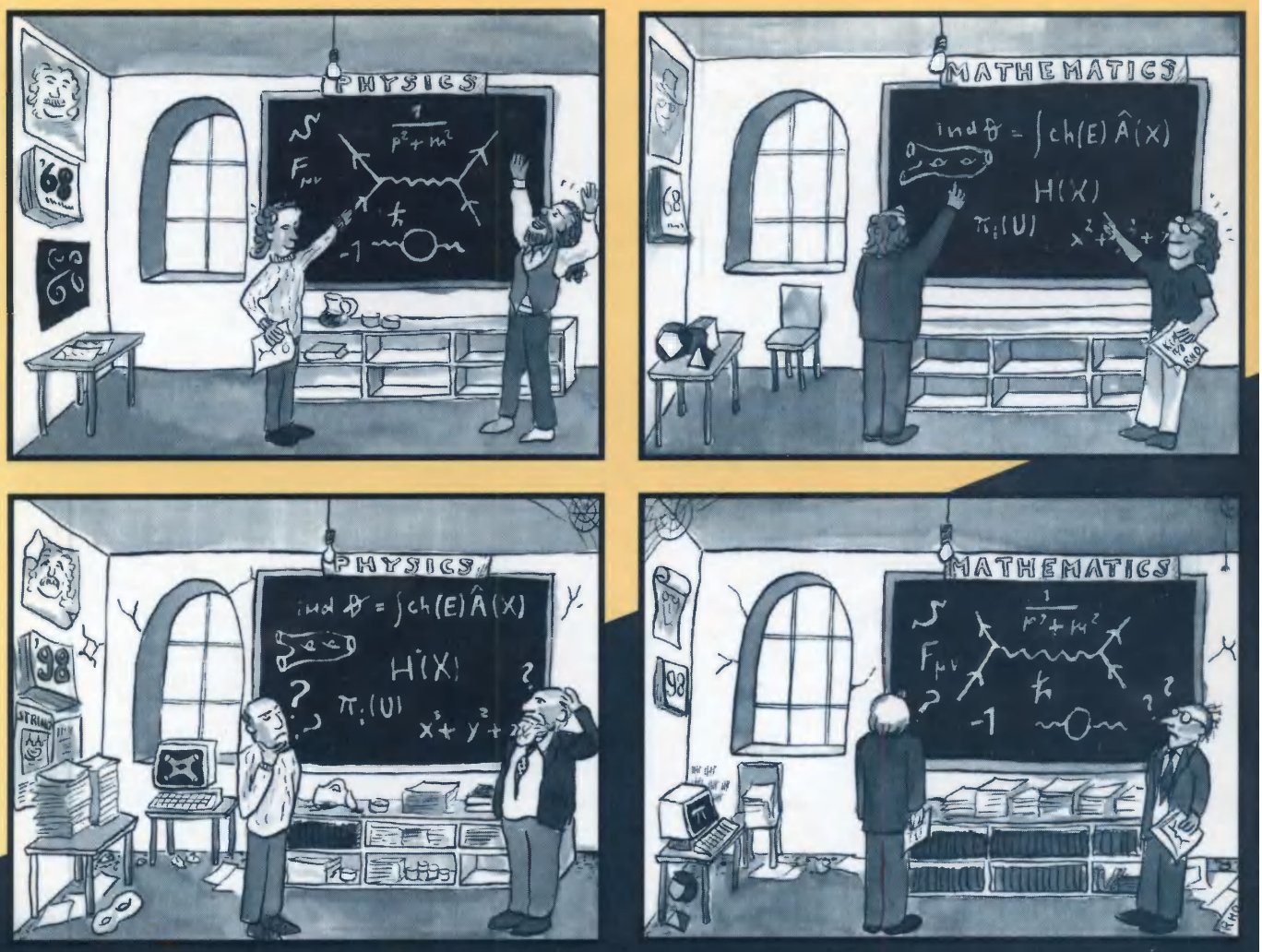
Quantum Fields and Strings: A Course for Mathematicians

VOLUME 2



Pierre Deligne
Pavel Etingof
Daniel S. Freed
Lisa C. Jeffrey
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Editors

American Mathematical Society
Institute for Advanced Study

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ABSTRACT. These two volumes are the written record of the 1996–1997 Special Year in Quantum Field Theory held at the Institute for Advanced Study. The goal of that program was to introduce mathematicians to the basic intuitions and modern developments in quantum field theory and string theory. Included in this publication are lecture notes from courses held during the Special Year, supplementary mathematical essays and texts, and problem sets with solutions.

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子曰：「性相近也，習相遠也。」

— CONFUCIUS, *Analects 17:2*

Preface

The 1996–1997 Special Year at the Institute for Advanced Study was a program in Quantum Field Theory. It was a year of intense activity. There were about four lectures per week, each lecture lasting two hours. In addition, there were weekly problem sessions, as well as supplementary lectures and remedial study meetings at all hours. The students ranged from the obscenely young to those less so.

In these two volumes we share our struggles with a broader audience.

The Special Year would not have been possible without the generous support of the National Science Foundation, the Friends of the Institute for Advanced Study, the J. Seward Johnson Sr. Charitable Trusts, and the Harmon Duncombe Foundation. It is a pleasure to thank these organizations for their support.

We warmly thank Dottie Phares for her infinite patience and skill in dealing with a noxious mixture of TEX flavors. We are indebted to Arthur Greenspoon, who not only proofread the entire manuscript but also prepared the index. The reader will join us in saying: thank you, Arthur. As always, Robbert Dijkgraaf's drawings—enjoy the covers!—liven up the proceedings, and we thank him for his contribution.

Most of all, we thank our teachers. They worked hard to educate us about beautiful ideas in physics, and we hope that they will be rewarded when one day this all becomes a beautiful part of mathematics. We will be pleased if these volumes can aid in that development. Finally, our thanksgiving would be amiss if we did not single out Edward Witten (and so exclude him from the signatories below). He gave generously of his time, energy, and abilities while serving as lecturer, teaching assistant, tutor, grader, organizer, and editor.

Pierre Deligne
Pavel Etingof
Dan Freed
Lisa Jeffrey
David Kazhdan
John Morgan
David Morrison

January, 1999

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Part 3

Conformal Field Theory and Strings

Lectures on Conformal Field Theory

Krzysztof Gawędzki

Introduction

Over the last decade and a half, conformal field theory (CFT) has been one of the main domains of interaction between theoretical physics and mathematics. The present review is designed as an introduction to the subject aimed at mathematicians. Its scope is limited to certain simple aspects of the theory of conformally invariant quantum fields in two space-time dimensions. Two-dimensional CFT experienced an explosive development following the seminal 1984 paper of Belavin, Polyakov and Zamolodchikov, although many of its concepts were introduced before that date. It still plays a very important role in numerous recent developments concerning higher-dimensional quantum fields. From the mathematical point of view, CFT may be defined as a study of the Virasoro algebra (or algebras containing it), of its representations and of their intertwiners. The theory defies, however, such restrictive definitions which obstruct the much wider landscape that it opens and into which we offer here only some glimpses. In four lectures we discuss:

1. conformal free fields,
2. axiomatic approach to conformal field theory,
3. perturbative analysis of two-dimensional sigma models,
4. exact solutions of the Wess-Zumino-Witten and coset theories.

To signal the omissions, whose full list would be much longer, let us point out that almost no mention is made of lattice models whose critical points are described by CFT's, of the perturbative approach to string theory, based on two-dimensional CFT, or of superconformal theories. The modest goal of these lectures is to make the physical literature on CFT, both the original papers and the textbooks (e.g. *Conformal Field Theory* by Di Francesco-Mathieu-Sénéchal, Springer 1996) more accessible to mathematicians.

LECTURE 1
Simple Functional Integrals

§1.1. What is quantum field theory?

Field theory deals with maps ϕ between a space Σ (the space-time) and a space M (the target). These spaces come with additional structure, e.g. they may be Riemannian or pseudo-Riemannian manifolds. The case of Minkowski signature on Σ is the one of field theory proper whereas the Euclidean signature corresponds to static (equilibrium) situations. In many cases, however (for example for flat Σ), the passage from one signature to the other may be obtained by analytic continuation in the time variable ("Wick rotation" $t \mapsto it$) and both situations may be studied interchangeably, the Euclidean setup being sometimes more convenient.

An important datum of the field theory is the action, a local functional of ϕ. For example, one may consider $S(\phi) = \int_\Sigma |d\phi|^2 dv$ (where the metric structures on Σ and M and a volume on Σ must be used to give sense to the right hand side).

In the classical field theory one studies the extrema of the action functional, i.e. maps ϕ_{cl} satisfying

$$\delta S(\phi_{\mathrm{cl}}) = 0.$$

The extremality condition is a PDE for ϕ_{cl}, e.g. the wave or Laplace equation or the Maxwell, Yang-Mills, or Einstein ones, to mention only the most famous cases. One should bear in mind that non-linear PDE's is a complicated subject, where our ignorance exceeds our knowledge.

Following an extremely intuitive reformulation of quantum field theory (QFT) by Feynman, the latter consists in studying functional integrals

$$\int_{\mathrm{Map}(\Sigma,M)} F(\phi)\, \mathrm{e}^{-\frac{1}{\hbar}S(\phi)}\, D\phi \tag{1.1}$$

where $F(\phi)$ is a functional of ϕ (an "insertion") and $D\phi$ stands for a local product $\prod_{x\in\Sigma} d\mu(\phi(x))$ of measures on M. The above expression is formal and one of the aims of these lectures is to show that it may be given sense and even calculated in some simple situations. More generally, however, the functional integral written above should be considered as an approximate expression for structures which live their own lives, different and in some aspects more interesting then the lives of the

objects whose symbols appear in the integral. Still, although formal and approximate, the functional integral language has proved extremely useful in studying the QFT structures. It also has made the relation of QFT to classical field theory quite intuitive: unlike in the case of the latter, all maps ϕ (called often but somewhat abusively classical field configurations or classical fields) give contributions to QFT, each with the probability amplitude $p(\phi) \sim \mathrm{e}^{-\frac{1}{\hbar}S(\phi)}$ (in the Minkowski case, these are not probabilities since $S(\phi)$ should be taken imaginary, but never mind). The classical physics corresponds to the stationary phase or saddle point approximation in which all contributions to the integral (1.1) except for those of the stationary points of S are discarded. Such an approximation is justified when the Planck constant $\hbar$ may be treated as very small (as in the usual macro-scale physics, but not for example in superfluid helium).

It should be stressed that, in general, the relation of quantum to classical is not one to one (as the formulation (1.1) could suggest) and not even many to many, except for especial situations, e.g. with lots of symmetries on both levels. Such situations are of special interest for mathematicians because they allow the reduction of QFT to the more familiar classical structures. The QFT approach has afforded in many such cases new insights; recall, for example, the use of topological or quasi-topological field theories as factories of invariants. This motivates the utilitarian interest of mathematicians in QFT. The very difficulty in making sense out of the integrals (1.1) is at the origin of a deeper source of mathematical interest of QFT, namely in the mathematics of the new structures carried by QFT. The mathematical structures in integrable or conformal two-dimensional field theories or in four-dimensional SUSY gauge theories just emerging provide here the examples (not even mentioning mathematics which promises to underlie the panoply of string theory dualities).

It may be good to recall briefly what physicists use QFT for.

(i) It provides a relativistic theory of interactions of elementary particles. Thus Quantum Electrodynamics (QED) describes interactions of electrons, their anti-particles positrons and photons, the Glashow-Weinberg-Salam theory of electro-weak interactions describes at the same time beta decays and QED, and Quantum Chromodynamics (QCD) deals with the strong interactions (quarks forming nucleons). The latter two comprise what is called the standard model of particle physics.

(ii) QFT in its Euclidean version describes critical phenomena at the second order phase transition points like that in H_2O at temperature $T_c \cong 374°C$ and pressure $p_c \cong 2.2\times 10^7$ pascals ($\sim$ 20atm), or that in Fe at $T_c \cong 770°C$ or in the Ising model at its critical temperature. The criticality is characterized by slow decay with distance of statistical correlations whose asymptotics is described by Euclidean QFT.

(iii) In string theory, aiming at unification of gravity with the other interactions (from point (i)), two-dimensional conformal QFT provides the classical (and perturbative) solutions and the the quantum string theory proper (still to be non-perturbatively defined) may be considered as a deformation of quantum field theory where particles are replaced by strings (the typical size of the string being the deformation parameter).

(iv) Finally, many QFT techniques are used in the theory of non-relativistic condensed matter.

§1.2. Euclidean free field and Gaussian functional integrals

In the rest of this lecture we shall describe how one may give sense to the functional integral (1.1) in the simplest case of a free field. This is the case where the space of maps $\mathrm{Map}(\Sigma, M)$ is a vector space (inheriting the linear structure from that of M) or is a union of affine spaces and where the action functional S is quadratic. The corresponding functional integral is Gaussian plus, in the 2nd case, an easy but interesting decoration (in theta functions). The adjective "free" refers to the absence of particle interactions, which is related to linearity of the classical equations.

Let (Σ, γ) be a Riemannian, $(d+1)$-dimensional, oriented, compact manifold and let $M = \mathbf{R}$ (we consider the Euclidean case). The action functional is taken as

$$S(\phi) = \frac{\beta}{4\pi} \int_{\Sigma} (|d\phi|^2 + m^2\phi^2) dv \equiv \tfrac{1}{2}(\phi, G^{-1}\phi)_{L^2} \tag{1.2}$$

where dv is the Riemannian volume and the operator $\frac{\beta}{2\pi} G = (-\Delta + m^2)^{-1}$ is often called the (Euclidean) propagator of the free field of mass m.

The simplest functional integral of the type (1.1) is the one with trivial insertion $F = 1$ giving what is called the "statistical sum" or "partition function" and denoted traditionally by Z:

$$Z = \int_{\mathrm{Map}(\Sigma,\mathbf{R})} \mathrm{e}^{-S(\phi)}\, D\phi = \int_{\mathrm{Map}(\Sigma,\mathbf{R})} \mathrm{e}^{-\frac{1}{2}(\phi, G^{-1}\phi)}\, D\phi$$

(we have put $\hbar = 1$ for simplicity). The names come from statistical physics: the integral sums the probabilities (probability amplitudes) $p(\phi)$ of all microscopic states ϕ of the system. The space $\mathrm{Map}(\Sigma, \mathbf{R})$ may be considered as a Hilbert space with the L^2 scalar product using the metric volume dv on Σ. Were this space finite dimensional, the integral would give

$$Z = (\det G)^{\frac{1}{2}}$$

if we normalized $D\phi = \prod_i \frac{d\phi_i}{\sqrt{2\pi}}$, where (ϕ_i) are any orthonormal coordinates on the Hilbert space of maps. It is sensible to maintain the above formula as a definition of the formal functional integral for the partition function even in the infinite-dimensional case. It is necessary then to give sense to the determinant of the positive operator G, whose (discrete) eigenvalues λ_n, $n = 1, 2, \ldots$, behave as $\mathcal{O}(n^{-2/(d+1)})$. A convenient (but nonunique) way to do this is by the zeta-function regularization defining

$$\det G = \mathrm{e}^{-\zeta_G'(0)}$$

where $\zeta_G(s)$, given as $\sum_n \lambda_n^{-s}$ for $\mathrm{Re}\, s < -\frac{d+1}{2}$, extends to a meromorphic function analytic in the vicinity of zero. We shall stick to this definition of infinite determinants throughout the present lectures.

The next functional integrals we may wish to compute are the ones for the correlation functions depending on a sequence $(x_i)_{i=1}^n$ of points in Σ:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle \equiv \frac{\int_{\mathrm{Map}(\Sigma,\mathbf{R})} \phi(x_1) \cdots \phi(x_n)\, \mathrm{e}^{-S(\phi)}\, D\phi}{\int_{\mathrm{Map}(\Sigma,\mathbf{R})} \mathrm{e}^{-S(\phi)}\, D\phi}.$$

Again mimicking the case of finite-dimensional Gaussian integrals, we may define the formal functional integral on the right hand side by setting

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \begin{cases} 0 & \text{for } n \text{ odd,} \\ G(x_1,x_2) & \text{for } n=2, \\ G(x_1,x_2)G(x_3,x_4)+G(x_1,x_3)G(x_2,x_4) & \\ \quad +G(x_1,x_4)G(x_2,x_3) & \text{for } n=4, \\ \sum\limits_{\substack{\text{pairings} \\ \{(i_+,i_-)\}}} \prod\limits_{(i_+,i_-)} G(x_{i_+},x_{i_-}) & \text{for } n \text{ even} \end{cases}$$

where $G(x,y)$ denotes the kernel of operator G which is smooth for $x \neq y$ and exhibits a coinciding points singularity $\sim \ln \operatorname{dist}(x,y)$ for $d=1$ and $\sim \operatorname{dist}(x,y)^{-d+1}$ for $d>1$. Such a definition of the correlation functions is additionally substantiated by the fact that there exists a probability measure $d\mu_G$ on the space of distributions $\mathcal{D}'(\Sigma)$ such that

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \int\limits_{\mathcal{D}'(\Sigma)} \phi(x_1)\cdots\phi(x_n)\, d\mu_G(\phi)$$

where the equality is understood in the sense of distributions. $\phi(x)$ may then be considered as a random distribution.

§1.3. Feynman-Kac formula

Some people in the audience may wonder what this all has to do with Minkowski space field theory involving the Hilbert space $\mathcal{H}$, quantum Hamiltonian H and quantum field operators acting in $\mathcal{H}$ since the (Euclidean) functional integral scheme has led us to entirely commutative structures such as the Euclidean random distributions $\phi(x)$ which may, at most, be considered as a distribution with values in commuting multiplication operators acting in $L^2(\mathcal{D}'(\Sigma), d\mu_G)$. The relation of the two schemes is provided by the so-called *Feynman-Kac formula*. Let us start from a simple quantum mechanical example.

Example 1. $d=0$, $\Sigma=[0,L]$ with the periodic identification of the ends and the standard metric. In this case, $d\mu_G$ is supported by the space of continuous functions $\mathcal{C}_{\text{per}}([0,L])$ and is essentially a version of the Wiener measure. More exactly, it differs from the latter by the density $\sim \mathrm{e}^{-\frac{\beta m^2}{2\pi}\int_0^L \phi(x)^2}$. Suppose that $0 \le x_1 \le x_2 \le \cdots \le x_n \le L$. Then

$$\int\limits_{\mathcal{C}_{\text{per}}([0,L])} \phi(x_1)\cdots\phi(x_n)d\mu_G(\phi) = \frac{\operatorname{tr} \mathrm{e}^{-x_1H}\varphi \mathrm{e}^{(x_1-x_2)H}\varphi\cdots\varphi \mathrm{e}^{(x_n-L)H}}{\operatorname{tr} \mathrm{e}^{-LH}} \tag{1.3}$$

where

$$H = -\frac{\pi}{\beta}\frac{d^2}{d\varphi^2} + \frac{\beta m^2}{4\pi}\varphi^2 - \frac{m}{2}$$
$$= m\left(-\sqrt{\frac{\pi}{\beta m}}\frac{d}{d\varphi} + \sqrt{\frac{\beta m}{4\pi}}\varphi\right)\left(\sqrt{\frac{\pi}{\beta m}}\frac{d}{d\varphi} + \sqrt{\frac{\beta m}{4\pi}}\varphi\right) \equiv ma^*a$$

is the Hamiltonian of a harmonic oscillator acting in $L^2(\mathbf{R}, d\varphi)$. a and its adjoint a^* satisfy the canonical commutation relation

$$[a, a^*] = 1.$$

The ground state Ω of H is proportional to $\mathrm{e}^{-\frac{\beta m}{4\pi}\varphi^2}$ and corresponds to the zero eigenvalue. Ω is annihilated by a and the higher H-eigenstates are obtained by applying powers of a^* to Ω, each a^* raising the energy (i.e. the eigenvalue of H) by m (a is called the annihilation and a^* the creation operator). Hence the spectrum of H is $\{0, m, 2m, \dots\} = m\mathbf{Z}_+$. With the use of the orthonormal basis $(\frac{1}{\sqrt{n!}}(a^*)^n\Omega)_{n=1}^{\infty}$ composed, up to normalizations, from the Hermite polynomials $H_n(\sqrt{\frac{\beta m}{\pi}}\varphi)$ times Ω, $L^2(\mathbf{R}, d\varphi)$ may be identified with (the Hilbert-space completion of) the symmetric algebra $S\mathbf{C}$ (the bosonic Fock space over $\mathbf{C}$). Note that

$$\varphi = \sqrt{\tfrac{\pi}{\beta m}}(a + a^*).$$

Problem 1. Consider formula (1.3) for the 2-point function.

(a) Use the Fourier transform to write the left hand side. Show that its $L \to \infty$ limit $G_\infty(x_1, x_2)$ exists.

(b) Prove that for $x_1, \dots, x_n > 0$ and complex numbers $\lambda_1, \dots, \lambda_n$,

$$\sum_{k,l=1}^{n} \bar{\lambda}_k \lambda_l \, G_\infty(-x_k, x_l) \;\geq\; 0. \tag{1.4}$$

(c) What is the $L \to \infty$ limit of the right hand side of Eq. (1.3) for $n = 2$?

(d) Show that both sides of Eq. (1.3) with $n = 2$ coincide at $L = \infty$. Prove (b) using this result.

(e) Prove relation (1.3) for $n = 2$ and finite L.

It may be more natural to read the Feynman-Kac formula from the right to left. $\mathrm{e}^{-xH}(\phi, \phi')$ is nothing else but the transition probability to pass from ϕ to ϕ' in time x, which may be used to define the Markov process $\phi(x)$ with the measure on the space of continuous realizations coinciding with $d\mu_G$.

Example 2. $d > 0$, $\Sigma = [0, L]^{d+1}$ with periodic identifications. Now $d\mu_G$ is carried by genuinely distributional ϕ's. Let $(x_i = (x_i^0, \mathbf{x}_i))_{i=1}^n$ satisfy $0 < x_1^0 < x_2^0 < \cdots < x_n^0 < L$. The Feynman-Kac formula now takes the form

$$\int_{\mathcal{D}'([0,L]^{d+1})} \phi(x_1) \cdots \phi(x_n) \, d\mu_G(\phi) = \frac{\operatorname{tr} \mathrm{e}^{-x_1^0 H} \varphi(\mathbf{x}_1) \mathrm{e}^{(x_1^0 - x_2^0)H} \varphi(\mathbf{x}_2) \cdots \varphi(\mathbf{x}_n) \mathrm{e}^{(x_n^0 - L)H}}{\operatorname{tr} \mathrm{e}^{-LH}} \tag{1.5}$$

where the quantum Hamiltonian H is a positive self-adjoint operator in the Hilbert space $\mathcal{H}$, a tensor product of an infinite number of harmonic oscillators, one for each Fourier mode $\varphi_{\mathbf{k}}$ of the classical time zero field $\varphi_{\mathbf{k}} = \int_{[0,L]^d} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \phi(0, \mathbf{x}) d\mathbf{x}$:

$$\mathcal{H} = \bigotimes_{\pm\mathbf{k} \in \frac{2\pi}{L}\mathbf{Z}^d} L^2(\mathbf{C}, d^2\varphi_{\mathbf{k}}).$$

The annihilation operators

$$a_{\mathbf{k}} = \sqrt{\frac{\pi L^d}{\beta k_0}}\frac{d}{d\varphi_{\mathbf{k}}} + \sqrt{\frac{\beta k_0}{4\pi L^d}}\varphi_{-\mathbf{k}},$$

where $k_0 = \sqrt{\mathbf{k}^2 + m^2}$, and the creation operators $a^*_{\mathbf{k}}$ adjoint to them satisfy the canonical commutation relations

$$[a_{\mathbf{k}}, a^*_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$$

with all the other commutators vanishing. The quantum (time zero) field is

$$\varphi(\mathbf{x}) = \sum_{\mathbf{k}} \sqrt{\frac{\pi}{\beta k_0 L^d}}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}(a_{-\mathbf{k}} + a^*_{\mathbf{k}}).$$

The Hamiltonian

$$H = \sum_{\mathbf{k}} k_0 a^*_{\mathbf{k}} a_{\mathbf{k}}$$

has $\Omega \sim \mathrm{e}^{\sum_{\mathbf{k}} \frac{\beta k_0}{4\pi L^d}|\varphi_{\mathbf{k}}|^2}$ as the ground state annihilated by all $a_{\mathbf{k}}$. The spectrum of H is $\sum_{\mathbf{k}} k_0 \mathbf{Z}_+$.
There are three natural ways to look at the Hilbert space $\mathcal{H}$:

i. $\mathcal{H}$ is an infinite tensor product of oscillator spaces $L^2(\mathbf{C}, d^2\varphi_{\mathbf{k}})$ (how should it be defined?);

ii. $\mathcal{H}$ is the Hilbert space completion of the symmetric algebra $S(l^2(\frac{2\pi}{L}\mathbf{Z}^d)) \cong S(L^2([0,L]^d))$; this is the Fock space picture;

iii. $\mathcal{H}$ is a space of functionals of the variables $\varphi_{\mathbf{k}}$ or of the time zero classical field $\phi(0,\mathbf{x})$ obtained by acting by creation operators $a^*_{\mathbf{k}}$ on the vacuum functional Ω.

One may introduce the (Minkowski) time dependence of the quantum field by defining

$$\varphi(t,\mathbf{x}) = \mathrm{e}^{itH}\varphi(\mathbf{x})\mathrm{e}^{-itH}.$$

Problem 2. Show that

$$\varphi(t,\mathbf{x}) = \sum_{\mathbf{k}} \sqrt{\frac{\pi}{\beta k_0 L^d}}(\mathrm{e}^{-\imath tk_0 + \imath\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + \mathrm{e}^{\imath tk_0 - i\mathbf{k}\cdot\mathbf{x}}a^*_{\mathbf{k}}). \tag{1.6}$$

The infinite volume limit $L \to \infty$ of the formulae for H and $\varphi(t,\mathbf{x})$ may be easily taken if we introduce operators $a(\mathbf{k}) = L^{d/2}a_{\mathbf{k}}$. In the limit one obtains the operator-valued distributions $a(k)$ and their adjoints $a^*(k)$ acting in the Fock space $\overline{S(L^2(\mathbf{R}^d, \bar{d}\,\mathbf{k}))}$ ($\bar{d}\,\mathbf{k} \equiv d\mathbf{k}/(2\pi)^d$) and satisfying the commutation relations

$$[a(\mathbf{k}), a^*(\mathbf{k}')] = (2\pi)^d\delta(\mathbf{k} - \mathbf{k}').$$

By identifying $L^2(\mathbf{R}^d, \bar{d}\,\mathbf{k})$ (by multiplication by $\sqrt{2k_0}$) with the space of functions on the upper mass hyperboloid $\{(k_0, \mathbf{k})\}$ square-integrable with the Lorentz-invariant measure $\frac{\bar{d}\,\mathbf{k}}{2k_0}$, one obtains the Minkowski scalar free field of mass m constructed in a more abstract and explicitly Poincaré-covariant way in [I-**Axiomatics**].

§1.4. Massless free field with values in S^1

Let us pass to the next case where (Σ, γ) is again a general compact $(d+1)$-dimensional manifold, $M = \mathbf{R}/2\pi\mathbf{Z} \cong S^1$ and the action functional is that of the massless free field: $S(\phi) = \frac{\beta}{4\pi}\int_\Sigma |d\phi|^2 dv$ (note that β has a natural interpretation as the square of the radius ρ of the circle if we rewrite the classical action as $\frac{1}{4\pi}\int |d\phi|^2 dv$ but use the metric $\rho^2 d\phi^2$ on the target). This is the case of a conformal (invariant) field theory with the conformal group acting (projectively) in the corresponding Hilbert space of states, transforming field operators covariantly. We shall get there slowly discussing in more detail the $d = 1$ case where the conformal group is infinite-dimensional, essentially $= \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. Let us start with an elementary treatment of the functional integral.

How should we view the space of maps from Σ to S^1? A convenient way is to define

$$\mathrm{Map}(\Sigma, \mathbf{R}/2\pi\mathbf{Z}) = \bigcup_{\chi \in \mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbf{Z})} \mathrm{Map}(\tilde{\Sigma}, \mathbf{R})_\chi / 2\pi\mathbf{Z}$$

where $\phi_\chi \in Map(\tilde{\Sigma}, \mathbf{R})_\chi$ is a a function on the universal cover $\tilde{\Sigma}$ of Σ equivariant with respect to the action of the fundamental group:

$$\phi_\chi(ax) = \phi_\chi(x) + \chi(a) \qquad \text{for} \qquad a \in \pi_1(\Sigma).$$

Note that this definition makes sense for maps of arbitrary class (smooth, continuous or distributional). $\mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbf{Z}) \cong H^1(\Sigma, 2\pi\mathbf{Z})$ with χ given by the periods of $\alpha \in H^1(\Sigma, 2\pi\mathbf{Z})$. Each $\phi_\chi \in \mathrm{Map}_\chi$ may be uniquely decomposed according to

$$\phi_\chi = \int_{x_0}^{x} \alpha_h + \psi \equiv \phi_h + \psi \tag{1.7}$$

where α_h is the harmonic representative of $\alpha \in H^1$ corresponding to χ, x_0 is the base point of Σ and ψ is a single-valued function on Σ. For the free field action we obtain

$$S(\phi_\chi) = \tfrac{\beta}{4\pi}\|\alpha_h\|^2_{L^2} + \tfrac{\beta}{4\pi}(\psi, -\Delta\psi)_{L^2}$$

(there is no mixed term, why?). This suggests the following definition of the functional integral for the partition function of the system:

$$\begin{aligned} Z = \int_{\mathrm{Map}(\Sigma, S^1)} \mathrm{e}^{-S(\phi)} D\phi &= \sum_{\alpha \in H^1(\Sigma, 2\pi\mathbf{Z})} \mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}} \int_{\mathrm{Map}(\Sigma, \mathbf{R})} \mathrm{e}^{-\frac{\beta}{4\pi}(\psi, -\Delta\psi)_{L^2}} D\psi \\ &= \sum_{\alpha \in H^1(\Sigma, 2\pi\mathbf{Z})} \mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}} \left(\frac{2\pi \mathrm{vol}_\Sigma}{\det'(-\frac{\beta}{2\pi}\Delta)}\right)^{1/2} \end{aligned} \tag{1.8}$$

where in $\det'$ the zero mode should be omitted (it contributes the factor $\sqrt{2\pi \mathrm{vol}_\Sigma}$, where $\mathrm{vol}_\Sigma = \int_\Sigma dv$, to the functional integral, why?).

Example 3. $d = 0$, $\Sigma = [0, L]_{\mathrm{per}}$. In this case, $\alpha_h = \frac{2\pi}{L} n dx$ and an easy calculation (see Problem 4 below) shows that

$$\det'(-\tfrac{\beta}{2\pi}\tfrac{d^2}{dx^2}) = 2\pi L^2/\beta.$$

Hence the $d = 0$ partition function

$$Z = \sum_{n\in\mathbf{Z}} \mathrm{e}^{-\pi\beta L^{-1}n^2} \left(\frac{2\pi L}{\det'(-\frac{\beta}{2\pi}\frac{d^2}{dx^2})}\right)^{1/2} \underset{\text{resummation}}{\overset{\text{Poisson}}{=}} \sum_{n\in\mathbf{Z}} \mathrm{e}^{-\pi\beta^{-1}Ln^2} = \operatorname{tr} \mathrm{e}^{-LH}$$

where now $H = -\frac{\pi}{\beta}\frac{d^2}{d\varphi^2}$ is the operator acting in $L^2(\mathbf{R}/2\pi\mathbf{Z}, d\varphi)$ with the eigenvectors $\mathrm{e}^{in\varphi}$, $n \in \mathbf{Z}$, corresponding to eigenvalues $\pi n^2/\beta$.

Problem 3. Prove for $0 \le x_1 \le \cdots \le x_n \le L$ and $q_i \in \mathbf{Z}$ the Feynman-Kac formula

$$\int \prod_{i=i}^{n} \mathrm{e}^{iq_i\phi(x_i)} \mathrm{e}^{-\frac{\beta}{4\pi}\int_0^L (d\phi/dx)^2} D\phi = \operatorname{tr} \mathrm{e}^{x_1 H} \mathrm{e}^{iq_1\varphi} \mathrm{e}^{(x_1-x_2)H} \mathrm{e}^{iq_2\varphi} \cdots \mathrm{e}^{iq_n\varphi} \mathrm{e}^{(x_n-L)H}$$

where the functional integral over $\mathrm{Map}([0, L], S^1)$ on the left hand side is computed as the one for the partition function Z treated above. Infer that the left hand side may be also expressed as the expectation $\langle \mathrm{e}^{iq_1\phi(x_1)} \cdots \mathrm{e}^{iq_n\phi(x_n)} \rangle$ w. r. t. the Wiener measure on the periodic paths on S^1 constructed from the transition probabilities $\mathrm{e}^{-xH}(\varphi, \varphi')$.

Let us discuss in greater detail the case $d = 1$ when (Σ, γ) is a Riemann surface of genus h_Σ with a fixed metric γ (inducing the complex structure of Σ). Let us choose a marking of Σ (a symplectic basis $(a_i, b_j)_{i,j=1}^{h_\Sigma}$ of $H_1(\Sigma, \mathbf{Z})$) with the corresponding basis $(\omega^i)_{i=1}^{h_\Sigma}$ of holomorphic $(1,0)$-forms, $\int_{a_i} \omega^j = \delta^{ij}$, $\int_{b_i} \omega^j = \tau^{ij}$. The imaginary part τ_2 of the period matrix $\tau = (\tau^{ij})$ is positive. The equation

$$\alpha_h = \frac{\pi}{i}(\bar\tau\mathbf{m} + \mathbf{n})^t \tau_2^{-1}\omega + c.c.$$

for $\mathbf{m}, \mathbf{n} \in \mathbf{Z}^{h_\sigma}$ gives the harmonic forms in $H^1(\Sigma, 2\pi\mathbf{Z})$ (with a_i-periods $-2\pi m_i$ and b_j-periods $2\pi n_j$).

$$\|\alpha_h\|^2_{L^2} = (2\pi)^2(\bar\tau\mathbf{m} + \mathbf{n})^t \tau_2^{-1}(\tau\mathbf{m} + \mathbf{n})$$

and the sum over α in eq.(1.8) may be done explicitly. After Poisson resummation over $\mathbf{n}$, one obtains the following result

$$\sum_{\alpha\in H^1(\Sigma,2\pi\mathbf{Z})} \mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}} = \beta^{-h_\Sigma/2}(\det\tau_2)^{1/2}\vartheta_{Q_\beta}(\tau, \bar\tau) \tag{1.9}$$

where the "theta function" ϑ_{Q_β} is defined as follows. Let $\mathbf{E}_s$ be the s-dimensional Euclidean space. Let Q be a lattice in $\mathbf{E}_{s_+,s_-} = \mathbf{E}_{s_+} \oplus \mathbf{E}_{s_-}$ considered with the indefinite metric $|\cdot|^2_{\mathbf{E}_{s_+}} - |\cdot|^2_{\mathbf{E}_{s_-}}$. Then

$$\vartheta_Q(\tau, \bar\tau) = \sum_{(q_+,q_-)\in Q^{h_\Sigma}} \mathrm{e}^{\pi i(q_+,\tau q_+) - \pi i(q_-,\bar\tau q_-)}.$$

where the decomposition $q = (q_+, q_-)$ is according to that of $\mathbf{E}_{s_+,s_-}$. Above,

$$Q_\beta = \{\, (\tfrac{\beta^{1/2}m+\beta^{-1/2}n}{\sqrt{2}}, \tfrac{\beta^{1/2}m-\beta^{-1/2}n}{\sqrt{2}}) \mid m, n \in \mathbf{Z} \,\} \subset \mathbf{R}\oplus\mathbf{R}.$$

Inserting the relation (1.9) into eq.(1.8), we obtain

$$Z \equiv Z_\beta = \mathrm{e}^{(-6\ln 2\pi + 11\ln\beta/2)(h_\Sigma - 1)}\, \vartheta_{Q_\beta}(\tau, \bar\tau) \left(\frac{\mathrm{vol}_\Sigma \det\tau_2}{\det'(-\Delta)}\right)^{1/2}.$$

From eq.(1.8) it is obvious that the right hand side is marking-independent. Technically, this is due to the fact that, in the indefinite scalar product, the lattice Q_β is even (scalar products are integers, scalar squares are even) and self-dual.

We may discard from Z any factor of the form $(\text{const.})^{h_\Sigma - 1}$ by the addition to the action of a term proportional to the integral of the scalar curvature r of Σ since $\int_\Sigma r dv = 4\pi(1 - h_\Sigma)$. Doing that we discover a somewhat miraculous equality

$$Z_\beta = Z_{1/\beta}, \tag{1.10}$$

a consequence of the obvious identity $Q_\beta = Q_{1/\beta}$. More directly, identity (1.10) follows from the Poisson resummation formula applied to the left hand side of eq.(1.9) and the fact that the lattice $H^1(\Sigma, 2\pi\mathbf{Z})$ with the L^2 scalar product is isomorphic to its dual (the isomorphism is induced by the intersection form). Eq.(1.10) is the simplest manifestation of the so called T-**duality** which states that the $1+1$-dimensional massless free fields with values in the circles of radius ρ and of radius ρ^{-1} are indistinguishable. This identification of inverse radii of free field compactification has a deep meaning in the string theory context and we shall return to it in a later discussion.

On the genus 1 curve $T_\tau = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ with τ in the upper half plane and the standard metric,

$$\det'(-\Delta) = \tau_2^2 |\eta(\tau)|^4$$

where $\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$ is the Dedekind eta function.

Problem 4 (a relatively complex calculation, going back to Kronecker, 1890).

(a) Using the identity $\lambda^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-\lambda t} dt$ show that $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{12}$ where ζ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (for $\operatorname{Re} s > 1$, analytically continued elsewhere).

(b) For $\tau = \tau_1 + i\tau_2$ with τ_i real, $\tau_2 > 0$ show using the identity from (a) and Poisson resummation that for $\operatorname{Re} s$ sufficiently large

$$\sum_{n=-\infty}^{\infty} |\tau + n|^{-2s} = \frac{\sqrt{\pi}}{\Gamma(s)} \left(\sum_{n \neq 0} e^{2\pi i n \tau_1} \int_0^\infty t^{s-3/2} e^{-\tau_2^2 t - \pi^2 n^2 / t} dt + \tau_2^{-2s+1} \Gamma(s - 1/2) \right).$$

Note that the right hand side is analytic in s around $s = 0$. Using (easy) relations $\Gamma(s)^{-1} = s + \mathcal{O}(s^2)$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $\int_0^\infty t^{-3/2} e^{-x(t+t^{-1})} dt = \sqrt{\pi} x^{-1/2} e^{-2x}$ obtain:

$$\left. \frac{d}{ds} \right|_{s=0} \sum_n |\tau + n|^{-2s} = -\ln |1 - q|^2 - 2\pi\tau_2$$

with the standard notation $q \equiv e^{2\pi i \tau}$.

(c) By taking $\tau \to 0$ in the last formula show that $\zeta'(0) = -\frac{1}{2} \ln(2\pi)$.

(d) Prove that for the periodic b.c. operator $\frac{d^2}{dx^2}$ on $[0, L]$ the zeta-regularized determinant

$$\det'\left(-\frac{\beta}{2\pi}\frac{d^2}{dx^2}\right) = 2\pi L^2/\beta.$$

(e) Show that the spectrum of the Laplacian Δ_τ on the torus $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ in the metric $|dz|^2$ is given by $\lambda_{m,n} = -(\frac{2\pi}{\tau_2})^2|\tau m + n|^2$ for $n, m \in \mathbf{Z}$.

(f) Proceeding as in (b) decompose

$$\sum_{(m,n)\neq(0,0)} |\tau m + n|^{-2s} = \sum_{m\neq 0,\, n} |\tau m + n|^{-2s} + 2\zeta(2s)$$
$$= \frac{\sqrt{\pi}}{\Gamma(s)}\Bigg(\sum_{m,n\neq 0} \mathrm{e}^{2\pi i m n \tau_1} \int_0^\infty t^{s-3/2}\mathrm{e}^{-m^2\tau_2^2 t - \pi^2 n^2/t} dt$$
$$+ \sum_{m\neq 0} m^{-2s+1}\tau_2^{-2s+1}\Gamma(s-1/2)\Bigg) + 2\zeta(2s)$$

and show that (after analytic continuation)

$$\zeta_{-\Delta'_\tau}(s) \equiv \sum_{(m,n)\neq(0,0)} (-\lambda_{m,n})^{-s} = -1 - 2s\ln|\prod_{m=1}^\infty (1-q^m)|^2 - 2s\ln\tau_2 + \tfrac{1}{3}\pi\tau_2 s.$$

Infer that

$$\zeta_{-\Delta'_\tau}(0) = -1 \qquad \zeta'_{-\Delta'_\tau}(0) = -2\ln|\prod_{m=1}^\infty (1-q^m)|^2 - 2\ln\tau_2 + \tfrac{1}{3}\pi\tau_2,$$

and that

$$\det'(-\Delta_\tau) = \tau_2^2|\eta(\tau)|^4$$

where the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^\infty (1-q^n)$.

Hence in the genus 1 case,

$$Z \equiv Z_\beta(\tau) = \vartheta_{Q_\beta}(\tau,\bar\tau)\,|\eta(\tau)|^{-2} = Z_{1/\beta}(\tau). \tag{1.11}$$

The marking independence (together with the independence of Z_β on the normalization of the flat metric on T_τ, see below) implies that $Z(\tau)$ is a *modular invariant* function

$$Z_\beta(\tau) = Z_\beta\left(\tfrac{a\tau+b}{c\tau+d}\right)$$

for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbf{Z})$.

§1.5. Toroidal compactifications: the partition functions

The above discussions may be easily generalized to the case of "toroidal compactifications" i.e. to the case of a massless free field on (Σ, γ) with values in the N-dimensional torus $T^N = (\mathbf{R}/2\pi\mathbf{Z})^N$. Fix a constant metric $g = \sum_{ij} g_{ij} d\phi^i d\phi^j$

and a constant 2-form $\omega = \sum_{ij} b_{ij} d\phi^i \wedge d\phi^j$ on T^N and define the classical action of the field $\phi : \Sigma \to T^N$ as

$$S(\phi) = \frac{1}{4\pi}(\|d\phi\|^2_{L^2} + i\int_\Sigma \phi^*\omega). \tag{1.12}$$

Applying the same method as before (do it!) results in the formula

$$Z \equiv Z_d = \vartheta_{Q_d}(\tau,\bar\tau)\left(\frac{\mathrm{vol}_\Sigma \det \tau_2}{\det'(-\Delta)}\right)^{N/2} = Z_{d^{-1}}$$

(T-duality again!) where $d = (d_{ij} = g_{ij} + b_{ij})$ and the lattice

$$Q_d = Q_{d^{-1}} = \{\ (\tfrac{dm+n}{\sqrt{2}}, \tfrac{d^t m - n}{\sqrt{2}}) \mid m, n \in \mathbf{Z}^N\ \} \subset \mathbf{R}^N \oplus \mathbf{R}^N$$

is an even self-dual lattice in $\mathbf{R}^N \oplus \mathbf{R}^N$ with the indefinite scalar product $|(x,y)|^2 = (x, g^{-1}x) - (y, g^{-1}y)$. At genus 1

$$Z \equiv Z_d(\tau) = \vartheta_{Q_d}(\tau,\bar\tau)|\eta(\tau)|^{-2N} = Z_{d^{-1}}(\tau) = Z_d(\tfrac{a\tau+b}{c\tau+d}).$$

Example 4. Let T be the Cartan torus of a simply-laced, simple, simply-connected Lie group (the compact form of the A, D, E groups). By spanning the Lie algebra of T by the coroots $\alpha_i^\vee$, we may identify T with T^N where N is the rank of the group. Let $g_{ij} = \frac{1}{2}\mathrm{tr}\,\alpha_i^\vee \alpha_j^\vee$ where tr is the Killing form normalized so that $g_{ii} = 1$. We may write

$$2g_{ij} = d_{ij} + d_{ji}$$

for some integers d_{ij} and set

$$2b_{ij} = d_{ij} - d_{ji}$$

so that $d_{ij} = g_{ij} + b_{ij}$. The corresponding action of the toroidal compactification coincides (mod $2\pi i$) with the action of the WZW model with fields taking values in the corresponding simple group (the $\sim \int \phi^*\omega$ term is the remnant of the topological WZ term). Defining for $p^\vee \in (P^\vee)^{h_\Sigma}$, where $P^\vee$ is the coweight lattice dual to the root lattice,

$$\vartheta_{Q^\vee, p^\vee}(\tau) = \sum_{q^\vee \in (Q^\vee)^{h_\Sigma}} \mathrm{e}^{\pi i\,\mathrm{tr}((p^\vee+q^\vee)^t, \tau(p^\vee+q^\vee))}$$

one obtains

$$\vartheta_{Q_b}(\tau,\bar\tau) = \sum_{[p^\vee]\in(P^\vee/Q^\vee)^{h_\Sigma}} |\vartheta_{Q^\vee,p^\vee}(\tau)|^2.$$

In particular at genus 1.

$$Z_d(\tau) = \sum_{[p^\vee]\in P^\vee/Q^\vee} \left|\frac{\vartheta_{Q^\vee,p^\vee}(\tau)}{\eta(\tau)^N}\right|^2 = \sum_{[p^\vee]\in P^\vee/Q^\vee} |\mathrm{ch}^1_{[p^\vee]}(\tau)|^2 \tag{1.13}$$

where $\mathrm{ch}^1_{[p^\vee]}(\tau) = \frac{\vartheta_{Q^\vee,p^\vee}(\tau)}{\eta(\tau)^N}$ runs through the characters of the level 1 representations of the corresponding Kac-Moody algebra. In particular, for the E_8 case $P^\vee = Q^\vee$ and the partition function is the absolute value squared of $\mathrm{ch}^1_0(\tau)$ which is a cubic root of the modular invariant function $j(\tau)$. In general, the right hand side of eq.(1.13) coincides with the genus 1 partition function of the level 1 WZW model. This remains true for higher genera and for the complete CFT's, which is another miraculous coincidence of field theories with fields taking values in quite

different target spaces (e.g. the $SU(2)$ WZW model at level 1 is equivalent to the free field with values in S^1 of radius 1).

The fact that the toroidal partition function (1.13) is a finite sesquilinear combination of expressions holomorphic in τ is a characteristic feature of rational conformal theories.

Problem 5. Show that the free field compactified on a circle of rational radius squared ($= \beta$) is rational in the above sense.

For free fields with values in the Cartan tori of simply laced groups described above, the general partition functions are Hermitian squares with respect to the Quillen-like metric of holomorphic sections of projectively flat vector bundles over the moduli spaces of curves. We shall return to these issues during a more detailed discussion of the WZW models.

Example 5. Consider the toroidal compactification to T^2 equipped with the complex structure induced by the complex variable $\psi = \phi^1 + T\phi^2$ (T is in the upper half plane) and with a constant Kähler metric $g = (R_2/T_2)d\psi d\bar{\psi}$ with $R_2 > 0$ and a constant 2-form $\omega = i(R_1/T_2)d\psi \wedge d\bar{\psi}$. Set $R = R_1 + iR_2$. The partition function of the corresponding free field is

$$Z \equiv Z_{R,T} = \vartheta_{Q_{R,T}}(\tau,\bar{\tau})\left(\frac{\mathrm{vol}_\Sigma \det \tau_2}{\det'(-\Delta)}\right)$$

where

$$Q_{R,T} = \{\, (\tfrac{Rm^1+TRm^2+Tn^1-n^2}{\sqrt{2R_2T_2}}, \tfrac{\bar{R}m^1+T\bar{R}m^2+Tn^1-n^2}{\sqrt{2R_2T_2}}) \mid m^i, n^i \in \mathbf{Z} \,\} \subset \mathbf{C}\oplus\mathbf{C}$$

with the indefinite quadratic form $|(z_1,z_2)|^2 = |z_1|^2 - |z_2|^2$. Note that $Q_{T,R}$ may be obtained from $Q_{R,T}$ by complex conjugation on the second $\mathbf{C}$. We infer that

$$Z_{R,T} = Z_{T,R} \tag{1.14}$$

which is the simplest instance of **mirror symmetry** claiming identity of CFT's with fields in two different Calabi-Yau manifolds with the role of modular parameters of complex and (polarized) Kähler structures interchanged.

Problem 6. Show that, besides the relation (1.14), the partition function satisfies the identities

$$Z_{R,T}(\tau) = Z_{R+1,T}(\tau) = Z_{R,T+1}(\tau) = Z_{R,-T^{-1}}(\tau) = Z_{-R^{-1},-T^{-1}}(\tau),$$

which imply the separate $SL_2(\mathbf{Z})$ invariance in R and T.
In general, the moduli space of N-dimensional toroidal compactifications is a double coset

$$O(N)\times O(N)\backslash O\left(\mathbf{R}^N\oplus\mathbf{R}^N, \left(\begin{smallmatrix} I & 0 \\ 0 & -I\end{smallmatrix}\right); \mathbf{R}^N\oplus\mathbf{R}^N, \left(\begin{smallmatrix} 0 & I \\ I & 0\end{smallmatrix}\right)\right) / O\left(\mathbf{R}^N\oplus\mathbf{R}^N, \left(\begin{smallmatrix} 0 & I \\ I & 0\end{smallmatrix}\right)|\mathbf{Z}\right)$$

(in a, hopefully, self-explanatory notation) which coincides with the moduli of even self-dual lattices in $\mathbf{R}\oplus\mathbf{R}$ with the indefinite scalar product.

The action functional (1.12) of the (compactified) two-dimensional massless free field uses only the conformal class of the metric γ on Σ. The regularization of the free field determinants reintroduces however the dependence on the conformal factor of the metric, an effect called the **conformal anomaly**. More exactly, one has

$$\frac{\delta}{\delta\sigma(x)}\bigg|_{\sigma=0} \ln\left(\frac{\det'(-\Delta)}{\mathrm{vol}_\Sigma}\right) = -\frac{1}{12\pi}r(x), \tag{1.15}$$

where r is the scalar curvature of Σ, or, denoting the metric dependence of the partition function by the subscript,

$$\frac{\delta}{\delta\sigma(x)}\Big|_{\sigma=0} Z_{e^{\sigma}\gamma} = \frac{N}{24\pi} r(x)\, Z_{\gamma}. \tag{1.16}$$

Problem 7. Prove the relation (1.15) using the identity $\zeta_{-\Delta}(s) = \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} \mathrm{tr} e^{t\Delta}$ and the short time expansion of the heat kernel of $-\Delta$:

$$e^{t\Delta}(x,x) = \frac{1}{4\pi t} + \frac{1}{12\pi} r(x) + \mathcal{O}(t)$$

Problem 8. Prove that the infinitesimal relation (1.16) is equivalent to the global one

$$Z_{e^{\sigma}\gamma} = e^{\frac{N}{96\pi}(\|d\sigma\|_{L^2}^2 + 4\int_{\Sigma}\sigma r dv)}\, Z_{\gamma} \tag{1.17}$$

§1.6. Toroidal compactifications: the correlation functions

Besides the functional integrals for the partition functions, we would like to study the ones for the correlation functions of the massless field with values in S^1 of the type

$$\int \prod_{i=1}^{n} e^{iq_i\phi(x_i)}\, e^{-\frac{\beta}{4\pi}\int_{\Sigma}|d\phi|^2 dv}\, D\phi,$$

for integer q_i, see Problem 3 for the $d = 0$ example. This may be attempted by the same strategy as before by separating the field into the harmonic and single-valued part, as in eq.(1.7), and then summing over the first and integrating over the second. This gives the expression

$$\sum_{\alpha \in H^1(\Sigma, 2\pi\mathbf{Z})} e^{-\frac{\beta}{4\pi}\|\alpha_h\|_{L^2}^2 + i\sum_i q_i \phi_h(x_i)} \int e^{-\frac{\beta}{4\pi}(\psi, -\Delta\psi)_{L^2} + i\sum_i q_i \psi(x_i)}\, D\psi.$$

The sum over H^1 may be expressed by partial Poisson resummation as an explicit theta-function. As for the functional integral, it may be formally performed by mimicking the finite-dimensional formulae:

$$\int e^{-\frac{\beta}{4\pi}(\psi, -\Delta\psi)_{L^2} + i\sum_i q_i\psi(x_i)}\, D\psi = \delta_{\sum_i q_i, 0}\, e^{\frac{\pi}{\beta}\sum_{i,j=1}^{n} q_i q_j G(x_i, x_j)} \left(\frac{2\pi \mathrm{vol}_{\Sigma}}{\det'(-\frac{\beta}{2\pi}\Delta)}\right)^{1/2}$$

where the Kronecker delta is contributed by the integral over the constant mode of ψ and $G(x,y) = G(y,x)$ is a Green function of Δ satisfying $\Delta_x G(x,y) = \delta(x,y) - \frac{1}{\mathrm{vol}_{\Sigma}}$ (the constant ambiguity should drop out above due to the vanishing of $\sum q_i$). The obvious problem with the above formula is that

$$G(x,y) = \frac{1}{2\pi} \ln \mathrm{dist}(x,y) + \text{finite}$$

when $y \to x$, so that $G(x,x)$ is not defined. This is a standard problem with the short-distance singularities due to the distributional character of typical configurations in the Gaussian free field measure. A possible treatment is to renormalize the above expression by replacing the divergent contributions by their finite parts

$$\tilde{G}(x,x) = \lim_{y \to x} G(x,y) - \frac{1}{2\pi} \ln \mathrm{dist}(x,y) \tag{1.18}$$

Upon division by the partition function, all this leads to a well defined renormalized expression for the correlation functions which we shall denote by

$$\langle : \mathrm{e}^{iq_1\phi(x_1)} : \cdots : \mathrm{e}^{iq_n\phi(x_n)} : \rangle_\gamma$$

where the colons recall the renormalization procedure (which is closely related to the Wick ordering discussed in [**I-Axiomatics**]). The correlations depend on the "charges" q_i and positions x_i but also on the metric γ on Σ which is signaled by the subscript. In particular, on $\mathbf{CP}^1$ with $H^1 = 0$, one may take $G(x,y) = \frac{1}{2\pi}\ln|z(x) - z(y)|$ in the standard complex variable and for the metric $g = \mathrm{e}^\sigma dzd\bar{z}$ with a conformal factor e^σ; we obtain, setting $z_i \equiv z(x_i)$,

$$\begin{aligned}\langle : \mathrm{e}^{iq_1\phi(x_1)} : \cdots : \mathrm{e}^{iq_n\phi(x_n)} : \rangle_\gamma \\ = \delta_{\sum_i q_i,0}\ \mathrm{e}^{-\sum_i \frac{q_i^2}{4\beta}\sigma(x_i)} \prod_{i<j} |z_i - z_j|^{\frac{q_iq_j}{\beta}}.\end{aligned} \tag{1.19}$$

The dependence on the conformal factor of the metric is solely due to the renormalization (1.18) and persists in general:

$$\begin{aligned}\langle : \mathrm{e}^{iq_1\phi(x_1)} : \cdots : \mathrm{e}^{iq_n\phi(x_n)} : \rangle_{\mathrm{e}^\sigma\gamma} \\ = \mathrm{e}^{-\sum_i \Delta_i\sigma(x_i)} \langle : \mathrm{e}^{iq_1\phi(x_1)} : \cdots : \mathrm{e}^{iq_n\phi(x_n)} : \rangle_\gamma\end{aligned} \tag{1.20}$$

where $\Delta_i = \frac{q_i^2}{4\beta}$ are the conformal dimensions of the (Euclidean) fields $: \mathrm{e}^{iq_i\phi} :$. The generalization to the toroidal compactifications is straightforward. The conformal dimensions of the fields $: \mathrm{e}^{iq\phi} :$ where $q \in \mathbf{Z}^N$ is now $\frac{1}{4}(q, g^{-1}q)$.

The operator picture of the the free field compactified on S^1 is as follows. The quantum Hibert space is

$$\mathcal{H} = L^2(S^1, d\varphi_0)^{\mathbf{Z}} \otimes \mathcal{F} \otimes \tilde{\mathcal{F}}.$$

Above, $L^2(S^1, d\varphi_0)^{\mathbf{Z}}$ is the infinite sum of copies of $L^2(S^1)$, each labeled by an integer ("winding number") w. The Fock space $\mathcal{F}$ is generated by applying operators α_n, $n = -1, -2, \ldots$, to a vector annihilated by α_n with $n = 1, 2, \ldots$,

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}, \qquad \alpha_n^* = \alpha_{-n},$$

and $\tilde{\mathcal{F}}$ is another copy of $\mathcal{F}$. Let $|p, w\rangle$ denote the function $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ipx}$ in the wth copy of $L^2(S^1, d\varphi_0)^{\mathbf{Z}}$. The integer p is the eigenvalue of the operator $p_0 = \frac{1}{i}\frac{d}{d\varphi_0}$. $\mathcal{H}$ may be generated by applying operators $\alpha_n, \tilde{\alpha}_n$ with negative n to vectors $|p, w\rangle$. The (multivalued) free field operator (with time dependence) is given by

$$\varphi(t,x) = \varphi_0 + \beta^{-1}p_0 t + wx + \frac{i}{\sqrt{2\beta}}\sum_{n\neq 0}\Big(\frac{\alpha_n}{n}\mathrm{e}^{-i(t+x)n} - \frac{\tilde{\alpha}_n}{n}\mathrm{e}^{-i(t-x)n}\Big).$$

The relation to the $m \to 0$ limit of the massive free field on a periodic interval of length $L = 2\pi$ given by eq.(1.6) should be evident. Modulo the constant and winding modes, for $n = 1, 2, \ldots$,

$$\alpha_n = -i\sqrt{n}a_{-n}, \quad \alpha_{-n} = i\sqrt{n}a_{-n}^*, \quad \tilde{\alpha}_n = i\sqrt{n}a_n, \quad \tilde{\alpha}_{-n} = -i\sqrt{n}a_n^*.$$

The relabeling allows us to separate the left-moving part (involving α_n) from the right-moving one (containing $\tilde{\alpha}$). The left- or right-moving parts of $\varphi(t,x)$ are called chiral fields (the zero modes can be separated too). For the uncompactified massless free field, the $\mathbf{k} = 0$ mode contributes to the Hamiltonian the term $\sim -d^2/d\varphi_0^2$ acting on $L^2(\mathbf{R}, d\varphi_0)$, which has continuous spectrum. The compactification of the

field (and consequently of its zero mode), restores the discreteness of the energy spectrum.

The Hilbert space $\mathcal{H}$ carries a representation of two commuting copies of the Virasoro algebra with generators L_n and $\tilde{L}_n$. Explicitly,

$$L_n = \tfrac{1}{2} \sum_{m \in \mathbf{Z}} : \alpha_m \alpha_{n-m} :$$

(creators to the left of annihilators) and similarly for $\tilde{L}_n$, where we have set $\alpha_0 = \frac{1}{\sqrt{2}}(\beta^{1/2} w + \beta^{-1/2} p_0)$ and $\tilde{\alpha}_0 = \frac{1}{\sqrt{2}}(\beta^{1/2} w - \beta^{-1/2} p_0)$. The quantum Hamiltonian $H = L_0 + \tilde{L}_0$. $P = L_0 - \tilde{L}_0$ generates the space translations. $\Omega = |0,0\rangle$ is the ground state of H (it is also annihilated by P).

Problem 9. Show that L_n's indeed satisfy the Virasoro algebra relations (with unit central charge)

$$[L_n, L_m] = (n-m) L_{n+m} + \tfrac{1}{12}(n^3 - n)\delta_{n,-m}.$$

Let us introduce the "vertex operators"

$$V_q(t,x) \; = : \mathrm{e}^{iq\varphi(t,x)} :$$

defined as the (formal) power series in α_n and $\tilde{\alpha}_n$ reordered by putting the creation operators with negative n indices to the left of the annihilation operators corresponding to positive n and also the φ_0 operators to the left of p_0.

Problem 10. Using the operator relation $\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^B \mathrm{e}^A \mathrm{e}^{[A,B]}$ holding if $[A,B]$ commutes with A and B, show that for $t_1 < t_2 < \cdots < t_n$,

$$(\Omega, V_{q_1}(it_1, x_1) \; \cdots \; V_{q_n}(it_n, x_n)\Omega) = \delta_{\sum_i q_i, 0} \prod_{i<j} |z_i - z_j|^{\frac{q_i q_j}{\beta}}$$

where $z_j = \mathrm{e}^{-t_j + i x_j}$. This, together with eq.(1.19), provides a spherical version of the Feynman-Kac formula.

In variables $z = \mathrm{e}^{i(t+x)}$ and $\tilde{z} = \mathrm{e}^{i(t-x)}$, the commutation relations of L_n's and $\tilde{L}_n$'s with the vertex operators take the form

$$\begin{aligned} [L_n, V_q(z,\tilde{z})] &= (n+1)\Delta z^n V_q(z,\tilde{z}) + z^{n+1} \partial_z V_q(z,\tilde{z}), \\ [\tilde{L}_n, V_q(z,\tilde{z})] &= (n+1)\Delta \tilde{z}^n V_q(z,\tilde{z}) + \tilde{z}^{n+1} \partial_{\tilde{z}} V_q(z,\tilde{z}) \end{aligned} \tag{1.21}$$

where above $\Delta = \frac{q^2}{4\beta}$ or $\Delta = \frac{1}{4}(q, g^{-1} q)$ stands for the conformal dimension of the operator. Later on, we shall see that these relations essentially follow from eq.(1.20) and the general covariance of the corresponding correlation functions. In the professional jargon, the fields satisfying such commutation relations are called primary Virasoro operators.

On the level of the Hilbert space $\mathcal{H} \equiv \mathcal{H}_\beta$, T-duality becomes the unitary transformation $U_T : \mathcal{H}_\beta \to \mathcal{H}_{1/\beta}$ such that

$$U_T |u,w\rangle = (-1)^{uw} |w,u\rangle \quad \text{and} \quad U_T \alpha_n = \alpha_n U_T, \quad U_T \tilde{\alpha}_n = -\tilde{\alpha}_n U_T,$$

where, for $u, w \in \mathbf{Z}$, $|u,w\rangle$ denotes the function $\frac{1}{\sqrt{2\pi}} \mathrm{e}^{iu\phi_0}$ in the w-component of $L^2(S^1, d\phi_0)^{\mathbf{Z}}$. U_T intertwines the action of the Virasoro algebras in $\mathcal{H}_\beta$ and $\mathcal{H}_{1/\beta}$ and maps the vertex operators in $\mathcal{H}_\beta$ to new operators which should be considered on an equal footing with the original ones.

Up to now, we have considered only S^1-valued free fields with periodic boundary conditions. In string theory applications, one also considers fields on space

with boundaries and with fixed boundary conditions like the Neumann ones (open strings). Quantizing such fields on space which is an interval $[0,\pi]$ ($d=1$) one obtains the quantum field

$$\varphi^N(t,x)=\varphi_0+\beta^{-1}pt+\frac{i}{\sqrt{2\beta}}\sum_{n\neq 0}\Big(\frac{\alpha_n^N}{n}\mathrm{e}^{-i(t+x)n}+\frac{\alpha_n^N}{n}\mathrm{e}^{-i(t-x)n}\Big)$$

which may be realized in the subspace of the periodic b.c. The Hilbert space $\mathcal{H}_\beta$ is generated by applying operators $\frac{1}{\sqrt{2}}(\alpha_n-\tilde\alpha_n)=\alpha_n^N$ with negative n to vectors $|u,0\rangle$. The T-duality maps this field into the one corresponding to the Dirichlet boundary conditions

$$\varphi^D(t,x)=\varphi_0+wx+\frac{i}{\sqrt{2\beta}}\sum_{n\neq 0}\Big(\frac{\alpha_n^D}{n}\mathrm{e}^{-i(t+x)n}-\frac{\alpha_n^D}{n}\mathrm{e}^{-i(t-x)n}\Big)$$

where φ_0 is fixed modulo $\pi\mathbf{Z}$. $\varphi^D(t,x)$ acts in the subspace of $\mathcal{H}_{1/\beta}$ generated by applying $\frac{1}{\sqrt{2}}(\alpha_n+\tilde\alpha_n)=\alpha_n^D$ to vectors $|0,w\rangle$. In toroidal compactifications with more dimensions of the target, one may have mixed "D-brane"-type boundary conditions with some coordinates of the field fixed to prescribed values at the ends of the space-interval $[0,\pi]$.

Problem 11 (Massless fermions on a Riemann surface).
Let (Σ,γ) be a Riemann surface. A spin structure on Σ may be identified with the square root L of the canonical bundle $K=T^{*1,0}(\Sigma)$. A Dirac spinor $\Psi=(\psi,\tilde\psi)$ is an element of $\Gamma(L\oplus\bar L)$ where $\bar L$ is the bundle complex conjugate to L. The conjugate spinor is $\bar\Psi=(\tilde\chi,\chi)\in\Gamma(\bar L\oplus L)$ and in the Euclidean Dirac theory it should be treated as an independent field ($\chi=\psi$, $\tilde\chi=\tilde\psi$ for Majorana fermions). Denote by $\bar\partial_L$ the $\bar\partial$ operator of L and by $\partial_{\bar L}$ its complex conjugate, which may be naturally identified with $\bar\partial_L^*$. The action is a function on the odd vector space $\Pi(\Gamma(L\oplus\bar L)\oplus\Gamma(\bar L\oplus L))$:

$$S(\Psi,\bar\Psi)=-\frac{1}{\pi}\int_\Sigma(\chi\bar\partial_L\psi+\tilde\chi\partial_{\bar L}\tilde\psi)$$

(note that the integrand is naturally a 2-form). Partition functions of the Dirac fermions are given by the formal Berezin integral

$$Z_L=\int\mathrm{e}^{-S(\Psi,\bar\Psi)}D\bar\Psi D\Psi=\det(\partial_{\bar L})\det(\bar\partial_L)=\det(\bar\partial_L^*\bar\partial_L).$$

The last determinant may be zeta-regularized giving a precise sense to the partition function Z_L of the Dirac field on Σ.

On the elliptic curve $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ with τ in the upper half-plane, the canonical bundle K may be trivialized by the section dz and spin structures correspond to the choice of periodic or anti-periodic boundary conditions under $z\to z+1$ and $z\to z+\tau$:

$$L=pp,\ pa,\ ap,\ aa.$$

(a). Show that the eigenvalues of $\bar\partial_L^*\bar\partial_L$ are

$$\lambda_{m,n}=(\tfrac{\pi}{\tau_2})^2|\tau m+n|^2$$

with

$$\begin{array}{lll} m\in\mathbf{Z}, & n\in\mathbf{Z} & \text{for}\quad L=pp,\\ m\in\mathbf{Z}, & n\in\mathbf{Z}+\tfrac12 & \text{for}\quad L=pa,\end{array}$$

$$m \in \mathbf{Z} + \tfrac{1}{2}, \qquad n \in \mathbf{Z} \qquad \text{for} \quad L = ap,$$
$$m \in \mathbf{Z} + \tfrac{1}{2}, \qquad n \in \mathbf{Z} + \tfrac{1}{2} \qquad \text{for} \quad L = aa.$$

(b). Infer that

$$Z_{pp}(\tau) = 0.$$

(c). Show that

$$\zeta_{\bar{\partial}^*_{pa}\bar{\partial}_{pa}}(s) = 2^{2s}(\zeta_{-\Delta'_{2\tau}}(s) - \zeta_{-\Delta'_{\tau}}(s)).$$

Infer from Eq.(1.11) that

$$Z_{pa}(\tau) = 4|q^{1/24}\prod_{n=1}^{\infty}(1+q^n)|^4.$$

In the Hilbert space picture

$$Z_{pa}(\tau) = \mathrm{tr}_{\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R} q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}. \tag{1.22}$$

The "Ramond sector" Hilbert space is $\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R$ with

$$\mathcal{H}_R = \mathbf{C}^2 \otimes \left(\wedge(\bigoplus_{n=1}^{\infty}\mathbf{C})\right)^{\otimes 2} \tag{1.23}$$

and $\tilde{\mathcal{H}}_R$ is another copy of $\mathcal{H}_R$. L_0 acts in the first copy. It has eigenvalue $\frac{1}{8}$ on $\mathbf{C}^2$ (the "Ramond ground states") and the occupied n^{th} mode in the fermionic Fock space adds n to it.
The periodic partition function is

(1.24)
$$Z_{pp}(\tau) = \mathrm{tr}_{\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R}(-1)^{F+\tilde{F}} q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24} \equiv \mathrm{str}_{\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R} q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}$$

where $(1,0),(0,1) \in \mathbf{C}^2$ correspond to the eigenvalues $+1,-1$ of $(-1)^F$ and each occupied fermionic Fock space mode adds 1 to F. $Z_{pp}(\tau)$ vanishes since modes with odd and even Fermi numbers are paired.
(d). Show that

$$\zeta_{\bar{\partial}^*_{ap}\bar{\partial}_{ap}}(s) = 2^{4s}\zeta_{-\Delta'_{\tau/2}}(s) - 2^{2s}\zeta_{-\Delta'_{\tau}}(s) \qquad \text{and}$$
$$Z_{ap}(\tau) = |q^{-1/48}\prod_{n=0}^{\infty}(1-q^{n+1/2})|^4.$$

The Hilbert space interpretation is

$$Z_{ap}(\tau) = \mathrm{str}_{\mathcal{H}_{NS}\otimes\tilde{\mathcal{H}}_{NS}} q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24} \tag{1.25}$$

where the "Neveu-Schwarz sector" Hilbert space is

$$\mathcal{H}_{NS} = \left(\wedge(\bigoplus_{n=0}^{\infty}\mathbf{C})\right)^{\otimes 2}. \tag{1.26}$$

The "Neveu-Schwarz ground state" has eigenvalue zero of L_0 and the nth occupied zero mode contributes $(n+\frac{1}{2})$ to it. The fermion number of the NS-ground state vanishes and each occupied fermionic mode adds 1 to it.

(e). Show that

$$\zeta_{\bar\partial^*_{aa}\bar\partial_{aa}}(s) = 2^{2s}\zeta_{\bar\partial^*_{pa}\bar\partial_{pa}}(s)\Big|_{\tau/2} - \zeta_{\bar\partial^*_{pa}\bar\partial_{pa}}(s)\Big|_{\tau} \qquad \text{and}$$

$$Z_{aa}(\tau) = |q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2})|^4$$

and that

$$Z_{aa}(\tau) = \mathrm{tr}_{\mathcal{H}_{NS}\otimes\tilde{\mathcal{H}}_{NS}} q^{L_0-1/24}\bar q^{\tilde L_0-1/24}. \tag{1.27}$$

(f). Prove the modular properties:

$$\begin{aligned} &Z_{pa}(\tau+1) = Z_{pa}(\tau), \quad && Z_{ap}(\tau+1) = Z_{aa}(\tau), \quad && Z_{aa}(\tau+1) = Z_{ap}(\tau),\\ &Z_{pa}(-1/\tau) = Z_{ap}(\tau), \quad && Z_{ap}(-1/\tau) = Z_{pa}(\tau), \quad && Z_{aa}(-1/\tau) = Z_{aa}(\tau). \end{aligned}$$

Problem 12 Bosonization.
The spin structure is called even (odd) if the dimension of the kernel of $\bar\partial_L$ is even (odd). Denote by $\sigma(L)$ the parity of L. The bosonization formula asserts that

$$\tfrac{1}{2}\sum_L (-1)^{\sigma(L)} Z_L = C^{h_\Sigma - 1} Z_{1/2} \tag{1.28}$$

where on the right hand side we have the partition function of the bosonic free field with values in the circle of radius squared $\frac{1}{2}$, C is a constant and h_Σ the genus of the Riemann surface Σ. These equalities extend to correlations. For example, the fermionic fields $(\psi\tilde\psi)(x)$ correspond to bosonic fields $:\mathrm{e}^{i\phi(x)}:$ and $(\tilde\chi\chi)(x)$ to $:\mathrm{e}^{-i\phi(x)}:$. What are their conformal weights? Prove identity (1.28) for $\Sigma = \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ using the expression (1.11) for $Z_{1/2}(\tau)$ and the classical product expressions for the theta functions

$$\vartheta(z|\tau) \equiv \sum_{n\in\mathbf{Z}} \mathrm{e}^{\pi i\tau n^2 + 2\pi i n z} = \prod_{n=1}^{\infty}(1-q^n)(1+\mathrm{e}^{2\pi i z}q^{n-1/2})(1+\mathrm{e}^{-2\pi i z}q^{n-1/2}).$$

What is the Hilbert space interpretation of the left hand side of Eq.(1.28) on the elliptic curve?

In summary, by "calculating" the functional integrals for compactified massless free fields we have constructed models of two-dimensional CFT specified by giving the partition functions and correlation functions on general Riemann surfaces. In the next lecture(s), we shall examine the emerging CFT structure on a more abstract level.

References and cross references

A set of Gaussian integration formulae may be found e.g. in Sects. 1.1 and 1.2 of

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, 1989.

See also Sects. 2.0-2.2 and 2.5 for the discussion of of the path integral and the Feynman-Kac formula. Of course, for the latter topic,

R. P. Feynman and A. R. Hibbs *Quantum Mechanics and Path Integrals*, McGraw-Hill, 1965.

is the physics classic. Rigorous theory of infinite-dimensional Gaussian integrals may be found e.g. in

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B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press, 1974.

The free fields with values in S^1 are discussed briefly, e.g., in either of the following

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C. Itzykson and J.-M. Drouffe, *Statistical Field Theory*, vol. 1, Cambridge University Press, 1989 (Sects. 3.2 and 3.6).

For the case of fields with values in a torus, resp. complex torus, see

K. S. Narain, M. H. Sarmadi, and E. Witten, *A note on toroidal compactification of heterotic string theory*, Nucl. Phys. B **279** (1987), 369–379.

C. Vafa, *Topological mirrors and quantum rings*, Essays on Mirror Symmetry (S.-T. Yau, ed.), International Press, 1992, pp. 96–119.

Free fermions and bosonization on a Riemann surface are discussed in

L. Alvarez-Gaumé, J.-B. Bost, G. Moore, P. Nelson, C. Vafa, *Bosonization on higher genus Riemann surfaces*, Comm. Math. Phys. **112** (1987), 503–552.

In these volumes, see Lecture 1 of [I-**Faddeev**] for another discussion of path integrals. Free fields are treated in Lecture 3 of [I-**Axiomatics**] and circle-valued free fields in [II-**Dynamics of QFT**, §6.5].

LECTURE 2
Axiomatic Approaches to Conformal Field Theory

§2.1. Conformal field theory data

In the first lecture, we have discussed a functional-integral construction of the simplest models of CFT: the toroidal compactifications (of massless free fields). In this lecture we shall present a more general approach to CFT which, although not overly formalized, will be axiomatic in spirit using only the most general properties of the free field models. We shall assume that the basic data of a CFT model specify for each compact Riemann surface (Σ, γ) its partition function $Z_\gamma > 0$ and a set of its correlation functions $\langle \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n) \rangle_\gamma$ of the "primary fields" from a fixed set $\{\phi_l\}$. The correlation functions are symmetric in the pairs of arguments (x_i, l_i), are defined for non-coincident insertion points $x_i \in \Sigma$ and are assumed smooth. We shall also need later some knowledge of their short distance singularities. The dependence of both the partition and the correlation functions on the Riemannian metric γ will be assumed regular enough to assure existence of distributional functional derivatives of arbitrary order. The basic hypothesis are the following symmetry properties:

(i) diffeomorphism covariance

$$Z_\gamma = Z_{D^*\gamma}\,, \tag{2.1}$$

$$\langle \phi_{l_1}(D(x_1)) \cdots \phi_{l_n}(D(x_n)) \rangle_\gamma = \langle \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n) \rangle_{D^*\gamma}\,, \tag{2.2}$$

(ii) local scale covariance

$$Z_{\mathrm{e}^\sigma\gamma} = \mathrm{e}^{\frac{c}{96\pi}(\|d\sigma\|^2_{L^2} + 4\int_\Sigma \sigma r\, dv)} Z_\gamma\,, \tag{2.3}$$

$$\langle \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n) \rangle_{\mathrm{e}^\sigma\gamma} = \prod_{i=1}^{n} \mathrm{e}^{-\Delta_{l_i}\sigma(x_i)} \langle \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n) \rangle_\gamma \tag{2.4}$$

where c is the central charge of the theory. In (**i**), we limit ourselves to orientation preserving diffeomorphism, assuming that under the change of orientation of the surface,

$$Z_\gamma \mapsto Z_\gamma\,, \tag{2.5}$$

$$\langle \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n) \rangle_\gamma \mapsto \overline{\langle \phi_{\bar{l}_1}(x_1) \cdots \phi_{\bar{l}_n}(x_n) \rangle_\gamma}\,, \tag{2.6}$$

where $\phi_l \mapsto \phi_{\bar{l}}$ is an involution of the set of primary fields preserving their conformal weights ($:\mathrm{e}^{iq\phi}: \mapsto :\mathrm{e}^{-iq\phi}:$ for the toroidal compactifications). In what follows we shall first explore the implications of the above identities which we shall, jointly, call **conformal symmetries**. Other important properties of the correlation functions, for example those responsible for the Hilbert space interpretation of the theory, will be introduced and analyzed later.

We define new correlation functions with insertions of the **energy-momentum tensor**[1] by setting

$$\langle T_{\mu_1\nu_1}(y_1)\cdots T_{\mu_m\nu_m}(y_m)\phi_{l_1}(x_1)\cdots\phi_{l_n}(x_n)\rangle_\gamma$$

$$= Z_\gamma^{-1}\frac{(4\pi)^m\delta^m}{\delta\gamma^{\mu_1\nu_1}(y_1)\cdots\delta\gamma^{\mu_m\nu_m}(y_m)}Z_\gamma\langle\phi_{l_1}(x_1)\cdots\phi_{l_n}(x_n)\rangle_\gamma\ , \tag{2.7}$$

where $\gamma^{\mu\nu}\partial_\mu\partial_\nu \equiv \gamma^{-1}$ is the inverse metric. In complex coordinates, the energy-momentum tensor has the components

$$T_{zz} = \overline{T_{\bar{z}\bar{z}}} \qquad \text{and} \qquad T_{z\bar{z}} = T_{\bar{z}z} = \overline{T_{z\bar{z}}}\ .$$

By definition, the correlation functions

$$\langle T_{\mu_1\nu_1}(y_1)\cdots T_{\mu_m\nu_m}(y_m)\phi_{l_1}(x_1)\cdots\phi_{l_n}(x_n)\rangle_\gamma$$

are distributions in their dependence on $y_1,\dots,y_m$. As we shall see below, they are given by smooth functions for non-coincident arguments and away from x_i's, but we shall also have to study their distributional behavior at coinciding points.

§2.2. Conformal Ward identities

Symmetries in QFT are expressed as **Ward identities** between correlation functions. Eqs.(2.2) and (2.4) are examples of such relations for group-like conformal symmetries. It is often useful to also work out Ward identities corresponding to infinitesimal, Lie algebra version of symmetries. We shall do this here for the infinitesimal conformal symmetries. The resulting formalism was the starting point of the 1984 Belavin-Polyakov-Zamolodchikov's paper. The approach presented here is close in spirit to the 1987 article by Eguchi-Ooguri (to some extent also to Friedan's 1982 Les Houches lecture notes). The general strategy is to expand the global symmetry identities to the second order in infinitesimal symmetries. This will be a little bit technical so you might wish to first see the results listed at the end of this section.

Let us start by exploring the infinitesimal version of the local scale covariance (2.3). Using the definition (2.7), we obtain the relation

$$4\pi Z_\gamma^{-1}\frac{\delta}{\delta\sigma}\Big|_{\sigma=0} Z_{\mathrm{e}^\sigma\gamma} = -\gamma^{zz}\langle T_{zz}\rangle_\gamma - 2\gamma^{z\bar{z}}\langle T_{z\bar{z}}\rangle_\gamma - \gamma^{\bar{z}\bar{z}}\langle T_{\bar{z}\bar{z}}\rangle_\gamma = \tfrac{c}{6}r\ . \tag{2.8}$$

Note that if $\gamma = |dz|^2$ then $\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = \gamma^{zz} = \gamma^{\bar{z}\bar{z}} = 0$, $\gamma_{z\bar{z}} = \frac{1}{2}$ and $\gamma^{z\bar{z}} = 2$. Besides, the scalar curvature of γ vanishes. In such a metric, eq.(2.8) reduces to the equality

$$\langle T_{z\bar{z}}\rangle = 0 \tag{2.9}$$

which states that the energy-momentum tensor in a CFT is traceless (in the flat metric, $\mathrm{tr}\,T_{\mu\nu} = 4T_{z\bar{z}}$). This is the first example of Ward identities expressing

[1] called also stress tensor in a static view of the Euclidean field theory

the infinitesimal conformal invariance on the quantum level. We shall see further identities of this type below.

Notice that if $\gamma \mapsto e^{\sigma}\gamma$ with $\sigma = 1$ around the insertion points then the correlation functions do not change. Let us fix the complex structure of Σ and holomorphic complex coordinates around the insertion points of a correlation function. Call a metric γ locally flat if it is compatible with the complex structure of Σ and of the form $|dz|^2$ around the insertions. For such a choice of γ we shall drop the subscript "γ" in the notation for the correlation functions, as in eq.(2.9). We may restore the full dependence on the conformal factor by using the covariance relations (2.3) and (2.4). For example, for $\langle T_{zz}\rangle_{\gamma}$, we obtain

$$\langle T_{zz}\rangle_{e^{\sigma}\gamma} = \langle T_{zz}\rangle_{\gamma} + \frac{c}{24}\frac{\delta}{\delta\gamma^{zz}}(\|\partial\sigma\|^2_{L^2} + 4\int_{\Sigma}\sigma r dv) \ . \tag{2.10}$$

In order to compute the functional derivative on the right hand side, we shall need the following

Lemma. *Let $\gamma^{z\bar{z}} = \gamma^{\bar{z}z} = 2$. To the first order in γ^{zz},*

$$r = -\tfrac{1}{2}(\partial_z^2\gamma^{zz} + \partial_{\bar{z}}^2\gamma^{\bar{z}\bar{z}}) \ . \tag{2.11}$$

Proof. Consider the inverse metric $\gamma^{-1} = \gamma^{zz}\partial_z^2 + 4\partial_z\partial_{\bar{z}} + \gamma^{\bar{z}\bar{z}}\partial_{\bar{z}}^2$. To compute the curvature to the first order in γ^{zz} we shall change the variables to $z' = z + \zeta(z,\bar{z})$ so that in the new coordinate the metric is $(4+\rho)\partial_{z'}\partial_{\bar{z}'}$. Since

$$\partial_z = (1+\partial_z\zeta)\partial_{z'} + (\partial_z\bar{\zeta})\partial_{\bar{z}'} \ , \qquad \partial_{\bar{z}} = (\partial_{\bar{z}}\zeta)\partial_{z'} + (1+\partial_{\bar{z}}\bar{\zeta})\partial_{\bar{z}'}$$

then, retaining only the terms of the first order in γ^{zz}, ζ (and their complex conjugates), we obtain

$$\begin{aligned}
\gamma^{-1} &= (\gamma^{zz} - (\partial_z\gamma^{zz})\zeta - (\partial_{\bar{z}}\gamma^{zz})\bar{\zeta} + 2\gamma^{zz}\partial_z\zeta + 4\partial_{\bar{z}}\zeta)\partial_{z'}^2 \\
&+ (4 + 4\partial_z\zeta + 4\partial_{\bar{z}}\bar{\zeta} + 2\gamma^{zz}\partial_z\bar{\zeta} + 2\gamma^{\bar{z}\bar{z}}\partial_{\bar{z}}\zeta)\partial_{z'}\partial_{\bar{z}'} \\
&+ (\gamma^{\bar{z}\bar{z}} - (\partial_{\bar{z}}\gamma^{\bar{z}\bar{z}})\bar{\zeta} - (\partial_z\gamma^{\bar{z}\bar{z}})\zeta + 2\gamma^{\bar{z}\bar{z}}\partial_{\bar{z}}\bar{\zeta} + 4\partial_z\bar{\zeta})\partial_{\bar{z}'}^2
\end{aligned} \tag{2.12}$$

(we have kept more terms then needed for the Lemma for future use). The requirement that $\gamma^{-1} = (4+\rho)\partial_{z'}\partial_{\bar{z}'}$ means to leading order that $\partial_{\bar{z}}\zeta = -\frac{1}{4}\gamma^{zz}$. Hence to the first order in γ^{zz}

$$\begin{aligned}
r'v' = -i\bar{\partial}'\partial'\log(1+\partial_z\zeta+\partial_{\bar{z}}\bar{\zeta}) &= i\partial_{\bar{z}}\partial_z(\partial_z\zeta+\partial_{\bar{z}}\bar{\zeta})dz\wedge d\bar{z} \\
&= -\tfrac{1}{2}(\partial_z^2\gamma^{zz} + \partial_{\bar{z}}^2\gamma^{\bar{z}\bar{z}})v'
\end{aligned}$$

where v' is the new volume form equal to $\frac{i}{2}dz\wedge d\bar{z}$ 0th order.
Using the Lemma and the relation $\|d\sigma\|^2_{L^2} = \int_{\Sigma}(\partial_\mu\sigma)(\partial_\nu\sigma)\gamma^{\mu\nu}dv$, we obtain the relation

$$\frac{\delta}{\delta\gamma^{zz}}\left(\|d\sigma\|^2_{L^2} + 4\int_{\Sigma}\sigma r dv\right) = -2\partial_z^2\sigma + (\partial_z\sigma)^2 \tag{2.13}$$

which, substituted into eq.(2.10), gives the dependence on the conformal factors of the expectation value of T_{zz}:

$$\langle T_{zz}\rangle_{e^{\sigma}dzd\bar{z}} = \langle T_{zz}\rangle - \frac{c}{12}\left(\partial_z^2\sigma - \tfrac{1}{2}(\partial_z\sigma)^2\right) \ . \tag{2.14}$$

What are the transformation properties of $\langle T_{zz}\rangle$ under holomorphic changes $z \mapsto z' = f(z)$ of the local coordinate? Under such replacements the notion of a locally

flat metric changes accordingly. By the diffeomorphism covariance and eq.(2.14), we have

$$\begin{aligned}\left(\frac{dz'}{dz}\right)^2 \langle T_{z'z'}\rangle &= \langle T_{zz}\rangle_{|dz'/dz|^2 dzd\bar z}\\ &= \langle T_{zz}\rangle - \frac{c}{12}\left(\partial_z^2 \log(dz'/dz) - \frac{1}{2}(\partial_z \log(dz'/dz))^2\right)\\ &= \langle T_{zz}\rangle - \frac{c}{12}\left(\frac{d^3z'/dz^3}{dz'/dz} - \frac{3}{2}\left(\frac{d^2z'/dz^2}{dz'/dz}\right)^2\right) \equiv \langle T_{zz}\rangle - \frac{c}{12}\{z';z\}\ .\end{aligned} \tag{2.15}$$

The function $\{z';z\}$ is the **Schwarzian derivative** of the change of variables. As we see, in the correlation functions with locally flat metric, T_{zz} does not transform as a pure quadratic differential under general holomorphic changes of variables. The transformation law (2.15) defines what is called a **projective connection** on Σ.

Problem 1. Show that the Schwarzian derivative $\{z';z\}$ vanishes if and only if $z' = \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbf{C})$, i.e. for the Möbius transformations.

Further information about the correlation functions with energy-momentum tensor insertions will be obtained by studying deviations of the metric from the locally flat one. Applying to eq.(2.8) the operator $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\gamma^{ww}}Z_\gamma$ at γ locally flat[2] and using eq.(2.11), we obtain

$$\pi\delta^{(2)}(z-w)\langle T_{zz}\rangle + \langle T_{ww}T_{z\bar z}\rangle = \frac{\pi c}{12}\partial_z^2\delta^{(2)}(z-w)\ , \tag{2.16}$$

where $\delta^{(2)}$ stands for the two-dimensional δ-function. Let us explore now the implications of the diffeomorphism covariance (2.1) and (2.2). Under an infinitesimal transformation

$$D(z) = z + \zeta(z,\bar z) \equiv z'\ , \tag{2.17}$$

the change in the inverse metric $\delta\gamma^{-1} = \gamma'^{-1} - \gamma^{-1}$, where $D^*\gamma' = \gamma$, may be read off from eq.(2.12):

$$\delta\gamma^{zz} = -(\partial_z\gamma^{zz})\zeta - (\partial_{\bar z}\gamma^{zz})\bar\zeta + 2\gamma^{zz}\partial_z\zeta + 4\partial_{\bar z}\zeta\ ,$$

$$\delta\gamma^{z\bar z} = 2\partial_z\zeta + 2\partial_{\bar z}\bar\zeta + \gamma^{zz}\partial_z\bar\zeta + \gamma^{\bar z\bar z}\partial_{\bar z}\zeta\ ,$$

$$\delta\gamma^{\bar z\bar z} = -(\partial_{\bar z}\gamma^{\bar z\bar z})\bar\zeta - (\partial_z\gamma^{\bar z\bar z})\zeta + 2\gamma^{\bar z\bar z}\partial_{\bar z}\bar\zeta + 4\partial_z\bar\zeta\ .$$

The diffeomorphism covariance implies that

$$\int_\Sigma \left(\langle T_{zz}\rangle_\gamma\delta\gamma^{zz} + 2\langle T_{z\bar z}\rangle_\gamma\delta\gamma^{z\bar z} + \langle T_{\bar z\bar z}\rangle_\gamma\delta\gamma^{\bar z\bar z}\right)dv = 0\ .$$

Inserting the expressions for $\delta\gamma^{-1}$, stripping from the resulting equation the arbitrary function ζ and retaining only the first order terms in γ^{zz} around a locally flat metric, we obtain

$$\begin{aligned}&(\partial_z\gamma^{zz})\langle T_{zz}\rangle_\gamma + 2\partial_z(\gamma^{zz}\langle T_{zz}\rangle_\gamma) + 4\partial_{\bar z}\langle T_{zz}\rangle_\gamma\\ &+4\partial_z\langle T_{z\bar z}\rangle_\gamma + 2\partial_{\bar z}(\gamma^{\bar z\bar z}\langle T_{z\bar z}\rangle) + (\partial_z\gamma^{\bar z\bar z})\langle T_{\bar z\bar z}\rangle = 0\ .\end{aligned} \tag{2.18}$$

[2] w and z refer to the complex coordinates of two nearby insertions taken in the same holomorphic chart

Specializing to $\gamma^{zz} = 0$, we infer that

$$\partial_{\bar{z}}\langle T_{zz}\rangle = 0 = \partial_z\langle T_{\bar{z}\bar{z}}\rangle \ . \tag{2.19}$$

More generally, the component T_{zz} ($T_{\bar{z}\bar{z}}$) of the energy-momentum tensor is analytic (anti-analytic) in correlation functions in a locally flat metric and away from other insertions. Eq.(2.19) is another conformal Ward identity.

At coinciding points, the correlation functions of the energy-momentum tensor give rise to singularities, which we shall now study. Application of $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\gamma^{ww}}Z_\gamma$ at γ locally flat to eq.(2.18) gives:

$$\pi\left(\partial_z\delta^{(2)}(z-w)\right)\langle T_{zz}\rangle + 2\pi\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle + \partial_{\bar{z}}\langle T_{zz}T_{ww}\rangle + \partial_z\langle T_{z\bar{z}}T_{ww}\rangle = 0 \ .$$

Using eq.(2.16) differentiated with respect to z in order to replace $\partial_z\langle T_{z\bar{z}}T_{ww}\rangle$ in the last relation, we obtain

$$\begin{aligned}\partial_{\bar{z}}\langle T_{zz}T_{ww}\rangle &= -\pi\left(\partial_z\delta^{(2)}(z-w)\right)\langle T_{zz}\rangle - \pi\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle - \tfrac{\pi c}{12}\partial_z^3\delta^{(2)}(z-w) \\ &= -\tfrac{\pi c}{12}\partial_z^3\delta^{(2)}(z-w) - 2\pi\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle + \pi\delta^{(2)}(z-w)\partial_w\langle T_{ww}\rangle \ .\end{aligned}$$

This is a distributional equation. Since $\delta^{(2)}(z-w) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w}$ in the sense of distributions, it follows that

$$\partial_{\bar{z}}\langle T_{zz}T_{ww}\rangle = \partial_{\bar{z}}\left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\langle T_{ww}\rangle + \frac{1}{z-w}\partial_w\langle T_{ww}\rangle\right) \ , \tag{2.20}$$

which is still another conformal Ward identity. Since the only solutions of the distributional equation $\partial_{\bar{z}}f = 0$ are analytic functions, one may rewrite the identity (2.20) as a short distance expansion encoding the ultraviolet properties of the CFT:

$$\langle T_{zz}T_{ww}\rangle = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\langle T_{ww}\rangle + \frac{1}{z-w}\partial_w\langle T_{ww}\rangle + \dots \ , \tag{2.21}$$

where "..." stands for terms analytic in z around $z = w$. The complex conjugation of eq.(2.21) gives the singular terms of $\langle T_{\bar{z}\bar{z}}T_{\bar{w}\bar{w}}\rangle$, this time up to anti-analytic terms. Expansions of the type (2.21) are usually called the **operator product expansion** (OPE) in accordance with the operator interpretation of correlation functions to be discussed in the next section. We shall follow this terminology.

What about the mixed insertions? Differentiating ($Z_\gamma\times$) eq.(2.18) with respect to $\gamma^{\bar{w}\bar{w}}$ at γ locally flat, we obtain

$$\partial_{\bar{z}}\langle T_{zz}T_{\bar{w}\bar{w}}\rangle + \partial_z\langle T_{z\bar{z}}T_{\bar{w}\bar{w}}\rangle + \pi(\partial_z\delta^{(2)}(z-w))\langle T_{\bar{z}\bar{z}}\rangle = 0 \ .$$

With the use of the complex conjugate version of eq.(2.16) to eliminate $\langle T_{z\bar{z}}T_{\bar{w}\bar{w}}\rangle$, this reduces to

$$\partial_{\bar{z}}\langle T_{zz}T_{\bar{w}\bar{w}}\rangle = -\tfrac{\pi c}{12}\partial_z\partial_{\bar{z}}^2\delta^{(2)}(z-w)$$

which, stripped of $\partial_{\bar{z}}$, gives

$$\langle T_{zz}T_{\bar{w}\bar{w}}\rangle = -\tfrac{\pi c}{12}\partial_z\partial_{\bar{z}}\delta^{(2)}(z-w) + \dots \ , \tag{2.22}$$

i.e. a contact term with support at $z = w$ plus a function analytic in z and anti-analytic in w.

The other source of singular contributions to the correlation functions of T_{zz} or $T_{\bar{z}\bar{z}}$ are insertions of the (primary) fields $\phi_l(x)$. Let us compute these singularities.

Proceeding similarly as before, we apply $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\sigma}$ to $Z_{e^\sigma\gamma}\langle\phi_l(x)\rangle_{e^\sigma\gamma}$ at $\sigma = 0$ and γ locally flat obtaining with the help of eqs.(2.3) and (2.4) the relation

$$\langle T_{z\bar z}\phi_l(w,\bar w)\rangle = \pi\Delta_l\delta^{(2)}(z-w)\langle\phi_l(w,\bar w)\rangle \tag{2.23}$$

(we have replaced the point x in the argument of ϕ_l by its local coordinate w and its complex conjugate to stress the non-holomorphic dependence on x of the $\phi_l(x)$ insertion). Next we exploit the diffeomorphism covariance. For $D(z) = z+\zeta(z,\bar z) \equiv z'$ and $\gamma = D^*\gamma'$,

$$\langle\phi_l(w',\bar w')\rangle_{\gamma'}Z_{\gamma'} = \langle\phi_l(w,\bar w)\rangle_\gamma Z_\gamma\ .$$

Since for $\gamma = |dz|^2$

$$\delta\gamma^{-1} \equiv (\gamma')^{-1} - \gamma^{-1} = 4(\partial_{\bar z}\zeta)\partial_z^2 + 4(\partial_z\zeta + \partial_{\bar z}\bar\zeta)\partial_z\partial_{\bar z} + 4(\partial_z\bar\zeta)\partial_{\bar z}^2,$$

to the first order in ζ, see eq.(2.12), we infer that

$$\pi\delta^{(2)}(z-w)\partial_w\langle\phi_l(w,\bar w)\rangle - \partial_{\bar z}\langle T_{zz}\phi_l(w,\bar w)\rangle - \partial_z\langle T_{z\bar z}\phi_l(w,\bar w)\rangle = 0\ .$$

Using the last equation to eliminate $\partial_z\langle T_{z\bar z}\phi_l(w,\bar w)\rangle$ from eq.(2.23) acted upon by ∂_z, we obtain the relation

$$\partial_{\bar z}\langle T_{zz}\phi_l(w,\bar w)\rangle = -\pi\Delta_l\delta^{(2)}(z-w)\langle\phi_l(w,\bar w)\rangle + \pi\delta^{(2)}(z-w)\partial_w\langle\phi_l(w,\bar w)\rangle$$

which may be conveniently rewritten as an OPE of the product of the T_{zz} component of the energy-momentum tensor with a primary field:

$$\langle T_{zz}\phi_l(w,\bar w)\rangle = \left(\frac{\Delta_l}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)\langle\phi_l(w,\bar w)\rangle + \dots\ . \tag{2.24}$$

Finally note that under the holomorphic change of the local coordinate $z \mapsto z' = f(z)$

$$\langle\phi_l(z',\bar z')\rangle = \langle\phi_l(z,\bar z)\rangle_{|dz'/dz|^2dzd\bar z} = |\tfrac{dz'}{dz}|^{-2\Delta_l}\langle\phi_l(z,\bar z)\rangle$$

or

$$\langle\phi_l(z',\bar z')\rangle(dz')^{\Delta_l}(d\bar z')^{\Delta_l} = \langle\phi_l(z,\bar z)\rangle(dz)^{\Delta_l}(d\bar z)^{\Delta_l}$$

so that ϕ_l behaves like a (Δ_l,Δ_l)-form in the correlation functions with locally flat metric. One often needs to consider also primary fields with weights $(\Delta_l,\tilde\Delta_l)$ and $\Delta_l - \tilde\Delta_l$ integer (or half-integer). $d_l = \Delta_l + \tilde\Delta_l$ is the scaling dimension of such a field and $s_l = \Delta_l - \tilde\Delta_l$ its spin. Geometrically, the correlation functions of such fields are sections of the sth power of the sphere subbundle in the cotangent bundle $T^*\Sigma$.

Let us collect the relations obtained in this section for low point insertions in the correlation functions. Since all the considerations were local, they are not changed in the presence of other fields, i.e. the same equalities hold in correlation functions with other insertions as long as their points stay away from the insertions taken together. Adding also the relations involving the complex conjugate components of the energy-momentum tensor and introducing the simplified notation $T \equiv T_{zz}$, $\bar T = T_{\bar z\bar z}$, we obtain:

i. **identities**

$$T_{z\bar z} = 0 = T_{\bar z z}\ , \tag{2.25}$$

$$\partial_{\bar z}T = 0 = \partial_z\bar T\ , \tag{2.26}$$

ii. **operator product expansions**

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \ldots,$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{c/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\bar{T}(\bar{w}) + \ldots,$$

$$T(z)\bar{T}(\bar{w}) = -\frac{\pi c}{12}\partial_z\partial_{\bar{z}}\delta^{(2)}(z-w) + \ldots, \tag{2.27}$$

$$T(z)\phi_l(w,\bar{w}) = \left(\frac{\Delta_l}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)\phi_l(w,\bar{w}) + \ldots, \tag{2.28}$$

$$\bar{T}(\bar{z})\phi_l(w,\bar{w}) = \left(\frac{\Delta_l}{(\bar{z}-\bar{w})^2} + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\right)\phi_l(w,\bar{w}) + \ldots, \tag{2.29}$$

iii. **transformation laws**

$$T(z')(dz')^2 = T(z)(dz)^2 - \frac{c}{12}\{z';z\}(dz)^2\ , \tag{2.30}$$

$$\bar{T}(\bar{z}')(d\bar{z}')^2 = \bar{T}(\bar{z})(d\bar{z})^2 - \frac{c}{12}\overline{\{z';z\}}(d\bar{z})^2\ , \tag{2.31}$$

$$\phi_l(z',\bar{z}')(dz')^{\Delta_l}(d\bar{z}')^{\tilde{\Delta}_l} = \phi_l(z,\bar{z})(dz)^{\Delta_l}(d\bar{z})^{\tilde{\Delta}_l}. \tag{2.32}$$

§2.3. Physical positivity and Hilbert space picture

Up to now we have analyzed abstract conformal fields in the Euclidean formalism, probabilistic in its nature and distinct from the traditional operator approach. The operator formalism of QFT fits into the general quantum mechanical scheme with the

1. Hilbert space of states,
2. representation of the symmetry group or algebra,
3. distinguished family of operators

as its basic triad. This is a fundamental fact of QFT, namely that the passage between the Euclidean and the operator formalisms, which we have discussed already for free fields, may be done in quite general circumstances. This fact is responsible for the deep relation between critical phenomena and quantum fields and it has strongly influenced the developments of QFT. CFT, which is not an exception in this respect, has largely profited from the unity of the two approaches. In the present section we shall discuss how the operator picture may be recovered from the Euclidean formulation of CFT presented above assuming the **physical** (or Osterwalder-Schrader) **positivity** formulated as a condition on correlation functions on the Riemann sphere $\mathbf{C}P^1$. Analysis of the genus zero situation will allow us to recover the Hilbert space of states and to translate the operator product expansions of the last section into an action of the Lie algebra of conformal symmetries and of the primary field operators in the space of states. Later we shall describe the operator formalism on higher genus Riemann surfaces which permits us to relate naturally CFT in different space-time topologies.

Let us consider the map $\vartheta : \mathbf{C}P^1 \to \mathbf{C}P^1$, $\vartheta(z) = \bar{z}^{-1}$. ϑ interchanges the disc $D = \{|z| \leq 1\}$ with $D' \equiv \{|z| \geq 1\}$ and leaves invariant their common boundary $\{|z| = 1\}$. Suppose that we are given a Riemannian metric γ on D, compatible with the complex structure, which is of the form $|z|^{-2}|dz|^2$ around ∂D (we shall call such

a metric flat at the boundary). $\vartheta^*\gamma$ is a metric on D' and it glues smoothly with γ on D to the metric $\vartheta^*\gamma \vee \gamma$ on $\mathbf{C}P^1$. Consider formal expressions

$$X = \prod_i \phi_{l_i}(z_i, \bar{z}_i) \tag{2.33}$$

for distinct points z_i in the interior of D (with the empty product case included). Denote

$$\Theta\phi_l(z,\bar{z}) \equiv (-\bar{z}^{-2})^{\Delta_l}(-z^{-2})^{\tilde{\Delta}_l}\phi_{\bar{l}}(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) \tag{2.34}$$

where $l \mapsto \bar{l}$ is the same involution that appeared in eq.(2.6). For X as above, we set

$$\Theta X = \prod_i \Theta\phi_{l_i}(z_i, \bar{z}_i)\ . \tag{2.35}$$

The physical positivity requires that for each family (λ_α) of complex numbers, each family (X_α) of expressions (2.33) and each family (γ_α) of metrics on D flat at the boundary

$$\sum_{\alpha_1,\alpha_2} \bar{\lambda}_{\alpha_2}\lambda_{\alpha_1} Z_{\vartheta^*\gamma_{\alpha_2}\vee\gamma_{\alpha_1}}\langle(\Theta X_{\alpha_2})X_{\alpha_1}\rangle_{\vartheta^*\gamma_{\alpha_2}\vee\gamma_{\alpha_1}} \ \geq\ 0\ . \tag{2.36}$$

These properties hold for the free field compactifications. The condition (2.36) may be rewritten using correlation functions with energy-momentum insertions and a fixed metric. Set

$$\begin{aligned}\Theta T(z) &\equiv \bar{z}^{-4}T(\tfrac{1}{\bar{z}})\ ,\\ \Theta\bar{T}(\bar{z}) &\equiv z^{-4}\bar{T}(\tfrac{1}{z})\end{aligned} \tag{2.37}$$

and extend the definition (2.33) to expressions

$$Y = \prod_m T(z_m)\prod_n \bar{T}(\bar{z}_n)\prod_i \phi_{l_i}(z_i,\bar{z}_i) \tag{2.38}$$

(with all points in the interior of D and distinct), for which

$$\Theta Y = \prod_m \Theta T(z_m)\prod_n \Theta\bar{T}(\bar{z}_n)\prod_i \Theta\phi_{l_i}(z_i,\bar{z}_i)\ . \tag{2.39}$$

One may infer from the property (2.36) that

$$\sum_{\alpha_1,\alpha_2} \bar{\lambda}_{\alpha_2}\lambda_{\alpha_1}\langle(\Theta Y_{\alpha_2})Y_{\alpha_1}\rangle \ \geq\ 0 \tag{2.40}$$

where $\langle\cdots\rangle$ denotes the correlation functions in a locally flat metric.

Problem 2. Show that (2.36) implies (2.40).

The construction of the Hilbert space $\mathcal{H}$ of states is now simple. The expression

$$\sum_{\alpha,\beta} \bar{\lambda}'_\beta\lambda_\alpha\langle(\Theta Y'_\beta)Y_\alpha\rangle$$

defines a Hermitian form on the space V_D of formal linear combinations of products (2.38). Due to (2.40), this form is positive and becomes positive definite on the quotient by its null subspace $V_D^{\rm null}$. One sets

$$\mathcal{H} = \overline{V_D/V_D^{\rm null}}. \tag{2.41}$$

We shall denote by ι the canonical map from V_D to $\mathcal{H}$, by $\mathcal{H}_0$ its image $\subset \mathcal{H}$ and by $\mathcal{Y}$ the image $\iota(Y)$ of Y. The empty product in (2.38) gives rise to the "**vacuum vector**" Ω. The scalar product is given by

$$(\mathcal{Y}', \mathcal{Y}) = \langle (\Theta Y')Y \rangle.$$

$\mathcal{H}$ carries an anti-unitary involution $\mathcal{I}$ mapping the vector $\mathcal{Y}$ to $\bar{\mathcal{Y}}$ where $\bar{\mathcal{Y}}$ corresponds to

$$\bar{Y} = \prod_m T(\bar{z}_m) \prod_n \bar{T}(z_n) \prod_l \phi_{\bar{l}_i}(\bar{z}_i, z_i) \ . \tag{2.42}$$

§2.4. Virasoro algebra and its primary fields

Define the action of dilations by $q \in \mathbf{C}$, $0 < |q| \leq 1$, on the fields by setting

$$S_q T(z) = q^2 T(qz), \quad S_q \bar{T}(\bar{z}) = \bar{q}^2 T(\bar{q}\bar{z}), \quad S_q \phi_l(z, \bar{z}) = q^{\Delta_l} \bar{q}^{\bar{\Delta}_l} \phi_l(qz, \bar{q}\bar{z}).$$

For Y given by (2.38), we put

$$S_q Y = \prod_m S_q T(z_m) \prod_n S_q \bar{T}(\bar{z}_n) \prod_l S_q \phi_{l_i}(z_l, \bar{z}_l) \ .$$

Problem 3. Using the conformal symmetries of the correlation functions, verify that

$$\langle (\Theta Y') S_q Y \rangle = \langle (\Theta S_{\bar{q}} Y') Y \rangle \ . \tag{2.43}$$

In the Hilbert space, we may define the dilation operator $\mathcal{S}_q$ by the equality

$$\mathcal{S}_q \mathcal{Y} = \iota(S_q Y).$$

Note that eq.(2.43) implies that $\mathcal{S}_q$ is well defined on the dense invariant domain $\mathcal{H}_0$. In fact, the family of operators $(\mathcal{S}_q)$ forms a semigroup: $\mathcal{S}_{q_1} \mathcal{S}_{q_2} = \mathcal{S}_{q_1 q_2}$. Applying many times the Schwarz inequality, identity (2.43) and the semigroup property of $\mathcal{S}_q$, one obtains the following Osterwalder-Schrader property:

$$|(\mathcal{Y}', \mathcal{S}_q \mathcal{Y})| \leq \|\mathcal{Y}'\| \|\mathcal{S}_q \mathcal{Y}\| = \|\mathcal{Y}'\| (\mathcal{Y}, \mathcal{S}_{\bar{q}q} \mathcal{Y})^{1/2}$$

$$\leq \cdot \cdot \cdot \cdot \cdot \leq \|\mathcal{Y}'\| \|\mathcal{Y}\|^{\frac{1}{2} + \cdots + \frac{1}{2^{n-1}}} (\mathcal{Y}, \mathcal{S}_{(\bar{q}q)^{2^{n-1}}} \mathcal{Y})^{1/2^n} \ . \tag{2.44}$$

Assume now that for each $\epsilon > 0$, there exists a constant C_ϵ such that

$$|\langle (\Theta Y') S_t Y \rangle| \leq C_\epsilon t^{-\epsilon} \tag{2.45}$$

when $t \to 0$. What this means is that when the distances of a group of insertions are uniformly shrunk to zero the singularity of the correlation functions is not stronger then the power law given by the overall scaling dimension of the group. Using the bound (2.45) on the right hand side of (2.44) and taking n to infinity, we infer that

$$|(\mathcal{Y}', \mathcal{S}_q \mathcal{Y})| \leq \|\mathcal{Y}'\| \|\mathcal{Y}\| \ ,$$

i.e. that the dilation semigroup $\mathcal{S}_q$ is composed of contractions of $\mathcal{H}$. Eq.(2.43) now implies that $\mathcal{S}_q^* = \mathcal{S}_{\bar{q}}$. The weak continuity of the semigroup $(\mathcal{S}_q)$ on $\mathcal{H}$ follows from that on $\mathcal{H}_0$, which is evident. By abstract semigroup theory

$$\mathcal{S}_q = q^{L_0} \bar{q}^{\tilde{L}_0} \tag{2.46}$$

for strongly commuting self-adjoint operators L_0 and $\tilde{L}_0$ sucht that $L_0 + \tilde{L}_0 \geq 0$. Clearly, $\mathcal{H}_0$ is inside the domain of L_0 and of $\tilde{L}_0$ and

$$L_0 \mathcal{Y} = \partial_q|_{q=1} \mathcal{S}_q \mathcal{Y} \ , \qquad \tilde{L}_0 \mathcal{Y} = \partial_{\bar{q}}|_{q=1} \mathcal{S}_q \mathcal{Y} \ . \tag{2.47}$$

It also follows that $\mathcal{S}_q\mathcal{H}_0$ is dense in $\mathcal{H}$ for all q.

L_0, $\tilde{L}_0$ are only the tip of an operator iceberg. To see more of it, define operators $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$ and $\varphi_l(z,\bar{z})$, with $\mathcal{S}_z\mathcal{H}_0$ as the (dense) domain ($|z|<1$), by setting

$$\mathcal{T}(z)\mathcal{Y} = \iota(T(z)Y), \qquad \bar{\mathcal{T}}(\bar{z})\mathcal{Y} = \iota(\bar{T}(\bar{z})Y), \qquad \varphi_l(z,\bar{z})\mathcal{Y} = \iota(\phi_l(z,\bar{z})Y).$$

It is easy to see that the operators $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$ and $\varphi_l(z,\bar{z})$ are well defined. Note that for Y given by eq.(2.38) and with the absolute values of all insertion points different,

$$\mathcal{Y} = R\left(\prod_m \mathcal{T}(z_m)\prod_n \bar{\mathcal{T}}(\bar{z}_n)\prod_i \varphi_{l_i}(z_i,\bar{z}_i)\right)\Omega \tag{2.48}$$

where $R(\cdots)$ reorders the operators so that they act in the order of increasing $|z|$. This is the reason why the operator scheme described here is often called **radial quantization**. Under the conjugation by the anti-unitary involution $\mathcal{I}$ of $\mathcal{H}$,

$$\mathcal{I}\varphi_l(z,\bar{z})\mathcal{I} = \varphi_{\bar{l}}(\bar{z},z), \quad \mathcal{I}\mathcal{T}(z)\mathcal{I} = \mathcal{T}(\bar{z}), \quad \mathcal{I}\bar{\mathcal{T}}(\bar{z})\mathcal{I} = \bar{\mathcal{T}}(z)\ . \tag{2.49}$$

It will be useful to introduce Fourier components of the operators $\mathcal{T}(z)$ and $\bar{\mathcal{T}}(\bar{z})$:

$$L_n = \frac{1}{2\pi i}\oint_{|z|=r<1} z^{n+1}\mathcal{T}(z)dz\ , \tag{2.50}$$

$$\tilde{L}_n = -\frac{1}{2\pi i}\oint_{|z|=r<1} \bar{z}^{n+1}\bar{\mathcal{T}}(\bar{z})d\bar{z}\ . \tag{2.51}$$

Since the insertion of $T(z)$ in the correlation functions $\langle\cdots\rangle$ is analytic in z as long as the other insertions are not met, the matrix elements $(\mathcal{Y}', L_n\mathcal{Y})$ (and hence the vector $L_n\mathcal{Y}$ itself) does not depend on r as long as $r<1$ and the contour $|z|=r$ surrounds the insertions of Y (similarly for $\tilde{L}_n$). Notice that

$$\begin{aligned}(\mathcal{Y}', L_n\mathcal{Y}) &= \frac{1}{2\pi i}\oint_{|z|=1-\epsilon} z^{n+1}\langle(\Theta Y')T(z)Y\rangle dz \\ &= \frac{1}{2\pi i}\oint_{|z|=1+\epsilon} z^{n-3}\langle(\Theta(Y'T(\tfrac{1}{z}))Y\rangle dz\end{aligned} \tag{2.52}$$

where we have moved the integration contour slightly, representing $T(z)$ with $|z| = 1+\epsilon$ as $z^{-4}\Theta T(\frac{1}{z})$. The right hand side is equal to

$$\begin{aligned}&(-\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} \bar{z}^{n-3}\mathcal{T}_{zz}(\tfrac{1}{\bar{z}})d\bar{z}\mathcal{Y}', \mathcal{Y}) \\ &= (\frac{1}{2\pi i}\oint_{|w|=(1+\epsilon)^{-1}} w^{-n+1}\mathcal{T}(w)dw\mathcal{Y}', \mathcal{Y}) = (L_{-n}\mathcal{Y}', \mathcal{Y})\ .\end{aligned}$$

It follows that the operators L_n (and $\tilde{L}_n$) are closable[3] and their adjoints satisfy

$$L_n^* = L_{-n}\ , \qquad \tilde{L}_n^* = \tilde{L}_{-n}\ . \tag{2.53}$$

The L_n's and $\tilde{L}_n$'s commute with the anti-involution $\mathcal{I}$ of $\mathcal{H}$. It will be convenient to somewhat extend the domain of definition of the operators introduced above. Let us admit in the expressions Y of (2.38) integrated insertions $\oint_{|z|=r} z^{n+1}T(z)dz$ and similarly for $\bar{T}(\bar{z})$. Denote by $\mathcal{H}_1$ the resulting subspace of $\mathcal{H}$. Of course $\mathcal{H}_1$

[3] we keep the same symbols for their closures

contains $\mathcal{H}_0$ and is invariant under the L_n's and $\tilde{L}_n$'s. The operators $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$ and $\varphi_l(z,\bar{z})$ may clearly be extended to $\mathcal{S}_z\mathcal{H}_1$ and we shall assume below that this has been done[4].

The calculation which we shall do now is an example of an argument which translates (certain) OPE's into commutation relations and is used in CFT again and again. A devoted student should memorize its idea once for all. We start with the OPE (2.27) for $T(z)$ which will give commutation relations between L_n's. Let us consider the matrix element

$$(\mathcal{Y}', [L_n, \mathcal{T}(w)]\mathcal{Y}) \tag{2.54}$$

$$= \frac{1}{2\pi i}\left(\oint_{|z|=|w|+\epsilon} dz - \oint_{|z|=|w|-\epsilon} dz\right) z^{n+1}\langle(\Theta Y')T(z)T(w)Y\rangle \tag{2.55}$$

$$= \frac{1}{2\pi i}\oint_{|z-w|=\epsilon} z^{n+1}dz\langle(\Theta Y')(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w))Y\rangle \tag{2.56}$$

where we have used the fact (2.48) that the order of operators is determined by the radial order of insertions in the correlation function. In the last line we have collapsed the contour of integration to a small circle around w and inserted the OPE (2.27). Expanding z^{n+1} around $z = w$

$$z^{n+1} = ((z-w)+w)^{n+1} = \frac{n^3-n}{6}(z-w)^3w^{n-2} + \frac{n^2+n}{2}(z-w)^2w^{n-1} + (n+1)(z-w)w^n + w^{n+1} + \dots$$

and retaining only the terms which contribute to the residue at $z = w$ in the last integral of eq.(2.56), we obtain

$$(\mathcal{Y}', [L_n, \mathcal{T}(w)]\mathcal{Y}) = \langle(\Theta Y')\{\frac{c}{12}(n^3-n)w^{n-2} + 2(n+1)w^nT(w) + w^{n+1}\partial_w T(w)\}Y\rangle$$

which is the weak form of the relations

$$[L_n, \mathcal{T}(w)] = \frac{c}{12}(n^3-n)w^{n-2} + 2(n+1)w^n\mathcal{T}(w) + w^{n+1}\partial_w\mathcal{T}(w)\ . \tag{2.57}$$

Similarly, the OPE (2.27) implies that

$$[\tilde{L}_n, \bar{\mathcal{T}}(\bar{w})] = \frac{c}{12}(n^3-n)\bar{w}^{n-2} + 2(n+1)\bar{w}^n\bar{\mathcal{T}}(\bar{w}) + \bar{w}^{n+1}\partial_{\bar{w}}\bar{\mathcal{T}}(\bar{w}). \tag{2.58}$$

By virtue of eq.(2.27), the mixed commutators $[L_n, \bar{\mathcal{T}}(\bar{w})]$ and $[\tilde{L}_n, \mathcal{T}(w)]$ vanish.

Performing a contour integral over w on both sides of eq.(2.57) multiplied by z^{m+1}, we obtain the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\ . \tag{2.59}$$

The (infinite-dimensional) Lie algebra with generators L_n and a central element $\mathcal{C}$ (called the **central charge**) and with relations (2.59), where c is replaced replaced by $\mathcal{C}$, is known as the **Virasoro algebra**. We shall denote it by Vir. It is closely related to the (Witt) Lie algebra of polynomial vector fields $\mathrm{Vect}(S^1)$ on the circle $\{|z| = 1\}$ with generators $l_n = -z^{n+1}\partial_z$ and relations

$$[l_n, l_m] = (n-m)l_{n+m}\ .$$

[4]let us remark that these operators are not closable so their domains should be handled with special care

More exactly, Vir is a central extension of $\mathrm{Vect}(S^1)$, i. e. we have an exact sequence of Lie algebras

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathrm{Vir} \longrightarrow \mathrm{Vect}(S^1) \longrightarrow 0\,,$$

where the second arrow sends 1 to C and the third one maps L_n to l_n.

Eq.(2.58) gives rise to another set of Virasoro commutation relations

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \tfrac{c}{12}(n^3-n)\delta_{n+m,0}\,. \tag{2.60}$$

The L_n's and $\tilde{L}_m$'s commute. Both (2.59) and (2.60) hold on the invariant dense domain $\mathcal{H}_1 \in \mathcal{H}$, As we see, the Hilbert space of states $\mathcal{H}$ of a CFT carries a densely defined unitary (i.e. with property (2.53)) representation of the algebra $\mathrm{Vir} \oplus \mathrm{Vir}$ with central charges acting as multiplication by c.

The representation theory of the Virasoro algebra has played an important role in the construction of models of CFT. We shall include for completeness a brief sketch of its elements in the next section. But why did the Virasoro algebra appear in CFT in the first place? As we have mentioned, Vir is the central extension of an algebra of vector fields on the circle. But $\mathrm{Vect}(S^1) \oplus \mathrm{Vect}(S^1)$ may be identified with the Lie algebra of (polynomial) conformal vector fields on the two-dimensional cylinder $\{(t,x) \mid x \bmod 2\pi\}$ with the Minkowski metric $\gamma_M \equiv dt^2 - dx^2$. By definition, the conformal vector fields X satisfy $\mathcal{L}_X\gamma_M = f_X\gamma_M$ for some function f_X, where $\mathcal{L}_X$ denotes the Lie derivative w.r.t. X. The identification assigns to generators l_n and $\bar{l}_n$ the conformal vector fields $-z^{n+1}\partial_z$ and $-\bar{z}^{n+1}\partial_{\bar{z}}$, respectively, with $z \equiv \mathrm{e}^{i(t+x)}$ and $\bar{z} \equiv \mathrm{e}^{i(t-x)}$. In particular, $i(l_0 + \bar{l}_0)$ is the infinitesimal shift of the Minkowski time t and $i(l_0 - \bar{l}_0)$ the infinitesimal shift of x. Hence $\mathrm{Vect}(S^1) \oplus \mathrm{Vect}(S^1)$ is the Lie algebra of Minkowskian conformal symmetries and representations of $\mathrm{Vir} \oplus \mathrm{Vir}$ describe its projective actions realizing such conformal symmetries on the quantum-mechanical level (projective representations correspond to genuine actions of symmetries on the rays in the Hilbert space representing (pure) quantum states). $H \equiv L_0 + \tilde{L}_0$ is the quantum Hamiltonian[5] and $P = L_0 - \tilde{L}_0$ is the quantum momentum operator. The unitarity conditions (2.53) correspond to the natural real form of the algebra composed of real vector fields: such vector fields are represented by skew-adjoint operators so that the corresponding global conformal transformations act by unitary operators. $\mathrm{Vect}(S^1)$ may be viewed as the Lie algebra of the group $\mathrm{Diff}_+(S^1)$ of orientation preserving diffeomorphisms of the circle. Let $\widetilde{\mathrm{Diff}}_+(S^1)$ denote the group of diffeomorphisms of the line commuting with the shifts by 2π.

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Diff}}_+(S^1) \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 0\,.$$

The group $\mathcal{D} \equiv (\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1))/\mathbf{Z}_{\mathrm{diag}}$ (which acts on the light-cone variables $x^\pm \equiv t \pm x$) is the group of conformal, orientation and time-arrow preserving diffeomorphisms of the Minkowski cylinder. The $\mathrm{Vir} \oplus \mathrm{Vir}$ action in $\mathcal{H}$ integrates to the projective unitary representation of $\mathcal{D}$.

We shall need more information about the representations of $\mathrm{Vir} \times \mathrm{Vir}$ which appear in CFT. This may be obtained by studying the behavior of the primary field operators with respect to the Virasoro algebra action.

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[5] later, we shall see that it is more natural to shift H by a constant

Problem 4. (a). Show by employing the contour integral technique that

$$[L_n, \varphi_l(w,\bar{w})] = \Delta_l(n+1)w^n\varphi_l(w,\bar{w}) + w^{n+1}\partial_w\varphi_l(w,\bar{w}) \ , \tag{2.61}$$

$$[\tilde{L}_n, \varphi_l(w,\bar{w})] = \bar{\Delta}_l(n+1)\bar{w}^n\varphi_l(w,\bar{w}) + \bar{w}^{n+1}\partial_{\bar{w}}\varphi_l(w,\bar{w}) \ . \tag{2.62}$$

(b). Using the above relations and eqs.(2.57) and (2.58) prove that the operators L_0, $\tilde{L}_0$ given by (2.50) and (2.51) coincide with the generators of the semigroup $(\mathcal{S}_q)$ introduced earlier.

Eqs.(2.61) and (2.62) express on the operator level the properties of the (Virasoro) primary fields of conformal weights $(\Delta_l, \tilde{\Delta}_l)$. Comparing them to the last two equations of Lecture 1, we infer that the operators $\varphi_l(w,\bar{w})$ for $(w,\bar{w}) = (\mathrm{e}^{-t+ix}, \mathrm{e}^{-t-ix})$ should be interpreted as the imaginary time versions of Minkowski fields. Note that the components $\mathcal{T}(z)$ and $\bar{\mathcal{T}}(\bar{z})$ fail to be Virasoro primary fields of weights $(2,0)$ and $(0,2)$, respectively, due to the anomalous term proportional to c in the relations (2.57) and (2.58).

Recall that (as a generator of a self-adjoint semigroup of contractions) the self-adjoint operator $H = L_0 + \tilde{L}_0$ has to be positive. In Minkowskian QFT with Poincaré invariance the positivity of the Hamiltonian implies the spectral condition $H \pm P \geq 0$, where P is the momentum operator. The same is true in CFT with its Hilbert space corresponding to cylindrical Minkowski space. The Virasoro commutation relations imply

$$L_0 \geq 0 \ , \qquad \tilde{L}_0 \geq 0 \ . \tag{2.63}$$

Indeed, let E_B be a non-vanishing joint spectral projector of L_0 and $\tilde{L}_0$ corresponding to eigenvalues in a small ball B, with the L_0 eigenvalues negative and such that $E_{B-(1,0)} = 0$. Then, for any normalized vector ψ with $E_B\psi = \psi$, we have

$$L_1\psi = L_1E_B\psi = E_{B-(1,0)}L_1\psi = 0 \ .$$

On the other hand,

$$0 \leq (L_{-1}\psi, L_{-1}\psi) = (\psi, L_1L_{-1}\psi) = (\psi, [L_1, L_{-1}]\psi) = 2(\psi, L_0\psi) < 0$$

which shows that L_0 cannot have negative spectrum. Similarly for $\tilde{L}_0$. Hence only **positive energy representations** of the Virasoro algebra with $L_0 \geq 0$ $(\tilde{L}_0 \geq 0)$ appear in CFT with the Hilbert space interpretation. The techniques of CFT also apply, however, to certain scaling limits of statistical mechanical models without physical positivity where a wider class of Virasoro representations intervenes.

Relations (2.61) and (2.62) provide further spectral information about L_0 and $\tilde{L}_0$. They imply the equalities

$$L_n\Omega = 0 \ , n \geq -1 \qquad \tilde{L}_n\Omega = 0 \ , n \geq -1 \tag{2.64}$$

$$L_n\varphi_l(0)\Omega = 0 \ , n > 0 \ , \quad \tilde{L}_n\varphi_l(0)\Omega = 0 \ , n > 0 \ , \tag{2.65}$$

$$L_0\varphi_l(0)\Omega = \Delta_l\varphi_l(0)\Omega \ , \quad \tilde{L}_0\varphi_l(0)\Omega = \tilde{\Delta}_l\varphi_l(0)\Omega$$

where, by definition, $\varphi_l(0)\Omega = \lim_{z\to 0} \varphi_l(z,\bar{z})\Omega$. In particular, it follows that the vacuum vector Ω is an eigenvector of L_0 and $\tilde{L}_0$ with the lowest possible eigenvalues 0. We shall assume that it is a unique vector, up to normalization, with this property (although there are CFT's without this property). In fact, Ω is annihilated by the $sl_2 \times sl_2$ Lie subalgebra generated by $L_0, L_{\pm 1}, \tilde{L}_0$ and $\tilde{L}_{\pm 1}$ but not by the entire symmetry algebra Vir$\times$Vir of the theory: the conformal symmetry is spontaneously broken.

$\varphi_l(0)\Omega$ are also eigenvectors of L_0 and $\tilde{L}_0$, with eigenvalues $(\Delta_l, \tilde{\Delta}_l)$ and it follows that $\Delta_l,\ \tilde{\Delta}_l > 0$. Also $\mathcal{I}\Omega = \Omega$ and $\mathcal{I}\varphi_l(0)\Omega = \varphi_{\bar{l}}(0)\Omega$. In fact, the vectors $\varphi_l(0)\Omega$ are annihilated by all Virasoro generators with positive indices. The eigenvectors of L_0, $\tilde{L}_0$ (and $\mathcal{C}$) with this property are called Virasoro highest weight (HW) vectors.

Summarizing, we have shown that the Hilbert space of states in a CFT carries a (densely defined) positive energy representation of two commuting copies of the Virasoro algebra with the same central charge c. The primary field operators $\varphi_l(z, \bar{z})$ applied to the vacuum become in the limit $z \to 0$ highest weight vectors of the Virasoro representations.

§2.5. Highest weight representations of Vir

For completeness, we include a brief sketch of the representation theory of the Virasoro algebra.

An important class of representations of the Virasoro algebra is constituted by the so called **highest weight** (HW) representations. Let $\theta = \mathbf{C}L_0 \oplus \mathbf{C}\mathcal{C}$, $\mathcal{N}_+ = \bigoplus_{n=1}^{\infty} \mathbf{C}L_n$, $\mathcal{N}_- = \bigoplus_{n=1}^{\infty} \mathbf{C}L_{-n}$. Vir $= \mathcal{N}_- \oplus \theta \oplus \mathcal{N}_+$ is the triangular decomposition of the Virasoro algebra. Let $\lambda \in \theta^*$, the dual space to θ, $\lambda(\mathcal{C}) = c$, $\lambda(L_0) = \Delta$. A Vir-module (representation) $M_{c,\Delta}$ is called a HW module of HW λ if there exists a vector $v_0 \in V$ such that

$$\begin{aligned} \mathcal{N}_+ v_0 &= 0\,, \\ \mathcal{U}(\mathcal{N}_-)v_0 &= M_{c,\Delta}\,, \\ xv_0 &= \lambda(x)v_0 \quad \text{for} \quad x \in \theta \end{aligned}$$

where $\mathcal{U}(\cdot)$ denotes the enveloping algebra. v_0 is called the HW vector, c the central charge and Δ the conformal weight of the HW representation. It follows that $M_{c,\Delta}$ is the linear span of the vectors $L_{-n_r}L_{-n_{r-1}}\cdots L_{-n_1}v_0$ with $0 < n_1 \le n_2 \le \ldots \le n_r$, but these vectors are not necessarily linearly independent. $N = \sum_{i=1}^{r} n_i$ is called the level of the vector $L_{-n_r}L_{-n_{r-1}}\cdots L_{-n_1}v_0$. A level N vector is an eigenvector of L_0 with eigenvalue $N + \Delta$. We shall denote the subspace of level N vectors by $M^{(N)}_{c,\Delta}$. Clearly vectors of different levels are linearly independent; thus we have

$$M_{c,\Delta} = \bigoplus_{N=1}^{\infty} M^{(N)}_{c,\Delta}$$

with $\dim(M^{(N)}_{c,\Delta}) \le p(N)$, the partition number of N. A HW module for which $\dim(M^{(N)}_{c,\Delta}) = p(N)$ for all N, i. e. where all the vectors of the form $L_{-n_r}L_{-n_{r-1}}\cdots L_{-n_1}v_0$ (with ordered n_i's) are linearly independent is called the Verma module $V_{c,\Delta}$. It exists for all complex c, Δ and is unique up to isomorphism. In the analysis of the HW modules of the Virasoro algebra an important role is played by **singular vectors**. A non-zero vector $v^{(N)}_s$ of level N is called singular if $L_n v^{(N)}_s = 0$ for all $n > 0$. Any vector v_s with $L_n v_s = 0$ for all $n > 0$ is a a sum of singular vectors (its non-zero homogeneous components). Any singular vector $v^{(N)}_s$ generates a submodule in $V_{c,\Delta}$ isomorphic to $V_{c,\Delta+N}$. We have:

(i) Any submodule of $V_{c,\Delta}$ is generated by its singular vectors.

(ii) Any HW module $M_{c,\Delta}$ is isomorphic to a quotient of the Verma module $V_{c,\Delta}$.
(iii) The factor module of the Verma module by the maximal proper submodule is the unique (up to isomorphism) irreducible HW module $H_{c,\Delta}$.

A Vir-module M is called unitary if there exists a (positive, Hermitian) scalar product $(\cdot,\cdot)$ on M such that

(2.66) $\quad (v, L_n w) = (L_{-n} v, w)$ for all $n \in \mathbf{Z}$ and for all $v, w \in V$.

It follows then that $(v, \mathcal{C}w) = (\mathcal{C}v, w)$ for all $v, w \in V$ and that the eigenvalues of $\mathcal{C}$ and L_0 are real. On each Verma module $V_{c,\Delta}$ with c and Δ real there exists a unique Hermitian (Shapovalov) form $\langle \cdot, \cdot \rangle$ such that $\langle v, L_n w\rangle = \langle L_{-n} v, w\rangle$ for all $v, w \in V_{c,\Delta}$ and that $\langle v_0, v_0\rangle = 1$. Define $Null_{c,\Delta} = \{v \in V_{c,\Delta} | \langle v, w\rangle = 0$ for all $w \in V_{c,\Delta}\}$.

(i) $\langle v^{(N)}, v^{(N')}\rangle = 0$ if $v^{(N)}$ ($v^{(N')}$) is a level N (N') vector and $N \neq N'$,
(ii) any singular vector of positive level belongs to $Null_{c,\Delta}$.
(iii) $Null_{c,\Delta}$ is the maximal proper submodule of $V_{c,\Delta}$, i.e. $H_{c,\Delta} = V_{c,\Delta}/Null_{c,\Delta}$.

Let us investigate the conditions under which $H_{c,\Delta}$ is a unitary Vir-module, or, equivalently, under which the Hermitian form on $H_{c,\Delta}$ induced by the Shapovalov form is positive. Since $\langle L_{-n} v_0, L_{-n} v_0\rangle = 2n\Delta + \frac{c}{12}(n^3 - n)$, necessarily $c \geq 0$ and $\Delta \geq 0$. Now consider the two $2n$ level vectors $v = L_n^2 v_0$ and $w = L_{-2n} v_0$ and suppose that $c = 0$. The Gram determinant

$$\det \begin{pmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{pmatrix} = -20n^4\Delta^2 + 32n^3\Delta^3 \ .$$

Thus for $c = 0$ a necessary (and sufficient) condition is $\Delta = 0$. $c, \Delta = 0$ correspond to the trivial one-dimensional representations. So there exists no non-trivial unitary HW representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the (polynomial) vector fields on the circle. It is enough to study the positivity of the Shapovalov form restricted the the subspaces $H^{(N)}_{c,\Delta}$ of given level. Let $m^{(N)}_{c,\Delta}$ denote the $p(N) \times p(N)$ matrix with entries $\langle L_{-n_r} L_{-n_{r-1}} \cdots L_{-n_1} v_0, L_{-n'_r} L_{-n'_{r-1}} \cdots L_{-n'_1} v_0\rangle$ where (n_i) and (n'_i) are ordered and $\sum n_i = \sum n'_i = N$. Unitarity of $H_{c,\Delta}$ is equivalent to the conditions $m^{(N)}_{c,\Delta} \geq 0$ for all N. In particular, $\det(m^{(N)}_{c,\Delta})$ has to be non-negative for all N when $H_{c,\Delta}$ is unitary. Direct calculation for level 1 and 2 gives

$$\begin{aligned} \det(m^{(1)}_{c,\Delta}) &= 2\Delta \ , \\ \det(m^{(2)}_{c,\Delta}) &= 4\Delta\left(\tfrac{1}{2}c + (c-5)\Delta + 8\Delta^2\right) \ . \end{aligned}$$

A general formula for the determinant of $m^{(N)}_{c,\Delta}$ was given by Kac and was proved by Feigin and Fuchs:

$$\det(m^{(N)}_{c,\Delta}) = \kappa_N \prod_{\substack{1 \leq rs \leq N \\ r,s \in \mathbf{N}}} (\Delta - \Delta_{r,s}(m))^{p(N-rs)}$$

where $\Delta_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$, m is a root of the equation $c = 1 - \frac{6}{m(m+1)}$, and $\kappa_N = \prod_{\substack{1 \leq rs \leq N \\ r,s \in \mathbf{N}}} ((2r)^s s!)^{n(r,s)}$ with $n(r,s)$ denoting the number of partitions of N in which r appears s times.

For $\Delta \to \infty$, $m^{(N)}_{c,\Delta}$ goes to a diagonal matrix with positive entries. From the Kac formula it follows that $\det(m^{(N)}_{c,\Delta}) > 0$ for $c > 1$, $\Delta > 0$. Therefore $m^{(N)}_{c,\Delta}$ is non-degenerate and positive for $c > 1$, $\Delta > 0$ and non-negative for $c \geq 1$, $\Delta \geq 0$. Thus $V_{c,\Delta}$ is irreducible for $c > 1$, $\Delta > 0$ and $H_{c,\Delta}$ is unitary for $c \geq 1$, $\Delta \geq 0$. Since, for $c = 1$

$$\det(m^{(N)}_{1,\Delta}) = \kappa_N \prod_{1 \leq rs \leq N} \left(\Delta - \frac{(r-s)^2}{4}\right)^{p(N-rs)},$$

it follows that $V_{1,\Delta}$ is irreducible if and only if $\Delta \neq \frac{n^2}{4}$, $n \in \mathbf{Z}$.

For $0 \leq c < 1$, the situation is more interesting, providing the first example of the selective power of conformal invariance.

Theorem. *For (c, Δ) with $0 \leq c < 1$ and $\Delta \geq 0$, unitarity of the irreducible HW representation $H_{c,\Delta}$ requires that (c, Δ) belong to the following discrete series of points:*

$$\left.\begin{aligned} c &= 1 - \tfrac{6}{m(m+1)} \\ \Delta_{r,s}(m) &= \tfrac{((m+1)r - ms)^2 - 1}{4m(m+1)} \end{aligned}\right\} \begin{array}{l} m = 2, 3, \ldots, \\ 1 \leq r \leq m-1, \\ 1 \leq s \leq r. \end{array}$$

The theorem was proven by Friedan-Qiu-Shenker by a careful analysis of the geometry of lines $(c(m), \Delta_{r,s}(m))$ in the (c, Δ) plane. This, in conjunction with the Kac determinant formula, allowed subsequent elimination of portions of the (c, Δ) plane where negative norm vectors appear, by an induction on N. At the end, only the points listed above were left. Goddard-Kent-Olive constructed explicitly unitary irreducible HW representations of the Virasoro algebra for the above series of (c, Δ) by employing the so-called "coset construction".

All unitary HW representations integrate to projective unitary representations of $\mathrm{Diff}_+(S^1)$ in the Hilbert space completion of $H_{c,\Delta}$.

Any unitary representation M of the Virasoro algebra such that (the closure of) L_0 is a positive self-adjoint operator with a discrete spectrum of finite multiplicity in the Hilbert space completion $\overline{M}$ of M is essentially isomorphic to a direct sum of unitary representations (the isomorphism may require a different choice of the common invariant dense domain for the L_n's in $\overline{M}$). We shall see in the next section that in CFT the operators $\mathcal{S}_q = q^{L_0}\bar{q}^{\tilde{L}_0}$ should be trace class for $|q| < 1$, from which it follows that L_0 and $\tilde{L}_0$ have a discrete spectrum of finite multiplicity. Hence the Virasoro algebra representations which appear in CFT are direct sums of the unitary HW representations.

The algebra $\mathrm{Vect}(S^1) \oplus \mathrm{Vect}(S^1)$, and hence also $\mathrm{Vir} \oplus \mathrm{Vir}$, acts naturally on the space $\Lambda^{\Delta\tilde{\Delta}}$ of $(\Delta, \tilde{\Delta})$-forms $f(dz)^{\Delta}(d\bar{z})^{\tilde{\Delta}}$ on $D \setminus \{0\}$ by Lie derivatives:

$$l_n f = -\Delta(n+1)z^n f - z^{n+1}\partial_z f, \qquad \bar{l}_n f = -\tilde{\Delta}(n+1)\bar{z}^n f - \bar{z}^{n+1}\partial_{\bar{z}} f.$$

The commutation relations (2.61) and (2.62) express the fact that the operators φ_l intertwine the action of $\mathrm{Vir} \oplus \mathrm{Vir}$ in $\mathcal{H}_1 \subset \mathcal{H}$ and $\mathcal{H}_1 \otimes \Lambda^{\Delta_l\tilde{\Delta}_l}$. This gives a representation theory interpretation of the primary fields of CFT.

§2.6. Segal's axioms and vertex operator algebras

Up to now we have studied QFT on closed Riemann surfaces. Let us consider now a compact Riemann surface Σ (connected or disconnected) with the boundary composed of the connected components $\mathcal{C}_i$, $i \in I$. We shall parametrize each $\mathcal{C}_i$ in a real analytic way by the standard unit circle $S^1 \subset \mathbf{C}$. The components $\mathcal{C}_i$ may be divided into "in" and "out" ones, depending on whether the parameterization disagrees or agrees with the orientation of $\mathcal{C}_i$ inherited from Σ. This induces the splitting $I = I_{\rm in} \cup I_{\rm out}$ of the set of indices i. We may invert the orientation of $\mathcal{C}_i$ by composing its parameterization with the map $z \mapsto z^{-1}$ of S^1. To Σ with parameterized boundary, we may uniquely assign a compact surface $\hat{\Sigma}$ without boundary by gluing a copy of the disc D to each boundary component $\mathcal{C}_i$ with $i \in I_{\rm in}$ and a copy of the disc D' to each $\mathcal{C}_i$ with $i \in I_{\rm out}$. Conversely, given a closed Riemann surface $\hat{\Sigma}$ with holomorphically embedded disjoint discs D and D' ("local parameters"), by removing their interiors we obtain the surface Σ with boundary parameterized by the standard circles $\{|z| = 1\}$. We shall call a metric h on Σ (compatible with its complex structure) flat at the boundary if, for each i, it is of the form $|z|^{-2}|dz|^2$ in the local holomorphic coordinate around $\mathcal{C}_i$ extending its parameterization. We shall consider only such metrics on surfaces with boundary.

First, let us note that a metric γ on Σ and metrics γ_i on D, all flat at the boundary, give rise by gluing to the metric

$$\tilde{\gamma} = (\vee_{i\in I_{\rm in}}\gamma_i) \vee \gamma \vee (\vee_{i\in I_{\rm out}}\vartheta^*\gamma_i)$$

on $\hat{\Sigma}$. It follows from the local scale covariance formula (2.3) that the combination of partition functions

$$Z_{\tilde{\gamma}} \prod_{i\in I}(Z_{\vartheta^*\gamma_i\vee\gamma_i})^{-1/2} \equiv Z_\gamma \tag{2.67}$$

is independent of (the conformal factors) of γ_i. Besides, the transformation of Z_γ under $\gamma \mapsto \mathrm{e}^\sigma\gamma$ with σ vanishing around the boundary is still given by eq.(2.3). It is sensible to call Z_γ the partition function of the Riemann surface Σ with boundary. Let $\mathcal{Y}_i \in \mathcal{H}_0$. Consider the matrix elements defined by

$$(\otimes_{i\in I_{\rm out}}\mathcal{Y}_i,\ A_{\Sigma,\gamma}\otimes_{i\in I_{\rm in}}\mathcal{Y}_i) = Z_\gamma\langle \prod_{i\in I_{\rm out}}(\Theta Y_i)\prod_{i\in I_{\rm in}} Y_i\rangle\ . \tag{2.68}$$

We shall postulate that the above formula defines operator "amplitudes" $A_{\Sigma,\gamma}$ mapping the tensor products of the CFT Hilbert spaces associated to the boundary components of Σ. In particular, for Σ without boundary, eq.(2.68) should be read as the identity $A_{\Sigma,\gamma} = Z_\gamma$. The amplitudes $A_{\Sigma,\gamma}$ (or rather their holomorphic counterparts) were considered as the defining data of CFT in the work of Segal. Let us state the Segal axioms (adapted to the real setup).

(i) We are given the Hilbert space of states $\mathcal{H}$ with an anti-unitary involution $\mathcal{I}$ and for each compact Riemann surface (with parameterized boundary or without boundary, connected or disconnected) the operator[6]

$$A_{\Sigma,\gamma} : \bigotimes_{i\in I_{\rm in}}\mathcal{H} \longrightarrow \bigotimes_{i\in I_{\rm out}}\mathcal{H}\ , \tag{2.69}$$

which we assume trace class.

[6]the empty tensor product should be interpreted as $\mathbf{C}$

(ii) If Σ is a disjoint union of Σ_1 and Σ_2, then

$$A_{\Sigma,\gamma} = A_{\Sigma_1,\gamma} \otimes A_{\Sigma_2,\gamma} \ .$$

(iii) If $D : \Sigma_1 \to \Sigma_2$ is a diffeomorphism restricting to the identity around the (parameterized) boundary then

$$A_{\Sigma_2,\gamma} = A_{\Sigma_1,D^*\gamma} \ .$$

(iv) If $\bar{\Sigma}$ denotes the Riemann surface with conjugate complex structure (and opposite orientation) then

$$A_{\bar{\Sigma},\gamma} = A^\dagger_{\Sigma,\gamma}$$

(v) The inversion of boundary parameterization acts on the amplitude $A_{\Sigma,\gamma}$ by the isomorphism $\mathcal{H} \cong \mathcal{H}^*$ induced by $\mathcal{I}$ in the corresponding Hilbert space factor.

(vi) If Σ' is obtained from Σ by gluing of the $\mathcal{C}_{i_0}$ and $\mathcal{C}_{i_1}$ boundary components then

$$A_{\Sigma',\gamma} = \mathrm{tr}_{i_0,i_1} \ A_{\Sigma,\gamma} \ ,$$

where tr_{i_0,i_1} denotes the partial trace in the $\mathcal{H}$ factors corresponding to $\mathcal{C}_{i_0}$ and $\mathcal{C}_{i_1}$.

(vii) For any function σ on Σ vanishing around the boundary

$$A_{\Sigma,\mathrm{e}^\sigma\gamma} = \mathrm{e}^{\frac{c}{96\pi}(\|d\sigma\|^2_{L^2}+4\int_\Sigma \sigma r dv)} \ A_{\Sigma,\gamma}$$

In the approach in which we start from the data specifying partition functions and correlation functions on surfaces without boundaries, property (**i**), in conjunction with eq.(2.68), imposes certain new regularity conditions on the correlation functions. Property (**ii**) may be viewed as a definition of the amplitudes for disconnected surfaces. All the remaining properties except for (**vi**) follow easily. The gluing axiom brings an essential novel element allowing us to compare the correlation functions of different surfaces (with different complex structures and different topologies). In the heuristic approach employing functional integrals, it expresses the locality of the latter. Let us explain this point in more detail.

We shall think about the partition function of a QFT on a closed Riemann surface as being given by a functional integral of the type

$$Z_\gamma = \int \mathrm{e}^{-S_\Sigma(\phi)} D\phi \tag{2.70}$$

where $S_\Sigma(\phi)$ is the action functional and $D\phi = \prod_{x\in\Sigma} d\nu(\phi(x))$ is the formal volume element on the space of fields. On a Riemann surface with boundary Σ, we could consider an analogous functional integral with fixed boundary values $\phi|_{\mathcal{C}_i} \equiv \varphi_i$ of the fields:

$$A_{\Sigma,\gamma}((\varphi_i)_{i\in I}) \ = \int_{\phi|_{\mathcal{C}_i}=\varphi_i} \mathrm{e}^{-S_\Sigma(\phi)} D\phi \ . \tag{2.71}$$

It is a function of the field boundary values. The space of functionals $\mathcal{F}(\varphi)$ of the fields on the circle with the formal L^2 scalar product may be thought of as the Hilbert space $\mathcal{H}$ of states of the system (we have seen a concrete realization of this idea for the free compactified field). We may then interpret the functional $A_{\Sigma,\gamma}((\varphi_i)_{i\in I})$ as the kernel of an operator mapping the tensor product of state

spaces, one for each "in" component of the boundary, to the similar product for the "out" components.

The space $\mathcal{H}$ may be equipped with a formally anti-unitary involution $\mathcal{I}$, $(\mathcal{I}\mathcal{F})(\varphi) \equiv \overline{\mathcal{F}(\varphi^\vee)}$, where $\varphi^\vee(z) \equiv \varphi(z^{-1})$, which allows us to turn the operators in $\mathcal{H}$ into bilinear forms and vice versa and to identify the operator amplitudes when we invert the orientation of some boundary components.

The basic formal property of the functional integral (2.71) is that one should be able to compute it iteratively. Suppose that the surface Σ' is obtained by identifying a (parameterized) "out" component $\mathcal{C}_{i_0}$ with an "in" component $\mathcal{C}_{i_1}$ of a (connected or disconnected) surface Σ. The functional integral on Σ' may be done by keeping first the value of the field on the gluing circle fixed and integrating over it only subsequently. In other words, the equality

$$\int\limits_{\substack{\phi|_{\mathcal{C}_i}=\varphi_i,\\ i\neq i_0,i_1}} \mathrm{e}^{-S_{\Sigma'}(\phi)}D\phi \;=\; \int\Bigg(\int\limits_{\substack{\phi|_{\mathcal{C}_i}=\varphi_i,\ i\neq i_0,i_1\\ \phi|_{\mathcal{C}_{i_0}}=\varphi_0=\phi|_{\mathcal{C}_{i_1}}}} \mathrm{e}^{-S_{\Sigma}(\phi)}D\phi\Bigg)D\varphi_0$$

should hold and this is exactly a formal version of the gluing property (**vi**).

What is the functional integral interpretation of eq.(2.68)? One should interpret vectors $\mathcal{Y} \in \mathcal{H}$ as corresponding to functions

$$\mathcal{F}_Y(\varphi) = (Z_{\vartheta^*\gamma\vee\gamma})^{-1/2} \int\limits_{\phi|_{\partial D}=\varphi} Y\mathrm{e}^{-S_D(\phi)}D\phi$$

of fields on the circle. The complex conjugate functions are then given by the D'-functional integrals

$$(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2} \int\limits_{\phi|_{\partial D}=\varphi} \Theta Y\mathrm{e}^{-S_{D'}(\phi)}D\phi$$

so that the formal L^2 scalar product for the functions of fields φ on the boundary circle gives

$$\begin{aligned}\int\overline{\mathcal{F}_Y(\varphi)}\mathcal{F}_Y(\varphi)D\varphi &= (Z_{\vartheta^*\gamma\vee\gamma})^{-1}\int(\Theta Y)Y\mathrm{e}^{-S_{D\cup D'}(\phi)}D\phi\\ &= \langle(\Theta Y)Y\rangle = (\mathcal{Y},\mathcal{Y})\end{aligned}$$

(γ has been assumed locally flat above). Formula (2.68) now follows by iterative calculation of the functional integral

$$Z_{\tilde\gamma}\langle\prod_{i\in I_{\mathrm{out}}}(\Theta Y_i)\prod_{i\in I_{\mathrm{in}}}Y_i\rangle = \int\prod_{i\in I_{\mathrm{out}}}(\Theta Y_i)\prod_{i\in I_{\mathrm{in}}}Y_i\ \mathrm{e}^{-S_{\hat\Sigma}(\phi)}D\phi$$

by first fixing the values of ϕ on $\partial\Sigma$ and then integrating over them.

Any two-dimensional QFT should give rise to operator amplitudes with properties (**i**) to (**vi**). In particular, the so-called $P(\phi)_2$ theories corresponding to the actions $S(\phi) \sim \|\phi\|^2_{L^2} + \int P(\phi)dv$, where P is a bounded below polynomial, undoubtedly give rise to such structures. The special property which distinguishes the CFT models from other two-dimensional QFT's is, of course, the local scale covariance (**vii**) (conjecturally, special limits of the $P(\phi)_2$ theories should possess this property).

As already stressed, if we consider eq.(2.68) as the definition of the amplitudes $A_{\Sigma,\gamma}$ then the properties (**i**) to (**vii**) above become further requirements on the

correlation functions. The point of Segal was, however, that the amplitudes $A_{\Sigma,\gamma}$ satisfying (**i**) to (**vii**) may be taken as the defining objects of CFT from which the rest of the structure, like the Virasoro algebra action, the primary fields and their correlation functions, etc. may be extracted. In this sense, the requirements (**i**) to (**vii**) may be viewed as the basic axioms of CFT, encapsulating its mathematical structure. Let us briefly sketch Segal's arguments. They require looking at the amplitudes $A_{\Sigma,\gamma}$ for the simplest geometries.

Discs For $\Sigma = D$, $A_{D,\gamma} \in \mathcal{H}$ and $(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2}A_{D,\gamma}$ is a metric independent vector which, according to eq.(2.68) we should interpret as the vacuum vector:

$$(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2}A_{D,\gamma} \equiv \Omega \,. \tag{2.72}$$

Similarly, $A_{D',\vartheta^*\gamma}$ is the linear functional on $\mathcal{H}$,

$$(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2}A_{D',\vartheta^*\gamma} = (\Omega,\ \cdot\ ) \,.$$

It follows from the properties (**iv**) and (**v**) that $\Omega = \mathcal{I}\Omega$.

Annuli Let us pass to surfaces Σ with annular topology. Under gluing of "out" and "in" boundary components of two such surfaces their amplitudes $A_{\Sigma,\gamma}$ form a semigroup due to the property (**vi**). This is the semigroup which encodes the Virasoro action on $\mathcal{H}$. In particular, for the annuli $\Sigma = \{|q| \le |z| \le 1\} = C_q$ with the "out" boundary parametrized by $z \mapsto z$ and the "in" one by $z \mapsto qz$, using the metric $\gamma_0 = |z|^{-2}|dz|^2$ on C_q, we obtain a semigroup almost identical to the semigroup $(\mathcal{S}_q)$ considered previously

$$(\mathcal{Y}', A_{C_q,\gamma_0}\mathcal{Y}) = Z_{\gamma_0}\langle(\Theta Y)S_qY\rangle = Z_{\gamma_0}(\mathcal{Y}', \mathcal{S}_q\mathcal{Y}) \tag{2.73}$$

where Z_{γ_0} is given by eq.(2.67).

Problem 5. Show that $Z_{\gamma_0} = (q\bar{q})^{-c/24}$.

The immediate consequence of this relation and of eqs.(2.73) and (2.46) is the identity

$$A_{C_q,\gamma_0} = q^{L_0-c/24}\bar{q}^{\tilde{L}_0-c/24} \,. \tag{2.74}$$

Note that the mapping $z \mapsto \mathrm{e}^{iz}$ establishes a holomorphic diffeomorphism between the finite cylinder $C_\tau = \{z \mid 0 \le \mathrm{Im}\, z \le 2\pi\tau_2\}/2\pi\mathbf{Z}$ and C_q, where $q = \mathrm{e}^{2\pi i\tau}$. The pullback of the metric γ_0 by $z \mapsto \mathrm{e}^{iz}$ is $|dz|^2$. The amplitude $A_{C_\tau,|dz|^2}$ is equal to $\mathrm{e}^{-2\pi\tau_2 H}\mathrm{e}^{2\pi i\tau_1 P}$, where $\tau = \tau_1 + i\tau_2$ and H is the Hamiltonian of the theory quantized in periodic volume $\mathbf{R}/2\pi\mathbf{Z}$ and P is the momentum operator. Comparison with eq.(2.74) yields

$$H = L_0 + \tilde{L}_0 - \tfrac{c}{12} \,, \qquad P = L_0 - \tilde{L}_0 \,.$$

The requirement that the amplitudes be trace class operators implies that L_0 and $\tilde{L}_0$ have discrete spectrum with finite multiplicity, so that their eigenvalue zero corresponding to the eigenvector Ω is isolated (with possible multiplicity). Note that energy of the vacuum state now becomes equal to $\frac{c}{12}$. If we work on the space $\mathbf{R}/L\mathbf{Z}$ instead, the energy spectrum is multiplied by $\frac{2\pi}{L}$ so that the lowest energy becomes $\frac{\pi c}{6L}$. This permits us to calculate (following Cardy) the central charge of the conformal models from the finite size dependence of the ground state energy in the periodic interval.

Gluing together the boundaries of C_τ, one obtains the elliptic curve $T_\tau = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ (with metric $4\pi^2|dz|^2$). The properties (**iii**) and (**vi**) of the operator amplitudes then imply the following expression for the **toroidal partition function**:

$$Z(\tau) \equiv A_{T_\tau, |dz|^2} = \operatorname{tr} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - c/24} \,. \tag{2.75}$$

which is necessarily a modular invariant function (recall why?). The modular invariance of the partition functions of CFT has played an important role in the search for and the classification of models.

