Chapter 13

D-Branes

Following *"String Theory"* by J. Polchinski, Vol.II.
Notes written by students (work still in progress).
For more information contact George Siopsis
gsiopsis@utk.edu

13.1 T -Duality

13.1.1 Type II (Closed) Strings

Recall that type two strings were divided into two further categories, A and B, depending on the combination of the Ramond and Neveu-Schwzrz sectors:

IIA : NS+NS+,R+NS+,NS+R-,R+R-
IIB : NS+NS+,R+NS+,NS+R+,R+R+

T-Duality was revealed by compactifying one dimension. Compactifying the dimension X^9:

$$X^9 \equiv X^9 + 2\pi R \tag{13.1}$$

In the bosonic case, we saw that taking the limit $R \to 0$ is equivalent to taking the the $R \to \infty$ limit in the dual co-ordinate ($R \equiv \alpha'/R$), and the right moving part of that being reflected:

$$X^9_R(\bar{z}) \to -X^9_R(\bar{z}) \tag{13.2}$$

Since we now have to fulfill superconformal invariance, the co-ordinate $\tilde{\psi}^9$ must be similarly reflected:

$$\tilde{\psi}^9(\bar{z}) \to -\tilde{\psi}^9(\bar{z}) \tag{13.3}$$

Thus the zero modes are also reflected, $\tilde{\psi}^9_0 \to -\tilde{\psi}^9_0$. A ground state in the R sector is given by any number of zero modes acting on the state $|\vec{s}, \vec{k}>$. This means, under this reflection, the 'spin' operator Σ^{89} acting on a ground state transforms as

$$\Sigma^{89}\tilde{\psi}^9_0\tilde{\psi}^9_0...|\vec{s}, \vec{k}> \to -\Sigma^{89}\tilde{\psi}^9_0\tilde{\psi}^9_0...|\vec{s}, \vec{k}> \tag{13.4}$$

i.e. the R_+ sector turns into the R_- sector. Thus the **IIA** theory at R is equivalent to the **IIB** theory at α'/R. Let's look at the particle states of the two theories:

IIA :

$$\begin{aligned} \text{(photon)} \qquad & C_\mu \psi_0^\mu |k> \\ & C_{\mu\nu\rho} \psi_0^{[\mu,} \psi_0^{[\nu,} \psi_0^{\sigma]} |k> \end{aligned} \tag{13.5}$$

IIB :

$$\begin{aligned} \text{(scalar)} \qquad & C|k> \\ & C_{\mu\nu} \psi_0^{[\mu,} \psi_0^{\nu]} |k> \\ & C_{\mu\nu\rho\sigma} \psi_0^{[\mu,} \psi_0^{[\nu,} \psi_0^{[\rho,} \psi_0^{\sigma]} |k> \end{aligned} \tag{13.6}$$

Now, let us look at the field C_μ in the **IIA** theory. Starting in 10 dimensions, the index μ runs from zero to 9. Now compactify a dimension: the field now splits into two: $C_\mu, \mu = 0, 1, ..., 8$ and C_9, where 9 represents the compactified dimension. Then C_9 is a scalar, and can be mapped to the scalar field C in **IIB**.
Similarly, compactifying $C_{\mu\nu}$ in the **IIB** theory results in the vector fields $C_{\mu 9}$ and $C_{9\nu}$ which are mapped to C_μ fields in **IIA**.

13.1.2 Type I (Open) Strings and Branes

We compactify as before, and again the limit $R \to 0$ results in the dual condition (**??**).

Recall that before, when we compactified one dimension, we saw the open strings were fixed a D8-brane, which was at position X^9. Let's examine effects of states on a brane:
Take the NS sector open string state $A_\mu \psi^\mu_{-1/2}|k>$: when we compactify a dimension, we get the fields $A_\mu, \mu = 0, 1, ..., 8$ which are tangent to the brane, and the scalar A_9. Thus the photon lives on the brane.
The field A_9 generates a change in the position of the brane. Now, $A_9 = A_9(k^\mu)$ has a fourier transform $A_9(X^\mu)$, so at each point on the brane, the motion can change - the brane is completely dynamical. So open strings force the brane to move in the extra dimension.
As well as translational invariance, supersymmetry is broken. The original theory has 32 SUSYs, and the brane breaks 16 of them. So the brane has 16 SUSYs, corresponding to 16 conserved charges.

13.2 D-Brane Charges and the Dirac Quantization Condition

To define a charge in the $8 + 1$ dimensions let us first consider volume elements in lower dimensions.
In 1 dimension, we have a straightforward line element $d\vec{l}$. In two dimensions we have the surface element $d\vec{S}$, which can be defined in the following way:

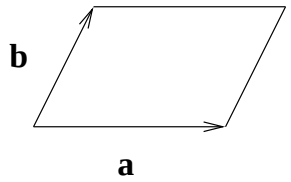


Figure 1

$$dS^i = (\vec{a} \times \vec{b})_i = \epsilon_{ijk} a^j a^k \tag{13.7}$$

We can extend this to 9 dimensions: then the integral that defines a charge can be written:

$$\int \epsilon^{\mu_1 \mu_2 \ldots \mu_9} C_{\mu_1 \mu_2 \ldots \mu_9} \tag{13.8}$$

This type of integral is called a topological integral, i.e. it is independent of the metric of the space. To show this, consider two examples:

Example 1 The action for the motion of a charged particle:

$$S = q \int d\tau v^\mu A_\mu \tag{13.9}$$

where $v^\mu = dx^\mu / d\tau$, so

$$S = q \int dx^\mu A_\mu \tag{13.10}$$

so this is a topological integral.

Example 2 The Magnetic Flux is given by

$$\Phi = \int_S \vec{B} \cdot d\vec{S} \tag{13.11}$$

around the closed surface S. Using $\vec{B} = \vec{\nabla} \times \vec{A}$ and $F_{ij} = \partial_i A_j - \partial_j A_i$ we can write:

$$B_i = \tfrac{1}{2} \epsilon_{ijk} F^{jk} \tag{13.12}$$

so

$$\Phi = \int_S F_{ij} dS^{ij} = 0 \tag{13.13}$$

where $dS_{ij} = dS_i \epsilon^{ijk}$
Generalizing this to include the electric field $F_{ij} \to F_{\mu\nu}$, $F_{0i} = E_i$:

$$\Phi = \int_\Sigma F_{0i} d\Sigma^{0i} = q \tag{13.14}$$

So this is a topological integral that defines a charge q (in fact it's Gauss' law).

