Chapter 11

The Heterotic String

Following *"String Theory"* by J. Polchinski, Vol.II.
Notes written by students (work still in progress).
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The $SO(32)$ and $E_8 \times E_8$ heterotic strings

The $(0,1)$ heterotic string combines the whole left-moving side of the bosonic string, with 26 flat dimensions, with the ten-dimensional right-moving side of the type II string. In 1985, Gross, Harvey, Martinec and Rohm introduced the string quartet in which left- and right-moving modes are decoupled. If we consider holomorphic and anti-holomorphic function of z, then

holomorphic	anti-holomorphic
$\partial X^\mu(z), \quad \mu = 0, 1, \cdots, 9$	$\bar\partial X^\mu(\bar z), \quad \mu = 0, 1, \cdots, 9$
$\partial X^A(z), \quad A = 10, \cdots, 25$	$\tilde\psi^\mu(\bar z), \quad \mu = 0, 1, \cdots, 9$

Here, we can take 32 left-moving spin-$\frac{1}{2}$ fields ψ^A from the relation $\partial X^A(z) \to (\psi_1^A, \psi_2^A)$ which have the central charge $c = 16$ and $c = \frac{1}{2}$ respectively. Therefore, the result can be witten as

$$\begin{aligned} \partial X^\mu(z), \quad \bar\partial X^\mu(\bar z), \quad \tilde\psi^\mu(\bar z) \quad &: \quad \mu \quad = \quad 0, 1, \cdots, 9 \\ \psi^A(z) \quad &: \quad A \quad = \quad 1, 2, \cdots, 32 \end{aligned} \tag{11.1}$$

The total matter action is

$$S = \frac{1}{4\pi} \int d^2 z \left(\frac{2}{\alpha'} \partial X^\mu \bar\partial X_\mu + \tilde\psi^\mu \partial \tilde\psi_\mu + \psi^A \bar\partial \psi_A \right) \tag{11.2}$$

where ψ^A carry an $SO(32)$ gauge symmetry. The OPE become

$$\begin{aligned} X^\mu(z, \bar z) X^\nu(0, 0) \quad &\sim \quad -\eta^{\mu\nu} \frac{\alpha'}{2} ln|z|^2 \\ \tilde\psi^\mu(\bar z) \tilde\psi^\nu(0) \quad &\sim \quad \eta^{\mu\nu} \frac{1}{z} \\ \psi^A(z) \psi^B(0) \quad &\sim \quad \delta^{AB} \frac{1}{z} \end{aligned} \tag{11.3}$$

The energy momentum tensor and supercurrent are

$$T(z) \quad = \quad -\frac{1}{\alpha'} \partial X^\mu \partial X_\mu - \frac{1}{2} \psi^A \partial \psi_A$$

$$\begin{aligned}
\tilde{T}(\bar{z}) &= -\frac{1}{\alpha'}\bar{\partial}X^\mu\bar{\partial}X_\mu - \frac{1}{2}\tilde{\psi}^\mu\bar{\partial}\tilde{\psi}_\mu \\
\tilde{T}_F &= i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu\bar{\partial}X_\mu, \quad (\tilde{N}=1) \\
T_F &= 0, \quad (N=0).
\end{aligned} \tag{11.4}$$

Hibert space

Let us give the boundary condition on the field $\tilde{\psi}^\mu$ and ψ^A and specify which sectors are in the spectrum.

$$\begin{aligned}
\tilde{\psi}^\mu(\sigma+2\pi) = -\tilde{\psi}^\mu(\sigma) \quad &: \quad \tilde{NS}, \\
\tilde{\psi}^\mu(\sigma+2\pi) = \tilde{\psi}^\mu(\sigma) \quad &: \quad \tilde{R}, \\
\psi^A(\sigma+2\pi) = -\psi^A(\sigma) \quad &: \quad NS, \\
\psi^A(\sigma+2\pi) = \psi^A(\sigma) \quad &: \quad R. \\
\Longrightarrow \quad NS\times\tilde{NS}, \quad NS\times\tilde{R}, \quad R\times\tilde{NS}, \quad &R\times\tilde{R}.
\end{aligned} \tag{11.5}$$