The amplitudes for other annular surfaces allow us to obtain the action of other generators of the double Virasoro algebra in the Hilbert space of states $\mathcal{H}$. If $D \ni z \mapsto f(z) \in D$ is a holomorphic embedding of D into its interior preserving the origin then $\Sigma_f := D \setminus f(\mathrm{int}(D))$ is in a natural way an annulus with parameterized boundary (one "in" $f(\partial D)$ and one "out" ∂D component). Such annuli form a semigroup. In particular, for $f_{q,\alpha,n}(z) =\mapsto q^{z\partial_z} \mathrm{e}^{\alpha z^{n+1}\partial_z} z$ for $n > 0$, $|q| < 1$ and $|\alpha|$ sufficiently small,

$$A_{\Sigma_{f_{q,\alpha,n}},\gamma} = Z_\gamma \mathrm{e}^{\alpha L_n} q^{L_0} \mathrm{e}^{\bar{\alpha}\tilde{L}_n} \bar{q}^{\tilde{L}_0}$$

encoding the action of the operators L_n, $\tilde{L}_n$ for $n > 0$ (the complex-conjugate annuli $\bar{\Sigma}_{f_{q,n,\alpha}}$ give amplitudes encoding L_n, $\tilde{L}_n$ for $n < 0$). The Virasoro HW vectors $\mathcal{X}_l \in \mathcal{H}$ of conformal weights $(\Delta_l, \tilde{\Delta}_l)$ may be characterized by the equalities

$$Z_\gamma^{-1} A_{\Sigma_f,\gamma} \mathcal{X}_l = (\tfrac{df(0)}{dz})^{\Delta_l} (\tfrac{d\bar{f}(0)}{d\bar{z}})^{\tilde{\Delta}_l} \mathcal{X}_l. \tag{2.76}$$

In particular, $L_0\mathcal{X}_l = \Delta_l\mathcal{X}_l$, $L_n\mathcal{X}_l = 0$ for $n > 0$ and similarly for the $\tilde{L}_n$'s.

Each Virasoro HW vector gives rise to a primary field and the correlation functions of the primary fields may be recovered from the operator amplitudes by the following construction. Let $(x_1, \ldots, x_n)$ be a sequence of non-coincident points in the surface $\hat{\Sigma}$ without boundary. Specify local parameters at the points x_j by embedding discs D into $\hat{\Sigma}$ so that their images centered at the points x_j do not intersect. $\hat{\Sigma}$ may be then viewed as a surface Σ with boundary capped with discs D. The correlation functions of the primary fields φ_{l_i} corresponding to the HW vectors $\mathcal{X}_{l_i}$ may be defined in a locally flat metric by the formula

$$\langle \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n) \rangle = \tfrac{1}{Z_\gamma} A_{\Sigma,\gamma} \otimes_i \mathcal{X}_{l_i} \tag{2.77}$$

(in accordance with relation (2.68)). The characteristic property (2.76) of the HW vectors ensures that the left hand side is unambiguously defined as a $(\Delta_{l_j}, \tilde{\Delta}_{l_j})$-form in the variable x_j.

Problem 6. Using relation (2.76) and the gluing axiom show that the right hand side of eq. (2.77) picks up only a factor $(\frac{df(0)}{dz})^{\Delta_{l_j}} (\frac{d\bar{f}(0)}{d\bar{z}})^{\tilde{\Delta}_{l_j}}$ under the change $z \mapsto f(z)$ of the jth local parameter of the surface $\hat{\Sigma}$.

As we see, the relation between the primary fields and the HW vectors $\phi_l \mapsto \varphi_l(0)\Omega$ discovered earlier is one to one (if we include among the primary fields the trivial "identity" field whose insertions have no effect in correlation functions and which corresponds to the HW vector Ω).

Pants More generally, it is possible to associate to each vector $\mathcal{X}$ in the Hilbert space of states $\mathcal{H}$, in the domain of $q_1^{-L_0} \bar{q}_1^{-\tilde{L}_0}$, for some q_1 with $|q_1| < 1$, an

operator-valued field $\varphi(\mathcal{X};w,\bar{w})$ defined for $0<|w|<1$ in the following way. For $0<|q|<|w|-|q_1|<1-2|q_1|$, consider the Riemann surface

$$P_{q,q_1,w}=\{|q|\le|z|\le 1,\ |z-w|\ge|q_1|\}$$

with the "out" boundary component $\{|z|=1\}$ parameterized by $z\mapsto z$ and the "in" ones by $z\mapsto qz$ and by $z\mapsto w+q_1z$. Such a surface is often called "pants" for obvious reasons. The operator $\varphi(\mathcal{X};w,\bar{w})$ will be essentially defined as the amplitude of the pants. More exactly, we shall put

$$\varphi(\mathcal{X};w,\bar{w})\mathcal{Y}\ =\frac{1}{Z_\gamma}A_{P_{q,q_1,w},\gamma}(q^{-L_0}\bar{q}^{-\tilde{L}_0}\mathcal{Y}\otimes {q_1}^{-L_0}\bar{q_1}^{-\tilde{L}_0}\mathcal{X})\ .$$

Note that $\varphi(\mathcal{X};w,\bar{w})\mathcal{Y}$ is independent of q_1 and of q as long as $\mathcal{X}$ is in the domain of ${q_1}^{-L_0}\bar{q_1}^{-\tilde{L}_0}$ and $\mathcal{Y}$ in the domain of $q^{-L_0}\bar{q}^{-\tilde{L}_0}$.

Problem 7. (a). Show that $\varphi(\mathcal{X};0)\Omega\equiv\lim_{w\to 0}\ \varphi(\mathcal{X};w,\bar{w})\Omega=\mathcal{X}$.
(b). Show that for a HW vector $\mathcal{X}_l$, $\varphi(\mathcal{X}_l;w,\bar{w})$ coincides with the operator $\varphi_l(w,\bar{w})$ assigned to the primary field ϕ_l corresponding to $\mathcal{X}_l$.
(c). Show that $\varphi(L_{-2}\Omega;w,\bar{w})=\mathcal{T}(w)$ and that $\varphi(\tilde{L}_{-2}\Omega;w,\bar{w})=\bar{\mathcal{T}}(\bar{w})$.
The operators $\varphi(\mathcal{X};w,\bar{w})$ satisfy an important identity

$$\varphi(\mathcal{Y};z,\bar{z})\varphi(\mathcal{X};w,\bar{w})\ =\ \varphi(\varphi(\mathcal{Y};z-w,\bar{z}-\bar{w})\mathcal{X};w,\bar{w}) \tag{2.78}$$

holding for $0<|w|<|z|$, $0<|z-w|<1$. Eq. (2.78) follows from the two ways that one may obtain the disc with three holes by gluing two discs with two holes.

Problem 8. Prove eq. (2.78) using the gluing property (**vi**) of the operator amplitudes and treating with care the normalizing factors Z_γ^{-1}.

The relation (2.78) may be viewed as a global form of the OPE's, the local forms following from it by expanding the vector $\varphi(\mathcal{Y};z-w,\bar{z}-\bar{w})\mathcal{X}$ into terms homogeneous in $z-w$ and $\bar{z}-\bar{w}$.

Since any Riemann surface can be built from discs, annuli and pants the general amplitudes $A_{\Sigma,\gamma}$ may be expressed by the Virasoro generators and operators $\varphi(\mathcal{X};w,\bar{w})$. This permits us to formulate the basic mathematical structure of CFT in an even more economical (and more algebraic) way than through the amplitudes $A_{\Sigma,h}$ with the properties (**i**) to (**vii**), getting rid of the Riemann surface burden. Instead, one obtains a set of axioms for the action of the Vir $\times$ Vir algebra and of the **vertex operators** $\varphi(\mathcal{X};w,\bar{w})$ in the Hilbert space $\mathcal{H}$, with the relation (2.78), the main consistency condition, playing a prominent role. This was, essentially, the idea which, in the holomorphic version of CFT, has led to the concept of a **vertex operator algebra** developed by Frenkel-Lepowsky-Meurman and by Borcherds. The latter, by studying the algebras arising in the context of toroidal compactifications (on Minkowki targets) was led to the concept of generalized Kac-Moody or (Borcherds) algebras which promise to play an important role in physics and mathematics.

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In these volumes, conformal field theory plays a large role in [II-**Strings**]. See §2.1 in particular.

LECTURE 3
Sigma Models

§3.1. 1PI effective action and large deviations

It will be useful to describe another relation between a field theoretic and a probabilistic concept. Consider a positive measure

$$d\mu(\xi) = \mathrm{e}^{-S(\xi)} D\xi.$$

on a (finite dimensional) Euclidean space $\mathbf{E}$. We shall assume that $S(\xi)$ grows faster than linearly at infinity and, for convenience, that the measure is normalized. The characteristic functional of $d\mu$

$$\mathrm{e}^{W(J)} = \int \mathrm{e}^{\langle \xi, J\rangle} d\mu(\xi)$$

is then an analytic function of J in the complexified dual of $\mathbf{E}$. Let $N\zeta = \sum_1^N \xi_j$ be the sum of N independent random variables equally distributed with measure $d\mu$. The probability distribution of $N\zeta$ is

$$\begin{aligned} P_N(\zeta) &= \int \delta(N\zeta - \sum_{j=1}^N \xi_j) \prod_{j=1}^N d\mu(\xi_j) \\ &= \int DJ \int \mathrm{e}^{-\langle N\zeta - \sum \xi_j, J\rangle} \prod_{j=1}^N d\mu(\xi_j) = \int \mathrm{e}^{-N\langle \zeta, J\rangle + NW(J)} DJ. \end{aligned} \tag{3.1}$$

where the J-integration is over an imaginary section of $\mathbf{E}_{\mathbf{C}}^*$ with DJ denoting the properly normalized Lebesgue measure on it[1]. We shall be interested in the behavior of P_N for large N. It is not difficult to see that

$$P_N(\zeta) = \mathrm{e}^{N \inf_{J\in\mathbf{E}^*}[-\langle \zeta, J\rangle + W(J)] + o(N)} \equiv \mathrm{e}^{-N\Gamma(\zeta) + o(N)}. \tag{3.2}$$

In probability theory,

$$\Gamma(\zeta) = \sup_{J\in\mathbf{E}^*} [\langle \zeta, J\rangle - W(J)] = [\langle \zeta, J\rangle - W(J)]\Big|_{\zeta = W'(J)} \tag{3.3}$$

(W' denotes the derivative of W) is called the "large deviation (rate) function". It is the Legendre transform of $W(J)$ which is a strictly convex function on $\mathbf{E}^*$. It controls the regime where $\sum \xi_j = \mathcal{O}(N)$, as opposed to the central limit theorem

[1] another reading of eq. (3.1) says that the N-fold convolution becomes the Nth power in the Fourier language

which probes $\sum(\xi_j - \langle\xi\rangle) = \mathcal{O}(N^{1/2})$, where $\langle\xi\rangle$ is the mean value of ξ_j. Since $\Gamma'(\zeta) = J$, the minimum of Γ occurs at $\zeta = W'(0) = \langle\xi\rangle$. The central limit theorem sees only the second derivative of Γ at $\langle\xi\rangle$:

$$\lim_{N\to\infty} P_N(\langle\xi\rangle + N^{-1/2}\psi) \sim \mathrm{e}^{-\frac{1}{2}\Gamma''(\langle\xi\rangle)(\psi^2)}.$$

W may be recovered from Γ by the inverse Legendre transform:

$$W(J) = \sup_{\zeta\in\mathbf{E}}[\langle\zeta, J\rangle - \Gamma(\zeta)] = [\langle\zeta, J\rangle - \Gamma(\zeta)]\Big|_{J=\Gamma'(\zeta)}. \tag{3.4}$$

We may compute $W(J)$ and $\Gamma(\zeta)$ as formal power series introducing a coefficient $\hbar$ (the Planck constant), Taylor expanding $S(\xi)$ and separating the quadratic contribution to it:

$$\mathrm{e}^{W(J)} = \int \mathrm{e}^{\frac{1}{\hbar}[\langle\xi,J\rangle - S(\xi)]} D\xi\Big|_{\hbar=1} = \Big[\mathrm{e}^{-\frac{1}{\hbar}S(0)} \int \mathrm{e}^{\frac{1}{\hbar}\langle\xi, J - S'(0)\rangle - \frac{1}{3!\hbar}S'''(0)(\xi^3) - \ldots\ldots}$$

$$d\mu_{\hbar S''(0)^{-1}}(\xi) \int \mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)} D\xi\Big]\Big|_{\hbar=1} \tag{3.5}$$

where the Gaussian measure

$$d\mu_{\hbar S''(0)^{-1}}(\xi) = \frac{\mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)} D\xi}{\int \mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)} D\xi}. \tag{3.6}$$

Expanding the exponential under the $d\mu$ integral into the power series and performing the Gaussian integration, we obtain an expansion in powers of $\hbar$ which, as discussed in [I-**Axiomatics**], gives upon exponentiation the relation

$$\mathrm{e}^{W(J)} = \Big[\mathrm{e}^{-\frac{1}{\hbar}S(0) + \sum\limits_{\substack{\text{vacuum}\\ \text{graphs } G}} \hbar^{\#\{\text{loops of } G\}-1} I_G(J,S)} \int \mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)} D\xi\Big]\Big|_{\hbar=1} \tag{3.7}$$

where, by definition, the graphs G are connected graphs[2] made up of 1-leg vertices J or $-S'(0)$, 3-leg vertices $-S'''(0)$, 4-leg vertices $-S^{(4)}(0)$ etc., with propagators $S''(0)^{-1}$ on the internal lines[3] and no propagators on the external lines. The vacuum graphs are the ones without external lines. The amplitudes $I_G(J,S)$ are associated to the graphs in a natural way, with the symmetry factors of the graphs included. If S is a polynomial and $S'(0) = 0$, then there is only a finite number of graphs with a given number of J-vertices and a given number of loops and we may view $W(J)$ as a formal series in J and in the number of loops. More exactly, comparing the left and the right hand sides of eq. (3.7), we infer that

$$W(J) = -S(0) + \sum_{\substack{\text{vacuum}\\ \text{graphs } G}} I_G(J,S) + \ln\left(\int \mathrm{e}^{-\frac{1}{2}S''(0)(\xi^2)} D\xi\right).$$

As discussed in [I-**Axiomatics**], by cutting all the lines of the graphs whose removal makes the graph disconnected, we obtain the second representation for $W(J)$:

$$W(J) = -\Gamma(0) + \sum_{\substack{\text{vacuum}\\ \text{trees } T}} I_T(J,\Gamma)$$

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[2] recall that the exponential of the sum over connected graphs is the sum over connected and disconnected graphs

[3] we count lines ending at the 1-leg vertices as internal

where the "1PI effective action" $\Gamma(\zeta)$ is defined by its formal Taylor series

$$\Gamma(0) = S(0) - \sum_{\substack{\text{1PI vacuum}\\ \text{graphs } G}} I_G(S) - \ln\Big(\int \mathrm{e}^{-\frac{1}{2}S''(0)(\xi^2)} D\xi\Big), \quad \Gamma'(0) = -\sum_{\substack{\text{1PI graphs}G\\ \text{with 1 external line}}} I_G(S),$$

(3.8)

$$\tfrac{1}{2}\Gamma''(0) = \tfrac{1}{2}S''(0) - \sum_{\substack{\text{1PI graphs } G\\ \text{with 2 external lines}}} I_G(S), \qquad \tfrac{1}{3!}\Gamma'''(0) = -\sum_{\substack{\text{1 PI graphs } G\\ \text{with 3 external lines}}} I_G(S),$$

.....

where "1PI" stands for (amputated) 1-particle irreducible graphs without J-vertices. Rewriting eq.(3.7) with S replaced by Γ, i.e. as an expansion for $\int \mathrm{e}^{\frac{1}{\hbar}[\langle\zeta,J\rangle - \Gamma(\zeta)]} D\zeta$, but keeping only the leading terms at $\hbar$ small, we obtain finally the equality

$$\sup_{\zeta\in\mathbf{E}}[\langle\zeta, J\rangle - \Gamma(\zeta)] = -\Gamma(0) + \sum_{\substack{\text{vacuum}\\ \text{trees } T}} I_T(J,\Gamma) \;=\; W(J).$$

Comparing this to eq.(3.4) we see that eqs.(3.8) provide a perturbative interpretation of the large deviation function Γ.

§3.2. Geometric sigma models

We have already discussed a simple way to write down a conformally invariant action for maps $\phi : \Sigma \to M$, where (Σ, γ) is a Riemann surface and (M, g) a Riemannian manifold. The functional

$$S_g(\phi) = \tfrac{1}{4\pi}\|d\phi\|^2_{L^2} = \tfrac{i}{2\pi}\int_\Sigma g_{ij}(\phi)\partial\phi^i \wedge \bar\partial\phi^j \tag{3.9}$$

(summation convention!) depends only on the conformal class of γ. We could add to $S_g(\phi)$ also a "topological" term

$$S_{\mathrm{top}}(\phi) = \frac{i}{4\pi}\int_\Sigma \phi^*\omega = \tfrac{i}{2\pi}\int_\Sigma b_{ij}(\phi)\partial\phi^i \wedge \bar\partial\phi^j \tag{3.10}$$

involving a 2-form $\omega = b_{ij}(\phi)d\phi^i \wedge d\phi^j$ on M which does not depend on γ but only on the orientation of Σ. Renormalizability forces the addition of two more terms to the action which break classical conformal invariance:

$$S_{\mathrm{tach}}(\phi) = \frac{1}{4\pi}\int_\Sigma u\circ\phi\; dv \qquad \text{and} \qquad S_{\mathrm{dil}} = \frac{1}{4\pi}\int_\Sigma w\circ\phi\; r dv \tag{3.11}$$

where u, w are functions on M called, respectively, the tachyon and the dilaton potentials[4], dv stands for the volume measure and r for the scalar curvature of Σ.

One may also consider a supersymmetric version of the model with the action

$$S_g^{\mathrm{SUSY}}(\Phi) \;=\; \tfrac{1}{2\pi i}\int g_{ij}(\Phi)D\Phi\tilde D\Phi\; dz\wedge d\bar z\wedge d\theta\wedge d\tilde\theta \tag{3.12}$$

where the superfield

$$\Phi = \phi + \theta\psi + \tilde\theta\tilde\psi + \theta\tilde\theta F.$$

[4]the names come from the string theory context

and $D = \partial_\theta + \theta\partial_z$, $\tilde{D} = \partial_{\tilde{\theta}} + \tilde{\theta}\partial_{\bar{z}}$. In components, after elimination of the auxiliary field F through its equation of motion, one obtains

$$S_g^{\mathrm{SUSY}}(\phi,\psi,\tilde{\psi}) = \frac{i}{2\pi}\int \Big(g_{ij}(\phi)\partial_z\phi^i\partial_{\bar{z}}\phi^j - g_{ij}(\phi)\left(\psi^i\nabla_z\psi^j + \tilde{\psi}^i\nabla_{\bar{z}}\tilde{\psi}^j\right) - \tfrac{1}{2}R_{ijkl}(\phi)\psi^i\psi^j\tilde{\psi}^k\tilde{\psi}^l\Big)dz\wedge d\bar{z} \tag{3.13}$$

where $\nabla_z\psi^j = \partial_z\psi^i + \Gamma^j_{kl}\partial_z\phi^k\psi^l$ with $\Gamma^j_{kl} = \{{}^{j}_{kl}\}$ standing for the Levi-Civita connection symbols and similarly for $\nabla_{\bar{z}}\tilde{\psi}^i$ and where R_{ijkl} denotes the curvature tensor of M. Addition of the 2-form $\omega = b_{ij}(\phi)d\phi^i\wedge d\phi^j$ term corresponds to the change $g_{ij}\mapsto g_{ij}+b_{ij}$ in eq. (3.12). In the component formula (3.13) it results in the same replacement of g_{ij} and, additionally, in the replacement of the Levi-Civita symbols in $\nabla_z\psi^j$ ($\nabla_{\bar{z}}\tilde{\psi}^j$) by symbols of a metric connection with torsion $\{{}^{j}_{kl}\}\pm\frac{3}{2}g^{jm}H_{klm}$, respectively, where H_{klm} is the antisymmetric tensor representing $d\omega$. The curvature R_{ijkl} in eq. (3.13) becomes that of the connection with the plus sign[5].

The two-dimensional field theory with action (3.9) is usually called a **sigma model**. The stationary points of $S(\phi)$ are harmonic maps from Σ to M and correspond to the classical solutions. Can one quantize sigma models by giving sense to functional integrals

$$\int_{\mathrm{Map}(\Sigma,M)} F(\phi)\,\mathrm{e}^{-S(\phi)}D\phi \tag{3.14}$$

with $S = S_g + S_{\mathrm{top}} + S_{\mathrm{tach}} + S_{\mathrm{dil}}$ where for $F(\phi)$ one may for example take $\prod_j u_j(\phi(x_j))$ for some functions u_j on M? We have already seen that this was easily doable for M a torus with a constant metric and a constant 2-form ω, with vanishing (or constant) tachyon and dilaton potentials. The corresponding functional integral was essentially Gaussian and the resulting theory was a little decoration of the free massless field. Here we would like to examine the case with an arbitrary topology and geometry of the target by treating the functional integrals of the type (3.14) in perturbation theory and also, possibly, going beyond the purely perturbative considerations employing powerful methods of the renormalization group approach to quantum field theories.

§3.3. Regularization and renormalization

We may anticipate problems with the definition of the functional integrals (3.14) even in a perturbative approach. We shall attempt to remove these problems by using the freedom to change the parameters of the theory, namely the metric on M and the tachyon and dilaton potentials (for the sake of simplicity, we shall discard the topological term in the action). The strategy to make sense of functional integrals of type (3.14) will then be as follows:

1. (**regularization**) we modify the theory by introducing a (short distance) cutoff Λ into it to make the functional integral exist;

2. (**renormalization**) we try to choose the metric g and the tachyon and dilaton potentials entering the action in a Λ-dependent way so that the cutoff

[5]it is equal to R_{klij} for the minus sign connection

versions of integrals (possibly after further multiplication by a Λ-dependent factor) converge to a non-trivial limit.

There are many ways to introduce a short distance (ultraviolet) cutoff into the theory. To simplify the problem further let us assume that Σ is the periodic box $[-0, L]^2$ (that will do away with the contribution of the dilaton potential). One possibility to introduce the UV cutoff is to consider the lattice version of the sigma model. Let $\Sigma_\Lambda \subset [0, L]^2$ be composed of points with coordinates in $\frac{1}{\Lambda}\mathbf{Z}$ where ΛL is a power of 2. The lattice version of $\phi : [0, L]^2 \to M$ is the map $\phi : \Sigma_\Lambda \to M$ and for the cutoff action we may take

$$S^\Lambda_{g,u}(\phi) = \frac{1}{8\pi} \sum_{\substack{x,y\in\Sigma_\Lambda \\ |x-y|=\Lambda^{-1}}} d_g^2(\phi(x), \phi(y)) + \frac{1}{4\pi} \sum_{x\in\Sigma_\Lambda} \Lambda^{-2} u(\phi(x))$$

where d_g stands for the metric distance on M. If ϕ is the restriction of a fixed smooth (periodic) M-valued map on $[0, L]^2$ and u is continuous then in the limit $\Lambda \to \infty$ we recover the value of the original action $S_g + S_{\text{tach}} \equiv S_{g,u}$. The cutoff version of the normalized integral (3.14) with $F(\phi) = \prod u_j(\phi(x_j))$ now becomes

$$\frac{\int \prod_{j=1}^{n} u_j(\phi(x_j))\ \mathrm{e}^{-S^\Lambda_{g,u}(\phi)} D_g\phi}{\int \mathrm{e}^{-S^\Lambda_{g,u}(\phi)} D_g\phi} \tag{3.15}$$

where $D_g\phi = \prod_{x\in\Sigma_\Lambda} dv_g(\phi(x))$ with dv_g denoting the metric volume measure on M. The integral is finite, e.g., for compact M and u_j, say, continuous. The lattice sigma models for $M = S^{N-1}$ with a metric proportional to that of the unit sphere in $\mathbf{R}^N$ are essentially the well known "spin" models in classical (as opposed to quantum) statistical mechanics ($N = 1$ corresponds to the Ising model, $N = 2$ to a version of the XY model, $N = 3$ to a slightly modified classical Heisenberg model; u proportional to a coordinate in $\mathbf{R}^N$ describes the coupling to the magnetic field).

We would like to study whether, after renormalization, the cutoff may be removed in the correlation functions[6] (3.15). More precisely we would like to show that the limits

$$\lim_{\Lambda\to\infty} \frac{\int \prod_{j=1}^{n} Z(\Lambda) u_j(\phi(x_j))\ \mathrm{e}^{-S^\Lambda_{g(\Lambda),u(\Lambda)}(\phi)}\ D_{g(\Lambda)}\phi}{\int\ \mathrm{e}^{-S^\Lambda_{g(\Lambda),u(\Lambda)}(\phi)}\ D_{g(\Lambda)}\phi} \tag{3.16}$$

exist for a cutoff-dependent linear map[7] $Z(\Lambda)$ on the space of functions on M and for cutoff-dependent choices of the metric $g(\Lambda)$ and of the tachyon potential $u(\Lambda)$ on M. We would also like to parameterize the possible limits (3.16) defining the correlation functions of the quantum two-dimensional sigma models.

§3.4. Renormalization group effective actions

One could study the questions raised above first by perturbative methods applied directly to the lattice correlation functions (3.15). It is important, however, to set

[6] from the point of view of statistical mechanics which reformulates the problem in terms of the system with a fixed lattice spacing, this is a question about the large distance behavior of correlation functions

[7] recall that even the free field case required a multiplicative renormalization of the correlation functions of exponents of the field ϕ

the perturbative scheme in a way that does not destroy the geometric features of the model (i.e. in a way covariant under diffeomorphisms of M). One way to assure this is to study, instead of correlation functions, objects known under the name of renormalization group effective actions.

Fix μ such that Λ/μ is a power of 2 and for $y \in \Sigma_\mu$ denote by $B(y)$ the set of $x \in \Sigma_\Lambda$ in the square $y+[0, \frac{\mu}{\Lambda}L[^2$, Call a point $\phi \in M$ a barycenter of a set of points $\phi_j \in M, j = 1, \dots, N$, if ϕ extremizes $\sum_j d_g^2(\phi, \phi_j)$. Clearly, if M is a Euclidean space then $\phi = \frac{1}{N}\sum_j \phi_j$. Suppose that we fix a map $\phi : \Sigma_\mu \to M$ and compute the integral

$$(3.17)\quad \mathrm{e}^{-S^\mu_{eff}(\phi)} D\phi = \int \prod_{y\in\Sigma_\mu} \delta(\tfrac{1}{2}\nabla_{\phi(y)} \sum_{x\in B(y)} d_g^2(\phi(y), \varphi(x)))\mathrm{e}^{-S^\Lambda_{g,u}(\varphi)}\, D_g\varphi.$$

The right hand side is naturally a measure on M^{Σ_μ}. It essentially computes the probability distribution of the barycenters $\phi(y)$ of "spins" $\varphi(x)$ with x in blocks $B(y)$[8]. $S^\mu_{\mathrm{eff}}(\phi)$, the logarithm of the the density of the right hand side w.r.t. to some reference measure $D\phi$ on M^{Σ_μ}, is called the ("block spin") **renormalization group** (RG) **effective action** on scale μ.

The renormalizability problem may now be reformulated as the question about existence of the $\Lambda \to \infty$ limit of S^μ_{eff}, more exactly, of the normalized measure

$$(3.18)\qquad d\nu^\mu_{\mathrm{eff}}(\phi) = \frac{\mathrm{e}^{-S^\mu_{\mathrm{eff}}(\phi)} D\phi}{\int \mathrm{e}^{-S^\mu_{\mathrm{eff}}(\phi)} D\phi}$$

on M^{Σ_μ}, if we choose "bare" g and u in a Λ-dependent way and keep μ fixed. With the modification of the definition of S^μ_{eff} described in the footnote, one may show that the two formulations of the renormalizability problem are essentially equivalent.

The limiting measures $d\nu^\mu_{\mathrm{eff}}$ may be viewed as describing the $\Lambda = \infty$ theory averaged over variations of the fields on distance scales $\leq \mu^{-1}$. Pictorially, they describe the system viewed from far away when we do not distinguish details of length $\lesssim \frac{1}{\mu}$. An important observation at the core of the RG analysis is that this averaging may be done inductively by first eliminating the variations on the smallest scales, then on the larger ones, and so on until the scale μ is reached[9]. In the infinite volume ($L = \infty$), the process may be viewed as a repeated application of a map on a space of unit lattice actions. If under the iterations the effective actions are driven to a simple attractor (like an unstable manifold of a fixed point) then the renormalization consists of choosing the initial "bare" actions so that the $\Lambda = \infty$ effective actions S^μ_{eff} end up on the attractor. In the vicinity of a fixed point this would be possible if the family of the bare actions (parameterized by bare couplings) crosses the stable manifold transversally. The renormalized couplings then parameterize then the unstable manifold (a drawing would be helpful here). This dynamical system view of renormalization developed by K. G. Wilson is extremely important and will hopefully be explained in much more detail in future lectures.

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[8] this would hold if the barycenters were unique and with the normalizing factor $\frac{1}{\int \delta(\frac{1}{2}\nabla_{\phi(y)} \sum_{x\in B(y)} d_g^2(\phi(y),\varphi(x)))D_g\phi}$ under the integral on the right hand side of eq. (3.17)

[9] this is not exactly the case for our definition of S^μ_{eff} but ignore this for a moment

We have suppressed in the notation the dependence on the size L of the box. If the choice of $g(\Lambda)$ and $u(\Lambda)$ involved in the $\Lambda \to \infty$ limit can be done in an L independent way, we automatically obtain a family of measures parameterized by μ and L. The measures with the product $L\mu$ fixed (to a power of 2) are related by the rescaling of space-time distances[10]. If the infinite volume limit $L \to \infty$ of the theory exists, μ becomes a continuous parameter. Suppose that the effective actions S^μ_{eff} of possible continuum limits (i.e. the attractor of the RG map in the dynamical system view) may be parametrized by (dimensionless) "renormalized" metrics g and tachyon potentials u. You should think that S^μ_{eff} is equal to $S^\mu_{g,\mu^2 u}$ plus less important (higher dimension) terms separated by a precise rule. We would then say that the theory is renormalizable by a metric and a tachyon potential renormalization. This is the scenario realized in perturbation theory; see below. In such a situation the $\Lambda, L = \infty$ theories are characterized by the "running" metric $\mu \mapsto g(\mu)$ and tachyon potential $\mu \mapsto u(\mu)$ describing S^μ_{eff} on different scales μ in the passive view of the scale-dependence of the renormalized theory. In the active view, the μ-dependence of g and u is generated by the action of rescalings of distances on the limiting theory. The infinitesimal scale transformations generate a vector field $\beta\partial_g + \gamma\partial_u$ in the space of (g, u) defined by

$$\beta(g,u) = \mu\frac{\partial}{\partial\mu}g, \qquad \gamma(g,u) = \mu\frac{\partial}{\partial\mu}u. \tag{3.19}$$

$\beta(g,u)$ and $\gamma(g,u)$ are called in the physicists' jargon the RG "**beta**" and "**gamma functions**". In the dynamical system language, $\beta\partial_g + \gamma\partial_u$ is a vector field on the attractor of the RG map and it extends the map to a flow. The importance of the RG functions lies in the fact that, computed in the perturbation expansion, they allow us to go beyond it, providing for example a consistency check on the latter: by solving the RG eqs. (3.19) with β and γ given by a few perturbative terms we may check whether the trajectories $\mu \mapsto (g(\mu), u(\mu))$ stay or are driven out for large μ (that is, at short distances) from the region of the (g,u)-space where the perturbative calculation may be trusted. It is clear that the zeros of the (β,γ) vector field should play an important role. They correspond to scale invariant (and hence conformallly invariant) field theories and, in the dynamical system picture, to fixed points of the RG map (since they already lie on the attractor).

How to generate the perturbation expansion for the RG effective actions $\mathrm{e}^{-S^\mu_{\text{eff}}}$? A helpful observation is that the delta-function in the definition (3.17) can be rewritten in simple terms if we use the exponential map $\mathrm{e} : T_\phi M \to M$.

Problem 1 (geometric). Show that for vectors ξ_j in a small ball in $T_\phi M$,

$$\tfrac{1}{2}\nabla_\phi\Big(\sum_{j=1}^N d_g^2(\phi, \mathrm{e}^{\xi_j}\phi) = \sum_{j=1}^N \xi_j.$$

It follows that ϕ is a barycenter of the set of points $\{\mathrm{e}^{\xi_j}\phi\}$ iff $\sum \xi_j = 0$.

Substituting in eq. (3.17) $\varphi(x) = \mathrm{e}^{\xi(x)}\phi(y)$ for $\xi(x) \in T_{\phi(y)}M$ if $x \in B(y)$, or, in a shorthand notation, $\varphi = \mathrm{e}^\xi\tilde\phi$ where $\tilde\phi(x) = \phi(y)$ for $x \in B(y)$, we obtain[11]

$$\mathrm{e}^{-S^\mu_{\text{eff}}(\phi)} = \Big[\int \prod_{y\in\Sigma_\mu} \delta\Big(\sum_{x\in B(y)} \xi(x)\Big)\, \mathrm{e}^{-\frac{1}{\hbar}S^\Lambda_{g,u}(\mathrm{e}^\xi\tilde\phi)}\, D_g\mathrm{e}^\xi\tilde\phi\Big]\Big|_{\hbar=1}. \tag{3.20}$$

[10] i. e. of the metric on Σ

[11] the fact that the exponential parameterization may work only locally does not impede the perturbative analysis

Note that the lattice field ξ takes values in a vector space $\{\xi|\xi(x) \in T_{\phi(y)}M, \sum_{x\in B(y)} \xi(x) = 0\}$. The loop expansion for $S^\mu_{\rm eff}$ may in fact be generated in the standard way by expanding in powers of ξ on the right hand side of eq. (3.20) all terms except for the quadratic contribution to $S_{g,u}$ which is used to produce a Gaussian measure. At each loop order the result will be invariant under the simultaneous action of diffeomorphisms of M on ϕ, g and u. When $\Lambda \to \infty$, divergences will appear in the perturbative expressions. The perturbative renormalizability of the theory may be studied by replacing the "bare" g and u on the right hand side of eq. (3.20) by

(3.21)
$$g(\Lambda) = g + \sum_{n=1}^\infty \hbar^n \delta g_n(g,u,\Lambda/\mu), \quad u(\Lambda) = \mu^2 \left(u + \sum_{n=1}^\infty \hbar^n \delta u_n(g,u,\Lambda/\mu) \right).$$

We may attempt to fix the above series by choosing some way to extract the renormalized metric g and the renormalized potential u from the effective actions $S^\mu_{\rm eff}$. We would then like to show that the above substitution cancels the $\Lambda \to \infty$ divergences in each loop order of $S^\mu_{\rm eff}$ resulting in a family of perturbative RG effective actions parameterized by the "running" metric $g(\mu)$ and tachyon potential $u(\mu)$. Differentiation of the series (3.21) over $\ln\mu$ with $g(\Lambda), u(\Lambda)$ fixed would then produce in the $\Lambda \to \infty$ limit the loop expansion for the beta and gamma RG functions.

§3.5. Background field effective action

In practice, the lattice perturbative calculations are prohibitively complicated. It would be easier to work with continuum regularization and renormalization, which allow us to calculate the diagram amplitudes by momentum space integrals and to make use of rotational invariance. We have seen in Witten's lectures [**I-Perturbative QFT**] on perturbative renormalization of the scalar field theories with the ϕ^3 or ϕ^4 interactions that it was convenient to analyze directly the "1PI effective action" Γ given by the Legendre transform of the free energy functional W. The latter was defined as the logarithm of the integral of type of (3.14), with $F(\phi) = {\rm e}^{\langle\phi,J\rangle}$. The definition of both W and Γ, however, as well as their perturbative analysis, used heavily the linear structure in the space of maps from the space-time to the target M, inherited from the linear structure of M. Such structure is missing if M is a general manifold. It is possible, nevertheless, to introduce for sigma models an effective action mimicking the construction of the large deviation function (see eq. (3.2)) and somewhat similar in spirit to the RG effective actions for the lattice version of sigma models discussed in the previous section. Instead of fixing the block barycenters in a single lattice theory, we shall take N independent copies of continuum theories with fields ϕ_j and shall fix for each x the barycenters $\phi(x)$ of $\phi_j(x)$ defining the functional

$$P_N(\phi) = \int \prod_x \delta(\tfrac{1}{2}\nabla_{\phi(x)} \sum_{j=1}^N d_g^2(\phi(x),\phi_j(x))) \prod_j {\rm e}^{-S_{g,u}(\phi_j)}\, D_g\phi_j \tag{3.22}$$

by a formal functional integral. Note that the right hand side reduces to a well defined integral for a lattice version of the theory. For a map $\phi : \Sigma \to M$ and for a section ξ of the pullback ϕ^*TM of the bundle tangent to M, denote by ${\rm e}^\xi\phi$ the

map from Σ to M whose value at point x is obtained by applying the exponential map to $\xi(x) \in T_{\phi(x)}M$. Reparameterizing in eq. (3.22) $\phi_j = \mathrm{e}^{\xi_j}\phi$, we obtain

$$P_N(\phi) = \int \delta\Big(\sum_{j=1}^{N} \xi_j\Big) \prod_j \mathrm{e}^{-S(\mathrm{e}^{\xi_j}\phi)} D(\mathrm{e}^{\xi_j}\phi). \tag{3.23}$$

We may try to extract the "**background field effective action**" $\Gamma_b(\phi)$ from the leading contribution to P_N at large N:

$$P_N(\phi) = \mathrm{e}^{N\Gamma_b(\phi)+o(N)}. \tag{3.24}$$

It should be clear that $\Gamma_b(\phi)$ coincides then with the effective action $\Gamma_\phi(\zeta = 0)$ of the ξ-field theory (depending on ϕ as a parameter) corresponding to the functional integral

$$\int -\mathrm{e}^{-S(\mathrm{e}^{\xi}\phi)}\, D(\mathrm{e}^{\xi}\phi). \tag{3.25}$$

The fields ξ take values in a vector space of sections of ϕ^*TM, so that the perturbative treatment of the ξ-theory is more standard. Note that only the geometric structure on M was used in the formal definition of $\Gamma_b(\phi)$.

We shall reformulate the renormalizability problem (3.16) for the second time as the question about the existence of the $\Lambda \to \infty$ limit of the regularized version of the background-field effective action $\Gamma_b^\Lambda(\phi)$ for a cutoff-dependent theory with the action $S_{g(\Lambda),u(\Lambda)}(\phi)$. We could regularize the functional integral (3.23) by putting the fields ξ_j on a lattice with spacing $\frac{1}{\Lambda}$ while keeping ϕ as a continuum field. This would not produce a big computational gain in comparison to the perturbative calculation of the RG effective actions. It is possible, however, to regularize the loop expansion of the background field effective action just by introducing an ultraviolet cutoff in the momentum space integrals for the 1PI vacuum amplitudes in the ξ-field theory whose fields form a vector space of sections of ϕ^*TM. $\Gamma_{b,\Lambda}(\phi)$ obtained in this way will be covariant under the diffeomorphisms of M in each order of the loop expansion. The perturbative renormalization will consist of choosing the "bare" parameters of the theory in a cutoff-dependent way as in eqs. (3.21) and such that the $\Lambda \to \infty$ limit of $\Gamma_b^\Lambda(\phi)$ exists order by order in the loop expansion. The perturbative limits will be parameterized by the "running" metric $g(\mu)$ and potential $u(\mu)$ on M, with the change of μ induced by rescaling of distances on Σ.

§3.6. Dimensional regularization

We shall prove the perturbative renormalizability of the background field effective action in the $2D$ sigma model only in the leading order of the loop expansion, concentrating instead on the discussion of the renormalization group aspects of the 1-loop result. To avoid calculational (but not conceptual) difficulties, we shall work in the flat Euclidean space-time $\Sigma = \mathbf{E}^2$. We shall also use a specific scheme for regularization of divergent diagrams: the **dimensional regularization**, and a particular way to renormalize the theory (i e. to choose $g(\Lambda)$ and $u(\Lambda)$): the **minimal subtraction**. Briefly, the idea is to

(i) regularize the momentum space integrals by rewriting them as integrals into which the space-time dimension D enters as an analytic (complex) parameter, then

(ii) to calculate the integral for the values of D where it converges and, finally,

(iii) to analytically continue to the physical values of D, extracting the pole parts of the result at the physical dimension as the divergence to be removed by the renormalization.

In order to gain some practice let us compare how the simplest divergent diagram of the 4-dimensional ϕ^3 theory –O– is regularized and renormalized first in the more conventional momentum space regularization used in [I-**Perturbative QFT**] and then in the dimensional regularization - minimal subtraction scheme. The momentum space amplitude $\hat{I}(k)$ corresponding to the (amputated) graph was given by the integral

$$\hat{I}_D(k) = \frac{g^2}{4}\int \frac{đ\, q}{(q^2+m^2)((q-k)^2+m^2)} = \frac{g^2}{4}\int_0^1 d\alpha \int \frac{đ\, q}{(q^2+\alpha(1-\alpha)k^2+m^2)^2} \tag{3.26}$$

where k is the external momentum, q is that of the loop (both Euclidean) and $đ\, q \equiv \frac{d^D q}{(2\pi)^D}$. In space-time dimension $D=4$ the q-integral diverges logarithmically. It may be regularized by restricting the integration to $|q|\le\Lambda$.

$$\hat{I}_4^\Lambda(k) = \frac{g^2}{4}\int_0^1 d\alpha \int\limits_{|q|\le\Lambda} \frac{đ\, q}{(q^2+\alpha(1-\alpha)k^2+m^2)^2} = \frac{g^2}{32\pi^2}\ln\frac{\Lambda}{\mu} + \hat{K}_4^\Lambda(\tfrac{k^2}{\mu^2},\tfrac{m}{\mu})$$

where

$$\lim_{\Lambda\to\infty}\hat{K}_4^\Lambda(\tfrac{k^2}{\mu^2},\tfrac{m}{\mu}) = -\frac{g^2}{64\pi^2}\left(\int_0^1 d\alpha \ln[\alpha(1-\alpha)\tfrac{k^2}{\mu^2}+\tfrac{m^2}{\mu^2}]+1\right) \tag{3.27}$$

The renormalization idea is then to substitute

$$g(\Lambda) = \mu\lambda, \qquad m^2(\Lambda) = \mu^2\left(r + \sum_{n=1}^{\infty}\hbar^n \delta r_n(\lambda, r, \tfrac{\Lambda}{\mu})\right) \tag{3.28}$$

for the coupling constant and the mass squared in the initial action (recall that the theory does not need renormalization of g and only a 1-loop counterterm would do). The powers of μ make λ and r dimensionless. With the choice

$$\delta r_1 = \frac{\lambda^2}{16\pi^2}\ln\frac{\Lambda}{\mu}$$

the contribution to the 1-loop amplitude $I_4^\Lambda(k)$ diverging when $\Lambda\to\infty$ is canceled resulting in the renormalized value of the amplitude

$$\hat{I}^\Lambda_{4,\mathrm{ren}}(k) = -\frac{\mu^2\lambda^2}{64\pi^2}\left(\int_0^1 d\alpha \ln[\alpha(1-\alpha)\tfrac{k^2}{\mu^2}+r]+1\right)$$

The RG functions $\beta(\lambda,r) = \mu\frac{d}{d\mu}\lambda, \gamma_2(\lambda,r) = \mu\frac{d}{d\mu}r$ describing the scale-dependence of the renormalized couplings are obtained by differentiating eq. (3.28) with respect to $\ln\mu$ with $\Lambda, g(\Lambda)$ and $m^2(\Lambda)$ held fixed:

$$0 = \mu\lambda + \mu^2\frac{d\lambda}{d\mu},$$

$$0 = \mu\frac{d}{d\mu}[\mu^2 r + \hbar\frac{g^2}{16\pi^2}\ln\frac{\Lambda}{\mu} + \mathcal{O}(\hbar^2)] = 2\mu^2 r + \mu^3\frac{dr}{d\mu} - \hbar\mu^2\frac{\lambda^2}{16\pi^2} + \mathcal{O}(\hbar^2)$$

so that

$$\beta(\lambda,r)=-\lambda,\qquad \gamma_2(\lambda,r)=-2r+\hbar\tfrac{\lambda^2}{16\pi^2}+\mathcal{O}(\hbar^2). \tag{3.29}$$

Let us see how the same problem is treated in the dimensional regularization - minimal subtraction scheme. Using the relation $\int_0^\infty \mathrm{e}^{-a\sigma}\sigma d\sigma = a^{-2}$, we may rewrite the integral for $\hat{I}_D(k)$ in the form[12]

$$\hat{I}_D(k) = \tfrac{g^2}{4}\int_0^1 d\alpha \int_0^\infty d\sigma \int \sigma \mathrm{e}^{-[q^2+\alpha(1-\alpha)k^2+m^2]\sigma}\,\vec{d}\,q. \tag{3.30}$$

Performing the q-integral first, we obtain

$$\hat{I}_D(k) = \tfrac{g^2}{2^{D+2}\pi^{D/2}}\int_0^1 d\alpha \int_0^\infty d\sigma \sigma^{1-D/2}\mathrm{e}^{-[\alpha(1-\alpha)k^2+m^2]\sigma}.$$

The latter integral converges for any complex D with $\mathrm{Re}\, D<4$. It gives explicitly

$$\hat{I}_D(k) = \tfrac{g^2}{2^{D+2}\pi^{D/2}}\Gamma(2-\tfrac{D}{2})\int_0^1 d\alpha[\alpha(1-\alpha)k^2+m^2]^{\frac{D}{2}-2}.$$

The divergence in four dimensions manifests itself as a pole in the expression at $D=4$;

$$\hat{I}_D(k) = \tfrac{g^2}{32\pi^2}\tfrac{1}{4-D} - \tfrac{g^2}{64\pi^2}\left(\int_0^1 d\alpha \ln[\alpha(1-\alpha)k^2+m^2] + \ln 4\pi + C\right) + \mathcal{O}(4-D)$$

where $C=-\Gamma'(1)$ is the Euler constant. In the minimal subtraction renormalization scheme, the amplitude is renormalized by substituting for the original coupling and mass squared

$$g=\mu^{3-D/2}\lambda,\qquad m^2=\mu^2\left(r+\sum_{n=1}^{\infty}\hbar^n\delta r_n(\lambda,r)\right) \tag{3.31}$$

with the pure pole dependence of δr_n on the dimension:

$$\delta r_n=\sum_{j=1}^{k_n}\delta r_{n,j}(\lambda,r)\tfrac{1}{(4-D)^j}$$

chosen to cancel exactly the pole part of the dimensionally regularized amplitudes. In reality, only the 1-loop amplitude has a pole which is simple. With

$$\delta r_1 = \tfrac{\lambda^2}{16\pi^2(4-D)},$$

we obtain the renormalized amplitude

$$\hat{I}^{\Lambda}_{4,ren}(k) = -\tfrac{\mu^2\lambda^2}{64\pi^2}\left(\int_0^1 d\alpha \ln[\alpha(1-\alpha)\tfrac{k^2}{\mu^2}+r] + \ln(4\pi) + C\right)$$

(note how μ has entered under the logarithm). The difference between the two renormalizations may be absorbed into a finite redefinition of the renormalized parameters λ, r; see Problem 2 below.

Now the RG functions $\beta(\lambda,r),\gamma_2(\lambda,r)$ are obtained by differentiating eqs. (3.31) with respect to $\ln\mu$ while keeping g and m^2 fixed:

$$0 = (3-\tfrac{D}{2})\mu^{3-D/2}\lambda + \mu^{4-D/2}\tfrac{d\lambda}{d\mu},$$

[12] for those who do not remember Feynman's famous formula (I don't), we could have used twice the identity $\int_0^\infty \mathrm{e}^{-a_i\sigma_i}d\sigma_i = a_i^{-1}$ in the original expression for $\hat{I}_D(k)$, changing then the variables to (α,σ) where $\sigma_1=\alpha\sigma$, $\sigma_2=(1-\alpha)\sigma$

$$\begin{aligned} 0 &= \mu\tfrac{d}{d\mu}[\mu^2 r + \hbar\tfrac{\mu^2\lambda^2}{16\pi^2(4-D)}] = \mu\tfrac{d}{d\mu}[\mu^2 r + \hbar\tfrac{\mu^{D-4}g^2}{16\pi^2(4-D)}] \\ &= 2\mu^2 r + \mu^3\tfrac{dr}{d\mu} - \hbar\mu^2\tfrac{\lambda^2}{16\pi^2} \end{aligned}$$

so that

$$\beta(\lambda,r) = (\tfrac{D}{2}-3)\lambda, \qquad \gamma_2(\lambda,r) = -2r + \hbar\tfrac{\lambda^2}{16\pi^2}. \tag{3.32}$$

The gain is that we have obtained formulae for β and γ_2 in general dimension. They may serve as an indication of how the field theory behaves in smaller or larger dimension then the one considered. For example, the vanishing of the linear contribution to $\beta(\lambda)$ in 6 dimensions signals that the theory becomes only renormalizable there. Note that eqs. (3.32) reduce to (3.29) at $D = 4$. This did not have to be the case since we have changed the parameterization of the limiting theories and what is geometrically defined is the vector field $\beta\partial_\lambda + \gamma_2\partial_r$.

Problem 2. Find to 1-loop order the transformation between the coordinates (λ, r) of the renormalized 4-dimensional ϕ^3 theory corresponding to the passage between the two renormalization schemes discussed above. Show that it preserves the form of the vector field $\beta\partial_\lambda + \gamma_2\partial_r$.

Problem 3. Find the running couplings $\lambda(\mu), r(\mu)$ using the 1 loop approximations to the β, γ_2 functions. What can one tell about the effect of higher loop corrections to the large μ (UV) behavior of the running couplings?

The four-dimensional ϕ^3 theory, in spite of its super-renormalizability (only a finite number of divergent 1PI graphs) and self-consistency of its perturbative calculations, has a non-perturbative stability problem related to the lack of a lower bound for the cubic polynomial. This should serve as a warning that even the RG improved perturbative analysis is not enough to assure existence of a renormalized QFT.

§3.7. Renormalization of the sigma models to 1 loop

As mentioned above, the background field effective action $\Gamma_b(\phi)$ of the sigma model is equal to the $\zeta = 0$ value of the effective action of the ξ-theory with the action $S_\phi(\xi)$ given by the relation

$$\mathrm{e}^{-S_{g,u}(\mathrm{e}^\xi\phi)}\, D_g(\mathrm{e}^\xi\phi) \;=\; \mathrm{e}^{-S_\phi(\xi)}\, D\xi.$$

The 1st of eqs. (3.8) then implies that in the perturbation expansion

$$\Gamma_b(\phi) \;=\; S_\phi(0) - \sum_{\substack{\text{1PI vacuum}\\ \text{graphs } G}} I_G(S_\phi) - \ln\Big(\int \mathrm{e}^{-\frac{1}{2}S''_\phi(0)(\xi^2)} D\xi\Big). \tag{3.33}$$

One of the simplifying features of the dimensional regularization is that we may disregard the terms in $S_\phi(\xi)$ coming from the logarithm of the "Radon-Nikodým derivative" $\frac{D_g(\mathrm{e}^\xi\phi)}{D\xi}$. Formally, they are proportional to $\delta^{(2)}(0) = \int \mathrm{d}\!\!\!^{-}\, q$ which vanishes in the dimensional regularization.

Problem 4. (a). Show by calculating the D-dimensional integral $\int \mathrm{e}^{-q^2} dq$ in two ways that the volume of the unit sphere in D dimensions is equal to $2\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2})$.

(b). Show that in the radial variables the D-dimensional integral $\int\limits_{|q|\geq\epsilon} dq$ converges for $\operatorname{Re} D < 0$:

$$\int\limits_{|q|\geq\epsilon} \bar{d}\, q = -\frac{2^{1-D}\pi^{-D/2}}{D\Gamma(D/2)}\epsilon^D.$$

Defining the value of the integral by analytic continuation for integer $D \geq 0$ and taking ϵ to zero we infer that $\int \bar{d}\, q$ vanishes in positive dimensions in the dimensional regularization.

Expanding in local coordinates

$$(\mathrm{e}^{\xi}\phi)^i \;=\; \phi^i + \xi^i - \tfrac{1}{2}\Gamma^i_{jk}(\phi)\xi^j\xi^k + \mathcal{O}(\xi^3)$$

where $\Gamma^i_{jk} = \{{}^{\,i}_{jk}\}$ is the Levi-Civita symbol, we obtain after a little calculation (you may do it!)

$$\begin{aligned}(3.34)\quad S_{g,u}(\mathrm{e}^{\xi}\phi) = \tfrac{1}{4\pi}\int\Big(& g_{ij}(\phi)\partial_\nu\phi^i\partial_\nu\phi^j + u(\phi) \;+ 2g_{ij}(\phi)\partial_\nu\phi^i\nabla_\nu\xi^j + \partial_i u(\phi)\xi^i \\ & + g_{ij}(\phi)\nabla_\nu\xi^i\nabla_\nu\xi^j - R_{ijkl}(\phi)\partial_\nu\phi^i\partial_\nu\phi^k\xi^j\xi^l + \tfrac{1}{2}\nabla_i\partial_j u(\phi)\xi^i\xi^j\Big)dx + \mathcal{O}(\xi^3).\end{aligned}$$

Above, ∇_i denotes the covariant derivative over the ith coordinate and

$$\nabla_\nu\xi^i = \partial_\nu\xi^i + \Gamma^i_{jk}(\phi)\partial_\nu\phi^j\xi^k.$$

The vector bundle ϕ^*TM has a natural metric given by ϕ^*g. It will be convenient to choose a global orthonormal frame $(e_a)_{a=1}^d$ of ϕ^*TM (d denotes the dimension on M). In coordinates, $e_a = e^i_a\partial_{\phi^i}$. Different choices of e_a are related by gauge transformations $e'_a = \lambda^b_a e_b$ with λ an $SO(d)$-valued function on the plane $\mathbf{E}^2$. Of course e^i_a also depend on the field ϕ. It will be more convenient to rewrite

$$\xi = \xi^a e_a \qquad \text{or} \qquad \xi^i = \xi^a e^i_a$$

with (ξ^a) a sequence of functions on $\mathbf{E}^2$. The expansion (3.34) becomes then

$$\begin{aligned}(3.35)\quad S_{g,u}(\mathrm{e}^{\xi}\phi) = \tfrac{1}{4\pi}\int\Big(& g_{ij}(\phi)\partial_\nu\phi^i\partial_\nu\phi^j + u(\phi) + 2e^a_i\partial_\nu\phi^i\nabla_\nu\xi^a + e^i_a\partial_i u(\phi)\xi^a \\ & + (\nabla_\nu\xi^a)^2 - R_{iakb}(\phi)\partial_\nu\phi^i\partial_\nu\phi^k\xi^a\xi^b + \tfrac{1}{2}e^i_a e^j_b\nabla_i\partial_j u(\phi)\xi^a\xi^b\Big)dx + \mathcal{O}(\xi^3).\end{aligned}$$

where (e^a_i) is the matrix inverse to (e^i_a), $R_{iakb} = e^j_a e^l_b R_{ijkl}$ and

$$\nabla_\nu\xi^a = \partial_\nu\xi^a + A^a_{b\nu}\xi^b \qquad \text{with} \qquad A^a_{b\nu} = e^a_i\left(\partial_\nu e^i_b + \Gamma^i_{jk}\partial_\nu\phi^j e^k_b\right).$$

Clearly, $A^a_{b\nu}dx^\nu$ transforms as an $SO(d)$ connection form under the gauge transformations $e_a \mapsto \lambda^b_a e_b$. The perturbation expansion (3.33) for $\Gamma_b(\phi)$ becomes now to the 1-loop order:

$$\Gamma_b(\phi) = \tfrac{1}{4\pi\hbar}\int\left(g_{ij}(\phi)\partial_\nu\phi^i\partial_\nu\phi^j + u(\phi)\right)dx$$

$$(3.36)\qquad -\ln\left(\int \mathrm{e}^{-\frac{1}{4\pi}\int\left((\nabla_\nu\xi^a)^2 - R_{iakb}(\phi)\partial_\nu\phi^i\partial_\nu\phi^k\xi^a\xi^b + \frac{1}{2}e^i_a e^j_b\nabla_i\partial_j u(\phi)\xi^a\xi^b\right)dx} D\xi\right) + \mathcal{O}(\hbar)$$

(the terms linear in ξ on the right hand side of (3.34) do not contribute to Γ_b). The functional integral gives a determinant and we could use the zeta-function

prescription to make sense out of it in an $SO(d)$-gauge-invariant way. We shall be, however, more interested in the divergent part of the determinant than in its renormalized value. The dimensional regularization will allow us to extract the divergence in a convenient ($SO(d)$-gauge-invariant) manner and, besides, it works to all orders.

We shall obtain the expression for the 1-loop contribution to $\Gamma_b(\phi)$ expanded around a constant value ϕ_0 of ϕ in the form (in coordinates around ϕ_0)

$$[\Gamma_b(\phi)]^{1-\text{loop}} = \sum_n \int K_{n,D}(x_1,\dots,x_n;\phi_0) \prod_{j=1}^{n} (\phi(x_j)-\phi_0)dx_j \tag{3.37}$$

with the translationally invariant kernels $K_{n,D}$ regularized dimensionally, i.e. meromorphically dependent on D, with possible poles at $D=2$ and with $\phi-\phi_0$ vanishing rapidly at infinity. In order to generate the expansion (3.37) for the 1-loop contribution to $\Gamma_b(\phi)$, we shall separate the term

$$\tfrac{1}{4\pi}\int \left((\partial_\nu \xi^a)^2 + \tfrac{1}{2}(e^i_a e^j_b \nabla_i \partial_j u)|_{\phi_0}\xi^a\xi^b\right) dx$$

in the action as producing the Gaussian measure[13] from

$$\tfrac{1}{4\pi}\int \Big(2A^a_{b\nu}\partial_\nu \xi^a \xi^b + A^a_{b\nu}A^a_{c\nu}\xi^b\xi^c + \tfrac{1}{2}\left(e^i_a e^j_b \nabla_i\partial_j u(\phi) - (e^i_a e^j_b \nabla_i\partial_j u)|_{\phi_0}\right)\xi^a\xi^b \\ -R_{iakb}(\phi)\partial_\nu\phi^i\partial_\nu\phi^k\xi^a\xi^b\Big)dx$$

treated as an interaction. Now it is easy to enumerate the divergent graphs. First there are logarithmically divergent contributions coming from the graphs



They cancel each other (there are no divergences in 2D gauge theory). The only divergent terms we are left with are

(3.38)



and $\frac{1}{2}\ln\det\frac{1}{2\pi}\left(-\delta_{ab}\Delta + \frac{1}{2}(e^i_a e^j_b\nabla_i\partial_j u)|_{\phi_0}\right)$.

All other contributions are easily checked to have finite limits at $D=2$. Since in the dimensional regularization

$$\int \frac{\bar{d}q}{q^2+m^2} = \int_0^\infty d\sigma \int \mathrm{e}^{-\sigma(q^2+m^2)}\bar{d}q = 2^{-D}\pi^{-D/2}\int_0^\infty d\sigma\sigma^{-D/2}\mathrm{e}^{-\sigma m^2} \\ = 2^{-D}\pi^{-D/2}m^{D-2}\Gamma(1-D/2) = \frac{1}{2\pi}\frac{1}{2-D} + \text{part regular at } D=2,$$

the pole part of the loops in (3.38) is equal to

$$\tfrac{1}{4\pi}\tfrac{1}{2-D}\int\left(-R_{iaka}(\phi)\partial_\nu\phi^i\partial_\nu\phi^k + \tfrac{1}{2}(\Delta_g u(\phi) - \Delta_g u(\phi_0))\right)dx$$

---

[13] we assume that the matrix $((e^i_a e^j_b\nabla_i\partial_j u)|_{\phi_0})$ is positive which is the case, for example, in the vicinity of a minimum of u

(since $e^i_a e^j_a \nabla_i \partial_j u = g^{ij} \nabla_i \partial_j u = \Delta_g u$ where Δ_g denotes the Laplacian on M). Similarly,

$$\frac{1}{2} \ln \det \frac{1}{2\pi} \left(-\Delta \delta_{ab} + \frac{1}{2} (e^i_a e^j_b \nabla_i \partial_j u)|_{\phi_0} \right)$$

$$= \frac{1}{2} \int dx \int \mathrm{tr} \ln \frac{1}{2\pi} \left(q^2 + \frac{1}{2} e\, e\, \nabla \partial\, u|_{\phi_0} \right) đ\, q$$

$$= \text{const.} + \frac{1}{2} \int dx \int_0^1 dt \mathrm{tr} \frac{1}{q^2 + \frac{t}{2} e\, e\, \nabla \partial\, u|_{\phi_0}} \frac{1}{2} e\, e\, \nabla \partial\, u|_{\phi_0} đ\, q$$

$$= \text{const.} + \frac{1}{8\pi} \frac{1}{2 - D} \int \Delta_g u(\phi_0) dx.$$

We infer that the pole part of $\Gamma_g(\phi)$ to 1-loop order is

$$[\Gamma_b(\phi)]^{1-\text{loop}}_{\text{div}} = -\frac{1}{4\pi} \frac{1}{2-D} \int R_{ij}(\phi) \partial_\nu \phi^i \partial_\nu \phi^j dx + \frac{1}{8\pi} \frac{1}{2-D} \int \Delta_g u(\phi) dx$$

where $R_{ij} = R_{iaja}$ is the Ricci tensor on M. $[\Gamma_b(\phi)]^{1-\text{loop}}_{\text{div}}$ is an integral of dimension 2 and dimension 0 operators and this result, in accord with a simple power counting, remains true at higher orders.

The minimal subtraction renormalization scheme adds counterterms to the bare metric g and the bare potential u which cancel the above poles. More exactly, one substitutes in the initial action $S_{g_0, u_0}(\phi)$ of the model with the bare metric g_0 and bare tachyon potential u_0,

$$g_0 = \mu^{D-2} (g + \frac{\hbar}{2-D} \delta g_1 + \mathcal{O}(\hbar^2)), \tag{3.39}$$

$$u_0 = \mu^D (u + \frac{\hbar}{2-D} \delta u_1 + \mathcal{O}(\hbar^2)). \tag{3.40}$$

The added 1-loop counterterms change the effective action by

$$\delta \Gamma_b(\phi) = \frac{1}{4\pi} \frac{\mu^{D-2}}{2-D} \int \left(\delta g_{1\, ij}(\phi) \partial_\nu \phi^i \partial_\nu \phi^j + \mu^2 \delta u_1(\phi) \right) dx + \mathcal{O}(\hbar)$$

and we put

$$\delta g_{1\, ij} = R_{ij}, \qquad \delta u_1 = -\frac{1}{2} \Delta_g u.$$

canceling the poles at $D = 2$. This proves the renormalizability of the sigma model (background field effective action) to 1 loop.

§3.8. Renormalization group analysis of sigma models

Let us compute the vector field $\beta \partial_g + \gamma \partial_u$ on the space of metrics and potentials as given by the minimal subtraction version of eqs. (3.19):

$$\beta(g, u) = \mu \frac{\partial}{d\mu} g \Big|_{\substack{g_0 = \text{const} \\ u_0 = \text{const.}}}, \qquad \gamma(g, u) = \mu \frac{\partial}{d\mu} u \Big|_{\substack{g_0 = \text{const} \\ u_0 = \text{const.}}}. \tag{3.41}$$

Applying the derivative $\frac{d}{d\ln\mu}$ to eq.(3.39), we obtain

$$\begin{aligned} 0 &= \mu\frac{d}{d\mu}\left[\mu^{D-2}\left(g_{ij}+\frac{\hbar}{2-D}R_{ij}+\mathcal{O}(\hbar^2)\right)\right] \\ &= \mu\frac{d}{d\mu}\left[\mu^{D-2}\left(g_{ij}+\frac{\hbar}{2-D}R_{0ij}+\mathcal{O}(\hbar^2)\right)\right] \\ &= \mu^{D-2}\Big[\beta_{ij}(g)-(2-D)g_{ij}-\hbar R_{ij}+\mathcal{O}(\hbar^2)\Big] \end{aligned}$$

from which we infer that

$$\beta_{ij}(g) = (2-D)g_{ij}+\hbar R_{ij}+\mathcal{O}(\hbar^2). \tag{3.42}$$

Similarly,

$$\mu\frac{d}{d\mu}\left[\mu^D\left(u-\frac{\hbar}{2(2-D)}\Delta_g u+\mathcal{O}(\hbar^2)\right)\right] \tag{3.43}$$

$$= \mu\frac{d}{d\mu}\left[\mu^D u-\mu^{D-2}\frac{\hbar}{2(2-D)}\Delta_{g_0}u_0+\mathcal{O}(\hbar^2)\right] = \mu^D\left[\gamma(u)+Du+\frac{\hbar}{2}\Delta_g u+\mathcal{O}(\hbar^2)\right]$$

so that

$$\gamma(u) = -Du-\tfrac{1}{2}\hbar\Delta_g u+\ \mathcal{O}(\hbar^2) \tag{3.44}$$

The vector field β in the space of metrics may be used to find out in which situations we may expect the perturbative calculation to be self-consistent. The condition is that the running metric $g(\mu)$ satisfying the RG equation

$$\mu\tfrac{d}{d\mu}g = \beta(g) \tag{3.45}$$

stays on all scales $\mu \geq \mu_0$ in the perturbative regime. Let us illustrate this on the example where $M = S^{N-1}$ with the metric $\frac{1}{\alpha'}$ times the induced metric g of the unit sphere in $\mathbf{R}^N$. Due to the rotational symmetry, the renormalized metric is $\frac{1}{\alpha'(\mu)}g$ and eq. (3.45) in its μ-dependence reduces in $D=2$ to

$$\mu\tfrac{d}{d\mu}\alpha' = \hbar(2-N)(\alpha')^2+\ \mathcal{O}(\hbar^2).$$

Clearly, for $N>2$, α' is driven to zero for large μ approximately as $\mathcal{O}(\frac{1}{\ln\mu})$. The perturbative regime corresponds to small α' so that the perturbative expansion is self-consistent for the sigma model with $M=S^{N-1}$ for $N>2$. This phenomenon is called the **asymptotic freedom** of the spherical sigma model since $\alpha'=0$ corresponds to a free theory. It leads us to expect that the theory may be constructed non-perturbatively, at least in finite volume. Such a non-perturbative theory would break the conformal invariance of the classical sigma model. In fact, there are strong reasons to believe that its infinite volume version is massive (an exact expression for its S-matrix is, conjecturally, known).

The property of asymptotic freedom is shared by all the sigma models which have compact symmetric spaces as their targets (and also, more importantly, by the non-abelian 4-dimensional gauge theories with not too many fermion species, like Quantum Chromodynamics (QCD) describing the strong interactions of quarks, mediated by SU_3 gauge fields).

Problem 5. Consider the flow

$$\mu\tfrac{d}{d\mu}\alpha = -\alpha^2, \qquad \mu\tfrac{d}{d\mu}u = -2u+u\alpha \tag{3.46}$$

in $\mathbf{R}^2$. Show that there exist only one perturbative solution

$$u = \sum_{n=0}^{\infty} a_n \alpha^n \tag{3.47}$$

for the invariant manifold of the flow. Study the (non-perturbative) invariant manifolds.

Especially interesting cases correspond to manifolds with vanishing Ricci curvature. The $N = 2$ spherical sigma model is the simplest example (coinciding with free field with values in S^1). As we know, it corresponds to a CFT. One may then read off from the γ function the dimensions equal to $d_q = \frac{1}{2}q^2$ of the composite operators given by the exponential functions $e^{iq\phi}$ on S^1. d_g are equal to the eigenvalues of $-\frac{1}{2}\Delta_g$ (the tree contribution to γ comes from the fact that we have considered integrated insertions of the composite operator into the action). For Ricci-flat targets there is no renormalization of the metric in 1-loop order and, in the supersymmetric versions, up to 4 loops (4 loops excluded). No renormalization of the metric means that the beta function vanishes and scale invariance is preserved (to 4 loops). One may then argue in perturbation theory (studying the Hessian of the 1-loop β) that in the Kählerian case, the Ricci-flat metric may be perturbed so as to give rise to a scale invariant quantum sigma model with $N = 2$ superconformal symmetry (as discussed by Ed Witten during his lectures). Thus Calabi-Yau ($\cong$ Kähler, Ricci-flat) manifolds should correspond to superconformal $N = 2$ field theories. This observation resulted in a conjectured mirror symmetry between Calabi-Yau manifolds, now established in many instances.

In the case of SUSY sigma models with hyper-Kähler targets (i.e. with $N = 4$ supersymmetry), β vanishes to all orders of the loop expansion.

The inclusion of the 2-form term (3.10) into the action of the sigma models modifies the above results. In 1-loop order, the beta function $\mu\frac{d}{d\mu}(g_{ij} + b_{ij})$ is given by the Ricci curvature of the metric connection with torsion $\Gamma^i_{jk} = \{\,{}^{i}_{jk}\,\} + \frac{3}{2}g^{il}H_{jkl}$, where the antisymmetric tensor H_{jkl} corresponds to $\pm d\omega$. In models in which the connections with torsion are globally flat, the beta function vanishes to all orders (even without supersymmetry). The WZW model of CFT, which we shall discuss in the next lecture, corresponds to such a situation. Addition of a 2-form which is closed does not modify the β function but may change the long-distance behavior of the model (this seems to happen for the sigma model with S^2 target where the addition of the term with ω equal to π times the volume form of the unit sphere should render the model massless).

As for the renormalization of the potentials whose scale-dependence is governed by the RG equation

$$\mu\frac{d}{d\mu}u = -2u - \tfrac{1}{2}\hbar\Delta_g u + \mathcal{O}(\hbar^2)$$

note that, on a symmetric space, u (approximately) reproduces itself up to a normalization if it belongs to an eigen-subspace of the Laplacian. The RG analysis allows then allows us to predict the short distance behavior of the correlation functions involving insertions of the corresponding composite operators (somewhat similarly as for $M = S^1$).

References and cross references

For the rudiments of the perturbative approach to functional integrals, Feynman graphs etc. see Sects. 5.1-5.3 of

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L. Alvarez-Gaumé and P. Ginsparg, *Finiteness of Ricci flat supersymmetric nonlinear* σ*-models*, Comm. Math. Phys. **102** (1985), 311–326.

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(The latter papers discovered 4th order contributions to the supersymmetric beta function). For the case of sigma models with a 2-form in the action see

E. Braaten, T. L. Curtright, and C. K. Zachos, *Torsion and geometrostasis in nonlinear sigma models*, Nucl. Phys. B **260** (1985), 630–688. *Errata:* ibid., **266** (1986), 748.

C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, *Strings in background fields*, Nucl. Phys. B **262** (1985), 593–609.

In these volumes, see [I-**Axiomatics**, §5.3] for another discussion of the 1PI effective action. Regularization and renormalization are treated further in [I-**Perturbative QFT**], and the theory of the renormalization group in [I-**Renorm Groups**]. For another discussion of the background field method, see Lecture 6 of [II-**Strings**]. For a mathematical treatment of dimensional regularization, see [I-**Dim'l Regularization**].

LECTURE 4
Constructive Conformal Field Theory

Let us recall the logical structure of this course. In the first lecture we studied the free field examples of CFT's. In the second one, we analyzed the scheme of (two-dimensional) CFT from a more abstract, axiomatic point of view. In the third one, we searched perturbatively among geometric sigma models for non-free examples of CFT's. Finally, in the present lecture, compressed due to lack of time, we shall analyze an especially important sigma model, the Wess-Zumino-Witten (WZW) model, whose correlation functions may be constructed non-perturbatively, with a degree of explicitness comparable to that attained for toroidal compactifications of free fields (constituting the simplest examples of WZW theories). The WZW model appears to be a generating theory of a vast family of CFT's whose correlations can be expressed in terms of the WZW ones. The comparison of the non-perturbative models obtained this way with the perturbative constructions of sigma models allows for highly non-trivial tests of differences between the geometry of Ricci-flat (Einstein) spaces and that of CFT's, replacing the Einstein geometry in the stringy approach to gravity.

§4.1. WZW model

The target space of the WZW sigma model is a compact Lie group manifold G and the two-dimensional theory may be considered as a generalization of quantum mechanics of a particle moving on G. In the latter case the (Euclidean) action functional is

$$S(g) = -\tfrac{1}{2}\int \mathrm{tr}\,(g^{-1}\partial_x g)^2 dx \tag{4.1}$$

where "tr" denotes the Killing form[1]. Let R denote an (irreducible) unitary representation $g \mapsto g_R$ of G in a (finite dimensional) Hilbert space V_R. The path integral for the quantum-mechanical particle on G, corresponding to the Wiener measure on G, may be solved with the use of the Feynman-Kac formula taking the form

$$\int_{\mathrm{Map}([0,L]_{per},G)} \bigotimes_{i=1}^{n} g_{R_i}(x_i)\ \mathrm{e}^{-kS(g)} \prod_x dg(x) \Big/ \int_{\mathrm{Map}([0,L]_{\mathrm{per}},G)} \mathrm{e}^{-kS(g)} \prod_x dg(x)$$

[1]normalized so that the long roots have length squared 2

$$= \operatorname{Tr} e^{-x_1 H} g_{R_1} e^{(x_2-x_1)H} g_{R_2} \cdots g_{R_n} e^{(L-x_n)H} \Big/ \operatorname{Tr} e^{-LH} \tag{4.2}$$

where $0 \le x_1 \le x_2 \le \cdots \le x_n \le L$, $2kH$ is the Laplacian on G and g_R is viewed as a matrix of multiplication operators, both acting in $L^2(G, dg)$ (dg is the Haar measure). Compare Problem 3 in Lecture 1 dealing with the case $G = S^1$. The theory possesses the $G \times G$ symmetry which may be used to solve it: the right hand side of (4.2) is calculable in terms of harmonic analysis on G.

Problem 1. *Compute explicitly the 1-,2- and 3-point functions in* (4.2).

The Euclidean action of the WZW model is a functional on $\mathrm{Map}(\Sigma, G)$ where Σ is a compact Riemann surface. If, for simplicity, we assume G to be connected and simply connected then

$$S(g) = -\frac{\imath}{4\pi} \int_\Sigma \operatorname{tr} g^{-1}\partial g \wedge g^{-1}\bar\partial g + \frac{i}{4\pi} \int_\Sigma g^*\omega \tag{4.3}$$

where the 2-form ω is defined on open subsets $\mathcal{O} \subset G$ with $H_2(\mathcal{O}) = 0$ and satisfies there $d\omega = -\frac{1}{3}\mathrm{tr}(g^{-1}dg)^{\wedge 3}$. The dependence of the second term of $S(g)$ on the choice of ω makes $S(g)$ defined modulo $2\pi i\mathbf{Z}$ so that $e^{-kS(g)}$ is well defined for integer k. To have the energy bounded below, we shall take k, called the level of the WZW model, positive.

Problem 2. *Assuming, more generally, that g takes values in the complexified group $G^{\mathbf{C}}$ find the equations for stationary points of $S(g)$.*

The correlation functions of the WZW model are formally given by the functional integrals:

$$\Big\langle \bigotimes_{i=1}^n g_{R_i}(x_i) \Big\rangle = \int_{\mathrm{Map}(\Sigma,G)} \bigotimes_{i=1}^n g_{R_i}(x_i) e^{-kS(g)} Dg \Big/ \int_{\mathrm{Map}(\Sigma,G)} e^{-kS(g)} Dg \in \bigotimes_{i=1}^n \mathrm{End}(V_{R_i}) \tag{4.4}$$

where Dg stands for the formal product of the Haar measures $dg(x)$ over $x \in \Sigma$.

As we have mentioned at the end of Lecture 3, the renormalization group beta function computed for the WZW sigma model vanishes to all orders due to the flatness of the connections with torsion generated from the metric and the 2-form ω on G. Thus the model is conformally invariant and does not need renormalization of the action (4.3) in perturbation theory. The conformal invariance holds, in fact, also non-perturbatively due to the $LG \times LG$ symmetry of the theory, where LG denotes the loop group $\mathrm{Map}(S^1, G)$ of G. The WZW model may be solved exactly by

1. harmonic analysis on LG

or by

2. exact functional integration.

As we shall see, the (matrix-valued) composite operators $g_R(x)$ need multiplicative renormalization and acquire scaling dimensions $\frac{2c_R}{k+h^\vee}$, where c_R denotes the quadratic Casimir of R and $h^\vee$ stands for the dual Coxeter number of G (the quadratic Casimir of the adjoint representation).

§4.2. Gauge symmetry Ward identities

We shall sketch here the functional integral approach to the WZW theory. It will be convenient to extend the model a little by coupling it to an external gauge field[2] $A = A^{10} + A^{01}$, a 1-form with values in the complexified Lie algebra $\mathbf{g}^{\mathbf{C}}$ of G. Define[3]

(4.5)

$$S(g,A) = S(g) + \frac{ik}{2\pi}\int_{\Sigma} \mathrm{tr}[A^{10}\wedge g^{-1}\bar\partial g + g\partial g^{-1}\wedge A^{01} + gA^{10}g^{-1}\wedge A^{01}].$$

Under the ("chiral") gauge transformations corresponding to maps $h_{1,2}:\Sigma\to G^{\mathbf{C}}$ the action $S(g,A)$ transforms according to the Polyakov-Wiegmann formula

$$S(h_2gh_1^{-1}, {}^{h_1}\!A^{10} + {}^{h_2}\!A^{01}) = S(g, A^{10}+A^{01}) - S(h_1, A^{10}) - S(h_2^{-1}, A^{01}) \tag{4.6}$$

where ${}^{h_1}\!A^{10} = h_1A^{10}h_1^{-1} + h_1\partial h_1^{-1}$ and ${}^{h_2}\!A^{01} = h_2A^{01}h_2^{-1} + h_2\bar\partial h_2^{-1}$.

Problem 3. *Prove the Polyakov-Wiegmann formula.*

In the presence of the external gauge field A, the partition function of the WZW theory will be formally defined as

$$Z_A = \int_{\mathrm{Map}(\Sigma,G)} \mathrm{e}^{-kS(g,A)}\,Dg \tag{4.7}$$

and the correlation functions $\langle\otimes g_{R_i}(x_i)\rangle_A$ by eq. (4.4) with $S(g)$ replaced by $S(g,A)$ (no functional integration over A). Using the formal extension to functional integrals of the simple invariance property

$$\int_G f(h_2gh_1^{-1})dg = \int_G f(g)dg$$

holding for $h_{1,2}\in G^{\mathbf{C}}$ if f is an analytic function on $G^{\mathbf{C}}$, we obtain

$$\begin{aligned} Z_{{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01}}\langle\otimes g_{R_i}(x_i)\rangle_{{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01}} &= \int \bigotimes_{i=1}^{n}(h_2gh_1^{-1})_{R_i}(x_i)\mathrm{e}^{-kS(h_2gh_1^{-1},\,{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01})}Dg \\ &= \mathrm{e}^{kS(h_1,A^{10})}\mathrm{e}^{kS(h_2^{-1},A^{01})}\bigotimes_i(h_2)_{R_i}(x_i)Z_A\langle\otimes g_{R_i}(x_i)\rangle_A\bigotimes_i(h_1)^{-1}_{R_i}(x_i). \end{aligned} \tag{4.8}$$

This is the chiral gauge symmetry Ward identity for the correlation functions (recall the diffeomorphism group and the local rescaling Ward identities discussed in Lecture 2).

The identity (4.8) factorizes into a holomorphic (A^{01}-dependent) and an anti-holomorphic (A^{10}-dependent) ones. Hence in order to study the A^{01}-dependence of the correlation functions it is enough to look for holomorphic maps on a Sobolev space of (0,1)-forms[4] A^{01} with values in $\mathbf{g}$

$$\Psi:\ \mathcal{A}^{01}\longrightarrow\bigotimes_{i=1}^{n}V_{R_i}\ \equiv\ V_{\mathbf{R}}$$

satisfying the "factorized" Ward identity

$$\Psi({}^{h}\!A^{01}) = \mathrm{e}^{kS(h^{-1},A^{01})}\bigotimes_{i=1}^{n}h_{R_i}(x_i)\ \Psi(A^{01}). \tag{4.9}$$

---

[2] in general, we shall not assume the unitarity $A=-A^*$ of the gauge field

[3] a more standard definition subtracts also $A^{10}\wedge A^{01}$ inside [...]

[4] what follows does not depend on the assumed degree of smoothness of forms provided it is high enough

The relation (4.9) describes the behavior of Ψ along the orbits of the group $\mathcal{G}^{\mathbf{C}}$ of complex (Sobolev-class) gauge transformations in $\mathcal{A}^{01}$. The orbit space $\mathcal{A}^{01}/\mathcal{G}^{\mathbf{C}}$ is the moduli space of holomorphic $G^{\mathbf{C}}$ bundles which, upon restriction to semi-stable bundles and appropriate treatment of semi-stable but not stable ones, becomes a compact variety $\mathcal{N}$ of complex dimension 0, $\mathrm{rank}(G)$ and $\dim(G)(h_\Sigma - 1)$ for genus h_Σ equal to 0, 1 and > 1, respectively. The space $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ of Ψ's coincides with the space $H^0(\mathcal{V})$ of holomorphic sections of a vector bundle $\mathcal{V}$ over $\mathcal{N}$ with typical fiber $V_{\mathbf{R}}$ ($\mathcal{V} = \mathcal{A}^{01} \times_{\mathcal{G}^{\mathbf{C}}} V_{\mathbf{R}}$ essentially). In another description, $W(\Sigma, \mathbf{x}, \mathbf{R}, k) = H^0(\mathcal{L})$ where $\mathcal{L}$ is a line bundle over the moduli space of holomorphic $G^{\mathbf{C}}$-bundles with parabolic structures at the points x_i and the Ψ's may be interpreted as a non-abelian generalization of theta functions. The essential implication of these identifications is that $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ is a finite-dimensional space. Its dimension depends, in fact, only on h_Σ, k and $\mathbf{R}$ and is given by the celebrated Verlinde formula. $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ may be also identified with the space of quantum states of the Chern-Simons theory.

Out of the global Ward identities (4.8) one may extract the infinitesimal ones by taking $h_i = \mathrm{e}^{\Lambda_i}$ and Taylor-expanding in Λ_i similarly as we analyzed the infinitesimal consequences of the diffeomorphism and rescaling Ward identities in Lecture 2. Define the insertions of currents into the correlations by

$$\langle J^a_z \dots \rangle_A = -\pi \frac{1}{Z_A} \frac{\delta}{\delta A^a_{\bar z}} Z_A \langle \dots \rangle_A,$$
$$\langle J^a_{\bar z} \dots \rangle_A = -\pi \frac{1}{Z_A} \frac{\delta}{\delta A^a_{z}} Z_A \langle \dots \rangle_A$$

(the subscript "a" corresponds to a basis (t^a) of the Lie algebra $\mathbf{g}$ such that $\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$). Denote by $J(z)$ ($\bar J(\bar z)$) the insertions of J_z ($\bar J_{\bar z}$) into correlations with A vanishing around the insertion point and the metric locally flat. $J(z)$ ($\bar J(\bar z)$) depends holomorphically (anti-holomorphically) on z away from other insertions. Expanding to the second order, one obtains from the Ward identities (4.8) the operator product expansions

$$J^a(z)J^b(w) = \frac{\delta^{ab}k/2}{(z-w)^2} + \frac{if^{abc}}{z-w} J^c(w) + \dots, \tag{4.10}$$

$$\bar J^a(\bar z)\bar J^b(\bar w) = \frac{\delta^{ab}k/2}{(\bar z-\bar w)^2} - \frac{if^{abc}}{\bar z-\bar w} \bar J^c(\bar w) + \dots, \tag{4.11}$$

$$J^a(z)\bar J^b(\bar w) = \dots \tag{4.12}$$

which imply for the modes of the corresponding Hilbert space operators $\mathcal{J}(z) = \sum_n J_n z^{-n-1}$, $\bar{\mathcal{J}}(\bar z) = \sum_n \bar J_n \bar z^{-n-1}$ the Kac-Moody algebra relations

$$[J^a_n, J^b_m] = if^{abc}J^c_{n+m} + \tfrac{1}{2}kn\delta^{ab}\delta_{n+m,0}$$

and similarly for $\bar J_n$ with the commutators between J_n and $\bar J_m$ vanishing.

Problem 4. *Prove the operator product expansions* (4.10-4.12).

Subtraction of the singular part from the expression $\mathrm{tr}\, J(z)J(w)$ gives the Sugawara construction of the energy-momentum tensor of the WZW theory:

$$T(w) = \frac{2}{k+h^\vee} \lim_{z\to w} \left(\mathrm{tr}\, J(z)J(w) - \frac{\dim(G)k}{4(z-w)^2} \right)$$

and similarly for $\bar{T}(\bar{w})$. In modes, this becomes

$$L_n = \frac{2}{k+h^\vee} \sum_{m=-\infty}^{\infty} \mathrm{tr} : J_{n-m} J_m :$$

where the normal ordering puts J_p with positive p to the right of the ones with negative p.

§4.3. Scalar product of non-abelian theta functions

Since the A^{10}-dependence of the unnormalized correlation functions $Z_A\langle \otimes g_{R_i}(x_i)\rangle_A$ coincides with that of $\overline{\Psi(-(A^{10})^*)}$ (recall that the complex conjugate space $\overline{V_R} \cong V_R^*$), we must have

$$Z_A\langle \otimes g_{R_i}(x_i)\rangle_A \quad \in \quad W(\Sigma, \mathbf{x}, \mathbf{R}, k) \otimes \overline{W(\Sigma, \mathbf{x}, \mathbf{R}, k)}$$

as a function of A or, more explicitly,

$$Z_A\langle \otimes g_{R_i}(x_i)\rangle_A \;=\; H^{\alpha\beta}\, \Psi_\alpha(A^{01}) \otimes \overline{\Psi_\beta(-(A^{10})^*)} \tag{4.13}$$

where (Ψ_α) is a basis of $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$, $(H^{\alpha\beta})$ is an $\mathbf{x}$-dependent matrix and the summation convention is assumed. From formal reality properties of the functional integral defining $Z_A\langle \otimes g_{R_i}(x_i)\rangle_A$ one may see that $(H^{\alpha\beta})$ should be a Hermitian matrix. In fact one may argue that the inverse matrix $(H_{\beta\alpha})$ corresponds to the scalar product on the space $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ of non-abelian theta functions:

$$H_{\beta\alpha} \;=\; (\Psi_\beta, \Psi_\alpha) \tag{4.14}$$

where $(\cdot, \cdot)$ is formally given by

$$(\Psi, \Psi') = \int (\Psi(A^{01}), \Psi'(A^{01}))_{V_{\mathbf{R}}}\, \mathrm{e}^{-\frac{k}{2\pi}\|A\|^2_{L^2}}\, DA \tag{4.15}$$

with the integration over the unitary gauge fields A with $A^{10} = -(A^{01})^*$. The scalar product (4.15) is exactly the one which gives the probability amplitudes between the states of the Chern-Simons theory. Expressions (4.13) and (4.14) for the correlation functions may be expressed in a basis-independent way as follows. Let $e_{A^{01}}$ denote the evaluation map

$$W(\Sigma, \mathbf{x}, \mathbf{R}, k) \ni \Psi \xrightarrow{e_{A^{01}}} \Psi(A^{01}) \in V_{\mathbf{R}}.$$

$e_{A^{01}}$ may be considered as an element of $V_{\mathbf{R}} \otimes W(\Sigma, \mathbf{x}, \mathbf{R}, k)^*$ and using the scalar product dual to (4.15) on the second factor, we obtain the equality

$$Z_A\langle \otimes g_{R_i}(x_i)\rangle_A \;=\; \overline{(e_{A^{01}}, e_{-(A^{10})^*})} \tag{4.16}$$

viewed as a relation between the $V_{\mathbf{R}} \otimes \overline{V_{\mathbf{R}}}$-valued functionals of A.

Let us present a functional integral proof of the relation (4.15). Denote $Z_A\langle\otimes g_{R_i}(x_i)\rangle_A \equiv \Gamma(A)$. Consider the integral over the unitary gauge fields B

$$\int \Gamma(B^{10}+A^{01})\Gamma(A^{10}+B^{01})\mathrm{e}^{-\frac{k}{2\pi}\|B\|^2_{L^2}}DB$$

$$= \int \bigotimes_{i=1}^{n}(g_1g_2)_{R_i}(x_i)\mathrm{e}^{-kS(g_1)-kS(g_2)}\cdot \mathrm{e}^{-\frac{ik}{2\pi}\int \mathrm{tr}[A^{10}\wedge g_2^{-1}\bar\partial g_2+g_1\partial g_1^{-1}\wedge A^{01}]}$$

$$\cdot \mathrm{e}^{-\frac{ik}{2\pi}\int \mathrm{tr}[B^{10}\wedge g_1^{-1}\bar\partial g_1+g_1B^{10}g_1^{-1}\wedge A^{01}+g_2\partial g_2^{-1}\wedge B^{01}+g_1A^{10}g_2^{-1}\wedge B^{01}-B^{10}\wedge B^{01}]}Dg_1Dg_2DB$$

$$= \int \bigotimes_{i=1}^{n}(g_1g_2)_{R_i}(x_i)\mathrm{e}^{-kS(g_1)-kS(g_2)-\frac{ik}{2\pi}\int \mathrm{tr}[A^{10}\wedge g_2^{-1}\bar\partial g_2+g_1\partial g_1^{-1}\wedge A^{01}]}$$

$$\cdot \mathrm{e}^{-\frac{ik}{2\pi}\int \mathrm{tr}[(g_2\partial g_2^{-1}+g_2A^{10}g_2^{-1})\wedge(g_1^{-1}\bar\partial g_1+g_1^{-1}A^{01}g_1)]}Dg_1Dg_2$$

$$= \int \bigotimes_{i=1}^{n}(g_1g_2)_{R_i}(x_i)\mathrm{e}^{-kS(g_1g_2,A)}\,Dg_1Dg_2 = \Gamma(A)$$

where the 2nd equality is obtained by a straightforward Gaussian integration over B. Upon the substitution of relations (4.13) and (4.15), the last identity becomes

$$H^{\alpha\beta}H^{\gamma\delta}\,(\Psi_\beta,\Psi_\gamma)\,\Psi_\alpha(A^{01})\otimes\overline{\Psi_\delta(-(A^{10})^*)} \;=\; H^{\alpha\delta}\,\Psi_\alpha(A^{01})\otimes\overline{\Psi_\delta(-(A^{10})^*)}$$

or $H^{\alpha\beta}H^{\gamma\delta}\,(\Psi_\beta,\Psi_\gamma) = H^{\alpha\delta}$, from which the relation (4.14) follows if we also assume that $(H^{\alpha\beta})$ is an invertible matrix.

The above expressions reduce the calculation of the correlation functions of the WZW model to that of the functional integral (4.15). The latter appears easier to calculate then the original functional integral (4.4). In the first step, the integral (4.15) may be rewritten by a trick resembling the Faddeev-Popov treatment of gauge theory functional integrals. The reparametrization of the gauge fields

$$A^{01} = {}^{h^{-1}}A^{01}(n) \tag{4.17}$$

by chiral gauge transforms of a (local) slice $n\mapsto A^{01}(n)$ in $\mathcal{A}^{01}$ cutting each $\mathcal{G}^{\mathbf{C}}$-orbit once[5] gives

(4.18)

$$\|\Psi\|^2 = \int (\Psi(A^{01}(n),\otimes(hh^*)^{-1}_{R_i}\Psi(A^{01}(n))_{V_{\mathbf{R}}}\,\mathrm{e}^{(k+2h^\vee)S(hh^*,A(n))}D(hh^*)d\mu_Q(n).$$

The term $2h^\vee S(hh^*)$ in the action comes from the Jacobian of the change of variables (4.17) contributing also to the measure $d\mu_Q(n)$. The latter is defined as follows. Denote by S the composition of the derivative of the map $n\mapsto A^{01}(n)$ with the canonical projection of $\mathcal{A}^{01}$ onto the cokernel of $\bar\partial+[A^{01}(n),\cdot]$. Then the volume form $\mu_Q(n)$ on the slice is the composition of the determinant ($\equiv$ the maximal exterior power) of S with the Quillen metric on the determinant bundle of the family $(\bar\partial+[A^{01}(n),\cdot])$ of $\bar\partial$-operators[6].

Unlike in the standard Faddeev-Popov setup, the integral over the group of gauge transformations did not drop out since the integrand in (4.15) is invariant only under the G-valued gauge transformations. Instead we are left with a functional integral (4.18) similar to the one (4.4) for the original correlation functions, except that it is over fields hh^* which may be considered as taking values in the hyperbolic space $G^{\mathbf{C}}/G$. $D(hh^*)$ is the formal product of $G^{\mathbf{C}}$-invariant measures on $G^{\mathbf{C}}/G$.

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[5] in genus 0 and 1, $h\in\mathcal{G}^{\mathbf{C}}$ should be additionally restricted

[6] again, the cases of genus 0 or 1 require minor modifications

The gain is that the functional integral (4.18) may be reduced to an explicitly doable iterative Gaussian integral. For example, for $G = SU(2)$ and at genus 0 where we may take $A^{01}(n) \equiv 0$,

$$S(hh^*) = -\tfrac{i}{2\pi}\int \partial\phi \wedge \bar\partial\phi - \tfrac{i}{2\pi}\int (\partial + \partial\phi)\bar v \wedge (\bar\partial + \bar\partial\phi)v$$

in the Iwasawa parameterization $h = \left(\begin{smallmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2}\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & v \\ 0 & 1\end{smallmatrix}\right) u$ of the 3-dimensional hyperboloid $SL_2(\mathbf{C})/SU_2$ by $\phi \in \mathbf{R}$ and $v \in \mathbf{C}$ ($u \in SU_2$). The field v enters quadratically into the action and polynomially into insertions. Hence the v-integral is Gaussian and its explicit calculation requires the knowledge of the determinant of the operator $(-\partial + \partial\phi)(\bar\partial + \bar\partial\phi) = -e^{\phi}\partial e^{-2\phi}\bar\partial e^{\phi}$ and of the propagator

$$((-\partial + \partial\phi)(\bar\partial + \bar\partial\phi))^{-1}(z_1, z_2) \sim e^{-\phi(z_1)-\phi(z_2)}\int \frac{e^{2\phi(y)}d^2y}{(\bar z_1 - \bar y)(y - z_2)}. \tag{4.19}$$

The ϕ-dependence of $\ln\det((-\partial + \partial\phi)(\bar\partial + \bar\partial\phi))$ is given by the chiral anomaly (or local index theorem) and is the sum of a local quadratic and a linear term. The resulting ϕ-field integral appears to be also Gaussian (of the type encountered in functional-integral representations of 2-dimensional Coulomb gas correlation functions in statistical mechanics). A similar iterative procedure based on the Iwasawa parametrization of $G^{\mathbf{C}}/G$ works for arbitrary G and also at higher genera. The result becomes a finite-dimensional integral over parameters $y_a \in \Sigma$ in the expressions of the type (4.19) for the v-field propagators (positions of the "screening charges" in the Coulomb gas interpretation) and, at genus $h_\Sigma > 0$, over (a part of) the moduli parameters n.

At genus 0, the $\mathcal{G}^{\mathbf{C}}$-orbit of $A^{01} = 0$ is dense in $\mathcal{A}^{01}$. As a result $\Psi \in W(\mathbf{C}P^1, \mathbf{x}, \mathbf{R}, k)$ is fully determined by $\Psi(0) \in (V_{\mathbf{R}})^G$, the G-invariant subspace of $V_{\mathbf{R}}$. Hence

$$W(\mathbf{C}P^1, \mathbf{x}, \mathbf{R}, k) \subset (V_{\mathbf{R}})^G$$

canonically. For $G = SU_2$ the representations R_i are labeled by integer or half-integer spins j_i and the representation spaces V_{j_i} are spanned by vectors $(f^l v_{j_i})_{l=0,1,\dots,2j_i}$ where v_{j_i} is the highest weight (HW) vector annihilated by e, with (e, f, h) the usual basis of sl_2. One has, using the standard complex variable z on $\mathbf{C}P^1$ to label the insertion points,

$$W(\mathbf{C}P^1, \mathbf{z}, \mathbf{j}, k) = \{v \in (V_{\mathbf{j}})^{SU_2} \mid (\otimes v_{j_i}, \prod_i e_i^{n_i} e^{z_i e_i} v) = 0 \text{ if } N \le J - k - 1\}$$

where $e_i = 1 \otimes 1 \cdots \otimes 1 \otimes \underset{\hat i}{e} \otimes 1 \otimes \cdots \otimes 1$, $N \equiv \sum_i n_i$ and $J \equiv \sum_i j_i$. In particular, for 2 or 3 points,

$$W(\mathbf{C}P^1, \mathbf{z}, \mathbf{j}, k) = \begin{cases} (V_{\mathbf{j}})^{SU_2} & \text{if } J \le k, \\ \{0\} & \text{if } J > k \end{cases}$$

and does not depend on $\mathbf{z}$. The scalar product (4.15) is given by

$$\|v\|^2 = f(\sigma, \mathbf{z}, \mathbf{j}, k)\int_{\mathbf{C}^J} \left|(v, \omega(\mathbf{z}, \mathbf{y}))e^{-\frac{1}{k+2}U(\mathbf{z},\mathbf{y})}\right|^2 \prod_{a=1}^{J} d^2y_a \tag{4.20}$$

where

$$f(\sigma, \mathbf{z}, \mathbf{j}, k) = e^{\sum_i \frac{j_i(j_i+1)}{k+2}\sigma(z_i) + \frac{1}{16\pi(k+2)}\|d\sigma\|^2_{L^2}}\left(\frac{\det'(-\Delta)}{\mathrm{area}_{\mathbf{C}P^1}}\right)^{3/2}$$

carries the dependence on the metric $e^{\sigma}|dz|^2$ on $\mathbf{C}P^1$, $\mathbf{y} = (y_1, \dots, y_J)$, $\omega(\mathbf{z}, \mathbf{y})$ is a meromorphic V_{j}-valued function

$$\omega(\mathbf{z}, \mathbf{y}) = \prod_{a=1}^{J} \sum_{i=1}^{n} \frac{1}{y_a - z_i} f_i \bigotimes_{i=1}^{n} v_{j_i}$$

and $U(\mathbf{z}, \mathbf{y})$ is a multivalued function

$$\tfrac{1}{2} U(\mathbf{z}, \mathbf{y}) = \sum_{i<i'} j_i j_{i'} \ln(z_i - z_{i'}) - \sum_{i,a} j_i \ln(z_i - y_a) + \sum_{a<a'} \ln(y_a - y_{a'}).$$

The integral (4.20) is over a positive density with singularities at coinciding y_a and the question arises as to whether it converges. A natural conjecture is that the integral is convergent if and only if $v \in W(\mathbf{C}P^1, \mathbf{x}, \mathbf{R}, k) \subset (V_{\mathbf{R}})^G$ (the only if part is easy). For 2- or 3-point functions the integrals can indeed be computed explicitly, confirming the conjecture. Numerous other special cases have been checked. However, the general case of the conjecture remains to be verified. Note that the dependence of the scalar product (4.20) on the conformal factor σ agrees with the value $\frac{3k}{k+2}$ of the central charge of the SU_2 WZW theory and with the values $\Delta_j = \bar{\Delta}_j = \frac{j(j+1)}{k+2}$ of the conformal dimensions of the fields $g_j(x)$ (it is the inverse of $f(\sigma, \mathbf{x}, \mathbf{j}, k)$ which enters the WZW correlation functions).

Explicit finite-dimensional integral formulae for the scalar product (4.15) have been also obtained for general groups and at genus 1 and, for $G = SU_2$, for higher genera[7]. The proof of the convergence of the corresponding integrals is the only missing element in the explicit construction of all correlation functions of the WZW theory although several special cases have been settled completely.

§4.4. KZB connection

The spaces $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ of non-abelian theta functions depend on the complex structures of the surface Σ and on the insertion points. The complex structures $J \in \Gamma(\mathrm{End}\, T\Sigma)$, $J^2 = -1$, form a complex (infinite dimensional Fréchet) manifold on which the group of diffeomorphisms of Σ acts naturally. The holomorphic tangent vectors to the quotient moduli space $\delta J = \delta\mu$ correspond to sections of $\mathrm{End}\, T^{\mathbf{C}}\Sigma$ satisfying $J\delta\mu = -\delta\mu J = i\delta\mu$. Locally, $\delta\mu$ may be represented as $\delta\mu^z_{\bar{z}} \partial_z \otimes d\bar{z}$ in J-complex coordinates. The family of spaces $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ forms a complex finite-dimensional bundle $\mathcal{W}(\mathbf{R}, k)$ over the space of complex structures and n-tuples of noncoincident points $\mathbf{x}$ in Σ.

The bundle $\mathcal{W}(\mathbf{R}, k)$ may be supplied with a natural (w.r.t. the action of diffeomorphisms of Σ) connection ∇ provided that we choose (smoothly) for each J a compatible metric on Σ. The connections for different choices of the metric are related by the conformal anomaly. If $(J, \mathbf{x}, A) \mapsto \Psi(J, \mathbf{x}, A)$ depending holomorphically on $A^{01} = A(1 + iJ)/2$ ($A = -A^*$ is assumed) represents a local section of

[7] it is clear that the case of general group and genus >1 could be treated along the same lines

$\mathcal{W}(\mathbf{R},k)$, then

$$\nabla_{\overline{\delta\mu}}\Psi = d_{\overline{\delta\mu}}\Psi + \frac{k}{8\pi}\left(\int \mathrm{tr}\, A^{01}\wedge A^{01}\overline{\delta\mu}\right)\Psi, \tag{4.21}$$

$$\nabla_{\bar z_i}\Psi = \bar\partial_{\bar z_i}\Psi + (A_{\bar z_i})_i\Psi, \tag{4.22}$$

$$\nabla_{\delta\mu}\Psi = d_{\delta\mu}\Psi - \frac{1}{2\pi i}\left(\int T(z)\delta\mu^z_{\bar z}d^2z\right)\Psi, \tag{4.23}$$

$$\nabla_{z_i}\Psi = \partial_{z_i} + \lim_{z\to z_i}\frac{2t_i^a}{k+h^\vee}\left(J^a(z)+\frac{t_i^a}{z-z_i}\right)\Psi. \tag{4.24}$$

Above z denotes a J-complex coordinate on Σ and $d_{\delta\mu}\Psi$ or $d_{\overline{\delta\mu}}\Psi$ stands for the directional derivatives of Ψ when the points $\mathbf{x}$ and A are kept constant. The first two equations equip $\mathcal{W}(\mathbf{R},k)$ with a structure of a holomorphic vector bundle. In the last 2 equations, the metric on Σ is assumed for simplicity to satisfy $\gamma^{z\bar z}=2$, $\delta\gamma^{zz}=\frac{2}{i}\delta\mu^z_{\bar z}$ and A is taken vanishing around the support of $\delta\mu$ or around the insertion point x_i and $\delta\mu=\mathcal{O}((z-z_i)^2)$.

In the genus 0 case, $\mathcal{W}(\mathbf{R},k)$ is a subbundle of the trivial bundle with the fiber $(V_{\mathbf{R}})^G$ and the connection ∇ extends to the bigger bundle and is given by

$$\nabla_{\bar z_i} = \partial_{\bar z_i},\qquad \nabla_{z_i} = \partial_{z_i} + \frac{2}{k+h^\vee}\sum_{i'\neq i}\frac{t_i^a t_{i'}^a}{z_{i'}-z_i} \equiv \partial_{z_i} + \frac{1}{k+h^\vee}H_i(\mathbf{z})$$

for the metric flat around the insertions. The commuting operators $H_i(\mathbf{z})\in \mathrm{End}(V_{\mathbf{R}})$ are known as the Gaudin Hamiltonians. The corresponding flat connection appeared (implicitly) in the work of Knizhnik-Zamolodchikov on WZW theory. The higher genus generalizations of the KZ connection were first studied by Bernard. We shall call the connection defined by eqs. (4.21-4.24) the KZB connection. In general, it is only projectively flat.

One of the basic open questions concerning the KZB connection is whether it is unitarizable. In other words, whether there exists a Hermitian structure on the bundle $\mathcal{W}(\mathbf{R},k)$ preserved by ∇. It was conjectured that the answer to this question is positive and that it is exactly the scalar product on the spaces $W(\Sigma,\mathbf{x},\mathbf{R},k)$ discussed above that provides the required Hermitian structure. Note that a $(0,1)$ unitary connection on a holomorphic Hermitian vector bundle is uniquely determined. Recall that the scalar product, given formally by the gauge field functional integral (4.15), may be reduced to a finite-dimensional integral which, if convergent, defines a positive Hermitian form on $W(\Sigma,\mathbf{x},\mathbf{R},k)$ and determines the unitary connection (and the energy-momentum tensor of the WZW theory). For genus 0 and $G=SU_2$, where the scalar product is given by the integral (4.20), the unitarity of the KZ connection requires that

$$\partial_{z_i}(v,v) = (v,(\partial_{z_i}-\frac{1}{k+2}H_i)v) \tag{4.25}$$

for a holomorphic family $\mathbf{z}\mapsto v(\mathbf{z})\in W(\mathbf{C}P^1,\mathbf{z},\mathbf{j},k)\subset V_{\mathbf{j}}^{SU_2}$. Assuming the convergence of the integrals permitting one to differentiate under the integral and to integrate by parts, the above is a consequence of the relation

$$\left(\partial_{z_i}+\frac{1}{k+2}H_i(\mathbf{z})\right)\left(\omega(\mathbf{z},\mathbf{y})\mathrm{e}^{-\frac{1}{k+2}U(\mathbf{z},\mathbf{y})}\right) = \partial_{y_a}\left(\eta_{i,a}\mathrm{e}^{-\frac{1}{k+2}U(\mathbf{z},\mathbf{y})}\right) \tag{4.26}$$

where

$$\eta_{i,a}(\mathbf{z},\mathbf{y}) = \frac{1}{z_i-y_a} f_i \prod_{a'\neq a} \sum_{i'=1}^{n} \frac{1}{y_{a'}-z_{i'}} f_{i'} \bigotimes_{i=1}^{n} v_{j_i}.$$

Identity (4.26) is equivalent to two relations:

$$\partial_{z_i}\omega(\mathbf{z},\mathbf{y}) = \partial_{y_a}\eta_{i,a}(\mathbf{z},\mathbf{y}),$$

$$\partial_{z_i}U(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) - \partial_{y_a}U(\mathbf{z},\mathbf{y})\eta_{i,a}(\mathbf{z},\mathbf{y}) - H_i(\mathbf{z})\omega(\mathbf{z},\mathbf{y}) = 0. \tag{4.27}$$

The first one is immediate whereas the second, more involved one implies that

$$H_i(\mathbf{z})\omega(\mathbf{z},\mathbf{y}) = \partial_{z_i}U(\mathbf{x},\mathbf{y})\ \omega(\mathbf{z},\mathbf{y}) \qquad \text{if} \quad \partial_{y_a}U(\mathbf{x},\mathbf{y}) = 0$$

i.e. the **Bethe Ansatz** diagonalization of the Gaudin Hamiltonians $H_i(\mathbf{z})$: the vectors $\omega(\mathbf{z},\mathbf{y})$ are common eigenvectors of $H_i(\mathbf{z})$, $i = 1,\dots,n$, with eigenvalues $\partial_{z_i}U(\mathbf{z},\mathbf{y})$ provided that $\mathbf{y}$ satisfies the Bethe Ansatz equations $\partial_{y_a}U(\mathbf{x},\mathbf{y}) = 0$. The relations between the Bethe Ansatz and the limit of the KZB connection when $k \to -h^\vee$ appear in the context of the Langlands geometric correspondence. These relations seem also to be at the heart of the question about the unitarity of the KZB connection for positive integer k.

§4.5. Coset theories

There is a rich family of CFT's which may be obtained from the WZW models by a simple procedure known under the name of a **coset construction**. On the functional integral level, the procedure consists of coupling the G-group WZW theory to a subgroup $H \subset G$ unitary gauge field B which is also integrated over with gauge-invariant insertions. Let us assume, for simplicity, that H is connected and simply connected, as G. Let $t_i \in (\mathrm{Hom}(V_{R_i}, V_{r_i}))^H$ be intertwiners of the action of H in the irreducible G- and H-representation spaces, respectively. The simplest correlation functions of the G/H coset theory take the form

$$\langle \prod_{i=1}^{n} \mathrm{tr}\, t_i g_{R_i}(x_i) t_i^* \rangle = \int \prod_{i=1}^{n} \mathrm{tr}_{V_{r_i}} t_i g_{R_i}(x_i) t_i^* \mathrm{e}^{-kS(g,B)} Dg\, DB \Big/ \int \mathrm{e}^{-kS(g,B)} Dg\, DB. \tag{4.28}$$

Note that the g-field integrals are the ones of the WZW theory and are given by eq. (4.13). Denoting $Z_{G/H} = \int \mathrm{e}^{-kS(g,B)} DgDB$, we obtain

$$Z_{G/H} \langle \prod_{i=1}^{n} \mathrm{tr}\, t_i g_{R_i}(x_i) t_i^* \rangle = H^{\alpha\beta} \int (\otimes t_i \Psi_\beta(B^{01}), \otimes t_i \Psi_\alpha(B^{01}))_{V_{\mathbf{r}}}\ \mathrm{e}^{-\frac{k}{2\pi}\|B\|^2_{L^2}} DB. \tag{4.29}$$

For $\Psi \in W(\Sigma, \mathbf{x}, \mathbf{R}, k)$, the map $B^{01} \mapsto \otimes t_i \Psi_\alpha(B^{01}) \in V_{\mathbf{r}}$ is a group H non-abelian theta function belonging to $W(\Sigma, \mathbf{x}, \mathbf{r}, \tilde{k})$ (the normalization of the Killing forms of G and H may differ, hence the replacement $k \to \tilde{k}$). Denote by T the corresponding map from $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ to $W(\Sigma, \mathbf{x}, \mathbf{r}, \tilde{k})$. Eq. (4.29) may be rewritten as

$$Z_{G/H} \langle \prod_{i=1}^{n} \mathrm{tr}\, t_i g_{R_i}(x_i) t_i^* \rangle = H^{\alpha\beta}\ (T\Psi_\beta, T\Psi_\alpha) = \mathrm{Tr}\, T^* T, \tag{4.30}$$

or choosing a basis (ψ_λ) of $W(\Sigma, \mathbf{x}, \mathbf{r}, \tilde{k})$,

$$Z_{G/H}\langle\prod_{i=1}^{n} \operatorname{tr} t_i g_{R_i}(x_i) t_i^*\rangle = H^{\alpha\beta}\overline{T_\beta^\lambda} h_{\lambda\nu} T_\alpha^\nu \tag{4.31}$$

where (T_α^λ) is the ("branching") matrix of the linear map T in bases (Ψ_α), (ψ_λ) and $h_{\lambda,\nu} = (\psi_\lambda, \psi_\nu)$. Since the above formula holds also for the partition function itself, it follows that the calculation of the coset theory correlation functions (4.28) reduces to that of the scalar products of group G and group H non-abelian theta functions, both given by explicit, finite-dimensional integrals.

Among the simplest examples of the coset theories is the case with $G = SU_2 \times SU_2$ with level $(k, 1)$ (for product groups, the levels may be taken independently for each group) and with H being the diagonal SU_2 subgroup. The resulting theories coincide with the unitary "minimal" series of CFT's with (Virasoro) central charges $c = 1 - \frac{6}{(k+2)(k+3)}$ first considered by Belavin-Polyakov-Zamolodchikov. The Hilbert spaces of these theories are built from the unitary highest weight representations of the Virasoro algebras with $0 < c < 1$ discussed in Lecture 2. The simplest one of them with $k = 1$ and $c = \frac{1}{2}$ is believed to describe the continuum limit of the Ising model at critical temperature or the scaling limit of the massless ϕ_2^4 theory. In particular, in the continuum limit the spins in the critical Ising model are represented by fields $\operatorname{tr} g_{1/2}(x)$ where g takes values in the first SU_2. The corresponding correlation functions may be computed as above. One obtains this way for the 4-point function an explicit expression in terms of hypergeometric functions.

Similar coset theories but at level $(k, 2)$ give rise to the supersymmetric $N = 1$ minimal unitary series of CFT's; the simplest one with $k = 1$ (appearing also at $k = 2$ in the previous series) corresponds to the so-called 3-critical Ising model.

The G/H coset theory with $H = G$ is a prototype of a two-dimensional topological field theory. As follows from eq. (4.30), the correlation functions of fields $\operatorname{tr} g_R(x)$ are equal to the dimension of spaces $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$, normalized by the dimension of $W(\Sigma, \emptyset, \emptyset, k)$ (and are given by the Verlinde formula). In particular, they do not depend on the position of the insertion points.

§4.6. WZW factory

As we have seen above, the coset construction allows us to obtain new soluble CFT's from the WZW models. Let us briefly discuss further refinements which permit an "assembly line production" of conformal models whose partition functions and correlation functions may be computed exactly, at least in principle. The most interesting cases of such models correspond to situations when two different constructions give rise to the same CFT, as in T-duality, mirror symmetry and other numerous instances.

1. If the group G is **not simply connected**, the original definition (4.3) of the action of the WZW model requires a modification. The result is possible further restrictions on the levels and the appearance, in some cases, of different quantizations of the same classical theory ("θ-vacua" or "discrete torsion"). The models are still exactly soluble although only the partition functions and the correlations of 'untwisted" fields have been worked out in detail for general G.

2. Let $H \subset G$ and $Z \subset H$ be a subgroup of the center Z_G of G. Let $P_{H'}$ be a principal H'-bundle where $H' = H/Z$ and $Q_G = P_{H'} \times_{\mathrm{Ad}_{H'}} G$ be the G-bundle associated to $P_{H'}$ via the adjoint action of H' on G. For appropriate k, and for a section g of Q_G and a connection B on $P_{H'}$ one may define the amplitude $\mathrm{e}^{-kS(g,B)}$. The (unnormalized) correlation functions of the coset G/H'-model may then be obtained by integrating gauge invariant insertions, weighted with $\mathrm{e}^{-kS(g,B)}$, over g and B and summing the result over inequivalent H'-bundles $P_{H'}$. Hence, for given $H \subset G$, there are as many coset theories as subgroups of $H \cap Z_G$ (some of them might have a non-unique vacuum).

3. If H is a discrete subgroup of G, then P_H carries a unique canonical flat connection and is given by a conjugation class of homomorphisms of the fundamental group of is Σ into H. The construction from the preceding point gives rise to the **orbifolds** of the WZW models.

4. Supersymmetric WZW models. One adds to the G-valued field g the (anticommuting) Majorana Fermi fields $\psi, \tilde{\psi}$ in the adjoint representation (i.e. sections of $L \otimes \mathbf{g}$ and $\bar{L} \otimes \mathbf{g}$, respectively, where L is a square root of the canonical bundle of Σ) and one considers the action

$$(4.32)\quad S(g,\psi,\tilde{\psi},A) = kS(g,A) - \frac{2}{\pi}\int \mathrm{tr}\left(\psi(\bar{\partial}_L + [A^{01},\cdot])\psi + \tilde{\psi}(\partial_{\bar{L}} + [A^{10},\cdot])\tilde{\psi}\right)$$

with the external group G gauge field A. The fermionic part of the theory is free and the complete theory may be easily solved.

5. Supersymmetric coset models. The action is as in eq. (4.32) except that A is replaced by a group H gauge field B and the Majorana fields $\psi, \tilde{\psi}$ are taken with values in $\mathbf{g}/\mathbf{h}$ rather than in $\mathbf{g}$. Both the supersymmetric WZW models and the supersymmetric coset models possess $N = 1$ superconformal symmetry.

6. $N = 2$ coset models. If G/H is a Kähler symmetric space then the supersymmetric G/H coset model possesses the $N = 2$ superconformal symmetry including the $U(1)$ loop group symmetry. The simplest examples are provided by the $SU(2)/U(1)$ models which, at level k, give rise to the minimal $N = 2$ superconformal theory with central charge $c = \frac{3k}{k+2}$.

7. Orbifolds of tensor products of conformal field theories may give rise to essentially new models. The famous examples are the $(\mathbf{Z}/5\mathbf{Z})^3$ orbifolds of the product of five $k = 3$ minimal $N = 2$ superconformal models. Two different orbifolds may give equivalent conformal sigma models corresponding to a mirror pair of Calabi-Yau quintic targets.

8. The theories with $U(1)$ loop group symmetries like the $N = 2$ supersymmetric coset models may be **twisted** by considering their fields as taking values in bundles associated with the sphere subbundle of the spin bundle L and coupled to the spin connection. Such twisting of $N = 2$ superconformal models may be done in two essentially different ways (A- and B-twist) and it produces topological field theories. The genus 0 correlations of the A-twisted $N = 2$ sigma models compute the quantum cohomology of the target.

9. For **non-compact groups** G, the WZW action $S(g)$ is not bounded below but one may try to stabilize the Euclidean functional integral by analytic continuation or/and coset-type gauging of subgroups of G. Such stabilization procedures may

however destroy the physical positivity (Hilbert-space picture) of the theory. The best studied models of non-compact type correspond to finite coverings of $SL_2(\mathbf{R})$ with the $U(1)$ subgroup twisted and the nilpotent subgroup gauged away (the construction of minimal models *à la* Drinfeld-Sokolov), the $SL_N(\mathbf{R})$ generalizations thereof, the Liouville and Toda theories and the $SL_2(\mathbf{R})/U(1)$ black hole model. Our knowledge of non-compact WZW models is certainly much less complete than that of the compact case (note that this is true also on the level of quantum mechanics, where we know everything about the harmonic analysis of compact Lie groups but the harmonic analysis of non-compact ones still has open problems).

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String Theory

Eric D'Hoker

Introduction

One of the most ambitious goals in physics is the unification of the laws governing the matter and the forces in Nature. The physical principles of *Quantum Mechanics* and *Relativity* were formulated in the first quarter of this century and, remarkably, have not required alteration since. By the late 1960's, the *electro magnetic* and *weak forces* were described by a *Quantum Field Theory* whose predictive power rests on the smallness of its couplings. The description of the *strong force* by quantum field theory remained intractable at that time because the effective couplings measured at nuclear energies always appeared to be large, forcing any candidate theory into a highly non-linear regime.

String Theory was invented in those years as an attempt to capture the empirical dynamics of the strong force, but it quickly presented obstacles of its own to becoming a viable theory of this force. By the mid 1970's, *Quantum Chromodynamics*, a theory based on *Yang-Mills gauge fields*, emerged as the uncontested quantum field theory description of the strong force. The predictive power of this theory at very high energies is guaranteed by the smallness of the coupling in view of asymptotic freedom. At low energies, this theory effectively becomes strongly coupled, and is widely expected to produce phenomena such as the permanent confinement of quarks and gluons, whose quantitative understanding remains however largely an open issue to this day.

Just as string theory was about to enter a 10 year period of virtual hibernation in the mid 1970's, it was recognized that the very obstacles that rendered it unsuccessful as a theory of the strong force made it an outstanding candidate for a theory of the unification of all matter and forces in Nature. Indeed, string theory was found to invariably encompass in its low energy limit the theories of *Einstein gravity* and of Yang-Mills gauge fields, albeit in dimension ten, which is larger than the physical four. By the early 1980's, it was realized that the newly discovered *supersymmetry* and *supergravity* models are also invevitably tied to string theory, and the first *superstring theories* (so-called *Type IIA,B and Type I*) were formulated.

The first string revolution of the mid 1980's was sparked by the dramatic discovery that superstring theory is anomaly-free and appears to be well-behaved at short distances, in contrast with field theories of gravity. Furthermore, the newly discovered *heterotic string theory* was found to possess many of the key ingredients necessary to reproduce the standard model of strong and electro weak interactions

at low energy along with a consistent theory of gravity. While string theory exists naturally in ten space-time dimensions, string solutions were obtained in which six of these dimensions are compact, of very small size, and thus practically unobservable at long distance scales, where space-time effectively appears four-dimensional. Tori, orbifolds and Calabi-Yau manifolds were put forward as possible such six-dimensional spaces. The techniques of weak coupling string perturbation theory and conformal field theory allowed one to check whether any given compactification space is a string solution, and large classes of solutions were thus classified. Yet, the fundamental principles of string theory which would govern the selection of the compactification spaces as viable string solutions remained elusive.

In the second string revolution of the mid 1990's, progress was achieved in a totally unexpected direction. Amongst the extended superstring solutions, discovered in the late 1980's, it was found that the Dirichlet particle or *D0 brane* of Type IIA superstring theory unmasks the existence of an eleventh dimension. As the radius of this extra dimension vanishes with the string coupling, the eleventh dimension is quintessentially non-perturbative in the string coupling. This is why its existence had been hidden from string theory for so long. The appearance of the extra dimension came to many as a happy surprise, as eleven is the maximum dimension in which supergravity can exist. The enlarged theory has been named *M-theory*, in part because the fundamental objects in the eleven dimensional theory appear to be *membranes* rather than strings. As of yet, a concrete and practical definition of M-theory is not available, and our best approach to it appears to still be from its string theory and supergravity limits.

This material was originally presented in 15 lectures, but the written account has been rearranged by subject into 10 sections. We present an introduction to some of the physics and the mathematics of string theory and supergravity. Many of the basics are most easily explained in a string theory of bosons only, and the first few lectures are devoted to the bosonic string. A systematic construction is given of the five known superstring theories in ten dimensions : Type IIA,B, Type I and Heterotic $E_8 \times E_8$ and $Spin(32)/Z_2$, and their weak coupling perturbation theory in terms of quantum field theory and conformal field theory on Riemann surfaces is presented. The equations describing space-times in which strings can propagate consistently are derived, and their relation with supergravity is made clear. Finally, we give a definition and brief discussion of supergravity in ten and eleven dimensions.

After each lecture, a brief bibliography will be given. Of course, no attempt is being made to give a list of references that is in any way complete. Instead, we wish to direct the reader to those texts that would appear most suited to members of a mathematically minded audience, and most efficient in communicating the basics of these vast subjects.

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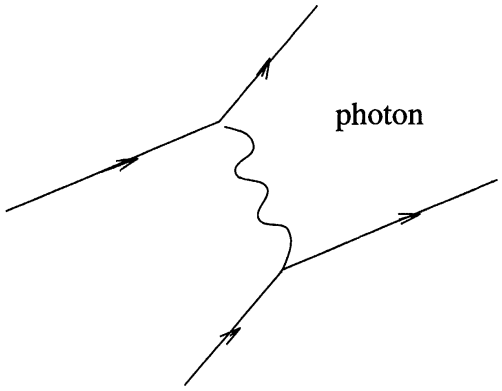
LECTURE 1
Point Particles vs Strings

In physics, the structure of matter is understood in terms of the dynamics of smaller constituents. Atoms are composed of nuclei and electrons, nuclei are made up of protons and neutrons, which in turn are bound states of quarks. To the present day limit of distance scales up to approximately 10^{-15} cm, electrons and quarks do not show any substructure, and are viewed as "elementary particles". (Which particles are elementary in this way may change over time, since at smaller distance scales substructure may be found in terms of more elementary particles.) In quantum field theory, such elementary particles are described as *points* in space.

Forces between particles result from exchanges between them of other elementary particles, and the fundamental interactions occur in a *local* way, at coincident time at the same point in space. For example, electro magnetic interaction forces between charged particles result from the exchange of photons, with local couplings to the charged particles.



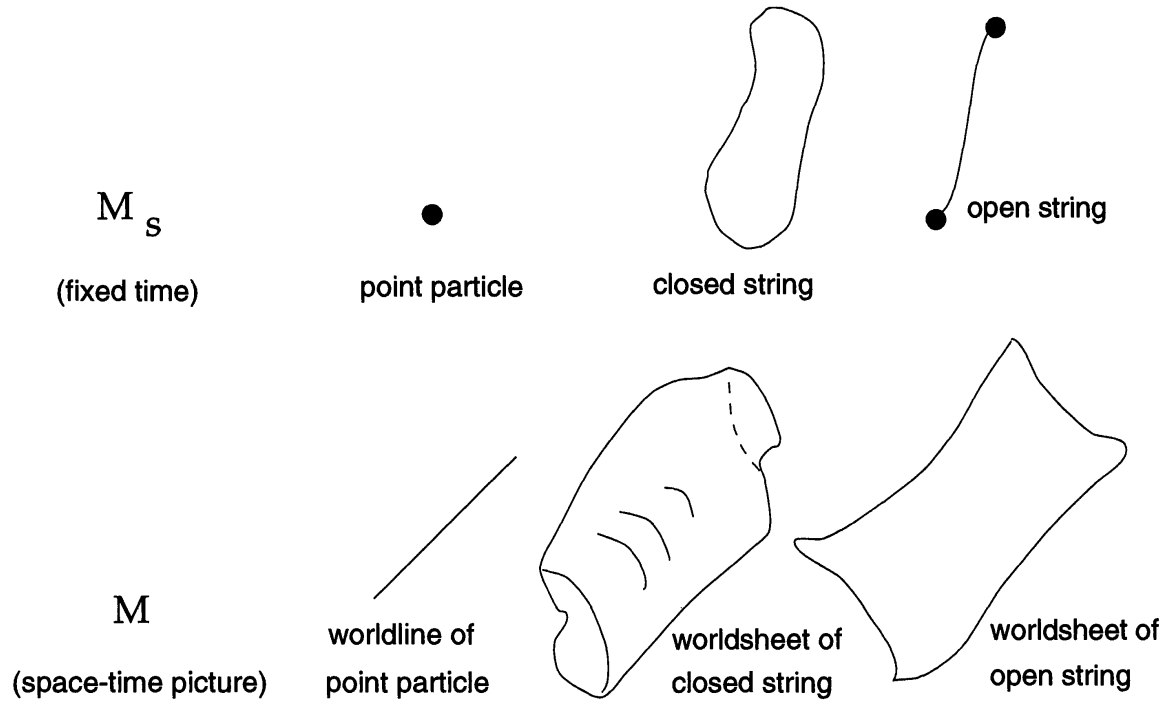
The photon thereby transfers momentum and produces a force, the Coulomb force.

Quantum field theory of point particles (QFT) has been extremely successful in describing quantitatively three out of the four fundamental forces of Nature by *Yang–Mills theory*: electro magnetic, weak and strong interactions. However, no consistent quantum field theory description appears to be available (or to exist?) for the force of gravity. One thus seeks an enlargement of the framework of QFT in which all forces of nature fit consistently with the known physical principles of quantum mechanics and (special and general) relativity.

String theory to date seems to be the only such candidate. In fact my task will be to show that consistent string theories automatically contain gravity, supersymmetry and under certain circumstances Yang–Mills theory.

§1.1. Strings

In string theory, particles are not described as points, but instead as strings: one-dimensional extended objects. If M_s and $M(\sim \mathbb{R} \times M_s)$ denote the space and space-time manifolds respectively, then we picture strings as follows:



While the point particle sweeps out a one-dimensional *worldline*, the string sweeps out a *worldsheet*, i.e. a 2-dimensional real surface. For a *free string*, the topology of the worldsheet is a *cylinder* (in the case of a closed string) or a *sheet* (for an open string).

Roughly, different elementary particles correspond to different vibration modes of the string just as different minimal notes correspond to different vibrational modes of musical string instruments.

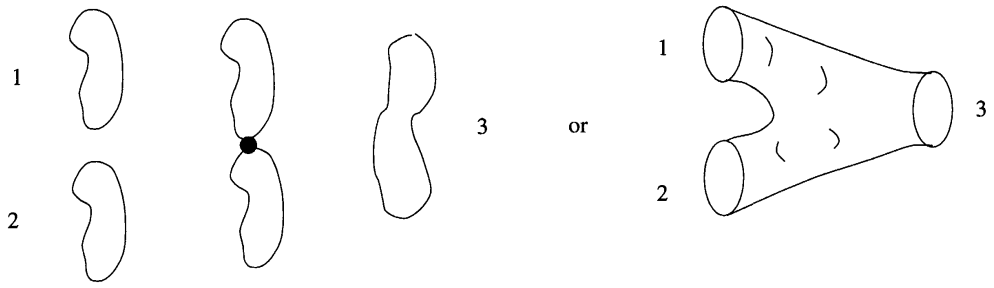
It will turn out that the physical size of strings is set by gravity, more precisely the Planck length $\ell_P = \sqrt{G_N \hbar c^{-3}} \sim 10^{-33}$ cm. This scale is so small that we effectively only see point particles at our distance scales. Thus, for length scales much larger than ℓ_P, we expect to recover a quantum field theory description of point particles, plus typical string corrections that represent physics at the Planck scale.

Extended objects arise in many areas of physics. For example, *flux tubes* arise in superconductivity, and are used to describe the physics of cosmic strings and vortices. However, those objects do have a transverse size, and are only *approximately one-dimensional.* String theory considered here will always involve strictly one-dimensional objects, suitable for the problems of quantum gravity and particle physics. In certain instances, it may be possible to use the strictly one-dimensional string models as approximations to the dynamics of flux tubes.

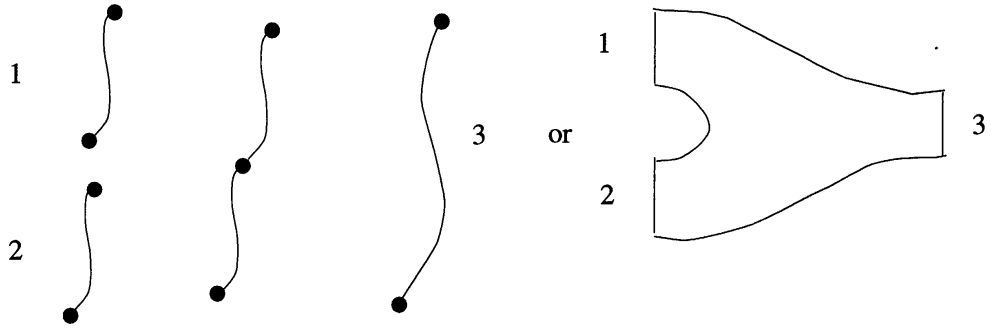
§1.2. Interactions

While the string itself is extended, the fundamental interactions are *local*, just as for point particles. The interaction takes place when strings overlap in space at the same time.

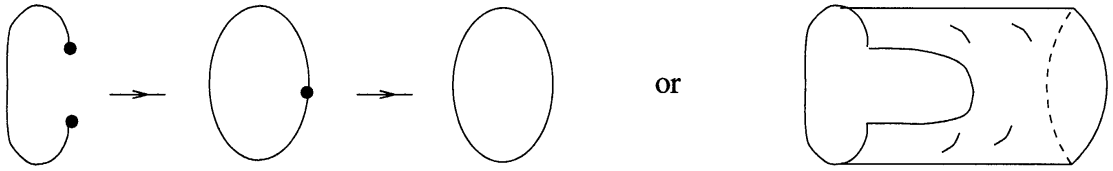
For *closed* strings



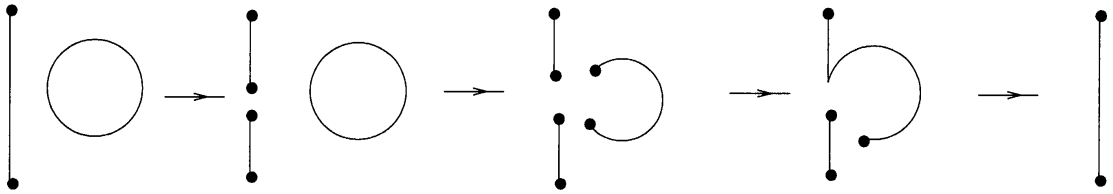
For *open* strings, end points can join



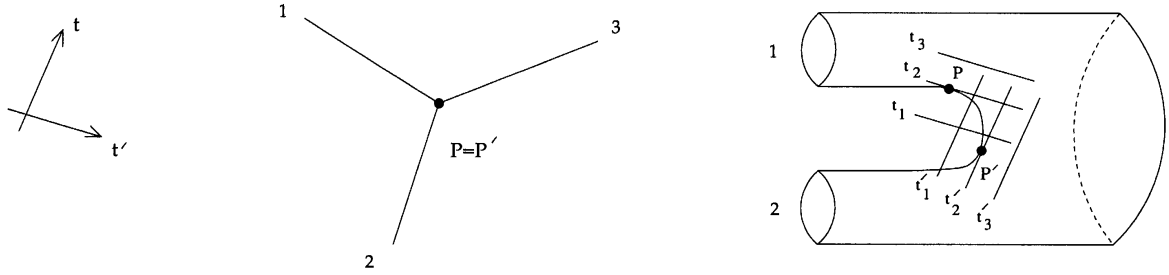
For consistency, as soon as an interacting theory contains open strings, it must also contain closed strings, since end points can join. Such theories are "*open string theories*".



On the other hand, it is consistent to have a theory of only closed strings, called *closed string theories.* Other interactions result from combining the interactions defined above



In point particle theories, the fundamental interactions are read off from the QFT Lagrangian. An interaction occurs at a *geometrical point*, where the world-lines join and cease to be a manifold. In Lorentz invariant theories (with M flat Minkowski space-time), the interaction point is Lorentz invariant. To specify how the point particles interact, additional data must be supplied at the interaction point, giving rise to *many possible* distinct quantum field theories.



In string theory, the interaction point *depends* upon the Lorentz frame chosen to observe the process. In the figure above, equal time slices are indicated from the point of view of two different Lorentz frames, schematically indicated by t and t'. The closed string interaction, as seen from frames t and t', occurs at times t_2 and t_2' and at (distinct) points P and P' respectively.

Lorentz invariance of interaction forbids that any point on the worldsheet be singled out as interaction point. Instead, the interaction results purely from the joining and splitting of strings. While free closed strings are characterized by their topology being that of a cylinder, interacting strings are characterized by the fact that their associated worldsheet is connected to at least 3 strings, incoming and/or outgoing.

As a result, the free string determines the nature of the interactions completely, leaving only the string coupling constant undetermined.

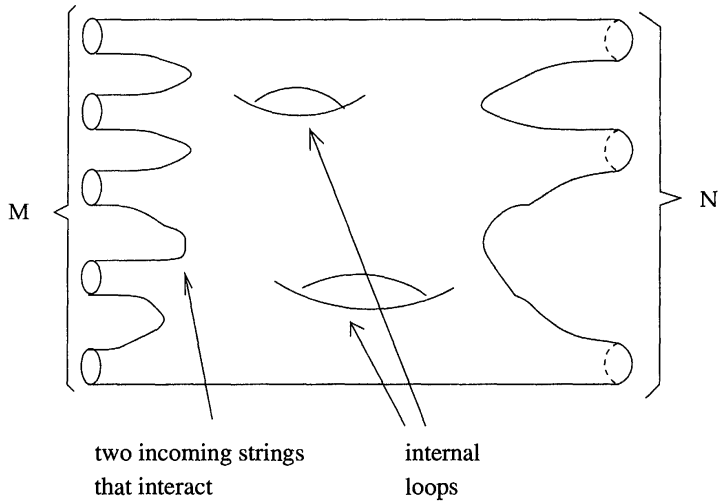
Additional structure

- Oriented strings: all worldsheets are assumed to be orientable.
- Non-oriented strings: worldsheets that are non-orientable, such as the Möbius strip and Klein bottle, are included as possible configurations as well.

We shall almost exclusively deal with *oriented* strings only.

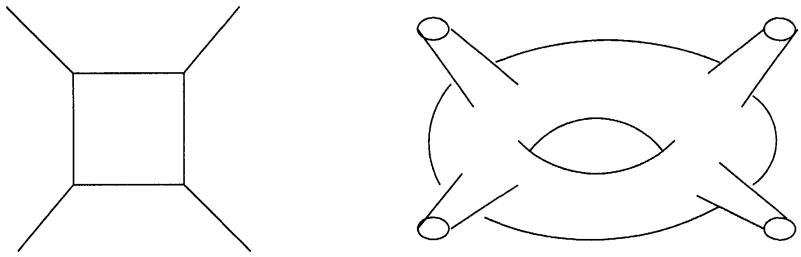
§1.3. Loop expansion–topology of closed surfaces

For simplicity consider closed oriented strings only, so that the associated worldsheet is also oriented. A general string configuration describing the process in which M incoming strings interact and produce N outgoing strings looks at the topological level like a closed surface with $M + N = E$ boundary components and any number of handles.



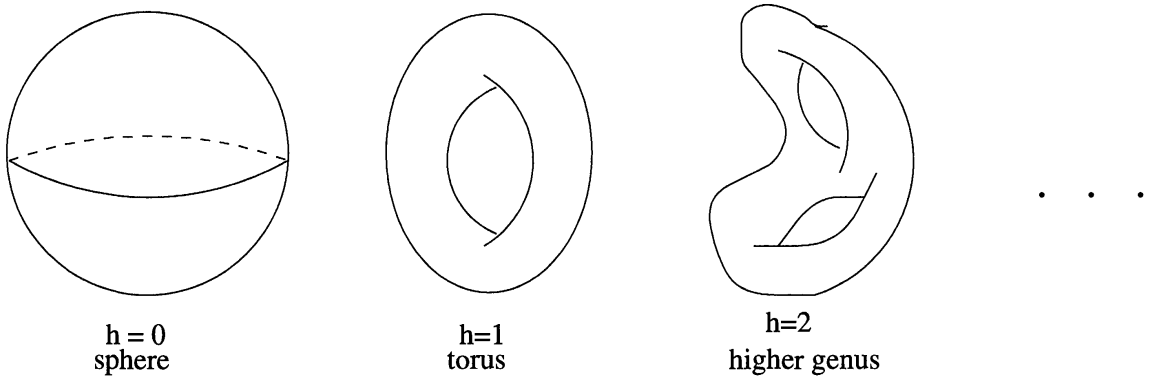
The internal loops may arise when virtual particle pairs are produced, just as in quantum field theory. For example, a Feynman diagram in quantum field theory that involves a loop is shown below together with its corresponding configuration

in string theory:



Surfaces associated with closed oriented strings have two topological invariants: the *number of boundary components* E (as we shall see later, boundary components may be shrunk to punctures, under certain conditions), and the number h of *handles* on the surface, which equals the *genus*.

When $E = 0$, we just have the topological classification of compact oriented surfaces without boundary:



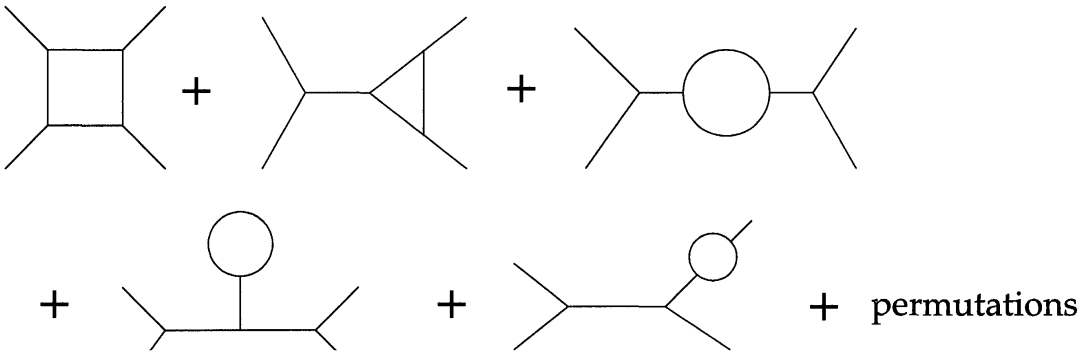
Rendering $E > 0$ is achieved by removing E discs from the surface.

Recall that in quantum field theory an expansion in powers of Planck's constant $\hbar$ yields an expansion in the number of *loops* of the associated Feynman diagram, for a given number of external states:

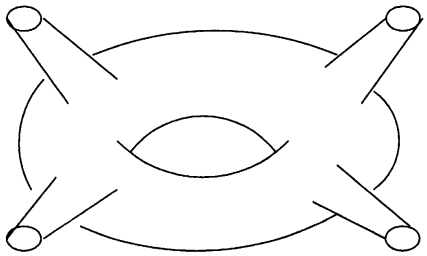
$$\hbar^{E+h-1} \begin{cases} \hbar & \text{for every propagator} \\ \hbar^{-1} & \text{for every vertex} \\ -1 & \text{for overall momentum conservation.} \end{cases}$$

Thus, in string theory we expect that, for a given number of external strings E, the topological expansion genus by genus should correspond to a *loop* expansion as well. (See Lecture 5 of [I-**Axiomatics**].)

In quantum field theory, there are in general many Feynman graphs that correspond to an amplitude with a given number of external particles and a given number of loops, e.g. for $E = 4$ external particles and $h = 1$ loop in ϕ^3 theory we have



The same process in string theory (for closed oriented strings) is described by just a single diagram

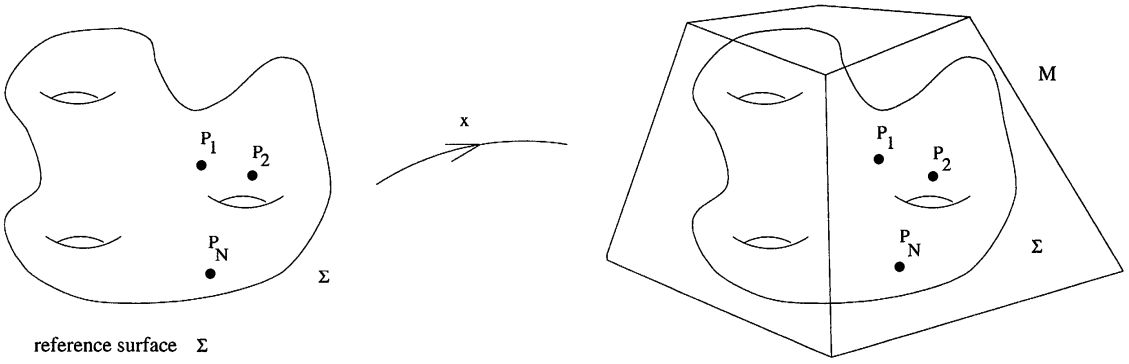


Later on in the lectures we will show that the single string diagram reproduces all the field theory diagrams. The different field theory diagrams arise from different cells in a *cell-decomposition of the moduli space of surfaces.*

§1.4. Transition amplitudes for strings

The fundamental quantities we are interested in examining in a quantum field theory are the transition amplitudes for processes in which a number M of incoming particles scatter to produce N outgoing particles. (The squared modulus of the amplitude yields the probability for this process to take place.)

The only way we have today to define string theory is by giving a *rule* for the evaluation of transition amplitudes, order by order in the loop expansion, i.e. genus by genus. The rule is to assign a relative weight to a given configuration and then to sum over all configurations. To make this more precise, we first describe the space of configurations.



reference surface Σ

space-time manifold M

We assume that Σ and M are differentiable manifolds, of dimensions 2 and D respectively, and that x is a continuous map from Σ to M. If ξ^m, $m = 1, 2$, are local coordinates on Σ and x^μ, $\mu = 1, \ldots, D$, are local coordinates on M then the map x may be described by functions $x^\mu(\xi^m)$ which are continuous.

To each configuration is associated a weight

$$e^{-S[x,\Sigma,M]}, \qquad S \in \mathbb{C}$$

and the transition amplitude for specified external strings (incoming and outgoing) is obtained by summing over all surfaces Σ and all possible maps x.

$$A = \sum_{\text{surfaces } \Sigma} \sum_{x} e^{-S[x,\Sigma,M]}$$

We now need to specify each of these ingredients:

1) We assume M to be a D-dimensional Riemannian manifold, with metric $(\ ,\ )_G$. A special case is flat Euclidean space-time $\mathbb{R}^D$. The space-time manifold and metric are assumed *fixed.*

$$ds^2 = (dx, dx)_G = \sum_{\mu,\nu=1}^{D} G_{\mu\nu}(x) dx^\mu \otimes dx^\nu \tag{1.1}$$

The physical world is of course not described by a Riemannian manifold, but rather by a pseudo-Riemannian space, such as Minkowski space-time. We shall always take as a starting definition the Riemannian situation, and analytically continue to pseudo-Riemannian spaces to obtain physical transition amplitudes. For flat Euclidean space-time, this continuation is analogous to the correspondence between Schwinger and Wightman functions in QFT (see [I-**Axiomatics**]).

2) The metric G on M induces a metric on Σ: $\gamma = x^*(G)$

$$\gamma = \gamma_{mn} d\xi^m \otimes d\xi^n$$
$$\gamma_{mn} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^m} \frac{\partial x^\nu}{\partial \xi^n}$$

This metric is non-negative, but depends upon x. It is advantageous to introduce an intrinsic Riemannian metric g on Σ, independently of x; in local coordinates, we have

$$g = g_{mn}(\xi) d\xi^m \otimes d\xi^n. \tag{1.2}$$

3) The worldsheet action $S[x, \Sigma, M]$ will in general depend on the map x. We choose S to be intrinsic, with the following properties

$$\begin{cases} Diff^+(\Sigma) & \text{invariant} \\ Diff(M) & \text{invariant} \\ \text{local in its dependence on } x, g, \text{ and } G \\ \text{renormalizable as a QFT.} \end{cases}$$

A natural intrinsic candidate for S is the area of $x(\Sigma)$, which gives the *Nambu-Goto action*

$$\text{Area}\,(x\,(\Sigma)) = \int_\Sigma d\mu_\gamma = \int_\Sigma d^2\xi \sqrt{\det\, \gamma_{mn}} \tag{1.3}$$

It depends only upon G and x, but not on g. The difficulty with this action is that it is non-polynomial in x and its derivatives, even in flat space-time. Its quantization is difficult to define unambiguously.

Instead, we take as starting point the *Polyakov action*

$$\begin{aligned} S[x, g, G] &= \kappa \int_\Sigma (dx, *dx)_G \\ &= \kappa \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \end{aligned} \tag{1.4}$$

Here κ is the *string tension*, which is a positive constant with dimension of inverse length squared. The stationary points of S with respect to g are at $g^0 = e^{\phi}\gamma$ for some function ϕ on Σ, and thus $S[x, g^0, G] \sim \text{Area}\,(x(\Sigma))$.

The Polyakov action leads to well-defined transition amplitudes, obtained by integration over the space $\text{Met}(\Sigma)$ of all positive metrics on Σ for a given topology, as well as over the space of all maps $\text{Map}\,(\Sigma, M)$: We *define*

$$A = \sum_{\substack{\text{topologies} \\ \Sigma}} \int_{\text{Met}(\Sigma)} \frac{1}{\mathcal{N}(g)} \int_{\text{Map}(\Sigma,M)} Dx\, e^{-S[x,g,G]}, \tag{1.5}$$

and make the following remarks.

- $\mathcal{N}$ is a normalization factor, to be determined later.
- The measures Dg and Dx are constructed from $Diff^+(\Sigma)$ and $Diff(M)$ invariant L^2 norms on Σ and M.
- For fixed metric g, the action S is well-known: its stationary points are the harmonic maps x: $\Sigma \to M$. Here, however, g varies and in fact is to be integrated over.
- For a general metric G, the action S defines a *non-linear sigma model*, which is renormalizable because the dimension of Σ is 2. It would not in general be renormalizable in dimension higher than 2, which is usually regarded as an argument against the existence of fundamental membrane theories.

§1.5. Weyl invariance and vertex operator formulation

The action S is also invariant under *Weyl rescalings* of the metric g by a positive function on Σ:

$$g \to e^{2\sigma} g \qquad\qquad \sigma\colon \Sigma \to \mathbb{R}.$$

In general, Weyl invariance of the full amplitude may be spoiled by anomalies. We shall establish later on that Weyl invariance is a crucial ingredient to obtain a consistent theory.

Assuming Weyl invariance of the full amplitude, the integral defining A may be *simplified* in two ways.

1) The integration over $\text{Met}(\Sigma)$ effectively collapses to an integration over the *moduli space of surfaces*, which is finite dimensional, for each genus h.

2) The boundary components of Σ – characterizing external string states – may be mapped to regular points on an underlying compact surface without boundary by conformal transformations. The *data*, such as momenta and other quantum numbers of the external states, are mapped into *vertex operators*. The amplitudes are now

$$A = \sum_{h=0}^{\infty} \int_{\text{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int_{\text{Maps}(\Sigma,M)} Dx\, V_1 \ldots V_N e^{-S} \tag{1.6}$$

for suitable vertex operators $V_1, \ldots, V_N$. We shall construct such vertex operators explicitly in Lecture 3.

§1.6. More general actions

The action S given above is not the most general one that satisfies the criteria we listed. Generalizations are possible when M carries extra structure.

1) M carries a 2-form $B \in \Omega^{(2)}(M)$, the so-called *anti-symmetric tensor field.* The resulting contribution to the action is also that of a "non-linear sigma model"

$$S_B[x, B] = \int_\Sigma x^*(B) = \int_\Sigma dx^\mu \wedge dx^\nu B_{\mu\nu}(x) \tag{1.7}$$

2) M may carry a *dilaton field* $\Phi \in \Omega^{(0)}(M)$ so that

$$S_\Phi[x, \Phi] = \int_\Sigma d\mu_g R_g \Phi(x). \tag{1.8}$$

where R_g is the Gaussian curvature of Σ for the metric g.

3) There may be a *tachyon field* $T \in \Omega^{(0)}(M)$ contributing

$$S_T[x, T] = \int_\Sigma d\mu_g T(x). \tag{1.9}$$

The physical significance of these fields will be discussed in Lecture 6.

§1.7. Transition amplitude for a single point particle

The transition amplitude for a single point particle could in fact be obtained in a way analogous to how we prescribed string amplitudes. Let space-time be again a Riemannian manifold M, with metric $(\ ,\ )_G$. The prescription for the transition amplitude of a particle traveling from a point $y \in M$ to a point y' to M is expressible in terms of a sum over all (continuous) paths connecting y to y':

$$A(y, y') = \sum_{\substack{\text{paths} \\ \text{joining } y \text{ and } y'}} e^{-S[\text{path}]}$$

Paths may be parametrized by maps from $C = [0, 1]$ into M with $x(0) = y$, $x(1) = y'$. A simple worldline action for a massless particle is obtained by introducing a metric g on $[0, 1]$

$$S[x, g] = \frac{1}{2} \int_C d\tau\, g(\tau)^{-1} \dot{x}^\mu \dot{x}^\nu G_{\mu\nu}(x) \qquad \dot{x} = \frac{dx}{d\tau}$$

which is invariant under $Diff^+(C)$ and $Diff(M)$.

The analogous prescription for the point particle transition amplitude is

$$A(y, y') = \int_{\text{Met}(C)} Dg \frac{1}{\mathcal{N}} \int_{\text{Map}(C,M)} Dx\, e^{-S[x,g]}$$

Notice that for string theory, we had a prescription for transition amplitudes valid for all topologies of the worldsheet. For point particles, there is only the

topology of the interval C, and we can only describe a single point particle, but not interactions with other point particles. To put those in, we would have to supply additional information.

It is very instructive to work out the amplitude A by carrying out the integrations. The only $Diff^+(C)$ invariant of g is the length $L = \int_0^1 d\tau\, g(\tau)$; all else is generated by $Diff^+(C)$, and may be gauge-fixed following the standard Faddeev-Popov procedure. Defining the normalization factor to be the volume of $Diff(C)$: $\mathcal{N} = \mathrm{Vol}(Diff(C))$ we have $Dg = Dv\, dL$ and the transition amplitude becomes

$$\begin{aligned} A(y,y') &= \int_0^\infty dL \int Dx\, e^{-\frac{1}{2L}\int_0^1 d\tau (\dot{x},\dot{x})_G} \\ &= \int_0^\infty dL \Big\langle y' | e^{-L\Box} | y \Big\rangle = \Big\langle y' | \frac{1}{\Box} | y \Big\rangle. \end{aligned}$$

Thus, the amplitude is just the Green function at (y,y') for the Laplacian (for Euclidean space-time) or the d'Alembertian (in the Minkowski case), and corresponds to the propagation of a massless particle.

§1.8. Generalized point particle propagation

The manifold M may carry structure in addition to the metric $(\ ,\ )_G$, which is reflected in the point particle action.

1. A $U(1)$ line bundle on M with connection A gives rise to

$$S_A = ie \int_C x^*(A) = ie \int_C d\tau\, \dot{x}^\mu\, A_\mu(x), \tag{1.10}$$

   i.e. point particle with charge e in the presence of a $U(1)$ gauge field.

2. An SU(N) vector bundle on M with connection A. To describe a particle in the defining representation ρ of SU(N), introduce worldline fermions $\lambda^i(\tau)$ $i = 1, \ldots, N$ and the action

$$S_{A_\rho} = \int_C \bar{\lambda}(d + x^*(A_\rho))\lambda \tag{1.11}$$

3. The spin bundle over M with spin connection ∂ describes space-time spinors of M. Introduce worldline fermions ψ and the action

$$S_\omega = \int_C (\psi, \partial\psi)_G \tag{1.12}$$

   The total particle worldline action $S + S_\omega$ has worldline supersymmetry. (See [I-**Homework**].)

References and cross references

First, very useful general references on the subject of string theory are

M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vols. I, II, Cambridge University Press, 1988.

A. M. Polyakov, *Gauge Fields and Strings*, Harcourt Brace Jovanovich, 1987.

A systematic exposition of 2-dimensional quantum field theory and especially conformal field theory has recently appeared in

P. Di Francesco, P. Mathieu and P. Sénéchal, *Conformal Field Theory*, Springer Verlag, 1997.

There is also a book, which is scheduled to appear in 1998,

J. Polchinski, *String Theory*, vols. I, II, Cambridge University Press, 1998, to appear.

Much of the material in these lectures is directly inspired by the review article

E. D'Hoker and D. H. Phong, *The geometry of string perturbation theory*, Rev. Modern Phys. **60** (1988), 917–1065.

Finally, accounts of at least some of the recent developments in duality and string theory may be found in various proceedings of summer schools, such as those given at Trieste and TASI. We shall just list one of those here

C. Efthimiou and B. Greene, eds., *TASI 1996*, World Scientific Publishers, 1998.

LECTURE 2
Spectrum of Free Bosonic Strings

We begin by constructing the Hilbert space of physical states and the spectrum of a single free bosonic closed oriented string, whose worldsheet has the topology of a cylinder (or equivalently an annulus) and of a single free open oriented string, whose worldsheet has the topology of half a cylinder (or equivalently half an annulus).

A free string should not couple to any other strings, or to any non-trivial background fields in the space-time manifold M. Thus we take $M = \mathbb{R}^D$, with constant $G_{\mu\nu}$ and $B_{\mu\nu} = \Psi = T = 0$ to start. We wish to examine the properties of physical strings in physical space-time, so it is essential that the space-time metric be Minkowskian, with $G_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(- + + \cdots +)$ and $\mu, \nu = 0, 1, \ldots, D-1$. In quantum field theory, the problem of constructing the Hilbert space of states and the spectrum of a single particle amounts to finding the 1-particle unitary representations of the Poincaré group – a problem for which the Minkowski nature of space-time is again essential.

A single string has an infinite number of excitation states, each of which corresponds to a representation of the Poincaré group. In this lecture, we wish to identify which representations of the Poincaré group occur (i.e. find the spectrum) and determine the conditions under which all these representations are unitary. We will find that conformal invariance plays an essential rôle.

Before we discuss the details of this fundamental analysis of string theory, let us outline the main steps of our strategy.

1) In the first lecture, we postulated a prescription for the transition amplitudes of N external string states – represented by vertex operators V_i, $i = 1, \ldots, N$ – in a general Riemannian space-time M with metric G (as well as possibly other fields $B_{\mu\nu}$, Φ, T)

$$A = \sum_{h=0}^{\infty} \int_{\text{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int_{\text{Map}(\Sigma, M)} Dx\, V_1 \ldots V_N\, e^{-S[x,g]} \tag{2.1}$$

2) To investigate the spectrum of the string in Minkowski $\mathbb{R}^D_M$ we shall start with the above amplitudes in Euclidean $\mathbb{R}^D_E$ with $G_{\mu\nu} = \delta_{\mu\nu}$, $B_{\mu\nu} = \Phi = T = 0$ and action

$$S[x, g] = \frac{1}{8\pi} \int_{\Sigma} d\mu_g g^{mn} \partial_m x \cdot \partial_n x \ . \tag{2.2}$$

(For flat Euclidean or Minkowskian metric, we have $x \in \mathbb{R}^D_E$ or $\mathbb{R}^D_M$ respectively and we use the notation $(v, v') = v \cdot v'$.) The worldsheet metric g is always kept Riemannian. The string tension κ introduced in §1.4 has been set to $\kappa = 1/8\pi$. In the physics literature one often finds the string tension expressed in terms of α' (the Regge slope parameter) by $\kappa = \frac{1}{4\pi\alpha'}$; so in our conventions $\alpha' = 2$.