So we have started with 9+1 dimensions, compactified one and obtained a D8-brane. We can repeat this process, compactifying more dimensions, going down to 7+1, 6+1,etc spaces, and getting D7,D6... branes. Eventually we will get down to a D0 brane, a point 'particle' with strings attached. The D0-brane has a charge, given by the 1-d integral

$$\int dx^{\mu} C_{\mu} \tag{13.15}$$

where the vector potential here is the R-R sector potential. Similarly the charges on a D1 or D2-brane are given by the integrals

$$\int d\Sigma^{\mu\nu} C_{\mu\nu} \tag{13.16}$$

$$\int d\Sigma^{\mu\nu\rho} C_{\mu\nu\rho} \tag{13.17}$$

So the potential for a Dn-brane, is $C^{\mu_1 \cdots \mu_n}$. For n even, these correspond to the R potentials for the **IIA** theory, and for n odd, these correspond to the R potentials for the **IIB**. So if we start from type I theories and compactify dimensions, we get alternately **IIA** and **IIB**.

Let us take a closer look at the point particle D0-brane, by examining a feature of 4-d Electromagnetism. The Maxwell Field Tensor $F_{\mu\nu}$ has a dual:

$$\tilde{F}_{\mu\nu} = \tfrac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{13.18}$$

The effect is switching the fields $\vec{B}$ and $\vec{E}$, and thus switching the electric and magnetic charges. What is the magnetic charge?

We can write $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ because $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. But when it comes to writing the dual of F in the same way, $\tilde{F}_{\mu\nu} = \partial_{\mu} \tilde{A}_{\nu} - \partial_{\nu} \tilde{A}_{\mu}$, we run in to the problem that if we have a magnetic charge, Gauss' law for the magnetic field becomes $\vec{\nabla} \cdot \vec{B} = \rho$.

The solution, from Dirac, is to define the Gaussian sphere around the magnetic charge m with a small hole 'cut' in it, and to define $\tilde{F}$ everywhere in it except on a line drawn from the charge through the hole - see below.

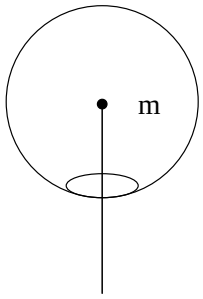


Figure 2

This line is called a Dirac string. No physics should depend on the string, nor should it matter where we put the string. m is a magnetic monopole. Let's look at the consequences: suppose an electric charge q moves from position $\vec{x}_1$ to $\vec{x}_2$ in the field of the magnetic monopole:

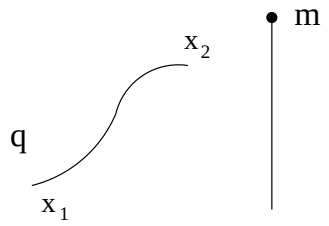


Figure 3

Then the amplitude for the particle to move from $\vec{x}_1$ to $\vec{x}_2$ is given by

$$A = \psi^*(\vec{x}_2)\psi(\vec{x}_1) \tag{13.19}$$

where ψ is the wavefunction of the particle. However, looking at the gauge transformation:

$$\begin{aligned} A_\mu &\to A_\mu - \partial_\mu \lambda \\ \psi(x) &\to e^{iq\lambda(x)}\psi(x) \end{aligned}$$

we see that

$$A \to \psi^*(\vec{x}_2)\psi(\vec{x}_1)e^{-iq(\lambda(x_2)-\lambda(x_1))} \tag{13.20}$$

So this amplitude is only invariant under gauge transformations when $\vec{x}_1 = \vec{x}_2$, and $A = |\psi(\vec{x}_1)|^2$. This situation is obviously unphysical. We must take into account the effect of the Dirac string in the amplitude. We modify it in the following way:

$$A = \psi^*(\vec{x}_2)e^{i\int_{x_1}^{x_2}\vec{A}\cdot d\vec{l}}\psi(\vec{x}_1) \tag{13.21}$$

Now performing the guage transformation, we obtain an extra factor

$$e^{i\int_{x_1}^{x^2}\vec{\nabla}\lambda\cdot d\vec{l}} = e^{iq(\lambda(x_2)-\lambda(x_1))} \tag{13.22}$$

Which cancels the factor introduced to the wavefunctions by the gauge transformation.
Now if we let the charge go round a closed loop ($\vec{x}_1 = \vec{x}_2$):

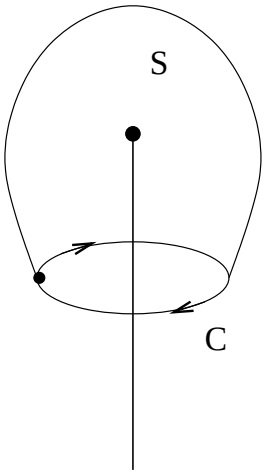


Figure3

We get the amplitude:

$$A = \psi^*(\vec{x}_1)e^{i\oint_C \vec{A}\cdot d\vec{l}}\psi(\vec{x}_1) = \psi^*(\vec{x}_1)\psi(\vec{x}_1) \tag{13.23}$$

$$\therefore e^{i\oint_C \vec{A}\cdot d\vec{l}} = 1 \tag{13.24}$$

By Stokes' theorem

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S \vec{B}\cdot d\vec{S} \tag{13.25}$$

If we let $C \to 0$ then S tends to a closed surface, and we get

$$\int_S \vec{B}\cdot d\vec{S} = m \tag{13.26}$$

Where m is the magnetic charge. We also have

$$\oint_C \vec{A}\cdot d\vec{l} = e^{iqm} = 1 \tag{13.27}$$

$$\therefore \quad qm = 2\pi n \qquad \text{n = integer}$$
$$\therefore \quad q = \frac{2\pi n}{m}$$

Thus the electric charge q is quantized according to this formula. This is the Dirac quantization condition. Note the discussion has been topological, so the magnetic monopole could be arbitrarily far away.

Let us look at how this result comes from the treatment of D0-Branes as point charges: The potential is now C_μ causing fields given by $F_{\mu\nu} = \partial_\mu C_\nu - \partial_{nu} C_\mu$. As in EM, we can define a dual Field tensor $\tilde{F}$. As we are now in 10 dimensions the dual will be defined as follows:

$$\epsilon^{\mu_0 \dots \mu_9} F_{\mu_8 \mu_9} = \tilde{F}^{\mu_0 \dots \mu_7} \tag{13.28}$$

Where $\tilde{F}$ is now derived from potentials $C_{\mu_0 \dots \mu_6}$. These are just the potentials of a D6 brane - i.e. the D6-brane is dual to the D0-brane in the same way as electric charge is dual to magnetic charge. Similarly, the D1-brane is dual to the D5-brane, and in general, the Dp-Brane is dual to the D(6-p)-Brane. (Interestingly, the D3-Brane is dual to the D3-Brane - and that is our space!).

