^2 = (m - \tau_1 w)^2 + \tau_2^2 w^2 \end{aligned} \tag{8.108}$$

Have to show sum is modular invariant:

$$Z = |\eta(\tau)|^{-2} \frac{R}{\sqrt{\tau_2 \alpha'}} \sum_{m,w} e^{\frac{-\pi R^2}{\alpha' \tau_2}|m-\tau w|^2} \qquad \left[e^{-S}\right] \tag{8.109}$$

1)$\tau \to \tau + 1$

$$\begin{aligned} |m-\tau w|^2 &\to |m-\tau w - w|^2 \quad \text{Let } m \to m+w \\ &\to |m-\tau w|^2 \end{aligned} \tag{8.110}$$

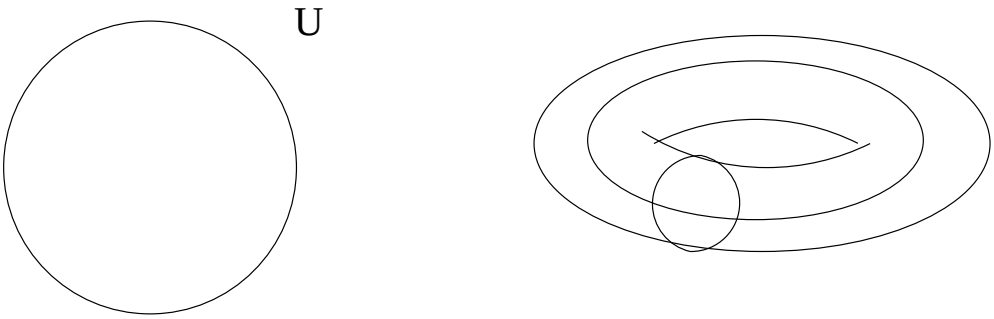
2)$\tau \to -1/\tau$

$$|m-\tau w|^2 \to |m + \frac{1}{\tau} w|^2 = \frac{|w+m\tau|^2}{|\tau|^2} \tag{8.111}$$

$$\text{Let } \; m \to -w, w \to m \tag{8.112}$$

$$\tau = \tau_1 + i\tau_2 \to -\frac{1}{\tau_1 + i\tau_2}, \qquad \tau_2 \to \frac{\tau_2}{|\tau|^2} \tag{8.113}$$

$$\frac{|m-\tau w|^2}{\tau_2} \to |\tau|^2 \frac{|m + \frac{1}{\tau} w|^2}{\tau_2} = \frac{|w+m\tau|^2}{\tau_2 |\tau|^2} |\tau|^2 \tag{8.114}$$



$$U(\sigma + 2\pi) = U(\sigma) + 2\pi R w \tag{8.115}$$

$$U(\sigma^1 + 2\pi, \sigma^2) = U(\sigma^1, \sigma^2) + 2\pi R w \tag{8.116}$$

$$U(\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) = U(\sigma^1, \sigma^2) = 2\pi R m \tag{8.117}$$

$$(\partial_1^2 + \partial_2^2) U = 0 \tag{8.118}$$

$$U = \frac{wR}{\tau_2}(\tau_2 \sigma^1 - \tau_1 \sigma^2) + \frac{mR}{\tau_2} \sigma^2 \tag{8.119}$$

$$\partial_1 U = wR \qquad \partial_2 U = \frac{R}{\tau_2}(m - w\tau_1) \tag{8.120}$$

$$S = \frac{1}{2\pi\alpha'} \int \frac{d\sigma_1 d\sigma_2}{2} \left[(\partial_1 U)^2 + (\partial_2 U)^2\right] = \frac{1}{2\pi\alpha'} \int \frac{d\sigma_1 d\sigma_2}{2} \frac{R^2}{\tau_2^2} |m - w\tau|^2 \tag{8.121}$$

$$= \frac{1}{2\pi\alpha'} * 2\pi * 2\pi\tau_2 \frac{1}{2} \frac{R^2}{\tau_2^2} |m - w\tau|^2 = \frac{\pi R^2}{\tau_2 \alpha'} |m - w\tau|^2 \tag{8.122}$$

8.3 Partition Function in Thermodynamics

$$Z = \sum_E e^{-E/T} \qquad \text{with } \beta = 1/T \in \mathbb{R} \tag{8.123}$$

$$H|E\rangle = E|E\rangle, \qquad \langle E|E\rangle = 1 \tag{8.124}$$

$$Z = \sum_E \langle E|e^{-\beta H}|E\rangle = Tr(e^{-\beta E}) \tag{8.125}$$

$$Z = \sum_{E,x,y} \langle E|x\rangle\langle x|e^{-\beta H}|y\rangle\langle y|E\rangle = \sum_{E,x,y} \psi_E(x) G(t;x,y)\psi_E(y) \tag{8.126}$$

Let $\beta \to it$

$$e^{-\beta H}|y\rangle = e^{-itH}|y\rangle, \qquad \langle x|e^{-itH} = \langle x(t)| \tag{8.127}$$

$$G(t;x,y) = \langle x(t)|y(0)\rangle \text{ - Green Function (propagator)} \tag{8.128}$$

$$H\psi = i\frac{\partial\psi}{\partial t} \tag{8.129}$$

Suppose t is small:

$$y - x = \dot{q}t \tag{8.130}$$

$$G(t;x,y) = \langle x(t)|y(0\rangle = \langle x|e^{-itH}|y\rangle = \sum_p \langle x|p\rangle\langle p|e^{-itH}|y\rangle \tag{8.131}$$

$$= \int \frac{dp}{2\pi} e^{-ipx} e^{-itH(p,q)} e^{ipy} = \int \frac{dp}{2\pi} e^{it(\dot{q}p - H(p,q))} \tag{8.132}$$

Dominant contribution:

$$\delta(\dot{q}p - H) = 0 = \delta(L) \tag{8.133}$$

$$\frac{\partial}{\partial p}(\dot{q}p - H) = 0 \tag{8.134}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{– Hamilton-Jacobi} \tag{8.135}$$

$$G(t;x,y) \sim e^{itL} = e^{iS} \qquad S = \int_0^t dt' L \quad \text{-(valid for finite time)} \tag{8.136}$$

$$Z = \sum_{E,x,y} \langle E|x\rangle e^{iS}\langle y|E\rangle = \sum_{x,y} e^{iS}\langle y|x\rangle = \int dx e^{iS} \quad \text{with y=x} \tag{8.137}$$

8.4 Vertex Operators

$$U = u - \frac{i\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) lnz - \frac{i\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right) ln\bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m \bar{z}^{-m}) \tag{8.138}$$

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'} \tag{8.139}$$

$$p_R = \frac{n}{R} - \frac{wR}{\alpha'} \tag{8.140}$$

$$p = \frac{1}{2}(p_L + p_R) \tag{8.141}$$

$$[u,p] = i \tag{8.142}$$

$$u = u_L + u_R \tag{8.143}$$

$$[u_L, p_L] = [u_R, p_R] = i \tag{8.144}$$

$$[u_L, p_R] = [u_R, p_L] = [u_L, u_R] = [p_L, p_R] = 0 \tag{8.145}$$

$$U = U_L + U_R \tag{8.146}$$

$$U_L = u_L - \frac{i\alpha'}{2} p_L ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m z^{-m} \tag{8.147}$$

$$U_R = u_R - \frac{i\alpha'}{2} p_R ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \tilde{\alpha}_m \bar{z}^{-m} \tag{8.148}$$

OPE

$$U_L(z) U_L(0) \sim \frac{\alpha'}{2} ln z \tag{8.149}$$

$$U_R(\bar{z}) U_R(0) \sim -\frac{\alpha'}{2} ln \bar{z} \tag{8.150}$$

$$U_L(z) U_R(0) \sim 0 \tag{8.151}$$

Vertex Operator

$$V = e^{ik\cdot X} =: e^{ik\cdot X_L} :: e^{ik\cdot X_R} : \tag{8.152}$$

$$V_{k_L,k_R} = V_L(z) V_R(\bar{z}) \tag{8.153}$$

$$k_L = \left(\frac{n}{R} + \frac{wR}{\alpha'} \right) \tag{8.154}$$

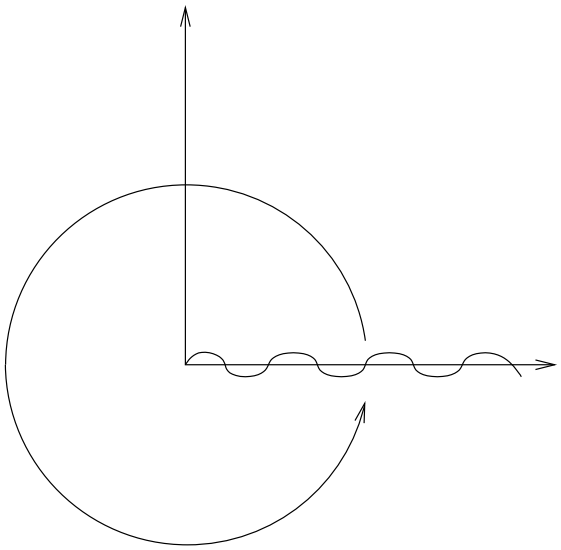
$$k_R = \left(\frac{n}{R} - \frac{wR}{\alpha'} \right) \tag{8.155}$$

$$V_L(z) =: e^{ik_L \cdot U_L} : \tag{8.156}$$

$$V_R(\bar{z}) =: e^{ik_R \cdot U_R} : \tag{8.157}$$

$$: e^{ik\cdot X_L}(z) :: e^{ik'\cdot X_L}(0) : \simeq e^{k\cdot k' \frac{\alpha'}{2} ln z} : e^{i(k+k')\cdot X_L}(0) := z^{\frac{\alpha'}{2} k\cdot k'} : e^{i(k+k')\cdot X_L}(0) : \tag{8.158}$$