3) At fixed g, the quantum field theory defined by $S[x, g]$ provides an example of a *conformal field theory* (see also [II-**CFT**]). All we shall need of the conformal field theory to solve for the free string spectrum is contained in the operator product expansions.

4) Once we have derived the OPE's for Euclidean $\mathbb{R}^D_E$, we shall continue the time component of the map x, let $x^0 \to ix^0$ and obtain well-defined OPE's for the operators in Minkowskian $\mathbb{R}^D_M$. In particular, we obtain two Virasoro algebras $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$ for the closed string, and a single Virasoro algebra Vir_L for the open string.

5) From the Minkowskian OPE's and associated algebra, we construct Fock spaces $\mathcal{F}_{\mathrm{open}}$ and $\mathcal{F}_{\mathrm{closed}}$ for open and closed strings respectively along the same lines that a Hilbert space was constructed in [II-**CFT**]. Due to the Minkowski signature of the space-time metric, however, the Fock space we obtain does *not* carry a positive definite inner product. $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$ acts on $\mathcal{F}_{\mathrm{closed}}$ and Vir_L acts on $\mathcal{F}_{\mathrm{open}}$.

6) The remaining integration over g – which is *not* present in conformal field theory, but characteristic to string theory – effectively selects out of $\mathcal{F}$ (either $\mathcal{F}_{\mathrm{closed}}$ or $\mathcal{F}_{\mathrm{open}}$) a subspace $\mathcal{F}^+$ on which we have a positive definite inner product, under certain conditions which we shall determine. In particular, this is how the critical dimension $D = 26$ arises.

(The occurrence of negative norm states in a standard Fock space construction is not unique to string theory. It will take place for gauge theories and for gravity as well, or more generally in any theory with particles of spin ≥ 1.)

§2.1. Basics of conformal field theory

• An oriented surface Σ, with a (fixed) Riemannian metric g automatically carries a complex structure, and we choose local complex coordinates z and $\bar{z}$ in which g is conformally flat, and given by

$$g = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = 2g_{z\bar{z}}|dz|^2 \tag{2.3}$$

This metric is left invariant under analytic or *conformal* diffeomorphisms $z \to z' = f(z)$, with f complex analytic, combined with Weyl transformations on g. A quantum theory invariant under this transformation is a conformal field theory (abbreviated CFT).

• Fundamental operators in CFT are conformal tensors or *primary fields* ϕ of conformal weight $(h, \bar{h})$ satisfying the transformation law

$$\phi = \phi(z, \bar{z})(dz)^h(d\bar{z})^{\bar{h}} = \phi'(z', \bar{z}')(dz')^h(d\bar{z}')^{\bar{h}} \ . \tag{2.4}$$

For integer weights h and $\bar{h}$, these are operator-valued sections of $K^h \otimes \bar{K}^{\bar{h}}$, where K is the holomorphic cotangent (or canonical) bundle of Σ. In general, h and $\bar{h}$

don't have to be integers, but their difference is the spin of the field ϕ and must satisfy $h - \bar{h} = \text{spin} \in \frac{1}{2}\mathbb{Z}$.

• In the conformal field theories we shall deal with in these lectures, the matrix elements of products of primary fields (and other fields) may be expressed in terms of functional integrals:

$$\langle \phi_1(\xi_1) \dots \phi_N(\xi_N) \rangle_g = \int Dx \ \phi_1(\xi_1) \dots \phi_N(\xi_N) e^{-S[x,g]} \tag{2.5}$$

where it is understood that ϕ_i can be expressed in terms of the canonical fields x. Notice that here we do not normalize correlation functions by dividing out by the partition function $\langle 1 \rangle_g$.

• The stress tensor is defined as the variation of the action S with respect to the inverse metric $g^{-1} = g^{mn}\partial_m \otimes \partial_n$, where the metric g is viewed as a background source field.

$$\delta_g S = -\frac{1}{4\pi} \int_\Sigma d\mu_g T_{mn} \delta g^{mn} \tag{2.6}$$

Assuming that the variation g has support away from the operator positions $\xi_1, \dots, \xi_N$, the variation of correlation functions is given by

$$\delta_g \langle \phi_1(\xi_1) \dots \phi_N(\xi_N) \rangle = \frac{1}{4\pi} \int_\Sigma d\mu_g \delta g^{mn} \langle T_{mn} \phi_1(\xi_1) \dots \phi_N(\xi_N) \rangle \tag{2.7}$$

The latter equation may be used as a definition of T_{mn}, even in conformal field theories where no Lagrangian formulation is available.

• Applying Noether's theorem to conformal symmetry yields the following *conserved currents*:

$$j_z = v^z T_{zz} \qquad j_{\bar{z}} = v^{\bar{z}} T_{\bar{z}\bar{z}} \tag{2.8}$$

where v^z (resp. $v^{\bar{z}}$) is any (locally) holomorphic (resp. anti-holomorphic) vector field. Conformal symmetry requires $\partial_{\bar{z}} j_z = 0$, $\partial_z j_{\bar{z}} = 0$, so that we must also have

$$\partial_{\bar{z}} T_{zz} = 0 \qquad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{2.9}$$

Thus, $T(z) = T_{zz}$ and $\widetilde{T}(\bar{z}) = T_{\bar{z}\bar{z}}$ are complex analytic and anti-analytic respectively. In correlation functions, however, a product of $T(z)$ and another field $\phi(w)$ may produce singularities, as $z \to w$, so the above analyticity condition (2.9) is to be understood as valid when T is away from other operators.

• It was shown in [II-**CFT**] how to extract these OPE singularities from conformal Ward identities for locally flat metrics g; for the case of general metric g, the derivation is the subject of Problem #4. One finds

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\frac{\partial}{\partial w} \right) \phi(w,\bar{w}) \\ T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T_{ww} + \frac{1}{z-w} \partial_w T_{ww} \end{aligned} \tag{2.10}$$

where c is the *central charge* of the CFT and h is the holomorphic conformal weight of ϕ. (Analogous relations hold for $\widetilde{T}$.) As always, these relations are to be understood as valid when inserted into any correlation function. The $\sim$ sign used in (2.10) stands for the fact that we have retained only the singular part on the r.h.s. When $c = 0$, $T(z)$ is a conformal tensor of weight $(2,0)$, but for $c \neq 0$, T transforms as a projective connection instead, with

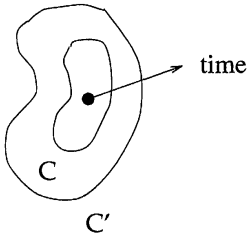
$$T'(z')(dz')^2 = T(z)(dz)^2 - \frac{c}{12}\{z';z\}(dz)^2 \tag{2.11a}$$

where the Schwarzian derivative $\{\ ;\ \}$ is defined by

$$\{f;z\} \equiv f'''/f' - (3/2)(f''/f')^2, \qquad f' = \partial f/\partial z. \tag{2.11b}$$

• Analytic currents $j_\alpha = j_{\alpha z}dz$, α labeling different species of currents, give rise to time-independent or conserved charges:

$$Q_\alpha(C) = \frac{1}{2\pi i}\oint_C j_\alpha$$

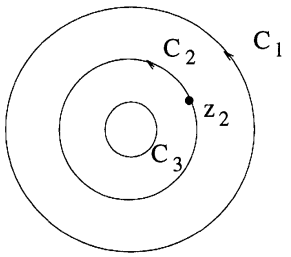


Here $Q_\alpha(C)$ is the charge enclosed in the region bounded by C and equals $Q_\alpha(C')$ provided no sources for Q_α occur between C and C'. The action of charges on fields ϕ may be evaluated using only the residues in the OPE of j_α with ϕ:

$$Q_\alpha(C)\phi(w,\bar w) = \frac{1}{2\pi i}\oint_C j_\alpha(z)\phi(w,\bar w) = \operatorname*{Res}_{z\to w} j_\alpha(z)\phi(w,\bar w) \tag{2.12}$$

provided no other sources for j_α lie within C. The commutator between two charges is recovered as follows:

$$\begin{aligned} &Q_\alpha(C_1)Q_\beta(C_2) - Q_\alpha(C_3)Q_\beta(C_2) \\ &\quad = [Q_\alpha, Q_\beta](C_2) \\ &\quad = \frac{1}{2\pi i}\oint_{C_2} \operatorname*{Res}_{z\to z_2} j_\alpha(z)j_\beta(z_2) \end{aligned} \tag{2.13}$$



• The conserved charges associated with conformal transformations are obtained from vector fields $v^z_{(m)} = z^{m+1}$ and $v^{\bar z}_{(m)} = \bar z^{m+1}$:

$$\begin{aligned} L_m &\equiv \oint \frac{dz}{2\pi i} z^{m+1}T(z) \\ \tilde L_m &\equiv \oint \frac{d\bar z}{2\pi i} \bar z^{m+1}\widetilde{T}(\bar z). \end{aligned} \tag{2.14}$$

From the OPE of two T's (and two $\widetilde{T}$'s), we recover two commuting Virasoro algebras

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}.
\end{aligned} \tag{2.15}
$$

From the OPE of T (and $\widetilde{T}$) with a primary field ϕ of conformal weight $(h, \bar{h})$, we obtain

$$
\begin{aligned}
[L_m, \phi(w,\bar{w})] &= w^{m+1}\partial_w \phi(w,\bar{w}) + h(m+1)w^m \phi(w,\bar{w}) \\
[\tilde{L}_m, \phi(w,\bar{w})] &= \bar{w}^{m+1}\partial_{\bar{w}} \phi(w,\bar{w}) + \bar{h}(m+1)\bar{w}^m \phi(w,\bar{w})
\end{aligned} \tag{2.16}
$$

• An important example of a CFT is the free scalar field $x\colon \Sigma \to \mathbb{R}$ (or $\mathbb{R}/\lambda\mathbb{Z}$ as discussed in Lecture 1 of [II-**Dynamics of QFT**]), with action

$$
S[x] = \frac{1}{4\pi} \int_\Sigma d^2 z \, \partial_z x \partial_{\bar{z}} x \tag{2.17}
$$

The OPE is derived from the Green function for x:

$$
x(z,\bar{z})x(w,\bar{w}) \sim -\ln |z-w|^2 \tag{2.18}
$$

It is customary to define normal ordering by

$$
:x(z,\bar{z})x(w,\bar{w}): \equiv x(z,\bar{z})x(w,\bar{w}) + \ln |z-w|^2 \tag{2.19}
$$

The normal ordered product admits a smooth limit as $w \to z$. The renormalized (also called the quantum) stress tensor may now be defined by

$$
T(z) = -\frac{1}{2} :(\partial_z x)^2: \equiv -\frac{1}{2} \lim_{w\to z} \left(\partial_z x \partial_w x + \frac{1}{(z-w)^2} \right) . \tag{2.20}
$$

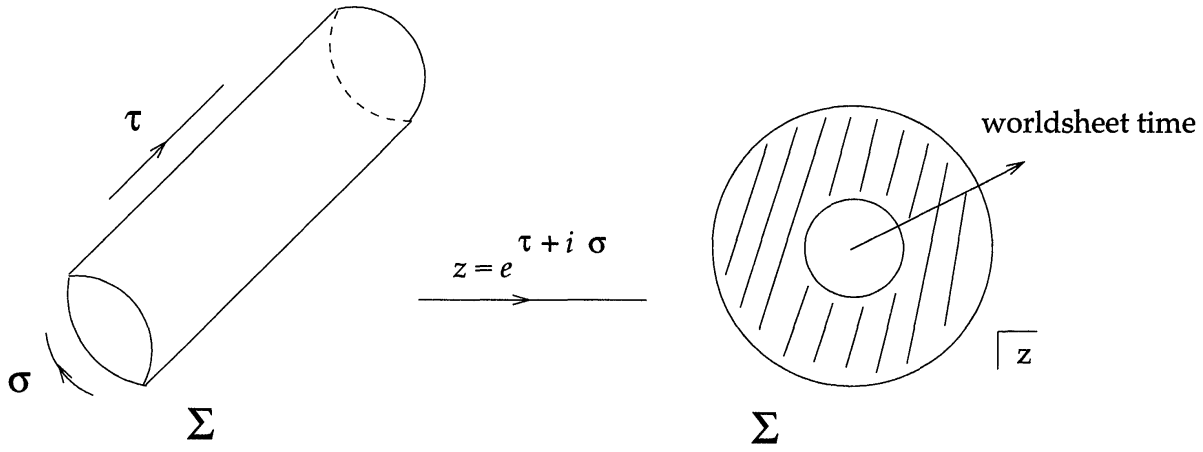
Elementary primary fields in this CFT are listed below, with their corresponding conformal weight.

$$
\begin{array}{ll}
\partial_z x & (1,0) \\
\partial_{\bar{z}} x & (0,1) \\
:\exp ikx: & \left(\frac{1}{2}k^2, \frac{1}{2}k^2 \right)
\end{array} \tag{2.21}
$$

The central charge is $c = 1$. Notice that while the derivative of x and the exponential are primary fields, x itself is not a well-defined quantum field (cf. Lecture 1 of [II-**Dynamics of QFT**]).

§2.2. The free closed bosonic string conformal field theory

The worldsheet topology for the free closed bosonic string is that of the cylinder or equivalently the annulus.



The metric g determines *global* complex coordinates z, $\bar{z}$ in which $g = 2g_{z\bar{z}}|dz|^2$.

Recall from 2) that our starting point is the flat Euclidean $\mathbb{R}^D_E$ worldsheet action

$$S = \frac{1}{4\pi}\int_\Sigma d^2z\, \partial_z x \cdot \partial_{\bar{z}} x \tag{2.22}$$

which for fixed metric g defines a CFT of D scalar fields x^μ, which are the coordinates of the map $x\colon \Sigma \to \mathbb{R}^D_E$. The quantum stress tensor is

$$T(z) = -\frac{1}{2} :\partial_z x \cdot \partial_z x: \tag{2.23}$$

and the central charge is $c = D$.

Following point 3), we obtain the OPE's and associated algebraic structure. To do this, we take advantage of the fact that $\partial_z x(z)$ and $T(z)$ are holomorphic, and thus admit a Laurent expansion inside the annulus:

$$\begin{aligned} \partial_z x(z) &= -i\sum_{m\in\mathbb{Z}} x_m z^{-m-1} & T(z) &= \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \\ \partial_{\bar{z}} x(\bar{z}) &= -i\sum_{m\in\mathbb{Z}} \tilde{x}_m \bar{z}^{-m-1} & \tilde{T}(\bar{z}) &= \sum_{m\in\mathbb{Z}} \tilde{L}_m \bar{z}^{-m-2} \end{aligned} \tag{2.24}$$

From the definition of $T \equiv -\frac{1}{2} :\partial_z x \cdot \partial_z x:$, we have

$$\begin{aligned} L_m &= \frac{1}{2}\sum_n x_{m-n}\cdot x_n \qquad\qquad m \neq 0 \\ L_0 &= \frac{1}{2}x_0^2 + \sum_{n=1}^{\infty} x_{-n}\cdot x_n \end{aligned} \tag{2.25}$$

From the OPE of $\partial_z x$ with itself and with $T(z)$, we get

$$[x^\mu_m, x^\nu_n] = m\delta_{m+n,0}\delta^{\mu\nu} \qquad [L_m, x_n] = -n x_{m+n}\ , \tag{2.26a}$$

also

$$[\tilde{x}^\mu_m, \tilde{x}^\nu_n] = m\delta_{m+n,0}\delta^{\mu\nu} \qquad [\tilde{L}_m, \tilde{x}_n] = -n\tilde{x}_{m+n} \ , \tag{2.26b}$$

where tilded and untilded operators mutually commute. L_m and $\tilde{L}_m$ satisfy Virasoro algebras with central charge $c = D$.

Following point 4), we find that the above OPE's and their algebra contents can safely be continued to $x^0 \to ix^0$ where x^0 is the time component of the map $x\colon \Sigma \to \mathbb{R}^D_M$. It is clear that this means

$$\left.\begin{aligned} x^0_m &\to ix^0_m \\ \tilde{x}^0_m &\to i\tilde{x}^0_m \end{aligned}\right\} \qquad \text{all } m \in \mathbb{Z}. \tag{2.27}$$

The stress tensor is now

$$T(z) = -\frac{1}{2} : \partial_z x^\mu \partial_z x^\nu :\ \eta_{\mu\nu} \equiv -\frac{1}{2} : \partial_z x \cdot \partial_z x : \tag{2.28}$$

and the algebraic relations are

$$\begin{aligned} &[x^\mu_m, x^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \quad [L_m, x^\mu_n] = -nx^\mu_{m+n} \\ &L_m = \frac{1}{2}\sum_n : x^\mu_{m-n} x^\nu_n :\ \eta_{\mu\nu} \end{aligned} \tag{2.29}$$

and analogously for $\tilde{x}$ and $\tilde{L}$. The central charge is unchanged, and we still have $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$.

The field x itself is not a conformal tensor, but may be obtained by integrating $\partial_z x$ and $\partial_{\bar{z}} x$:

$$x(z, \bar{z}) = x_L(z) + x_R(\bar{z}) \tag{2.30}$$

with

$$\begin{aligned} x_L(z) &= q_L - ix_0 \ln z + i\sum_{m\neq 0} x_m \frac{z^{-m}}{m} \\ x_R(\bar{z}) &= q_R - i\tilde{x}_0 \ln \bar{z} + i\sum_{m\neq 0} \tilde{x}_m \frac{\bar{z}^{-m}}{m}. \end{aligned} \tag{2.31}$$

Since $x : \Sigma \to \mathbb{R}^D_M$, x must be a single-valued map as a function of z. This will require that $x_0 = \tilde{x}_0 = p$, and p is to be identified with the *momentum* operator of the string, while $q = q_L + q_R$ is the overall position operator of the string.

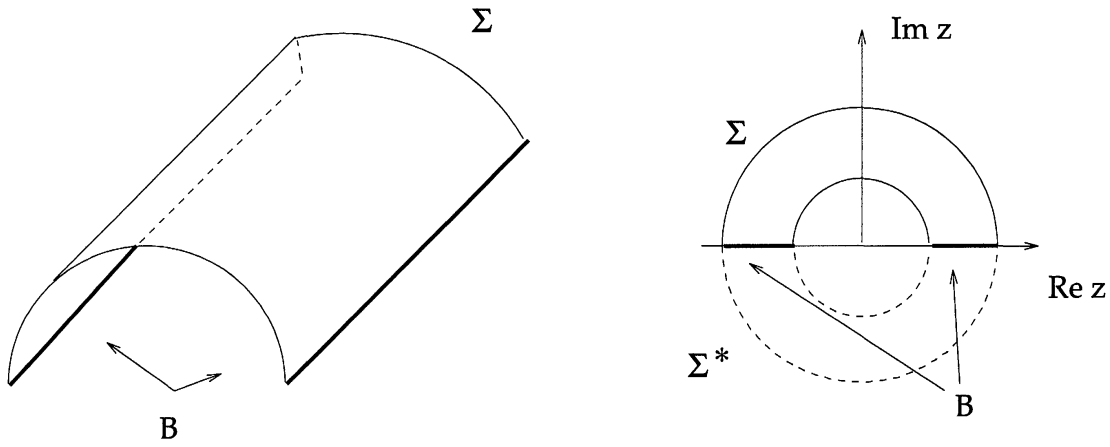
It will turn out to be useful to record the explicit form of the operators L_0 and $\tilde{L}_0$, expressed in terms of p.

$$\begin{aligned} L_0 &= \frac{1}{2}p^2 + \sum_{n=1}^{\infty} x_{-n} \cdot x_n \\ \tilde{L}_0 &= \frac{1}{2}p^2 + \sum_{n=1}^{\infty} \tilde{x}_{-n} \cdot \tilde{x}_n \end{aligned} \tag{2.32}$$

For non-simply connected space-time manifolds M, the map x need not be single-valued, since the string can wrap around non-trivial 1-cycles in M. As a result, x_0 need not be equal to $\tilde{x}_0$, but the difference $x_0 - \tilde{x}_0$ belongs to a lattice, dual to $H^1(M, \mathbb{Z})$.

§2.3. The free open bosonic string conformal field theory

The worldsheet of a free open string has the topology of a strip or a half annulus.



Fundamental open strings should have freely moving boundary points. Thus Neumann conditions $\partial_{\mathrm{Im}\, z} x = 0$ at the boundary B are appropriate. (Dirichlet b.c. would fix the end points of the string on some submanifold $\mathcal{D}$ of M. Such submanifolds are D-branes or, depending upon their space dimension p, p-branes. In certain string theories, D-branes become dynamical (extended) objects, perhaps to be discussed later on.)

It is standard to double Σ by reflecting Σ onto the lower half plane into Σ^* and then considering the union $\Sigma' = \Sigma \cup \Sigma^*$ which is the full annulus. Functions on Σ with Neumann boundary conditions on B are then even functions under $z \to \bar{z}$. Thus, the mode expansion of the open string is obtained from this restriction applied to $x(z, \bar{z})$

$$x(z, \bar{z}) = x_L(z) + x_L(\bar{z}) \tag{2.33}$$

and only involves the oscillators x_m, but there are no oscillators $\tilde{x}_m$. Similarly, we only have Virasoro generators L_n but there are no $\tilde{L}_n$. As a result, the energy-momentum tensor satisfies

$$T(z) = \widetilde{T}(\bar{z}) = 0 \qquad \text{on} \qquad \mathrm{Im}(z) = 0. \tag{2.34}$$

This condition guarantees conservation of T across B, and results from the fact that $Diff(\Sigma)$ leaves B invariant. We have the mode expansion

$$x(z, \bar{z}) = x + 2ip \ln |z|^2 - i \sum_{m \neq 0} x_m \left(\frac{z^{-m}}{m} + \frac{\bar{z}^{-m}}{m} \right) \tag{2.35}$$

Notice that $x_0 = 2p$, in contrast with the closed string. It is useful to also record the Virasoro generators

$$\begin{aligned} L_0 &= 2p^2 + \sum_{n=1}^{\infty} x_{-n} \cdot x_n \\ L_m &= \sum_{n \in \mathbb{Z}} x_{m-n} \cdot x_n \qquad\qquad m \neq 0 \end{aligned} \tag{2.36}$$

Thus the open string only has a single Virasoro algebra Vir_L.

§2.4. Fock space, negative norm states

We now follow point 5) and construct the Fock space of the conformal field theory of x, continued to Minkowski space-time. (This parallels the construction in [II-**CFT**]).

To do so, we must choose a polarization and split the operators $\{x_m^\mu\}$ and $\{\tilde{x}_m^\mu\}$ into a group of raising, lowering and central generators. This choice must be consistent with Poincaré invariance. (Also, here we must use a property inherited from the Minkowski nature of the worldsheet, namely that x_m and $\tilde{x}_m$ oscillators are independent, except at $m = 0$. We shall not expand upon this issue now, but come back to this part later on.) The correct choice is $(x_0^\mu)^\dagger = x_0^\mu$, while x_m^μ and $\tilde{x}_m^\mu$ are raising (resp. lowering) for $m < 0$ (resp. $m > 0$). Also

$$(x_m^\mu)^\dagger = x_{-m}^\mu \qquad (\tilde{x}_m^\mu)^\dagger = \tilde{x}_{-m}^\mu \tag{2.37}$$

and this implies

$$L_m^\dagger = L_{-m} \qquad \tilde{L}_m^\dagger = \tilde{L}_{-m}. \tag{2.38}$$

The Fock space $\mathcal{F}_k$ is constructed from x_m^μ oscillators only. The ground state $|0,k\rangle_L$ (also called the highest weight state) is labeled by the momentum k (the eigenvalue of the momentum operator p)

$$\begin{cases} p|0,k\rangle_L = k|0,k\rangle_L \\ x_m^\mu|0,k\rangle_L = 0, \quad m > 0. \end{cases} \tag{2.39}$$

We may choose its normalization as follows:

$${}_L\langle 0,k|0,k'\rangle_L = \delta(k-k') \ . \tag{2.40}$$

Arbitrary vectors in $\mathcal{F}_k$ are obtained by applying x_{-m}^μ's for $m > 0$ to the ground state

$$|\varepsilon,k\rangle_L = \varepsilon_{\mu_1\dots\mu_n}(k;m_1,\dots,m_n)x_{-m_1}^{\mu_1},\dots x_{-m_n}^{\mu_n}|0,k\rangle_L \tag{2.41}$$

for all possible Lorentz *polarization tensors* $\varepsilon_{\mu_1\dots\mu_n}(k;m_1,\dots,m_n)$; $n \in \mathbb{N}$, with all possible $m_i \in \mathbb{N}$.

- The open string Fock space is a sum over all momenta k of $\mathcal{F}_k$:

$$\mathcal{F}_{\text{open}} = \bigoplus_{k\in\mathbb{R}_M^D} \mathcal{F}_k$$

- The closed string Fock space is a sum over all momenta – which must be the same for left and right movers, in view of the constraint $x_0 = \tilde{x}_0$ – of the tensor product of $\mathcal{F}_k$ and $\widetilde{\mathcal{F}}_k$, where $\widetilde{\mathcal{F}}_k$ is built from $\tilde{x}_m^\mu$:

$$\mathcal{F}_{\text{closed}} = \bigoplus_{k\in\mathbb{R}_M^D} \mathcal{F}_k \otimes \widetilde{\mathcal{F}}_k.$$

- Using the norm on $|0,k\rangle_L$ introduced in (2.40), $\mathcal{F}_k$ naturally inherits a quadratic form. However, $\mathcal{F}_k$ automatically contains *negative norm states.* To see this, consider states $|\varepsilon,k\rangle_L = \varepsilon_\mu x_{-m}^\mu|0,k\rangle_L$ and with $m > 0$ evaluate their norm:

$${}_L\langle\varepsilon,k|\varepsilon,k'\rangle_L = \langle 0,k|\varepsilon\cdot x_{-m}^\dagger\varepsilon\cdot x_{-m}|0,k'\rangle = m\varepsilon^2\delta(k-k') \tag{2.42}$$

For $\varepsilon^2 > 0$, the norm of the state $|\varepsilon,k\rangle_L$ is positive. But, for $\varepsilon^2 < 0$, which is allowed with Minkowski metric, the norm of $|\varepsilon,k\rangle_L$ is negative. Such states are called *ghosts* and their presence violates quantum mechanics: they cannot be present in a physically sensible theory.

§2.5. Integration over Met(Σ)–Virasoro constraints

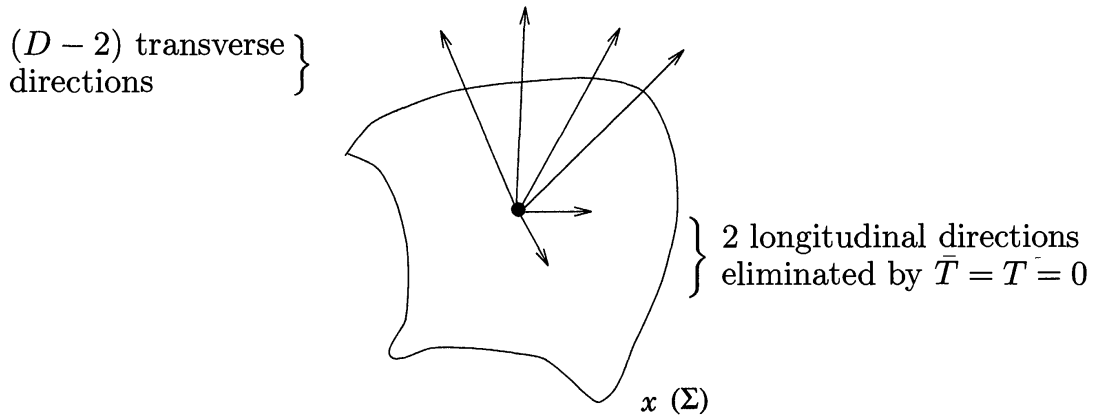
We now deal with point 6): the integration over Met(Σ), and the elimination of ghosts from the spectrum. We shall perform this integration with great care in forthcoming lectures, but we can already here extract its key effects.

The metric g enters in the action S as a non-dynamical field, i.e. without derivatives. The rôle of the metric g here is completely analogous to that of the time-like component A_0 of the gauge field in Yang–Mills theory (see also [I-**Faddeev**]). The effect of such non-dynamical fields is to supply a constraint on the dynamics of the remaining degrees of freedom, and to reduce their phase space. (You may loosely view the effect in the functional integral as generating a delta function $\int dA_0 e^{iA_0 x} = \delta(x)$.) The constraints also generate a group action on the phase space, and the reduced phase space may be viewed as the quotient by this group action.

Let us analyze the effect of the constraints at the *classical level* first. The variational equations are

$$\text{(2.43)} \qquad \begin{cases} \partial_{\bar z}\partial_z x = 0 & \text{by varying } x \\ \\ T(z) = \bar T(\bar z) = 0 & \text{by varying } g \end{cases}$$

The second equation is a constraint on x since $T(z) = -\frac{1}{2}\partial_z x \cdot \partial_z x$ (similarly for $\widetilde{T}(\bar z)$). Since $\partial_{\bar z} T(z) = 0$, the first equation shows that T will vanish throughout Σ as long as $T = 0$ has been imposed on the boundary. The action of the constraints at the classical level is something we have already analyzed, namely the Virasoro algebra but with zero central charge. Geometrically, the constraint reflects the fact that vibrations of $x(\Sigma)$ tangent to the surface $x(\Sigma)$ are eliminated, leaving only transverse directions:



We now analyze the effects of the constraints at the *quantum level*. There are basically three ways of dealing with the constraints:

1) Eliminate the longitudinal degrees of freedom at the classical level by solving for them in terms of the transverse degrees of freedom, and then quantize the transverse degrees of freedom. This procedure goes under the name of *light-cone gauge* quantization. It cannot be achieved in a manifestly Lorentz invariant parametrization. We shall discuss it very briefly later on in this lecture.

2) Ultimately, in Lecture 4, we shall reformulate this constraint problem in terms of the powerful BRST quantization method. This is a more complex scheme based

on introducing extra, unphysical degrees of freedom, called Faddeev-Popov ghosts, and enforcing BRST (Becchi-Rouet-Stora-Tyutin) invariance. This method is manifestly Lorentz invariant. The physical Hilbert space will in fact arise as a cohomology group of a semi-infinite cohomology complex, as developed by Feigin and later on by Freeman and Olive in physics and Frenkel, Garland and Zuckerman in mathematics.

3) Before launching into BRST, however, we shall analyze the quantum effects of the constraints in a more direct fashion, which will also yield immediate results on the spectrum. In this third method, we impose the constraint as an invariance condition and use it to select an invariant subspace out of the full Fock space $\mathcal{F}$ (either $\mathcal{F}_{\text{open}}$ for open strings or $\mathcal{F}_{\text{closed}}$ for closed strings). In Yang–Mills theory, the corresponding constraint was just Gauss' law, enforcing gauge invariance on physical states. In string theory, the constraints are the vanishing of the stress tensor, enforcing $Diff(\Sigma)$ invariance on physical states. Due to the central extension in Vir, the reduced Fock space $\mathcal{F}^+ = \mathcal{F}/\text{Vir}$ will have a positive definite norm only when certain extra conditions are satisfied, which we now determine.

§2.6. Physical spectrum, no-ghost theorem

We use the Virasoro constraints to select an invariant subspace out of the full Fock space $\mathcal{F}$. It is tempting to try and impose $L_n|\phi\rangle = \tilde{L}_n|\phi\rangle >= 0$ for all $n \in \mathbb{Z}$. However, unless $c = 0$, these conditions imply that $|\phi\rangle >= 0$. Instead, we impose invariance under half of Vir. The correct physical state conditions are therefore

$$
\begin{aligned}
|\phi\rangle_L &\in \mathcal{F}_k^{\text{phys}} \quad \text{if} \quad (L_m - a\delta_{m,0})|\phi\rangle_L = 0 \qquad m \in \mathbb{N} \\
|\phi\rangle >_R &\in \widetilde{\mathcal{F}}_k^{\text{phys}} \quad \text{if} \quad (\tilde{L}_m - a\delta_{m,0})|\phi\rangle_R = 0 \qquad m \in \mathbb{N}.
\end{aligned} \tag{2.44}
$$

and the full physical Fock spaces are obtained by

$$
\begin{aligned}
\mathcal{F}_{\text{open}}^{\text{phys}} &= \bigoplus_k \mathcal{F}_k^{\text{phys}} \ , \\
\mathcal{F}_{\text{closed}}^{\text{phys}} &= \bigoplus_k \mathcal{F}_k^{\text{phys}} \otimes \widetilde{\mathcal{F}}_k^{\text{phys}} \ .
\end{aligned}
$$

The constant a is undetermined at this point, and is allowed for since L_0 and $\tilde{L}_0$ were naturally defined only up to an additive constant in view of the ordering choice we made in their definition. For closed strings, the value of a is taken the same for L_0 and $\tilde{L}_0$ so that $L_0 - \tilde{L}_0 = \frac{\partial}{\partial\sigma}$ enforces rotation invariance on the annulus. Notice that states $|\phi\rangle$, $|\psi\rangle >\in \mathcal{F}^{\text{phys}}$ automatically yield zero matrix elements of Virasoro generators $L_m - a\delta_{m,0}$ for all $m \in \mathbb{Z}$ (similarly for $\tilde{L}_m - a\delta_{m,0}$ in the case of the closed string). Hence, we have now realized the physical state condition $\langle\phi|T|\psi\rangle = \langle\phi|\widetilde{T}|\psi\rangle = 0$!

States of the form $|\phi\rangle = L_{-m}|\psi\rangle >$, for $m > 0$, are called *spurious*, and we define

$$
\mathcal{F}_k^{\text{spurious}} = \left\{ |\phi\rangle = \bigoplus_{m=1}^{\infty} L_{-m}|\chi_m\rangle \right\}
$$

It is easy to see that $\mathcal{F}_k^{\text{spurious}}$ is generated by states of the form $L_{-1}|\chi_1\rangle \oplus L_{-2}|\chi_2\rangle$, using the commutation relations of the Virasoro algebra which imply that

$[L_{-1}, L_{-n}]$ is a non-zero multiple of L_{-1-n} for $n \geq 2$. Notice that spurious states are orthogonal to all physical states. States that are both physical and spurious are *null*,

$$\mathcal{F}_k^{\text{spurious}} \cap \mathcal{F}_k^{\text{phys}} \subset \mathcal{F}_k^{\text{null}} ,$$

where $\mathcal{F}_k^{\text{null}}$ stands for the space of null states of momentum k.

The special Virasoro conditions $(L_0 - a)|\phi\rangle = (\tilde{L}_0 - a)|\phi\rangle = 0$ determine the value of k^2 for a given level of operators, and thus determines the mass of the state. To see this, introduce the number (also called level) operators

$$N \equiv \sum_{n=1}^{\infty} x_{-n} \cdot x_n \qquad \widetilde{N} \equiv \sum_{n-1}^{\infty} \tilde{x}_{-n} \cdot \tilde{x}_n \ . \tag{2.45}$$

The operators N and $\widetilde{N}$ count the number of operators x_{-n} and $\tilde{x}_{-n}$, $n \geq 1$ with weight n, applied to the ground state $|0, k\rangle$. The special Virasoro conditions thus become

$$\begin{array}{lllll} \text{closed} & (k^2 + M^2)|\phi\rangle = 0 & M^2 = 2N - 2a & \text{and } N = \widetilde{N}|\phi\rangle = 0 \\ \text{open} & (k^2 + M^2)|\phi\rangle = 0 & M^2 = \frac{1}{2}N - \frac{1}{2}a \ . & \end{array} \tag{2.46}$$

Thus, the mass squared of various string states are spread by $2\times$ integers for closed strings and $\frac{1}{2}\times$ integers for open strings.

The mass squared of a state is one of the Casimir values that label the representation of the Poincaré group under which the state transforms. The remaining data that fully specify the Poincaré representation are those given by the representation of the stabilizer group (i.e. the subgroup that leaves the momentum invariant). This information is contained in the expression of the Lorentz generators of the string.

$$\begin{aligned} J^{\mu\nu} = {} & q^u p^\nu - q^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} (x_{-n}^\mu x_n^\nu - x_{-n}^\nu x_n^\mu) \\ & \left(-i \sum_{n=1}^{\infty} \frac{1}{n} (\tilde{x}_{-n}^\mu \tilde{x}_n^\nu - \tilde{x}_{-n}^\nu \tilde{x}_{-n}^\mu) \quad \text{for closed strings} \right) \end{aligned} \tag{2.47}$$

It follows that the string ground state $|0, k\rangle$ transforms as a scàlar (i.e. trivial representation under the stabilizer).

We observed in the first lecture that, at long distances compared to the Planck length ℓ_P, the size of strings effectively becomes unobservable. We can then approximate the dynamics of the string by that of point particles, and string theory by quantum field theory. Thus, we shall seek to identify string states with states in quantum field theory. To do so, we require that string and quantum field states transform under the same representations of the Poincaré group, as well as of other symmetry groups that are available. Also, their interactions should agree in this long distance limit.

We examine under what conditions the Virasoro constraints eliminate all negative norm states, so that $\mathcal{F}_k^{\text{phys}}$ has positive norm. Before stating the general result, let's look at the lowest level cases (for the closed string; for the open string it suffices to let $M^2 \to M^2/4$).

(1) Ground state $|0,k\rangle_L$ with mass square $M^2 = -2a$ (actually $|0,k\rangle_L \otimes |0,k\rangle_R$ for the closed string). The Virasoro conditions are automatically satisfied.

(2) First excited state $|\varepsilon,k\rangle_L = \varepsilon \cdot x_{-1}|0,k\rangle_L$ with mass squared $M^2 = -2a+2$:

$$L_1|\varepsilon,k\rangle_L = 0 \Longrightarrow \varepsilon \cdot k = 0\ .$$

The remaining Virasoro conditions $L_n|\varepsilon,k\rangle_L = 0$, $n \geq 2$, are automatically satisfied. There are three possibilities:

(i) $a > 1$, so that k^2 is space-like. If we pick $k = (0,k^1,0,\ldots,0)$ then we are allowed to choose $\varepsilon = (\varepsilon^0,0,\ldots,0)$, and we see that ghosts are still present.

(ii) $a < 1$, so that k^2 is time-like. If we pick $k = (k^0,0,\ldots,0)$ we see that we always have $\varepsilon^2 > 0$ and there are $D-1$ independent states with positive norm.

(iii) $a = 1$, so that k^2 is light-like. If we pick $k = (k^1,k^1,0,\ldots,0)$, we see that there are $(D-2)$ massless states of positive norm, and one of zero norm.

Case (i) is clearly eliminated: Virasoro constraints do not suffice to eliminate all negative norm states. Case (ii) has massive $M^2 > 0$ vector states (open string), or massive rank 2 tensor states (closed string). Case (iii) has massless $M^2 = 0$ vector states (open strings), which suggests the presence of massless Yang–Mills states, and has massless $M^2 = 0$ rank 2 tensor states (closed strings) which suggests the presence of gravity (for symmetric tensors).

3) At the next level $|\varepsilon,k\rangle_L = \{\varepsilon_{\mu\nu}x^\mu_{-1}x^\nu_{-1} + \eta_\mu x^\mu_{-2}\}|0,k\rangle_L$, with mass squared $M^2 = -2a+4$ and

$$\left.\begin{array}{l} L_1|\varepsilon,k\rangle_L = 0 \\ L_2|\varepsilon,k\rangle_L = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \varepsilon_{\mu\nu}k^\nu + \eta_\mu = 0 \\ \\ \varepsilon^\mu_\mu + 2k_\mu\eta^\mu = 0\,. \end{array}\right.$$

The norm of the state is

$${}_L\langle\varepsilon,k|\varepsilon,k'\rangle_L = 2\mathcal{N}_L\langle 0,k|0,k'\rangle_L$$

with

$$\begin{aligned}\mathcal{N} &= \varepsilon_{\mu\nu}\varepsilon^{\mu\nu} + \eta_\mu\eta^\mu \\ &= (\varepsilon_{00})^2 - 2(\varepsilon_{0i})^2 + (\varepsilon_{ij})^2 - (\eta_0)^2 + (\eta_i)^2.\end{aligned}$$

Choosing a rest frame where $k^0 = M$, $k^i = 0$, the Virasoro conditions become

$$\left\{\begin{array}{l} \varepsilon_{00}M + \eta_0 = 0 \\ \varepsilon_{i0}M + \eta_i = 0 \\ -\varepsilon_{00} + \varepsilon_{ii} + 2M\eta_0 = 0\ . \end{array}\right.$$

The Virasoro conditions impose no constraints on the traceless part of ε_{ij}, and its contribution to $\mathcal{N}$ is ≥ 0. Hence, we may retain only the trace part: $\varepsilon_{ij} = \varepsilon\delta_{ij}$. Eliminating η_i and η_0 and ε_{00} in favor of ε_{0i} and ε, we have

$$\mathcal{N} = \frac{\varepsilon^2(D-1)}{(2M^2+1)^2}\left\{(2M^2+1)^2 - (M^2-1)(D-1)\right\} + (M^2-2)(\varepsilon_{0i})^2.$$

Assuming $D > 1$, we find the conditions

$$\begin{cases} M^2 \geq 2 \\ \\ (2M^2+1)^2 - (M^2-1)(D-1) \geq 0 \ . \end{cases}$$

the first of which is already satisfied since $M^2 = -2a + 4$ and we have seen before that we must have $a \leq 1$. For $M^2 = 2$ (corresponding to massless states at level 1), we find from the second inequality that $D \leq 26$. In fact, for $D = 26$, the second inequality implies $M^2 \leq 2$, so that we must have $M^2 = 2$ for $D = 26$.

For $D \leq 25$, the quadratic form $(2M^2+1)^2 - (M^2-1)(D-1)$ is non-negative for all $M^2 \geq 2$, so the state $|\varepsilon, k\rangle$ always has non-negative norm.

For $D \geq 27$, the quadratic form is negative for a region in which $M^2 \geq 2$. In fact, by analyzing higher level states it can be shown that for $D \geq 27$ no value of a can be chosen so as to eliminate all ghosts. The complete result can be summarized in the following

No Ghost Theorem (R. Brower, P. Goddard and C. Thorn).

- *For* $D > 26$ **or** $a > 1$

  $\mathcal{F}^{\text{phys}}$ *contains negative norm states;*

- *For* $D = 26$ **and** $a < 1$

  $\mathcal{F}^{\text{phys}}$ *contains negative norm states;*

- *For* $D = 26$ **and** $a = 1$

  $\mathcal{F}^{\text{phys}}$ *contains only non-negative norm states, in particular, two towers of null states and* $\mathcal{F}^+ \equiv \mathcal{F}^{\text{phys}}/\mathcal{F}^{\text{null}}$ *has positive definite norm;*

$$\mathcal{F}^{\text{null}} = \left\{ (L_{-2} + \frac{3}{2} L_{-1}^2)|\chi_2\rangle \oplus L_{-1}|\chi_1\rangle \right\}$$

- *For* $D < 26$ **and** $a \leq 1$

  $\mathcal{F}^{\text{phys}}$ *contains only positive norm states for* $a < 1$ *(contains one tower of null states* $L_{-1}|\chi_1\rangle$ *for* $a = 1$*).*

We shall not prove the no ghost theorem here, since later on, using BRST quantization, a simple proof can be given.

The case $D \leq 25$ and $a \leq 1$ is contained in $D = 26$, $a = 1$ by setting $26 - D$ space-like operators to zero, and adjusting the $26 - D$ positive squared momenta p_i^2, $i = 1, \ldots, 26 - D$, so that

$$\frac{1}{2} \sum_i p_i^2 - a + 1 = 0 \ .$$

§2.7. Spectrum of the critical bosonic string with $D = 26$, $a = 1$

Based on the masses and Lorentz tensor structure of the lowest lying states of the spectrum, we may identify which particles they naturally correspond to in field theory.

It is easy to see that the spectrum consists entirely of bosons.

Open strings

(1) States corresponding to $|0,k\rangle$ have $M^2 = -\frac{1}{2}$, hence they are *tachyons* (particles traveling faster than light), and Lorentz scalars;

(2) States of the form $\varepsilon \cdot x|0,k\rangle$ with $M^2 = 0$ are vectors under $\mathrm{SO}(1, D-1)$. There are $(D-2)$ positive norm states and one null state, which constitute a massless gauge boson in $\mathbb{R}^D$. Gauge transformations have the form $\varepsilon_\mu \to \varepsilon_\mu + k_\mu \gamma$ (invariance under these transformations is clear since they amount to shifting the state by a null state $\gamma L_{-1}|0,k\rangle = \gamma k \cdot x_{-1}|0,k\rangle$);

(3) $M^2 > 0$ states, with $M^2 = \frac{1}{2}n$, $n \in \mathbb{N}$, transform under various tensor representations of $\mathrm{SO}(1, D-1)$.

Closed strings

(1) States corresponding to $|0,k\rangle = |0,k\rangle_L \otimes |0,k\rangle_R$ have $M^2 = -2$, so they are *tachyons*, and Lorentz scalars;

(2) States of the form $\varepsilon_{\mu\nu} x^\mu_{-1} \tilde{x}^\nu_{-1}|0,k\rangle$ have $M^2 = 0$. Corresponding to the trace-part of ε there is a Lorentz scalar, the *dilaton*, with positive norm. The symmetric traceless part of ε gives a Lorentz rank 2 tensor, the *graviton*. There are $\frac{D(D+1)}{2} - 1$ such states, $\frac{D(D-1)}{2} - 1$ of them having positive norm. As for the antisymmetric part of ε, it gives a Lorentz rank 2 tensor called the *antisymmetric tensor* with $\frac{D(D-1)}{2}$ states, out of which $\frac{(D-1)(D-2)}{2}$ have positive norm. These states exhibit gauge invariance under

$$\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \gamma_\mu k_\nu + \gamma'_\nu k_\mu \tag{2.48}$$

(which amounts to shifting by null states);

(3) $M^2 > 0$ states, with $M^2 = -2n$, $n \in \mathbb{N}$, transform under various tensor representations of $\mathrm{SO}(1, D-1)$.

§2.8. Lightcone gauge, density of states

For $D = 26$ and $a = 1$, the null states in $\mathcal{F}^{\mathrm{null}}$ may be ignored completely. We may retain positive norm states only at the cost of giving up manifest Lorentz invariance, $\mathrm{SO}(1,25)$, by retaining $D - 2 = 24$ transverse oscillators only. Let M' be a space-like sub-vector space of M, with $\dim M' = 24$, and let ε, $\tilde{\varepsilon}$ be in M'; then $\varepsilon \cdot x_{-n}$ and $\tilde{\varepsilon} \cdot \tilde{x}_{-n}$ for $n > 0$ create only positive norm states. This construction has manifest $\mathrm{SO}(24)$ invariance. It can be checked that the full Lorentz algebra $\mathrm{SO}(1,25)$ closes in this realization. Also, it can be shown in this case that there is a one-to-one correspondence between those states constructed above and the states in $\mathcal{F}^+ = \mathcal{F}^{\mathrm{phys}}/\mathcal{F}^{\mathrm{null}}$ in the covariant approach.

If the same construction were attempted for $D \neq 26$ or $a \neq 1$, one would find that the Lorentz algebra in this realization does not close.

Using the above correspondence, it is very easy to estimate the total number of states at each mass level. Introducing a generating function

$$G(w) = \mathrm{Tr}_{\mathcal{F}^+}(w^{2M^2+2}) = \prod_{n=1}^{\infty}(1-w^n)^{-24} \equiv \sum_{n=1}^{\infty} d_n w^{2n} \ . \tag{2.49}$$

We find the number of states asymptotically for large M^2: $d_{M^2} = M^{-25/2} e^{M/4\pi\sqrt{2}}$.

§2.9. Primary fields and vertex operators for physical states

The construction given above for the physical Hilbert space has the disadvantage that the operators x^μ_{-n} with $n > 0$ create states in Fock space that are not in general physical. We now construct *vertex operators*, which have the property that they send physical states into physical states, and spurious states into other spurious states.

Since physical states obey $(L_n - \delta_{n,0})|\psi\rangle = (\tilde{L}_n - \delta_{n,0})|\psi\rangle = 0$, $n \geq 0$, it is clear that any operator commuting with the entire Virasoro algebra will yield such a vertex operator. Also, since $[L_n, V] = [\tilde{L}_n, V] = 0$, V automatically sends spurious states into spurious states.

Since L_n generates conformal reparametrizations, V should be conformally invariant. To construct V's, we consider general tensors, then construct invariants. We use the action of L_n on any *primary field*:

$$\begin{aligned} [L_n, \phi(z,\bar{z})] &= z^{n+1}\partial_z\phi(z,\bar{z}) + h(n+1)z^n\phi(z,\bar{z}) \\ &= \partial_z(z^{n+1}\phi(z,\bar{z})) + (h-1)(n+1)z^n\phi(z,\bar{z}) \\ [\tilde{L}_n, \phi(z,\bar{z})] &= \partial_{\bar{z}}(\bar{z}^{n+1}\phi(z,\bar{z})) + (\bar{h}-1)(n+1)\bar{z}^n\phi(z,\bar{z}) \ . \end{aligned} \tag{2.50}$$

It is clear that amongst primary fields, the only operator that commutes with all of L_n and $\tilde{L}_n$ is the unit operator $\phi = 1$.

However, integrals of ϕ may be invariant. For any primary field of conformal weight $(h,\bar{h}) = (1,1)$, its commutators with L_n, $\tilde{L}_n$ are total derivatives, and so its integral will be Virasoro invariant. In addition, primary fields ϕ of weight $(1,1)$ form natural volume forms on Σ. Hence the integral of ϕ

$$V = \int \phi_{(1,1)} \qquad \text{satisfies} \qquad [L_n, V] = [\tilde{L}_n, V] = 0 \ . \tag{2.51}$$

This is not the only way to construct vertex operators though: if ϕ is a holomorphic primary field (e.g. $\partial_z x^\mu$), then it will commute with $\tilde{L}_n$ provided $\bar{h} = 0$ and its commutator with L_n is a total derivative if $h = 1$. In addition $(1,0)$ and $(0,1)$ forms are naturally integrated along contours. Thus, the operators

$$\oint dz\, \phi_{(1,0)}$$

and

$$\oint d\bar{z}\, \phi_{(0,1)}$$

are also vertex operators provided that $\partial_{\bar{z}}\phi_{(1,0)} = 0$ and $\partial_z\phi_{(0,1)} = 0$. The operators $\phi_{(1,0)}$ and $\phi_{(0,1)}$ are precisely what we defined previously as conformal currents; their charges are conserved. Such operators do not in fact create new physical states, as will become clear by considering examples below.

Vertex operators for the closed bosonic string are now easily reconstructed from the basic field x^μ associated with D copies of the $c = 1$ CFT. We have the following primary fields

$$\begin{aligned} &\partial_z x^\mu && \text{holomorphic, weight} && (1,0) \\ &\partial_{\bar{z}} x^\mu && \text{anti-holomorphic, weight} && (0,1) \\ &:e^{ik\cdot x}: && \text{weight} && \left(\tfrac{k^2}{2}, \tfrac{k^2}{2}\right) \end{aligned}$$

From the holomorphic current $\partial_x x^\mu$ we get the conserved charge of *momentum*:

$$p^\mu = \frac{1}{2\pi}\oint dz\, \partial_z x^\mu \ . \tag{2.52}$$

Physical states in $\mathcal{F}_k^{\text{phys}}$ will just be eigenstates of p^μ, so p^μ does not really create any new states, but simply measures their momentum k.

A single exponential yields a vertex operator for $k^2 = 2$, i.e. when the conformal weight is $(1,1)$:

$$V_T(k) = \int_\Sigma d^2z\ :e^{ikx}: \tag{2.53}$$

This scalar operator creates a state of mass $M^2 = -2$, which is the tachyon discovered previously. But the exponential may also be combined with derivatives for $k^2 = 0$

$$V_{G,B,\Phi}(k) = \int_\Sigma d^2z\, \varepsilon_{\mu\bar{\mu}}(k)\, \partial_z x^\mu \partial_{\bar{z}} x^{\bar{\mu}}\ :e^{ik\cdot x}: \tag{2.54}$$

which reproduces the states for the graviton, the antisymmetric tensor, and the dilaton as symmetric traceless, antisymmetric, and trace part respectively, with $k_\mu \varepsilon^{\mu\bar{\mu}}(k) = k_{\bar{\mu}}\varepsilon^{\mu\bar{\mu}}(k) = 0$. For higher powers of $\partial_z x^\mu$ and $\partial_{\bar{z}} x^{\bar{\mu}}$, we will have masses of the form $M^2 = 2n - 2$ in general. However, higher derivatives $\partial_z^2 x^\mu$ may also appear. While these are not primary fields (they mix with $\partial_z x$ under conformal reparametrizations), they may combine with other operators and still yield primary fields.

When we obtained the Virasoro conditions on the polarization tensors $\varepsilon_{\mu\bar{\mu}}$, we established an invariance of the states in $\mathcal{F}^{\text{phys}}$ under the transformations

$$\varepsilon_{\mu\bar{\mu}} \to \varepsilon_{\mu\bar{\mu}} + k_\mu \bar{\gamma}_{\bar{\mu}} + k_{\bar{\mu}}\gamma_\mu \tag{2.55}$$

(We shall shortly interpret this invariance as a manifestation of $Diff(M)$-invariance for the closed string.) This invariance is also easily established for the vertex operators: for example a shift by $\bar{\gamma}$ yields

$$\begin{aligned} \int_\Sigma d^2z\, k_\mu \bar{\gamma}_{\bar{\mu}} \partial_z x^\mu \partial_{\bar{z}} x^{\bar{\mu}}\ :e^{ikx}: &= -i\int_\Sigma d^2z \bar{\gamma}_{\bar{\mu}} \partial_{\bar{z}} x^{\bar{\mu}} (\partial_z e^{ikx}) \\ &= i\int_\Sigma d^2z \bar{\gamma}_{\bar{\mu}} (\partial_z \partial_{\bar{z}} x^{\bar{\mu}}) e^{ik\cdot x} \end{aligned} \tag{2.56}$$

But we have $\partial_z\partial_{\bar{z}} x^\mu = 0$ by the field equations – away from any other singularities – and thus the integral vanishes. (The effects of the singularities may also be taken into account, as we shall show later on, and do not modify the conclusion above.)

§2.10. Identifying the graviton: vertex operators from background fields

One of the most important discoveries in string theory is that the $k^2 = 0$, $\varepsilon_{\mu\bar{\mu}}(k)$-symmetric traceless physical state, found above, should be identified with the graviton. This is the particle in quantum field theory that represents excitations of the space-time metric and that mediates the force of gravity. Here, we provide further evidence for this identification. Along the same lines, we provide evidence for the identification of the open string $k^2 = 0$, $\varepsilon_\mu(k)$-state with the Yang–Mills particle in this section and in §2.11.

Consider a space-time M which is almost flat with the following metric

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \varepsilon_{\mu\nu}(x) \tag{2.57}$$

where $\varepsilon_{\mu\nu}(x)$ is everywhere small compared to $\eta_{\mu\nu}$. The tensor $\varepsilon_{\mu\nu}(x)$ may be viewed as a superposition of plane waves by Fourier transform:

$$\varepsilon_{\mu\nu}(x) = \int d^{26}k\ \varepsilon_{\mu\nu}(k) e^{ik\cdot x} \tag{2.58}$$

Now, consider string theory in this background: the action on the worldsheet is given by

$$S_G = \frac{1}{8\pi} \int_\Sigma d\mu_g\, g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \tag{2.59}$$

and the transition amplitude involves the exponential of S_G. Since $\varepsilon_{\mu\nu}$ is small, we may expand in powers of $\varepsilon_{\mu\nu}$:

$$e^{-S_G} = e^{-S_\eta} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_n \frac{1}{(8\pi)^n} \tag{2.60}$$

with S_η the action corresponding to the flat metric and S_n defined by

$$S_n = \int d^{26}k_1 \ldots d^{26}k_n V(k_1) \ldots V(k_n) \tag{2.61}$$

where

$$V(k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu e^{\imath kx}$$

The operator $V(k)$ is precisely the vertex operator we constructed for the graviton – written now in general metric g – for a definite momentum k.

Thus, we see that strings couple to the fluctuations in the background metric precisely by the graviton vertex operator. The plane waves may be regarded as creating excitations in the background space-time metric, which make it deviate from the flat metric. (A completely analogous correspondence holds for the other massless states, the dilaton and the antisymmetric tensor.)

We have however established more precise information: in order to maintain conformal invariance and eliminate negative norm states, the momentum k and the polarization tensor $\varepsilon_{\mu\nu}$ had to satisfy *on-shell conditions* of §2.6

$$\begin{aligned} k^2 \varepsilon_{\mu\nu}(k) = 0 &\Longrightarrow \Box \varepsilon_{\mu\nu}(x) = \Box G_{\mu\nu}(x) = 0 \\ k^\mu \varepsilon_{\mu\nu}(k) = 0 &\Longrightarrow \partial^\mu \varepsilon_{\mu\nu}(x) = \partial^\mu G_{\mu\nu}(x) = 0 \\ \varepsilon^\mu_\mu(k) = 0 &\Longrightarrow \varepsilon^\mu_\mu(x) = 0\ , \end{aligned} \tag{2.62}$$

where we have listed to the right of the double arrows the restatement of the conditions on the metric.

Now, since $G_{\mu\nu}$ is almost flat (ε being small), corrections quadratic and higher in ε may be neglected. Considering the Ricci tensor in this approximation, we have

$$\begin{aligned} R_{\mu\nu} &= R^{\lambda}_{\mu\lambda\nu} = \partial_\nu \Gamma^{\lambda}_{\mu\lambda} - \partial_\lambda \Gamma^{\lambda}_{\mu\nu} + O(\varepsilon^2) \\ 2R_{\mu\nu} &= \partial_\mu \partial_\nu \varepsilon^{\lambda}_{\lambda} - \partial_\lambda \partial_\mu \varepsilon^{\lambda}_{\nu} - \partial_\lambda \partial_\nu \varepsilon^{\lambda}_{\mu} + \Box \varepsilon_{\mu\nu} \end{aligned} \tag{2.63}$$

The conditions on $G_{\mu\nu}$ obtained from the on-shell conditions of $\varepsilon_{\mu\nu}(k)$ now imply that

$$R_{\mu\nu} = 0 \ , \tag{2.64}$$

within the almost flat (or linearized) approximation.

In addition, the fact that shifts by null states are physically unobservable implies the gauge invariance of the physical states in $\mathcal{F}^+$ under the shifts

$$\varepsilon_{\mu\nu}(k) \longrightarrow \varepsilon_{\mu\nu}(k) + k_\mu \gamma_\nu + k_\nu \gamma_\mu \tag{2.65}$$

(with $k^\mu \gamma_\mu = 0$ for on-shell states). In terms of the metric, this gauge invariance translates into

$$\varepsilon_{\mu\nu}(x) \to \varepsilon_{\mu\nu}(x) + \partial_\mu \gamma_\nu(x) + \partial_\nu \gamma_\mu(x). \tag{2.66}$$

We recognize this transformation as an infinitesimal diffeomorphism generated by the vector field $\gamma^\mu(x)$, in the approximation where $O(\varepsilon^2)$ terms are neglected. $R_{\mu\nu} = 0$ is invariant under these transformations, as can be checked explicitly.

In fact $R_{\mu\nu} = 0$ is the only $Diff(M)$-invariant equation that reduces to $\Box G_{\mu\nu}(x) = 0$ in the linearized limit. Thus, we find that – to leading order in metric fluctuations – the conformal invariance constraints translate into Ricci flatness, i.e. Einstein's equations in the absence of matter. This discovery supports our proposed identification of $\varepsilon_{\mu\nu}(k)$, $k^2 = 0$ string states with gravitons.

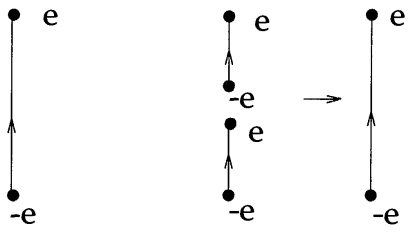
Later on (Lecture 6), we shall further confirm this identification in another important limit, that of small variations in the derivatives of the metric, the so-called low energy limit.

§2.11. Internal degrees of freedom of open strings: Chan–Paton rules

The end points of the open string are distinguished points. It is consistent with Poincaré and 2-d conformal invariance to put degrees of freedom or *charges* at the end points. During interactions of several strings, the charges simply flow along the boundary, and should thus be *conserved.*



The simplest case corresponds to adding *electric charge* to the end points of oriented strings:



Consistency with the interactions requires that the charges be opposite (i.e. corresponding to conjugate representations) at the two ends. An external $U(1)$ gauge field A_μ couples to these charges just in the same way as it couples to point particles with electric charge $\pm e$, moving on worldlines:

$$S_A = ie \int_B d\tau\, \dot{x}^\mu(\tau) A_\mu(x(\tau)). \tag{2.67}$$

Following the derivation of vertex operators from the coupling to background fields that we used for the closed string, we easily obtain the vertex operators for the open string gauge states, by decomposing A_μ into Fourier modes

$$A_\mu(x) = \int d^D k\, \varepsilon_\mu(k) e^{ik\cdot x}\ , \tag{2.68}$$

where $\varepsilon_\mu(k)$ is the polarization tensor. Thus, the vertex operators are

$$V(\varepsilon, k) = \varepsilon_\mu(k) \int_B d\tau\, \dot{x}^\mu(\tau) e^{ik\cdot x(\tau)} \tag{2.69}$$

with conformal invariance for $k^2 = 0$.

We may add further degrees of freedom to the end points of the string in the following way. Let λ^i, $i = 1, \ldots, N$ be Grassmann valued fields that live only on the boundary B of the string worldsheet (recall that Neumann b.c. were imposed on B for the map x: $\Sigma \to M$). A natural candidate for the action for such degrees of freedom is

$$S_0(\lambda) = \int_B d\tau\, \bar{\lambda}^i \frac{d}{d\tau} \lambda_i\ . \tag{2.70}$$

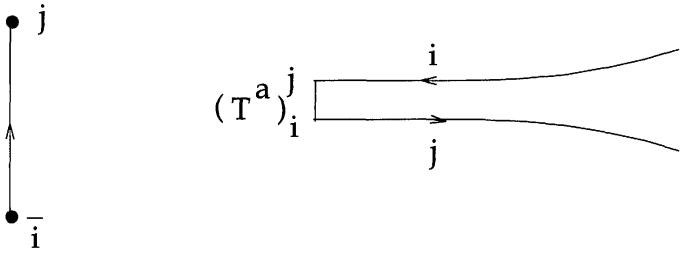
This action is $Diff(B)$ invariant, and does not depend upon the Σ metric. The equation of motion for λ_i simply requires that λ_i be τ-independent, so that λ_i may indeed be interpreted as conserved charges. The action S_0 automatically has $SO(N)$ symmetry when λ_i are real (and $\bar{\lambda}^i = \delta^{ij}\lambda_j$) and $SO(2N)$ symmetry when λ_i is complex. As a result, quantization automatically produces a Hilbert space of states on which a spinor representation of $SO(N)$ (for λ_i real) or $SO(2N)$ (for λ_i complex) acts.

To restrict to a subgroup G of $SO(N)$ or $SO(2N)$, and/or a subrepresentation Λ of the corresponding spinor representations, it is necessary to restrict the Hilbert space of states. This is conveniently achieved by introducing a Lagrange multiplier, σ: $B \to \mathbb{R}$. For $G = U(N)$, for example, the restriction to the rank p $(0 \leq p \leq N)$

totally antisymmetric tensor representation of $U(N)$, we use the following modification of S_0:

$$S(\lambda) = \int_B d\tau \left(\bar{\lambda}^i \frac{d}{d\tau} \lambda_i + \sigma(\tau)(\bar{\lambda}^i \lambda_i - p) \right) \tag{2.71}$$

Analogous modifications may be used for $G = \mathrm{SO}(N)$ and $\mathrm{Sp}(N)$. Let us now restrict to $G = U(N)$, and $p = 1$, so that Λ equals the defining representation. Quantization of λ_i generates N conserved charges attached to the boundaries of the string



The string for $U(N)$ gauge group should be *oriented*, since the charges at both ends of the string transform under inequivalent representations of $U(N)$. The string state now transforms under the representation $\bar{\Lambda} \otimes \Lambda$ of $U(N)$, which is the adjoint of $U(N)$. The representation matrices T^a, with $a = 1, \ldots, \dim U(N) = N^2$ describe the string state belonging to the representation $\bar{\Lambda} \otimes \Lambda$, by the index a. The matrix elements $(T^a)^j_i$ of T^a specify which λ_i and $\bar{\lambda}^j$ are created at the boundary B of the open string. The full Fock space of the open string is given by $\mathcal{F}_{\mathrm{open}} \otimes \bar{\Lambda} \otimes \Lambda$, and the physical state space is $\mathcal{F}^{\mathrm{phys}}_{\mathrm{open}} \otimes \bar{\Lambda} \otimes \Lambda$.

We observe that for $G = U(N)$ and Λ the defining representation, $\bar{\Lambda} \otimes \Lambda$ transforms in the adjoint representation and the open string states are thus natural candidates for Yang–Mills particles with gauge group $G = U(N)$. Had we considered any other representation for Λ (other than the defining or its complex conjugate), the string states would not have transformed precisely under the adjoint representation of $U(N)$, and could not have been interpreted as Yang–Mills particles. Thus the defining representation is singled out by its relevance to describing Yang–Mills particles.

The strings for $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ should be unoriented, since the charges at both ends transform under equivalent representations of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, respectively. Considering Λ to be the defining representation of $\mathrm{SO}(N)$, we obtain string states in the adjoint of $\mathrm{SO}(N)$ by the antisymmetrized tensor product of Λ by itself, $(\Lambda \otimes \Lambda)_A$. On the other hand, for Λ the defining representation of $\mathrm{Sp}(N)$, we obtain the string states in the adjoint of $\mathrm{Sp}(N)$ by the symmetrized tensor product of Λ by itself, $(\Lambda \otimes \Lambda)_S$. In each case we recover precisely the string states suitable for Yang–Mills particles.

For exceptional groups, no construction is known that produces string states in the adjoint representation only, and it is widely believed that it is impossible to obtain Yang–Mills particles from open strings with these groups.

A non-Abelian gauge field can couple to the boundaries of the open string, much in the same way as this happens for Abelian gauge fields. Here, however, we must include the coupling of the charges of the gauge field A^a_μ to those of the string. Let us restrict again to the defining representation of $U(N)$. The coupling

of A^a_μ is given as follows:

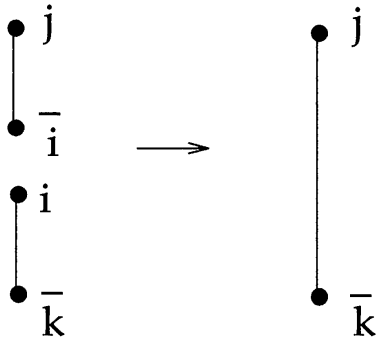
$$S_A(\lambda) = \int_B d\tau \left\{ \bar\lambda^i \left(\delta^j_i \frac{d}{d\tau} + i A^a_\mu \dot x^\mu (T^a)^j_i \right) \lambda_j + \sigma(\tau)(\bar\lambda^i \lambda_i - 1) \right\} . \tag{2.72}$$

The coupling to the gauge field A^a_μ may again be used to construct the vertex operators for the creation and annihilation of gauge states of the string. One finds upon quantization that the vertex operator for the $U(1)$ case is simply to be multiplied by the matrices $(T^a)^j_i$:

$$V^a(\varepsilon, k) = T^a V(\varepsilon, k) . \tag{2.73}$$

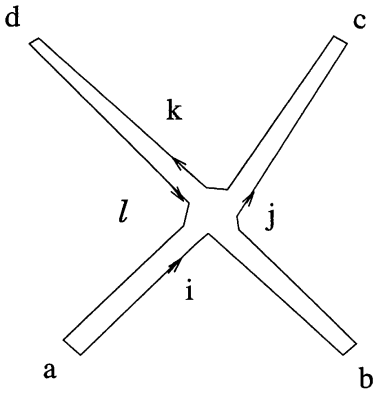
As the internal degree of freedom and the matrices T^a commute with Virasoro, the physical state conditions on V^a are satisfied whenever they are on $V(\varepsilon, k)$.

It is instructive to examine the effects of these degrees of freedom on interactions as well. For simplicity, we restrict to $A^a_\mu = 0$. Charges then simply continue to be translated along the boundary B in a conserved way, as shown schematically in the figure below.



Interactions are only allowed if charges are conserved.

In a full string diagram, the effect will be to include a trace over the matrices T, just as in Yang–Mills Feynman rules, as shown schematically below.



$\sim \operatorname{tr} T^a T^b T^c T^d$

With the help of the Yang–Mills vertex operators V^a of (2.73) and (2.69), one recovers Yang–Mills scattering amplitudes in the limit where momenta are small compared to the Planck length.

References and cross references

For an introduction to conformal field theory, see the lecture notes by K. Gawędzski in this book [**II-CFT**], as well as the following set of lecture notes [**Ginsparg (1990)**], reprint volume [**Itzykson, Saleur and Zuber (1988)**] and encyclopedic textbook [**Di Francesco, Mathieu and Sénéchal (1987)**],

P. Ginsparg, *Applied conformal field theory*, Fields, Strings, and Critical Phenomena, Les Houches, Session 49 (E. Brezin and J. Zinn-Justin, eds.), North Holland, 1990.

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V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess–Zumino models in two dimensions*, Nucl. Phys. B **247** (1984), 83.

R. Dijkgraaf, E. Verlinde and H. Verlinde, $c = 1$ *conformal field theories on Riemann surfaces*, Comm. Math. Phys. **115** (1988), 649.

The quantization of free bosonic strings is discussed in virtually any introduction to string theory, see for example

J. Scherk, Rev. Modern Phys. **47** (1975), 123.

M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vols. I, II, Cambridge University Press, 1988, and references therein.

LECTURE 3
String Amplitudes and Moduli Space of Curves

In the previous lecture, we have quantized free bosonic strings and determined their spectrum and Hilbert space of states under certain conditions: $D = 26$ and the presence of massless gravitons and Yang–Mills particles. (The case $D \leq 25$, allowed by the no-ghost theorem, was shown to be a subcase.)

Now, we shall quantize interacting strings, and investigate quantum mechanical transition amplitudes for the scattering of string states. While operator methods were convenient for determining the spectrum of free strings, the functional integral formulation in terms of a summation over surfaces is better suited for dealing with interactions. Thus, our starting point is the transition amplitude

$$A = \sum_{\substack{\text{topologies } \Sigma \\ (h=0,\dots,\infty)}} \int_{\mathrm{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int_{\mathrm{Map}(\Sigma,M)} V_1 \dots V_N e^{-S[x,g]} \tag{3.1}$$

with Riemannian space-time M, and external string states represented by vertex operators V_i, $i = 1, \dots, N$. This was precisely our original starting point with Euclidean signature metric on M. Minkowski amplitudes (for example, for flat $M = \mathbb{R}^D_M$) are obtained by analytic continuation in external momenta and polarization tensors. The vertex operators were determined in Lecture 2 in terms of conformal primary fields of weight $(1,1)$. The summation in the transition amplitude A is over topologies of compact oriented surfaces, since we shall limit ourselves here to the study of the closed oriented string.

The strategy is as follows:

1) For fixed topology of Σ, we use the product structure of $\mathrm{Met}(\Sigma) = Weyl(\Sigma) \times$ conformal classes (Σ) to decompose the measure Dg on $\mathrm{Met}(\Sigma)$ into a product of measures on the orbits of Weyl, on the orbits of $Diff(\Sigma)$ and on the moduli space $\mathcal{M}_h = \mathrm{Met}(\Sigma)/Diff(\Sigma) \ltimes Weyl(\Sigma)$. A Jacobian determinant arises, which is essentially the Faddeev-Popov determinant. This decomposition holds for any matter conformal field theory, in particular for any space-time M, defining a conformal field theory.

2) For flat Euclidean space-time M, the combined measure – including the Faddeev-Popov determinant – and the correlation functions are found to be Weyl-invariant (i.e. $Weyl(\Sigma)$ is a symmetry group) if and only if $D = 26$ and gravitons are massless on surfaces of any topology (equivalently, the $a = 1$ condition of Lecture 2 holds).

3) Under the assumptions of 2) the string theory is usually called critical. In this case, the formal measure $Dg/\mathcal{N}(g)$ on $\mathrm{Met}(\Sigma)$, divided by the volume of the orbits of the symmetry group $Diff(\Sigma) \ltimes Weyl(\Sigma)$, *naturally projects down* to an *intrinsically defined measure on* $\mathcal{M}_h$. Thus, the transition amplitudes for critical strings reduce to well-defined integrals over $\mathcal{M}_h$ of the matter conformal field theory correlation functions.

4) More generally, the assumptions of 2) may be relaxed, and the matter part may be any conformal field theory of central charge $c = 26$ and with correlation functions satisfying certain Weyl invariance conditions.

5) In particular, in Lecture 6, we shall generalize the Weyl invariance conditions to the case of curved space-time manifolds M – in certain limits.

6) Under the assumptions of 2), we shall obtain in this chapter 0- and 1-loop transition amplitudes for critical bosonic string theory.

§3.1. Finite-dimensional case

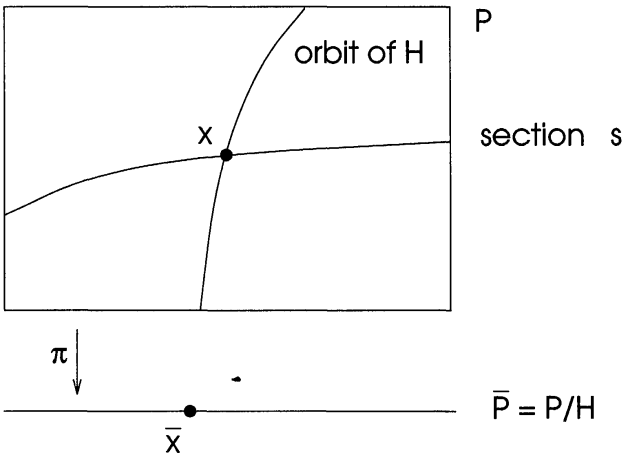
Let P be a finite-dimensional manifold, with Riemannian metric G, and associated measure $d\mu_G$. Let H be a Lie group acting on P, and assume that $d\mu_G$ is H-invariant. We wish to reduce the integrals of H-invariant functions on P to integrals on the quotient $\bar{P} = P/H$. Let $f(x)$: $P \to \mathbb{C}$ be H-invariant,

$$f(xh) = f(x) \qquad \text{for all} \qquad x \in P,\ h \in H.$$

Then for any section s: $\bar{P} \to P$, we have

$$\int_P d\mu_G f = \int_{\bar{P}} d\bar{\mu}\ s^*(f) J \int_H d\nu_H\ ,$$

where $d\bar{\mu}$ is the measure on $\bar{P}$ induced from $d\mu_G$, $d\nu_H$ is the left-invariant measure on H, and J is a Jacobian (or Faddeev-Popov) determinant which we define below, as follows. First, let us present a figure:



The tangent space at x to P may be decomposed into the part tangent to the orbit of H, generated by the Lie algebra $\mathcal{H}$,

$$i\colon \mathcal{H} \to T_x P$$

and a part from the push-forward of the tangent space to $T_{\bar{x}}\bar{P}$:

$$s_*\colon T_{\bar{x}}\bar{P} \to T_x(P) \ ,$$

as follows:

$$s_* \oplus i\colon T_{\bar{x}}\bar{P} \oplus \mathcal{H} \to T_x(P).$$

The Jacobian J is now obtained as

$$J = \det(T_{\bar{x}}\bar{P} \oplus \mathcal{H} \to T_x P).$$

§3.2. Basic notation: tensors, derivatives

Fix a positive definite metric g on Σ and choose local complex coordinates z, $\bar{z}$ in which

$$g = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = 2g_{z\bar{z}}|dz|^2$$

Tensors decompose into one-dimensional components of weight (m, n), which are sections of a line bundle $K^{(m,n)}$:

$$\phi = \phi_{\underbrace{z\ldots z}_{m}\,\underbrace{\bar{z}\ldots\bar{z}}_{n}}(dz)^m(d\bar{z})^n \in K^{(m,n)} \ .$$

Using the metric g, we identify tensors of weight (m, n) with tensors of weight $(m-n, 0)$. The space of tensors of weight $(m, 0)$ is denoted K^m, with K^1 the space of sections of the canonical line bundle $T^*_{(1,0)}(\Sigma)$. (Generally, we do not distinguish between a bundle and the space of its sections.)

On sections of K^m we have the L^2 inner product

$$(\phi_1, \phi_2)_g = \int_\Sigma d\mu_g (g_{z\bar{z}})^{-m} \phi_1^* \phi_2 \tag{3.2}$$

as well as the covariant derivatives

$$\nabla\phi = \nabla^{(m)}_{\bar{z}}\phi + \nabla^{(m)}_{z}\phi \tag{3.3a}$$

where $\nabla^{(m)}_{\bar{z}} = \bar{\partial}_{(m)}$ is the Cauchy–Riemannn operator

$$\nabla^{(m)}_{\bar{z}}\colon K^m \to K^{(m,1)} \ \nabla^{(m)}_{\bar{z}}\phi = \frac{\partial}{\partial\bar{z}}\,\phi \otimes d\bar{z} \ . \tag{3.3b}$$

Equivalently, identifying $(m, 1) \sim (m-1, 0)$, we have

$$\begin{cases} \nabla^z_{(m)}\colon K^m \to K^{m-1} & \nabla^z_{(m)}\phi = (g_{z\bar{z}})^{-1}\frac{\partial}{\partial\bar{z}}\phi \otimes (dz)^{-1} \\ \nabla^{(m)}_z\colon K^m \to K^{m+1} & \nabla^{(m)}_z\phi = (g_{z\bar{z}})^m\frac{\partial}{\partial z}(g_{z\bar{z}})^{-m}\phi \otimes dz \ . \end{cases} \tag{3.4}$$

(In general, we shall not exhibit the differentials dz and $d\bar{z}$ in ∇.) The adjoints of these operators with respect to $(\ ,\ )_g$ are

$$\left(\nabla^z_{(m)}\right)^\dagger = -\nabla^{(m-1)}_z; \quad \left(\nabla^{(m)}_z\right)^\dagger = -\nabla^z_{(m+1)} \tag{3.5}$$

and the Laplace operators are

$$\begin{aligned} \Delta^{+}_{(m)} &= -2\nabla^{z}_{(m+1)}\nabla^{(m)}_{z} \\ \Delta^{-}_{(m)} &= -2\nabla^{(m-1)}_{z}\nabla^{z}_{(m)} \end{aligned} \tag{3.6}$$

We also make use of the Riemann-Roch-Atiyah-Singer theorem

$$\dim \mathrm{Ker}\nabla^{(m)}_{z} - \dim \mathrm{Ker}\nabla^{z}_{(m+1)} = \left(m+\frac{1}{2}\right)\chi(\Sigma) = (2m+1)(1-h) \ , \tag{3.7}$$

as well as the vanishing theorems:

$$\begin{aligned} &\bullet \quad h=0, m\geq 1, && \mathrm{Ker}\,\nabla^{(m)}_{\bar{z}} = 0 \\ &\qquad && \text{(no holomorphic forms on the sphere)} \\ &\bullet \quad h\geq 2, m\leq -1, && \mathrm{Ker}\,\nabla^{(m)}_{\bar{z}} = 0 \ . \\ &\qquad && \text{(no holomorphic vector fields for } h\geq 2) \end{aligned} \tag{3.8}$$

§3.3. Space of metrics – moduli space of Riemann surfaces

Let $\mathrm{Met}(\Sigma)$ denote the space of positive definite metrics on Σ, and $T_g\mathrm{Met}(\Sigma)$ its tangent space at g, consisting of infinitesimal deformations $\delta g \in K^{(1,1)} \oplus K^{(2,0)} \oplus K^{(0,2)}$. Following the results of §3.2, we have an L^2 norm $\|\delta g\|^2_g = (\delta g, \delta g)_g$ on $T_g\mathrm{Met}(\Sigma)$, which makes $\mathrm{Met}(\Sigma)$ into an infinite dimensional Riemannian manifold. The measure Dg is defined with respect to $\|\delta g\|$.

$Diff(\Sigma)$ acts isometrically on $\mathrm{Met}(\Sigma)$, but $Weyl(\Sigma)$ does not. The space of orbits of $Diff(\Sigma) \ltimes Weyl(\Sigma)$ in $\mathrm{Met}(\Sigma)$ is the *moduli space* of Riemann surfaces (or complex curves) of genus h:

$$\mathcal{M}_h \equiv \mathrm{Met}(\Sigma)/Diff(\Sigma) \ltimes Weyl(\Sigma) \ . \tag{3.9}$$

Actually, the group $Diff(\Sigma)$ is not connected. The group of connected components is the *mapping class group* G_Σ:

$$G_\Sigma = \pi_0(Diff(\Sigma)) \ ,$$

which is an infinite discrete group for $h\geq 1$. The component connected to the identity in $Diff(\Sigma)$ is denoted by $Diff_0(\Sigma)$. Thus, there arises another natural space: $\mathcal{T}_h$, which is Teichmüller space

$$\mathcal{T}_h \equiv \mathrm{Met}(\Sigma)/Diff_0(\Sigma) \ltimes Weyl(\Sigma) \ ,$$

which is the simply connected covering space of $\mathcal{M}_h$:

$$\mathcal{M}_h = \mathcal{T}_h/G_\Sigma \ .$$

Teichmüller space is a complex finite dimensional manifold, while moduli space is a complex finite dimensional orbifold (the action of G_Σ on $\mathcal{T}_h$ produces singularities), both of dimension

$$\dim_{\mathbb{C}} \mathcal{T}_h = \dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h=0 \\ 1 & h=1 \\ 3h-3 & h\geq 2 \ . \end{cases} \tag{3.10}$$

We have explained that string transition amplitudes can be reduced to an integral over moduli space. We wish to factorize the measure Dg into its components along $Diff(\Sigma)$, $Weyl(\Sigma)$ and $\mathcal{M}_h$, i.e. we want to fix the gauge. To examine this factorization concretely, we decompose $T_g\mathrm{Met}(\Sigma)$ into the tangent spaces to the orbits of $Diff_0(\Sigma)$ and $Weyl(\Sigma)$, as explained in the finite dimensional example. We perform this factorization at an arbitrary metric $g \in \mathrm{Met}(\Sigma)$, and shall choose a section later on.

We use the fact that $Diff_0(\Sigma)$ is generated by vector fields $v = v^z \oplus v^{\bar{z}} \in K^{(-1,0)} \oplus K^{(0,-1)}$. A Weyl transformation will in general be denoted by $e^{2\sigma}$, where σ takes real values.

Let the reference metric be $g = 2g_{z\bar{z}}|dz|^2$; an arbitrary metric $g + \delta g$ in the neighborhood of g may then be conveniently parametrized by a change in $g_{z\bar{z}}$ as well as by Beltrami differentials $\delta\eta^z_{\bar{z}} \in K^{(-1,1)}$ and $\delta\bar{\eta}^{\bar{z}}_z \in K^{(1,-1)}$.

$$g + \delta g = 2\left(g_{z\bar{z}} + \delta g_{z\bar{z}}\right)|dz + \delta\eta^z_{\bar{z}} d\bar{z}|^2 \ . \tag{3.11}$$

The space $T_g\mathrm{Met}(\Sigma)$ admits the orthogonal decomposition

$$\begin{aligned} T_g\mathrm{Met}(\Sigma) &= K^{(1,1)} \oplus K^{(2,0)} \oplus K^{(0,2)} \\ \delta g &= 2\delta g_{z\bar{z}}|dz|^2 + 2g_{z\bar{z}}\delta\eta^z_{\bar{z}}(d\bar{z})^2 + 2g_{z\bar{z}}\delta\bar{\eta}^{\bar{z}}_z(dz)^2 \ . \end{aligned} \tag{3.12}$$

The infinitesimal action of $(\delta v^z, \delta v^{\bar{z}}, 2\delta\sigma)$ in the Lie algebra of $Diff_0(\Sigma) \ltimes Weyl(\Sigma)$ is then simply

$$\begin{cases} \delta g_{z\bar{z}} = (2\delta\sigma + \nabla_z \delta v^z + \nabla_{\bar{z}} \delta v^{\bar{z}}) g_{z\bar{z}} \\ \delta\eta^z_{\bar{z}} = \nabla_{\bar{z}} \delta v^z \\ \delta\bar{\eta}^{\bar{z}}_z = \nabla_z \delta v^{\bar{z}} \ . \end{cases} \tag{3.13}$$

While the range of $Weyl(\Sigma)$ is the full $K^{(1,1)}$, in general the range of $Diff_0(\Sigma)$ need not be all of $K^{(2,0)} \oplus K^{(0,2)}$. Instead, we have the orthogonal decomposition

$$K^{(2,0)} = \mathrm{Range}\, \nabla^{(1)}_z \oplus \mathrm{Ker}(\nabla^{(1)}_z)^\dagger \ . \tag{3.14}$$

The space $\mathrm{Ker}\,(\nabla^{(1)}_z)^\dagger = \mathrm{Ker}\,\nabla^{(2)}_{\bar{z}}$ consists of holomorphic quadratic differentials on Σ, and can be identified with the holomorphic cotangent space of $\mathcal{M}_h$:

$$T^*_{(1,0)}(\mathcal{M}_h) = \mathrm{Ker}\,\nabla^{(2)}_{\bar{z}} \tag{3.15}$$

(Analogously, we have $T^*_{(0,1)}(\mathcal{M}_h) = \mathrm{Ker}\,\nabla^{(-2)}_z$.) To see this, we notice that holomorphic differentials $\phi_j \in \mathrm{Ker}\,\nabla^{(2)}_{\bar{z}}$ provide linear forms on the space of differentials $\delta\eta$:

$$(\delta\eta, \phi_j) = \int_\Sigma dz\, d\bar{z}\ \delta\eta^z_{\bar{z}} \phi_{jzz} \tag{3.16}$$

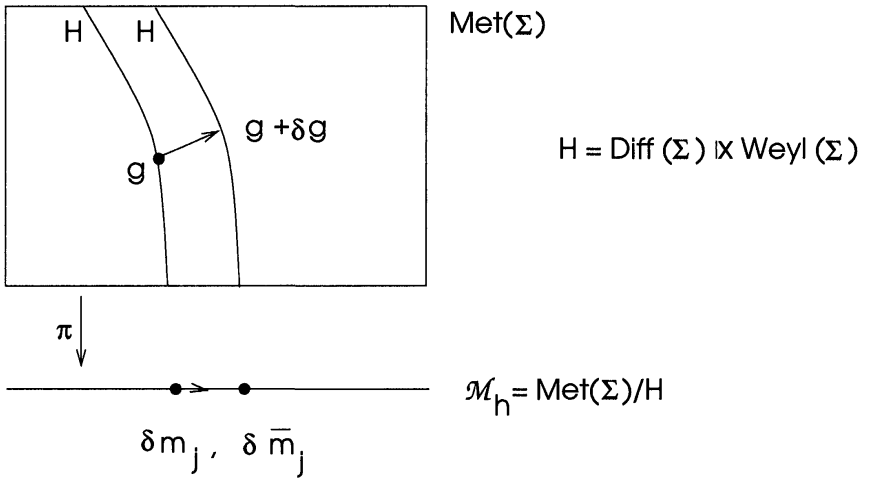
The above pairing is $Weyl(\Sigma)$-invariant and the kernel of $(\ \cdot\ , \phi_j)$ (and $(\ \cdot\ , \phi^*_j)$) is precisely the tangent space to the orbits of $Diff_0(\Sigma)$ and $Weyl(\Sigma)$; hence the identification. By the Riemann-Roch theorem, and the vanishing theorems listed

above, it is easy to see that the dimensions of $\mathcal{M}_h$ and of $\mathrm{Ker}\,\nabla_{\bar{z}}^{(2)} \oplus \mathrm{Ker}\,\nabla_z^{(-2)}$ coincide.

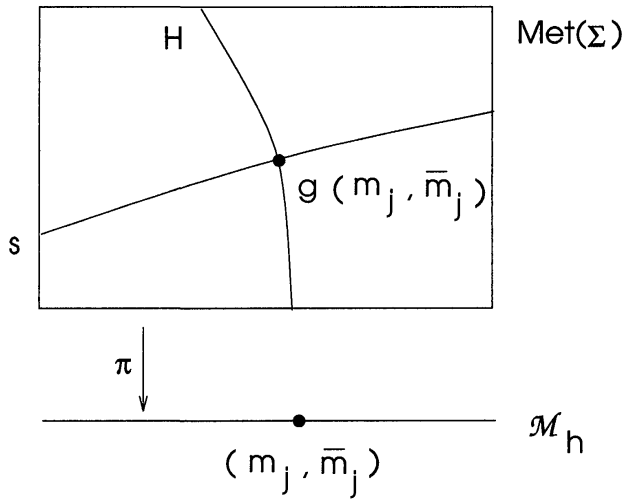
More precisely, the holomorphic quadratic differentials $\phi_j \in \mathrm{Ker}\,\nabla_{\bar{z}}^{(2)}$ form a complex vector space, thus providing a natural almost complex structure on the tangent space of $\mathcal{M}_h$. In fact, this almost complex structure is integrable and results in a complex structure on $\mathcal{M}_h$; for a given choice of ϕ, the above linear forms define local complex coordinates m_j, $\bar{m}_j$ on $\mathcal{M}_h$ given by

$$\delta m_j = (\delta\eta, \phi_j) \qquad \delta\bar{m}_j = (\delta\bar{\eta}, \bar{\phi}_j) \ . \tag{3.17}$$

We may represent this projection onto $\mathcal{M}_h$ by a picture analogous to the one used to discuss the finite dimensional example at the beginning of this lecture.



Vice versa, for a given set of complex coordinates $(m_j, \bar{m}_j)$ on $\mathcal{M}_h$, we may choose a section s of $\mathcal{M}_h$ in $\mathrm{Met}(\Sigma)$ – also called a slice – of dimension dim $\mathcal{M}_h$, and parametrized by $(m_j, \bar{m}_j)$:



The tangent space at $g(m_j, \bar{m}_j)$ then decomposes into the tangent directions to the action of $Weyl(\Sigma)$, the action of $Diff_0(\Sigma)$ on the differentials $\delta\eta$, $\delta\bar{\eta}$, and moduli deformations, given by a basis of *Beltrami differentials* μ_j and $\bar{\mu}_j$:

$$\begin{cases} \delta\eta_{\bar{z}}^{z} = \nabla_{\bar{z}}^{(-1)}\delta v^z + \sum_{j=1}^{\dim \mathcal{M}_h} \delta m_j \mu_{j\bar{z}}^{z} \\ \delta\bar{\eta}_{z}^{\bar{z}} = \nabla_{z}^{(1)}\delta v^{\bar{z}} + \sum_{j=1}^{\dim \mathcal{M}_h} \delta\bar{m}_j \bar{\mu}_{jz}^{\bar{z}} \ . \end{cases} \tag{3.18}$$

We may choose the Beltrami differentials as follows

$$\begin{cases} \mu^z_{j\bar z} = g^{z\bar z}\frac{\partial g_{\bar z\bar z}}{\partial m_j}(m,\bar m) \\ \bar\mu^{\bar z}_{jz} = g^{z\bar z}\frac{\partial g_{zz}}{\partial \bar m_j}(m,\bar m)\ , \end{cases} \tag{3.19}$$

so that μ_j and $\bar\mu_j$ are unchanged under Weyl transformations $g \to e^{2\sigma}g$: $\mu_j \to \mu_j$; $\bar\mu_j \to \bar\mu_j$. Thus, under a change of section s, μ_j and $\bar\mu_j$ will be shifted only by the action of a vector field in the Lie algebra of $Diff_0(\Sigma)$.

Finally, for Σ with the topology of either the sphere or the torus, we also have holomorphic vector fields in $\mathrm{Ker}\,\nabla^{(-1)}_{\bar z}$. These generate conformal automorphisms, and do not act on η at all. We have

$$\begin{cases} h=0 & \dim \mathrm{Ker}\nabla^{(-1)}_{\bar z} = 3 & \mathrm{SL}(2,\mathbb{C}) \\ h=1 & \dim \mathrm{Ker}\nabla^{(-1)}_{\bar z} = 1 & T^2 \\ h\geq 2 & \dim \mathrm{Ker}\nabla^{(-1)}_{\bar z} = 0 & \text{possibly finite automorphism group}\ . \end{cases} \tag{3.20}$$

§3.4. Factorizing the integration measure

We decompose the integration measure Dg on $\mathrm{Met}(\Sigma)$ into measures on the orbits of $Diff(\Sigma)$ and $Weyl(\Sigma)$ and on $\mathcal{M}_h$ by using the parametrization developed above for δg in terms of Weyl changes $\delta\sigma$ and differentials $\delta\eta$, $\delta\bar\eta$:

$$\delta g = 2\delta\sigma g + 2g_{z\bar z}\delta\eta^z_{\bar z}(d\bar z)^2 + 2g_{z\bar z}\delta\bar\eta^{\bar z}_z(dz)^2\ . \tag{3.21}$$

The relation

$$\|\delta g\|^2_g = 4\|\delta\sigma\|^2_g + 2\|\delta\eta\|^2_g + 2\|\delta\bar\eta\|^2_g \tag{3.22}$$

implies that $Dg = D\sigma\, D\eta\, D\bar\eta$, in obvious notation.

Next, we make use of the decomposition of $\delta\eta$ and $\delta\bar\eta$ into orbits of $Diff_0(\Sigma)$ and moduli deformations, which we parametrize by coordinates $(\delta m_j, \delta\bar m_j)$ on $\mathcal{M}_h$ and a choice of Beltrami differentials $\mu^z_{j\bar z}$ and $\bar\mu^{\bar z}_{jz}$, as in (3.18). Using the orthogonal decomposition of the differentials $\delta\eta^{\bar z}_z\ g_{z\bar z}$ according to

$$K^{(2,0)} = \mathrm{Range}\,\nabla^{(1)}_z \oplus \mathrm{Ker}\,\nabla^{(2)}_{\bar z}\ , \tag{3.23}$$

we evaluate $\|\delta\eta\|^2_g$ and find

$$2\|\delta\eta\|^2_g = 2\|\nabla^{(-1)}_{\bar z}\delta v^z\|^2_g + \|\delta\eta\perp \text{ projection of } \delta\eta^{\bar z}_z\ g_{z\bar z} \text{ onto } \mathrm{Ker}\,\nabla^{(2)}_{\bar z}\|^2_g\ . \tag{3.24}$$

The second term on the right hand side is easily worked out by choosing an orthogonal basis for $\mathrm{Ker}\,\nabla^{(2)}_{\bar z}$, with basis vectors ϕ_j, $j = 1,\dots,\dim\mathcal{M}_h$:

$$\begin{aligned} &\|\delta\eta\perp \text{ projection of } \delta\eta^{\bar z}_z\ g_{z\bar z} \text{ onto } \mathrm{Ker}\,\nabla^{(2)}_{\bar z}\|^2_g \\ &\quad = \sum_j (\delta\eta,\phi_j)_g(\phi_j,\phi_j)^{-1}_g(\phi_j,\delta\eta)_g \\ &\quad = \sum_{j,k,\ell} \delta m_k\delta\bar m_\ell(\mu_k,\phi_j)_g(\phi_j,\phi_j)^{-1}_g(\phi_j,\mu_\ell)_g \end{aligned} \tag{3.25}$$

The first term on the right hand side of (3.24) is conveniently rewritten in terms of the Laplace operator $\Delta^-_{(-1)}$:

$$2\|\nabla^{(-1)}_{\bar{z}}\delta v^z\|^2_g = (\delta v^z, \Delta^-_{(-1)}\delta v^z)_g$$

A last point must be clarified before evaluating the measure $D\eta\, D\bar{\eta}$. For the special cases of the sphere ($h = 0$) and the torus ($h = 1$), $\mathrm{Ker}\,\nabla^{(-1)}_{\bar{z}} \neq 0$, and there are (continuous families of) conformal automorphisms of Σ: $\mathrm{SL}(2,\mathbb{C})$ for $h = 0$ and T^2 for $h = 1$. The corresponding vector fields leave η and $\bar{\eta}$ invariant. Thus, in the change of variables, $\delta\eta \to (\delta v, \delta m_j)$, we must restrict to vector fields δv that are orthogonal to $\mathrm{Ker}\,\nabla^{(-1)}_{\bar{z}}$. The corresponding restricted volume element on vector fields v is denoted by $D'v$. The associated determinant of $\Delta^-_{(-1)}$ must be similarly restricted to non-zero modes.

Putting this all together, we find

$$Dg = \det{}'\Delta^-_{(-1)}\, \frac{|\det(\mu_j,\phi_k)_g|^2}{\det(\phi_j,\phi_k)_g} \prod_j dm_j\, d\bar{m}_j\, D'v\cdot D\sigma\ . \tag{3.26}$$

While originally we chose $\phi_j \in \mathrm{Ker}\,\nabla^{(2)}_{\bar{z}}$ to form an orthogonal basis, it is now clear that the above formula holds for any basis $\mathrm{Ker}\,\nabla^{(2)}_{\bar{z}}$. (The finite dimensional determinants are of dim $\mathcal{M}_h \times$ dim $\mathcal{M}_h$ dimensional matrices.)

It is standard practice – and extremely useful – to introduce the following combination of determinants

$$Z^{\pm}_{(n)}(g) \equiv \frac{\mathrm{Det}'\Delta^{\pm}_{(n)}}{\det(\phi_j,\phi_k)_g\,\det(\psi_a,\psi_b)_g} \tag{3.27}$$

where

- $\phi_j \in \mathrm{Ker}\,\nabla^z_{(n+1)}$ for $+$; $\phi_j \in \nabla^{(n-1)}_z$ for $-$
- $\psi_a \in \mathrm{Ker}\,\nabla^{(n)}_z$ for $+$; $\psi_a \in \nabla^z_{(n)}$ for $-$.

While $Z^{\pm}_{(n)}$ have non-trivial Weyl transformation properties, we shall show in the next section that this particular combination has simple transformation properties.

Putting this all together, we have the following expression for the measure Dg:

Theorem 3.1.

$$Dg = D\sigma\, \widetilde{D}v\, Z^-_{(-1)}(g)\, |\det(\mu_j,\phi_k)_g|^2 \prod_{j=1}^{\dim \mathcal{M}_h} dm_j\, d\bar{m}_j\ , \tag{3.28}$$

where we have

1) $\widetilde{D}v \equiv D'v\, \det(\psi_a,\psi_b)_g$. For $h \geq 2$, we have $\widetilde{D}v = Dv$, i.e. the measure over all vector fields of $Diff_0(\Sigma)$. For $h = 0, 1$, we shall postpone the explicit construction of $\widetilde{D}v$ until §3.9.
2) The measure

$$|\det(\mu_j,\phi_k)_g|^2 \prod_j dm_j\, d\bar{m}_j \tag{3.29}$$

is intrinsically defined on $\mathcal{M}_h$. To see this, notice that we chose μ_j to be Weyl independent, that ϕ_j can be chosen Weyl independent, and that the pairing $(\mu_j, \phi_k)_g$ is Weyl invariant. Furthermore, we defined Beltrami differentials by

$$\mu^z_{j\bar{z}} = g^{z\bar{z}} \frac{\partial g_{\bar{z}\bar{z}}}{\partial m_j} \tag{3.30}$$

so that the above combination is manifestly unchanged under reparametrizations of m_j and $\bar{m}_j$.

3) $Z^-_{(-1)}(g)$ will be Weyl dependent, as we shall establish in the next section. This object is called the Faddeev-Popov determinant (up to the finite dimensional determinants), expressing the fact that we gauge fixed the $Diff_0(\Sigma)$-invariance.

4) To complete the construction outlined in the finite-dimensional case, for the projection down onto $\mathcal{M}_h$ of the measure Dg on $\mathrm{Met}(\Sigma)$, we will have to understand the Weyl dependence of the combined $Z^-_{(-1)}(g)$ and matter contributions, which we do in the next section.

§3.5. Weyl rescalings of functional determinants

The determinant combinations $Z^{\pm}_{(n)}$ (including those with $n \in \frac{1}{2} + \mathbb{Z}$, which will enter when dealing with the superstring, as we shall see later on) will be considered repeatedly, and we shall need their behavior under $Weyl(\Sigma)$ rescalings of the metric. Throughout, we shall adopt a regularization and renormalization scheme that preserves $Diff(\Sigma)$ invariance.

Theorem 3.2.

$$Z^{\pm}_{(n)}(ge^{2\sigma}) = Z^{\pm}_{(n)}(g) e^{-2c^{\pm}_{(n)} S_L(\sigma, g)} \tag{3.31}$$

where $c^{\pm}_{(n)}$ *is given by*

$$c^{\pm}_{(n)} \equiv 6n^2 \pm 6n + 1 \tag{3.32}$$

and the Liouville action *is independent of* n *and given by*

$$S_L(\sigma, g) = \frac{1}{12\pi} \int_\Sigma d\mu_g \left[\frac{1}{2} g^{mn} \partial_m \sigma \partial_n \sigma + R_g \sigma + \beta^2 (e^{2\sigma} - 1) \right] . \tag{3.33}$$

The constant β *depends upon the renormalization scheme used, and will vanish when determinants are defined with* ζ*-function regularization, or with dimensional regularization.*

Since special care must be paid to the effects of the zero modes, we outline the proof. (Various intermediate steps will be proven in Problem 6.) First, it suffices to prove the formula for the Weyl transformation of $Z^{\pm}_{(n)}$ by infinitesimal σ; the result for general σ follows by straightforward integration ([II-**CFT**], Prob. 8). Thus, we must prove

$$\delta_\sigma \ln Z^{\pm}_{(n)}(g) = -\frac{c^{\pm}_{(n)}}{6\pi} \int_\Sigma d\mu_g [R_g \delta\sigma + 2\beta^2 \delta\sigma] \tag{3.34}$$

(Notice that from the form of this infinitesimal variation, it is clear that the determinant $Z^{\pm}_{(n)}$ defines a conformal field theory with central charge related to $\pm c^{\pm}_{(n)}$.)

We define determinants by introducing the ζ-function associated with the eigenvalues λ_α of $\Delta^{\pm}_{(n)}$:

$$\zeta(s) \equiv \sum_\alpha{}' \lambda_\alpha^{-s}$$

Here, $\sum'$ denotes the sum over all non-zero eigenvalues λ_α of $\Delta^{\pm}_{(n)}$. It is standard to represent this sum in terms of the heat-kernel of $\Delta^{\pm}_{(n)}$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} (\mathrm{Tr}\, e^{-t\Delta^{\pm}_{(n)}} - N^{\pm}_{(n)}) \tag{3.35}$$

where $N^{\pm}_{(n)} = \dim \mathrm{Ker}\, \Delta^{\pm}_{(n)}$.

The $\zeta(s)$ function series and the heat-kernel integral representation are absolutely convergent for $\mathrm{Re}(s) > 1$. $\zeta(s)$ is holomorphic for $\mathrm{Re}(s) > 1$, but may be analytically continued throughout $\mathbb{C}$, with a simple pole at $s = 1$. The determinants are then defined by

$$\ln \mathrm{Det}' \Delta^{\pm}_{(n)} = -\zeta'(0) \tag{3.36}$$

The non-zero spectra of $\Delta^{+}_{(n)}$ and $\Delta^{-}_{(n+1)}$ coincide and thus their determinants are equal, so we shall henceforth focus on $\Delta^{-}_{(n)}$.

To evaluate the Weyl dependence of determinants, we compute that of $\zeta(s)$ and analytically continue to $s = 0$. We use the following intermediate results (proven in Problem 6).

$$\begin{aligned}
&1) \quad N^{\pm}_{(n)} \text{ are Weyl independent} \\
&2) \quad \delta_\sigma \mathrm{Tr}\, e^{-t\Delta^{-}_{(n)}} = -t\tfrac{\partial}{\partial t} \mathrm{Tr}\, [2n\delta\sigma e^{-t\Delta^{+}_{(n-1)}} - (2n-2)\delta\sigma e^{-t\Delta^{-}_{(n)}}] \\
&3) \quad \mathrm{Tr}\, \delta\sigma e^{-t\Delta^{\pm}_{(n)}} = \tfrac{1}{4\pi t} \textstyle\int_\Sigma d\mu_g \delta\sigma + \tfrac{1\pm 3n}{12\pi} \int_\Sigma d\mu_g\, R_g\, \delta\sigma + O(t) \\
&4) \quad \mathrm{Tr}\, \delta\sigma e^{-t\Delta^{-}_{(n)}} = -\tfrac{1}{2n-2} \delta_\sigma \ln \det(\phi_j, \phi_k)_g + O(e^{-\lambda_1 t}) \\
&5) \quad \mathrm{Tr}\, \delta\sigma e^{-t\Delta^{+}_{(n-1)}} = \tfrac{1}{2n} \delta_\sigma \ln \det(\psi_a, \psi_b)_g + O(e^{-\lambda_1 t})
\end{aligned} \tag{3.37}$$

where λ_1 is the smallest non-zero eigenvalue of $\Delta^{-}_{(n)}$.

We make use of the above results and carry out the analytic continuation of $\delta_\sigma \zeta(s)$ to $s = 0$, and then consider $\delta_\sigma \zeta'(0)$. For $\mathrm{Re}(s) > 1$, using 1) and 2), we have

$$\delta_\sigma \zeta(s) = -\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^s \frac{\partial}{\partial t} \mathrm{Tr}[\,\cdot\,](t) \tag{3.38}$$

where we introduced the abbreviation

$$\mathrm{Tr}[\,\cdot\,](t) \equiv \mathrm{tr}[2n\delta\sigma e^{-t\Delta^{+}_{(n-1)}} - (2n-2)\delta\sigma e^{-t\Delta^{-}_{(n)}}] \tag{3.39}$$

We shall make use of the asymptotic behaviors of $\mathrm{Tr}[\,\cdot\,](t)$ as $t \to 0$ and $t \to \infty$, which are obtained from 3), 4), 5):

$$(3.40)\qquad \begin{cases} \mathrm{Tr}[\,\cdot\,](t) = \frac{1}{2\pi t}\int_\Sigma d\mu_g \delta\sigma + \frac{c^-_{(n)}}{6\pi}\int_\Sigma d\mu_g \delta\sigma R_g + O(t) \\ \\ \mathrm{Tr}[\,\cdot\,](t) = \delta_\sigma \ln \det(\phi_j,\phi_k)_g \det(\psi_a,\psi_b)_g + O(e^{-\lambda_1 t})\ . \end{cases}$$

Here, $c^-_{(n)}$ was defined in (3.32).

To separate the asymptotic behaviors as $t \to 0$, $t \to \infty$, we cut $[0,\infty]$ at $t = t_0 > 0$, and define $\delta_\sigma\zeta(s) = \delta_\sigma\zeta_0(s) + \delta_\sigma\zeta_\infty(s)$

$$(3.41)\qquad \begin{cases} \delta_\sigma\zeta_0(s) = -\frac{1}{\Gamma(s)}\int_0^{t_0} dt\, t^s \frac{\partial}{\partial t}\mathrm{Tr}[\,\cdot\,](t) \\ \\ \delta_\sigma\zeta_\infty(s) = -\frac{1}{\Gamma(s)}\int_{t_0}^{\infty} dt\, t^s \frac{\partial}{\partial t}\mathrm{Tr}[\,\cdot\,](t)\,. \end{cases}$$

The first term fails to converge at $s = 0$, so we subtract the asymptotic behavior of $\mathrm{Tr}[\,\cdot\,](t)$ in $1/t$ as $t \to 0$:

$$\begin{aligned} \delta_\sigma\zeta_0(s) = &-\frac{1}{\Gamma(s)}\int_0^{t_0} dt\, t^s \frac{\partial}{\partial t}\left(\mathrm{Tr}[\,\cdot\,](t) - \frac{1}{2\pi t}\int_\Sigma d\mu_g\delta\sigma\right) \\ &+ \frac{1}{\Gamma(s)}\frac{1}{2\pi}\int_\Sigma d\mu_g\delta\sigma\int_0^{t_0} dt\, t^{s-2}\ . \end{aligned}$$

The first integral on the right hand side is convergent at $s = 0$, while the second may be explicitly analytically continued to $s = 0$. Taking the derivative at $s = 0$:

$$\begin{aligned} \delta_\sigma\zeta_0'(0) &= -\int_0^{t_0} dt\frac{\partial}{\partial t}\left(\mathrm{Tr}[\,\cdot\,](t) - \frac{1}{2\pi t}\int_\Sigma d\mu_g\delta\sigma\right) - \frac{1}{2\pi t_0}\int_\Sigma d\mu_g\delta\sigma \\ &= -\mathrm{Tr}[\,\cdot\,](t_0) + \left(\mathrm{Tr}[\,\cdot\,](t) - \frac{1}{2\pi t}\int_\Sigma d\mu_g\delta\sigma\right)\Big|_{t=0} \\ &= -\mathrm{Tr}[\,\cdot\,](t_0) + \frac{c^-_{(n)}}{6\pi}\int_\Sigma d\mu_g R_g\delta\sigma\ . \end{aligned}$$

The term $\delta_\sigma\zeta_\infty(s)$ is given by

$$\delta_\sigma\zeta_\infty(s) = -\frac{1}{\Gamma(s)}\int_{t_0}^{\infty} dt\, t^s \frac{\partial}{\partial t}\mathrm{Tr}[\,\cdot\,](t)$$

As $t \to \infty$, $\mathrm{Tr}[\,\cdot\,] \to$ constant $+ O(e^{-\lambda_1 t})$, $\lambda_1 > 0$, so that the integral is exponentially convergent, and analytic around $s = 0$. Hence, using the asymptotics of $\mathrm{Tr}[\,\cdot\,](t)$ as $t \to \infty$:

$$\begin{aligned} \delta_\sigma\zeta_\infty'(0) &= -\int_{t_0}^{\infty} dt\frac{\partial}{\partial t}\mathrm{Tr}[\quad](t) \\ &= -\delta_\sigma \ln \det(\phi_j,\phi_k)_g \det(\psi_a,\psi_b)_g + \mathrm{Tr}[\,\cdot\,](t_0)\ . \end{aligned}$$

Putting all together, we have

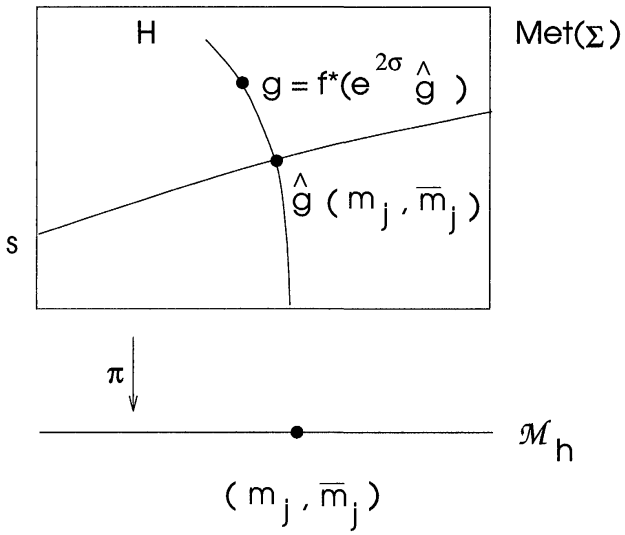
$$\begin{aligned} \delta_\sigma \ln \mathrm{Det}'\Delta^-_{(n)} &= -\delta_\sigma\zeta'(0) \\ &= \delta_\sigma \ln \det(\phi_j,\phi_k)_g \det(\psi_a,\psi_b)_g + \frac{c^-_{(n)}}{6\pi}\int_\Sigma d\mu_g R_g\delta\sigma\ , \end{aligned}$$

as announced previously. Notice that in ζ-function definitions of the determinants, no "area term" involving the additional constant β arises in (3.33).

§3.6. Critical central charge–critical dimension

We complete the reduction of the measure Dg to a measure on $\mathcal{M}_h$, making use of the Weyl transformation properties of functional determinants, derived in §3.5.

Just as we did in the case of the finite-dimensional example, we choose a section s of $\mathrm{Met}(\Sigma)$ and denote the corresponding metric $\hat{g}(m_j, \bar{m}_j)$. A general metric $g \in \mathrm{Met}(\Sigma)$ may then be parametrized in terms of the action of $Diff(\Sigma)$ and $Weyl(\Sigma)$ on $\hat{g}(m_j, \bar{m}_j)$: let $f \in Diff(\Sigma)$ and $e^{2\sigma} \in Weyl(\Sigma)$; then



The action of $Diff(\Sigma)$ is by isometries, and our definition of functional determinants was invariant with respect to $Diff(\Sigma)$. Thus, the effect of f^* is immaterial and may be dropped. The effect of Weyl transformations is obtained by combining the results of §§3.4, 3.5, and we have

$$Dg = \widetilde{D}v D\sigma Z^{-}_{(-1)}(\hat{g}) e^{-26 S_L(\hat{g},\sigma)} |\det(\hat{\mu}_j, \phi_k)_{\hat{g}}|^2 \, dm_j \, d\bar{m}_j \ . \tag{3.42}$$

We now consider a full transition amplitude A_h, evaluated on surfaces Σ of genus h. (To avoid certain complications due to conformal automorphisms, we postpone the cases $h = 0, 1$ and restrict to $h \geq 2$. The cases $h = 0, 1$ are completely analogous and will be dealt with separately in §3.9.) Recall that

$$A_h = \int_{\mathrm{Met}(\Sigma)} Dg \, \frac{1}{\mathcal{N}(g)} \langle V_1 \dots V_N \rangle_g \ , \tag{3.43}$$

where

$$\langle V_1 \dots V_N \rangle_g = \int_{\mathrm{Map}(\Sigma, M)} Dx \, V_1 \dots V_N e^{-S[x,g]} \ . \tag{3.44}$$

Given the Weyl transformation properties of Dg established above, it is clear that a *matter theory* (i.e. the quantum field theory of x fields for given metric g) is singled out that makes the Weyl dependence of the amplitude trivial. We define a *critical* (*bosonic*) *string theory* as one governed by a matter conformal field theory with central charge $c = 26$, and with physical vertex operators V_i, $i = 1, \dots, N$ such that

$$\langle V_1 \dots V_N \rangle_{e^{2\sigma}\hat{g}} = e^{26 S_L(\hat{g},\sigma)} \langle V_1 \dots V_N \rangle_{\hat{g}} \ . \tag{3.45}$$

(This last condition puts restrictions on the allowed vertex operators in the theory.) For critical string theories, the amplitudes at fixed topology ($h \geq 2$) reduce to

$$A_h = \int_{\mathcal{M}_h} \prod_j dm_j \, d\bar{m}_j |\det(\hat{\mu}_j, \phi_k)_{\hat{g}}|^2 Z^-_{(-1)}(\hat{g}) \, \langle V_1 \dots V_N \rangle_{\hat{g}} \cdot \int_{Weyl(\Sigma)} D\sigma \int_{Diff_0(\Sigma)} Dv / \mathcal{N}_0(g) \tag{3.46}$$

(Recall that for $h \geq 2$, $\widetilde{Dv} = Dv$.) Now, recall that the formal measure $Dg/\mathcal{N}(g)$ was the measure on $\mathrm{Met}(\Sigma)$, divided by the volume of the orbits of the group $Weyl(\Sigma) \times Diff(\Sigma)$. Since we restricted our integration to one over $\mathcal{M}_h$ (and not Teichmüller space T_h), we effectively quotiented by the mapping class group. The remaining normalization factor is $\mathcal{N}_0(g)$, formally the volume of $Weyl \times Diff_0(\Sigma)$. Thus, the formal integrals over $Weyl(\Sigma)$ and $Diff_0(\Sigma)$ cancel $\mathcal{N}_0(g)$:

$$\int_{Weyl(\Sigma)} D\sigma \int_{Diff_0(\Sigma)} Dv / \mathcal{N}_0(g) = 1 \, . \tag{3.47}$$

Thus, we finally obtain an ultimate expression for amplitudes in *critical string theory* at fixed topology ($h \geq 2$); the expression reduces to integrals over moduli of Riemann surfaces:

$$A_h = \int_{\mathcal{M}_h} \prod_j dm_j \, d\bar{m}_j |\det(\hat{\mu}_j, \phi_k)_{\hat{g}}|^2 Z^-_{(-1)}(\hat{g}) \, \langle V_1 \dots V_N \rangle_{\hat{g}} \tag{3.48}$$

The cases for $h = 0, 1$ will be discussed in §3.9, 3.10.

§3.7. Flat space-time manifold M

For flat Euclidean space-time $M = \mathbb{R}^D$, we have already established that the central charge of the conformal field theory defined by maps $x \in \mathrm{Map}(\Sigma, M)$ is $c = D$. Thus, the flat space-time string theory is a critical string theory precisely when $D = 26$. (We now notice that this condition is the same as the one found for the free string: this is rather fortunate!) Let us work out in detail under what conditions the vertex operators also satisfy the criteria of a critical string.

To do so, it is convenient to start with a study of the Weyl transformation properties of correlation functions of unintegrated exponents:

$$\Big\langle \prod_{i=1}^N e^{ik_i \cdot x(z_i)} \Big\rangle_g = \int_{\mathrm{Map}(\Sigma, M)} Dx \prod_{i=1}^N e^{ik_i \cdot x(z_i)} e^{-S_0[x,g]} \tag{3.49}$$

where S_0 is the Gaussian action, including the constant dilaton Φ_0. (We ignore the constant tachyon.)

$$S_0[x, g] = \frac{1}{8\pi} \int_\Sigma d\mu_g \, x \cdot \Delta_{(0)} x + \frac{\Phi_0}{4\pi} \int_\Sigma d\mu_g R_g \tag{3.50}$$

The Gaussian integral is carried out by separating the constant mode in x:

$$x = x_0 + x' \, , \qquad \int_\Sigma d\mu_g \, x' = 0$$

and making use of the Green functions for x':

$$\begin{cases} \Delta_{(0)} G_g(z,w) = 4\pi\delta_g(z,w) - \dfrac{4\pi}{\int_\Sigma d\mu_g} \\ \int_\Sigma d\mu_g G_g(z,w) = 0 \end{cases} . \tag{3.51}$$

The Green function at coincident points must be regularized and renormalized in a $Diff(\Sigma)$-invariant way. This may be carried out with the help of heat-kernel methods, or simply by renormalizing by the $Diff(\Sigma)$-invariant distance:

$$G_g^R(z,z) \equiv \lim_{w\to z} (G_g(z,w) + 2\ln \mathrm{dist}_g(z,w)) . \tag{3.52}$$

The Weyl scaling properties of the Green function are then

$$\begin{cases} G_{e^{2\sigma}\hat{g}}(z,w) = G_{\hat{g}}(z,w) + f(z) + f(w) \\ G^R_{e^{2\sigma}\hat{g}}(z,z) = G^R_{\hat{g}}(z,z) + 2f(z) + 2\sigma(z) , \end{cases} \tag{3.53}$$

where $f(z)$ is a function required to maintain $\int_\Sigma d\mu_g G_g = 0$ under Weyl rescalings. We do not need its explicit form here. One finds

$$\left\langle \prod_i e^{k_i\cdot x(z_i)} \right\rangle_g = \delta(k) e^{-\frac{1}{2}\chi\Phi_0} \left(\frac{\mathrm{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_g} \right)^{-D/2} \prod_{i,j} e^{\frac{1}{2}k_i\cdot k_j G_g(z_i,z_j)} , \tag{3.54}$$

where $k \equiv \sum_i k_i$, $\delta(k)$ is the D-dimensional δ-function, and the factor $\int_\Sigma d\mu_g$ under the determinant arises from the normalization of the zero mode x_0. As a result, under $g = e^{2\sigma}\hat{g}$, we have

$$\prod_{i,j} e^{-\frac{1}{2}k_i\cdot k_j G_g(z_i,z_j)} = \prod_{i,j} e^{-\frac{1}{2}k_i\cdot k_j G_{\hat{g}}(z_i,z_j)} e^{-k_i^2\sigma(z_i)} . \tag{3.55}$$

All dependence on $f(z)$ cancels in view of momentum conservation $k = \sum_i k_i = 0$, as guaranteed by the overall $\delta(k)$ factor in (3.54). The Weyl scaling property of the determinant factor is precisely that of $Z_{(0)}(g)^{-D/2}$. ($Z_{(0)}(g)$ contains an additional factor involving holomorphic Abelian differentials, but this factor is manifestly Weyl invariant.)

Combining all of the above, we find for $D = 26$

$$\left\langle \prod_i^N e^{ik_i\cdot x(z_i)} \right\rangle_{\hat{g}e^{2\sigma}} = \prod_{i=1}^N e^{-k_i^2\sigma(z_i)} e^{26 S_L(\hat{g},\sigma)} \left\langle \prod_{i=1}^N e^{ik_i\cdot x(z_i)} \right\rangle_{\hat{g}} . \tag{3.56}$$

It is now straightforward to examine the Weyl covariance conditions of vertex operators for low mass cases.

1) Tachyon vertex operator

$$V_T(k) = \varepsilon(k) \int_\Sigma d\mu_g e^{ik\cdot x} .$$

In order to cancel the explicit Weyl dependence of the measure $d\mu_g$ on Σ:

$$d\mu_{e^{2\sigma}\hat{g}} = d\mu_{\hat{g}} e^{2\sigma}$$

against the Weyl dependence of the correlation function, we recover the familiar condition $k^2 = 2$ for the tachyon. (This condition also corresponded to $a = 1$ in §2.6.)

2) The massless vertex operators

$$V_\varepsilon(k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ik\cdot x} \ ;$$

the Weyl dependence of the part $d\mu_g g^{mn}$ is trivial, so we need the familiar condition $k^2 = 0$. In addition, Weyl dependence appears through the contraction of $\partial_m x^\mu$ with the exponential, and is cancelled by the transversality conditions

$$k^\mu \varepsilon_{\mu\nu}(k) = k^\nu \varepsilon_{\mu\nu}(k) = 0 \ .$$

Finally, from the contraction of $\partial_m x^\mu$ and $\partial_n x^\nu$, (actually only the trace part contributes), a curvature R_g term appears, and Weyl invariance for the trace part is achieved through the addition of a (classically non-Weyl invariant) part involving the curvature (see Prob. 8):

$$V_\varepsilon(k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ikx} + \alpha \varepsilon^\mu_\mu \int_\Sigma d\mu_g R_g e^{ik\cdot x} \ .$$

The physical particle corresponding to this vertex operator is the dilaton. Recall that the dilaton field Φ entered the generalized sigma model action with a Gaussian curvature term R_g as well.

3) Higher mass states: the general structure of vertex operators is given by

$$V_\varepsilon(k) = \varepsilon_{\mu_1 \dots \mu_{2p}} \int_\Sigma d\mu_g g^{m_1 n_1} \dots g^{m_p n_p} \partial_{m_1} x^{\mu_1} \dots \partial_{n_p} x^{\mu_{2p}} e^{ik\cdot x}$$

plus terms involving higher derivatives on x. To cancel the overall exponential Weyl dependence, we must require $k^2 = -2p + 2$, yielding the same mass spectrum as in the free string case. In complete analogy with the case 2), there will be further conditions of transversality on ε and and further terms involving R_g will have to be added to render the trace parts Weyl invariant.

§3.8. Non-critical strings

Conformal matter field theories with $c \neq 26$ are called *non-critical string theories.* The amplitudes for these theories retain a non-trivial Weyl dependence through the Liouville action, and through the vertex operators. Its direct quantization requires understanding the dynamics of Liouville theory. Despite much effort, there is no complete picture to date of how this can be achieved consistently, and without the presence of a tachyon.

A tremendous number of results have been obtained indirectly (by matrix model techniques) when c is less than 1 and rational.

§3.9. Tree level amplitudes

Tree level amplitudes correspond to the worldsheet topology of a sphere S^2, which we stereographically project onto the complex plane $\mathbb{C}$. The moduli space $\mathcal{M}_0$ of the sphere consists of just a single point, which we may choose to represent by the *round metric*, of curvature $R_{\hat{g}} = 1$, and area 4π:

$$\hat{g} = \frac{4|dz|^2}{(1+|z|^2)^2} \qquad z \in \mathbb{C} \ . \tag{3.57}$$

The group of conformal automorphisms is $\mathrm{PSL}(2,\mathbb{C})$ and acts by Möbius transformations

$$\zeta\colon z \to \zeta(z) = \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \ .$$

A convenient basis for the associated vector fields (called conformal Killing vectors) is

$$\psi_1^z = 1 \qquad \psi_2^z = z \qquad \psi_3^z = z^2 \ ,$$

and similarly for $\psi_a^{\bar{z}}$, $a = 1,2,3$.

To obtain the general expression for tree level transition amplitudes, we must first complete the treatments of conformal automorphisms, which we had postponed in §3.6, and of Weyl transformation of the measure. The starting point is our expression (3.28) for the measure Dg, considered now for $h = 0$

$$Dg = D\sigma \widetilde{D}v Z^-_{(-1)}(\hat{g}) e^{-26 S_L(\hat{g},\sigma)} \tag{3.58}$$

where $\widetilde{D}v = D'v \det(\psi_a, \psi_b)_g$, $\psi_a \in \mathrm{Ker}\nabla_{\bar{z}}^{(-1)}$, and the measure $D'v$ is over $Diff_0(\Sigma)$ vector fields orthogonal to $\mathrm{Ker}\nabla_{\bar{z}}^{(-1)} \oplus \mathrm{Ker}\nabla_z^{(1)}$. Tree level amplitudes are then given by

$$A_0 = \int_{Weyl(\Sigma)} D\sigma \int_{Diff_0(\Sigma)/\mathrm{PSL}(2,\mathbb{C})} (\widetilde{D}v/\mathcal{N}(g)) Z^-_{(-1)}(\hat{g}) \langle V_1 \ldots V_N \rangle_{\hat{g}} \tag{3.59}$$

using the same notations as in §3.6.

While the group $\mathrm{PSL}(2,\mathbb{C})$ leaves the metric g invariant (up to Weyl rescaling), it does transform the positions of vertex operator insertions non-trivially. It is natural and – as we shall establish later on – required for unitarity of the transition amplitudes, to divide by the (formal) volume of the *full* $Diff_0(\Sigma)$ group, and not just of $Diff_0(\Sigma)/\mathrm{PSL}(2,\mathbb{C})$. To achieve this, we proceed as follows.

1) For $N \geq 3$ cases

Let the last three vertex operators be given by (of course, we may choose *any* three vertex operators)

$$V_i = \int_{\mathbb{C}} d^2 z_i \, W_i(z_i) \ , \qquad i = N-2, \, N-1, \, N \ .$$

We propose to make a change of variables from the vertex insertion points z_i, $\bar{z}_i$ $i = N-2, N-1, N$ to the group elements of $\mathrm{PSL}(2,\mathbb{C})$, thus completing the

measure $D'v$ into Dv. This is possible since, given any three points $z_i \in \mathbb{C}$, there is a unique element $\zeta \in \mathrm{PSL}(2,\mathbb{C})$, such that ζ maps z_i^0 onto any triplet $z_i \in \mathbb{C}$, with

$$z_i = \zeta(z_i^0) \ .$$

to carry this out in detail, let ψ_a^z be a basis of the $\mathrm{Ker}\nabla_{\bar{z}}^{(-1)}$, $a = 1,2,3$, chosen to be Weyl invariant. We parametrize general vector fields v in $\mathrm{PSL}(2,\mathbb{C})$ by complex parameters λ_a, $\bar{\lambda}_a$ with

$$v^z = \sum_{a=1}^{3} \lambda_a \psi_a^z \qquad\qquad v^{\bar{z}} = \sum_{a=1}^{3} \bar{\lambda}_a \psi_a^{\bar{z}} \ .$$

The measure on $\mathrm{PSL}(2,\mathbb{C})$ vector fields is then

$$d\mu_v^{(6)} = Dv/D'v = \prod_{a=1}^{3} d\lambda_a \, d\bar{\lambda}_a \ \det(\psi_a, \psi_b)_g$$

where the determinant $\det(\psi_a,\psi_b)_g$ arises as the Jacobian of the coordinate change from v to λ. On the other hand, from the action of $\mathrm{PSL}(2,\mathbb{C})$ on the points z_i^0: $z_i = \zeta(z_i^0) = e^v z_i^0$, we have $dz_i = \sum_a d\lambda_a \, \psi_a(z_i)$, so that the measure on $d^2 z_i$ becomes:

$$\prod_{i=N-2}^{N} d^2 z_i = \prod_{a=1}^{3} d\lambda_a \, d\bar{\lambda}_a \, |\det \, \psi_a(z_i)|^2 \ .$$

Clearly, all factors in this identity are Weyl-invariant. Combining all the results obtained, we have

$$\begin{aligned} \widetilde{Dv} \prod_{i=N-2}^{N} d^2 z_i &= D'v \ \det(\psi_a,\psi_b)_g \prod_{a=1}^{3} d\lambda_a \, d\bar{\lambda}_a \, |\det \, \psi_a(z_i)|^2 \\ &= Dv \, |\det \, \psi_a(z_i)|^2 \end{aligned} \tag{3.60}$$

We may now divide by the formal volume of $Weyl(\Sigma) \ltimes Diff(\Sigma)$ just as in the cases $h \geq 2$, and we obtain the final expression for tree level amplitudes with $N \geq 3$:

$$A_0 = Z_{(-1)}^{-}(\hat{g}) |\det \, \psi_a(z_i)|^2 \langle V_1 \ldots V_{N-3} \prod_{i=N-2}^{N} W(z_i) \rangle_{\hat{g}} \ , \tag{3.61}$$

valid for any 3 points z_i. We shall evaluate some of these amplitudes momentarily for flat space-time.

2) $N = 0$, 1, 2 cases

Clearly, here, no three vertex operators are available to complete the integration $D'v$ into Dv. In fact, these amplitudes vanish! To justify this, notice that $D'v$ can at most be completed by the vector fields in $\mathrm{PSL}(2,\mathbb{C})/S_N$ where S_N is the stabilizer of N points on the sphere. Thus, we are left to divide out by the volume of S_N

$$\int_{Weyl(\Sigma)} D\sigma \int_{Diff_0/(\mathrm{PSL}(2,\mathbb{C})/S_N)} D'v/\mathcal{N}(g) \sim \frac{1}{\mathrm{Vol}(S_N)} = 0 \ . \tag{3.62}$$

For $N = 2$, fix the two points at 0 and ∞, and S_2 is the group of dilations, whose volume is ∞.

An alternative argument that the amplitudes should vanish is that no conformally invariant amplitudes can be constructed, except with value 0. The physical interpretations of these vanishing amplitudes are as follows:

1)	$N = 0$	tree-level vacuum energy is zero "vanishing of the cosmological constant".
2)	$N = 1$	the "tadpole" amplitudes vanish, i.e. conformal field theory provides solutions (no linear terms) to string theory, at tree level.
3)	$N = 2$	no mass corrections to tree level.

3) Green function and determinants

The scalar Green function on the sphere, with metric $\hat{g}$, is given by

$$G(z, z') = -\ln \frac{|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)} \,. \tag{3.63}$$

The determinants involving $\Delta_{(0)}$ and $\Delta^-_{(-1)}$ are just constants. They may be computed explicitly, but we shall not do so here. Instead, we abbreviate

$$\mathfrak{z} \equiv Z^-_{(-1)}(\hat{g}) \left(\frac{\mathrm{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_{\hat{g}}} \right)^{-13} \,. \tag{3.64}$$

Furthermore, the only remaining finite-dimensional determinant is

$$\begin{aligned} |\det \psi_a(z_i)|^2 &= \left| \det \begin{pmatrix} 1 & 1 & 1 \\ z_{N-2} & z_{N-1} & z_N \\ z_{N-2}^2 & z_{N-1}^2 & z_N^2 \end{pmatrix} \right|^2 \\ &= |(z_{N-2} - z_{N-1})(z_{N-1} - z_N)(z_N - z_{N-2})|^2 \,. \end{aligned} \tag{3.65}$$

4) Tachyon Amplitudes

The tachyon vertex operator is particularly simple:

$$V_T(k) = \int_\Sigma d^2 z \, W_T(z, k) \qquad W_T(z, k) = \varepsilon \sqrt{\hat{g}} \, e^{ik \cdot x} \,. \tag{3.66}$$

Here, ε is a constant, independent of k, and plays the same rôle as the field normalization Z-factors in the Lehmann-Symanzik-Zimmermann (LSZ) formalism in quantum field theory: $\langle k|\varphi(x)|0\rangle = Z\, e^{ik\cdot x}$ for a scalar field $\varphi(x)$ and its associated 1-particle state $|k\rangle$ momentum k, and $|0\rangle$ the ground state.

We now make use of the results in §2.10 on the correlation function of exponential operators. We introduce the renormalized ε_R by absorbing a multiplicative renormalization factor of the exponential

$$\varepsilon_R = \varepsilon \, e^{-G_R(z_i, z_i)_{\hat{g}}} \tag{3.67}$$

The amplitude now takes the form

(3.68)

$$A_0 = \delta(k) e^{-2\Phi_0} \varepsilon_R^N \, \mathfrak{z} \prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i \prod_{i=1}^{N} \sqrt{\hat{g}(z_i)}\; e^{-\frac{1}{2}\sum_{i\neq j} k_i \cdot k_j G(z_i, z_j)} \cdot |\det(\psi_a(z_i))|^2 \, .$$

The contributions to G of the form $\ln(1+|z_i|^2)$ sum up and, using momentum conservation and the fact that $k_i^2 = 2$, precisely cancel the $\sqrt{\hat{g}}$ prefactors. We are left with

$$A_0(k_1 \ldots k_N) = \delta(k) e^{-2\Phi_0} \varepsilon_R^N \, \mathfrak{z} |\det \psi_a(z_i)|^2 \prod_{i=1}^{N-3} d^2 z_i \prod_{i<j}^{N} |z_i - z_j|^{2k_i \cdot k_j} \tag{3.69}$$

Now, z_{N-2}, z_{N-1}, z_N were *any* 3 points on the space, so we may let them equal 0, 1, ∞, respectively, and we find

$$A_0(k_1, \ldots, k_N) = \delta(k) e^{-2\Phi_0} \varepsilon_R^N \, \mathfrak{z} \prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i \prod_{i<j}^{N-1} |z_i - z_j|^{2k_i \cdot k_j} \, . \tag{3.70}$$

It is very instructive to examine amplitudes for small N:

(1) three-point function

$$A_0(k_1, k_2, k_3) = \mathfrak{z}\, e^{-2\Phi_0} \varepsilon_R^3 \delta(k) \, . \tag{3.71}$$

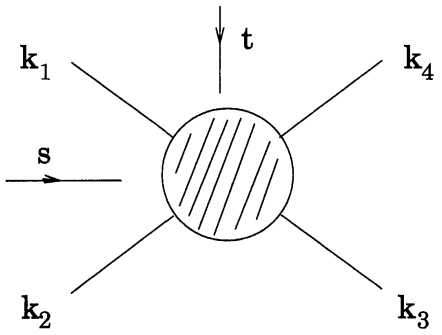
This 3-point function provides the 3-tachyon on shell coupling.

(2) four-point function

$$A_0(k_1, \ldots, k_4) = \mathfrak{z}\, e^{-2\Phi_0} \varepsilon_R^4 \delta(k) \int_{\mathbb{C}} d^2 z \; |z|^{2k_1 \cdot k_2} |1-z|^{2k_1 \cdot k_3} \, . \tag{3.72}$$

It is standard to introduce the 3 Lorentz invariants s, t, u that characterize the kinematics of the 4-point function:

(3.73)



$$\begin{cases} s \equiv -(k_1 + k_2)^2 \\ t \equiv -(k_2 + k_3)^2 \\ u \equiv -(k_1 + k_3)^2 \, . \end{cases}$$

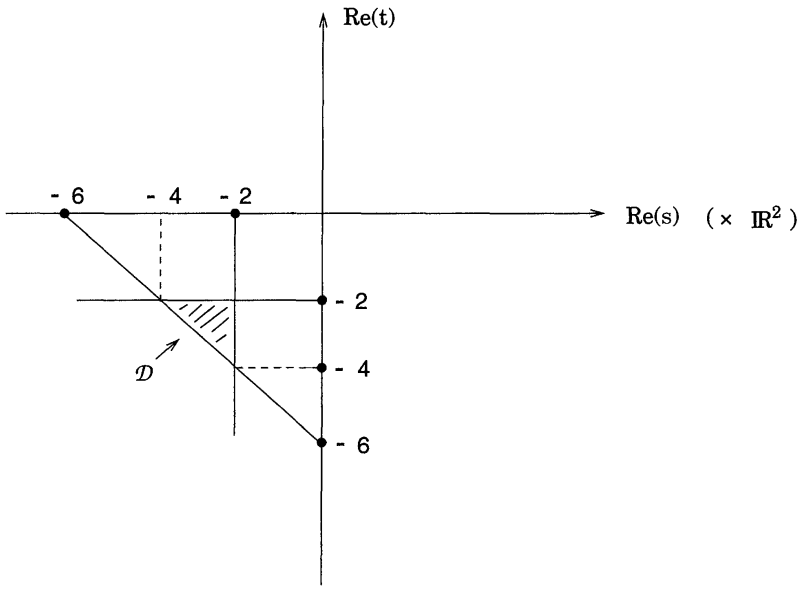
(Note that $s + t + u = -8$.) In terms of s, t, u the four-point function is

$$A_0(k_1, \ldots, k_u) = \mathfrak{z}\, e^{-2\Phi_0} \varepsilon_R^4 \delta(k) \int_{\mathbb{C}} d^2 z |z|^{-s-4} |z-1|^{-u-4} \, . \tag{3.74}$$

This integral expression is absolutely convergent in the following region $\mathcal{D}$, defined by

$$(3.75)\qquad \begin{cases} \mathrm{Re}(s) < -2 \\ \mathrm{Re}(u) < -2 \\ \mathrm{Re}(t) < -2 \end{cases} \quad \text{or} \quad \begin{cases} \mathrm{Re}(s) < -2 \\ \mathrm{Re}(t) < -2 \\ \mathrm{Re}(s+t) > -6\ . \end{cases}$$

Taking just the real parts:

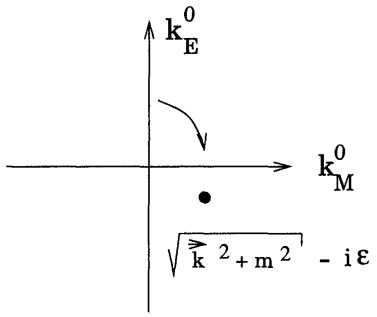


Let $s, t \in \mathcal{D}$; then we may evaluate the amplitude by elementary methods, and we find the *Virasoro-Shapiro* amplitude:

$$(3.76)\qquad A_0(s,t) = \mathfrak{z} e^{-2\Phi_0} \varepsilon_R^4 \delta(k) \pi \frac{\Gamma(-1-s/2)\Gamma(-1-t/2)\Gamma(-1-u/2)}{\Gamma(2+s/2)(\Gamma(2+t/2)\Gamma(2+u/2)}\ .$$

Since we know the analytic continuation of $\Gamma(z)$ throughout $\mathbb{C}$, with poles only at $z \in -\mathbb{N}$, we automatically have an analytic continuation of the full amplitude $A_0(s,t)$, throughout $(s,t) \in \mathbb{C} \times \mathbb{C}$. As a byproduct, we also immediately have the amplitude in Minkowski space-time momenta, by letting

$$k_i^0 \to -i k_i^0$$



$$\begin{cases} s \to s + i\epsilon \\ t \to t + i\epsilon \\ u \to u + i\epsilon \end{cases} \qquad \delta(k) \to i\delta(k)$$

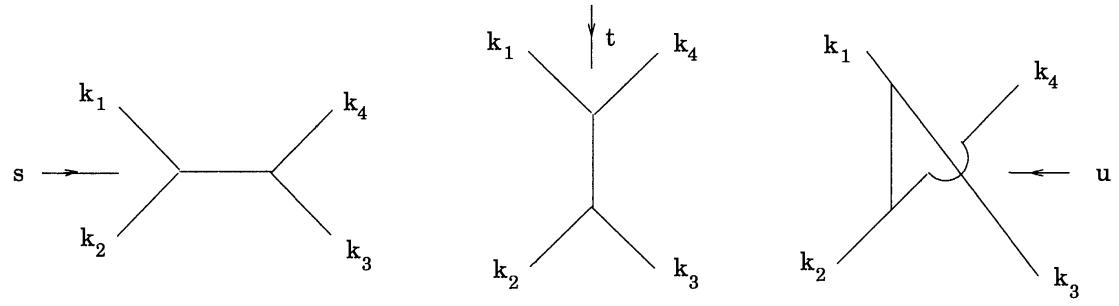
where it is understood that $\epsilon > 0$ and that we only consider the limit in which $\epsilon \to 0$. Thus, the 4-point amplitude in Minkowski space-time is given by

$$(3.77)\qquad A_0(s,t) = \mathfrak{z} e^{-2\Phi_0} \epsilon_R^4 \delta(k) \pi i \frac{\Gamma(-1-s/2-i\epsilon)\Gamma(-1-t/2-i\epsilon)\Gamma(-1-u/2-i\epsilon)}{\Gamma(2+s/2+i\epsilon)\Gamma(2+t/2+i\epsilon)\Gamma(2+u/2+i\epsilon)}\ .$$

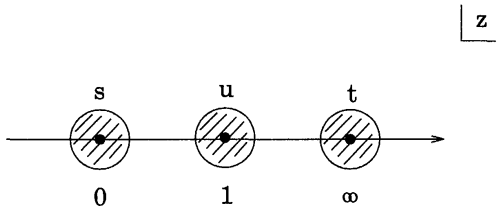
The amplitude has only pole singularities – as expected at tree level – and their location is easily identified. The Γ-function contributions from the denominator are entire. From the numerator, we have poles at

$$\text{(3.78)} \qquad \begin{cases} s = -2 + 2n_s & n_s \in \mathbb{N} \\ t = -2 + 2n_t & n_t \in \mathbb{N} \\ u = -2 + 2n_u & n_u \in \mathbb{N} \, . \end{cases}$$

These poles precisely correspond to the creation of intermediate string states whose mass is the above value for s, t or u. The states may occur either in the *s, t or u channel*:



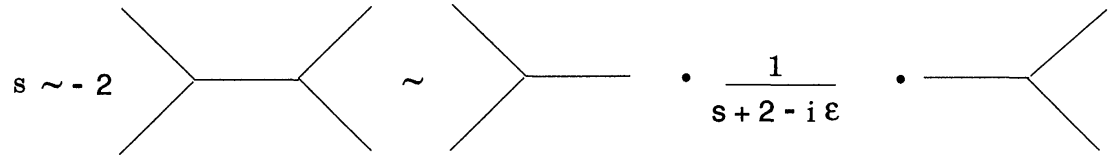
It is instructive to identify where these singularities arise from in the integral representation for the amplitude:



s poles: $z \to 0$
t poles: $z \to \infty$
u poles: $z \to 1$.

The fact that three QFT diagrams arise from a single string diagram is the simplest example of a general phenomenon mentioned in the introduction.

Let us concentrate on the $s = -2$ pole, which corresponds to a tachyon intermediate state. The diagram there *factorizes* into a propagator, and 2 three-point functions for external tachyon:



Insisting upon this factorization, we obtain a non-linear equation between the 3-point function and 4-point function, expressing factorization – or unitarity – of the transition amplitudes (or so-called S-matrix elements). This relation fixes ε_R:

$$\text{(3.79)} \qquad \varepsilon_R^2 = \frac{2\pi}{\mathfrak{z}} e^{2\Phi_0} \, .$$

§3.10. One loop amplitudes

At one loop, the worksheet has the topology of a torus, and there is a single complex modulus $\tau \in \mathbb{C}$. We represent Σ by

$$\Sigma = \mathbb{C}/(\mathbb{Z} + \tau Z) \qquad\qquad \mathrm{Im}\,\tau > 0$$

with the flat metric $\hat{g} = 2|dz|^2$. Teichmüller space is $\mathcal{T}_1 = \{\tau \in \mathbb{C},\ \mathrm{Im}\,\tau > 0\}$, the mapping class group is $G_\Sigma = \mathrm{SL}(2,\mathbb{Z})$, acting on τ as $\mathrm{PSL}(2,\mathbb{Z})$ by

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \qquad\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{3.80}$$

Moduli space is conveniently taken as $\mathcal{T}_1/\mathrm{PSL}(2,\mathbb{Z}) = \mathcal{M}_1$, but we shall have to recall that the full mapping class group also divides out by $\{\pm I\}$ in $\mathrm{SL}(2,\mathbb{Z})$. It is standard to take the fundamental domain

$$\mathcal{M}_1 = \left\{\tau = \tau_1 + i\tau_2, \tau_{1,2} \in \mathbb{R},\ \tau_2 > 0,\ |\tau_1| \le \tfrac{1}{2},\ |\tau| \ge 1\right\} . \tag{3.81}$$

Determinants of Laplace operators on tensors of any integer weight are given by the case of Laplacians on functions, which was computed in [II-**CFT**]:

$$\begin{aligned} \mathrm{Det}'\Delta_{(0)} &= \tau_2^2\,|\eta(\tau)|^4 \\ \eta(\tau) &\equiv e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau}) \ . \end{aligned} \tag{3.82}$$

The Green function on the scalar field x is given by

$$G(z,w|\tau) = -\ln\left|\frac{\vartheta_1(z-w|\tau)}{\vartheta_1'(0|\tau)}\right|^2 - \frac{\pi}{2\tau_2}(z - w - \bar{z} + \bar{w})^2 \ , \tag{3.83}$$

where $\vartheta_1(z|\tau)$ is the Jacobi theta function that has a zero at $z = 0$.

The N-string transition amplitude is given by the following expression

$$A_1(k_1,\ldots,k_N) = \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2^2}\,\frac{1}{(4\pi^2\tau_2)^{12}}\,\frac{1}{|\eta(\tau)|^{48}} \ll V_1,\ldots,V_N \gg \tag{3.84}$$

where $\ll V_1 \ldots V_N \gg$ stands for the *normalized* correlation function of the vertex operators $V_1 \ldots V_N$: $\ll V_1 \ldots V_N \gg = \langle V_1 \ldots V_N\rangle\,/\,\langle 1\rangle$.

For example, considering only tachyon vertex operators, we have

$$\ll V_1 \ldots V_N \gg = \delta(k)\varepsilon_R^N \prod_{i=1}^{N}\int d^2 z_i \exp\left(-\frac{1}{2}\sum_{i\neq j} k_i \cdot k_j G(z_i, z_j|\tau)\right) \tag{3.85}$$

There is no dependence on the dilaton constant Φ_0 here, because the Euler number of the torus vanishes. Other correlation functions may be similarly derived.

To one-loop order, we may now also have non-zero $N = 0$-, 1-, 2-point functions. Let us consider $N = 0$ first: $\ll 1 \gg = 1$.

$$A_1(\,\cdot\,) = \mathrm{Vol}(M)\int_{\mathcal{M}_1}\frac{d^2\tau}{2\tau_2^2}\,\frac{1}{(4\pi^2\tau_2)^{12}}\,\frac{1}{|\eta(\tau)|^{48}} \ . \tag{3.86}$$

Using the asymptotics for $\mathrm{Im}\,\tau \to \infty$,

$$\eta(\tau) \sim e^{i\pi\tau/12} = e^{i\pi\tau_1/12} e^{-\pi\tau_2/12} \tag{3.87}$$

we see immediately that this amplitude diverges for large τ_2. In fact, this $\eta(\tau)$ factor is common to all amplitudes, for any type of vertex operators, and produces a divergence in any amplitude, for any arrangement of external momenta. We shall not show this here in general, but leave this point to be checked by the reader. Thus, bosonic closed oriented string theory is divergent in flat space-time.

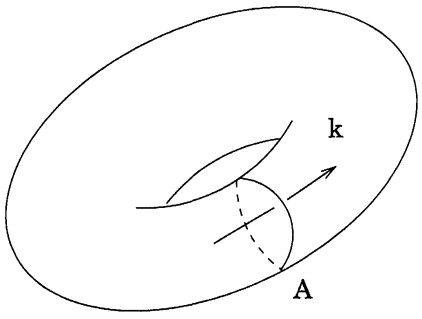
Nature of Divergence

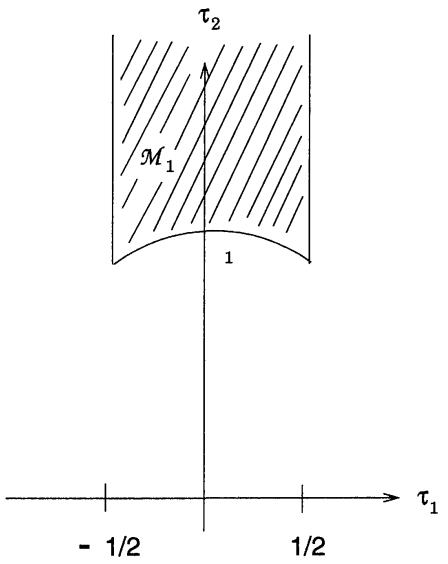
Let us try to gain additional insight into the physical origin of this divergence. To do so, it is helpful to exhibit the dependence on internal momenta by inserting a factor:

$$1 = \int d^{26}k\, \delta\left(k - \frac{1}{2\pi i}\oint_A dz\, \partial_z x\right) , \tag{3.88}$$

where A is the cycle in Σ whose pre-image in $\mathbb{C}$ runs from $z = 0$ to $z = 1$. We find

$$A_1(\,\cdot\,) = \mathrm{Vol}(M) \int_{\mathbb{R}^{26}} d^{26}k \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2}\, \frac{1}{(4\pi^2)^{12}}\, \frac{1}{|\eta(\tau)|^{48}} e^{-2\pi\tau_2 k^2} . \tag{3.89}$$





Substituting the asymptotics of the $\eta(\tau)$-function

$$A_1(\,\cdot\,) = \mathrm{Vol}(M) \int_{\mathbb{R}^{26}} d^{26}k \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2}\, \frac{1}{(4\pi^2)^{12}} e^{-2\pi\tau_2(k^2-2)} (1 + \#e^{-2\pi\tau_2} + \cdots) , \tag{3.90}$$

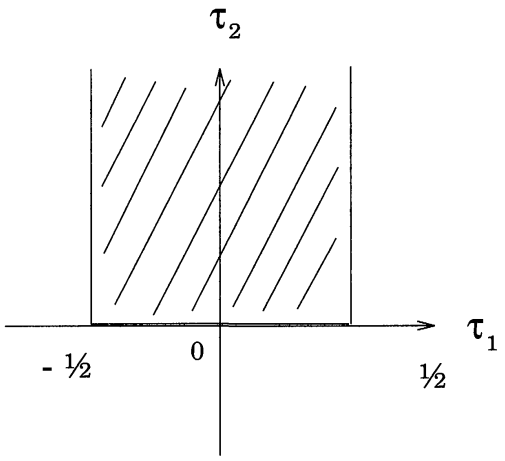
it is now immediately apparent why a divergence occurs, and where it occurs. For $\tau_2 \to \infty$, the momentum range $0 \leq k^2 < 2$ contributes an exponentially growing factor. In fact, integrating out τ_2 for large τ_2:

$$\int \frac{d\tau_2}{\tau_2} e^{-2\pi\tau_2(k^2-2)} \sim \frac{1}{k^2 - 2}$$

This contribution arises from the tachyon! It is an I.R. problem related to the stability of flat space-time.

It is equally instructive to see where the integral is convergent: at large k^2, since the range of τ_2 is cut off from below by the choice of $\mathcal{M}_1$, the integral is exponentially suppressed for large momenta k^2. Thus, this loop amplitude is superconvergent at high energy, in contrast with field theory.

In fact, we can use this simple example to exhibit the distinction between what this partition function would look like in quantum field theory and what it looks like in string theory. In field theory, we may imagine the partition function for the same states, given by the same generating function. The only difference is the region $\mathcal{M}_1$:



Integration region for QFT with the same states as string theory.

Now, since τ_2 reaches down to $\tau_2 = 0$, the superconvergence disappears and the integrand is power behaved in momenta. Thus, we see here that a crucial difference between string theory and quantum field theory arises from modular invariance, producing ultraviolet finite answers in the case of strings

References and cross references

The subject is extensively reviewed in

E. D'Hoker and D. H. Phong, *The geometry of string perturbation theory*, Rev. Modern Phys. **60** (1988), 917–1065.

Let us mention here in addition three important foundational papers,

A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B **103** (1981), 207.

D. H. Friedan, *Introduction to Polyakov's string theory*, Recent Advances in Field Theory and Statistical Mechanics, Les Houches, Session 39 (R. Stora and J. B. Zuber, eds.), North Holland Publishers, 1982.

O. Alvarez, *Theory of strings with boundaries : fluctuations, topology and quantum geometry*, Nucl. Phys. B **216** (1983), 125.

LECTURE 4
Faddeev–Popov Ghosts–BRST Quantization

It is standard practice to recast the Faddeev-Popov determinant, which arose from fixing $Diff(\Sigma)$ gauge invariance, in terms of ghost fields. (See also [I-**Faddeev**].) One introduces Grassmann-valued *ghost fields* $(c,\bar{c}) = (c^z(dz)^{-1}, c^{\bar{z}}(d\bar{z})^{-1})$ and *anti-ghost* fields $(b,\bar{b}) = (b_{zz}(dz)^2, b_{\bar{z}\bar{z}}(d\bar{z})^2)$ transforming as the gauge transformation vector fields $(v,\bar{v})$ and as the metric deformations $(\delta g_{zz}, \delta g_{\bar{z}\bar{z}})$, respectively. It is convenient to study such b-c systems for general weight first, since we shall make use of such systems as well: $c \in K^{1-n}; b \in K^n$. For now n is an integer, but later n will also be allowed to be a half-integer.

§4.1. Determinants and b-c systems

We therefore generalize by considering a system of a pair of Grassmann-valued fields $(c,\bar{c}) = (c(dz)^{1-n}, \bar{c}(d\bar{z})^{1-n})$ and $(b,\bar{b}) = (b(dz)^n, \bar{b}(d\bar{z})^n)$, so that the above case is $n = 2$. In this notation, b and c are independent whereas $\bar{b}$, $\bar{c}$ are their complex conjugates. The action is first order in derivatives:

$$S_n(b,c) = \frac{1}{2\pi}\int_\Sigma d\mu_g \left(b\nabla^z_{(1-n)}c + \bar{b}\nabla^{\bar{z}}_{(n-1)}\bar{c}\right) . \tag{4.1}$$

We have the following Theorem (4.1):

$$\begin{aligned} &Z^-_{(n)}(g)|\det\,\phi_j(z_k)|^2\;|\det\,\psi_a(w_b)|^2 \\ &\quad = \int D(b\bar{b})D(c\bar{c})e^{-S_n(b,c)}\prod_{j=1}^{N^-_{(n)}} b(z_j)\bar{b}(z_j)\prod_{a=1}^{N^-_{(n-1)}} c(w_a)\bar{c}(w_a)\ , \end{aligned} \tag{4.2}$$

where the measures $D(b\bar{b})$ and $D(c\bar{c})$ are obtained from the L^2 norms on K^n, $\bar{K}^n$, $K^{(1-n)}$ and $\bar{K}^{(1-n)}$; $\phi_j \in \mathrm{Ker}\,\nabla^z_{(n)}$, $\psi_a \in \mathrm{Ker}\,\nabla^z_{(1-n)}$ and $N^\pm_{(n)} = \dim\,\mathrm{Ker}\,\Delta^\pm_{(n)}$. Note that for $n \geq 2$ there are no ψ_α's whereas for $h = 0$ there are no ϕ_j's.