Going back to the D0-brane: let us define an action for it:

$$S = -\tfrac{1}{2} \int d^{10}x \sqrt{-\det g_{\mu\nu}} F_{\mu\nu} F^{\mu\nu} + q \int dx^\mu A_\mu \tag{13.29}$$

Note the second term is topological. Now the Coulomb potential in 4 dimensions is $V(y) \sim q^2/y$. So in 10 dimensions we might expect it to be of the form $V(y) \sim q^2/y^7$. How do find this from the brane? We obtained the 4-d Coulomb potential from the action, specifically looking at scattering problems:

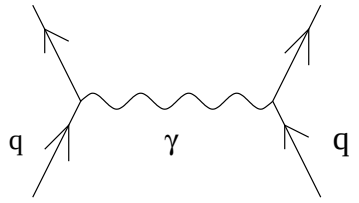


Figure 4

Here the photon propagator is $-i/k^2$ and the amplitude for the scattering event goes as:

$$A(k) = \frac{-iq^2}{k^2} \tag{13.30}$$

The potential is the fourier transform of the amplitude over momentum space:

$$V(y) = \frac{-iq^2}{V} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\vec{k}\cdot\vec{y}}}{k^2} \tag{13.31}$$

where

$$d^D k = d\omega d^{D-1} k \tag{13.32}$$

and

$$\int d\omega = V \tag{13.33}$$

where V is the volume of space over which the integration is performed. Then for D = 4, this reproduces the Coulomb potential. For D = 10, we get

$$V(y) = -i\frac{15}{32\pi^4}\frac{q^2}{y^7} \tag{13.34}$$

Now let us try and derive this from String Theory as the potential between two D-branes, parallel to each other, seperated by distance y:

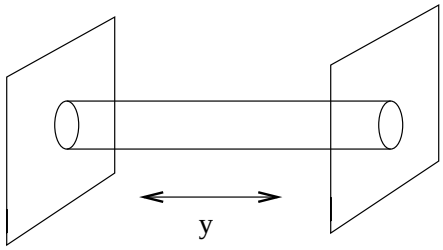


Figure 5

A closed string is propagated between them, causing a force between the D-branes. This can be calculated as an open with ends on each brane moving in a circle, radius t:



Figure 6

Recall that the partition function was given by:

$$Z = \int_0^\infty \frac{dt}{2t} Z(t) \tag{13.35}$$

$$Z(t) = \mathrm{Tr} e^{-2\pi t L_0} \tag{13.36}$$

and we found $Z_{NS} + Z_R = 0$. For Z_{NS} we calculated

$$Z_{NS} = \frac{iV}{8\pi(8\pi\alpha')^5}\int_0^\infty ds(16 + O(e^{-2s})) \tag{13.37}$$

This is not quite right for out case. We have compactified 9 dimensions. The factor $(8\pi\alpha')^{-5}$ comes from $(8\pi\alpha' t)^{-D/2}$. In our case D = 1, so to correct for this we must multiply equation (**??**) by a factor $(8\pi\alpha' t)^{-9/2}$. Next we need to multiply equation (**??**) by a factor of two since there are two possibilities as to which end of the string to attach to which brane. Then there is an additional factor of 2 as there is no need to orient the

strings.
Finally, we require that:

$$Z(\sigma = \pi) - Z(\sigma = 0) = y \tag{13.38}$$

so

$$Z(\sigma) = \frac{y\sigma}{\pi} + ... \tag{13.39}$$

$$\begin{aligned} L_0 &\sim (\partial_\sigma Z)^2 + ... \\ &= \frac{y^2}{4\pi^2\alpha'} + ... \end{aligned} \tag{13.40}$$

Thus, from equation (**??**), we get an extra factor

$$e^{-2\pi t(\frac{y^2}{4\pi^2\alpha'})} = e^{\frac{-y^2 t}{2\pi\alpha'}} \tag{13.41}$$

Putting all these extra factors together, we get the Partition function:

$$\begin{aligned} Z_R &= \frac{iV(64)}{8\pi(8\pi^2\alpha')^5}\int_0^\infty \frac{\pi dt}{t^2}(8\pi\alpha' t)^{9/2} e^{\frac{-y^2 t}{2\pi\alpha'}} \\ &= iV(2\pi)(4\pi^2\alpha')^3 \frac{15}{32\pi^4}\frac{1}{y^7} \end{aligned} \tag{13.42}$$

So the potential is

$$V(y) = \frac{-iZ_R}{V} = -i(2\pi)(4\pi^2\alpha')^3 \frac{15}{32\pi^4}\frac{1}{y^7} \tag{13.43}$$

Comparing this with equation (**??**), we can read off the charge of the D0-brane: it is given by

$$q^2 = (2\pi)(4\pi^2\alpha')^3 \tag{13.44}$$

We can generalize this to Dp-branes. Again, we look to EM: the potential between two point charges goes as $1/y$. Between two parallel line charges, it goes as $\ln(y)$, between two parallel charged planes, as y. The charge gets generalized to distributions over d dimensional surfaces. So we expect the electric potential between two Dp-branes to be of the form

$$V_p(y) \sim \frac{q^2}{y^{7-p}} \tag{13.45}$$

Where q is now a distribution over the brane:

$$q^2 = (2\pi)(4\pi^2\alpha')^{3-p} \tag{13.46}$$

The D(6-p)-brane is dual to the Dp-brane, so its charge is dual to the Dp-brane's charge - i.e. it has a magnetic charge:

$$m^2 = 2(\pi)(4\pi^2\alpha')^{p-3} \tag{13.47}$$

Combining (**??**) and (**??**)

$$qm = 2\pi \tag{13.48}$$

This is the Dirac quantization condition for n = 1!. This treatment is valid for parallel branes of the same dimension. What about two differently dimensioned branes at an angle to each other?