Singularity has branch cut



$$lnz = ln|z| + iArgz \tag{8.159}$$

$$z^{\frac{\alpha'}{2}k\cdot k'} = |z|^{\frac{\alpha'}{2}k\cdot k'} e^{ik\cdot k'\frac{\alpha'}{2}Argz} \rightarrow |z|^{\frac{\alpha'}{2}k\cdot k'} e^{ik\cdot k'\alpha'\pi} \tag{8.160}$$

For Compactified Case:

$$V_L(z)V'_L(0) \sim z^{\frac{\alpha'}{2}k_L\cdot k'_L} : e^{i(k_L+k'_L)U_L}(0) : \rightarrow |z|^{\frac{\alpha'}{2}k_L\cdot k'_L} e^{ik_Lk'_L\alpha'\pi} : e^{i(k_L+k'_L)U_L}(0) : \tag{8.161}$$

$$V_R(\bar{z})V_{R'}(0) \sim \bar{z}^{\frac{\alpha'}{2}k_R\cdot k'_R} : e^{i(k_R+k'_R)U_R}(0) : \rightarrow |\bar{z}|^{\frac{\alpha'}{2}k_R\cdot k'_R} e^{-ik_Rk'_R\alpha'\pi} : e^{i(k_R+k'_R)U_R}(0) : \tag{8.162}$$

$$z^{\frac{\alpha'}{2}k_L\cdot k'_L}\bar{z}^{\frac{\alpha'}{2}k_R\cdot k'_R} \rightarrow |z|^{\frac{\alpha'}{2}k_L\cdot k'_L}|\bar{z}|^{\frac{\alpha'}{2}k_R\cdot k'_R} e^{i(k_Lk'_L-k_Rk'_R)\alpha'\pi} \tag{8.163}$$

$$k_Lk'_L - k_Rk'_R = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\left(\frac{n'}{R} + \frac{w'R}{\alpha'}\right) - \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\left(\frac{n'}{R} - \frac{w'R}{\alpha'}\right) \tag{8.164}$$

$$= 2\frac{nw'}{\alpha'} + 2\frac{n'w}{\alpha'} = \frac{2}{\alpha'}(nw' + n'w) \tag{8.165}$$

$e^{i\alpha'\pi(k_Lk'_L-k_Rk'_R)} = e^{2\pi i(nw'+n'w)} = 1$, which it must be to be a physical quantity (8.166)

Subtlety

Consider VV' and $V'V$, $(z \rightarrow -z)$

$$e^{i\frac{\alpha'}{2}k_Lk'_L\pi}e^{-i\frac{\alpha'}{2}k_Rk'_R\pi} = e^{i\pi(nw'+wn')} =^{+}_{-} 1 \tag{8.167}$$

A cocycle must be inserted to ensure a physical quantity:

$$V_k = c(p)V_L^{k_L}V_R^{k_R} \tag{8.168}$$

$$c_k(p)c_{k'}(p) = c_{k+k'}(p) \tag{8.169}$$

$$c_k(p) = e^{i\pi\frac{\alpha'}{2}(k_L-k_R)p} \tag{8.170}$$

8.5 Amplitudes

Uncompactified:

$$V_k(z,\bar{z}) =: e^{ik\cdot X(z,\bar{z})} : \text{(Tachyons)} \tag{8.171}$$

$$A \sim \langle V_{k_1}(z_1)V_{k_2}(z_2)...V_{k_n}(z_n)\rangle \tag{8.172}$$

$$\begin{aligned} &\sim \quad \delta(k_1+k_2+...+k_n)\prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j} \\ &= \quad \prod_{i<j}(z_i-z_j)^{\frac{\alpha'}{2}k_i\cdot k_j}\prod_{i<j}(\bar{z}_i-\bar{z}_j)^{\frac{\alpha'}{2}k_i\cdot k_j}\delta(k_1+k_2+...+k_n) \end{aligned} \tag{8.173}$$

Compactified:

$$A \sim \binom{+}{-}\prod_{i<j}(z_i-z_j)^{\frac{\alpha'}{2}k_{L_i}\cdot k_{L_j}}\prod_{i<j}(\bar{z}_i-\bar{z}_j)^{\frac{\alpha'}{2}k_{R_i}\cdot k_{R_j}}\delta(k_{L_1}+k_{L_2}+...+k_{L_n})\delta(k_{R_1}+k_{R_2}+...+k_{R_n}) \tag{8.174}$$

$$\delta(\alpha)\delta(\beta) \sim \delta\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) \tag{8.175}$$

$$\therefore \delta(k_{L_1}+k_{L_2}+...+k_{L_n})\delta(k_{R_1}+k_{R_2}+...+k_{R_n}) \tag{8.176}$$

$$\sim \delta(n_1+n_2+...+n_N)\delta(w_1+w_2+...+w_N) \tag{8.177}$$

8.6 Spectrum

$$(t,x,y,z,...,u) = (x^\mu, U) \tag{8.178}$$

Uncompactified:

$$L_0 = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{\mu n} = \frac{\alpha' p^2}{4} + N \tag{8.179}$$

$$N = \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{\mu n} + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n \tag{8.180}$$

$$\tilde{L}_0 = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty}\tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{\mu n} = \frac{\alpha' p^2}{4} + \tilde{N} \tag{8.181}$$

Compactified:

$$L_0 = \frac{\alpha'(p^2+{p_L}^2)}{4} + N \tag{8.182}$$

$$\tilde{L}_0 = \frac{\alpha'(p^2+{p_R}^2)}{4} + \tilde{N} \tag{8.183}$$

$$\begin{aligned}(L_0-1)|phys\rangle &= 0 \quad \text{- Hamiltonian = 0, invariance under } \tau \\ (\tilde{L}_0-1)|phys\rangle &= 0 \\ (\tilde{L}_0-L_0)|phys\rangle &= 0 \quad \text{- invariance under } \sigma \end{aligned} \tag{8.184}$$

$$\begin{aligned} L_0-1=0 \;\;\therefore\;\; \frac{\alpha'}{4}(-m^2+k_L^2)+N-1=0 \rightarrow m^2 &= k_L^2+\frac{4}{\alpha'}(N-1) \\ \tilde{L}_0-1=0 \qquad\qquad\qquad\qquad \rightarrow m^2 &= k_R^2+\frac{4}{\alpha'}(\tilde{N}-1) \end{aligned} \tag{8.185}$$

1)Subtracting

$$0=k_L^2-k_R^2+\frac{4}{\alpha'}(N-\tilde{N}) \tag{8.186}$$

$$\frac{2n}{R}*\frac{2wR}{\alpha'}+\frac{4}{\alpha'}(N-\tilde{N})=0 \tag{8.187}$$

$$\tilde{N}-N=nw \tag{8.188}$$

2)Adding

$$\begin{aligned} m^2 &= \frac{k_L^2+k_R^2}{2}+\frac{2}{\alpha'}(N+\tilde{N}-2) \\ m^2 &= \left(\frac{n}{R}\right)^2+\left(\frac{wR}{\alpha'}\right)^2+\frac{2}{\alpha'}(N+\tilde{N}-2) \end{aligned} \tag{8.189}$$

