Chapter 10

Type I and Type II Superstrings

Following *"String Theory"* by J. Polchinski, Vol.II.
Notes written by students (work still in progress).
For more information contact George Siopsis
gsiopsis@utk.edu

So far we have dealt with the Bosonic string. This theory by itself can not be physical because of the existance of the tachyon. We wish to find a theory in which the lowest mass is a nonnegative value. One idea is to add a new symmetry into the theory and see if it will eliminate the tachyon. This new symmetry, called supersymmetry, will exchange Bosons and Fermions.

10.1 The superconformal algebra

The Polyakov action describes the world-sheet for bosons.

$$S_X = \frac{1}{2\pi\alpha'}\int d^2z\ \partial X^\mu \bar{\partial} X_\mu \tag{10.1}$$

From this action we derived the stress tensor and central charge of the theory. Since this central charge was not zero, there wasn't a physical theory.

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu \tag{10.2}$$

$$T(z)T(0) \sim \frac{c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms} \qquad \text{c} \neq 0 \tag{10.3}$$

In chapter 2 we studied the bc theory (where b and c are holomorphic anticommuting fields). Once again we calculated the stress tensor and central charge from the action.

$$S_{bc} = \frac{1}{2\pi}\int dz\ b\bar{\partial} c \tag{10.4}$$

$$T(z) = (\partial b)c - \lambda\partial(bc) \tag{10.5}$$

$$T(z)T(0) \sim \frac{c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms :} \qquad \text{c} = -3(2\lambda-1)^2 + 1 \tag{10.6}$$

where λ is the weight of b and $1-\lambda$ is the weight of c.
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For the special case $\lambda = 2$, c=-26 and D=26. Combining this theory with the theory of the X's should give a theory with a vanishing central charge. From chapter 4 it was shown that in twenty-six dimensions the BRST charge is nilpotent. For reasons that should be clear later, we will look at the case when b and c have the same weight.

What happens in the case when $h_b = h_c$?

$$h_b = h_c \qquad \lambda = \frac{1}{2} \qquad c = 1 \tag{10.7}$$

Break b and c in to linear combinations of two new fields ψ_1 and ψ_2.

$$b = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \qquad c = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \tag{10.8}$$

The action and stress tensor in terms of the new fields:

$$S = \frac{1}{2\pi}\int d^z \frac{1}{2}\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2 \tag{10.9}$$

$$T = -\frac{1}{2}\psi_1\partial\psi_1 - \psi_2\partial\psi_2 \tag{10.10}$$

As we can see, this is just two copies of the same theory. We will define a theory from a single anticommuting field ψ.

Defing the action and stress tensor:

$$S_\psi = \frac{1}{4\pi}\int d^2z\ \psi^\mu\bar{\partial}\psi_\mu \qquad T = -\frac{1}{2}\psi\partial\psi \tag{10.11}$$

Summing over the index μ will just give D copies of ψ.

10.1.1 World-sheet supercurrent

We can put the Ts for the X and ψ theories together into a new T, and find the new TT OPE.

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \tag{10.12}$$

$$T(z)T(0) = \frac{D + D/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms...} : \qquad \mathrm{c_{X+\psi}} = \frac{3}{2}\mathrm{D} \tag{10.13}$$

Define world-sheet supercurrents T_F:

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\partial X_\mu(z) \qquad \tilde{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu(\bar{z})\partial X_\mu(\bar{z}) \tag{10.14}$$

Look at the transformations on X and ψ to check that the supercurrents create supersymmetric transformations. In order to do this one must know the OPE of T with X or ψ.

$$\begin{aligned} T_F(z)X^\mu &\sim i\sqrt{\frac{2}{\alpha'}}\left(-\frac{\alpha'}{2}\partial\ln|z|^2\right)\psi^\mu(0) \\ &= -i\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\psi^\mu(0) \end{aligned} \tag{10.15}$$

$$T_F(z)\psi^\mu \sim -i\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\partial X^\mu(0) \tag{10.16}$$

Now one can calculate the transformations of X and ψ.

$$\begin{aligned}\delta X^\mu &= -i\epsilon \oint \frac{dz}{2\pi i}\, j(z) X^\mu(0,0) \\ &= -i\epsilon \left(-1\sqrt{\frac{\alpha'}{2}}\right) \eta\psi^\mu \\ &= -\epsilon\sqrt{\frac{\alpha'}{2}}\eta\psi^\mu \qquad (10.17)\end{aligned}$$

$$\begin{aligned}\delta\psi^\mu &= -i\epsilon \oint \frac{dz}{2\pi i}\, j(z) \psi^\mu(0,0) \\ &= \epsilon\sqrt{\frac{\alpha'}{2}}\eta\partial X^\mu \qquad (10.18)\end{aligned}$$

We can see these transformations exchange X with ψ making them superconformal.
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Evaluate the $T T_F$ and $T_F T_F$ OPEs.

$$\begin{aligned}T(z)T_F(0) &\sim 2\left(\frac{\alpha'}{2}\partial^2 \ln|z|^2\right)\left(i\sqrt{\frac{2}{\alpha'}}\right)\left(-\frac{1}{\alpha'}\right)\psi^\mu(0)\partial X^\mu(z) + \\ &\quad \partial\left(\frac{1}{z}\right)\left(-\frac{1}{2}\right)\left(i\sqrt{\frac{2}{\alpha'}}\right)\partial X^\mu(0)\psi^\mu(z) + \\ &\quad \left(-\frac{1}{2}\right)\left(i\sqrt{\frac{2}{\alpha'}}\right)\frac{1}{z}\partial\psi^\mu(z)X^\mu(0) \\ &\sim i\sqrt{\frac{2}{\alpha'}}\frac{1}{z^2}\partial X^\mu(z)\psi_\mu(0) + \frac{i}{2}\sqrt{\frac{2}{\alpha'}}\frac{1}{z^2}\partial X_\mu(0)\psi^\mu(z) - \frac{i}{2}\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\partial\psi_\mu(z)\partial X^\mu(0)\end{aligned}$$

Expand around 0.

$$T(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \qquad (10.19)$$

$$\begin{aligned}T_F(z)T_F(0) &\sim \left(i\sqrt{\frac{2}{\alpha'}}\right)^2\left(\frac{\alpha'}{2}\partial^2\ln|z|^2\right)\frac{D}{z} + \\ &\quad \frac{1}{z}i\sqrt{\frac{2}{\alpha'}}\partial X_\mu(0)\partial X^\mu(0) + \left(i\sqrt{\frac{2}{\alpha'}}\right)^2\left(\frac{\alpha'}{2}\partial^2\ln|z|^2\right)\psi^\mu(z)\psi_\mu(0) \\ &\sim \frac{D}{z^3} - \frac{2}{\alpha'}\frac{1}{z}\partial X_\mu(0)\partial X^\mu(0) + \frac{1}{z^2}\psi^\mu(z)\psi_\mu(0) \\ &\text{but}\quad \psi^\mu(z) \sim \psi^\mu(0) + z\partial\psi^\mu(0) + ... \\ &\sim \frac{D}{z^3} - \frac{2}{\alpha'}\frac{1}{z}\partial X_\mu(0)\partial X^\mu(0) + \frac{1}{z}\partial\partial si^\mu(0)\psi_\mu(0) \\ T_F(z)T_F(0) &\sim \frac{D}{z^3} + \frac{2}{z}T(0) \qquad (10.20)\end{aligned}$$

We can see the OPEs form a closed algrebra over T and T_F.

The number of supersymmetries is dependent on the number of supercurrents. For now we have described one supercurrent (T_F), and therefore our theory is an N=1 supersymmetric theory.

So far we have created a supersymmetric theory consisting of X's and ψ's. We will create one more supersymmetric theory consisting of the bc and $\beta\gamma$ theories and see what comes about when we combine them. In describing the supersymmetric bc $\beta\gamma$ theory, all the steps will be the same as earlier. We will go through the process of constructing T and T_F, then calculate the OPEs to show that the algebra is closed.