We begin the proof by recalling a finite-dimensional Grassmann integral, over variables b_i, c_i, $i = 1,\ldots,N$, such that $\{b_i,b_j\} = \{c_i,c_j\} = \{b_i,c_j\} = 0$. We use the standard Grassmann integration rules:

$$\int dc_i\, c_j = \delta_{ij} \qquad\qquad \int db_i\, b_j = \delta_{ij} \tag{4.3}$$

and $\int dc_i\, 1 = \int db_i\, 1 = 0$. Then, for any matrix M_{ij}, we have

$$\prod_{i=1}^{N} \int db_i dc_i\, e^{b_i M_{ij} c_j} = \det M \ . \tag{4.4}$$

To prove the Theorem, we begin by expanding b and c in an orthonormal basis of eigenfunctions of $\Delta^-_{(n)}$ and $\Delta^-_{(1-n)}$ as follows. Let Ψ_α be an orthonormal basis of eigenfunctions of $\Delta^-_{(1-n)}$ corresponding to the non-zero eigenvalues λ^2_α, and let ψ_a be an orthonormal basis of zero modes of $\Delta^-_{(1-n)}$. We have

$$\begin{cases} \Delta^-_{(1-n)} \Psi_\alpha = \lambda^2_\alpha \Psi_\alpha \\ \Delta^-_{(1-n)} \psi_a = 0 \ , \end{cases} \tag{4.5}$$

and we define Φ_α by

$$\begin{cases} \sqrt{2}\, \nabla^z_{(1-n)} \Psi_\alpha = \lambda_\alpha \Phi_\alpha \\ -\sqrt{2}\, \nabla^{(1-n)}_z \Phi_\alpha = \lambda_\alpha \Psi_\alpha \ , \end{cases} \tag{4.6}$$

so that Φ_α provides an orthonormal basis of eigenfunctions of $\Delta^+_{(-n)}$ for non-zero eigenvalues. Let ϕ_j be an orthonormal basis for $\mathrm{Ker}\, \nabla^{(-n)}_z$. As a result, $(g_{z\bar z})^n \Phi^*_\alpha$ together with the zero modes $(g_{z\bar z})^n \phi^*_j$ provides an orthonormal basis of eigenfunctions for $\Delta^-_{(n)}$. Thus, we have the decompositions

$$\begin{aligned} c &= \sum_\alpha c_\alpha \Psi_\alpha + \sum_a c_a \psi_a \\ b &= \sum_\alpha b_\alpha (g_{z\bar z})^n \Phi^*_\alpha + \sum_j b_j (g_{z\bar z})^n \phi^*_j \ . \end{aligned} \tag{4.7}$$

The L^2 norm then induces the following measures:

$$\begin{cases} D(b\bar b) = \prod_\alpha db_\alpha\, d\bar b_\alpha \prod_j d\, b_j\ d\, \bar b_j \\ D(c\bar c) = \prod_\alpha dc_\alpha\, d\bar c_\alpha \prod_a dc_a\, d\bar c_a \end{cases} \tag{4.8}$$

and it is straightforward to evaluate the action:

$$\begin{aligned} S_n(b,c) &= \frac{1}{2\pi} \sum_{\alpha,\beta} b_\alpha c_\beta \int_\Sigma d\mu_g (g_{z\bar z})^n \Phi^*_\alpha \nabla^z_{(1-n)} \Psi_\beta + c.c. \\ &= \frac{1}{2\pi\sqrt{2}} \sum_{\alpha,\beta} b_\alpha c_\beta \lambda_\beta \int_\Sigma d\mu_g (g_{z\bar z})^n \Phi^*_\alpha \Phi_\beta + c.c. \\ &= \frac{1}{2\pi\sqrt{2}} \sum_\alpha (b_\alpha c_\alpha \lambda_\alpha + \bar b_\alpha \bar c_\alpha \lambda_\alpha) \ . \end{aligned} \tag{4.9}$$

Notice that the zero mode coefficients b_j, c_a do not enter the action. Modulo a constant $\left(\frac{1}{2\pi\sqrt{2}}\right)^{\zeta(0)}$ coming from the ζ-function regularization,

$$\int \prod_{\alpha} db_\alpha d\bar{b}_\alpha dc_\alpha d\bar{c}_\alpha \, e^{-S_n(b,c)} = \prod_{\alpha} \lambda_\alpha^2 = \mathrm{Det}'\Delta^-_{(1-n)} \, . \tag{4.10}$$

The remaining finite product of factors can only involve the zero modes, since the number of factors of b and c precisely equals the number of zero-mode integrations.

$$\begin{aligned} \int \prod_{a} dc_a d\bar{c}_a \prod_{j} db_j d\bar{b}_j \prod_{j=1}^{N^-_{(n)}} b(z_j)\bar{b}(z_j) \prod_{a=1}^{N^-_{(1-n)}} c(w_a)\bar{c}(w_a) \\ = |\det \phi_j(z_k)|^2 \, |\det \psi_a(w_b)|^2 \, . \end{aligned} \tag{4.11}$$

Recall that our bases ϕ_j, ψ_a had been chosen orthonormal. In an arbitrary basis, the result would be

$$\frac{|\det \phi_j(z_k)|^2}{\det(\phi_j, \phi_k)} \, \frac{|\det \psi_a(w_b)|^2}{\det(\psi_a, \psi_b)} \, . \tag{4.12}$$

Putting all together, we obtain the result in the Theorem by using the definition of the determinant combination $Z^-_{(n)}(g)$.

Notice that the functional integral without b or c insertions,

$$\int D(b\bar{b})D(c\bar{c})e^{-S_n(b,c)} \, ,$$

vanishes because of the presence of zero modes which do not enter the action.

§4.2. Ghost representation of the Faddeev-Popov determinant

For bosonic string theory, $n = 2$ is the suitable weight, and we have the following general form for the amplitudes

$$A = \sum_{h=0}^{\infty} e^{-\chi\Phi_0} \int_{\mathcal{M}_h} dm_j d\bar{m}_j \int D(b\bar{b})D(c\bar{c})e^{-S_2(b,c)} \prod_{j} |(\mu_j, b)|^2 \int_{\mathrm{Map}(\Sigma;M)} Dx \, \mathcal{O} \, e^{-S_{\mathrm{m}}(x)} \tag{4.13}$$

where

$$\mathcal{O} = \begin{cases} \bullet \quad \prod_{a=N-2}^{N} c(z_a)\bar{c}(z_a) V_1 \dots V_{N-3} W(z_{N-2}) W(z_{N-1}) W(z_N) & (N \geq 3;\ h = 0) \\ \bullet \quad 0 & (N = 0, 1, 2;\ h = 0) \\ \bullet \quad c(z_N)\bar{c}(z_N) V_1 \dots V_{N-1} W(z_N) & (N \geq 1;\ h = 1) \\ \bullet \quad V_1 \dots V_N \, , & (h \geq 2) \end{cases} \tag{4.14}$$

and $S_2(b, c)$ is the action for the ghost fields in the case $n = 2$. This representation is valid for any matter action S_{m}, and any set of vertex operators V_i. The fields c and b are called the Faddeev-Popov ghosts and anti-ghosts.

§4.3. Conformal field theory of the b-c system

From the Weyl transformation law of the determinant combinations $Z^-_{(n)}(g)$, and the Weyl invariance of $\det \phi_j(z_k)$ and $\det \psi_a(w_b)$, we know that the b-c system is a conformal field theory with central charge

$$-2c^-_{(n)} = -2(6n^2 - 6n + 1) \ .$$

In particular, we confirm that for the ghost system of the bosonic string (i.e. $n = 2$) the central charge is $c_{\text{ghosts}} = -26$, cancelling the central charge of the matter part. We now study this conformal field theory for any $n \in \frac{1}{2}\mathbb{Z}$, assuming that a spin structure has been chosen on Σ. In particular, for $n = \frac{1}{2}$, we have a Dirac fermion $\tilde{\psi}_+ = b$, $\psi_+ = c$; $\tilde{\psi}_- = \bar{b}$, $\psi_- = \bar{c}$ (in which case b, c are in the same space of Dirac spinors), or two Majorana spinors: $(\psi_+, \tilde{\psi}_+)$ and $(\psi_-, \tilde{\psi}_-) = (\psi_+, \tilde{\psi}_+)^*$.

From the classical action, we obtain the stress tensor by varying with respect to δg^{zz} and using

$$\delta \nabla^z_{(1-n)} = \frac{1}{2}\delta g^{zz} \nabla_z^{(1-n)} + \frac{1-n}{2} \nabla_z(\delta g^{zz}) \ . \tag{4.15}$$

We find for the classical stress tensor:

$$\begin{aligned} T^{(bc)}(z) = T^{(bc)}_{zz} &= (1-n)(\nabla_z b)c - nb\nabla_z c \\ &= (1-n)(\partial_z b - n\Gamma b)c - nb(\partial_z c - (1-n)\Gamma c) \\ &= (1-n)(\partial_z b)c - nb\partial_z c \ . \end{aligned} \tag{4.16}$$

The affine connection Γ associated with the metric g cancels out in the final expression for $T^{(bc)}$.

Using Weyl rescaling, we may study the system on a surface with locally flat metric $\hat{g} = 2|dz|^2$. At the same time, covariant derivatives become just Cauchy-Riemann operators: $\nabla^z_{(n)}$ becomes $\bar{\partial}_{(n)}$ and $\nabla_z^{(n)}$ becomes $\partial_{(n)}$ ($\bar{\partial}$ and ∂ for short). The action is then

$$S_n(b,c) = \frac{1}{2\pi} \int_\Sigma d^2 z (b\bar{\partial}c + \bar{b}\partial\bar{c}) \ . \tag{4.17}$$

The field equations are $\bar{\partial}c = \bar{\partial}b = 0$, $\partial\bar{c} = \partial\bar{b} = 0$. The OPE of b and c is given by the Green function of $\bar{\partial}$:

$$b(z)c(w) \sim \frac{1}{z-w} \sim c(z)b(w) \ . \tag{4.18}$$

The quantum stress tensor may now be defined by subtracting the OPE singularities from the classical stress tensor. The holomorphic stress tensor is defined by

$$T^{(bc)}(z) = \lim_{w\to z} \left\{ (1-n)\partial b(z)c(w) - nb(z)\partial c(w) + \frac{1}{(z-w)^2} \right\} \ . \tag{4.19}$$

One immediately verifies

(4.20)

$$\begin{aligned} T^{(bc)}(z)T^{(bc)}(w) &\sim -\frac{(6n^2-6n+1)}{(z-w)^4} + \frac{2}{(z-w)^2}T^{(bc)}(w) + \frac{1}{z-w}\partial_w T^{(bc)}(w) \\ T^{(bc)}(z)c(w) &\sim \frac{1-n}{(z-w)^2}c(w) + \frac{1}{z-w}\partial_w c(w) \\ T^{(bc)}(z)b(w) &\sim \frac{n}{(z-w)^2}b(w) + \frac{1}{(z-w)}\partial_w b(w) \ . \end{aligned}$$

(There is no singularity in the expansion of $b(z)b(w)$ or $c(z)c(w)$.) The classical action S_n is invariant under $U(1)_{bc} \times U(1)_{\bar{b}\bar{c}}$ phase rotations:

$$\text{(4.21)} \qquad \begin{cases} c \to e^{i\theta} c \\ b \to e^{-i\theta} b \end{cases} \qquad \begin{cases} \bar{c} \to e^{-i\theta'} \bar{c} \\ \bar{b} \to e^{i\theta'} \bar{b} \end{cases}$$

Only the diagonal subgroup with $\theta = \theta'$ leaves the measure $D(b\bar{b})D(c\bar{c})$ invariant, and results in a quantum symmetry.

The current associated with $U(1)_{bc}$ is given by

$$\text{(4.22)} \qquad j_z^{(bc)} = :cb: = \lim_{w \to z} \left\{ c(w)b(z) - \frac{1}{w-z} \right\} ,$$

and we find the OPE's:

$$\text{(4.23)} \qquad j_z^{(bc)} c(w) \sim -\frac{1}{z-w}\, c(w) \; ; \qquad j_z^{(bc)} b(w) \sim \frac{1}{z-w}\, b(w) \; .$$

The current $j_z^{(bc)}$ has an anomaly, as can be established from the fact that $j_z^{(bc)}$ does not transform as a tensor (for $n \neq \frac{1}{2}$):

$$\text{(4.24)} \qquad T^{(c)}(z) j_w^{(bc)} \sim \frac{1-2n}{(z-w)^3} + \frac{1}{(z-w)^2} j_w^{(bc)} + \frac{1}{z-w} \partial_w j_w^{(bc)} \; .$$

Thus, under analytic diffeomorphisms $z \to z' = f(z)$, we have

$$\text{(4.25)} \qquad j'^{(bc)}(z')dz' = j^{(bc)}(z)dz + \left(n - \frac{1}{2} \right) d(\ln f'(z))$$

Note that there is no anomaly for the case of spinors when $n = 1/2$. Also, on a worldsheet with general metric g, $j^{(bc)}$ is not conserved; instead $\nabla^z j_z^{(bc)} = \left(\frac{1}{2} - n\right) R_g$.

§4.4. Bosonization of the b-c system

In Lecture 6 of [II-**Dynamics of QFT**] a Bose-Fermi correspondence was established between a 2-dimensional Dirac fermion and a real scalar boson field. The Dirac fermion corresponds here to the case $n = \frac{1}{2}$. We will generalize the Bose-Fermi correspondence to the case of any $n \in \frac{1}{2}\mathbb{Z}$. We begin by noticing that the b-c stress tensor can be expressed in terms of the $U(1)_{bc}$ current only:

$$\text{(4.26)} \qquad T^{(bc)}(z) = \frac{1}{2} j^{(bc)}(z)^2 + \left(n - \frac{1}{2} \right) \partial_z j^{(bc)}(z) \; ,$$

where the product of operators is always renormalized à la OPE. Using

$$\text{(4.27)} \qquad j^{(bc)}(z) j^{(bc)}(w) \sim \frac{1}{(z-w)^2} \; ,$$

it is straightforward to verify that $T^{(bc)}$ satisfies all identities listed above.

We now introduce a scalar field ϕ, with action (for general metric on Σ)

$$S_Q(x) = \frac{1}{4\pi}\int_\Sigma d\mu_g \left\{\frac{1}{2}g^{mn}\partial_m\phi\partial_n\phi + QR_g\phi\right\} . \tag{4.28}$$

This system was studied extensively in Problem #1. Let us consider it only on locally flat metrics. Its central charge is $c = 1 + 3Q^2$, and there is a current $\partial_z\phi$. Identifying the two systems will require that they have equal central charge:

$$1 + 3Q^2 = -12n^2 + 12n - 2 \qquad \text{or} \qquad Q = i(2n-1) . \tag{4.29}$$

(The second solution $Q = -i(2n-1)$ is equivalent upon changing the sign of ϕ.) The identification of the currents gives

$$\begin{aligned} j^{(bc)}(z) &= i\partial_z\phi \\ T^{(bc)}(z) &= -\frac{1}{2}(\partial_z\phi)^2 + \frac{1}{2}Q\partial_z^2\phi . \end{aligned} \tag{4.30}$$

The fields b, c and $\bar{b}$, $\bar{c}$ may be identified in terms of the *chiral boson fields* $\phi_+(z)$,$\phi_-(\bar{z})$, with $\phi(z,\bar{z}) = \phi_+(z) + \phi_-(\bar{z})$ by

$$\begin{cases} c(z) = e^{i\phi_+(z)} \\ b(z) = e^{-i\phi_+(z)} \end{cases} \qquad \begin{cases} \bar{c}(\bar{z}) = e^{+i\phi_-(\bar{z})} \\ \bar{b}(\bar{z}) = e^{-i\phi_-(\bar{z})} . \end{cases} \tag{4.31}$$

The $U(1)_{bc} \times U(1)_{\bar{b}\bar{c}}$ transformations now simply shift ϕ_+ and ϕ_- by θ and $-\theta'$, respectively.

It is now easy to see that the number (N_b, N_c) of b and c operator insertions yielding non-zero correlation functions is constrained by the Riemann-Roch-Atiyah-Singer theorem:

$$N_b - N_c - \frac{i}{2}Q\chi = 0 \qquad \text{or} \qquad N_b - N_c = \left(\frac{1}{2} - n\right)\chi ,$$

where χ is the Euler number of Σ.

§4.5. The b-c Fock space

The full Fock space is obtained as a tensor product of the bc Fock space $\mathcal{F}^{(bc)}$ and the $\bar{b}\bar{c}$ Fock space $\mathcal{F}^{(\bar{b}\bar{c})}$:

$$\text{Fock}(b,c,\bar{b},\bar{c}) = \mathcal{F}^{(bc)} \otimes \mathcal{F}^{(\bar{b}\bar{c})} . \tag{4.32}$$

We proceed to construct $\mathcal{F}^{(bc)}$, and $\mathcal{F}^{(\bar{b}\bar{c})}$ is constructed analogously. To do so, we consider the system on an annulus Σ, centered at $z = 0$, with flat metric $g = 2|dz|^2$, and we expand the holomorphic fields $b(z)$ and $c(z)$ in a Laurent series:

$$\begin{aligned} c(z) &= \sum_r c_r\, z^{-r-(1-n)} \\ b(z) &= \sum_r b_r\, z^{-r-n} \end{aligned} \qquad r \in \mathbb{Z} . \tag{4.33}$$

The shifts in the powers of z reflect the conformal weights of the fields. The OPE of b and c yields

$$\{b_r, b_s\} = \{c_r, c_s\} = 0 \qquad \{b_r, c_s\} = \delta_{r+s,0} \ . \tag{4.34}$$

The stress tensor (renormalized as in §4.4) yields the Virasoro generators

$$\begin{aligned} L_m^{(bc)} &= \sum_{r\in\mathbb{Z}} (mn - r) b_r c_{m-r} \qquad m \neq 0 \\ L_0^{(bc)} &= \sum_{r=1}^{\infty} r(b_{-r}c_r + c_{-r}b_r) - \frac{1}{2} n(n-1) \ . \end{aligned} \tag{4.35}$$

The $U(1)_{bc}$ current gives the bc number operator:

$$U = \oint \frac{dz}{2\pi i} j_z^{(bc)} = \sum_{r=1}^{\infty} (c_{-r}b_r - b_{-r}c_r) + \frac{1}{2}(c_0 b_0 - b_0 c_0) + n - \frac{1}{2} \ . \tag{4.36}$$

(The constant shifts in $L_0^{(bc)}$ and U_0 require great care, and are fundamental in the sequel.)

The polarization here is again inherited from Minkowski worldsheet signature:

$$b_r^\dagger = b_{-r} \qquad c_r^\dagger = c_{-r} \ ; \qquad\qquad r \in \mathbb{Z} \ . \tag{4.37}$$

The ground state of the b-c system is doubly degenerate, since b_0 and c_0 commute with L_0 (in fact they do not enter into the expression for L_0), and since b_0, c_0 generate an algebra $\{b_0, c_0\} = 1$, $b_0^2 = c_0^2 = 0$ of which the ground state is a two-dimensional representation. We define the basis vectors by $|\uparrow\rangle$ and $|\downarrow\rangle$, with

$$\begin{cases} b_r \, |\uparrow\rangle = c_r \, |\uparrow\rangle = 0 \\ b_r \, |\downarrow\rangle = c_r \, |\downarrow\rangle = 0 \end{cases} \qquad\qquad r > 0 \ . \tag{4.38}$$

We choose the states so that

$$\begin{aligned} c_0 \, |\uparrow\rangle &= 0 \\ b_0 \, |\downarrow\rangle &= 0 \ . \end{aligned} \tag{4.39}$$

It follows that

$$\begin{aligned} b_0 \, |\uparrow\rangle &= |\downarrow\rangle & \qquad U \, |\uparrow\rangle &= n \, |\uparrow\rangle \\ c_0 \, |\downarrow\rangle &= |\uparrow\rangle & \qquad U \, |\downarrow\rangle &= (n-1) \, |\downarrow\rangle \ . \end{aligned} \tag{4.40}$$

It is natural to choose the following inner products

$$\langle\downarrow|\uparrow\rangle = \langle\uparrow|\downarrow\rangle = 1 \tag{4.41}$$

(Notice that $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0$ by their properties under b_0 and c_0.)

The b-c Fock space is naturally obtained by applying the creation operators $\{b_{-r}, c_{-r}\}_{r\in\mathbb{N}}$ to either one of the ground states $|\uparrow\rangle$ or $|\downarrow\rangle$. The inner product thus inherited on $\mathcal{F}^{(bc)}$ is not positive definite.

§4.6. BRST quantization

We now return to string theory, where the relevant b-c system is the one with $n = 2$, in the notation used above. The combined matter-ghost action is (in a locally flat g)

$$S = \frac{1}{2\pi}\int d^2z \left(\frac{1}{2}\partial x \cdot \bar{\partial} x + b\bar{\partial}c + \bar{b}\partial\bar{c}\right) . \tag{4.42}$$

The combined stress tensor (we denote the x stress tensor by $T^{(x)}$)

$$T(z) = T^{(x)}(z) + T^{(bc)}(z) \tag{4.43}$$

has zero central charge in $D = 26$ and is thus a primary field of weight $(2, 0)$. The associated Virasoro generators are

$$\begin{cases} L_m = \sum_{r\in\mathbb{Z}}(2m - r)b_r c_{m-r} + \frac{1}{2}\sum_{r\in\mathbb{Z}} x_r \cdot x_{m-r} \quad (m \neq 0) \\ L_0 = \frac{1}{2}p^2 - 1 + \sum_{r=1}^{\infty} r(x_{-r}x_r + b_{-r}c_r + c_{-r}b_r) \end{cases} \tag{4.44}$$

The second term (-1) in the second equation arises from the ghost stress tensor in (4.35) for $n = 2$. The Virasoro algebra of L_m reduces to the Witt algebra (i.e. with zero central charge)

$$[L_\ell, L_m] = (\ell - m)L_{\ell+m} \tag{4.45}$$

Recall the ghost number

$$U = 1 + c_0 b_0 + \sum_{r=1}^{\infty}(c_{-r}b_r - b_{-r}c_r) \tag{4.46}$$

and recall that the ground states $|\uparrow\rangle$ and $|\downarrow\rangle$ have ghost number 2 and 1, respectively.

The full Fock space of states of the closed bosonic string, including the ghost states, is

$$\mathcal{F}_{\text{closed}} = \left(\bigoplus_{k\in\mathbb{R}^{26}} \mathcal{F}_k \otimes \widetilde{\mathcal{F}}_k\right) \otimes \mathcal{F}^{(bc)} \otimes \mathcal{F}^{(\bar{b}\bar{c})} \tag{4.47}$$

We now wish to identify, inside $\mathcal{F}$, the subspace of physical states. We shall concentrate on $\mathcal{F}_k \otimes \mathcal{F}^{(bc)}$; the treatment of $\widetilde{\mathcal{F}}_k \otimes \mathcal{F}^{(\bar{b}\bar{c})}$ is analogous.

Since the Virasoro algebra of L_ℓ generators holds with vanishing central charge, it is natural to impose

$$L_\ell|\phi\rangle = 0 \qquad \forall\ \ell \in \mathbb{Z} .$$

However, since the Fock space now also contains ghost states, this condition is insufficient. One would like to supplement it with a condition of "no ghosts". But from the treatment above, we see that ghost number changes under conformal mappings, so the no ghost condition is not invariant. The proper framework for

imposing physical state conditions, when ghosts are present, is BRST quantization. It permits us to recast the physical state conditions in terms of cohomology.

We begin by discussing a finite-dimensional case, with finite-dimensional Lie group G and associated Lie algebra $\mathcal{G}$. Let Λ be a representation of $\mathcal{G}$, with representation matrices Λ_i, $i = 1, \ldots, \dim G$, and let f_{ij}^k be the structure constants of $\mathcal{G}$:

$$[\Lambda_i, \Lambda_j] = f_{ij}^k \Lambda_k \tag{4.48}$$

One introduces anti-ghosts b_i in the adjoint representation of $\mathcal{G}$ and ghosts c^i in the dual to the adjoint representation of $\mathcal{G}$. (For compact G, these are identified, but in general, it may not be possible to do so.) These satisfy

$$\{b_i, c^j\} = \delta_i^j \qquad \{c^i, c^j\} = \{b_i, b_j\} = 0 \ . \tag{4.49}$$

The number operator is the analogue of the ghost number

$$U = c^i b_i \qquad [U, c^i] = c^i \ ; \ \ [U, b_i] = -b_i \tag{4.50}$$

and is invariant under $\mathcal{G}$. The Fock space $\mathcal{F}$ is determined by choosing a polarization. The natural polarization here is as follows

$$b_i|0\rangle = 0 \qquad \mathcal{F} = \{c^i\}|0\rangle \ , \tag{4.51}$$

and $\mathcal{F}$ contains $2^{\dim G}$ states.

This model may be realized concretely by taking c^i, $i = 1, \ldots, \dim G$, to be a basis of left-invariant 1-forms on G, and b_i to be the *contraction* with the form c^i: $b_i = \partial/\partial c^i$. The operator U gives the degree of the form and the ground state $|0\rangle$ corresponds to forms of degree 0, i.e. constants. The Fock space $\mathcal{F}$ is the vector space of all left-invariant forms on G: $\Omega^{\text{inv}}(G)$.

In BRST theory, one introduces the *BRST operator* (still for our finite-dimensional group G):

$$Q \equiv c^i \Lambda_i - \frac{1}{2} f_{ij}^k c^i c^j b_k \ . \tag{4.52}$$

This operator acts on $\mathcal{F} \otimes \Lambda$, where Λ is the representation vector space of Λ. This operator is *nilpotent*. To see this, first show that $Q^2 = 0$ when $\Lambda_i = 0$; the remaining terms with Λ present are

$$Q^2 = \frac{1}{2} c^i c^j \left([\Lambda_i, \Lambda_j] - f_{ij}^k \Lambda_k \right) \ , \tag{4.53}$$

which vanishes in view of the structure relations on Λ.

In terms of left-invariant forms on the group G, the operator Q has a natural interpretation as well. This is understood by examining how a differential acts on a left-invariant differential form in $\Omega^p(G)$: if $\omega \in \Omega^{(p)}(G)$ then $\omega = \omega_{i_1 \ldots i_p} \, c^{i_1} \wedge \ldots \wedge c^{i_p}$, with constant $\omega_{i_1 \ldots i_p}$. The differential on c^i is given by the Cartan structure equations

$$dc^i + \frac{1}{2} f_{jk}^i c^j \wedge c^k = 0 \tag{4.54}$$

and thus its action on ω may be recast in a purely algebraic form

$$d\omega = -\frac{1}{2} f^i_{jk} c^j c^k b_i \ . \tag{4.55}$$

Furthermore, when ω transforms under the representation Λ of $\mathcal{G}$, then Q is nothing but the covariant derivative acting on $\mathcal{F} \otimes \Lambda$, and its square vanishes in view of the flatness of the connection c^i.

We now consider states in the Fock space $\mathcal{F}_{(m)}$ of degree $m = 0, 1, \ldots$, dim G. The BRST operator Q has degree 1, and maps Q: $\mathcal{F}_{(m)} \to \mathcal{F}_{(m+1)}$ for $m \leq$ dim $G-1$, and Q: $\mathcal{F}_{(\dim G)} \to 0$. A state $|\psi\rangle$ is *BRST invariant* if

$$Q|\psi\rangle = 0 \qquad \text{(i.e. } Q\text{-closed)}$$

and *BRST trivial* if for some $|x\rangle \in \mathcal{F}_{(m-1)}$, we have

$$|\psi\rangle = Q|\chi\rangle \qquad \text{(i.e. } Q\text{-exact)}$$

BRST cohomology is defined by the following equivalence relation: $|\psi\rangle \sim |\psi'\rangle$ provided there is some $|\chi\rangle$ such that

$$|\psi'\rangle = |\psi\rangle + Q|\chi\rangle \ .$$

BRST cohomology classes of degree m are denoted by $H^{(m)}(Q)$. The BRST cohomology of degree 0 is easily computed. First, $|\psi\rangle \in \mathcal{F}_{(0)}$ means that $b_j|\psi\rangle = 0$ for all j, so that the condition of BRST invariance becomes

$$Q|\psi\rangle = c^i \Lambda_i |\psi\rangle = 0 \qquad \Rightarrow \qquad \Lambda_i|\psi\rangle = 0$$

Here, we have used the fact that, since $\{b_i, c^j\} = \delta^j_i$, we have $c^i|\psi\rangle \neq 0$ for all i, and these vectors are linearly independent. Thus, all BRST invariant states of ghost number 0 are $\mathcal{G}$-invariant. Clearly no states in $\mathcal{F}_{(0)}$ are BRST trivial. Vice versa, all $\mathcal{G}$-invariant states of ghost number 0 are BRST invariant.

In string theory, the BRST formalism is implemented as follows. The Lie algebra $\mathcal{G}$ is the infinite-dimensional Virasoro algebra, with structure constants

$$[L_i, L_j] = (i-j)L_{i+j} + \text{ central}; \quad f^k_{ij} = (i-j)\delta^k_{i+j} \ . \tag{4.56}$$

We do not specify the central term, since we shall not infer its implications from the finite-dimensional model. The representation Λ of $\mathcal{G}$ is given by the matter fields x, with generators $L^{(x)}_i$, and central charge 26. The ghost and anti-ghost modes are

$$c^\ell = c_{-\ell} \qquad \text{and} \qquad b_\ell \qquad\qquad \ell \in \mathbb{Z} \ . \tag{4.57}$$

The polarization for the string theory BRST is different from that for the finite-dimensional model

$$\begin{cases} b_\ell|\downarrow\rangle = c_\ell|\downarrow\rangle = 0 \\ b_\ell|\uparrow\rangle = c_\ell|\uparrow\rangle = 0 \end{cases} \qquad \ell > 0 \ . \tag{4.58}$$

The BRST operator is defined by the structure constants f^k_{ij}, but the precise ordering prescription and the effects of the central terms cannot be obtained from

the finite-dimensional model. It will turn out (see next paragraph) that the correct prescription is given by

$$Q \equiv \sum_{r\in\mathbb{Z}} (L_r^{(x)} - \delta_{r,0}) c_{-r} - \frac{1}{2} \sum_{r,s\in\mathbb{Z}} (r-s) \; : c_{-r} c_{-s} b_{r+s} : \tag{4.59}$$

The correct prescription may be derived most efficiently from the fact that the BRST charge derives from a conserved (in fact analytic) BRST current, which arises as the Noether current of BRST symmetry of the action S. The transformation rules are (they may be derived from the commutation of the operator λQ, for some constant Grassmann parameter λ, with the fields x^μ, b, c):

$$\begin{cases} \delta x^\mu = \lambda c \partial x^\mu \\ \delta c = \lambda c \partial c \\ \delta b = 2\lambda (T^{(x)} + T^{(bc)}) \; . \end{cases} \tag{4.60}$$

The resulting BRST current is

$$\begin{aligned} j^{\mathrm{BRST}}(z) &= -\frac{1}{2} c \partial x \cdot \partial x + b c \partial c + \frac{3}{2} \partial^2 c \\ &= c T^{(x)} + \frac{1}{2} c T^{(bc)} + \frac{3}{2} \partial^2 c \; . \end{aligned} \tag{4.61}$$

The last term on the right hand side ensures that j^{BRST} is a $(1,0)$ form as a quantum operator. The quantum (renormalized) current is defined as usual by OPE singularity subtractions. The BRST charge is then

$$Q = \oint \frac{dz}{2\pi i} j^{\mathrm{BRST}}(z) \; , \tag{4.62}$$

and agrees with the expression of (4.59).

Using the OPE algebra for the b-c system, derived in §4.3, we find that for $D = 26$ we have $Q^2 = 0$. Conversely, if $Q^2 = 0$, then the Virasoro algebra of $L_m = L_m^{(x)} + L_m^{(\mathrm{gh})}$ must be satisfied with zero central charge. This is easily established from the fact that

$$L_m = \{Q, b_m\} \qquad \Longrightarrow \qquad [L_m, Q] = 0 \tag{4.63}$$

and hence

$$[L_m, L_n] = \{Q, [L_m, b_n]\} = (m-n)\{Q, b_{m+n}\} = (m-n) L_{m+n} \; .$$

The BRST invariance condition $Q|\psi\rangle = 0$ now reproduces the physical state conditions $L_m|\psi\rangle = 0$, $m \in \mathbb{Z}$, in the following way. First, we define a map i from $\mathcal{F}_k^{\mathrm{phys}}$ into $\mathcal{F}_k^{\mathrm{phys}} \otimes \mathcal{F}^{(bc)}$. which maps physical states in $\mathcal{F}_k^{\mathrm{phys}}$ into states of ghost number 1 as follows:

$$i\colon \mathcal{F}_k^{\mathrm{phys}} \to \mathcal{F}_k \otimes \mathcal{F}^{(bc)}, \; |\varepsilon, k\rangle \to |\varepsilon, k, \downarrow\rangle = |\varepsilon, k\rangle \otimes |\downarrow\rangle \tag{4.64}$$

Since $b_m|\downarrow\rangle = 0$ for all $m \in \mathbb{N}$ and $c_n|\downarrow\rangle = 0$ for all $n \in \mathbb{N}^+$, it follows that if $|\epsilon, k\rangle$ is physical, then $Q|\varepsilon, k, \downarrow\rangle = 0$. Thus, i is really a map into $\operatorname{Ker} Q$

$$i\colon \mathcal{F}_k^{\text{phys}} \to \operatorname{Ker} Q \ . \tag{4.65}$$

Next, let us consider a state $|\psi\rangle \in \mathcal{F}_k \otimes \mathcal{F}^{(bc)}$, of the form

$$|\psi\rangle = |\epsilon, k\rangle \otimes |\downarrow\rangle \tag{4.66}$$

with ghost number 1, and belonging to $\operatorname{Ker} Q$: $Q|\psi\rangle = 0$. Then, in view of $b_m|\downarrow\rangle = 0$ for $m \in \mathbb{N}$ and $c_n|\downarrow\rangle = 0$ for $n \in \mathbb{N}^+$, we have

$$\begin{aligned} Q|\psi\rangle &= \sum_{r\in\mathbb{Z}} (L_r^{(x)} - \delta_{r,0}) c_{-r} |\varepsilon, k\rangle \otimes |\downarrow\rangle \\ &= \left\{ (L_0^{(x)} - 1)c_0 + \sum_{r=1}^{\infty} L_r^{(x)} c_{-r} \right\} |\varepsilon, k\rangle \otimes |\downarrow\rangle \ , \end{aligned} \tag{4.67}$$

The condition $Q|\psi\rangle = 0$ implies $(L_0^{(x)} - 1)|\varepsilon, k\rangle = 0$ and $L_r^{(x)}|\varepsilon, k\rangle = 0$ for $r \in \mathbb{N}^+$, which are the correct physical state conditions on $\mathcal{F}_k$. (The ground state $|\uparrow\rangle$ would miss the conditions $(L_0^{(x)} - 1)|\varepsilon, k\rangle = 0$.)

Which of these states are BRST trivial? Again, from the definition of Q and the fact that $b_m|\psi\rangle = c_m|\psi\rangle = 0$ for $m \in \mathbb{N}^+$, it follows that if $|\psi\rangle = Q|\lambda\rangle$, with $|\lambda\rangle$ of ghost number 0 then

$$|\psi\rangle = \sum_{r=1}^{\infty} L_{-r}^{(x)} |\lambda_r\rangle \tag{4.68}$$

for some states $|\lambda_r\rangle$. But then $|\lambda\rangle$ is spurious.

In fact, one can show that any state $|\psi\rangle$ in $\operatorname{Ker} Q$, of ghost number 1, is of the form

$$|\psi\rangle = |\varepsilon, k\rangle \otimes |\downarrow\rangle + Q|x\rangle$$

where $|\varepsilon, k\rangle \in \mathcal{F}_k^{\text{phys}}$ and $|x\rangle$ is some state of ghost number 0. Thus, the cohomology of Q of ghost number 1, i.e. space of states of ghost number 1 which are BRST invariant, modulo BRST trivial states, is isomorphic to the physical state space $\mathcal{F}^+$ defined previously:

$$\mathcal{F}^+ = \mathcal{F}^{\text{phys}} / \mathcal{F}^{\text{spurious}} \sim H^{(1)}(Q) \tag{4.69}$$

References and cross references

Systematic discussions of ghosts in bosonic string theory are given in

D. H. Friedan, *Introduction to Polyakov's string theory*, Recent Advances in Field Theory and Statistical Mechanics, Les Houches, Session 39 (R. Stora and J. B. Zuber, eds.), North Holland Publishers, 1982.

D. Friedan, E. Martinec and S. Shenker, *Conformal invariance, supersymmetry and string theory*, Nucl. Phys. B **271** (1986), 93.

E. D'Hoker and D. H. Phong, *The geometry of string perturbation theory*, Rev. Modern Phys. **60** (1988), 917–1065.

BRST cohomology and its role in string quantization is very nicely discussed in

M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. I, Cambridge University Press, 1988, pp. 131–156.

LECTURE 5
Moduli Dependence of Determinants and Green Functions

In Lectures 3 and 4, we have reduced the calculation of string transition amplitudes to integrals of combinations involving certain determinants and Green functions over the finite-dimensional moduli spaces $\mathcal{M}_h$. In this lecture, we complete the analysis by producing explicit formulas for these determinants and Green functions. We shall begin by collecting results on surfaces with constant curvature metrics. Then, we proceed and exploit the complex structure of $\mathcal{M}_h$ to obtain explicit expressions in terms of Jacobi theta functions. (Derivations of the results stated in this section will not be given here; they may be found in E. D'Hoker, D. H. Phong, Rev. Mod. Phys. Vol. 60 (1988), p. 917–1065, and references therein.)

§5.1. Worldsheets with constant curvature metric ($h \geq 2$)

Let Σ be a compact surface of genus $h \geq 2$. By the uniformization theorem, the simply connected cover of Σ is the upper half plane H, and $\Sigma = H/\Gamma$, where $\Gamma = \pi_1(\Sigma)$ is the fundamental group of Σ. For any metric g on Σ, there is a unique Weyl transformation σ such that $g = e^{2\sigma}\hat{g}$ with $\hat{g}$ of constant curvature $R_{\hat{g}} = -1$. The metric $\hat{g}$ lifts to the Poincaré metric on H and Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting on H, with compact quotient $\Sigma = H/\Gamma$. Moduli space may now be identified as follows:

$$\mathcal{M}_h = \{\hat{g}, R_{\hat{g}} = -1\}\,/Diff(\Sigma)\ . \tag{5.1}$$

We now have the following results for determinants and for the measure on $\mathcal{M}_h$ arising in string amplitudes:

$$\mathrm{Det}\,\Delta^+_{(n)} = S(n+1)\,; \quad \mathrm{Det}'\Delta_{(0)} = S'(1) \tag{5.2}$$

$$\prod_J dm_J d\bar{m}_J |\det(\mu_J, \phi_k)|^2 \det(\phi_J, \phi_k)^{-1} = d(\text{Weil-Petersson})\ . \tag{5.3}$$

Here $S(s)$ is the *Selberg zeta function*, defined by

$$S(s) \equiv \prod_{\gamma\ \text{primitive}} \prod_{k=1}^{\infty} (1 - e^{-(s+k)\ell_\gamma}) \tag{5.4}$$

for ℓ_γ the lengths of the *simple closed* (*i.e. primitive*) *geodesics* γ on $\Sigma = H/\Gamma$.

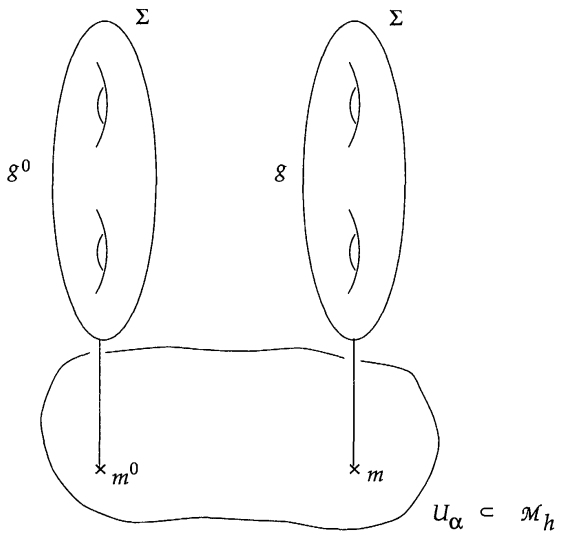
§5.2. Holomorphicity in moduli

The most powerful results on Green functions and determinants are obtained by exploiting the fact that $\mathcal{M}_h$ is a complex orbifold, and that Green functions and determinants primarily involve complex analytic dependence on $\mathcal{M}_h$.

To examine this complex dependence, we first work locally on $\mathcal{M}_h$, in a neighborhood of a point $m^0 \in \mathcal{M}_h$, represented by a metric g^0. We now parametrize the neighborhood U_α of m^0 and thus of g^0 by Beltrami differentials $\mu = \mu_{\bar{z}}^z$, which provide complex coordinates on $\mathcal{M}_h$, as established in Lecture 3. We have a general metric g in the neighborhood of g^0:

$$g = 2g_{z\bar{z}}|dz + \mu d\bar{z}|^2 \qquad \text{where} \qquad g^0 = 2g_{z\bar{z}}|dz|^2 \ . \tag{5.6}$$

We retain the Weyl factor $g_{z\bar{z}}$ for now, since we shall also study the interplay of Weyl dependence and moduli dependence.



The dependence of the functional determinant combinations $Z^{\pm}_{(n)}(g)$ on the complex coordinates of $\mathcal{M}_h$ is simple, and may be evaluated explicitly. This result is the content of the so-called *Belavin-Knizhnik theorem*, or holomorphic anomaly theorem:

$$\delta_\mu \delta_{\bar{\mu}} \ln Z^{\pm}_{(n)}(g) = -\frac{c^{\pm}_{(n)}}{12\pi} \int_\Sigma d\mu_g \, (\nabla_z \delta\mu \nabla_{\bar{z}} \delta\bar{\mu} - R_g \delta\mu\delta\bar{\mu}) \tag{5.7}$$

where the coefficients $c^{\pm}_{(n)} = 6n^2 \pm 6n + 1$ are the same as those entering the Weyl anomaly.

It follows immediately from the Belavin-Knizhnik theorem that Weyl invariant combinations of $Z^{\pm}_{(n)}(g)$'s are also especially simple in their dependence on moduli. Indeed, the $\delta_\mu \delta_{\bar{\mu}}$ variation of any Weyl invariant combination vanishes, so that such a combination is a sum of a holomorphic and an anti-holomorphic part. We have, e.g.

$$Z^{\pm}_{(n)}(g)\, Z_{(0)}(g)^{-c^{\pm}_{(n)}} = |\mathcal{Z}^{\pm}_{(n)}(m_j)|^2 \ . \tag{5.8}$$

where $\mathcal{Z}^{\pm}_{(n)}(m_j)$ is holomorphic inside $\mathcal{M}_h$. In particular, the combination entering the integrand of the bosonic string amplitudes is precisely of this form, for superscript $-$ and $n = -1$:

$$Z^{-}_{(-1)}(g)\, Z_{(0)}(g)^{-13} = |\mathcal{Z}^{-}_{(-1)}(m_j)|^2 \ . \tag{5.9}$$

We shall make use of this powerful result in string theory in §5.3.

The BK Theorem may be understood geometrically in the language of determinant line bundles over $\mathcal{M}_h$, where the right hand side of BK – the holomorphic anomaly – arises as a non-zero curvature of the determinant line bundles $\mathrm{Det}(\nabla^z_{(n)})$ and $\mathrm{Det}(\nabla^{(n)}_z)$ with certain Hermitian metrics, the so-called Quillen metrics. The vanishing $\delta_\mu\delta_{\bar\mu}$ of the BK anomaly $\mathcal{Z}^\pm_{(n)}$ then indicates that the corresponding line bundles

$$\mathrm{Det}\nabla^z_{(n)} \otimes (\mathrm{Det}\nabla^z_{(1)})^{-\otimes c^-_{(n)}} \ , \tag{5.10}$$

are flat. In string theory, $n = 2$, and this case is related to the flatness of $\mathcal{K} \otimes \lambda^{-13}$, where $\mathcal{K}$ is the canonical bundle of $\mathcal{M}_h$ and λ is the Hodge bundle.

We shall not prove the BK Theorem here (it can be proven using heat-kernel methods, just as in the derivation of the Weyl anomaly), but only indicate the origin of the holomorphic anomaly. The key observation is that the covariant derivatives $\nabla^z_{(n)}$ depend only on $\delta\mu$, but not on $\delta\bar\mu$ to first order in $\delta\mu$ and $\delta\bar\mu$; thus $\nabla^z_{(n)}$ is holomorphic in $\delta\mu$ to first order.

$$\begin{cases} \delta_\mu \nabla^z_{(n)} = \delta\mu \nabla^{\bar z}_{(n)} + n(\nabla^{\bar z}\delta\mu) \\ \delta_{\bar\mu}\nabla^z_{(n)} = 0 \ . \end{cases} \tag{5.11}$$

Functional determinants of Laplace operators would then naively be expected to be products of a holomorphic times anti-holomorphic factor:

$$\mathrm{Det}\Delta^-_{(n)} \overset{?}{\sim} \mathrm{Det}\nabla^{(n-1)}_z \mathrm{Det}\nabla^z_{(n)} \ , \tag{5.12}$$

with $\mathrm{Det}\nabla^z_{(n)}$ holomorphic on $\mathcal{M}_h$ and $\mathrm{Det}\nabla^{(n-1)}_z$ anti-holomorphic. Actually, determinants must be properly defined (e.g. using ζ-function techniques, as in Lecture 3.), in a fashion that preserves their $Diff(\Sigma)$ invariance. When this analysis is carried out, anomalies, such as the Weyl anomaly, may appear. Here, the above holomorphic factorization of determinants is also beset by an anomaly. In fact, these anomalies are intimately related, as can be seen from the argument below. As was explained in Lecture 3, the variations of μ, $\bar\mu$ and Weyl rescalings are independent. Hence, using the Weyl anomaly of $Z^\pm_{(n)}(g)$:

$$\begin{aligned} \delta_\sigma\delta_\mu\delta_{\bar\mu} \ln Z^\pm_{(n)}(g) &= \delta_\mu\delta_{\bar\mu}\delta_\sigma \ln Z^\pm_{(n)}(g) \\ &= -\frac{c^\pm_{(n)}}{6\pi}\,\delta_\mu\delta_{\bar\mu}\int_\Sigma d\mu_g\, R_g \ . \end{aligned} \tag{5.13}$$

But, the variation of the Gaussian curvature is easily computed

$$\delta_\mu\delta_{\bar\mu}(d\mu_g\, R_g) = (\nabla_z\delta\mu\nabla_{\bar z}\delta\bar\mu)d\mu_g \neq 0 \ . \tag{5.14}$$

Thus, clearly, the holomorphic anomaly $\delta_\mu\delta_{\bar\mu} \ln Z^\pm_{(n)}(g)$ cannot cancel unless also the Weyl anomaly cancels.

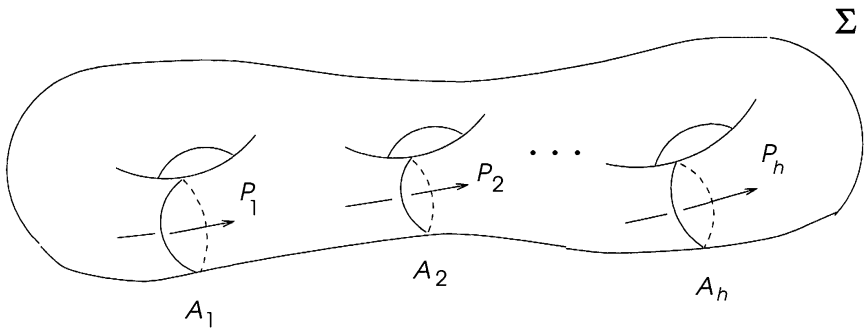
§5.3. The chiral splitting theorem

In critical string theory, we also have strong results on the holomorphicity properties of the correlation functions for vertex operators of physical on-shell states, given by the Chiral Splitting Theorem. We begin by assembling the data we need to formulate this theorem.

Let $V_i = \int_\Sigma d^2 z_i \, W_i(z_i, \bar{z}_i)$ be vertex operators of physical on-shell states whose polarization tensors are factorized in left and right movers. Let A_I, B_I, $I = 1, \ldots, h$ be a canonical basis of homology 1-cycles on Σ, with

$$\#(A_I, A_J) = \#(B_I, B_J) = 0 \qquad \#(A_I, B_J) - \#(B_J, A_I) = \delta_{IJ} \; . \tag{5.15}$$

We define internal loop momenta $p_I \in \mathbb{R}^{26}$ associated with each A_I cycle. Because the A_I cycles have trivial intersection numbers, these internal loop momenta are independent.



The correlation functions of unintegrated vertex operators $W_i(z_i, \bar{z}_i)$ at fixed internal loop momenta p_I are defined by

$$\begin{aligned} \langle W_1 \ldots W_N \rangle_g (p_I) \equiv & \int_{\mathrm{Map}(\Sigma, M)} Dx \, W_1(z_1, \bar{z}_1) \ldots W_N(z_N, \bar{z}_N) \\ & \prod_{I=1}^{h} \delta \left(p_I^\mu - \int_{A_I} \frac{dz}{2\pi i} \, \partial_z x^\mu \right) e^{-S[x,g]} \end{aligned} \tag{5.16}$$

The *Chiral Splitting Theorem* states that

$$\langle W_1 \ldots W_N \rangle_g (p_I) = \delta(k) Z_0(g)^{-13} \mathcal{F} \widetilde{\mathcal{F}} \tag{5.17}$$

where $\mathcal{F}(z_i, m_j; p_I, k_i)$ is a complex analytic function of z_i and m_j, and $\widetilde{\mathcal{F}}$ is its complex conjugate

$$\widetilde{\mathcal{F}}(\bar{z}_i, \bar{m}_j; p_I, k_i) = \mathcal{F}(z_i, m_j; p_I, k_i)^* \; . \tag{5.18}$$

Here, we have suppressed the dependence of W and $\mathcal{F}$ on polarization tensors occurring in the construction of W. Also, the function $\mathcal{F}$ depends upon the choice of curves representing the A_I boundary cycles, and this dependence has been subsumed in the definition of the internal momenta p_I.

An outline of the proof of this Theorem will be given in §5.5. Let us simply remark here that the proper conformal field theory interpretation of the integration

over loop momenta is the following. At fixed values of the internal loop momenta, p_I through all A_I cycles, only a single conformal family traverses each A_I-cycle. The contributions from left and right movers are factorized, so that the entire amplitude – at fixed momenta p_I – is factorized.

Mathematically, we have gained considerably through the Belavin-Knizhnik and Chiral Splitting Theorems, since we may now apply techniques of complex function theory, theta-functions and algebraic geometry to study amplitudes.

Physically, we have uncovered a method for separating the contributions of left and right movers in the integrand of transition amplitudes, via the holomorphic versus anti-holomorphic dependence on $\mathcal{M}_h$ and Σ. (This will be crucial in the study of superstrings.)

An immediate result from the combination of the Belavin-Knizhnik and Chiral Splitting Theorems is that the full transition amplitude at genus h is an integral over a product of a holomorphic times anti-holomorphic function.

$$A_h = \delta(k) \int_{\mathbb{R}^{26h}} \prod_I dp_I^{26} \int_{\mathcal{M}_h} \prod_J dm_J \bar{m}_J \int_\Sigma d^2 z_1 \dots \int_\Sigma d^2 z_N \, C \, \widetilde{C} \ , \tag{5.19}$$

where

$$C(z_i, m_J; p_I, k_i) = \det(\mu_J, \phi_k) \mathcal{Z}^-_{(-1)}(m_J) \mathcal{F}(z_i, m_J; p, k) \tag{5.20}$$

and

$$\widetilde{C}(\bar{z}_i, \bar{m}_J; p_I, k_i) = C(z_i, m_J; p_I, k_i)^* \ . \tag{5.21}$$

(Of course, for $h \geq 1$, these expressions are only formal, since the integration ultimately diverges. The generalization to the case of the superstring, on the other hand, will be well-defined.)

§5.4. Holomorphic and meromorphic differentials (a brief review of basics)

We begin by recalling some basic tools of complex function theory on Riemann surfaces. The space of holomorphic line bundles of degree d is the *Picard variety*

$$\mathrm{Pic}_d(\Sigma) = \{\text{holomorphic line bundles } L \text{ on } \Sigma \, , \, c_1(L) = d\} \ .$$

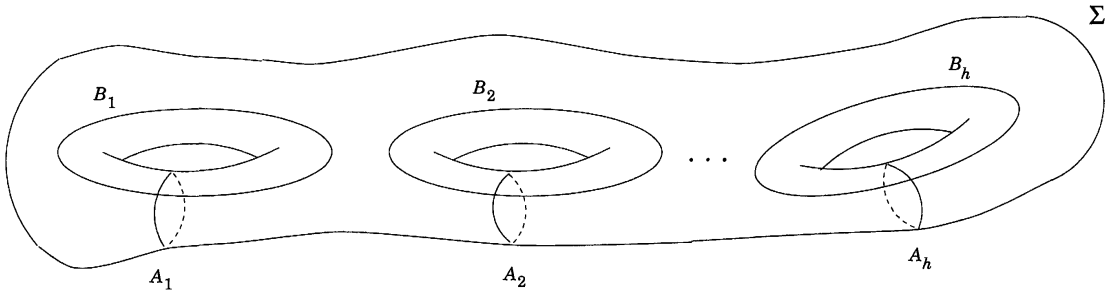
The subspace of line bundles L of degree 0 is the *Jacobian variety*: $J(\Sigma) = \mathrm{Pic}_0(\Sigma)$, which may also be viewed as

$$J(\Sigma) = H^1(\Sigma, \mathbb{R}) / H^1(\Sigma, \mathbb{Z}) \ . \tag{5.22}$$

Choosing a reference $L_0 \in \mathrm{Pic}_d(\Sigma)$, any other $L \in \mathrm{Pic}_d(\Sigma)$ may be obtained as a tensor product $L = L_0 \otimes \ell$, $\ell \in J(\Sigma)$.

A completely explicit realization is obtained by making a choice of homology basis (A_I, B_I), $I = 1, \dots, h$, with canonical intersection matrix:

$$\#(A_I, A_J) = \#(B_I, B_J) = 0 \qquad \#(A_I, B_J) = -\#(B_J, A_I) = \delta_{IJ} \ . \tag{5.23}$$



To this choice corresponds a unique basis of holomorphic Abelian differentials (holomorphic sections of K), $(\omega_1, \ldots, \omega_h) = \vec{\omega}$, or denoted simply ω, by

$$\oint_{A_I} \omega_J = \delta_{IJ} , \qquad I, J = 1, \ldots, h . \tag{5.24}$$

The *period matrix* is then defined by

$$\oint_{B_I} \omega_J = \Omega_{IJ} \tag{5.25}$$

and the Jacobian is a complex torus of dimension h:

$$J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h + \Omega\mathbb{Z}^h) . \tag{5.26}$$

The intersection from $\#(\ ,\ )$ is left invariant under the action of the *modular group* on the homology cycles

$$\begin{pmatrix} B \\ A \end{pmatrix}_I \longrightarrow \begin{pmatrix} B' \\ A' \end{pmatrix}_I = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{IJ}}_{\in \mathrm{Sp}(2h;\mathbb{Z})} \begin{pmatrix} B \\ A \end{pmatrix}_J \tag{5.27}$$

under which

$$\Omega \to \Omega' = (a\Omega + b)(c\Omega + d)^{-1} . \tag{5.28}$$

The Riemann bilinear relations guarantee that Ω lies in the *Siegel upper half-space*: $\Omega \in \mathcal{H}_h$, where

$$\mathcal{H}_h = \{h \times h \text{ complex } \Omega;\ \Omega_{IJ} = \Omega_{JI};\ \mathrm{Im}\,\Omega > 0\} . \tag{5.29}$$

In fact, Torelli's theorem guarantees that to each Riemann surface in $\mathcal{M}_h$ there corresponds a unique point in the fundamental domain of the modular group

$$\mathcal{A}_h \equiv \mathcal{H}_h/\mathrm{Sp}(2h;\ \mathbb{Z}) , \tag{5.30}$$

(the converse holds for the torus, but not for $h \geq 2$.)

$\mathcal{H}_h$ carries a $\mathrm{Sp}(2h;\ \mathbb{Z})$-invariant metric

$$ds^2 = \mathrm{Tr}((\mathrm{Im}\,\Omega)^{-1} d\Omega (\mathrm{Im}\,\Omega)^{-1} d\Omega^*) . \tag{5.31}$$

Since ω_I depends holomorphically on $\mathcal{M}_h$, so does Ω_{IJ}; in fact, we have an explicit formula for the variation of Ω_{IJ} with holomorphic Beltrami differentials $\mu_{\bar{z}}^z$ associated with a Riemann surface deformation:

$$\delta\Omega_{IJ} = -i \int_\Sigma d^2z\, \mu_{\bar{z}}^z\, \omega_{Iz}\, \omega_{Jz} \tag{5.32}$$

The induced complex structure on $\mathcal{M}_h$ is the one associated with the metric ds^2 of (5.31).

The *Jacobi ϑ-function*, with characteristics $\delta = (\delta', \delta'') \in [0,1]^{2h}$, is defined as a function on $\mathbb{C}^h \times \mathcal{H}_h$:

$$\vartheta[\delta](\zeta, \Omega) \equiv \sum_{n \in \mathbb{Z}^h + \delta'} e^{\pi i n^T \Omega n + 2\pi i n^T (\zeta + \delta'')} \,. \tag{5.33}$$

It obeys, for any $M, N \in \mathbb{Z}^h$:

$$\vartheta[\delta](\zeta + M + \Omega N, \Omega) = \vartheta[\delta](\zeta, \Omega) e^{-i\pi N^T \Omega N - 2\pi i N^T (\zeta + \delta'') + 2\pi i \delta' M} \,, \tag{5.34}$$

and this property may be used to define ϑ as the unique (up to constant factor) holomorphic section of a holomorphic line bundle on $J(\Sigma)$, with the above transition functions. One also often uses $\vartheta(\zeta, \Omega) \equiv \vartheta[0](\zeta, \Omega)$.

Using the Jacobi ϑ-functions, and the Riemann vanishing theorem (which yields the zeros of ϑ as a function of ζ, but which we shall not discuss here), one may uniquely (up to constant factor) reconstruct a meromorphic function on Σ from its divisor $D = z_1 + \cdots + z_d - w_1 - \cdots - w_d$:

$$f_D(z) = \prod_{i=1}^d \vartheta\left(\zeta + \int_{z_i}^z \omega, \Omega\right) \vartheta\left(\zeta + \int_{w_i}^z \omega, \Omega\right)^{-1} \,. \tag{5.35}$$

Here ζ is such that $\vartheta(\zeta, \Omega) = 0$, but so that the above ϑ functions do not vanish identically; $f_D(z)$ is then independent of ζ.

Spin bundles on Σ belong to $\mathrm{Pic}_{h-1}(\Sigma)$, and may be described in terms of their spin structure, of which there are 2^{2h}. A choice of homology basis singles out a particular spin bundle S_0, and all other spin structures may be labeled by points in the Jacobian, as indicated earlier:

$$S = S_0 \otimes \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \qquad\qquad \delta'' + \Omega\delta' \in J(\Sigma) \,, \tag{5.36}$$

where δ' and δ'' are half-points, i.e. δ'_I, $\delta''_I \in \left\{0, \frac{1}{2}\right\}$. Spin structures are partitioned into *even* or *odd spin structures*, according to the parity of the ϑ-function

$$\vartheta[\delta](-\zeta, \Omega) = (-)^{4\delta' \cdot \delta''} \vartheta[\delta](\zeta, \Omega) \,. \tag{5.37}$$

Thus, there are $2^{h-1}(2^h \pm 1)$ even/odd spin structure with $4\delta' \cdot \delta''$ even/odd, and the number of holomorphic sections is then even/odd as well, in fact generally 0/1. The generic holomorphic section $h_\delta(z)$ for an odd spin structure δ is obtained from

the square root of an Abelian differential with exactly $h-1$ double zeros (this $h_\delta(z)$ is a Dirac zero mode)

$$h_\delta(z) \equiv \left(\sum_I \partial_I \vartheta[\delta](0,\Omega)\,\omega_I(z)\right)^{1/2} . \tag{5.38}$$

Finally, we define one of the fundamental objects that enter Green functions and determinants in string theory. The *prime form* is a function on the universal cover of $\Sigma \times \Sigma$ and $\mathcal{H}_h$, given by

$$E(z,w) \equiv \frac{\vartheta[\delta]\left(\int_w^z \omega, \Omega\right)}{h_\delta(z)\; h_\delta(w)} \tag{5.39}$$

where δ is any *odd* half characteristic. $E(z,w)$ is a holomorphic $\left(-\frac{1}{2},0\right)$ differential form in both z and w, with a single zero at $z=w$. As such, $E(z,w)$ generalizes the function $z-w$ on $\mathbb{C}$ to higher genus surfaces. On the torus, we have the well-known expression

$$E(z,w) = \frac{\vartheta_1(z-w,\Omega)}{\vartheta_1'(0,\Omega)} . \tag{5.40}$$

The monodromies of $E(z,w)$ around A_I homology cycles are simple: $E(z,w)$ picks up a $-$ sign; the monodromy as z moves around B_I once is

$$E(z,w) \to -\exp\left[-i\pi\Omega_{II} - 2\pi i \int_z^w \omega_I\right] E(z,w) . \tag{5.41}$$

Under modular transformations (represented by their action on the homology cycles as in (5.27)), we have

$$E(z,w) \to \exp\left[\pi i \int_z^w \omega^T (c\Omega+d)^{-1} c \int_z^w \omega\right] E(z,w) . \tag{5.42}$$

Meromorphic Abelian differentials (meromorphic sections of K) are readily obtained

$$\begin{aligned} \omega_{xy}(z) &= \partial_z \ln \frac{E(z,x)}{E(z,y)}\, dz \\ \omega_w(z) &= \partial_z \partial_w \ln E(z,w) dz . \end{aligned} \tag{5.43}$$

Meromorphic $\frac{1}{2}$ differentials (meromorphic sections of S_δ) are

$$\begin{aligned} \delta \text{ even} \qquad S_\delta(z,w) &= \frac{1}{E(z,w)} \frac{\vartheta[\delta]\left(\int_z^w \omega, \Omega\right)}{\vartheta[\delta](0,\Omega)} \\ \delta \text{ odd} \qquad S_\delta(z,w) &= \frac{1}{E(z,w)} \frac{\partial_I \vartheta[\delta]\left(\int_z^w \omega, \Omega\right) \omega_I(y)}{\partial_I \vartheta[\delta](0,\Omega)\omega_I(y)} \end{aligned} \tag{5.44}$$

For δ even, S_δ is the Szegö kernel, while for δ odd, S_δ is not single-valued on Σ as defined.

§5.5. Green functions, determinants and chiral splitting

The Green function of scalar fields on Σ is simply expressed in terms of the prime form and Abelian differentials

$$G(z,w) = -\ln|E(z,w)|^2 + 2\pi \operatorname{Im}\int_w^z \omega_I (\operatorname{Im}\Omega)^{-1}_{IJ} \operatorname{Im}\int_w^z \omega_J \tag{5.45}$$

The second term on the right hand side is required to render $G(z,w)$ single-valued on $\Sigma\times\Sigma$, as well as to render it modular invariant. This expression is readily seen to generalize the 1-loop result (and of course the 0-loop result as well.)

We use this result to prove the Chiral Splitting theorem for the case of vertex operators for tachyon states. (Higher mass states and their vertex operators may be obtained by factorization or equivalently by using the OPE for exponential operators.) To restrict the loop momenta and bring out an integration over them, we can insert the identity (as we did in Lecture 3 for one loop amplitudes)

$$1 = \prod_I \int_{\mathbb{R}^{26}} dp_I\, \delta\left(p_I^\mu - \frac{1}{2\pi i}\int_{A_I} dz\, \partial_z x^\mu\right) \tag{5.46}$$

into the correlation function $\langle W_1(z_1,\bar z_1)\dots W_N(z_N,\bar z_N)\rangle_g$. In fact here, we may use a short-cut, yielding the same result, carried out as follows. If $W_i(z_i,\bar z_i) = \exp ik_i\cdot x(z_i,\bar z_i)$ then we have

$$\begin{aligned}\left\langle \prod_{i=1}^N e^{ik_i\cdot x(z_i,\bar z_i)}\right\rangle_g &= \delta(k)\left(\frac{\operatorname{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_g}\right)^{-13}\prod_{i<j} e^{-k_i\cdot k_j G(z_i,z_j)}\\ &= \delta(k) Z_{(0)}(g)^{-13}\prod_{i<j}|E(z_i,z_j)|^{2k_i\cdot k_j} X\end{aligned} \tag{5.47}$$

where

$$X = \det(\omega_I,\omega_J)^{-13}\prod_{i<j} e^{-2\pi k_i\cdot k_j \operatorname{Im}\int_w^z \omega\,(\operatorname{Im}\Omega)^{-1}\operatorname{Im}\int_w^z\omega}\,. \tag{5.48}$$

Using the Riemann bilinear relations, we have

$$\det(\omega_I,\omega_J) = \det(2\operatorname{Im}\Omega_{IJ}) \tag{5.49}$$

and we may now represent X by a single Gaussian integral over real parameters p_I^μ, which are just the internal loop momenta:

$$X = \prod_I \int_{\mathbb{R}^{26}} d^{26}p_I \left| e^{i\pi p_I^\mu \Omega_{IJ} p_J^\mu + 2\pi i p_I^\mu k_i^\mu \int_P^{z_i}\omega_I}\right|^2 . \tag{5.50}$$

It is now straightforward to read off the function $\mathcal{F}$ that enters the Chiral Splitting Theorem:

$$\mathcal{F}(z_i,m_j;\,p_I,k_i) = \prod_{i<j} E(z_i,z_j)^{k_i\cdot k_j}\cdot\exp\left[i\pi p_I^\mu\Omega_{IJ}p_j^\mu + 2\pi i p_I^\mu k_i^\mu\int_P^{z_i}\omega_I\right] \tag{5.51}$$

The point P is arbitrary in view of the momentum conservation $\delta(k)$-function factor. $\mathcal{F}$ above is determined only up to phases. In particular, $\mathcal{F}$ is multiple-valued, with monodromies that may be easily computed from the monodromies of $E(z,w)$, as given in (5.41). $\mathcal{F}$ also has branch cuts along the homology cycles: the effect of letting vertex insertion point z_i cross a cycle A_I is to add the external momentum k_i to the loop momentum p_I. Modular transformation properties of $\mathcal{F}$ are also non-trivial, and may be read off from (5.42) and (5.28).

It remains to express the functional determinant combinations $\mathcal{Z}^-_{(n)}(m_j)$ in terms of ϑ-functions and differentials. This may be achieved by a variety of techniques, inducing bosonization and integration over $\mathcal{M}_h$ of the stress tensor. We shall not develop these techniques here but limit ourselves to stating the result for $\mathcal{Z}^-_{(-1)}$:

$$\mathcal{Z}^-_{(-1)}(m_j) = \frac{\vartheta(3\Delta_w|\Omega)}{\det \partial_w^i \phi_j(w)} \left(\frac{\partial_w^h \vartheta(\Delta_w|\Omega)}{\det \partial_w^I \omega_J(w)} \right)^{-9} .$$

Here, w is an arbitrary point on Σ, ∂_w^α stands for the derivative of order α in w and Δ_w is the Riemann vector

$$\Delta_w^I = \tfrac{1}{2} - \tfrac{1}{2}\Omega_{II} + \sum_{J \neq I} \oint_{A_J} \omega_J(z) \int_w^z \omega_I .$$

References and cross references

This part is discussed in detail in

E. D'Hoker and D. H. Phong, *The geometry of string perturbation theory*, Rev. Modern Phys. **60** (1988), 917–1065.

which also contains numerous references to the mathematical literature.

LECTURE 6
Strings on General Manifolds

In the preceding lectures, we have quantized strings in flat Minkowski space-time. The existence of a Hilbert space of physical string states (i.e. the consistent elimination of negative norm states) requires $Weyl(\Sigma)$ and $Diff(\Sigma)$ invariance of the full quantum theory. This condition fixes the dimension of space-time to be $D = 26$ and results in the appearance of massless spin 1 and spin 2 string states which behave like Yang–Mills particles and gravitons, respectively. The spectrum also contains massless dilatons and antisymmetric tensor states.

In the present lecture, we wish to extend the analysis of the consistent elimination of negative norm states to the case of arbitrary space-time manifolds, with general metric $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ and dilaton Φ, and possible other background fields. On physical grounds, we shall assume that the space-times of interest have Minkowski signature $(-,+,\cdots,+)$. For simplicity, we assume that the dimension of space-time D is such that $D \geq 4$ in order to avoid dealing with certain peculiarities that arise in $D = 2, 3$. We expect again that the space-time dimension D will be fixed by the condition that all negative norm states be consistently eliminated. The precise value of D where this happens may, however, depend on the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, so we shall leave D to be determined later on. We shall also assume that space-time M has at least 2 non-compact dimensions, including the time direction, so that scattering can be defined in M.

Given the signature of the space-time manifold M, the Fock space of states produced by the quantization of the map x will again contain negative norm states. These states are to be eliminated before a Hilbert space of string states can be constructed. To achieve elimination of negative norm states, it is clearly necessary that the worldsheet quantum field theory of the map x, defined in terms of the background $G_{\mu\nu}$, $B_{\mu\nu}$, Φ (and perhaps other fields), together with the Faddeev-Popov ghost fields b, c, $\bar{b}$, $\bar{c}$, be invariant under $Diff(\Sigma)$ and $Weyl(\Sigma)$ transformations. In this case, the BRST quantization scheme of Lecture 4 may be set up, just as in the case of flat Minkowski M.

For certain classes of space-time manifolds M, this condition can be shown to be also sufficient. When $M = \mathbb{R}^2 \times M_{D-2}$, where $\mathbb{R}^2$ includes the time direction, so that the metric G on M_{D-2} is positive, we have the usual Hilbert space construction for the states arising from the M_{D-2} part of space-time, since these states have positive norm. One can show that the BRST quantization scheme for all of M leads to a Hilbert space. The proof proceeds by passing to the lightcone gauge variables, which may be defined for the $\mathbb{R}^2$ part of M just as in the case of flat M.

In lightcone gauge, only positive norm, physical states survive, and may be mapped onto BRST cohomology classes in a one-to-one fashion, just as for flat M. For more general M, I am not aware of a proof that a Hilbert space may be constructed, but it is generally believed to be the case, at least "under certain conditions". For example, a space-time M with a black hole is expected to cause problems.

Henceforth, we shall restrict to space-time manifolds M and background fields, such that $Weyl(\Sigma) \ltimes Diff(\Sigma)$ invariance of the x, b, c, $\bar{b}$, $\bar{c}$ system leads to a Hilbert space. Equivalently, the quantum field theory of the x field is a conformal field theory of central charge $c = 26$. Thus, we have reduced the conditions of consistent string propagation in space-time manifolds M background fields to a problem in quantum field theory: that of identifying and parametrizing conformal field theories of the map x with central charge $c = 26$.

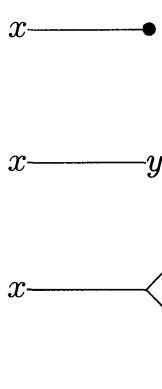
§6.1. Perturbation theory around general field configurations

Before launching into a study of the conditions for conformal invariance of the field theory of the map x with general space-time M, general $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, (and possibly other background fields), we wish to clarify the physical significance of string quantization in non-flat background field configurations. We proceed by comparison with perturbation theory in quantum field theory (QFT).

In QFT, we start with canonical fields ϕ and a classical action $S[\phi]$ which we assume to be independent of $\hbar$. Perturbation theory (in the loop expansion) is carried out by choosing a field configuration ϕ_0, and expanding the field ϕ in powers of $\sqrt{\hbar}$: $\phi = \phi_0 + \sqrt{\hbar}\,\varphi$. The Feynman rules are then obtained by an expansion of the action $S[\phi]$ in powers of $\sqrt{\hbar}$:

$$\frac{1}{\hbar} S[\phi] = \frac{1}{\hbar} S[\phi_0] + \frac{1}{\sqrt{\hbar}} \int \varphi \frac{\delta S}{\delta \phi}[\phi_0] + \frac{1}{2} \iint \varphi\varphi \frac{\delta^2 S}{\delta \phi^2}[\phi_0] + \\ + \frac{1}{6}\sqrt{\hbar} \iiint \varphi\varphi\varphi \frac{\delta^3 S}{\delta \phi^3}[\phi_0] + \cdots$$

(6.1)



$$\frac{1}{\sqrt{\hbar}} \frac{\delta S}{\delta \phi(x)}[\delta_0]$$

$$\left(\frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)}[\phi_0] \right)^{-1}$$

$$\sqrt{\hbar} \frac{\delta^3 S}{\delta \phi(x) \delta \phi(y) \delta \phi(z)}[\phi_0]$$

$$\vdots$$

Both the Feynman rules for the φ propagator and for the φ interaction vertices depend upon the background field ϕ_0 in a non-trivial way, as is evident from the Feynman rule expansion in (6.1).

In string theory, we have no equivalent of the canonical field ϕ and of the classical action $S[\phi]$ that is as well understood as in QFT. (Many attempts at constructing a string field theory formulation have been made over the years, but have met with only limited success.) In string theory, our starting point consists

only of the Feynman rules, which are given in terms of a summation over surfaces Σ, and maps $\Sigma \to M$, and which yield on-shell transition amplitudes only. The rôle of ϕ_0 in QFT is played in string theory by the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ Φ, and possibly other fields. And indeed, the Feynman rules of string theory depend upon these background fields, just as the Feynman rules of QFT depended upon ϕ_0.

Fortunately, much crucial physical information may be reconstructed from the Feynman rules: we already know that they yield the transition amplitudes for physical states, i.e. the S-matrix (at least in perturbation theory). In particular, it is clear from QFT that certain transition amplitudes yield direct information on the background ϕ_0. The most important one for our purposes is that ϕ_0 *is a classical solution*, i.e. $(\delta S/\delta\phi)[\phi_0] = 0$ if and only if all *tadpole graphs are zero to tree level.* The string theory equivalent of this criterion is that a given background field configuration $G_{\mu\nu}$, $B_{\mu\nu}$, Φ (and possibly other fields) are a *solution to the string equations of motion to tree level* (i.e. classically) *if and only if all tadpole graphs vanish.* This criterion gives us a powerful tool to investigate whether a certain configuration is a classical solution or not. In QFT, we do not usually investigate the Feynman rules to decide whether a configuration ϕ_0 is a solution, because we can directly check whether $(\delta S/\delta\phi)[\phi_0]$ vanishes or not. In string theory, this alternative is not available, at present.

We have established previously that the existence of a Hilbert space of string states requires the map x to define a conformal field theory of central charge $c = 26$. It is a simple consequence of conformal invariance on the sphere (established for x in flat space-time M in Lecture 3) that the 1-point function of any primary conformal field of weight $(1, 1)$, describing an on-shell physical state, must vanish to tree level. Combining this result with that of the preceding paragraph, we discover that if x defines a conformal field theory of central charge 26, for a certain background configuration $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, (and possibly other fields), then this configuration is a classical (tree level) solution to the string equation! In other words, demanding that the backgrounds $G_{\mu\nu}$, $B_{\mu\nu}$, Φ , ... define a $c = 26$ CFT will enforce on $G_{\mu\nu}$, $B_{\mu\nu}$, Φ (and possibly other fields), the string equations of motion and guarantee the existence of a Hilbert space of physical states. While in QFT we knew the equations of motion for ϕ_0 directly from the action, in string theory the equations on the background fields will be novel.

We finish this subsection by noticing that other useful information may be obtained from the Feynman rules. First, a configuration $G_{\mu\nu}$, $B_{\mu\nu}$, Φ ,... is a solution to the full (perturbative) quantum string equations if the sum of all tadpoles cancels to all orders. This is in analogy with QFT, where a solution to the full (perturbative) quantum fields equations is an extremum of the full effective action, so that the sum of all tadpole graphs cancels. Secondly, the presence of tachyons in the spectrum signals that the background around which the perturbative expansion was formulated is not a local minimum.

§6.2. Renormalization of generalized non-linear sigma models

Let M be a Riemannian manifold of dimension D, with metric $G_{\mu\nu}$, and fields $B_{\mu\nu}$, Φ defined in Lecture 1. The starting point for string dynamics in the background of these fields is the generalized non-linear sigma model action for the maps x: $\Sigma \to M$,

given by $S = S_{G,B} + S_\Phi$ and

$$
\begin{aligned}
S_{G,B}[x,g] &= \frac{1}{8\pi\ell^2}\int_\Sigma d\mu_g \partial_m x^\mu \partial_n x^\nu \{g^{mn} G_{\mu\nu}(x) + \varepsilon^{mn} B_{\mu\nu}(x)\} \\
S_\Phi[x,g] &= \frac{1}{2\pi}\int_\Sigma d\mu_g R_g \Phi(x)
\end{aligned}
\tag{6.2}
$$

Recall that the term involving $B_{\mu\nu}$ is the pullback of a two-form $B = B_{\mu\nu} dx^\mu \wedge dx^\nu$ under the map x, written out in coordinates. Here, ε^{mn} is the tensor defined by

$$
\varepsilon^{mn} \equiv \bar{\varepsilon}^{mn}/\sqrt{\det\, g_{pq}} \qquad\qquad \bar{\varepsilon}^{01} = -\bar{\varepsilon}^{10} = 1 \ . \tag{6.3}
$$

We have re-exhibited the dependence on the string tension κ, introduced in Lecture 1 via the parameter ℓ. It is standard notation to express κ in terms of the *Regge slope parameter* α'; we find it convenient to use the Planck length scale ℓ, instead, with the following relations:

$$
\kappa = \frac{1}{4\pi\alpha'} = \frac{1}{8\pi\ell^2} \ , \qquad\qquad \alpha' = 2\ell^2 \ , \qquad \ell > 0 \ . \tag{6.4}
$$

It is possible to include a further term in S corresponding to the dynamics of the tachyon state, to which we shall associate the field U:

$$
S_U[x,g] = \frac{1}{4\pi}\int_\Sigma d\mu_g U(x) \tag{6.5}
$$

Recall from Lecture 1 that the classical action S is invariant under

(*a*) $Diff(\Sigma)$

(*b*) $Diff(M)$

(*c*) $U(1)_B$ gauge invariance, acting by $B \to B + d\gamma$, for some $\gamma \in \Omega^{(1)}(M)$.

$Weyl(\Sigma)$ transformations on g leave $S_{G,B}$ invariant, but will develop anomalies upon quantization. S_Φ and S_U fail to be Weyl invariant, even classically.

We naturally have two notions of dimension in the problem: the worldsheet or Σ-dimension, and the space-time or M-dimension. The Σ-dimension of x, g, G, B, Φ and ℓ vanishes, while that of ∂ and R_g is 1 and 2, respectively. The M-dimension of x and ℓ is -1, and that of g, G, B, Φ vanishes.

We begin by discussing the renormalization of the quantum field theory defined by S, for general fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ. We shall not discuss here the effects of including the tachyon field in S. We shall see that in certain regularization schemes, such as dimensional regularization, its contribution can be ignored. Also, in the superstring, the tachyon will be absent anyway.

1) *Low Energy or α'-expansion.*.

The parameter ℓ is a very small length-scale (see Prob. 9), compared to scales accessible at present (whose order of magnitude we denote by L_{expt})

$$
\ell \sim 10^{-33}\text{cm} \qquad\qquad L_{\text{expt}} \sim 10^{-17}\text{cm} \ . \tag{6.6}
$$

Thus, we consider the quantum theory defined by S in an expansion in powers of ℓ (or equivalently in powers of α'), and we are guaranteed that this approximation will be reliable if we just retain the leading contributions. The $\alpha' = 2\ell^2$ expansion is an expansion in increasing powers of M-derivatives on the fields, hence the name *low-energy expansion.*

If it were only for $S_{G,B}$, the parameter ℓ^2 would play precisely the same rôle as $\hbar$ plays in quantum field theory, and the expansion is powers of ℓ would yield a loop expansion. S_Φ has a different dependence in ℓ though, as each factor of Φ effectively contributes an extra factor of ℓ^2, and this will rearrange the orders of the loop expansion. Perturbation theory may be carried out as usual by letting $x = x_0 + \ell y$ for some fixed reference configuration x_0. Expansion of the fields G, B, Φ, for example of

$$G_{\mu\nu}(x) = G_{\mu\nu}(x_0) + \ell\partial_\kappa G_{\mu\nu}(x_0)y^\kappa + \frac{1}{2}\ell^2\partial_\kappa\partial_\lambda G_{\mu\nu}(x_0)y^\kappa y^\lambda + \cdots \tag{6.7}$$

reveals that S is a quantum field theory with an infinite number of couplings. It is seen from (6.7) that the independent couplings of this quantum field theory correspond to the successive derivatives of the fields G, B and Φ at the expansion point x_0.

2) *Regularization schemes..*

It is possible to choose regularization schemes that preserve the classical symmetries $(Diff(\Sigma), Diff(M), U(1)_B)$ of S. We shall always assume that such a regulator is used, and give here two examples. It is of course not possible to choose a regularization scheme that also preserves $Weyl(\Sigma)$ invariance. This failure precisely results in the appearance of the Weyl anomaly, as was already discovered for flat M and vanishing B and Φ in Lecture 4.

a) Dimensional regularization consists in replacing the dimensionality of Σ by an arbitrary complex number $2 - \varepsilon$, and multiplying the action S by a factor of $\Lambda^{-\varepsilon}$. Here Λ is the dimensional regularization renormalization scale with Σ-dimension 1. Dimensional regularization has the advantage that it never generates quadratic divergences, so that if we start with $S_U = 0$ (no contribution of the tachyon to the action), then no tachyon dependence will ever be generated upon renormalization. Dimensional regularization has the disadvantage that the ϵ^{mn} tensor of (6.3) is specifically 2-dimensional, so that an additional ad hoc prescription for its regularization must be provided, and the results should be verified using the Ward identities to make sure that the ad hoc prescription is consistent with all symmetries.

b) Heat-kernel methods have already been used in the definition of determinants resulting from Gaussian quantum theories. They may also be used for QFT with non-trivial interactions by replacing its propagator by the heat-kernel regularization propagator. Let Δ_0 be the second order operator on the field y at x_0, and Δ_0^{-1} the associated propagator for y. We define the heat-kernel propagator by using a small real constant $\eta > 0$:

$$\Delta_\eta^{-1} = \Delta_0^{-1} \cdot e^{-\Delta_0\eta^2} \ . \tag{6.8}$$

3) *$Diff(M)$ and $U(1)_B$ invariant renormalization..*

Since x and y have zero Σ-dimension, all terms listed in (6.2) for S have Σ-dimension 2 and thus S is power counting renormalizable. Under the renormalization group flow, the change in each one of the infinite number of coupling constants (i.e. the Taylor coefficients of the expansion of G, B, Φ in powers of y as shown for G in (6.7)) will be given by a beta function associated with that coupling constant. An infinite number of renormalization conditions is required to specify the initial condition of the theory completely. These conditions amount to providing the functions $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ and $\Phi(x)$ at some reference renormalization scale, which we denote by Λ_0. For the metric, for example, the renormalization group equations will read:

$$\begin{cases} \Lambda \frac{d}{d\Lambda} G_{\mu\nu}(x) = \beta_{\mu\nu}(x) \\ \\ G_{\mu\nu}(x)\big|_{\Lambda_0} = G^0_{\mu\nu}(x) \end{cases} \tag{6.9}$$

Here $\beta_{\mu\nu}(x)$ depends on x only through the dependence of $\beta_{\mu\nu}$ on the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, as well as the derivatives of these fields.

Now, we impose upon the renormalization procedure that it preserve the $Diff(M)$ and $U(1)_B$ invariance of S. The existence of a regulator that preserves these symmetries guarantees that there will be no anomalies in these symmetries. This implies that $\beta_{\mu\nu}$ must be a $U(1)_B$-invariant, symmetric rank 2 $Diff(M)$ tensor. Similarly for the β-functions of the fields B and Φ.

We conclude with a remark on the nature of the renormalizability of the action S. Customarily, renormalizable QFT's are power counting renormalizable, but depend only upon a finite number of input parameters, which may be viewed as initial conditions on the renormalization group equations. In four dimensional QFT's of particle dynamics, the fact that only a finite number of input parameters is needed renders the theory physically predictive. Indeed, once those parameters have been determined (even approximately), the theory is specified and all physical quantities are in principle calculable. Here, however, the number of input parameters in the theory of action S is infinite. In string theory, this is precisely what is needed in order to be able to formulate string theory in infinite dimensional families of backgrounds, which will correspond to solutions to the string equations of motion.

4) *Irrelevance of Higher Mass Couplings..*

The fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ are associated with the graviton, anti-symmetric tensor and dilaton states in string theory, respectively. Those are precisely all the *massless states* of the bosonic string. A natural question arises as to whether fields associated with *massive string states* should also be included in S. In other words, we wish to find out whether S is the most general action possible for the dynamics of bosonic string theory.

Using the correspondence (established in Lecture 2) between variations in the background fields (such as $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ) and vertex operators for the associated physical states, we may infer the structure of possible contributions to S from massive states directly from the corresponding vertex operators. A vertex operator of a string state of squared mass $M^2 = -2 + 2p$ is of the form

$$V(\varepsilon, k) = \int_\Sigma d\mu_g \epsilon_{\mu_1 \dots \mu_{2p}}(k) P^{\mu_1 \dots \mu_{2p}}(\partial x, \partial^2 x, \dots ; k) e^{ik\cdot x} \tag{6.10}$$

where ϵ is a polarization tensor characterizing the string state and P is a polynomial of degree $2p$ in derivatives of x, so that P has Σ-dimension $2p$. Denoting the field associated with this state by $E_{\mu_1 \dots \mu_{2p}}(x)$, the contribution to the action S will take the form

$$S_E[x,g] = \int_\Sigma d\mu_g \, E_{\mu_1 \dots \mu_{2p}}(x) \widetilde{P}^{\mu_1 \dots \mu_{2p}}(\partial x, \partial^2 x, \dots) \ . \tag{6.11}$$

Here $\widetilde{P}$ is again a polynomial of worldsheet dimension $2p$. For $p = 0$, we recover the tachyon action, which we omit. For $p = 1$, we recover the action S, associated with massless string states.

For $p \geq 2$, S_E produces a non-renormalizable, or *irrelevant*, contribution to the worldsheet action. Addition of S_E to S yields a quantum field theory that can also be defined solely from the action S, but with modified background field values G, B, Φ. In other words, the family of renormalized QFT's defined by $S + S_E$ coincides with the family of renormalized QFT's defined solely by S.

This means that bosonic string theory is completely specified by the background fields associated with massless states only (we continue to ignore the tachyon). Once those have been specified, the dynamics of all massive states is a consequence of string dynamics and is completely determined. Of course, as argued in the beginning of this lecture, the condition of consistent string propagation will further limit the possible backgrounds to those yielding a conformal field theory of central charge $c = 26$.

§6.3. General structure of Weyl dependence

We now determine the general structure of the Weyl dependence of Green functions in the quantum field theory specified by the action S in terms of fixed background fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ. We consider unnormalized correlation functions of operators $\phi_1, \dots, \phi_n$, inserted at points $\xi_1, \dots, \xi_n \in \Sigma$.

In §6.5, we shall provide a concrete and useful reformulation of correlation functions in terms of the effective action, via the covariant background field method. From this reformulation, it will be immediate to establish in perturbation theory the properties that we shall use formally in this section and in the next one (§6.4).

Under a $Weyl(\Sigma)$ transformation $g \to e^{2\delta\sigma} g$, with support away from $\xi_1, \dots, \xi_n$, we have by definition of the stress tensor T_{mn}:

$$\delta_\sigma \langle \phi_1 \dots \phi_n \rangle_g = \frac{1}{2\pi} \int_\Sigma d\mu_g \delta\sigma \, \langle T_m{}^m \phi_1 \dots \phi_n \rangle_g \ . \tag{6.12}$$

We assume throughout that correlation functions are covariant under $Diff(\Sigma)$, so that

$$\nabla^m T_{mn} = 0 \ . \tag{6.13}$$

$T_m{}^m$ receives contributions from the explicit Weyl non-invariance of S_Φ, from Weyl anomalies, and from the interplay between non-invariance of S_Φ and Weyl anomalies. All such contributions to $T_m{}^m$ are *local functions of* x, (i.e., dependent on x and on finite order derivatives on x) and *polynomial in the derivatives of* x (see [II-**Dynamics of QFT**, §3.7]).

Collecting the various criteria, we find that $T_m{}^m$ has to be

1) local in x and polynomial in derivatives of x;
2) $Diff(\Sigma)$ scalar;
3) of Σ-dimension 2;
4) of M-dimension 0;
5) $Diff(M)$-invariant;
6) $U(1)_B$ invariant;
7) invariant under shifts in Φ by a constant.

Only the last point may need justification: a shift in Φ by a constant yields a shift in the action S by a term proportional to the Euler number, which is Weyl independent.

Making use of a combination of these criteria, we find that $T_m{}^m$ must involve precisely 2 derivatives on Σ, and must be determined in terms of 4 unknown local functions β^G, β^B, β^Φ and β^V of x:

$$T_m{}^m = \partial_m x^\mu \partial_n x^\nu \left(\beta^G_{\mu\nu}(x) g^{mn} + \beta^B_{\mu\nu}(x)\epsilon^{mn}\right) + \beta^\Phi(x) R_g + \beta^V_\mu(x) g^{mn} D^*_m \partial_n x^\mu \ . \tag{6.14}$$

Here, D^*_m is the covariant derivative on $T^*\Sigma \otimes TM$ pulled back to Σ by the map x, and is defined by*

$$D^*_m \partial_n x^\mu = \nabla_m \partial_n x^\mu + \Gamma^\mu_{\nu\rho} \partial_m x^\nu \partial_n x^\rho \ , \tag{6.15}$$

where ∇_m is the standard covariant derivative on $T^*\Sigma$, as defined in Lecture 2, and Γ is the affine connection of the metric $G_{\mu\nu}$. The functions β^G, β^B, β^Φ, β^V are called *beta functions*, even though in this context they govern a change of Weyl scale rather than a change of renormalization scale.

The structure of $T_m{}^m$ reveals a puzzle: the action S involves only the three fields G, B, Φ, but the Weyl variation of the correlation functions produces four beta functions. Indeed, there is an additional beta function β^V. If only *constant* Weyl transformations δ_σ are performed in (6.12), the term involving β^V may be integrated by parts and effectively absorbed into β^G. For general Weyl transformations, however, the term involving β^V is truly independent.

The resolution of this puzzle lies in the fact that upon performing a Weyl transformation on g, we are free to simultaneously transform x. On general grounds, the form of this transformation is given by $\delta_\sigma x^\mu = \ell^2 V^\mu(x)\delta\sigma$, where $V^\mu(x)$ is a vector field on M. We shall make it clear later on how to implement this additional effect of a Weyl transformation in actual calculations. Using the freedom to make this additional transformation, we see that $V^\mu(x)$ may be chosen so as to compensate for the β^V-function. The remaining β-functions $\beta^G_{\mu,\nu}$, $\beta^B_{\mu\nu}$, β^Φ then have precisely as many components as the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ whose Weyl variation they describe.

Assuming now that the β^V_μ term has been cancelled by a suitable transformation $\delta_\sigma x^\mu$, we have the most general expression for $T_m{}^m$ as follows (we continue to use the same notation for β's):

$$T_m{}^m = \partial_m x^\mu \partial_n x^\nu \left(\beta^G_{\mu\nu}(x) g^{mn} + \beta^B_{\mu\nu}(x)\varepsilon^{mn}\right) + \beta^\Phi(x) R_g \ . \tag{6.16}$$

*We shall denote the pullback under the map x of covariant derivatives D on TM by D^*.

(This phenomenon of augmenting Weyl transformations on g by further transformations on the fields was already encountered in Problem 1. There, we studied a special case of action S where $D = 1$, $G = 1$, $B = 0$, $\Phi = Qx$ and

$$S[x,g] = \frac{1}{4\pi}\int_\Sigma d\mu_g \left(\tfrac{1}{2}x\Delta_g x + QR_g x\right) \tag{6.17}$$

and discovered that upon a Weyl transformation, $\delta_\sigma x = Q\delta\sigma$.)

Classical contribution to $T_m{}^m$. .

The contribution to $T_m{}^m$ arising from the variation of the classical action with respect to $\delta_\sigma x^\mu$ is given by

$$T_m{}^m(\delta_\sigma x^\mu) = g^{mn}\mathcal{D}^*_m\partial_n x^\mu V^\nu(x)G_{\mu\nu}(x)\ . \tag{6.18}$$

To define the covariant derivative $\mathcal{D}^*_m$, we need the *torsion tensor*, $H_{\mu\nu\rho}$, which is the field strength of the field $B_{\mu\nu}$, given by $H = dB$, or in component notation

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\ . \tag{6.19}$$

Then, we have

$$\mathcal{D}^*_m\partial_n x^\mu \equiv D^*_m\partial_n x^\mu - \tfrac{1}{2}H^\mu_{\nu\rho}g_{mp}\epsilon^{qp}\partial_q x^\nu\partial_n x^\rho \tag{6.20}$$

so that $\mathcal{D}^*_m$ is a covariant derivative with respect to a connection with torsion $\Gamma + \frac{1}{2}H$ on TM.

It is now clear that a vector field $V^\mu(x)$ may be chosen so as to cancel β^V_μ in the expression for $T_m{}^m$ in (6.14). To lowest order in ℓ^2, we have

$$V^\mu(x)G_{\mu\nu}(x) = -\beta^V_\nu(x) + O(\ell^2)\ . \tag{6.21}$$

Notice that this cancellation of the β^V term produces modifications in the functions β^G and β^B and possibly also β^Φ.

§6.4. General structure of Weyl anomaly in low energy expansion

An exact computation of β^G, β^B and β^Φ is possible only if G, B and Φ exhibit a high degree of symmetry. (e.g. flat $G_{\mu\nu}$, $B_{\mu\nu} = \Phi = 0$.) For general G, B and Φ, only approximate expressions can be obtained. Fortunately, the constant ℓ is exceedingly small compared to L_{expt}, so an expansion in power of ℓ should be very reliable. On general grounds, the expansion of the functions β^G, β^B and β^Φ has the following structure:

$$\beta(x) = \sum_{p=0}^{\infty}\ell^{2p}\beta^{(p)}(x) \tag{6.22}$$

where the coefficients $\beta^{(p)}(x)$ are

1) local in x and functions of x, but not of derivatives of x;
2) independent of g;

3) of Σ-dimension 0;
4) of M-dimension $2+2p$ for β^G, β^B; $2p$ for β^Φ;
5) $Diff(M)$ tensors;
6) $U(1)_B$ invariant, dependent upon B only through $H = dB$;
7) invariant under shifts in Φ by a constant.

Expanding up to and including terms with two derivatives (total) on the fields G, B, Φ requires expansion up to order $p = 0$ for β^G, β^B and orders $p = 0, 2$ for β^Φ. Up to this order, we find the most general possible expression satisfying the above criteria:

$$\begin{aligned}
\beta^G_{\mu\nu} &= a_1 R^G_{\mu\nu} + a_2 G_{\mu\nu} + a_3 G_{\mu\nu} R^G + a_4 H_{\mu\rho\sigma} H^{\rho\sigma}_\nu + a_5 G_{\mu\nu} H_{\rho\sigma\tau} H^{\rho\sigma\tau} \\
&\quad + a_6 D_\mu D_\nu \Phi + a_7 G_{\mu\nu} D^2\Phi + a_8 G_{\mu\nu} D^\rho \Phi \, D_\rho \Phi \\
\beta^B_{\mu\nu} &= b_1 D^\kappa H_{\kappa\mu\nu} + b_2 (D^\kappa \Phi) H_{\kappa\mu\nu} \\
\beta^\Phi &= c_0 + \ell^2 \left\{ c_1 R^G + c_2 D^2\Phi + c_3 (D^\kappa \Phi) D_\kappa \Phi + c_4 H_{\rho\sigma\tau} H^{\rho\sigma\tau} \right\}
\end{aligned} \tag{6.23}$$

Here, R^G and $R^G_{\mu\nu}$ are the Ricci scalar and tensor of the metric G, indices are raised and lowered with the help of the metric G, and D_μ is the covariant derivative with respect to the affine connection Γ. On a vector field V^μ, D_μ acts by

$$D_\mu V^\kappa \equiv \partial_\mu V^\kappa + \Gamma^\kappa_{\mu\nu} V^\nu \ . \tag{6.24}$$

The coefficients $a_1, \ldots, a_8$, b_1, b_2, $c_0, c_1, \ldots, c_4$ are now the only remaining unknowns, and we shall determine them shortly.

It is very useful to investigate the ℓ-dependence of the graphs contributing to each coefficient. This counting is given by the usual loop expansion, except that each factor of Φ contributes a factor of ℓ^2 additionally. Within this counting, a_8 would arise from a graph with -1 loops, and thus must be absent: $a_8 = 0$. We have furthermore

$$\begin{aligned}
&\text{0-loops:} \quad a_6, a_7, b_2, c_3 \\
&\text{1-loop:} \quad a_1, \ldots, a_5, b_1, c_0, c_2 \\
&\text{2-loops:} \quad c_1, c_4 \ .
\end{aligned}$$

Two of these coefficients may be obtained from the Weyl anomaly calculation in flat $G_{\mu\nu}$, with $B_{\mu\nu} = \Phi = 0$. We have

- $a_2 = 0$: no field renormalization in free x-field theory;
- $c_0 = +D/6$: the central charge of D scalars x^μ.

(6.25)

The tree-level coefficients may be obtained from classical calculations, and are thus also easily gotten. One finds (see Problem 13)

- $a_6 = 1$, $\quad a_7 = 0$
- $b_2 = 1/2$.
- $c_3 = 2$.

(6.26)

It remains to obtain the 1-loop coefficients a_1, a_3, a_4, a_5, b_1, c_2 and the 2-loop coefficients c_1 and c_4. To do so, we use the background field method.

§6.5. Background field quantization method

The background field quantization method is a powerful tool for obtaining the effective action, which is the generating functional for 1-particle irreducible (1PI) Feynman diagrams, in terms of a functional integral. When local symmetries are present, the method may be organized in a manifestly covariant way, thereby greatly enhancing the effectiveness of perturbative expansion methods. We start by explaining the background field method for a general scalar field ϕ, and then adapt the method to the case of sigma models.

Let S be the classical action for the classical field ϕ, given in terms of classical coupling constants. We shall designate by coupling constants all the parameters entering the definition of S. Their dependence will not be exhibited. By replacing classical field and coupling constants by *bare* quantities, we obtain the *bare action* $S_0[\phi_0]$ in terms of the *bare field* ϕ_0 and *bare coupling constants.* We shall assume that the field ϕ_0 is multiplicatively renormalized by a field renormalization factor Z_ϕ, which depends only upon the coupling constants and the renormalization scale Λ, but not on the field ϕ. Let J be a source function. Then the generating functional $W[J]$, which is the generating functional for *renormalized connected Feynman diagrams*, is defined by

$$e^{W[J]} \equiv \int D\phi_0 \, e^{-S_0[\phi_0] + Z_\phi \int J\phi_0} \, . \tag{6.27}$$

Alternatively, the above functional integral may be expressed directly in terms of the *renormalized field* $\phi \equiv Z_\phi \phi_0$. It is standard to introduce the *renormalized action* $S_R[\phi]$ by

$$S_0[\phi \, Z_\phi^{-1}] \equiv S_R[\phi] \tag{6.28}$$

(S_R consists of the classical action, plus renormalization counterterms.) Then, we have

$$e^{W[J]} = \int D\phi \, e^{-S_R[\phi] + \int J\phi} \, . \tag{6.29}$$

The effective action for connected, 1PI renormalized Feynman diagrams is denoted by $\Gamma[\varphi]$ and is defined as the Legendre transform of $W[J]$:

$$\Gamma[\varphi] \equiv \int J\varphi - W[J] \qquad\qquad J(x) = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} \, . \tag{6.30}$$

Expressing $W[J]$ in terms of $\Gamma[\varphi]$, eliminating J, and shifting ϕ by φ in (6.29), we obtain the final formula for $\Gamma[\varphi]$:

$$e^{-\Gamma[\varphi]} = \int D\phi \, \exp\left\{ -S_R[\varphi + \phi] + \int \phi \frac{\delta\Gamma[\varphi]}{\delta\varphi} \right\} \tag{6.31}$$

At first sight, it would appear that we have gained little: the right hand side depends upon $\Gamma[\varphi]$ itself, and we have only obtained an implicit equation for $\Gamma[\varphi]$. However, a loop expansion (in powers of $\hbar$ in QFT, which we shall perform in terms of ℓ^2 here, for later use) reveals that the above expression yields a powerful recursive

equation for $\Gamma[\varphi]$. To see this, redefine $\phi \to \ell\phi$, and expand S_R and $\Gamma[\varphi]$ around the classical action. We make use of the fact that $\Gamma[\varphi]$ admits a Laurent expansion in powers of ℓ^2 (not just ℓ). We have

$$S_R[\varphi + \ell\phi] = \frac{1}{\ell^2} S[\varphi] + \frac{1}{\ell} \int \phi \frac{\delta S}{\delta \varphi}[\varphi] + \bar{S}[\varphi;\phi](\ell) \tag{6.32}$$

and

$$\Gamma[\varphi] = \frac{1}{\ell^2} S[\varphi] + \bar{\Gamma}[\varphi](\ell) \tag{6.33}$$

Here, $\bar{S}$ and $\bar{\Gamma}$ admit a Taylor series expansion in ℓ and ℓ^2, respectively. Substituting these general forms into (6.31), we find:

$$e^{-\bar{\Gamma}[\varphi](\ell)} = \int D\phi \exp\left\{ -\bar{S}[\varphi;\phi](\ell) + \ell \int \phi \frac{\delta \bar{\Gamma}[\varphi](\ell)}{\delta \varphi} \right\} \tag{6.34}$$

from which it is transparent that $\bar{\Gamma}$ is determined by a nice recursive equation. For example, to order ℓ^0, we have a standard formula for the 1-loop effective action

$$e^{-\bar{\Gamma}[\varphi](0)} = \int D\phi\, e^{-\bar{S}[\varphi;\phi](0)} \,. \tag{6.35}$$

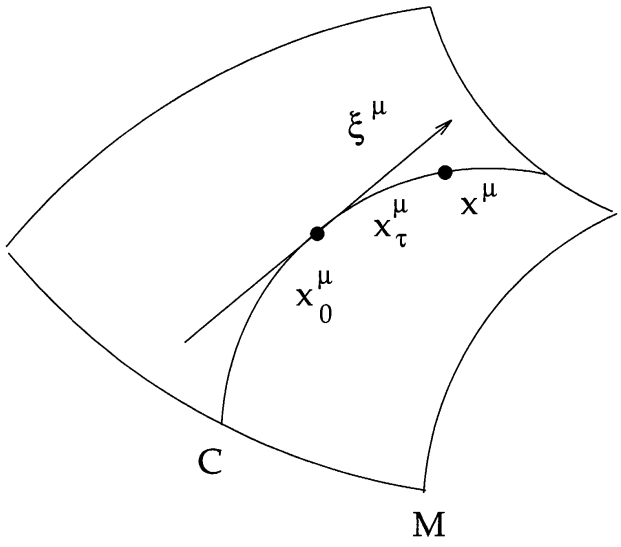
§6.6. Covariant expansion methods

Applying the background field method to the non-linear sigma model with action S, given in (6.2), the quantum field ϕ becomes the coordinate x^μ of a point on the Riemannian manifold M, while the classical field φ becomes a reference point ${x_0}^\mu$ on M. Addition of coordinates – as would apparently be needed in the background field method developed above – is not covariant under $Diff(M)$. It is beneficial to make use of the Riemann normal coordinate expansion instead, since this method yields $Diff(M)$ covariant results.

For x_0^μ and x^μ sufficiently close to one another, there is a unique shortest geodesic curve C, parametrized by x_τ^μ, that interpolates between x_0^μ for $\tau = 0$ and x^μ for $\tau = \ell$. The function x_τ^μ obeys the geodesic equation

$$\frac{D}{D\tau} \dot{x}_\tau^\mu = \ddot{x}_\tau^\mu + \Gamma^\mu_{\nu\rho}(x_\tau) \dot{x}_\tau^\nu \dot{x}_\tau^\rho = 0 \tag{6.36}$$

The tangent vector to C at $\tau = 0$ is a vector field $\xi^\mu \equiv \dot{x}_0^\mu$ and x^μ may be parametrized (in a neighborhood of x_0^μ) by x_0^μ and ξ^μ: $x^\mu = e^{\ell\xi} x_0^\mu$.



$Diff(M)$-tensors admit covariant expansions in powers of ℓ. For example, the expansion of rank 0 tensors (scalars, such as Φ) is given by

$$\Phi(x) = \Phi(x_0) + D_\kappa \Phi(x_0) \ell \xi^\kappa + \frac{1}{2} D_\kappa D_\lambda \Phi(x_0) \ell^2 \xi^\kappa \xi^\lambda + O(\ell^3). \tag{6.37}$$

while the expansion of rank 2 tensors (such as G, B) is given by

$$\begin{aligned} T_{\mu\nu}(x) = T_{\mu\nu}(x_0) + D_\kappa T_{\mu\nu}(x_0) \ell \xi^\kappa + \tfrac{1}{2}\{D_\kappa D_\lambda T_{\mu\nu}(x_0) \\ - \tfrac{1}{3} R^\rho_{\kappa\mu\lambda} T_{\rho\nu}(x_0) - \tfrac{1}{3} R^\rho_{\kappa\nu\lambda} T_{\mu\rho}(x_0)\} \ell^2 \xi^\kappa \xi^\lambda + O(\ell^3) \ . \end{aligned} \tag{6.38}$$

Here, D_κ is the covariant derivative with respect to the affine connection $\Gamma^\mu_{\nu\rho}$, and $R^\rho_{\kappa\mu\lambda}$ is the associated Riemann curvature.

As a result, the generalized non-linear sigma model action $S[x, g]$ of §6.2 may be similarly expanded around x_0, in a power series in ℓ, by setting $x = e^{\ell\xi} x_0$.

$$\begin{aligned} S[x, g] =& S[x_0, g] + \ell \int_\Sigma d\mu_g \, \xi^\mu \, S^{(1)}_\mu[x_0, g] + \bar{S}[x_0, \xi; g] \\ S^{(1)}_\mu[x_0, g] =& \frac{1}{\sqrt{\det g}} \frac{\delta S[e^{\xi'} x_0, g]}{\delta \xi'^\mu} \Big|_{\xi'=0} \end{aligned} \tag{6.39}$$

Here, $\bar{S}$ admits a Taylor series expansion in ℓ. Notice that S and $S^{(1)}_\mu$ are each the sum of a term of order ℓ^{-2} and a term of order ℓ^0 arising from the dilaton action S_Φ. The effective action $\Gamma[x_0, g]$ is now obtained just as in (6.34), except with the field φ replaced by x_0 and ϕ replaced by ξ.

$$e^{-\Gamma[x_0, g]} = \int D\xi^\mu \exp\left\{ -S[e^{\ell\xi} x_0, g] + \ell \int_\Sigma d\mu_g \xi^\mu \Gamma^{(1)}_\mu[x, g] \right\} \tag{6.40}$$

It will be understood throughout that S stands for the renormalized action, denoted by S_R previously. We have also used the definition

$$\Gamma^{(1)}_\mu[x_0, g] \equiv \frac{1}{\sqrt{\det g}} \frac{\delta \Gamma[e^{\xi'} x_0, g]}{\delta \xi'^\mu} \Big|_{\xi'=0} \ . \tag{6.41}$$

The functional measure $D\xi^\mu$ is defined through the L^2 norm on TM, which is given explicitly by

$$\|\xi^\mu\|^2 \equiv \int_\Sigma d\mu_g \, G_{\mu\nu}(x_0) \xi^\mu \xi^\nu \ . \tag{6.42}$$

Using again the property that $\Gamma[x_0, g]$ admits a Laurent expansion in terms of ℓ^2 rather than ℓ, we have

$$\Gamma[x_0, g] = S[x_0, g] + \bar{\Gamma}[x_0, g] \tag{6.43}$$

where $\bar{\Gamma}$ admits a Taylor series expansion in ℓ^2, and is given by

$$e^{-\bar{\Gamma}[x_0, g]} = \int D\xi^\mu \, e^{-\bar{S}[x, g] + \ell \int_\Sigma d\mu_g \xi^\mu \, \bar{\Gamma}^{(1)}_\mu[x_0, g]} \ . \tag{6.44}$$

Here, $\bar{S}$ is defined as in (6.39), and $\bar{\Gamma}^{(1)}_{\mu}$ is defined as in (6.41) by replacing Γ by $\bar{\Gamma}$.

It remains to list the explicit expansion terms that we shall need to evaluate β^G, β^B and β^Φ, up to order ℓ^0 in β^G and β^B and ℓ^0, ℓ^2 in β^Φ. They are given by

$$\bar{S} = \bar{S}_0 + \ell\,\bar{S}_1 + \ell^2 \bar{S}_2 + O(\ell^3) \tag{6.45}$$

where

(6.46)

$$\begin{aligned}
\bar{S}_0 &= \frac{1}{8\pi}\int_\Sigma d\mu_g \Big\{ \mathcal{D}^*_m \xi^\mu \mathcal{D}^*_n \xi^\nu G_{\mu\nu}(x_0) g^{mn} + \mathcal{R}_{\mu\nu\rho\sigma}(x_0)\partial_m x_0^\mu \partial_n x_0^\rho \xi^\nu \xi^\sigma (g^{mn} - \epsilon^{mn}) \Big\} \\
\bar{S}_1 &= \frac{1}{8\pi}\int_\Sigma d\mu_g \Big\{ \tfrac{1}{3} H_{\mu\nu\rho} \xi^\mu \mathcal{D}^*_m \xi^\nu \mathcal{D}^*_n \xi^\rho \epsilon^{mn} \Big\} \\
\bar{S}_2 &= \frac{1}{8\pi}\int_\Sigma d\mu_g \Big\{ \tfrac{1}{3} R_{\mu\nu\rho\sigma} \xi^\nu \xi^\rho \mathcal{D}^*_m \xi^\mu \mathcal{D}^*_n \xi^\sigma g^{mn} - \tfrac{1}{2} \mathcal{R}_{\mu\nu\rho\sigma} \xi^\nu \xi^\rho \mathcal{D}^*_m \xi^\mu \mathcal{D}^*_n \xi^\sigma \epsilon^{mn} \\
&\qquad + 2 D_\mu D_\nu \Phi(x_0) R_g \xi^\mu \xi^\nu \Big\} .
\end{aligned}$$

Here, the covariant derivative with torsion on TM and pulled back to Σ is defined by

$$\mathcal{D}^*_m \xi^\mu \equiv D^*_m \xi^\mu + \tfrac{1}{2} H^\sigma{}_{\nu\rho} g_{mp} \epsilon^{pq} \partial_q x^\nu \xi^\rho \ . \tag{6.47}$$

(This definition agrees with that of (6.20).) Furthermore, we have the following covariant derivatives with torsion on TM

$$\mathcal{D}_\mu \xi^\nu \equiv D_\mu \xi^\nu + \tfrac{1}{2} H^\sigma{}_{\mu\rho} \xi^\rho \tag{6.48}$$

whose structure relations yield the torsion H and the curvature tensor $\mathcal{R}$ of the connection with torsion $\Gamma + \frac{1}{2} H$:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\, \xi^\rho = H^\sigma{}_{\mu\nu} \mathcal{D}_\sigma \xi^\rho + \mathcal{R}^\rho_{\sigma\mu\nu} \xi^\sigma \ . \tag{6.49}$$

It is easy to express $\mathcal{R}$ in terms of the Riemann tensor R and the torsion tensor H:

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} &+ \tfrac{1}{2} D_\rho H_{\sigma\mu\nu} - \tfrac{1}{2} D_\sigma H_{\rho\mu\nu} \\
&+ \tfrac{1}{4} H_{\rho\mu\alpha} H_{\sigma\nu}{}^\alpha - \tfrac{1}{4} H_{\sigma\mu\alpha} H_{\rho\nu}{}^\alpha
\end{aligned} \tag{6.50}$$

§6.7. Reformulation as an $\mathrm{SO}(1, D-1)$ gauge theory

The difficulty encountered in the above formulation is that the quadratic form representing the kinetic term for the field ξ^μ in (6.46) depends in a non-trivial way on the metric $G_{\mu\nu}$ and on the background field x_0^μ. As a result, perturbation theory will involve determining the ξ-propagator in arbitrary x_0 background, which is an impossible task. One may get around this complication by changing variables and absorbing the $G_{\mu\nu}$ and x_0^μ dependence into the new field.

One introduces an orthonormal frame, which we denote by $e_\mu{}^a$, with $a = 0, 1, \ldots, D-1$ representing the tangent space direction of TM. We have, by definition

$$G_{\mu\nu} = e_\mu{}^a\, e_\nu{}^b\, \eta_{ab} \tag{6.51}$$

where $\eta_{ab} = \mathrm{diag}(- + \cdots +)$ is the $\mathrm{SO}(1, D-1)$ invariant flat metric. The inverse frame is denoted by $e_a{}^\mu$, so that $e_a{}^\mu\, e_\mu{}^b = \delta_a{}^b$ and $e_\mu{}^a\, e_a{}^\nu = \delta_\mu{}^\nu$. We also introduce the $\mathrm{SO}(1, D-1)$-valued connection ω_μ, with components $\omega_\mu{}^a{}_b$, such that under the covariant derivative with respect to the affine connection Γ and the spin connection ω_μ, the frame $e_\mu{}^a$ is covariantly constant

$$D_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - \Gamma^\kappa_{\mu\nu} e_\kappa{}^a + \omega_\mu{}^a{}_b\, e_\nu{}^b = 0 \tag{6.52}$$

The covariant derivative $\mathcal{D}$, including the torsion connection $\frac{1}{2}H$, is similarly defined. For example, on $e_\nu{}^a$, we have

$$\mathcal{D}_\mu e_\nu{}^a = D_\mu e_\nu{}^a + \frac{1}{2} H_{\mu\nu}{}^\rho e_\rho{}^a \tag{6.53}$$

We now rewrite all tensors with respect to the orthonormal frame basis. Thus, we have

$$\begin{aligned} H_{\mu\nu\rho} &= e_\mu{}^a e_\nu{}^b e_\rho{}^c H_{abc} \\ \mathcal{R}_{\mu\nu\rho\sigma} &= e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d \mathcal{R}_{abcd} \\ \xi^\mu &= \xi^a e_a{}^\mu \\ \mathcal{D}_a &= e_a{}^\mu \mathcal{D}_\mu \end{aligned} \tag{6.54}$$

where all tensors are evaluated at the same point on M, which in our background calculation will be $x_0{}^\mu$. We also introduce the pullback of the tangent vector $\partial_m x_0^\mu$, expressed in orthonormal frame basis:

$$e^*_m{}^a \equiv \partial_m x_0{}^\mu e_\mu{}^a(x_0) \tag{6.55}$$

The expression for the quantum contributions to the effective action $\bar{\Gamma}[x_0, g]$ of (6.44) is now

$$e^{-\Gamma[x_0,g]} = \int D\xi^a \exp\left\{ -\bar{S}[e^{\ell\xi} x_0, g] + \ell \int_\Sigma d\mu_g \xi^a \Gamma^{(1)}_a \right\} \tag{6.56}$$

The functional measure $D\xi^a$ is again defined with respect to the L^2 norm

$$\|\xi^a\|^2 = \int_\Sigma d\mu_g \xi^a \xi^b \eta_{ab} \tag{6.57}$$

which does not depend upon x_0 or $G_{\mu\nu}$ anymore. The action $\bar{S}$ may be expanded, just as in (6.46), and we obtain

(6.58a)
$$\bar{S}_0 = \frac{1}{8\pi} \int_\Sigma d\mu_g \{ \mathcal{D}^*_m \xi^a \mathcal{D}^*_n \xi^b \eta_{ab} g^{mn} + \mathcal{R}_{abcd}\, e^*_m{}^a e^*_n{}^c \xi^b \xi^d (g^{mn} - \varepsilon^{mn}) \}$$

(6.58b)
$$\bar{S}_1 = \frac{1}{8\pi} \int_\Sigma d\mu_g \{ \tfrac{1}{3} H_{abc} \xi^a \mathcal{D}^*_m \xi^b \mathcal{D}^*_n \xi^c \epsilon^{mn} \}$$

(6.58c)
$$\begin{aligned} \bar{S}_2 = \frac{1}{8\pi} \int_\Sigma d\mu_g \{ & \tfrac{1}{3} (\mathcal{R}_{abcd} - \tfrac{1}{4} H_{caf} H_{db}{}^f) \xi^b \xi^c \mathcal{D}^*_m \xi^a \mathcal{D}^*_n \xi^d g^{mn} \\ & - \tfrac{1}{2} \mathcal{R}_{abcd} \xi^b \xi^c \mathcal{D}^*_m \xi^a \mathcal{D}^*_n \xi^d \varepsilon^{mn} + 2 \mathcal{D}_a \mathcal{D}_b \Phi\, R_g\, \xi^a \xi^b \} \,. \end{aligned}$$

The pullback covariant derivative is now very simple:

$$\mathcal{D}^*_m \xi^a = \nabla_m \xi^a + A^a_{mb}\, \xi^b \tag{6.59}$$

The effective $SO(1, D-1)$ gauge field A_m is defined by the pullback of the spin connection ω_μ:

$$A^a_{mb}(\xi) \equiv \partial_m x_0^\mu(\xi) \omega_\mu{}^a{}_b \tag{6.60}$$

In particular, the kinetic term for the ξ^a-field is now also independent of the metric $G_{\mu\nu}$ and the background field ${x_0}^\mu$. As a result, standard perturbation methods in ℓ may be used to evaluate the effective action $\Gamma[x_0, g]$, with a standard propagation for ξ^a.

§6.8. Weyl variation of the effective action

The definition of the trace of the stress tensor for Green functions introduced in (6.12) may be translated into an equation for the Weyl variation of the effective action, $\Gamma[x_0, g]$. One finds

$$\delta_\sigma \Gamma[x_0, g] = -\frac{1}{2\pi} \int_\Sigma d\mu_g\, \delta\sigma\, T_m{}^m \ . \tag{6.61}$$

where $T_m{}^m$ denotes the expectation value of the operator $T_m{}^m$ in the background field x_0, and metric g. The derivation of (6.61) from (6.12) is given in Appendix A. It is understood that x_0 is also allowed to transform under $Weyl(\Sigma)$, as explained in §6.3. The arguments developed in §6.3 for what the most general structure of $T_m{}^m$ is carry over here, and we find, for a suitable $Weyl(\Sigma)$ action on x_0:

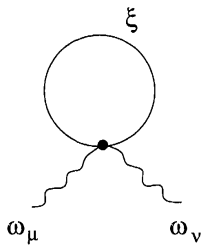
$$T_m{}^m = \partial_m x_0^\mu \partial_n x_0^\nu (\beta^G_{\mu\nu} g^{mn} + \beta^B_{\mu\nu} \epsilon^{mn}) + \beta^\Phi R_g \tag{6.62}$$

where β^G, β^B and β^Φ are evaluated at x_0. This definition of $T_m{}^m$ and β^G, β^B, and β^Φ is now completely precise, and these quantities may be directly evaluated from $\Gamma[x_0, g]$. Notice that the variation $\delta_\sigma \Gamma$ obtained by substituting (6.62) into (6.61) must satisfy an integrability condition, known as the Wess-Zumino consistency condition. This condition provides a relation between β^G, β^B and β^Φ, which we shall discuss in §6.9.

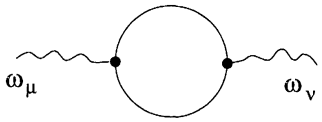
The effective action $\Gamma[x_0, g]$ is the sum of the classical action $S[x_0, g]$ and the quantum corrections $\bar{\Gamma}[x_0, g]$. The contribution of S to β^G, β^B and β^Φ was already evaluated previously (Problem 13). It remains to evaluate the Weyl transformation of $\bar{\Gamma}$ to the order we are computing. We begin by evaluating the $O(\ell^0)$ contribution, which arises solely from $\bar{S}_0$. We shall treat the contributions to $T_m{}^m$ perturbatively in the number of M-derivatives applied to the fields G, B and Φ. This expansion is reliable since we know from general considerations that we should only retain contributions to $T_m{}^m$ with two derivatives. For example, the $\mathcal{R}$-term in $\bar{S}_0$ can contribute at most to first order.

The form of the first term in $\bar{S}_0$ is that of a gauge field – the spin connection ω_μ – minimally coupled to ξ^a. By power counting its one-loop graphs with one

insertion of $\omega_\mu\,\omega_\nu$ and with two insertions of $\omega_\mu\,\partial_\nu$ are the only ones that could potentially contribute to the Weyl anomaly.



graph $\omega_\mu\omega_\nu$



graph $\omega_\mu\partial_\nu$

All other graphs are absolutely convergent and obey anomaly-free Ward identities. By gauge invariance, however, the degree of divergence is lowered by 2 and the sum of the two graphs does not contribute to the Weyl anomaly either (see also [II-**CFT**]). Thus, the only contribution from the first term in $\bar{S}_0$ is to the central charge: $+D/6$ contribution to β^Φ, which we have already discussed previously by fixing $c_0 = +D/6$.

It remains to evaluate the contributions from the first order expansion of the $\mathcal{R}$-term in $\bar{S}_0$ to β^G and β^B. We directly use the Ansatz for these functions in terms of G, B, Φ: we denote their contributions to this order by $\tilde{\beta}^G$, $\tilde{\beta}^B$, $\tilde{\beta}^\Phi$. Clearly $\tilde{\beta}^\Phi = 0$, and the Weyl transformation of this part is given by

$$\begin{aligned} &-\frac{1}{2\pi}\int_\Sigma d\mu_g\,\delta\sigma\,\partial_m x_0^\mu \partial_n x_0^\nu \left(\tilde{\beta}^G_{\mu\nu}(x_0) g^{mn} + \tilde{\beta}^B_{\mu\nu}(x_0)\epsilon^{mn}\right) \\ &= \delta_\sigma\left\langle -\frac{1}{8\pi}\int_\Sigma d\mu_g\,\mathcal{R}_{\mu a\rho b}(x_0)\partial_m x_0^\mu \partial_n x_0^\rho \xi^a\xi^b (g^{mn} - \epsilon^{mn})\right\rangle\,, \end{aligned} \tag{6.63}$$

where $\langle\quad\rangle$ stands for the normalized correlation function taken with respect to the action given by only the first term in $\bar{S}_0$. But, since x_0 was arbitrary throughout, we may identify the symmetric and antisymmetric tensor parts:

$$\begin{aligned} \tilde{\beta}^G_{\mu\nu}(x_0)\delta\sigma &= \delta_\sigma\left\langle \xi^a\xi^b\right\rangle \frac{1}{8}(\mathcal{R}_{\mu a\nu b} + \mathcal{R}_{\nu a\mu b}) \\ \tilde{\beta}^B_{\mu\nu}(x_0)\delta\sigma &= -\delta_\sigma\left\langle \xi^a\xi^b\right\rangle \frac{1}{8}(\mathcal{R}_{\mu a\nu b} - \mathcal{R}_{\nu a\mu b}) \end{aligned} \tag{6.64}$$

The correlation function of ξ's to this order is independent of x_0, and we we find

$$\delta_\sigma\left\langle \xi^a\xi^b\right\rangle = 2\delta\sigma\eta^{ab}\,. \tag{6.65}$$

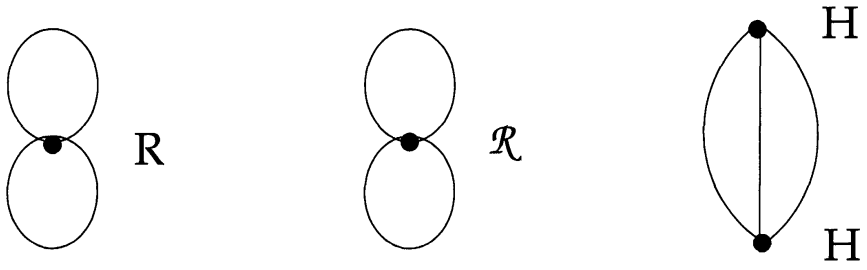
Making use of the expression for $\mathcal{R}$ in terms of R^G and H of (6.50), we find

$$\begin{aligned} \tilde{\beta}^G_{\mu\nu} &= \tfrac{1}{2}R^G_{\mu\nu} - \tfrac{1}{8}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} \\ \tilde{\beta}^B_{\mu\nu} &= -\tfrac{1}{4}D^\alpha H_{\alpha\mu\nu}\,. \end{aligned} \tag{6.66}$$

We now combine this result with the one obtained to tree level: this yields the full β-functions β^G, β^B to order ℓ^0; we also collect the results obtained thus far for β^Φ:

$$\begin{aligned} \beta^G_{\mu\nu} &= \tfrac{1}{2}R^G_{\mu\nu} - \tfrac{1}{8}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} + D_\mu D_\nu\Phi \\ \beta^B_{\mu\nu} &= -\tfrac{1}{4}D^\alpha H_{\alpha\mu\nu} + \tfrac{1}{2}D^\alpha\Phi\, H_{\alpha\mu\nu} \\ \beta^\Phi &= D/6 + \ell^2\{2D_\alpha\Phi D^\alpha\Phi - 2D_\alpha D^\alpha\Phi + c_1 R^G + c_4 H^2\}\,. \end{aligned} \tag{6.67}$$

Here, we have included the coefficient of $D_\alpha D^\alpha \Phi$, which is obtained from a simple 1-loop calculation from the last term in $\bar{S}_2$. The calculation of c_1, c_4 requires a full 2-loop computation, with the following Feynman diagram contributions:



The result is

$$c_1 = -\frac{1}{2}; \qquad c_4 = \frac{1}{24} \,. \tag{6.68}$$

§6.9. Low energy string field equations and string effective action

The string field equations are obtained by including also the effects of the Faddeev-Popov ghosts b, c; this effect is limited to an addition of $-13/3$ to β^Φ. We then obtain the following equations for string dynamics:

$$\begin{cases} \beta^G_{\mu\nu} = \frac{1}{2} R^G_{\mu\nu} - \frac{1}{8} H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} + D_\mu D_\nu \Phi = 0 \\ \\ \beta^B_{\mu\nu} = -\frac{1}{4} D^\alpha H_{\alpha\mu\nu} + \frac{1}{2} D^\alpha \Phi \, H_{\alpha\mu\nu} = 0 \\ \\ \beta^\Phi = (D-26)/6 + \ell^2 \{ 2 D_\alpha \Phi \, D^\alpha \Phi - 2 D_\alpha D^\alpha \Phi - \frac{1}{2} R^G + \frac{1}{24} H^2 \} = 0 \,. \end{cases} \tag{6.69}$$

to leading order in the expansion in powers of derivatives on G, B and Φ.

There is an interesting consistency condition, whose origin may be understood as follows. On a (locally) flat worldsheet, conformal invariance requires only that $\beta^G = \beta^B = 0$. One expects that these conditions should suffice to make the model conformal on non-flat worldsheets as well. This in turn suggests that the equation $\beta^\Phi = 0$ is not independent of the equations $\beta^G = \beta^B = 0$. In fact, this interdependence may indeed be established with the help of the Bianchi identities:

$$\begin{aligned} D^\mu R^G_{\mu\nu} &= \tfrac{1}{2} D_\nu \, R^G \\ D^\mu (H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta}) &= D^\mu H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} + \tfrac{1}{6} D_\nu H^2 \,, \end{aligned} \tag{6.70}$$

where $H^2 \equiv H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}$. The following relation is then found to hold to this order in powers of ℓ:

$$D^\mu \beta^G_{\mu\nu} = -\frac{1}{2\ell^2} D_\nu \beta^\Phi + \tfrac{1}{2} \beta^B_{\alpha\beta} H_\nu{}^{\alpha\beta} + 2 \beta^G_{\mu\nu} D^\mu \Phi \,, \tag{6.71}$$

so that the equations $\beta^G = \beta^B = 0$ imply that β^Φ is constant. We see that it suffices indeed to render the quantum field theory of the x-field conformal on a

(locally) flat worldsheet. The only quantity left undetermined is then the central charge. There are arguments in the physics literature that such an equation for β^Φ holds to all orders in ℓ.

The only way to satisfy $\beta^\Phi = 0$, given that D must be an integer, and that ℓ is an expansion parameter, is to set $D = 26$.

The above equations can be put in a more standard form, which reflects the fact that $B_{\mu\nu}$ and Φ may be viewed as matter fields, coupling to the metric $G_{\mu\nu}$ by their energy-momentum tensor. Alternatively, the above field equations are seen to derive from an action

$$I(G,B,\Phi) = \frac{1}{2\kappa^2}\int d^{26}x\sqrt{\det G}\; e^{-2\Phi}\Big\{R^G + 4D_\mu\Phi D^\mu\Phi - \frac{1}{12}H^2\Big\}\,, \tag{6.72}$$

provided the dimension is critical, $D = 26$. The way the dilaton coupling enters exposes the fact that $e^{2\Phi_0}$ is the string loop expansion parameter, or string coupling constant, when the dilaton vacuum expectation value is Φ_0. The presence of the factor $e^{-2\Phi_0}$ reveals that this effective action arises from string tree level effects. (One-loop contributions would have no exponential Φ dependence.) It is possible to put the action in a more standard form, by performing a Weyl transformation on G. (For later use, we consider arbitrary dimension D; our results will then have application also to superstring theory, where $D = 10$.)

$$G_{\mu\nu} \to G'_{\mu\nu} = e^{-4\Phi/(D-2)}G_{\mu\nu}\,. \tag{6.73}$$

The action in this metric assumes the form

$$I(G',B,\Phi) = \frac{1}{2\kappa^2}\int d^Dx\sqrt{\det G'}\,\Big\{R^{G'} - \frac{4}{D-2}D_\mu\Phi D^\mu\Phi - \frac{1}{12}e^{-8\Phi/(D-2)}H^2\Big\}\,. \tag{6.74}$$

The metrics G and G' are usually called the *string metric* and the *Einstein metric*, respectively.

The action $I(G',B,\Phi)$ is in fact well-known in field theory. If we ignore the peculiar value of the $D = 26$ dimension, and arbitrarily set $D = 10$ instead, we recover the action for the bosonic part of the $N = 1$ supergravity multiplet in $D = 10$ which is part of the so-called Chapline-Manton action (see Lecture 10). When we deal will with superstrings, we shall see that the superstring calculation to this order yields the same as the bosonic string, except for the dimension!

Finally, we point out that the field equations (6.69) are consistent with the low energy limit of transition amplitudes of strings, scattering in flat Minkowski space-time. The simplest example of such a low energy limit was encountered in Lecture 2, where we argued that the transversality conditions on the graviton polarization tensor required conditions that arise as the low energy limit of the linearized limit of $R^G_{\mu\nu} = 0$. In fact, string scattering amplitudes may be used directly to compute the string field equations. Unfortunately, it is difficult to compute the effective action directly from string transition amplitudes, since the latter are always evaluated on-shell only.

§6.10. A first look at compactification

One of the most remarkable results of string theory is that the dimension of space-time (at long distances) is dynamically determined: $D = 26$ for bosonic strings and

$D = 10$ for superstrings in flat (or nearly flat) space-time. The nagging aspect of this result, however, is that the number of dimensions we observe is only 4. Well, at least 10 is *larger* than 4!

In attempts at unifying general relativity and Maxwell's electromagnetism, Kaluza and Klein proposed in the 1920's that space-time could really be of higher dimension. And that as long as space-time effectively appears to be 4-dimensional at all length scales where present day experiments can probe it, the extra dimensions would not be directly observable. This scheme is usually termed Kaluza-Klein theory.

The most straightforward way to realize such unobservable extra dimensions is to consider space-times M of the form $M = \mathbb{R}^4 \times K$. Here, $\dim K = \dim M - 4$, and K is a compact manifold, whose (largest) size will be denoted by R_K. As long as $R_K \ll L_{\text{expt}}$, the K manifold should not be directly observable. More generally, we may consider M to be of the form $M = M_4 \times K$, where M_4 is approximately Minkowskian (such as would be the case for our universe, where M_4 is a manifold curved by the presence of matter).

The low energy approximation to string theory yields information on the possible solutions for the compact manifold K, provided the size R_K of K is much larger than the Planck scale:

$$R_K \gg \ell = \sqrt{\alpha'/2} \tag{6.75}$$

In this case the approximation in which higher powers of ℓ are neglected is reliable. The string field equations (6.69) restrict the fields G, B and Φ that are allowed on K, and thus restrict K.

If it is assumed that space-time supersymmetry remains partially unbroken at distance scales of order R_K, then K must be a Kähler manifold, and Φ must be constant. A particular solution corresponds to $H = 0$, so that K must be Kähler and Ricci flat. For superstrings, where the critical dimension is $D = 10$, the combination of these conditions amounts to demanding that K be a Calabi–Yau manifold of complex dimension 3.

Appendix A

The correlation functions in (6.12) are defined by

$$\begin{aligned}\langle \phi_1 \dots \phi_n \rangle_g &= \int D\phi\, \phi_1 \dots \phi_n\, e^{-S[\phi,g]} \\ &= \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n}\, e^{W[J,g]}\Big|_{J=0}\end{aligned}$$

Here, we consider correlation functions of a canonical field ϕ, which in our case is just the map x or the field ξ and J is some function, just as in (6.27). We now perform an infinitesimal Weyl rescaling δ_σ, whose support does not contain the points where the operators $\phi_1 \dots \phi_n$ are applied. Thus the Weyl variation becomes:

$$\delta_\sigma \langle \phi_1 \dots \phi_n \rangle_g = \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n}\, \delta_\sigma W[J,g]\, e^{-W[J,g]}\Big|_{J=0}$$

and using the definition of the effective action by Legendre transform

$$\delta_\sigma W[J,g]\big|_{J=0} = -\delta_\sigma \Gamma[\varphi,g]\big|_{\varphi}$$

where φ is such that $J = 0$. Equation (6.61) follows immediately.

References and cross references

A very clear exposition is given in the original work on the subject,

D. H. Friedan, *Non-linear models in* $2 + \epsilon$ *dimensions*, Ann. Physics **163** (1985), 318.

L. Alvarez-Gaumé and D. Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, Comm. Math. Phys. **80** (1980), 443.

T. L. Curtright, L. Mezincescu and C. K. Zachos, *Geometrostasis and torsion in covariant superstrings*, Phys. Lett. B **161** (1985), 79.

The further inclusion of the dilaton is discussed in

C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Strings in background fields*, Nucl. Phys. B **262** (1985), 593.

LECTURE 7
Free Superstrings

Purely bosonic string theory, as defined and studied in the preceding lectures, has a number of fundamental flaws, which we shall now briefly summarize.

(i) Bosonic string theory in flat – or nearly flat – space-time possesses a tachyonic state. The presence of a tachyon leads to a violation of the physical principle of causality. It signals an instability of (nearly) flat space-time, which may be detected from the fact that the 1-loop order dilaton tadpole amplitude is non-zero (in fact it is infinite). Ultimately, the presence of the tachyon leads to the divergence of transition amplitudes beyond tree level. Thus, bosonic string theory in (nearly) flat space-time is in itself inconsistent.

(ii) Bosonic string theory does not possess any fermionic states (i.e. transforming under a spinor representation of the space-time Lorentz group). This fact by itself does not a priori lead to an inconsistency of the bosonic string. Rather, since Nature clearly displays fermionic states (electrons, quarks, etc.), any viable physical theory aimed at encompassing all physical particles in Nature, as well as their interactions, must contain fermions. From this point of view, the bosonic string cannot be a viable theory of Nature.

(iii) In addition, it is desirable to have a theory of closed strings only that contains Yang–Mills gauge states. (We have shown already how Yang–Mills particles arise in open string theories.)

(iv) Finally, we do not perceive 26 dimensions of space-time, but only 4, at distance scales accessible to present day experiments. Thus, it must be understood why and how the remaining dimensions have remained invisible.

Problems (i), (ii) and in some cases also (iii) may all be resolved by introducing certain additional dynamical degrees of freedom on the string worldsheet. In particular, extra degrees of freedom may be introduced that will result in fermionic string states in the physical Hilbert space. This extension will in general modify the critical dimension D of space-time.

There are several ways in which one can proceed to include fermionic string states in the Hilbert space of string states.

(1) *The Green-Schwarz (GS) formulation.*

The theory has space-time vector fields x^μ, transforming under the vector representation of $\mathrm{SO}(1, D-1)$ as well as space-time spinor fields θ^α transforming

under a spinor representation of $\mathrm{Spin}(1, D-1)$. The ground state (i.e. lowest mass state of a single string) is in general degenerate, with both vector and spinor states. Bosonic and fermionic states are obtained by applying x^μ or θ^α to the ground states.

(2) *The Neveu-Schwarz-Ramond (RNS) formulation.*

The fundamental fields in the theory are space-time vectors x^μ, ψ^μ. There are two sectors in the Hilbert space of states, built on two distinct ground states: The Neveu-Schwarz ground state is a boson, while the Ramond ground state is a fermion. All bosonic states are obtained by applying x^μ and ψ^μ to the Neveu-Schwarz ground state; this generates the Neveu-Schwarz sector of the theory. All fermionic states are obtained by applying x^μ and ψ^μ to the fermionic ground state; this generates the Ramond sector of the theory.

By quantizing each formulation in the lightcone gauge, one demonstrates that (under certain conditions) the GS and RNS formulations are equivalent. The Green-Schwarz formulation appears difficult (if not impossible) to quantize in a manifestly Lorentz covariant way.

One distinguishes the following five, consistent, tachyon free, space-time supersymmetric string theories in flat Minkowski space-time.

• Type I	Open and closed unoriented strings, with Yang–Mills degrees of freedom via Chan–Paton rules: SO(32) gauge group only.
• Type II A,B	Closed oriented strings only; A: opposite chirality Ramond ground states B: same chirality Ramond ground states.
• Heterotic $\mathrm{Spin}(32)/Z_2$, $E_8 \times E_8$	Type II left-movers, bosonic right movers, including gauge degrees of freedom: $\mathrm{Spin}(32)/Z_2$ or $E_8 \times E_8$ gauge groups only.

The goal of this lecture (Lecture 7) and the next one (Lecture 8) is to define these theories, show their uniqueness and analyze the Hilbert space of states, in the RNS formulation. Space-time supersymmetry and the GS formulation will be discussed extensively in Lecture 10.

§7.1. Degrees of freedom of the RNS string

To obtain fermionic physical states, we add further degrees of freedom which are *spinors on the worldsheet* Σ. We begin by focusing on closed oriented worldsheets, without boundary.

Worldsheet spinors exist on Σ provided a spin bundle $S = K^{1/2}$ can be constructed as the square root of the canonical bundle K. The worldsheet spinor fields are then sections of S. Spin bundles exist on Σ provided the second Stiefel-Whitney class $H^2(\Sigma, Z_2)$, which is the first Chern class $\bmod 2$ of K, vanishes. This is indeed the case, since $c_1(K) = -\chi(\Sigma) = 2h - 2$, which is even. The different spin bundles are labeled by $H^1(\Sigma, Z_2)$ and may be distinguished by $\pm$ sign assignments around

the homology cycles of Σ. For compact Σ without boundary, there are 2^{2h} different spin structures.

Following the introductory remarks, the worldsheet spinors transform as Lorentz $SO(1, D-1)$ vectors. We denote them by $\psi^\mu_\pm(z, \bar{z})$, where the subscript $\pm$ is the worldsheet spinor index. In addition, ψ^μ_+ and ψ^μ_- are to be constrained by a requirement inherited from the Minkowski signature worldsheet: each is a worldsheet Majorana-Weyl spinor, which means that each is a one-component real spinor in a suitable basis.

On worldsheets with Euclidean signature, there are no Majorana-Weyl spinors. However, one may use a complex Weyl spinor ψ^μ_+, and take ψ^μ_- to be its complex conjugate. This construction preserves the number of degrees of freedom. Chiral splitting techniques, relying on the holomorphic dependence on moduli – as introduced and applied in Lecture 5 – may then be invoked to separate correlation functions into parts arising from ψ_+ and ψ_- separately. Thus, henceforth, ψ^μ_+ is a complex Weyl spinor, which is a section of a spin bundle $K^{1/2}$ for some given spin structure; ψ^μ_- is the complex conjugate of ψ^μ_+, and a section of $\bar{K}^{1/2}$ for the same spin structure.

It is natural (but, as we shall find later, it is incomplete) to let the dynamics of $\psi^\mu_\pm$ be governed by the action for massless spinors obtained from the bc system of Lecture 4 for $n = 1/2$. For $n = 1/2$, the bc system decomposes into two identical spinor systems in a conformally invariant way:

$$b = \frac{1}{\sqrt{2}}(\psi_+ + i\psi'_+) \qquad c = \frac{1}{\sqrt{2}}(\psi_+ - i\psi'_+) \ . \tag{7.1}$$

The action for $\psi^\mu_\pm$ is then found to be

$$S_\psi[\psi] = \frac{1}{4\pi}\int_\Sigma d\mu_g \left(\psi^\mu_+ \nabla^z_{(1/2)}\psi_{+\mu} + \psi^\mu_- \nabla^{\bar{z}}_{(-1/2)}\psi_{-\mu}\right) \ . \tag{7.2}$$

It describes a conformal field theory with central charge $D \times \frac{1}{2} = \frac{1}{2}D$, namely $c = 1/2$ for each Majorana fermion. We record here the stress tensor for this system:

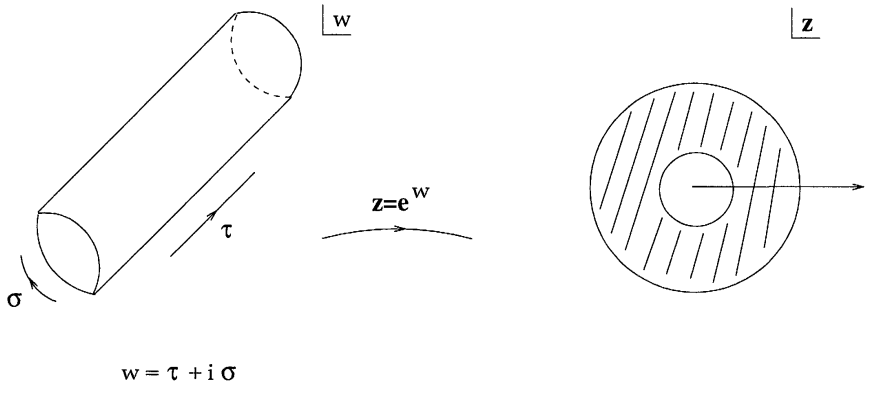
$$T^{(\psi)}_{zz} = \tfrac{1}{2}\partial_z \psi^\mu_+ \psi_{+\mu} \tag{7.3}$$

as well as the field equations and the OPE (both can be deduced from the bc system)

$$\nabla^z \psi^\mu_+ = 0\,; \qquad \psi^\mu_+(z)\psi^\nu_+(w) \sim \frac{1}{z-w}\,\eta^{\mu\nu} \tag{7.4}$$

and analogously for ψ^μ_-.

We now specialize to the case of a flat cylindrical worldsheet, or by conformal mapping the annular worldsheet Σ.



First of all, this topology supports a single homology cycle, and there are two different spin bundles, corresponding to $\pm$ assignments around this cycle. To preserve the action of Lorentz transformations under which ψ^μ transforms as a vector, all components of ψ^μ, $\mu = 0, \ldots, D-1$ must carry the same spin structure, i.e. be sections of the same spin bundle. It is conventional to denote these spin structures by Ramond (R) and Neveu-Schwarz (NS), defined as follows. (The correspondence between the assignments on the cylinder and annulus is unintuitive, so we shall be completely explicit here.)

On the Cylinder

$$\begin{aligned} &\psi^\mu_\pm(\tau, \sigma + 2\pi) = +\psi^\mu_\pm(\tau, \sigma) && \text{R: periodic on cylinder} \\ &\psi^\mu_\pm(\tau, \sigma + 2\pi) = -\psi^\mu_\pm(\tau, \sigma) && \text{NS: anti-periodic on cylinder} \end{aligned} \tag{7.5}$$

To obtain the corresponding boundary conditions on the annulus, we use the conformal mapping $z = e^w$. On spinors ψ_+, the effect is

$$\begin{aligned} \psi_+^{\text{annulus}}(z)(dz)^{1/2} &= \psi_+^{\text{cylinder}}(w)(dw)^{1/2} \\ \left(\frac{dz}{dw}\right)^{1/2} &= e^{w/2} \end{aligned} \tag{7.6}$$

Under a 2π rotation on the cylinder, which corresponds to a 2π rotation on the annulus, the transition factor $e^{\frac{1}{2}w}$ changes by a sign. Hence the (anti-) periodicity assignments are reversed between the cylinder and the annulus.

On the Annulus

$$\begin{aligned} &\psi^\mu_\pm(e^{2\pi i} z) = -\psi^\mu_\pm(z) && \text{R: anti-periodic on annulus} \\ &\psi^\mu_\pm(e^{2\pi i} z) = +\psi^\mu_\pm(z) && \text{NS: periodic on annulus} \end{aligned} \tag{7.7}$$

We are now ready to decompose $\psi^\mu_\pm$ in modes and deduce the algebra of the modes, and the Virasoro algebra. We work on the annulus; translation to the cylinder is straightforward. In the Ramond sectors, we have

$$\begin{cases} \psi^\mu_+(z) = \sum_{n\in\mathbb{Z}} d^\mu_n z^{-n-1/2} \\ \psi^\mu_-(\bar{z}) = \sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n \bar{z}^{-n-1/2} \end{cases} \qquad \text{R (annulus)} \tag{7.8}$$

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \qquad \{\tilde{d}^\mu_m, \tilde{d}^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}\ . \tag{7.9}$$

And in the Neveu-Schwarz sector, we have*

$$\begin{cases} \psi^\mu_+(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} b^\mu_r\, z^{-r-1/2} \\ \psi^\mu_-(\bar{z}) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \tilde{b}^\mu_r\, \bar{z}^{-r-1/2} \end{cases} \qquad \text{NS (annulus)} \tag{7.10}$$

$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \qquad \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{7.11}$$

*It is conventional to denote the NS mode operators by b^μ_r, but there is no relation with the b operators of the bc system.

§7.2. Ramond and Neveu-Schwarz Fock spaces

Let $\mathcal{F}_k$ and $\widetilde{\mathcal{F}}_k$ denote the Fock spaces of the bosonic degrees of freedom at momentum k for left and right movers, respectively. Let $\mathcal{F}^{\mathrm{R}}$ and $\mathcal{F}^{\mathrm{NS}}$ be the Fock spaces for Ramond and Neveu-Schwarz degrees of freedom for left-movers and $\widetilde{\mathcal{F}}^{\mathrm{R}}$ and $\widetilde{\mathcal{F}}^{\mathrm{NS}}$ their right-moving counterparts. Then, the full RNS Fock spaces for open and closed RNS strings are

$$\begin{aligned} \mathcal{F}_{\mathrm{open}} &= \bigoplus_k \mathcal{F}_k^{\mathrm{RNS}} \\ \mathcal{F}_{\mathrm{closed}} &= \bigoplus_k \mathcal{F}_k^{\mathrm{RNS}} \otimes \widetilde{\mathcal{F}}_k^{\mathrm{RNS}} \qquad\qquad k \in \mathbb{R}^D \ . \end{aligned} \tag{7.12}$$

Here we have defined the left and right moving RNS Fock spaces by

$$\begin{aligned} \mathcal{F}_k^{\mathrm{RNS}} &= \mathcal{F}_k \otimes (\mathcal{F}^{\mathrm{R}} \oplus \mathcal{F}^{\mathrm{NS}}) \\ \widetilde{\mathcal{F}}_k^{\mathrm{RNS}} &= \widetilde{\mathcal{F}}_k \otimes (\widetilde{\mathcal{F}}^{\mathrm{R}} \oplus \widetilde{\mathcal{F}}^{\mathrm{NS}}) \ . \end{aligned} \tag{7.13}$$

It thus suffices to construct $\mathcal{F}^{\mathrm{R}}$ and $\mathcal{F}^{\mathrm{NS}}$.

The *Neveu-Schwarz ground state* $|0;\mathrm{NS}\rangle \in \mathcal{F}^{\mathrm{NS}}$ is defined by**

$$b_r^\mu |0;\ \mathrm{NS}\rangle = 0 \quad r \in \frac{1}{2} + \mathbb{Z}^+ \tag{7.14}$$

The full ground state in $\mathcal{F}_k \otimes \mathcal{F}^{\mathrm{NS}}$ will be denoted by $|0,k;\ \mathrm{NS}\rangle \equiv |0,k\rangle \otimes |0;\ \mathrm{NS}\rangle$. We normalize the ground state by $\langle 0;\ \mathrm{NS}|0;\ \mathrm{NS}\rangle = 1$, and the same for $\widetilde{\ }$. By examining the Lorentz spin operator in the NS sector

$$S^{\mu\nu}_{\mathrm{NS}} = \sum_{r \in \frac{1}{2}+\mathbb{Z}^+} \left(b^\mu_{-r} b^\nu_r - b^\nu_{-r} b^\mu_r \right) \ , \tag{7.15}$$

we see that $|0;\ \mathrm{NS}\rangle$ is a scalar under $\mathrm{SO}(1, D-1)$, and thus a *boson*. The NS Fock space $\mathcal{F}^{\mathrm{NS}}$ is now obtained by applying the algebra of raising operators $\{b_r^\mu, r < 0\}$ to $|0;\ \mathrm{NS}\rangle$. All the states obtained this way transform under tensor representations of $\mathrm{SO}(1, D-1)$ and hence are *bosons*.

The *Ramond ground state* is degenerate, just as the ground state of the *bc* system was degenerate. We label the degenerate states in the ground state multiplet by an index α, and we have as usual

$$d_n^\mu |0, \alpha;\ \mathrm{R}\rangle = 0 \qquad\qquad n \in \mathbb{N} \tag{7.16}$$

The full ground state in $\mathcal{F}_k \otimes \mathcal{F}^{\mathrm{R}}$ will be denoted by $|0,k;\alpha;\mathrm{R}\rangle \equiv |0,k\rangle \otimes |0,\alpha;\mathrm{R}\rangle$. The mode d_0^μ satisfies a Clifford algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \tag{7.17}$$

so that $|0,\alpha;\mathrm{R}\rangle$ transforms under a spinor representation of the Lorentz group $\mathrm{Spin}(1, D-1)$. Here, α ranges over the weights of the Dirac spinor $\alpha = 1, \ldots, 2^{D/2}$

**We use the notation $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ for the sets of positive, positive or zero and negative or zero integers, respectively.

(we assume D even). The fact that the ground state $|0,\alpha;\mathrm{R}\rangle$ is a spinor may also be confirmed from examining the expression for the Lorentz spin operator in the R sector:

$$S_{\mathrm{R}}^{\mu\nu} = \tfrac{1}{2}[d_0^\mu, d_0^\nu] + \sum_{n\in\mathbb{N}} (d_{-n}^\mu d_n^\nu - d_{-n}^\nu d_n^\mu) \tag{7.18}$$

On $|0,\alpha;\mathrm{R}\rangle$, the operator $S_{\mathrm{R}}^{\mu\nu}$ reduces to the familiar Lorentz generator in the spinor representation associated with the Clifford algebra of d_0^μ. Thus, the Ramond ground state is a $\mathrm{Spin}(1,D-1)$ spinor, and a space-time *fermion*. $\mathcal{F}^{\mathrm{R}}$ is obtained by applying the algebra of raising operators $\{d_n^\mu,\ n<0\}$ to $|0,\alpha;\mathrm{R}\rangle$ and all states obtained this way are spinors of $\mathrm{SO}(1,D-1)$ and space-time fermions! The Ramond construction achieves the goal of introducing fermions into string theory.

Both Fock spaces $\mathcal{F}^{\mathrm{R}}$ and $\mathcal{F}^{\mathrm{NS}}$ contain *negative norm states*, just as $\mathcal{F}_k$ contained negative norm states in the bosonic string. The reason is simply that both b_r^μ and d_n^μ transform under vector representations, just as the bosonic modes x_n^μ did. Thus, we have for some vector $\epsilon \in \mathbb{R}^D$:

$$\begin{aligned} \|\epsilon\cdot b_r|0;\mathrm{NS}\rangle\|^2 &= \epsilon^2 \\ \|\epsilon\cdot d_{-n}|0,\alpha;\mathrm{R}\rangle\|^2 &= \epsilon^2\ \||0;\alpha;\mathrm{R}\rangle\|^2 \end{aligned} \tag{7.19}$$

which may become negative when ϵ is time-like and $\epsilon^2<0$.

§7.3. Local supersymmetry on the worldsheet

In bosonic string theory, the negative norm states of x^μ are eliminated by the constraints of diffeomorphism invariance (expressed through the Virasoro algebra) which result from the quantization of the worldsheet metric.

In the RNS string, the bosonic fields x^μ of course still produce negative norm states, which are still eliminated by the Virasoro algebra constraints. But now, the RNS fields $\psi_\pm^\mu$ produce further negative norm states (independent of those of x^μ) which are not eliminated by the Virasoro algebra. In fact, since the fields $\psi_\pm^\mu$ are Grassmannian, the constraints needed to eliminate the states created by the time component $\psi_\pm^0$ will have to be Grassmannian. In addition, since we are to eliminate roughly an entire field component $\psi_\pm^0$, we need this invariance to be *local on* Σ. Grassmann odd local symmetries are so-called *local supersymmetries on* Σ, and actions invariant under those symmetries are so-called *supergravity actions*. They involve, in addition to x^μ, $\psi_\pm^\mu$ and the worldsheet metric g_{mn}, also a *gravitino field* $\chi_m^\pm$, which is a spinor-vector, i.e. $\chi \in K^{3/2}\oplus K^{1/2}\oplus \bar{K}^{1/2}\oplus \bar{K}^{3/2}$. This type of field was discovered by Rarita and Schwinger in their study of spin 3/2 fields in 4 space-time dimensions, and is therefore sometimes called the Rarita-Schwinger field.

We shall here give the appropriate supergravity action on any worldsheet, with arbitrary metric g_{mn} and χ_m^σ-field with the spinor index $\sigma=\pm$. To do so, we introduce a local frame $e_m{}^a$, where the m-indices refer to coordinate vectors, while the a-indices refer to the frame vectors $a=1,2$. We denote the inverse frame by $e_a{}^m$. We then have

$$g_{mn} = e_m{}^a e_n{}^b \delta_{ab}\,; \qquad e_a{}^m e_m{}^b = \delta_a{}^b \tag{7.20}$$

where δ_{ab} is the $O(2)$-invariant frame metric. Replacing the metric by the frame introduces an additional field. However, there is now also an entire group of local frame $O(2)$ rotations, which can always be used to reduce the number of components to that of the metric. We shall denote the spin connection by ω_m.

We also have a 2-dimension Clifford algebra, defined by

$$\{\gamma^a, \gamma^b\} = -\delta^{ab} \ . \tag{7.21}$$

The $N = 1$ supergravity action, with a single supersymmetry, is given by (for flat space-time M):

$$\begin{aligned} S[x, \psi; g, \chi] = \frac{1}{4\pi} \int_\Sigma d\mu_g \Big[& \tfrac{1}{2} g^{mn} \partial_m x^\mu \partial_n x_\mu + \psi^\mu \gamma^a e_a{}^m \partial_m \psi_\mu \\ & - \psi^\mu \gamma^a \gamma^b \chi_a e_b{}^m \partial_m x_\mu - \tfrac{1}{4} \psi^\mu \gamma^a \gamma^b \chi_a (\chi_b \psi_\mu) \Big] \end{aligned} \tag{7.22}$$

(The generalization to arbitrary space-time manifolds M is straightforward.) For this action to make sense, and be the integral of a single-valued function, it is necessary that the spin structures of ψ^μ and χ_m^α be the same. The action is invariant under the following transformations.