<u>Massless States</u> (n=w=0):

$$n=w=0 \;\;,\;\; N=\tilde{N}=1 \tag{8.190}$$

$$g_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0,k\rangle-\text{graviton} \tag{8.191}$$

$$B_{\mu}(k)\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}|0,k\rangle \;\;,\;\; C_{\mu}(k)\alpha_{-1}\tilde{\alpha}_{-1}^{\mu}|0,k\rangle-\text{vectors} \tag{8.192}$$

$$\Phi(k)\alpha_{-1}\tilde{\alpha}_{-1}|0,k\rangle-\text{scalar} \tag{8.193}$$

Amplitudes

$$\langle V(z_1)V(z_2)V(z_3)\rangle \tag{8.194}$$

Again, freedom to choose 3 points, choose $z_1=\infty, z_2=1, z_3=0$

$$Amp\sim\langle 0,k_1|V_{k_2}(1)B_{\mu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}|0,k\rangle, \quad \langle 0,k_1|=\langle 0,k_1,k_{1L},k_{1R}| \;\text{ and }\; |0,k_1\rangle=|0,k_1,0,0\rangle \tag{8.195}$$

$$V_{k_2}(z,\bar{z}) = e^{ik\cdot X + ik_L U_L + ik_R U_R} \tag{8.196}$$

$$U_L = u_L - i\frac{\alpha' p_L}{2} lnz + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_m}{m} z^{-m} = u_L - i\frac{\alpha' p_L}{2} lnz + i\sqrt{\frac{\alpha'}{2}}\alpha_1 z^{-1} + ... \tag{8.197}$$

$$U_L(1) = u_L + i\sqrt{\frac{\alpha'}{2}}\alpha_1 + ... \tag{8.198}$$

$$U_R = u_R - i\frac{\alpha' p_R}{2} ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_m}{m} \bar{z}^{-m} = u_R - i\frac{\alpha' p_R}{2} ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_1 \bar{z}^{-1} + ... \tag{8.199}$$

$$U_R(1) = u_R + i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha_1} + ... \tag{8.200}$$

$$X^\mu = x^\mu - i\frac{\alpha'}{2} p^\mu ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha^\mu_m}{m}(z^{-m} + \bar{z}^{-m}) = x^\mu - i\frac{\alpha'}{2} p^\mu ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\alpha^\mu_1 + ... \tag{8.201}$$

$$X^\mu(1) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\alpha^\mu_1 + ... \tag{8.202}$$

$$\therefore V_{k_2}(1) = e^{ik\cdot x + ik_L\cdot u_L + ik_R\cdot u_R + i\sqrt{\frac{\alpha'}{2}}(k_R\tilde{\alpha}_1 + k_L\alpha_1 + k_\mu\alpha^\mu_1) + ...} \tag{8.203}$$

$$[\tilde{\alpha}_1, \tilde{\alpha}_{-1}] = 1 \tag{8.204}$$

$$\begin{aligned} Amp &\sim \langle 0; k_1+k_2, k_{1L}+k_{2L}, k_{1R}+k_{2R}|B_\mu\left(i\sqrt{\frac{\alpha'}{2}}k_{2R}\right) i\sqrt{\frac{\alpha'}{2}}k_2^\mu|0,k\rangle \\ &\sim k_2\cdot B\frac{\alpha'}{2}k_{2R}\delta(k_1+k_2+k)\delta(n_1+n_2)\delta(w_1+w_2) \end{aligned} \tag{8.205}$$

$$k_1 = -k_2 - k \tag{8.206}$$

$$k_1\cdot B = -k_2\cdot B - k\cdot B = -k_2\cdot B \tag{8.207}$$

$$\therefore k_2\cdot B = \frac{1}{2}(k_2 - k_1)\cdot B \tag{8.208}$$

$$Amp_C \sim k_C\cdot C\frac{\alpha'}{2}k_{2L}\delta()\delta()\delta() \tag{8.209}$$

The photon is the sum of two states ($B_\mu and C_\mu$)

$$\begin{aligned} B_\mu(k)\alpha^\mu_{-1}\tilde{\alpha}_{-1} + C_\mu(k)\alpha_{-1}\tilde{\alpha}^\mu_{-1} &\Rightarrow \text{couples to } \tfrac{n}{R} : \text{photon} \\ B_\mu(k)\alpha^\mu_{-1}\tilde{\alpha}_{-1} - C_\mu(k)\alpha_{-1}\tilde{\alpha}^\mu_{-1} &\Rightarrow \text{couples to w : magnetic charge} \end{aligned} \tag{8.210}$$

Massless States (n,w $\neq 0$):
Scalars:

$$N = \tilde{N} = 0 \tag{8.211}$$

$$m^2 = 0 \Rightarrow \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 - \frac{4}{\alpha'} = 0 \tag{8.212}$$

$$\text{If } R = \sqrt{\alpha'}: \qquad n^2 + w^2 = 4 \tag{8.213}$$

$$\begin{aligned} n &= 0, w = \overset{+}{-} 2 \\ n &= \overset{+}{-} 2, w = 0 \end{aligned} \tag{8.214}$$

$$\text{If } R = \frac{\sqrt{\alpha'}}{k}: \qquad n^2k^2 + \frac{w^2}{k^2} = 4 \tag{8.215}$$

Vectors:

$$R = \sqrt{\alpha'} \tag{8.216}$$

$$N = 1 \ \ or \ \ \tilde{N} = 1 \tag{8.217}$$

1. $n = w = \overset{+}{-} 1 \qquad N = 0, \tilde{N} = 1 - vector$ (8.218)
2. $n = -w = \overset{+}{-} 1 \qquad N = 1, \tilde{N} = 0 - vector$ (8.219)
3. $n = \overset{+}{-} 2, w = 0 \qquad N = 0, \tilde{N} = 0 - scalar$ (8.220)
4. $n = 0, w = \overset{+}{-} 2, \qquad N = 0, \tilde{N} = 0 - scalar$ (8.221)

8.7 Weak Interactions

(W^+, W^-, W^o) - 3D vector in abstract space

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} e \\ \nu_e \end{pmatrix} \quad \text{-spinors (spin in abstract space) can rotate into each other} \quad \text{Gauge SU(2)} \tag{8.222}$$

From Electromagnetism:

$$S = \int d^4x \left[\frac{-1}{4} F^{\mu\nu} F_{\mu\nu} + |D_\mu \Phi|^2 + m^2 |\Phi|^2 \right] \tag{8.223}$$

$$\begin{aligned} D_\mu &= \partial_\mu + iqA_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{8.224}$$

Gauge Invariance

$$\begin{aligned} \Phi &\rightarrow e^{iq\lambda}\Phi \\ A_\mu &\rightarrow A_\mu - \partial_\mu \lambda \end{aligned} \tag{8.225}$$

$$qA_\mu(\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*) \tag{8.226}$$

- EM is a U(1) Gauge Theory

Extend to Weak Interactions:

$$(A^1_\mu, A^2_\mu, A^3_\mu) = A^i_\mu \tag{8.227}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\mu \tag{8.228}$$

$$-\frac{1}{4}\sum_{i=1}^{3} F^i_{\mu\nu} F^{i\mu\nu} + A^i_\mu J^{i\mu} \quad \text{Guess that this is a piece of the Action} \tag{8.229}$$

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu \qquad \text{Last term is nonlinear} \tag{8.230}$$

$$\partial_\mu \vec{F}_{\mu\nu} = \vec{A}_\mu \times \vec{F}^{\mu\nu} \tag{8.231}$$

Gauge Invariance:

$$\vec{A}_\mu \to \vec{A}_\mu - \partial_\mu \vec{\lambda} - \vec{A}_\mu \times \vec{\lambda} \qquad \text{(For small } \vec{\lambda}) \tag{8.232}$$

$$\begin{aligned}
\vec{F}_{\mu\nu} \quad \to \quad & \partial_\mu(\vec{A}_\nu - \partial_\nu\vec{\lambda} - \vec{A}_\nu \times \vec{\lambda}) - \partial_\nu(\vec{A}_\mu - \partial_\mu\vec{\lambda} - \vec{A}_\mu \times \vec{\lambda}) \\
+ \quad & (\vec{A}_\mu - \partial_\mu\vec{\lambda} - \vec{A}_\mu \times \vec{\lambda}) \times (\vec{A}_\nu - \partial_\nu\vec{\lambda} - \vec{A}_\nu \times \vec{\lambda}) \\
= \quad & \partial_\mu\vec{A}_\nu - \partial_\mu\partial_\nu\vec{\lambda} - \partial_\mu(\vec{A}_\nu \times \vec{\lambda}) - \partial_\nu\vec{A}_\mu + \partial_\nu\partial_\mu\vec{\lambda} + \partial_\nu(\vec{A}_\mu \times \vec{\lambda}) \\
+ \quad & \vec{A}_\mu \times \vec{A}_\nu - \partial_\mu\vec{\lambda} \times \vec{A}_\nu - (\vec{A}_\mu \times \vec{\lambda}) \times \vec{A}_\nu + \partial_\nu\vec{\lambda} \times \vec{A}_\mu + (\partial_\mu\vec{\lambda} \times \partial_\nu\vec{\lambda}) + \partial_\mu\vec{\lambda} \times (\vec{A}_\nu \times \vec{\lambda}) \\
- \quad & \vec{A}_\mu \times (\vec{A}_\nu \times \vec{\lambda}) + \partial_\mu\vec{\lambda} \times (\vec{A}_\nu \times \vec{\lambda}) + (\vec{A}_\mu \times \vec{\lambda}) \times (\vec{A}_\nu \times \vec{\lambda}) \\
= \quad & \vec{F}_{\mu\nu} - \partial_\mu(\vec{A}_\nu \times \vec{\lambda}) + \partial_\nu(\vec{A}_\mu \times \vec{\lambda}) - \partial_\mu\vec{\lambda} \times \vec{A}_\nu - (\vec{A}_\mu \times \vec{\lambda}) \times \vec{A}_\nu + \partial_\nu\lambda \times \vec{A}_\mu \\
- \quad & \vec{A}_\mu \times (\vec{A}_\nu \times \vec{\lambda}) + (\vec{A}_\mu \times \vec{\lambda}) \times \vec{A}_\nu \qquad \text{keeping only } 1^{st} \text{ order terms in } \lambda \\
= \quad & \vec{F}_{\mu\nu} - \partial_\mu\vec{A}_\nu \times \vec{\lambda} + \partial_\nu\vec{A}_\mu \times \vec{\lambda} - (\vec{A}_\mu \times \vec{A}_\nu) \times \vec{\lambda} \\
= \quad & \vec{F}_{\mu\nu} - \vec{F}_{\mu\nu} \times \vec{\lambda} \to \qquad \vec{F}_{\mu\nu} \text{ is not invariant}
\end{aligned} \tag{8.233}$$