We start with the $\beta\gamma$ action.

$$S_{\beta\gamma} = \frac{1}{2\pi}\int d^2z\ \beta\bar{\partial}\gamma \tag{10.21}$$

where β and γ have weights λ' and $1-\lambda'$ respectively.

The only way we can get the supercurrent is by making a good guess and checking it. We will make a lucky guess and say it is of the form:

$$T_F \supset -\frac{1}{2}(\partial\beta)c \qquad\qquad h_{T_F} = \frac{3}{2} \tag{10.22}$$

We can look at the weight and find conditions on λ and λ'.

$$\begin{aligned} 1+\lambda'+1-\lambda &= \frac{3}{2} \\ \lambda' &= \lambda - \frac{1}{2} \end{aligned} \tag{10.23}$$

Now that we know how the λs relate we know the central charge the this new theory. Since these fields commute the will be an overall minus sign difference in the central charge compared to the bc theory.

$$c_{\beta\gamma} = 3(2\lambda'-1)^2 - 1 = 3(2\lambda-2)^2 - 1 \tag{10.24}$$

$$c_{Total} = c_{bc} + c_{\beta\gamma} = -3(4\lambda - 3) \tag{10.25}$$

And our new T_F becomes

$$T_F = -\frac{1}{2}(\partial\beta)c + \lambda'\partial(\beta c) - 2c\gamma \tag{10.26}$$

One can look at the interesting case when λ=2. Recall from chapter 2, when c=0, T is a tensor. Recall from chapter 4, this allowed us to find a BRST charge and quantize the system for D=26. In this new supersymmetric case c=-15. We can now plug in the value of c and see how many dimensions are needed to give zero central charge.

$$c = \frac{3}{2}D - 15 = 0 \Rightarrow D = 10. \tag{10.27}$$

For c=2, we have a quantizable theory if we include the two supersymmetric theories and D=10.
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The last theory we looked at in chapter 2 was the Linear Dilaton. We can look at various properties of this this and once again calculate the supercurrent.

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu X_\mu + V_\mu\partial^2 X^\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \tag{10.28}$$

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial\psi_\mu - i\sqrt{2\alpha'}V_\mu\partial\psi^\mu \tag{10.29}$$

We find the total central charge to be

$$c = \frac{3}{2}D + 6\alpha' V^\mu V_\mu. \tag{10.30}$$

10.2 Ramond and Neveu-Schwarz sectors

Recall in chapter 2 we looked at a CFT on the circle. This was mapped to the cylinder through w=$\sigma+i\tau$. From this mapping, one can see w=w+2π. We can apply this condition on the fields and get two different values. These two conditions produce sectors which we will name the Ramond and Neveu-Schwarz sectors.

$$\begin{aligned} \text{Ramond (R) } \psi^\mu(\mathrm{w}+2\pi) &= \psi^\mu(w) \quad \text{"anti}-\text{periodic...natural choice"} \quad (10.31)\\ \text{Neveu}-\text{Schwarz (NS) } \psi^\mu(\mathrm{w}+2\pi) &= -\psi^\mu(w) \quad \text{"periodic...unnatural choice"} \quad (10.32)\end{aligned}$$

When taking the antiholomorphic piece into account, there are four different combinations of sectors. Each of these combinations will be a diferent Hilbert space. We would like to find which combination of sectors meets all the conditions for a valid theory along with removing the tachyon.

Holomorphic		Antiholomorphic
NS	–	NS
NS	–	R
R	–	NS
R	–	R

The Neveu-Schwarz Sector:

To meet the approprite boundary conditions above for the NS sector we will look at w. As w translates by 2π, there should be an overall minus sign. If we guess to sum over odd modes, the desired result is achieved.

$$w = \mathrm{e}^{-\mathrm{i}(2\mathrm{m}+1)(\sigma+\tau)/2}$$

Let $\sigma \to \sigma + 2\pi$

$$\begin{aligned} w &\to \mathrm{e}^{-\mathrm{i}(2\mathrm{m}+1)[(\sigma+2\pi)+\tau)]/2}\\ &= \mathrm{e}^{-\mathrm{i}(2\mathrm{m}+1)(\sigma+\tau)/2}\mathrm{e}^{-\mathrm{i}(2\mathrm{m}+1)\pi}\\ &= -\mathrm{e}^{-\mathrm{i}(2\mathrm{m}+1)(\sigma+\tau)/2} \quad (10.33)\end{aligned}$$

Now expand the eigenfunctions in these odd modes.

$$\psi^\mu(\sigma+\tau) = \sqrt{i}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu e^{-ir(\sigma+\tau)} : \qquad r = \frac{2m+1}{2} \quad (10.34)$$

Transform into the z basis.

$$\begin{aligned} \psi^\mu(z) &= \left(\frac{\partial w}{\partial z}\right)^h \psi^\mu(\sigma+\tau), \qquad h=\frac{1}{2}, \qquad w=\sigma+\tau, \qquad \frac{\partial z}{\partial w} = iz\\ &= (iz)^{-h}\psi^\mu(\sigma+\tau)\\ &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu z^{-r-1/2} \quad (10.35)\end{aligned}$$

The $\psi\psi$ OPE and algebra are given by:

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z} \quad (10.36)$$

$$\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{10.37}$$

As before we could construct the L operator by finding the residue of T. We will do the same for the supercurrent, and show that the operator algebra is closed. This is expected since the set of OPEs closes. Recall:

$$T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \tag{10.38}$$

$$T(z)T(0) \sim \frac{c}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + ... \tag{10.39}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \qquad \text{"Virasoro Algebra"} \tag{10.40}$$

$$T(z)T_F(0) \sim \frac{3}{2}\frac{1}{z^2}T_F(0) + \frac{1}{z}\partial T_F(0)... \tag{10.41}$$

$$T_F(z) = \sum_{r\in\mathbb{Z}+1/2} G_r z^{-r-3/2} \tag{10.42}$$

Now look at the algebra of the Ls and Gs.

$$[L_m, O_n] = [(h-1)m - n]O_{m+n} \tag{10.43}$$

$$[L_m, G_r] = (\frac{1}{2}m - r)G_{m+r} \tag{10.44}$$

$$T_F(z)T_F(z') \sim \frac{D}{(z-z')^3} + \frac{2}{z-z'}T(z')... \tag{10.45}$$

$$G_r = \oint \frac{dz}{2\pi i}\, z^{r+\frac{1}{2}} T_F(z) \tag{10.46}$$

Choose contour that encloses z' and integrate over z first.

$$G_r = \oint \frac{dz}{2\pi i}\, z^{r+\frac{1}{2}} \left[\frac{D}{(z-z')^3}\right] + 2z'^{r+\frac{1}{2}}T(z') \tag{10.47}$$

$$f(z) = z^{r+\frac{1}{2}} \qquad f'(z) = (r+\frac{1}{2})z^{r-\frac{1}{2}} \qquad f''(z) = (r+\frac{1}{2})(r-\frac{1}{2})z^{r-\frac{3}{2}} \tag{10.48}$$

$$= \frac{D}{2}(r^2 - \frac{1}{4})z'^{r-\frac{3}{2}} + 2z'^{r+\frac{1}{2}}T(z') \tag{10.49}$$

Now integrate over z'.

$$\oint \frac{dz}{2\pi i}\, z'^{s+\frac{1}{2}} \left[\frac{D}{2}(r^2 - \frac{1}{4})z'^{r-\frac{3}{2}} + 2z'^{r+\frac{1}{2}}T(z')\right] \tag{10.50}$$

$$\{G_r, G_s\} = \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r+s,0} + 2L_{r+s} : \qquad \text{Neveu} - \text{SchwarzAlgebra} \tag{10.51}$$

The mode algebra is given as:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} && (10.52)\\
\{G_r, G_s\} &= \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r+s,0} + 2L_{r+s} && (10.53)\\
[L_m, G_r] &= (\frac{1}{2}m - r)G_{m+r} && (10.54)
\end{aligned}
$$

Plug in the expansions for X and ψ to calculate L and G from the residue of T.