13.3 Interactions of Stationary Branes

Let us consider two D8-branes at perpendicular to each other: one along X^9, obtained by taking the dual of X^8, and one along X^8, obtained by taking the dual of X^9:

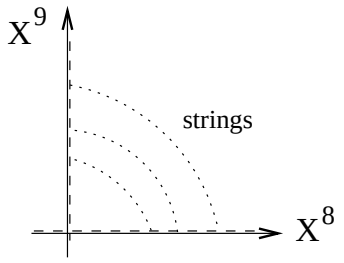


Figure 7

We imagine strings connecting the two branes. How do we deal with this? In the uncompactified case, setting $\tau = 0$, the mode expansion is:

$$X^\mu(\sigma) = x^\mu + i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha_n^\mu \cos(n\sigma) \tag{13.49}$$

This was a good choice of expansion because $\partial_\sigma X^\mu = ...\sum ... \sin n\sigma$, so at $\sigma = 0, \pi, \partial_\sigma X^\mu = 0$. These are Neumann boundary conditions. We shall call this expansion $X_{NN}(\sigma)$.
For the D-Brane, we had X fixed at each end to the brane. In this case the mode expansion for the compactified dimension

$$X(\sigma) = x + i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha_n \sin(n\sigma) \tag{13.50}$$

So now $X(0) = X(\pi) = x$. Generalizing to two parallel branes, separation y, we write

$$X(\sigma) = \frac{y\sigma}{\pi} + i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha_n \sin(n\sigma) \tag{13.51}$$

So $X(0) = 0$, $X(\pi) = y$. These are Dirichlet boundary dimensions. Call this mode expansion X_{DD}.
We can split X_{NN}, X_{DD} into holomorphic and anti holomorphic parts. For X_{DD}, $\sin(n\sigma) \to e^{in\sigma} - e^{-in\sigma}$, $X_{DD} \to X_L - X_R$, which is what we got when we took the dual of X.

Now what will the mode expansion be with the D-branes perpendicular? When we take the dual of X^9, we take D boundary conditions on X^9, N boundary conditions on $X^\mu, \mu \neq 9$. The other end of the string is on the X^8 D-brane, so it has D boundary conditions on X^8, N boundary conditions on $X^\mu, \mu \neq 8$. The string has mixed boundary conditions. The mode expansions that satisfy these are as follows:

$$X^8(\sigma) = X_{DN}(\sigma) = -i\sqrt{2\alpha'} \sum_{r \in Z + \frac{1}{2}} \frac{1}{r} \alpha_r \sin(r\sigma) \tag{13.52}$$

$$X^9(\sigma) = X_{ND}(\sigma) = -i\sqrt{2\alpha'} \sum_{r \in Z + \frac{1}{2}} \frac{1}{r} \alpha_r \cos(r\sigma) \tag{13.53}$$

And the same for $\psi, \tilde{\psi}$. So at $\sigma = 0, X^8 = 0, \partial_\sigma X^9 = 0$, and at $\sigma = \pi, X^9 = 0, \partial_\sigma X^8 = 0$.
Now what happens when we put the branes at an angle?

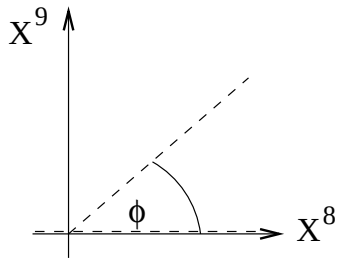


Figure 8

Think of this as the complex plane: $Z = X^8 + iX^9$. then at $\sigma = 0$ we require:

$$\mathrm{Im} Z = 0 \qquad\qquad \mathrm{Re}\partial_\sigma Z = 0 \tag{13.54}$$

And, noting that multiplying Z by $e^{-i\phi}$ brings it to the X^8 axis,

$$\mathrm{Im} Z e^{-i\phi} = 0 \qquad\qquad \mathrm{Re}\partial_\sigma Z e^{-i\phi} = 0 \tag{13.55}$$

A mode expansion for for Z that satisfies these conditions is

$$Z(\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{r \in Z + \frac{1}{2}\phi} \frac{1}{r} \alpha_r e^{ir\sigma} + i\sqrt{\frac{\alpha'}{2}} \sum_{r \in Z - \frac{1}{2}\phi} \frac{1}{r} \alpha_r^\dagger e^{-ir\sigma} \tag{13.56}$$

where

$$\begin{aligned} \alpha_r &\sim \alpha_r^8 + \alpha_r^9 \\ \alpha_r^\dagger &\sim \alpha_r^8 - \alpha_r^9 \end{aligned} \tag{13.57}$$

are treated as independent.
So at $\sigma = 0$:

$$\begin{aligned} Z &\sim \alpha_r + \alpha_r^\dagger & \mathrm{Im}Z &= 0 \\ \partial_\sigma Z &\sim i(\alpha_r - \alpha_r^\dagger) & \mathrm{Re}\partial_\sigma Z &= 0 \end{aligned} \tag{13.58}$$

and at $\sigma = \pi$

$$\begin{aligned} Ze^{-i\phi} &\sim \alpha_r + \alpha_r^\dagger & \mathrm{Im}Ze^{-i\phi} &= 0 \\ \partial_\sigma Ze^{-i\phi} &\sim i(\alpha_r - \alpha_r^\dagger) & \mathrm{Re}\partial_\sigma Ze^{-i\phi} &= 0 \end{aligned} \tag{13.59}$$

Partition Function As α_r and $\alpha_r^\dagger$ are independent, so are their respective L_0s. Thus the partition function is of the form:

$$Z = q^a(\mathrm{Tr}q^{L_0^\alpha}\mathrm{Tr}q^{L_0^{\alpha^\dagger}}) \tag{13.60}$$

where a is the zero point energy $L =: L : +a$.
Recall that L_0 is just the number operator, which acts like

$$N\alpha_{-r}|0>= r\alpha_{-r}|0> \tag{13.61}$$

We can take eigenvalues $2r$, $3r$, ... for states $\alpha_{-r}\alpha_{-r}|0>$, $\alpha_{-r}\alpha_{-r}\alpha_{-r}|0>$.... If we fix r, the trace is

$$q^0 + q^r + q^{2r} + ... = (1-q^r)^{-1} \tag{13.62}$$

Thus the total trace in our case consists of, for $r > 0$

$$\begin{aligned} \mathrm{Tr}q^{L_0^\alpha} &= \Pi_r(1-q^r)^{-1} & r &\in Z + \tfrac{1}{2}\phi \\ \mathrm{Tr}q^{L_0^{\alpha^\dagger}} &= \Pi_r(1-q^r)^{-1} & r &\in Z - \tfrac{1}{2}\phi \end{aligned} \tag{13.63}$$

For $r \in Z + \frac{1}{2}\phi$, the smallest r is ϕ/π, so we can write that part as

$$\mathrm{Tr}q^{L_0^\alpha} = \Pi_{m=0}^\infty(1-q^{m+\phi/\pi})^{-1} \tag{13.64}$$

where m is an integer. Similarly

$$\mathrm{Tr}q^{L_0^{\alpha^\dagger}} = \Pi_{m=1}^\infty(1-q^{m-\phi/\pi})^{-1} \tag{13.65}$$