$$(\vec{F}_{\mu\nu} - \vec{F}_{\mu\nu} \times \vec{\lambda})^2 = \vec{F}^2_{\mu\nu} - \vec{F}_{\mu\nu} \cdot (\vec{F}_{\mu\nu} \times \vec{\lambda}) = \vec{F}^2_{\mu\nu} \to \qquad \text{Action is invariant and a physical quantity} \tag{8.234}$$

$$A_\mu = \frac{1}{2} A^i_\mu \sigma^i - \qquad 2\times 2 \text{ matrix} \tag{8.235}$$

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \tag{8.236}$$

$$F_{\mu\nu} = \frac{1}{2} F^i_{\mu\nu}\sigma^i \tag{8.237}$$

$$S = e^{\frac{i}{2}\lambda^i\sigma^i} \in SU(2) \tag{8.238}$$

$$SS^\dagger = \mathbb{I} \tag{8.239}$$

$$\text{For small } \lambda : S = \mathbb{I} + \frac{i}{2}\lambda^i\sigma^i \tag{8.240}$$

$$A_{\mu\nu} \to -iS\partial_\mu S^\dagger + SA_\mu S^\dagger \qquad - \text{finite transformation} \tag{8.241}$$

$$F_{\mu\nu} \to SF_{\mu\nu}S^\dagger \quad Tr(F_{\mu\nu}F^{\mu\nu}) \to Tr(SF_{\mu\nu}S^\dagger SF^{\mu\nu}S^\dagger) \tag{8.242}$$

$$\begin{aligned} \Phi &\rightarrow S\Phi \\ A_\mu &\rightarrow -iS\partial_\mu S^\dagger + SA_\mu S^\dagger \end{aligned} \tag{8.243}$$

$$\begin{aligned} D_\mu\Phi &\rightarrow \partial_\mu(S\Phi) + iA'_\mu S\Phi \\ &= S(\partial_\mu\Phi + S^\dagger\partial_\mu S\Phi + iS^\dagger A'_\mu S\Phi) \\ &= SD_\mu\Phi \end{aligned} \tag{8.244}$$

$$|D_\mu\Phi|^2 \rightarrow (D_\mu\Phi)^\dagger S^\dagger S D_\mu\Phi = (D_\mu\Phi)^\dagger D_\mu\Phi \quad \text{Invariant} \tag{8.245}$$

There is no freedom in choosing q $\therefore$ SU(2)-compact

Strings

$$(n, w) \quad (k_l, k_R) \tag{8.246}$$

$$\text{Two Quantum Numbers} - \quad U(1) \times U(1) \tag{8.247}$$

$$Q = \int d^3x j^0 \tag{8.248}$$

$$\partial_\mu j^\mu = 0 \tag{8.249}$$

$$\bar{\partial}T(z) = 0 - \quad \text{Conformal Transformations} \tag{8.250}$$

$$\oint dz T(z) = L_0 \qquad \oint dz z^n T(z) = L_n - \quad \text{Virasora Operators} \tag{8.251}$$

$$\delta A = -i\epsilon[Q, A] \qquad i.e.\ i\frac{\partial A}{\partial t} = [H, A] - \quad \text{H is a charge, generates translations in time} \tag{8.252}$$

$$\bar{\partial}(\partial U(z)) = 0 \tag{8.253}$$

$$\partial(\bar{\partial}U(\bar{z})) = 0 \tag{8.254}$$

$$|n, w\rangle = |k_L, k_R\rangle - \quad \text{state which corresponds to an operator} \tag{8.255}$$

$$V(z, \bar{z}) =: e^{ik_L U_L} e^{ik_R U_R} : \tag{8.256}$$

$$\lim_{z\to 0} V(z, \bar{z})|0\rangle = |k_L, k_R\rangle - \quad \text{infinite past} \tag{8.257}$$

Calculate OPE of $\partial U(z)V(0,0)$:

$$\begin{aligned} \partial U(z)V(0,0) &\sim \partial U_L(z) : e^{ik_L U_L} : \ \sim ik_L \partial\left(\frac{-\alpha'}{2} lnz\right) : e^{ik_L U_L}(0) : \\ &\sim -ik_L\frac{\alpha'}{2} \cdot \frac{1}{z} V(0,0) \end{aligned} \tag{8.258, 8.259}$$

$$\bar{\partial}U(\bar{z})V(0,0) \sim -ik_R\frac{\alpha'}{2}\cdot\frac{1}{\bar{z}}V(0,0) \tag{8.260}$$

$$Q_L = \frac{1}{2\pi i}\oint_C dz\partial U(z) \tag{8.261}$$

$$Q_R = -\frac{1}{2\pi i}\oint_C d\bar{z}\partial U(\bar{z}) \tag{8.262}$$

$$\int\frac{dz}{2\pi i}j(z)V(0) = [Q,V] \tag{8.263}$$

$$[Q_L,V] = -ik_L\frac{\alpha'}{2}V \tag{8.264}$$

$$[Q_R,V] = -ik_R\frac{\alpha'}{2}V \tag{8.265}$$

$$\Rightarrow [Q_L+Q_R,V] = -i\frac{n}{R}\alpha' V \tag{8.266}$$

$$\delta V = -\lambda[Q_L,V] = -ik_L\frac{\alpha'}{2}\lambda|n,w\rangle \tag{8.267}$$

$$\delta_L|n,w\rangle = ik_L\frac{\alpha'}{2}\lambda|n,w\rangle \tag{8.268}$$

$$\therefore |n,w\rangle \to |n,w\rangle + ik_L\frac{\alpha'}{2}\lambda|n,w\rangle = \left(1+ik_L\frac{\alpha'}{2}\lambda\right)|n,w\rangle \simeq e^{ik_L\frac{\alpha'}{2}\lambda}|n,w\rangle \tag{8.269}$$

Currents $\partial U, \partial\bar{U}$ generate $U(1)\times U(1)$ for all R

Vectors ($R=\sqrt{\alpha'}$):

$$1.n = w = \overset{+}{-}1 \qquad N=0,\tilde{N}=1- \quad \text{2 photons} \tag{8.270}$$

$$2.n = -w = \overset{+}{-}1 \qquad N=1,\tilde{N}=0- \quad \text{2 photons} \tag{8.271}$$

$$3.n = -w = 0 \qquad\qquad -\text{ 2 photons} \tag{8.272}$$

$$= 6 \text{ photons} = (2\times 3) - \text{Guess: 2 copies of weak interaction} \tag{8.273}$$

$$k_L = \frac{n}{R}+\frac{wR}{\alpha'} = \frac{n+w}{\sqrt{\alpha'}} \tag{8.274}$$

$$k_R = \frac{n}{R}-\frac{wR}{\alpha'} = \frac{n-w}{\sqrt{\alpha'}} \tag{8.275}$$