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{4}\psi^\mu(\partial\psi_\mu) + \frac{1}{4}(\partial\psi_\mu)\psi^\mu \qquad (10.55)$$

$$
\begin{aligned}
L_m &= \sum_{n,n'} \oint \frac{dz}{2\pi i} z^{m+1} \frac{1}{\alpha'} \alpha^\mu_{n'} \alpha_{\mu n} z^{-n'-1} z^{-n-1} \left(\frac{\alpha'}{2}\right)\\
&\quad + \sum_{r,r'} \oint \frac{dz}{2\pi i} z^{m+1} \psi^\mu_{r'} \psi_{\mu r} z^{-r'-\frac{3}{2}} z^{-r-\frac{1}{2}} (r' - r)\\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{m-n}\alpha_{\mu n} - \frac{1}{4}\sum_{r\in\mathbb{Z}} \psi^\mu_{m-r}\psi_{\mu r}(m - 2r) && (10.56)
\end{aligned}
$$

$$
\begin{aligned}
G_r &= \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} + i\sqrt{\frac{2}{\alpha'}}\psi^\mu \partial X_\mu\\
&= \sum_{n,r'} \oint \frac{dz}{2\pi i} \alpha^\mu_m \psi_{\mu r'} z^{-r'-\frac{1}{2}} z^{-n-1} \Rightarrow r - r' = n\\
&= \sum_{n\in\mathbb{Z}} \alpha^\mu_n \psi_{\mu(r-n)} && (10.57)
\end{aligned}
$$

We still have to worry about normal ordering of the operators, but one can see only L_0 has problems.

$$
\begin{aligned}
[L_1, L_{-1}] &= 2(L_0 + a)\\
L_0|0\rangle &= 0\\
\langle 0|[L_1, L_{-1}]|0\rangle &= 2a\\
\langle 0|L_1 L_{-1}|0\rangle &= 2a && (10.58)
\end{aligned}
$$

$$L_{-1} = \frac{1}{2}\sum \alpha^\mu_{-1-n}\alpha_{\mu n} + \frac{1}{4}\sum (2r+1)\psi^\mu_{-1-r}\psi_{\mu r} \qquad (10.59)$$

Need to act on $|0\rangle$. We know :

$$\alpha^\mu_n|0\rangle = 0$$

$$\alpha^\mu_0|0\rangle \sim p^\mu|0\rangle = 0$$

Which leads us to:

$$L_{-1}|0\rangle = 0 \qquad \Rightarrow a = 0 \qquad (10.60)$$

10.2.1 The Ghosts

Earlier in chapter 2 we looked at two theories dealing with anticommuting and commuting operators. These two theories were the b,c and β, γ theories. The algebras are given as:

$$\{b_m, c_n\} = \delta_{m+n,0} \qquad [\gamma_r, \beta_s] = \delta_{r+s,0} \tag{10.61}$$

We found the OPEs and weights of the fields.

$$\begin{array}{ll} h_b = \lambda & h_c = 1 - \lambda\text{'} \\ h_\beta = \lambda & h_\gamma = 1 - \lambda\text{'} \end{array}$$

In chapter 2 we found that when $\lambda' = \lambda - \frac{1}{2}$ the theory is supersymmetric. When $\lambda = 2$, it was shown the central charge vanishes and a BRST charge can be calculated. For this case the weights of the fields are:

$$\begin{array}{ll} h_b = 2 & h_c = -1 \\ h_\beta = \frac{3}{2} & h_\gamma = -\frac{1}{2} \end{array}$$

We can write the expansions of the fields in terms of z.

$$b = \sum_{m\in\mathbb{Z}} b_m z^{-m-2} \qquad c = \sum_{m\in\mathbb{Z}} c_m z^{-m+1}$$

$$\beta = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \beta_r z^{-r-\frac{3}{2}} \qquad \gamma = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \gamma_r z^{-r+\frac{1}{2}}$$

With these expansions we want to find T in terms of z and then integrate to get the Ls. We will also look for a new generator "G" that will come from integrating T_F.

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z) \tag{10.62}$$

$$\begin{aligned} T(z) &= (\partial b)c - \lambda\partial(bc) + (\partial b)\gamma - \lambda'\partial(\beta\gamma) \\ &= \sum_{n',n} b_{n'}c_n(-n'-2)z^{-n'-n-2} - 2\sum_{n',n} b_{n'}c_n(-n'-n-1)z^{-n'-n-2} + \\ &\quad \sum_{r',r} \beta_{r'}\gamma_r(-r'-\frac{3}{2})z^{-r'-r-\frac{3}{2}} - \frac{3}{2}\sum_{r',r}\beta_{r'}\gamma_r(-r'-r-1)z^{-r'-r-2} \\ &= \sum_{n',n} b_{n'}c_n(n'+2n)z^{-n'-n-2} + \sum_{r',r}\beta_{r'}\gamma_r(3r+r')z^{-r'-r-2} \end{aligned} \tag{10.63}$$

To find the simple poles the conditions on n,n',r,r' are:

$$\begin{aligned} m - n' - n &= 0 \\ m - r' - r &= 0 \end{aligned}$$

After the integration L is found to be:

$$L_m = \sum_n b_{m-n}c_n(m+n) + \frac{1}{2}\sum_r \beta_{m-r}\gamma_r(m+2r) \tag{10.64}$$

Do the same for the new operator G.

$$G_r = \oint \frac{dz}{2\pi i}\, z^{r+\frac{1}{2}} T_F(z) \tag{10.65}$$

$$\begin{aligned} T_F(z) &= -\frac{1}{2}(\partial\beta)c + \lambda'\partial(\beta c) - 2b\gamma \\ &= -\frac{1}{2}\sum_{s,n}\beta_s c_n(-s-\frac{3}{2})z^{-s-n-\frac{3}{2}} + \frac{3}{2}\sum_{s,n}\beta_s c_n(-s-n-\frac{1}{2})z^{-s-n-\frac{3}{2}} \\ &\quad -2\sum_{s,n} b_n\gamma_s z^{-n-s-\frac{3}{2}} \\ &= -\frac{1}{2}\sum_{n,s}\beta_s c_n(2s+3n)z^{-s-n-\frac{3}{2}} - 2\sum_{n,s} b_n\gamma_s z^{-n-s-\frac{3}{2}} \end{aligned} \tag{10.66}$$

We find the conditions on r,n,s.

$$r - n - s = 0 \tag{10.67}$$

$$G_r = -\frac{1}{2}\sum_n \beta_{r-n}c_n(2r+n) - 2\sum_n b_n\gamma_{r-n} \tag{10.68}$$

Once again we have to normal order the operators.

$$[L_-, L_{-1}] = 2(L_0 + a)$$

What is the constant "a" equal to?

$$\langle|L_1, L_{-1}|\rangle = 2a$$

where L_{-1} is given by:

$$L_{-1} = \sum_n b_{-n-1}c_n(n-1) + \frac{1}{2}\sum_r (2r-1)\beta_{-r-1}\gamma_r \tag{10.69}$$

From this we can see conditions on n and r.

$$\begin{aligned} -1 < n \leq 0 \quad &: \qquad n = -1 \text{ term is present.} \\ -1 < r < 0 \quad &: \qquad r = -\frac{1}{2} \text{ term is present.} \end{aligned}$$

Now act on the ground state.

$$L_{-1}|0\rangle = -2b_0c_{-1}|0\rangle - \beta_{-\frac{1}{2}}\gamma_{-\frac{1}{2}}|0\rangle \tag{10.70}$$

$$\langle 0|l_1 = \langle 0|b_1c_0 + \langle 0|\beta_{\frac{1}{2}}\gamma_{\frac{1}{2}} \tag{10.71}$$

$$\langle 0|L_1L_{-1}|0\rangle = -2 + 1 = -1 \quad \Rightarrow \quad a = -1 + \frac{1}{2} = -\frac{1}{2} \tag{10.72}$$