1) Poincaré symmetry in M.
2) $Diff(\Sigma)$.
3) $Weyl(\Sigma)$ $\quad (\delta\psi^\mu = -\frac{1}{2}\delta\sigma\psi^\mu \qquad \delta\chi_m^\sigma = \frac{1}{2}\delta\sigma\chi_m^\sigma)$.
4) Lorentz(Σ) $\quad$ local $O(2)$ frame rotations.
5) super-Weyl $\quad (\delta e^a = 0 \qquad \delta\chi_m = \gamma_m \delta\lambda)$

(7.23) $\quad$ (The spin structure of λ is the same as that of χ.)

6) Local supersymmetry, generated by a spinor ζ

$$\begin{cases} \delta x^\mu = \zeta\psi^\mu \\ \delta\psi^\mu = \gamma^m(\partial_m x^\mu - \frac{1}{2}\chi_m\psi^\mu)\zeta \\ \delta e_m{}^a = \zeta\gamma^a\chi_m \\ \delta\chi_m^\sigma = -2\nabla_m\zeta^\sigma \end{cases}$$

The spin structure of ζ is the same as that of ψ and χ. In view of $Diff(\Sigma)$ and $Weyl(\Sigma)$ invariance, the above action defines a conformal field theory; as we shall explain shortly it will in fact be *super conformal.*

§7.4. Functional integral representation of transition amplitudes

It will turn out to be useful in discussing the quantization of free strings to have available the general functional integral representation of transition amplitudes for interacting strings as well, on general worldsheet topologies. Thus, we briefly interrupt the discussion of free strings to include the functional integral representation.

We give first a definition of the transition amplitudes in terms of the fields x, ψ, g, χ. Later on, we shall make use of superfields to simplify the expressions. Let V_i be a collection of vertex operators for the RNS string (we shall postpone

the construction of the vertex operators). The scattering amplitude of N strings represented by vertex operators $V_1, \ldots, V_N$ is given by:

$$A = \sum_{\substack{\text{topologies} \\ h=0,\ldots,\infty}} \sum_{\substack{\text{spin} \\ \text{structures} \\ \nu,\bar{\nu}}} w(\nu,\bar{\nu}) \int_{\text{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int D\chi \int_{\text{Map}(\Sigma,M)} D\chi \int D\psi \cdot V_1 \ldots V_N e^{-S[x,\psi;g,\chi]} \tag{7.24}$$

The measures are constructed with respect to the L^2 norms on the corresponding spaces of sections of $K^r \otimes \bar{K}^{\bar{r}}$ for a suitable choice of r, and $w(\nu,\bar{\nu})$ is a weight factor that depends upon the spin structure, to be determined later.

1) *A quick derivation of the critical dimension*

Using our experience from the bosonic string, we may collect the various determinant factors that arise from integrating out x and ψ

$$\left(\frac{\text{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_g}\right)^{-D/2} \left(\text{Det}\,\Delta^+_{(-1/2)}\right)^{D/2} \tag{7.25}$$

as well as from the Faddeev-Popov determinants arising from gauge fixing $Diff(\Sigma)$ and local supersymmetry.

$$\left(\text{Det}'\Delta^-_{(-1)}\right)\left(\text{Det}'\,\Delta^-_{(-1/2)}\right)^{-1} \tag{7.26}$$

The inverse power occurs for the local supersymmetry determinant because local SUSY is Grassmann valued. (We have ignored finite-dimensional determinant contributions.) Now, we have the following contribution to the Weyl anomaly from each determinant

$$\begin{cases} C_{(0)} &= 1 \\ C^+_{(-1/2)} &= -1/2 \\ C^-_{(-1)} &= 13 \\ C^-_{(-1/2)} &= 11/2\,. \end{cases} \qquad \begin{cases} (-D/2)(-2C_{(0)}) &= D \\ (D/2)(-2C^+_{(-1/2)}) &= D/2 \\ 1(-2C^-_{(-1)}) &= -26 \\ -1(-2\,C^-_{(-1/2)}) &= 11 \end{cases} \tag{7.27}$$

Hence the total Weyl scaling factor of the determinant parts is $\frac{3}{2}(D-10)$, and the critical dimension is $D = 10$. The super-Weyl symmetry (item 5) in (7.23) also has an anomaly, and one can show that it cancels in $D = 10$.

2) *A quick look at super-moduli*

Later on, we shall carry out a careful study of the space of *super geometries*, which is parametrized by (g_{mn}, χ^σ_m). For the time being, let us take a quick approach that will suffice for the study of free strings.

Given a metric g, and local complex coordinates z, $\bar{z}$, with $g = 2g_{z\bar{z}}|dz|^2$, let us analyze the action of super-Weyl and supersymmetry on $\chi^\pm_m$:

$$\begin{cases} \delta\chi^+_z = -2\nabla_z\zeta^+ + \delta\lambda^- \\ \delta\chi^+_{\bar{z}} = -2\nabla_{\bar{z}}\zeta^+ \end{cases} \tag{7.28}$$

and similarly for their complex conjugates. We see that by a super-Weyl transformation, we may always set $\chi_z^+ = 0$. Locally, we may also set $\chi_{\bar{z}}^+ = 0$, but globally, there will be obstructions, which are most easily identified when $\delta\chi$ is parametrized by

$$\begin{cases} \delta\chi_{\bar{z}}^+ = \text{Range}\,\nabla_{\bar{z}}^{(-1/2)} \oplus \text{Ker}\,\nabla_z^{(-3/2)} \\ \delta\chi_z^- = \text{Range}\,\nabla_z^{(1/2)} \oplus \text{Ker}\,\nabla_{\bar{z}}^{(3/2)} \ . \end{cases} \tag{7.29}$$

The space $\text{Ker}\,\nabla_{\bar{z}}^{(3/2)}$ is spanned by holomorphic $3/2$ differentials.

Odd moduli are defined as the Grassmann odd coordinates that parametrize those parts of χ that are in the quotient of $\delta\chi$ by super-Weyl and supersymmetry:

$$\begin{cases} \delta m_\kappa = (\delta\chi_z^-, \phi_\kappa) & \phi_\kappa \in \text{Ker}\,\nabla_{\bar{z}}^{(3/2)} \\ \delta\bar{m}_\kappa = (\delta\chi_{\bar{z}}^+, \phi_\kappa^*) & \phi_\kappa^* \in \text{Ker}\,\nabla_z^{(-3/2)} \ . \end{cases} \tag{7.30}$$

The number of odd moduli is given by the dimension of $\text{Ker}\,\nabla_{\bar{z}}^{(3/2)}$:

$$\dim \text{Ker}\,\nabla_{\bar{z}}^{(3/2)} = \begin{cases} 0 & h = 0 \\ \begin{cases} 0 & h = 1, \text{ even } \nu \\ 1 & h = 1, \text{ odd } \nu \end{cases} \\ 2h - 2 & h \geq 2 \ . \end{cases} \tag{7.31}$$

Here, ν denotes the spin structure of χ.

Later on, we shall show that, together with the even moduli m_k, $\bar{m}_k$, the odd moduli m_κ, $\bar{m}_\kappa$ parametrize a so-called *super moduli space* $s\mathcal{M}_k$, with supermoduli coordinates $(m_k, m_\kappa; \bar{m}_k, \bar{m}_\kappa)$. Here, $k = 1, \ldots, \dim \text{Ker}\,\nabla_{\bar{z}}{}^{(2)}$ and $\kappa = 1, \ldots, \dim \text{Ker}\,\nabla_{\bar{z}}{}^{(3/2)}$. This space may be defined as the quotient of the space of all $\{e_m^a, \chi_m^\sigma\}$, denoted by $s\text{Met}(\Sigma)$, by the group of superdiffeomorphisms $sDiff(\Sigma) = \{Diff(\Sigma), \text{local SUSY }\}$, of super-Weyl transformations $sWeyl(\Sigma) = \{Weyl(\Sigma), \text{super } \delta\lambda\}$ and local $O(2)$ frame rotations $\text{Lorentz}(\Sigma)$:

$$s\mathcal{M}_h = s\text{Met}(\Sigma)/(sDiff(\Sigma) \rtimes sWeyl(\Sigma) \times \text{Lorentz}(\Sigma)) \tag{7.32}$$

§7.5. Super-Virasoro algebra and physical spectrum

Having clarified the rôle of χ_m^σ, we now apply the results obtained in §7.4 to the free string with Σ of cylindrical topology. Using $Diff(\Sigma)$, local supersymmetry, Weyl and super-Weyl invariance, in the critical dimension, we may choose a flat metric on Σ, $g = 2|dz|^2$ with vanishing $\chi_m^\sigma = 0$. However, just as was the case in quantizing the free bosonic string, we must retain the constraints arising from the integration over g and χ. They are given by

$$\begin{cases} T(z) = -\frac{1}{2}\partial_z x \cdot \partial_z x + \frac{1}{2}\partial_z \psi_+ \cdot \psi_+ \\ G(z) = \psi_+ \cdot \partial_z x \end{cases} \tag{7.33}$$

along with their complex conjugates $\bar{T}$, $\bar{G}$. Their OPE's are easily obtained.

$$
\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\
T(z)G(w) &\sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial_w G(w) \\
G(z)G(w) &\sim \frac{2c/3}{(z-w)^2} + \frac{2}{z-w}T(w) \ .
\end{aligned} \tag{7.34}
$$

Here $c = (3/2)D$. The mode expansion must be carried out with care here, since we must distinguish between Ramond and Neveu-Schwarz sectors.

$$
\begin{aligned}
T(z) &= \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \\
G(z) &= \sum_r G_r z^{-r-3/2}
\end{aligned}
\qquad
\begin{cases}
r \in \frac{1}{2}+\mathbb{Z} & \underline{\text{NS}} \\
r \in \mathbb{Z} & \underline{\text{R}} \ .
\end{cases} \tag{7.35}
$$

The (super) algebra formed by L_m and G_r is the $N = 1$ *super conformal* or *super-Virasoro algebra*, with the following structure relations

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[L_m, G_r] &= \left(\frac{1}{2}m - r\right)G_{r+m} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2-\varepsilon)\delta_{r+s,0}
\end{aligned} \tag{7.36}
$$

where $\varepsilon = 0$ for $\underline{\text{R}}$ and $\varepsilon = 1/2$ for $\underline{\text{NS}}$. (Notice that G_r and G_s must be both either $\underline{\text{R}}$ or $\underline{\text{NS}}$!) It will be convenient in the sequel to have the expressions for L_n and G_r in terms of the modes x_n^μ, d_n^μ and b_r^μ. One readily finds by combining (7.33) and (7.35):

(7.37)

$$
m \neq 0 \qquad
\begin{cases}
L_m = \frac{1}{2}\sum_n x_{m-n}\cdot x_n + \frac{1}{2}\sum_r r b_{m-r}\cdot b_r & \underline{\text{NS}} \ . \\
L_m = \frac{1}{2}\sum_n x_{m-n}\cdot x_n + \frac{1}{2}\sum_n n d_{m-n}\cdot d_n & \underline{\text{R}} \ .
\end{cases}
$$

$$
\begin{cases}
L_0 = \frac{1}{2}p^2 + \sum_{n=1}^{\infty} x_{-n}\cdot x_n + \sum_{r\in 1/2+\mathbb{Z}^+} r b_{-r}\cdot b_r & \underline{\text{NS}} \ . \\
L_0 = \frac{1}{2}p^2 + \sum_{n=1}^{\infty}(x_{-n}\cdot x_n + n d_{-n}\cdot d_n) & \underline{\text{R}} \ .
\end{cases} \tag{7.38}
$$

$$
\begin{cases}
G_r = \sum_m x_{-m}\cdot b_{r+m} & \underline{\text{NS}} \quad r \in \frac{1}{2}+\mathbb{Z} \\
G_r = \sum_m x_{-m}\cdot d_{r+m} & \underline{\text{R}} \quad r \in \mathbb{Z} \ .
\end{cases} \tag{7.39}
$$

The physical state conditions are enforced separately in the Ramond and Neveu-Schwarz sectors and may be formulated separately for left and right movers at fixed momentum. We have the following definitions for physical states in the left moving sector. (An analogous definition holds for the right-moving sector.)

Ramond sector

A left-moving state $|\psi; k; \mathrm{R}\rangle_L$ is physical of momentum k in the Ramond sector ($|\psi; k; \mathrm{R}\rangle_L \in \mathcal{F}^{\mathrm{phys}}_{\mathrm{R},k}$) if

$$\begin{aligned} &|\psi; k; \mathrm{R}\rangle_L \in \mathcal{F}_k \otimes \mathcal{F}^{\mathrm{R}}, \quad \text{for some } k \in \mathbb{R}^{10} \\ &L_n|\psi; k; \mathrm{R}\rangle_L = G_n|\psi; k; \mathrm{R}\rangle = 0, \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{7.40}$$

and similarly $\widetilde{\mathcal{F}}^{\mathrm{phys}}_{\mathrm{R},k}$ is defined from $\widetilde{\mathcal{F}}_k \otimes \widetilde{\mathcal{F}}^{\mathrm{R}}$.

Neveu-Schwarz

A left-moving state $|\psi; k; \mathrm{NS}\rangle_L$ is physical of momentum k in the Neveu-Schwarz sector ($|\psi; k; \mathrm{NS}\rangle_L \in \mathcal{F}^{\mathrm{phys}}_{\mathrm{NS},k}$) if

$$\begin{aligned} &|\psi; k; \mathrm{NS}\rangle_L \in \mathcal{F}_k \otimes \mathcal{F}^{\mathrm{NS}}, \quad \text{for some } k \in \mathbb{R}^{10} \\ &\left(L_n - \tfrac{1}{2}a\delta_{n,0}\right)|\psi; k; \mathrm{NS}\rangle = 0, \quad \text{for all } n \in \mathbb{N} \\ &G_r|\psi; k; \mathrm{NS}\rangle = 0, \quad \text{for all } r \in \mathbb{Z} + \tfrac{1}{2} \end{aligned} \tag{7.41}$$

and similarly $\widetilde{\mathcal{F}}^{\mathrm{phys}}_{\mathrm{NS},k}$ is defined from $\widetilde{\mathcal{F}}_k \otimes \widetilde{\mathcal{F}}^{\mathrm{NS}}$. From the Fock spaces of physical states, $\mathcal{F}^{\mathrm{phys}}_{\mathrm{R},k}$ and $\mathcal{F}^{\mathrm{phys}}_{\mathrm{NS},k}$, we construct the Fock spaces of open and closed physical strings, in analogy with (7.12) and (7.13):

$$\begin{aligned} \mathcal{F}^{\mathrm{phys}}_{\mathrm{open}} &= \bigoplus_{k\in\mathbb{R}^{10}} \left(\mathcal{F}^{\mathrm{phys}}_{\mathrm{NS},k} \oplus \mathcal{F}^{\mathrm{phys}}_{\mathrm{R},k}\right) \\ \mathcal{F}^{\mathrm{phys}}_{\mathrm{closed}} &= \bigoplus_{k\in\mathbb{R}^{10}} \left(\mathcal{F}^{\mathrm{phys}}_{\mathrm{NS},k} \oplus \mathcal{F}^{\mathrm{phys}}_{\mathrm{R},k}\right) \otimes \left(\widetilde{\mathcal{F}}^{\mathrm{phys}}_{\mathrm{NS},k} \oplus \widetilde{\mathcal{F}}^{\mathrm{phys}}_{\mathrm{R},k}\right) . \end{aligned} \tag{7.42}$$

In complete analogy with the bosonic theory, we have a

No Ghost Theorem. *$\mathcal{F}^{\mathrm{phys}}_{\mathrm{open}}$ and $\mathcal{F}^{\mathrm{phys}}_{\mathrm{closed}}$ have non-negative norm for $D = 10$, $a = 1$. The positive definite Hilbert space of the RNS string is obtained by taking the quotient by the null space, which we denote by $\mathcal{F}^{\mathrm{null}}$:*

$$\begin{aligned} \mathcal{F}^{+}_{\mathrm{open}} &= \mathcal{F}^{\mathrm{phys}}_{\mathrm{open}}/\mathcal{F}^{\mathrm{null}}_{\mathrm{open}} \\ \mathcal{F}^{+}_{\mathrm{closed}} &= \mathcal{F}^{\mathrm{phys}}_{\mathrm{closed}}/\mathcal{F}^{\mathrm{null}}_{\mathrm{closed}} \end{aligned} \tag{7.43}$$

We shall not prove this theorem here.

§7.6. The spectrum of physical states at low mass

In the *Ramond sector*, the ground state obeys $x^\mu_n, d^\mu_n|0, k; \alpha; \mathrm{R}\rangle = 0$ for all $n \in \mathbb{N}$ as well as the physical state condition from (7.40)

$$G_0|0, k; \alpha; \mathrm{R}\rangle = 0 \tag{7.44}$$

Since $L_0 = G_0^2$, the constraint by L_0 is automatically satisfied; similarly the constraints by G_n, L_n, $n \geq 1$ are automatically satisfied. Denoting the generators d^μ_0 of the 10-dimensional Clifford algebra by standard Dirac notation $d^\mu_0 = \frac{1}{\sqrt{2}}\Gamma^\mu$, we find

that the G_0 condition amounts to the fact that the Ramond ground state satisfies the 10-dimensional massless Dirac equation:

$$\Gamma_\mu k^\mu |0,k;\alpha;\mathrm{R}\rangle = 0 \ . \tag{7.45}$$

This equation on the Ramond ground state confirms once more that it is a fermion. Higher mass states transform under higher spinor representations of Spin(1,9), and are all *fermions* as well.

In the *Neveu-Schwarz sector*, the ground state obeys x_n^μ, $b_r^\mu|0,k;\mathrm{NS}\rangle = 0$ for all $n \in \mathbb{N}$, $r \in \frac{1}{2} + \mathbb{Z}^+$ as well as the physical state condition from (7.41)

$$\left(L_0 - \tfrac{1}{2}\right)|0,k;\mathrm{NS}\rangle = 0 \tag{7.46}$$

The constraints by L_n, G_r, $n,r \geq \frac{1}{2}$ are again automatically satisfied. These conditions imply that $k^2 = 1$, i.e. $M^2 = -1$, so that this state is a *tachyon*! It is also a Lorentz scalar and thus a boson. The first excited state in the NS sector is obtained by applying $b^\mu_{-1/2}\,|\zeta,k;\mathrm{NS}\rangle \equiv \zeta_\mu b^\mu_{-1/2}|0,k;\mathrm{NS}\rangle$ and obeys the constraints

$$\left(L_0 - \tfrac{1}{2}\right), L_1|\zeta,k;\mathrm{NS}\rangle = 0 \tag{7.47}$$

We find $k^2 = 0$ and $\zeta \cdot k = 0$, so $|\zeta,k;\mathrm{NS}\rangle$ is a *massless vector state*, such as we need for Yang–Mills particles. The transversality condition $\zeta \cdot k = 0$ is also what we shall need for gauge invariance. It implies that the negative norm state is eliminated, and because $k^2 = 0$ that one further state is null. Thus, we are left with 8 components with positive norm, just as expected for a gauge particle. Higher mass states transform under higher tensor representations of SO(1,9), and are all bosons as well.

Let us note that by using lightcone gauge, we may construct all positive norm states by applying only the *transverse operators*

$$x_n^i, \quad d_n^i, \quad b_r^i \qquad\qquad i = 1,\ldots,8 \tag{7.48}$$

to the ground states in each respective sector for $n,r < 0$. This method in particular allows for an easy counting of the number of physical states modulo null states, at each mass level (see Problems 17 and 18).

Actually, we have looked only at the left-moving sector of the Hilbert space. This suffices to construct the Hilbert space of the *open strings*. To summarize, we have found that this sector contains a NS tachyon, a massless Dirac spinor and a massless vector. Adding Chan–Paton rules to the open string, the single vector may be promoted to a Yang–Mills gauge field multiplet under the Chan–Paton gauge group. Similarly, the Dirac spinor (and the tachyon) will form multiplets under this group. We shall work out this construction for Type I strings shortly.

For *closed strings*, we must put left and right moving sectors together, in order to obtain the full spectrum, with the condition that left and right momenta should match $p_L = p_R$, i.e. $x_0^\mu = \tilde{x}_0^\mu$. We have four sectors:

NS-NS*: bosons.*

- The ground state $|0,k;\mathrm{NS}\rangle_L \otimes |0,k;\mathrm{NS}\rangle_R$ is a tachyon with mass squared equal to $M^2 = -1$.

- The massless states are $\epsilon_{\mu\nu}(k)b^{\mu}_{-1/2}\tilde{b}^{\nu}_{-1/2}|0,k;\mathrm{NS}\rangle_L \otimes |0,k;\mathrm{NS}\rangle_R$ and contain
  the graviton (symmetric μ, ν),
  the anti symmetric tensor (anti symmetric μ, ν),
  the dilaton (trace part).
- massive excited states.

R-NS *(and* NS-R*): fermions.*

The Ramond sector always has mass$^2 \geq 0$, and thus no tachyons occur in the R-NS and NS-R sectors.

- The ground state $|0,k,\alpha;\mathrm{R}\rangle_L \otimes \zeta_\mu(k)\tilde{b}^{\mu}_{-1/2}|0,k;\mathrm{NS}\rangle_R$ is massless. This state is reducible into a spinor λ called the *dilatino* and a spinor-vector, called the *gravitino.*
- massive excited states.

R-R*: more bosons.*

- The ground state $|0,k,\beta;\mathrm{R}\rangle_L \otimes |0,k,\alpha;\mathrm{R}\rangle_R$ is massless.

  These states decompose into a sum of tensor representations of SO(1, 9), that may be obtained as antisymmetrized products of the vector representation. In other words, these states transform as differential forms of degree $0,\ldots,10$.
- massive excited states.

We shall establish in Lecture 10 that the RNS spectrum (after the GSO projection has been carried out in §7.7) exhibits space-time supersymmetry, under which bosons and fermions are interchanged. We shall then find that the dilaton and the dilatino on the one hand, and the graviton and the gravitino on other other hand, form supersymmetry multiplets.

§7.7. The GSO projection, space-time supersymmetry

By combining the NS and R sectors of the RNS string, we have obtained a string theory with both bosons and fermions, with massless graviton, anti symmetric tensor, dilaton and anti symmetric tensor R-R states which are all bosons, plus gravitino and dilatino states which are fermions. This theory is still inconsistent, for two reasons. First, in the NS sector, there is still a *tachyon* present, which gives rise to the same type of problems as the tachyon did in the bosonic string. Second, there is a *massless gravitino*, which can interact consistently only by coupling to a conserved supercurrent. (The gravitino shares many properties with a gauge field, which must couple to a conserved current.) But conservation of a supercurrent means that the theory must exhibit invariance under supersymmetry transformations. Clearly, the RNS theory is not supersymmetric as it stands. The easiest way to see this is by noticing that the tachyon has no fermionic counterpart. On the other hand, if the theory could be made supersymmetric, the tachyon problem would be eliminated right away and then the gravitino could be coupled to a conserved supercurrent, thus eliminating both of the above problems.

Gliozzi-Scherk and Olive (1976) proposed a truncation of the RNS string theory that produces a spectrum with space-time ($\mathbb{R}^{10}$) supersymmetry. We shall introduce the GSO projection first at the level of the free string Fock space, and later on show how to enfore it in the functional integral formulation. It will turn out that the GSO projection corresponds there to carrying out a summation over all spin structures

of Σ, with certain weight factors. The weight factors will be arranged in such a way that the amplitudes will be invariant under the action of the modular group of the worldsheet Σ. Such spin structure weight factors were already included in the functional integral formulation in (7.24) on general grounds.

The GSO projection is defined by using the *worldsheet fermion number operator* F in the form $(-1)^F$. This operator has the property

$$\{(-1)^F, \psi^\mu_\sigma(z,\bar{z})\} = 0 \tag{7.49}$$

both in the R and NS sectors. This property defines the operator $(-1)^F$ only up to a factor of ± 1, which one may think of as the assignment of the eigenvalue of $(-1)^F$ on the string ground state in each sector. We shall now provide a careful definition of $(-1)^F$ in each sector.

In the *Neveu-Schwarz sector*, the renormalized worldsheet fermion number operators in the left- and right-moving sectors are given by

$$F_{\rm NS} \equiv \sum_{r\in 1/2+\mathbb{Z}^+} b_{-r}\cdot b_r \qquad \widetilde{F}_{\rm NS} \equiv \sum_{r\in 1/2+\mathbb{Z}^+} \tilde{b}_{-r}\cdot \tilde{b}_r \ . \tag{7.50}$$

and the assignment of $(-1)^F$ on the NS ground state is -1. Thus, we may represent $(-1)^F$ by

$$(-1)^F \equiv (-1)^{F_{\rm NS}-1} \qquad (-1)^{\widetilde{F}} \equiv (-1)^{\widetilde{F}_{\rm NS}-1} \ . \tag{7.51}$$

The GSO *projection* in the Fock space of the NS sector is then given by

$$\mathcal{F}_{\rm NS,GSO} \equiv \{|\psi\rangle_L \in \mathcal{F}^{\rm NS}, (-1)^F|\psi\rangle_L = |\psi\rangle_L\} \tag{7.52}$$

and similarly

$$\widetilde{\mathcal{F}}_{\rm NS,GSO} \equiv \left\{|\psi\rangle_R \in \widetilde{\mathcal{F}}^{\rm NS}, (-1)^{\widetilde{F}}|\psi\rangle_R = |\psi\rangle_R\right\} \ . \tag{7.53}$$

Alternatively, $\mathcal{F}_{\rm NS,GSO}$ consists of states obtained from the ground state in the NS sector by applying only an *even* number of b operators. (Similarly for $\widetilde{\mathcal{F}}_{\rm NS,GSO}$). It is straightforward to see that the GSO projection is compatible with the physical state conditions, since

$$\left[(-1)^F, L_m\right] = 0 \qquad \left\{(-1)^F, G_r\right\} = 0 \ . \tag{7.54}$$

Thus, we define $\mathcal{F}^{\rm phys}_{\rm NS,GSO}$ and $\widetilde{\mathcal{F}}^{\rm phys}_{\rm NS,GSO}$ by imposing physical state conditions on $\mathcal{F}_{\rm NS,GSO}$ and $\widetilde{\mathcal{F}}_{\rm NS,GSO}$.

The tachyon ground state (as well as any state obtained by applying an even number of b-operators to it) is projected out by GSO, and the lowest mass state in $\mathcal{F}^{\rm phys}_{\rm GSO,NS}$ is massless. It consists of a vector of SO(1, 9):

$$b^\mu_{-1/2}|0,k;{\rm NS}\rangle$$

with 8 physical, positive norm components. The first excited states have the form

$$b^{\mu_1}_{-1/2}b^{\mu_2}_{-1/2}b^{\mu_3}_{-1/2}|0,k;{\rm NS}\rangle \qquad b^\mu_{-3/2}|0,k;{\rm NS}\rangle \tag{7.55}$$

and have mass $M^2 = 2$. (There are $8 \cdot 7 \cdot 6/6 = 56, +8 = 64$ such physical states.)

We now proceed to define the GSO projection in the *Ramond sector*. Supersymmetry in $\mathbb{R}^{10}$ will require that we have equal numbers of bosonic and fermionic physical states at each mass level. (We assume here that supersymmetry is manifest, and not spontaneously broken.) The original Ramond sector of the RNS string has a massless Dirac spinor ground state. In $D = 10$, such a spinor has $2^{D/2} = 2^5 = 32$ components, which does not match the 8 bosonic components of the NS sector. To have equality, we ought to reduce the number of components of the spinor from 32 to 8, of course, in a way consistent with Spin$(1,9)$ invariance!

For groups of the form Spin$(1, D-1)$, with D even, we have the following reducibility criteria for various dimensions D. The Clifford algebra is defined by $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. The chirality matrix $\bar{\Gamma}$, defined by

$$\bar{\Gamma} \equiv \Gamma^0\Gamma^1 \dots \Gamma^{D-1} \qquad \bar{\Gamma}^2 = (-1)^{D/2-1} \tag{7.56}$$

commutes with all the generators of Spin$(1, D-1)$ in the Dirac spinor representation – which we denote by S. Thus, S may be reduced to a sum of *Weyl spinors* $S = S_+ \oplus S_-$, where both $S_\pm$ are of dimension $2^{D/2-1}$.

For $D = 2, 4 \pmod 8$, there is an independent reducibility criterion on Dirac spinors, which decomposes S into a sum of two *Majorana spinors* $S = S_M + iS'_M$. A Majorana spinor is of dimension $2^{D/2}$, but is defined to be *real in some basis*:

$$S^*_M = BS_M \qquad \text{with} \qquad \Gamma^{\mu *} = B\Gamma^\mu B^{-1} \tag{7.57}$$

A spinor may be reduced to *Majorana-Weyl* spinors provided the Weyl and Majorana conditions are compatible (i.e. $\bar{\Gamma}$ and B commute). This is possible only in $D = 2 \pmod 8$. A Majorana-Weyl spinor is a real spinor of dimension $2^{D/2-1}$. In $D = 10$, we see that the Majorana-Weyl condition reduces the dimensionality of the R-ground state precisely to an 8-component spinor, just as is needed for supersymmetry!

In the *Ramond sector*, the GSO projection is defined in terms of the worldsheet fermion number operator as well, with the following normalization.

$$\begin{aligned} (-1)^F &\equiv \bar{\Gamma}(-1)^{F_{\rm R}} & (-1)^F &= \bar{\Gamma}(-1)^{\widetilde{F}_{\rm R}} \\ F_{\rm R} &\equiv \sum_{n=1}^{\infty} d_{-n} \cdot d_n & \widetilde{F}_{\rm R} &= \sum_{n=1}^{\infty} \tilde{d}_{-n} \cdot \tilde{d}_n \end{aligned} \tag{7.58}$$

The GSO projection in the Fock space of the R sector involves a choice of chirality of the R ground state in each sector (left movers and right movers). Let us denote these chiralities by χ_L and χ_R, respectively, with values $\pm$ each. We now define

$$\mathcal{F}^{\rm R}_{\rm GSO}(\chi_L) \equiv \left\{ |\psi, {\rm R}\rangle_L \in \mathcal{F}^{\rm R}; \qquad (-1)^F |\psi, {\rm R}\rangle_L = \chi_L |\psi; {\rm R}\rangle_L \right\} \tag{7.59a}$$

and

$$\widetilde{\mathcal{F}}^{\rm R}_{\rm GSO}(\chi_{\rm R}) \equiv \left\{ |\psi; {\rm R}\rangle_{\rm R} \in \widetilde{\mathcal{F}}^{\rm R}; \qquad (-1)^{\widetilde{F}} |\psi; R\rangle_R = \chi_{\rm R} |\psi; {\rm R}\rangle_R \right\} . \tag{7.59b}$$

This means that, given the chirality of the ground state, the chirality of the higher states are determined by the GSO projection.

1) *Chirality and Massive States in the Ramond Sector*

The above GSO projection in the Ramond sector selects the chirality of the massless ground state. But it also appears to select the chirality of massive states, which would appear to be in contradiction with the fact that they are massive. The resolution of this apparent contradiction lies in the fact that opposite chirality components of a massive state arise in different ways. Consider the first excited level for left movers; it has states (in the lightcone gauge)

$$x^i_{-1}|0,k;\alpha;\mathrm{R}\rangle_L \qquad d^i_{-1}|0,k;\beta;\mathrm{R}\rangle_L \tag{7.60}$$

The GSO condition (for example $\chi = +$) requires that

$$\begin{aligned}\bar{\Gamma}x^i_{-1}|0,k;\alpha;\mathrm{R}\rangle_L &= +x^i_{-1}|0,k;\alpha;\mathrm{R}\rangle_L\\ \bar{\Gamma}d^i_{-1}|0,k;\beta;\mathrm{R}\rangle_L &= -d^i_{-1}|0,k;\beta;\mathrm{R}\rangle_L\end{aligned} \tag{7.61}$$

Hence these two states have opposite chiralities and together form the 16 components of a massive Majorana spinor! The same phenomenon persists at higher mass levels, and is responsible for correctly filling out the chirality components of all massive states.

2) *Space-time supersymmetry*

Now that we have defined the GSO projection in all sectors of the RNS string, it remains to investigate whether the spectrum thus obtained is supersymmetric. To carry out this analysis, we would need to construct the supersymmetry operators, and this requires technology that we shall develop only in Lecture 10. Instead, we can check here that the GSO projection passes an elementary test necessary to have a supersymmetric spectrum. It is an elementary consequence of the structure relations of supersymmetry that the supercharges commute with the mass^2 matrix. Thus, the number of bosonic and fermionic states at each mass level must be the same. We now show that this is indeed the case for the RNS spectrum, after GSO projection.

To do so, we count the number of fermionic states (in the R sector) and compare with the number of bosonic states (in the NS sector). To count states, it suffices to use the lightcone construction, in terms of transverse operators x^i, b^i, d^i, with $i = 1,\dots 8$, and we calculate the generating function in each sector. In the Ramond sector, we have

$$\begin{aligned}f_{\mathrm{R}}(q) = 8\mathrm{Tr}\, q^{2M^2} &= \sum_{n=1}^{\infty} d_R(n) q^{2n}\\ &= 8\prod_{m=1}^{\infty}(1-q^{2m})^{-8}(1+q^{2m})^8\end{aligned} \tag{7.62}$$

Here, the factor $(1-q^{2m})^{-8}$ arises from the x^i oscillators and the factor $(1+q^{2m})^8$ from the d^i modes. The prefactor of 8 arises from the eight-fold degeneracy of the R ground state. Similarly, in the Neveu-Schwarz sector, we have

$$f_{\mathrm{NS}}(q) = \prod_{m=1}^{\infty}(1-q^{2m})^{-8}\frac{1}{2q}\left\{\prod_{n=1}^{\infty}(1+q^{2n-1})^8 - \prod_{n=1}^{\infty}(1-q^{2n-1})^8\right\} \tag{7.63}$$

Using the product formulas for Jacobi ϑ-functions, and the equality

$$\vartheta_2(0|\tau)^4 + \vartheta_4(0|\tau)^4 = \vartheta_3(0|\tau)^4 \tag{7.64}$$

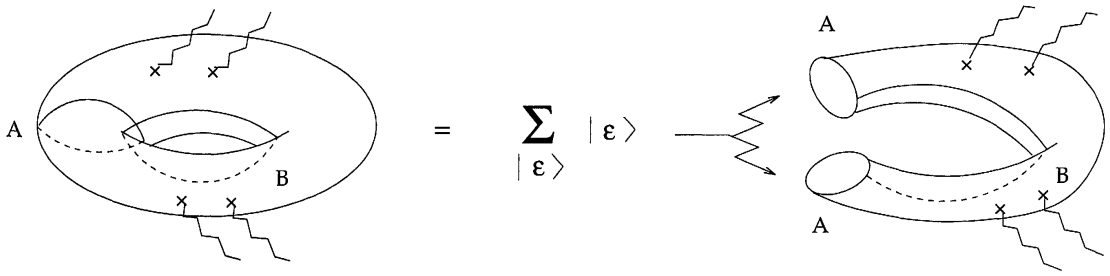
with $q = e^{2\pi i\tau}$, we see that $f_{\rm R}(q) = f_{\rm NS}(q)$. Thus, at each mass level, the number of bosonic and fermionic states agrees.

Notice that we have established evidence for the presence of a space-time supersymmetry in each sector, left-moving and right-moving independently. It is conventional to denote the number of independent supersymmetries by the letter N. In this sense, the open GSO projected RNS string has $\mathcal{N} = 1$, while the closed GSO projected RNS string has $\mathcal{N} = 2$.

3) *Summation over spin structures*

We are now ready to exhibit the relation between the GSO projection of the states in string transition amplitudes to any loop order, and summation over spin structures of string worldsheets. We concentrate on processes involving only NS-NS *external states*, which are created by vertex operators that are single-valued on the worldsheet. (Vertex operators for R states produce quadratic branch cuts and will be introduced in Lecture 9.) Of course, the internal states will include both the R and the NS sectors.

We consider 1-loop amplitudes for closed, oriented strings first, and choose a homology basis of 1-cycles A and B. We analyze the contribution from a given spin structure.



The one-loop amplitude may be *factorized* and expressed as a sum over intermediate states, as shown in the diagram above. In the functional integral, factorization is carried out by integrating over all field values (for x^μ and ψ^μ_σ) on the curve A. Integration over all RNS fields x^μ and ψ^μ_σ, with given spin structure, will give rise to the following structure.

The spin structure on the A-cycle determines the boundary condition on ψ^μ_+ (resp. ψ^μ_-) as either periodic (Ramond) or anti-periodic (Neveu-Schwarz). This information determines whether the state $|\varepsilon\rangle$ in the factorization is a fermion (R) or a boson (NS). In the full 1-loop amplitude, both bosons and fermions will contribute; thus, the spin structure on the A-cycle must be summed over with unit coefficients.

What is the physical significance of the spin structure along a B-cycle? Functional integration over worldsheet spinors ψ^μ_+ (or ψ^μ_-) with *anti-periodic boundary conditions* along the B-cycle produces a sum over states $|\varepsilon\rangle$ in the factorization. On the other hand, functional integration over spinors ψ^μ_+ (or ψ^μ_-) with *periodic boundary conditions* along the B-cycle produces a sum over states $|\varepsilon\rangle$ with an insertion of the operator $(-1)^F$ in the factorization. Here, F stands for worldsheet fermion number, just as we encountered in the definition of the GSO projection. This result

may be established via the usual derivation of the Grassmann functional integral from canonical quantization of spinors. A summation over spin structures on the B-cycles with coefficients (± 1) produces the insertion of the projection operator

$$\frac{1}{2}(1+(-1)^F) \tag{7.65}$$

in the summation over states $|\varepsilon\rangle$ in the factorization, and this operator precisely enforces the GSO projection. It is straightforward to extend the reasoning presented above to the case of worldsheets of arbitrary genus h. We shall leave it as an exercise to the reader.

Just as the GSO projection had to be enforced separately on left and right movers, we shall also have to separate left and right movers in loop amplitudes, using the chiral splitting theorem for the RNS string (which generalizes the results of Lecture 5), and sum independently over spin structures of left and right movers.

Finally, let us remark that we have shown that the GSO projection is enforced by separate summation over all spin structures of left and right movers with weights ± 1. The actual sign choice for each spin structure must be consistent with modular invariance of the full amplitude. There may be different consistent choices, yielding different string theories, as we shall see in §7.8.

§7.8. Type IIA, B superstrings and their spectra

So far, we have analyzed the GSO projection in a single sector of RNS string theory: either left- or right-movers. We now consider closed oriented string theories built from GSO projected RNS strings. We shall temporarily ignore the physical state conditions, which may be easily imposed independently. There are 4 possible combinations, with the following Hilbert spaces:

$$\begin{aligned}\mathcal{F}^{\rm phys}_{\rm GSO}(\chi_L;\chi_R) = \bigoplus_{k\in\mathbb{R}^{10}} \mathcal{F}_k \otimes (\mathcal{F}^{\rm NS}_{\rm GSO} \oplus \mathcal{F}^{\rm R}_{\rm GSO}(\chi_L)) \\ \otimes \widetilde{\mathcal{F}}_k \otimes (\widetilde{\mathcal{F}}^{\rm NS}_{\rm GSO} \oplus \mathcal{F}^{\rm R}_{\rm GSO}(\chi_R))\ .\end{aligned} \tag{7.66}$$

Now, a space-time parity operation reverses the chirality of all spinors and in particular, of the R ground states: $\chi_L \to -\chi_L$, $\chi_R \to -\chi_R$. Thus the string theory with assignment (χ_L,χ_R) defines the same string theory as the one with assignment $(-\chi_L,-\chi_R)$. So, instead of 4 combinations, we are left with only 2 physically inequivalent string theories:

Type IIA : $\chi_L = -\chi_R$

$$\mathcal{F}^{\rm phys}_{\rm IIA} = \mathcal{F}^{\rm phys}_{\rm GSO}(+,-) \qquad (\sim \mathcal{F}^{\rm phys}_{\rm GSO}(-,+))\ . \tag{7.67}$$

Type IIB : $\chi_L = \chi_R$

$$\mathcal{F}^{\rm phys}_{\rm IIB} = \mathcal{F}^{\rm phys}_{\rm GSO}(+,+) \qquad (\sim \mathcal{F}^{\rm phys}_{\rm SO}(-,-))\ . \tag{7.68}$$

The Type IIA theory is parity invariant upon exchange of left and right movers, while Type IIB is parity violating. One also says that *Type* IIB *is chiral*, since it singles out one of the chiralities, while *Type* II A *is non-chiral.*

To gain insight into these theories, we derive their physical spectra, with the help of the lightcone gauge formulation, in terms of the modes x^i_n, b^i_r, d^i_n with

$i = 1, \ldots, D-2 = 8$. (See Problems 17 and 18.) We restrict attention first to massless states. We begin by noticing that the GSO projection eliminates the tachyon, but leaves the massless states unchanged in the NS sector. In the Ramond sector, the GSO projection selects the chirality of the ground state, which is the only state needed for massless string states. We shall choose the chiralities to be + in the left moving sector and − for Type IIA and + for Type IIB in the right moving sectors, respectively. A list of massless states in each sector, together with their standard identifications in terms of fields familiar from quantum field theory, is given in the table below. In parentheses, we indicate the number of physical degrees of freedom present in each state.

$L-R$	Type IIA	Type IIB
NS-NS (bosons)	$b^i_{-1/2}\tilde{b}^j_{-1/2}\|0,k;\mathrm{NS}\rangle_L \otimes \|0,k;\mathrm{NS}\rangle_R$ • (35) $G_{\mu\nu}$ (graviton) • (28) $B_{\mu\nu}$ • (1) Φ (dilaton)	idem
R-NS (fermions)	$\|0,k;\alpha;R\rangle_L \otimes \tilde{b}^i_{-1/2}\|0,k;\mathrm{NS}\rangle_R$ • (56) χ^α_μ (gravitinos) chirality + • (8) λ_α (dilatinos) chirality −	idem chirality + chirality −
NS-R (fermions)	$b^i_{-1/2}\|0,k;\mathrm{NS}\rangle_L \otimes \|0,k;\alpha;R\rangle_R$ • (56) χ^α_μ (gravitinos) chirality − • (8) λ_α (dilatinos) chirality +	idem chirality + chirality −
R-R (bosons)	$\|0,k;\alpha;R\rangle_L \otimes \|0,k;\beta;R\rangle_R$ • (8) $A^{(1)}_\mu$ • (56) $A^{(3)}_{\mu\nu\rho}$	$\|0,k;\alpha;R\rangle_L \otimes \|0,k;\beta;R\rangle_R$ • (1) $A^{(0)}$ • (28) $A^{(2)}_{\mu\nu}$ • (35) $A^{(4)}_{\mu\nu\rho\sigma}$ self-dual F

The first entry that requires clarification is the R-NS (and NS-R) sector: the dilatino and gravitino have opposite chiralities. To see how this happens, consider the tensor product of a spinor of definite chirality (say + as in R-NS) and a vector, which yields a field $\bar{\chi}^\alpha_\mu$ of chirality + with 64 physical components. It decomposes into irreducible spinors by

$$\bar{\chi}^\alpha_\mu = \chi^\alpha_\mu + (\Gamma_\mu)^{\alpha\beta}\lambda_\beta \tag{7.69}$$

where χ^α_μ is the gravitino field, obeying $(\Gamma^\mu)_{\beta\alpha}\chi^\alpha_\mu = 0$, of chirality +, and λ is the dilatino of chirality −, since $\{\bar{\Gamma}, \Gamma_\mu\} = 0$.

The next entry that requires clarification is the R-R sector. To see why the tensors that are indicated are the ones that arise, we study the tensor product

decomposition properties of two spinors. This may be done either on SO(1, 9) spinors, or on the reduced SO(8) spinors. Notice that a Majorana-Weyl spinor of the Poincaré group of SO(1, 9) is *irreducible* under its SO(8) stabilizer subgroup, where it is simply a Weyl spinor.

If S_1 and S_2 are two Majorana-Weyl spinors of *the same chirality*, then they reduce to SO(8) Weyl spinors of the same chirality. Their product decomposes into anti symmetric tensors of *even* rank

$$S_1 \otimes S_2 = A^{(0)} \oplus A^{(2)} \oplus A^{(4)}_+ \tag{7.70}$$

(For Weyl spinors, the components $A^{(6)}$, $A^{(8)}$ are dependent on $A^{(0)}$ and $A^{(2)}$, respectively.) $A^{(4)}_+$ stands for the part with self-dual field strength, $F = dA^{(4)}$ with ${}^*F = F$.

If S_1 and S_2 are two Majorana-Weyl spinors of *opposite chirality*, they reduce to SO(8) Weyl spinors cf opposite chirality. Their product decomposes into anti-symmetric tensors of *odd* rank

$$S_1 \otimes S_2 = A^{(1)} \oplus A^{(3)} \ . \tag{7.71}$$

(For Weyl spinors, the components $A^{(5)}$ and $A^{(7)}$ are dependent on $A^{(3)}$ and $A^{(1)}$.)

The counting of the number of physical states for each of the fields $A^{(1)}_\mu$, $A^{(2)}_{\mu\nu}$, $A^{(3)}_{\mu\nu\kappa}$ and $A^{(4)}_{\mu\nu\kappa\rho}$ indeed confirms that they are all to be viewed as *gauge fields*, in fact, as differential forms $A^{(p)}$, invariant under

$$A^{(p)} \to A^{(p)} + dC^{(p-1)} \tag{7.72}$$

where $C^{(p-1)}$ is a form of degree $p-1$. Thus, they are all Abelian gauge fields.

D-branes

It can be shown that the fundamental strings are neutral under these gauge fields, and do not carry their quantum numbers. A gauge field of the type $A^{(p)}$ couples naturally, however, to an extended object of space-time dimension p called a $D(p-1)$ brane. By analogy with electromagnetism, one says that the $D(p-1)$ brane carries the electric charge of $A^{(p)}$. By Poincaré duality of their field strength $F^{(p)} = dA^{(p)}$, the field $*F^{(p)} \equiv F^{(10-p)}$ may also naturally couple to extended objects of space-time dimensions $8-p$, again called a $D(8-p)$-brane.

$$\begin{aligned} A^{(1)} &\sim D0 \text{ branes} \\ A^{(2)} &\sim D1 \text{ branes} \quad \text{(strings)} \\ A^{(3)} &\sim D2 \text{ branes} \quad \text{(membranes)} \\ A^{(4)} &\sim D3 \text{ branes} \end{aligned} \tag{7.73}$$

Polchinski showed that D-branes indeed carry the electric charges of these gauge fields.

The massive spectra of Type IIA and Type IIB coincide.

We conclude this section on Type IIA,B superstrings with a discussion of their massive spectra. The full spectra are obtained by tensor product of left- and right-moving sectors as usual. The spectrum of each sector (say left movers) is identical

to the spectrum of an $N = 1$ chiral open superstring. While its massless spinor states are chiral, its massive states cannot be chiral. Instead, we have already exhibited the chiral pairing of states of opposite chirality previously. For example, at the first excited level in the R sector, we have

$$x^i_{-1}|0,k;\alpha;\mathrm{R}\rangle \qquad\qquad d^i_{-1}|0,k;\beta;\mathrm{R}\rangle \tag{7.74}$$

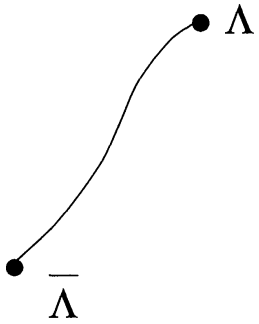
which are of opposite chirality and combine into a representation of massive Majorana fermions. Thus, the massive spectrum is insensitive to the choice of chirality for the massless ground state. As a result *the massive spectra of* Type IIA *and* Type IIB *superstrings coincide.* In other words, Type IIA and Type IIB superstrings differ only in their massless spectrum.

In fact, one can show that upon compactifying Type IIA on a circle of radius R_A and Type IIB on a circle of radius R_B, one obtains identical theories, including the massless sector, when $R_A R_B = \alpha'$. This identity may be seen as a result from a transformation between the two theories, called T-duality, which we investigate in Problem Set #20.

§7.9. Type I superstring

The type I superstring is constructed from open and closed strings, including orientable as well as unorientable worldsheets, and with Chan–Paton Yang–Mills charges on the open string end points.

We begin by reviewing the results from Lecture 2 on Chan–Paton charges: the end points of open strings carry a multiplet of charges, transforming under representations Λ and $\bar{\Lambda}$ of a compact gauge group G. These charges may be obtained by quantization of Grassmann variables λ^i that move at the end points of the open strings.



For the open string excitation states to be precisely those of Yang–Mills gauge states, it is necessary that $\Lambda \otimes \bar{\Lambda}$ be equal to the adjoint representation of G, possibly after imposing certain reality and/or symmetrization rules. In fact, there are only 3 series of solutions, given by classical groups, and the following rules:

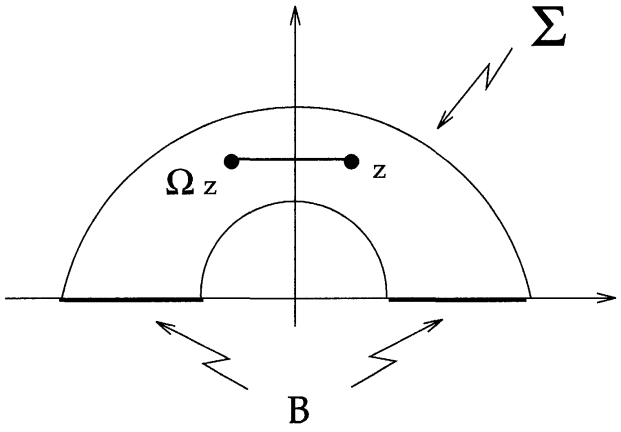
$U(N)$	$\bar{\Lambda} = \Lambda^*$	orientable strings
$\mathrm{SO}(N)$	$\bar{\Lambda} = \Lambda$	unorientable strings
$\mathrm{Sp}(N)$	$\bar{\Lambda} = \Lambda$	unorientable strings

(We shall examine the interplay between Chan–Paton charges and orientability shortly.)

Next, we need to clarify the construction of string theories with unorientable strings. One defines the *worldsheet parity operator* Ω (sometimes also called the twist operator) by its action on worldsheet coordinates:

$$\begin{cases} \Omega z = -\bar{z} \\ \Omega \bar{z} = -z \\ \Omega^2 = 1 \end{cases} \tag{7.75}$$

The action of Ω is exhibited in the figure below for an open string represented as a half annulus, with boundary B.



We begin by considering the bosonic string. Worldsheet parity is a symmetry of the bosonic string action, when x^μ transforms as

$$\Omega x^\mu(z,\bar{z}) = x^\mu(-\bar{z},-z) \ . \tag{7.76}$$

For *open strings*, we have the mode expansion

$$x^\mu(z,\bar{z}) = q^\mu - ip^\mu \ln|z|^2 + i\sum_{n\neq 0}^{\infty} \frac{1}{n} x_n^\mu (z^{-n} + \bar{z}^{-n}) \tag{7.77}$$

so that

$$\Omega q^\mu = q^\mu; \quad \Omega p^\mu = p^\mu; \quad \Omega x_m^\mu = (-1)^m x_m^\mu \ . \tag{7.78}$$

Thus, to define the worldsheet parity of each state in $\mathcal{F}_{\text{open}}$, it suffices to have the Ω assignment of the (tachyonic) ground state. Now, Ω is a symmetry of the worldsheet action and thus is preserved in string interactions. Since there is a non-zero 3-tachyon vertex, we must have $\Omega^3 = 1$, so that $\Omega = 1$ on the tachyon:

$$\Omega|0,k\rangle = +|0,k\rangle \ . \tag{7.79}$$

As a result, the Ω-assignment of Yang–Mills states is negative:

$$\Omega \ x_{-1}^\mu|0,k\rangle = -x_{-1}^\mu|0,k\rangle \tag{7.80}$$

For *closed strings*, we have the mode expansion

$$x^\mu(z,\bar{z}) = q^\mu - ip^\mu \ln|z|^2 + i\sum_{m\neq 0}\frac{1}{m}(x_m^\mu z^{-m} + \tilde{x}_m^\mu \bar{z}^{-m}) \tag{7.81}$$

and

$$\Omega q^\mu = q^\mu;\ \Omega p^\mu = p^\mu \qquad \begin{cases} \Omega x_m^\mu = (-1)^m \tilde{x}_m^\mu \\ \Omega \tilde{x}_m^\mu = (-1)^m x_m^\mu \end{cases} . \tag{7.82}$$

The tachyon, dilaton and graviton now have $\Omega = 1$, but the antisymmetric tensor state $x_{-1}^{[\mu}\,\tilde{x}_{-1}^{\nu]}|0;k\rangle$ has $\Omega = -1$, because Ω interchanges the indices μ and ν.

In addition to acting on the string coordinate x^μ, Ω will also act on the Chan–Paton charges, since the end points of the open string are exchanged. We consider here SO(N) or Sp(N) Chan–Paton charges, where the representations Λ are real or pseudo-real. We introduce a basis of open string states with Chan–Paton charges

$$|\varepsilon, k; ij\rangle \tag{7.83}$$

where k denotes the momentum, the indices $i,j = 1,\ldots,\dim\Lambda = \dim\bar{\Lambda} = N$ label the Chan–Paton charges, and ε label the remaining quantum numbers, such as polarizations. A general state may be described by a matrix of Chan–Paton charges λ_{ij}, and we shall specialize here to the ground state:

$$\lambda_{ij}|0,k;ij\rangle\ . \tag{7.84}$$

Now apply Ω to this state, and use the fact that charges ij are interchanged,

$$\Omega\,\lambda_{ij}|0,k;ij\rangle = \lambda'_{ij}|0,k;ij\rangle\ . \tag{7.85}$$

Clearly, λ' must be linear in λ^T, so we should have

$$\lambda' = M\lambda^T M'\ . \tag{7.86}$$

Now, parity is a symmetry of the worldsheet action, and thus of the transition amplitudes, so we must have an additional requirement on the Chan–Paton charges:

$$\mathrm{tr}\lambda_1\ldots\lambda_p = \mathrm{tr}\ \lambda_1^T\ldots\lambda_p^T\ . \tag{7.87}$$

Hence $M' = M^{-1}$. In addition, from $\Omega^2 = 1$, we have

$$\lambda = M(M\lambda^T M^{-1})^T = M(M^{-1})^T\lambda(M(M^{-1})^T)^{-1}\ . \tag{7.88}$$

Assuming that the set of matrices λ is complete, we have by Schur's lemma that $M^T = \pm M$. Up to changes of bases, we then have 2 possibilities

$$\begin{aligned} &1)\quad M^T = M = I_N: \quad \Omega\lambda_{ij}|0,k;ij\rangle = \lambda_{ij}^T|0,k;ij\rangle \\ &\qquad\qquad\qquad\quad \Omega\lambda_{ij}x_{-1}^\mu|0,k;ij\rangle = -\lambda_{ij}^T x_{-1}^\mu|0,k;ij\rangle \end{aligned} \tag{7.89}$$

2) $M^T = -M = J = \mathrm{diag}(\sigma_2 \ldots \sigma_2)$:

$$\Omega\lambda_{ij}|0,k;ij\rangle = (J\lambda^T J)_{ij}|0,k;ij\rangle \tag{7.90}$$

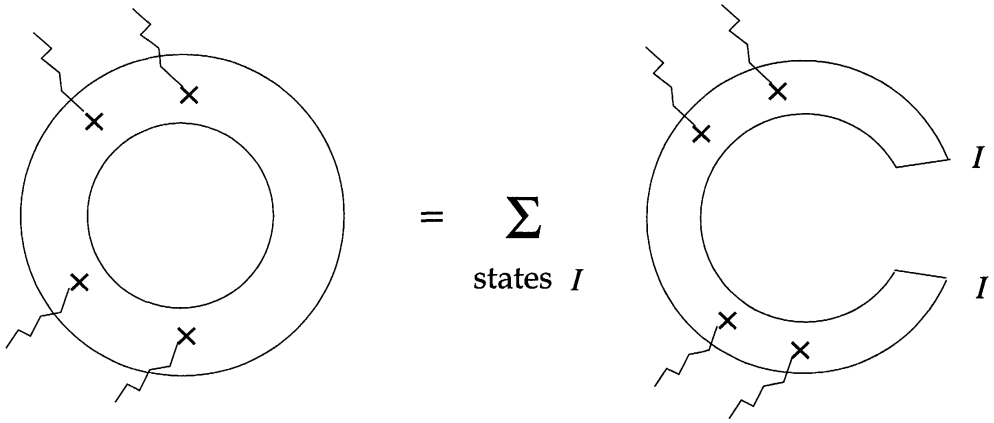
$$\Omega\lambda_{ij}x^\mu_{-1}|0,k;ij\rangle = -(J\lambda^T J)_{ij}x^\mu_{-1}|0,k;ij\rangle \ .$$

Now, we are in a position to identify the rôle of *unorientable* strings. Ω is a (global) symmetry of the worldsheet action and of string interactions. This symmetry may be *gauged*, which means that the Hilbert space of states is projected to states that are invariant under Ω. The relevant projection operator is

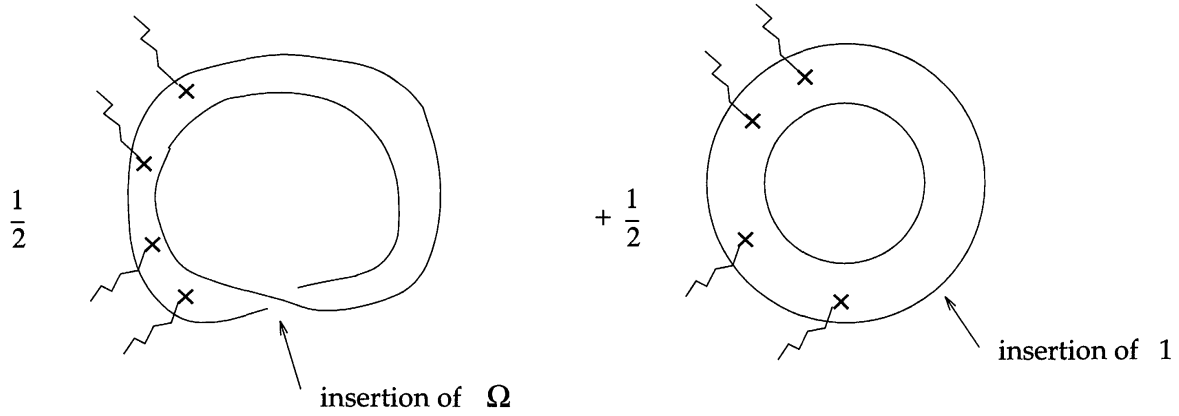
$$\tfrac{1}{2}(1+\Omega) \ . \tag{7.91}$$

In the closed (unoriented) string, this projection eliminates the anti symmetric tensor; in the open string, with Chan–Paton charges, as described above, it will project out the tachyon in 1) and 2) provided we require $\lambda^T = -\lambda$ in 1), i.e. $\mathrm{SO}(N)$ gauge group, and $\lambda^T = -J\lambda J$ in 2), i.e. $\mathrm{Sp}(N)$ gauge group. We see that we then retain the Yang–Mills vector states in the spectrum.

Non-orientable worldsheets now arise in the following way. An open string worldsheet, to tree level, with initial state I and final state F and possibly some vertex operators inserted, will give rise to a one-loop diagram by summing over all possible intermediate states. For oriented open strings, we have



But when the open string spectrum is projected onto $\Omega = 1$ states only, then the projection operator $\frac{1}{2}(1+\Omega)$ must be inserted in the trace. Then, the contributions arise from the annulus and the Möbius string:



In the GSO projected RNS superstring, the worldsheet parity operator exchanges left and right moving ψ_σ's and thus interchanges left and right moving sectors of the GSO projected RNS spectrum. The unoriented ($\Omega = 1$) projection of the closed string is achieved by imposing symmetry between left and right moving sectors. This can be achieved starting from the Type IIB superstring, and retaining only states invariant under interchange of left and right movers. There remains in the *Type I closed string sector*

$$
\begin{array}{llllll}
\text{NS-NS} & \text{bosons} & (35) & G_{\mu\nu}; & (1) & \Phi \\
\text{R-R} & \text{bosons} & (28) & A^{(2)}_{\mu\nu} & & \\
 & & & & & \\
\text{NS-R} & \text{fermions} & & & & \\
\text{R-NS} & \text{fermions} & (56) & \chi^\alpha_\mu, & (8) & \lambda_\alpha \ .
\end{array}
\tag{7.92}
$$

In the *open superstring sector*, we have an $N = 1$ super-Yang–Mills multiplet in $D = 10$, with gauge group SO(32). (The restriction to SO(32) arises only at 1-loop level, and will not be justified further here.)

$$
\begin{array}{lll}
\text{NS}: & 8 \times 496 & A_\mu \\
\text{R}: & 8 \times 496 & \psi^\sigma
\end{array}
\tag{7.93}
$$

References and cross references

A complete definition and description of the free superstrings may be found in

J. H. Schwarz, *Superstring Theory*, Phys. Rep. **89** (1982), 223.

M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. I, Cambridge University Press, 1988.

D-branes, which were only very briefly mentioned in §7.8, are extensively discussed in

J. Polchinski, *TASI lectures on D-branes*, TASI 1996 (C. Efthimiou and B. Greene, eds.), World Scientific Publishers, 1998, `hep-th/9611050`.

LECTURE 8
Heterotic Strings

So far, the only method we have discussed for obtaining massless Yang–Mills states was via the Chan–Paton rules in open string theory, where gauge charges were appended to the end points of open strings.

Heterotic string theories utilize a second method for obtaining massless Yang–Mills states. Heterotic string theories are constructed from *closed oriented strings* only. Internal degrees of freedom – required to obtain Yang–Mills states – are provided by degrees of freedom that live in the bulk of the worldsheet. The heterotic string construction relies on the independence of the oscillator modes of left and right-moving sectors on closed oriented string worldsheets (the left and right momenta obey certain relations that couple left and right degrees of freedom). The left-moving sector of the heterotic string is built out of the (left-moving sector of the) Type II superstring, while the right-moving sector of the heterotic string is built out of the (right-moving sector of the) bosonic string.

At first sight, this kind of construction would appear inconsistent. The Type II superstring part (left movers) exhibits $N = 1$ worldsheet supersymmetry, and $N = 1$ super-Virasoro invariance and has critical space-time dimension $D = 10$, while the bosonic string part (right-movers) exhibits only Virasoro invariance, and has critical space-time dimension $D = 26$. This mismatch of dimensions is put to excellent use in the following way. We let the heterotic string live in $D = 10$ space-time dimensions, and we let the remaining 16 dimensions of right movers emerge as internal degrees of freedom, which are to be metamorphosed into Yang–Mills degrees of freedom.

This system of 16 free internal right movers forms a conformal field theory of central charge 16. It is a unitary CFT, since the 16 scalar fields emerge out of this construction with positive definite internal metric. (We shall later establish that the 16 scalars actually live on a torus, with constant internal metric.) An alternative way of seeing that this CFT must be unitary is by the fact that in the right-moving sector, the Virasoro algebra has already been used up to eliminate the negative norm states from the 10-dimensional space-time part of the x-fields. Consistency of the construction will only be possible if the remaining internal part of the x-fields is unitary all by itself.

Poincaré invariance is required to hold only in the 10-dimensional subspace of the bosonic right movers which is naturally complemented by the 10-dimensional space of the bosonic left movers from the superstring. The internal degrees of freedom in the right moving sector are invariant under this 10-dimensional Poincaré

symmetry, and are thus not constrained to be free fields on $\mathbb{R}^{16}$. In fact, it is consistent with conformal invariance in the right moving sector to consider a system of internal degrees of freedom described by a general, unitary conformal field theory of central charge 16.

On worldsheets of general topology, the separation of left and right movers, which is used in the very definition of the heterotic string, leads to anomalies in each sector separately under diffeomorphisms of Σ. Actually, the subgroup of diffeomorphisms connected to the identity, $Diff_0(\Sigma)$, is anomaly free as long as the combined central charge of matter and ghosts vanishes in each sector. However, global diffeomorphisms, belonging to the modular group $\mathrm{Sp}(2h;\mathbb{Z})$, in general lead to global anomalies. Unitarity of the heterotic string (on worldsheets of genus $h \geq 1$) will require that such anomalies cancel, and this will impose severe restrictions on the allowed $c = 16$ CFT's that can govern the right moving sector of internal degrees of freedom. In fact, only gauge groups $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ will be allowed, as we shall show.

The benefits of the heterotic string construction should be quite evident. Due to space-time supersymmetry of the left movers, we automatically have a string theory with space-time fermions, but without a tachyon! In addition, the theory will possess large – and phenomenologically interesting – gauge symmetries. We now begin the construction of the heterotic string.

First, we shall investigate unitary $c = 16$ conformal field theories involving free fields only. Using Bose-Fermi correspondence, free fields may be formulated either in terms of 16 real scalar fields x^I, $I = 1, \ldots, 16$, which map Σ into a flat torus $\mathbb{R}^{16}/\Lambda$, for some lattice Λ, with Euclidean signature metric, or in terms of 32 Majorana spinors λ^α, $\alpha = 1, \ldots, 32$. Of course, we shall have to retain only the right-moving sector of these degrees of freedom. At the level of conformal field theory operators, one may simply retain the right moving operators, while at the level of transition amplitudes, we shall make use of a generalization of the chiral splitting theorem, established in Lecture 6.

The actions in local complex coordinates z, $\bar{z}$ are given by

$$S_I[x^I, g] \equiv \frac{1}{4\pi} \int_\sigma d^2z \partial_z x^I \partial_{\bar{z}} x^I \tag{8.1}$$

$$S_I[\lambda^\alpha, g] \equiv \frac{1}{4\pi} \int_\Sigma d^2z (\lambda_+^\alpha \partial_{\bar{z}} \lambda_+^\alpha + \lambda_-^\alpha \partial_z \lambda_-^\alpha) \tag{8.2}$$

Here, we represent Majorana spinors by a complex Weyl spinor λ_+^α and its complex conjugate $\lambda_-^\alpha = (\lambda_+^\alpha)^*$. Summation over repeated indices I and α is assumed throughout.

§8.1. Free fermion realization of internal degrees of freedom

We start by analyzing the local properties (on Σ) of the conformal field theory described by $S_I[\lambda, g]$. We shall discuss the global issues, such as the spin structure of λ^α, shortly. We restrict attention to λ_-^α, since λ_+^α may be gotten by complex conjugation.

The field equations $\partial_z \lambda_-^\alpha = 0$ imply that $\lambda_-^\alpha(\bar{z})$ is anti-holomorphic, and the OPE is given as usual. (We assume that all λ_-^α have the same spin structure.)

$$\lambda_-^\alpha(\bar{z}) \lambda_-^\beta(\bar{w}) \sim \frac{\delta^{\alpha\beta}}{\bar{z} - \bar{w}} \tag{8.3}$$

The stress tensor is

$$T_{\bar{z}\bar{z}} = \frac{1}{2}(\partial_{\bar{z}}\lambda_-^\alpha)\lambda_-^\alpha \tag{8.4}$$

The action S_I is also invariant under SO(32) rotations of the spinor λ_-^α. We denote the structure constants of SO(32) by f^{abc} and the representation matrices in the fundamental (32-dimensional) representation of SO(32) by t_a, $a, b, c = 1, \ldots, \dim \mathrm{SO}(32) = 496$:

$$[t_a, t_b] = i f^{abc} t_c \qquad \mathrm{tr}\, t_a t_b = +\frac{1}{2}\delta_{ab} \ . \tag{8.5}$$

The SO(32) Noether current $J_{\bar{z}}^a$ is anti-analytic and given by

$$J_{\bar{z}}^a(\bar{z}) = (t^a)_{\alpha\beta}\lambda_-^\alpha(\bar{z})\lambda_-^\beta(\bar{z}) \tag{8.6}$$

and satisfies the following OPE:

$$J_{\bar{z}}^a(\bar{z})J_{\bar{z}}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + i\frac{f^{abc}}{\bar{z}-\bar{w}} J_{\bar{z}}^c(\bar{w}) \ . \tag{8.7}$$

Here, we have $k\delta^{ab} \equiv 2\,\mathrm{tr}\, t^a t^b$ and thus $k = 1$. Notice that this OPE is precisely the one encountered in Problem 16. Here again, the stress tensor of (8.4) may be expressed in terms of the SO(32) current by the Sugawara construction:

$$T_{\bar{z}\bar{z}} = \frac{1}{2\kappa} : J_{\bar{z}}^a J_{\bar{z}}^a : \tag{8.8}$$

with $\kappa = +\frac{1}{2}C_2(G) + k$, where $C_2(G)$ is the quadratic Casimir of the group G. The central charge is given by

$$c = \frac{2k \cdot \dim G}{2k + C_2(G)} \ . \tag{8.9}$$

For $G = \mathrm{SO}(32)$, $\dim G = 496$, $C_2(G) = 60$ and we have $k = 1$. (Witten established that Wess-Zumino-Witten models with $k = 1$ are equivalent to free fermions for general G.)[1]

We now make use of this free fermion model to introduce internal degrees of freedom in the right-moving sector of the heterotic string. Thus, we retain from λ^α only λ_-^α. We begin by investigating free heterotic strings, and work on a worldsheet with the topology of an annulus (equivalently a cylinder), where two spin structures are possible. We denote these by P and A, and define them in analogy with R and NS spin structures:

P :	λ_-^α	periodic on cylinder	(cf. R-sector)
A :	λ_-^α	anti-periodic on cylinder	(cf. NS-sector)

(8.10)

A priori, not all λ_-^α have to carry the same spin structures. If all λ_-^α carry the same spin structure, then the free fermion theory will be invariant under SO(32).

[1] E. Witten, On holomoprhic factorization of WZW and coset models, Comm. Math. Phys. 144 (1992), 189–212.

More generally, one may consider spinors λ_-^α belonging to different groups, where the spin structure in each group is independently chosen. For example if λ_-^α fall into 2 groups of size k and $32-k$, with $1 \leq k \leq 31$, respectively, then the remaining symmetry will be $SO(k) \times SO(32-k)$, with four possible combinations of spin structures; PP, AP, PA, AA. We shall investigate these possibilities shortly.

The mode expansions of $\lambda_-^\alpha(\bar{z})$ are given in terms of integer (P) or integer $+\frac{1}{2}$ (A) modes:

$$\begin{aligned} &\text{P}: && \lambda_-^\alpha(\bar{z}) = \sum_{n\in\mathbb{Z}} \lambda_n^\alpha \, \bar{z}^{-n-1/2} \\ &\text{A}: && \lambda_-^\alpha(\bar{z}) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \lambda_r^\alpha \, \bar{z}^{-r-1/2} \end{aligned} \tag{8.11}$$

with

$$\begin{aligned} &\text{P}: && \{\lambda_m^\alpha, \lambda_n^\beta\} = \delta^{\alpha\beta}\delta_{m+n,0} \qquad m,n \in \mathbb{Z} \\ &\text{A}: && \{\lambda_r^\alpha, \lambda_s^\beta\} = \delta^{\alpha\beta}\delta_{r+s,0} \qquad r,s \in \tfrac{1}{2}+\mathbb{Z} \end{aligned} \tag{8.12}$$

As is expected in analogy with the R sector, there are zero modes in the P sector, generated by λ_0^α and obeying the Clifford algebra of SO(32). As a result, the P-ground state is a Spin(32) spinor, which we shall denote by $|\sigma;\text{P}\rangle_R$, with σ running over the weights of the Spin(32) spinor representation (we shall see later that this spinor must be Weyl), and $\lambda_n^\alpha|\sigma;\text{P}\rangle_R = 0$, for all $n \in \mathbb{N}$. (If several independent groups of λ_-^α's arise, SO(32) is reduced accordingly.) The P sector produces a Hilbert space

$$\mathcal{F}^{\text{P}} \equiv \left\{ \{\lambda_{-n}^\alpha\}_{n\in\mathbb{N}} |\sigma;\text{P}\rangle_R \right\} . \tag{8.13}$$

The A sector has a unique ground state denoted by $|0,\text{A}\rangle_R$ and obeying $\lambda_r^\alpha|0,\text{A}\rangle_R = 0$ for all $r \in -\frac{1}{2}+\mathbb{N}$, and produces a Hilbert space

$$\mathcal{F}^{\text{A}} \equiv \left\{ \{\lambda_{-r}^\alpha\}_{r\in-\frac{1}{2}+\mathbb{N}} |0,\text{A}\rangle_R \right\} . \tag{8.14}$$

The Virasoro generators are given as usual, and we shall only need L_0 and $\tilde{L}_0$ of these. We define those generators by normal ordering as usual:

$$L_0 = \tfrac{1}{2}p^2 + N; \qquad \tilde{L}_0 = \tfrac{1}{2}p^2 + \widetilde{N} \tag{8.15}$$

where the number operators are given by

$$\begin{aligned} &\text{R}: && N \equiv \sum_{n=1}^{\infty} (x_{-n}^\mu x_{n\mu} + n d_{-n}^\mu d_{n\mu}) \\ &\text{NS}: && N \equiv \sum_{n=1}^{\infty} x_{-n}^\mu x_{n\mu} + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^\mu b_{r\mu} \end{aligned} \tag{8.16}$$

and

$$\begin{aligned} &\text{P}: && \widetilde{N} = \sum_{n=1}^{\infty} (\tilde{x}_{-n}^\mu \tilde{x}_{n\mu} + n\lambda_{-n}^\alpha \lambda_n^\alpha) \\ &\text{A}: && \widetilde{N} = \sum_{n=1}^{\infty} \tilde{x}_{-n}^\mu \tilde{x}_{n\mu} + \sum_{r=1/2}^{\infty} r\lambda_{-r}^\alpha \lambda_r^\alpha \end{aligned} \tag{8.17}$$

It is understood in the above that the summations over the α-index run over the number of λ^α's that belong either to the P or A sectors. We shall denote the eigenvalues in each sector by N_{R}, N_{NS}, N_{P}, N_{A}.

§8.2. Free fermion realization of the $\mathrm{Spin}(32)/Z_2$ heterotic string

We now assume that S_I has SO(32) symmetry, so that there are only 2 sectors: either all λ_-^α are P or all λ_-^α are A. The Virasoro conditions are easily worked out

$$(L_0 - a)|\psi\rangle_L = 0 \qquad (\tilde{L}_0 - \tilde{a})|\psi\rangle_R = 0 \tag{8.18}$$

The constants a and $\tilde{a}$ are given by:

$$a = \begin{cases} 0 & \mathrm{R} \\ 1/2 & \mathrm{NS} \end{cases} \qquad \tilde{a} = \begin{cases} 1 & \mathrm{A} \\ -1 & \mathrm{P} \end{cases} \tag{8.19}$$

The values for a were obtained in Lecture 7. The values for $\tilde{a}$ follow in complete analogy with the corresponding values of a in the R, NS sectors, namely $1/24$ for each physical bosonic x, $-1/24$ for each integer moded physical fermionic ψ or λ, and $+1/48$ for each half integer moded physical fermionic ψ or λ.

We are now in a position to determine the massless spectrum. We have the following states

$$\begin{pmatrix} N_{\mathrm{R}} = 0 \\ N_{\mathrm{NS}} = 1/2 \end{pmatrix}_L \otimes \begin{pmatrix} N_{\mathrm{A}} = 1 \\ \text{no P states allowed} \end{pmatrix}_R \tag{8.20}$$

This may be seen from the expression for M^2 in each sector by combining (8.15–17) and (8.19)

$$\begin{aligned} M^2 &= 2N_{\mathrm{R}} = 2N_{\mathrm{NS}} - 1 && \text{left-movers} \\ &= 2N_{\mathrm{A}} - 2 = 2N_{\mathrm{P}} + 2 && \text{right-movers} \end{aligned} \tag{8.21}$$

For reasons that will become fully apparent at string loop amplitudes only, a GSO projection must be carried out on the λ^α's, just as it was required in the R and NS sectors. In the functional integral formulations, the GSO summation will again correspond to a summation over the spin structures of the worldsheet spinor λ^α.

In the P sector, the projection is carried out with the help of the $(-1)^F$ operator, defined by

$$\begin{aligned} (-1)^F &\equiv \bar{\lambda}_0 (-1)^{F_{\mathrm{P}}} \qquad F_{\mathrm{P}} \equiv \sum_{n=1}^{\infty} \lambda_{-n}^\alpha \lambda_n^\alpha \\ \bar{\lambda}_0 &\equiv \lambda_0^1 \lambda_0^2 \ldots \lambda_0^{32} \end{aligned} \tag{8.22}$$

which anti-commutes with all λ_n^α. We define the GSO projection on the Fock space of λ_-^α by requiring that only one eigenvalue of $(-1)^F$ be retained. Which eigenvalue is chosen is immaterial.

$$\widetilde{\mathcal{F}}^{\mathrm{P}}_{\mathrm{GSO}} \equiv \left\{ |\psi\rangle_R \in \widetilde{\mathcal{F}}^p \quad \text{such that} \quad (-1)^F |\psi\rangle_R = |\psi\rangle_R \right\} . \tag{8.23}$$

In the A sector, the projection uses

$$(-1)^F \equiv (-1)^{F_{\mathrm{A}}} \qquad F_{\mathrm{A}} \equiv \sum_{r=1/2}^{\infty} \lambda_{-r}^\alpha \lambda_r^\alpha \tag{8.24}$$

which also anti-commutes with all λ^α_{-r}. We define the GSO projection on the Fock space of λ^α_- by requiring that only the + eigenvalue of $(-1)^F$ is retained. This guarantees that the A-ground state is retained by the GSO projection.

$$\widetilde{\mathcal{F}}^{\rm A}_{\rm GSO} \equiv \left\{ |\psi\rangle_R \in \widetilde{\mathcal{F}}^{\rm A} \quad \text{such that} \quad (-1)^F |\psi\rangle_R = |\psi\rangle_R \right\} . \tag{8.25}$$

Notice that in the NS and A sectors, the GSO projection is compatible with the physical state conditions. In the R and P sectors, the GSO projection simply determines the chirality.

It is now easy to list the massless states. In parentheses, we list the number of physical states in each sector.

Bosons NS sector only $N_{\rm NS} = 1/2,\ N_{\rm A} = 1$

(1) $b^\mu_{-1/2}\tilde{x}^\nu_{-1}|0,k;{\rm NS}\rangle_{\rm L} \otimes |0,k\rangle_R \otimes |0,{\rm A}\rangle_R$

- graviton $G_{\mu\nu}$ (35)
- anti-symmetric tensor $B_{\mu\nu}$ (28)
- dilaton Φ (1)

(2) $b^\mu_{-1/2}\lambda^\alpha_{-1/2}\lambda^\alpha_{-1/2}|0,k;{\rm NS}\rangle_L \otimes |0,k\rangle_R \otimes |0,{\rm A}\rangle_R$

- SO(32) Yang–Mills field (8×496)
  (adjoint representation)

Fermions R sector only $N_{\rm R} = 0,\ N_{\rm A} = 1$

(1) $\tilde{x}^\nu_{-1}|0,\alpha;{\rm R}\rangle_L \otimes |0,k\rangle_R \otimes |0;{\rm A}\rangle_R$

- gravitino χ^α_μ (56)
- dilatino λ_α (8)

(2) $\lambda^\alpha_{-1/2}\lambda^\beta_{-1/2}|0,\alpha;k;{\rm R}\rangle_L \otimes |0,k\rangle_R \otimes |0;{\rm A}\rangle_R$

- SO(32) gaugino $\psi^\alpha(8 \times 496)$
  (adjoint representation)

Next, we examine the first massive level, obtained for $M^2 = 2$:

$$\begin{pmatrix} N_{\rm R} = 1 \\ N_{\rm NS} = 3/2 \end{pmatrix}_L \otimes \begin{pmatrix} N_{\rm A} = 2 \\ N_{\rm P} = 0 \end{pmatrix}_R \tag{8.26}$$

The $N_{\rm R} = 1$ multiplet of left-movers has the structure

$$\begin{aligned} &x^\mu_{-1}|0,k,\alpha,{\rm R}\rangle_L \\ &d^\mu_{-1}|0,k,\alpha;{\rm R}\rangle_L \end{aligned} \tag{8.27}$$

and spans a massive Majorana fermion multiplet, with $2 \times 8 \times 8 = 128$ physical states. The $N_{\rm NS} = 3/2$ multiplet of left-movers has the following states

$$
\begin{aligned}
& x^\mu_{-1} b^\nu_{-1/2} |0, k, \mathrm{NS}\rangle_L && (64) \\
& b^\mu_{-1/2} b^\nu_{-1/2} b^\rho_{-1/2} |0, k; \mathrm{NS}\rangle_L && (56) \\
& b^\mu_{-3/2} |0, k; \mathrm{NS}\rangle_L && (8)
\end{aligned}
\tag{8.28}
$$

and has 128 physical bosonic states. Thus, the number of fermionic and bosonic physical states in the left-moving sector are equal, as is expected from space-time supersymmetry, as discussed also in Lecture 7, The $N_{\rm A} = 2$ multiplet of right-movers has the following states

$$
\begin{aligned}
& \tilde{x}^\mu_{-1} \tilde{x}^\nu_{-1} |0, k; \mathrm{A}\rangle_R && (36) \\
& \tilde{x}^\mu_{-2} |0, k; \mathrm{A}\rangle_R && (8) \\
& \tilde{x}^\mu_{-1} \lambda^\alpha_{-1/2} \lambda^\beta_{-1/2} |0, k; A\rangle_R && (8 \times 496) \\
& \lambda^\alpha_{-1/2} \lambda^\beta_{-1/2} \lambda^\gamma_{-1/2} \lambda^\delta_{-1/2} |0, k; \mathrm{A}\rangle_R && (32 \times 31 \times 30 \times 29/24) \\
& \lambda^\alpha_{-1/2} \lambda^\beta_{-3/2} |0, k; \mathrm{A}\rangle_R && (32 \times 32)
\end{aligned}
\tag{8.29}
$$

with a total of 41016 states. Finally, the novel sector here is the P sector, whose analogue did not occur at the massless level. It produces a single Weyl spinor of Spin(32) with dimension $2^{15} = 32768$.

Now, consider the representations allowed in the spectrum:

- only 1 Weyl spinor of Spin(32);
- even numbers of λ's applied to ground state.

Thus, we obtain representations not of Spin(32), but only of $\mathrm{Spin}(32)/Z_2$.

Physically, the spinor representation of $\mathrm{Spin}(32)/Z_2$ is a very interesting object: it is a massive string state, but it is *stable* (in the 10-dimensional theory) since there are no massless states that are spinors, and since a spinor cannot decay into only vector representations.

§8.3. Free fermion realization of the $E_8 \times E_8$ heterotic string

In the free fermion construction of the $\mathrm{Spin}(32)/Z_2$ theory, we have restricted $\lambda^\alpha_\pm$'s to have either all P or all A spin structure in keeping with SO(32) invariance. We now investigate in what way this restriction may be relaxed, and we shall find one other consistent string theory: the $E_8 \times E_8$ model.

We assume that λ^α's fall into two groups of sizes k and $32 - k$ respectively and $0 < k < 32$, with all λ^α's in each group carrying the same spin structure, either A or P. The manifest symmetry of the action S_I with these spin structure assignments is then only $\mathrm{SO}(k) \times \mathrm{SO}(32 - k)$, and the possible spin structures are AA, AP, PA, and PP. Here the first and second assignments refer to the spin structures of the first group of $\lambda^\alpha_\pm$'s transforming under $\mathrm{SO}(k)$, and the second

group of $\lambda_{\pm}^{\alpha}$'s transforming under $SO(32-k)$, respectively. (The case of more than two groups of $\lambda_{\pm}^{\alpha}$'s with independent spin structures may be treated analogously, but will ultimately not yield any consistent string theories.

The first modification in the setup of §8.2 arises from the fact that the $\tilde{a}$ constants of the Virasoro conditions are now given by

$$(8.30)\qquad \begin{cases} \tilde{a}_{\mathrm{AA}} = 1 \\ \tilde{a}_{\mathrm{PP}} = -1 \end{cases} \qquad \begin{cases} \tilde{a}_{\mathrm{AP}} = \dfrac{8}{24} + \dfrac{k}{24} - \dfrac{32-k}{24} = -1 + \dfrac{k}{16} \\ \tilde{a}_{\mathrm{PA}} = \dfrac{8}{24} - \dfrac{k}{24} + \dfrac{32-k}{48} = 1 - \dfrac{k}{16} \end{cases}.$$

To investigate which states are allowed, we examine the mass^2 conditions, by matching the M^2 eigenvalue in the left- and right-moving sectors

$$(8.31)\qquad \begin{aligned} M^2 &= 2N_{\mathrm{R}} = 2N_{\mathrm{NS}} - 1 && \text{(left movers)} \\ &= 2\widetilde{N} - 2\tilde{a} && \text{(right movers)} \end{aligned}$$

Here, $\widetilde{N}$ denotes the eigenvalue of the number operator introduced in (8.17). Now, in view of the fact that N_{R} is an integer ≥ 0 and that the GSO projection in the NS sector restricts to $2N_{\mathrm{NS}} - 1$ integral, it follows that M^2 is an even integer ≥ 0. By (8.31), $\widetilde{N} - \tilde{a}$ must be an integer as well. Now, $\widetilde{N}$ is half-integral by construction; so if there are to be any states in the AP and PA sectors, we must have $\tilde{a}_{\mathrm{AP}} = -\tilde{a}_{\mathrm{PA}}$ half-integral.

If $k \neq 0 \pmod 8$, then we find from the above reasoning that there are no physical states in the AP and PA states. Thus, the only remaining sectors are AA and PP. But, then all $\lambda_{\pm}^{\alpha}$'s always have the same spin structure, either A or P, and the theory really has SO(32) symmetry, and reduces to the case studied in §8.2.

There remain 2 new, inequivalent cases: $SO(16) \times SO(16)$ and $SO(8) \times SO(24)$. The latter turns out to be inconsistent at the 1-loop level (as will be demonstrated in Lecture 9), and we shall not investigate it any further here.

We now analyze the $SO(16) \times SO(16)$ model, and concentrate on the representations of $SO(16) \times SO(16)$, under which the massless gauge fields (space-time vectors) transform. These states belong to the multiplet with $N_{\mathrm{NS}} = 1/2$ in the left-moving sector, and to those states with $\widetilde{N} = \tilde{a}$ and *with zero $\tilde{x}$ oscillator number*, in the right moving sector. There are now 4 contributions, from the AA, AP, PA and PP sectors. Below, we list the allowed states and indicate their respective representations of $SO(16) \times SO(16)$ (by listing the dimension of the representations)

under each set of states.

$$
\begin{aligned}
&\text{AA}: && \lambda^{\alpha}_{-1/2}\lambda^{\beta}_{-1/2}|0,\mathrm{A}\rangle_R \otimes |0,\mathrm{A}\rangle_R \\
& && (120,1)\oplus(1,120)\oplus(16,16) \\
&\text{AP}: && |0,\mathrm{A}\rangle_R \otimes |0,\sigma,\mathrm{P}\rangle_R \\
& && (1,128)\oplus(1,128') \\
&\text{PA}: && |0,\sigma,\mathrm{P}\rangle_R \otimes |0,\mathrm{A}\rangle_R \\
& && (128,1)\oplus(128',1) \\
&\text{PP}: && \text{no massless states}
\end{aligned}
\tag{8.32}
$$

Here, we have decomposed the Dirac spinor of SO(16) (actually Spin(16)) into its two irreducible Weyl components denoted by 128 and 128′.

Now, all these states are to correspond to massless Yang–Mills fields, and they can form a consistent quantum field theory only if they correspond to the adjoint representation of some compact Lie algebra. (A theory of vector fields that does not arise as a Yang–Mills theory will not produce sufficient gauge invariance to eliminate the negative norm states created by the vector field.) But, there is no Lie algebra that contains all the states listed in (8.32). From the multiplet structure, it is clear that the Lie algebra has rank 16, with SO(16) × SO(16) as a maximal subalgebra. If all states were to occur, then the Lie algebra should have dimension 1008, but there is no such Lie algebra.

Again, to obtain a consistent theory, we have to project à la GSO. Of course, we can project out the AP and PA sectors, to recover the $\mathrm{Spin}(32)/Z_2$ theory. But, there is another solution: if we project out the (16, 16) representation, then we ought to obtain a factorized group, as is evident from the representation contents. Projecting out the (16, 16) representation is the only possible consistent projection that will not lead back to $\mathrm{Spin}(32)/Z_2$. This can be seen as follows. The representation $(1,120)\oplus(120,1)$ in the AA sector is the adjoint representation of SO(16) × SO(16), and cannot be projected out if the Yang–Mills multiplet has to have at least SO(16) × SO(16) symmetry. But then, if we were to retain the (16, 16) representation, the AA sector by itself would produce the gauge group SO(32). Including any of the representations in the AP and PA sectors would upset the Lie algebra structure. Thus, it remains only to project out the (16, 16).

Projecting out the (16, 16) representation may be achieved by using the following construction. We have naturally 2 SO(16) × SO(16)-invariant fermion number operators.

$$
\begin{aligned}
(-1)^{F_1} &\equiv (-1)^{\sum_{r>0}\lambda^{\alpha_1}_{-r}\lambda^{\alpha_1}_{r}} && \alpha_1 = 1,\ldots,16 \\
(-1)^{F_2} &\equiv (-1)^{\sum_{r>0}\lambda^{\alpha_2}_{-r}\lambda^{\alpha_2}_{r}} && \alpha_2 = 17,\ldots,32
\end{aligned}
\tag{8.33}
$$

The product $(-1)^{F_1}(-1)^{F_2}$ is just the operator $(-1)^F$ of (8.24), in the A sector. The states (16, 16) are now eliminated by imposing the GSO projection

$$
(-1)^{F_1} = 1 \quad \underline{\text{and}} \quad (-1)^{F_2} = 1\ . \tag{8.34}
$$

This projection preserves all of the AP and PA sectors.

Having projected out (16, 16), we are now left with the following representation content in each factor:

$$120 \oplus 128 \oplus 128' \tag{8.35}$$

This content still does not close into a Lie algebra. But if we retain only one of the two SO(16) Weyl spinors, say 128, then we obtain the SO(16) contents of the adjoint representation of the exceptional Lie algebra E_8. Projecting out (128′) is easily achieved by the following GSO projection in the P sector. We define

$$\begin{aligned} (-1)^{F_1} &\equiv \lambda_0^1 \lambda_0^2 \dots \lambda_0^{16} \quad (-1)^{\sum_{n>0} \lambda_{-n}^{\alpha_1} \lambda_n^{\alpha_1}} && \alpha_1 = 1, \dots, 16 \\ (-1)^{F_2} &\equiv \lambda_0^{17} \dots \lambda_0^{32} \quad (-1)^{\sum_{n>0} \lambda_{-n}^{\alpha_2} \lambda_n^{\alpha_2}} && \alpha_2 = 17, \dots, 32 \ , \end{aligned} \tag{8.36}$$

and require in the P sector

$$(-1)^{F_1} = 1 \qquad (-1)^{F_2} = 1 \ . \tag{8.37}$$

The Yang–Mills states that we are left with after this projection now correspond to the Lie algebra $E_8 \times E_8$.

In the functional integral formulation, the above GSO projections correspond to an *independent* summation over all spin structures, separately for the two groups of 16 spinors λ^{α_1} and λ^{α_2}. We shall show in the next chapter that this summation leads to modular invariant amplitudes. Insistence on modular invariance will ultimately be what selects out the two gauge groups $\mathrm{Spin}(32)/Z_2$ and $E_8 \times E_8$, as we shall start to see in the next section, and prove in full in Lecture 9.

We leave it as an exercise to the reader to show that the GSO projected states at the massless level are those of the SO(32) heterotic string, but with the gauge group $\mathrm{Spin}(32)/Z_2$ replaced by $E_8 \times E_8$. Notice that the adjoint representations of both groups have the same dimensions 496, and that both groups have the same rank 16.

§8.4. Bosonic realizations of the $\mathrm{Spin}(32)/Z_2$ and $E_8 \times E_8$ strings

For the $\mathrm{Spin}(32)/Z_2$ heterotic string, we exhibited all the generators of the associated Kac-Moody algebra, in the form of fermion bilinear operators

$$J^a(\bar{z}) = (t^a)_{\alpha\beta} \lambda_-^\alpha \lambda_-^\beta(\bar{z}) \ . \tag{8.38}$$

Indeed, all λ_- have the same spin structure, and so the current $J^a(\bar{z})$ is a single valued 1-form. For the $E_8 \times E_8$ construction, however, we can only exhibit an $\mathrm{SO}(16) \times \mathrm{SO}(16)$ current algebra this way, since products of λ_-'s with opposite spin structure will not produce single-valued currents. Thus, while the adjoint representation of SO(16) is represented this way, the spinor of SO(16) cannot be represented by currents of the form (8.38).

A general construction of vertex operators for simply laced algebras (all of whose roots have equal length) allows us to represent all current generators. This construction was developed by Lepowsky and Wilson and by Frenkel, Kac and

Segal; we shall not reproduce it here in detail. The key ingredient is that scalar fields compactified on a torus may produce enhanced symmetries at special radii of the torus. For a single scalar, this is familiar from Problem 6 of [II-**Dynamics of QFT, Exercises**].

The starting point is a bosonized realization of the internal degrees of freedom λ_-^α in terms of right-moving scalars $x_R^I(\bar{z})$, $I = 1, \ldots, 16$. These x^I are assumed to be propagating on a 16-dimensional torus $\mathbb{R}^{16}/2\pi\Gamma$, specified by a lattice Γ, with basis vectors e_i^I, $i = 1, \ldots, 16$. We define the metric of the lattice Γ by the matrix of inner products of basis vectors ${e_i}^I$:

$$g_{ij} = \sum_{I=1}^{16} e_i^I e_j^I \ . \tag{8.39}$$

By construction, this lattice has positive definite metric.

To describe the $x_R^I(\bar{z})$, we begin by describing a full $x^I = x_L^I + x_R^I$ field propagating on $\mathbb{R}^{16}/2\pi\Gamma$. The solution for x^I on a worldsheet of a cylinder is:

$$x^I(z, \bar{z}) = q^I - i x_0^I \ln z - i \tilde{x}_0^I \ln \bar{z} + \text{ oscillators} \ . \tag{8.40}$$

Now, x^I is valued in $\mathbb{R}^{16}/2\pi\Gamma$, and this allows x^I to be multiple-valued by shifts in $2\pi\Gamma$ as $z \to e^{2\pi i} z$:

$$x^I \left(e^{2\pi i} z, e^{-2\pi i} \bar{z}\right) = x^I(z, \bar{z}) + 2\pi \left(x_0^I - \tilde{x}_0^I\right) \tag{8.41}$$

with

$$x_0^I - \tilde{x}_0^I = \sum_{i=1}^{16} n_i e_i^I \in \Gamma \qquad\qquad n_i \in \mathbb{Z} \ . \tag{8.42}$$

To see the effect of this multi-valuedness, it is helpful to consider the topology of the cylinder instead of the torus. Thus we set $z = \exp(\tau + i\sigma)$ with $\sigma \in [0, 2\pi]$, so that

$$x^I(\tau, \sigma) = q^I - i(x_0^I + \tilde{x}_0^I)\tau + (x_0^I - \tilde{x}_0^I)\sigma + \text{oscillators} \ . \tag{8.43}$$

Clearly now, as $\sigma \to \sigma + 2\pi$, and we wind around the string once, the integers n_i indicate how many times the string wraps around direction i of the torus $\mathbb{R}^{16}/2\pi\Gamma$. Such winding configurations can occur because the torus $\mathbb{R}^{16}/2\pi\Gamma$ is not simply connected, and these configurations are called *winding modes*, with winding numbers n_i.

Now x_0^I and $\tilde{x}_0^I$ are momenta of the string, and $\frac{1}{2}(x_0^I + \tilde{x}_0^I)$ is the total momentum. It must be valued in the lattice dual to Γ, which we denote by Γ^*, and which is generated by the basis of vectors e_i^{*I}, dual to the e_i^I:

$$\sum_I e_i^{*I} e_j^I = \delta_{ij} \ . \tag{8.44}$$

Thus, we must have

$$x_0^I + \tilde{x}_0^I = \sum_{i=1}^{16} m_i e_i^{*I} \in \Gamma^* \qquad\qquad m_i \in \mathbb{Z} \ . \tag{8.45}$$

The discreteness of the momenta results from the compactness of the torus $\mathbb{R}^{16}/2\pi\Gamma$. These modes are called *Kaluza-Klein modes*, and the integers m_i are the Kaluza-Klein momentum mode numbers. Now, the Virasoro constraints are just as in the bosonic string, except for the fact that we have split momenta and oscillators into 10 dimensions of space-time and 16 internal bosons. We reflect this splitting in the Virasoro

$$\begin{aligned} L_0 &= \tfrac{1}{2}p^2 + \tfrac{1}{2}x_0^I x_0^I + N \qquad & N &= \sum_{n=1}^{\infty}(x_{-n}\cdot x_n + x_{-n}^I x_n^I) \\ \tilde{L}_0 &= \tfrac{1}{2}p^2 + \tfrac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + \widetilde{N} \qquad & \widetilde{N} &= \sum_{n=1}^{\infty}(\tilde{x}_{-n}\cdot \tilde{x}_n + \tilde{x}_{-n}^I \tilde{x}_n^I) \end{aligned} \tag{8.46}$$

The relevant Virasoro constraints are unmodified: $L_0 = \tilde{L}_0 = 1$, so that we have

$$\begin{aligned} &\tfrac{1}{2}x_0^I x_0^I - \tfrac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + N - \widetilde{N} = 0 \\ &M^2 = \tfrac{1}{2}x_0^I x_0^I + \tfrac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + N + \widetilde{N} - 2\ . \end{aligned} \tag{8.47}$$

Now, we replace left-movers of the bosonic string by the left-movers of the RNS, GSO projected string. Thus, the Virasoro and mass constraints become:

$$L_0 = \tfrac{1}{2}p^2 + N \quad \begin{cases} N_{\mathrm{R}} = \sum_{n=1}^{\infty}(x_{-n}\cdot x_n + n d_{-n}\cdot d_n) & \text{(R)} \\ N_{\mathrm{NS}} = \sum_{n=1}^{\infty}(x_{-n}\cdot x_n) + \sum_{r=1/2}^{\infty} r b_{-r}\cdot b_r & \text{(NS)} \end{cases} \tag{8.48}$$

$$\tilde{L}_0 = \tfrac{1}{2}p^2 + \tfrac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + \widetilde{N} \qquad \widetilde{N} = \sum_{i=1}^{\infty}(\tilde{x}_{-n}\cdot \tilde{x}_n + \tilde{x}_{-n}^I \tilde{x}_n^I)$$

Thus, we have

$$\begin{aligned} M^2 &= 2N_{\mathrm{R}} = 2N_{\mathrm{NS}} - 1 \\ &= \tilde{x}_0^I \tilde{x}_0^I + 2\widetilde{N} - 2 \end{aligned} \tag{8.49}$$

Furthermore, $\tilde{x}_0^I$ must now live on Γ and on Γ^*, and the length of any allowed $\tilde{x}_0^I \tilde{x}_0^I$ must be *even*. For general lattices Γ, Γ and Γ^* will not have any common points. Thus, the above condition puts severe constraints on what lattices are allowed for Γ. We assume that 16 independent directions for $\tilde{x}_0^I$ are allowed, which essentially implies that Γ must be *self-dual*. (We cannot fully prove at this point that self-duality is necessary; the latter condition will be firmly established from the requirement of modular invariance of the transition amplitudes at loop order $h \geq 1$, as we shall see in Lecture 9. For the time being, we assume self-duality.)

There are then only two possible lattices:

$$\begin{aligned} &E_8 \times E_8 && \text{root lattice,} && g_{ij} && \text{Cartan matrix} \\ &\mathrm{Spin}(32)/Z_2 && \text{root lattice,} && g_{ij} && \text{Cartan matrix} \end{aligned} \tag{8.50}$$

The states of the heterotic strings are reproduced as follows: for massless states, we have

$$\begin{pmatrix} N_{\mathrm{R}} = 0 \\ N_{\mathrm{NS}} = 1/2 \end{pmatrix}_L \otimes \left\{ \begin{pmatrix} \widetilde{N} = 1 \\ \tilde{x}_0^I \tilde{x}_0^I = 0 \end{pmatrix}_R \oplus \begin{pmatrix} \widetilde{N} = 0 \\ \tilde{x}_0^I \tilde{x}_0^I = 2 \end{pmatrix}_R \right\}$$

$$\begin{pmatrix} \widetilde{N} = 1 \\ \tilde{x}_0^I \tilde{x}_0^I = 0 \end{pmatrix} \qquad \begin{array}{l} \tilde{x}_{-1}^{\mu} \text{ states} \rightarrow N = 1 \text{ supergravity multiplet} \\ \tilde{x}_{-1}^{I} \text{ states} \rightarrow 16 \text{ states of super-Yang–Mills multiplet} \end{array}$$

$$\begin{pmatrix} \widetilde{N} = 0 \\ \tilde{x}_0^I \tilde{x}_0^I = 2 \end{pmatrix} \qquad \begin{array}{l} 480 \text{ states of } N = 1 \text{ super-Yang–Mills multiplet} \\ \text{obtained as winding states} \end{array}$$

The total number of Yang–Mills states is $480 + 16 = 496$, which coincides with the dimension of $E_8 \times E_8$ and $\mathrm{Spin}(32)/Z_2$.

In Problem 20, toroidal compactifications of the heterotic strings, their relation by T-duality and their moduli spaces are discussed.

References and cross references

Heterotic strings were introduced in

D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, *Heterotic string theory (I). The free heterotic string*, Nucl. Phys. B **256** (1985), 253.

D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, *Heterotic string theory (I). The interacting heterotic string*, Nucl. Phys. B **267** (1986), 75.

A very helpful discussion is given in

M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. I, Cambridge University Press, 1988.

LECTURE 9
Superstring Perturbation Theory

We shall now develop a systematic treatment of superstring perturbation theory in the RNS formulation. Local $\mathcal{N} = 1$ supersymmetry (SUSY) on the worldsheet plays a fundamental rôle, but this symmetry is not manifest in the component formalism, set up in Lecture 7. The $\mathcal{N} = 1$ superfield formalism, formulated in the language of $\mathcal{N} = 1$ supergeometry, treats diffeomorphisms and local supersymmetry transformations on an equal footing, and renders both manifest. Their combined action makes up the group of superdiffeomorphisms of the worldsheet supermanifold, and local supersymmetry invariants can be constructed from super-diffeomorphism tensors. We first present the different ingredients needed and then apply these methods to the RNS superstrings.

§9.1. $N = 1$ supergeometry (also called $\mathcal{N} = 1$, $D = 2$ supergravity)

We consider a supermanifold Σ of dimension $(2|2)$, and use local coordinates $\xi^M = (\xi^m, \theta^\mu)$, $m = 1, 2$, $\mu = 1, 2$ with the grading

$$\xi^M \xi^N = (-)^{MN} \xi^N \xi^M \tag{9.1}$$

where $(-)^{MN}$ equals 1 unless $M = \mu$, $N = \nu$, in which case it equals -1. We introduce a local orthonormal *frame*

$$E^A \equiv d\xi^M E_M{}^A \qquad A = (a, \alpha) \quad a = z, \bar{z};\ \alpha = +, - \ , \tag{9.2}$$

where it is understood that a has the same grading as m, while α has the same grading as μ. We denote the inverse frame by $E_A{}^M$, and we have $E_A{}^M E_M{}^B = \delta_A{}^B$ and $E_M{}^A E_A{}^N = \delta_M{}^N$.

Next, we consider frame rotations by a group of $U(1)$ gauge transformations, under which the fields E^z, $E^{\bar{z}}$, E^+, E^- have weights -1, $+1$, $-\frac{1}{2}$, $+\frac{1}{2}$, respectively. A general superfield V has $U(1)$ weight $-n$ if it transforms as $(E^z)^{\otimes n}$. We introduce a $U(1)$ gauge field or *connection*

$$\Omega = d\xi^M \Omega_M \tag{9.3}$$

with associated covariant derivatives

$$\mathcal{D}_A^{(n)} V \equiv E_A{}^M (\partial_M + in\Omega_M) V \ . \tag{9.4}$$

Here $\partial_M \equiv \partial/\partial\xi^M$ and $i = \sqrt{-1}$. Torsion $T_{AB}{}^C$ and curvature R_{AB} superfield tensors are defined by the graded commutators

$$[\mathcal{D}_A, \mathcal{D}_B]V = T_{AB}{}^C \mathcal{D}_C V + in\, R_{AB} V \ . \tag{9.5}$$

The data of $\mathcal{N} = 1$ supergeometry is the frame E^A and the $U(1)$ connection Ω, together with the following *constraints* on the torsion and curvature, arising in $[\mathcal{D}_\alpha, \mathcal{D}_\beta]$:

$$T_{\alpha\beta}^{\gamma} = 0; \quad T_{\alpha\beta}^{c} = 2\gamma_{\alpha\beta}^{c}; \quad R_{++} = R_{--} = 0 \tag{9.6}$$

Here, γ^c are the 2-dimensional Dirac matrices, satisfying $\{\gamma^a, \gamma^b\} = -\eta^{ab}$, with a convenient representation given by $\gamma_{++}^{z} = \gamma_{--}^{\bar{z}} = 1$, all other components $= 0$. The constraints of (9.6) are equivalent to the following set of constraints

$$\begin{aligned} &\mathcal{D}_+^2 = \mathcal{D}_z, \quad \mathcal{D}_-^2 = \mathcal{D}_{\bar{z}} \\ &\{\mathcal{D}_+, \mathcal{D}_-\}V = in\, R_{+-} V \ , \end{aligned} \tag{9.7}$$

where V is any superfield of weight n. The structure relations of the remaining covariant derivatives are defined by using the (graded) Jacobi identities. The only free quantity in this supergeometry is the *curvature superfield* $R_{+-} = R_{-+} \equiv R$.

The symmetry transformations on the data E^A and Ω that preserve the constraints (9.6) or equivalently (9.7) are

1) $sDiff(\Sigma)$: super-diffeomorphisms $\xi^M \to \xi'^M(\xi^N)$
contains $Diff(\Sigma)$, local $\mathcal{N} = 1$ SUSY, and further *algebraic* transformations.

2) a) $sWeyl(\Sigma)$: generated by a superfield Σ, contains $Weyl(\Sigma)$ and further algebraic transformations.

b) $sU(1)$: generated by a superfield L, contains $U(1)$ gauge transformations and further algebraic transformations.

It is convenient to combine transformations a) and b) into complexified Weyl transformations, given in terms of a complex superfield Λ, defined by

$$\Lambda \equiv \Sigma - iL, \quad \Lambda^* \equiv \Sigma + iL \ . \tag{9.8}$$

The superderivatives transform under $sWeyl \times sU(1)$ as

$$\begin{aligned} \mathcal{D}_+^{(n)} &= e^{n\Lambda - \frac{1}{2}\Lambda^*} \widehat{\mathcal{D}}_+^{(n)} e^{-n\Lambda} \\ \mathcal{D}_-^{(n)} &= e^{-n\Lambda^* - \frac{1}{2}\Lambda} \widehat{\mathcal{D}}_-^{(n)} e^{n\Lambda^*} \end{aligned} \tag{9.9}$$

and the curvature transforms as

$$R_{+-} = e^{-\Sigma}\left(\widehat{R}_{+-} - 2i\widehat{\mathcal{D}}_+\widehat{\mathcal{D}}_-\Sigma\right) \ . \tag{9.10}$$

From (9.9), the transformation rules for $E_\pm$ and $\Omega_\pm$ may be deduced, and from these combined with the torsion constraints (9.6), the transformation laws of E_z, $E_{\bar{z}}$, Ω_z and $\Omega_{\bar{z}}$ follow.

$\mathcal{N} = 1$ supergeometry admits an almost complex structure

$$J_M{}^N = E_M{}^a \epsilon_a{}^b E_b{}^N + E_M{}^\alpha \bar{\gamma}_\alpha{}^\beta E_\beta{}^N \tag{9.11}$$

where $\epsilon^2 = \bar{\gamma}^2 = -1$. In fact, J is a *complex structure* provided the constraints given in (9.6) are satisfied. (This may be viewed as one of the motivations for choosing those constraints.) We shall denote local complex coordinates by $(\xi, \bar{\xi}, \theta, \bar{\theta})$ with $\xi = (1/\sqrt{2})(\xi^1 + i\xi^2)$, $\theta = (1/\sqrt{2})(\theta^1 + i\theta^2)$, and $\bar{\xi}$ and $\bar{\theta}$ their complex conjugates.

By a combined $sDiff$, $sWeyl$ and $sU(1)$ transformation, any $\mathcal{N} = 1$ supergeometry may be mapped on a locally flat (Euclidean) supergeometry, characterized by the superderivatives

$$\begin{aligned} \mathcal{D}_+ &= \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial\xi} & \mathcal{D}_z = \mathcal{D}_+^2 &= \frac{\partial}{\partial\xi} \\ \mathcal{D}_- &= \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\frac{\partial}{\partial\bar{\xi}} & \mathcal{D}_{\bar{z}} = \mathcal{D}_-^2 &= \frac{\partial}{\partial\bar{\xi}} \ . \end{aligned} \tag{9.12}$$

The residual symmetry group of $sDiff \times sU(1) \times sWeyl$ that leaves this supergeometry invariant is the group of *superconformal* transformations, which consists of superanalytic diffeomorphisms, $\xi \to \xi'(\xi, \theta)$, $\bar{\xi} \to \bar{\xi}'(\bar{\xi}, \bar{\theta})$, $\theta \to \theta'(\xi, \theta)$; $\bar{\theta} \to \bar{\theta}'(\bar{\xi}, \bar{\theta})$, combined with $sWeyl$ and $sU(1)$ transformations.

Wess-Zumino gauge is obtained by fixing a gauge for the action of the *algebraic* parts of $sDiff(\Sigma)$, $sU(1)$ and $sWeyl$. The remaining independent fields are $e_m{}^a$, $\chi_m{}^\alpha$ and an auxiliary field A, and we have

$$\begin{aligned} E_m{}^a &= e_m{}^a + \theta\gamma^a\chi_m - \tfrac{1}{2}\theta\bar{\theta}e_m{}^a A \\ E_m{}^\alpha &= -\tfrac{1}{2}\chi_m{}^\alpha - \frac{i}{2}\theta^\beta(\gamma^a)_\beta{}^\alpha e_m{}^a A + \cdots \\ E_\mu{}^a &= (\gamma^a)_\mu{}^\beta\theta_\beta \\ E_\mu{}^\alpha &= \delta_\mu{}^\alpha\Big(1 + \frac{i}{4}\theta\bar{\theta}A\Big) \end{aligned} \tag{9.13}$$

This presentation of the supergeometry is so-called *in components.*

Integration over Σ is performed with the help of a super-measure $d\mu_E$, defined as follows:

$$d\mu_E \equiv s\det E_M{}^A d\xi\, d\bar{\xi}\, d\theta\, d\bar{\theta} \tag{9.14}$$

where the superdeterminant (also sometimes referred to as the Berezinian) is defined by

$$s\det E_M{}^A \equiv \det(E_m{}^a - E_m{}^\alpha (E_\mu{}^\alpha)^{-1} E_\mu{}^a) \cdot (\det E_\mu{}^\alpha)^{-1} \tag{9.15}$$

Of course, $d\mu_E$ is $sDiff$ invariant. Inner products on superfields Ψ_1 and Φ_2 of weight n are defined by

$$\langle \Phi_1, \Phi_2 \rangle = \int_\Sigma d\mu_E\, \Phi_1^* \Phi_2 \ . \tag{9.16}$$

§9.2. Functional integral representation of transition amplitudes

The RNS worldsheet action, introduced in Lecture 7 in components, admits a simple expression in superfield form. We introduce a string coordinate superfield of $sU(1)$ weight 0; $X = \Sigma \to \mathbb{R}^{10}$

$$X^\mu = x^\mu + \theta\psi_+{}^\mu + \bar\theta\psi_-^\mu + \theta\bar\theta\, F^\mu \qquad \mu = 0, 1, \ldots, 9 \,. \tag{9.17}$$

The action

$$S_X = \frac{1}{4\pi}\int_\Sigma d\mu_E \mathcal{D}_- X \cdot \mathcal{D}_+ X \tag{9.18}$$

reduces to the RNS action,* except for the fact that it has an entire term of the form F^2. This term is non-dynamical and the F field may be integrated out without any consequences.

The next action we need is that of the super (B, C) system: let B and C be superfields of $sU(1)$ weight n and $\frac{1}{2} - n$, respectively. The (B, C) action is then

$$S_{BC} = \frac{1}{2\pi}\int_\Sigma d\mu_E (B\mathcal{D}_-^{(n)} C + \bar B \mathcal{D}_+^{(-n)} \bar C) \,. \tag{9.19}$$

To make contact with the (b, c) action, we decompose the superfields B, C in components

$$\begin{aligned} B &= \beta + \theta b + \bar\theta b' + \theta\bar\theta\beta' \\ C &= c + \theta\gamma + \bar\theta\gamma' + \theta\bar\theta c' \end{aligned} \tag{9.20}$$

The fields b', β', γ' and c' are auxiliary, just as F was an auxiliary field in S_X, and we shall drop their contribution to S_{BC}. The $U(1)$ weight assignments in Wess-Zumino gauge are now obtained by using the fact that θ (resp. $\bar\theta$) have weight $-\frac{1}{2}$ (resp. $\frac{1}{2}$). Hence, the weights are, for $n = -1$:

field	β	b	γ	c
$U(1)$ weight	3/2	2	−1/2	−1

(9.21)

We recognize the (b, c) system of anti-commuting ghost fields, and we now also find a (β, γ) system of commuting ghost fields (called superghosts) which has precisely the correct weight assignments to be the ghost system for local supersymmetry. The $(\beta\gamma)$ fields are worldsheet spinors, and their spin structure must be the same

*In locally flat coordinates, this action equals in components

$$S_X = \frac{1}{4\pi}\int_\Sigma d^2z(\partial_z x \cdot \partial_{\bar z} x - \psi_+ \cdot \partial_{\bar z}\psi_+ - \psi_- \cdot \partial_z \psi_- - F^2)$$

so that the sign of the ψ-kinetic term is opposite to the convention we used previously. Notice that this reverses the sign of the ψ stress tensor.

as that of ${\psi_+}^\mu$, in order for $(\beta\gamma)$ to be $\mathcal{N} = 1$ local supersymmetry ghosts. The BC action may be worked out in components and becomes in WZ gauge:

$$S_{BC} = \frac{1}{2\pi}\int_\Sigma d^2z[b\nabla^z_{(-1)}c + \beta\nabla^z_{(-1/2)}\gamma - {\chi_{\bar z}}^+ S^{\bar z}_+ + c.c.] \tag{9.22}$$

where the BC supercurrent is

$$S_{z+} \equiv +\frac{1}{2}b\gamma - \frac{3}{2}\beta\nabla_z c - (\nabla_z\beta)c \tag{9.23}$$

This current is the $N = 1$ superpartner of the stress tensor and it is conserved; in fact it is analytic.

The combined X, (B, C) system for $D = 10$ components of X is superconformal invariant, and so the superconformal field theory associated with this system depends upon $\mathcal{N} = 1$ supergeometry only through its *supermoduli.* Recall that supermoduli space is defined by

$$s\mathcal{M}_h \equiv \frac{\{{E_M}^A, \Omega_M \text{ satisfying constraints of (9.6)}\}}{sDiff \times sWeyl \times sU(1)} \tag{9.24}$$

and is a space of dimension

$$\dim s\mathcal{M}_h = \begin{cases} (0|0) & h = 0 \\ \begin{cases} (1|0) & h = 1 \quad \text{even spin structure} \\ (1|1) & h = 1 \quad \text{odd spin structure} \end{cases} & \\ (3h-3|2h-2) & n \geq 2 \end{cases} \tag{9.25}$$

The holomorphic cotangent space to $s\mathcal{M}_h$ is spanned by super-holomorphic 3/2 differentials Φ_K, which satisfy

$$\mathcal{D}^{(3/2}_-\Phi_K = 0 \qquad \begin{matrix} \Phi_K \text{ odd graded} \\ K = 1, \ldots, \dim s\mathcal{M}_h \end{matrix} \tag{9.26}$$

Super-Beltrami differentials are defined as the duals to Φ_K, and may be parametrized by the supermoduli variations of ${E_M}^A$:

$$\mu_K \equiv {E_-}^M \frac{\partial {E_M}^z}{\partial m_K} \qquad K = 1, \ldots, \dim s\mathcal{M}_h \ . \tag{9.27}$$

It is instructive to work out the μ_K in Wess-Zumino gauge:

$$\mu_K = \bar\theta\left({e_{\bar z}}^n \frac{\partial {e_n}^z}{\partial m_K} - \theta\frac{\partial {\chi_{\bar z}}^+}{\partial m_K}\right) \ . \tag{9.28}$$

The first term on the r.h.s. is recognized as the ordinary Beltrami differential, for m_K an even modulus, while the second term corresponds to a novel odd Beltrami differential for m_K an odd modulus.

We are now ready to state a preliminary definition of transition amplitudes for RNS superstring amplitudes. This preliminary definition is the natural generalization of the "summation over surfaces" definition adopted for bosonic string

amplitudes in Lecture 1. It is preliminary in the sense that it treats left and right chirality worldsheet spinors with the same spin structure. In the final definition of transition amplitudes, these spin structures will have to be summed over independently for left and right movers, and this will be presented in §9.6, after the Chiral Splitting Theorem has been established. The preliminary definition will be useful, however, in order to introduce superghosts, and it is needed in order to set up the Chiral Splitting Theorem. Transition amplitudes are defined by

$$A_h = \int DE_M{}^A D\Omega_M \delta \text{ (constraints)} \int DX^\mu V_1 \dots V_N e^{-S_X} \tag{9.29}$$

for certain vertex operators V_i, to be specified shortly.

In the critical dimension, $D = 10$, and for super-conformally invariant vertex operators V_i, we have the following Theorem:

$$\begin{aligned} A_h = \int_{s\mathcal{M}_h} \prod_K dm_K \int DX^u \int D(B\bar{B}) \int D(C\bar{C}) V_1 \dots V_N e^{-S_X - S_{BC}} \\ \cdot \prod_\kappa \delta(\langle \mu_\kappa, B \rangle) \delta(\langle \bar{\mu}_\kappa, \bar{B} \rangle) \end{aligned} \tag{9.30}$$

The proof of this result is much more involved than in the case or bosonic strings, because the inner products $\langle \Phi_1, \Phi_2 \rangle$ are no longer positive on superfields, and extra care must be used in dealing with the $\mathcal{D}_+$ and $\mathcal{D}_-$ operators. We shall not prove it here; we refer the reader to Rev. Mod. Phys. **60** (1988) p. 917.

§9.3. **Superconformal field theory** (some basics)

A conformal field theory with an additional $\mathcal{N} = 1$ local supersymmetry invariance is automatically invariant under superconformal transformations, and is an $\mathcal{N} = 1$ *superconformal field theory.* Each conformal field has a superpartner (which is also a conformal field), and together both fields fit into a superconformal field; X^μ, B, C are examples of such fields. Conserved currents are other examples of conformal fields that generalize to supercurrents. Conservation of a supercurrent J is given by

$$\mathcal{D}_- J = 0 \tag{9.31}$$

In local supercomplex coordinates, this means that

$$J = J(z, \theta) = J_{1/2}(z) + \theta J_1(z) \tag{9.32}$$

with $J_{1/2}$ and J_1 holomorphic in z. The associated charge is defined by

$$Q_J = \oint dz\, d\theta\, J = \oint dz\, J_1(z) \tag{9.33}$$

Important examples of supercurrents include the stress tensor, which is built up out of the ordinary stress tensor T_{zz} and the (worldsheet) supercurrent S_{z+}:

$$T(z, \theta) \equiv S_{z+}(z) + \theta T_{zz}(z) \ . \tag{9.34}$$

For the X-system, T is given by

$$\begin{aligned} T^{(X)} &= -\tfrac{1}{2}\mathcal{D}_+X\cdot\mathcal{D}_+^2X \\ &= -\tfrac{1}{2}\psi_+\cdot\partial_z x+\theta\left(-\tfrac{1}{2}\partial_z x\cdot\partial_z x-\tfrac{1}{2}\partial_z\psi_+\cdot\psi_+\right) \end{aligned} \tag{9.35}$$

We recognize indeed the supercurrent and the $x\psi$ stress tensor. For the (B,C) system, T is given by

$$\begin{aligned} T^{(BC)} &= -C\mathcal{D}_+^2B+\frac{1}{2}\mathcal{D}_+C\mathcal{D}_+B-\frac{3}{2}\mathcal{D}_+^2CB \\ &= -c\partial_z\beta+\frac{1}{2}\gamma b-\frac{3}{2}\partial_z c\beta+\theta(T^{(bc)}+T^{(\beta\gamma)}) \end{aligned} \tag{9.36}$$

The OPE of T with superfields of weight $(h,0)$ is determined by Ward identities, and we have the asymptotic expansions

$$\begin{aligned} T(z,\theta)\Phi(w,\zeta)\sim n\frac{\theta-\zeta}{(z-w-\theta\zeta)^2}\Phi(w,\zeta)+\tfrac{1}{2}\frac{1}{z-w-\theta\zeta}\mathcal{D}_+\Phi(w,\zeta) \\ +\frac{\theta-\zeta}{z-w-\theta\zeta}\mathcal{D}_+^2\Phi(w,\zeta) \end{aligned} \tag{9.37}$$

For example $\mathcal{D}_+X$, B and C have weights $n=1/2,3/2,-1$ respectively. The super-Virasoro algebra is similarly expressed. Denote $s\equiv z-w-\theta\zeta$

$$T(z,\theta)T(w,\zeta)\sim\frac{\frac{1}{4}\hat{c}}{s^3}+\frac{3}{2}\frac{\theta-\zeta}{s^2}T(w,\zeta)+\tfrac{1}{2}\frac{1}{s}\mathcal{D}_+T(w,\zeta)+\frac{\theta-\zeta}{s}\mathcal{D}_+^2T(w,\zeta) \tag{9.38}$$

where $\hat{c}=\frac{2}{3}c$ represents the central charge.

§9.4. BRST quantization

The combined action S_X+S_{BC} is BRST invariant. For completeness, we record the BRST transformations on the fields

$$\begin{aligned} \delta X^\mu &= \lambda C\mathcal{D}_+^2X^\mu-\tfrac{1}{2}\lambda\,\mathcal{D}_+C\mathcal{D}_+X^\mu+\text{ complex conjugate} \\ \delta C &= \lambda C\mathcal{D}_+^2C-\frac{1}{4}\lambda\mathcal{D}_+C\mathcal{D}_+C \\ \delta B &= -\lambda T \end{aligned} \tag{9.39}$$

where λ is a constant Grassmann-valued parameter and T is the stress tensor superfield, defined in C). The associated BRST supercurrent is

$$J_{\text{BRST}}=C(T^{(X)}+\tfrac{1}{2}T^{(BC)})-\frac{3}{4}\mathcal{D}_+(C\mathcal{D}_+CB) \tag{9.40}$$

The BRST charge

$$Q_{\text{BRST}}=\oint dz\,d\theta\,J_{\text{BRST}} \tag{9.41}$$

is conserved and nilpotent precisely in $D=10$, in analogy with the bosonic string. Vertex operators for physical states commute with Q_{BRST}, while vertex operators of spurious states are Q_{BRST}-commutators of some operator. We shall not carry out BRST quantization any further here.

§9.5. Vertex operators for physical states

We distinguish two kinds of vertex operators: those for the NS sector, creating space-time bosons, and those for the R sector, creating space-time fermions (also sometimes called the fermion vertex operator). For closed strings, this produces four different kinds of vertex operators: NS-NS and R-R creating bosons, and NS-R and R-NS creating fermions. The R-vertex cannot be constructed directly out of the fields X^μ, since it must produce a branch cut starting at its insertion points.

(1) The NS vertex operator

For simplicity, we specialize to the closed RNS string, and consider NS-NS vertex operators first; the left-moving part may be constructed using the chiral splitting theorem for the RNS string, to be derived in the next section, §9.6. We begin by constructing RNS vertex operators for NS-NS states without ghost dependence.

Vertex operators in the NS-NS sector are given in a fashion completely analogous to the bosonic string

$$V(\varepsilon,k)=\int_\Sigma d\mu_E P_n(\varepsilon,\mathcal{D}_+X,\mathcal{D}_-X,\mathcal{D}_+^2X,\dots)e^{ik\cdot X} \tag{9.42}$$

Here, P_n is a polynomial involving a sum over terms, each of which has a total number n of $\mathcal{D}_+$ derivatives. By $sU(1)$ invariance of V, P_n must then also have a total number n of $\mathcal{D}_-$ derivatives. (On non-flat $N=1$ supergeometries, the curvature superfield R_{+-} may also enter, just as the Gaussian curvature R_g entered for the bosonic string in the dilaton vertex. R_{+-} effectively counts for one $\mathcal{D}_+$ and for one $\mathcal{D}_-$.) By construction, $V(\varepsilon,k)$ is invariant under $sDiff$ transformations. It remains to guarantee that $V(\varepsilon,k)$ is $sWeyl$ invariant.

Under constant $sWeyl$ transformations, $\mathcal{D}_+$ and $\mathcal{D}_-$ have weight $1/2$, $d\mu_E$ has weight -1 and the exponential has weight k^2. Given that $\#\mathcal{D}_+ = n_\mathcal{D} = \#\mathcal{D}_-$, we have

$$M^2=-k^2=n_\mathcal{D}-1 \tag{9.43}$$

The GSO projection forces n to be odd only, since the number of NS fields applied to the ground state in each sector (left and right moving) must be odd. Thus, we have $n_\mathcal{D}=2n+1$, with $n=0,1,2,\dots$ and the masses of NS-NS states are given by

$$M^2=2n \qquad\qquad n=0,1,2,\dots \tag{9.44}$$

Non-constant $sWeyl$ transformations will require in addition that certain conditions be fulfilled on the polarization tensors ε, such as transversality.

The massless string NS-NS vertex operator is

$$V(\varepsilon,k)=\varepsilon_{\mu\nu}(k)\int_\Sigma d\mu_E\mathcal{D}_+X^\mu\mathcal{D}_-X^\nu e^{ik\cdot X} \tag{9.45}$$

$sWeyl$ invariance requires $\varepsilon_{\mu\nu}(k)k^\mu=\varepsilon_{\mu\nu}(k)k^\nu=0$, and we recover the vertex operator for dilaton, graviton and $B_{\mu\nu}$ fields. It is useful to work out this vertex

operator in components, dropping the auxiliary field F, and other terms that vanish by the field equations, such as $\partial_{\bar{z}}\psi_+$. (The full justification for dropping terms that vanish by the field equations may be given with the help of the Chiral Splitting Theorem in the next section, §9.6.) Restricting to left-movers only, we use the fact that

$$X^\mu = {X_L}^\mu + {X_R}^\mu \tag{9.46}$$

and obtain the left-moving NS part of the vertex operator (9.45) as follows:

$$\int d\theta\, \mathcal{D}_+ X_L^\mu e^{\imath k\cdot X_L} = (\partial_z x_L^\mu - i\psi_+^\mu \psi_+^\nu k_\nu) e^{\imath k\cdot x_L} \tag{9.47}$$

This vertex contains the bosonic vertex operator, as well as an operator that allows for an application of an even number of NS fermions only.

(2) The R vertex operator

Ramond states transform under spinor representations of the (double cover of the) Lorentz group, Spin(1, 9). This group has two inequivalent Majorana-Weyl spinors S_α and S'^α, each of dimension 16, so that we let $\alpha = 1, \ldots, 16$. To describe the representations S_α and S'^α, we introduce the Clifford algebra of Spin(1, 9)

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} I_{(32)} \tag{9.48}$$

where $I_{(32)}$ is the identity matrix of dimension 32. The chirality matrix $\bar{\Gamma}$ is defined by

$$\bar{\Gamma} \equiv \Gamma^0 \Gamma^1 \cdots \Gamma^9$$

and satisfies

$$\bar{\Gamma}^2 = I_{(32)} \quad \text{and} \quad \{\Gamma^\mu, \bar{\Gamma}\} = 0 \tag{9.49}$$

Weyl spinors of left (resp. right) chirality are eigenvectors of $\bar{\Gamma}$ with eigenvalues $+1$ (resp. -1). The charge conjugation matrix C is defined by the relation

$$(\Gamma^\mu)^T = -C\Gamma^\mu C^{-1} \tag{9.50}$$

and anti-commutes with $\bar{\Gamma}$:

$$\{\bar{\Gamma}, C\} = 0\ . \tag{9.51}$$

The charge conjugate ψ^c of ψ is defined by

$$\psi^c \equiv C\bar{\psi}^T \qquad \bar{\psi} = \psi^\dagger \Gamma^0 \tag{9.52}$$

A Majorana spinor is a spinor whose charge conjugate is equal to itself, $\psi^c = \psi$. It is consistent (and convenient) to choose $C = \Gamma^0$, so that a Majorana spinor is *real*: $\psi^c = \psi^* = \psi$. We shall choose a basis in which

$$\bar{\Gamma} = \begin{pmatrix} I_{(16)} & 0 \\ 0 & -I_{(16)} \end{pmatrix}\ , \qquad C = \Gamma^0 = \begin{pmatrix} 0 & I_{(16)} \\ -I_{(16)} & 0 \end{pmatrix} \tag{9.53}$$

A 32-component Majorana spinor ψ now decomposes into two 16-component Majorana-Weyl spinors S_α and S'^α. It will be convenient to work with Γ-matrices in this basis:

$$\psi = \begin{pmatrix} S_\alpha \\ S'^\alpha \end{pmatrix} \qquad \Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)_{\alpha\beta} \\ (\gamma^\mu)^{\alpha\beta} & 0 \end{pmatrix} \tag{9.54}$$

with $\alpha, \beta = 1, \ldots, 16$. It then follows from (9.50), (9.53) that $\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha}$ and $\gamma^{\mu\alpha\beta} = \gamma^{\mu\beta\alpha}$. The generator of Spin$(1,9)$ transformations in this representation is $\frac{1}{2}\Gamma^{\mu\nu}$, with

$$\begin{aligned} \Gamma^{\mu\nu} &\equiv \tfrac{1}{2}[\Gamma^\mu, \Gamma^\nu] = \begin{pmatrix} \gamma^{\mu\nu} & 0 \\ 0 & \gamma^{\mu\nu} \end{pmatrix} \\ \gamma^{\mu\nu} &\equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \, . \end{aligned} \tag{9.55}$$

Clearly, $\Gamma^{\mu\nu}$ commutes with $\bar{\Gamma}$, and $\frac{1}{2}\gamma^{\mu\nu}$ are the representation generators of Spin$(1,9)$ in one of the Weyl representations.

The SO$(1,9)$ transformation properties in the Ramond sector are governed by the SO$(1,9)$ currents (we omit the contribution from the x-field)

$$j^{\mu\nu}(z) = \psi^\mu_+ \psi^\nu_+(z) \tag{9.56}$$

Clearly, the field ψ^μ_+ itself transforms under the vector representation of SO$(1,9)$:

$$j^{\mu\nu}(z)\psi^\rho_+(w) \sim \frac{1}{z-w}(\eta^{\mu\rho}\psi^\nu_+(w) - \eta^{\nu\rho}\psi^\mu_+(w)) \, . \tag{9.57}$$

We seek to construct fields $S_\alpha(z)$ and $S'^\alpha(z)$ such that they transform as spinors under Spin$(1,9)$ with representation matrices $\gamma^{\mu\nu}$:

$$j^{\mu\nu}(z)S_\alpha(w) \sim \frac{1}{z-w}\left(-\tfrac{1}{2}\right)(\gamma^{\mu\nu})_\alpha{}^\beta S_\beta(w) \tag{9.58a}$$

$$j^{\mu\nu}(z)S'^\alpha(w) \sim \frac{1}{z-w}\left(-\tfrac{1}{2}\right)(\gamma^{\mu\nu})^\alpha{}_\beta S'^\beta(w) \, . \tag{9.58b}$$

From these OPE's, and from the fact that $j^{\mu\nu}$ is bilinear in ψ^μ_+, we may infer the OPE of ψ_+ with S:

$$\psi^\mu_+(z)S_\alpha(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\frac{1}{\sqrt{2}}(\gamma^\mu)_{\alpha\beta}S'^\beta(w) \tag{9.59a}$$

$$\psi^\mu_+(z)S'^\alpha(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\frac{1}{\sqrt{2}}(\gamma^\mu)^{\alpha\beta}S_\beta(w) \, . \tag{9.59b}$$

The relative normalization between S and S' is arbitrary, and has been fixed above on symmetry grounds. The conformal weights of $S_\alpha(z)$ and $S'^\alpha(z)$ are computed from their OPE with the stress tensor $T^{(\psi)} = -\frac{1}{2}\partial_z\psi_+ \cdot \psi_+$:

$$\begin{aligned} T^{(\psi)}(z)S_\alpha(w) &\sim \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\frac{1}{\sqrt{2}}\frac{1}{(z-w)^{3/2}}\gamma^\mu_{\alpha\beta}\,\psi^\mu_+(z)S'^\beta(w) \\ &\sim \frac{1}{16}(\gamma^\mu\gamma_\mu)^\beta_\alpha S_\beta(w)\frac{1}{(z-w)^2} \, . \end{aligned} \tag{9.60}$$

Since $\gamma^\mu \gamma_\mu = 10\, I_{(16)}$, we find that the conformal weight of S_α (and similarly of S'^α) is $5/8$.

To obtain the spin fields S_α and S'^α, we make use first of the action of the currents $j^{\mu\nu}$ in the Cartan subalgebra of Spin$(1,9)$ only. These components mutually commute and may be bosonized in terms of 5 independent free scalar Bose fields:

$$j^{2a,2a+1}(z) = -i\partial_z \phi^a(z) \qquad a = 0,1,2,3,4\ . \tag{9.61}$$

The R-fields ${\psi_+}^\mu$ are exponentials of ϕ^a with ± 1 weights:

$$\frac{1}{\sqrt{2}}(\psi_+^{2a} \pm i\psi_+^{2a+1}) = e^{\pm \imath \phi^a}\ . \tag{9.62}$$

The spin fields are exponentials of ϕ^a with $\pm\frac{1}{2}$ weights:

$$S_\alpha(z) = e^{i\alpha\cdot\phi}\ . \tag{9.63}$$

where we identify the space-time indices with the spinor representation weights, given by $\alpha = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. The spinors S_α are obtained for α with an even number of $-$ signs, while S'^α is obtained with an odd number of $-$ signs. The conformal dimension of this field is easily verified to be $\frac{1}{2}\alpha^2 = 5/8$, with the help of the standard formula for the conformal weight of an exponential of a $c = 1$ conformal field.

One may complete the analysis by showing that the currents $j^{\mu\nu}$ corresponding to roots may also be constructed in terms of exponentials (multiplied by certain cocycle factors) and that $j^{\mu\nu}$ indeed satisfies the correct OPE with S_α and S'^α.

With the above normalization, one also obtains the OPE's of S with S and with S':

$$\begin{aligned} S'^\alpha(z)S_\beta(w) &\sim \frac{{\delta_\beta}^\alpha}{(z-w)^{5/4}} + \frac{1}{(z-w)^{1/4}}\frac{1}{2}\gamma^{\mu\nu} j_{\mu\nu}(w) \\ S_\alpha(z)S_\beta(w) &\sim \frac{1}{(z-w)^{3/4}}\,\gamma^\mu_{\alpha\beta}\,\psi_\mu \end{aligned} \tag{9.64}$$

(Notice that the construction of spinor fields that we have just given for Spin$(1,9)$ may be carried over in a straightforward way to Spin$(2n)$. In particular, this type of construction is needed to obtain the spinor of SO(16) necessary to build the full current algebra of $E_8 \times E_8$ in the heterotic string. There, the spin fields have precisely conformal dimension $(1,0)$, since the weight vectors all have length $\alpha^2 = 2$.)

It remains to construct the Ramond vertex operator. The spin field by itself cannot produce a vertex operator for massless fermions, since its conformal weight is $5/8 \neq 1$. The resolution of this issue is found in a contribution to the R vertex operator from the (β,γ) and possibly (b,c) ghost system. The relevant conformal primary field is obtained as follows. (We consider only the left-moving part of the vertex operator.)

$$W_-(z,u,k) = u^\alpha(k)\mathcal{O}(z)S_\alpha(z)e^{ik\cdot x_L(z)} \tag{9.65}$$

Here, u^α is a spinor analogue of the polarization tensors ε that entered the construction of NS states and vertex operators. The operator $\mathcal{O}(z)$ depends on the β and γ ghosts, and is a primary field of weight 3/8, so that W_- is of weight 1 for massless states (with $k^2 = 0$). $\mathcal{O}(z)$ is constructed in such a way that W_- is BRST invariant; it is an analogue of the spin field, but now for the superghost system.

The (β, γ) ghost system may be represented in terms of two free boson fields ϕ and χ, such that

$$\gamma = e^{\phi - \chi} \qquad \beta = e^{-\phi + \chi} \partial_z \chi \tag{9.66}$$

with

$$\begin{aligned} \phi(z)\,\phi(w) &\sim -\ln(z-w) \quad c_\phi = 13 \\ \chi(z)\chi(w) &\sim \ln(z-w) \quad c_\chi = -2 \end{aligned} \tag{9.67}$$

The ghost number current is $j_z = -\partial_z \phi$. The operator $\mathcal{O}(z)$ is again an exponential with weight $\frac{1}{2}$:

$$\mathcal{O}(z) = e^{-\frac{1}{2}\phi(z)} \,. \tag{9.68}$$

Because the central charge of the ϕ system is 13, the dimension of $\mathcal{O}(z)$ is 3/8, instead of 1/8, as would be the case for a field ϕ of central charge 1.

The above construction of W_- implies, though, that W_- has ghost number $-1/2$! But overall ghost number is given in terms of the index theorem, and this would suggest that the number of Ramond vertex operators that can be inserted on any Riemann surface is determined in terms of the topology of the surface, which is clearly absurd.

The resolution of this problem lies in the subtleties of the quantization of the (β, γ) system, which we shall not discuss here. The final result is that the quantization of the (β, γ) system yields vertex operators in different possible pictures, labeled by their ghost number. One shows, for example, that there is an equivalent vertex operator for W_-, which had ghost number $-1/2$. This operator is W_+ with ghost number $+\frac{1}{2}$ and is given by

$$W_+(z, u, k) = e^{\phi/2} u^\alpha (\gamma_\mu)_{\alpha\beta} S'^\beta(z)(\partial_z x^\mu - i\psi_+^\mu k \cdot \psi_+) e^{ik\cdot x} \tag{9.69}$$

Vertex operators for R states may now be inserted on any worldsheets in any arbitrary (even) numbers.

§9.6. The chiral splitting theorem

In critical RNS string theory, we have powerful results on the superholomorphicity properties of correlation functions for (factorized) vertex operators, at fixed internal momenta. This result is contained in the *Chiral Splitting Theorem* for the RNS string.

We begin by defining *factorized vertex operators.* A vertex operator

$$V_i = \int d^2 z_i \, d^2 \theta_i \, W_i(z_i, \theta_i, \bar{z}_i, \bar{\theta}_i)$$

is factorized provided W_i is the product of holomorphic and anti-holomorphic factors

$$W_i(z_i,\theta_i,\bar{z}_i,\bar{\theta}_i) = W_i(z_i,\theta_i)\widetilde{W}_i(\bar{z}_i,\bar{\theta}_i)$$

Any vertex operator may be written as a linear combination of factorized vertex operators. For example, for NS-NS massless states, we have

$$W_i(z_i,\theta_i;\bar{z}_i,\bar{\theta}_i) = \left(\xi_{i\mu}\mathcal{D}_+X^\mu e^{ik\cdot X_L}\right)\left(\bar{\xi}_{i\bar{\mu}}\mathcal{D}_-X^{\bar{\mu}}e^{ik\cdot X_R}\right) \tag{9.70}$$

Next, let A_I and B_I, $I = 1,\ldots,h$ be any canonical basis of homology 1-cycles with canonical intersection form. We define h independent internal loop momenta $p_I \in \mathbb{R}^{10}$, associated with each A_I-cycle, by

$$p_I = \oint_{A_I} dz\, d\theta\, \mathcal{D}_+X \tag{9.71}$$

Let the spin structures for left and right chirality spinors be equal, and equal to ν.

The correlation function of the unintegrated vertex operators W_i at fixed internal loop momenta p_I and fixed supermoduli are defined by*

$$\begin{aligned}&\langle W_1\ldots W_N\rangle_E(p_I)\\ &\equiv \int DX\int DBDC\; W_1\ldots W_N\prod_{I=1}^{h}\delta\left(p_I^\mu - \int_{A_I} dz\, d\theta\; \mathcal{D}_+X^\mu\right)\\ &\qquad \prod_{K=1}^{\dim\, s\mathcal{M}_h} |\,\langle\mu_K, B\rangle\,|^2 e^{-S_X-S_{BC}}\end{aligned} \tag{9.72}$$

(I) The first part of the *Chiral Splitting Theorem* states that the RNS amplitudes factorize at fixed internal loop momenta:

$$\langle W_1\ldots W_N\rangle_E(p_I) = \delta(k)\mathcal{C}_\nu^F\bar{\mathcal{C}}_\nu^F \tag{9.73}$$

Here, $\mathcal{C}_\nu{}^F = \mathcal{C}_\nu{}^F(z_i,\theta_i;\zeta_K;m_K;p_I,k_i)$ is a complex analytic function of supermoduli m_K, $K = 1,\ldots,\dim\, s\mathcal{M}_h$, of the insertion points (z_i,θ_i), and of the left-moving factors $\zeta_{i\mu}$ of the polarization tensors. $\bar{\mathcal{C}}_\nu^F$ is the complex conjugate of $\mathcal{C}_\nu{}^F$, with the same spin structure

$$\bar{\mathcal{C}}_\nu^F(\bar{z}_i,\bar{\theta}_i;\bar{\zeta}_i;\bar{m}_K;p_I,k_i) = \mathcal{C}_\nu{}^F(z_i,\theta_i;\zeta_i;m_K;p_I,k_i)^* \tag{9.74}$$

The functions C_ν^F may be evaluated explicitly using Green functions and determinants, using the prime form and the Szegö kernel introduced in Lecture 5.

(II) The second part of the Chiral Splitting Theorem prescribes the transition amplitudes for Type II and heterotic strings. We shall denote the analogous complex functions $\mathcal{C}^B = \mathcal{C}^B(z_i;m_k;p_I,k_i)$ for the bosonic string, compactified on the 16-dimensional lattices of $E_8\times E_8$ or $\mathrm{Spin}(32)/Z_2$.

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*It is understood that for genus $h = 0$ (resp. $h = 1$) three (resp. 1) of the vertex operators should not be integrated over, and should be multiplied by a factor of $\delta((C,\psi))$, where ψ is a superconformal Killing vector.

Type IIA, B:

$$A_h = \delta(k) \sum_{\nu,\bar{\nu}} Q_{\nu\bar{\nu}} \int_{\mathbb{R}^{10h}} d^{10}p_I \int_{s\mathcal{M}_h} dm_K \, d\bar{m}_K \prod_{i=1}^{N} \int_\Sigma d^2 z_i \, d^2\theta_i \, C_\nu^F \bar{C}_{\bar{\nu}}^F \tag{9.75}$$

Heterotic $E_8 \times E_8$, Spin(32)/Z_2

(9.76)

$$A_h = \delta(k) \sum_{\nu} Q_\nu \int_{\mathbb{R}^{10\,h}} d^{10}p_I \int_{s\mathcal{M}_h} dm_K \int_{\mathcal{M}_h} d\bar{m}_k \prod_{i=1}^{N} \int_\Sigma dz_i \, d\theta_i \int_\Sigma d\bar{z}_i \, C_\nu^F \bar{C}^B$$

Here $Q_{\nu\bar{\nu}}$ and Q_ν are spin structure dependent weight factors which realize the GSO projection. If carried out in the supermoduli picture, ν and $\bar{\nu}$ should run only over even or odd. But, if the integrals are understood in components, then one should sum over all individual spin structures in each class as well. In general $Q_{\nu\bar{\nu}} = \pm 1$, and $Q_\nu = \pm 1$.

§9.7. Tree-level amplitudes for NS-NS states

To tree level, the worldsheet has the topology of the sphere, or stereographically projected plane, and there are neither moduli nor odd moduli. Thus, we let $g = 2|dz|^2$ and $\chi = 0$. We concentrate on the scattering of massless NS-NS states, which can be represented by the vertex operator

$$V(\varepsilon, k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d^2 z \, d^2\theta \, \mathcal{D}_+ X^\mu \mathcal{D}_- X^\nu e^{ik\cdot x} \, . \tag{9.77}$$

By factorizing $\varepsilon_{\mu\nu}(k) = \zeta_\mu(k)\bar{\zeta}_\nu(k)$, we may start from a more convenient vertex operator

$$V(\zeta, \bar{\zeta}; k) = \int_\Sigma d^2 z \, d^2\theta \, e^{ik\cdot X + \xi\cdot\mathcal{D}_+ X + \bar{\xi}\cdot\mathcal{D}_- X} \tag{9.78}$$

and restrict to its contributions linear in ζ and linear in $\bar{\zeta}$. It is understood that ζ and $\bar{\zeta}$ are Grassmann parameters. The X-propagator is easily calculated:

$$\langle X^\mu(z,\theta) X^\nu(z',\theta') \rangle = -\ln |z - z' - \theta\theta'|^2 \, \eta^{\mu\nu} \tag{9.79}$$

The determinants of x, ψ, b, c and β, γ fields all combine into an overall constant, which we denote by $\mathfrak{z}$.

The chiral splitting theorem is easily used here: there are no internal loop momenta, so that the correlation function of the unintegrated vertex operators

$$W(z_i, \theta_i; \bar{z}_i, \bar{\theta}_i) = W_L(z_i, \theta_i) W_R(\bar{z}_i, \bar{\theta}_i) \tag{9.80}$$

will yield a product of a function $\mathcal{C}^F$, complex analytic in z_i, θ_i, times its complex conjugate. Here, the chiral vertex operators are

$$W_L(z_i, \theta_i) = e^{ik_i\cdot X_L(z_i,\theta_i) + \zeta_i\cdot\mathcal{D}_+ X_L(z_i,\theta_i)} \tag{9.81}$$

and analogously for W_R, where the chiral fields X_L are contracted with the effective propagator

$$\langle X_L^\mu(z,\theta) X_L^\nu(z',\theta')\rangle = -\ln(z - z' - \theta\theta')\eta^{\mu\nu} \ . \tag{9.82}$$

One readily finds

$$\begin{aligned} &\langle W_L(z_1,\theta_1)\cdots W_L(z_N,\theta_N)\rangle = \mathcal{C}^F(z_i,\theta_i) \\ &\mathcal{C}^F = \exp\sum_{i\neq j}^{N}\left\{ik_i\cdot\zeta_j\,\frac{\theta_{ij}}{z_{ij}} + \tfrac{1}{2}\zeta_i\cdot\zeta_j\,\frac{1}{z_{ij}} + \tfrac{1}{2}k_i\cdot k_j \ln z_{ij}\right\} \end{aligned} \tag{9.83}$$

where we have used the standard notation

$$\begin{cases} \theta_{ij} = \theta_i - \theta_j \ ; \\ z_{ij} = z_i - z_j - \theta_i\theta_j \ . \end{cases} \tag{9.84}$$

At tree level, we have invariance under superconformal automorphisms in $O\mathrm{Sp}(1,2)$; the integration measure must be properly treated to account for this invariance. This group of transformations is defined as follows; consider the matrices

$$K = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & A \end{pmatrix} \tag{9.85}$$

$$z \to \frac{az + b + \alpha\theta}{cz + d + \beta\theta} \qquad \theta \to \frac{\gamma z + \delta + A\theta}{cz + d + \beta\theta} \tag{9.86}$$

where Latin entries are commuting and Greek entries are anticommuting. To obtain a *superconformal* transformation T, the line element $dz = dz + \theta\, d\theta$ must be transformed into itself up to a conformal scaling. Equivalently, the form

$$z_{12} = z_1 - z_2 - \theta_1\theta_2 = \frac{v_1 w_2 - v_2 w_1 - \psi_1\psi_2}{w_1 w_2} \tag{9.87}$$

should transform into itself up to a conformal scaling. This is uniquely achieved when the orthosymplectic form is left invariant under T:

$$T^T K T = K \tag{9.88}$$

The weight under which the difference transforms is easily derived, and we have

$$T\colon\ z_{12} \to \bar{z}_{12} = \frac{z_{12}}{(cz_1 + d + \beta\theta_1)(cz_2 + d + \beta\theta_2)} \tag{9.89}$$

Similarly the line element transforms as

$$dz \to d\bar{z} = \frac{dz}{(cz + d + \beta\theta)^2} \tag{9.90}$$

and the volume element as

$$dz \wedge d\theta \to \frac{dz \wedge d\theta}{cz + d + \beta\theta} \,. \tag{9.91}$$

Elements in $O\mathrm{Sp}(1,2)$ are in unique correspondence with a triplet of points in the superplane (z_1,θ_1), (z_2,θ_2), (z_3,θ_3) obeying one single (Grassmann valued) constraint. The counting works out because $O\mathrm{Sp}(1,2)$ has 3 commuting and 2 anticommuting parameters. The constraint is an $O\mathrm{Sp}(1,2)$ invariant Grassmann valued function dependent on three points, given by

$$\Delta = \frac{z_{12}\theta_3 + z_{31}\theta_2 + z_{23}\theta_1 + \theta_1\theta_2\theta_3}{(z_{12}z_{23}z_{31})^{1/2}} \tag{9.92}$$

The natural value for Δ is of course 0, which implies that one θ is dependent. With this value for Δ, it is easy to see that there is a unique correspondence between triplets of points satisfying $\Delta = 0$ and elements of $O\mathrm{Sp}(1,2)$, so that the latter may be accordingly parametrized. The invariant volume element induced on $O\mathrm{Sp}(1,2)$ is then

$$d\mu = \frac{dz_1 dz_2 dz_3 d\theta_1 d\theta_2 d\theta_3}{(z_{12}z_{23}z_{31})^{1/2}}\,\Delta \tag{9.93}$$

As for the bosonic string, the volume of the superconformal transformation group must be extracted from the amplitudes. As a result of $O\mathrm{Sp}(1,2)$ invariance, the 0-, 1-, and 2-point functions vanish.

We now work out the three-point amplitude. In evaluating $\mathcal{C}^F$, one retains terms proportional to $\zeta_1\zeta_2\zeta_3$; however, the term with three θ's vanishes because $\theta_{12}\theta_{23}\theta_{31} = 0$. Defining the tensors

$$K_{\mu_1\mu_2\mu_3} = \eta_{\mu_1\mu_2}k_{1\mu_3} + \eta_{\mu_2\mu_3}k_{2\mu_1} + \eta_{\mu_3\mu_1}k_{3\mu_2} \tag{9.94}$$

we find for the three-point function

$$\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)\rangle = 4(2\pi)^{10}\delta(k)\varepsilon_1^{\mu_1\bar{\mu}_1}\varepsilon_2^{\mu_2\bar{\mu}_2}\varepsilon_3^{\mu_3\bar{\mu}_3}K_{\mu_1\mu_2\mu_3}K_{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3} \tag{9.95}$$

As a result of transversality of the polarization tensors, the 3-point function for massless external particles also vanishes.

To compute the four-point amplitude, we shall fix superconformal invariance as follows: $z = z_1$, $z_2 = 0$, $z_3 = 1$, $z_4 = \infty$, $\theta_1,\theta_2,\theta_3 = \theta_4 = 0$, and then have

$$\begin{aligned}\exp \mathcal{G}_4^{\zeta} = &\left\{\zeta_1\cdot\zeta_2\zeta_3\cdot\zeta_4\frac{1}{z_{12}z_{34}} + \zeta_1\cdot\zeta_3\zeta_2\cdot\zeta_4\frac{1}{z_{13}z_{24}} + \zeta_1\cdot\zeta_4\zeta_2\cdot\zeta_3\frac{1}{z_{14}z_{23}}\right\}\\ &+\left\{\zeta_1\cdot\zeta_2\left(k_1\cdot\zeta_3k_2\cdot\zeta_4\frac{\theta_2\theta_1}{z_{12}z_{13}z_{24}} + k_1\cdot\zeta_4k_2\cdot\zeta_3\frac{\theta_2\theta_1}{z_{12}z_{14}z_{23}}\right) + \text{perm.}\right\}\end{aligned} \tag{9.96}$$

In principle, one should now multiply this whole expression by the one involving the $\bar{\zeta}$'s, perform the integrals over z and θ and regroup terms, clearly a daunting

task. The calculation is enormously simplified by the factorization properties of the Veneziano integrals. Recall that we have the ordinary integrals

$$\int \frac{d^2z}{\pi} z^A \bar{z}^{\tilde{A}} (1-z)^B (1-\bar{z})^{\widetilde{B}} = \frac{\Gamma(-1-\tilde{A}-\widetilde{B})}{\Gamma(-\tilde{A})\Gamma(-\widetilde{B})} \quad \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+2)} \tag{9.97}$$

provided $A-\tilde{A}$ and $B-\widetilde{B}$ are integers, which is always the case in string theory. Using the reciprocity formula for Γ-functions and the fact that $A-\tilde{A}$ and $B-\widetilde{B}$ are integers, this expression is actually symmetric under $(A,B) \leftrightarrow (\tilde{A},\widetilde{B})$, as one might expect from complex conjugation. More importantly, the answer factorizes into a product of factors each only dependent either on the parameters for the z or $\bar{z}$ coordinates. This product property implies that one must only consider say the z coordinates to find the full amplitude, which by the same token will also completely factorize as a function of ζ's and $\bar{\zeta}$'s. An analogous formula is derived for the super-integrals we need:

(9.98)

$$\int \frac{d^2z_1}{\pi} d^2\theta_2 [\theta_1\theta_2]^a [\bar{\theta}_1\bar{\theta}_2]^{\tilde{a}} z_{12}^A \bar{z}_{12}^{\tilde{A}} (1-z_1)^B (1-\bar{z}_1)^{\widetilde{B}}$$
$$= (-2i)^{1-a}(+2i)^{1-\tilde{a}} \frac{\Gamma(-\tilde{a}-\tilde{A}-\widetilde{B})}{\Gamma(-\tilde{A})\Gamma(-\widetilde{B})} \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+1+a)}$$

Here a and $\tilde{a}$ are either 0 or 1, and the integrals are symmetric under $(aAB) \leftrightarrow (\tilde{a}\tilde{A}\widetilde{B})$. With the help of (9.98), it is now straightforward to evaluate the four-point function

(9.99)

$$\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)V(\varepsilon_4,k_4)\rangle$$
$$= (2\pi)^{10}\delta(k)g^4 \int d^2z_1 d^2\theta_2 |z_{12}|^{-s}|z_1-1|^{-u} e^{\mathcal{G}_4^{\zeta}+\mathcal{G}_4^{\bar{\zeta}}}$$
$$= \pi(2\pi)^{10}\delta(k)g^4 \frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)\gamma\left(+\frac{u}{2}\right)} \varepsilon^{1\bar{1}}\varepsilon^{2\bar{2}}\varepsilon^{3\bar{3}}\varepsilon^{4\bar{4}} K_{1234} K_{\bar{1}\bar{2}\bar{3}\bar{4}}$$

Using the abbreviation i for μ_i to save some writing, we have

$$\begin{aligned} K_{1234} &= (st\eta_{13}\eta_{24} - su\eta_{14}\eta_{23} - tu\eta_{12}\eta_{34}) \\ &\quad - s(k_1^4 k_2^2 \eta_{24} + k_2^3 k_4^1 \eta_{13} - k_1^3 k_4^2 \eta_{23} - k_2^4 k_3^1 \eta_{14}) \\ &\quad + t(k_2^1 k_4^3 \eta_{13} + k_3^4 k_1^2 \eta_{24} - k_2^4 k_1^3 \eta_{34} - k_3^1 k_4^2 \eta_{12}) \\ &\quad - u(k_1^2 k_4^3 \eta_{23} + k_3^4 k_2^1 \eta_{14} - k_1^4 k_2^3 \eta_{34} - k_3^2 k_4^1 \eta_{12}) \end{aligned} \tag{9.100}$$

§9.8. One-loop amplitudes for NS-NS states in Type IIA, B

We begin by using part (I) of the Chiral Splitting Theorem, and we compute the Chiral Amplitudes $\mathcal{C}_\nu^F$. The spin structure for left and right movers is the same here.