1. $\tilde{\alpha}^\mu_{-1}|0,k,+1,+1\rangle \sim :\bar{\partial}X^\mu e^{ik\cdot X}e^{ik_LU_L}e^{ik_RU_R}: \sim :\bar{\partial}X^\mu e^{ik\cdot X}e^{i\frac{2}{\sqrt{\alpha'}}U_L(z)}:$
   $\tilde{\alpha}^\mu_{-1}|0,k,-1,-1\rangle \sim :\partial X^\mu e^{ik\cdot X}e^{-i\frac{2}{\sqrt{\alpha'}}U_L(z)}:$
2. $\alpha^\mu_{-1}|0,k,+1,-1\rangle \sim :\partial X^\mu e^{ik\cdot X}e^{i\frac{2}{\sqrt{\alpha'}}U_R(\bar{z})}:$
   $\alpha^\mu_{-1}|0,k,-1,+1\rangle \sim :\partial X^\mu e^{ik\cdot X}e^{-i\frac{2}{\sqrt{\alpha'}}U_R(\bar{z})}:$

$$j^+(z) =: e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z) \tag{8.276}$$

$$j^-(z) =: e^{-i\frac{2}{\sqrt{\alpha'}}U_L}(z) \tag{8.277}$$

$$\begin{aligned} j^+(z)j^+(0) &\sim : e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z) :: e^{i\frac{2}{\sqrt{\alpha'}}U_L}(0) : \\ &\sim e^{\frac{-4}{\alpha'}ln(z)\frac{-\alpha'}{2}} : e^{i\frac{4}{\sqrt{\alpha'}}U_L}(0) \\ &\sim z^2 : e^{i\frac{4}{\sqrt{\alpha'}}U_L}(0) : \sim 0 \\ j^-(z)j^-(0) &\sim 0 \end{aligned} \tag{8.278}$$

(8.279)

$$j^+(z)j^-(0) \sim z^{-2} : e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z)e^{-i\frac{2}{\sqrt{\alpha'}}U_L}(0) : \tag{8.280}$$

$$= z^{-2}\left[1 + z\, i\frac{2}{\sqrt{\alpha'}}\partial U_L + z^2(\ \ ) + ...\right] \tag{8.281}$$

$$\sim \frac{1}{z^2} + i\frac{2}{\sqrt{\alpha'}}\partial U_L\frac{1}{z} \tag{8.282}$$

$$Q^{\pm} = \oint \frac{dz}{2\pi i} j^{\pm}(z) \tag{8.283}$$

$$[Q^+, Q^+] = \oint \frac{dz}{2\pi i} j^+(z)j^-(0) = 0 \tag{8.284}$$

$$[Q^-, Q^-] = 0 \tag{8.285}$$

$$[Q^+, Q^-] = \oint i\frac{2}{\sqrt{\alpha'}}\partial U_L\frac{dz}{2\pi i} = iQ \tag{8.286}$$

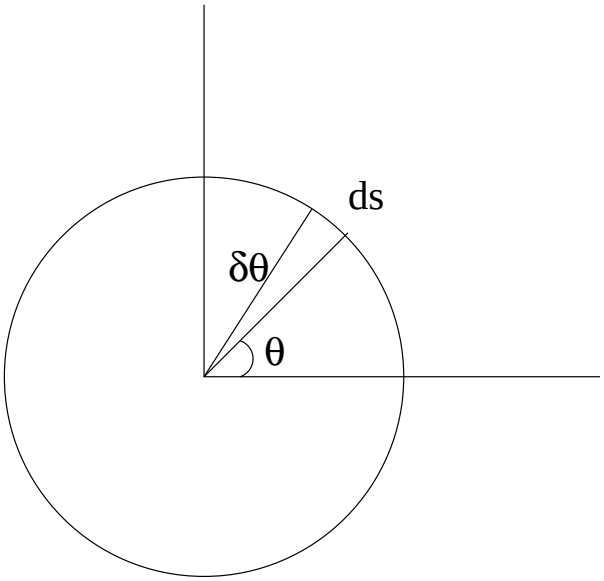
$$Q = \oint \frac{dz}{2\pi i}\frac{1}{\sqrt{\alpha'}}\partial U \tag{8.287}$$

$$[Q^+, Q^-] = \frac{1}{2i}\left[Q^+, [Q^+, Q^-]\right] \tag{8.288}$$

$$j^+\frac{1}{\sqrt{\alpha'}}\partial U \sim \frac{1}{\sqrt{\alpha'}}i\frac{2}{\sqrt{\alpha'}}\partial\left(\frac{\alpha'}{2}ln(z)\right)j^+ = \frac{i}{z}j^+ \tag{8.289}$$

$$[Q^+, Q] = iQ^+ \qquad [Q^-, Q] = -iQ^- \tag{8.290}$$

$$Q^{\pm} = L_1 \pm iL_2 \tag{8.291}$$



$$ds = Rd\theta \tag{8.292}$$

$$ds^2 = R^2 d\theta^2 \tag{8.293}$$

$$ds^2 = G_{uu}du^2 = R^2 G_{uu} d\theta^2 \tag{8.294}$$

$$\text{Define } \theta = \frac{u}{R}, \quad 0 \le \theta \le 2\pi \tag{8.295}$$

$$R_{phys} = R\sqrt{G_{uu}} \;\; \therefore \;\; G_{uu} \text{ is a dynamical variable} \tag{8.296}$$

Currents

$$j^+(z) =: e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z): \quad \tilde{j}^+(\bar{z}) =: e^{i\frac{2}{\sqrt{\alpha'}}U_R}(\bar{z}): \tag{8.297}$$

$$j^-(z) =: e^{-i\frac{2}{\sqrt{\alpha'}}U_L}(z): \quad \tilde{j}^-(\bar{z}) =: e^{-i\frac{2}{\sqrt{\alpha'}}U_R}(\bar{z}): \tag{8.298}$$

$$j^3(z) = \frac{i}{\sqrt{\alpha'}}\partial U(z): \quad \tilde{j}^3(\bar{z}) = \frac{i}{\sqrt{\alpha'}}\bar{\partial}U(\bar{z}) \tag{8.299}$$

$$j^+(z)j^-(0) \sim \frac{1}{z^2} + \frac{2}{z}j^3(0) \tag{8.300}$$

$$j^3(z)\overset{+}{j^-}(0) \sim^+_- \frac{1}{z}\overset{+}{j^-}(0) \tag{8.301}$$

$$Q^a = \oint \frac{dz}{2\pi i} j^a(z) \qquad L_0 = \oint \frac{dz}{2\pi i} zT(z) \tag{8.302}$$

$$[Q^a, Q^b] = \epsilon^{abc}Q^c \tag{8.303}$$

$$j^a(z) = \sum j^a_m z^{-m-1} \tag{8.304}$$

$$[j^a_m, j^b_n] = k\frac{m}{2}\delta_{m+n,0}\delta^{ab} + i\epsilon^{abc}j^c_{m+n}, \;\; k \in \mathbb{N} \quad \text{- Current Algebra} \tag{8.305}$$

Zero Modes ($m = n = 0$)

$$[j^a_0, j^b_0] = i\epsilon^{abc}j_0 \tag{8.306}$$

$$\int |D_\mu \Phi|^2 + m^2 |\Phi|^2 - \quad \text{Action} \tag{8.307}$$

Higgs realized that you can have any function of Φ for m^2

$$V\left(|\Phi|^2\right) \tag{8.308}$$

$$\Phi \to S\Phi \tag{8.309}$$

$$|\Phi|^2 \to \Phi^\dagger S^\dagger S \Phi = |\Phi|^2 \tag{8.310}$$

$R = \sqrt{\alpha'}$
Photons:

$$: \bar{\partial} X^\mu e^{ik\cdot X} e^{ik_L U_L} : \tag{8.311}$$

Massless Scalars:

$$: \bar{\partial} U e^{ik\cdot X} e^{ik_L U_L} : \to \quad : \quad \tilde{j}^3 \overset{+}{j^-} e^{ik\cdot X} : \tag{8.312}$$

$$: \quad j^3 \overset{+}{\tilde{j}^-} e^{ik\cdot X} : \tag{8.313}$$

$$3.\ \ n = \overset{+}{-} 2, w = 0 \to k_L = k_R = \frac{n}{R} = \overset{+}{-} \frac{2}{\sqrt{\alpha'}} \tag{8.314}$$

$$4.\ \ n = 0, w = \overset{+}{-} 2 \to k_L = -k_R = \frac{wR}{\alpha'} = \overset{+}{-} \frac{2}{\sqrt{\alpha'}} \tag{8.315}$$

$$3.\text{ and } 4. \quad e^{ik_L U_L} e^{ik_R U_R} e^{ik\cdot X} \to : \overset{+}{j^-}(z) \overset{+}{\tilde{j}^-}(\bar{z}) e^{ik\cdot X} \tag{8.316}$$

$$\partial U \bar{\partial} U e^{ik\cdot X} \tag{8.317}$$

$$\Phi \alpha_{-1} \tilde{\alpha}_{-1} |k\rangle \tag{8.318}$$

9 scalars:

$$: j^a \tilde{j}^b e^{ik\cdot X} : \tag{8.319}$$

massless:

$$j^3 \tilde{j}^3 \tag{8.320}$$

8 others:

$$m^2 = \left(\frac{R^2 - \alpha'^2}{R\alpha'}\right)^2 \ \text{ from } \ m^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{8.321}$$