What about q^a? Recall, for X_{NN}

$$L_0 = \tfrac{1}{2}\sum_{-\infty}^{\infty}\alpha_n\alpha_{-n} = \tfrac{1}{2}\sum_{-\infty}^{\infty} :\alpha_n\alpha_{-n}: + \tfrac{1}{2}\sum n \tag{13.66}$$

where $\frac{1}{2}\sum n = a = -1/24$. In our case we have a sum over non integers. We have seen similar sums before:

$$\begin{aligned}
a_1 &= \tfrac{1}{2}\sum r \\
&= \tfrac{1}{2}\sum_{m=1}^{\infty}(m-\frac{\phi}{\pi}) \\
&= \tfrac{1}{2}\left[\frac{1}{24}-\frac{1}{8}\left(\frac{2\phi}{\pi}-1\right)^2\right] \\
a_2 &= \tfrac{1}{2}\sum_{m=0}^{\infty}(m+\frac{\phi}{\pi}) \\
&= \tfrac{1}{2}\sum_{m=1}^{\infty}(m-1+\frac{\phi}{\pi}) \\
&= \tfrac{1}{2}\left[\frac{1}{24}-\frac{1}{8}\left(2-\frac{2\phi}{\pi}-1\right)^2\right] \\
a &= 2a_1 = \frac{1}{24}\cdot\frac{1}{8}\left(\frac{2\phi}{\pi}-1\right)^2
\end{aligned} \tag{13.67}$$

So now we are in a position to calculate the partition function:

$$\begin{aligned}
Z &= q^a\Pi_{m=0}^{\infty}(1-q^{m+\phi/\pi})^{-1}\Pi_{m=1}^{\infty}(1-q^{m-\phi/\pi})^{-1} \\
&= q^a(1-q^{\phi/\pi})^{-1}\Pi_{m=1}^{\infty}(1-q^{m+\phi/\pi})^{-1}(1-q^{m-\phi/\pi})^{-1}
\end{aligned} \tag{13.68}$$

With $z = q^{\phi/\pi}$,

$$Z = q^a(1-z)^{-1}\left[\Pi_{m=1}^{\infty}(1-zq^m)(1-\frac{1}{z}q^m)\right]^{-1} \tag{13.69}$$

Recall the theta function Θ_{11}:

$$\Theta_{11}(\nu,it) = -2q^{1/8}\sin(\pi\nu)\Pi_{m=1}^{\infty}(1-q^m)(1-q^{m+\phi/\pi})(1-q^{m-\phi/\pi}) \tag{13.70}$$

and

$$\eta(it) = q^{1/24}\Pi_{m=1}^{\infty}(1-q^m) \tag{13.71}$$

therefore

$$\begin{aligned}
\frac{\eta}{\Theta_{11}} &= \frac{q^{\frac{1}{24}-\frac{1}{8}}}{-2\sin(\pi\nu)}\left[\Pi_{m=1}^{\infty}(1-zq^m)(1-\frac{1}{z}q^m)\right]^{-1} \\
&= -\frac{q^{\frac{-1}{12}}}{2\sin(\pi\nu)}\frac{Z}{q^a(1-z)^{-1}}
\end{aligned} \tag{13.72}$$

And using $2i\sin(\pi\nu) = e^{i\pi\nu}-e^{-i\pi\nu} = (z-1)/\sqrt{(}z) = (z-1)/q^{-\phi/2\pi}$, we finally obtain (putting a in too)

$$Z_{X^8X^9} = -ie^{\frac{\phi^2 t}{\pi}}\frac{\eta(it)}{\Theta_{11}(\nu,it)} \tag{13.73}$$

Similarly for fermions ψ^8, ψ^9, we obtain

$$Z_{ab} = \frac{\Theta_{ab}(\nu, it)}{e^{\frac{\phi^2 t}{\pi}}\eta(it)} \tag{13.74}$$

Where a,b = 0,1 for the N,R sectors. Now, as $\nu \to \infty, \Theta_{11} \to 0$, and $Z_{X^8 X^9} \to \infty$. Physically this is the case $\phi \to 0$, i.e. the branes are becoming parallel, and the strings obtain the freedom to move along the branes, that is to posses continuous momentum. This adds an extra factor to Z from the zero modes and oscillators which correspond to continuous momentum. We have calculated this factor before: it is

$$\frac{V}{\sqrt{8\pi^2\alpha' t}}(\eta(it))^{-2} \tag{13.75}$$

Z_{ab} doesn't change as there is no momentum associated with the fermions.
Now take the dual of X^8 as $\phi \to 0$. What happens? A brane on X^8 means Dirichlet boundary conditions on X^9, Neumann on X^8. When the dual is taken, the boundary conditions on X^8 become Dirichlet, so the strings are fixed on X^8 too. This defines a point on the brane, or taking into account the rest of the dimensions, a D(p-1)-brane:



Figure 9

If we have 2 branes parallel on X^8, we have the following picture:

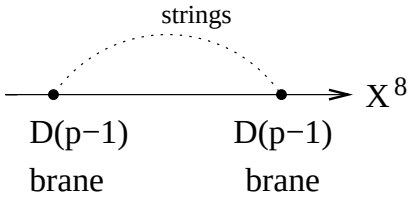


Figure 10

i.e. L_0 has an extra term we worked out before, $y^2/4\pi^2\alpha'$. So when we expand X^8 we get

$$X^8 = \frac{y\sigma}{\pi} + ... \tag{13.76}$$

Note: We could have chosen our branes along any axis.
Now let's look at an example of using this developement:

Example Calculate the force between 2 D4-branes. The picture in all dimensions is:

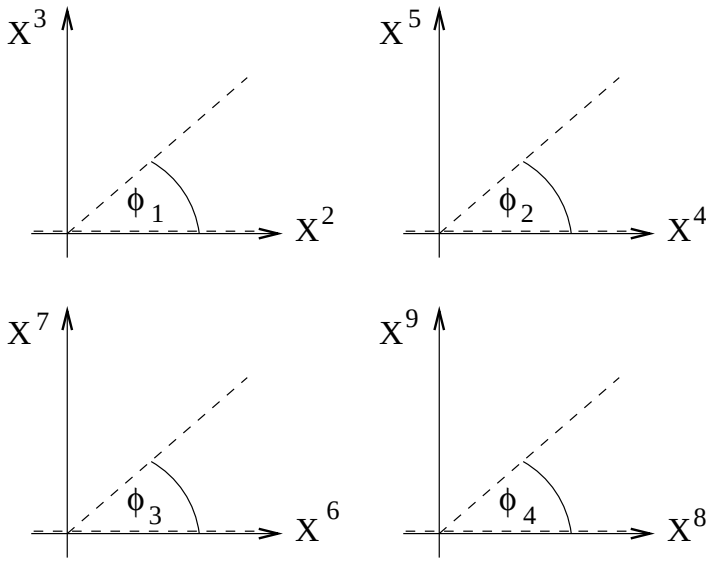


Figure 11

Looking at the NS-NS sector: using equation (**??**):

$$Z_{00} = \Pi_{i=1}^{4} \frac{\Theta_{00}(\nu_i, it)}{e^{\frac{\phi_i^2 t}{\pi}} \eta(it)} \tag{13.77}$$

where $\nu_i = i\phi_i t/\pi$. The total fermion wavefunction is

$$Z_\phi = \tfrac{1}{2}\left[\Pi_{i=1}^{4} \frac{\Theta_{00}(\nu_i, it)}{e^{\frac{\phi_i^2 t}{\pi}} \nu(it)} - \Pi_{i=1}^{4} \frac{\Theta_{01}(\nu_i, it)}{e^{\frac{\phi_i^2 t}{\pi}} \eta(it)} - \Pi_{i=1}^{4} \frac{\Theta_{10}(\nu_i, it)}{e^{\frac{\phi_i^2 t}{\pi}} \nu(it)} - \Pi_{i=1}^{4} \frac{\Theta_{11}(\nu_i, it)}{e^{\frac{\phi_i^2 t}{\pi}} \eta(it)}\right] \tag{13.78}$$

which is rather complicated. We need to make use of the generalized abstruse identity:

$$\Pi_{i=1}^{4}\Theta_{00}(\nu_i, it) - \Pi_{i=1}^{4}\Theta_{10}(\nu_i, it) - \Pi_{i=1}^{4}\Theta_{01}(\nu_i, it) - \Pi_{i=1}^{4}\Theta_{11}(\nu_i, it) = 2\Pi_{i=1}^{4}\Theta_{11}(\nu_i', it) \tag{13.79}$$

where $\nu_i' = i\phi_i' t/\pi$ and

$$\begin{aligned}
\phi_1' &= \tfrac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4) \\
\phi_2' &= \tfrac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4) \\
\phi_3' &= \tfrac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4) \\
\phi_4' &= \tfrac{1}{2}(\phi_1 - \phi_2 - \phi_3 + \phi_4)
\end{aligned} \tag{13.80}$$

So

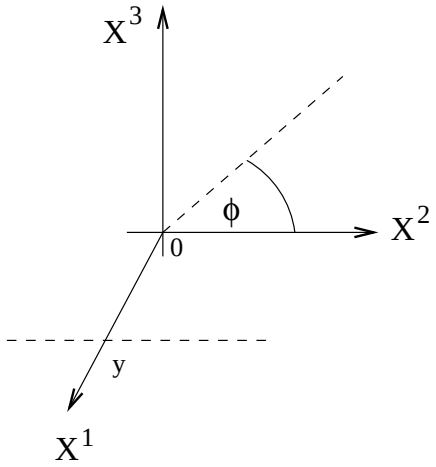
$$Z_\psi = \Pi_{i=1}^4 \frac{\Theta_{11}(\nu_i', it) e^{\frac{\phi_i'^2 t}{\pi}}}{\eta^4(it)} \tag{13.81}$$

The boson partition function we worked out before to be:

$$\begin{aligned} Z_X &= \Pi_{i=1}^4 - ie^{\frac{\phi_i^2 t}{\pi}} \frac{\eta(it)}{\Theta_{11}(\nu_i, it)} \\ &= \eta^4(it)\Pi_{i=1}^4 \frac{e^{\frac{\phi_i^2 t}{\pi}}}{\Theta_{11}(\nu_i, it)} \end{aligned} \tag{13.82}$$

Let us picture this interaction, which takes place in the (0,1) plane. 0 is the time dimension. The branes are points in the X^1 direction:

Figure 12



What is the force between the two branes? The partition function in the 0, 1 directions are

$$Z_{time} = \frac{1}{\sqrt{8\pi^2 \alpha' t}} \tag{13.83}$$

$$Z_{X^1} = e^{\frac{-ty^2}{2\pi\alpha'}} \tag{13.84}$$

Then the total partition function is given by:

$$\begin{aligned} Z &= -\int_0^\infty \frac{dt}{t} Z_\psi Z_X Z_{time} Z_{X^1} \\ &= -\int_0^\infty \frac{dt}{t} \frac{T}{\sqrt{8\pi^2\alpha' t}} e^{\frac{-ty^2}{2\pi\alpha'}} \Pi_{i=1}^4 \frac{\Theta_{11}(\nu_i', it)}{\Theta_{11}(\nu_i, it)} \end{aligned} \tag{13.85}$$

Where T is the 'volume' in time. The potential is given by $V(y) = Z/T$. We can see that as $t \to 0$, Z and V diverge. However, V is not a physical quantity. Let us calculate the force:

$$F = -\frac{\partial V}{\partial y} = -\frac{2y}{\pi\alpha'}\int_0^\infty dt \frac{T}{\sqrt{8\pi^2\alpha' t}} e^{\frac{-ty^2}{2\pi\alpha'}} \Pi_{i=1}^4 \frac{\Theta_{11}(\nu_i', it)}{\Theta_{11}(\nu_i, it)} \tag{13.86}$$

The leading contribution in the product as $t \to 0$ is equivalent to the limit of large t in the formula:

$$\Theta_{11}(-i\nu/t, i/t) = -i\sqrt{t} e^{\pi\nu^2/t} \Theta_{11}(\nu, it) \tag{13.87}$$

Then

$$\begin{aligned} \Pi_{i=1}^4 \frac{\Theta_{11}(\nu_i', it)}{\Theta_{11}(\nu_i, it)} &= \Pi_{i=1}^4 \frac{\Theta_{11}(-i\nu_i'/t, i/t)}{\Theta_{11}(-i\nu_i/t, i/t)} \\ &= \Pi_{i=1}^4 \frac{\sin(i\pi\nu_i'/t)}{\sin(i\pi\nu_i/t)} \\ &= \Pi_{i=1}^4 \frac{\sin(\phi_i')}{\sin(\phi_i')} \end{aligned} \tag{13.88}$$

Now at large y,

$$F \sim y \int_0^\infty \frac{dt}{\sqrt{t}} e^{-ty^2/2\pi\alpha'} \tag{13.89}$$

To get the dependence on y, we make the transformation $t \to t/y^2$, in which case:

$$F \sim y \cdot \frac{1}{y} \int_0^\infty \frac{dt}{\sqrt{t}} e^{-t/2\pi\alpha'} = \text{constant} \tag{13.90}$$

Thus $V(y) \sim y$, like a parallel plate capacitor.