For *even spin structures* ν, there are no odd moduli, and there are no Dirac zero modes. The metric on the torus is taken to be $g = 2|dz|^2$, and the torus is represented as usual by a parallelogram with sides 0, 1, τ, $1+\tau$, with $\tau \in H$

in the upper half complex plane H. The S_X and S_{BC} actions are free actions in components:

$$S_X + S_{BC} = \frac{1}{4\pi}\int_\Sigma d^2z\Big(\partial_z x\cdot\partial_{\bar z}x - \psi_+\cdot\partial_{\bar z}\psi_+ - \psi_-\cdot\partial_z\psi_- + 2b\partial_{\bar z}c + 2\bar b\partial_z\bar c + 2\beta\partial_{\bar z}\gamma + 2\bar\beta\partial_z\bar\gamma\Big)\ . \tag{9.101}$$

The ghost and anti-ghost insertions are reduced to

$$b\bar b\, c\bar c\ . \tag{9.102}$$

No anti-superghost insertions occur for even spin structures, since there are no zero modes there. In the NS-NS sector, vertex operators do not involve ghosts, so we may integrate out b, c, $\bar b$, $\bar c$, β, γ, $\bar\beta$ and $\bar\gamma$. We shall also retain the determinants of x and $\psi_\pm$ integrations. Thus

$$\delta(k)\mathcal{C}_\nu\mathcal{C}_\nu = M_\nu\bar M_\nu\,\mathcal{F}_\nu\bar{\mathcal{F}}_\nu \tag{9.103}$$

where $M_\nu\,\bar M_\nu$ arises from determinants and $\mathcal{F}_\nu\,\bar{\mathcal{F}}_\nu$ arises from correlation functions of vertex operators. We find

$$M_\nu\bar M_\nu = \left(\frac{\det'\Delta_0}{(\operatorname{Im}\tau)^2}\right)^{-5}\left(\det\not{D}_+\right)_\nu^5\left(\det\not{D}_-\right)_\nu^5 \left(\frac{\det'\Delta_{-1}}{(\operatorname{Im}\tau)^2}\right)^{+1}(\det\Delta^-_{-1/2})_\nu^{-1} \tag{9.104}$$

As usual, on the torus, these determinants are related to one another, and we find

$$\frac{\det'\Delta_0}{(\operatorname{Im}\tau)^2} = \frac{\det'\Delta_{-1}}{(\operatorname{Im}\tau)^2} = |\eta(\tau)|^4$$
$$(\det\not{D}_+\not{D}_-)_\nu = (\det\Delta^-_{1/2})_\nu = \left|\frac{\vartheta[\nu](0|\tau)}{\eta(\tau)}\right|^2 \tag{9.105}$$

Putting all together:

$$M_\nu = \frac{\vartheta[\nu](0|\tau)^4}{\eta(\tau)^{12}} \qquad \bar M_\nu = \frac{\overline{\vartheta[\nu](0|\tau)^4}}{\overline{\eta(\tau)}^{12}} \tag{9.106}$$

up to an overall phase which is independent of τ, but which may depend upon ν. Clearly such a phase can be absorbed into the spin structure weights $Q_{\nu\bar\nu}$. The vertex contributions $\mathcal{F}_\nu$ are easily evaluated:

$$\mathcal{F}_\nu(z_i,\theta_i,\zeta_i,k,p) = \exp\{i\pi p^2\tau + 2\pi p\cdot\sum_i(-\zeta_i\theta_i + ik_iz_i) - \tfrac12\sum_{i\neq j}(k_i\cdot k_jG_\nu(z_i,\theta_i;z_j,\theta_j) + \zeta_i\cdot\zeta_j\mathcal{D}_+^i\mathcal{D}_+^jG_\nu + 2ik_i\cdot\zeta_j\mathcal{D}_+^jG_\nu)\}\ . \tag{9.107}$$

Here, the correlation function G_ν is given by

$$
\begin{aligned}
G_\nu(z_i,\theta_i;z_j,\theta_j) &= \langle X_L(z_i,\theta_i)X_L(z_j,\theta_j)\rangle \\
&= -\ln E(z_i,z_j) + \theta_i\theta_j S_\nu(z_i,z_j)
\end{aligned}
\tag{9.108}
$$

where $E(z,w)$ is the prime form, defined in Lecture 5, and S_ν is the Szegö kernel. For the torus, these functions are particularly simple:

$$
\begin{aligned}
E(z,w) &= \frac{\vartheta_1(z-w|\tau)}{\vartheta_1'(0|\tau)}\ ; \\
S_\nu(z-w) &= \frac{\vartheta[\nu](z-w|\tau)\vartheta_1'(0|\tau)}{\vartheta[\nu](0|\tau)\vartheta_1(z-w|\tau)}\ .
\end{aligned}
\tag{9.109}
$$

For *odd spin structures*, ν, there is now one complex odd modulus χ and there are 10 Dirac zero modes, one for each space-time direction. The action $S_X + S_{BC}$ is the one given on in (9.101) but there is now also an insertion of the worldsheet supercurrent involving

$$
\chi S_{z+}\ S_{z+} = \left(\psi_+\cdot\partial_z x - \tfrac{1}{2}b\gamma + \frac{3}{2}\beta\partial_z c + \partial_z\beta\, c\right) + c.c\ .
\tag{9.110}
$$

Integrating out χ will bring down an insertion of $\psi_+\cdot\partial_z x$ and of $\psi_-\cdot\partial_{\bar z}x$. Combining this with the ghost and anti-ghost insertions, we have

$$
b\bar b\, c\bar c\,\delta(\beta)\delta(\bar\beta)\delta(\gamma)\delta(\bar\gamma)\int S_{z+}\int S_{\bar z-}\ .
\tag{9.111}
$$

Now, we specialize to correlation functions of massless NS-NS states. As shown in (9.47), these operators involve at most 2 factors of ψ_+. To obtain a non-zero contribution from odd spin structures, we need to have at least 5 external vertex operators. For example, the anomaly diagrams with 6 external vertex operators receives contributions solely from odd spin structures. We shall concentrate on amplitudes with 4 or fewer external vertex operators, and odd spin structures will never contribute to those. Notice that this means that for these amplitudes, there is no difference between Type IIA, B. For completeness, we record that for ν odd, we have $M_\nu = M_{\bar\nu} = 1$.

It remains to work out the amplitudes with 0, 1, 2, 3, 4 external vertex operators. We begin with determining the factors $Q_{\nu\bar\nu}$ of spin structure weights. Following Mumford, we label ϑ-functions as follows:

$$
\begin{aligned}
\vartheta_{00}(z,\tau) &= \vartheta_3(z,\tau) = \vartheta(z,\tau) \\
\vartheta_{01}(z,\tau) &= \vartheta_4(z,\tau) = \vartheta\left(z+\tfrac{1}{2},\tau\right) \\
\vartheta_{10}(z,\tau) &= \vartheta_2(z,\tau) = e^{i\pi\tau/4+\pi iz}\vartheta\left(z+\frac{\tau}{2},\tau\right) \\
\vartheta_{11}(z,\tau) &= \vartheta_1(z,\tau) = e^{i\pi\tau/4+\pi iz}\vartheta\left(z+\tfrac{1}{2}+\frac{\tau}{2},\tau\right) \\
\vartheta(z,\tau) &\equiv \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau+2\pi inz}
\end{aligned}
\tag{9.112}
$$

Modular transformations of ϑ_2, ϑ_3, ϑ_4 obey

$$\begin{aligned} \vartheta_{ab}(0,\tau+1) &= e^{\frac{i\pi}{4}a}\vartheta_{a(b+a)}(0,\tau) \\ \vartheta_{ab}\left(0,-\frac{1}{\tau}\right) &= (-)^{ab}\sqrt{-i\tau}\,\vartheta_{ba}(0,\tau) \end{aligned} \tag{9.113}$$

The Dedekind η-function satisfies

$$\begin{aligned} \eta(\tau+1)^{12} &= -\eta(\tau)^{12} \\ \eta\left(-\frac{1}{\tau}\right)^{12} &= \tau^6\eta(\tau)^{12} \end{aligned} \tag{9.114}$$

Thus

$$\begin{aligned} M_{ab}(\tau+1) &= -e^{i\pi a}M_{a(b+a+1)}(\tau) \\ M_{ab}\left(-\frac{1}{\tau}\right) &= -\frac{1}{\tau^4}M_{ba}(\tau)\ . \end{aligned} \tag{9.115}$$

We must now require modular invariance of the amplitude:

(9.116a)

$$\sum_{(a,b)} Q_{(ab)\bar\nu}M_{ab}(\tau+1) = \sum_{(a,b)} Q_{(a,b)\bar\nu}M_{ab}(\tau)$$

(9.116b)

$$\sum_{(a,b)} Q_{(a,b)\bar\nu}M_{ab}(-1/\tau) = -\frac{1}{\tau^4}\sum_{(a,b)} Q_{(a,b)\bar\nu}M_{ab}(\tau)\ .$$

(The factor of $-1/\tau^4$ is compensated for by the p-integral.) Clearly from $\tau \to -1/\tau$, we get

$$Q_{(01)\bar\nu} = Q_{(10)\bar\nu} \tag{9.117}$$

while from $\tau \to \tau+1$, we get

$$Q_{(01)\bar\nu} = -Q_{(00)\bar\nu} \tag{9.118}$$

Setting

$$Q_{(01)\bar\nu} = Q_{(10)\bar\nu} = -Q_{(00)\bar\nu} \tag{9.119}$$

we fix the overall phase of the amplitude, which is unobservable anyway. One proceeds similarly for $\bar\nu$.

It is now straightforward to compute the 0-*point function.* (Recall that in the bosonic string this object produced the first divergence we had encountered.) Since there are no vertex operator insertions, we simply have

$$A_1(0-pt) = \mathrm{Vol}(M)\int_{\mathcal{M}_1}\frac{d^2\tau}{\tau_2^4}\sum_{\nu\bar\nu} Q_{\nu\bar\nu}M_\nu M_{\bar\nu} \tag{9.120}$$

It suffices to examine the combination

$$\sum_{\nu} Q_{\nu\bar{\nu}} M_{\nu} = Q_{(01)\bar{\nu}} M_{01} + Q_{(10)\bar{\nu}} M_{10} + Q_{(00)\bar{\nu}} M_{00} \tag{9.121}$$

Notice that since $M_{(11)} = 0$ the corresponding contribution has been omitted in (9.121). Now, using the expressions for M_{ν}, and the equality for Q's, we find

$$\sum_{\nu} Q_{\nu\bar{\nu}} = \frac{Q_{(01)\bar{\nu}}}{\eta(\tau)^{12}} \left[\vartheta_4(0|\tau)^4 + \vartheta_2(0|\tau)^4 - \vartheta_3(0|\tau)^4\right] = 0 \tag{9.122}$$

by a famous Jacobi ϑ-function identity. This identity arose once before in matching bosonic and fermionic multiplicities at all mass levels. In fact, this cancellation is the result of space-time supersymmetry. The 0-point function, and thus the 1-point functions vanish. Recall from the discussion of the bosonic string that the vanishing of the 1-point function means that Minkowski flat space-time is a one-loop solution to the superstring equations!

Notice that our argument for cancellation was effected only in the left-moving sector; thus, the same argument holds true in heterotic string theory as well.

The above summation identity on ϑ-functions is only one of a series, usually called the Riemann identities on theta functions. It is most useful here to express two more of these identities in terms of the Dirac propagator (i.e. the Szegö kernel):

$$\begin{aligned} &\sum_{\nu} Q_{\nu\bar{\nu}} M_{\nu} S_{\nu}(z_1 - z_2) S_{\nu}(z_2 - z_1) = 0 \\ &\sum_{\nu} Q_{\nu\bar{\nu}} M_{\nu} S_{\nu}(z_1 - z_2) S_{\nu}(z_2 - z_3) S_{\nu}(z_3 - z_1) = 0 \end{aligned} \tag{9.123}$$

Without giving details here, one similarly shows with the help of these identities that the two- and three-point functions vanish. In physics terms, this means that to one-loop order, there is neither mass renormalization (massless particles remain massless) nor coupling constant renormalization. These results again arise from space-time supersymmetry, and are usually referred to as *non-renormalization theorems.*

§9.9. One-loop amplitudes in the heterotic string

Let us consider the free fermion representation of the heterotic strings, starting from 32 Majorana-Weyl fermions λ^{α}, represented by 32 Euclidean Weyl spinors $\lambda^{\alpha}_{-}(\bar{z})$. For simplicity, consider amplitudes with external states that are neutral under the gauge group. Then the one-loop contribution of the internal degree of freedom will result entirely from the determinantal factors arising from integrating out λ^{α}_{-}. These depend upon the spin structures, just as the ones for space-time degrees of freedom ψ^{μ}_{-}.

For the gauge group $\mathrm{Spin}(32)/Z_2$, all λ^{α}_{-} have the same spin structure, and the determinant for given spin structure contributes a factor

$$\bar{M}^4_{\bar{\nu}} = \frac{\overline{\vartheta[\bar{\nu}](0|\tau)}^{16}}{\overline{\eta(\tau)}^{48}} \tag{9.124}$$

The only modular invariant we can form is with all coefficients $Q_{\bar{\nu}}$ equal for all 3 even spin structures.

For the gauge group $E_8 \times E_8$, λ_-^α are divided into two classes of 16. For each class, the determinant is

$$\bar{M}_{\bar{\nu}}^2 = \frac{\overline{\vartheta[\nu](0|\tau)}^8}{\overline{\eta(\tau)}^{24}} \,. \tag{9.125}$$

The only modular invariant is

$$\sum_{\bar{\nu}} \bar{M}_\nu^2 = \frac{\overline{\vartheta_2(0|\tau)}^8 + \overline{\vartheta_3(0|\tau)}^8 + \overline{\vartheta_4(0|\tau)}^8}{\overline{\eta(\tau)}^{24}} \,. \tag{9.126}$$

One can show, using relations on the number of representations of an integer as a sum of squares, that this function equals the ϑ-function for the E_8 lattice: $\vartheta_{E_8}(0|\tau)$. Furthermore, the partition functions for $\mathrm{Spin}(32)/Z_2$ and $E_8 \times E_8$ are the same in view of

$$\sum_{\bar{\nu}} \bar{M}_{\bar{\nu}}^4 = \left(\sum_{\bar{\nu}} \bar{M}_{\bar{\nu}}^2 \right)^2 \,. \tag{9.127}$$

We see that no other modular invariant combinations are allowed. You may think that $\sum_\nu Q_\nu M_\nu = 0$ is allowed, but this would mean that some states enter with negative weight, and this is impossible since there are no fermions here! This argument shows now that $E_8 \times E_8$ and $\mathrm{Spin}(32)Z_2$ are indeed the only possibilities, as we had promised.

§9.10. The NS-NS 4-point function

It remains to discuss the 4-point functions, again for massless NS-NS states. Clearly, the 4-point function cannot vanish, lest the theory be non-interacting. It is evaluated by making use of a Riemann identity on ϑ-functions with 4 Szegö kernels:

$$\sum_\nu Q_{\nu\bar{\nu}} M_\nu S_\nu(z_1 - z_{\sigma(1)}) S_\nu(z_2 - z_{\sigma(2)}) S_\nu(z_3 - z_{\sigma(3)}) S_\nu(z_1 - z_{\sigma(4)}) = 16\pi^4 Q_{(00)\bar{\nu}} \tag{9.128}$$

where $\sigma(i)$ is a permutation of i, *without fixed points.*

The final result for the 4-point function can be conveniently expressed as follows:

$$A_1(k_i, \varepsilon_i) = \delta(k) \varepsilon^{1\bar{1}} \varepsilon^{2\bar{2}} \varepsilon^{3\bar{3}} \varepsilon^{4\bar{4}} K^{1234} K^{\bar{1}\bar{2}\bar{3}\bar{4}} A_1(s,t) \tag{9.129}$$

where the single Lorentz-invariant function $A_1(s,t)$ is given by

$$A_1(s,t) = \frac{1}{2} \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \int_\Sigma \frac{d^2 z_1}{\tau_2} \int_\Sigma \frac{d^2 z_2}{\tau_2} \int_\Sigma \frac{d^2 z_3}{\tau_2} e^{\frac{s}{2}(G_{12}+G_{34}-G_{13}-G_{24})} e^{\frac{t}{2}(G_{23}+G_{14}-G_{13}-G_{24})} \tag{9.130}$$

where G_{ij} stands for the 1-loop propagator of the x-field:

$$G_{ij} = G(z_i - z_j|\tau) \equiv -\ln \left| \frac{\vartheta_1(z_i - z_j|\tau)}{\theta_1'(0|\tau)} \right|^2 + \frac{2\pi}{\tau_2} \operatorname{Im}(z_i - z_j)^2 \tag{9.131}$$

and K is the kinematical factor that also arose in the tree-level amplitudes $A_0(k_i, \epsilon_i)$ for NS-NS transition amplitudes. Recall that K is polynomial in momenta. By translation invariance on the torus, the point z_4 may be chosen at will, say at $z_4 = 0$.

It remains to study the function $A_1(s,t)$! We begin by examining the convergence structure of the above functional representation as a function of s and t.

(1) For fixed τ, and $z_i \to z_j$, $i \neq j$, we have

$$G(z_i - z_j|\tau) \to -\ln |z_i - z_j|^2 \to +\infty \tag{9.132}$$

(2) while for z_i's well separated, but $\operatorname{Im}\tau \to \infty$, we have

$$G(z_i - z_j|\tau) \to -\tau_2 \to -\infty\ . \tag{9.133}$$

The latter limit imposes the strictest convergence condition:

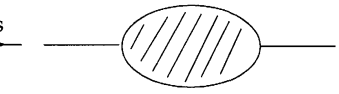
$$\operatorname{Re}(s),\ \operatorname{Re}(t),\ \operatorname{Re}(u) \geq 0 \tag{9.134}$$

But, since we also have $s + t + u = 0$ for on-shell massless external states, the convergence condition (9.134) requires $\operatorname{Re}(s) = \operatorname{Re}(t) = \operatorname{Re}(u) = 0$. It is easily seen that this condition is sufficient for convergence, since the integral is bounded by the value at $s = t = 0$. This means that the integral is absolutely convergent for at least some values of the external momenta, in contrast with the case of the bosonic string, where the integral representation is never convergent for any momenta. Yet, the integral representation is defined only for unphysical momentum configurations where s, t, u are purely imaginary.

A second shortcoming of the integral representation is that formally, when s, t, u are close to the physical region corresponding to real values of s and t, $A_1(s,t)$ would be real. This would be inconsistent in an interacting theory, where the 1-loop 4-point function cannot be real, since its imaginary part is related by unitarity to the square of the tree level amplitude, which cannot vanish in any interacting theory.

Our task is thus to properly define the amplitude throughout $(s,t) \in \mathbb{C} \times \mathbb{C}$, including around the physical region where $(s,t) \in \mathbb{R} \times \mathbb{R}$. Naturally, one seeks an analytic continuation of the amplitude in both s and t. (Recall that even at tree level, the integral representation $A_0(s,t)$ was absolutely convergent only in a small region, and the full amplitude had to be analytically continued in s and t as well.)

To appreciate the physical meaning of such an analytic continuation, we compare with the situation in quantum field theory. There, the physical principle of causality is equivalent to the requirement of locality of observable fields, and this property, combined with Lorentz invariance, implies analytic dependence of the Green functions on the external momenta. Often, this analytic behavior is conveniently summarized in the form of a spectral representation. For example, the 2-point function (off-shell) may be expressed as

(9.135) 

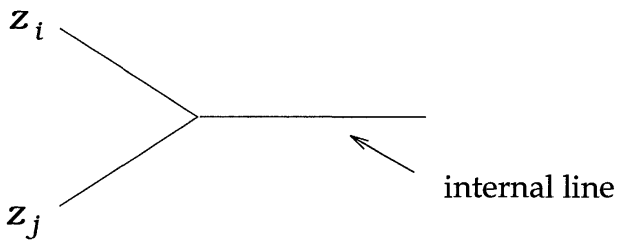
$$H(s) = \int_0^\infty dM^2 \, \frac{\rho(M^2)}{s - M^2 + i\varepsilon} \qquad \varepsilon > 0$$

where the real and positive function $\rho(M^2)$ is the density of states at mass M, also called the spectral density. Provided the integral over M^2 converges, it follows that $H(s)$ is analytic throughout $\mathbb{C}$, except possibly on the positive real axis, where it may have poles and branch cuts.

In string theory, we do not have available at present a formulation of the amplitudes in terms of local field observables (i.e. commuting at space-like separations). It is thus not known a priori whether analyticity of the amplitudes can be derived solely from Lorentz invariance and causality. (Probably, in a good second quantized formulation, one should be able to derive this.) Thus, analyticity of string amplitudes must be established, and the questions that must be answered are

(1) does an analytic continuation exist?
(2) is the analytic continuation unique?
(3) is the analytic continuation physically acceptable?

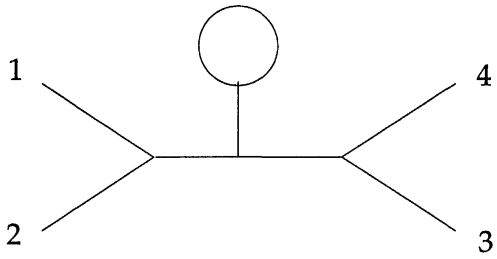
The present problem of analytic continuation of $A_1(s,t)$ is a difficult matter of pure analysis. Yet, all three questions above have been answered in the affirmative, so that one has a full analytically continued amplitude $A_1(s,t)$ available, defined throughout $(s,t) \in \mathbb{C} \times \mathbb{C}$, with cuts and poles on $\mathbb{R} \times \mathbb{R}$, but nowhere else. What we have achieved here is the construction of a *finite* and *unitary* amplitude, at least to one loop-order, that defines NS-NS string scattering! We shall not reproduce the details of the analytic continuation procedure here. We shall limit ourselves to describing the physical singularities that appear in $A_1(s,t)$ upon analytic continuation. To do so it is most convenient to represent the different singularities that appear as a function of s and t in terms of QFT Feynman diagrams. The Feynman diagrams indicate the region in moduli space (including the positions of vertex operators z_i) that produce the singularities. The singularities essentially arise from combinations of the singular limits of G_{ij} in (9.132) and (9.133), and fall into five categories, schematically indicated by five different types of Feynman diagrams below. Whenever two external z_i and z_j approach one another the Feynman diagram representation will show that their external lines merge



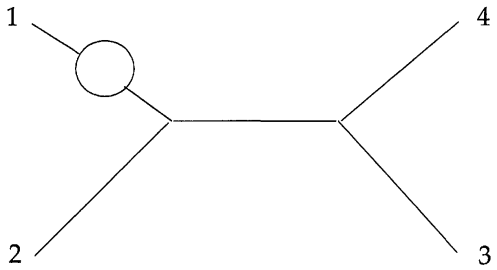
This singularity may be treated at the operator level by considering the operator product expansion (OPE) of the two external vertex operators. The internal line then represents all the operators that arise in the OPE. This type of singularity produces a pole in the total momentum traversing the internal line, i.e. in $-(k_i + k_j)^2$, and the pole turns out to be at the value of the mass^2 of the physical

string states, i.e. $2n = M^2$, $n \in \mathbb{Z}^+$. The five different categories represented by diagrams (a), (b), (c), (d) and (e) arise, respectively, from 4, 3, 2, 1 and 0 external lines coming close together. Within each category, there are additional singularities (represented by a dotted line across the loop) that arise from the asymptotic behavior in τ, given by (9.133).

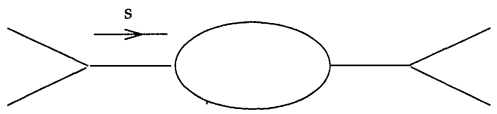
(a)



(b)

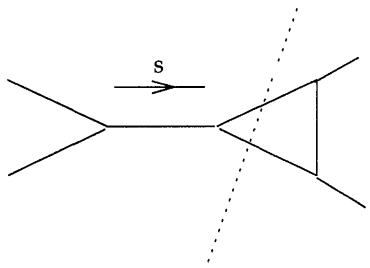


(c)



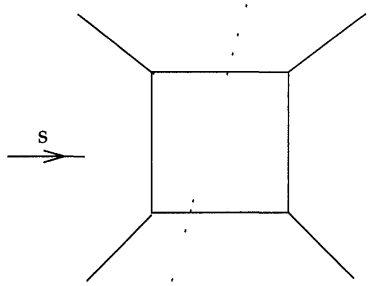
- double poles in s at $s = 2n$
- branch cuts in s, starting at $s = 2n, \quad n \in \mathbb{Z}^+$.

(d)



- single poles in s at $s = 2n$
- branch cuts in s, starting at $s = 2n, \quad n \in \mathbb{Z}^+$.

(e)



- no poles
- branch cuts in s, starting at $s = 2n, \quad n \in \mathbb{Z}^+$ with spectral density $\rho(2n, t)$
- $\rho(2n, t)$ has branch cuts in t, starting at $t = 2m$, $m \in \mathbb{N}^0$.

It is understood that within each category, we also include permutation of the diagrams shown, for example by adding diagrams on which the external legs have been permuted.

Now, for superstrings, diagrams in categories (a) and (b) vanish identically. This follows from the fact that the on-shell 1- and 2-point functions for massless

states vanish identically. The singularities of the remaining diagrams are precisely the ones that one would expect on physical grounds, and their structure confirms the consistency of the string theory up to this order. A complete proof is in **[E. D'Hoker and D. H. Phong (1995)]**.

References and cross references

The super Riemann surface approach to superstring quantization was initiated in

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and developed in

E. D'Hoker and D. H. Phong, *The geometry of string perturbation theory*, Rev. Modern Phys. **60** (1988), 917–1065.

The construction of the Ramond vertex operator and the BRST quantization of the superstring using superconformal field theory was carried out in

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The Chriral Splitting Theorem was stated and proven in part in **[D'Hoker and Phong (1988)]** with a detailed proof given in

E. D'Hoker and D. H. Phong, *Conformal scalar fields and chiral splitting on super Riemann surfaces*, Comm. Math. Phys. **125** (1989), 469.

K. Aoki, E. D'Hoker and D. H. Phong, *Unitarity of closed superstring perturbation theory*, Nucl. Phys. B **342** (1990), 149–230.

In **[Aoki, D'Hoker and Phong (1990)]**, a proof of the unitarity of superstring perturbation theory was given in flat Minkowski space-time or more generally in any space-time of the form $\mathbf{R}^2 \times M^8$, where M^8 has positive definite metric. The proof of the existence of the analytic continuations of the NS-NS four-point functions discussed in §9.10, as well as the explicit construction of the analytically continued amplitudes was given in

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The fact that Type IIA and B superstring theories are expected to be ultraviolet finite was first advocated in the following seminal paper

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LECTURE 10
Supersymmetry and Supergravity

On many occasions during the discussion of superstring theory we have obtained results that are consistent with the presence of space-time supersymmetry. These include the counting of the number of bosonic versus fermionic states at all mass levels, as well as the study of the spectrum at zero mass. In this section, we shall exhibit supersymmetry on the spectrum and on the scattering amplitudes in an explicit fashion. We begin by completing the study of the RNS formulation of string theory, and use the Ramond (fermion) vertex operator to produce a conserved supercharge that generates space-time supersymmetry. Next, we introduce and study the Green-Schwarz formulation of superstring theory, in which space-time supersymmetry is manifestly realized. Finally, we exhibit the low energy field theory approximations to the various superstring theories we have introduced, and identify them with known supergravity theories, in which we have local or gauged supersymmetry. We show how the Green-Schwarz formulation of superstrings may be used to couple strings to supergravity in a covariant way.

§10.1. Global space-time supersymmetry in the RNS formulation

In the previous chapter, we have provided a complete prescription for the vertex operators and for the calculation of transition amplitudes in the RNS formulation of superstrings. The RNS formulation does not exhibit space-time supersymmetry in a manifest fashion however: the construction of bosonic and fermionic vertex operators is asymmetrical.

Nonetheless, it is possible to construct a conserved current $j_\alpha(z)$ on the world-sheet, whose charge Q_α, defined by

$$Q_\alpha = \oint \frac{dz}{2\pi}\, j_\alpha(z) \tag{10.1}$$

obeys the supersymmetry algebra of 10-dimensional space-time:

$$\begin{aligned} [P_\mu, P_\nu] &= [P_\mu, Q_\alpha] = 0 \\ \{Q_\alpha, Q_\beta\} &= 2\,\Gamma^\mu_{\alpha\gamma}(C^{-1})^\gamma{}_\beta\, P_\mu = 2\,\gamma^\mu_{\alpha\beta}\, P_\mu \end{aligned} \tag{10.2}$$

Here P_μ is the translation operator, and we use the Dirac algebra defined in Lecture 9. For $\mathcal{N} = 2$ supersymmetric theories such as Type IIA, B, there will be an

independent supercharge Q' arising from the $\bar{z}$ sector of the theory, whose discussion is analogous to that of Q. The current $j_\alpha(z)$ is related to the Ramond vertex operator. To see this, we notice that Q_α should map states of given M^2 into states of the same M^2, and should transform as a spinor. The R vertex operator, at zero space-time momentum, precisely has these properties. Thus, we identify $j_\alpha(z)$ with

$$j_\alpha(z) = e^{-\phi/2}(z) S_\alpha(z) \tag{10.3}$$

of ghost number $-1/2$, or

$$j_\alpha(z) = e^{\phi/2}(\gamma_\mu)_{\alpha\beta} S'^\beta(z) \partial_z x^\mu \tag{10.4}$$

of ghost number $1/2$. Both are holomorphic and, with the help of

$$\begin{aligned} e^{-\phi/2}(z) e^{\phi/2}(w) &\sim \frac{1}{(z-w)^{-1/4}} \\ S_\alpha(z) S'^\beta &\sim \frac{\delta_\alpha^\beta}{(z-w)^{5/4}} \end{aligned} \tag{10.5}$$

we find

$$j_\alpha(z) j_\beta(w) \sim (\gamma_\mu)_{\alpha\beta} \partial_z x^\mu \frac{1}{z-w} \tag{10.6}$$

and recover the superalgebra of Q_α and P_μ upon integration.

Under the action of Q_α, vertex operators in the NS and R sectors are mapped into one another. Considering left moving vertex operators for massless states, for example,

$$\begin{aligned} W_{\mathrm{NS}}(z, \varepsilon, k) &= \varepsilon_\mu \partial_z x^\mu e^{ik\cdot x} \\ W_{\mathrm{R}}(z, u, k) &= e^{-\phi/2} S_\alpha u^\alpha e^{ik\cdot x} \end{aligned} \tag{10.7}$$

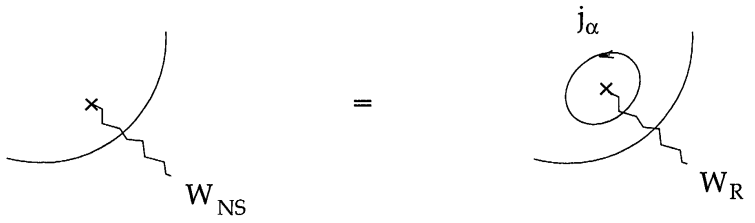
where we have considered W_{NS} in the picture with zero ghost number and W_{R} in the picture with ghost number $-1/2$. We have

$$\begin{aligned} [\zeta^\alpha Q_\alpha, W_{\mathrm{NS}}(z, \varepsilon, k)] &= W_{\mathrm{R}}(z, u, k) \\ [\zeta^\alpha Q_\alpha, W_{\mathrm{R}}(z, v, k)] &= W_{\mathrm{NS}}(z, \eta, k) \end{aligned} \tag{10.8}$$

with

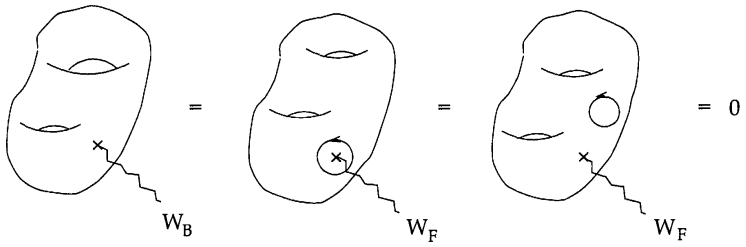
$$\begin{aligned} u^\alpha &= k^\mu \epsilon^\nu (\gamma_{\mu\nu})^\alpha{}_\beta \zeta^\beta \\ \eta^\mu &= v^\alpha (\gamma^\mu)_{\alpha\beta} \zeta^\beta \end{aligned} \tag{10.9}$$

Pictorially, we may represent this action by



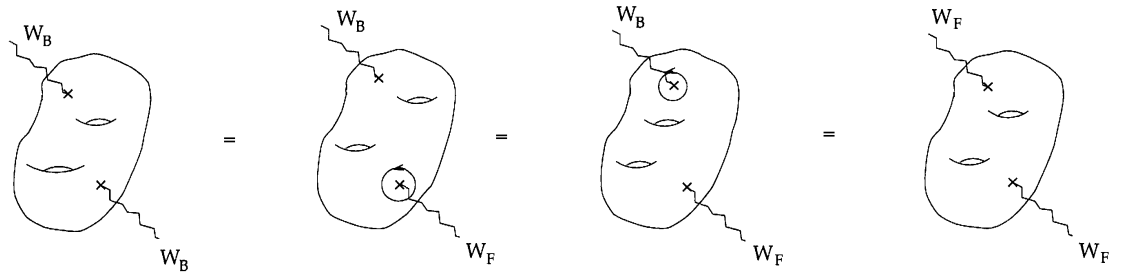
and similarly for W_{R} insertions.

From the action of supersymmetry, we obtain several important relations between transition amplitudes. For the one-point function, we have



by conservation of the current $j_\alpha(z)$. The only vertex operator that enters is the dilaton at zero momentum, and the above argument now shows directly that this amplitude must vanish, as we have indeed established to tree and one-loop level, by explicit calculation.

For the two-point function, we obtain



By chiral symmetry, the W_F two-point function must vanish for massless fermions, and hence the W_B two-point function must also vanish, as we have indeed established to tree and one-loop level, by explicit calculation.

It is hoped that the use of the supersymmetry currents will ultimately allow one to show that superstring theory in the RNS formulation is finite to all orders, but subtleties with the superghost system have prevented us so far from carrying out this program in full.

§10.2. The Green-Schwarz formulation

The string degrees of freedom in the Green-Schwarz (GS) formulation of superstring theory (in D space-time dimensions) are

(1) the bosonic coordinate x^μ, $\mu = 0, 1, \ldots, D-1$

(2) the fermionic coordinate $\theta^{\alpha I}$

The x^μ coordinates are as before. The $\theta^{\alpha I}$ coordinates are spinors under $\mathrm{Spin}(1, D-1)$, labeled by the spin index α, with $I = 1, 2, \ldots, \mathcal{N}$ describing the degree of extension of the supersymmetry algebra of the Poincaré group. $\theta^{\alpha I}$ are Grassmann valued worldsheet scalars. For simplicity, we shall assume that $\mathrm{Spin}(1, D-1)$ admits Majorana spinors, and we shall use Majorana notation. (For $D = 10$, $\theta^{\alpha I}$ are Majorana-Weyl spinors, with $\mathcal{N} = 1$ for Type I and heterotic and $\mathcal{N} = 2$ for Type II.)

Poincaré supersymmetry acts on the coordinates x^μ and $\theta^{\alpha I}$ as follows (we use the conventions of §9):

$$\text{(10.10)} \qquad \begin{cases} \delta\theta^I = \varepsilon^I \\ \delta\bar{\theta}^I = \bar{\varepsilon}^I \\ \delta x^\mu = i\bar{\varepsilon}^I \Gamma^\mu \theta^I \end{cases}$$

where ε^I is a constant set of Majorana spinors, labeled by $I = 1, \ldots, \mathcal{N}$.

We seek a worldsheet action in terms of x^μ and θ^I for the various superstring theories, that possesses manifest space-time supersymmetry as defined above.

As a warm-up, we obtain the action for a massless point particle, propagating on a worldline L, parametrized by τ (and $\dot{} = \frac{d}{d\tau}$). To do so, we need a generalization of $\dot{x}^\mu$ that has good supersymmetry transformation properties. The following combination is easily seen to be *invariant* under supersymmetry

$$\Pi^\mu \equiv \dot{x}^\mu - i\bar{\theta}^I \Gamma^\mu \dot{\theta}^I \tag{10.11}$$

and the following worldline action is SUSY invariant as well as $Diff(L)$-invariant:

$$S = \int_L d\tau\, e^{-1}(\tau)(\dot{x}^\mu - i\bar{\theta}^I \Gamma^\mu \dot{\theta}^I)^2 \tag{10.12}$$

Here, e is the worldline metric. The variational equations are

$$\Pi^2 = 0, \quad \dot{\Pi}^\mu = 0, \quad \Gamma_\mu \Pi^\mu \dot{\theta}^I = 0 \tag{10.13}$$

Interpreting Π^μ as particle momentum, we find that the particle propagates with constant momentum, that it is massless, and that because $(\Gamma \cdot \Pi)^2 = 0$, half of the θ^J's are unconstrained. This last property is due to the fact that S exhibits a further symmetry, called *κ-symmetry*:

$$\begin{cases} \delta\theta^I = i\Gamma \cdot \Pi \kappa^I \\ \delta x^\mu = i\bar{\theta}^I \Gamma^\mu \delta\theta^I \\ \delta e = 4e\dot{\bar{\theta}}^I \kappa^I \end{cases} \tag{10.14}$$

The presence of κ-symmetry guarantees that only half of the θ's are physical degrees of freedom of the particle.

We now generalize the above construction to obtain a worldsheet action with space-time supersymmetry. There is an obvious generalization of momentum, which is space-time supersymmetric:

$$\Pi_m{}^\mu \equiv \partial_m x^\mu - i\bar{\theta}^I \Gamma^\mu \partial_m \theta^I \tag{10.15}$$

Thus there is an obvious worldsheet action (with worldsheet metric g, which does not transform under supersymmetry)

$$S_1[x, \theta] = \frac{1}{8\pi} \int_\Sigma d\mu_g g^{mn} \Pi_m{}^\mu \Pi_n{}^\nu \eta_{\mu\nu} \tag{10.16}$$

invariant under $Diff(\Sigma)$, $Weyl(\Sigma)$ and space-time supersymmetry. However, the κ-symmetry of the point particle does not extend to S_1. This is a problem, because in $D = 10$, $N = 1$ superstrings, e.g., we will now have 8 physical degrees of freedom of x^μ, but 16 for θ. The κ-symmetry is required to have the correct on-shell particle content of the string spectrum.

When $\mathcal{N} = 1$ or $\mathcal{N} = 2$, the action S_1 may be modified to $S = S_1 + S_2$ in such a way that S is $Diff(\Sigma)$, $Weyl(\Sigma)$, space-time supersymmetric, but also invariant under κ-supersymmetry. The additional part S_2 is found to be

$$\begin{aligned} S_2[x, \theta] = \frac{1}{4\pi} \int_\Sigma \{ & - i\, dx^\mu \wedge (\bar{\theta}^1 \Gamma_\mu d\theta^1 - \bar{\theta}^2 \Gamma_\mu d\theta^2) \\ & \bar{\theta}^1 \Gamma_\mu d\theta^1 \wedge \bar{\theta}^2 \Gamma^\mu d\bar{\theta}^2 \} \end{aligned} \tag{10.17}$$

Actually, the space-time supersymmetry of S_2 is not automatic. Restricting to $\mathcal{N} = 1$ only for simplicity, the ε-transform of S_2 yields

$$\delta S_2 = \frac{1}{4\pi}\int_\Sigma (\bar\varepsilon\Gamma^\mu d\theta \wedge \bar\theta\Gamma_\mu d\theta - i\, dx^\mu \wedge \bar\varepsilon\Gamma_\mu d\theta) \tag{10.18}$$

The second term in δS_2 is a total derivative and leads to no further requirements. The vanishing of the first term in δS_2 requires that a certain Γ-matrix identity be satisfied:

$$\Gamma^\mu\psi_1(\bar\psi_2\Gamma_\mu\psi_3) + \Gamma^\mu\psi_2(\bar\psi_3\Gamma_\mu\psi_1) + \Gamma^\mu\psi_3(\bar\psi_1\Gamma_\mu\psi_2) = 0 \tag{10.19}$$

This identity can be realized in the following situations:

$$\begin{array}{llllll} & \text{(i)} & D = 10, & \theta & \text{Majorana-Weyl} & 8\ \theta\text{'s},\ 8\ x\text{'s} \\ & \text{(ii)} & D = 6, & \theta & \text{Weyl} & 4\ \theta\text{'s},\ 4\ x\text{'s} \\ (10.20) & \text{(iii)} & D = 4, & \theta & \text{Majorana} & 2\ \theta\text{'s},\ 2\ x\text{'s} \\ & \text{(iv)} & D = 3, & \theta & \text{Majorana} & 1\ \theta,\ 1\ x \quad . \end{array}$$

(Note, for $D = 6$, certain formulas must be adapted, since the corresponding spinors θ are not Majorana.)

The κ-symmetry may be expressed in terms of a worldsheet vector κ_m^J, which is also a space-time spinor:

$$\begin{cases} \delta\theta^I = 2i\,\Gamma\cdot\Pi_m\kappa^{Im} \\ \delta x^\mu = \bar\theta^I\Gamma^\mu\delta\theta^I \\ \delta g_{mn} = \quad \dots \end{cases} \tag{10.21}$$

The worldsheet vector κ_m^I produces a 1-form $\kappa^I \equiv \kappa_m^I d\xi^m$ with the following self-duality properties

$$\begin{cases} {}^*\kappa^1 = -i\,\kappa^1 \\ {}^*\kappa^2 = i\,\kappa^2 \end{cases} \tag{10.22}$$

The κ-symmetry closes onto a combination involving a κ-symmetry, a $Diff(\Sigma)$, $Weyl(\Sigma)$ transformation, and an additional bosonic symmetry that has no dynamical consequences.

On a worldsheet with flat metric g, the field equations for g, x and θ are

$$\begin{array}{ll} \bullet & \Pi_z{}^2 = \Pi_{\bar z}{}^2 = 0 \\ (10.23) \quad \bullet & \partial_z(\partial_{\bar z}x^\mu - 2i\bar\theta^1\Gamma^\mu\partial_{\bar z}\theta^1) + \partial_{\bar z}(\partial_z x^\mu - 2i\bar\theta^2\Gamma^\mu\partial_z\theta^2) = 0 \\ \bullet & \Gamma_\mu\Pi_z^\mu\partial_{\bar z}\theta^1 = \Gamma_\mu\Pi_{\bar z}^\mu\partial_z\theta^2 = 0\ . \end{array}$$

These equations of motion are highly nonlinear. As a result, quantization of the GS string in covariant form is fraught with difficulties, arising in particular from the fact that θ's do not have a standard kinetic term. No satisfactory approach exists to date.

§10.3. Lightcone gauge quantization of the GS formulation

In the lightcone gauge, the nonlinearities of the GS field equations disappear, and quantization may be carried out in complete parallel with the lightcone quantization of the RNS string (Problems 17 and 18). As usual, we impose

$$x^+ = q^+ + p^+(z+\bar{z}) \tag{10.24}$$

and assume that $p^+ \neq 0$. (If $p^+ = 0$, then by positivity of energy, we must have $p = 0$, and we only have the vacuum state.) Hence

$$\Pi_z^+ = \Pi_{\bar{z}}^+ = p^+ \tag{10.25}$$

On-shell, we have $\Pi_z^2 = \Pi_{\bar{z}}^2 = 0$, so at a given point on the worldsheet, we may rotate Π so that $\Pi^- = \Pi^i = 0$, with only p^+ remaining as the only non-vanishing component. At that point, the κ-transformation on θ^I becomes

$$\delta\theta^I = 2i\,\Gamma^- p^+(\kappa^{Iz} + \kappa^{I\bar{z}})\ . \tag{10.26}$$

Thus, by suitable κ-transformation, all θ^I's in the image of Γ^- may be rotated away. Thus, by κ-symmetry, we may impose the additional gauge choice

$$\Gamma^+\theta^I = 0 \tag{10.27}$$

This choice implies that the equations for $\partial_{\bar{z}}\theta^1$ and $\partial_z\theta^2$ simplify; to see this, multiply each to the left by Γ^+, and use $\{\Gamma^+,\Gamma^-\} = 2$:

$$\partial_{\bar{z}}\theta^1 = \partial_z\theta^2 = 0 \tag{10.28}$$

As a result, the equation for x^μ linearizes as well

$$\partial_z\partial_{\bar{z}}x^\mu = 0\ . \tag{10.29}$$

The only remaining non-linearity is the mass-shell constraint, given by

$$\begin{aligned} \Pi_z^2 &= 2p^+\Pi_z^- + \Pi_z^i\Pi_z^i = 0 \\ \Pi_{\bar{z}}^2 &= 2p^+\Pi_{\bar{z}}^- + \Pi_{\bar{z}}^i\Pi_{\bar{z}}^i = 0\ . \end{aligned} \tag{10.30}$$

As always in the lightcone gauge, the Lorentz group is reduced (in $D = 10$) to SO(8), under which x^i transforms as a vector 8_v, while θ^I (or more precisely, its eight dimensional restriction S^I) transforms as a spinor of SO(8): 8_s or 8_c. These representations are permuted into one another under *triality* of SO(8).

The lightcone quantization of the GS string is equivalent to the lightcone quantization of the RNS superstring. Let us exhibit how this works on a cylinder, where the RNS field ψ_+^μ has periodic boundary conditions, suitable to a space-time spinor state. In complete parallel with the covariant construction of the Ramond vertex operator, we may construct $S^{\alpha 1}(z)$ from $\psi_+^i(z)$ (and $S^{\alpha 2}(\bar{z})$ out of $\psi_-^i(\bar{z})$). One begins by bosonizing the Cartan generators of SO(8) in the RNS representation:

$$\psi_+^{2i-1}(z)\psi_+^{2i}(z) = -i\partial_z\phi^i \qquad i = 1,2,3,4 \tag{10.31}$$

Recall that the fields ψ_+^i then have an exponential form in terms of ϕ^i, with $\pm$ weights

$$\frac{1}{\sqrt{2}}(\psi_+^{2i-1} \pm i\psi_+^{2i})(z) = e^{\pm i\phi^i(z)} \tag{10.32}$$

and the spin fields are also exponentials with $\pm\frac{1}{2}$ weights:

$$\begin{aligned} S_\alpha^1(z) &= e^{i\alpha\cdot\phi(z)} \\ \alpha &= \left(\pm\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \pm\tfrac{1}{2}\right) \end{aligned} \tag{10.33}$$

The conformal weight of $S_\alpha^1(z)$ is now $\frac{1}{2}\alpha^2 = \frac{1}{2}$! This means that the lightcone gauge has made $S_\alpha^1(z)$ into a worldsheet spinor.

§10.4. Flat superspace GS formulation

A more intrinsic formulation of the GS worldsheet action is obtained by expressing it in superspace We concentrate on the $D = 10$, $\mathcal{N} = 2$ case, for which superspace is of dimension (10|32), with 32 Majorana spinor components. Coordinates are $X^M = (x^m, \theta^{\mu I})$, where $m = 0, 1, \ldots, 9$ and $\mu = 1, \ldots, 16$ while $I = 1, 2$ labels the two irreducible Majorana-Weyl (MW) multiplets.

Flat superspace is characterized by a frame $E^A = dX^M E_M{}^A$, where $A = (a, \alpha I)$ are frame indices, with $a = 0, 1, \ldots, 9$ labeling the defining representation of SO(1, 9), α labeling the MW spinor of Spin(1, 9) and $I = 1, 2$. The Spin(1, 9) connection vanishes in flat superspace, and the superderivatives are

$$\mathcal{D}_A = E_A{}^M \partial_M \qquad \partial_M \equiv \frac{\partial}{\partial X^M} \tag{10.34}$$

and they satisfy

$$\{\mathcal{D}_{\alpha I}, \mathcal{D}_{\beta J}\} = 2(\Gamma^a)_{\alpha\gamma}(C^{-1})^\gamma{}_\beta \mathcal{D}_a \delta_{IJ} \tag{10.35}$$

with all others commuting. The combination Π_m^r, encountered previously, has a very natural interpretation as a pull-back of the superspace frame E^A to the worldsheet. Indeed

$$\begin{aligned} X^*(E^a) &\equiv E_p{}^a d\xi^p = (\partial_p x^a - i\bar{\theta}^I \Gamma^a \partial_p \theta^I) d\xi^p \\ X^*(E^{\alpha I}) &\equiv E_p{}^{\alpha I} d\xi^p = \partial_p \theta^{\alpha I} d\xi^p \end{aligned} \tag{10.36}$$

so that $\Pi_p{}^a = E_p{}^a$. This identification readily allows us to write the first part of the GS superstring action in superspace:

$$S_1[X] = \frac{1}{8\pi}\int_\Sigma d\mu_g g^{pq} E_p{}^a E_q{}^b \eta_{ab} \tag{10.37}$$

The second part of the GS action may be recast in the form of a Wess-Zumino-Witten action. To show that this is indeed the case, we consider the following 3-form

$$H \equiv E^a \wedge (\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a \wedge E^2) \tag{10.38}$$

where we have used the notation $\bar{E}^I \equiv (E^I)^T \Gamma^0$ and suppressed spinor index contraction. To construct a Wess-Zumino-Witten action, we require as usual that the form H be closed; $dH = 0$. This condition implies

$$
\begin{aligned}
0 = dH &= dE^a \wedge (\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a \wedge E^2) \\
&= -i\bar{\theta}^I \Gamma^a d\theta^I (\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a \wedge E^2) \\
&= -i(\bar{E}^1 \Gamma^a E^1 + \bar{E}^2 \Gamma^a E^2)(\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a E^2) \\
&= -i(\bar{E}^1 \Gamma^a E^1 \bar{E}^1 \Gamma_a E^1 - (1 \to 2)) \\
&= -(\Gamma^0 \Gamma^a)_{\alpha\beta} (\Gamma^0 \Gamma_a)_{\gamma\delta} E^{1\alpha} E^{1\beta} E^{1\gamma} E^{1\delta} - (1 \to 2) \ .
\end{aligned}
\tag{10.39}
$$

The last line vanishes since the wedge product of the E^1's is symmetry in $(\alpha\beta\gamma\delta)$ in view of the identity (10.19) we used to show that the S_2 action is supersymmetric.

It is easy to show that $H = db$, where b is the 2-dimensional integrand of S_2. Indeed,

$$
\begin{aligned}
H &= (dx^a - i\bar{\theta}^j \Gamma^a d\theta^j) \wedge (d\bar{\theta}^1 \Gamma_a d\theta^1 - d\bar{\theta}^2 \Gamma_a d\theta^2) \\
&= d(dx^a (\bar{\theta}^1 \Gamma_a d\theta^1 - \bar{\theta}^2 \Gamma_a d\theta^2)) \\
&\quad - i(\bar{\theta}^1 \Gamma^a \theta^1 d\bar{\theta}^1 \Gamma_a d\theta^1 - \bar{\theta}^2 \Gamma^a d\theta^2 d\bar{\theta}^2 \Gamma_a d\theta^2) \\
&\quad - i\bar{\theta}^2 \Gamma^a d\theta^2 d\bar{\theta}^1 \Gamma_a d\theta^1 + \bar{\theta}^1 \Gamma^a d\theta^1 d\bar{\theta}^2 \Gamma_a d\theta^2
\end{aligned}
\tag{10.40}
$$

Now, the second line again vanishes by the Γ-identity (10.19), and the last term is a total differential:

$$
H = d\{dx^a (\theta^1 \Gamma_a d\theta^1 - \bar{\theta}^2 \Gamma_a d\theta^2) + i\bar{\theta}^1 \Gamma^a d\theta^1 \bar{\theta}^2 \Gamma_a d\theta^2\}
\tag{10.41}
$$

Identifying with the term in S_2, we find

$$
S_2[x] = \frac{-i}{4\pi} \int_B X^*(H) \ .
\tag{10.42}
$$

Finally, it is very instructive to exhibit the action of the κ-symmetry in the superspace formulation. It turns out that the κ-symmetry is most easily expressed in terms of a worldsheet scalar (space-time spinor)

$$
\kappa \equiv i E_p^a \Gamma_a (\kappa^{p1} + \kappa^{p2})
\tag{10.43}
$$

In terms of this parameter, one has

$$
\begin{aligned}
\delta X^M E_M{}^a &\equiv \delta X^a = 0 \\
\delta X^M E_M{}^1 &= (1 + \Gamma)\kappa \\
\delta X^M E_M{}^2 &= (1 - \Gamma)\kappa
\end{aligned}
\tag{10.44}
$$

where the operator Γ is defined by

$$
\Gamma \equiv \frac{1}{2\sqrt{h}} \epsilon^{pq} E_p{}^a E_q{}^b \epsilon_{ab}
\tag{10.45}
$$

and h is such that $\Gamma^2 = I$.

§10.5. Supergravity and low energy superstrings

We have shown in the first part of this chapter that the RNS and GS formulations of superstring theory exhibit 10-dimensional (global) Poincaré supersymmetry, in flat Minkowski space-time $M = \mathbb{R}^{10}$.

Actually, superstring theory also exhibits $Diff(M)$ invariance when formulated on a general manifold M. For the bosonic string, we had shown this in §VI by coupling bosonic string theory to background fields, including the metric $G_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$ and the dilaton Φ, in a $Diff(M)$ covariant fashion. For the superstring, we may again couple superstring theory to $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ in a $Diff(M)$ covariant way. The relevant ($\mathcal{N} = 1$ on the worldsheet) supersymmetric nonlinear sigma model for the type II superstrings is given by

$$\begin{aligned} S[X; G, B, \Phi] = &\frac{1}{4\pi} \int_\Sigma d\mu_E \mathcal{D}_- X^\mu \mathcal{D}_+ X^\nu (G_{\mu\nu}(X) + iB_{\mu\nu}(X)) \\ &+ \frac{1}{4\pi} \int_\sigma d\mu_E R_{+-} \Phi(X) \end{aligned} \tag{10.46}$$

This worldsheet action defines the RNS superstring in an arbitrary space-time manifold M, with the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. (For the $\mathcal{N} = 1$ theories, this sigma model must be modified: for heterotic theories, one replaces $\mathcal{D}_-$ by $\partial_{\bar{z}}$ and R_{+-} by $R_{+\bar{z}}$, while for Type I theories, the $B_{\mu\nu}$ coupling will not occur here, but will result from the R-R sector.)

Of course, it must be realized that for any of the superstrings, the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ do *not* correspond to the full set of massless states discussed in §VII and VIII. First of all, there are additional space-time fermions: the gravitino and the dilatino. But, there are also additional boson fields: in Type II theories, these are the R-R gauge fields; in heterotic and Type I theories these are the Yang–Mills bosons. Following the discussion provided for the study of bosonic strings, on general manifolds, with general background fields, we should include in the nonlinear sigma model all space-time background fields corresponding to massless string states. The inclusion of fermions or R-R bose fields is complicated by the fact that it requires the use of the Ramond vertex operator. Nonetheless, let us imagine that such a formulation were used. (We shall exhibit it explicitly later, with the use of the GS formulation.)

Now, since the superstring theory is covariant under $Diff(M)$ as well as under global Poincaré supersymmetry, it is fully expected that it will in fact be invariant under *local supersymmetry transformations* (also called supergravity transformations). This is expected because the composition of $Diff(M)$ and Poincaré supersymmetry transformations does *not* close onto these, but does close onto local supersymmetry. (We have already examined in §9 how this happens in $D = 2$ within the context of super-Riemann surfaces.) Field theories that are invariant under local supersymmetry transformations are so-called *supergravities.*

In the low energy approximation, superstring theory can be approximated by the spectrum and dynamics of its massless states only, summarized in terms of a *local field theory.* The massless sector of superstring theory will inherit invariance under local supersymmetry, and will then be a supergravity.

The dynamics of the NS-NS sector of Type IIA, B superstrings, in the massless sector, will yield the part of the corresponding supergravity involving only the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ (with the fermion fields and R-R fields set to zero). This

dynamics may be deduced directly from the above nonlinear sigma model, and results from demanding that its quantum dynamics be superconformal invariant. It turns out that the conditions of superconformal invariance of $S[X;G,B,\Phi]$ to leading order (i.e. two derivatives or fewer on fermion fields) are precisely the same as the condition found for the bosonic string, except that $D = 10$ now. The field equations for G, B and Φ could be deduced from an action for those fields on the space-time manifold M:

$$I_1(G,B,\Phi) = -\frac{1}{2K^2}\int_M d\mu_G \left\{ R_G - 4D_\mu\Phi D^\mu\Phi + \frac{1}{12}H^2 \right\} e^{-2\Phi} \tag{10.47}$$

Here, we have expressed the metric $G_{\mu\nu}$ in the *string* convention, so that an extra overall factor of the dilaton coupling appears.

For the heterotic strings, the fields G, B and Φ result from the NS sector of the left-movers coupled to the space-time part of the right movers, and the resulting action is still given by I_1.

§10.6. Type IIA, $D = 10$, $\mathcal{N} = 2$ and $D = 11$, $\mathcal{N} = 1$ supergravities

We begin by discussing the low energy field theory (at most 2 derivatives on boson fields, at most 1 on fermion fields) associated with Type IIA superstring theory, i.e. Type IIA, $\mathcal{N} = 2$, $D = 10$ supergravity. The existence of this theory was predicted by Nahm, based on representation theory of supersymmetry algebras, and was constructed by Cremmer, Julia and Scherk, by dimensional reduction from the $D = 11$, $\mathcal{N} = 1$ supergravity. The Type IIA field content is readily identified from §VII, 4:

• $G_{\mu\nu}$	NS-NS	space-time metric (equivalently frame $e_m{}^a$)
• $B_{\mu\nu}$		antisymmetric NS-NS tensor
• Φ		dilaton
• $A^{(1)}_\mu$	R-R	graviphoton
• $A^{(3)}_{\mu\nu\rho}$		rank 3 antisymmetric tensor
• $\chi_\mu{}^\alpha$	R-NS and NS-R	32-component Majorana gravitino
• λ_α		32-component Majorana dilatino
		(both contain both chiralities.)

The action I_1 corresponds to the dynamics in which $A^{(1)} = A^{(3)} = \chi = \lambda = 0$, and we now seek the generalization of I_1 that also describes the remaining fields.

It turns out that the easiest way to do this is to reorganize the field contents in such a way that it manifestly results from dimensional reduction of an $D = 11$ supergravity theory with $\mathcal{N} = 1$ supersymmetry. The precise statement is that the low energy limit of the $D = 11$ supergravity, compactified on a circle with small radius, is precisely the Type IIA supergravity. This re-organization of field variables is advantageous because it will turn out that the $\mathcal{N} = 1$, $D = 11$ supergravity is described by a relatively simple Lagrangian in terms of relatively few fields.

We begin by showing that the field contents may be organized according to multiplets of Spin$(1,10)$. We use space-time indices $\mu,\nu,\ldots = 0,1,\ldots,9$ and $\bar\mu,\bar\nu =$

$0, 1, \ldots, 9, 10$. We denote the compactified coordinate by x^{10}. Upon dimensional reduction on a circle, the only parts that survive to the low energy approximation of the $D = 10$ theory are the fields that are independent of x^{10}. Indeed, x^{10}-dependence renders the fields massive, with mass on the order of $1/R$.

A metric tensor in $D = 11$ decomposes as follows:

$$G_{\bar{\mu}\bar{\nu}}(x^\mu, x^{10}) \longrightarrow G_{\bar{\mu}\bar{\nu}}(x^\mu) \tag{10.48}$$

Under the action of SO(1, 10), this multiplet is irreducible, but under SO(1, 9), we find $\square\!\square \to \square\!\square \oplus \square \oplus \cdot$, so that the objects of Type IIA theory that naturally fit into the $D = 11$ metric multiplet are

$$(G_{\mu\nu}, A^{(1)}_\mu, \Phi) \to G_{\bar{\mu}\bar{\nu}} \tag{10.49}$$

Proceeding analogously for an antisymmetric rank 3 tensor

$$A_{\bar{\mu}\bar{\nu}\bar{\rho}}(x^\mu, x^{10}) \to A_{\bar{\mu}\bar{\nu}\bar{\rho}}(x^\mu) \tag{10.50}$$

and under SO(1, 10), this tensor is again irreducible, but decomposes under SO(1, 9) as

$$\begin{array}{|c|c|}\hline \\ \hline \\ \hline \end{array} \to \begin{array}{|c|c|}\hline \\ \hline \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \end{array}\,,$$

so that

$$(A^{(3)}_{\mu\nu\rho}, B_{\mu\nu}) \to A_{\bar{\mu}\bar{\nu}\bar{\rho}} \tag{10.51}$$

Thus, all the NS-NS and R-R bosons fit into two fields: $G_{\bar{\mu}\bar{\nu}}$ and $A_{\bar{\mu}\bar{\nu}\bar{\rho}}$.

The spinors also fit together. The smallest dimensional representation of a spinor of Spin(1, 10) is a Majorana spinor of dimension 32. Under Spin(1, 9), it will reduce into 2 Majorana-Weyl spinors of opposite chirality, and this is precisely what we need for Type IIA, where the gravitino as well as the dilatino are non-chiral Majorana spinors. Furthermore, the dilatino may be appended to ${\chi_\mu}^\alpha$ as its 11-th component:

$$({\chi_\mu}^\alpha, \lambda_\alpha) \to \chi^\alpha_{\bar{\mu}}\ . \tag{10.52}$$

$\mathcal{N} = 1$, $D = 11$ supergravity

Based upon general arguments by Nahm, the largest possible dimension D in which supersymmetric multiplets can exist with spin ≤ 2 only is $D = 11$, with a single local supersymmetry. Its field contents are precisely the ones that we have described above:

- $G_{\bar{\mu}\bar{\nu}}$ (equiv. $e_{\bar{\mu}}{}^{\bar{a}}$) graviton: 44 states
- $A_{\bar{\mu}\bar{\nu}\bar{\rho}}$ $U(1)$ gauge field: 84 states
- $\chi^\alpha_{\bar{\mu}}$ gravitino: 128 states.

The dimensional reduction is carried out by setting x^{10} dependence to zero, and by letting $G_{\bar{\mu}\bar{\nu}}$ decompose as

$$G_{\mu\nu}\,; \qquad G_{\mu 10} = G_{10\mu} = A^{(1)}_\mu \qquad G_{10\,10} = e^{-2\Phi} \tag{10.53}$$

which yields the action I_1 when $A^{(1)}_\mu = 0$, and $A^{(3)}_{\mu\nu\rho} = \chi^\alpha_\mu = \lambda_\alpha = 0$. The normalization of the field strength $H = dB$ of B in I_1 provides a normalization of the field strength $F = dA$ in the $D = 11$ supergravity theory. The remaining terms are fixed by local supersymmetry invariance.

The full Lagrangian is obtained by seeking $\mathcal{N} = 1$ local supersymmetry and was established by Cremmer, Julia and Scherk:

$$I(G, A, \chi) = -\frac{1}{2\kappa^2}\int_{\bar{M}} d\mu_G \mathcal{L} - \frac{\sqrt{2}\,\kappa}{3^4 5 6}\int_{\bar{M}} A \wedge F \wedge F \tag{10.54}$$

where $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L} &= R_G + \frac{1}{12}\kappa^2 F^2 + \kappa^2 \bar{\chi}_{\bar{\mu}} \Gamma^{\bar{\mu}\bar{\nu}\bar{\rho}} D_{\bar{\nu}} \chi_{\bar{\rho}} \\ &\quad + \frac{\sqrt{2}\,\kappa^3}{384}\left(\bar{\chi}_{\bar{\mu}}\Gamma^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}\bar{\omega}}\chi_{\bar{\omega}} + 12\bar{\chi}^{\bar{\nu}}\Gamma^{\bar{\rho}\bar{\nu}\bar{\sigma}}\chi^{\bar{\tau}}\right)(F + \widehat{F})_{\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}}\end{aligned} \tag{10.55}$$

In this action, we have used the standard notation

$$\Gamma^{\mu_1 \cdots \mu_\rho} \equiv \Gamma^{[\mu_1}\Gamma^{\mu_2} \ldots \Gamma^{\mu_\rho]} \tag{10.56}$$

where the square brackets stand for complete anti-symmetrization of the indices. $F = dA$ was defined previously, and

$$\widehat{F} = F + \chi\text{-terms} \tag{10.57}$$

such that the supersymmetry variation of $\widehat{F}$ does not contain derivatives of η – the SUSY parameter. $D_{\bar{\nu}}$ is the covariant derivative with respect to $\frac{1}{2}(\omega_{\bar{M}} + \widehat{\omega}_{\bar{M}})$ where

$$\widehat{\omega}_{\bar{\mu}\bar{a}\bar{b}} = \omega_{\bar{\mu}\bar{a}\bar{b}} + \frac{1}{8}\bar{\chi}^{\bar{\nu}}\Gamma_{\bar{\nu}\bar{\mu}\bar{a}\bar{b}\bar{\rho}}\chi^{\bar{\rho}} \tag{10.58}$$

and $\omega_{\bar{\mu}}$ is the spin connection, which is determined by its equation of motion. For completeness, we record the supergravity transformations:

$$\begin{aligned}\delta e_{\bar{\mu}}{}^{\bar{a}} &= \frac{\kappa}{2}\bar{\eta}\Gamma^{\bar{a}}\chi_{\bar{\mu}} \\ \delta A_{\bar{\mu}\bar{\nu}\bar{\rho}} &= -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[\bar{\mu}\bar{\nu}}\chi_{\bar{\rho}]} \\ \delta\chi_{\bar{\mu}} &= \frac{1}{\kappa}D'_{\bar{\mu}}\eta + \frac{\sqrt{2}}{288}\left(\Gamma_{\bar{\mu}}{}^{\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}} - 8\delta_{\bar{\mu}}{}^{\bar{\nu}}\Gamma^{\bar{\rho}\bar{\sigma}\bar{\tau}}\right)\widehat{F}_{\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}}\end{aligned} \tag{10.59}$$

The covariant derivative D' is here defined with the connections $\widehat{\omega}$.

Type IIA supergravity

The Type IIA supergravity Lagrangian may now be deduced from $D = 11$ supergravity by the compactification and dimensional reduction discussed previously. The reduction of the metric $G_{\bar{\mu}\bar{\nu}}$ was already given; the reduction of the $A_{\bar{\mu}\bar{\nu}\bar{\kappa}}$ field proceeds by

$$\begin{aligned}A_{\mu\nu\kappa} &= \kappa^{\frac{1}{4}} A^{(3)}_{\mu\nu\kappa} \\ A_{\mu\nu 10} &= \frac{1}{\kappa}\,B_{\mu\nu}\end{aligned} \tag{10.60}$$

The bosonic part of the Type II A supergravity is then

$$
\begin{aligned}
&I_{IIA}(G,B,\Phi;A^{(1)},A^{(3)}) \\
&\quad = -\frac{1}{2\kappa^2}\int_M d\mu_G\Big\{e^{-2\Phi}\Big(R_G - 4D_\mu\Phi D^\mu\Phi + \frac{1}{12}H^2\Big) \\
&\qquad\qquad + \sqrt{\kappa}\,G_A^2 + \sqrt{\kappa}\,\frac{1}{12}F^2\Big\} \\
&\quad - \frac{\sqrt{\kappa}}{288\kappa^2}\int_M B\wedge F\wedge F
\end{aligned}
\tag{10.61}
$$

here $G_A = dA^{(1)}$, $H = dB$, $F = dA^{(3)}$. Notice that the R-R part of the action does not occur with the dilaton rescaling factor $e^{-2\Phi}$; thus, these contributions are in some sense of order 1 in the string loop expansion.

§10.7. Type IIB, $D = 10$, $\mathcal{N} = 2$ supergravity

The Type IIB field contents are as follows (see Lecture 7).

- $G_{\mu\nu}$(equiv. $e_\mu{}^a$) graviton.
- $B_{\mu\nu} + i\,A^{(2)}_{\mu\nu}$ complex rank 2 antisymmetric tensor.
- $\Phi + i\,A^{(0)}$ complex dilaton-axion.
- $A^{(4)}_{\mu\nu K\lambda}$ real, self-dual $F = dA^{(4)}$.

- χ^α_μ (chirality+) complex Majorana-Weyl spinor gravitino.
- λ_α (chirality−) complex Majorana-Weyl spinor dilatino.

Clearly, the field contents do not arise from a simple dimensional reduction, since the spinor content is chiral here. In fact, since the theory contains a self-dual field, F, no off-shell Lorentz-covariant formulation is known for this theory.

The field equations may be written covariantly though, either in components or in superspace. The gauge group is $\mathrm{Spin}(1,9)\times U(1)_R$ and the complex dilaton-axion field $\Phi + i\,A^{(0)}$ lives in $\mathrm{SU}(1,1)/U(1)_R$.

§10.8. Type I − Heterotic, $D = 10$, $\mathcal{N} = 1$ supergravities

Both for the Type I and for the heterotic theories, we have only a single space-time supersymmetry: $\mathcal{N} = 1$. However, both theories now also have an $\mathcal{N} = 1$ super-Yang–Mills multiplet, which we shall have to include.

The field contents of the $\mathcal{N} = 1$ *supergravity multiplet* are obtained (say for the heterotic string) by considering the left-moving part of Type II, tensored with the right-moving 10-dimensional part of the bosonic string. The corresponding fields are

- $G_{\mu\nu}$ space-time metric
- $B_{\mu\nu}$ real rank 2 antisymmetric tensor
- Φ real dilaton
- $\chi_\mu{}^\alpha$ 1 Majorana-Weyl (real) gravitino (+ chirality)
- λ_α 1 Majorana-Weyl (real) dilatino (− chirality)

the Lagrangian for this theory exists (since there are no fields with self-dual field strength) and may be obtained from the Type IIA theory by setting

$$A_\mu^{(1)} = A_{\mu\nu\rho}^{(3)} = 0 \qquad \chi_\mu^\alpha\big|_- = \lambda_\alpha\big|_+ = 0 \tag{10.62}$$

where the $|_\pm$ indicates the chirality. This truncation is consistent with the field equations of Type IIA, since both $A^{(1)}$ and $A^{(3)}$ occur quadratically in the bosonic Lagrangian, and since the fermionic Lagrangian does not mix between chiralities. We limit ourselves to examining only the bosonic action of the supergravity multiplet, and notice immediately that it must be just I_1. The fermionic part may be deduced from the $D = 11$ supergravity Lagrangian by the above dimensional reductions and restrictions.

The field contents of the $\mathcal{N} = 1$ *super Yang–Mills multiplet* are obtained (again for the heterotic string) by considering the left-moving part of the Type II string, tensored with the right-moving internal part of the bosonic string. The corresponding fields are

- A gauge field of $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2 = G$ (adjoint rep.)
- ψ gaugino of $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2 = G$ (adjoint rep.)

The dynamics are governed by the action (in flat M, with $G = \eta$)

$$I_{YM}^0 = \int_M d\mu_G \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \bar\psi^a \Gamma^\mu (D_\mu \psi)^a \right) \tag{10.63}$$

which is *global* $N = 1$ Poincaré supersymmetry invariant. Here, we use the notation ($a = 1, \ldots, \dim G$) and

$$A = A_\mu^a \, t^a \, dx^\mu \qquad F = dA + gA \wedge A = \frac{1}{2} F_{\mu\nu}^a \, t^a \, dx^\mu \, dx^\nu \ . \tag{10.64}$$

It remains to construct an action with $N = 1$ local supersymmetry, that combines both the supergravity multiplet and the super-Yang–Mills multiplet. First, the action I_{YM}^0 is adapted as follows.

$$I_{YM} = \frac{1}{\kappa} \int_M d\mu_G e^{-2\Phi} \left\{ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \bar\psi^a \Gamma^\mu D_\mu \psi^a \right\} \tag{10.65}$$

For $G = U(1)$ gauge theory, $I_1 + I_{YM}+$ the fermionic terms of the supergravity multiplet yield an $\mathcal{N} = 1$ SUSY action. For non-Abelian G, this is not so, and the form of the B-field strength H in I_1 must be modified as follows:

$$\widetilde{H} = dB - \frac{\kappa}{\sqrt{2}} \omega_3(A) \tag{10.66}$$

Here, $\omega_3(A)$ is the so-called Chern-Simons form of degree 3:

$$\omega_3(A) \equiv \mathrm{tr} \left(A \wedge dA + \frac{2}{3} gA \wedge A \wedge A \right) \ , \tag{10.67}$$

with the properties that

$$d\omega_3(A) = \operatorname{tr} F \wedge F$$

(10.68)
$$\delta_\Lambda \omega_3(A) = \operatorname{tr}(d\Lambda \wedge dA) \qquad \text{under } \delta_\Lambda A = d\Lambda + [A, \Lambda] \ .$$

To render $\widetilde{H}$ gauge invariant under non-Abelian transformations in G, B must transform as

$$\delta_\Lambda B = \frac{\kappa}{\sqrt{2}} \operatorname{tr} \Lambda dA$$

The supergravity + super Yang–Mills action is now (for bosons)

(10.69)
$$I = -\frac{1}{2K^2} \int_M d\mu_G \left\{ e^{-2\Phi} \left(R_G + 4 D_\mu \Phi D^\mu \Phi - \frac{1}{12} \widetilde{H}^2 \right) + e^{-2\Phi} \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right) \right\}$$

where the first part coincides with I_1, except for the fact that H has been replaced with its G-invariant version $\widetilde{H}$.

§10.9. Superspace formulation of supergravities in $D = 11$ and $D = 10$

In §10.6, §10.7 and §10.8, we have discussed supergravities in $D = 11$ and $D = 10$ in the component formulation. While supersymmetry invariance transformations of these actions are known, there is the problem that the composition of two supergravity transformations closes onto $Diff(M)$ only upon the use of the equations of motion. Also, the supergravity invariance is not manifest.

The superspace approach provides a formulation in which $Diff(M)$ and local supersymmetry transformations appear on the same footing and are united into the group of diffeomorphisms of superspace. However, the superspace formulation in $D = 11$ and $D = 10$ supergravity – and super Yang-Mills-theory – is known only for the equations of motion. No superspace action is known that describes the off-shell configurations as well.

To exhibit local supersymmetric covariant coupling of superstrings to supergravity, it is most convenient to make use of the superspace formulation of supergravity, as we shall show in the next section. For this issue, the fact that only the on-shell formulation is available causes no problem. In fact, already superstring theory itself, and as we showed previously, Type IIB supergravity, do not admit an off-shell formulation; it is thus perhaps not so surprising that superspace supergravity does not either.

$\mathcal{N} = 1$, $D = 11$ *supergravity in superspace*

From many points of view, the $D = 11$ theory is the simplest theory, and we shall begin by giving its formulation. The superspace data are as follows: local supercoordinates $X^M = (x^m, \theta^\mu)$, with $m = 0, 1, \ldots, 10$ and $\mu = 1, \ldots, 32$ parametrize a real supermanifold M of dimension $(11|32)$. The fields are most conveniently expressed as

(10.70)
- $E^A = dX^M E_M{}^A$
- $\Omega_A{}^B = dX^M \Omega_{MA}{}^B$
- $X = \frac{1}{3!} E^C \wedge E^B \wedge E^A X_{ABC}$

Here, A, B, C sum over frame indices: (a,α), (b,β), (c,γ), $a,b,c = 0,1,\ldots 10$; $\alpha,\beta,\gamma = 1,\ldots,32$, and transform under the defining representation and the Majorana spinor representations of Spin$(1,10)$, respectively. The field E^A is a frame, belonging to $\Omega^1(M)$; the field $\Omega_A{}^B$ is a Spin$(1,10)$ connection, which admits the following decomposition under Spin$(1,10)$:

$$\begin{aligned} &\Omega_{ab} = -\Omega_{ba} \qquad\qquad \Omega_{a\beta} = \Omega_{\alpha\beta} = 0 \\ &\Omega_{\alpha\beta} = \tfrac{1}{4}(\Gamma^{ab})_{\alpha\beta}\Omega_{ab} \end{aligned} \tag{10.71}$$

Finally X is a 3 form in superspace, and a Spin$(1,10)$ invariant. It contains the rank 3 field $A^{(3)}_{\bar\mu\bar\nu\bar\rho}$.

Curvature, torsion and X field strength are defined by

$$\begin{aligned} T^A &\equiv dE^A + E^B \wedge \Omega_B{}^A && \left(\equiv \tfrac{1}{2} E^C \wedge E^B T_{BC}{}^A\right) \\ R_A{}^B &\equiv d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B && \left(\equiv \tfrac{1}{2} E^D \wedge E^C R_{CD,A}{}^B\right) \\ H &\equiv dX && \left(\equiv \frac{1}{4!} E^D \wedge E^C \wedge E^B \wedge E^A \wedge H_{ABCD}\right) . \end{aligned} \tag{10.72}$$

These fields satisfy Bianchi identities

$$\begin{aligned} &\text{(a)} \quad DT^A = E^B \wedge R_B{}^A \\ &\text{(b)} \quad DR_A{}^B = 0 \\ &\text{(c)} \quad dH = 0. \end{aligned} \tag{10.73}$$

written in differential form notation. It is most convenient to write these Bianchi identities out in terms of the Spin$(1,10)$ tensors $T_{AB}{}^C$, $R_{AB,C}{}^D$ and H_{ABCD}; (parentheses $(ABC\ldots)$ stand for cyclic, graded summation of indices)

$$\begin{aligned} &\text{(a)} \quad \sum_{(ABC)} (R_{ABC}{}^D - D_A T_{BC}{}^D - T_{AB}{}^E T_{EC}{}^D) = 0 \\ &\text{(b)} \quad \sum_{(ABC)} (D_A R_{BCD}{}^E + T_{AB}{}^F R_{FCD}{}^E) = 0 \\ &\text{(c)} \quad \sum_{(ABCDE)} (D_A H_{BCDE} + T_{AB}{}^F H_{FCDE}) \\ &\qquad\qquad - \sum_{(ADBEC)} (-)^{BC+BD+CE} T_{AD}{}^F H_{FBEC} = 0 \end{aligned} \tag{10.74}$$

This geometry is too general, and contains many more fields than are necessary for the $D = 11$ supergravity theory that we constructed in components. We must impose suitable "constraint equations". (We already encountered this problem in $D = 2$, $\mathcal{N} = 1$ supergravity, where certain constraints were also to be imposed, see §9.1.) The choice of constraints is dictated by the structure of the component fields that one expects to remain after the constraints have been imposed, by dimensional

analysis, and of course by Lorentz invariance. We shall limit ourselves here to quoting the results from Brink and Howe (1980):

$$
\begin{aligned}
&T_{\alpha\beta}{}^{\gamma} = T_{\alpha b}{}^{c} = T_{ab}{}^{c} = 0; \; T_{\alpha\beta}{}^{c} = -i(\Gamma^{c})_{\alpha\beta} \\
&H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma c} = H_{\alpha bcd} = 0; \; H_{\alpha\beta cd} = i(\Gamma_{cd})_{\alpha\beta}
\end{aligned}
\tag{10.75}
$$

It can be shown that all the torsion constraints can be solved in terms of a single superfield H_{abcd}. We shall not list out the various components here, but limit ourselves to providing the two field equations that result. We introduce

$$
R_{ab} = \eta^{cd} R_{acbd} \qquad R = \eta^{ab} R_{ab}
\tag{10.76}
$$

and have the equations of motion

$$
\begin{aligned}
R_{ab} = \frac{1}{2}\eta_{ab} R \;\; &= -\tfrac{1}{12} H_{acde} H_{b}{}^{cde} + \tfrac{1}{96}\eta_{ab} H_{cdef} H^{cdef} \\
D^{a} H_{abcd} \;\; &= -\tfrac{1}{1728}\epsilon_{bcde_1\ldots e_4 f_1\ldots f_4} H^{e_1\ldots e_4} H^{f_1\ldots f_4}
\end{aligned}
\tag{10.77}
$$

Retaining only the $\theta = 0$ part of these equations, we readily observe that they are the bosonic field equations that would also have been obtained from the component formulation.

We conclude by remarking that the $D = 10$, $\mathcal{N} = 2$ Type IIA supergravity may be obtained again by dimensional reduction. The resulting formulation is rather involved. There also exist a superspace formulation for Type IIB supergravity.

$\mathcal{N} = 1$, $D = 10$ *supergravity in superspace*

The superspace data are as follows: local supercoordinates $X^{M} = (x^{m}, \theta^{\mu})$, with $m = 0, 1, \ldots, 9$ and $\mu = 1, \ldots 16$, parametrize a real supermanifold of dimension (10|16). The fundamental fields are

$$
\begin{aligned}
&\bullet \quad E^{A} = dX^{M} E_{M}{}^{A} \\
&\bullet \quad \Omega_{A}{}^{B} = dX^{M} \Omega_{MA}{}^{B}
\end{aligned}
\tag{2.78}
$$

where A,B run over frame indices: (a, α), (b, β), $a, b = 0, 1, \ldots, 9$ and $\alpha, \beta = 1, \ldots, 16$. The spinors here transform under a single Majorana-Weyl representation of Spin$(1, 9)$. Notice that there is *no* further anti-symmetric tensor as there was in $D = 11$. The $B_{\mu\nu}$ field that occurs in $D = 10$, $\mathcal{N} = 1$ supergravity is contained in the fields E^{A} and $\Omega_{A}{}^{B}$. The field $\Omega_{A}{}^{B}$ is again a spin connection, but now for Spin$(1, 9)$, acting on the direct sum of the defining representation and the Majorana-Weyl representation:

$$
\begin{aligned}
&\Omega_{ab} = -\Omega_{ba} \qquad\qquad \Omega_{\alpha b} = \Omega_{a\beta} = 0 \\
&\Omega_{\alpha\beta} = \tfrac{1}{4}(\Gamma^{ab})_{\alpha\beta}\Omega_{ab}
\end{aligned}
\tag{10.79}
$$

Torsion and curvature are defined by

$$
\begin{aligned}
&T^{A} \equiv dE^{A} + E^{B} \wedge \Omega_{B}{}^{A} \qquad \left(\equiv \tfrac{1}{2} E^{C} \wedge E^{B} T_{BC}{}^{A}\right) \\
&R_{A}{}^{B} \equiv d\Omega_{A}{}^{B} + \Omega_{A}{}^{C} \wedge \Omega_{C}{}^{B} \quad \left(\equiv \tfrac{1}{2} E^{D} \wedge E^{C} R_{CDA}{}^{B}\right) .
\end{aligned}
\tag{10.80}
$$

As in the case of $D = 11$, $\mathcal{N} = 1$, these quantities satisfy Bianchi identities

$$DT^A = E^B \wedge R_B{}^A \qquad DR_A{}^B = 0 \ . \tag{10.81}$$

(A component decomposition for these equations is given in (10.74).)

Again, this geometry is too general, and torsion constraints should be imposed. The choice of constraints is not unique, and depends upon the choice of connection, among others. The constraints here were first derived by **[Nilsson (1981)]**, and reformulated by **[Witten (1986)]**:

$$\begin{aligned} &T_{\alpha\beta}{}^{\gamma} = T_{\alpha b}{}^{c} = 0 \qquad T_{\alpha\beta}{}^{c} = 2\Gamma^{c}_{\alpha\beta} \\ &T_{\alpha\beta}{}^{\gamma} = (\Gamma_a)_{\beta\delta}\phi^{\delta\gamma} \quad \text{for some } \phi^{\delta\gamma} \end{aligned} \tag{10.82}$$

One may show that the resulting fields, after elimination of all auxiliary fields, are precisely those of the $\mathcal{N} = 1$ supergravity multiplet in $D = 10$, with the same equations of motion as those that we obtained from the component formulation.

§10.10. Local supersymmetric coupling of superstrings to supergravity

It is now very easy to produce a GS worldsheet action, formulated in superspace, that couples in a manifestly locally supersymmetric way to the superspace fields of supergravity in $D = 10$. For simplicity, we consider the case with $\mathcal{N} = 1$ supersymmetry only.

Recall that the flat superspace action was

$$S[X] = \frac{1}{8\pi}\int_\Sigma d\mu_g g^{pq} E_p{}^a E_q{}^b \eta_{ab} - \frac{i}{4\pi}\int_B X^*(H)$$

where $E_p{}^a d\xi^p \equiv X^*(E^a)$ is the pullback to X of E^a, and where $\partial B = \Sigma$, and H was a closed 3-form on M. The form of H was dictated by the condition that we had to have κ-symmetry in order to obtain the correct spectrum.

In general curved superspace backgrounds, we may again propose $S[X]$ as a suitable action, provided the superspace geometry admits a closed 3-form H, and upon inserting a factor of a scalar superfield k:

$$S[X] = \frac{1}{8\pi}\int_\Sigma d\mu_g g^{pq} E_p{}^a E_q{}^b \eta_{ab} e^{4k} - \frac{i}{4\pi}\int_B X^*(H)$$

where $E_p{}^a d\xi^p = X^*(E^a)$, just as in the flat superspace case. The superfield k cannot be arbitrary, of course.

It can be shown that when the torsion constraints for $\mathcal{N} = 1$, $D = 10$ supergravity are satisfied, then there exists a natural closed 3-form H, such that $S[X]$ is invariant under κ-symmetry. In other words, if we assume (locally) that $H = dB$, then the requirement of κ-invariance demands that certain torsion constraints are satisfied. Up to equivalences, these constraints are precisely the torsion constraints.

References and cross references

Almost all introductory texts on supersymmetry in physics deal exclusively with supersymmetry in four space-time dimensions. An extremely useful introduction to supersymmetry algebras and representation theory in general dimensions is provided in two foundational articles,

J. Strathdee, *Extended Poincaré supersymmetry*, Internat. J. Modern Phys. A **2** (1987), 273.

W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. B **135** (1978), 149.

A very useful account of supergravity and super-Yang-Mills in D=10 and D=11, as well as their relation to superstring theory is given in

M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. II, Cambridge University Press, 1988, and references therein,

where also the Green-Schwarz formulation in flat Minkowski space-time is nicely discussed. The superspace formulations of supergravity in D=11, and D=10 were constructed in a series of papers,

L. Brink and P. S. Howe, *Eleven-dimensional supergravity on the mass-shell in superspace*, Phys. Lett. B **91** (1980), 384.

B. E. W. Nilsson, *Simple ten-dimensional supergravity in superspace*, Nucl. Phys. B **188** (1981), 176.
E. Witten, *Twistor-like transform in ten-dimensions*, Nucl. Phys. B **266** (1986), 245.

P. S. Howe and P. C. West, *The complete N=2, D=10 supergravity*, Nucl. Phys. B **238** (1984), 181.

Finally, let us indicate that many papers on supergravity theories in various dimensions are reprinted in a set of two volumes (that is notoriously hard to obtain in libraries),

A. Salam, E. Sezgin, eds., *Supergravities in Diverse Dimensions*, vols. I, II, World Scientific, 1989.

EXERCISES

Problem 1. Consider the quantum field theory of a single real scalar field x, on a 2-dimensional surface Σ with Riemannian metric g and action

$$S[x,g] = \frac{1}{4\pi}\int_\Sigma d\mu_g \left\{ \frac{1}{2} g^{mn}\partial_m x \partial_n x + Q R_g x \right\}.$$

Here, $Q \in \mathbf{C}$ and R_g is the Gaussian curvature of g. Introduce local complex coordinates z, $\bar{z}$ in which $g = 2g_{z\bar{z}}|dz|^2$.
For $Q = 0$, define the renormalized stress tensor by

$$T(z) = -\frac{1}{2}\lim_{w\to z}\left(\partial_z x \partial_w x + \frac{1}{(z-w)^2}\right).$$

(a) Using the OPE of $\partial_z x$ and $\partial_w x$, derive all singular terms in the OPE of $T(z)$ and $T(w)$, and confirm the form derived on general grounds

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w).$$

(b) Show, by explicit conformal reparametrization $z \to z' = f(z)$, that $T(z)$ transforms as a projective connection.

(c) Use the result of (b) to relate the stress tensor $T(z)$ and the Virasoro generators L_m on the cylinder and on the annulus.

Problem 2. We continue the setup of Problem 1. For $Q \neq 0$, consider the flat metric $g = 2|dz|^2$.

(a) Derive the OPE of $\partial_z x$ and $\partial_w x$.

(b) Compute the stress tensor and show that $T = T_{zz}$ is complex analytic at the classical level. Define a suitable renormalized stress tensor.

(c) Use the result of (b) to obtain the central charge of the theory.

(d) Show that $:\exp\{\beta x(z,\bar{z})\}:$ is a primary field for any value of $\beta \in \mathbf{C}$, and compute its conformal weight $(h,\bar{h})$.

(e) Is the field $\partial_z x$ primary ? Derive its transformation law under conformal reparametrizations $z \to z' = f(z)$.

Problem 3. The purpose of Problems 3 and 4 is to establish that the singular parts of the OPE of two stress tensors

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \ , \qquad (1)$$

and of a stress tensor and a primary field, are valid for any worldsheet metric g in any complex coordinate system in which $g = 2g_{z\bar{z}}|dz|^2$, and are thus covariant under analytic diffeomorphisms of the complex coordinates.
Using the fact that $T(z)$ transforms as a projective connection, establish directly that the OPE in (1) transforms covariantly under analytic diffeomorphisms.

Problem 4. An alternative derivation of the covariance of (1), and in addition of the validity of (1) on any surface with any metric, may be obtained by retracing the derivation of the OPE using Diff(Σ) and Weyl(Σ) Ward identities. The derivation was carried out in [II-**CFT**], but the actual OPE seems to have been derived only for locally flat metrics. Here, we propose to complete the analysis for general metric. For the sake of clarity, I shall go through all the steps of the derivation.
Let ϕ_i be a set of primary fields, with conformal weights $(h_i,\bar{h}_i)$, $i = 1,\cdots,n$, and consider their (unnormalized) correlation functions $\langle\phi_1\cdots\phi_n\rangle_g$ for worldsheet metric g. We define the (covariant) quantum stress tensor $T^{\rm cov}_{mn}$ by

$$\delta_g\langle\phi_1\cdots\phi_n\rangle_g = \frac{1}{4\pi}\int_\Sigma d\mu_g \delta g^{mn}(w,\bar{w})\langle T^{\rm cov}_{mn}(w,\bar{w})\phi_1\cdots\phi_n\rangle_g$$

and the variation of the correlation function on the r.h.s is given by

$$\delta_g\langle T^{\rm cov}_{mn}(w,\bar{w})\phi_1\cdots\phi_n\rangle_g = \frac{1}{4\pi}\int_\Sigma d\mu_g \delta g^{pq}(z,\bar{z})\langle T^{\rm cov}_{pq}(z,\bar{z})T^{\rm cov}_{mn}(w,\bar{w})\phi_1\cdots\phi_n\rangle_g$$

We define a conformal field theory on any surface Σ, and for any metric g by the following requirement on its full (covariant) quantum stress tensor $T^{\rm cov}_{mn}$

$$\nabla^m T^{\rm cov}_{mn} = 0 \qquad\qquad T^{\rm cov}{}_m{}^m = -\frac{c}{3}R_g$$

where ∇ is the covariant derivative with respect to the affine connection. This definition is identical to the axiomatic definition used in [II-**CFT**].

(a) Use the covariance of the correlation function $\langle\phi_1\cdots\phi_n\rangle_g$ under general infinitesimal diffeomorphisms (not just analytic ones) to derive a first order differential equation relating $\langle T^{\rm cov}_{ww}(w)\phi_1\cdots\phi_n\rangle_g$ to $\langle\phi_1\cdots\phi_n\rangle_g$. Deduce the OPE of $T^{\rm cov}_{ww}$ with $\phi_i(z)$ from this equation by solving for its singular part as $z \to w$.

(b) By evaluating the variation of $\langle T^{\rm cov}_{mn}(w,\bar w)\phi_1\cdots\phi_n\rangle_g$ under a change δg^{zz}, derive an equation for $\langle T^{\rm cov}_{zz}(z,\bar z)T^{\rm cov}_{ww}(w,\bar w)\phi_1\cdots\phi_n\rangle_g$. Deduce the OPE $T^{\rm cov}_{zz}T^{\rm cov}_{ww}$ from this equation by solving for its singular part as $z\to w$.

(c) Show that there exists a quantity Q_{zz} such that $T(z)$ defined by

$$T(z)=T^{\rm cov}_{zz}(z,\bar z)+Q_{zz}(z,\bar z)$$

is analytic and obeys the standard OPE in (1). Determine Q, and compute it explicitly when $g_{z\bar z}=1$.

(d) Deduce from the results of (c) for general metric that $T(z)$ transforms as a projective connection under analytic diffeomorphisms.

Problem 5. Transition amplitudes of a single point particle in Quantum Field Theory may be described by summation over the space of paths, in complete analogy with the summation over surfaces that we prescribed for transition amplitudes of strings. In this problem, we propose to show that certain point particle amplitudes, prescribed this way, precisely agree with those obtained from corresponding Feynman diagrams.

Let $M=\mathbf{R}^D$ be flat Euclidean space-time, and define the world-line action for a single point particle of electric charge e in the presence of a fixed external $U(1)$ gauge field A by

$$S[x,g,A]=\frac12\int_0^1 d\tau\left\{g(\tau)^{-1}\dot x\cdot\dot x+g(\tau)m_0^2\right\}+ie\int_0^1 d\tau\dot x\cdot A(x) \tag{2}$$

We define amplitudes for transition between points $y,\ y'\in M$ by

$$A(y,y')=\int_{\mathrm{Met}([0,1])}Dg\frac{1}{\mathcal{N}(g)}\int_{\mathrm{Map}([0,1];M)}Dxe^{-S}$$

The measures Dg and Dx are defined with respect to the Diff([0,1]) invariant L^2 norms on δg and δx, and $\mathcal{N}(g)$ is formally equal to the volume of Diff([0,1]).

(a) Derive the infinitesimal transformations under Diff([0,1]) (generated by a vector field v) of g and x, verify that S is invariant and find the Diff([0,1]) invariant L^2 norms of δg and δx.

(b) Derive an orthogonal decomposition of an arbitrary δg into infinitesimal diffeomorphisms and a change in length $L=\int_0^1 d\tau g(\tau)$.

(c) Use the above decomposition to factorize the measure Dg into $Dv\ dL$, and compute the associated Jacobian.

(d) Show that $A(y,y')$ coincides with the Green function for a scalar field of mass m in the presence of an Abelian gauge field A. (The relation between m_0 and m may depend upon the regularization scheme used to calculate the Jacobian in (c).)

(e) Generalize the above prescription and adapt it to the calculation of one-loop amplitudes for charged scalar particles in the presence of an external $U(1)$ gauge field.

(f) Using (e), show that the one-loop amplitudes obtained this way agree with those derived from Feynman diagrams, in a perturbation expansion in powers of A. (You may restrict attention to the first non-trivial order in A.) Show that the length parameters (L, introduced in (b)) of the worldlines between successive insertions of A in this expansion are related to the Feynman parameters used to evaluate the corresponding Feynman diagrams in momentum space.

Problem 6. In this problem, we propose to prove a certain number of intermediate steps in the calculation of the Weyl anomaly of the determinant combinations

$$Z^-_{(n)}(g) = \frac{\mathrm{Det}'\Delta^-_{(n)}}{\det(\phi_j,\phi_k)_g \det(\psi_a,\psi_b)_g}$$

Here, we use the following expressions for covariant derivatives and Laplacian operators on sections f of K^n (K is the canonical bundle of Σ and z, $\bar{z}$ are local complex coordinates such that $g = 2g_{z\bar{z}}|dz|^2$)

$$\nabla^z_{(n)} f = (g_{z\bar{z}})^{-1}\partial_{\bar{z}} f \qquad \Delta^+_{(n)} f = -2\nabla^z_{(n+1)}\nabla^{(n)}_z f$$
$$\nabla^{(n)}_z f = (g_{z\bar{z}})^n \partial_z (g_{z\bar{z}})^{-n} f \qquad \Delta^-_{(n)} f = -2\nabla^{(n-1)}_z \nabla^z_{(n)} f$$

Also, $\phi_j \in \mathrm{Ker}\nabla^{(n-1)}_z$, $\psi_a \in \mathrm{Ker}\nabla^z_{(n)}$ and these kernels have dimensions $N^+_{(n-1)}$ and $N^-_{(n)}$ respectively. Show that under infinitesimal Weyl transformations of the metric $\delta_\sigma g = 2\delta\sigma g$, we have

(a) $N^\pm_{(n)}$ are invariant,

(b) and the trace of the heat-kernel changes by

$$\delta_\sigma \mathrm{Tr}\, e^{-t\Delta^-_{(n)}} = -t\frac{\partial}{\partial t}\mathrm{Tr}\left[2n\delta\sigma e^{-t\Delta^+_{(n-1)}} - (2n-2)\delta\sigma e^{-t\Delta^-_{(n)}}\right].$$

(c) Establish the following short time (t) expansion of the heat-kernels

$$\mathrm{Tr}\,\delta\sigma e^{-t\Delta^\pm_{(n)}} = \frac{1}{4\pi t}\int_\Sigma d\mu_g \delta\sigma + \frac{1\pm 3n}{12\pi}\int_\Sigma d\mu_g R_g \delta\sigma + \mathcal{O}(t),$$

(d) as well as their long time limits

$$\mathrm{Tr}\,\delta\sigma e^{-t\Delta^-_{(n)}} = -\frac{1}{2n-2}\delta_\sigma \ln\det(\phi_j,\phi_k)_g + \mathcal{O}(e^{-\lambda_1 t})$$
$$\mathrm{Tr}\,\delta\sigma e^{-t\Delta^+_{(n)}} = \frac{1}{2n}\delta_\sigma \ln\det(\psi_a,\psi_b)_g + \mathcal{O}(e^{-\lambda_1 t}),$$

where $\lambda_1 > 0$.

(e) Check that the result in (c) reproduces the Riemann-Roch-Atiyah-Singer index theorem for the operators $\nabla^z_{(n)}$ and $\nabla^{(n+1)}_z$.

Problem 7. **(a)** Construct the vertex operators for all physical states with mass squared $M^2 = 2$, in closed (oriented) string theory in flat space-time $M = \mathbf{R}^{26}$, by requiring that vertex operators are integrals of primary fields of conformal dimension $(1,1)$.

(b) Verify that the vertex operators you find are in one to one correspondence with the physical states $|\phi\rangle$, obtained by imposing the Virasoro conditions $L_n - \delta_{n,0}$ and $\tilde{L}_n - \delta_{n,0}$ on $|\phi\rangle$, (as was carried out in Lecture 2).

Problem 8. In this problem, we determine the dilaton vertex operator on a worldsheet Σ with general metric g. Show that the operator

$$V^{\Phi}(k) = \int_{\Sigma} d\mu_g \{ g^{mn} \epsilon_{\mu\nu}(k) \partial_m x^\mu \partial_n x^\nu + \alpha R_g \} e^{ik\cdot x}$$

is Weyl invariant (when inserted into correlation functions) provided certain conditions on the polarization tensor $\epsilon_{\mu\nu}$ and the parameter α are satisfied. (Here, R_g is the Gaussian curvature of the metric g.) Derive these conditions and solve for them (i.e. find $\epsilon_{\mu\nu}$ and α, up to an overall constant rescaling) for the vertex operator describing the dilaton.

Problem 9. In this problem, we perform a number of calculations in closed oriented string theory, in flat space-time, in the critical dimension $D = 26$, and to tree level.

(a) Using the transition amplitude for the scattering of four tachyons, deduce the strength of the interaction between two tachyons as a function of distance, in the limit where they are far apart.

(b) Assuming that the effects of the couplings of the dilaton and anti-symmetric tensor states can be ignored, use the results of (a) to derive a relation between Newton's constant of gravity, the string coupling $e^{-2\Phi_0}$, and the string tension $\kappa = 1/4\pi\alpha'$. (We have mostly set $\alpha' = 2$ during the lectures, so you will have to restore the α' dependence in your answer.)

(c) Compute the transition amplitude for the scattering of 3 tachyons and one graviton with polarization tensor $\epsilon_{\mu\nu}$. Verify that the amplitude is invariant under infinitesimal diffeomorphism transformations on the polarization tensor $\epsilon_{\mu\nu}$. Verify that the amplitude exhibits the singularities (as a function of external momenta) expected on physical grounds.

Problem 10. In this problem, we perform a number of calculations in open oriented boson string theory, in flat space-time, in the critical dimension $D = 26$, and to tree level.

(a) Construct a general expression for the transition amplitude for the scattering of N open string tachyons (to tree level). (Hint : make use of the fact that open string worldsheets may be conveniently obtained from closed string worldsheets that have an involution.)

(b) Compute the four (open string) tachyon amplitude and verify that it exhibits the singularities (as a function of external momenta) expected on physical grounds.

(c) Compute the transition amplitude for the scattering of one graviton and two photons. Give the simplest interaction in quantum field theory that would produce this scattering amplitude.

Problem 11. In Problems 11 and 12 we investigate a number of properties of the combined matter (for flat space-time of dimension D) and ghost system. Recall that the worldsheet action in locally flat complex coordinates z and $\bar{z}$ is given by

$$S = \frac{1}{2\pi} \int_{\Sigma} d^2 z \left(\frac{1}{2} \partial x \cdot \bar{\partial} x + b \bar{\partial} c + \bar{b} \partial \bar{c} \right).$$

The matter stress tensor is denoted by $T^{(x)}(z)$ and the quantum ghost stress tensor is given by

$$T^{(bc)}(z) = \lim_{w\to z}\big(-\partial b(z)c(w) - 2b(z)\partial c(w) + \frac{1}{(z-w)^2}\big).$$

The classical BRST current is given by

$$j^B(z) = c(z)\big(T^{(x)}(z) + \frac{1}{2}T^{(gh)}(z)\big) + \frac{3}{2}\partial^2 c(z).$$

(a) Give a suitable definition of the quantum BRST current, by subtracting OPE singularities.

(b) Compute the OPE of j^B with b, c and x, and deduce the (anti-) commutators of the BRST operator

$$Q^B = \oint \frac{dz}{2\pi i} j^B_z$$

with these fields.

(c) Show that the resulting transformation laws on the fields b, c and x give rise to a classical symmetry of S for any dimension D of space-time, and that j^B is related to the Noether current of this symmetry.

(d) Show that Q^B is nilpotent as a quantum mechanical operator if and only if $D = 26$, using your favorite method. (Hint : It is easy to show that when $D \neq 26$, Q^B cannot be nilpotent, by using the relation between the full stress tensor $T = T^{(x)} + T^{(bc)}$ and the anti-commutator of Q^B with b.)

Problem 12. Now set $D = 26$ and consider correlation functions of vertex operators V_i, including ghosts :

$$\langle V_1 \cdots V_N\rangle = \int Dx \int D(b\bar{b})D(c\bar{c})V_1\cdots V_N \prod_{j=1}^{\nu} |(\mu_j, b)|^2 e^{-S}$$

For simplicity, we restrict to surfaces Σ of genus $h > 1$; μ_j is a basis of Beltrami differentials, associated with moduli coordinates m_j, and $\nu = 3h - 3$.

Assume that V_i are physical vertex operators, so that V_i commutes with the BRST operator Q^B. Now assume that one vertex operator, say V_N, is BRST exact, i.e.

$$V_N = \int_\Sigma d^2w \oint_C \frac{dz}{2\pi i} j^B(z) W_N(w)$$

for *some* operator W_N, which is not necessarily a physical vertex operator. The curve C is a simple contour around the point w. By deforming the contour C, show that $\langle V_1 \cdots V_N\rangle$ is a total derivative in the moduli m_j.

Problem 13. Let M be a Riemannian manifold with local coordinates x^μ, metric $G = G_{\mu\nu}dx^\mu \otimes dx^\nu$, two-form $B = B_{\mu\nu}dx^\mu \wedge dx^\nu$, and scalar field Φ, and let l be a real positive parameter. The generalized nonlinear sigma model action on a surface

Σ, with metric g, Gaussian curvature R_g, volume form $d\mu_g$, and anti-symmetric tensor ϵ^{mn}, is given by

$$S[x;g] = \frac{1}{8\pi l^2}\int_\Sigma d\mu_g \left[g^{mn}G_{\mu\nu} + \epsilon^{mn}B_{\mu\nu}\right]\partial_m x^\mu \partial_n x^\nu + \frac{1}{2\pi}\int_\Sigma d\mu_g R_g \Phi$$

(a) Derive the form of the vector field $V^\mu(x)$ in TM, such that the infinitesimal Weyl transformation $\delta g = 2\delta\sigma g$, combined with the transformation $\delta x^\mu = l^2\delta\sigma V^\mu(x)$ yields a trace of the stress tensor $T_{mn}g^{mn}$ which is of the same form as the Lagrangian density of $S[x;g]$, i.e. such that

$$\delta S[x;g] = -\frac{1}{2\pi}\int_\Sigma d\mu_g \left[g^{mn}\beta^G_{\mu\nu} + \epsilon^{mn}\beta^B_{\mu\nu}\right]\partial_m x^\mu \partial_n x^\nu + \frac{1}{2\pi}\int_\Sigma d\mu_g R_g \beta^\Phi$$

(b) Derive the resulting expressions for β^G, β^B and β^Φ. (These are the tree level contributions to the β-functions defined for Weyl transformations.)

Problem 14. Let ϕ be a real scalar field on a Riemann surface Σ, with action

$$S[\phi;g] = \frac{1}{8\pi}\int_\Sigma d\mu_g \left[g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\right]$$

where m is an IR regulator mass, which we let go to zero whenever the limit is well-defined.

(a) Use dimensional regularization to derive the Weyl transformation properties of the Green function $G(\xi,\xi') = \langle\phi(\xi)\phi(\xi')\rangle$ and its first and second derivatives, evaluated on the diagonal (i.e. at coincident points).

(b) Repeat the calculation of (a), but now with a heat-kernel regularization of the Green function. Compare your result with that of (a).

Problem 15. The next two problems concern the Wess-Zumino-Witten model. Problem 15 deals with the classical part and Problem 16 with the quantum part. Let G be a semi-simple compact Lie group, with Lie algebra $\mathcal{G}$, and structure constants f_{abc}, $a,b,c = 1,\cdots,\dim G$. Let Σ be a surface with the topology of the plane, with a flat metric. The Wess-Zumino-Witten model is defined in terms of maps $g:\ \Sigma \to G$ and classical action

$$S[g] = \frac{k}{16\pi}\int_\Sigma \operatorname{tr}\theta\wedge *\theta + \frac{k}{24\pi}\int_B \operatorname{tr}\tilde\theta\wedge\tilde\theta\wedge\tilde\theta.$$

Here, θ denotes the $\mathcal{G}$-valued left-invariant one-form $\theta = g^{-1}dg$, B is a 3-dimensional manifold whose boundary is Σ. Also, we have $\tilde\theta = \tilde g^{-1}d\tilde g$ where $\tilde g:\ B\to G$ is any differentiable extension of g to B such that $\tilde g|_\Sigma = g$.

(a) Show that the action $S[g]$ is of the same form as the generalized nonlinear sigma models studied in the lectures, with action (for definitions, see Problem 13.)

$$S_{G,B}[x] = \frac{1}{8\pi l^2}\int_\Sigma d\mu_\delta \left[\delta^{mn}G_{\mu\nu} + \epsilon^{mn}B_{\mu\nu}\right]\partial_m x^\mu \partial_n x^\nu$$

(b) Show that the following multiplication formula holds

$$S[gh] = S[g] + S[h] + \frac{k}{16\pi}\int_\Sigma \operatorname{tr} g^{-1}\bar\partial g h^{-1}\partial h$$

and use it to show that S is well-defined, up to shifts by integer multiples of $2\pi k$.

(c) Using the formula of (b), show that $S[g]$ is invariant under the infinite dimensional current (Kac-Moody) algebra $\hat{\mathcal{G}} \times \hat{\mathcal{G}}$:

$$g(z,\bar{z}) \to \Omega(z) g(z,\bar{z}) \bar{\Omega}(\bar{z})^{-1}$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are holomorphic and anti-holomorphic maps from Σ into G (actually into the complexification of G) respectively. Derive the $\mathcal{G}$-valued conserved current $J = J^a t_a$ resulting from the transformations Ω. (t_a are generators of $\mathcal{G}$, satisfying the structure relations $[t_a, t_b] = f^{abc} t_c$.) Derive the transformation law of J under Ω.

(d) Derive the stress tensor T, show that it is holomorphic and express T as a quadratic form in terms of J.

Problem 16. We now quantize the theory of Problem 15. Assume that the quantum current J and stress tensor T continue to be holomorphic. We also assume in part (b) that the relation between the quantum current J and the quantum stress tensor T is given by the relation derived in (d), except for an unknown multiplicative constant κ :

$$2\kappa T(z) =: J^a(z) J^a(z) :$$

(a) Show that the Ward identities for Diff(Σ) and $\hat{\mathcal{G}}$ yield the following OPE's on T and J:

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) \\
T(z)J^a(w) &\sim \frac{1}{(z-w)^2} J^a(w) + \frac{1}{z-w} \partial_w J^a(w) \\
J^a(z)J^b(w) &\sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{abc}}{z-w} J^c(w)
\end{aligned}$$

(b) By considering the current correlation functions

$$\langle J^{a_1}(z_1) \cdots J^{a_n}(z_n) \rangle$$

and limits thereof when different z_i approach one another, show that the OPE's for J and T are consistent only if

$$c = \frac{k \dim G}{k + C_2(G)},$$

where $C_2(G)$ is the quadratic Casimir value of the adjoint representation of $\mathcal{G}$, defined by $f^{acd} f^{bcd} = C_2(G) \delta^{ab}$.

(c) Show that, in the large k approximation, the metric G and the antisymmetric tensor field B of the Wess-Zumino-Witten model satisfy the equations for conformal invariance and for the central charge that were derived (in the small l^2 limit) in the lectures.

Problem 17. In the next two problems we investigate the so-called lightcone gauge quantization method. The key ingredient of this method is to eliminate at the classical level the constraints resulting from Diff(Σ) invariance (and also other symmetries for superstring theories), by solving for the constraints explicitly. Quantization is then carried out on the remaining "transverse degrees of freedom" only, and automatically results in a Hilbert space, with positive definite norm. Solving the constraints is achieved by choosing a null vector ξ in Minkowski space-time (i.e. a vector on the lightcone, whence the name of the method). Poincaré symmetry will be realized in the quantum theory provided Poincaré generators can be constructed that obey the Poincaré algebra, and this is possible again only in the critical dimension.

Let $M = \mathbf{R}^D$ be flat Minkowski space-time of dimension D, with flat metric $\eta = \mathrm{diag}(-,+,\cdots,+,+)$. We choose the null vector $\xi = (1,0,\cdots,0,1)$, introduce associated lightcone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1})$$

and denote the remaining "transverse coordinates" by x^i, $i = 1,\cdots,D-2$. The scalar product in M reads

$$v \cdot w = v^\mu w^\nu \eta_{\mu\nu} = -v^+ w^- - v^- w^+ + v^i w^i.$$

The string worldsheet Σ is taken to be an annulus with global complex coordinates $z, \bar{z}$ and worldsheet metric $g = 2|dz|^2$.

We specialize in this problem to the closed oriented bosonic string; the open and unoriented strings may be constructed by truncation. The classical field equations are $\partial_z \partial_{\bar{z}} x^\mu(z,\bar{z}) = 0$, and are solved by

$$x^\mu(z,\bar{z}) = q^\mu + ip^\mu \ln|z|^2 + i\sum_{n\neq 0}\frac{1}{n}(x_n^\mu z^{-n} + \tilde{x}_n^\mu \bar{z}^{-n}) \tag{3}$$

The classical constraints are

$$\partial_z x \cdot \partial_z x = \partial_{\bar{z}} x \cdot \partial_{\bar{z}} x = 0 \tag{4}$$

The generators of the Poincaré algebra are given by p^μ and by the Lorentz generators

$$J^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu + S^{\mu\nu} + \tilde{S}^{\mu\nu} \tag{5}$$

where the spin operators of the massive states are defined by

$$S^{\mu\nu} = -i\sum_{n=1}^{\infty}\{x_{-n}^\mu x_n^\nu - x_{-n}^\nu x_n^\mu\} \qquad \tilde{S}^{\mu\nu} = -i\sum_{n=1}^{\infty}\{\tilde{x}_{-n}^\mu \tilde{x}_n^\nu - \tilde{x}_{-n}^\nu \tilde{x}_n^\mu\} \tag{6}$$

(a) Show that it is consistent with the field equations for x^μ to make a choice of coordinates $z, \bar{z}$ such that $x_n^+ = \tilde{x}_n^+ = 0$ for all $n \in \mathbf{Z}$, $n \neq 0$.

(b) In these coordinates, solve for the classical constraint equations (4), by solving explicitly for $x^-(z,\bar{z})$ in terms of $x^i(z,\bar{z})$, for x_n^- in terms of x_n^i and for $\tilde{x}_n^-$ in terms of $\tilde{x}_n^-$.

(c) Quantize the independent degrees of freedom x_n^i and $\tilde{x}_n^i$ by requiring for $m, n \in \mathbf{Z}$ and $i, j = 1, \cdots, D-2$ that

$$[x_m^i, \tilde{x}_n^j] = 0 \qquad [x_m^i, x_n^j] = [\tilde{x}_m^i, \tilde{x}_n^j] = \delta^{ij}\delta_{m+n,0}. \tag{7}$$

It is convenient to work on a sector of Fock space with definite eigenvalue of the operator p^+, so that p^+ may effectively be regarded as a c-number. Define a suitable renormalization for the composite field x^-. This renormalization will be unique up to a constant, which we denote by a for later reference. Show that the oscillators $p^+x_n^-$ (and similarly $p^+\tilde{x}_n^-$) obey a Virasoro algebra with central charge $D-2$.

(d) Show that the algebra of the transverse Lorentz group, SO(24), generated by J^{ij} closes for any D and any value of a. Show that the commutators $[J^{ij}, J^{k-}]$ obey the Lorentz algebra of $\mathrm{SO}(1, D-1)$, for all values of a and D.

(e) From the general structure of the Lorentz generators, show that the commutators $[J^{i-}, J^{j-}]$ must be of the form

$$[J^{i-}, J^{j-}] = \frac{1}{(p^+)^2} \sum_{n=1}^{\infty} \Delta_n (x_{-n}^i x_n^j - x_{-n}^j x_n^i) + \{x \to \tilde{x}\}. \tag{8}$$

Determine Δ_n, and find the conditions under which Δ_n vanishes for all $n \in \mathbf{Z}$, $n \neq 0$. (This is a necessary condition for the closure of the Lorentz and Poincaré algebras.)

(f) Show that the remaining Poincare structure relations are obeyed.

(g) We define the Del Giudice-Di Vecchia-Fubini (DDF) operators (for simplicity only in the left-moving sector) by

$$V^i(k) = \oint dz \partial_z x_+^i e^{ik\cdot x_+}$$

where

$$x^i(z, \bar{z}) = x_+^i(z) + x_-^i(\bar{z}).$$

Show that, for a suitable choice of momenta k, the operators map positive norm states into positive norm states, and physical states into physical states.

Problem 18. Generalize the set-up and the analysis carried out in Problem 17 to the case of the Ramond-Neveu-Schwarz string.

Problem 19. In the following three problems, we shall derive certain properties, under so-called T-duality, of closed oriented string theories, compactified in one dimension on a circle of radius R.

Consider the closed oriented bosonic string on the space-time manifold $M = \mathbf{R}^{25} \times S^1$, with Minkowskian metric $\eta = \mathrm{diag}\,(- + \cdots +)$ and with S^1 of radius $R > 0$ (lying in a space direction of η).

(a) Obtain the mode expansion of the most general solution to the free theory on a worldsheet with the topology of an annulus. (Make sure to include the effects of the fact that the space-time manifold M is not simply connected.)

(b) Show that the states of the theory for radius R are in one to one correspondence with the states for the theory with radius $R' = b/R$ for some (R-independent) constant $b > 0$. This relation is usually called T-duality.

(c) Show that transition amplitudes for the theory with radius R and R' – as defined in (b) – are similarly related. One says that the bosonic closed oriented string is self-dual under T-duality.

Problem 20. Consider now the Type IIA and Type IIB closed oriented superstring theories on the space-time manifold $M = \mathbf{R}^9 \times S^1$, with Minkowskian metric $\eta =$ diag $(- + \cdots +)$ and with S^1 of radius $R > 0$ (lying in a space direction of η).

(a) Begin by taking $R = \infty$ (the original superstring theories) and show that the massive spectra of the Type IIA and Type IIB theories are identical.

(b) Now, for finite R, obtain the mode expansion of the most general solution to the free theories on a worldsheet with the topology of an annulus.

(c) Show that the states which are annihilated by all the oscillator annihilation operators of level > 0 (these are the pure momentum and winding states) for the Type IIA string at radius R are in one to one correspondence with states for the Type IIB string at radius $\tilde{R} = c/R$ for some constant c. (You will have to explain how the relative chirality of left and right movers can be changed upon $R \to \tilde{R}$.)

(d) Generalize the argument of (c) to show that the full spectra of the Type IIA superstring at radius R and of the Type IIB superstring at radius $\tilde{R}$ are identical. One says that the Type IIA and Type IIB superstrings are mapped into one another under T-duality.

Problem 21. Finally, consider the heterotic strings with gauge groups $E_8 \times E_8$ and $Spin(32)/Z_2$, on the space-time manifold $M = \mathbf{R}^9 \times S^1$, with Minkowskian metric $\eta =$ diag $(- + \cdots +)$ and with S^1 of radius $R > 0$ (lying in a space direction of η).

The circle compactification now allows for a non-trivial holonomy of the gauge field along S^1 – in physics language : a non-zero value for the Wilson loop around S^1. We assume that this holonomy element is such that it breaks both $E_8 \times E_8$ and $Spin(32)/Z_2$ down to $SO(16) \times SO(16)$.

Show that the corresponding $SO(16) \times SO(16)$ theories are again related to one another upon $R \to \tilde{R}$, where $\tilde{R}$ was defined in Problem 20(c). One says that the two heterotic theories are mapped into one another under T-duality.

Super Space Descriptions of Super Gravity

Pierre Deligne

§1. General remarks

The notion of a symmetric tensor field g_{ij} has an obvious analogue in the super world. This obvious analogue is not what one has in mind when speaking of super gravity. In super space formulations, two methods are in use for describing relevant analogues in the super world of pseudo Riemannian structures. One is used in [II-**Strings**, §§9.1, 10.9]. The other is used in Chapter 5, §7 of Yu. Manin, Gauge field theory and Complex Geometry, Grundlehren 289, Springer Verlag 1988.

We will work in Minkowski signature. In each method, one starts by fixing a dimension D, giving rise to the group $\mathrm{Spin}(1, D-1)$ and its vector representation V, with positive cone C. One also chooses a real spinorial representation S, and a morphism of representations Γ: $S \otimes S \to V$. It is assumed that Γ is symmetric and that for v in C^0, $\langle v, \Gamma(s,t)\rangle$ is positive definite. We define $s := \dim S$.

Only a few (V, S, Γ) are interesting, all with $D \leq 11$ and $s \leq 32$. For some, no super space Lagrangian formulation of super gravity is available.

Whatever the method used, it is not enough to define what is the chosen analogue of a pseudo Riemannian structure on a fixed super manifold M of dimension $D|s$. One has to define what is a family of such structures, on a family of super manifolds M_b parametrized by a super manifold B. Such a family (M_b) is by definition a submersive morphism $M \to B$ of relative dimension $D|s$. As is traditional, we will speak as if working on a fixed M, but in a way making the extension to families obvious. In practice, this means we are forbidden to use the notion of "the reduced space of M".

In the first method, the data consists of

(a) a super manifold M of dimension $D|s$.

(b) a reduction of the structural group of the tangent bundle of M to $\mathrm{Spin}(1, D-1)$, in its representation $V \oplus \Pi S$.

(c) a $\mathrm{Spin}(1, D-1)$-connection ∇ on the tangent bundle.

The connection ∇ gives rise to a torsion tensor T: a 2-form with values in the tangent bundle, and to a curvature tensor R. The values of suitable components of T and R are prescribed.

If bases e_a $(a = 1, \dots, D)$ of V and e_α $(\alpha = 1, \dots, \dim S)$ of V and S are chosen (in which Γ becomes $\Gamma^a_{\alpha\beta}$), a structure (b) can locally be described by a basis E_A ($A = a$ or α) of the tangent bundle, with E_a even and E_α odd. One may prefer to use the dual basis E^A of Ω^1_M: $\langle E_A, E^B \rangle = \delta^B_A$. The structure (c) is then described as a 1-form Ω with values in the Lie algebra of the Spin group, i.e. in $\wedge^2 V$: $\Omega = \frac{1}{2}\,\Omega^{ab} E_a \wedge E_b$, with $\Omega^{ab} + \Omega^{ba} = 0$.

One of the conditions imposed on the torsion tensor is that, in the moving frame E_A, $T^a_{\alpha\beta}$ be some fixed multiple of $\Gamma^a_{\alpha\beta}$. As the subspace τ of the tangent bundle spanned by the E_α is stable by the connection, this amounts to saying that the Lie bracket $[E_\alpha, E_\beta]$ is, modulo τ, some fixed multiple of $\Gamma^a_{\alpha\beta} E_a$.

In the second method, the data consist of (a), of

(d) a subbundle τ of the tangent bundle, of dimension $0|s$; and

(e) additional data, typically a section of some Berezinian line bundle.

The bracket of vector fields induces the Frobenius pairing F: $\tau \otimes \tau \to T/\tau$. It is required that the pair of vector bundles τ, T/τ, with F, be isomorphic to ($\Pi S, V$, some fixed multiple of Γ). If one forgets (e), one has a notion of super conformal structure.

When going from a description by the first method to a description by the second, one takes τ to be spanned by the E_α. The condition on the $T^a_{\alpha\beta}$ component of the torsion implies the condition on the Frobenius pairing.

In the best cases, one can attach to a structure of the type considered a Lagrangian density $\mathcal{L}$, which is a section of $\mathrm{Ber}(\Omega^1_M)$. It gives rise to the action integral $\int \mathcal{L}$. There is in fact no reason to insist that the action be given by integration of a density on super space. In a pure theory of super gravity, one then considers the extremals of the action. The Euler-Lagrange equations (equations of motion) are the analogue of Einstein equation.

A component formalism does the following. Let M^0 be a submanifold of M of dimension $D|0$. As we insist on working in a basis B, such a submanifold is not unique. The super manifold M, with its structure, is then to be described (up to unique isomorphism) in terms of fields on M^0.

One of those fields is a pseudo-Riemannian structure on M^0: the tangent bundle of M^0 maps isomorphically to T_M/τ (restricted to M^0), and the restriction of the structural group of T_M to $\mathrm{Spin}(1, D-1)$ induces a restriction of the structural group of the tangent bundle of M^0 to $\mathrm{Spin}(1, D-1)$. Similarly, the normal bundle T_M/T_{M^0} of M^0 in M is the spin bundle of M^0 defined by the representation S of $\mathrm{Spin}(1, D-1)$: τ (restricted to M^0) maps isomorphically to the quotient of T_M (restricted to M^0) by T_{M^0}.

§2. Example: $D = 2$, $N = 1$ in the notations of §9.1 of D'Hoker's lectures

In this case, V is $\mathbb{R}^2$, with basis e_1, e_{-1}, positive cone spanned by e_1 and e_{-1}, and quadratic form xy. The spin representation S is $\mathbb{R}^2$, with basis $e_{1/2}$, $e_{-1/2}$. The spin group is $\mathbb{R}^*$, with $\lambda \in \mathbb{R}^*$ acting by multiplication by λ^{2n} on e_n. The morphism Γ: $S \otimes S \to V$ is given by $\Gamma(e_{\pm 1/2}, e_{\pm 1/2}) = e_{\pm 1}$, $\Gamma(e_{\pm 1/2}, e_{\mp 1/2}) = 0$.

Let us write $+$, $-$ for $+1/2$, $-1/2$; a will run over 1, -1, α on $+$, $-$, and A on the values of a or α. In the first method, the conditions on the torsion and

curvature are that

$$R_{++} = R_{--} = 0$$
$$T^{\gamma}_{\alpha\beta} = 0$$

and for the $T^{a}_{\alpha\beta}$ to be such that modulo τ spanned by E_+, E_-,

$$\frac{1}{2}[E_+, E_+] = E_1,\ \frac{1}{2}[E_-, E_-] = E_{-1},\ [E_+, E_-] = 0 \pmod{\tau}\ .$$

In the second method, the condition on the Frobenius pairing implies, at least locally, that τ has a unique decomposition $\tau = \tau_+ \oplus \tau_-$ into the sum of subbundles of dimension $0|1$, with $[\tau_+, \tau_-] = 0$ modulo τ, and that for X_+, X_- bases of τ_+ and τ_-, X_+^2, X_-^2, X_+ and X_- span the tangent bundle. The distribution spanned by X_+ and X_+^2 (resp. X_- and X_-^2) does not depend on the particular choice of X_+ (resp. X_-). It is stable under bracket, and by the Frobenius theorem defines a foliation with leaves of dimension $1|1$. We get two transversal foliations, i.e. locally a decomposition of M as $M_+ \times M_-$, with M_+ and M_- of dimension $1|1$.

Locally, X_- generates an action of the group with Lie algebra of dimension $1|1$ freely generated by one odd generator. As $[X_-, \tau + \mathcal{O}X_-^2] \subset \tau \oplus \mathcal{O}X_-^2$, the distribution $\tau \oplus \mathcal{O}X_-^2$ is respected by this action, and is the pullback of a distribution $\tau_+[M_+]$ on the quotient M_+. The distribution τ_+ on M, being the intersection of $\mathcal{O}X_+ \oplus \mathcal{O}X_+^2$ and $\tau \oplus \mathcal{O}X_-^2$, it comes from $\tau_+[M_+]$: the distributions τ_+ and τ_- on $M = M_+ \times M_-$ come from distributions, still denoted τ_+ (or $\tau_+[M_+]$) and τ_- (or $\tau_-[M_-]$) on M_+ and M_-, via the decomposition of the tangent bundle T_M of M as the sum of the pullback $T_M^{\pm}$ of the tangent bundle of M_+ and M_-.

The additional data (e) is a duality between τ_+ and τ_-, or equivalently: a trivialization of $\mathrm{Ber}(\tau)$; a trivialization of $\mathrm{Ber}(T) \simeq \mathrm{Ber}(T/\tau) \otimes \mathrm{Ber}(\tau) \simeq \mathrm{Ber}(\tau)^{\vee}$.

We now show the equivalence of the two methods.

Suppose (d), (e) are given. The odd line τ_+ on $M = M_+ \times M_-$ comes from M_+, hence carries a connection in the direction of M_-. It is the dual of τ_-, which similarly carries a connection in the direction of M_+. This gives on τ_+ a connection in the direction of M_+. Putting the two together, we get a connection on τ_+. One similarly defines a connection on τ_-, and the connections on τ_+ and τ_- are in duality. By construction, those connections are integrable in the direction of M_+ as well as in the direction of M_-: the components R_{++} and R_{--} of the curvature vanish.

There is a unique morphism $f\colon \tau^{+\otimes 2} \to T$ lifting the Frobenius pairing, such that if X_+, Y_+ are sections of τ_+, horizontal in the direction of M_+, one has

$$f(X_+, Y_+) = [X_+, Y_+]$$

Indeed, if X_+^0 is a basis of τ^+, any X_+, Y_+ are of the form fX_+, gX_+ with $X_+(f) = X_+(g) = 0$, and

$$[fX_+, gX_+] = (-1)^{p(X_+)p(g)} fg[X_+, X_+]\ .$$

One could also say that wherever X_+ and Y_+ are horizontal, $f(X_+, Y_+) = [X_+, Y_+]$. This, and the same for τ^-, give a decomposition of T as $\tau^+ \oplus \tau^- \oplus (\tau^+)^{\otimes 2} \oplus (\tau^-)^{\otimes 2}$. This composition, the duality between τ^+ and τ^- and the connection on τ^+ define

a reduction of the structural group of T to $\mathbb{R}^*$, and an $\mathbb{R}^*$-connection: we have obtained a structure as in the first method. The condition on $T^a_{\alpha\beta}$ holds because the Frobenius pairing is what it should be. The vanishing of $T^\gamma_{\alpha\beta}$ is checked as follows.

a) $T^\pm_{++} = 0$: vanishing of the $\pm$ components of $2\nabla_{X_+}(X_+) - [X_+, X_+]$.

For X_+ horizontal in the direction of M^+, it is indeed in $f(\tau^{+\otimes 2})$ by definition of f. Similarly for $T^\pm_{--}$.

b) $T_{+-} = 0$: vanishing of $\nabla_{X_+}(X_-) + \nabla - X_-(X_+) - [X_+, X_-]$. For X_+ horizontal in the M_- direction, i.e. coming from a section of τ_+ on M_+, and X_- horizontal in the M_+ direction, i.e. coming from a section of τ_-on M_-, the three terms indeed vanish.

Conversely, suppose that (b), (c) are given. The restriction (b) of the structural group to $\mathbb{R}^*$ gives the odd distribution τ, corresponding to ΠS, and the condition on the components $T^c_{\alpha\beta}$ of the torsion tensor ensures that the Frobenius pairing $\tau \otimes \tau \to T/\tau$ is as required, with τ_+ and τ_- given by the restriction of the structural group to $\mathbb{R}^*$, which also gives the pairing between τ_+ and τ_-. It remains to check, using the curvature and torsion conditions, that the even lines in the decomposition of T as well as the connection which are obtained by the second method all are what we started with. The vanishing of $T^\pm_{++}$ and $T^\pm_{--}$ ensures that the even lines are what they should be. If X_+ and X_- are sections of τ_+ and τ_-, as $\nabla_{X_+}(X_-)$ (resp. $\nabla_{X_-}(X_+)$) are in τ_+ (resp. τ_-), the torsion condition $T_{+-} = 0$: $\nabla_{X_+}(X_-) + \nabla_{X_-}(X_+) = [X_+, X_-]$, ensures that ∇_{X^+} and ∇_{X^-} are as they should be, while the curvature conditions $R_{++} = R_{--} = 0$ express $\nabla_{X^{+2}}$ (resp. $\nabla_{X^{-2}}$) as $(\nabla_{X^+})^2$ (resp. $(\nabla_{X^-})^2$).

In [II-**Strings**, §9.1], D'Hoker works in the Euclidean context, in which the decomposition of M as $M^+ \times M^-$ becomes that M is the $2|2$ real super space underlying a $1|1$ complex super space.

Notes on $2d$ Conformal Field Theory and String Theory

Dennis Gaitsgory

Introduction

§0.1. Contents of these notes

§0.1.1.

These notes appeared out of an attempt to write down explanations and solutions of home assignments for the course on String Theory which was given by E. D'Hoker in Spring 1997 at IAS.

Our original plan was to deal only with questions that admitted a rigorous mathematical formulation or axiomatization. A variety of such questions arises in the most basic ingredient of String Theory, namely in 2-dimensional conformal field theory (CFT).

We shall restrict our attention to a tiny part of 2D CFT: our goal is to explain the mathematical formalism that describes the operator product expansion (OPE) of quantum fields. To make things as simple as possible (and since we wish to stay within the realm of algebraic geometry) we will deal only with fields from the so-called holomorphic sector.

Fortunately, appropriate mathematical objects that imitate the OPE operation on quantum fields have been found several years ago by A. Beilinson and V. Drinfeld (they called them "chiral algebras"). We shall adopt their point of view completely and the present paper (and especially Chapters 1, 3 and 4) can be regarded as an example-oriented digest of the long-awaited text of Beilinson and Drinfeld.

The language of chiral algebras is in many ways equivalent to the language of vertex operator algebras (VOA's). More precisely, a vertex operator algebra is the same as a "universal chiral algebra", i.e. one that is defined naturally over an arbitrary Riemann surface. Since we will be interested in chiral algebras that are universal in the above sense, our treatment of such matters as the Sugawara construction (Section 3.1), fermion algebras and boson-fermion correspondence (Sect. 3.3), the BRST complex (Sect. 4.1) and of many others, is nothing more than a reinterpretation in terms of algebraic geometry of constructions that have been known to the founders of the theory of vertex operator algebras for many years (cf. [**10**],[**9**],[**13**], [**14**],[**15**]).

Therefore, we wish to emphasize that hardly any part of the contents of these notes is the invention of their author. Besides the classical sources mentioned just

above we have borrowed the material from the partially written manuscripts [**2**] and [**3**].

§0.1.2.

Let us discuss in some detail the mathematical contents of the paper.

Chapter 1 deals with the basics of the theory of chiral algebras.

In Section 1.1 we introduce the main character, namely the notion of a chiral algebra. A chiral algebra on a curve X is a D-module with a binary operation that satisfies the axioms of a Lie-* bracket. One of the features of chiral algebras is that these objects are not so easy to construct. The more elementary constructions go through the procedure of taking the chiral universal enveloping algebra of a more accessible object– a Lie-* algebra (the latter notion is introduced in Section 1.2). Finally, in Section 1.3. we introduce the notion of the space of conformal blocks of a chiral algebra (the dual to the space of conformal blocks should be thought of as a set of possible correlation functions of a quantum field theory (QFT) with a given OPE of the fields).

In Chapter 2 we make an attempt to define axiomatically what a 2D CFT is. However, it should be clear form the very beginning that our list of axioms is far from being complete: we are dealing with the short-distance singularities of quantum fields only and we do not mention at all what happens when a Riemann surface approaches the boundary of the moduli space. We hope to be able to address this more subtle and interesting issue in the future.

In Section 2.1 we introduce the notion of a local $\mathcal{O}$-module over the universal curve $\mathfrak{X}$ over the moduli space of smooth curves. A local $\mathcal{O}$-module is a quasi-coherent sheaf which is "universal" (in the same sense as a VOA is a "universal" chiral algebra). We need this notion since fields of a CFT are clearly local in the above sense, i.e. the fields and the OPE structure on them over a given region of a Riemann surface do not "feel" a change of a metric at a remote point. In Section 2.2 we give the axioms of a CFT data in the simplest case of central charge 0 and in Section 2.3 we generalize the discussion to the case of an arbitrary central charge. In particular, we show (what should be a standard thing by now) that the space of conformal blocks of a CFT chiral algebra acquires a (projective) connection along the moduli space of smooth curves.

Chapter 3 is devoted to the examples. (Unfortunately, all our examples are trivial or almost trivial from the point of view of physics. They correspond to either non-existing physical theories or to theories that are free.)

We discuss three types of examples here: Kac-Moody chiral algebras, i.e. those associated to Lie algebras with an invariant quadratic form, in particular, the Heisenberg chiral algebra which corresponds to a Lie algebra with a zero bracket (all this in Section 3.1); the dilaton twist of the Heisenberg chiral algebra (Sect. 3.2) and the fermionic chiral algebra, which is referred to as the *bc*-system in the physics literature (Sect. 3.3). In all these examples we write down explicitly the energy-momentum tensor (or equivalently, the Sugawara construction) and perform explicitly the corresponding calculations. We should mention that there are many more ways to make the same calculations and ours are not always the optimal ones: we just wanted to exhibit as many techniques as possible of manipulating

with chiral algebras. Also, the reader should notice that the way we present these calculations is very close to how a physicist would have done it.

In Chapter 4 we discuss the BRST complex and related matters.

In Section 4.1, given a Lie-* algebra $\mathcal{B}$, we construct (following Beilinson) its central extension $\mathcal{B}'$ and define a BRST differential on a chiral algebra obtained by tensoring the twisted chiral universal enveloping algebra of $\mathcal{B}'$ and the corresponding fermionic algebra. This enables one to write a BRST reduction of any chiral algebra endowed with an action of $\mathcal{B}$ (as usual in such situations, this action must be of "Harish-Chandra" type). In Section 4.2 we apply the discussion of Sect. 4.1 in the case of the BRST reduction of a chiral algebra corresponding to a CFT of central charge 26 with respect to the Virasoro algebra. This procedure is used by physicists for what can be mathematically reformulated as writing down "explicitly" the connection along $\mathfrak{M}$ on the space of conformal blocks. However, we do not pretend to having really understood why the physicists really need the BRST procedure.

Starting from Chapter 5, this paper becomes much more involved mathematically and sometimes we are forced to omit proofs of certain important statements. Our goals in Chapters 5 and 6 are the construction of the bosonic chiral algebra (in the VOA terminology it is sometimes called the lattice VOA) and the proof of Theorem 6.2.2 that computes its space of conformal blocks on a given curve in terms of the canonical line bundle over the Jacobian (or, rather a direct product of several copies of the Jacobian) of this curve. Usually, in the theory of VOA's, the sought-for bosonic chiral algebra is constructed by an explicit formula (cf. [**13**]) (which is the most classical example of a "vertex operator") and the fact that this formula works remains for us a miracle.

The construction of the bosonic chiral algebra described in this paper is quite different: it is very geometric and uses a sophisticated algebro-geometric object called the Beilinson-Drinfeld Grassmannian (cf. [**16**], [**11**]). In fact, our exposition covers a part of a baby (=abelian) version of a new construction of the geometric Langlands correspondence due again to Beilinson and Drinfeld.

In Section 5.1 we describe a new way of thinking about chiral algebras (due to Beilinson): now a chiral algebra is interpreted as a system of $\mathcal{O}$-modules on various powers of our curve endowed with what we shall call "factorization isomorphisms". In Section 5.3 (after some geometric preliminaries in Section 5.2) we demonstrate in a relatively simple situation an interplay between the two pictures.

In Section 6.1 we describe the canonical line bundle over the B.-D. Grassmannian corresponding to a torus, which arises from an even quadratic form on the lattice of 1-parameter subgroups of this torus. Finally, in Section 6.2 we introduce the bosonic chiral algebra and prove Theorem 6.2.2.

§0.1.3.

For the convenience of the reader we shall now single out the subsections in our text where one can find solutions to problems of E. D'Hoker's course.

Problem 1: The fact that T is indeed a stress tensor for this theory follows from Section 3.1. Points a) and c) follow from the general properties of the stress tensor (Section 2.3.2). Point b) is explained in Section 2.3.1.

Problem 2: The corresponding theory is studied in detail in Section 3.2.

Problem 3: The corresponding property follows from the definition of a CFT given in Section 2.3.2.

Problem 11: See Sections 4.1 and 4.2.

Problem 12: Our response is Section 4.2, especially Proposition 4.2.2.

Problem 16: See Section 3.1.

§0.2. Some background on D-modules

§0.2.1.

First, some notation:

For a scheme (or a stack) Z, $\mathcal{O}_Z$ (resp., Θ_Z, Ω_Z, D_Z) will denote the structure sheaf (resp., the tangent sheaf, the sheaf of 1-forms, the sheaf of differential operators) on Z. The terms "an $\mathcal{O}$-module on Z" and "a quasi-coherent sheaf on Z" will mean for us the same thing (with a few exceptions in Sections 2.2-2.3, where we will be dealing with projective limits of coherent sheaves). If $f : Z_1 \to Z_2$ is a morphism of algebraic varieties, we shall denote by f_* and f^* the direct and the inverse image functors respectively, in the category of quasi-coherent sheaves. The functor of the inverse image in the category of sheaves (and not $\mathcal{O}$-modules) will be used only once (in Section 2.3.3) and will be denoted by $f^\bullet$.

For an algebraic curve X, Δ will denote the closed embedding of the diagonal divisor in X^n and $j : X^n \setminus \Delta \hookrightarrow X^n$ will denote the open embedding of its complement. A symbol like $\Delta_{x_i=x_j}$ will designate an embedding of the closed subset of the diagonal divisor corresponding to points $(x_1, \ldots, x_n) \in X^n$ with $x_i = x_j$. By $p_i : X^n \to X$ we shall denote the projection on the i-th factor.

$\mathfrak{M}$ will denote the moduli stack of smooth complete curves and $\pi : \mathfrak{X} \to \mathfrak{M}$ will denote the universal curve over $\mathfrak{M}$. $\mathfrak{X}^n$ will denote the n-fold Cartesian product of $\mathfrak{X}$ with itself over $\mathfrak{M}$; Δ will as above denote the embedding of the divisor of diagonals into $\mathfrak{X}^n$ and $p_i : \mathfrak{X}^n \to \mathfrak{X}$ will stand for the corresponding projections. If L_1 and L_2 are quasi-coherent sheaves on $\mathfrak{X}^{k_1}$ and $\mathfrak{X}^{k_2}$ respectively we let $L_1 \boxtimes L_2$ denote (by slightly abusing the notation) the sheaf on $\mathfrak{X}^{k_1+k_2}$ obtained by restricting the corresponding sheaf on $\mathfrak{X}^{k_1} \times \mathfrak{X}^{k_2}$.

§0.2.2.

In this paper we will be extensively using the language of D-modules. Unfortunately, there are very few sources with an appropriate exposition of the theory, the standard reference being the book [**7**].[1]

For a smooth variety Z a (right) D-module is a (right) module over the sheaf D_Z of differential operators on Z. By definition, there exists a natural forgetful functor D-modules $\to$ $\mathcal{O}$-modules.

When Z is a singular scheme, one proceeds as follows: we embed Z (locally) as a closed subscheme into a non-singular variety Z' and consider D-modules on Z' that are set-theoretically supported on Z. (The correctness of this definition follows from the well-known Kashiwara theorem). In this case, there still is a forgetful functor D-modules $\to$ $\mathcal{O}$-modules which sends a D-module $\mathcal{M}$ over Z' as above to the $\mathcal{O}'_Z$-module of its sections that are scheme-theoretically supported on

[1] One of the best readings on D-modules is an unpublished series of lectures by J. Bernstein ([**6**]) (obtainable by e-mail from the author of these notes).

Z. This enables one to make sense of right D-modules over a strict ind-scheme that are set-theoretically supported on a closed finite-dimensional subscheme (cf. Section 6).

For a smooth variety Z there exists a functor

$$h: \{\text{ D-modules on } Z\} \to \{\text{ Sheaves in the étale topology on } Z\}$$

defined as follows:

$$\mathcal{M} \to \mathcal{M}/\mathcal{M} \cdot \Theta_Z$$

(we shall often abuse the notation and denote by h also the projection $\mathcal{M} \to h(\mathcal{M})$.)

The functor h is right-exact and we let Lh denote the corresponding left derived functor. (the functor Lh applied to holonomic D-modules with regular singularities realizes the celebrated Riemann-Hilbert correspondence). The functor

$$R\Gamma \circ Lh: \{\text{ D-modules }\} \to \{\text{ Vector spaces }\}$$

is called the De Rham cohomology functor and we shall denote it by $\mathcal{M} \to DR^\bullet(Z, \mathcal{M})$.

When $f: Z_1 \to Z_2$ is a map between smooth algebraic varieties, one defines the direct image functor

$$f_*: \{\text{ D-modules on } Z_1\} \to \{\text{ D-modules on } Z_2\}$$

(to distinguish it from the corresponding functor on the category of $\mathcal{O}$-modules; in cases where confusion can occur, we shall specify explicitly which one we are using.) When f is a projection $Z_1 = Z_2 \times Y \to Z_2$ the definition of f_* is a straightforward generalization of that of $DR^\bullet$ (the latter corresponds to the case $Z_2 = \mathrm{pt}$). When f is a closed embedding, f_* is defined using Kashiwara's theorem (see above); in this case we shall replace the notation f_* by $f_!$ in order to avoid confusion with the functor f_* for $\mathcal{O}$-modules.

For instance, when f is affine, the functor f_* is right-exact and when f is an open embedding, f_* is left exact.

For (f, Z_1, Z_2) as above one defines also the inverse image functor

$$f^!: \{\text{ D-modules on } Z_2\} \to \{\text{ D-modules on } Z_1\}.$$

We refer the reader to [7] for the definition.

When Z is a smooth variety, in addition to right D-modules one can consider left D-modules (it may seem surprising, but one cannot define left D-modules for non-smooth schemes). The categories of left and right D-modules are equivalent and the functor that realizes this equivalence sends a left D-module ${}^l\mathcal{M}$ to $\mathcal{M} := {}^l\mathcal{M} \otimes \Omega_Z^{top}$;in particular, the basic left D-module $\mathcal{O}_Z$ goes over to the basic right D-module Ω_Z^{top}.

If ${}^l\mathcal{M}$ is a left D-module on Z and $\mathcal{N}$ is another left (resp., right) D-module, the tensor product ${}^l\mathcal{M} \otimes \mathcal{N}$ is naturally a left (resp., right) D-module. If now $\mathcal{M}_1$ and $\mathcal{M}_2$ are two right D-modules, we shall denote by $\mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2$ the right D-module

$$({}^l\mathcal{M}_1 \otimes {}^l\mathcal{M}_2) \otimes \Omega_Z^{top}.$$

The operation $\overset{!}{\otimes}$ makes the category of right D-modules over a smooth variety into a symmetric tensor category. For instance, one can talk about Lie algebras in the category of right D-modules and of their universal enveloping algebras (this will be important in Section 1.2.4).

In addition, in this paper we will be dealing with D-modules on algebraic stacks in the smooth topology ($\mathfrak{M}$ is an example of such). It follows from the definition of a stack that these objects are perfectly suitable for having D-modules on them. For example, if a stack has the form Z/G , where Z is a scheme and G is a group acting on Z, a D-module on Z/G is the same as a strongly G-equivariant D-module on Z (cf. [**1**], Section 1 for the discussion of equivariant D-modules.)

The following observation will be used several times in this paper:

For a smooth variety Z consider the co-induction functor

$$\{ \mathcal{O}\text{-modules on } Z\} \to \{ \text{D-modules on } Z\}$$

which sends an $\mathcal{O}$-module $\mathcal{L}$ to $\mathcal{L} \underset{\mathcal{O}_Z}{\otimes} D_Z$.

Lemma 0.2.1. *We have:*

(a) *For a closed embedding $f : Z_1 \to Z_2$ and an $\mathcal{O}$-module $\mathcal{L}$ over Z_1 there is a canonical isomorphism:*

$$f_!(\mathcal{L} \underset{\mathcal{O}_{Z_1}}{\otimes} D_{Z_1}) \simeq f_*(\mathcal{L}) \underset{\mathcal{O}_{Z_2}}{\otimes} D_{Z_2}.$$

(b) *For two $\mathcal{O}$-modules $\mathcal{L}_1$ and $\mathcal{L}_2$ over Z the space*

$$\mathrm{Hom}_{\mathcal{O}-\mathrm{mod}}(\mathcal{L}_1, \mathcal{L}_2 \underset{\mathcal{O}_Z}{\otimes} D_Z) \simeq \mathrm{Hom}_{\mathrm{D}-\mathrm{mod}}(\mathcal{L}_1 \underset{\mathcal{O}_Z}{\otimes} D_Z, \mathcal{L}_2 \underset{\mathcal{O}_Z}{\otimes} D_Z)$$

is canonically isomorphic to the space of differential operators from $\mathcal{L}_1$ to $\mathcal{L}_2$.

§0.2.3.

Another notion that will be important for us is that of a Lie algebroid.

A Lie algebroid $\mathfrak{a}$ over a smooth variety Z is a quasi-coherent sheaf $\mathfrak{a}$ over Z with the following additional structure:

- An $\mathcal{O}$-module map $\phi : \mathfrak{a} \to \Theta_Z$.
- A bi-differential operator $[\ ,\ ] : \mathfrak{a} \underset{\mathbb{C}}{\otimes} \mathfrak{a} \to \mathfrak{a}$ satisfying the axioms of a Lie-* bracket.

The above structures should be compatible in the following way: if a_1 and a_2 are sections of $\mathfrak{a}$ and f is a a function over Z, we must have:

$$[f \cdot a_1, a_2] = f \cdot [a_1, a_2] + a_1 \cdot \mathrm{Lie}_{\phi(a_2)}(f).$$

The most basic example of a Lie algebroid is $\mathfrak{a} = \Theta_Z$.

In a natural way one defines a notion of a (left) module over a Lie algebroid. In particular, if $\mathfrak{a}$ is as above, any left D-module on Z is automatically an $\mathfrak{a}$-module.

Let now $f : Z_1 \to Z_2$ be a map between smooth varieties and let $\mathfrak{a}$ be a Lie algebroid on Z_2. Consider the $\mathcal{O}$-module $\mathfrak{a}'$ over Z_1 given by

$$\ker(f^*(\mathfrak{a}) \oplus \Theta_{Z_1} \to f^*(\Theta_{Z_2})),$$

where the maps $f^*(\mathfrak{a}) \to f^*(\Theta_{Z_2})$ and $\Theta_{Z_1} \to f^*(\Theta_{Z_2})$ are $f^*(\phi)$ and df, respectively.

Lemma 0.2.2. *The $\mathcal{O}$-module $\mathfrak{a}'$ on Z_1 has a canonical structure of Lie algebroid. Moreover, the functor f^* maps naturally left $\mathfrak{a}$-modules to left $\mathfrak{a}'$-modules.*

We shall call $\mathfrak{a}'$ the pull-back of $\mathfrak{a}$ with respect to f.

There are several other notions and constructions connected to Lie algebroids that will be used in this paper (such as Picard Lie algebroids and their connection with rings of twisted differential operators; Lie algebroids in the equivariant setting, in particular, Harish-Chandra Lie algebroids). They all are discussed in a clear and self-contained way in the first two sections of [**1**].

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CHAPTER 1
Chiral Algebras

In this section we will be working over a fixed smooth curve X (in Section 1.3 we will assume, in addition, that X is complete). The symbols $\mathcal{O}$ (resp., Ω, Θ) with an omitted subscript will mean the corresponding objects for X. However, all the results of this section are valid in the case when X is in fact a family of curves of the above type over an arbitrary base; in particular, X can be taken to be the universal curve $\mathfrak{X}$ over the moduli stack $\mathfrak{M}$ of smooth complete curves.

§1.1. Definition of chiral algebras

§1.1.1.

Let ${}^l\mathcal{A}$ be a (left) D-module on X, which we shall think of as consisting of "fields" of our CFT. Consider the corresponding right D-module $\mathcal{A} = {}^l\mathcal{A} \otimes \Omega$; its sections should be viewed as fields with values in 1-forms, i.e. as currents.

A chiral algebra structure on $\mathcal{A}$ is a D-module map (called "chiral bracket"):

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\{,\}} \Delta_!(\mathcal{A}),$$

that satisfies the following two conditions:

- Antisymmetry:

  If $f(x,y) \cdot a \boxtimes b$ is a section of $j_*j^*(\mathcal{A} \boxtimes \mathcal{A})$, then

  $$\{f(x,y) \cdot a \boxtimes b\} = -\sigma_{1,2}(\{f(y,x) \cdot b \boxtimes a\}),$$

  where $\sigma_{1,2}$ is the transposition acting on $\Delta_!(\mathcal{A})$.
- Jacobi identity:

  If $a \boxtimes b \boxtimes c \cdot f(x,y,z)$ is a section of the restriction of $\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}$ to the complement of the divisor of diagonals in $X \times X \times X$, then the element

$$\{\{f(x,y,z)\cdot a\boxtimes b\}\boxtimes c\}+\sigma_{1,2,3}(\{\{f(z,x,y)\cdot b\boxtimes c\}\boxtimes a\})+\sigma_{1,2,3}(\{\{f(y,z,x)\cdot c\boxtimes a\}\boxtimes b\})$$

  of $\Delta_{x=y=z!}(\mathcal{A})$ vanishes. (Here $\sigma_{1,2,3}$ denotes the lift of the cyclic automorphism of $X \times X \times X$: $(x,y,z) \to (y,z,x)$ to $\Delta_{x=y=z!}(\mathcal{A})$.)

A basic example of a chiral algebra is $\mathcal{A} = \Omega$. The bracket operation is given by the canonical map $\mathrm{can}_\Omega : j_*j^*(\Omega \boxtimes \Omega) \to \Delta_!(\Omega)$ (see below).

In most examples we shall deal with chiral algebras that possess a unit: a unit in a chiral algebra $\mathcal{A}$ is a map $\mathbf{unit} : \Omega \to \mathcal{A}$ such that the square below becomes

commutative.

$$\begin{array}{ccc} j_*j^*(\Omega \boxtimes \mathcal{A}) & \xrightarrow{\mathbf{unit}\boxtimes \mathrm{id}} & j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \\ {\scriptstyle \mathrm{can}_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \{,\}} \\ \Delta_!(\mathcal{A}) & \xrightarrow{\mathrm{id}} & \Delta_!(\mathcal{A}), \end{array}$$

where the left vertical arrow is the canonical map that comes from the short exact sequence

$$0 \to \Omega \boxtimes \mathcal{A} \to j_*j^*(\Omega \boxtimes \mathcal{A}) \to \Delta_!(\mathcal{A}) \to 0.$$

Along with ordinary chiral algebras one can consider the corresponding super-objects. In this case, the above antisymmetry and Jacobi identity conditions get transformed according to the sign rules of the super world.

§1.1.2.

Now let $\mathcal{A}$ be a chiral algebra and let $\mathcal{M}$ be a (right) D-module on X.

A chiral $\mathcal{A}$-module structure on $\mathcal{M}$ is a D-module map

$$\rho : j_*j^*(\mathcal{A} \boxtimes \mathcal{M}) \to \Delta_!(\mathcal{M})$$

such that if $f(x,y,z) \cdot a \boxtimes b \boxtimes m$ is a section of the restriction of $\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{M}$ to the complement of the divisor of diagonals in $X \times X \times X$, we have:

$$\rho(\{f(x,y,z) \cdot a \boxtimes b\}, m) = \rho(a, \rho(f(x,y,z) \cdot b, m)) - \sigma_{1,2}(\rho(b, \rho(f(y,x,z) \cdot a, m)),$$

as sections of $\Delta_{x=y=z!}(\mathcal{M})$.

When $\mathcal{A}$ has a unit we require moreover that the induced map $j_*j^*(\Omega \boxtimes \mathcal{M}) \to \Delta_!(\mathcal{M})$ coincide with the map $\mathrm{can}_{\mathcal{M}}$.

It is easy to check that the chiral bracket makes $\mathcal{A}$ into a chiral module over itself.

§1.1.3.

Let us now explain the concept of the "mode expansion" of a field.

Let x be a point of X and let $\mathcal{M}$ be a chiral $\mathcal{A}$-module supported at x; we shall denote by M the underlying vector space (i.e. $M \simeq h(\mathcal{M}) \simeq DR^0(\mathcal{M})$).

A typical example of such an $\mathcal{A}$-module is $\mathcal{M} := i_{x!}i_x^!(\mathcal{A})[1]$, called the vacuum module at x, in this case $A_x \simeq i_x^!(\mathcal{A})[1]$.

Let $\mathfrak{z}$ be a coordinate near x, i.e. $\mathfrak{z}$ is a regular function on some U with $\mathfrak{z}(x) = 0$, $d\mathfrak{z}(x) \neq 0$. Then any section $a \in \Gamma(U, \mathcal{A})$ gives rise to a collection a_n, $n \in \mathbb{Z}$, of elements in $\mathrm{End}(M)$:

By passing to the De Rham cohomology, the structure map ρ gives rise to a map $DR^0(U \setminus x, \mathcal{A}) \otimes M \to M$ and we take a_n to be the endomorphism of M corresponding to the image of $a \cdot \mathfrak{z}^n$ under the projection

$$\Gamma(U \setminus x, \mathcal{A}) \to DR^0(U \setminus x, \mathcal{A}).$$

Remark 1.1.1. We shall see shortly that for any $(x \in U)$ as above, $DR^0(U \setminus x, \mathcal{A})$ is naturally a Lie algebra. The vector space A_x together with the action of $DR^0(U \setminus x, \mathcal{A})$ on it is a prototype of the Hilbert space of our CFT.