10.2.2 Open String

Recall from the previous sections.

$$\alpha_n^\mu = \tilde{\alpha}_n^\mu$$

$$\begin{aligned} \partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1} \\ \bar{\partial} X^\mu(\bar{z}) &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu \bar{z}^{-m-1} \end{aligned}$$

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu : \qquad \text{(closed strings } \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\mathrm{p}^\mu)$$

$$\begin{aligned} \psi^\mu(z) &= \sum_r \psi_r^\mu z^{-r-\frac{1}{2}} \\ \tilde{\psi}^\mu(z) &= \sum_r \tilde{\psi}_r^\mu z^{-r-\frac{1}{2}} \end{aligned}$$

Closed strings consist of left-movers and right-movers. These two particles can be treated independently allowing us to look at combinations of boundary condidions, ie periodic for right-movers, antiperiodic for left-movers. For open strings this isn't the case. Therefore one can only have either the R or NS sector. Table shown below:

Closed String		**Open String**
NS	NS	NS
NS	R	R
R	NS	
R	R	

Spectrum

$$L_m|phys\rangle = 0 \text{ for } \mathrm{m} > 0 \tag{10.73}$$

$$L_{-m}|phys\rangle \cong 0 \tag{10.74}$$

Define

$$L_{-m}|phys\rangle = |\psi\rangle = \text{"null state"} \qquad \langle\psi|\mathrm{L_m L_{-m}}|\psi\rangle = 0 \tag{10.75}$$

$$\begin{aligned} \langle phys'||\psi\rangle &= \langle phys'|L_-m|phys\rangle \\ &= \langle phys|L_m|phys'\rangle^* \\ &= 0 \end{aligned} \tag{10.76}$$

Next, find H and we know the particle spectrum.

$$H|phys\rangle = 0 \qquad H = L_0 - \frac{1}{2} \tag{10.77}$$

$$\begin{aligned} H &= \frac{1}{2}\sum_n : \alpha_{\mu-n}\alpha_n^\mu : + \frac{1}{2}\sum_r : \psi_{-r}^\mu \psi_{\mu r} : - \frac{1}{2} \\ &= \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu} + \sum_{r=\frac{1}{2}}^{\infty} \psi_{-r}^\mu \psi_{r\mu} - \frac{1}{2} \\ &= \alpha' p^2 + N - \frac{1}{2} \qquad\qquad N_\alpha + N_\psi = N \\ &= 0 \end{aligned} \tag{10.78}$$

N=0 gives the lowest state.

$$\alpha' p^2 = \frac{1}{2} \qquad\qquad m^2 = -p^2 \tag{10.79}$$

$$m^2 = -\frac{1}{2\alpha'} < 0 \tag{10.80}$$

We see that the tachyon state is still there giving a bad theory. We expected this since all we did was add states to a bad theory which would not eliminate this state.

Look at the next state (N=$\frac{1}{2}$):

$$A_\mu(z)\psi_{-\frac{1}{2}}^\mu |0,k\rangle \qquad\qquad \alpha' p^2 = 0, \;\; m^2 = 0 \tag{10.81}$$

$$\begin{aligned} L_n|phys\rangle &= 0, \qquad\qquad n > 0 \\ G_r|phys\rangle &= 0, \qquad\qquad r > 0 \end{aligned} \tag{10.82}$$

Look at the $r = \frac{1}{2},\ n = 0$ term:

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_n^\mu \psi_{\mu(r-n)}$$

$$\begin{aligned} \alpha_0^\nu \psi_{\nu\frac{1}{2}} \psi_{-\frac{1}{2}}^\mu |0;k\rangle &= \alpha_0^\mu |0;k\rangle \qquad\qquad \alpha_0^\mu = \sqrt{2\alpha'} p^\mu \\ &= \sqrt{2\alpha'} k^\mu |0;k\rangle A_\mu \\ &\rightarrow A_\mu k^\mu = 0 \end{aligned} \tag{10.83}$$

This implies that the transverse polarization is only in eight dimensions.

$$L_{-n}|phys\rangle \simeq 0 \qquad\qquad G_{-r}|phys\rangle \simeq 0 \tag{10.84}$$

$$G_{-\frac{1}{2}}|0;k\rangle = \alpha_0^\mu \psi_{\mu-\frac{1}{2}}|0;k\rangle = \sqrt{2\alpha'} k\dot{\psi}_{-\frac{1}{2}}|0;k\rangle \tag{10.85}$$

Show the norm equals 0.

$$\langle 0;k|k\dot{\psi}_{\frac{1}{2}} k\dot{\psi}_{-\frac{1}{2}}|0;k\rangle \propto k^2 = 0 \tag{10.86}$$

$$\lambda k\dot{\psi}_{-\frac{1}{2}}|0;k\rangle + A\dot{\psi}_{-\frac{1}{2}}|0;k\rangle = A\dot{\psi}_{-\frac{1}{2}}|0;k\rangle \tag{10.87}$$

This is just a gauge transformation.

$$A^\mu = A^\mu + \lambda k^\mu : \qquad\qquad A^\mu = A^\mu + \partial^\mu \lambda$$

10.2.3 Closed String

$$L_0 = \tilde{L}_0 = 0$$

$$\begin{aligned} \frac{\alpha'}{4}p^2 + N - \frac{1}{2} &= 0 \\ \frac{\alpha'}{4}p^2 + \tilde{N} - \frac{1}{2} &= 0 \end{aligned} \tag{10.88}$$

We can see that with the N=0 state we get the tachyon. So let's look at the case when we get the massless state.

$$A_{\mu\nu}\psi^{\mu}_{-\frac{1}{2}}\psi^{\tilde{\mu}}_{-\frac{1}{2}}|0;k\rangle \qquad\qquad m^2 = 0 \tag{10.89}$$

Since each index has eight degrees of freedom, this case gives 8×8=64 degrees of freedom. We can this easily by writing A in terms of the trace, a symmetric tensor and an antisymmetric tensor and cound the independent components.

$$\begin{aligned} A_{\mu\nu} = \quad & \frac{1}{D-2}A^{\rho}_{\rho}\eta_{\mu\nu} + \qquad \text{"1 component"} \\ & \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) + \qquad \text{"}\frac{8\times 7}{2}\text{components} \\ & \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu} - \frac{2}{D-2}A^{\rho}_{\rho}\eta_{\mu\nu}) \qquad \text{"}\frac{8\times 9}{2} - 1\text{" components"} \end{aligned} \tag{10.90}$$

The next thing we have to do is eliminate the tachyon. Unless this particle can be eliminated, the theory is useless. The first thing we will do is look at an example of the harmonic oscillator to give us insight on the problem.

What happens if we put an infinite barrier right down the middle of the oscillator, ie. splitting it in half. What we see is that only the odd states of the oscillator will survive sice the even states don't meet the boundary conditions at x=0. Next, look at the parity operator. We can see the operator commutes with Hamiltonian, ie:

$$[H, P] = 0.$$

From this we can diagonalize the Parity and Hamiltonian simultaneously. When doing so, we notice if the states evolve they don't switch from a state of differing parity. An even state stays an even state and so on. Now we see we can break the Hilbert space into two seperate Hilbert subspaces (symmetric, antisymmetric).

$$\mathcal{H} \to \mathcal{H}_s \oplus \mathcal{H}_a$$

10.2.4 Lorentz Group

In three-dimensions the angular momentum, $\vec{L} = \vec{r} \times \vec{p}$ is described by the group SO(3).

$$\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= zp_y - yp_x \end{aligned}$$

L can be written as an antisymmetric tensor.

$$L_{ij} = x_i p_j - x_j p_i \tag{10.91}$$

In three dimensions any antisymmetric tensor can be written as a vector.

$$L^i = \frac{1}{2}\epsilon^{ijk} L_{jk} \tag{10.92}$$

Each L describes a rotation about the ith component.
We can generalize L to n dimensions.