8.8 T-Duality (Closed Strings)

$$R \to \infty : n = 0, \ \ m^2 \to \infty \tag{8.322}$$

$$w = 0 \qquad \to \text{form continuum} \tag{8.323}$$

$$R \to 0 : w = 0, \ \ m^2 \to \infty \tag{8.324}$$

$$n = 0 \qquad \to \text{form continuum} \tag{8.325}$$

$$R \to R' = \frac{\alpha'}{R} \Longrightarrow \text{for} R' = R \Rightarrow R = \frac{\alpha'}{R} \Rightarrow R = \sqrt{\alpha'} \tag{8.326}$$

$$m^2 = \left(\frac{nR'}{\alpha'}\right)^2 + \left(\frac{w}{R'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{8.327}$$

$$n \leftrightarrow w \tag{8.328}$$

Theorem: $R, R' = \frac{\alpha'}{R}$ are identicle theories
Proof:

$$U = U_L + U_R \tag{8.329}$$

$$Z = U_L - U_R \tag{8.330}$$

Both have the same operator product expansions, Z has no physical meaning

$$U_L(z)U_L(0) \sim -\frac{\alpha'}{2} ln(z) \tag{8.331}$$

$$U_R(\bar{z})U_R(0) \sim -\frac{\alpha'}{2} ln(\bar{z}) \tag{8.332}$$

$$U_L(z)U_R(0) \sim 0 \tag{8.333}$$

$$U_L = u_L - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) ln(z) + ... \tag{8.334}$$

$$U_R = u_R - i\frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right) ln(\bar{z}) + ... \tag{8.335}$$

$$Z = u_L - u_R - i\frac{\alpha'}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) ln(z) - \left(\frac{n}{R} - \frac{wR}{\alpha'}\right) ln(\bar{z})\right] + ... \tag{8.336}$$

$$U = u_L + u_R - i\frac{\alpha'}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) ln(z) + \left(\frac{n}{R} - \frac{wR}{\alpha'}\right) ln(\bar{z})\right] + ... \tag{8.337}$$

$$\begin{aligned} U = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right) ln(z) + \left(\frac{n}{R} - \frac{wR}{\alpha'}\right) ln(\bar{z}) \quad &\longrightarrow \quad \left(\frac{w}{R'} + \frac{nR'}{\alpha'}\right) ln(z) + \left(\frac{w}{R'} - \frac{nR}{\alpha'}\right) ln(\bar{z}) \; n \overset{R \to R'}{\leftrightarrow} w \\ &\longrightarrow \quad \left(\frac{w}{R'} + \frac{nR'}{\alpha'}\right) ln(z) - \left(\frac{-w}{R'} + \frac{nR}{\alpha'}\right) ln(\bar{z}) \\ &\longrightarrow \quad \left(\frac{w\alpha'}{R} + \frac{n}{R}\right) ln(z) - \left(\frac{n}{R} - \frac{wR}{\alpha'}\right) ln(\bar{z}) = Z \end{aligned} \tag{8.338}$$

Z is identical to U provided $R \leftrightarrow R'$

$$T : R \to R' \tag{8.339}$$

$$T^2 : I \tag{8.340}$$

$$\mathbb{Z}_2 = \{I, T\} \subseteq SU(2) \times SU(2) \tag{8.341}$$

Consider:

$$R = \sqrt{\alpha'} + \delta R \tag{8.342}$$

$$R' = \frac{\alpha'}{R} = \frac{\alpha'}{\sqrt{\alpha'} + \delta R} = \frac{\sqrt{\alpha'}}{1 + \frac{\delta R}{\sqrt{\alpha'}}} \simeq \sqrt{\alpha'}\left(1 - \frac{\delta R}{\sqrt{\alpha'}}\right) = \sqrt{\alpha'} - \delta R \tag{8.343}$$

$$\therefore \delta R' = -\delta R \tag{8.344}$$

$$(j^1, j^2, j^3) \times (\tilde{j}^1, \tilde{j}^2, \tilde{j}) \tag{8.345}$$

$$R = \sqrt{\alpha'} \qquad R = \frac{\sqrt{\alpha'}}{k}\ ,\ k \in \mathbb{N} \tag{8.346}$$

$$0 = \frac{n^2 k^2}{\alpha'} + \frac{w^2}{\alpha' k^2} - \frac{4}{\alpha'} \tag{8.347}$$

$$\text{For } \ n = 0, \quad \frac{w^2}{k^2} = 4 \Rightarrow w = \overset{+}{-}\, 2k - \ \text{ 2 massless scalars} \tag{8.348}$$

$$: j^3 \tilde{j}^3 e^{ik\cdot X} : - \ \text{ 1 massless scalar} \tag{8.349}$$

$$U_L = u_L - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) ln(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m z^{-m} \tag{8.350}$$

$$U_R = u_R - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) ln(\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \tilde{\alpha}_m \bar{z}^{-m} \tag{8.351}$$

Compare:

$$R = \sqrt{\alpha'}\ (k=1) \ \text{ to } \ R = \frac{\sqrt{\alpha'}}{2}\ (k=2) \ \simeq \ R = 2\sqrt{\alpha'} \tag{8.352}$$

$$U_L = u_L - i\frac{\sqrt{\alpha'}}{2}(n+w) ln(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m z^{-m} \ \ (k=1) \tag{8.353}$$

$$U_L = u_L - i\frac{\sqrt{\alpha'}}{2}(2n + \frac{w}{2}) ln(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m z^{-m} \ \ (k=2) \tag{8.354}$$

$$n \to 2n \quad w \to \frac{w}{2} \qquad \text{n is now even} \tag{8.355}$$

Example: Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad , \quad E = (n + 1/2)\hbar\omega \tag{8.356}$$

Wavefunctions are odd or even:

$$\begin{aligned} x \to -x \ : \quad \psi_0 \ &\to \ \psi_0 \\ \psi_1 \ &\to \ -\psi_1 \end{aligned} \tag{8.357}$$

Restrict hilbert space to even functions:

$$\psi(x) = \sum_{\text{n even}} a_n \psi_n(x) \tag{8.358}$$

$$\psi(x,t) = \text{even} \tag{8.359}$$

$$p \to -p \tag{8.360}$$

$$[H, p] = 0 \tag{8.361}$$

For general $\psi(x)$:

$$\begin{aligned} \psi_{even}(x) &= \frac{1}{2}\left(\psi(x) + \psi(-x)\right) = \frac{1}{2}\left(\psi(x) + P(\psi(x))\right) \\ &= \frac{1}{2}(1+P)\psi(x) \quad \text{P is the Projection Operator} \end{aligned} \tag{8.362}$$

$$\left(\frac{1}{2}(1+P)\right)^2 = \frac{1}{4}(1+2p+p^2) = \frac{1}{4}(2+2p) = \frac{1}{2}(1+p) \tag{8.363}$$

In String Theory, you have to look at the OPE:

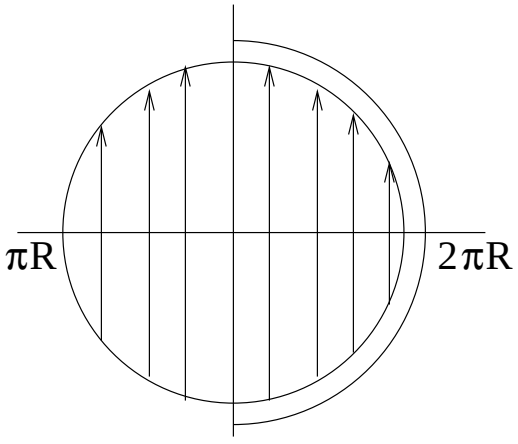
$$V_{k_L,k_R} : e^{ik_L U_L} e^{ik_R U_R} e^{ik\cdot X} : \tag{8.364}$$

$$V_{\substack{k_L,k_R \\ 2n}}(z,\bar{z}) V_{\substack{k'_L,k'_R \\ 2n'}}(0,0) \sim z^{\frac{\alpha'}{2}k_L k'_L} \bar{z}^{\frac{\alpha'}{2}k_R k'_R} V_{\substack{k_L+k'_L,k_R+k'_R \\ 2n+2n'}} \tag{8.365}$$

$$\text{Parity Operator}: \quad (-1)^n \qquad P : U \to -U \tag{8.366}$$

$$\frac{n}{R} = \quad \text{Charge} = \text{momentum in compact direction} \tag{8.367}$$

For $w \to \frac{w}{2}$: winding$\to$ 1/2 winding, string folds onto inself



-This is not a manifold, it's an orbifold

8.9 T-Duality (Open Strings)

$$\alpha^{\mu}_{-n}|0,k\rangle \quad \text{fell out of favor because it is so massive} \tag{8.368}$$

$$g_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0\rangle \quad \text{-massless (closed strings)} \tag{8.369}$$