Lemma 1.1.1. *For any $m' \in M$ and a as above we have*

$$a_n \cdot m' = 0 \text{ for } n >> 0.$$

Now let a and b be two sections of $\mathcal{A}$ over U; we can produce a third section $c \in \Gamma(U, \mathcal{A})$ by setting

$$c = (\mathrm{id} \boxtimes h)(\{\frac{1}{\mathfrak{z}_1 - \mathfrak{z}_2} \cdot a \boxtimes b\}) \in (h \boxtimes \mathrm{id})(\Delta_!(\mathcal{A})) \simeq \mathcal{A},$$

where $\mathfrak{z}_1 = \mathfrak{z} \boxtimes 1$ and $\mathfrak{z}_2 = 1 \boxtimes \mathfrak{z}$ are the corresponding functions on $X \times X$.

Proposition 1.1.1. *Modes of c can be expressed through the modes of a and b by the "normal ordering" formula:*

$$c_k =: a \cdot b :_k = \sum_{n \geq 0} a_{k-1-n} \cdot b_n + \sum_{n \geq 0} b_{k-1-n} \cdot a_n.$$

Note that both sides of the above formula depend on the choice of the coordinate $\mathfrak{z}$.

Proof. Choose a section $m \in \mathcal{M}$ which goes over to $m' \in M$ under the identification $DR^0(X, \mathcal{M}) \simeq M$.

The element $c_k \cdot m'$ is the image of

$$\rho(\{\frac{z_1^k \boxtimes 1}{\mathfrak{z}_1 - \mathfrak{z}_2} \cdot a \boxtimes b\}, m) \in \Delta_!(\mathcal{M})$$

under the identification $DR^0(X \times X \times X, \Delta_{1,2,3!}(\mathcal{M})) \simeq DR^0(X, \mathcal{M}) \simeq M$.

By definition, the above expression can be rewritten as

$$\rho(\mathfrak{z}_1^{k-1} \cdot a, \rho(\frac{1}{1 - \mathfrak{z}_2/\mathfrak{z}_1} \cdot b, m)) + \rho(\mathfrak{z}_1^{k-1} \cdot b, \rho(\frac{1}{1 - \mathfrak{z}_2/\mathfrak{z}_1} \cdot a, m)).$$

(In this formula we have omitted $\sigma_{1,2}$ since it does not affect the image of our element in De Rham cohomology.)

Now, Lemma 1.1.1 implies that we can expand $\frac{1}{1-\mathfrak{z}_2/\mathfrak{z}_1}$ into a power series $1 + \mathfrak{z}_2/\mathfrak{z}_1 + \mathfrak{z}_2^2/\mathfrak{z}_1^2 + \ldots$ and our assertion follows.

□

§1.2. Lie-* algebras and construction of chiral algebras

We shall now make a brief detour and discuss the auxiliary notion of a Lie-* algebra. These are objects that do not appear naturally in QFT; however, they are useful (and even necessary) for a construction of any non-trivial chiral algebra.

§1.2.1.

A Lie-* algebra on X is a (right) D-module $\mathcal{B}$ with a map (called "Lie-* bracket"):

$$\mathcal{B} \boxtimes \mathcal{B} \xrightarrow{\{,\}} \Delta_!(\mathcal{B}),$$

which is antisymmetric and satisfies the Jacobi identity in the sense similar to what we had in the definition of chiral algebras.

If $\mathcal{B}$ is a Lie-* algebra, it follows from the definition that $h(\mathcal{B})$ is a sheaf (in the étale topology) of ordinary Lie algebras; moreover it acts on $\mathcal{B}$ by endomorphisms of the D-module structure that are derivations of the Lie-* structure.

In particular, for an affine subset $U \subset X$, $DR^0(U, \mathcal{B})$ is a Lie algebra.

§1.2.2.

As for modules over a Lie-* algebra, there are two types of these:

A Lie-* module over a Lie-* algebra $\mathcal{B}$ is a D-module $\mathcal{M}$ with a map

$$\rho : \mathcal{B} \boxtimes \mathcal{M} \to \Delta_!(\mathcal{M})$$

such that for a section $a \boxtimes b \boxtimes m$ of $\mathcal{B} \boxtimes \mathcal{B} \boxtimes \mathcal{M}$ the two sections

$$\rho(\{a \boxtimes b\}, m) \text{ and } \rho(a, \rho(b, m)) - \sigma_{1,2}(\rho(b, \rho(a, m)))$$

of $\Delta_{1,2,3!}(\mathcal{B})$ coincide.

A chiral module over a Lie-* algebra is again a D-module $\mathcal{M}$, but with an operation

$$\rho : j_* j^*(\mathcal{B} \boxtimes \mathcal{M}) \to \Delta_!(\mathcal{M})$$

such that every section

$$f(x, y, z) \cdot a \boxtimes b \boxtimes m \in \Gamma(X \times X \times X \setminus (\Delta_{x=z} \cup \Delta_{y=z}), \mathcal{B} \boxtimes \mathcal{B} \boxtimes \mathcal{M})$$

satisfies an identity similar to the one in the definition of chiral modules over a chiral algebra.

For example, if $\mathcal{M}$ is a Lie-* (resp., chiral) module over a Lie-* algebra $\mathcal{B}$ which is supported at $x \in U \subset X$, the Lie algebra $DR^0(U, \mathcal{B})$ (resp., $DR^0(U \setminus x, \mathcal{B})$) acts on the underlying vector space $M = DR^0(U, \mathcal{M})$.

We have an obvious forgetful functor from the category of chiral modules over a Lie-* algebra $\mathcal{B}$ to the category of Lie-* modules.

In addition, there is a forgetful functor from the category of chiral algebras to the category of Lie-* algebras. If $\mathcal{M}$ is a chiral module over a chiral algebra $\mathcal{A}$ it is automatically a chiral module over the corresponding Lie-* algebra.

§1.2.3.

Let us now consider a few examples of Lie-* algebras.

Example 1

Let $\mathfrak{g}$ be a Lie algebra with an ad-invariant quadratic form Q on it. Consider the D-module $\mathcal{B}(\mathfrak{g}, Q) := \mathfrak{g} \otimes D_X \oplus \Omega$. We define a Lie-* algebra structure on $\mathcal{B}(\mathfrak{g}, Q)$ in the following way:

The Lie-* bracket $\{,\}$ factors through $\mathcal{B}(\mathfrak{g}, Q) \boxtimes \mathcal{B}(\mathfrak{g}, Q) \to \mathfrak{g} \otimes D_X \boxtimes \mathfrak{g} \otimes D_X$ and

$$\{\xi_1 \otimes 1 \boxtimes \xi_2 \otimes 1\} = [\xi_1, \xi_2] \otimes \mathbf{1} \oplus Q(\xi_1, \xi_2) \otimes \mathbf{1}',$$

where $\mathbf{1}$ is the unit section of $D_X \subset \Delta_!(D_X)$ and $\mathbf{1}'$ is the canonical antisymmetric section of $\Omega \boxtimes \Omega(2 \cdot \Delta)/\Omega \boxtimes \Omega$.

If $\mathrm{Spec}(\hat{\mathcal{O}}_x)$ (resp., $\mathrm{Spec}(\hat{\mathcal{K}}_x)$) is the formal disc (resp., formal punctured disc) around $x \in X$, the Lie algebras $DR^0(\mathrm{Spec}(\hat{\mathcal{O}}_x), \mathcal{B}(\mathfrak{g}, Q))$ and $DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{B}(\mathfrak{g}, Q))$ are identified with $\mathfrak{g} \otimes \hat{\mathcal{O}}_x$ and $\mathfrak{g} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}$. For instance, when $\mathfrak{g}$ is semisimple, $DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{B}(\mathfrak{g}, Q))$ is the corresponding Kac-Moody algebra.

Example 2

Let $\mathcal{T}$ denote the D-module $\Theta \otimes D_X$. The action of Θ on itself by Lie derivations is a bi-differential operator $\Theta \underset{\mathbb{C}}{\otimes} \Theta \to \Theta$; hence, according to Section 0.2, it gives rise to a D-module map

$$\mathcal{T} \boxtimes \mathcal{T} \simeq \Theta \otimes D_X \boxtimes \Theta \otimes D_X \to \Theta \underset{\mathcal{O}_{X\times X}}{\otimes} D_{X\times X} \simeq \Delta_!(\mathcal{T}),$$

which we take to be minus[1] the structure map for the Lie-* algebra $\mathcal{T}$.

It is easy to check that the resulting map satisfies the axioms of a Lie-* bracket. The Lie algebras $DR^0(\mathrm{Spec}(\hat{\mathcal{O}}_x), \mathcal{T})$ and $DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{T})$ identify with $\mathrm{Vir}^+ := \Theta \underset{\mathcal{O}}{\otimes} \hat{\mathcal{O}}_x$ and $\mathrm{Vir} := \Theta \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x$, respectively (the bracket being minus the usual one).

Example 3

Let Θ' be an extension $0 \to \Omega \to \Theta' \to \Theta \to 0$. Assume that the sheaf Θ of vector fields on X acts on Θ' "by Lie derivations"[2] in a way compatible with the above filtration and with its action on Ω and on Θ. We shall endow the D-module

$$\mathcal{T}' := \Theta' \otimes D_X / \ker(\Omega \otimes D_X \to \Omega)$$

with a Lie-* algebra structure. As a D-module $\mathcal{T}'$ is an extension

$$0 \to \Omega \to \mathcal{T}' \to \mathcal{T} \to 0$$

and the Lie-* algebra structure on it will be such that Ω is central (i.e. is annihilated by the Lie-* bracket) and $\mathcal{T}'$ will be a central extension of the Lie-* algebra $\mathcal{T}$ considered in Ex.2.

According to Section 0.2, the action of Θ on Θ' can be viewed as a map

$$\Theta \boxtimes \Theta' \to \Delta_*(\Theta') \underset{\mathcal{O}_{X\times X}}{\otimes} D_{X\times X} \simeq \Delta_!(\Theta' \otimes D_X).$$

Therefore, we obtain a D-module map

$$(\Theta \otimes D_X) \boxtimes (\Theta' \otimes D_X) \to \Delta_!(\Theta' \otimes D_X) \to \Delta_!(\mathcal{T}').$$

It is easy to see that $\Theta \otimes D_X \boxtimes \Omega \otimes D_X$ lies in the kernel of the above map, which means that it factors as $\mathcal{T} \boxtimes \mathcal{T} \to \Delta_!(\mathcal{T}')$. We define now the Lie-* bracket on $\mathcal{T}'$ as minus the composition:

$$\mathcal{T}' \boxtimes \mathcal{T}' \to \mathcal{T} \boxtimes \mathcal{T} \to \Delta_!(\mathcal{T}').$$

We shall see in Section 2.3.1 that the set of isomorphism classes of extensions Θ' as above is in bijection with $\mathbb{C}$. For any Θ' of this form, the Lie algebra $DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{T}')$ is the corresponding Virasoro extension

$$0 \to \mathbb{C} \to \mathrm{Vir}' \to \mathrm{Vir} \to 0.$$

---

[1] The minus sign is chosen in order to be consistent with the physics literature.

[2] We say that Θ acts on an $\mathcal{O}$-module $\mathcal{L}$ by Lie derivations if the action map $\Theta \underset{\mathbb{C}}{\otimes} \mathcal{L} \to \mathcal{L}$ is a bi-differential operator that satisfies

$$\xi(f \cdot l) = f \cdot \xi(l) + \mathrm{Lie}_\xi(f) \cdot l,$$

for $\xi \in \Theta$, $l \in \Theta'$ and $f \in \mathcal{O}$.

§1.2.4.

A fact of crucial importance for us is that the forgetful functor from chiral algebras to Lie-* algebras admits a left adjoint, i.e. that for every Lie-* algebra $\mathcal{B}$ there exists a (unital) chiral algebra $\mathcal{U}(\mathcal{B})$ (called the universal enveloping algebra of $\mathcal{B}$) such that

$$\mathrm{Hom}_{\mathrm{Lie}-*}(\mathcal{B},\mathcal{A}) \simeq \mathrm{Hom}_{\mathrm{ch}}(\mathcal{U}(\mathcal{B}),\mathcal{A}),$$

functorially in $\mathcal{A}$.

Let us sketch the construction of $\mathcal{U}(\mathcal{B})$ (we shall assume for simplicity that $\mathcal{B}$ is torsion-free as an $\mathcal{O}$-module).

The construction will be local and we can assume that X is affine. Consider the D-module direct images $B_1 := p_{1*}(\Omega \boxtimes \mathcal{B})$ and $B_2 := p_{1*}(j_*j^*(\Omega \boxtimes \mathcal{B}))$ on X (here p_1 is the projection $(x,y) \in X \times X \longrightarrow x \in X$). There is an obvious map $B_1 \to B_2$ and from the fact that $\mathcal{B}$ is a Lie-* algebra we infer that both B_1 and B_2 are Lie algebras in the category of D-modules.[3]

We define $\mathcal{U}(\mathcal{B})$ to be the "vacuum representation", i.e.

$$\mathcal{U}(\mathcal{B}) := U(B_2) \underset{U(B_1)}{\otimes} \mathbb{C},$$

where "U" is the universal enveloping algebra in the ordinary sense. By definition, for $x \in X$, the fiber of $\mathcal{U}(\mathcal{B})$ at x is identified with $U(DR^0(X \setminus x, \mathcal{B})) \underset{U(DR^0(X,\mathcal{B}))}{\otimes} \mathbb{C}$.

Since X is affine, the cokernel

$$\mathrm{coker}(DR^0(X,\mathcal{B}) \to DR^0(X \setminus x, \mathcal{B}))$$

is identified with

$$\mathrm{coker}(DR^0(\mathrm{Spec}(\hat{\mathcal{O}}_x),\mathcal{B}) \to DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x),\mathcal{B})) \simeq B_x := i_x^!(\mathcal{B})[1].$$

This description implies that, first of all, $\mathcal{U}(\mathcal{B})$ constructed in the above way will not change if we shrink X to a smaller open affine subset, and secondly, that $\mathcal{U}(\mathcal{B})$ is naturally a chiral module over the Lie-* algebra $\mathcal{B}$.

Note, that by the construction, $\mathcal{U}(\mathcal{B})$ carries a filtration $\mathcal{U}(\mathcal{B}) \simeq \underset{n}{\cup}\mathcal{U}(\mathcal{B})_n$, $n \geq 0$ induced from the filtration on $U(B_2)$. We have $\mathcal{U}(\mathcal{B})_0 \simeq \Omega$, $\mathcal{U}(\mathcal{B})_1 \simeq \mathcal{U}(\mathcal{B})_0 \oplus \mathcal{B}$.

Lemma 1.2.1. *Consider the chiral action of $j_*j^*(\mathcal{B} \boxtimes \mathcal{U}(\mathcal{B})) \to \Delta_!(\mathcal{U}(\mathcal{B}))$. We have:*

(a) $\mathrm{Im}(\mathcal{B} \boxtimes \mathcal{U}(\mathcal{B})_n) = \Delta_!(\mathcal{U}(\mathcal{B})_n)$.

(b) $\mathrm{Im}(j_*j^*(\mathcal{B} \boxtimes \mathcal{U}(\mathcal{B})_n)) = \Delta_!(\mathcal{U}(\mathcal{B})_{n+1})$.

(c) $\mathcal{U}(\mathcal{B})_n/\mathcal{U}(\mathcal{B})_{n-1} \simeq \mathrm{Sym}^n(\mathcal{B})$.

§1.2.5.

We shall now endow $\mathcal{U}(\mathcal{B})$ with a chiral algebra structure. If fact, we will show that whenever $\mathcal{M}$ is a chiral $\mathcal{B}$-module, there exists a canonical map

$$j_*j^*(\mathcal{U}(\mathcal{B}) \boxtimes \mathcal{M}) \to \Delta_!(\mathcal{M}).$$

---

[3]The category of right D-modules is a tensor category under the operation $\overset{!}{\otimes}$, cf. Section 0.2.

Consider the D-modules[4]

$$j_{y\neq z*}j^*_{y\neq z}(\Omega\boxtimes\Omega\boxtimes\mathcal{B}) \text{ and } j_{x\neq z\neq y*}j^*_{x\neq z\neq y}(\Omega\boxtimes\Omega\boxtimes\mathcal{B})$$

on $X\times X\times X$ and let B'_1 and B'_2 denote their direct (D-module) images with respect to the projection $(x,y,z)\to(x,y)$.

We have an embedding $B'_1\to B'_2$ and the Lie-* algebra structure on $\mathcal{B}$ makes B'_1 and B'_2 into Lie algebras in the category of D-modules on $X\times X$. For any $(x,y)\in X\times X$, the fiber of B'_1 (resp., of B'_2) at this point is identified with $DR^0(X\setminus y,\mathcal{B})$ (resp., with $DR^0(X\setminus\{x,y\},\mathcal{B})$). In particular, for a chiral $\mathcal{B}$-module $\mathcal{M}$ and for (x,y) as above, the action of $DR^0(X\setminus y,\mathcal{B})$ on $M_y:=i^!_y(\mathcal{M})[1]$ defines on $\Omega\boxtimes\mathcal{M}$ a structure of module over B'_1.

Consider now the induced module $\mathcal{M}':=U(B'_2)\underset{U(B'_1)}{\otimes}(\Omega\boxtimes\mathcal{M})$. It is clear that $j^*(\mathcal{M}')\simeq j^*(\mathcal{U}(\mathcal{B})\boxtimes\mathcal{M})$ and that $\Delta^!(\mathcal{M}')[1]\simeq\mathcal{M}$. The canonical map

$$j_*j^*(\mathcal{M}')\to\Delta_!\Delta^!(\mathcal{M}')[1]$$

yields, therefore, a map $j_*j^*(\mathcal{U}(\mathcal{B})\boxtimes\mathcal{M})\to\Delta_!(\mathcal{M})$.

It is easy to show now that if we set $\mathcal{M}$ to be $\mathcal{U}(\mathcal{B})$, then the resulting operation on $\mathcal{U}(\mathcal{B})$ will satisfy the axioms of a chiral bracket. Moreover, the above construction shows that any $\mathcal{M}$ which is chiral module over $\mathcal{B}$ acquires a structure of chiral module over $\mathcal{U}(\mathcal{B})$.

§1.3. Conformal blocks, correlation functions

From now on we shall assume that $\mathcal{A}$ is a chiral algebra possessing a unit.

§1.3.1.

We have a surjection of D-modules $j_*j^*(\mathcal{A}\boxtimes\mathcal{A})\to\Delta_!(\mathcal{A})$; let $\mathcal{A}^{(2)}$ denote its kernel.

For a complete curve X the space of conformal blocks $H_\nabla(X,\mathcal{A})$ of $\mathcal{A}$ is by definition the vector space $DR^2(X\times X,\mathcal{A}^{(2)})$. From the long exact cohomology sequence we obtain that

$$H_\nabla(X,\mathcal{A})\simeq\operatorname{coker}(DR^1(X\times X\setminus\Delta(X),j^*(\mathcal{A}\boxtimes\mathcal{A}))\to DR^1(X,\mathcal{A})).$$

Let x be a point of X and recall that A_x is a $DR^0(X\setminus x,\mathcal{A})$-module.

Proposition 1.3.1. *The space $H_\nabla(X,\mathcal{A})$ is identified naturally with the space of coinvariants*

$$\operatorname{coker}(DR^0(X\setminus x,\mathcal{A})\otimes A_x\to A_x).$$

Proof. We have a commutative square:

$$\begin{array}{ccc} DR^0(X\setminus x,\mathcal{A})\otimes A_x & \longrightarrow & A_x \\ \downarrow & & \downarrow \\ DR^1(X\times X,j_*j^*(\mathcal{A}\boxtimes\mathcal{A})) & \longrightarrow & DR^1(X,\mathcal{A}) \end{array}$$

such that the vertical arrows are surjective. To prove the statement it remains to show that the kernel of the map $A_x\to DR^1(X,\mathcal{A})$ contains the image of $DR^0(X\setminus x,\mathcal{A})\otimes A_x$.

---

[4]Here $j_{y\neq z}$ and $j_{x\neq z\neq y}$ denote the open embeddings of the complements to the divisors $\Delta_{y=z}$ and $\Delta_{x=z}\cup\Delta_{y=z}$ in $X\times X\times X$ respectively.

The unit in $\mathcal{A}$ provides a canonical element $\mathbf{unit}_x \in \mathcal{A}_x$ and the map $DR^0(X \setminus x, \mathcal{A}) \simeq DR^0(X \setminus x, \mathcal{A}) \otimes \mathbf{unit}_x \to A_x$ coincides with the canonical map $DR^0(X \setminus x, \mathcal{A}) \to A_x$.

Since the image of the latter is precisely the kernel of the map $A_x \to DR^1(X, \mathcal{A})$ our assertion follows.

□

The following observation is useful for actual computations of the space of conformal blocks:

Proposition 1.3.2. *Let $\mathcal{A}$ be a chiral universal enveloping algebra of a Lie-* algebra $\mathcal{B}$. Then the natural map*

$$A_x/(DR^0(X \setminus x, \mathcal{B}) \otimes A_x) \to A_x/(DR^0(X \setminus x, \mathcal{A}) \otimes A_x) \simeq H_\nabla(X, \mathcal{A})$$

is an isomorphism.

Proof. We will prove a slightly more general assertion: we can replace A_x in the formulation of the proposition by the underlying space M of any chiral $\mathcal{A}$-module $\mathcal{M}$ supported at x.

Let $\mathcal{A}_n$ be the filtration on $\mathcal{A}$ as in Section 1.2.4. We must show that the surjection

$$M/(DR^0(X \setminus x, \mathcal{B}) \otimes M) \twoheadrightarrow M/(DR^0(X \setminus x, \mathcal{A}_n) \otimes M)$$

is an isomorphism for any $n \geq 1$. We shall argue by induction.

For $a \in \Gamma(X \setminus x, \mathcal{A}_{n+1})$ and $m \in M$ consider the element $\rho(h(a), m) \in M$. By the construction, it belongs to $\mathrm{Im}(DR^0(X \setminus x, \mathcal{A}_{n+1}) \otimes M)$ and it would suffice to show that it also belongs to $\mathrm{Im}(DR^0(X \setminus x, \mathcal{A}_n) \otimes M)$.

Using the Lemma 1.2.1 and the fact that $X \setminus x$ is affine, for $a \in \Gamma(X \setminus x, \mathcal{A}_{n+1})$ as above, we can find a section $b \boxtimes a' \cdot f(x,y) \in \Gamma(X \setminus x \times X \setminus x, j_* j^*(\mathcal{B} \boxtimes \mathcal{A}_n))$ such that

$$(h \boxtimes \mathrm{id})(\{b \boxtimes a' \cdot f(x,y)\}) = a.$$

Using the Jacobi identity, the element $\rho(h(a), m) \in M$ is the image under the projection

$$h \boxtimes h \boxtimes h : \Delta_{1,2,3!}(\mathcal{M}) \to M$$

of the element

$$\rho(a', \rho(b, m \cdot f(x,y))) - \rho(b, \rho(a', m \cdot f(y,x))).$$

Now, the first term obviously belongs to $\mathrm{Im}(DR^0(X \setminus x, \mathcal{A}_n) \otimes M)$ and the second term even belongs to

$$\mathrm{Im}(DR^0(X \setminus x, \mathcal{B}) \otimes M) \subset \mathrm{Im}(DR^0(X \setminus x, \mathcal{A}_n) \otimes M).$$

□

§1.3.2.

Now let $x_1, \ldots, x_k$ be a non-empty collection of distinct points of X. Each A_{x_i} is a module over the Lie algebra $DR^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A})$; hence, so is the tensor product $A_{x_1} \otimes \ldots \otimes A_{x_k}$.

Proposition 1.3.3. *The space $H_\nabla(X, \mathcal{A})$ is canonically isomorphic to the space of coinvariants*

$$\mathrm{coker}(DR^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A}) \otimes (A_{x_1} \otimes \ldots \otimes A_{x_k}) \to A_{x_1} \otimes \ldots \otimes A_{x_k}).$$

Proof. When $k = 1$ our assertion coincides with that of Proposition 1.3.1 above. To treat the case $k \geq 2$ we shall proceed by induction, so we can assume that the assertion is true for $k - 1$.

We shall construct two mutually inverse maps ϕ and ψ between

$$(A_{x_1} \otimes \ldots \otimes A_{x_{k-1}})_{DR^0(X \setminus \{x_1, \ldots, x_{k-1}\}, \mathcal{A})} \text{ and } (A_{x_1} \otimes \ldots \otimes A_{x_k})_{DR^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A})} :$$

For $a_1 \otimes \ldots \otimes a_{k-1}$ in $A_{x_1} \otimes \ldots \otimes A_{x_{k-1}}$ we set $\phi(a_1 \otimes \ldots \otimes a_{k-1}) \in A_{x_1} \otimes \ldots \otimes A_{x_k}$ to be $a_1 \otimes \ldots \otimes a_{k-1} \otimes \mathbf{unit}_{x_k}$. It is clear that ϕ induces a well-defined map between the spaces of coinvariants.

Now let $a_1 \otimes \ldots \otimes a_{k-1} \otimes b'$ be an element of $A_{x_1} \otimes \ldots \otimes A_{x_{k-1}} \otimes A_{x_k}$. Since $X \setminus \{x_1, \ldots, x_{k-1}\}$ is affine, we can find an element $b \in DR^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A})$ that maps to b' under the canonical map $DR^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A}) \to A_{x_k}$.

We set $\psi(a_1 \otimes \ldots \otimes a_{k-1} \otimes b') \in A_{x_1} \otimes \ldots \otimes A_{x_{k-1}}$ to be $-b(a_1 \otimes \ldots \otimes a_{k-1})$. It is again easy to see that ψ is well-defined as a map between the coinvariants and that ϕ and ψ are mutually inverse.

It remains to show that the isomorphism

$$\mathrm{coker}(DR^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A}) \otimes (A_{x_1} \otimes \ldots \otimes A_{x_k}) \to A_{x_1} \otimes \ldots \otimes A_{x_k}) \simeq H_\nabla(X, \mathcal{A})$$

does not depend on the ordering of the points $x_1, \ldots, x_k$. Obviously, it is enough to do this when $k = 2$.

However, for two distinct points $x_1, x_2 \in X$, our map $A_{x_1} \otimes A_{x_2} \to H_\nabla(X, \mathcal{A})$ coincides with the canonical map

$$A_{x_1} \otimes A_{x_2} \simeq i^*_{x_1 \times x_2}(\mathcal{A} \boxtimes \mathcal{A})[2] \simeq i^*_{x_1 \times x_2}(\mathcal{A}^{(2)})[2] \to DR^2(X \times X, \mathcal{A}^{(2)}) \simeq H_\nabla(X, \mathcal{A})$$

and our assertion follows, since the fact that $\{,\}$ is antisymmetric implies that the isomorphism $DR^2(X \times X, \mathcal{A}^{(2)}) \simeq H_\nabla(X, \mathcal{A})$ is invariant with respect to the transposition.

□

§1.3.3.

By globalizing the argument of Proposition 1.3.3 we obtain that there exists a canonical map of D-modules on X^n:

$$\langle \ldots \rangle : j_* j^*(\mathcal{A} \boxtimes \ldots \boxtimes \mathcal{A}) \to j_* j^*(\Omega \boxtimes \ldots \boxtimes \Omega) \otimes H_\nabla(X, \mathcal{A})$$

with the following properties:

- For $x_1, \ldots, x_n$ distinct points on X the induced map $\langle \ldots \rangle : A_{x_1} \otimes \ldots \otimes A_{x_n} \to H_\nabla(X, \mathcal{A})$ coincides with the composition

  $$A_{x_1} \otimes \ldots \otimes A_{x_1} \to (A_{x_1} \otimes \ldots \otimes A_{x_n})_{DR^0(X \setminus \{x_1, \ldots, x_n\}, \mathcal{A})} \simeq H_\nabla(X, \mathcal{A}).$$

  In particular, the map $\langle \ldots \rangle$ is equivariant with respect to the action of the symmetric group S^n on $j_* j^*(\mathcal{A}^{\boxtimes n})$ and on $j_* j^*(\Omega^{\boxtimes n})$.

- The square

$$\begin{array}{ccc} j_*j^*(\mathcal{A}^{\boxtimes n-1}\boxtimes\Omega) & \xrightarrow{\langle\dots\rangle\boxtimes\mathrm{id}} & j_*j^*(\Omega^{\boxtimes n})\otimes H_\nabla(X,\mathcal{A}) \\ {\scriptstyle \mathrm{id}^{\boxtimes n-1}\boxtimes\mathbf{unit}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}^{\boxtimes n}} \\ j_*j^*(\mathcal{A}^{\boxtimes n}) & \xrightarrow{\langle\dots\rangle} & j_*j^*(\Omega^{\boxtimes n})\otimes H_\nabla(X,\mathcal{A}) \end{array}$$

  is commutative.
- The square

$$\begin{array}{ccc} j_*j^*(\mathcal{A}^{\boxtimes n}) & \xrightarrow{\langle\dots\rangle} & j_*j^*(\Omega^{\boxtimes n})\otimes H_\nabla(X,\mathcal{A}) \\ {\scriptstyle \{,\}\boxtimes\mathrm{id}^{\boxtimes n-1}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}_\Omega\boxtimes\mathrm{id}^{\boxtimes n-1}} \\ \Delta_{x_1=x_2!}(j_*j^*(\mathcal{A}^{\boxtimes n-1})) & \xrightarrow{\langle\dots\rangle} & \Delta_{x_1=x_2!}(j_*j^*(\Omega^{\boxtimes n-1}))\otimes H_\nabla(X,\mathcal{A}). \end{array}$$

  commutes as well.

(The first two of the above properties are evident from the definitions. The third one can be easily deduced from the second one using the fact that the upper horizontal arrow in the first commutative diagram is surjective).

By passing to the corresponding left D-modules, we obtain a canonical S^n-invariant map

$$\langle\dots\rangle : j_*j^*({}^l\mathcal{A}\boxtimes\dots\boxtimes{}^l\mathcal{A})\to j_*j^*(\mathcal{O}\boxtimes\dots\boxtimes\mathcal{O})\otimes H_\nabla(X,\mathcal{A}).$$

Remark 1.3.1. For a functional χ on $H_\nabla(X,\mathcal{A})$ and a section $a_1\boxtimes\dots\boxtimes a_n$ of $j_*j^*({}^l\mathcal{A}^{\boxtimes n})$ we can produce, therefore, a function $\langle a_1,\dots,a_n\rangle_\chi$ on X^n with poles on the diagonal divisor. This function should be thought of as an analog of the n-point correlation function of the fields $a_1,\dots,a_n$.

CHAPTER 2
CFT Data (Algebraic Version)

Recall that $\mathfrak{X}$ denotes the universal curve over the moduli stack $\mathfrak{M}$ of smooth complete curves. In this section the default meaning of $\mathcal{O}$ (resp., Ω, Θ, D) is the structure sheaf (resp., the sheaf of relative 1-forms, the sheaf of vertical vector fields, vertical differential operators) on $\mathfrak{X}$; D-modules (unless specified otherwise) are right modules over the sheaf of rings D.

§2.1. Local $\mathcal{O}$-modules on $\mathfrak{X}$

On the intuitive level, we want to call a sheaf $\mathcal{L}$ on $\mathfrak{X}$ local if the following holds: whenever X and X' are two curves such that their (analytic) open subsets $U \subset X$ and $U' \subset X'$ are identified, the sheaves $(\mathcal{L}|X)|U$ and $(\mathcal{L}|X')|U'$ are identified as well. For instance, if we are dealing with a 2-dimensional CFT which is defined for every Riemann surface, its fields form naturally a local sheaf over $\mathfrak{X}$.

§2.1.1.

Let $\mathcal{L}$ be an $\mathcal{O}$-module on $\mathfrak{X}$, i.e. for a scheme S with a map to $\mathfrak{M}$ we have an $\mathcal{O}$-module $\mathcal{L}^S$ on the corresponding family of curves X^S. We say that $\mathcal{L}$ is local if it possesses the following additional structure:

Let I be a local Artinian scheme and let $X^{S\times I}$ be a family of curves over $X \times I$ extending X^S. Let, in addition, $x_1^{S\times I}, \ldots, x_n^{S\times I}$ be sections $S \times I \to X^{S\times I}$ (we shall denote by $x_1^S, \ldots, x_n^S$ their restrictions to $S \hookrightarrow S \times I$). Assume now that $\phi^{S,I}$ is an isomorphism:

$$X^{S\times I} \setminus \{x_1^{S\times I}, \ldots, x_n^{S\times I}\} \simeq (X^S \setminus \{x_1^S, \ldots, x_n^S\}) \times I.$$

A local structure on $\mathcal{L}$ is lifting of $\phi^{S,I}$ to an isomorphism of sheaves

$$\beta^{S,I} : \phi^{S,I*}(\mathcal{L}^S|X^S \setminus \{x_1^S, \ldots, x_n^S\} \otimes \mathcal{O}(I)) \to \mathcal{L}^{S\times I}|X^{S\times I} \setminus \{x_1^{S\times I}, \ldots, x_n^{S\times I}\}.$$

These $\beta^{S,I}$'s for various $X^{S\times I}, x_1^{S\times I}, \ldots, x_n^{S\times I}, \phi^I$ must satisfy two conditions. The first condition is a compatibility in the obvious sense in the situation when a map $S \times I \to \mathfrak{M}$ factors as $S \times I \to S' \times I' \to \mathfrak{M}$. To formulate the second condition we need to introduce some notation.

Let X^S be again a curve over the base S and let $y_1^S, \ldots, y_n^S$ and $z_1^S, \ldots, z_k^S$ be two disjoint collections of sections $S \to X^S$. Let I and J be two local Artinian schemes and let $(X^{S\times I}, y_1^{S\times I}, \ldots, y_n^{S\times I}, \phi^I)$ and $(X^{S\times J}, z_1^{S\times J}, \ldots, z_k^{S\times J}, \phi^J)$

be two pieces of data as above. (In what follows, to simplify the notation, we shall abbreviate $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_m\}$ to y and z respectively.)

We have sections $\widetilde{y}^J : S \times J \to X^{S \times J}$ and $\widetilde{z}^I : S \times I \to X^{S \times I}$ and we can construct a curve $X^{S \times I \times J}$ over $S \times I \times J$ equipped with sections $y^{S \times I \times J}, z^{S \times I \times J}$ such that:

$$X^{S \times I \times J} \setminus y^{S \times I \times J} \overset{\phi^{S \times J, I}}{\simeq} (X^J \setminus \widetilde{y}^J) \times I \tag{2.1}$$

$$X^{S \times I \times J} \setminus z^{S \times I \times J} \overset{\phi^{S \times I, J}}{\simeq} (X^I \setminus \widetilde{z}^I) \times J \tag{2.2}$$

$$X^{S \times I \times J} \setminus \{y^{S \times I \times J}, z^{S \times I \times J}\} \overset{\phi^{S, I \times J}}{\simeq} (X^S \setminus \{y^S, z^S\}) \times I \times J. \tag{2.3}$$

Note that the composition:

$$(\phi^{S,J} \times \mathrm{id}_I) \circ \phi^{S \times J, I} : X^{S \times I \times J} \setminus \{y^{S \times I \times J}, z^{S \times I \times J}\} \to (X^S \setminus \{y^S, z^S\}) \times I \times J$$

coincides with $\phi^{S, I \times J}$.

The second condition on β reads as follows:

We need that the two isomorphisms between

$$\phi^{S, I \times J *}((\mathcal{L}^S) | X \setminus \{y^S, z^S\} \otimes \mathcal{O}(I) \otimes \mathcal{O}(J))$$

and

$$(\mathcal{L}^{S \times I \times J}) | X^{S \times I \times J} \setminus \{y^{S \times I \times J}, z^{S \times I \times J}\},$$

namely, $\beta^{S, I \times J}$ and $(\beta^{S,J} \times \mathrm{id}_I) \circ ((\phi^{S,J} \times \mathrm{id}_I)^*(\beta^{S \times J, I}))$, coincide.

The most basic example of a local $\mathcal{O}$-module on $\mathfrak{X}$ is the sheaf $\mathcal{O}$ itself. Local $\mathcal{O}$-modules on $\mathfrak{X}$ form an abelian category (morphisms in this category are, by definition, maps of $\mathcal{O}$-modules compatible with the $\beta^{S,I}$'s). It is easy to see that the sheaves Ω^i, D, etc. are all local $\mathcal{O}$-modules in a natural way.

In a similar way one defines local $\mathcal{O}$-modules on $\mathfrak{X}^n$.

An $\mathcal{O}$-module $\mathcal{L}$ on $\mathfrak{X}^n$ assigns, by definition, to every scheme S mapping to $\mathfrak{M}$ an $\mathcal{O}$-module $\mathcal{L}^S$ over

$$X^S \underset{S}{\times} \dots \underset{S}{\times} X^S,$$

where X^S is as before. A local structure on $\mathcal{L}$ attaches to each data $X^{S,I}, x_k^{S,I}, \phi^{S,I}$ as above an isomorphism

$$\beta^{S,I} : \phi^{S,I *}(\mathcal{L}^S | (X^S \setminus \{x_1^S, \dots, x_n^S\}) \underset{S}{\times} \dots \underset{S}{\times} (X^S \setminus \{x_1^S, \dots, x_n^S\}) \otimes \mathcal{O}(I)) \simeq$$
$$\mathcal{L}^{S \times I} | (X^{S \times I} \setminus \{x_1^{S \times I}, \dots, x_n^{S \times I}\}) \underset{S \times I}{\times} \dots \underset{S \times I}{\times} (X^{S \times I} \setminus \{x_1^{S \times I}, \dots, x_n^{S \times I}\}).$$

These isomorphisms should satisfy two conditions analogous to what we had in the case $n = 1$.

For example, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are local $\mathcal{O}$-modules on $\mathfrak{X}$, then $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a local $\mathcal{O}$-module on $\mathfrak{X}^2$. In addition, if $\mathcal{L}$ is a local $\mathcal{O}$-module on $\mathfrak{X}^2$, so is $j_* j^*(\mathcal{L})$, and $\Delta^*(\mathcal{L})$ is a local $\mathcal{O}$-module back on $\mathfrak{X}$.

In particular, we have the following assertion:

Lemma 2.1.1. *Let $\mathcal{B}$ be a local $\mathcal{O}$-module on $\mathfrak{X}$ with the following additional structure: $\mathcal{B}$ is a D-module and a Lie-* algebra such that:*

(a) *The map $\mathcal{B} \otimes D \to \mathcal{B}$ is a map of local $\mathcal{O}$-modules on $\mathfrak{X}$.*

(b) *The Lie-* bracket $\mathcal{B} \boxtimes \mathcal{B} \to \Delta_!(\mathcal{B})$ is a map of local $\mathcal{O}$-modules on $\mathfrak{X}^2$ (the RHS is a local $\mathcal{O}$-module on $\mathfrak{X}^2$ in a natural way).*

Then the chiral universal enveloping algebra $\mathcal{U}(\mathcal{B})$ is a local $\mathcal{O}$-module on $\mathfrak{X}$ and conditions (a) *and* (b) *hold for $\mathcal{U}(\mathcal{B})$ as well.*

§2.1.2.

We shall now describe a typical way of producing local $\mathcal{O}$-modules.

Consider the standard 1-dimensional local ring $\hat{\mathcal{O}} := \mathbb{C}[[t]]$ and the corresponding local field $\hat{\mathcal{K}} := \mathbb{C}((t))$. Let Aut_0^+ (resp., Aut^+, Aut) be the group scheme of automorphisms of $\hat{\mathcal{O}}$ that preserve the maximal ideal (resp., the group ind-scheme of all automorphisms of $\hat{\mathcal{O}}$, the group ind-scheme of all automorphisms of $\hat{\mathcal{K}}$). We shall denote by $\mathrm{Vir}_0^+ \subset \mathrm{Vir}^+ \subset \mathrm{Vir}$ the corresponding Lie algebras.

Consider the scheme $\hat{\mathfrak{X}}$ that classifies the following data:

$$(X \in \mathfrak{M},\, x \in X, \alpha : \hat{\mathcal{O}}_x \to \hat{\mathcal{O}}),$$

where α is an isomorphism that preserves the maximal ideal. The group Aut_0^+ acts on this scheme by changing α and the quotient is identified with $\mathfrak{X}$. Let q denote the projection of $\hat{\mathfrak{X}}$ to $\mathfrak{X}$.

If now V is a representation of the group Aut_0^+, the $\mathcal{O}_{\mathfrak{X}}$-module $V \otimes \mathcal{O}_{\hat{\mathfrak{X}}}$ is Aut_0^+-equivariant and hence descends to an $\mathcal{O}$-module $\mathrm{Ass}(V)$ over $\mathfrak{X}$. The next assertion is evident from the definitions:

Lemma 2.1.2. *For V as above, the $\mathcal{O}$-module $\mathrm{Ass}(V)$ has a natural structure of local $\mathcal{O}$-module.*

For instance, for $i \in \mathbb{Z}$ let L_i be the 1-dimensional Aut_0^+-module of Section 2.3.1. (Aut_0^+ acts on L_i via the i-th power of the standard character.) We have:

$$\mathrm{Ass}(L_i) \simeq \Omega^i.$$

§2.1.3.

Let $\mathcal{L}$ be a local $\mathcal{O}$-module on $\mathfrak{X}$. We shall now construct a map[1]

$$\nabla_{loc} : \mathcal{L} \to p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)).$$

Let $x \in X \in \mathfrak{M}$ be a point of $\mathfrak{X}$, let l be a local section of $\mathcal{L}$ on $\mathfrak{X}$ and let ξ be an element of $\hat{\mathcal{K}}_x/\hat{\mathcal{O}}_x \otimes \Theta$.

Constructing ∇_{loc} amounts to our being able to evaluate $(\mathrm{id} \boxtimes \mathrm{Res}_x)(\nabla_{loc}(l) \otimes (1 \boxtimes \xi))$ as a local section of $\mathcal{L}|X$ defined outside of x. (The triple (X, x, ξ) should be thought of as living over a base S, where S is a scheme mapping to $\mathfrak{M}$; we shall denote by $\mathcal{L}|X$ the $\mathcal{O}$-module $\mathcal{L}^S$ over X^S.)

---

[1]For two $\mathcal{O}$-modules $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathfrak{X}$, it will be sometimes more convenient to use the notation $\mathcal{L}_1 \boxtimes \mathcal{L}_2(\infty \cdot \Delta)$ for $j_* j^*(\mathcal{L}_1 \boxtimes \mathcal{L}_2)$.

However, ξ as above is the same as a deformation of X over the scheme $I = \mathrm{Spec}(\mathbb{C}[t]/t^2)$ which is trivialized outside of x. As $\mathcal{L}$ is local, we have a section $\beta^{I-1}(l)$ of $((\mathcal{L}|X)|X \setminus x) \otimes \mathbb{C}[t]/t^2$ and we set

$$(\mathrm{id} \times \mathrm{Res}_x)(\nabla_{loc}(l) \otimes (1 \boxtimes \xi)) := \partial_t(\beta^{I-1}(l)).$$

Since the isomorphism β^I above respects the $\mathcal{O}$-module structure, the map ∇_{loc} satisfies an analog of the Leibnitz rule: for a function f on $\mathfrak{X}$ and a local section l of $\mathcal{L}$ we have:

$$\nabla_{loc}(f \cdot l) = (f \boxtimes 1) \cdot \nabla_{loc}(l) + \nabla_{loc}(f) \otimes (l \boxtimes 1) \in p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)),$$

(here $\nabla_{loc}(f)$ is the corresponding operation for $\mathcal{L} = \mathcal{O}$).

Analogously, when $\mathcal{L}$ is a local $\mathcal{O}$-module on $\mathfrak{X}^n$ we have a map

$$\nabla_{loc} : \mathcal{L} \to p_{1,\ldots,n*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta_{x_{n+1} \neq x_k})),$$

that satisfies the Leibnitz rule in the same sense as above.

In particular, for a local $\mathcal{O}$-module $\mathcal{L}$ on $\mathfrak{X}$, the $\mathcal{O}$-module $\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)$ on $\mathfrak{X}^2$ is local as well and we can form a map

$$\nabla^2_{loc} : \mathcal{L} \xrightarrow{\nabla_{loc}} p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)) \xrightarrow{\nabla_{loc}} p_{1*}(\mathcal{L} \boxtimes \Omega^2 \boxtimes \Omega^2(\infty \cdot \Delta)).$$

Lemma 2.1.3. *Let $\sigma_{2,3}$ be the transposition of the last 2 coordinates acting on the sheaf $p_{1*}(\mathcal{L} \boxtimes \Omega^2 \boxtimes \Omega^2(\infty \cdot \Delta))$. We have:*

$$\sigma_{2,3} \circ \nabla^2_{loc} = \nabla^2_{loc},$$

i.e. the image of ∇^2_{loc} is symmetric in the last two variables.

The proof follows immediately from the second condition on the β's in the definition of local $\mathcal{O}$-modules.

Let us consider a few examples:

Note that the cotangent bundle of $\hat{\mathfrak{X}}$ is identified in an Aut_0^+-equivariant way with

$$q^* p_{1*}(\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)).$$

Therefore, the De Rham differential on $\hat{\mathfrak{X}}$ yields maps

$$d^0 : \mathcal{O} \to \Omega_{\mathfrak{X}} \simeq p_{1*}(\mathcal{O} \boxtimes \Omega^2(\Delta)) \hookrightarrow p_{1*}(\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)) \text{ and}$$

$$d^1 : p_{1*}(\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)) \to p_{1*}(\mathcal{O} \boxtimes \Omega^2 \boxtimes \Omega^2(\infty \cdot \Delta)).$$

We leave it to the reader to check that d^0 is the map ∇_{loc} for $\mathcal{L} = \mathcal{O}$ and that d^1 is the map ∇_{loc} for the local $\mathcal{O}$-module $\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)$ on $\mathfrak{X}^2$.

We claim, in fact, that for an $\mathcal{O}$-module $\mathcal{L}$ on $\mathfrak{X}$, a structure of local $\mathcal{O}$-module on it is equivalent to the data of ∇_{loc}:

Theorem 2.1.1. *The following three categories are equivalent:*

- *The category of local $\mathcal{O}$-modules on $\mathfrak{X}$.*
- *The category of weakly Aut_0^+-equivariant left D-modules on $\hat{\mathfrak{X}}$.*

- *The category of $\mathcal{O}$-modules $\mathcal{L}$ on $\mathfrak{X}$ endowed with a map $\nabla_{loc} : \mathcal{L} \to p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta))$ such that:*

  (a) *For any function f on $\mathfrak{X}$ and a local section l of $\mathcal{L}$*

$$\nabla_{loc}(f \cdot l) = (f \boxtimes 1) \cdot \nabla_{loc}(l) + \nabla_{loc}(f) \otimes (l \boxtimes 1) \in p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)).$$

  (b) *If we define (assuming* (a)*)*

$$\nabla_{loc} : \mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta) \to p_{1*}(\mathcal{L} \boxtimes \Omega^2 \boxtimes \Omega^2)(\infty \cdot \Delta)$$

  according to the Leibnitz rule applied to ∇_{loc} acting on $\mathcal{L}$ and on Ω^2, then the composition

$$\nabla_{loc}{}^2 := \nabla_{loc} \circ \nabla_{loc} : \mathcal{L} \to p_{1*}(\mathcal{L} \boxtimes \Omega^2 \boxtimes \Omega^2)(\infty \cdot \Delta)$$

  satisfies $\sigma_{2,3} \circ \nabla_{loc}{}^2 = \nabla_{loc}{}^2$.

  Morphisms in this category are $\mathcal{O}$-module maps $\mathcal{L}_1 \to \mathcal{L}_2$ compatible with the ∇_{loc}'s.

The proof of this theorem is quite straightforward and we shall omit it, especially since this statement will not be used in the sequel.

We shall conclude this subsection by showing that if $\mathcal{L}$ is a local $\mathcal{O}$-module on $\mathfrak{X}$, there is a natural action of Θ on it "by Lie derivations" (cf. Section 1.2.3). Consider the composition

$$\nabla'_{loc} : \mathcal{L} \xrightarrow{\nabla_{loc}} p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)) \to p_{1*}(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)/\mathcal{L} \boxtimes \Omega^2).$$

This map is $\mathcal{O}_{\mathfrak{M}}$-linear and it satisfies:

$$\nabla'_{loc}(f \cdot l) = (f \boxtimes 1) \cdot \nabla'_{loc}(l) + (l \boxtimes 1) \otimes df,$$

where $df \in \Omega$ is viewed as an element of $\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)/\mathcal{O} \boxtimes \Omega^2$ via

$$\Omega \simeq \mathcal{O} \boxtimes \Omega^2(\Delta)/\mathcal{O} \boxtimes \Omega^2 \subset \mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)/\mathcal{O} \boxtimes \Omega^2.$$

If l is a local section of $\mathcal{L}$ and ξ is a vertical vector field on $\mathfrak{X}$ (i.e. a local section of Θ), we set

$$\mathrm{Lie}_\xi(l) := (\mathrm{id} \boxtimes \mathrm{Res}_\Delta)(\nabla'_n(l) \otimes (1 \boxtimes \xi)) \in \mathcal{L}.$$

The properties of ∇_{loc} imply that the map $(l, \xi) \to \mathrm{Lie}_\xi(l)$ is a bi-differential operator and is $\mathcal{O}_{\mathfrak{M}}$-linear in both variables. In addition, we have:

$$\mathrm{Lie}_\xi(f \cdot l) = f \cdot \mathrm{Lie}_\xi(l) + \mathrm{Lie}_\xi(f) \cdot l.$$

§2.2. A formulation of CFT (central charge 0)

§2.2.1.

A CFT data (of central charge 0) for us will consist of

(1) A local $\mathcal{O}$-module $\mathcal{A}$ on $\mathfrak{X}$ endowed with a structure of a chiral algebra (in particular, $\mathcal{A}$ is a right D-module).

(2) A map of local $\mathcal{O}$-modules $T : \Theta \to \mathcal{A}$ (called the energy-momentum tensor).

These data must satisfy the following conditions:

(a) The chiral bracket on $\mathcal{A}$:

$$j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\{,\}} \Delta_!(\mathcal{A})$$

is a map of local $\mathcal{O}$-modules on $\mathfrak{X}^2$. Moreover, the unit map **unit** : $\Omega \to \mathcal{A}$ is also a map of local $\mathcal{O}$-modules.

(b) Let ξ be a vertical vector field on $\mathfrak{X}$ and consider the map $\mathcal{A} \to \mathcal{A}$ given by

$$l \to (\mathrm{id} \boxtimes h)(\{l, T(\xi)\}) - \mathrm{Lie}_\xi(l).$$

We need this map to coincide with the action of ξ on $\mathcal{A}$ coming from the right D-module structure on $\mathcal{A}$.

It follows from (a) and (b) that the map $\mathcal{A} \otimes D \to \mathcal{A}$ defining the right D-module structure on $\mathcal{A}$ is a map of local $\mathcal{O}$-modules.

Recall the Lie-* algebra $\mathcal{T} = \Theta \otimes D$ that we constructed in Section 1.2.3. The map T gives rise to a map $\mathcal{T} \to \mathcal{A}$ and conditions (a) and (b) above guarantee that this is a homomorphism of Lie-* algebras.

Remark 2.2.1. In the physics literature one often encounters the term "primary field of weight i". In our language, this is the same as a map of local $\mathcal{O}$-modules $\Omega^{1-i} \to \mathcal{A}$.

§2.2.2.

Let ${}^l\mathcal{A}$ be the corresponding left D-module ${}^l\mathcal{A} := \mathcal{A} \otimes \Theta$ and let ${}^lT$ denote the corresponding map $\Omega^{-2} \to {}^l\mathcal{A}$. We will show now that ${}^l\mathcal{A}$ acquires a (left) module structure over some canonical Lie algebroid on $\mathfrak{X}$ (cf. Section 0.2 for the notion of a Lie algebroid). This will later on enable us to produce a connection along $\mathfrak{M}$ on the space of conformal blocks of $\mathcal{A}$.

Recall the scheme $\hat{\mathfrak{X}}$ that we introduced in the previous subsection. It is known that the natural Aut_0^+-action on it extends to an action of the whole of Aut. Let us recall how this action is constructed.

For a point $(x \in X \in \mathfrak{M}, \alpha)$ of $\hat{\mathfrak{X}}$ and a point $g \in \mathrm{Aut}$ with values in a local Artinian scheme $\mathrm{Spec}(I)$, we must produce a $\mathrm{Spec}(I)$-valued point $(x_I \in X_I, \alpha_I)$ of $\hat{\mathfrak{X}}$.

We let X_I be unchanged as a topological space; moreover, we set $X_I \setminus x_I$ to be $(X \setminus x) \times \mathrm{Spec}(I)$. To fix the triple $(x_I \in X_I, \alpha_I)$ it remains to specify which of the meromorphic functions on $(X \setminus x) \times \mathrm{Spec}(I)$ are regular at $x_I \in X_I$.

The set of meromorphic functions on $X \times \mathrm{Spec}(I)$ embeds into $\hat{\mathcal{K}} \otimes I$ by means of α. We declare a meromorphic function on $(X \setminus x) \times \mathrm{Spec}(I)$ to be regular for X_I at x_I if its image in $\hat{\mathcal{K}} \otimes I$ is transformed by $g : \hat{\mathcal{K}} \otimes I \to \hat{\mathcal{K}} \otimes I$ into an element of $\hat{\mathcal{O}} \otimes I$.

In particular, the Lie algebra Vir acts on $\hat{\mathfrak{X}}$ by vector fields and we can form the Lie algebroid $\widetilde{\mathfrak{v}} = \mathrm{Vir} \otimes \mathcal{O}_{\hat{\mathfrak{X}}}$ (resp., $\widetilde{\mathfrak{v}}^+ = \mathrm{Vir}^+ \otimes \mathcal{O}_{\hat{\mathfrak{X}}}$, $\widetilde{\mathfrak{v}}_0^+ = \mathrm{Vir}_0^+ \otimes \mathcal{O}_{\hat{\mathfrak{X}}}$) on $\hat{\mathfrak{X}}$; all the three are Harish-Chandra Lie algebroids with respect to Aut_0^+. Let $\mathfrak{v}$, $\mathfrak{v}^+$ and $\mathfrak{v}_0^+$ denote the corresponding Lie algebroids on $\mathfrak{X}$, i.e. $\widetilde{\mathfrak{v}} \simeq q^*(\mathfrak{v})$ and similarly for $\mathfrak{v}^+$ and $\mathfrak{v}_0^+$; the Harish-Chandra property implies that $\mathfrak{v}_0^+$ is an ideal in the other two.

We have:

$$\mathfrak{v} \simeq p_{1*}(\mathcal{O} \hat{\boxtimes} \Theta(\infty \cdot \Delta)) \text{ and } \mathfrak{v}/\mathfrak{v}_0^+ \simeq p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta)/\mathcal{O} \boxtimes \Theta(-\Delta)),$$

where the notation $\hat{\boxtimes}$ means completion along the diagonal divisor.

Moreover $\mathfrak{v}^+/\mathfrak{v}_0^+ \hookrightarrow \mathfrak{v}/\mathfrak{v}_0^+$ is identified with

$$\Theta \simeq \mathcal{O} \boxtimes \Theta/\mathcal{O} \boxtimes \Theta(-\Delta) \hookrightarrow p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta)/\mathcal{O} \boxtimes \Theta(-\Delta)).$$

There are two maps $\mathfrak{v} \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$:

The first map $\tau_{loc} : \mathfrak{v} \otimes {}^l\mathcal{A} \to {}^l\mathcal{A}$ is defined as follows:
A typical local section of $\mathfrak{v} \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A}$ can be written as $(f \cdot 1 \boxtimes \xi) \otimes a$, for $f \in p_{1*}(\mathcal{O}\hat{\boxtimes}\mathcal{O}(\infty \cdot \Delta))$, $\xi \in \Theta$ and $a \in {}^l\mathcal{A}$. We set[2]

$$\tau_{loc}((f \cdot 1 \boxtimes \xi), a) = (\mathrm{id} \boxtimes \mathrm{Res}_\Delta)(f \cdot \nabla_{loc}(a) \otimes (1 \boxtimes \xi)).$$

To define the second map τ_{em} (the subscript "em" here stands for "energy-momentum"), let $(f \cdot 1 \boxtimes \xi) \otimes a \in \mathfrak{v} \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A}$ be as above and we set[3]

$$\tau_{em}((f \cdot 1 \boxtimes \xi), a) = (\mathrm{id} \boxtimes h)(\{f \cdot a \boxtimes T(\xi)\}) \in {}^l\mathcal{A}.$$

Let τ be the difference:

$$\tau = \tau_{loc} - \tau_{em} : \mathfrak{v} \otimes {}^l\mathcal{A} \to {}^l\mathcal{A}.$$

Lemma 2.2.1. *The map τ is a left action of the algebroid $\mathfrak{v}$ on ${}^l\mathcal{A}$. Moreover, τ annihilates the ideal $\mathfrak{v}_0^+ \subset \mathfrak{v}$ and the restriction of τ to $\Theta \simeq \mathfrak{v}^+/\mathfrak{v}_0^+$ coincides with the action of Θ on ${}^l\mathcal{A}$ coming from the structure of a left D-module on ${}^l\mathcal{A}$.*

The proof follows from conditions (a) and (b) in Section 2.2.1 and we leave the verification to the reader.

§2.2.3.

Let $\mathcal{A}$ be as in Section 2.2.1. For each $X \in \mathfrak{M}$ we can construct a vector space $H_\nabla(X, \mathcal{A})$ and globally over $\mathfrak{M}$ this construction yields an $\mathcal{O}_\mathfrak{M}$-module $H_\nabla(\mathfrak{X}, \mathcal{A})$.

According to the discussion in Section 1.3.3, we have a map of left D-modules on $\mathfrak{X}$:

$$\langle \dots \rangle : {}^l\mathcal{A} \twoheadrightarrow \pi^*(H_\nabla(\mathfrak{X}, \mathcal{A})).$$

Proposition 2.2.1. *There exists a (unique) left $D_\mathfrak{M}$-module structure on $H_\nabla(\mathfrak{X}, \mathcal{A})$ such that when $\pi^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$ is viewed as an $\mathfrak{v}/\mathfrak{v}_0^+$-module via $\mathfrak{v}/\mathfrak{v}_0^+ \to \Theta_\mathfrak{X}$, the map $\langle \dots \rangle$ is compatible with the $\mathfrak{v}/\mathfrak{v}_0^+$-module structure.*

Proof. Since the map $\langle \dots \rangle$ is surjective, there is at most one $\mathfrak{v}/\mathfrak{v}_0^+$-module structure on $\pi^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$ compatible with that on ${}^l\mathcal{A}$.

To see that it is correctly defined, we must show that if c is a local section of ${}^l\mathcal{A}$ of the form

$$c = (h \boxtimes \mathrm{id})(\{f_0 \cdot a \boxtimes b\}),$$

where $f_0 \cdot a \boxtimes b \in p_{2*}(\mathcal{A} \boxtimes {}^l\mathcal{A}(\infty \cdot \Delta))$ and if $f \cdot 1 \boxtimes \xi \in p_{1*}(\mathcal{O}\hat{\boxtimes}\Theta(\infty \cdot \Delta))$ is a local section of $\mathfrak{v}/\mathfrak{v}_0^+$, the section $c' := \tau(f \cdot 1 \boxtimes \xi, c)$ is a sum of sections of the form

$$(h \boxtimes \mathrm{id})(\{f_0' \cdot a' \boxtimes b'\}),$$

---

[2]Using Proposition 2.1.1 one can rephrase the definition of τ_{loc} as follows: since $q^*({}^l\mathcal{A})$ is a weakly Aut_0^+-equivariant D-module on $\hat{\mathfrak{X}}$, it is also a weakly equivariant module over the algebroid $\widetilde{\mathfrak{v}}$; this defines on ${}^l\mathcal{A}$ a structure of $\mathfrak{v}$-module, which is the same as a map τ_{loc}.

[3]Here we have denoted by $(\mathrm{id} \boxtimes h)(\{,\})$ the map ${}^l\mathcal{A}\hat{\boxtimes}\mathcal{A}(\infty \cdot \Delta) \to {}^l\mathcal{A}$ that comes from the chiral bracket on $\mathcal{A}$.

for $f_0' \cdot a' \boxtimes b'$ a local section of $p_{2*}(\mathcal{A} \boxtimes {}^l\mathcal{A}(\infty \cdot \Delta))$.

This, however, follows from the next lemma.

Lemma 2.2.2. *Let $f_0 \cdot a \boxtimes b$ and $f \cdot 1 \boxtimes \xi$ and c' be as above and let a'' be the section of $p_{2*}(\mathcal{A} \boxtimes \mathcal{O}(\infty \cdot \Delta))$ defined as*

$$a'' = (\mathrm{id} \boxtimes \mathrm{id} \boxtimes \mathrm{Res}_{x_2=x_3})(\nabla_{loc}(f_0 \cdot a \boxtimes 1) \otimes (1 \boxtimes f \boxtimes \xi)).$$

Then

$$c' = (h \boxtimes \mathrm{id})(\{f_0 \cdot a \boxtimes \tau(f \cdot 1 \boxtimes \xi, b)\}) + (h \boxtimes \mathrm{id})(\{a'' \otimes (1 \boxtimes b\})).$$

Next, we must show that the action of $\mathfrak{v}/\mathfrak{v}_0^+$ on $\pi^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$ factors through $\Theta_{\mathfrak{X}}$. The map $\mathfrak{v}/\mathfrak{v}_0^+ \to \Theta_{\mathfrak{X}}$ is surjective and its kernel is identified with

$$p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta)) \subset p_{1*}(\mathcal{O} \hat{\boxtimes} \Theta(\infty \cdot \Delta)),$$

(in other words, the fiber of the above kernel at $x \in X \in \mathfrak{M}$ is identified with the subspace $H^0(X \setminus x, \Theta_X) \subset \Theta \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x$).

For $f \cdot 1 \boxtimes \xi \in p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta))$ and for a local section a of ${}^l\mathcal{A}$ we have by definition:

$$\langle \tau(f \cdot 1 \boxtimes \xi, a) \rangle = \langle (\mathrm{id} \boxtimes \mathrm{Res}_\Delta)(f \cdot \nabla_{loc}(a) \otimes (1 \boxtimes \xi)) \rangle - \langle (\mathrm{id} \boxtimes h)(\{f \cdot a \boxtimes T(\xi)\}) \rangle.$$

However, the first term vanishes by the residue formula, since $(f \cdot \nabla_{loc}(a) \otimes (1 \boxtimes \xi))$ is a section of $p_{1*}({}^l\mathcal{A} \boxtimes \Omega(\infty \cdot \Delta))$ and the second term vanishes by the very definition of $H_\nabla(\mathfrak{X}, \mathcal{A})$.

We have shown, therefore, that $\pi^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$ is a D-module on $\mathfrak{X}$ such that the action of vertical vector fields on it is the "tautological" one. Therefore, the $\mathcal{O}$-module $H_\nabla(\mathfrak{X}, \mathcal{A})$ on $\mathfrak{M}$ is naturally a $D_{\mathfrak{M}}$-module.

□

Here is another characterization of the connection on $H_\nabla(\mathfrak{X}, \mathcal{A})$:

Let ∇ denote the covariant derivative acting on $\pi_k^*(H_\nabla(\mathfrak{X}, A))$; in particular, there is a map[4]

$$\nabla : j_* j^*(\pi_k^*(H_\nabla(\mathfrak{X}, A))) \to p_{1,\dots,k*}(j_* j^*(\pi_k^*(H_\nabla(\mathfrak{X}, \mathcal{A})) \boxtimes \Omega^2)).$$

Proposition 2.2.2. *Let $f \cdot a_1 \boxtimes \dots \boxtimes a_k$ be a section of $j_* j^*({}^l\mathcal{A}^{\boxtimes k})$ on $\mathfrak{X}^k \setminus \Delta$. Then*

$$\nabla(\langle f \cdot a_1 \boxtimes \dots \boxtimes a_k \rangle) = (\langle \dots \rangle \boxtimes \mathrm{id})(\nabla_{loc}(f \cdot a_1 \boxtimes \dots \boxtimes a_k)) + \langle f \cdot a_1 \boxtimes \dots \boxtimes a_k \boxtimes {}^l T \rangle.$$

Proof. When $k = 1$, the statement follows immediately from the definition of the action τ.

To treat the case of arbitrary k one proceeds as follows:

Step 1:

---

[4]In the formula below, $\langle f \cdot a_1 \boxtimes \dots \boxtimes a_k \boxtimes {}^l T \rangle^{k+1}$ will denote a section of $j_* j^*(\mathcal{O}^{\boxtimes k} \boxtimes \Omega^2)$ characterized by the property that for $\eta \in \Omega^{-2}$:

$$\langle f \cdot a_1 \boxtimes \dots \boxtimes a_k \boxtimes {}^l T \rangle \otimes (1 \boxtimes \dots \boxtimes 1 \boxtimes \eta) := \langle f \cdot a_1 \boxtimes \dots \boxtimes a_k \boxtimes {}^l T(\eta) \rangle \in j_* j^*(\mathcal{O}^{\boxtimes k}).$$

Consider the diagram

$$\begin{array}{ccc} j_*j^*({}^l\mathcal{A}^{\boxtimes k}) & \longrightarrow & p_{1,\dots,k*}(j_*j^*({}^l\mathcal{A}^{\boxtimes k}\boxtimes\Omega^2)) \\ {\scriptstyle\langle\dots\rangle}\downarrow & & {\scriptstyle\langle\dots\rangle\boxtimes\mathrm{id}}\downarrow \\ j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X},\mathcal{A}))) & \xrightarrow{?} & p_{1,\dots,k*}(j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X},\mathcal{A}))\boxtimes\Omega^2)), \end{array}$$

where the upper horizontal arrow sends a section $f\cdot a_1\boxtimes\ldots\boxtimes a_k$ of $j_*j^*({}^l\mathcal{A}^{\boxtimes k})$ to

$$\nabla_{loc}(f\cdot a_1\boxtimes\ldots\boxtimes a_k)+f\cdot a_1\boxtimes\ldots\boxtimes a_k\boxtimes{}^lT.$$

Lemma 2.2.3. *There exists a unique map*

$$\nabla':j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X},\mathcal{A})))\to p_{1,\dots,k*}(j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X},\mathcal{A}))\boxtimes\Omega^2))$$

which completes the above diagram to a commutative diagram.

Step 2:
It remains to show that the map ∇' constructed above coincides with the covariant derivative ∇. Consider the diagram

$$\begin{array}{ccc} j_*j^*({}^l\mathcal{A}\boxtimes\mathcal{O}^{\boxtimes k-1}) & \longrightarrow & p_{1,\dots,k*}(j_*j^*({}^l\mathcal{A}\boxtimes\mathcal{O}^{\boxtimes k-1}\boxtimes\Omega^2)) \\ {\scriptstyle\langle\dots\rangle\boxtimes\mathrm{id}}\downarrow & & {\scriptstyle\langle\dots\rangle\boxtimes\mathrm{id}\boxtimes\mathrm{id}}\downarrow \\ j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X},\mathcal{A}))) & \xrightarrow{?} & p_{1,\dots,k*}(j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X},\mathcal{A}))\boxtimes\Omega^2)), \end{array}$$

where the upper horizontal arrow is given by the same formula as above.

On the one hand, according to Step 1 and Section 1.3.3, this diagram becomes commutative if we set the lower horizontal arrow to be ∇'. On the other hand, the assertion of our proposition for $k=1$ implies that this diagram becomes commutative if we set the lower horizontal arrow to be ∇. Since the left vertical arrow is a surjection (by Proposition 1.3.3), $\nabla'=\nabla$.

□

§2.3. Introducing the central charge

§2.3.1.

We shall start with some linear algebra preliminaries.

Consider the canonical decreasing sequence of ideals in the Lie algebra Vir_0^+:

$$0\subset\ldots\subset\mathrm{Vir}_n^+\subset\ldots\subset\mathrm{Vir}_1^+\subset\mathrm{Vir}_0^+.$$

For $k\in\mathbb{N}$ let L_k denote the Aut_0^+-module $\mathrm{Vir}_k^+/\mathrm{Vir}_{k+1}^+$. We have canonical isomorphisms $L_k\otimes L_{k'}\simeq L_{k+k'}$ and for $k\in\mathbb{N}$ we define L_{-k} as L_k^*.

Lemma 2.3.1. *The following two categories are equivalent:*

(a) *The category* $\mathrm{Ext}_{\mathrm{Aut}_0^+}(L_{-1},L_1)$ *of extensions* $0\to L_1\to E\to L_{-1}\to 0$ *of* Aut_0^+*-modules.*

(b) *The category of Harish-Chandra pairs* $(\mathrm{Vir}',\mathrm{Aut}_0^+)$, *where* Vir' *is a central extension of* Vir*:*

$$0\to\mathbb{C}\to\mathrm{Vir}'\to\mathrm{Vir}\to 0.$$

In fact, both categories are groupoids such that every object has no non-trivial automorphisms; moreover the set of isomorphism classes of objects in each of them is canonically identified with $\mathbb{C}$.[5] This implies the assertion of the lemma.

The above equivalence of categories has the following additional property:

To an object $0 \to L_1 \to E \to L_{-1} \to 0$ of $\mathrm{Ext}_{\mathrm{Aut}_0^+}(L_{-1}, L_1)$ one can attach a local $\mathcal{O}$-module $\Theta' := \mathrm{Ass}(E)$ on $\mathfrak{X}$ (cf. Section 2.1.2). By the construction, Θ' is an extension

$$0 \to \Omega \to \Theta' \to \Theta \to 0.$$

Now let $(x \in X)$ be a point of $\mathfrak{X}$. Consider the corresponding extension of $\hat{\mathcal{K}}_x$-modules:

$$0 \to \Omega \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \Theta' \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \Theta \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to 0$$

and consider the vector space

$$\Theta' \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x / \ker(\mathrm{Res} : \Omega \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \mathbb{C}).$$

The action of $\Theta \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x$ on $\Theta' \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x$ by Lie derivations (cf. Section 2.1.3) makes the above vector space into a Lie algebra.

Lemma 2.3.2. *Every choice of an identification $\alpha : \hat{\mathcal{O}} \simeq \hat{\mathcal{O}}_x$ gives rise to an isomorphism*

$$\mathrm{Vir}' \simeq \Theta' \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x / \ker(\mathrm{Res} : \Omega \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \mathbb{C}),$$

where Vir' *is the central extension of* Vir *corresponding to E. A modification of α by means of $g \in \mathrm{Aut}_0^+$ corresponds to the adjoint action of g on* Vir'.

Let us now describe explicitly the extension $0 \to \Omega \to \Theta_2' \to \Theta \to 0$ that corresponds to the object $E_2 \in \mathrm{Ext}_{\mathrm{Aut}_0^+}(L_{-1}, L_1)$ whose class in $\mathbb{C}$ equals 2:

Consider the local $\mathcal{O}$-module

$$p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)/\mathcal{O} \boxtimes \mathcal{O}(-2 \cdot \Delta));$$

the section $1 \boxtimes 1$ gives rise to an embedding of $\mathcal{O}$ into it.

Lemma 2.3.3. *The local $\mathcal{O}$-module Θ_2' is identified with*

$$(p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)/\mathcal{O} \boxtimes \mathcal{O}(-2 \cdot \Delta)))/\mathcal{O}.$$

In what follows, for $c \in \mathbb{C}$ we shall denote by Θ_c (resp., Vir_c') the corresponding extension of Θ by Ω (resp., the corresponding central extension of Vir). The construction of Example 2 of Section 1.2.3 produces from Θ_c' a Lie-* algebra $\mathcal{T}_c'$ on $\mathfrak{X}$ that satisfies the conditions of Lemma 2.1.1.

---

[5] These identifications are chosen as follows:

If $0 \to L_1 \to E \to L_{-1} \to 0$ is an object of $\mathrm{Ext}_{\mathrm{Aut}_0^+}(L_{-1}, L_1)$, the action of Vir_0^+ on E produces a map $L_2 \otimes L_{-1} \to L_1$, i.e. a scalar $c \in \mathbb{C}$, since $L_2 \otimes L_{-1}$ is apriori identified with L_1. We set the class of this E in $\mathbb{C}$ to be $-2c$.

If Vir' is an object of the second category, the adjoint action of $\mathbb{C}^* \subset \mathrm{Aut}_0^+$ on Vir' defines a decomposition of the latter as a vector space: $\mathrm{Vir}' = \underset{i}{\oplus} L_i \oplus \mathbb{C}$. The Lie bracket on Vir' induces a map $L_2 \otimes L_{-2} \to \mathbb{C}$, i.e. a scalar $c \in \mathbb{C}$. We set the class of this Vir' to be equal to be $-2c$.

Let $\mathcal{P}_c$ denote the Serre dual to Θ'_c. This is again a local $\mathcal{O}$-module on $\mathfrak{X}$, which is an extension of $\mathcal{O}$ by Ω^2. It is easy to see that $\mathcal{P}_c$ identifies with the Ω^2-torsor of c-projective connections. In particular, for a fixed $X \in \mathfrak{M}$, the extensions

$$0 \to \Omega \to \Theta'_c \to \Theta \to 0 \text{ and } 0 \to \Omega^2 \to \mathcal{P}_c \to \mathcal{O} \to 0$$

are (non-canonically) split.

§2.3.2.

A CFT of central charge c consists of:

(1) A local $\mathcal{O}$-module $\mathcal{A}$ on $\mathfrak{X}$ endowed with a structure of chiral algebra.

(2) A map of local $\mathcal{O}$-modules $T : \Theta'_c \to \mathcal{A}$ (called the energy-momentum tensor), such that the composition $\Omega \to \Theta'_c \to \mathcal{A}$ coincides with the map **unit**.

The pair $(\mathcal{A}, T)$ must satisfy the following conditions:

(a) The chiral bracket on $\mathcal{A}$:

$$j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\{,\}} \Delta_!(\mathcal{A})$$

is a map of local $\mathcal{O}$-modules on $\mathfrak{X}^2$.

(b) Let ξ be a vertical vector field on $\mathfrak{X}$ and let ξ' be some lift of ξ to a section of Θ'_c. Consider the map $\mathcal{A} \to \mathcal{A}$ given by

$$l \to (\mathrm{id} \boxtimes h)(\{l, T(\xi')\}) - \mathrm{Lie}_\xi(l).$$

We need this map to coincide with the action of ξ on $\mathcal{A}$ coming from the right D-module structure on $\mathcal{A}$. (Note that the above expression a priori does not depend on the choice of ξ'.)

As in the case of central charge 0, it follows that the map $\mathcal{A} \otimes D \to \mathcal{A}$ defining the D-module structure on $\mathcal{A}$ is a map of local $\mathcal{O}$-modules and that the D-module map $\mathcal{T}'_c \to \mathcal{A}$ induced by $T : \Theta'_c \to \mathcal{A}$ is a Lie-* algebra homomorphism.

§2.3.3.

Let $\widetilde{\mathfrak{v}}'_c$ denote the Lie algebroid $\mathrm{Vir}'_c \otimes \mathcal{O}_{\hat{\mathfrak{X}}}$ on $\hat{\mathfrak{X}}$. Let $\widetilde{\mathfrak{v}}_{glob}$ denote the kernel $\widetilde{\mathfrak{v}} \to \Theta_{\hat{\mathfrak{X}}}$; obviously $\widetilde{\mathfrak{v}}_{glob}$ is an ideal in $\widetilde{\mathfrak{v}}$ and its fiber at a point $(X, x, \alpha) \in \hat{\mathfrak{X}}$ is identified with $H^0(X \setminus x, \Theta)$.

Proposition 2.3.1. *There is a canonical lifting:*

$$\widetilde{\mathfrak{v}}_{glob} \to \widetilde{\mathfrak{v}}'_c.$$

Moreover, $\widetilde{\mathfrak{v}}_{glob}$ becomes an ideal in $\widetilde{\mathfrak{v}}'_c$.

Proof. To construct the embedding $\widetilde{\mathfrak{v}}_{glob} \hookrightarrow \widetilde{\mathfrak{v}}'_c$ we must exhibit for each $(X, x, \alpha) \in \hat{\mathfrak{X}}$ as above a map

$$H^0(X \setminus x, \Theta) \to \Theta'_c \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x / \ker(\mathrm{Res} : \Omega \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \mathbb{C}).$$

We obviously have a map $H^0(X \setminus x, \Theta'_c) \to \Theta'_c \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x$, and the composition

$$H^0(X \setminus x, \Omega) \to \Theta'_c \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \Theta'_c \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x / \ker(\mathrm{Res} : \Omega \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \mathbb{C})$$

vanishes by the residue formula.

Since the extension $0 \to \Omega \to \Theta'_c \to \Theta \to 0$ is (non-canonically) split, the projection $H^0(X \setminus x, \Theta'_c) \to H^0(X \setminus x, \Theta)$ is surjective and we obtain a well-defined map

$$H^0(X \setminus x, \Theta) \to \Theta'_c \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x / \ker(\mathrm{Res} : \Omega \underset{\mathcal{O}}{\otimes} \hat{\mathcal{K}}_x \to \mathbb{C})$$

as needed.

To prove that $\widetilde{\mathfrak{v}}_{glob}$ is an ideal in $\widetilde{\mathfrak{v}}'_c$ we must show that it is stable under the action of the formal group Aut on $\hat{\mathfrak{X}}$. This follows from the description of this action that was given in Section 2.2.2.

□

Obviously, $\widetilde{\mathfrak{v}}'_c$ is a Harish-Chandra Lie algebroid with respect to the group Aut_0^+; let $\mathfrak{v}'_c$ and $\mathfrak{v}'_c/\mathfrak{v}_0^+$ denote the corresponding Lie algebroids on $\mathfrak{X}$. Clearly, we have short exact sequences:

$$0 \to \mathcal{O} \to \mathfrak{v}'_c \to \mathfrak{v} \to 0 \text{ and } 0 \to \mathcal{O} \to \mathfrak{v}'_c/\mathfrak{v}_0^+ \to \mathfrak{v}/\mathfrak{v}_0^+ \to 0$$

with $\mathcal{O}$ being an ideal and the adjoint action of $\mathfrak{v}$ on $\mathcal{O}$ being the canonical one. As an $\mathcal{O}$-module, $\mathfrak{v}'_c$ is identified with

$$p_{1*}(\mathcal{O} \hat{\boxtimes} \Theta'_c(\infty \cdot \Delta)) / \ker(p_{1*}(\mathcal{O} \hat{\boxtimes} \Omega(\infty \cdot \Delta)) \to \mathcal{O})$$

and similarly for $\mathfrak{v}'_c/\mathfrak{v}_0^+$.

Now let $\mathfrak{v}_{glob}$ denote the kernel of $\mathfrak{v} \to \Theta_{\mathfrak{X}}$. It follows from the proposition above that $\mathfrak{v}_{glob}$ embeds naturally into $\mathfrak{v}'_c$ and is an ideal thereof. Moreover, the quotient

$$\Theta'_{\mathfrak{X}_c} := \mathfrak{v}'_c/(\mathfrak{v}_0^+ + \mathfrak{v}_{glob})$$

is a Picard Lie algebroid on $\mathfrak{X}$.

Note that we have a canonical embedding $\mathrm{Vir}^+ \to \mathrm{Vir}'_c$ which gives rise to an embedding of Lie algebroids $\Theta \to \mathfrak{v}'_c/\mathfrak{v}_0^+$. In particular, there is a homomorphism of Lie algebroids $\Theta \to \Theta'_{\mathfrak{X}_c}$.

Proposition 2.3.2. *There exists a canonical Picard Lie algebroid $\Theta'_{\mathfrak{M}_c}$ on $\mathfrak{M}$ such that $\Theta'_{\mathfrak{X}_c}$ is its pull-back (in the sense of Section 0.2) under the map π. As an $\mathcal{O}_{\mathfrak{M}}$-module, $\Theta'_{\mathfrak{M}_c}$ is identified with $R^1\pi_*(\Theta'_c)$.*

Proof. As an $\mathcal{O}$-module, the Picard Lie algebroid $\Theta'_{\mathfrak{X}_c}$ is identified with

$$R^1p_{1*}(\mathcal{O} \boxtimes \Theta'_c(-\Delta)) / \ker(R^1p_{1*}(\mathcal{O} \boxtimes \Omega(-\Delta)) \to R^1p_{1*}(\mathcal{O} \boxtimes \Omega))$$

and the quotient $\Theta'_{\mathfrak{X}_c}/\Theta$ is therefore identified with $\pi^*(R^1\pi_*(\Theta'_c))$.

Let $N(\Theta)$ denote the normalizer of Θ in $\Theta'_{\mathfrak{X}_c}$.

Lemma 2.3.4. *$N(\Theta)/\Theta \subset \Theta'_{\mathfrak{X}_c}/\Theta$ corresponds to $\pi^\bullet(R^1\pi_*(\Theta'_c)) \subset \pi^*(R^1\pi_*(\Theta'_c))$.*

Therefore, $\Theta'_{\mathfrak{M}_c} := R^1\pi_*(\Theta'_c)$ has a canonical structure of Lie algebroid on $\mathfrak{M}$ and we have a commutative square:

$$\begin{array}{ccccccc}
\Theta & \longrightarrow & \Theta'_{\mathfrak{X}_c} & \longrightarrow & \pi^*(\Theta'_{\mathfrak{M}_c}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Theta & \longrightarrow & \Theta_{\mathfrak{X}} & \longrightarrow & \pi^*(\Theta_{\mathfrak{M}}) & \longrightarrow & 0,
\end{array}$$

which is what we had to prove. □

§2.3.4.

Similarly to the case $c = 0$, we shall show now that our CFT chiral algebra ${}^l\mathcal{A}$ carries a natural action of the Lie algebroid $\mathfrak{v}'_c/\mathfrak{v}'_0{}^+$. To do this, we must construct a map $\tau : \mathfrak{v}'_c/\mathfrak{v}'_0{}^+ \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$ such that the composition

$$\mathcal{O} \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \hookrightarrow \mathfrak{v}'_c/\mathfrak{v}'_0{}^+ \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$$

coincides with the tautological map $\mathcal{O} \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$.

We have a map $\tau_{loc} : \mathfrak{v}'_c \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$ that factors through

$$\mathfrak{v}'_c \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to \mathfrak{v} \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A},$$

the second arrow being defined by the same formula as in Section 2.2.2 (it comes from the structure on ${}^l\mathcal{A}$ of a local $\mathcal{O}$-module).

We have a map $\tau_{em} : \mathfrak{v}'_c \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$ that attaches to a local section $f \cdot 1\hat{\boxtimes}\xi' \in p_{1*}(\mathcal{O}\hat{\boxtimes}\Theta'_c(\infty \cdot \Delta)) \simeq \mathfrak{v}'_c$ and a local section $a \in {}^l\mathcal{A}$ the section

$$(\mathrm{id} \boxtimes h)(\{f \cdot a \boxtimes T(\xi')\}) \in {}^l\mathcal{A}.$$

The difference $\tau := \tau_{loc} - \tau_{em}$ is therefore a map $\mathfrak{v}'_c \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$ which factors (due to condition (b) of Section 2.3.2) as

$$\mathfrak{v}'_c/\mathfrak{v}'_0{}^+ \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}.$$

As in the previous subsection we have the following assertion:

Lemma 2.3.5. *The map $\tau : \mathfrak{v}'_c/\mathfrak{v}'_0{}^+ \underset{\mathbb{C}}{\otimes} {}^l\mathcal{A} \to {}^l\mathcal{A}$ is a left action of the Lie algebroid $\mathfrak{v}'_c/\mathfrak{v}'_0{}^+$ on ${}^l\mathcal{A}$.*

Consider now the $\mathcal{O}_{\mathfrak{M}}$-module $H_\nabla(\mathfrak{X}, \mathcal{A})$.

Proposition 2.3.3. *The $\mathcal{O}_{\mathfrak{M}}$-module $H_\nabla(\mathfrak{X}, \mathcal{A})$ on $\mathfrak{M}$ has a (unique) structure of left $\Theta'_{\mathfrak{M}_c}$-module such that when we view $\pi^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$ as a $\mathfrak{v}'_c/\mathfrak{v}'_0{}^+$-module via $\mathfrak{v}'_c/\mathfrak{v}'_0{}^+ \to \Theta'_{\mathfrak{X}_c}$, the projection*

$$\langle \dots \rangle : {}^l\mathcal{A} \to \pi^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$$

is compatible with the $\mathfrak{v}'_c/\mathfrak{v}'_0{}^+$-action.

The proof of this assertion is the same as in the case $c = 0$.

For $k \in \mathbb{N}$ let $\Theta'_{\mathfrak{X}^k_c}$ denote the pull-back of the Picard Lie algebroid $\Theta'_{\mathfrak{M}_c}$ to $\mathfrak{X}^k$ (in the sense of Section 0.2); by definition, for $k = 1$ we re-obtain on $\mathfrak{X}$ the Picard Lie-algebroid $\Theta'_{\mathfrak{X}_c}$. Note that the corresponding twisted cotangent bundle on $\mathfrak{X}^k \setminus \Delta$ is identified with

$$\ker(p_{1,\dots,k*}(\mathcal{O}^{\boxtimes k} \boxtimes \mathcal{P}_c(\Delta)) \to p_{1,\dots,k*}(\mathcal{O}^{\boxtimes k+1}(\Delta))).$$

In particular, the $\Theta'_{\mathfrak{X}^k_c}$-action on $\pi_k^*(H_\nabla(\mathfrak{X}, \mathcal{A}))$ yields a covariant derivative map

$$\nabla : j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X}, \mathcal{A}))) \to p_{1,\dots,k*}(j_*j^*(\pi_k^*(H_\nabla(\mathfrak{X}, \mathcal{A})) \boxtimes \mathcal{P}_c)).$$

The following is a generalization to the case $c \neq 0$ of Proposition 2.2.2:

Proposition 2.3.4. *Let $f \cdot a_1 \boxtimes \ldots \boxtimes a_k$ be a local section of $j_* j^*({}^l\mathcal{A}^{\boxtimes k})$ on $\mathfrak{X}^k \setminus \Delta$. Then the section $\nabla(f \cdot a_1 \boxtimes \ldots \boxtimes a_k)$ of $j_* j^*(\pi_k^*(H_\nabla(\mathfrak{X}, \mathcal{A})) \boxtimes \mathcal{P}_c)$ equals*

$$(\langle \ldots \rangle \boxtimes \mathrm{id})(\nabla_{loc}(f \cdot a_1 \boxtimes \ldots \boxtimes a_k)) + \langle f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes {}^l T \rangle,$$

where the insertion of ${}^l T$ in the correlation function has the same meaning as in Proposition 2.2.2.

CHAPTER 3
Examples

§3.1. Heisenberg and Kac-Moody algebras

§3.1.1.

Let $\mathfrak{g}$ and Q be as in Example 1 of Section 1.2.3 and let $\mathcal{B}(\mathfrak{g}, Q)$ be the corresponding Lie-* algebra; consider its chiral universal enveloping algebra $\mathcal{U}(\mathcal{B}(\mathfrak{g}, Q))$.

There are two maps of Ω into $\mathcal{U}(\mathcal{B}(\mathfrak{g}, Q))$: one is the map $\mathbf{unit} : \Omega \to \mathcal{U}(\mathcal{B}(\mathfrak{g}, Q))$ and the other is the composition

$$\widetilde{\mathbf{unit}} : \Omega \to \mathcal{B}(\mathfrak{g}, Q) \to \mathcal{U}(\mathcal{B}(\mathfrak{g}, Q)).$$

We define the chiral algebra $\mathcal{A}(\mathfrak{g}, Q)$ as a quotient of $\mathcal{U}(\mathcal{B}(\mathfrak{g}, Q))/I$, where I is the ideal[1] generated by the image of $\mathbf{unit} - \widetilde{\mathbf{unit}} : \Omega \longrightarrow \mathcal{U}(\mathcal{B}(\mathfrak{g}, Q))$.

We are going to show now that under certain conditions, $\mathcal{A}(\mathfrak{g}, Q)$ defines a CFT data.

It is clear that $\mathcal{B}(\mathfrak{g}, Q)$ has a natural structure of local $\mathcal{O}$-module on $\mathfrak{X}$, that satisfies the conditions of Lemma 2.1.1. Hence, $\mathcal{A}(\mathfrak{g}, Q)$ is a chiral algebra on $\mathfrak{X}$ which has a structure of local $\mathcal{O}$-module. Therefore, in order for $\mathcal{A}(\mathfrak{g}, Q)$ to define a CFT structure, we must construct a map of local $\mathcal{O}$-modules

$$T_{\mathfrak{g},Q} : \Theta'_c \to \mathcal{A}(\mathfrak{g}, Q)$$

for some $c \in \mathbb{C}$ that satisfies condition (b) of Section 2.3.2.

All our constructions will be local, therefore, we will be working on a fixed curve X.

Let $B \in \mathfrak{g} \otimes \mathfrak{g}$ be an $\mathrm{ad}_{\mathfrak{g}}$-invariant symmetric tensor that satisfies the following conditions:

- If $B = \sum\limits_{i,j} b_{i,j} \cdot u_i \otimes u_j$, then the endomorphism of $\mathfrak{g}$ given by

$$v \to \sum_{i,j} b_{i,j} \cdot [u_i, [u_j, v]]$$

[1] An ideal in a chiral algebra $\mathcal{A}$ is by definition a chiral submodule of $\mathcal{A}$, when the latter is viewed as a module over itself. For a D-submodule of $\mathcal{A}$ there exists a minimal ideal that contains it, called the "ideal generated by the given D-sub-module". If $I \subset \mathcal{A}$ is an ideal, the quotient D-module $\mathcal{A}/I$ has a natural structure of chiral algebra.

is a scalar operator $\kappa(B) \cdot \mathrm{Id}_{\mathfrak{g}}$.

- The map $B \circ Q : \mathfrak{g} \to \mathfrak{g}^* \to \mathfrak{g}$ is a scalar operator that satisfies

$$2(B \circ Q) + \kappa(B) = \mathrm{Id}_{\mathfrak{g}}.$$

First, we define a map $\widetilde{T}_{\mathfrak{g},Q} : \Theta'_2 \to \mathcal{A}(\mathfrak{g}, Q)$ as follows:
Recall that in Section 2.3.2, Θ'_2 was constructed as a quotient of the $\mathcal{O}$-module $p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta))$. Let $\widetilde{\widetilde{T}}_{\mathfrak{g},Q}$ be the map

$$p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)) \to \mathcal{A}(\mathfrak{g}, Q)$$

that sends a section $f(x,y) \in \mathcal{O} \boxtimes \mathcal{O}(\Delta)$ to

$$(h \boxtimes \mathrm{id})(\sum_{i,j} b_{i,j} \cdot \{u_i \boxtimes u_j \cdot f(x,y)\}) \in \mathcal{A}(\mathfrak{g}, Q).$$

Lemma 3.1.1. *For any symmetric $B \in \mathfrak{g} \otimes \mathfrak{g}$, the map $\widetilde{\widetilde{T}}_{\mathfrak{g},Q} : p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)) \to \mathcal{A}(\mathfrak{g}, Q)$ factors as*

$$p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)) \twoheadrightarrow \Theta'_2 \xrightarrow{\widetilde{T}_{\mathfrak{g},Q}} \mathcal{A}(\mathfrak{g}, Q).$$

Moreover, the composition

$$\Omega \hookrightarrow \Theta'_2 \xrightarrow{\widetilde{T}_{\mathfrak{g},Q}} \mathcal{A}(\mathfrak{g}, Q)$$

coincides with $\mathrm{Tr}(B \circ Q) \cdot \mathbf{unit}$.

We now set $c = 2\,\mathrm{Tr}(B \circ Q)$ and define the map $T_{\mathfrak{g},Q} : \Theta'_c \to \mathcal{A}(\mathfrak{g}, Q)$ as a composition

$$\Theta'_c \xrightarrow{?} \Theta'_2 \xrightarrow{\widetilde{T}_{\mathfrak{g},Q}} \mathcal{A}(\mathfrak{g}, Q),$$

where the first arrow is the unique isomorphism that fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega & \longrightarrow & \Theta'_c & \longrightarrow & \Theta & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \frac{1}{\mathrm{Tr}(B \circ Q)}} & & \downarrow{\scriptstyle ?} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
0 & \longrightarrow & \Omega & \longrightarrow & \Theta'_2 & \longrightarrow & \Theta & \longrightarrow & 0.
\end{array}$$

Clearly, the map $T_{\mathfrak{g},Q}$ is compatible with the local $\mathcal{O}$-module structure.

Proposition 3.1.1. *For B and c as above, the map $T_{\mathfrak{g},Q} : \Theta'_c \to \mathcal{A}(\mathfrak{g}, Q)$ satisfies condition (b) of the CFT data.*

Before giving a proof of this proposition, let us consider two examples:

Example 1

Let $\mathfrak{g} = \mathfrak{h}$ be abelian; in this case B exists if and only if Q is non-degenerate and $B = (2 \cdot Q)^{-1}$. We have $c = \dim(\mathfrak{h})$.

Example 2

Now let $\mathfrak{g}$ be semisimple. The condition for the existence of B is that $\kappa(Q^{-1}) + 2 \neq 0$, i.e. that Q is not equal to $-\frac{1}{2} K$, where K is the Killing form.

Therefore, in this case the construction we described above is just an invariant way to write down Sugawara's construction. Moreover, if $Q = q \cdot K$ we see that the central charge in this case equals $q \cdot \dim(\mathfrak{g})/(q + 1/2)$.

§3.1.2.

Let us now prove Proposition 3.1.1:

Proof. We have to show that for a section $f(x,y)$ of $\mathcal{O}\boxtimes\mathcal{O}(\Delta)$ and for any section $l\in\mathcal{A}(\mathfrak{g},Q)$, we have:

$$(h\boxtimes\mathrm{id})(\{T_{\mathfrak{g},Q}(\xi_f),l\})=-\operatorname{Lie}_{\xi_f}(l)-l\cdot\xi_f,$$

where ξ_f is a vector field on X corresponding to $f(x,y)$ via the identification

$$\mathcal{O}\boxtimes\mathcal{O}(\Delta)/\mathcal{O}\boxtimes\mathcal{O}\simeq\Theta.$$

First of all, since $\mathcal{B}(\mathfrak{g},Q)$ generates $\mathcal{A}(\mathfrak{g},Q)$, the Jacobi identity implies that it is enough to prove the above equality for $l\in\mathcal{B}(\mathfrak{g},Q)\subset\mathcal{A}(\mathfrak{g},Q)$. Moreover, we can assume that l is of the form $l=v\otimes 1\in\mathfrak{g}\otimes D_X$. Therefore, we have to prove the equality

$$(h\boxtimes h\boxtimes\mathrm{id})(\sum_{i,j}b_{i,j}\{\{u_i\boxtimes u_j\cdot f(x,y)\},v\})=-v\otimes\xi_f\in\mathcal{B}(\mathfrak{g},Q)\subset\mathcal{A}(\mathfrak{g},Q).$$

It is easy to see that the assertion holds for $f(x,y)\in\mathcal{O}\boxtimes\mathcal{O}$, therefore, we can assume that $f(x,y)=-f(y,x)$. Using the fact that the tensor $\sum\limits_{i,j}b_{i,j}\cdot u_i\otimes u_j$ is symmetric and the Jacobi identity, the above expression can be rewritten as:

$$\sum_{i,j}2b_{i,j}\cdot(h\boxtimes h\boxtimes\mathrm{id})(\{u_i\boxtimes\{f(x,y)\cdot u_j\boxtimes v\}\}).$$

By the definition of the Lie-* bracket on $\mathcal{B}(\mathfrak{g},Q)$, the last expression equals

$$\sum_{i,j}2b_{i,j}\cdot(h\boxtimes\mathrm{id})(\{f(x,y)\cdot u_i\boxtimes[u_j,v]\})+$$
$$+\sum_{i,j}2b_{i,j}\cdot Q(u_j,v)\cdot(h\boxtimes h\boxtimes\mathrm{id})(\{f(x,y)\cdot u_i\boxtimes\mathbf{1}'_{y=z}\}),$$

where $\mathbf{1}'_{y,z}$ denotes the canonical antisymmetric section of $\Delta_!(\Omega)\subset\Delta_!(\mathcal{A}(\mathfrak{g},Q))$.

To prove the proposition we must show that

a) $\sum\limits_{i,j}2b_{i,j}\cdot(h\boxtimes\mathrm{id})(\{f(x,y)\cdot u_i\boxtimes[u_j,v]\})=-\sum\limits_{i,j}b_{i,j}\cdot[u_i,[u_j,v]]\otimes\xi_f$ and

b) that for $u\in\mathfrak{g}$, $(h\boxtimes h\boxtimes\mathrm{id})(\{f(x,y)\cdot u\boxtimes\mathbf{1}'_{y,z}\})=-u\otimes\xi_f$.

Using the fact that the tensor $\sum\limits_{i,j}b_{i,j}\cdot u_i\otimes u_j$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant, the expression

$$\sum_{i,j}2b_{i,j}\cdot(h\boxtimes\mathrm{id})(\{f(x,y)\cdot u_i\boxtimes[u_j,v]\})$$

can be rewritten as

$$\sum_{i,j}b_{i,j}\cdot(h\boxtimes\mathrm{id})(\{f(x,y)\cdot u_i\boxtimes[u_j,v]\}-\{f(x,y)\cdot[u_j,v]\boxtimes u_i\})$$

and assertion a) follows from the next lemma:

Lemma 3.1.2. *Let $\mathcal{A}$ be a chiral algebra and let $\sum\limits_{i,j}a_1^i\boxtimes a_2^j$ be a section of $\mathcal{A}\boxtimes\mathcal{A}$. Assume that $\sum\limits_{i,j}\{a_1^i\boxtimes a_2^j\}\in\mathcal{A}\subset\Delta_!(\mathcal{A})$. Now let $f(x,y)$ be an antisymmetric section of $\mathcal{O}\boxtimes\mathcal{O}(\Delta)$. We have:*

$$\sum_{i,j}(\{a_1^i\boxtimes a_2^j\cdot f(x,y)\}-\{a_2^j\boxtimes a_1^i\cdot f(x,y)\})=\sum_{i,j}\{a_1^i\boxtimes a_2^j\}\cdot(\xi_f,-\xi_f)\in\Delta_!(\mathcal{A}),$$

where ξ_f has the same meaning as above.

The expression $(h \boxtimes h \boxtimes \mathrm{id})(\{f(x,y) \cdot u \boxtimes \mathbf{1}'_{y=z}\})$ equals

$$(h \boxtimes \mathrm{id})(\{u \boxtimes d_y(f(x,y))\})$$

and assertion b) follows from the following general fact:

Lemma 3.1.3. *Let $\mathcal{M}$ be a right D-module on a curve X, let $m \in \mathcal{M}$ be its local section and let $f(x,y)$ be a section of $\mathcal{O} \boxtimes \mathcal{O}(\Delta)$. Then under the map*

$$(h \boxtimes \mathrm{id}) \circ \mathrm{can}_{\mathcal{M}} : j_* j^*(\mathcal{M} \boxtimes \Omega) \to \Delta_!(\mathcal{M}) \to \mathcal{M}$$

the section $m \boxtimes d_y(f(x,y))$ goes over to $-m \cdot \xi_f$.

□

§3.2. The linear dilaton

§3.2.1.

Now let $\mathfrak{g} = \mathbb{C}$. We are going to consider a twist of the CFT corresponding to $\mathcal{A}(\mathbb{C}, Q)$, called the linear dilaton.

We start with the following observation:

Lemma 3.2.1. *Consider the category $\mathrm{Ext}_{\mathrm{Aut}_0^+}(L_0, L_1)$ of extensions*

$$0 \to L_1 \to E \to L_0 \to 0$$

of Aut_0^+-modules. We have a canonical isomorphism[2] *$\pi_0(\mathrm{Ext}_{\mathrm{Aut}_0^+}(L_0, L_1)) \simeq \mathbb{C}$.*

Thus, for $\lambda \in \mathbb{C}$ one obtains a canonical extension F_λ:

$$0 \to \Omega \to F_\lambda \to \mathcal{O} \to 0$$

in the category of local $\mathcal{O}$-modules on $\mathfrak{X}$. For instance, if $\lambda = -2n$, where n is an integer, we have a canonical isomorphism

$$F_\lambda \simeq \mathrm{Diff}_1(\Omega^n, \Omega^{n+1}).$$

For a pair of complex numbers (λ, Q) we now define a Lie-* algebra structure on the D-module

$$\mathcal{B}(\lambda, Q) := F_\lambda \otimes D / \ker(\Omega \otimes D \to \Omega)$$

by the rule that the Lie-* bracket is the composition

$$\mathcal{B}(\lambda, Q) \boxtimes \mathcal{B}(\lambda, Q) \to D \boxtimes D \to \Delta_!(\Omega),$$

where the last arrow sends $1 \boxtimes 1 \in D \boxtimes D$ to $Q \cdot \mathbf{1}' \in \Delta_!(\Omega)$.

We define the chiral algebra $\mathcal{A}(\lambda, Q)$ as the quotient of $\mathcal{U}(\mathcal{B}(\lambda, Q))$ by the standard relation $\mathbf{unit} = \widetilde{\mathbf{unit}}$.

Here is another useful point of view on the chiral algebra $\mathcal{A}(\lambda, Q)$:

Proposition 3.2.1. *Consider the chiral algebra $\mathcal{A}(\mathbb{C}, Q)$ (Q is viewed as a quadratic form on $\mathbb{C}$).*

(a) *There is a natural map of sheaves $\Omega \to \mathrm{Aut}(\mathcal{A}(\mathbb{C}, Q))$.*

(b) *The chiral algebra $\mathcal{A}(\lambda, Q)$ is identified naturally with a twist of $\mathcal{A}(\mathbb{C}, Q)$ with respect to the Ω-torsor F_λ.*

---

[2] We fix this isomorphism in such a way that if $0 \to L_1 \to E \to L_0 \to 0$ is an object of $\mathrm{Ext}_{\mathrm{Aut}_0^+}(L_0, L_1)$ corresponding to $\lambda \in \mathbb{C}$, the map $L_1 \simeq L_1 \otimes L_0 \to L_1$ induced by the action of $L_1 \subset \mathrm{Vir}_0^+$ on E is multiplication by λ.

Proof. For a section $\omega \in \Omega$ we define the corresponding automorphism of $\mathcal{B}(\mathbb{C}, Q)$ by the rule that it acts as the identity on $\Omega \subset \mathcal{B}(\mathbb{C}, Q)$ and that it sends the section $1 \in D \in \mathcal{B}(\mathbb{C}, Q)$ to $\omega \in \Omega \subset \mathcal{B}(\mathbb{C}, Q)$; clearly, it extends to an automorphism of $\mathcal{A}(\mathbb{C}, Q)$. This establishes point (a) of the proposition, whereas point (b) follows immediately from the definitions.

□

§3.2.2.

We shall now show that the chiral algebra $\mathcal{A}(\lambda, Q)$ defines a CFT with $c = 1+3\lambda^2/Q$.

Obviously, $\mathcal{A}(\lambda, Q)$ carries a structure of local $\mathcal{O}$-module on $\mathfrak{X}$ which is compatible with the chiral bracket. Thus, to define the corresponding CFT data we must exhibit a map of local $\mathcal{O}$-modules $T_{\lambda,Q} : \Theta_c \to \mathcal{A}(\lambda, Q)$ with c as above which will satisfy condition (b) of Section 2.3.2.

Proposition 3.2.2. *There exists a unique map $\widetilde{T}_{\lambda,Q} : \Theta \to \mathcal{A}(\lambda, Q)/\Omega$ such that the following holds:*
For any local section $l \in \mathcal{A}(\lambda, Q)$, any vector field ξ on X and any section ξ' of $\mathcal{A}(\lambda, Q)$ that projects to $\widetilde{T}_{\lambda,Q}(\xi)$ we have:

$$(h \boxtimes \mathrm{id})(\{\xi', l\}) = -\operatorname{Lie}_\xi(l) - l \cdot \xi.$$

Proof. Note first of all that the left hand side of the above expression is a priori independent of the choice of ξ'. Moreover, the uniqueness statement is clear, since Ω is the only $\mathcal{O}$-submodule in $\mathcal{A}(\lambda, Q)$ that annihilates the Lie-* bracket (this follows, for example, from the corresponding fact in the untwisted case (i.e. when $\lambda = 0$) in view of Proposition 3.2.1(b)).

The existence assertion is local and let us choose a coordinate $\jmath$ on some open sub-set $U \subset X$; let $\partial_\jmath$ denote the corresponding vector field on U. It is clear that the choice of $\jmath$ as above trivializes the Ω-torsor F_λ over U; let $\phi_\jmath$ denote the corresponding isomorphism

$$\phi_\jmath : \mathcal{A}(\mathbb{C}, Q) \to \mathcal{A}(\lambda, Q)$$

defined over U.

We define the map $\widetilde{T}_{\lambda,Q}$ by the condition that

$$\widetilde{T}_{\lambda,Q}(\partial_\jmath) = \phi_\jmath(T_{\mathbb{C},Q}(\partial'_\jmath) - \lambda/2Q \cdot \partial_\jmath),$$

where $\partial'_\jmath$ in the first term is any lifting of $\partial_\jmath$ to a section of Θ'_1 and $\partial_\jmath$ in the second term is viewed as a section of $D_X \subset \mathcal{B}(\mathbb{C}, Q)$.

Let us check now that the map $\widetilde{T}_{\lambda,Q}$ defined in the above way satisfies the condition of the proposition.

Let f be a function on U. It is enough to show that

$$(h \boxtimes \mathrm{id})(\{\phi_\jmath(T_{\mathbb{C},Q}(\partial'_\jmath) \cdot f - \lambda/2Q \cdot \partial_\jmath f) \boxtimes \phi_\jmath(1)\}) = -\phi_\jmath(1) \cdot f\partial_\jmath - \operatorname{Lie}_{f\partial_\jmath}(\phi_\jmath(1)).$$

Since $\phi_\jmath$ is a homomorphism of chiral algebras and since $T_{\mathbb{C},Q}$ is the energy momentum tensor for $\mathcal{A}(\mathbb{C}, Q)$, we have:

$$(h \boxtimes \mathrm{id})(\{\phi_\jmath(T_{\mathbb{C},Q}(\partial'_\jmath) \cdot f) \boxtimes \phi_\jmath(1)\}) = -\phi_\jmath(1) \cdot f\partial_\jmath$$

and the above assertion reduces to the equality

$$\lambda \cdot d(\partial_\jmath(f)) = 2 \operatorname{Lie}_{f\partial_\jmath}(\phi_\jmath(1)),$$

which follows from the definition of F_λ.

□

The map $\widetilde{T}_{\lambda,Q}$ constructed above is compatible with the local $\mathcal{O}$-module structure due to the uniqueness part of the assertion of Proposition 3.2.2. We now define Θ' to be the extension $0 \to \Omega \to \Theta' \to \Theta \to 0$ induced from the extension $0 \to \Omega \to \mathcal{A}(\lambda, Q) \to \mathcal{A}(\lambda, Q)/\Omega \to 0$ by means of the map $\widetilde{T}_{\lambda,Q}$.

By the construction, Θ' carries in a natural way a structure of local $\mathcal{O}$-module on $\mathfrak{X}$ and we have a map $T_{\lambda,Q} : \Theta' \to \mathcal{A}(\lambda, Q)$ that satisfies condition (b) of Section 2.3.2.

Proposition 3.2.3. *The local $\mathcal{O}$-module Θ' constructed above is identified with Θ_c for $c = 1 + 3\lambda^2/Q$.*

Proof. The proof of the proposition will be based on the following lemma:

Lemma 3.2.2. *Let $0 \to \Omega \to \Theta' \to \Theta \to 0$ be an arbitrary extension in the category of local $\mathcal{O}$-modules on $\mathfrak{X}$. Assume that for any pair $(x \in X)$ the following holds: if $\mathfrak{z}$ is a coordinate around x and if $\widetilde{\partial}_\mathfrak{z}$ is a local section of Θ' that projects to $\partial_\mathfrak{z}$, the value of $\mathrm{Lie}_{\mathfrak{z}^3\partial_\mathfrak{z}}(\partial'_\mathfrak{z})$ at x equals $-c/2 \cdot d\mathfrak{z}$. Then Θ' is canonically isomorphic to Θ'_c.*

Let U, $\mathfrak{z}$ and $\phi_\mathfrak{z}$ be as in Proposition 3.2.2. For a function f on U consider the map from $\mathcal{A}(\mathbb{C}, Q)$ to itself given by

$$l \to \phi_\mathfrak{z}^{-1} \circ \mathrm{Lie}_{f\partial_\mathfrak{z}} \circ \phi_\mathfrak{z}(l) - \mathrm{Lie}_{f\partial_\mathfrak{z}}(l).$$

We claim that it coincides with the map

$$l \to \lambda/2Q(h \boxtimes \mathrm{id})(\{\partial_\mathfrak{z} \cdot f \boxtimes l\}).$$

Indeed, both maps are derivations of the chiral algebra structure on $\mathcal{A}(\mathbb{C}, Q)$ (in particular, they commute with the action of D_X) and coincide on the generators.

Let us now apply Lemma 3.2.2 to the extension Θ' as in the formulation of the proposition. By the construction, we can take $\widetilde{\partial}_\mathfrak{z}$ to be

$$\phi_\mathfrak{z}(T_{\mathbb{C},Q}(\partial'_\mathfrak{z}) - \lambda/2Q \cdot \partial_\mathfrak{z}) \in \mathcal{A}(\lambda, Q),$$

where $\partial'_\mathfrak{z}$ is the corresponding section of Θ'_1.

We have:

$$\phi_\mathfrak{z}^{-1}(\mathrm{Lie}_{\mathfrak{z}^3\partial_\mathfrak{z}}(\widetilde{\partial}_\mathfrak{z})) = \lambda/2Q(h \boxtimes \mathrm{id})(\{\partial_\mathfrak{z} \cdot \mathfrak{z}^3 \boxtimes T_{\mathbb{C},Q}(\partial'_\mathfrak{z})\}) -$$
$$- \lambda/2Q(h \boxtimes \mathrm{id})(\{\partial_\mathfrak{z} \cdot \mathfrak{z}^3 \boxtimes \lambda/2Q \cdot \partial_\mathfrak{z}\}) + \mathrm{Lie}_{\mathfrak{z}^3\partial_\mathfrak{z}}(T_{\mathbb{C},Q}(\partial'_\mathfrak{z})) - \mathrm{Lie}_{\mathfrak{z}^3\partial_\mathfrak{z}}(\lambda/2Q \cdot \partial_\mathfrak{z}).$$

It is easy to see that the values at x of the first and of the fourth terms of the above formula are 0. The value at x of the second term is $-3\lambda^2/2Q \cdot d\mathfrak{z}$, by definition. Since the chiral algebra $\mathcal{A}(\mathbb{C}, Q)$ defines a CFT of central charge 1, the value at x of the third term is $-\frac{1}{2}\, d\mathfrak{z}$.

□

§3.3. The bc-system

§3.3.1.

Let $\mathcal{M}$ be a locally free finitely generated D-module on a curve X (e.g. $\mathcal{M} = \mathcal{L} \otimes D_X$, where $\mathcal{L}$ is a torsion-free quasi-coherent sheaf). Let $\mathcal{M}^*$ denote the Verdier-dual D-module, i.e.

$$\mathcal{M}^* := \mathrm{Hom}_{D_X}(\mathcal{M}, D_X \otimes \Omega).$$

Consider the D-module $\mathcal{B}_{bc}(\mathcal{M}) := \Omega \oplus \mathcal{M} \oplus \mathcal{M}^*$. We define a $\mathbb{Z}$-grading (and hence a $\mathbb{Z}_2$-grading) on $\mathcal{B}_{bc}(\mathcal{M})$ by declaring that Ω is homogeneous of degree 0 and $\mathcal{M}$ and $\mathcal{M}^*$ are of degrees -1 and 1 respectively. We introduce a (super) Lie-* algebra structure on $\mathcal{B}_{bc}(\mathcal{M})$ as follows:
Ω is central and the map $\mathcal{M} \boxtimes \mathcal{M}^* \to \Delta_!(\mathcal{B}_{bc}(\mathcal{M}))$ is the composition

$$\mathcal{M} \boxtimes \mathcal{M}^* \to D_X \otimes \Omega \simeq \Delta_!(\Omega) \hookrightarrow \Delta_!(\mathcal{B}_{bc}(\mathcal{M})).$$

We define $\mathcal{A}_{bc}(\mathcal{M})$ to be the super chiral algebra defined as a quotient of $\mathcal{U}(\mathcal{B}_{bc}(\mathcal{M}))$ by the standard relation $\mathbf{unit} - \widetilde{\mathbf{unit}} = 0$.

The chiral algebra $\mathcal{A}_{bc}(\mathcal{M})$ carries a $\mathbb{Z}$-grading $\mathcal{A}_{bc}(\mathcal{M}) = \oplus_i \mathcal{A}_{bc}(\mathcal{M})^i$ that comes from the $\mathbb{Z}$-grading on $\mathcal{B}_{bc}(\mathcal{M})$ and a filtration $\mathcal{A}_{bc}(\mathcal{M}) = \bigcup_{i \in \mathbb{Z}^+} \mathcal{A}_{bc}(\mathcal{M})_i$ that comes from the canonical filtration on $\mathcal{U}(\mathcal{B}_{bc}(\mathcal{M}))$ (cf. Section 1.2.4).

In this subsection we will compute the space of conformal blocks of $\mathcal{A}_{bc}(\mathcal{M})$ (assuming that X is complete).

Note that $DR^{-1}(X, \mathcal{M}) = 0$, since $\mathcal{M}$ is locally free as a D-module. Let $\det DR(\mathcal{M})$ denote the 1-dimensional vector space $\Lambda^{top}(DR^0(X, \mathcal{M})) \otimes (\Lambda^{top}(DR^1(X, \mathcal{M})))^{-1}$.

Proposition 3.3.1. *The space $H_\nabla(X, \mathcal{A}_{bc}(\mathcal{M}))$ is 1-dimensional. It is concentrated in the homogeneity degree $\dim(DR^0(X, \mathcal{M})) - \dim(DR^1(X, \mathcal{M}))$ and is canonically isomorphic to $(\det DR(\mathcal{M}))^{-1}$.*

Proof. Let x be a point of X. We shall compute the space $H_\nabla(X, \mathcal{A}_{bc}(\mathcal{M}))$ using Proposition 1.3.2, which implies that

$$H_\nabla(X, \mathcal{A}_{bc}(\mathcal{M})) = \mathrm{coker}(DR^0(X \setminus x, \mathcal{M} \oplus \mathcal{M}^*) \otimes A_{bc}(\mathcal{M})_x) \to A_{bc}(\mathcal{M})_x).$$

The space $DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{M})$ is a topological vector space of Tate type (cf. [**3**]) and we have a canonical isomorphism

$$DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x, \mathcal{M}^*)) \simeq (DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{M})))^*.$$

Therefore, the direct sum $V := DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{M} \oplus \mathcal{M}^*)$ acquires a non-degenerate quadratic form such that $V_1 := DR^0(\mathrm{Spec}(\hat{\mathcal{O}}_x), \mathcal{M} \oplus \mathcal{M}^*)$ (resp., $V_2 := DR^0(X \setminus x, \mathcal{M} \oplus \mathcal{M}^*)$) is a Lagrangian compact (resp., co-compact) subspace of V.

We have the following general statement:

Lemma 3.3.1. *Let V be a Tate topological vector space endowed with a symmetric nondegenerate pairing $Q : V \otimes V \to \mathbb{C}$, let V_1 be a Lagrangian compact subspace of V and let S be the corresponding representation of the Clifford algebra. Let $V_2 \subset V$ be a Lagrangian co-compact subspace of V. We have a canonical isomorphism*

$$\mathrm{coker}(V_2 \otimes S \to S) \simeq \Lambda^{top}(V/V_1 + V_2).$$

This implies the existence of an isomorphism as in the statement of the proposition using Verdier duality:

$$DR^1(X,\mathcal{M}^*) \simeq (DR^0(X,\mathcal{M}))^*.$$

The fact that the isomorphism

$$H_\nabla(X,\mathcal{A}_{bc}(\mathcal{M})) \simeq (\det DR(\mathcal{M}))^{-1}$$

does not depend on the choice of $x \in X$ is easy to show by generalizing the above argument to the case of several points $x_1,\dots,x_n \in X$ and by establishing the appropriate compatibility when one of them is deleted.

□

Lemma 3.3.1 above implies also the following statement:

Let $x_1,\dots,x_{k_1},y_1,\dots,y_{k_2}$ be distinct points of X and consider the composition:

$$\overset{k_1}{\underset{i=1}{\otimes}} M_{x_i} \overset{k_2}{\underset{j=1}{\otimes}} M_{y_j} \to \overset{k_1}{\underset{i=1}{\otimes}} A_{bc}(\mathcal{M})_{x_i} \overset{k_2}{\underset{j=1}{\otimes}} A_{bc}(\mathcal{M}^*)_{y_j} \to H_\nabla(X,\mathcal{A}_{bc}(\mathcal{M})) \simeq (\det DR(\mathcal{M}))^{-1}.$$

(Here M_x denotes the fiber of $\mathcal{M}$ at $x \in X$ and similarly for $\mathcal{M}^*$.)

Corollary 3.3.1. *The above map* $M_{x_1} \otimes \dots \otimes M_{x_{k_1}} \otimes M^*_{x_1} \otimes \dots \otimes M^*_{x_{k_2}} \to (\det DR(\mathcal{M}))^{-1}$ *is zero unless* $k_1 = \dim(DR^1(X,\mathcal{M}))$ *and* $k_2 = \dim(DR^0(X,\mathcal{M})) = \dim(DR^0(X,\mathcal{M}^*))$ *and in the latter case it coincides with the composition*

$$\overset{k_1}{\underset{i=1}{\otimes}} M_{x_i} \overset{k_2}{\underset{j=1}{\otimes}} M_{y_j} \to DR^1(X,\mathcal{M})^{\otimes k_1} \otimes DR^1(X,\mathcal{M}^*)^{\otimes k_2} \to$$
$$\Lambda^{k_1}(DR^1(X,\mathcal{M})) \otimes \Lambda^{k_2}(DR^1(X,\mathcal{M}^*)) \simeq (\det DR(\mathcal{M}))^{-1}$$

§3.3.2.

Let $\mathcal{A}_{bc}(n)$ denote the chiral algebra $\mathcal{A}_{bc}(\mathcal{M})$ for $\mathcal{M} = \Omega^n \otimes D$. In this subsection we shall show that it defines a CFT with central charge equal to $c_n := -2(6n^2-6n+1)$. In the physics literature this CFT is often referred to as the *bc*-system.

According to Lemma 2.1.1 $\mathcal{A}_{bc}(n)$ carries a structure of local $\mathcal{O}$-module on $\mathfrak{X}$ that satisfies condition (a) of Section 2.3.2. Therefore, to define the corresponding CFT we must construct the energy-momentum tensor $T_{bc,n} : \Theta'_{c_n} \to \mathcal{A}_{bc}(n)$ that satisfies condition (b) of Section 2.3.2.

Proposition 3.3.2. *There exists a unique map* $\widetilde{T}_{bc,n} : \Theta \to \mathcal{A}_{bc}(n)/\Omega$ *such that for any section* $l \in \mathcal{A}_{bc}(n)$, *any vector field* ξ *and a section* ξ' *of* $\mathcal{A}_{bc}(n)$ *that projects to* $\widetilde{T}_{bc,n}(\xi)$ *we have:*

$$(h \boxtimes \mathrm{id})(\{\xi', l\}) = -l \cdot \xi - \mathrm{Lie}_\xi(l).$$

Proof. It is easy to see that Ω is the unique $\mathcal{O}$-sub-module of $\mathcal{A}_{bc}(n)$ which annihilates the Lie-* bracket. This implies the uniqueness part in the assertion of Proposition 3.3.2.

Therefore, the existence part is local; let $\mathfrak{z}$ be a coordinate over some $U \subset X$.

For $f(x,y) \in \mathcal{O} \boxtimes \mathcal{O}(\Delta)$, let ξ_f denote the corresponding vector field and set $\widetilde{T}_{bc,n}(\xi_f)$ to be equal to the image of

$$\xi'_f := -(h \boxtimes \mathrm{id})(n\{d\mathfrak{z}^{\otimes n} \cdot \partial_{\mathfrak{z}} \boxtimes d\mathfrak{z}^{\otimes 1-n} \cdot f(x,y)\} - (n-1)\{d\mathfrak{z}^{\otimes n} \boxtimes d\mathfrak{z}^{\otimes 1-n} \cdot \partial_{\mathfrak{z}} \cdot f(x,y)\})$$

under the projection $\mathcal{A}_{bc}(n) \to \mathcal{A}_{bc}(n)/\Omega$.

It is easy to see that the above expression is well-defined (i.e. does not depend on the choice of $f(x,y)$ that gives rise to ξ_f) and that, moreover, it equals the projection to $\mathcal{A}_{bc}(n)/\Omega$ of

$$-(\mathrm{id}\boxtimes h)(n\{d\mathfrak{z}^{\otimes n}\cdot\partial_{\mathfrak{z}}\boxtimes d\mathfrak{z}^{\otimes 1-n}\cdot f(x,y)\}-(n-1)\{d\mathfrak{z}^{\otimes n}\boxtimes d\mathfrak{z}^{\otimes 1-n}\cdot\partial_{\mathfrak{z}}\cdot f(x,y)\}).$$

Let us check that the map $\widetilde{T}_{bc,n}$ constructed above satisfies the condition of the proposition. It is enough to check that for ξ_f and ξ'_f as above

$$(h\boxtimes\mathrm{id})(\{\xi'_f,l\})=-l\cdot\xi_f-\mathrm{Lie}_{\xi_f}(l)$$

for l being a nonvanishing section of either Ω^n or Ω^{1-n}. As the situation is symmetric in n and $1-n$ we shall treat the case of $l=d\mathfrak{z}^{\otimes n}\in\Omega^n\subset\mathcal{A}_{bc}(n)$.

By applying the Jacobi identity we obtain:

$$(h\boxtimes\mathrm{id})(\{\xi'_f,l\})=-n(h\boxtimes h\boxtimes\mathrm{id})(\{d\mathfrak{z}^{\otimes n}\cdot\partial_{\mathfrak{z}}\boxtimes\{d\mathfrak{z}^{\otimes 1-n}\boxtimes d\mathfrak{z}^{\otimes n}\cdot f(x,y)\}\})+$$
$$(n-1)(h\boxtimes h\boxtimes\mathrm{id})(\{d\mathfrak{z}^{\otimes n}\boxtimes\{d\mathfrak{z}^{\otimes 1-n}\cdot\partial_{\mathfrak{z}}\boxtimes d\mathfrak{z}^{\otimes n}\cdot f(y,x)\}\}).$$

An easy computation shows that the first term in the above expression equals $-ndz^{\otimes n}\cdot\xi_f-\mathrm{Lie}_{\xi_f}(dz^{\otimes n})$. The second term equals $(n-1)dz^{\otimes n}\cdot\xi_f$, according to Lemma 3.1.3. This implies the assertion of the proposition.

□

The uniqueness assertion in Proposition 3.3.2 guarantees that the map $\widetilde{T}_{bc,n}:\Theta\to\mathcal{A}_{bc}(n)/\Omega$ is compatible with the $\mathcal{O}$-module structure. We define the local $\mathcal{O}$-module $\Theta'(n)$ to be the extension

$$0\to\Omega\to\Theta'(n)\to\Theta\to 0$$

induced from the extension $0\to\Omega\to\mathcal{A}_{bc}\to\mathcal{A}_{bc}(n)/\Omega\to 0$ by means of the map $\widetilde{T}_{bc,n}$. By definition, we have the energy-momentum tensor map

$$T_{bc,n}:\Theta'(n)\to\mathcal{A}_{bc}(n).$$

It can be shown by a direct computation (as in Proposition 3.2.2), that $\Theta'(n)\simeq\Theta'_{c_n}$. In the next subsection we shall show a way to establish such an isomorphism without making any computations: we shall deduce it from the so-called Bose-Fermi correspondence.

Corollary 3.3.2. *The line bundle on $\mathfrak{M}$ whose fiber at $X\in\mathfrak{M}$ is $\det(R\Gamma(X,\Omega^n))$ carries a canonical structure of a $\Theta'_{\mathfrak{M}_{-c_n}}$-module.*

Proof. This assertion is simply a combination of Proposition 3.3.1 and Proposition 2.3.3.

□

Remark 3.3.1. Take $n=-1$ (we have $c_{-1}=-26$) and consider the union of the connected components of $\mathfrak{M}$ that correspond to curves of genus $g\geq 2$. In this case the line bundle $X\to\det(R\Gamma(X,\Omega^{-1}))$ is identified with $\Omega^{top}_{\mathfrak{M}}$. Therefore, the ring of twisted differential operators on $\mathfrak{M}$ corresponding to the Picard Lie algebroid $\Theta'_{\mathfrak{M}_{26}}$ is identified in this case with $D^{op}_{\mathfrak{M}}$ in such a way that $(H_\nabla(\mathfrak{X},\mathcal{A}_{bc}(-1)))^*$ becomes the canonical right D-module over $\mathfrak{M}$.

§3.3.3.

Consider the local $\mathcal{O}$-module $p_{2*}(\Omega^n \boxtimes \Omega^{1-n}(\Delta)/\Omega^n \boxtimes \Omega^{1-n}(-\Delta)$. We have a short exact sequence:

$$0 \to \Omega \to p_{2*}(\Omega^n \boxtimes \Omega^{1-n}(\Delta)/\Omega^n \boxtimes \Omega^{1-n}(-\Delta) \to \mathcal{O} \to 0$$

and it is easy to show that we have in fact a canonical isomorphism

$$p_{2*}(\Omega^n \boxtimes \Omega^{1-n}(\Delta)/\Omega^n \boxtimes \Omega^{1-n}(-\Delta) \simeq F_{2n-1},$$

(cf. Section 3.2.1).

We define a map $\widetilde{J} : F_{2n-1} \to \mathcal{A}_{bc}(n)$ by setting

$$\widetilde{J}(\omega^n \boxtimes \omega^{1-n} \cdot f(x,y)) = (h \boxtimes \mathrm{id})(\{\omega^n \boxtimes \omega^{1-n} \cdot f(x,y)\}).$$

Clearly, $\widetilde{J}$ is a map of local $\mathcal{O}$-modules. It is commonly referred to as the "fermion number current" due to the following property:

Proposition 3.3.3. *Let $\mathfrak{f}_1$ be a section of F_{2n-1} that maps to 1 under the projection $F_{2n-1} \to \mathcal{O}$ and set $\mathfrak{j}_1 := \widetilde{J}(\mathfrak{f}_1)$. We have*

$$(h \boxtimes \mathrm{id})(\{\mathfrak{j}_1, l\}) = i \cdot l \text{ for } l \in \mathcal{A}_{bc}(n)^i.$$

Proof. It enough to check the assertion on the generators, i.e. for $l \in \Omega^n$ and for $l \in \Omega^{1-n}$, in which case it follows immediately from the Jacobi identity. □

The following result is a part of the twisted Bose-Fermi correspondence:

Theorem 3.3.1. (a) *The map $\widetilde{J}$ gives rise to a map of chiral algebras*

$$J : \mathcal{A}(2n-1,-1) \to \mathcal{A}_{bc}(n).$$

The map J is compatible with the local $\mathcal{O}$-module structure.
(b) *We have a canonical isomorphism $\Theta'(n) \simeq \Theta'_{c_n}$ and the composition*

$$\Theta'_{c_n} \overset{T_{2n-1,-1}}{\longrightarrow} \mathcal{A}(2n-1,-1) \overset{J}{\longrightarrow} \mathcal{A}_{bc}(n)$$

coincides under this identification with the map $T_{bc,n}$.[3]

Proof. To prove point (a) of the theorem it is enough to show that if $\mathfrak{f}_g \in F_{2n-2}$ is a section that maps to $g \in \mathcal{O}$ under the projection $F_{2n-1} \to \mathcal{O}$, then

$$(h \boxtimes \mathrm{id})(\{J(\mathfrak{f}_g), \mathfrak{j}_1\}) = -dg.$$

Set $\mathfrak{j}_g = J(\mathfrak{f}_g)$. As in Proposition 3.3.3 we have:

$$(h \boxtimes \mathrm{id})(\{\mathfrak{j}_g, l\}) = g \cdot l \text{ for } l \in \Omega^{1-n} \text{ and } (h \boxtimes \mathrm{id})(\{\mathfrak{j}_g, l\}) = -g \cdot l \text{ for } l \in \Omega^n.$$

The assertion of point (a) now follows from the Jacobi identity.

In order to prove point (b) observe that it is enough to show that the map

$$J \circ \widetilde{T}_{2n-1,-1} : \Theta \to \mathcal{A}(2n-1,-1)/\Omega \to \mathcal{A}_{bc}(n)/\Omega$$

coincides with the map $\widetilde{T}_{bc,n}$. This will be done in two steps:

Step 1: We first show that $J \circ \widetilde{T}_{2n-1,-1}$ is proportional to $\widetilde{T}_{bc,n}$.

Indeed, $\widetilde{T}_{bc,n}$ is obviously a map $\Theta \to \mathcal{A}_{bc}(n)_2 \cap \mathcal{A}_{bc}(n)^0$. We claim that the same is true for $J \circ \widetilde{T}_{2n-1,-1}$. A priori, the latter is a map $\Theta \to \mathcal{A}_{bc}(n)_4 \cap \mathcal{A}_{bc}(n)^0$.

[3] Of course, $c_n := -2(6n^2 - 6n + 1) = 1 - 3(2n-1)^2$.

However, it is easy to see that its symbol, i.e. the composition

$$\Theta \to \mathcal{A}_{bc}(n)_4 \cap \mathcal{A}_{bc}(n)^0/\mathcal{A}_{bc}(n)_3 \cap \mathcal{A}_{bc}(n)^0$$

vanishes. Therefore, the image of $J \circ \widetilde{T}_{2n-1,-1}$ lies in $\Theta \to \mathcal{A}_{bc}(n)_3 \cap \mathcal{A}_{bc}(n)^0$. However, by parity considerations, $\mathcal{A}_{bc}(n)_3 \cap \mathcal{A}_{bc}(n)^0 = \mathcal{A}_{bc}(n)_2 \cap \mathcal{A}_{bc}(n)^0$.

Observe now that as an $\mathcal{O}$-module, $\mathcal{A}_{bc}(n)_2 \cap \mathcal{A}_{bc}(n)^0$ is identified with

$$(\Omega^n \otimes D) \overset{!}{\otimes} (\Omega^{1-n} \otimes D),$$

according to Lemma 1.2.1(c).

The assertion of Step 1 now follows from the next lemma:

Lemma 3.3.2. *The space* $\mathrm{Hom}(\Theta, (\Omega^n \otimes D) \overset{!}{\otimes} (\Omega^{1-n} \otimes D))$ *in the category of local* $\mathcal{O}$*-modules is* 1*-dimensional.*

Step 2:

We know by now that $J \circ \widetilde{T}_{2n-1,-1} = \epsilon \cdot \widetilde{T}_{bc,n}$ for some $\epsilon \in \mathbb{C}$ and it remains to show that $\epsilon = 1$.

For $\xi \in \Theta$ let ξ' be any section of $\mathcal{A}_{bc}(n)$ that projects to $T_{bc,n}(\xi)$. Since $T_{bc,n}$ is an energy-momentum tensor for $\mathcal{A}_{bc}(n)$, on the one hand we have

$$(h \boxtimes \mathrm{id})(\{\epsilon \cdot \xi' \boxtimes J(l)\}) = \epsilon(-J(l) \cdot \xi - J(\mathrm{Lie}_\xi(l))) \ \forall l \in \mathcal{A}_{bc}(n).$$

On the other hand, since J is a homomorphism of chiral algebras and $T_{2n-1,-1}$ is an energy-momentum tensor for $\mathcal{A}(2n-1,-1)$, we have

$$(h \boxtimes \mathrm{id})(\{\epsilon \cdot \xi' \boxtimes J(l)\}) = -J(l) \cdot \xi - J(\mathrm{Lie}_\xi(l)) \ \forall l \in \mathcal{A}(2n-1,-1).$$

The two equations are compatible only when $\epsilon = 1$.

□

CHAPTER 4
BRST and String Amplitudes

§4.1. The BRST complex

§4.1.1.

Let $\mathcal{B}$ be a Lie-* algebra over a curve X, which is locally free and finitely generated as a D-module. We shall now attach to $\mathcal{B}$ a canonical central extension

$$0 \to \Omega \to \mathcal{B}'' \to \mathcal{B} \to 0$$

of Lie-* algebras.

The construction will be based on the following general assertion:

Lemma 4.1.1. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two D-modules on X with $\mathcal{M}_1$ being locally free and finitely generated. Then:*

(a) *For a third D-module $\mathcal{M}$ on X, there is a canonical isomorphism:*

$$\mathrm{Hom}_{D(X^2)}(\mathcal{M} \boxtimes \mathcal{M}_1, \Delta_!(\mathcal{M}_2)) \simeq \mathrm{Hom}_D(\mathcal{M}, \mathcal{M}_1^* \overset{!}{\otimes} \mathcal{M}_2).$$

(b) *The canonical map (from point (a)) $\mathcal{M}_1^* \overset{!}{\otimes} \mathcal{M}_2 \boxtimes \mathcal{M}_1 \to \Delta_!(\mathcal{M}_2)$ induces an isomorphism $h(\mathcal{M}_1^* \overset{!}{\otimes} \mathcal{M}_2) \to \mathrm{Hom}_D(\mathcal{M}_1, \mathcal{M}_2)$.*

Corollary 4.1.1. *For a Lie-* algebra $\mathcal{B}$ we have canonical maps*

$$\text{co-ad} : \mathcal{B} \boxtimes \mathcal{B}^* \to \Delta_!(\mathcal{B}^*) \text{ and } \text{co-br} : \mathcal{B}^* \to \mathcal{B}^* \overset{!}{\otimes} \mathcal{B}^*.$$

We shall call the two maps of the above corollary "the co-adjoint action" and "the co-bracket", respectively.

Let us for a moment view $\mathcal{B}$ as a plain D-module on X (i.e. forget the Lie-* algebra structure) and consider the chiral algebra $\mathcal{A}_{bc}(\mathcal{B})$ of Section 3.3.1. Let $i_{\mathcal{B}}$ (resp., $i_{\mathcal{B}}^*$) denote the canonical embedding of $\mathcal{B}$ (resp., of $\mathcal{B}^*$) into $\mathcal{A}_{bc}(\mathcal{B})$.

It follows from Lemma 1.2.1(c) that the intersection $\mathcal{A}_{bc}(\mathcal{B})_2 \cap \mathcal{A}_{bc}(\mathcal{B})^0$ is an extension of D-modules

$$0 \to \Omega \to \mathcal{A}_{bc}(\mathcal{B})_2 \cap \mathcal{A}_{bc}(\mathcal{B})^0 \to \mathcal{B} \overset{!}{\otimes} \mathcal{B}^* \to 0.$$

According to Lemma 4.1.1 above, the Lie-* bracket on $\mathcal{B}$ gives rise to a map

$$S_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^* \overset{!}{\otimes} \mathcal{B} \hookrightarrow \mathcal{A}_{bc}(\mathcal{B})/\Omega.$$

Proposition 4.1.1. *For two local sections b_1 and b_2 of $\mathcal{B}$ let $S_{\mathcal{B}}(b_1)'$ and $S_{\mathcal{B}}(b_2)'$ be local sections of $\mathcal{A}_{bc}(\mathcal{B})$ that project to $S_{\mathcal{B}}(b_1)$ and $S_{\mathcal{B}}(b_2)$, respectively. Then the section $\{S_{\mathcal{B}}(b_1)' \boxtimes S_{\mathcal{B}}(b_2)'\}$ of $\Delta_!(\mathcal{A}_{bc}(\mathcal{B}))$ projects to the section $S_{\mathcal{B}}(\{b_1, b_2\})$ of $\Delta_!(\mathcal{A}_{bc}(\mathcal{B})/\Omega)$.*

Proof. It is enough to prove that

$$(h \boxtimes h)(\{S_{\mathcal{B}}(b_1)' \boxtimes S_{\mathcal{B}}(b_2)'\}) = h(S_{\mathcal{B}}(\{b_1, b_2\})) \in h(\mathcal{A}_{bc}(\mathcal{B})/\Omega)$$

for any b_1 and b_2 as above.

For a section l of $h(\mathcal{B}^* \overset{!}{\otimes} \mathcal{B})$ and for some lift of l to a section l' of $h(\mathcal{A}_{bc}(\mathcal{B})_2 \cap \mathcal{A}_{bc}(\mathcal{B})^0)$ consider the map $\mathcal{B} \to \mathcal{A}_{bc}(\mathcal{B})$ given by

$$b \to (h \boxtimes \mathrm{id})(\{l' \boxtimes i_{\mathcal{B}}(b)\}).$$

It follows from the definition of the chiral bracket on $\mathcal{A}_{bc}(\mathcal{B})$ that this map takes values in $\mathcal{B} \subset \mathcal{A}_{bc}(\mathcal{B})$ and coincides with the canonical map $h(\mathcal{B}^* \overset{!}{\otimes} \mathcal{B}) \otimes \mathcal{B} \to \mathcal{B}$ of point (b) of Lemma 4.1.1.

The assertion of the proposition now follows from the Jacobi identity combined with Lemma 4.1.1(b), since both $(h \boxtimes h)(\{S_{\mathcal{B}}(b_1)' \boxtimes S_{\mathcal{B}}(b_2)'\})$ and $h(S_{\mathcal{B}}(\{b_1, b_2\}))$ belong to $h(\mathcal{B}^* \overset{!}{\otimes} \mathcal{B}) \subset h(\mathcal{A}_{bc}(\mathcal{B})/\Omega)$.

□

We now define $\mathcal{B}''$ to be the induced extension:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega & \longrightarrow & \mathcal{B}'' & \longrightarrow & \mathcal{B} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle S''_{\mathcal{B}}} & & \downarrow{\scriptstyle S_{\mathcal{B}}} & & \downarrow \\
0 & \longrightarrow & \Omega & \longrightarrow & \mathcal{A}_{bc}(\mathcal{B})_2 \cap \mathcal{A}_{bc}(\mathcal{B})^0 & \longrightarrow & \mathcal{B}^* \overset{!}{\otimes} \mathcal{B} & \longrightarrow & 0.
\end{array}$$

Proposition 4.1.1 above implies that $\mathcal{B}''$ acquires a canonical Lie-* algebra structure which is a central extension of that of $\mathcal{B}$.

§4.1.2.

Let $\mathcal{B}'$ denote the extension $0 \to \Omega \to \mathcal{B}' \to \mathcal{B} \to 0$ obtained from $\mathcal{B}''$ by acting by -1 on Ω. Let $\mathcal{U}(\mathcal{B})'$ denote the quotient of $\mathcal{U}(\mathcal{B}')$ by the standard relation $\mathbf{unit} - \widetilde{\mathbf{unit}} = 0$ and consider now the tensor product chiral algebra $\mathcal{A}_{BRST}(\mathcal{B}) := \mathcal{U}(\mathcal{B})' \overset{!}{\otimes} \mathcal{A}_{bc}(\mathcal{B})$.[1] The chiral algebra $\mathcal{A}_{BRST}(\mathcal{B})$ inherits from $\mathcal{A}_{bc}(\mathcal{B})$ a $\mathbb{Z}$– and a $\mathbb{Z}_2$–grading.

From the construction of $\mathcal{B}'$ it follows that we have a canonical map of Lie-* algebras $S_{\mathcal{B},BRST} : \mathcal{B} \to \mathcal{A}_{BRST}(\mathcal{B})$.

Our next goal will be to construct a canonical element $\delta_{\mathcal{B}} \in h(\mathcal{A}_{BRST}(\mathcal{B}))$ with the following properties:

---

[1] It follows from the definitions that, for two chiral algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, the tensor product $\mathcal{A}_1 \overset{!}{\otimes} \mathcal{A}_2$ acquires a natural chiral algebra structure.

- $\delta_{\mathcal{B}}$ has degree 1 with respect to the $\mathbb{Z}$-grading on $\mathcal{A}_{BRST}(\mathcal{B})$ and
$$(h \boxtimes h)(\{\delta_{\mathcal{B}}, \delta_{\mathcal{B}}\}) = 0.$$
- $(h \boxtimes \mathrm{id})(\{\delta_{\mathcal{B}}, S_{\mathcal{B},BRST}(b)\}) = 0$ for any $b \in \mathcal{B}$.
- $(h \boxtimes \mathrm{id})(\{\delta_{\mathcal{B}}, i_{\mathcal{B}}(b)\}) = -S_{\mathcal{B},BRST}(b)$ for any $b \in \mathcal{B}$.

In particular, for any chiral $\mathcal{B}'$-module $\mathcal{L}$, the map $(h \boxtimes \mathrm{id})(\{\delta_{\mathcal{B}}, \cdot\})$ would define a differential on the tensor product $\mathcal{L} \overset{!}{\otimes} \mathcal{A}_{bc}(\mathcal{B})$, which is homogeneous of degree 1 and whose square equals 0.

We have two maps $G_1, G_2 : j_*j^*(\mathcal{B} \boxtimes \mathcal{B}^*) \to \Delta_!(\mathcal{A}_{BRST}(\mathcal{B}))$:
The map G_1 is defined as minus the composition

$$j_*j^*(\mathcal{B} \boxtimes \mathcal{B}^*) \overset{S_{\mathcal{B},BRST} \boxtimes i^*_{\mathcal{B}}}{\longrightarrow} j_*j^*(\mathcal{A}_{BRST}(\mathcal{B}) \boxtimes \mathcal{A}_{BRST}(\mathcal{B})) \overset{\{,\}}{\longrightarrow} \Delta_!(\mathcal{A}_{BRST}(\mathcal{B})).$$

To define the map G_2 note that the chiral bracket on $\mathcal{A}_{bc}(\mathcal{B})$ gives rise to a map $\mathcal{B}^* \overset{!}{\otimes} \mathcal{B}^* \to \mathcal{A}_{bc}(\mathcal{B})$ and let $i^*_{\mathcal{B},2}$ denote the composition $\mathcal{B}^* \overset{\text{co-br}}{\longrightarrow} \mathcal{B}^* \overset{!}{\otimes} \mathcal{B}^* \to \mathcal{A}_{bc}(\mathcal{B})$. We set the map G_2 to be the composition:

$$j_*j^*(\mathcal{B} \boxtimes \mathcal{B}^*) \overset{i_{\mathcal{B}} \boxtimes i^*_{\mathcal{B},2}}{\longrightarrow} j_*j^*(\mathcal{A}_{bc}(\mathcal{B}) \boxtimes \mathcal{A}_{bc}(\mathcal{B})) \overset{\{,\}}{\longrightarrow} \Delta_!(\mathcal{A}_{bc}(\mathcal{B})) \to \Delta_!(\mathcal{A}_{BRST}(\mathcal{B})).$$

Lemma 4.1.2. *The restriction of G_1 (resp., of G_2) to $\mathcal{B} \boxtimes \mathcal{B}^* \subset j_*j^*(\mathcal{B} \boxtimes \mathcal{B}^*)$ coincides with the map*

$$\mathcal{B} \boxtimes \mathcal{B}^* \overset{\text{co-ad}}{\longrightarrow} \Delta_!(\mathcal{B}^*) \overset{i^*_{\mathcal{B}}}{\longrightarrow} \Delta_!(\mathcal{A}_{BRST}(\mathcal{B})).$$

In particular, the difference $G_1 - G_2$ is a well-defined map

$$\mathcal{B} \overset{!}{\otimes} \mathcal{B}^* \to \mathcal{A}_{BRST}(\mathcal{B}).$$

We set $\delta_{\mathcal{B}}$ to be the image under $G_1 - G_2$ of the canonical element $\mathrm{id}_{\mathcal{B}} \in \mathrm{End}_D(\mathcal{B}) \simeq h(\mathcal{B} \overset{!}{\otimes} \mathcal{B}^*)$

Proposition 4.1.2. *The element δ_{BRST} satisfies conditions (1)–(3) above.*

Proof. Property (3) is a straightforward calculation.

To prove property (2), it is enough to show that

$$(h \boxtimes h)(\{\delta_{BRST}, S_{\mathcal{B},BRST}(b)\}) = 0 \in h(\mathcal{A}_{BRST}(\mathcal{B}))$$

for any $b \in \mathcal{B}$.

Consider the action of $h(\mathcal{B})$ on the D-module $\mathcal{B} \boxtimes \mathcal{B}^*$ given by the formula:

$$b' \cdot (b \boxtimes b^*) \to (h \boxtimes \mathrm{id})(\{b', b\}) \boxtimes b^* + b \boxtimes (h \boxtimes \mathrm{id})(\text{co-ad}(b' \boxtimes b^*)).$$

This action extends to an action of $h(\mathcal{B})$ on $j_*j^*(\mathcal{B} \boxtimes \mathcal{B}^*)$.

Consider also the action of $\mathcal{B}$ on $\mathcal{A}_{BRST}(\mathcal{B})$ given by

$$b \cdot l \to (h \boxtimes \mathrm{id})(\{S_{\mathcal{B},BRST}(b), l\}).$$

Lemma 4.1.3. *The maps G_1 and G_2 commute with the above $h(\mathcal{B})$–actions.*

Property (2) now follows from the fact that the section $\mathrm{id}_{\mathcal{B}} \in h(\mathcal{B} \overset{!}{\otimes} \mathcal{B}^*)$ is $h(\mathcal{B})$–invariant.

To establish property (1) note that the element δ^2_{BRST} satisfies: $\{\delta^2_{BRST}, i_{\mathcal{B}}(b)\} = 0$ for all $b \in \mathcal{B}$. The assertion now follows from the next lemma:

Lemma 4.1.4. *Let $\mathcal{M}$ be as in Section 3.3.1 and let $l \in h(\mathcal{A}_{bc}(\mathcal{M}))$ be an element of positive degree. Assume that $(h \boxtimes \mathrm{id})(\{l, i_{\mathcal{M}}(m)\}) = 0 \ \forall m \in h(\mathcal{M})$. Then $l = 0$.*

□

§4.2. String amplitudes

§4.2.1.

Let us now apply the discussion of the previous subsection to the case $\mathcal{B} = \mathcal{T} = \Theta \otimes D_X$.

In this case the chiral algebra $\mathcal{A}_{bc}(\mathcal{B})$ is identified with the bc-system chiral algebra $\mathcal{A}_{bc}(n)$ for $n = -1$.

Lemma 4.2.1. *The canonical central extension $\mathcal{B}''$ of Section 4.1.1 is identified for $\mathcal{B} = \mathcal{T}$ with $\mathcal{T}'_{-26}$. Moreover, under this identification, the map $S''_{\mathcal{B}}$ goes over to the energy-momentum tensor $T_{bc,-1}$.*

This assertion follows immediately from the definitions (using the equality $c_{-1} := -2(6 \cdot (-1)^2 - 6 \cdot (-1) + 1 = -26)$.

Now let $(\mathcal{A}, T)$ be a CFT data of central charge 26. Consider the chiral algebra $\mathcal{A}_{comp} := \mathcal{A} \overset{!}{\otimes} \mathcal{A}_{bc}(-1)$. The energy momentum tensor for $\mathcal{A}$ gives rise to a homomorphism of chiral algebras:

$$\varphi : \mathcal{A}_{BRST}(\mathcal{T}) \to \mathcal{A}_{comp},$$

in particular, by composing it with the map $S_{\mathcal{T},BRST}$ we obtain a map $T_{comp} : \Theta \to \mathcal{T} \to \mathcal{A}_{comp}$.

Proposition 4.2.1. *The pair $\mathcal{A}_{comp}, T_{comp}$ defines a CFT data of central charge 0.*

Proof. Since both $\mathcal{A}$ and $\mathcal{A}_{bc}(-1)$ carry a structure of local $\mathcal{O}$-module over $\mathfrak{X}$, so does $\mathcal{A}_{comp}$.

The fact that the pair $(\mathcal{A}_{comp}, T_{comp})$ satisfies conditions (a) and (b) of Section 2.2.1 follows from the corresponding facts for $\mathcal{A}$ and for $\mathcal{A}_{bc}(-1)$.

□

§4.2.2.

Let $\delta_{comp} \in h(\mathcal{A}_{comp})$ denote the image of $\delta_{BRST} \in h(\mathcal{A}_{BRST}(\Theta))$ under the homomorphism φ.

Note that the $\mathcal{O}_{\mathfrak{M}}$-module $X \to H_{\nabla}(X, \mathcal{A}_{comp})$ carries a canonical structure of left $D_{\mathfrak{M}}$-module, according to Proposition 2.2.1. The next proposition describes the connection between the differential δ_{comp} and the De Rham differential on $H_{\nabla}(\mathfrak{X}, \mathcal{A}_{comp})$:

Proposition 4.2.2. *Let $f \cdot a_1 \boxtimes \ldots \boxtimes a_k$ be a $\mathbb{Z}_2$-homogeneous section of $j_* j^*({}^l\mathcal{A}_{comp}^{\boxtimes k})$ on $\mathfrak{X}^k \setminus \Delta$. Then*[2]

$$\nabla(\langle f \cdot a_1, \ldots, a_k \rangle) = (\langle \ldots \rangle \boxtimes \mathrm{id})(\nabla_{loc}(f \cdot a_1, \ldots, a_k)) +$$
$$+ \sum_{i=1}^{k} (-1)^{\deg(a_i)+\ldots+\deg(a_k)} \langle f \cdot a_1, \ldots, \delta_{BRST}(a_i), \ldots, a_k, {}^l i_{\mathcal{T}} \rangle.$$

Proof. Using Proposition 2.2.2, the assertion of the proposition reduces to the fact that

$$\sum_{i=1}^{k} (-1)^{\deg(a_i)+\ldots+\deg(a_k)} \langle f \cdot a_1, \ldots, \delta_{BRST}(a_i), \ldots, a_k, {}^l i_{\mathcal{T}} \rangle = \langle f \cdot a_1, \ldots, a_k, {}^l T \rangle.$$

This, however, follows from the equality

$$\sum_{i=1}^{k} (-1)^{\deg(a_1)+\ldots+\deg(a_{i-1})} \langle f \cdot a_1, \ldots, \delta_{BRST}(a_i), \ldots, a_k, {}^l i_{\mathcal{T}} \rangle +$$
$$+ (-1)^{\deg(a_1)+\ldots+\deg(a_k)} \langle f \cdot a_1, \ldots, a_k, \delta_{BRST}({}^l i_{\mathcal{T}}) \rangle = 0$$

and property (3) of the element δ_{BRST} (cf. Section 4.1.2).

□

Since $c = 26$, the space of conformal blocks $H_\nabla(\mathfrak{X}, \mathcal{A})$ carries a canonical right D-module structure over $\mathfrak{M}$ (cf. Remark 3.3.1). Moreover, over the connected components of $\mathfrak{M}$ corresponding to curves of genus $g \geq 2$, we have an isomorphism

$$H_\nabla(X, \mathcal{A}) \simeq H_\nabla(X, \mathcal{A}_{comp}) \otimes \Omega^{top}_{\mathfrak{M}},$$

in view of the following lemma:

Lemma 4.2.2. *For two chiral algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ and a complete curve X we have a canonical isomorphism*

$$H_\nabla(X, \mathcal{A}_1 \overset{!}{\otimes} \mathcal{A}_2) \simeq H_\nabla(X, \mathcal{A}_1) \otimes H_\nabla(X, \mathcal{A}_2).$$

Therefore, Proposition 4.2.2 expresses the right D-module structure on $H_\nabla(\mathfrak{X}, \mathcal{A})$ in terms of the fields of the composite theory (i.e. of $\mathcal{A}_{comp}$).

---

[2]Here ${}^l i_{\mathcal{T}}$ denotes the map $\Omega^{-2} \to {}^l\mathcal{A}_{comp}$ obtained from the canonical map $i_{\mathcal{T}} : \Theta \to \mathcal{A}_{comp}$ and its insertion into the correlation functions has the same meaning as in Proposition 2.2.2.

CHAPTER 5
Further Constructions

§5.1. Chiral algebras via the Ran space

The definition of chiral algebras that we used above was a rather straightforward axiomatization of the OPE operation on quantum fields. We shall now discuss a different approach to chiral algebras, which is much less obvious from the point of view of QFT.

In what follows we shall use the following conventions:

For a curve X and a surjection of finite sets $\phi : \mathcal{I} \twoheadrightarrow \mathcal{J}$, Δ_ϕ will denote the corresponding diagonal embedding $X^{\mathcal{J}} \to X^{\mathcal{I}}$.

If $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$ is a decomposition of I into a disjoint union of finite subsets we shall denote by $j_{\mathcal{I}_1,\ldots,\mathcal{I}_k}$ the embedding of the open subset $X^{\mathcal{I}_i \neq \mathcal{I}_j}$ of $X^{\mathcal{I}}$ that consists of points

$$x_1^1, \ldots, x_{|\mathcal{I}_1|}^1, \ldots, x_1^k, \ldots, x_{|\mathcal{I}_k|}^k \in X^{\mathcal{I}_1} \times \ldots \times X^{\mathcal{I}_k} \simeq X^{\mathcal{I}}$$

with $x_{i_1}^{j_1} \neq x_{i_2}^{j_2}$ whenever $j_1 \neq j_2$.

When $k = |\mathcal{I}|$, we shall replace the notation $j_{\mathcal{I}_1,\ldots,\mathcal{I}_k}$ by $j_{\mathcal{I}}$ or simply by j.

§5.1.1.

For a chiral algebra $\mathcal{A}$ consider the corresponding left D-module ${}^l\mathcal{A}$ and for a finite set $\mathcal{I}$ consider the left D-module ${}^l\mathcal{A}^{\mathcal{I}}$ over $X^{\mathcal{I}}$.

Now let $\phi : \mathcal{I} \twoheadrightarrow \mathcal{J}$ be a surjection of finite sets with $|\mathcal{J}| = |\mathcal{I}| - 1$. The chiral bracket on $\mathcal{A}$ yields a well-defined map

$$\{,\}_\phi : j_{\mathcal{I}*} j_{\mathcal{I}}^*({}^l\mathcal{A}^{\mathcal{I}}) \to \Delta_{\phi!}(j_{\mathcal{J}*} j_{\mathcal{J}}^*({}^l\mathcal{A}^{\mathcal{J}})).$$

The fact that the chiral bracket on $\mathcal{A}$ satisfies the Jacobi identity implies that we can form a complex $C^\bullet$ of left D-modules on X^n, with C^{-k} being

$$\bigoplus_{\phi:\{1,\ldots,n\}\twoheadrightarrow\mathcal{I},|\mathcal{I}|=k} \Delta_{\phi!}(j_{\mathcal{I}*} j_{\mathcal{I}}^*({}^l\mathcal{A}^{\mathcal{I}})).$$

Let ${}^l\mathcal{A}^{(n)}$ denote the $-n$-th cohomology of this complex, i.e. ${}^l\mathcal{A}^{(n)}$ is a sub-D-module of ${}^l\mathcal{A}^n$ obtained by intersecting the kernels of the maps $\{,\}_\phi$ for all possible

pairs $(\phi : \{1,\dots,n\} \to \mathcal{I}, |\mathcal{I}| = n-1)$. For example, ${}^l\mathcal{A}^{(2)}$ is the left D-module corresponding to $\mathcal{A}^{(2)}$ of Section 1.3.

Since ${}^l\mathcal{A}^{(n)}$ is equivariant with respect to the action of the symmetric group on X^n, we can form a D-module ${}^l\mathcal{A}^{(\mathcal{I})}$ for any finite non-empty set $\mathcal{I}$.

Proposition 5.1.1. *$H^i(C^\bullet) = 0$ for $i \neq -n$.*

Proof. To simplify the notation, we shall give a proof in the case $n = 3$. In the general case, the proof is completely analogous.