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \qquad \text{where } [\mathrm{x}_\mu, \mathrm{p}_\nu] = \mathrm{i}\eta_{\mu\nu} \tag{10.93}$$

The new Ls desribe the group SO(n-1,1) (Lorentz Group), where the 1 comes from the negative component in the metric.
$\vec{L}$ generates rotations:

$$\begin{aligned} \delta x_i &= [L_{kl}, x_i]\left(-\frac{i}{2}\omega^{kl}\right) = \omega^{ij} x_j \qquad\qquad \omega^{ij} = -\omega^{ji} \\ \delta\vec{x} &= \vec{\omega} \times \vec{x} \end{aligned} \tag{10.94}$$

Look at the 0,1 component:

$$\begin{aligned} L_0 1: \qquad \delta t &= -\omega_{01} x \qquad\qquad t \to t - \omega_{01} x \\ \delta x &= -\omega_{01} t \qquad\qquad x \to x - \omega_{01} t \end{aligned}$$

Write this in a more familiar form.

$$\begin{aligned} t &\to \gamma(t - vx) \\ x &\to \gamma(x - vt) \end{aligned}$$

These tranformations describe boosts.
The algebra of the Lorentz generators is given by:

$$\begin{aligned} [L_{\mu\nu}, L_{\sigma\rho}] &= [x_\mu p_\nu - x_\nu p_\mu, x_\rho p_\sigma - x_\sigma p_\rho] \\ &= i(\eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho}) \end{aligned} \tag{10.95}$$

We would like to add spin.

$$\vec{L} \to \vec{L} + \vec{S}$$

This is equivalent to saying:

$$L_{\mu\nu} \to L_{\mu\nu} + \Sigma_{\mu\nu} \qquad \text{where } \Sigma \text{ acts on spinors.} \tag{10.96}$$

The Σs obey the same algebra as the Ls and commute with the Ls.
We have built the Ls with X, we will build the Σs with ψ.

$$\Sigma^{\mu\nu} = -\frac{i}{2}\sum_r [\psi_r^\mu, \psi_r^\nu] \qquad\qquad \{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{10.97}$$

and

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$$

For ten dimensions there are five independent Σs: $\Sigma^{01}\Sigma^{23}\Sigma^{45}\Sigma^{67}\Sigma^{89}$.

These five operators commute with each other, so we can diagonalize them simultaneously. First see how they operate on the spinor.

$$\begin{aligned}
[\Sigma^{23}, \psi_r^2] &= -\frac{i}{2}\sum_s [\psi_s^2\psi_s^3, \psi_r^2] \\
&= -\frac{i}{2}\sum_s \left(\psi_s^2\psi_s^3\psi_r^2 - \psi_r^2\psi_s^2\psi_s^3\right) \\
&= i\sum_s \{\psi_s^2, \psi_r^2\}\psi_{-s}^3 \\
&= i\psi_r^3 \qquad (10.98)
\end{aligned}$$

$$\begin{aligned}
\left[\Sigma^{23}, \psi_r^3\right] &= -[\Sigma^{32}, \psi_r^3] \\
&= -i\psi_r^2 \qquad (10.99)
\end{aligned}$$

The algebra closes. Find an eigenvector of the spin operator.

$$[\Sigma^{23}, \psi_r^2 \pm i\psi_r^3] = i\psi_r^3 \pm \psi_r^2 \qquad \text{eigenvalue} = \pm 1 \qquad (10.100)$$

Finite Rotations

$$U(\theta) = e^{i\theta\Sigma^{23}}$$

$$U(\theta)(\psi_r^2 \pm i\psi_r^3)U^\dagger(\theta) = e^{\overset{+}{-i\theta}}(\psi_r^2 \pm i\psi_r^3) \qquad (10.101)$$

$$\begin{aligned}
[\Sigma^{23}, v] &= v: \qquad\qquad v = \psi_r^2 + i\psi_r^3 \\
\Sigma^{23}v - v\Sigma^{23} &= \\
\Sigma^{23}v &= v(1 + \Sigma^{23}) \qquad (10.102)
\end{aligned}$$

$$\sum_n \frac{(\Sigma^{23})^n}{n!}v = \sum_n v\frac{(1+\Sigma^{23})^n}{n!}(i\theta)^n \qquad (10.103)$$

$$\begin{aligned}
e^{i\theta\Sigma^{23}}v &= ve^{i\theta(1+\Sigma^{23})} \\
&= e^{i\theta}ve^{i\theta\Sigma^{23}} \\
e^{i\theta\Sigma^{23}}ve^{-i\theta\Sigma^{23}} &= e^{i\theta}v \qquad (10.104)
\end{aligned}$$

Special case ($\theta = \pi$):

$$\begin{aligned}
U(\pi)vU^\dagger(\pi) &= e^{i\pi}v = -v \\
U(\pi)v'U^\dagger(\pi) &= e^{-i\pi}v' = -v' \\
U(\pi)v &= -vU(\pi) \qquad (10.105)
\end{aligned}$$

$$\Rightarrow \{U(\pi), v\} = 0 \text{ for v} = \psi_{\mathrm{r}}^{2,3} \qquad (10.106)$$

and

$$[U(\pi), v] = 0 \text{ for v} = \psi^{\mu}_{\text{r}} : \text{ where } \mu \neq 2, 3 \tag{10.107}$$

Look at the case when r>0:

$$\begin{aligned} U(\pi)\psi^2_{-r}|0\rangle &= -\psi^2_{-r}U(\pi)|0\rangle : \qquad \Sigma^{23}|0\rangle = 0 \\ &= -\psi^2_{-r}|0\rangle \end{aligned} \tag{10.108}$$

$$\begin{aligned} U(\pi)\psi^2_{-r_1}\psi^3_{-r_2}|0\rangle &= -\psi^2_{-r_1}U(\pi)\psi^3_{-r_3}|0\rangle \\ &= -\psi^2_{-r_1}\psi^3_{-r_2}U(\pi)|0\rangle \\ &= -\psi^2_{-r_1}\psi^3_{-r_2}|0\rangle \end{aligned} \tag{10.109}$$

Can multiply all the Σs ($e^{i\pi\Sigma^{01}}e^{i\pi\Sigma^{23}}e^{i\pi\Sigma^{45}}e^{i\pi\Sigma^{67}}e^{i\pi\Sigma^{89}}$):

We will define $e^{i\pi\Sigma^{01}}$ as $e^{\pi\Sigma^{01}}$, due to the minus sign in the anticommutator for the zero component.

Define a new operator which is the product of the Σs.

$$e^{i\pi F} \ : \ F = -i\Sigma^{01} + \Sigma^{23} + \Sigma^{45} + \Sigma^{67} + \Sigma^{89} \tag{10.110}$$

$$e^{i\pi F}\psi^{\mu}_{r} = -\psi^{\mu}_{r}e^{i\pi F}$$

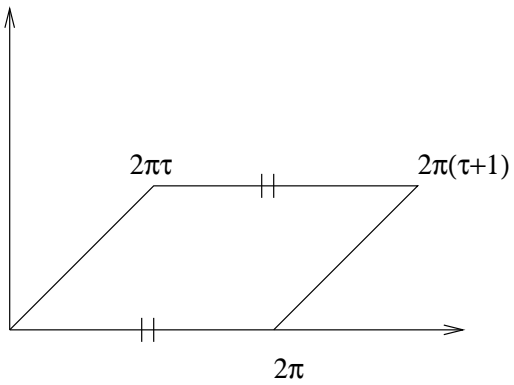
The eigenspaces of $e^{i\pi F}$ is +1,-1.

Subtlety:

$$e^{i\pi F} \rightarrow e^{i\pi F}U_{gh} \qquad\qquad U_{gh}|0\rangle = -|0\rangle$$

U_{gh} breaks the eigenspace into 2 subspaces of even and odd states. We can denote these two subspaces as NS+ and NS- for the +1 and -1 eigenvalue. Like the harmonic oscillator these two space do not talk to each other therefore we can look at a theory in one of the subspaces and throw the other away. We can see that in the NS- subspace the tachyon still exists therefore we will work with the NS+ subspace.