The partition function for the $\tilde{\psi}$ and ψ is

$$\begin{aligned}
Z_{\tilde{\psi}} &= \frac{1}{2}\left(\left(\frac{\theta_{00}}{\eta}\right)^4 - \left(\frac{\theta_{10}}{\eta}\right)^4 - \left(\frac{\theta_{01}}{\eta}\right)^4 + \left(\frac{\theta_{11}}{\eta}\right)^4\right) \\
Z_{\psi} &= \frac{1}{2}\left(\left(\frac{\theta_{00}}{\eta}\right)^{16} + \left(\frac{\theta_{10}}{\eta}\right)^{16} + \left(\frac{\theta_{01}}{\eta}\right)^{16} + \left(\frac{\theta_{11}}{\eta}\right)^{16}\right)
\end{aligned} \tag{11.6}$$

Under the modular transformations $\tau \to -\frac{1}{\tau}$,

$$\begin{aligned}
\theta_{00}(0,-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_{00}(0,\tau) \\
\theta_{01}(0,-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_{10}(0,\tau) \\
\theta_{10}(0,-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_{01}(0,\tau) \\
\eta(-\frac{1}{\tau}) &= \sqrt{-i\tau}\eta(\tau).
\end{aligned} \tag{11.7}$$

Therefore, the modular transformations just permute the four terms with no phase:

$$Z_{\tilde{\psi}}(-\frac{1}{\tau}) = Z_{\tilde{\psi}}(\tau), \quad Z_{\psi}(-\frac{1}{\tau}) = Z_{\psi}(\tau). \tag{11.8}$$

Under the transformations $\tau \to \tau+1$,

$$\begin{aligned}
\theta_{00}(0,\tau+1) &= \theta_{01}(0,\tau) \\
\theta_{01}(0,\tau+1) &= \theta_{00}(0,\tau) \\
\theta_{10}(0,\tau+1) &= exp(\frac{\pi i}{4})\theta_{01}(0,\tau) \\
\eta(\tau+1) &= exp(\frac{\pi i}{12})\eta(\tau)
\end{aligned} \tag{11.9}$$

give a phase of $exp(\frac{2\pi i}{3})$:

$$Z_{\tilde{\psi}}(\tau+1) = exp(\frac{2\pi i}{3})Z_{\tilde{\psi}}(\tau), \quad Z_{\psi}(\tau+1) = exp(\frac{2\pi i}{3})Z_{\psi}(\tau). \tag{11.10}$$

We now find the lightest states of $L_0 = \tilde{L}_0 = 0$.
1) $\tilde{NS}$ (anti-holomorphic NS):

$$\tilde{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} - \frac{1}{2} \tag{11.11}$$

For the ground state $\tilde{N} = 0$,

$$\textit{the lowest state} \quad |0;k>, \qquad m^2 = -k^2 = -\frac{2}{\alpha'} \quad : \quad \textit{tachyon.}$$

For the next excited state $\tilde{N} = \frac{1}{2}$,

$$\tilde{\psi}^\mu_{-\frac{1}{2}}|0;k>, \qquad m^2 = -k^2 = 0$$

$$\textit{GSO Projection } e^{i\pi\tilde{F}} = +1 \quad : \quad e^{i\pi\tilde{F}}\tilde{\psi}^\mu_{-\frac{1}{2}}|0;k> = +\tilde{\psi}^\mu_{-\frac{1}{2}}|0;k> .$$

2) $\tilde{R}$ (anti-holomorphic R):

$$\tilde{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} \tag{11.12}$$

For the ground state $\tilde{N} = 0$, $m^2 = -k^2 = 0$.
3) Holomorphic side:
On the left-moving side, the normal ordering constant is the left-moving transverse Hamiltonian is

$$NS : -\frac{1}{3} - \frac{32}{48} = -1, \qquad R : -\frac{1}{3} + \frac{32}{24} = +1. \tag{11.13}$$