13.4 D-Brane Scattering

So far we have been calculating the force between stationary branes. To treat scattering, one must be moving relative to the other with a velocity $\vec{v}$. Thus scattering formulas are just related to previous formulas by a Lorentz transformation. We can treat it by looking at the brane in the X^0, X^1 plane:

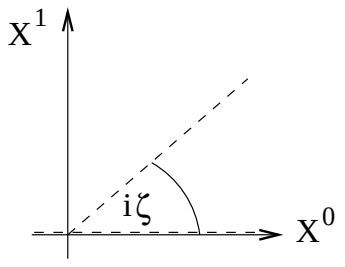


Figure 13

where ζ is the rapidity. Before we had $\tan(\phi) = X^9/X^8$. Now we have $\tanh(\zeta) = X^1/X^0 = v$, the speed. We also have:

$$\begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \to \begin{pmatrix} \cosh(\xi) & \sinh(\xi) \\ \sinh(\xi) & \cosh(\xi) \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \tag{13.91}$$

$$\cosh(\xi) = \gamma \tag{13.92}$$

So let us study the previous formulas with $\phi \to i\xi$ and the seperation generalized from y to $r(\tau) = \sqrt{y^2 + v^2\tau^2}$:

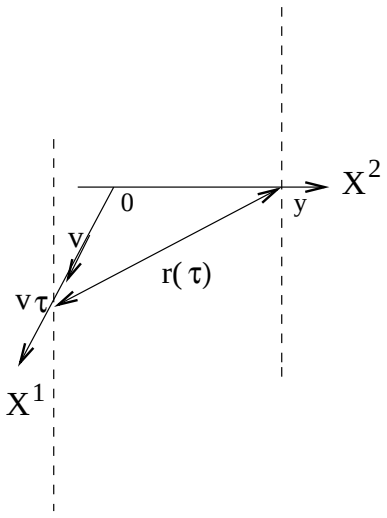


Figure 14

$$Z_{X^0X^1} = -ie^{\frac{\phi^2 t}{\pi}} \frac{\eta(it)}{\Theta_{11}(\nu, it)} \tag{13.93}$$

$$Z_{ab} = \frac{\Theta_{ab}(\nu, it)}{e^{\frac{\phi^2 t}{\pi}}\eta(it)} \tag{13.94}$$

Where $\nu = i\phi t/\pi = \xi t/\pi$. Let us say the branes are parallel in all the other directions. Then

$$\begin{aligned} Z_\psi &= \tfrac{1}{2}\left[\frac{\Theta_{00}(\nu, it)}{e^{\frac{\phi^2 t}{\pi}}\eta(it)}\left(\frac{\Theta_{00}(0, it)}{\eta(it)}\right)^3 - (01) - (10) - (11)\right] \\ &= \frac{e^{\frac{\phi^2 t}{\pi}}}{(\eta(it))^4}\Pi_{i=1}^4 \Theta_{11}(\nu_i', it) \\ &= \frac{e^{\frac{\phi^2 t}{\pi}}}{(\eta(it))^4}\Theta_{11}^4(\tfrac{1}{2}\nu, it) \end{aligned} \tag{13.95}$$

where $\nu_i' = \nu_i/2$. The partition function for the rest of the dimensions is given by:

$$Z_{X^2..X^9} = \frac{V_p}{(\sqrt{8\pi^2\alpha' t})^P}[\eta(it)]^{-6}e^{-ty^2/2\pi\alpha'} \tag{13.96}$$

where the successive parts come from the continuous momentum along the brane, the oscillators in the (8-2) transverse dimensions, and from the separation of the branes. Then

$$\begin{aligned} Z_{tot} &= \int_0^\infty \frac{dt}{t} Z_{X^0X^1} Z_\psi Z_{X^2\ldots X^9} \\ &= -iV_p \int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-p/2} e^{-ty^2/2\pi\alpha'} \frac{\Theta_{11}^4(\frac{1}{2}\nu, it)}{\Theta_{11}(\nu, it)}[\eta(it)]^{-9} \end{aligned} \tag{13.97}$$

Now the scattering amplitude is given by integrating over the whole trajectory:

$$Z/V_p = \int_{-\infty}^{\infty} d\tau V(r(\tau), v) \tag{13.98}$$

where the potential is found for $it \to i/t$.

$$V(r,v) = \frac{i2V_p}{(\sqrt{8\pi^2\alpha' t})^{p+1}} \int_0^t dt t^{(s-p)/2} e^{-tr^2/2\pi\alpha'} \frac{\Theta_{11}^4(\zeta/2\pi, i/t)}{\Theta_{11}^4(\zeta/\pi, i/t)}[\eta(i/t)]^{-9} \tag{13.99}$$

For $v \to 0, \zeta \to 0, \theta^4/\Theta \to 0, V \to 0, Z \to 1$. How fast do these terms disappear?

$$\Theta_{11}(\nu, it) = -2q^{-1/8}\sin(\pi\nu)\Pi(1-..)(1-..)(1-..) \tag{13.100}$$

$$\eta(it) = q^{1/24}\Pi(1-..) \tag{13.101}$$

Thus in the partition function the products exactly cancel, and we can clearly see that $Z \to 1$ as $\zeta \to 0$. For finite ζ:

$$\begin{aligned} \frac{\Theta_{11}^4(\zeta/2\pi, i/t)}{[\eta(i/t)]^9\Theta_{11}(\zeta/\pi, i/t)} &\to \frac{8q^{4/8}}{q^{9/24}q^{1/8}} \frac{\sin^4(\zeta/2)}{\sin(\zeta)} \\ &\to \frac{(\zeta)^3}{2} \\ &\to \frac{v^3}{2} \end{aligned} \tag{13.102}$$

Thus

$$\lim_{v\to 0} V(r,v) = v^4 \frac{2V_p}{(\sqrt{8\pi^2\alpha' t})^{p+1}} \int_0^\infty dt\, t^{(s-p)/2} e^{-tr^2/2\pi\alpha'} + O(v^6) \tag{13.103}$$

To find the r dependence, put $t \to t/r^2$ as before, and

$$t^{(s-p)/2}dt \to \frac{t^{(s-p)/2}dt}{r^2 r^{5-p}} \tag{13.104}$$

Thus

$$\lim_{v\to 0} V(r,v) = \frac{v^4}{r^{7-p}} \int_0^\infty dt\, t^{(s-p)/2} e^{-t/2\pi\alpha'} \tag{13.105}$$