We must check modular invariance once again.



Remembering the partition function:

$$\begin{aligned} Z_X &= Tr\ q^{L_0+\tilde{L}_0} \quad q = e^{2\pi i} \quad \tau = \tau_1 + i\tau_2 \\ &= \left(\frac{1}{2\pi\sqrt{\alpha'}\tau_2}|\eta(\tau)|^2\right)^D : \ \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \end{aligned} \tag{10.111}$$

Must show invariance under the modular transformations:

$$\tau \to \tau + 1, \quad \tau \to -\frac{1}{\tau}$$

It is easy to show the ψs are not modular invariant.

$$\begin{aligned} \psi^\mu(\sigma + 2\pi) &= -\psi^\mu(\sigma) \\ \psi^\mu(z + 2\pi) &= -\psi^\mu(z) \\ \psi^\mu(z + 2\pi\tau) &= -\psi^\mu(z) \\ \psi^\mu(z + 2\pi(\tau+1)) &= \psi^\mu(z) \end{aligned} \tag{10.112}$$

The ψ has periodic boundary conditions when they should be antiperiodic.To fix this problem we must include the Hilbert space with periodic boundary conditions, (Ramond Sector).

$$\psi^\mu(\sigma + 2\pi) = \psi(\sigma)$$

The mode expansion will change.

$$\psi^\mu(z) = \sum_{n\in\mathbb{Z}} \psi^\mu_n z^{-n-\frac{1}{2}} \tag{10.113}$$

This is no longer a Laurent expansion and involves a branch cut. Next look at the cooresponding operators L and G and find the normal ordering constant. We see there will be a problem when normal ordering L_0.

$$[L_1, L_{-1}] = 2L_0$$

$$\langle 0|L_1 L_{-1}|0\rangle = 2\langle 0|L_0|0\rangle = 2a$$

Find a to normal order.

$$L_{-1}|0\rangle = \left(\frac{1}{2}\sum_n \alpha^\mu_{-1-n}\alpha_{\mu n} + \frac{1}{4}\sum_n (2n+1)\psi^\mu_{-1-n}\psi_{\mu n}\right)|0\rangle \tag{10.114}$$

We must find the terms which do not kill the vacuum.

$$-1 - n < 0 \ \text{ and } \ -1 < \text{n} < 0 \ \text{ for } \alpha \text{ terms}$$

The α terms don't contribute.

$$-1 - n \leq 0 \ \text{ and } \ -1 \leq \text{n} \leq 0 \ \text{ for } \psi \text{ terms}$$

The ψ terms contribute.

$$\begin{aligned} L_{-1}|0\rangle &= \frac{1}{4}\left(\psi^\mu_{-1}\psi_{\mu 0} - \psi^\mu_0\psi_{\mu -1}\right)|0\rangle \\ &= \frac{1}{2}\psi^\mu_{-1}\psi_{\mu 0}|0\rangle \end{aligned} \tag{10.115}$$

$$\begin{aligned} 2a &= \frac{1}{4}\langle 0|\psi_{\nu 0}\psi_1^\nu \psi_{-1}^\mu \psi_{\mu 0}|0\rangle \\ &= \frac{1}{4}\langle 0|\psi_{\mu 0}\psi_0^\mu|0\rangle \\ &= \frac{1}{8}\langle 0|\{\psi_{\mu 0},\psi_0^\mu\}|0\rangle \\ &= \frac{D}{8} \\ a &= \frac{D}{16} \end{aligned} \tag{10.116}$$

The ghosts:

$$\begin{aligned} b(z) &= \sum b_m z^{-m-2} \qquad & c(z) &= \sum c_m z^{-m+1} \\ \beta(z) &= \sum \beta_m z^{-m-3/2} \qquad & \gamma(z) &= \sum \gamma_m z^{-m+1/2} \end{aligned}$$

These are not Laurent expansions and have the same commutation rules.
We now have a zero mode for $[\gamma_0, \beta_0] = 1$
Define:

$$\beta_0|0\rangle = 0 \tag{10.117}$$
$$b_0|0\rangle = 0 \tag{10.118}$$

Normal ordering

$$\langle 0|L_1 L_{-1}|0\rangle = 2\langle 0|L_0|0\rangle = 2a$$

$$L_{-1}|0\rangle = \left(\sum (n-1) b_{-1-n} c_n + \frac{1}{2}\sum (2n-1)\beta_{-n-1}\gamma_n\right)|0\rangle \tag{10.119}$$

We must find the terms which do not kill the vacuum.

$$n \leq 0 \text{ and } -1 - \mathrm{n} < \mathrm{n} < 0 \Rightarrow \mathrm{n} = 0 \text{ for bc terms}$$

$$-1 - n < 0 \text{ and } \mathrm{n} \leq 0 \Rightarrow \mathrm{n} = 0 \text{ for } \beta\gamma \text{ terms}$$

Both sets contribute.

$$L_{-1}|0\rangle = \left(-b_{-1}c_0 - \frac{1}{2}\beta_{-1}\gamma_0\right)|0\rangle \tag{10.120}$$

$$\begin{aligned} a &= \frac{1}{2}\langle 0|C_1 b_0 b_{-1} c_0|0\rangle - \frac{1}{4}\langle 0|\beta_1\gamma_1\beta_{-1}\gamma_0|0\rangle \\ &= -\frac{1}{2} - \frac{1}{8} \\ &= -\frac{5}{8} \end{aligned} \tag{10.121}$$

We see that when D=10 the a from the (X,ψ) give the same value of a.
Spectrum:

How do you define $|0\rangle$?

$$\psi_m^\mu|0\rangle = 0, \quad m > 0 \tag{10.122}$$

$$\psi_0^\mu|0\rangle =? \qquad\qquad \{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \tag{10.123}$$

$$\psi_m^\mu, \psi_0^\nu|0\rangle = -\psi_0^\nu, \psi_m^\mu|0\rangle = 0 \tag{10.124}$$

$|0\rangle$ is a spinor.
spins :

$$\Sigma^{01}_{s_1=\pm\frac{1}{2}}\Sigma^{23}_{s_2=\pm\frac{1}{2}}\Sigma^{45}_{s_3=\pm\frac{1}{2}}\Sigma^{67}_{s_4=\pm\frac{1}{2}}\Sigma^{89}_{s_5=\pm\frac{1}{2}}$$

Now we can describe the state as:

$$|s_1, s_2, s_3, s_4, s_5\rangle$$

$$L_0 : \quad \frac{D}{16} - \frac{5}{8} = \frac{10}{16} - \frac{5}{8} = 0 \tag{10.125}$$

$$H = \alpha' p^2 + N$$

The lowest level is when N=0.

$$-p^2 = 0 = m^2$$

The lowest level is 0. The tachyon is gone!
Next look at how L and G operate on the state.

$$L_m, G_m|\vec{s}; k\rangle = 0 \quad m \geq 0 \tag{10.126}$$

What about when m=0?

$$\begin{aligned} G_0|\vec{s}; k\rangle &= 0 \\ \alpha_{\mu 0}\psi_0^\mu|\vec{s}; k\rangle &= 0 \\ k_\mu\gamma^\mu|\vec{s}; k\rangle &= 0 \quad \text{Dirac equation} \end{aligned} \tag{10.127}$$

$$\{G_m, G_n\} = 2L_{m+n}$$

Look at when m=n=0:

$$\{G_0, G_0\} = 2L_0 \Rightarrow G_0^2 = L_0 \tag{10.128}$$

So from all this we see that the lowest eigenvalue of H is zero, which is very nice. Also we see that this predicts the cosmological constant to be zero, which may seem bad.