-There is no winding number for open strings

$$\begin{aligned} \text{As } R \to 0 \text{ (extra dimension disappears), for } n &\neq 0, \ m \to \infty \\ n &= 0, \ m^2 = \frac{1}{\alpha'}(N-1) \end{aligned} \tag{8.370}$$

Closed strings interact with open strings
If you have open strings, you must have closed strings

Do the same thing with open strings as we did with the closed strings:

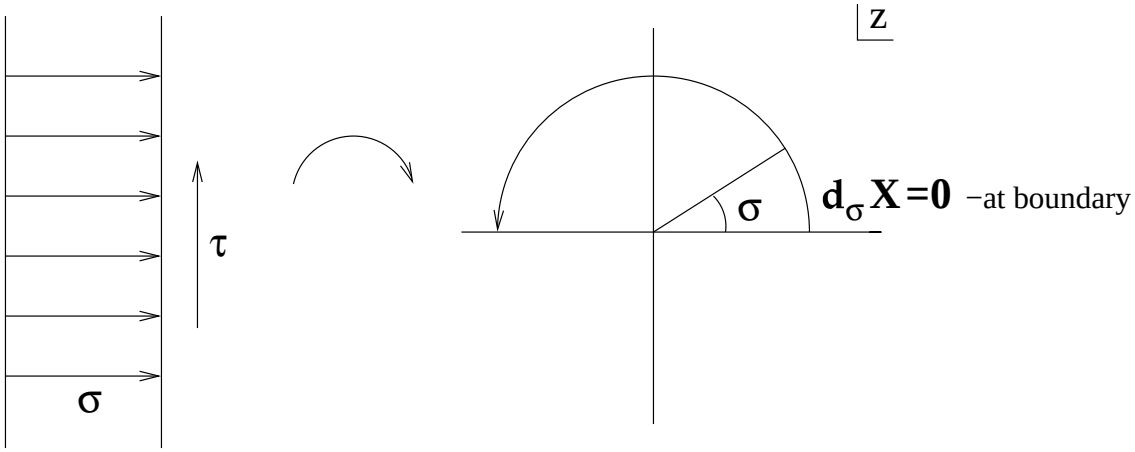
$$U = U_L + U_R \tag{8.371}$$

$$Z = U_L - U_R \tag{8.372}$$

$$U = u - i\alpha' p\, ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m\left(z^{-m} + \bar{z}^{-m}\right) \tag{8.373}$$

$$p = \frac{n}{R} \quad \text{-because compact dimension is quantized} \tag{8.374}$$

$$z = e^{\pi i\frac{(\sigma+\tau)}{l}}, \quad \bar{z} = e^{-\pi i\frac{(\sigma-\tau)}{l}}, \quad l = \pi \text{ for open strings } (l = 2\pi \text{ for closed strings}) \tag{8.375}$$



$$\partial_\sigma z = iz \tag{8.376}$$

$$\partial_\tau z = iz \tag{8.377}$$

$$\partial_\sigma \bar{z} = -i\bar{z} \tag{8.378}$$

$$\partial_\tau \bar{z} = i\bar{z} \tag{8.379}$$

$$\partial_\sigma U_L(z) = \partial_\sigma z \partial U = iz\partial U \tag{8.380}$$

$$\partial_\tau U_L(z) = iz\partial U \tag{8.381}$$

$$\partial_\sigma U_R(\bar{z}) = \partial_\sigma \bar{z}\bar{\partial} U = -i\bar{z}\bar{\partial} U \tag{8.382}$$

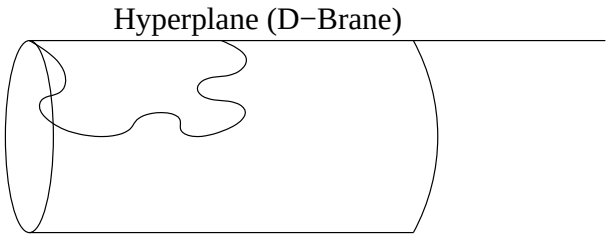
$$\partial_\tau U_R(\bar{z}) = i\bar{z}\bar{\partial} U \tag{8.383}$$

$$\partial_\tau U(z,\bar{z}) = iz\partial U + i\bar{z}\bar{\partial} U \tag{8.384}$$

$$\partial_\sigma Z(z,\bar{z}) = iz\partial U + i\bar{z}\bar{\partial} U = \partial_\tau U \tag{8.385}$$

$$\partial_\tau U(z,\bar{z}) = \partial_\sigma Z(z,\bar{z}) \tag{8.386}$$

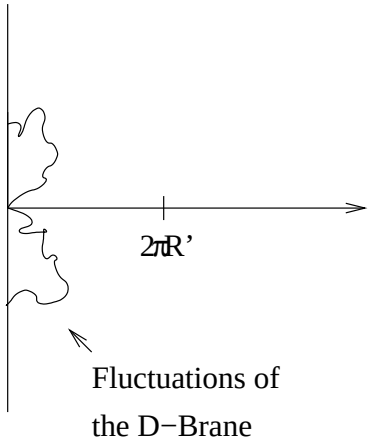
$$\begin{aligned} Z(\sigma=\pi) - Z(\sigma=0) &= \int_0^\pi d\sigma \partial_\sigma Z = \int_0^\pi d\sigma \partial_\tau U = \int_0^\pi d\sigma \partial_\tau (-i\alpha' \, p \, ln \, e^{2i\tau}) \\ &= \int_0^\pi d\sigma 2\alpha' p = 2\pi\alpha' p = 2\pi n \frac{\alpha'}{R} = 2\pi n R' \end{aligned} \tag{8.387}$$



Z at end = fixed (Dirchilet B.C.)
Endpoints see one less dimension
Massless Modes:

$$\alpha^\mu_{-1}|0,k\rangle \quad \alpha_{-1}|0,k\rangle \quad \mu \text{ means along uncompactified dimensions} \tag{8.388}$$

Photons never venture into extra dimensions



$$: A_\mu(k)\partial_\tau X^\mu e^{ik\cdot X} : \leftarrow \text{photon (transverse in uncompactified dimensions)} \tag{8.389}$$

$$\partial_\tau U e^{ik\cdot X} \tag{8.390}$$

$$\partial_\sigma Z e^{ik\cdot X} \tag{8.391}$$

-Brane is as fundamental as the string

8.10 Scattering

Closed Strings:

$$X(z,\bar{z}')X(0,0) \sim -\frac{\alpha'}{2} ln|z|^2 \tag{8.392}$$

$$\partial\bar{\partial}\left(X(z,\bar{z}')X(0,0)\right) \sim \partial\bar{\partial}\left(-\frac{\alpha'}{2} ln|z|^2\right) = -\frac{\alpha'}{2}\partial\bar{\partial} ln|z|^2 = -\pi\alpha'\delta^2(z,\bar{z}) \tag{8.393}$$

$$\therefore G(z,\bar{z}) = X(z,\bar{z}')X(0,0) \tag{8.394}$$

Electrostatics

$$\nabla^2 ln(r) \simeq \delta(r) \tag{8.395}$$

$$\partial_\tau G = 0 \quad - \text{Electrostatics} \tag{8.396}$$

$$\partial_\sigma G = 0 \quad - \text{Strings} \tag{8.397}$$

For E&M, put opposite charge at $\bar{z}$ (Method of Images)
For Strings, put same charge at $\bar{z}$

$$G(z,\bar{z},z',\bar{z}') = -\frac{\alpha'}{2} ln|z-z'|^2 - \frac{\alpha'}{2} ln|\bar{z}-z'|^2 \quad \text{-Interior of open strings} \tag{8.398}$$

$$\text{Let z,z' approach real axis} = -2\alpha' ln|z-z'| \quad \text{Open strings} \tag{8.399}$$