Now we encounter a problem: as $r \to 0, V \to \infty$. But String theory is meant to be free of infinities. What could be wrong? In Electromagnetism, the Coulomb potential $\sim 1/r$ diverges as $r \to 0$ - so EM is wrong at short distances. There are other forces at short distances ($\equiv$ High energies) to correct it - quantum corrections.
So what do we need to do with String theory? We took $t \to t/r^2$ where r^2 is a scale: as $r \to 0$, the scale goes to 0. Thus most of the integral comes from the large t. The integral is dominated by $t \to \infty$. Go back to the original ration of theta functions: as $t \to \infty, q \to 0$. Recall we had:

$$\frac{\Theta_{11}^4(\frac{\nu}{2,it})}{[\eta(it)]^9 \Theta_{11}(\nu, it)} \to \frac{\sin^4(\zeta t/2)}{\sin(\zeta t)} \tag{13.106}$$

Most of the integral (**??**) comes from $t \sim 2\pi\alpha'/r^2$ as $r \to 0$. As $t \to \infty$, the sin becomes rapidly oscllating, and we can't take a series expansion - thus the integral remains finite.

Now let's look at the small (but not zero) v limit: $\zeta \simeq v$, and as $t \to \infty$, $vt \to \infty$: The dominant contribution to the integral now comes from oscillations in sin that have period of order $vt \sim 1$.

$$\frac{v 2\pi\alpha'}{r^2} \lesssim 1 \tag{13.107}$$

$$r \gtrsim \sqrt{v\alpha'} \tag{13.108}$$

thus with scattering processes, we can't probe distances smaller than this. If v is sufficiently small,

$$r \lesssim \sqrt{\alpha'} \tag{13.109}$$

i.e. distances smaller than the string scale. The interaction lasts for a time:

$$\delta t \sim \frac{r}{v} \tag{13.110}$$

Therefore the distances we can resolve are

$$\delta x \gtrsim \sqrt{v\alpha'} \tag{13.111}$$

In time

$$\delta t \sim \frac{\delta x}{v} \tag{13.112}$$

i.e.

$$\begin{aligned}\delta x \delta t &\gtrsim \frac{\delta x}{v}\sqrt{v\alpha'} \\ &\gtrsim \frac{v\alpha'}{v} \\ &\gtrsim \alpha'\end{aligned} \tag{13.113}$$

We have a new uncertainty principle! We cannot know x and t at the same time! These are just co-ordinates, and it can be extended to any pair of co-ordinates. So this means co-ordinates do not commute! They must be represented by matrices/operators. So it would seem, at the string scale, Nature is based on a non commutative geometry. How is this possible? Let's look for a physical explanation:
Here are two branes, with strings attached in all possible ways:

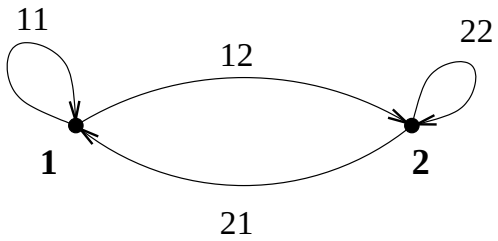


Figure 15

There are four ways to attach strings to the two branes. Now our everyday world is based on the massless modes of these strings. In the above diagram, we see that one string is associated with each brane, that is it has both its ends fixed to it. Thus there is one massless mode on each brane. There is a symmetry associated with the massless mode - $U(1)$ - just as in EM the photon has a gauge symmetry $U(1)$. So this system has 2 massless modes, each describing a $U(1)$ 'photon'. Overall we have

$$U(1) \otimes U(1) \tag{13.114}$$

What about the stretched strings? For them, we have $L_0 = y^2/4\pi^2\alpha' + ... > 0$ - there are no massless modes. $U(1) \otimes U(1)$ is all we have.
What if the seperation tends to zero? We get the following picture:

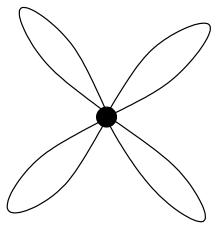


Figure 16

i.e we get 2 extra massless modes. Now all the strings look the same - we should be able to transform one into another. The symmetry now is

$$U(2) \tag{13.115}$$

This is the symmetry group of a sphere, and has 2 by 2 matrices as its elements. The positions are now all the same too, where before there were 4 labels for the strings, so one position is represented by a 2 by 2 matrix:

$$X \in U(2) \tag{13.116}$$

So they don't have to commute, and we get an uncertainty relation. The units of the relation are in terms of $\sqrt{\alpha'} = l_s$, the string scale length (as opposed to $\hbar$ in Quantum Mechanics).
How do we deal with this strange situation? Let's go back to elctroweak theory: The photon is described by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ A_\mu &\to A_\mu + \partial_\mu \lambda \end{aligned} \tag{13.117}$$

And the 3 weak bosons by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\ A_\mu &= \sum_{i=1}^{3} A_\mu^i \tau_i \end{aligned} \tag{13.118}$$

where τ_i are the Pauli matrices. This is an $SU(2)$ symmetry. With this, the interaction Lagrangian $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$ is invariant.
With gluons, a similar relation holds, but this time the symmetry is $SU(3)$:

$$A_\mu = \sum_{i=1}^{8} A_\mu^i \lambda_i \tag{13.119}$$

where λ_i are the Gell-Mann matrices. The gluon -gluon interactions are described by a similar $[A_\mu, A_\nu]$ term: The interaction Lagrangian (or Potential) is given by $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$ which contains $\mathrm{Tr}[A_\mu, A_\nu]^2$.
Now back to D-branes. Let us treat the co-ordinates transverse to the brane (i.e. at the position of the brane) X^μ as the potentials A_μ (See section 8.6 of Polchinski). The co-ordinate potential is then

$$V = \mathrm{Tr}[X_\mu, X_\nu]^2 \tag{13.120}$$

Expanding around the position of the brane $X^\mu = 0$, we get linear terms $\sim \partial V = 0$, and quadratic terms $\sim \partial^2 V = 0$ as these are the mass terms, and we are dealing with massless modes.
Now the number of fields we have is equal to the no. of transverse directions multiplied by n^2 where X is an $n \times n$ matrix. When we brought the two branes together we had a $U(2)$ symmetry, thus $1 \times 2 \times 2 = 4$ fields.
When $V = 0$, $[X_\mu, X_\nu] = 0$ and we can simultaneously diagonalize all the X^μ - we get n co-ordinates - the co-ordinates of n branes when they become distinct. Thus V correctly describes n free branes.