It is possible for superstring theories to have periodic or antiperiodic sectors, R or NS, and also different eigenvalues of $e^{i\pi F}$, ± 1,

$$e^{i\pi F} = \begin{cases} +1 & NS+ \quad R+ \\ -1 & NS- \quad R- \end{cases}$$

NS containing tachyon. All superstring thoeries can be categorized by NS$\pm$ and R$\pm$. For close strings

$$\begin{array}{llll} IIA & (NS+,NS+) & (R+,NS+) & (NS+,R-) & (R+,R-) \\ IIB & (NS+,NS+) & (R+,NS+) & (NS+,R+) & (R+,R+) \\ IIA' & (NS+,NS+) & (R-,NS+) & (NS+,R+) & (R-,R+) \\ IIB' & (NS+,NS+) & (R-,NS+) & (NS+,R-) & (R-,R-). \end{array}$$

Type IIA is actually the same theory as type IIA′, shown by e.g. letting $x^9 \to -x^9$, $\psi^9 \to -\psi^9$ and $\tilde{\psi}^9 \to -\tilde{\psi}^9$, then

$$\Sigma^{8,9}|0>=-\Sigma^{8,9}|0> \Longrightarrow R+ \longleftrightarrow R-$$

also the same as type IIB and IIB′. For open strings there is only type I NS+ R+.

Modular invariance

NS+ is not modular invariance.

$$Z_{NS+} = \mathrm{Tr} q^H, \quad q = e^{2\pi i \tau}$$

If $|\psi> \epsilon$ NS+ then $e^{i\pi F} = +|\psi>$ and the projection operator of $|\psi>$ is

$$P = \frac{1}{2}(1 + e^{i2\pi F})$$

and let define a parity operation $\bar{P}\psi(x) = \psi(-x)$ with $\bar{P}^2 = 1$ then

$$P = \frac{1}{2}(1+\bar{P}), \quad P^2 = P$$

$$P\psi(x) = \frac{1}{2}(\psi(x) + \psi(-x))$$

$$P^2 = \frac{1}{4}(1+e^{i\pi F})^2 = \frac{1}{4}(2+2e^{i\pi F}) = P$$

$$e^{i\pi F}P|A>=\frac{1}{2}(e^{i\pi F}+1)|A>=+P|A>$$

then the partition function is

$$\mathrm{Tr}_{NS+} q^H = \mathrm{Tr} P q^H = \frac{1}{2}\mathrm{Tr} q^H + \frac{1}{2}\mathrm{Tr} e^{i\pi F} q^H.$$

From a creation operation ψ^μ_{-r}, $r > 0$ and $r \epsilon \mathcal{Z} + \frac{1}{2}$, if fix μ and r, the corresponding vector states can be

$$...\psi^\mu_{-r}...|\psi>, \to H = 0, \ g^0 = 1, \text{or}$$

$$.....|\psi>, \to H = \alpha' p^2 + N - \frac{1}{2} = r, \ q^r$$

then

$$\mathrm{Tr}_{\mathrm{NS}}\ q^H = \left(q^{-\frac{1}{48}} \prod_r (1+q^r) \right)^8. \tag{10.129}$$

Jocobi ϑ-functions

$z = e^{2\pi\nu}$, $q = e^{2\pi\tau}$

$$\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-\frac{1}{2}})(1+\frac{1}{z}q^{m-\frac{1}{2}}) && (10.130)\\
\vartheta_{01}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-\frac{1}{2}})(1-\frac{1}{z}q^{m-\frac{1}{2}}) && (10.131)\\
\vartheta_{10}(\nu,\tau) &= 2e^{\pi i\tau/4}\cos\pi\nu\prod_{m=1}^{\infty}(1-q^m)(1+zq^m)(1+\frac{1}{z}q^m) && (10.132)\\
\vartheta_{11}(\nu,\tau) &= -2e^{\pi i\tau/4}\sin\pi\nu\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-\frac{1}{z}q^m) && (10.133)\\
\eta(\tau) &= q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m)
\end{aligned}$$

then ϑ_{00} can be writen as

$$\begin{aligned}
\vartheta_{00}(0,\tau) &= \prod_{m=1}^{\infty}(1-q^m)\left(\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}})\right)^2\\
&= \eta(z)\left(q^{-\frac{1}{48}}\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}})\right)^2 \quad \text{then} && (10.134)\\
\mathrm{Tr}_{\mathrm{NS}}q^H &= \left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right)^4\\
\mathrm{Tr}_{\mathrm{NS}}e^{-\pi F}q^H &= -\left(q^{\frac{1}{48}}\prod_{r=1/2}(1-q^r)\right)^8\\
&= -\left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^4. && (10.135)
\end{aligned}$$

The factor -1 in front of $q^{1/48}$ comes ghost. The partition function for $R+$ can be calculated the same way.

$$\begin{aligned}
Z_{R+} &= \mathrm{Tr}_{\mathrm{R}}Pq^H\\
&= \frac{1}{2}\mathrm{Tr}_{\mathrm{R}}\ q^H + \frac{1}{2}\mathrm{Tr}_{\mathrm{R}}\ e^{i\pi F}q^H. && (10.136)
\end{aligned}$$

From creation operators, fixing μ, m, ψ^{μ}_{-m}, $m > 0$, if there is no ψ , then $q^{0+1/16}$, where 1/16 comes from $L_0 - a$, $a = 1/16$ but if there is one ψ, then $q^{m+\frac{1}{16}}$,or $q^{\frac{1}{16}(1+q^m)}$ for both.

$$\mathrm{Tr}_{\mathrm{R}}q^H = -\left(q^{-\frac{1}{48}}q^{-\frac{1}{16}}\prod_{m=1}^{\infty}(1+q^m)\right)^8\cdot[\text{zero mode}]\,.$$

The ground state is spinor, $|\vec{s}>$ with two eigenvectors, $|\uparrow>$, $|\downarrow>$ corresponding to $\pm 1/2$ for each operator $\Sigma^{i,i+1}$. Then ϑ is

$$\begin{aligned}
\vartheta_{01}(0,\tau) &= 2q^{1/8}\prod_{m}^{\infty}(1-q^m)\left(\prod_{m}^{\infty}(1+q^m)\right)^2\\
&= 2q^{1/8}q^{-1/24}\eta(z)\left(\prod_{m}^{\infty}(1+q^m)\right)^2
\end{aligned}$$

$$\frac{\vartheta_{01}(0,\tau)}{\tau(\eta)} = 2q^{1/16}q^{-1/48}\left(\prod_m^\infty(1+q^m)\right)^2 \tag{10.137}$$

$$\mathrm{Tr}_{\mathrm{R}}\ q^H = -\left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)}\right)^4. \tag{10.138}$$

From $\Sigma^{2,3}$ with eigenvectors $|\uparrow>$, $|\downarrow>$, the operator $e^{i\pi\Sigma^{2,3}}$ will give eigenvalues,+1 and −1, then $\mathrm{Tr}_{\mathrm{R}}\ e^{i\pi F}q^H$ will be

$$\mathrm{Tr}_{\mathrm{R}}\ e^{i\pi F}q^H = \left(\frac{\vartheta_{11}(0,\tau)}{\eta(\tau)}\right)^4. \tag{10.139}$$

Then the partition function for ψ is

$$Z_\psi = \frac{1}{2\eta^4(\tau)}\left(\vartheta_{00}^4(0,\tau) - \vartheta_{01}^4(0,\tau) - \vartheta_{10}^4(0,\tau) + \vartheta_{11}^4(0,\tau)\right) \tag{10.140}$$

But $\vartheta_{00}^4(0,\tau) - \vartheta_{01}^4(0,\tau) - \vartheta_{10}^4(0,\tau) = 0$,'abstruse identity', and $\vartheta_{11}^4(0,\tau)$ is zero then the total partition function, $Z_\psi Z_{\tilde{\psi}}$, is zero ,also does the cosmological constant,ie.