8.11 Amplitudes

Closed strings:

$$A_n = \langle : e^{ik_1 X}(z_1,\bar{z}_1) : ... : e^{ik_n X}(z_n,\bar{z}_n) : \rangle \tag{8.400}$$

$$\partial_{z_1} A_n = \langle : ik_1 \partial X e^{ik_1 X}(z_1,\bar{z}_1) : ... : e^{ik_n X}(z_n,\bar{z}_n) : \rangle \tag{8.401}$$

$$\begin{aligned} \partial X(z_1) e^{ik_i X}(z_i,\bar{z}_i) &\sim ik_i \left(-\frac{\alpha'}{2}\partial ln|z_1-z_i|^2\right) e^{ik_i X}(z_i,\bar{z}_i) \\ &\sim -ik_i \frac{\alpha'}{2}\frac{1}{z_1-z_i} e^{ik_i X}(z_i,\bar{z}_i) \end{aligned} \tag{8.402}$$

$$\therefore \partial_{z_1} A_n = \frac{\alpha'}{2}\sum_i k_1 k_i \frac{1}{z_1-z_i} A_n \quad \text{-Singular point} \tag{8.403}$$

$$\frac{\partial_{z_1} A_n}{A_n} = \frac{\alpha'}{2}\sum_i k_1 k_i \frac{1}{z_1-z_i} \tag{8.404}$$

$$\partial_{z_1} ln(A_n) = \frac{\alpha'}{2}\sum_i k_1 k_i \frac{1}{z_1-z_i} \tag{8.405}$$

$$ln(A_n) = \int d_{z_1} \frac{\alpha'}{2}\sum_i k_1 k_i \frac{1}{z_1-z_i} \tag{8.406}$$

$$A_n = e^{\int d_{z_1} \frac{\alpha'}{2} \sum_i k_1 k_i \frac{1}{z_1-z_i}} \tag{8.407}$$

$$\begin{aligned} \therefore A_n &= (z_1-z_i)^{\frac{\alpha' k_1 k_i}{2}} (\bar{z}_1-\bar{z}_i)^{\frac{\alpha' k_1 k_i}{2}} \quad \text{-For } z_1 \\ &= \prod (z_j-z_i)^{\frac{\alpha' k_i k_j}{2}} (\bar{z}_j-\bar{z}_i)^{\frac{\alpha' k_i k_j}{2}} \quad \text{-For all } z_n \\ &= \prod_{i<j} |z_i-z_j|^{\frac{\alpha' k_i k_j}{2}} \end{aligned} \tag{8.408}$$

Open Strings:

$$A_n = \prod_{i<j} |z_i-z_j|^{2\alpha' k_i k_j} \tag{8.409}$$

Chapter 9

Type I and Type II Superstrings

Following *"String Theory"* by J. Polchinski, Vol.II.
Notes written by students (work still in progress).
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So far we have dealt with Bosonic strings. This theory by itself can not be a physical theory because we see the tachyon. What we hope to find is a theory which eliminates the tachyon. We will add a new symmetry (supersymmetry) into the theory and see if it will eliminate the tachyon. Supersymmetry is a symmetry which exchanges Bosons and Fermions.

9.1 The superconformal algebra

If we look at the Bosons we used the Polyakov action the describe our world-sheet.

$$S_X = \frac{1}{2\pi\alpha'}\int d^2z\ \partial X^\mu \bar{\partial} X_\mu \tag{9.1}$$

From this action we derived the stress tensor and found the central charge of the theory. Since this central charge was not zero, we knew this could not describe a physical theory.

$$T(z) = \frac{-1}{\alpha'}\partial X^\mu \partial X_\mu \tag{9.2}$$

$$T(z)T(0) \sim \frac{c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms} \qquad \text{c} \neq 0 \tag{9.3}$$

In chapter 2 we also looked at the bc theory, where b and c are holomorphic anticommuting fields. Once again we calculated the stress tensor from an action and found the central charge for this theory.

$$S_{bc} = \frac{1}{2\pi}\int d^z\ b\bar{\partial} c \tag{9.4}$$

$$T(z) = (\partial b)c - \lambda\partial(bc) \tag{9.5}$$

$$T(z)T(0) \sim \frac{c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms} \qquad \text{c} = -3(2\lambda-1)^2 + 1 \tag{9.6}$$

where λ is the weight of b and $1-\lambda$ is the weight of c.

The special case when $\lambda = 2$ c=-26 and D=26. If one combines this theory with the theory of the Xs, the central charge should vanish in the TT OPE. From chapter 4 it was shown that in twenty-six dimensions the BRST charge is nilpotent. For reasons that should be clear lat, we will look at the case when b and c have the same weight.

What happens in the case when $h_b = h_c$?

$$\begin{array}{cc} h_b = h_c & \lambda = 1 \\ \hline 2 & c = 1 \end{array} \tag{9.7}$$

Break b and c in to linear combinations of two new fields ψ_1 and ψ_2.

$$b = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \qquad c = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \tag{9.8}$$

The action and stress tensor in terms of the new fields:

$$\frac{S = \frac{1}{2\pi}\int d^z\ 1}{2\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2} \tag{9.9}$$

$$\frac{T = -1}{2\psi_1\partial\psi_1 - \psi_2\partial\psi_2} \tag{9.10}$$

As we can see, this is just two copies of the same theory. We will define a theory from a single anticommuting field ψ.

Defing the action and stress tensor:

$$S_\psi = \frac{1}{4\pi}\int d^2z\ \psi^\mu\bar{\partial}\psi_\mu \qquad T = -\frac{1}{2}\psi\partial\psi \tag{9.11}$$

Summing over the index μ will just give D copies of ψ.

We can put the Ts for the X theory and the ψ together into one T, and find the OPE with itself.

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \tag{9.12}$$

$$T(z)T(0) = \frac{D + D/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms...} \qquad \mathrm{c_{X+\psi}} = \frac{3}{2}\mathrm{D} \tag{9.13}$$

ψ^μ has an infinite number of new symmetries.

Define world-sheet supercurrents T_F.

$$T(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\partial X_\mu(z) \qquad \tilde{T}(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu(\bar{z})\partial X_\mu(\bar{z}) \tag{9.14}$$

Look at the transformations on X and ψ to check that this supercurrent create supersymmetric transformations. In order to do this one must know the OPE of T with X or ψ.

$$T_F(z)X^\mu \sim i\sqrt{\frac{2}{\alpha'}}\left(-\frac{\alpha'}{2}\partial\ln|z|^2\right)\psi^\mu(0) \tag{9.15}$$

$$= -i\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\psi^\mu(z) \tag{9.16}$$

$$T_F(z)\psi^\mu \sim -i\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\partial X^\mu(0) \tag{9.17}$$

Now one can calculate the transformations for X and ψ.

$$\delta X^\mu = -i\epsilon \oint \frac{dz}{2\pi i}\, j(z) X^\mu(0,0) = -i\epsilon \left(-1\sqrt{\frac{\alpha'}{2}}\right) \eta\psi^\mu = -\epsilon\sqrt{\frac{\alpha'}{2}}\eta\psi^\mu \tag{9.18}$$

$$\delta\psi^\mu = -i\epsilon \oint \frac{dz}{2\pi i}\, j(z) \psi^\mu(0,0) = \epsilon\sqrt{\frac{\alpha'}{2}}\eta\partial X^\mu \tag{9.19}$$

We can see these transformations exchange X with ψ making them superconformal.
Look at the OPE of T and T_F and T_F with itself.

$$\begin{aligned} T(z)T_F(0) &\sim 2\left(\frac{\alpha'}{2}\partial^2 \ln|z|^2\right)\left(i\sqrt{\frac{2}{\alpha'}}\right)\left(-\frac{1}{\alpha'}\right)\psi^\mu(0)\partial X^\mu(z) + \partial\left(\frac{1}{z}\right)\left(-\frac{1}{2}\right)\left(i\sqrt{\frac{2}{\alpha'}}\right)\partial X^\mu(0)\psi^\mu(z) + \left(-\frac{1}{2}\right)\left(i\sqrt{\frac{2}{\alpha'}}\right) \\ &\sim i\sqrt{\frac{2}{\alpha'}}\frac{1}{z^2}\partial X^\mu(z)\psi_\mu(0) + \frac{i}{2}\sqrt{\frac{2}{\alpha'}}\frac{1}{z^2}\partial X_\mu(0)\psi^\mu(z) - \frac{i}{2}\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\partial\psi_\mu(z)\partial X^\mu(0) \end{aligned}$$

Expand around 0.

$$\sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \tag{9.22}$$

$$\begin{aligned} T_F(z)T_F(0) &\sim \left(i\sqrt{\frac{2}{\alpha'}}\right)^2\left(\frac{\alpha'}{2}\partial^2\ln|z|^2\right)\frac{D}{z} + \frac{1}{z}i\sqrt{\frac{2}{\alpha'}}\partial X_\mu(0)\partial X^\mu(0) + \left(i\sqrt{\frac{2}{\alpha'}}\right)^2\left(\frac{\alpha'}{2}\partial^2\ln|z|^2\right)\psi^\mu(z)\psi_\mu(0) && (9.23)\\ &\sim \frac{D}{z^3} - \frac{2}{\alpha'}\frac{1}{z}\partial X_\mu(0)\partial X^\mu(0) + \frac{1}{z^2}\psi^\mu(z)\psi_\mu(0) && (9.24)\\ &\text{but}\quad \psi^\mu(z) \sim \psi^\mu(0) + z\partial\psi^\mu(0) + ... && (9.25)\\ &\sim \frac{D}{z^3} - \frac{2}{\alpha'}\frac{1}{z}\partial X_\mu(0)\partial X^\mu(0) + \frac{1}{z}\partial\partial si^\mu(0)\psi_\mu(0) && (9.26)\\ &\sim \frac{D}{z^3} + \frac{2}{z}T(0) && (9.27) \end{aligned}$$