The left-moving NS ground state ($N = 0$):

$$m^2 = -k^2 = -\frac{4}{\alpha'} \quad \Rightarrow \quad |0;k>: \ \textit{tychyon.}$$

The first excited states ($N = \frac{1}{2}$):

$$m^2 = -k^2 = -\frac{2}{\alpha'}, \quad e^{i\pi F}\psi^\mu_{-\frac{1}{2}}|0;k> = -\psi^\mu_{-\frac{1}{2}}|0;k> .$$

Therefore, they are removed by GSO projection.
For $N = 1$, $m^2 = -k^2 = 0$.

$$\Longrightarrow \qquad A_\mu(k)\alpha^\mu_{-1}|0;k>, \quad B_{AB}(k)\psi^A_{-\frac{1}{2}}\psi^B_{-\frac{1}{2}}|0;k>$$

where $A_\mu(k)$ has 8 transverse component and $B_{AB}(k)$ is 32×32 anti-symmetric matrix with demension $32 \times 31/2 = 496$.

$*$ Summary:

m^2	$NS+$	$R+$	$\tilde{NS}+$	$\tilde{R}+$
$-\frac{4}{\alpha'}$	$\|0;k>$	-	-	-
0	A_μ, B_{AB}	-	$\tilde{\psi}^\mu_{-\frac{1}{2}}\|0;k>$	$\|\vec{s};k>$

At the massless level, heterotic string states are $NS = (8_v, 1) + (1, 496)$, $\tilde{NS} = 8_v$ and $\tilde{R} = 8$. From consideration of these products, we can see that heterotic $SO(32)$ is precisely same as the type-I plus closed $SO(32)$ theory.

The second heterotic string theory is obtained by dividing the ψ^A into two set of 16 with independent boundary conditions:

$$\begin{aligned}
\psi^A(\sigma = +2\pi) &= -\psi^A(\sigma) : \ NS_1, \\
\psi^A(\sigma = +2\pi) &= +\psi^A(\sigma) : \ R_1, \\
\psi^B(\sigma = +2\pi) &= -\psi^B(\sigma) : \ NS_2, \\
\psi^B(\sigma = +2\pi) &= +\psi^B(\sigma) : \ R_2,
\end{aligned} \tag{11.14}$$

where $A = 1, 2, \cdots, 16$ and $B = 17, 18, \cdots, 32$. In this theory, there are 4 Hibert spaces such that

$$NS_1 + NS_2, \quad NS_1 + R_2, \quad R_1 + NS_2, \quad R_1 + R_2,$$

and there are the operators which satisfy the following relations:

$$e^{i\pi F_1} = e^{i\pi F_2} = +1.$$

Partition Function

$$\begin{aligned} Z_\psi &= Z_{\psi^A} Z_{\psi^B} = \left(Z_{\psi^A}\right)^2 \\ Z_{\psi^A} &= \frac{1}{2}\left(\left(\frac{\theta_{00}}{\eta}\right)^8 + \left(\frac{\theta_{10}}{\eta}\right)^8 + \left(\frac{\theta_{01}}{\eta}\right)^8 + \left(\frac{\theta_{11}}{\eta}\right)^8\right) \end{aligned} \tag{11.15}$$

Under the modular transformation $\tau \to -\frac{1}{\tau}$,

$$Z_{\psi^A} \longrightarrow Z_{\psi^A}.$$

Under the transformation $\tau \to \tau + 1$,

$$\theta_{00} \leftrightarrow \theta_{01}, \quad \theta_{10} \to e^{\frac{\pi i}{4}}\theta_{10}, \quad \eta \to e^{\frac{\pi i}{12}}\eta$$

$$\Longrightarrow \qquad Z_{\psi^A} \to e^{-\frac{2\pi i}{3}} Z_{\psi^A}, \quad Z_\psi \to e^{-\frac{4\pi i}{3}} Z_\psi, \quad Z^*_{\tilde{\psi}} \to e^{-\frac{2\pi i}{3}} Z^*_{\tilde{\psi}}.$$

The gauge group of this theory can be expected as $SO(16) \times SO(16)$ and the actual gauge group is $E_8 \times E_8$.