$$H = Q^2. \tag{10.141}$$

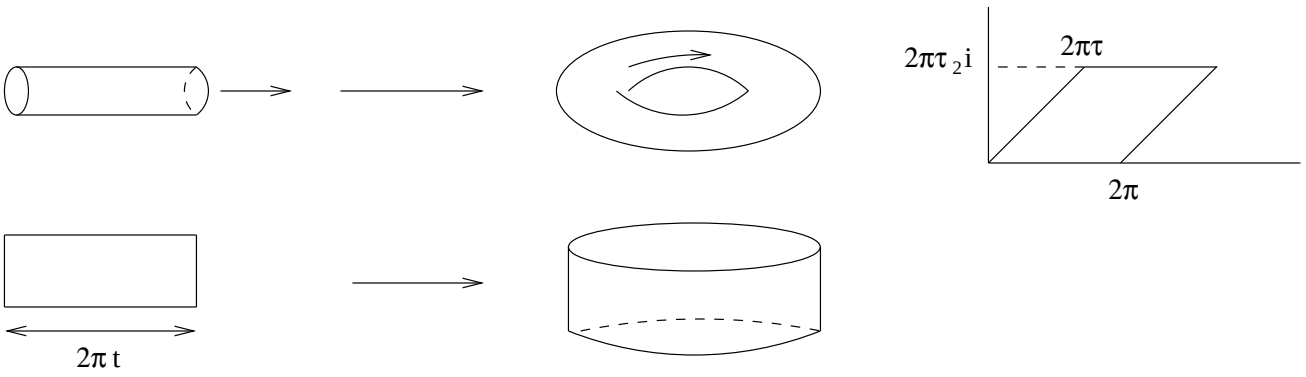
The partition function for type II A,B is

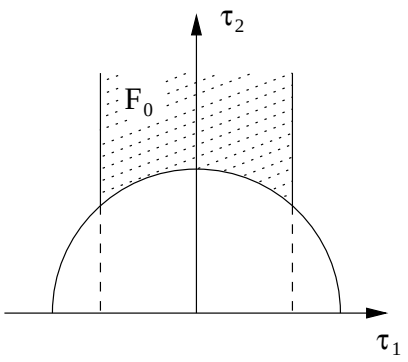
$$Z(\tau) = Z_x Z_\psi Z_{\tilde{\psi}} \tag{10.142}$$

and the total partition functionis

$$Z = \int_{F_0}\frac{d^2\tau}{\tau_2}Z(\tau)$$

For the type I (open strings) consider a periodic propergating open string as a close string moving from one end of open string to the other.





As $\tau_2 \to \infty$ contributes to long-distant effect or low-energy effct, IR(infared) and for the low mass case contributes to high-energy effect, UV, causing field theory diverged. From the picture, the area of F_0 does not contain UV region, then there is no UV divergences in string theory. From the picture the integration parameter can be changed to $0 < t < \infty$, ie.

$$Z = \int_0^\infty \frac{dt}{2t} Z(t) \tag{10.143}$$

the factor 2 comes from a group translation on a circle.

$$\begin{aligned} Z(t) &= \mathrm{Tr} q^H, \quad q = e^{2pi\tau} = e^{-2\pi t} \\ &= \mathrm{Tr} e^{-2\pi t L_0} \end{aligned} \tag{10.144}$$

where X^μ, $\mu = 0, 1, ..., D-1$. On torus the partition function $Z(t)$ is

$$\begin{aligned} Z(t) &= \mathrm{Tr} q^{L_0} q^{-\tilde{L}_0} \\ &= iV \left(\frac{1}{\sqrt{8\pi^2 \alpha' \tau_2}} \right)^D |\eta(\tau)|^{-2(D-2)} \end{aligned} \tag{10.145}$$

where V is the volume of space and time, $1/\sqrt{8\pi^2\alpha' t}$ is from zero mode(momentum). Then the open string the partition function $Z(t)$ is

$$Z(t) = \mathrm{Tr} q^{L_0} = iV \left(\frac{1}{\sqrt{8\pi^2 \alpha' t}} \right)^D (\eta(it))^{-(D-2)}. \tag{10.146}$$

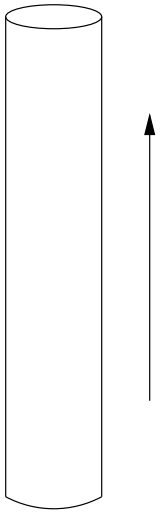
For bosonic case, $D = 26$,

$$Z = iV \int_0^\infty \frac{dt}{2t} \left(\frac{1}{\sqrt{8\pi^2 \alpha' t}} \right)^{26} (\eta(it))^{-24} \tag{10.147}$$

as $t \to \infty$ can be compared to temperature $T \to 0$ (just like $Z = \sum e^{-E_n/T}$)

$$\begin{aligned} \eta(\tau) &= q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) \\ \eta(\tau)^{-24} &= q^{-1}(1-q)^{-24} + .. \\ &= q^{-1}(1 + 24q) + ... \\ &= q^{-1} + 24 + ... \\ &= e^{2\pi t} + 24 + (\ ) e^{-24\pi t} + ... \end{aligned} \tag{10.148}$$

the first term, e^{2pit} represents tachyon and 24 represents twenty four tranverse photons.



Looks like a closed string propagating.

As $t \to 0$, it looks like UV case, but when considers a periodic propergating open string as a small close string moving for a long distance, ie IR singularity.

The modular properies of $\eta(\tau)$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau)$$
$$\tau = it\eta(i/t) = \sqrt{t}\eta(it)$$
$$\begin{aligned}\eta(it) &= \frac{1}{\sqrt{t}}\eta(i/t) \\ &= \sqrt{\frac{s}{\pi}}\eta(\frac{is}{\pi}), \quad s = \pi/t\end{aligned}$$

then

$$\begin{aligned}Z &\sim \int_0^\infty \frac{dt}{2t} t^{-13} \left(\eta(it)\right)^{-24} \\ &\sim \int_0^\infty \frac{dt}{t^2} \left(\eta(i/t)\right)^{-24} \\ &\sim \int_0^\infty ds \left(\eta(\frac{is}{\pi})\right)^{-24}, \\ &\sim \int_0^\infty ds \left(e^{2s} + 24 + ...\right)\end{aligned}$$

where $ds = -\frac{dt}{t^2}$.

Type-I (open strings, D=10)

The parity operator $\Omega : X^\mu(\sigma) \to X^\mu(\pi - \sigma)$ with $\Omega^2 = 1$, the eigenvalues are ± 1. $+1$ is the eigenvalue for even number in fourier terms in X^μ

$$x^\mu(\sigma) = \sum_n^\infty \frac{\alpha_n^\mu}{n} e^{in\sigma}$$

$$\begin{aligned} x^\mu(\pi-\sigma) &= \sum_n^\infty \frac{\alpha_n^\mu}{n}(-1)^n e^{-in\sigma} \\ \Omega\alpha_n^\mu\Omega^{-1} &= (-1)^n\alpha_n^\mu. \end{aligned}$$

The photon for type-I can be writen in the form of $\psi^\mu_{-\frac{1}{2}}|0;k>$ and the projection operation is $\frac{1}{2}(1+\Omega)$, then the partition function is

$$Z = \int_0^\infty \frac{1}{2}\frac{1}{2} Z_X(t) Z_\psi(t) \tag{10.149}$$

The two factors of 1/2 come Ω and $e^{i\pi F}$ and Z_ψ is

$$Z_\psi(t) = \frac{1}{\eta^4(it)}\left(\underbrace{\vartheta_{00}^4(0,it)-\vartheta_{10}^4(0,it)}_{NS} - \underbrace{\vartheta_{01}^4(0,it)-\vartheta_{11}^4(0,it)}_{R}\right) = 0.$$

The minus sign in front of ϑ_{10} comes from the ghost and the two minus signs in front of ϑ_{01} and ϑ_{11} come from fermion effect.

$$Z_{NS} = \frac{iV}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\left(\eta(\frac{is}{\pi})\right)^{-8} Z_\psi(\frac{\pi}{s})$$