Our complex (for the corresponding right D-modules) looks as follows:

$$j_{\{1,2,3\}*}j^*_{\{1,2,3\}}(\mathcal{A}^3) \xrightarrow{d_{-3}} \Delta_{x_1=x_2!}(j_*j^*(\mathcal{A}^2)) \oplus \Delta_{x_2=x_3!}(j_*j^*(\mathcal{A}^2))$$
$$\oplus \Delta_{x_3=x_1!}(j_*j^*(\mathcal{A}^2)) \xrightarrow{d_{-2}} \Delta_{x_1=x_2=x_3!}(\mathcal{A}) \to 0$$

and we have to prove that it is exact at the -2-nd term (the fact that it is exact at the -1-st term is obvious, since $\mathcal{A}$ has a unit).

We claim, first of all, that any section of C^{-2} is equivalent modulo the image of d_{-3} to a section that belongs to $\Delta_{x_2=x_3!}(j_*j^*(\mathcal{A}^2))$. This again follows from the fact that $\mathcal{A}$ has a unit.

Now let l be a section

$$l \in \Gamma(X \times X, j_*j^*(\mathcal{A}^2)) \subset \Gamma(X \times X \times X, \Delta_{x_2=x_3!}(j_*j^*(\mathcal{A}^2)))$$

with $d_{-2}(l) = 0$; it remains to show that l lies in the image of d_{-3}.

Choose (locally) a 1-form ω on X and a function $f(x_1, x_2) \in \mathcal{O} \boxtimes \mathcal{O}(\Delta)$ such that $f(x_1, x_2) \cdot \omega = 1 \bmod \mathcal{O} \boxtimes \mathcal{O}(-\Delta)$ (we have used here the identification $\Omega \simeq \mathcal{O} \boxtimes \mathcal{O}(-\Delta)/\mathcal{O} \boxtimes \mathcal{O}(-2 \cdot \Delta)$).

It now follows from the Jacobi identity that $l = d_{-3}(l \boxtimes \omega \cdot f(x_2, x_3))$.

□

Corollary 5.1.1. *For a surjection $\phi : \mathcal{I} \twoheadrightarrow \mathcal{J}$, we have a canonical isomorphism*

$$\Delta^!_\phi({}^l\mathcal{A}^{(\mathcal{I})}) \simeq {}^l\mathcal{A}^{(\mathcal{J})}[|\mathcal{I}| - |\mathcal{J}|]$$

and for a decomposition $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$ we have a canonical isomorphism

$$j^*_{\mathcal{I}_1,\ldots,\mathcal{I}_k}({}^l\mathcal{A}^{(\mathcal{I})}) \simeq j^*_{\mathcal{I}_1,\ldots,\mathcal{I}_k}({}^l\mathcal{A}^{(\mathcal{I}_1)} \boxtimes \ldots \boxtimes {}^l\mathcal{A}^{(\mathcal{I}_k)}).$$

Moreover, the above isomorphisms are compatible in the obvious sense.

We shall call the isomorphisms of Corollary 5.1.1 "factorization isomorphisms".

Remark 5.1.1. Consider the topological space $Ran(X)$ that consists of all finite non-empty subsets of X. The above corollary implies that the collection of ${}^l\mathcal{A}^{(\mathcal{I})}$'s can be viewed as "D-module" over $Ran(X)$: its fiber over a finite subset of X that corresponds to a map $\mathcal{I} \to X$ is, by definition, the stalk of ${}^l\mathcal{A}^{(\mathcal{I})}$ at the corresponding point of $X^{\mathcal{I}}$.

§5.1.2.

Next we want to show that the chiral algebra structure can be completely recovered from the system of the ${}^l\mathcal{A}^{(n)}$'s viewed as plain $\mathcal{O}_{X^n}$-modules.

Let ${}^l\mathcal{A}$ be an $\mathcal{O}$-module on X. Assume now that for every finite set $\mathcal{I}$ we are given an $\mathcal{O}_{X^{\mathcal{I}}}$-module ${}^l\mathcal{A}'^{(\mathcal{I})}$ (such that ${}^l\mathcal{A} = {}^l\mathcal{A}'^{(\{1\})}$) together with a compatible system of factorization isomorphisms:

$$\Delta_\phi^*({}^l\mathcal{A}'^{(\mathcal{I})}) \simeq {}^l\mathcal{A}'^{(\mathcal{J})} \text{ for a surjection } \phi : \mathcal{I} \twoheadrightarrow \mathcal{J} \text{ and}$$

$$j^*_{\mathcal{I}_1,\ldots,\mathcal{I}_k}({}^l\mathcal{A}^{(\mathcal{I})}) \simeq j^*_{\mathcal{I}_1,\ldots,\mathcal{I}_k}({}^l\mathcal{A}'^{(\mathcal{I}_1)} \boxtimes \ldots \boxtimes {}^l\mathcal{A}'^{(\mathcal{I}_k)}) \text{ for a decomposition } \mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k.$$

Assume also that there exists a map $\mathbf{unit} : \mathcal{O} \to {}^l\mathcal{A}$ with the following properties:

- For $\mathcal{I} = \mathcal{I}_0 \cup i$ the map

  $$\mathrm{id} \boxtimes \mathbf{unit} : j^*_{\mathcal{I}_0,\{i\}}({}^l\mathcal{A}'^{(\mathcal{I}_0)} \boxtimes \mathcal{O}) \to j^*_{\mathcal{I}_0,\{i\}}({}^l\mathcal{A}'^{(\mathcal{I}_0)} \boxtimes {}^l\mathcal{A}) \simeq j^*_{\mathcal{I}_0,\{i\}}({}^l\mathcal{A}'^{(\mathcal{I})})$$

  extends to a map $\mathrm{id} \boxtimes \mathbf{unit} : {}^l\mathcal{A}'^{(\mathcal{I}_0)} \boxtimes \mathcal{O} \to {}^l\mathcal{A}'^{(\mathcal{I})}$.
- Take $\mathcal{I} = \mathcal{I}_0 \cup i_1 \cup i_2$ and $\mathcal{J} = \mathcal{I}_0 \cup i_1$ and let $\phi : \mathcal{I} \to \mathcal{J}$ be a map that contracts $\{i_1, i_2\} \to i_1$. We need that the composition

  $$ {}^l\mathcal{A}'^{(\mathcal{J})} \simeq \Delta_\phi^*({}^l\mathcal{A}'^{(\mathcal{J})} \boxtimes \mathcal{O}) \overset{\mathrm{id} \boxtimes \mathbf{unit}}{\longrightarrow} \Delta_\phi^*({}^l\mathcal{A}'^{(\mathcal{I})}) \simeq {}^l\mathcal{A}'^{(\mathcal{J})}$$

  is the identity map.

Theorem 5.1.1. *Let $({}^l\mathcal{A}, {}^l\mathcal{A}'^{(\mathcal{I})}, \mathbf{unit})$ be data as above. Then there exists a (canonical) chiral algebra structure on $\mathcal{A} := {}^l\mathcal{A} \otimes \Omega^{-1}$ such that the corresponding D-modules ${}^l\mathcal{A}^{(\mathcal{I})}$ are identified as $\mathcal{O}_{X^{\mathcal{I}}}$-modules with the ${}^l\mathcal{A}'^{(\mathcal{I})}$'s.*

We shall not give here a complete proof of this theorem. Instead, we will explain the key points:

Proof. (sketch)

Let $\mathcal{A}'^{(n)}$ denote the $\mathcal{O}_{X^n}$-module corresponding to $\mathcal{I} = \{1, \ldots, n\}$.

We shall first show how to endow ${}^l\mathcal{A}$ with a left D-module structure:

The embedding of $\mathcal{O}_{X^2}$-modules $\mathbf{unit} \boxtimes \mathrm{id} : \mathcal{O} \boxtimes {}^l\mathcal{A} \hookrightarrow {}^l\mathcal{A}'^{(2)}$ induces an isomorphism between their restrictions to the diagonal. This fact implies that the restrictions of the above two $\mathcal{O}_{X^2}$-modules to the second infinitesimal neighborhood of $\Delta(X) \subset X \times X$ are isomorphic as well, i.e. that

$$\mathcal{O} \boxtimes {}^l\mathcal{A}/\mathcal{O} \boxtimes {}^l\mathcal{A}(-2 \cdot \Delta) \simeq {}^l\mathcal{A}'^{(2)}/{}^l\mathcal{A}'^{(2)}(-2 \cdot \Delta).$$

Symmetrically, we have an isomorphism

$${}^l\mathcal{A} \boxtimes \mathcal{O}/{}^l\mathcal{A} \boxtimes \mathcal{O}(-2 \cdot \Delta) \simeq {}^l\mathcal{A}'^{(2)}/{}^l\mathcal{A}'^{(2)}(-2 \cdot \Delta)$$

and by transitivity, we obtain an isomorphism of $\mathcal{O}_{X^2}$-modules:

$$\varphi : \mathcal{O} \boxtimes {}^l\mathcal{A}/\mathcal{O} \boxtimes {}^l\mathcal{A}(-2 \cdot \Delta) \to {}^l\mathcal{A} \boxtimes \mathcal{O}/{}^l\mathcal{A} \boxtimes \mathcal{O}(-2 \cdot \Delta).$$

This enables us to define a connection on ${}^l\mathcal{A}$: to a section $l \in {}^l\mathcal{A}$ we associate a section $\nabla(l) \in {}^l\mathcal{A} \otimes \Omega$ as follows:

Note that $l \boxtimes 1 - \varphi(1 \boxtimes l)$ is a section of ${}^l\mathcal{A} \boxtimes \mathcal{O}/{}^l\mathcal{A} \boxtimes \mathcal{O}(-2 \cdot \Delta)$ which vanishes modulo ${}^l\mathcal{A} \boxtimes \mathcal{O}(-\Delta)$. We set $\nabla(l)$ to be the image of $l \boxtimes 1 - \varphi(1 \boxtimes l)$ under the identification

$${}^l\mathcal{A} \boxtimes \mathcal{O}(-\Delta)/{}^l\mathcal{A} \boxtimes \mathcal{O}(-2 \cdot \Delta) \simeq {}^l\mathcal{A} \otimes \Omega.$$

We leave it to the reader to check the correctness of this definition (this can be done using ${}^l\mathcal{A}'^{(3)}$). Moreover, the above construction can be generalized to produce left D-module structures on all the ${}^l\mathcal{A}'^{(\mathcal{I})}$'s in a way compatible with the factorization isomorphisms.

Let us now show how $\mathcal{A}$ acquires a chiral algebra structure:

The short exact sequence of D-modules

$$0 \to {}^l\mathcal{A}'^{(2)} \to j_*j^*({}^l\mathcal{A}'^{(2)}) \to \Delta_!\Delta^!({}^l\mathcal{A}'^{(2)})[1] \to 0$$

gives rise under our identifications to a map $j_*j^*({}^l\mathcal{A} \boxtimes {}^l\mathcal{A}) \to \Delta_!({}^l\mathcal{A})$.

Using the D-module ${}^l\mathcal{A}'^{(3)}$, it can be checked that this operation satisfies the Jacobi identity when we pass to the corresponding right D-modules.

Finally, we have to identify ${}^l\mathcal{A}'^{(\mathcal{I})}$ with ${}^l\mathcal{A}^{(\mathcal{I})}$.

It follows from the construction that there is a natural embedding ${}^l\mathcal{A}'^{(\mathcal{I})} \hookrightarrow {}^l\mathcal{A}^{(\mathcal{I})}$. Moreover, for every $\phi : \mathcal{I} \twoheadrightarrow \mathcal{J}$ this map induces an isomorphism

$$j_{\mathcal{J}}^*\Delta_\phi^!({}^l\mathcal{A}'^{(\mathcal{I})}) \simeq j_{\mathcal{J}}^*({}^l\mathcal{A}'^{(\mathcal{J})}) \simeq j_{\mathcal{J}}^*({}^l\mathcal{A}^{(\mathcal{J})}) \simeq j_{\mathcal{J}}^*\Delta_\phi^!({}^l\mathcal{A}^{(\mathcal{I})}).$$

This implies that the embedding ${}^l\mathcal{A}'^{(\mathcal{I})} \hookrightarrow {}^l\mathcal{A}^{(\mathcal{I})}$ is an isomorphism, too.

□

Remark 5.1.2. One may have observed that the construction of the chiral universal enveloping algebra discussed in Section 1.2.4 was also in the spirit of Theorem 5.1.1. In fact, it can be completely reformulated in terms of the system of $\mathcal{O}_{X^n}$-modules $\mathcal{U}(\mathcal{B})^{(n)}$, which we leave as an exercise to the reader.

§5.1.3.

Let $\mathcal{A}$ be a chiral algebra over a complete curve X. It follows from Section 1.3.3 that the map

$$\langle \dots \rangle : j_*j^*({}^l\mathcal{A} \boxtimes \dots \boxtimes {}^l\mathcal{A}) \to j_*j^*(\mathcal{O} \boxtimes \dots \boxtimes \mathcal{O}) \otimes H_\nabla(X, \mathcal{A})$$

restricts to a D-module map

$${}^l\mathcal{A}^{(n)} \to \mathcal{O}_{X^n} \otimes H_\nabla(X, \mathcal{A}).$$

Hence, we obtain a map of vector spaces $DR^n(X^n, {}^l\mathcal{A}^{(n)}) \to H_\nabla(X, \mathcal{A})$.

Proposition 5.1.2. *For $n \geq 2$, the map $DR^n(X^n, {}^l\mathcal{A}^{(n)}) \to H_\nabla(X, \mathcal{A})$ is an isomorphism.*

Proof. We can compute $DR^n(X^n, {}^l\mathcal{A}^{(n)})$ using the resolution $C^\bullet$ of Section 5.1.1. We have:

$$DR^n(X^n, {}^l\mathcal{A}^{(n)}) \simeq \operatorname{coker}(DR^1(X^n, C^{-2}) \to DR^1(X^n, C^{-1})).$$

However, $DR^1(X^n, C^{-1})$ is just $DR^1(X, \mathcal{A})$ and $DR^1(X^n, C^{-2})$ is a direct sum of several copies of $DR^1(X \times X, j_*j^*(\mathcal{A} \boxtimes \mathcal{A}))$. The assertion of the proposition follows from the fact that

$$H_\nabla(X, \mathcal{A}) \simeq DR^1(X, \mathcal{A}^{(2)}) \simeq \operatorname{coker}(DR^1(X^2, j_*j^*(\mathcal{A} \boxtimes \mathcal{A})) \to DR^1(X, \mathcal{A})).$$

□

§5.2. Geometry of the affine Grassmannian

Let G be an algebraic group and let $\mathfrak{g}$ be the corresponding Lie algebra. Consider the Lie-* algebra $\mathcal{B}(\mathfrak{g}, 0)$ of Ex.1 in Section 1.2.3. In the next two subsections we shall describe a geometric construction of the corresponding chiral universal enveloping algebra $\mathcal{A}(\mathfrak{g}, 0)$ via the so-called affine Grassmannian of the group G. This description will be used later on for the construction of the free bosonic theory.

§5.2.1.

Consider the group $G(\hat{\mathcal{O}})$ and the ind-group $G(\hat{\mathcal{K}})$ that classify maps

$$\mathrm{Spec}(\hat{\mathcal{O}}) \to G \text{ and } \mathrm{Spec}(\hat{\mathcal{K}}) \to G,$$

respectively. Consider now the affine Grassmannian $\mathfrak{Gr}_G := G(\hat{\mathcal{K}})/G(\hat{\mathcal{O}})$ corresponding to the group G. The ind-scheme $\mathfrak{Gr}_G$ may be highly non-reduced; for instance, when $G = H$ is a torus, the corresponding reduced scheme is identified with the discrete set of co-characters of H, while $\mathfrak{Gr}_H$ is infinite-dimensional.

Note that the group Aut_0^+ of automorphisms of $\hat{\mathcal{O}}$ acts naturally on $\mathfrak{Gr}_G$.

For a curve X consider the group scheme $G_{\mathcal{O}}(X)$ over X whose fiber at $x \in X$ is $G(\hat{\mathcal{O}}_x)$ and a group ind-scheme $G_{\mathcal{K}}(X)$, whose fiber at $x \in X$ is $G(\hat{\mathcal{K}}_x)$. The quotient $\mathfrak{Gr}_G(X) := G_{\mathcal{K}}(X)/G_{\mathcal{O}}(X)$ is a global version of the affine Grassmannian $\mathfrak{Gr}_G$ considered above. Let $r : \mathfrak{Gr}_G(X) \to X$ denote the natural projection and let $\mathbf{unit}_{\mathfrak{Gr}}$ denote the unit section $X \to \mathfrak{Gr}_G(X)$.

Lemma 5.2.1. *The ind-scheme $\mathfrak{Gr}_G(X)$ represents the functor whose value on a scheme S is the data of a map $f : S \to X$, a G-torsor $\mathcal{P}$ on $S \times X$ and a trivialization*

$$\alpha : \mathcal{P}|(S \times X \setminus \Gamma_f) \simeq \mathcal{P}_0|(S \times X \setminus \Gamma_f),$$

where Γ_f is the graph of the map f and $\mathcal{P}_0$ is the trivial G-torsor.

This is a version of the Beauville-Laszlo theorem ([**5**]).

Proposition 5.2.1. *The ind-scheme $\mathfrak{Gr}_G(X)$ possesses a natural connection along X. The section $\mathbf{unit}_{\mathfrak{Gr}} : X \to \mathfrak{Gr}_G(X)$ is preserved by this connection.*

Proof. Let I be a local Artinian scheme and let f be a map $I \times S \to X$, where S is an arbitrary scheme. Let $\mathrm{Spec}(\mathbb{C}) \simeq I_0 \subset I$ be the reduced (point) scheme and let let $\widetilde{f}_0$ be a map $I_0 \times S \to \mathfrak{Gr}_G(X)$ such that the map $r \circ \widetilde{f}_0 : I_0 \times S \to X$ coincides with the composition $I_0 \times S \to I \times S \to X$.

To define a connection on $\mathfrak{Gr}_G(X)$ along X, we must associate to a triple $(I, f, \widetilde{f}_0)$ as above a map $\widetilde{f} : I \times S \to \mathfrak{Gr}_G(X)$ such that $r \circ \widetilde{f} = f$ that extends $\widetilde{f}_0$.

This can be done as follows:

The corresponding map $I \times S \to X$ has already been given to us (this is our f) and we set the G-torsor $\mathcal{P}^I$ over $I \times S \times X$ that would correspond to $\widetilde{f}$ to be pulled pack under $I \times S \times X \to S \times X$ from the G-torsor $\mathcal{P}$ that corresponds to $\widetilde{f}_0$. However, $I \times S \times X \setminus \Gamma_f \simeq (S \times X \setminus \Gamma_{f_0}) \times I$ and the data of trivialization for $\mathcal{P}^I$ comes from the corresponding data for $\mathcal{P}$.

□

When X is complete let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles over X. We have a natural projection $k : \mathfrak{Gr}_G(X) \to \mathrm{Bun}_G(X)$.

Lemma 5.2.2. *The projection $k : \mathfrak{Gr}_G(X) \to \mathrm{Bun}_G(X)$ is preserved by the connection on $\mathfrak{Gr}_G(X)$ along X.*

It follows from the definitions that the dependence of $\mathfrak{Gr}_G(X)$ on the curve X is local (in the sense of Section 2.1.1):

We have the ind-stack $\mathfrak{Gr}_G(\mathfrak{X})$ fibered over $\mathfrak{X}$, that is, for a family X^S of curves over a base S we can form a scheme $\mathfrak{Gr}_G(X^S)$ over X^S. Now let $X^{S,I}, x^{S,I}, \phi^{S,I}$ be as in Section 2.1.1. We have a canonical isomorphism:

$$\mathfrak{Gr}^n_G(X^{S^I}) \setminus r^{-1}(x^{S\times I}) \simeq (\mathfrak{Gr}_G(X^S) \setminus r^{-1}(x^S)) \times I.$$

The ind-scheme $\mathfrak{Gr}_G(\mathfrak{X})$ carries a connection along the fibers of the projection $\mathfrak{X} \to \mathfrak{M}$ and it is easy to see that the above isomorphism is compatible with this connection.

§5.2.2.

For a finite set $\mathcal{I}$ we define the Beilinson-Drinfeld Grassmannian $\mathfrak{Gr}^{\mathcal{I}}_G(X)$ to be the ind-scheme representing the following functor:

For a scheme S, $\mathrm{Hom}(S, \mathfrak{Gr}^{\mathcal{I}}_G(X))$ is the data consisting of a map $f^{\mathcal{I}} : S \to X^{\mathcal{I}}$, a G-torsor $\mathcal{P}$ on $S \times X$ and a trivialization

$$\alpha : \mathcal{P}|(S \times X \setminus \{\Gamma_{f^{\mathcal{I}}_{i_1}}, \dots, \Gamma_{f^{\mathcal{I}}_{i_n}}\}) \simeq \mathcal{P}_0|(S \times X \setminus \{\Gamma_{f^{\mathcal{I}}_{i_1}}, \dots, \Gamma_{f^{\mathcal{I}}_{i_n}}\}),$$

where for $i_k \in \mathcal{I}$, $f^{\mathcal{I}}_{i_k}$ is the composition of $f^{\mathcal{I}}$ with the projection on the i_k-th factor $X^{\mathcal{I}} \to X$.

Let $r^{\mathcal{I}}$ denote the natural projection $\mathfrak{Gr}^{\mathcal{I}}_G(X) \to X^{\mathcal{I}}$ and let $\mathbf{unit}^{\mathcal{I}}_{\mathfrak{Gr}} : X^{\mathcal{I}} \to \mathfrak{Gr}^{\mathcal{I}}_G(X)$ denote the corresponding unit section.

When $\mathcal{I} = \{1, \dots, n\}$ we shall denote $\mathfrak{Gr}^{\mathcal{I}}_G(X)$ by $\mathfrak{Gr}^n_G(X)$ and the corresponding maps $r^{\mathcal{I}}$ and $\mathbf{unit}^{\mathcal{I}}_{\mathfrak{Gr}}$ by r^n and $\mathbf{unit}^n_{\mathfrak{Gr}}$, respectively.

A remarkable feature of the ind-scheme $\mathfrak{Gr}^{\mathcal{I}}_H(X)$ is the following factorization property:

Proposition 5.2.2. *For a surjection of finite sets $\mathcal{I} \twoheadrightarrow \mathcal{J}$, we have a natural isomorphism:*

$$X^{\mathcal{J}} \underset{X^{\mathcal{I}}}{\times} \mathfrak{Gr}_G(X)^{\mathcal{I}} \simeq \mathfrak{Gr}_G(X)^{\mathcal{J}}$$

and for a decomposition $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$ we have a natural isomorphism

$$X^{\mathcal{I}_i \neq \mathcal{I}_j} \underset{X^{\mathcal{I}}}{\times} \mathfrak{Gr}_G(X)^{\mathcal{I}} \simeq X^{\mathcal{I}_i \neq \mathcal{I}_j} \underset{X^{\mathcal{I}_1} \times \ldots \times X^{\mathcal{I}_k}}{\times} \mathfrak{Gr}_G(X)^{\mathcal{I}_1} \times \ldots \times \mathfrak{Gr}_G(X)^{\mathcal{I}_k}.$$

Proof. The first assertion follows immediately from the definitions. To prove the second one let us assume for simplicity that $\mathcal{I} = \{1, 2\}$, $\mathcal{I}_1 = \{1\}$ and $\mathcal{I}_2 = \{2\}$.

Given an object $(x_1, x_2, \mathcal{P}, \alpha)$ of $\mathfrak{Gr}^2_H(X)$ we construct an object of

$$(x_1, \mathcal{P}_1, \alpha_1) \times (x_2, \mathcal{P}_2, \alpha_2) \in \mathfrak{Gr}_G(X) \times \mathfrak{Gr}_G(X)$$

as follows:

The G-torsor $\mathcal{P}_1$ (resp., $\mathcal{P}_2$) is set to be isomorphic to $\mathcal{P}_0$ over $X \setminus x_1$ (resp., over $X \setminus x_2$) and to be isomorphic to $\mathcal{P}$ over $X \setminus x_2$ (resp., over $X \setminus x_1$). The datum of α for $\mathcal{P}$ provides the gluing data for $\mathcal{P}_1$ and $\mathcal{P}_2$. The data of α_1 and α_2 follow from the construction.

Vice versa, given an object $(x_1, \mathcal{P}_1, \alpha_1, x_2, \mathcal{P}_2, \alpha_2) \in \mathfrak{Gr}_G(X) \times \mathfrak{Gr}_G(X)$ we define $(x_1, x_2, \mathcal{P}, \alpha) \in \mathfrak{Gr}^2_G(X))$ by setting $\mathcal{P}$ to equal $\mathcal{P}_1$ over $X \setminus x_2$ and to equal $\mathcal{P}_2$ over $X \setminus x_1$. The isomorphisms α_1 and α_2 provide the gluing data for $\mathcal{P}$ together with the trivialization α over $X \setminus \{x_1, x_2\}$

□

Remark 5.2.1. It follows from Proposition 5.2.2 above that the fiber of $\mathfrak{Gr}^2_G(X)$ over a point $(x_1, x_2) \in X \times X$ is identified with $\mathfrak{Gr}_G \times \mathfrak{Gr}_G$ whenever $x_1 \neq x_2$ and with $\mathfrak{Gr}_G$ when $x_1 = x_2$. This is a purely infinite-dimensional phenomenon (in the finite-dimensional situation the dimension of fibers cannot drop under a specialization). Moreover, it is not difficult to prove that the map $r^{\mathcal{I}}$ is formally smooth.

As in the case $|\mathcal{I}| = 1$, it is easy to show that $\mathfrak{Gr}^{\mathcal{I}}_G(X)$ possesses a natural connection along $X^{\mathcal{I}}$ and that the isomorphisms constructed in Proposition 5.2.2 are compatible with these connections. When X is complete we have a projection $k^{\mathcal{I}} : \mathfrak{Gr}^{\mathcal{I}}_G(X) \to \mathrm{Bun}_G(X)$ which (as in the case of $|\mathcal{I}| = 1$) is preserved by the connection on $\mathfrak{Gr}^{\mathcal{I}}_G(X)$ along $X^{\mathcal{I}}$. (When $\mathcal{I} = \{1, \ldots, n\}$ we shall replace the notation $k^{\mathcal{I}}$ simply by k^n.)

Again, as in the case of $\mathcal{I} = \{1\}$, there exists an ind-stack $\mathfrak{Gr}^{\mathcal{I}}_G(\mathfrak{X})$ fibered over $\mathfrak{X}^I$ which has a locality property in the sense that was specified at the end of Section 5.2.1. It follows from the definitions that the local structure on the $\mathfrak{Gr}^{\mathcal{I}}_G(X)$'s is compatible with the connections along the $X^{\mathcal{I}}$'s and with the factorization isomorphisms of Proposition 5.2.2.

§5.2.3.

We shall now introduce a twisted version of the Beilinson-Drinfeld Grassmannian, which will be used for the construction of the bosonic chiral algebra in the next section.

For an integer n we introduce the iterated Beilinson-Drinfeld Grassmannian $\widetilde{\mathfrak{Gr}}^n_G(X)$ as follows:

$\widetilde{\mathfrak{Gr}}^n_G(X)$ is the ind-scheme representing the functor whose value on a scheme S is the data consisting of a map $f^n : S \to X^n$ and a collection of n G-torsors $\mathcal{P}_i$ over $S \times X$ together with isomorphisms

$$\alpha_i : \mathcal{P}_i|(S \times X \setminus \Gamma_{f^n_i}) \simeq \mathcal{P}_{i-1}|(S \times X \setminus \Gamma_{f^n_{i-1}}),$$

(here f^n_i is the composition of the map f^n with the projection on the i-th factor $X^n \to X$.)

Let $\widetilde{r}^n$ denote the projection $\widetilde{\mathfrak{Gr}}^n_G(X) \to X^n$. It is easy to see that $\widetilde{\mathfrak{Gr}}^n_G(X)$ is a fibration over $\widetilde{\mathfrak{Gr}}^{n-1}_G(X) \times X$ with typical fiber $\mathfrak{Gr}_G$.

More precisely, consider the direct product $\widetilde{\mathfrak{Gr}}^{n-1}_G(X) \times X$ and let $\widetilde{G}^n_{\mathcal{O}}(X)$ denote the pull-back of the group scheme $G_{\mathcal{O}}(X)$ with respect to the projection

$$\widetilde{\mathfrak{Gr}}^{n-1}_G(X) \times X \to X.$$

On the one hand, there is a canonical $\widetilde{G}^n_{\mathcal{O}}(X)$-torsor over $\widetilde{\mathfrak{Gr}}^{n-1}_G(X) \times X$, whose fiber at a point $(x_1, \ldots, x_{n-1}, \mathcal{P}_1, \ldots, \mathcal{P}_{n-1}, x_n)$ is the restriction of $\mathcal{P}_{n-1}$ to the formal disc around x_n. On the other hand, $\widetilde{G}^n_{\mathcal{O}}(X)$ acts on the $\widetilde{\mathfrak{Gr}}^{n-1}_G(X) \times X$-scheme

$\widetilde{\mathfrak{Gr}}_G^{n-1}(X) \times \mathfrak{Gr}_G(X)$. It is easy to see that the $\widetilde{\mathfrak{Gr}}_G^{n-1}(X) \times X$-scheme $\widetilde{\mathfrak{Gr}}_G^n(X)$ is a twist of $\widetilde{\mathfrak{Gr}}_G^{n-1}(X) \times \mathfrak{Gr}_G(X)$ with respect to the above-mentioned $\widetilde{G}_{\mathcal{O}}^n(X)$-torsor.

We have a canonical projection $w_n : \widetilde{\mathfrak{Gr}}_G^n(X) \to \mathfrak{Gr}_G^n(X)$ that sends the data of $(x_1, \ldots, x_n, \mathcal{P}_1, \ldots, \mathcal{P}_n, \alpha_1, \ldots, \alpha_n)$ to $(x_1, \ldots, x_n, \mathcal{P}_n, \alpha)$, where α is obtained by composing $\alpha_1, \ldots, \alpha_n$. By repeating the proof of Proposition 5.2.2, we see that the map w_n is an isomorphism over $X^n \setminus \Delta$.

As in the case of $\mathfrak{Gr}_G^n(X)$, there exists a scheme $\widetilde{\mathfrak{Gr}}_G^n(\mathfrak{X})$ over $\mathfrak{X}^n$ which is local in the sense of Section 5.2.1.

§5.3. Chiral algebra attached to the affine Grassmannian

§5.3.1.

Consider the right D-module $\mathbf{unit}_{\mathfrak{Gr}!}(\Omega)$ on $\mathfrak{Gr}_G(X)$.[1] As an $\mathcal{O}_{\mathfrak{Gr}_G(X)}$-module, $\mathbf{unit}_{\mathfrak{Gr}!}(\Omega)$ is a union of $\mathcal{O}$-modules on $\mathfrak{Gr}_G(X)$ that are supported scheme-theoretically on an increasing family of finite-dimensional infinitesimal neighborhoods of $\mathbf{unit}_{\mathfrak{Gr}}(X)$ in $\mathfrak{Gr}_G(X)$.

Consider now the $\mathcal{O}$-module direct image

$$\mathcal{A}_{\mathfrak{Gr}_G} := r_*(\mathbf{unit}_{\mathfrak{Gr}!}(\Omega)).$$

The connection on $\mathfrak{Gr}_G(X)$ along X defines a right D-module structure on $\mathcal{A}_{\mathfrak{Gr}_G}$ and the section $\mathbf{unit}_{\mathfrak{Gr}}$ defines an embedding $\Omega \hookrightarrow \mathcal{A}_{\mathfrak{Gr}_G}$. Let ${}^l\mathcal{A}_{\mathfrak{Gr}_G}$ be the corresponding left D-module.

More generally, for a finite set $\mathcal{I}$ let $\mathcal{A}'^{(\mathcal{I})}_{\mathfrak{Gr}_G}$ be the right D-module on $X^{\mathcal{I}}$ defined as $r^{\mathcal{I}}_*(\mathbf{unit}^{\mathcal{I}}_{\mathfrak{Gr}!}(\Omega_{X^{\mathcal{I}}}))$ and let ${}^l\mathcal{A}'^{(\mathcal{I})}_{\mathfrak{Gr}_G}$ denote the corresponding left D-module.

Let us now view the ${}^l\mathcal{A}'^{(\mathcal{I})}_{\mathfrak{Gr}_G}$'s as plain $\mathcal{O}$-modules on $X^{\mathcal{I}}$. The next assertion follows from Proposition 5.2.2:

Lemma 5.3.1. *The system $\mathcal{I} \to {}^l\mathcal{A}'^{(\mathcal{I})}_{\mathfrak{Gr}_G}$ satisfies the conditions of Theorem 5.1.1.*

By applying Theorem 5.1.1 we obtain, therefore, a chiral algebra structure on $\mathcal{A}_{\mathfrak{Gr}_G}$ with the ${}^l\mathcal{A}'^{(\mathcal{I})}_{\mathfrak{Gr}_G}$'s being identified with the corresponding ${}^l\mathcal{A}^{(\mathcal{I})}_{\mathfrak{Gr}_G}$'s as $\mathcal{O}$-modules.

However, since the factorization isomorphisms of Proposition 5.2.2 are compatible with the connections along $X^{\mathcal{I}}$, it is easy to see that the isomorphisms $\mathcal{A}^{(\mathcal{I})}_{\mathfrak{Gr}_G} \simeq \mathcal{A}'^{(\mathcal{I})}_{\mathfrak{Gr}_G}$ are compatible with the D-module structure.

§5.3.2.

Consider the fiber $A_{\mathfrak{Gr}_G x}$ of the chiral algebra $\mathcal{A}_{\mathfrak{Gr}_G}$ at a point $x \in X$. The action of the group $G(\hat{\mathcal{K}}_x)$ on $\mathfrak{Gr}_G = G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x)$ defines a structure of $(\mathfrak{g} \otimes \hat{\mathcal{K}}_x, G(\hat{\mathcal{O}}_x))$-module on $A_{\mathfrak{Gr}_G x}$. We shall denote by

$$\mathrm{Act}_x : (\mathfrak{g} \otimes \hat{\mathcal{K}}_x) \otimes A_{\mathfrak{Gr}_G x} \to A_{\mathfrak{Gr}_G x}$$

the corresponding action map.

---

[1] When Z is a strict ind-scheme (i.e. Z is a union of closed finite dimensional subschemes) and $Z' \subset Z$ is a closed finite-dimensional subscheme of Z it makes perfect sense to talk about right D-modules on Z supported on Z' (cf. Section 0.2).

It is easy to see, moreover, that $A_{\mathfrak{Gr}_G x}$ is in fact canonically isomorphic to the vacuum module $\mathrm{Ind}^{\mathfrak{g}\otimes\hat{\mathcal{K}}_x}_{\mathfrak{g}\otimes\hat{\mathcal{O}}_x}(\mathbb{C})$.

Our goal now is to prove the following theorem:

Theorem 5.3.1. *There exists a natural isomorphism of chiral algebras $\mathcal{A}_{\mathfrak{Gr}_G} \simeq \mathcal{A}(\mathfrak{g},0)$. Moreover, for every $x \in X$ the map Act_x goes over to the action of $\mathfrak{g}\otimes\hat{\mathcal{K}}_x \simeq DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{B}(\mathfrak{g},0))$ on $A(\mathfrak{g},0)_x$.*

Remark 5.3.1. A similar description of the chiral algebra $\mathcal{A}(\mathfrak{g},Q)$ can be given when $Q \neq 0$. This will be done explicitly in Section 6.2.1 when $G = H$ is a torus.

We shall deduce the assertion of the theorem from the following result:

Proposition 5.3.1. *There is a natural map*

$$\mathrm{Act} : j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\mathcal{A}_{\mathfrak{Gr}_G}) \to \Delta_!(\mathcal{A}_{\mathfrak{Gr}_G})$$

such that

(a) $\mathcal{B}(\mathfrak{g},0)$ *acts on $\mathcal{A}_{\mathfrak{Gr}_G}$ by derivations of the chiral algebra structure.*

(b) *The induced map*

$$(h\boxtimes\mathrm{id})(\mathrm{Act}) : DR^0(\mathrm{Spec}(\hat{\mathcal{K}}_x), \mathcal{B}(\mathfrak{g},0))\otimes A_{\mathfrak{Gr}_G x} \to A_{\mathfrak{Gr}_G x}$$

coincides with the map Act_x.

Proof. (Of Theorem 5.3.1)

According to property (b) of the map Act, the composition

$$j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\Omega) \overset{\mathrm{id}\times\mathbf{unit}}{\longrightarrow} j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\mathcal{A}_{\mathfrak{Gr}_G}) \to \Delta_!(\mathcal{A}_{\mathfrak{Gr}_G})$$

factors as

$$j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\Omega) \twoheadrightarrow \mathcal{B}(\mathfrak{g},0) \to \mathcal{A}_{\mathfrak{Gr}_G},$$

i.e. we obtain a map of D-modules

$$emb : \mathcal{B}(\mathfrak{g},0) \to \mathcal{A}_{\mathfrak{Gr}_G}.$$

On the level of fibers, this map corresponds to the embedding $\hat{\mathcal{K}}_x/\hat{\mathcal{O}}_x \otimes \mathfrak{g} \hookrightarrow \mathrm{Ind}^{\mathfrak{g}\otimes\hat{\mathcal{K}}_x}_{\mathfrak{g}\otimes\hat{\mathcal{O}}_x}(\mathbb{C})$.

By again applying point (b) of Proposition 5.3.1, we obtain that $emb : \mathcal{B}(\mathfrak{g},0) \to \mathcal{A}_{\mathfrak{Gr}_G}$ is a map of chiral $\mathcal{B}(\mathfrak{g},0)$-modules.

Therefore, to prove the theorem it remains to show that the two maps

$$j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\mathcal{A}_{\mathfrak{Gr}_G}) \to \Delta_!(\mathcal{A}_{\mathfrak{Gr}_G}),$$

namely Act and $(\{,\})\circ(emb\times\mathrm{id})$, coincide. This is, however, an almost immediate consequence of point (a) of Proposition 5.3.1.

Let $b\boxtimes a\cdot f(x,y)$ be a section of $j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\mathcal{A}_{\mathfrak{Gr}_G})$. It is enough to show that

$$(\mathrm{id}\times h)(\mathrm{Act}(b\boxtimes a\cdot f(x,y))) = (\mathrm{id}\times h)(\{emb(b)\boxtimes a\cdot f(x,y)\})$$

for any a, b and $f(x,y)$ as above.

Let ω be a non-vanishing 1-form on X and let $h(x,y)$ be a function on $X\times X\setminus\Delta$ with a simple pole along the diagonal and with $\mathrm{Res}_\Delta(\omega\otimes h(x,y)) = 1$. Consider the section

$$b\boxtimes\omega\boxtimes a\cdot f(x,z)\cdot h(x,y) \in j_*j^*(\mathcal{B}(\mathfrak{g},0)\boxtimes\mathcal{A}_{\mathfrak{Gr}_G}\boxtimes\mathcal{A}_{\mathfrak{Gr}_G}).$$

The fact that $\mathcal{B}(\mathfrak{g},0)$ acts on $\mathcal{A}_{\mathfrak{Gr}_G}$ by derivations of the chiral algebra structure implies that

$$\{\mathrm{Act}(b\boxtimes\omega\cdot h(x,y))\boxtimes a\cdot f(x,z)\}=$$
$$\mathrm{Act}(b\boxtimes\{\omega\boxtimes a\cdot h(x,y)\cdot f(x,z)\})-\sigma_{1,2}\{\omega,\mathrm{Act}(b\boxtimes a\cdot h(y,x)\cdot f(y,z))\}.$$

Note, first of all, that the expression $\mathrm{Act}(b\boxtimes\omega\cdot h(x,y))$ is equal by definition to $emb(b)$ and that the first term on the RHS vanishes.

By applying $(\mathrm{id}\boxtimes h\boxtimes h)$ to both sides of the above formula we obtain the needed result.

□

§5.3.3.

The proof of Proposition 5.3.1 will be based on the following construction:

Assume that X is affine and consider the formal group $\mathrm{Maps}(X,G)$ that classifies maps $X\to G$. In addition, given a finite set $\mathcal{I}$ we shall consider a formal group-scheme $\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)$ over $X^{\mathcal{I}}$ that corresponds to the functor whose value on a pair $(S,f^{\mathcal{I}}:S\to X^{\mathcal{I}})$ is the group of regular maps

$$(S\times X\setminus\{\Gamma(f^{\mathcal{I}}_{i_1}),\dots,\Gamma(f^{\mathcal{I}}_{i_n})\})\to G.$$

It follows from the definitions that $\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)$ carries a natural connection along $X^{\mathcal{I}}$. The corresponding sheaf of Lie algebras $\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G))$ is a left D-module on $X^{\mathcal{I}}$ which is identified with $(p_{\mathcal{I}}\circ j_{\mathcal{I},\{i\}})_*(\mathfrak{g}\otimes\mathcal{O}_{X^{\mathcal{I}\cup i}})$,[2] where $p_{\mathcal{I}}$ denotes the projection $X^{\mathcal{I}\cup i}\to X^{\mathcal{I}}$.

For a surjection of finite sets $\phi:\mathcal{I}\twoheadrightarrow\mathcal{J}$ we have a natural map

$$X^{\mathcal{J}}\underset{X^{\mathcal{I}}}{\times}\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)\to\widetilde{\mathrm{Maps}}^{\mathcal{J}}(X,G)$$

and for a decomposition $\mathcal{I}=\mathcal{I}_1\cup\ldots\cup\mathcal{I}_k$, we have a map

$$\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)|(X^{\mathcal{I}_i\neq\mathcal{I}_j})\to\widetilde{\mathrm{Maps}}^{\mathcal{I}_1}(X,G)\times\ldots\times\widetilde{\mathrm{Maps}}^{\mathcal{I}_k}(X,G)|(X^{\mathcal{I}_i\neq\mathcal{I}_j}).$$

Observe now that there is a natural action

$$\mathrm{Act}^{\mathcal{I}}:\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)\underset{X^{\mathcal{I}}}{\times}\mathfrak{Gr}^{\mathcal{I}}_G(X)\to\mathfrak{Gr}^{\mathcal{I}}_G(X):$$

Take

$$(x^{\mathcal{I}}\in X^{\mathcal{I}},g:X\setminus x^{\mathcal{I}}\to G)\in\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)\text{ and }(x^{\mathcal{I}},\mathcal{P},\alpha:\mathcal{P}\simeq\mathcal{P}_0|X\setminus x^{\mathcal{I}})\in\mathfrak{Gr}^{\mathcal{I}}_G(X).$$

To define the corresponding new point $\mathrm{Act}^{\mathcal{I}}((x^{\mathcal{I}},g),(x^{\mathcal{I}},\mathcal{P},\alpha))$ of $\mathfrak{Gr}^{\mathcal{I}}_G(X)$ we need to specify a new G-torsor $\mathcal{P}'$ with a trivialization α' over $X\setminus x^{\mathcal{I}}$.

The G-torsor $\mathcal{P}'$ is set to be isomorphic to $\mathcal{P}$ over $X\setminus x^{\mathcal{I}}$ with $\alpha':(\mathcal{P}'\to\mathcal{P}_0)|X\setminus x^{\mathcal{I}}$ being obtained from α. A local section $s:X\setminus x^{\mathcal{I}}\to\mathcal{P}_0$ is now set to give rise to a regular section of $\mathcal{P}'$ (via α') if $\alpha^{-1}\circ g\circ s:X\setminus x^{\mathcal{I}}\to\mathcal{P}$ is regular.

It is easy to see, moreover, that the action

$$\mathrm{Act}^{\mathcal{I}}:\widetilde{\mathrm{Maps}}^{\mathcal{I}}(X,G)\underset{X^{\mathcal{I}}}{\times}\mathfrak{Gr}^{\mathcal{I}}_G(X)\to\mathfrak{Gr}^{\mathcal{I}}_G(X)$$

---

[2] In this formula, $(p_{\mathcal{I}}\circ j_{\mathcal{I},\{i\}})_*$ denotes the direct image in the category of $\mathcal{O}$-modules.

described above is compatible with the connection along $X^{\mathcal{J}}$ and with the factorization isomorphisms. Hence, the sheaves of Lie algebras $\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\mathcal{J}}(X,G))$ act on the D-modules $\mathcal{A}^{(\mathcal{J})}_{\mathfrak{Gr}_G}$ in a way compatible with the factorization isomorphisms. In particular, the chiral bracket $j_*j^*(\mathcal{A}_{\mathfrak{Gr}_G} \boxtimes \mathcal{A}_{\mathfrak{Gr}_G}) \to \Delta_!(\mathcal{A}_{\mathfrak{Gr}_G})$ is compatible with the action of $\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1,2\}}(X,G))$ in the sense that the diagram:

$$\begin{array}{ccc}
\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1,2\}}(X,G)) \otimes j_*j^*(\mathcal{A}_{\mathfrak{Gr}_G} \boxtimes \mathcal{A}_{\mathfrak{Gr}_G}) & \xrightarrow{\mathrm{Act}^{\{1,2\}}} & j_*j^*(\mathcal{A}_{\mathfrak{Gr}_G} \boxtimes \mathcal{A}_{\mathfrak{Gr}_G}) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\{,\}} & & \downarrow \\
\Delta_!(\Delta^!(\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1,2\}}(X,G)))[1] \otimes \mathcal{A}_{\mathfrak{Gr}_G}) & & \\
\downarrow & & \downarrow \\
\Delta_!(\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1\}}(X,G)) \otimes \mathcal{A}_{\mathfrak{Gr}_G}) & \xrightarrow{\mathrm{Act}^{\{1\}}} & \Delta_!(\mathcal{A}_{\mathfrak{Gr}_G})
\end{array}$$

commutes.

In the proof of Proposition 5.3.1 we shall use the following observation:

Let $\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1\}}(X,G))_x$ denote the fiber of the sheaf $\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1\}}(X,G))$ at $x \in X$.

Lemma 5.3.2. *The action* $\mathrm{Act}^{\{1\}} : \mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1\}}(X,G))_x \otimes A_{\mathfrak{Gr}_G x} \to A_{\mathfrak{Gr}_G x}$ *coincides with the one that comes from the embedding*

$$\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1\}}(X,G))_x \simeq \mathfrak{g} \otimes \mathcal{O}(X \setminus x) \hookrightarrow \mathfrak{g} \otimes \hat{\mathcal{K}}_x$$

and the canonical action $\mathrm{Act}_x : (\mathfrak{g} \otimes \hat{\mathcal{K}}_x) \otimes A_{\mathfrak{Gr}_G x} \to A_{\mathfrak{Gr}_G x}$.

Proof. (Of Proposition 5.3.1)

Consider the completion $p_{1*}(\mathfrak{g} \otimes \mathcal{O}\hat{\boxtimes}\mathcal{O})$ as a left D-module on X.

To specify a map $j_*j^*(\mathcal{B}(\mathfrak{g},0) \boxtimes \mathcal{A}_{\mathfrak{Gr}_g}) \to \Delta_!(\mathcal{A}_{\mathfrak{Gr}_g})$ is the same as to specify a continuous map $p_{1*}(\mathfrak{g} \otimes \mathcal{O}\hat{\boxtimes}\mathcal{O}) \otimes \mathcal{A}_{\mathfrak{Gr}_g} \to \mathcal{A}_{\mathfrak{Gr}_g}$ compatible with the right D-module structure.

Without restricting the generality, we can assume that X is affine. In this case $\mathrm{Lie}(\widetilde{\mathrm{Maps}}^{\{1\}}(X,G)) \simeq p_{1*}(\mathfrak{g} \otimes \mathcal{O} \boxtimes \mathcal{O})$ is dense in $p_{1*}(\mathfrak{g} \otimes \mathcal{O}\hat{\boxtimes}\mathcal{O})$ and it is enough to construct a continuous map

$$p_{1*}(\mathfrak{g} \otimes \mathcal{O} \boxtimes \mathcal{O}) \otimes \mathcal{A}_{\mathfrak{Gr}_g} \to \mathcal{A}_{\mathfrak{Gr}_g}.$$

However, such a map has been already constructed: this is the map $\mathrm{Act}^{\{1\}}$. The fact that property (a) of Proposition 5.3.1 holds follows from the fact that the map $\mathrm{Act}^{\{1,2\}}$ commutes with the chiral bracket (cf. the commutative diagram above). Property (b) of Proposition 5.3.1 follows from Lemma 5.3.2. □

CHAPTER 6
The Free Bosonic Theory

§6.1. The canonical line bundle

Let H be a torus, let Λ denote the lattice of 1-parameter subgroups in H and let $\mathfrak{h}$ be the Lie algebra of H. Consider the affine Grassmannians $\mathfrak{Gr}_H$, $\mathfrak{Gr}_H(X)$ and $\mathfrak{Gr}_H^{\mathcal{J}}(X)$ corresponding to the group H.

The reduced schemes ${}_{red}\mathfrak{Gr}_H$ and ${}_{red}\mathfrak{Gr}_H(X)$ are finite dimensional and are identified with Λ and $\Lambda \times X$, respectively. For $\lambda \in \Lambda$ we shall denote by $\mathfrak{Gr}_H^{\lambda}$ (resp., $\mathfrak{Gr}_H^{\lambda}(X)$) the corresponding connected component of $\mathfrak{Gr}_H$ (resp., of $\mathfrak{Gr}_H(X)$); we shall denote by $\mathbf{unit}^{\lambda}_{\mathfrak{Gr}}$ the canonical section $X \simeq {}_{red}\mathfrak{Gr}_H^{\lambda} \hookrightarrow \mathfrak{Gr}_H^{\lambda}$.

As for $\mathfrak{Gr}_H^{\mathcal{J}}(X)$, its set of connected (resp., irreducible) components is identified with Λ (resp., with $\Lambda^{\mathcal{J}}$). The reduced scheme corresponding to each irreducible component projects isomorphically onto $X^{\mathcal{J}}$. We leave it to the reader to work out the intersection pattern of various irreducible components of ${}_{red}\mathfrak{Gr}_H^{\mathcal{J}}(X)$. For $\overline{\lambda} \in \Lambda^{\mathcal{J}}$ we shall denote by $\mathbf{unit}^{\overline{\lambda}}_{\mathfrak{Gr}}$ the corresponding section $X^{\mathcal{J}} \to {}_{red}\mathfrak{Gr}_H^{\mathcal{J}}(X) \to \mathfrak{Gr}_H^{\mathcal{J}}(X)$.

§6.1.1.

Recall the groups $H(\hat{\mathcal{O}})$ and $H(\hat{\mathcal{K}})$ of Section 5.2.1. Now let now Q be an integral-valued even symmetric form on Λ. We shall denote by the same character Q the corresponding quadratic form on $\mathfrak{h}$. In addition, we shall choose once and for all a 2-cocycle ϵ of the group Λ with values in $\mathbb{Z}_2$ whose class in $H^2(\Lambda, \mathbb{Z}_2)$ corresponds to $Q \bmod 2$.

It is known (cf. [**8**], [**4**]) that to the data of (Λ, Q, ϵ) one can associate in a canonical way a central extension $H(\hat{\mathcal{K}})'$ of the ind-algebraic group $H(\hat{\mathcal{K}})$:

$$1 \to \mathbb{C}^* \to H(\hat{\mathcal{K}})' \to H(\hat{\mathcal{K}}) \to 1.$$

The group $H(\hat{\mathcal{K}})'$ has the following properties:

- We have a canonical lifting of the embedding $H(\hat{\mathcal{O}}) \to H(\hat{\mathcal{K}})$ to an embedding $H(\hat{\mathcal{O}}) \to H(\hat{\mathcal{K}})'$. In particular, we obtain an $H(\hat{\mathcal{K}})'$-equivariant line bundle R_Q on $\mathfrak{Gr}_H = H(\hat{\mathcal{K}})/H(\hat{\mathcal{O}})$.

- The Lie algebra of $H(\hat{\mathcal{K}})'$ is identified with the Heisenberg algebra $\mathrm{Heis}(\mathfrak{h}, Q) := \mathfrak{h} \otimes \hat{\mathcal{K}} \oplus \mathbb{C}$.
- For a complete curve curve X and n points $x_1, \ldots, x_n \in X$ consider the corresponding central extension

$$1 \to \mathbb{C}^* \to (H(\hat{\mathcal{K}}_{x_1}) \times \ldots \times H(\hat{\mathcal{K}}_{x_n}))' \to H(\hat{\mathcal{K}}_{x_1}) \times \ldots \times H(\hat{\mathcal{K}}_{x_n}) \to 1.$$

We have a canonical lifting of the embedding

$$\mathrm{Maps}(X \setminus \{x_1, \ldots, x_n\}, H) \subset H(\hat{\mathcal{K}}_{x_1}) \times \ldots \times H(\hat{\mathcal{K}}_{x_n})$$

to a homomorphism of formal group schemes $\mathrm{Maps}(X \setminus \{x_1, \ldots, x_n\}, H) \to H(\hat{\mathcal{K}})'_n$.

To state the additional property of $H(\hat{\mathcal{K}})'$ we need to introduce some notation. For a character $\mu \in \mathfrak{h}^*$, let V^μ denote the Weyl module over $\mathrm{Heis}(\mathfrak{h}, Q)$, i.e.

$$V^\mu \simeq \mathrm{Ind}^{\mathrm{Heis}(\mathfrak{h},Q)}_{\mathfrak{h}\otimes\hat{\mathcal{O}}\oplus\mathbb{C}}(\mathbb{C}^\mu),$$

where $\mathbb{C}^\mu$ denotes the 1-dimensional representation of $\mathfrak{h} \otimes \hat{\mathcal{O}} \oplus \mathbb{C}$ corresponding to $\mu : \mathfrak{h} \to \mathbb{C}$. When $\mu = Q(\lambda, \cdot)$ for $\lambda \in \Lambda \subset \Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{C} \simeq \mathfrak{h}$, we endow V^μ with an action of the group Aut_0^+ by requiring that it be compatible with the Aut_0^+-action on $\mathrm{Heis}(\mathfrak{h}, Q)$ and by setting $\mathbb{C}^\mu \simeq L_{-Q(\lambda,\lambda)/2}$ as an Aut_0^+-module (cf. Section 2.3.1).

Now note that the vector space $\Gamma(\mathfrak{Gr}_H^\lambda, R_Q)$ carries an Aut_0^+-action (this is due to the fact that the construction of $H(\hat{\mathcal{K}})'$ and hence of R_Q is canonical).

The fourth property of $H(\hat{\mathcal{K}})'$ reads as follows:

- We have an isomorphism of $\mathrm{Heis}(\mathfrak{h}, Q)$-modules: $\Gamma(\mathfrak{Gr}_H^\lambda, (R_Q)^{-1})^* \simeq V^\mu$, where $\mu = Q(\lambda, \cdot)$. This isomorphism is compatible with the Aut_0^+-action.

§6.1.2.

Globally over a (not necessarily complete) curve X, the form Q gives rise to a central extension

$$1 \to \mathbb{C}^* \to H'_{\mathcal{K}}(X) \to H_{\mathcal{K}}(X) \to 1$$

of the ind-group scheme $H_{\mathcal{K}}(X)$. Let $R_Q(X)$ denote the line bundle

$$R_Q(X) := H'_{\mathcal{K}}(X)/H_{\mathcal{O}}(X)$$

over $\mathfrak{Gr}_H(X) = H_{\mathcal{K}}(X)/H_{\mathcal{O}}(X)$. Property (2) of the extension $H(\hat{\mathcal{K}})'$ implies that $R_Q(X)$ is equivariant with respect to the action of the group scheme $H_{\mathcal{O}}(X)$ on $\mathfrak{Gr}_H(X)$.

Property (4) of the group $H(\hat{\mathcal{K}})'$ implies the following assertion:

Lemma 6.1.1. *We have a canonical isomorphism*

$$\mathbf{unit}^{\lambda}_{\mathfrak{Gr}}{}^*(R_Q(X)) \simeq \Omega^{-Q(\lambda,\lambda)/2}.$$

A generalization of the above construction yields a line bundle $R_Q^{\mathcal{I}}(X)$ over the pre-image in $\mathfrak{Gr}_H^{\mathcal{I}}(X)$ of $X^{\mathcal{I}} \setminus \Delta$. Our present goal is to extend this line bundle to the whole of $\mathfrak{Gr}_H^{\mathcal{I}}(X)$, i.e. to the locus where the base points $x_{i_1}^{\mathcal{I}}, \ldots, x_{i_n}^{\mathcal{I}} \in X$ collide with one another.

Assume now that X is complete. Property (3) of the extension $H(\hat{\mathcal{K}})'$ implies the following assertion:

Lemma 6.1.2. *For a complete curve X there exists a canonical line bundle $\mathcal{R}_Q(X)$ over $\mathrm{Bun}_H(X)$ such that for every $\mathcal{I}$, the restriction of the line bundle $k^{\mathcal{I}*}(\mathcal{R}_Q(X))$ to $\mathfrak{Gr}^{\mathcal{I}}_H(X) \setminus (r^{\mathcal{I}})^{-1}(\Delta)$ is identified with $R^{\mathcal{I}}_Q$.*

We now set $R^{\mathcal{I}}_Q(X)$ to be the line bundle over $\mathfrak{Gr}^{\mathcal{I}}_H(X)$ equal to $k^{\mathcal{I}*}(\mathcal{R}_Q(X))$. (When $\mathcal{I} = \{1, \ldots, n\}$, we shall replace the notation $R^{\mathcal{I}}_Q(X)$ simply by $R^n_Q(X)$.)

Since the projection $k^{\mathcal{I}} : \mathfrak{Gr}^{\mathcal{I}}_H(X) \to \mathrm{Bun}_H(X)$ respects the connection on $\mathfrak{Gr}^{\mathcal{I}}_H(X)$ along $X^{\mathcal{I}}$, we obtain a connection along $X^{\mathcal{I}}$ on the line bundle $R^{\mathcal{I}}(Q)$.

Proposition 6.1.1. *Under the isomorphisms of Proposition 5.2.2, the system of line bundles $\mathcal{I} \to R^{\mathcal{I}}_Q(X)$ satisfies the following factorization property:*

(a) *For a surjection of finite sets $\phi : \mathcal{I} \twoheadrightarrow \mathcal{J}$ the first isomorphism of Proposition 5.2.2 underlies an isomorphism of line bundles: $R^{\mathcal{I}}_Q(X)|\mathfrak{Gr}^{\mathcal{J}}_H(X) \simeq R^{\mathcal{J}}_Q(X)$.*

(b) *For a decomposition $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$ the second isomorphism of Proposition 5.2.2 underlies an isomorphism of line bundles:*

$$R^{\mathcal{I}}_Q(X)|(X^{\mathcal{I}_i \neq \mathcal{I}_j} \underset{X^{\mathcal{I}}}{\times} \mathfrak{Gr}^{\mathcal{I}}_H(X)) \simeq$$

$$\simeq R^{\mathcal{I}_1}_Q(X) \times \ldots \times R^{\mathcal{I}_1}_Q(X)|(X^{\mathcal{I}_i \neq \mathcal{I}_j} \underset{X^{\mathcal{I}_1} \times \ldots \times X^{\mathcal{I}_k}}{\times} \mathfrak{Gr}^{\mathcal{I}_1}_H(X) \times \ldots \times \mathfrak{Gr}^{\mathcal{I}_k}_H(X)).$$

Moreover, the above isomorphisms are compatible with the connections on the line bundles $R^{\mathcal{I}}_Q(X)$'s along the $X^{\mathcal{I}}$'s.

Proof. The isomorphism of point (a) follows from the definitions. To prove point (b) note that the two line bundles are obviously isomorphic over the pre-image of $X^{\mathcal{I}} \setminus \Delta$ in $\mathfrak{Gr}^{\mathcal{I}}_H(X)$ and we only have to show that this isomorphism extends to the whole of $\mathfrak{Gr}^{\mathcal{I}}_H(X)$. This follows easily (by degree considerations) from the following assertion:

Lemma 6.1.3. *For $\overline{\lambda} = \lambda_1, \ldots, \lambda_n$ the line bundle $\mathbf{unit}^{\overline{\lambda}}_{\mathfrak{Gr}}{}^*(R^n_Q(X))$ over X^n is identified canonically with*

$$\Omega^{-Q(\lambda_1,\lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{-Q(\lambda_n,\lambda_n)/2}(\sum_{i<j} -Q(\lambda_i, \lambda_j) \cdot \Delta_{x_i = x_j}).$$

□

§6.1.3.

We now want to show that the dependence of the line bundles $R^{\mathcal{I}}_Q(X)$ on X is local in the sense of Section 5.2.1. This fact is not immediately obvious, since in the definition of $R^{\mathcal{I}}_Q(X)$ we gave above we used the fact that X is complete. A way to overcome this difficulty will be via the ind-schemes $\widetilde{\mathfrak{Gr}}^n_H(X)$ introduced in Section 5.2.3.

Using the iterative description of the ind-scheme $\widetilde{\mathfrak{Gr}}^n_H(X)$ given in Section 5.2.3 we can produce a line bundle $\widetilde{R}^n_Q(X)$ over $\widetilde{\mathfrak{Gr}}^n_H(X)$. This construction works for curves that are not necessarily complete; therefore we can form line bundle $\widetilde{R}^n_Q(\mathfrak{X})$ over $\widetilde{\mathfrak{Gr}}^n_H(\mathfrak{X})$ which is local in the sense of Section 5.2.1. In more detail, let $X^{S,I}, x^{S,I}, \phi^{S,I}$ be as in Section 5.2.1 and let $\widetilde{\mathfrak{Gr}}^n_H(X^S) \setminus x^S$ denote the open subscheme of $\widetilde{\mathfrak{Gr}}^n_H(X^S)$ equal to the pre-image in $\widetilde{\mathfrak{Gr}}^n_H(X^S)$ of

$$(X^S \setminus x^S) \underset{S}{\times} \ldots \underset{S}{\times} (X^S \setminus x^S) \subset X \underset{S}{\times} \ldots \underset{S}{\times} X$$

under the projection $r^n \circ w_n$.

Then the isomorphism of ind-schemes

$$\widetilde{\mathfrak{Gr}}^n_H(X^{S\times I}) \setminus x^{S\times I} \simeq (\widetilde{\mathfrak{Gr}}^n_H(X^S) \setminus x^S) \times I$$

underlies an isomorphism of line bundles:

$$\widetilde{R}^n_Q(X^{S\times I})|\widetilde{\mathfrak{Gr}}^n_H(X^{S\times I}) \setminus x^{S\times I} \simeq (\widetilde{R}^n_Q(X^S)|\widetilde{\mathfrak{Gr}}^n_H(X^S) \setminus x^S) \times I.$$

Note that over the pre-image of $X^n \setminus \Delta$ in $\widetilde{\mathfrak{Gr}}^n_H(X)$ (which is identified with the pre-image of $X^n \setminus \Delta$ in $\mathfrak{Gr}^n_H(X)$) the line bundle $\widetilde{R}^n_Q(X)$ is identified naturally with $R^n_Q(X)$.

Proposition 6.1.2. *The line bundle $R^n_Q(X)$ is uniquely characterized by the following property:*

There exists a canonical isomorphism

$$w_n^*(R^n_Q) \simeq \widetilde{R}^n_Q$$

which reduces to the identity isomorphism over $X^n \setminus \Delta$.

The proof of this assertion is not difficult to deduce from the construction of the line bundle $\widetilde{R}^n_Q(X)$ and we will omit it.

Corollary 6.1.1. *The line bundle $R^{\mathcal{J}}_Q(\mathfrak{X})$ over $\mathfrak{Gr}^{\mathcal{J}}_H(\mathfrak{X})$ is local (in the above sense). Moreover, the factorization isomorphisms of Lemma 6.1.1 are compatible with the local structure.*

§6.2. Construction of the bosonic chiral algebra

§6.2.1.

For (H, Q) as above consider the chiral algebra $\mathcal{A}(\mathfrak{h}, Q)$ of Example 1 of Section 3.1.1. Our goal in this subsection is to construct a bigger chiral algebra $\mathcal{A}(H, Q)$ that will contain $\mathcal{A}(\mathfrak{h}, Q)$ as a subalgebra.

Recall that if Z is a strict ind-scheme and R is a line bundle over Z it makes sense to talk about R-twisted right D-modules on Z which are set-theoretically supported on finite-dimensional subschemes of Z.

Thus, consider the $R_Q(X)$-twisted right D-module

$$\mathbf{unit}^{\lambda}_{\mathfrak{Gr}!}(\Omega) \otimes R_Q(X)$$

on $\mathfrak{Gr}_H(X)$. Let $\mathcal{A}(H, Q)^{\lambda}$ denote its $\mathcal{O}$-module direct image onto X, i.e.

$$\mathcal{A}(H, Q)^{\lambda} := r_*(\mathbf{unit}^{\lambda}_{\mathfrak{Gr}!}(\Omega) \otimes R_Q(X)).$$

The connection on the pair $(\mathfrak{Gr}_H(X), R_Q(X))$ along X defines on $\mathcal{A}(H, Q)^{\lambda}$ a structure of right D-module on X. Finally, let $\mathcal{A}(H, Q)$ be the direct sum $\underset{\lambda\in\Lambda}{\oplus} \mathcal{A}(H, Q)^{\lambda}$. The sections $\mathbf{unit}^{\lambda}_{\mathfrak{Gr}} : X \to \mathfrak{Gr}_H(X)^{\lambda}$ give rise to $\mathcal{O}$-module maps

$$\mathbf{unit}^{\lambda} : \Omega^{-Q(\lambda,\lambda)/2+1} \hookrightarrow \mathcal{A}(H, Q);$$

for $\lambda = 0$ the map

$$\mathbf{unit} : \Omega \hookrightarrow \mathcal{A}(H, Q)^0 \hookrightarrow \mathcal{A}(H, Q)$$

is in fact a map of D-modules.

More generally, for a finite set $\mathcal{I}$ and for an element $\overline{\lambda} \in \Lambda^{\mathcal{I}}$ consider on $X^{\mathcal{I}}$ the right D-module

$$\mathcal{A}(H,Q)'^{\overline{\lambda}} := r^{\mathcal{I}}_*(\mathbf{unit}^{\overline{\lambda}}_{\mathfrak{Gr}!}(\Omega_{X^{\mathcal{I}}}) \otimes R^{\mathcal{I}}_Q(X)).$$

Let $\mathcal{A}(H,Q)'^{(\mathcal{I})}$ denote the direct sum $\underset{\overline{\lambda}\in\Lambda^{\mathcal{I}}}{\oplus} \mathcal{A}(H,Q)'^{\overline{\lambda}}$ and let ${}^l\mathcal{A}(H,Q)'^{(\mathcal{I})}$ be the corresponding left D-module on $X^{\mathcal{I}}$.

Consider the sheaf $r^{\mathcal{I}}_*((R^{\mathcal{I}}_Q(X))^{-1})$ over $X^{\mathcal{I}}$. Since $\mathfrak{Gr}^{\mathcal{I}}_H(X)$ is an inductive limit of schemes finite over $X^{\mathcal{I}}$, the above direct image is a projective limit of coherent sheaves on $X^{\mathcal{I}}$; moreover, it carries a natural left D-module structure due to the connection on $R^{\mathcal{I}}_Q(X)$ along $X^{\mathcal{I}}$. It is easy to see that we have in fact a canonical isomorphism of left D-modules:

$${}^l\mathcal{A}(H,Q)'^{(\mathcal{I})} \simeq \mathrm{Hom}_{\mathcal{O}_{X^{\mathcal{I}}}}(r^{\mathcal{I}}_*((R^{\mathcal{I}}_Q(X))^{-1}), \mathcal{O}_{X^{\mathcal{I}}}).$$

The following assertion follows from Proposition 6.1.1:

Lemma 6.2.1. *The system of $\mathcal{O}$-modules $\mathcal{I} \to {}^l\mathcal{A}(H,Q)'^{(\mathcal{I})}$ satisfies the conditions of Theorem 5.1.1.*

Thus, $\mathcal{A}(H,Q)$ has a natural structure of chiral algebra such that for every $\mathcal{I}$

$${}^l\mathcal{A}(H,Q)^{(\mathcal{I})} \simeq {}^l\mathcal{A}(H,Q)'^{(\mathcal{I})},$$

as $\mathcal{O}$-modules. The last assertion of Proposition 6.1.1 implies that the above isomorphisms preserve the D-module structure as well.

For $x \in X$, let $\mathfrak{Gr}_H(X)_x$ denote the fiber of $\mathfrak{Gr}_H(X)$ over x and let $R_Q(X)_x$ denote the restriction of $R_Q(X)$ to this fiber. (Once we choose an isomorphism $\hat{\mathcal{O}} \simeq \hat{\mathcal{O}}_x$, the pair $(\mathfrak{Gr}_H(X)_x, R_Q(X)_x)$ can be identified with $(\mathfrak{Gr}_H, R_Q)$.). By the construction, the fiber $A(H,Q)^{\lambda}_x$ of $\mathcal{A}(H,Q)^{\lambda}$ at x is identified with

$$\Gamma(\mathfrak{Gr}^{\lambda}_H(X)_x, (R_Q(X)_x)^{-1})^*.$$

Therefore, the Harish-Chandra pair $(\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}, H(\hat{\mathcal{O}}_x))$ acts on $A(H,Q)_x$; we shall denote by

$$\mathrm{Act} : (\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}) \otimes A(H,Q)_x \to A(H,Q)_x$$

the corresponding action map.

Property (4) from Section 6.1.1 implies that as a $(\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}, H(\hat{\mathcal{O}}_x))$-module, $A(H,Q)^{\lambda}_x$ is canonically isomorphic to the Weyl module V^{μ} with $\mu = Q(\lambda, \cdot)$.

Theorem 6.2.1. *We have:*

(a) *The chiral bracket on $\mathcal{A}(H,Q)$ preserves the Λ-grading, i.e.*

$$j_*j^*(\mathcal{A}(H,Q)^{\lambda} \boxtimes \mathcal{A}(H,Q)^{\mu}) \to \Delta_!(\mathcal{A}(H,Q)^{\lambda+\mu}).$$

In particular, $\mathcal{A}(H,Q)^0$ is a chiral subalgebra in $\mathcal{A}(H,Q)$ and each $\mathcal{A}(H,Q)^{\lambda}$ is a chiral module over $\mathcal{A}(H,Q)^0$.

(b) *There exists a canonical isomorphism of chiral algebras $\mathcal{A}(H,Q)^0 \simeq \mathcal{A}(\mathfrak{h},Q)$.*

(c) *For $x \in X$, the map $(\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}) \otimes A(H,Q)_x \to A(H,Q)_x$ induced by the chiral action of $\mathcal{B}(\mathfrak{h},Q)$ on $\mathcal{A}(H,Q)$ coincides with the map* Act *(cf. above).*

The proof of this theorem goes along the same lines as the proof of Theorem 5.3.1 and we will omit it.

§6.2.2.

We shall now show that the chiral algebra $\mathcal{A}(H,Q)$ defines a CFT of central charge $c = \dim(H)$.

It follows from Corollary 6.1.1 that the $\mathcal{O}_{X^J}$-modules ${}^l\mathcal{A}(H,Q)'^{(J)}$ carry a structure of local $\mathcal{O}$-modules over $\mathfrak{X}^J$. Moreover, the factorization isomorphisms are morphisms of local $\mathcal{O}$-modules. Hence, according to Proposition 5.1.1, the chiral algebra $\mathcal{A}(H,Q)$ satisfies condition (a) of Section 2.3.2 and it remains to endow it with an energy-momentum tensor such that condition (b) of Section 2.3.2 holds.

Consider the map $T_{H,Q} : \Theta'_{\dim(\mathfrak{h})} \to \mathcal{A}(H,Q)$ obtained as a composition:

$$\Theta_{\dim(\mathfrak{h})} \xrightarrow{T_{\mathfrak{h},Q}} \mathcal{A}(\mathfrak{h},Q) \simeq \mathcal{A}(H,Q)^0 \subset \mathcal{A}(H,Q).$$

Proposition 6.2.1. *The map $T_{H,Q}$ satisfies condition (b) of Section 2.3.2.*

Proof. For a vector field ξ on a curve X consider the two maps $\mathcal{A}(H,Q)^\lambda \to \mathcal{A}(H,Q)^\lambda$ given by $l \to l \cdot \xi + \mathrm{Lie}_\xi(l)$ and $(\mathrm{id} \boxtimes h)(\{l, T_{H,Q}(\xi')\})$, where ξ' is some lifting of ξ to a section of $\Theta'_{\dim(\mathfrak{h})}$. Both these maps are endomorphisms of the D-module structure on $\mathcal{A}(H,Q)^\lambda$; therefore, they induce endomorphisms of each fiber $A(H,Q)^\lambda_x$ for $x \in X$.

Now choose an identification $\hat{\mathcal{O}}_x \simeq \hat{\mathcal{O}}$. Since $T_{\mathfrak{h},Q}$ is an energy-momentum tensor for $\mathcal{A}(\mathfrak{h},Q)$, the above construction yields two actions of the Lie algebra Vir^+ on V^μ (here $\mu = Q(\lambda,\cdot)$) that are compatible with the Vir^+-action on $\mathrm{Heis}(\mathfrak{h},Q)$.

Moreover, we claim that the restrictions of these two actions to $\mathrm{Vir}_0^+ \subset \mathrm{Vir}^+$ coincide. Indeed, it is enough to check this fact on the generating space $\mathbb{C}^\mu \subset V^\mu$ and in both cases Vir_0^+ acts on $\mathbb{C}^\mu$ as on $L_{-Q(\lambda,\lambda)/2)}$ (this is an easy computation using Lemma 6.1.1 and Proposition 1.1.1).

The assertion of the proposition now follows from the following general fact:

Lemma 6.2.2. *There is at most one extension of the Vir_0^+-action on V^μ to an action of Vir^+ which is compatible with the action of the latter Lie algebra on $\mathrm{Heis}(\mathfrak{h},Q)$.*

□

§6.2.3.

Assume now that X is complete. Our present goal is to compute the space of conformal blocks of the chiral algebra $\mathcal{A}(H,Q)$.

Theorem 6.2.2. *For a complete curve X we have a canonical isomorphism*

$$H_\nabla(X, \mathcal{A}(H,Q)) \simeq H^0(\mathrm{Bun}_H(X), (\mathcal{R}_Q(X))^{-1})^*.$$

Before proving this theorem let us make a few observations:

Observation 1:

The stack $\mathrm{Bun}_H(X)$ splits into connected components

$$\mathrm{Bun}_H(X) = \underset{\lambda \in \Lambda}{\cup} \mathrm{Bun}_H(X)^\lambda$$

and each $\mathrm{Bun}_H(X)^\lambda$ is isomorphic to the quotient of the Jacobian of X by the trivial action of the group H; this corresponds to the fact that $\forall \mathcal{P} \in \mathrm{Bun}_H(X)$, $\mathrm{Aut}(\mathcal{P}) \simeq H$.

Observation 2:

It is easy to infer from the properties of the extension $H(\hat{\mathcal{K}})'$ in Section 6.1.1 that for every $\mathcal{P} \in \mathrm{Bun}_H(X)^\lambda$ the group H of automorphisms of $\mathcal{P}$ acts on the fiber of $\mathcal{R}_Q(X)$ at $\mathcal{P}$ via the character $Q(\lambda, \cdot)$. This implies, in particular, that the contribution to $H^0(\mathrm{Bun}_H(X), (\mathcal{R}_Q(X))^{-1})$ comes only from the 0-th component of $\mathrm{Bun}_H(X)$. This shows that the RHS of the isomorphism of Theorem 6.2.2 is finite-dimensional.

It is equally easy to see that if $x_1, \ldots, x_n$ are distinct points of X, then the canonical map $A(H,Q)^{\lambda_1}_{x_1} \otimes \ldots \otimes A(H,Q)^{\lambda_n}_{x_n} \to H_\nabla(X, \mathcal{A}(H,Q))$ is non-zero only if $\lambda_1 + \ldots + \lambda_n = 0$. (This is due to the fact that the subspace of "constant currents"

$$\mathfrak{h} \subset DR^0(X, \mathcal{A}(\mathfrak{h}, Q)) \subset DR^0(X, \mathcal{A}(H,Q))$$

acts on each $A(H,Q)^{\lambda_i}_{x_i} \simeq V^{\mu_i}$ by the character μ_i.)

Observation 3: The assertion of the theorem can be reformulated as an isomorphism between $H^0(\mathrm{Bun}_H(X), (\mathcal{R}_Q(X))^{-1})$ and $H_\nabla(X, \mathcal{A}(H,Q))^*$.

According to Proposition 5.1.2,

$$H_\nabla(X, \mathcal{A}(H,Q))^* \simeq \mathrm{Hom}_{D(X^n)}({}^l\mathcal{A}(H,Q)^{(n)}, \mathcal{O}_{X^n})$$

for any $n \geq 2$. Since

$${}^l\mathcal{A}(H,Q)^{(n)} \simeq \mathrm{Hom}_{\mathcal{O}_{X^n}}(r^n_*((R^n_Q(X))^{-1}), \mathcal{O}_{X^n}),$$

we obtain that the space $H_\nabla(X, \mathcal{A}(H,Q))^*$ is identified with the space

$$\Gamma(\mathfrak{Gr}^n_H(X), (R^n_Q(X))^{-1})^{D_{X^n}}$$

of D_{X^n}-invariant sections of $(R^n_Q(X))^{-1}$ for n as above.

Let us write out the above isomorphism

$$H_\nabla(X, \mathcal{A}(H,Q))^* \simeq \Gamma(\mathfrak{Gr}^n_H(X), (R^n_Q(X))^{-1})^{D_{X^n}}$$

more explicitly:

Let χ be an element of $H_\nabla(X, \mathcal{A}(H,Q))^*$. Let us on the one hand consider the composition

$$\Omega^{-Q(\lambda_1,\lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{-Q(\lambda_n,\lambda_n)/2} \xrightarrow{\mathbf{unit}^{\lambda_1} \boxtimes \ldots \boxtimes \mathbf{unit}^{\lambda_n}} {}^l\mathcal{A}(H,Q)^{\boxtimes n} \to$$

$$j_*j^*(\mathcal{O}_{X^n}) \otimes H_\nabla(X, \mathcal{A}(H,Q)) \xrightarrow{\chi} j_*j^*(\mathcal{O}_{X^n}).$$

The above map can be regarded as a section of $\Omega^{Q(\lambda_1,\lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n,\lambda_n)/2}$ over $X^n \setminus \Delta$.

On the other hand, an element $\chi' \in \Gamma(\mathfrak{Gr}^n_H(X), (R^n_Q(X))^{-1})^{D_{X^n}}$ can be regarded as a map from $\mathfrak{Gr}^n_H(X)$ to the total space of the line bundle $(\mathcal{R}_Q(X))^{-1}$ over $\mathrm{Bun}_H(X)$. According to Lemma 6.1.1, the pull-back of the line bundle $(\mathcal{R}_Q(X))^{-1}$ under the composition

$$k^n \circ \mathbf{unit}^{\overline{\lambda}}_{\mathfrak{Gr}} \circ j : X^n \setminus \Delta \hookrightarrow X^n \to \mathrm{Bun}_H(X)$$

is the line bundle $\Omega^{Q(\lambda_1,\lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n,\lambda_n)/2}$ over $X^n \setminus \Delta$. Therefore, the composition

$$\chi' \circ \mathbf{unit}^{\overline{\lambda}}_{\mathfrak{Gr}} \circ j : X^n \setminus \Delta \hookrightarrow X^n \to \mathrm{Tot}(\mathcal{R}_Q(X)^{-1})$$

is again a section of $\Omega^{Q(\lambda_1,\lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n,\lambda_n)/2}$ over $X^n \setminus \Delta$.

It follows from the definitions that the above two sections of $\Omega^{Q(\lambda_1,\lambda_1)/2}\boxtimes\ldots\boxtimes\Omega^{Q(\lambda_n,\lambda_n)/2}(\infty\cdot\Delta)$ coincide if the elements χ and χ' correspond to one another under the isomorphism

$$H_\nabla(X,\mathcal{A}(H,Q))^*\simeq\Gamma(\mathfrak{Gr}^n_H(X),(R^n_Q(X))^{-1})^{DX^n}.$$

Proof. (Of Theorem 6.2.2)

By the very definition of the connection on the line bundle $R^n_Q(X)$, we have a map

$$H^0(\mathrm{Bun}_H(X),(\mathcal{R}_Q(X))^{-1})\to\Gamma(\mathfrak{Gr}^n_H(X),(R^n_Q(X))^{-1})$$

whose image belongs to $\Gamma(\mathfrak{Gr}^{\mathcal{J}}_H(X),(R^n_Q(X))^{-1})^{DX^n}$. Moreover, the assertion of the above Observation 3 combined with Proposition 1.3.3 imply that the composition

$$H^0(\mathrm{Bun}_H(X),(\mathcal{R}_Q(X))^{-1})\to\Gamma(\mathfrak{Gr}^n_H(X),(R^n_Q(X))^{-1})^{DX^n}\to H_\nabla(X,\mathcal{A}(H,Q))^*$$

is the same for any n.

Moreover, the map k^n is dominant for any n. Therefore, the map

$$H^0(\mathrm{Bun}_H(X),(\mathcal{R}_Q(X))^{-1})\to H_\nabla(X,\mathcal{A}(H,Q))^*$$

is injective and it remains to show that it is surjective. This will be done in two steps.

Step 1

Let

$$\chi'\in\Gamma(\mathfrak{Gr}^n_H(X),(R^n_Q(X))^{-1})^{DX^n}\subset\Gamma(\mathfrak{Gr}^n_H(X),k^{n*}(\mathcal{R}^n_Q(X))^{-1})$$

be as above. We shall first show that it is locally constant along the fibers of the projection k^n.

Let $(x_1,\ldots,x_n,\mathcal{P},\alpha:\mathcal{P}\to\mathcal{P}_0|X\setminus\{x_1,\ldots,x_n\})$ be a $\mathbb{C}$-point of $\mathfrak{Gr}^n_H(X)$ and for a local Artinian scheme I let $(x^I_1,\ldots,x^I_n,\mathcal{P},\alpha^I:\mathcal{P}^I\to\mathcal{P}^I_0|X\setminus\{x^I_1,\ldots,x^I_n\})$ be its extension to an I-valued point of $\mathfrak{Gr}^n_H(X)$, such that $\mathcal{P}^I=\mathcal{P}\times I$.

We must show that the composition

$$I\to\mathfrak{Gr}^n_H(X)\xrightarrow{\chi'}\mathrm{Tot}(\mathcal{R}_Q(X)^{-1})$$

is a constant map

$$I\to\mathrm{Spec}(\mathbb{C})\to\mathfrak{Gr}^n_H(X)\xrightarrow{\chi'}\mathrm{Tot}(\mathcal{R}_Q(X)^{-1}).$$

Obviously, it is enough to check this property when $I=\mathrm{Spec}(\mathbb{C}[t]/t^2)$.

First of all, using the connection along X^n on $\mathfrak{Gr}^n_H(X)$ we can reduce the assertion to the case when the x_i are constant maps $I\to X$ and without loss of generality we can assume that the points $x_1,\ldots,x_n$ are distinct.

In this case the difference between the trivializations $\alpha\times I$ and α^I of $\mathcal{P}^I\simeq\mathcal{P}\times I$ over $(X\setminus\{x_1,\ldots,x_n\})\times I$ is given by an infinitesimal "gauge transformation", i.e. by an action on the fiber of $\mathfrak{Gr}^n_H(X)$ over $(x_1,\ldots,x_n)$ by an element of

$$H^0(X\setminus\{x_1,\ldots,x_n\},\mathfrak{h}\otimes\mathcal{O})\subset\mathfrak{h}\otimes\hat{\mathcal{K}}_{x_1}\oplus\ldots\oplus\mathfrak{h}\otimes\hat{\mathcal{K}}_{x_n}.$$

Therefore, in terms of the corresponding functional $\chi:H_\nabla(X,\mathcal{A}(H,Q))\to\mathbb{C}$, we must prove that the functional on $A(H,Q)_{x_1}\otimes\ldots\otimes A(H,Q)_{x_n}$ obtained as a composition

$$A(H,Q)_{x_1}\otimes\ldots\otimes A(H,Q)_{x_n}\to H_\nabla(X,\mathcal{A}(H,Q))\xrightarrow{\chi}\mathbb{C}$$

is $H^0(X \setminus \{x_1, \ldots, x_n\}, \mathfrak{h} \otimes \mathcal{O})$-invariant. However, this follows from the definition of the space of conformal blocks in view of point (c) of Theorem 6.2.1.

Step 2

To prove the theorem it remains to show that if $(x_1, \ldots, x_n, \mathcal{P}, \alpha)$ and $(x'_1, \ldots, x'_n, \mathcal{P}', \alpha')$ are two $\mathbb{C}$-points of $\mathfrak{Gr}^n_H(X)$ with $\mathcal{P} = \mathcal{P}'$, then χ' (we retain the notation of Observation 3) takes on these points the same value.

Let now $\nu_1, \ldots, \nu_k$ be a basis of Λ, let m be an integer satisfying $m > 2g(X) - 2$ and let $n = 2m \cdot k$. Take an element $\overline{\lambda}(m) \in \Lambda^n$ equal to

$$\overline{\lambda}(m) = \underbrace{\lambda_1, \ldots, \lambda_1}_{m \text{ times}}, \ldots, \underbrace{-\lambda_1, \ldots, -\lambda_1}_{m \text{ times}}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{m \text{ times}}, \ldots, \underbrace{-\lambda_k, \ldots, -\lambda_k}_{m \text{ times}}.$$

First of all, we can assume that the two points $(x_1, \ldots, x_n, \mathcal{P}, \alpha)$ and $(x'_1, \ldots, x'_n, \mathcal{P}', \alpha')$ belong to the irreducible component $\mathfrak{Gr}^{\overline{\lambda}(m)}_H(X)$, where $\overline{\lambda}(m)$ is as above, for some m (this can be arranged using Observations 2 and 3 by increasing n if necessary). Recall that $\mathbf{unit}^{\overline{\lambda}}_{\mathfrak{Gr}}(X) : X^n \to \mathfrak{Gr}^n_H(X)$ defines an isomorphism between X^n and the reduced scheme of the corresponding irreducible component of $\mathfrak{Gr}^n_H(X)$.

The composition

$$X^n \hookrightarrow {}_{red}\mathfrak{Gr}^n_H(X) \hookrightarrow \mathfrak{Gr}^n_H(X) \xrightarrow{k^n} \mathrm{Bun}_H(X)$$

factors in this case via

$$X^n \to \underbrace{X^{(m)} \times \ldots \times X^{(m)}}_{2k \text{ times}} \to \underbrace{\mathrm{Pic}^m(X) \times \ldots \times \mathrm{Pic}^m(X)}_{2k \text{ times}} \simeq \mathrm{Bun}_H(X),$$

where $X^{(m)}$ denotes the m-th symmetric power of X and the second arrow is the product of the Abel-Jacobi maps $X^{(m)} \to \mathrm{Pic}^m(X)$.

We claim that it is enough to show that the map $\chi' : X^n \to \mathrm{Tot}(\mathcal{R}_Q(X)^{-1})$ factors via

$$X^n \to X^{(m)} \times \ldots \times X^{(m)} \to \mathrm{Tot}(\mathcal{R}_Q(X)^{-1}) :$$

Indeed, the map

$$X^{(m)} \times \ldots \times X^{(m)} \to \mathrm{Bun}_H(X)$$

is smooth, proper and has connected fibers, therefore any section of the pull-back of $(\mathcal{R}_Q(X))^{-1}$ to $X^{(m)} \times \ldots \times X^{(m)}$ comes from a section of this line bundle over $\mathrm{Bun}_H(X)$.

To prove the required factorization property of χ', we can restrict our attention to $X^n \setminus \Delta \subset X^n$. Using Observation 3, all we need to show is that the composition

$$(\Omega^{-Q(\lambda_1,\lambda_1)/2})^{\boxtimes m} \boxtimes \ldots \boxtimes (\Omega^{-Q(\lambda_n,\lambda_n)/2})^{\boxtimes m} \to {}^l\mathcal{A}(H,Q)^{\boxtimes n} \to$$

$$\to j_* j^*(\mathcal{O}_{X^n}) \otimes H_\nabla(X, \mathcal{A}(H,Q)) \xrightarrow{\chi} j_* j^*(\mathcal{O}_{X^n})$$

is invariant under the group $\underbrace{S^m \times \ldots \times S^m}_{2k \text{ times}}$ of permutations. However, this follows from the fact that any correlation function is symmetric in the insertions (cf. Section 1.3.3).

□

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Kaluza-Klein Compactifications, Supersymmetry, and Calabi-Yau Spaces

Andrew Strominger

Introduction

As we have seen in [II-**Strings**], study of the ten-dimensional heterotic string dynamics leads us to a low energy effective action of the form:

$$S_{\text{eff}}[g, A, \phi, \dots] = \frac{1}{2\kappa^2} \int_{X^{10}} d^{10}x \sqrt{-\det(g)} e^{-2\phi} \left(R(g) + \frac{\alpha'}{30} \operatorname{Tr} F_A^2 + \cdots \right)$$

for a metric g of signature $(-,+,+,\cdots,+)$ on a ten-dimensional manifold X^{10}, a connection A on a principal G-bundle over X^{10}, a scalar field ϕ, called the *dilaton field*, and other fields that we have not written here including the supersymmetric partners of the given fields. Here $R(g)$ is the scalar curvature of g, G is compact, semisimple, F_A is the curvature of A and $\operatorname{Tr} F_A^2$ represents a G-invariant metric on the Lie algebra of G. The study of the heterotic string shows that it must be the case that the Cartan matrix of the Dynkin diagram for G is an even unimodular lattice in dimension 16. The only two possibilities for G are $E_8 \times E_8$ or $Spin(32)/(\mathbf{Z}/2\mathbf{Z})$ for a central $\mathbf{Z}/2\mathbf{Z}$ which is not in the kernel of the projection to $SO(32)$. At the level at which we are working in these lectures, the delicate distinction between various groups with Lie algebra $SO(32)$ cannot be seen. For this reason, and also to simplify the notation, we shall consider $SO(32)$ instead of the more exotic form. Lastly, the expression $d^{10}x\sqrt{-\det(g)}$ is the volume element associated to the metric. The minus sign arises from the fact that the metric is of signature $(1,9)$, and the sign is necessary to have a positive quantity under the square root.

Our goal in these lectures is to study and analyze this Lagrangian in its own right, not considering the small corrections arising from string theory. It turns out that, because of the topological nature of the analysis, our conclusions remain largely valid in the exact string theory. In fact, this Lagrangian was written down as a Lagrangian for supergravity before string theory was invented, and so is of interest in its own right. There are two basic problems that any such analysis should resolve if this theory is to be a reasonable model of nature.

Problem 1. This action exists for metrics, connections, etc. on a ten-dimensional manifold X^{10}, yet it seems fairly apparent that any theory of nature must have to do with the four-dimensional space-time of our universe. Consequently, there must be an explanation of the 'collapsing' of ten space-time dimensions to four.

Notes by John Morgan

Problem 2. The two gauge groups that occur for this action, $E_8 \times E_8$ and $SO(32)$, are not appropriate ones for the observed matter content of the universe. Somehow, by a mechanism of spontaneous symmetry-breaking, we must get down to a smaller group which is suitable for describing observed matter.

Our approach will be to first look for classical ground states (solutions to the classical equations of motion minimizing the action) and study their features. At a later stage one should consider quantum corrections.

Motivated by the two problems we listed above, here are the properties that we want our ground states (or classical solutions) to have:

1. The effective theory looks four-dimensional instead of ten-dimensional.

2. The gauge group G should be broken down to a 'good' gauge group, one that fits with the GUT (grand unification theory) scheme, so that it produces the matter content observed in nature. The gauge group of the standard model is $SU(3) \times SU(2) \times U(1)$ and accounts for all the matter that we know of, in the sense that all the 'elementary' particles are grouped together into a representation of this reductive group. The GUT schemes are to embed this group in a larger group, $SU(3) \times SU(2) \times U(1) \subset G$, so that the representations that occur in the standard model occur among those arising in the restriction of an irreducible representation of G to the subgroup $SU(3) \times SU(2) \times U(1)$. This scheme (among others) allows the following set of groups for G:

$$SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_6.$$

Of course, on the level of groups on can continue the series with

$$E_6 \subset E_7 \subset E_8,$$

but the irreducible representations that we must use have to be chiral and hence not isomorphic to their duals. This rules out E_7 and E_8, where the relevant representations are real. Somehow we must break either $E_8 \times E_8$ or $SO(32)$ down to one of these groups, most likely E_6.

3. Our solution should have $N = 1$ supersymmetry in dimension 4 (the $N = 1$ super Poincaré group should act as a supergroup of symmetries on the theory). We do not observe supersymmetry in nature, so whatever supersymmetry exists in our effective four-dimensional gravity theory must be capable of being entirely broken to $N = 0$, hopefully by small quantum corrections, which we shall not discuss here. There are no mechanisms for breaking $N \geq 2$ supersymmetry down to a chiral theory. This implies that we can have at most $N = 1$ supersymmetry in our model. On the other hand, we desire nontrivial supersymmetry in our model because it is the best mechanism we know which is capable of accounting for the very small ratios of masses of the observed low energy particles, i.e., for solving the 'hierarchy problem,' as it is known to physicists.

4. Our solution should yield a realistic matter content.

References and cross references

The results were worked out in the mid 1980's. More recently, with the advent of dualities, there are new wrinkles, but the basic story as it was understood ten years ago remains relevant. The original sources are

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LECTURE 1
Compactification to Dimension Four

§1.1. Kaluza-Klein model

Before compactifying from dimension ten to dimension four, let us begin with a much simpler and much older example, which goes back to the work of Kaluza-Klein in 1926. We consider the five-dimensional Einstein-Hilbert action

$$S[\widehat{g}] = \frac{1}{2\pi\kappa^2}\int_Y d^5z\sqrt{-\det(\widehat{g})}R(\widehat{g}).$$

(The variable z is real.) The action is for a metric $\widehat{g}$ on a five-dimensional space, and $R(\widehat{g})$ is the scalar curvature of g. Lastly, κ is Newton's constant. (The hats are simply to distinguish this from other metrics which will arise later.) The classical equations of motion for this action are Einstein's equations

$$R_{MN}(\widehat{g}) = 0,$$

the vanishing of the Ricci curvature tensor of $\widehat{g}$.

One solution $\widehat{\widehat{g}}$ is on the manifold $Y = \mathbf{R}^4 \times S^1$ where the metric is

$$\widehat{\widehat{g}}_{MN}(z)dz^M dz^N = \eta_{\mu\nu}dx^\mu dx^\nu + r^2 d\theta^2; \quad \text{where } z = (x,\theta).$$

Here we have introduced several conventions which we shall use consistently. The indices μ,ν range from 0 to 3 and are indices for coordinates x^μ in Minkowski four-dimensional space-time. The metric tensor $\eta_{\mu\nu}$ is the standard diagonal form $(-,+,+,+)$. The indices M,N range from 0 to 4 and index the coordinates z^M on Y. Here, r is a constant which determines the radius of the internal circle.

We begin with this basic solution $\widehat{\widehat{g}}$ and study the situation near this solution. It is convenient to write metrics (or symmetric two-tensors) in a slightly different basis:

$$\widehat{g}_{MN}(z)dz^M dz^N = e^{-\rho/3}\left(e^\rho\left(d\theta + \kappa A_\mu dx^\mu\right)^2 + g_{\mu\nu}dx^\mu dx^\nu\right).$$

In this description the metric on the five-dimensional space has become three fields: a four-dimensional metric tensor $g_{\mu\nu}$, a one-form on four-space $A_\mu dx^\mu$, and a scalar field ρ. Of course, at this point all these fields depend not only on the point in the four-space but also on the auxiliary (or *internal*) variable θ parameterizing the circle.

Now let us consider the expansion in Fourier modes with respect to the variable θ on the circle:

$$g_{\mu\nu}(z) = \sum_{n=-\infty}^{\infty} g_{\mu\nu;n}(x)e^{in\theta},$$

$$A_\mu(z) = \sum_{n=-\infty}^{\infty} A_{\mu;n}(x)e^{in\theta},$$

$$\rho(z) = \sum_{n=-\infty}^{\infty} \rho_n(x)e^{in\theta}.$$

At this point what we have done is to rewrite the five-dimensional metric as an infinite collection of purely four-dimensional fields (metrics, one-forms, scalars). As we shall see shortly, when taking the low energy effective limit we can ignore all but finitely many of these fields. (We keep only the zero modes in the above Fourier expansions.)

Now we are ready to consider excitations (or linearized fluctuations) around our solution $\widehat{\widehat{g}}$. Consider a general metric $\widehat{g}_{MN}(z)dz^M dz^N$ near $\widehat{\widehat{g}}$. The linearized fluctuation is

$$\tilde{h}_{MN}(z) = \widehat{g}_{MN}(z) - \widehat{\widehat{g}}_{MN}.$$

It is convenient to define

$$h_{MN} = \tilde{h}_{MN} - \frac{1}{2}\widehat{\widehat{g}}^{RS}\tilde{h}_{RS}\widehat{\widehat{g}}_{MN}.$$

We view the symmetric two-tensor $h = h_{MN}(z)dz^M dz^N$ as parameterizing the fluctuations of the metric. We must take some care here for there are gauge symmetries, the group of diffeomorphisms of Y. Metrics lying on the same orbit of this group action must be considered as equivalent since they differ by simply reparametrizing the space. The standard way around this problem is to pick a gauge (a slice perpendicular to the orbit of the gauge symmetries). In the current context the simplest gauge to pick is called the *transverse gauge*. It consists of considering metrics satisfying

$$\nabla^M h_{MN} = 0.$$

Now, as we observed above, the equations of motion are the Einstein equations

$$R_{MN}(\widehat{g}) = 0.$$

Expanding $R_{MN}(\widehat{\widehat{g}} + h) = 0$ to first order in h yields

$$\Box^5_{\widehat{\widehat{g}}} h_{MN}(z) = \left(\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{r^2}\partial_\theta^2\right) h_{MN}(x,\theta) = 0.$$

Combining this equation with the Fourier mode expansion we see that

$$\Box^4_{\widehat{\widehat{g}}} h_{MN;n}(x) - \frac{n^2}{r^2}h_{MN;n}(x) = 0.$$

That is to say, $h_{MN;n}(x)$ satisfies the wave equation in four-space for a field of mass $|n|/r$. Invoking a basic result from quantum mechanics, we know that the lowest energy excitation of a field of mass $|n|/r$ has energy at least $|n|/r$. Now it is time to say more about the size of the circle factor in Y. So far it has been a free parameter in the theory. We shall take r to be roughly the Planck length, which is 10^{-33} cm. At this length scale, all the excitations of $h_{MN;n}(x)$ have huge energy provided that $n \neq 0$. Since we are interested in the low energy effective theory, we

can safely ignore all the modes $h_{MN;n}(x)$ except the zero modes. Thus, at energy scales below $1/r$, our effective theory is purely four-dimensional. It has three fields on four-space which are now independent of the internal variable θ. The fields are: a metric $g_0 = g_{\mu\nu;0}(x)dx^\mu dx^\nu$ on $\mathbf{R}^4$, a one-form $A_0 = A_{\mu;0}(x)dx^\mu$ on four-space and a scalar field $\rho_0(x)$ on four-space.

Next, let us plug these fields into the Kaluza-Klein action and write the resulting effective four-dimensional action. It is actually a somewhat complicated, if routine, computation to carry out the substitution; what comes out is:

$$S_{\text{eff}}[g_0, A_0, \rho_0] = r\int_{\mathbf{R}^4} d^4x\sqrt{-g_0}\Big(\frac{1}{\kappa^2}R(g_0) - \frac{1}{4}e^{\rho_0}F^0_{\mu\nu}F_0^{\mu\nu} - \frac{1}{6\kappa^2}(\nabla\rho_0)^2 + \Big((\nabla\rho_1)^2 + \frac{1}{r^2}\rho_1^2 + \cdots\Big)\Big),$$

where $F^0_{\mu\nu}dx^\mu dx^\nu$ is the curvature of the connection A_0. All the terms enclosed in the last pair of parentheses represent massive particles and can be ignored in our low energy effective action. Thus, we are left with

$$S_{\text{eff}}[g_0, A_0, \rho_0] = r\int_{\mathbf{R}^4} d^4x\sqrt{-g_0}\left(\frac{1}{\kappa^2}R(g_0) - \frac{1}{4}e^{\rho_0}F^0_{\mu\nu}F_0^{\mu\nu} - \frac{1}{6\kappa^2}(\nabla\rho_0)^2\right).$$

We began with a five-dimensional gravity theory and for sufficiently small radius r we have found the low energy effective four-dimensional action and written this action in terms of three four-dimensional fields: a metric, a connection which is related to the Killing vector field on the five-manifold tangent to the circle directions, and a scalar field ρ which is related to the free parameter r in the five-dimensional theory.

Let us now consider the symmetries of this effective four-dimensional action. We claim that these are simply the symmetries of the original five-manifold $Y = \mathbf{R}^4 \times S^1$ which commute with the natural S^1-action coming from this product structure of Y. This group is generated by two types of diffeomorphisms. The first are simply diffeomorphisms of $\mathbf{R}^4$. These are generated by vector fields ζ on $\mathbf{R}^4$, independent of θ. It is easy to see that the action of these on our fields are given by:

$$\delta_\zeta g_{\mu\nu} = \mathcal{L}_\zeta(g_{\mu\nu}),$$
$$\delta_\zeta A_\mu = \mathcal{L}_\zeta(A_\mu),$$
$$\delta_\zeta \rho = \mathcal{L}_\zeta(\rho).$$

Here, in all cases $\mathcal{L}_\zeta$ denotes the Lie derivative in the ζ-direction. The second type of diffeomorphisms are those generated by vector fields of the form

$$x^\mu \mapsto x^\mu; \quad \theta \mapsto \theta + \zeta^5(x).$$

Under these vector fields we have

$$\delta_\zeta(A_\mu) = -\frac{1}{\kappa}\partial_\mu\zeta^5,$$
$$\delta_\zeta g = \delta_\zeta\rho = 0.$$

Notice, in particular, that $A = A_\mu dx^\mu$ transforms like an abelian (i.e., $U(1)$) gauge field.

Of course, there are other diffeomorphisms of the five-manifold Y, which do not preserve the S^1-action. But these diffeomorphisms mix the massive and low energy modes. Since we have thrown away the massive modes in forming our effective

four-dimensional Lagrangian, these symmetries are broken in the limiting process, leaving the unbroken group for the four-dimensional effective action exactly the one described in the previous paragraph.

§1.2. Compactifying Einstein's equation from dimension ten to dimension four

Now let us consider ten-dimensional pure gravity. The action is

$$S[\hat{g}] = \int_{X^{10}} d^{10}x\sqrt{-\det(\hat{g})}R(\hat{g}),$$

where $\hat{g}$ is a metric on X^{10} with signature $(-,+,\cdots,+)$. The classical solutions extremizing this Einstein-Hilbert action are Ricci-flat metrics

$$R_{MN}(\hat{g}) = 0.$$

In analogy with the Kaluza-Klein model, let us study ground states (classical solutions) of the following form: The manifold X^{10} is a product $\mathbf{R}^4 \times K^6$ where K^6 is compact, and the metric is of the form $\hat{g}_0 = (\hat{g}_0)_{MN}dz^M dz^N$ where

$$(\hat{g}_0)_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn}(y), \end{pmatrix}$$

where g_0 is a positive definite metric on K^6. Our notation is similar to the five-dimensional case. The indices μ, ν run from 0 to 3 and index the coordinates x^μ on Minkowski space. The indices m, n index the internal coordinates y^m which are local coordinates on K. The indices M, N run from 0 to 9 and index local coordinates $z = (x, y)$ on the product $\mathbf{R}^4 \times K^6$. The split metric $\hat{g}_0$ is Ricci-flat if and only if g_0 is a Ricci-flat metric on K^6. The existence of such a metric, of course, imposes a topological condition on Y. We fix now a classical solution $\hat{g}_0$ of this form.

As in the Kaluza-Klein model, let us study small fluctuations about this classical solution:

$$h_{MN}(z) = \hat{g}_{MN}(z) - (\hat{g}_0)_{MN}.$$

Again, there is the gauge symmetry of the group of diffeomorphisms of X^{10} to deal with, and again we use the transverse gauge. The Ricci-flat condition to first order in h_{MN} is

$$\left(\Box^4 + \Box^6\right) h_{MN}(z) = 0,$$

where

$$\Box^4 h_{\mu\nu} = \eta^{\mu\nu}\partial_\mu\partial_\nu h_{\mu\nu}$$

and

$$\Box^6 h_{mn} = (g_0)^{ab}\nabla_a\nabla_b h_{mn} + 2R^a{}_m{}^b{}_n h_{ab}.$$

Following the same line of reasoning as before, we expand in Fourier modes in harmonics of K^6. We write

$$h_{\mu\nu}(z) = \sum_k h_{\mu\nu;k}(x)Y_k(y)$$

where

$$\Box^6 Y_k(y) = -\lambda_k Y_k(y).$$

Here, since the metric on Y is positive definite, this d'Alembertian on scalar functions on Y is the (negative of the) usual Laplacian. It is a non-positive operator, so that $\lambda_k \geq 0$ for all k. The equation satisfied by $h_{\mu\nu;k}$ is

$$(\Box^4 - \lambda_k)h_{\mu\nu;k}(x) = 0.$$

Once again we find that $h_{\mu\nu;k}$ is a solution to the wave equation in four-space for a field of mass $\sqrt{\lambda_k}$. If we scale the metric on K^6 by multiplying it by $1/r$, then λ_k scales by $1/r^2$. In particular, if we scale by $1/r$ for r very small, then for all k for which $\lambda_k \neq 0$, all excitations of the field $h_{\mu\nu;k}(x)$ become extremely massive, and hence can be ignored in our low energy effective action.

There is a similar analysis for the fields $h_{mn}(z)$. We impose the transverse gauge condition on the metric in the K^6-direction and we expand the equations of motion in the harmonics of K^6:

$$h_{mn}(z) = \sum_k \varphi_k(x) Y_{mn;k}(y),$$

where this time $Y_{mn;k}(y)dy^m dy^n$ are sections of the symmetric square of the cotangent bundle of Y which are eigenvectors with eigenvalue λ'_k for the d'Alembertian:

$$\Box^6 Y_{mn;k}(y) = g^{ab}\nabla_a \nabla_b Y_{mn;k}(y) + 2R^a{}_m{}^b{}_n Y_{ab;k}(y) = -\lambda'_k Y_{mn;k}(y).$$

Thus, we obtain

$$\Box^4 \varphi_k(x) - \lambda'_k \varphi_k(x) = 0.$$

As before, this is the wave equation for a wave of mass $\sqrt{\lambda'_k}$.

Here, though, it is not necessarily the case that $\lambda'_k \geq 0$ for all k. There can be a finite number of negative eigenvalues for the operator on Y. A negative eigenvalue means that our solution g_0 is a critical point for the Einstein-Hilbert action for positive definite metrics on Y which is not a local minimum for this action, i.e., an unstable critical point. We wish to avoid this possibility, so for now we simply assume that our operator $\Box^6$ on Y has no negative eigenvalues. (This will be automatic by supersymmetry in the Calabi-Yau case.) As long as $\lambda'_k > 0$, all excitations of this field will have mass on the order of $1/r^2$ and can be ignored in our low energy limit. Thus, we keep only those $\varphi_k(x)$ for which $\lambda'_k = 0$, i.e., we keep only those $\varphi_k(x)$ for which the corresponding $Y_{mn;k}(y)dy^m dy^n$ is a harmonic section of the symmetric square of the co-tangent bundle of K^6. This space of harmonic forms is the formal tangent space at g_0 to the moduli space of Ricci-flat metrics on K^6. So, for each parameter in the moduli space of solutions to the Einstein equations on the internal space K^6 we have a massless scalar for the effective action on four-space. These fields are the analogues of the massless field ρ we found in the Kaluza-Klein model, a field which came from the one parameter, the radius, describing the metric on the internal space in that theory.

§1.3. Adding Matter to the Mix

We have made a fairly complete analysis of pure gravity compactified from dimension ten to dimension four by using a compact six-dimensional manifold K^6 with a Ricci-flat metric. It is time now to add other terms to the action so as to create a theory that has matter as well as gravity. Our goal here is to give the general nature of the terms that will add matter, both bosons and fermions, to the action that we have been studying. In the next section we will be more explicit as to the exact nature of the terms we are adding. Here the analysis is quite general and will

only be sketched, since much of it is similar to what we have done in great detail above for the metric.

First of all, we can add boson fields by adding to the action a term of the form $S(\varphi)$ where φ is a scalar or vector-valued function on ten-dimensional space-time. Once again, we expand φ in Fourier modes using the harmonics of Y, and we can disregard all but the zero modes. In this way, after compactification φ becomes a boson field on four-space.

Let us consider fermions. We fix a spin structure on $\mathbf{R}^4 \times Y^6$ leading to spinor fields ψ which can be sections of the spin bundles tensored with other auxiliary bundles with connections. The Clifford algebra for an inner product space of type $(1,9)$ is $\mathbf{R}[32]$, the 32×32 real matrix algebra. The even component of the Clifford algebra is the block 16×16 real matrices. This means that the spin bundle over Y has a real structure and decomposes as $S^+_{\mathbf{R}}(Y) \oplus S^-_{\mathbf{R}}(Y)$, each factor being a 16-dimensional real bundle. A section of $S^+_{\mathbf{R}}(Y)$ or $S^-_{\mathbf{R}}(Y)$ is called a Majorana-Weyl spinor. By definition, a section of $S^+_{\mathbf{R}}(Y)$ (resp., $S^-_{\mathbf{R}}(Y)$) is a spinor of $+$ (resp., $-$) chirality. The metric $\hat{g}_0$ leads to a spinor connection and thus to the real Dirac operator

$$\not{D}^{10}\colon \Gamma(S^{\pm}_{\mathbf{R}}(Y)) \to \Gamma(S^{\mp}_{\mathbf{R}}(Y)).$$

In terms of the gamma-matrices for Clifford multiplication we can write

$$\not{D}^{10} = \gamma^m D_m$$

where D_m is the spin covariant derivative in the direction of the mth coordinate. The Dirac equation on the ten-dimensional manifold is then written as

$$i\not{D}^{10}\psi = 0.$$

Since the metric $\hat{g}_0$ is split, so is the Dirac operator, $\not{D}^{10} = \not{D}^4 + \not{D}^6$, and thus the Dirac equation is

$$i(\not{D}^4 + \not{D}^6)\psi = 0,$$

where $\not{D}^4$ and $\not{D}^6$ are the Dirac operators on Minkowski four-space and K^6 respectively. The solutions to this equation can be written as a sum

$$\psi(z) = \sum_k \psi_k(x)\psi'_k(y)$$

with

$$i\not{D}^4\psi_k(x) = \lambda'_k\psi_k(x)$$

and

$$i\not{D}^6\psi'_k(y) = -\lambda'_k\psi'_k(y).$$

Again, throwing away massive fluctuations means keeping only the zero modes of the Dirac equation on the internal space K^6.

§1.4. The effective action from ten-dimensional heterotic string theory

The effective action that heterotic string theory in dimension ten gives us is

$$S_{\text{eff}}[g,\varphi,\psi,\lambda,\chi,A,H] = \\ = \frac{1}{2\kappa^2}\int_Y d^{10}x\sqrt{-\det(g)}e^{-2\varphi}\left(R(g)+4(\nabla\varphi)^2-\frac{1}{3}H^2\right. \\ \left.+\frac{\alpha'}{30}\operatorname{Tr}F_A^2-\overline{\psi}_M\gamma^{MNP}D_N\psi_P-\overline{\lambda}\not{D}\lambda-\operatorname{Tr}\overline{\chi}\not{D}\chi+(\cdots)\right).$$

The fields in question are as follows: There is a metric g of signature $(-,+,\cdots+)$ on Y. There is the dilaton field φ which is a real scalar field. There is a connection A on a principal $E_8\times E_8$ or $SO(32)$ bundle P over Y. Its curvature is denoted F_A. The axion form H is a three-form with the property that

$$dH=\alpha'(\operatorname{tr}R\wedge R-\frac{1}{30}\operatorname{Tr}F_A\wedge F_A).$$

This means that we can write H, at least locally, in the form

$$H=dB+\alpha'\left(CS(\nabla_g)-\frac{1}{30}CS(A)\right),$$

where B is a two-form and $CS(A)$ is the Chern-Simons three-form of the connection A. The boson fields g,B,φ are grouped together in a multiplet (i.e., they are various components of an irreducible representation). This multiplet has a supersymmetric partner which is a multiplet given by a pair of spinors (ψ,λ). The spinor ψ is a Majorana-Weyl spinor with values in the cotangent bundle of Y; that is to say, it is a section of $S^+_{\mathbf{R}}(Y)\otimes T^*(Y)$. It is called the *gravitino*. The spinor λ is a Majorana-Weyl spinor of opposite chirality to that of ψ. That is to say, it is a section of $S^-_{\mathbf{R}}(Y)$. It is called the *dilatino*. The gauge field A has its supersymmetric partner χ, called the *gaugino*. This field is a Majorana-Weyl spinor with values in the adjoint bundle of P and is of the same chirality as ψ. That is to say, it is a section of $S^+_{\mathbf{R}}(Y)\otimes\operatorname{ad}P$.

Having discussed the fields that occur, let us turn to the functions of them that appear in the action. As usual, $R(g)$ is the scalar curvature of g and $\operatorname{Tr}F_A^2$ is the norm of F_A in the G-invariant metric in the adjoint bundle. Lastly, γ^{MNP} is obtained from the gamma-matrices by skew-symmetrizing:

$$\gamma^{MNP}=\gamma^{[M}\gamma^N\gamma^{P]}.$$

These are the gamma-matrices that give the representation of three-forms on spinors. In language more familiar to mathematicians, all the spinor pairings in the action arise from the two basic pairings: the nondegenerate inner product between spinors of opposite chirality:

$$\langle\ \rangle\colon S^+_{\mathbf{R}}(Y)\otimes S^-_{\mathbf{R}}(Y)\to\mathbf{R}$$

and the adjoint of Clifford multiplication

$$S^+_{\mathbf{R}}(Y)\otimes S^+_{\mathbf{R}}(Y)\to T(Y). \tag{1.1}$$

Here we are viewing Clifford multiplication as a pairing $T(Y)\otimes S^+_{\mathbf{R}}(Y)\to S^-_{\mathbf{R}}(Y)$. To write its adjoint as we have above, we use the isomorphisms $T(Y)\cong T^*(Y)$, given by the metric, and $S^+_{\mathbf{R}}(Y)\cong S^-_{\mathbf{R}}(Y)^*$, given by the inner product.) The

term $\overline{\lambda}\not{D}\lambda$ in the action is simply the inner product $\langle\lambda, \not{D}\lambda\rangle$. Likewise, $\mathrm{Tr}\,\overline{\chi}\not{D}\chi$ is $\mathrm{Tr}\,\langle\chi, \not{D}\chi\rangle$ defined from the inner product on spinors and the trace in the adjoint representation. Lastly, the pairing $\overline{\psi}_M\gamma^{MNP}D_N\psi_P$ in the action means the pairing $\langle\psi, \nabla\psi\rangle$, where the pairing is between a one-form with values in the positive spin bundle and a two-form with values in the same spin bundle. The pairing is given by wedging together the forms to produce a three-form and then letting that three-form operate by Clifford multiplication on one of the spinors to produce a spinor of minus chirality which can then be paired with the plus chirality spinor to produce a real number.

In addition, there are other terms (indicated by $(\cdots)$) that we have not explicitly written down and which are not important for the analysis that we shall make. These extra terms come in two types. Some are required by supersymmetry. There are, for example, terms which are fourth power in the spinor fields such as $(\overline{\psi}\psi)^2$. There are also terms which are higher order in α', which is a dimensionful parameter. These can be ignored if we assume that, even though K^6 is on the Planck scale, it is much larger than the string length.

This action clearly has a group of symmetries: the group of orientation-preserving diffeomorphisms of the underlying manifold. But in addition, we have gone to great lengths to build it so that it has supersymmetry, $N = 1$ local supersymmetry to be more precise. For example, our boson fields form two multiplets (g, B, φ) and A, and each multiplet has its fermionic partner (ψ, λ) and χ, respectively. This of course, is only a necessary condition for the existence of supersymmetry. But in fact the $N = 1$ local supersymmetry algebra is generated by sections ϵ of the Majorana-Weyl spin bundle $S^+_{\mathbf{R}}(Y)$. These generators are odd elements in the super Lie algebra. The infinitesimal transformation laws are partially given by:

$$\begin{aligned}
\delta_\epsilon\chi &= F_{MN}\gamma^{MN}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon\psi_M &= \nabla_M\epsilon - \frac{1}{4}H_{MAB}\gamma^{AB}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon\lambda &= (\gamma^M\nabla_M\varphi)\epsilon + \frac{1}{24}H_{MNP}\gamma^{MNP}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon g_{MN} &= \bar{\epsilon}\gamma_N\psi_M + \bar{\epsilon}\gamma_M\psi_N + (\text{fermions})^2 \\
\delta_\epsilon A_M &= \bar{\epsilon}\gamma_M\chi + (\text{fermions})^2 \\
\delta_\epsilon B &= \cdots \\
\delta_\epsilon\phi &= \cdots
\end{aligned}$$

The complete laws can be written down, but we shall not do that here as we will not need them.

The space of classical solutions is invariant under the supersymmetry algebra generated by these infinitesimal (odd) symmetries, and on the space of solutions the algebra generated by these infinitessimal symmetries closes to yield the $N = 1$ local supersymmetry algebra. In fact, one has the following formula for the supercommutator:

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_v + T$$

where v is the vector field obtained by pairing ϵ and ϵ' under the pairing given in Equation 1.1. Written in terms of Clifford matrices we have

$$v^m = \overline{\epsilon'}\gamma^m\epsilon.$$

The term T is an expression which includes gauge transformations and terms which vanish when one imposes the equations of motion.

All of this was done without worrying about the compactification. The next step in the study is to show under what conditions we can compactify down to dimension four. In doing this compactification much of the $N = 1$ local supersymmetry will be broken, but in special circumstances we will be able to retain $N = 1$ super Poincaré symmetry in dimension four. This is where the next lecture will begin.

References and cross references

Translations of the original papers of Kaluza and Klein can be found in

L. O'Raifeartaigh, *The Dawning of Gauge Theory*, Princeton Series in Physics, Princeton University Press, 1997.

Supergravity is discussed in detail Lecture 10 of [II-**Strings**]. Ten dimensional heterotic supergravity including the dilatino, the gravitino and the axion is considered in [II-**Strings**, §10.8] Heterotic string theory (including the appearance of ten-dimensional space and the gauge group $E_8 \times E_8$ or a form of $SO(32)$) is explained in Lecture 8 of [II-**Strings**].

LECTURE 2
Supersymmetry and Calabi-Yau Manifolds

§2.1. Review of material from the first lecture

Let us recall the fields and $N = 1$ local supersymmetries that we developed last time for our ten-dimensional supergravity coupled to super Yang-Mills theory. We have bosonic fields: a metric g_{MN} on space-time X^{10} which is a ten-dimensional manifold, a two-form potential B, the dilaton ϕ which is a scalar field, and a gauge field in the adjoint bundle of a principal $E_8 \times E_8$ or $SO(32)$ bundle P over space-time. The fermion fields are the gravitino ψ_M, which is a Majorana-Weyl spinor of plus chirality with values in the tangent bundle, the dilatino λ, which is a Majorana-Weyl spinor of minus chirality, and the gaugino χ, which is a Majorana-Weyl spinor of plus chirality with values in the adjoint bundle of P. The fields $(g_{MN}, B, \phi, \psi_M, \lambda)$ form an $N = 1$ supermultiplet, called the gravity supermultiplet, and (A, χ) form the $N = 1$ Yang-Mills supermultiplet, which is a matter supermultiplet. Because of anomalies we are not free to have arbitrary numbers of these supermultiplets: there must be exactly one ($E_8 \times E_8$ or $SO(32)$) matter supermultiplet to go with the supergravity supermultiplet.

Recall from last time that the supersymmetric (odd) generators ϵ are Majorana-Weyl spinors on ten-dimensional space-time, i.e., sections of $S^+_{\mathbf{R}}(X^{10})$. Their action on the fields is given by;

$$\begin{aligned}
\delta_\epsilon \chi &= F_{MN}\gamma^{MN}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon \psi_M &= \nabla_M \epsilon - \frac{1}{4} H_{MAB}\gamma^{AB}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon \lambda &= (\gamma^M \nabla_M \varphi)\epsilon + \frac{1}{24} H_{MNP}\gamma^{MNP}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon g_{MN} &= \bar{\epsilon}\gamma_N \psi_M + \bar{\epsilon}\gamma_M \psi_N + (\text{fermions})^2 \\
\delta_\epsilon A_M &= \bar{\epsilon}\gamma_M \chi + (\text{fermions})^2 \\
\delta_\epsilon B &= \cdots \\
\delta_\epsilon \phi &= \cdots
\end{aligned}$$

§2.2. Partially breaking the supersymmetry by compactifying down to dimension four

Now we are ready to study what happens to these $N = 1$ local supersymmetries as we compactify from ten dimensions to four dimensions in the way we discussed last time. Let us recall what we hope to achieve. We wish to find a four-dimensional low energy effective theory which has $N = 1$ super Poincaré symmetry. The odd generators of the $N = 1$ super Poincaré group in dimension four are constant four-dimensional Majorana spinors over four-dimensional Minkowski space-time. (Recall that the spin group from Minkowski space-time is $SL_2(\mathbf{C})$ which has an obvious real four-dimensional representation which is the underlying real representation of the defining 2-dimenisonal complex representation. A section of this spin bundle is a Majorana spinor.) We begin with a supersymmetry group generated by the sections of a 16-dimensional real spin bundle over ten-dimensional space-time, and we must somehow break this down so that after compactification the remaining supersymmetry generators are the constant Majorana spinors over Minkowski four-space. We need to break the symmetries in three-quarters of the components as well as for the non-constant sections.

As a basic model of how this works, recall what happens in the Kaluza-Klein model, where we compactify along a circle from five dimensions to four dimensions. We began with a five-dimensional theory with the full five-dimensional diffeomorphism group as symmetries. But the only diffeomorphisms which remained after compactifying and taking the low energy effective action for small circle radius were the diffeomorphisms which did not mix up the massive and massless modes. Recall that the five-dimensional vacuum we consider has a metric of the form

$$\eta_{\mu\nu}dx^\mu dx^\nu + r^2 d\theta^2.$$

The group of global symmetries which remain unbroken (i.e., leave the vacuum invariant) are exactly the Poincaré group of Minkowski four-space times the group of constant $U(1)$ rotations. The infinitesimal generators of the original group of symmetries are the vector fields on the five-dimensional space-time. The action of a vector field v on a metric g on space-time is given by $\delta_v g = \mathcal{L}_v(g)$, the Lie derivative of g in the v-direction. The condition on a vector field that it preserve a given vacuum g_0 is that $\delta_v g_0 = \mathcal{L}_v(g_0) = 0$, in other words, that v be a Killing vector field for g_0.

Now let us return to our ten-dimensional theory. The situation is analogous to the Kaluza-Klein model. The main difference is that this time our generators are supersymmetric (odd) generators instead of (even) vector fields. Nevertheless, the condition on a generator ϵ that it be unbroken, i.e., preserve a given vacuum is that the first variation of all the fields with respect to ϵ must be trivial. As we develop the argument we will list the properties our solutions have. Some come by assumption: others are consequences of these assumptions. The first property comes from the basic set-up of the reduction as discussed last time:

Assumption (i): The ten-dimensional space-time manifold of the solution is of the form $\mathbf{R}^4 \times K^6$, where K^6 is compact, and the metric is of the form

$$\begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn} \end{pmatrix}$$

where g_0 is a Ricci-flat metric on K^6 and $\eta_{mu\nu}dx^\mu dx^\nu$ is the standard metric on Minkowski four-space. Furthermore, the obvious isometric action of the four-dimensional Poincaré group on this Riemannian manifold is covered by an $N = 1$ four-dimensional super Poincaré symmetry group of the solution.

Since the metric is split as a four-dimensional metric times a six-dimensional metric, the same is true of the spin bundles. They decompose

$$S^+(\mathbf{R}^4 \times K^6) = \left(S^+(\mathbf{R}^4) \otimes S^+(K^6)\right) \oplus \left(S^-(\mathbf{R}^4) \otimes S^-(K^6)\right).$$

Since there are no nonzero spinors in $S^\pm(\mathbf{R}^4)$ which are invariant under the Poincaré group, there are no nonzero spinors on the ten-dimensional manifold which are invariant under the Poincaré group.

Consequence (ii): All the fermionic fields (ψ_M, λ, χ) of our solution must be trivial.

Now, since all the fermionic fields vanish, it follows that all the boson fields are automatically invariant under any odd supersymmetry, simply by parity considerations. Thus, to understand which of the $N = 1$ local supersymmetry generators are unbroken in this vacuum (i.e., leave it invariant), we have only three equations to study:

$$\begin{aligned}
\delta_\epsilon\chi &= F_{MN}\gamma^{MN}\epsilon + (\text{fermions})^2 = 0 \\
\delta_\epsilon\psi_M &= \nabla_M\epsilon - \frac{1}{4}H_{MAB}\gamma^{AB}\epsilon + (\text{fermions})^2 = 0 \\
\delta_\epsilon\lambda &= (\gamma^M\nabla_M\varphi)\epsilon + \frac{1}{24}H_{MNP}\gamma^{MNP}\epsilon + (\text{fermions})^2 = 0.
\end{aligned}$$

Furthermore, in these equations we can now drop the terms $(\text{fermion})^2$ as well, leaving

$$\delta_\epsilon\chi = F_{MN}\gamma^{MN}\epsilon = 0 \tag{2.1}$$

$$\delta_\epsilon\psi_M = \nabla_M\epsilon - \frac{1}{4}H_{MAB}\gamma^{AB}\epsilon = 0 \tag{2.2}$$

$$\delta_\epsilon\lambda = \left(\gamma^M\nabla_M(\phi)\right)\epsilon + \frac{1}{24}H_{MNP}\gamma^{MNP}\epsilon = 0. \tag{2.3}$$

Our goal is, given the metric g_0 on K^6, to find A, H, ϕ so that there are exactly four spinor fields ϵ (Majorana spinors on Minkowski four-space) for which these equations are satisfied. One can then ask: If these equations are satisfied then are the equations of motion satisfied as well, or is there an extra condition? In this case, and in fact almost always, the (first-order) supersymmetry equations imply the equations of motion. As a simple example of this principle, let us suppose we are dealing with the case of four-dimensional supergravity. Thus, we have $H = F = 0$. Then the only equation for a Majorana spinor ϵ to be an unbroken supersymmetry is $\nabla_\mu\epsilon = 0$, i.e., ϵ is a supersymmetry if it is a covariantly constant spinor over four-dimensional space-time. But the existence of a non-zero covariantly constant spinor field implies that the metric on space-time must be Ricci-flat. The can be seen as follows: $\nabla_\mu\epsilon = 0$ implies that $\gamma^\nu[\nabla_\nu, \nabla_\mu]\epsilon = 0$ and hence that $R_{\mu\nu}\gamma^\mu\epsilon = 0$. Since ϵ is non-zero, this implies that the Ricci curvature $R_{\mu\nu} = 0$, i.e., that space-time is Ricci-flat.

§2.3. Geometric consequence of the unbroken supersymmetry

Our assumption is that we have a split metric on $\mathbf{R}^4 \times K^6$

$$g = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn} \end{pmatrix}$$

with g_0 being a Ricci-flat metric on K^6. We are also assuming that we have non-trivial supersymmetry generated by Majorana spinor fields ϵ on space-time, and that there is an unbroken $N = 1$ supersymmetry of our vacuum. We shall make further assumptions in order to keep the analysis simple.

Assumption (iii): The scalar field $\phi = \phi_0$ is a constant function and the three-form H is zero. Recall from last time that the axion field H and the B field are related by

$$H = \alpha' \left(CS(\nabla_g) - \frac{1}{30} CS(A) \right) + dB,$$

where CS is the Chern-Simons functional. Thus, this assumption is equivalent to assuming that the field B is a closed two-form.

Of course, as we have already argued, all the fermion fields must be zero. With all these assumptions, Equation 2.1 is automatically satisfied and Equations 2.2 and 2.3 simplify to

$$\delta_\epsilon \psi_M = \nabla_M \epsilon = 0 \tag{2.4}$$

$$\delta_\epsilon \chi = F_{MN} \gamma^{MN} \epsilon = 0 \tag{2.5}$$

In more invariant language, ϵ must be a covariantly constant Majorana-Weyl spinor of positive chirality and the effect of Clifford multiplying this spinor by the curvature two-form of the connection on the principal bundle is zero.

§2.3.1. Consequences of the first equation

The $N = 1$ supersymmetry that we are searching for is to be global super Poincaré symmetry. In particular, the four (odd) spinor generators of the superalgebra must make the standard four-dimensional representation of the four-dimensional Poincaré group acting in its usual way on the first factor of space-time. Since the spinor bundle $S^+(\mathbf{R}^4 \times K^6)$ decomposes as $\left(S^+(\mathbf{R}^4) \otimes S^+(K^6)\right) \oplus \left(S^-(\mathbf{R}^4) \otimes S^-(K^6)\right)$, the only Weyl-Majorana spinors of plus chirality on the ten-dimensional manifold which transform as required under the Poincaré group of Minkowski four-space are of the form

$$\epsilon = \sum_i \eta_+^i(y) \cdot (\epsilon_+^i)^{(4)} + \text{c.c.}$$

where $\eta_+^i(y)$ is a 6-dimensional Weyl spinor of positive chirality on K and $(\epsilon_+^i)^{(4)}$ is a constant Weyl spinor of positive chirality on Minkowski four-space. (Here, c.c. means the complex conjugate. The complex conjugate is a product of Weyl spinors of negative chirality.)

Equation 2.4 becomes two equations: one in the $\mathbf{R}^4$-direction and one in the K^6-direction:

$$\delta_\epsilon \psi_\mu = \sum_i \eta_+^i \partial_\mu (\epsilon_+^i)^{(4)} + \text{c.c.} = 0$$

$$\delta_\epsilon \psi_m = \sum_i \nabla_m (\eta_+^i)(\epsilon_+^i)^{(4)} + \text{c.c.} = 0.$$

The first of these equations is automatic since $(\epsilon_+^i)^{(4)}$ is constant. (We could also use it to prove that $(\epsilon_+^i)^{(4)}$ is constant.) As to the second, since the complex conjugate of η_+^i is a spinor of opposite chirality, and since covariant differentiation preserves chirality, it implies that $\nabla_m \eta_+^i(y) = 0$. This means that η_+^i must be a covariantly constant non-zero Weyl spinor of plus chirality on K^6. It is clear that then that this spinor has constant norm, which we can renormalize to be 1. In particular, η_+^i is nowhere zero.

In particular, we have seen that there must be at least one nonzero, covariantly constant spinor η_+ on K. Next, we study the consequences of this fact for the geometry of (K^6, g_0).

Claim 2.1. The fact that η_+ is a covariantly constant spinor field of norm one implies that the endomorphism of the tangent bundle of K given by

$$J_m{}^n = i(\eta_+)^\dagger \gamma_m{}^n \eta_+$$

is a real endomorphism whose square is minus the identity. Consequently, it defines an almost complex structure on K. This endomorphism is an isometry with respect to g_0 and is parallel under the metric. This means that it defines an integrable almost complex structure (i.e., a complex structure) for which the metric g_0 is a Kähler metric.

Here we are using the notation $\gamma_m{}^n = \frac{1}{2}(\gamma_m \gamma^n - \gamma^n \gamma_m)$ and $\gamma_n = \gamma^p g_{np}$.

Proof. In more invariant language, the element J that is determined by the spinor field η_+ is simply the image under the natural maps

$$S(K) \otimes S^*(K) \cong \mathrm{Cl}(K) \cong \wedge^* TK \to \wedge^2 TK \subset T^*K \otimes TK$$

of the element $\eta_+ \otimes \eta_+^\dagger$. (The inclusion $\wedge^2 TK \subset T^*K \otimes TK$ uses the metric to identify TK and T^*K.) It is clear from the defining formula that J is a real tensor since $\gamma_n{}^m = -(\gamma_n{}^m)^\dagger$. It follows from the Fierz identity among Clifford matrices that

$$J_m{}^n J_n{}^p = -\delta_m{}^p,$$

i.e., that J is an endomorphism whose square is minus the identity. Since η_+ and $\gamma_m{}^n$ are covariantly constant, it is clear that J is covariantly constant. It is now a well-known fact that this implies that J is integrable and that the metric is Kähler, but let us briefly go through the derivation of these facts. The obstruction to integrability is the Nijenhuis tensor

$$N_{mn}{}^p = J_m{}^q(\nabla_q J_n{}^p - \nabla_n J_q{}^p) - J_n{}^q(\nabla_q J_m{}^p - \nabla_m J_q{}^p).$$

Since J is covariantly constant, it follows that $N = 0$ and that J defines a complex structure on K. Next, let us see that J is an orthogonal transformation with respect to the metric g or equivalently that g defines a hermitian metric on the complex manifold determined by J. This is simply the equation $J_m{}^q J_n{}^p g_{pq} = g_{mn}$, which follows easily from the definitions, the fact that $|\eta_+|^2 = 1$, and the relations among the γ-matrices. Lastly, since J is covariantly constant, so is the two-form associated to the Hermitian metric determined by g. This means in particular, that the two-form is closed, and consequently that the metric is Kähler. □

Note that when it is not assumed that $H = 0$, the almost complex structure J is not covariantly constant. Nevertheless, it can still be shown that the Nijenhuis

tensor vanishes, so that J defines the structure of a complex manifold. Of course, in this case the complex structure it is not Kähler.

Let us keep the assumption that $H = 0$ and hence that the complex structure on K^6 is Kähler. Since the complex structure on K^6 which is parallel under the metric reduces the holonomy group of parallel translation for the metric from $SO(6)$ to $U(3)$. But in fact, the existence of the self-parallel spinor η_+ implies that the holonomy reduces further. Decomposing the eight-dimensional Weyl spin bundle $S^+(Y)$ under $U(3)$ we get

$$8 \mapsto 3_1 \oplus 1_1 \oplus \overline{3}_{-1} \oplus \overline{1}_{-1}$$

where the first integer indexes the representation of $SU(3)$ and the subscript indexes the action of $U(1)$. (Thus, 3_1 is the standard three-dimensional complex representation of $U(3)$ and $\overline{3}_{-1}$ is its dual, whereas 1_1 is the determinant representation of $U(3)$.) The existence of a parallel non-zero spinor means that there must be a non-zero vector in the representation which is invariant under the image of the holonomy representation. Clearly, the only vector in this space invariant under the $U(1)$ diagonal subgroup is the trivial vector. This means then that the image of holonomy lies in $SU(3) \subset U(3)$. There are several possible subgroups with this property, $SU(3), SU(2)$, etc, but $SU(3)$ is the only one which has exactly a one-dimensional space of invariant spinors of each chirality. Thus, if we want to have exactly $N = 1$ supersymmetry and no more, then the holonomy group for the metric must be $SU(3)$. In this case there is a one-dimensional space of covariantly constant spinor fields, generated by η_+, say.

There is another way to think about all of this. Consider the action of $Spin(6) = SU(4)$ on the complex four-dimensional space of Weyl spinors S^+ of positive chirality. The stabilizer of an point in the unit sphere in S^+ is conjugate to $SU(3) \subset SU(4) = Spin(6)$. The image in $SO(6)$ is the usual embedding of $SU(3) \subset SO(6)$. If there is a parallel non-zero spinor, then the holonomy of the metric must lie in this $SU(3)$. This reduces the structure group of K^6 to $SU(3)$ and the reduction is parallel under the metric. That is to say, we have produced a parallel almost complex structure J. The arguments above show that this structure is integrable and that the metric is Kähler with respect to this complex structure. Since the holonomy is reduced to $SU(3)$, the metric is Ricci-flat.

We have shown:

Consequence (iv): The odd generators of the super Poincaré algebra are covariantly constant spinors ϵ of the form $(\eta_+ \epsilon_+^{(4)} + \text{c.c})$ where η_+ is the unique (up to constant multiples) covariantly constant Weyl spinor of plus chirality on K^6 and $\epsilon_+^{(4)}$ is a constant Weyl spinor of plus chirality on Minkowski four-space.

In all cases, since the holonomy is $SU(3)$, the Ricci tensor of the metric is trivial, and in particular the first Chern class of the metric is trivial. According to Yau's theorem on the existence of Ricci-flat metrics on Kähler manifolds, $c_1 = 0$ is sufficient for the existence of such a metric (if the manifold is non-simply connected then the condition must be interpreted as saying that the canonical bundle of the Kähler manifold is trivial as a holomorphic line bundle.) According to the uniqueness part of Yau's theorem, the moduli space of Ricci-flat Kähler metrics is the moduli space of complex structures and Kähler classes. As we remarked in the last lecture, and as we shall explain in more detail below, tangent directions

to this moduli space are associated with massless fields in the low energy effective four-dimensional action.

This completes the study of Equations 2.2 and 2.3 for the existence of an unbroken supersymmetry of the solution equations under the assumption that $H = 0$ and $\phi = \phi_0$ is constant. We have found the necessary and sufficient conditions for the existence of non-zero supersymmetries satisfying these equations:

Consequence (v): The Riemannian manifold (K, g_0) must be a Calabi-Yau three-fold with holonomy $SU(3)$.

We have two more things to deal with: we must examine the consequences of assuming $H = 0$, and we must show that Equation 2.1 is automatically satisfied under the assumptions that we are making.

§2.3.2. The condition $H = 0$.

So far we have not mentioned the gauge field A of the solution. This field is constrained by our assumption that $H = 0$. We have

$$0 = dH = \mathrm{tr}|, R \wedge R - \frac{1}{30}\mathrm{Tr}\, F_A \wedge F_A,$$

where tr refers to the trace in the standard representation, that is to say the trace of the composite endomorphism of the tangent bundle, and Tr denotes the trace in the adjoint representation of the gauge group. This means that

$$\mathrm{tr}\, R \wedge R = \frac{1}{30}\mathrm{Tr}\, F_A \wedge F_A,$$

There are two possible gauge groups $SO(32)$ and $E_8 \times E_8$. First, let us consider $SO(32)$. The trace in the adjoint representation is equal to 30 times the trace in the fundamental representation of $SO(32)$. Thus, we are looking for an $SO(32)$ connection A with $\mathrm{tr}\, F_A \wedge F_A = \mathrm{tr}\, R \wedge R$, where tr on the left-hand-side refers to the trace in the defining $SO(32)$ representation. One way to arrange this is to 'embed the spin connection in the gauge connection'. This means that there is a natural embedding $SU(3) \subset SO(32)$ with the commutant being $SO(26) \times U(1)$. Using this embedding, we can view the connection on the tangent bundle (which has had its structure group reduced to $SU(3)$) as being a principal $SO(32)$ connection. Clearly, it satisfies the trace condition above. In this way, we can construct the last of the fields A necessary to specify a solution.

We can perform a similar construction with $E_8 \times E_8$, using only one of the factors. There is an embedding $SU(3) \subset E_8$ with the commutant being E_6. Once again we can use this to embed the spin connection as a gauge connection for the group $E_8 \times E_8$, with symmetry group $E_6 \times E_8$.

Assumption (vii): We obtain the gauge field by embedding the spin connection in the gauge connection.

With this assumption, $H = 0$ is consistent with the defining equation for H.

§2.3.3. The remaining supersymmetry equation

Now let us examine the remaining equation for the existence of an unbroken supersymmetry ϵ:

$$\delta_\epsilon \chi = F_{MN}\gamma^{MN}\epsilon = 0.$$

The two-form F with values in the adjoint bundle to P is acting on the spinor field ϵ by Clifford multiplication of the two-form to produce a section of the spin

bundle tensored with the adjoint bundle. We claim that with our hypotheses this equation is always satisfied. Since the gauge connection is, by assumption, obtained by embedding the spin connection, this equation will follow immediately from the analogous equation for the spin connection. Thus, the result we need to establish is

$$\frac{1}{2}R_{MNA}{}^{B}\gamma^{MN}\epsilon = 0, \tag{2.6}$$

where $R_{MNA}{}^{B}dx^{M} \wedge dx^{N}$ is the Riemann curvature tensor. The condition that ϵ is covariantly constant implies that the curvature two-form evaluates trivially on ϵ, i.e., that

$$R_{MNA}{}^{B}\gamma^{A}{}_{B}\epsilon = 0.$$

Using the metric to raise and lower indices, we can rewrite this as

$$R_{MNAB}\gamma^{AB}\epsilon = 0.$$

Now using the fact that our connection is torsion-free, we can switch the roles of AB and MN and rewrite

$$R_{MNAB}\gamma^{MN}(\epsilon) = 0,$$

which is equivalent to Equation 2.6.

Let us say a few words about what happens in the more general situation where do not require $H = 0$. The equation $F_{MN}\gamma^{MN}\epsilon = 0$ is related to the Hermitian Yang-Mills condition for the connection. That is to say, the connection must be a holomorphic connection and the curvature must be a primitive two-form, i.e., $\omega \wedge \omega \wedge F = 0$, where ω is the fundamental (Kähler) two-form. Then, using this, the relationship between F, R, and H and supersymmetry we can then determine the value of H.

Let us return to the case of present interest where $H = 0$. In this case we have completed the descriptions of all the fields and a study of these supersymmetry equations. We have shown:

Theorem 2.2. *Let (K^6, J, g_0) be a Ricci-flat compact Kähler manifold with holonomy $SU(3)$. Then there is a classical solution to ten-dimensional supergravity coupled with super Yang-Mills with gauge group either $E_8 \times E_8$ or $SO(32)$. Space-time for this solution is $\mathbf{R}^4 \times K^6$, the metric on space-time is*

$$\begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn} \end{pmatrix}.$$

The axion field H is zero and the dilaton field ϕ is constant. The fermion fields (the dilatino, the gravitino, and the gaugino) of this solution vanish and the gauge field A is obtained by embedding the spin connection on K^6 as a gauge connection. If the gauge group is $E_8 \times E_8$, then the commutant of the connection, (i.e., the automorphism group of P which leaves the connection invariant) is $E_6 \times E_8$. If the gauge group is $SO(32)$, then the commutant is $SO(26) \times U(1)$. Furthermore, this solution has $N = 1$ four-dimensional super Poincaré symmetry lifting the natural action of the four-dimensional Poincaré group on Minkowski space-time.

§2.4. Massless fields in the low energy effective Lagrangian

As we remarked when discussing the Kaluza-Klein theory, parameters for the internal space of the high-dimensional theory (ten-dimensional in this case) yield massless fields in the effective, low energy four-dimensional Lagrangian. By Yau's

existence and uniqueness theorem for Ricci-flat Kähler metrics, the moduli space of Calabi-Yau structures on K is the moduli space of isomorphism classes of pairs (J, ω) consisting of a holomorphic structure J on K with $c_1 = 0$ (or more precisely with trivial canonical bundle) and a Kähler class ω. Let us consider the space of holomorphic structures; a similar discussion can be made for the variation of the Kähler class. The massless fields in the low energy effective Lagrangian associated to the variations of the Calabi-Yau metric come from expanding the metric g_{MN} on ten-space in terms of the decomposition of space-time and keeping only the harmonic modes of eigenvalue zero on the internal space:

$$g_{MN}(z) = M^\alpha(x) h^\alpha_{mn}(y) + \cdots .$$

The other terms in this expansion are not relevant for the massless fields coming from the variations of the Calabi-Yau structure: They lead to massless fields of other types. Since we are working at a critical point, there will be no terms in the effective low energy action involving only first derivatives of g_{MN}. As we take second derivatives we get three types of terms from $M^\alpha(x) h^\alpha_{mn}(y)$. We can differentiate $M^\alpha(x)$ twice, we can differentiate $h^\alpha_{mn}(y)$ twice, or we can differentiate each of $M^\alpha(x)$ and $h^\alpha_{mn}(y)$ once. Terms of the last type cannot contribute to the action because we have to pair the two derivatives using the metric of the solution. Since this metric is split, the pairing between the cross terms vanishes. Applying two derivatives to $h^\alpha_{mn}(y)$ would lead to a term in the Lagrangian which involves $M(x)^2$. Such a term would give $M(x)$ a mass. We know that there are no such terms because $h^\alpha_{mn}(y)$ is a harmonic variation of the linearized equations of motion. Thus, the only terms that can contribute to the action are where both derivatives are applied to $M(x)$. This leads to a term in the low energy effective action of the form

$$\int_{\mathbf{R}^4} d^4x \sqrt{-\det(g)} g^{\mu\nu} \partial_\mu M^\alpha(x) \partial_\nu M^\beta(x) G_{\alpha\beta}(M(x))$$

for some symmetric two-tensor $G = G_{\alpha\beta} dM^\alpha dM^\beta$ on the moduli space.

In fact, it is possible to explicitly identity G. In general, the tangent space to the moduli space of holomorphic structures on a complex manifold K is $H^1(TK)$. In the case that K is a Calabi-Yau three-fold, we can use the non-vanishing holomorphic three-form Ω on K (normalized to norm 1) to give an identification between $H^1(TK)$ and $H^{2,1}(K)$. Let us suppose that we are at a smooth point of the moduli space of holomorphic structures and let $M^\alpha, \alpha = 1, \ldots, h^{2,1}$ be local coordinates on the moduli space of holomorphic structures centered at the given point. A direct computation shows that under the identification of the tangent space of the moduli space at M with $H^{2,1}(M)$ the pairing $G(M)$ is the usual pairing of $(2,1)$-forms on M, namely if $a, b \in H^{2,1}(M)$ then

$$\langle a, b \rangle = \int_M a \wedge \bar{b}.$$

If our manifold is simply connected, or in fact if $h^{1,0} = 0$, then this pairing is positive definite. (It is always non-degenerate.) But, because it arises from an $N = 1$ supersymmetric Lagrangian, we can deduce highly non-trivial facts about this pairing. It follows from a general $N = 1$ supersymmetric computation that this metric is in fact a Kähler metric. Indeed, it is a Hodge metric. Much more can be obtained using the fact that not only can we work with the heterotic string in ten dimensions, we can also work with the type II string. Putting these together, we are

able to extend this Lagrangian to a four-dimensional $N = 2$ supersymmetric Lagrangian. This imposes even more stringent restrictions, leading to what is referred to as 'special geometry'. For instance, the Kähler potential is locally determined by a certain holomorphic 'potential'. In this way supergravity makes contact with the theory of filtrations of Hodge structures.

§2.5. Relation to Grand Unification Theory

As we remarked in the first lecture, E_6 is a possible group for GUT's. In fact, the representation that we need to use of E_6 in order to do GUT is the 27-dimensional defining representation. In our context we begin with the adjoint representation of E_8 and break it down to $SU(3) \times E_6$. Under this subgroup the 248-dimensional adjoint representation of E_8 decomposes

$$(8,1) \oplus (3,27) \oplus (\overline{3},\overline{27}) \oplus (1,78).$$

Here, the first term is the adjoint representation of $SU(3)$ and the last is the adjoint representation of E_6. The second and third terms are tensor products of three-dimensional representations of $SU(3)$ with 27-dimensional representations of E_6. Thus, among the representations that we are dealing with is the 27-dimensional representation of E_6, which fits with what GUT's requires.

Now the number of families of massless modes in $(3,27)$ is given by an index theorem. Actually, the difference of massless modes of distinct (left and right) chiralities is computed by the equation:

$$n_{27}^L - n_{27}^R = \text{index}(i\not{D}_3^{(6)}) = \frac{1}{48}(2\pi)^3 \int \text{tr}_3 F \wedge F \wedge F = \frac{-\chi(K)}{2},$$

where the Dirac operator and the trace are taken in the vector bundle associated to the fundamental three-dimensional representation of $SU(3)$. The fact that it is the difference rather than the actual number of families is not too serious a problem. It turns out to be relatively easy for massless modes of opposite chirality to combine and become massive. Thus, one can believe that the absolute value of this index is the correct number for the massless families. Now experiment shows us that in the universe there are exactly three families (quarks and leptons) of these light modes. This means that we should be looking for Calabi-Yau's with Euler characteristic ± 6. A few of these have been found.

From the physics point of view, it is very exciting that the simplest compactifications lead naturally to realistic GUT gauge groups, realistic chiral representations of matter, and the structure of family replication. This remains a significant source of optimism for the eventual physical relevance of string theory. But important and difficult problems – for example, supersymmetry breaking – remain to be solved.

References and cross references

The general discussion here follows

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A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (1986), pp. 253–284.

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Yau's existence theorem for Ricci-flat Kähler metrics is found in

S.T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat'l Acad. Sci. **74** (1977), pp. 1798-1799.

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Part 4

Dynamical Aspects of QFT

Dynamics of Quantum Field Theory

Edward Witten

Introduction

The goal of my lectures this term will be to present an introduction to the dynamics of gauge theories. We deal for the most part no longer with formal questions such as what a quantum field theory is and which ones exist. Rather, given a quantum field theory, we want to know how it behaves – what are the most interesting physical and mathematical properties of a given theory; and what can be learned by studying it?

The transition from formal questions to dynamics has analogs in many areas of mathematics and physics. For example, if one is presented with Newton's equations in mechanics, or with the Navier-Stokes equations for a fluid, one question is to show that the problem is well-posed, that for given initial conditions there exists a unique solution. Having established this, one then wants to know how a given system behaves – which essentially means, how it behaves over long times. After all, having a well-posed differential equation governing a system means that one knows the behavior for short times. Deep questions of an entirely different nature arise when one wants to solve the equations and determine what happens over large times.

The analog in quantum field theory is that establishing that a given theory exists is largely a matter of controlling the short distance behavior, by tools such as the renormalization theory and, in subtle cases, asymptotic freedom. Since we are working relativistically, "short distance" means small times as well as short distance in space. Having controlled the short distance behavior and convinced ourselves that a given theory exists, we then want to "solve" the theory as fully as possible, which means to determine, to the extent that we can, what happens over large distances.

Such investigations require quite a different way of thinking from what we have developed so far. There are a wide range of interesting examples of different types, and a wide range of methods most effective for studying them. One must develop some experience at asking, for a given model, what are the most significant questions and methods. One must learn what kind of information is important and what kind of information is accessible. Practicing with some examples seems to be the only way to obtain such experience. In selecting examples for this term's lectures, I have tried to choose examples that are relatively simple, instructive from the physical point of view, and have some mathematical interest. Also, I adopted

as a boundary condition a goal of finishing with an explanation of the behavior of $N = 2$ supersymmetric Yang-Mills theory in four dimensions (which is relevant to Donaldson theory of four-manifolds); this goal helped in organizing some of the selections.

The problem sets on supersymmetry in [I-**Homework**] were given as background material for some of the lectures. Other problem sets are collated here.

LECTURE 1
Symmetry Breaking

In this semester we will continue the discussion of quantum field theory, but now mostly dynamics rather than perturbation theory and purely formal things. The topics we will discuss are much more difficult to deal with rigorously than the formal theory we studied in the fall semester. We will try to be as precise as possible with mathematical statements, but sometimes this will be hard. In general, we will try to form an intuitive picture of what is going on, and illustrate it by considering concrete examples.

§1.0. Theories and realizations

[This section is explanatory and was written by the preparers of these notes in order to create a false sense of security, i.e. an unsubstantiated feeling that we understand what we are talking about in the rest of the lecture. Unfortunately, this is not the case, at least if "understand" means what it usually means among mathematicians.]

For the purposes of the present and forthcoming lectures, it will be important for us to distinguish theories and their realizations. So let us explain what we mean by a theory and what we mean by its realization. This explanation is not a mathematical definition (in fact, it is hard to give a definition which is both rigorous and useful), but we hope that it will make more clear what we are talking about.

Recall that a classical physical system Σ is usually described by defining the space of states X of Σ (a symplectic manifold, maybe infinite-dimensional) and a 1-parameter group g^t of time translations which preserves the symplectic structure. Then g^t produces a Hamiltonian flow, which is defined by a Hamiltonian function H. (For physical reasons, the Hamiltonian function should exist globally.) H is called the Hamiltonian, or energy function of the system. One should remember that H is defined only up to adding a (locally) constant function.

In most examples, H is bounded from below. In such a case, H is always normalized in such a way that the infimum of H on (each connected component of) X is zero.

In this situation, by *a theory* we mean a pair (X, H), where X is a symplectic manifold, and $H : X \to \mathbb{R}$ the energy function, defined up to a (locally) constant function. By *a ground state of this theory* we mean a lowest energy equilibrium

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Notes by Pavel Etingof, David Kazhdan, and David Morrison

state $x \in X$ of the system Σ, i.e. a state where the energy functional H attains a global minimum.

The same theory can have different ground states. For example, if we have a particle on the line with potential energy $U(x) = \frac{g}{4!}(x^2 - a^2)^2$, then the space X is the phase plane $\mathbb{R}^2$ with coordinates (x, p), the Hamiltonian is $\frac{p^2}{2} + U(x)$, and there are two ground states $(a, 0)$ and $(-a, 0)$.

Often a classical system Σ can be described by a Lagrangian $\mathcal{L}$, defined on some space of fields S on the spacetime V. In this case its space of states is the space $X \subset S$ of extremals of $\mathcal{L}$. As we have seen before, the space X carries a natural closed two-form ω, which is nondegenerate, and thus defines a symplectic structure on X. Also, the group of time translations acts on X and preserves ω. Therefore, the flow on X generated by this 1-parameter group is Hamiltonian, and is defined by a Hamiltonian function H.

However, one should remember that the description of a theory by a Lagrangian is not intrinsic, since different Lagrangians defined on different spaces of fields S may define the same theory. Indeed, consider, for example, the Lagrangian $\mathcal{L}_1 = \int x'(t)^2 dt/2$ defined on $S_1 = C^\infty(\mathbb{R}, M)$, where M is a Riemannian manifold. This Lagrangian defines the geodesic flow. The space of states in this case is T^*M, and the Hamiltonian function is $p^2/2$. On the other hand, we can write the Lagrangian $\mathcal{L}_2 = \int (x'(t)p(t) - p^2(t)/2)dt$ defined on the space $S_2 = C^\infty(\mathbb{R}, T^*M)$. It is easy to see that these two Lagrangians define the same theory.

We will consider relativistically invariant theories (X, H), i.e. theories with an action of the Poincaré group $\mathbf{P}$ on X, which preserves the symplectic structure (in the case when the system is defined by a Lagrangian, the group $\mathbf{P}$ acts on S and preserves the Lagrangian, and therefore, it acts on X preserving the symplectic structure). The group of time translations is a subgroup of $\mathbf{P}$, so $\mathbf{P}$ also preserves the energy function H. In non-pathological examples (in fact, in any example in which the minimum of the energy function is actually achieved) the ground state of a relativistic field theory is $\mathbf{P}$-invariant.

Recall that to define a symplectic manifold X is the same thing as to define the Poisson algebra $A = C^\infty(X)$ (X can be reconstructed as the spectrum of A). Therefore, we may say that a theory is a pair (A, H), where A is a Poisson algebra and $H \in A$. The Poisson algebra A is called the algebra of observables of the system.

Now let us consider quantum systems. The definition of a theory in this case is similar to the classical case. Namely, we will define *a quantum theory* to be a pair (A, H), where A is a *-algebra (not necessarily commutative), and H is a selfadjoint element of A, defined up to adding a real number. The algebra A is called the algebra of quantum observables (operators). The element H, as before, is called the Hamiltonian. Everything here is dependent on a real positive parameter $\hbar$ (the Planck constant).

By *a realization (or solution)* of a quantum theory (A, H) we will mean an irreducible *-representation of the algebra A in some Hilbert space $\mathcal{H}$, such that the spectrum of the operator H is bounded from below (representations are considered up to an isomorphism which preserves H). We will always normalize H so that the lowest point of its spectrum is zero. The space $\mathcal{H}$ is called the quantum space of states of the system (in this realization). Of course the same theory can in principle have different realizations, as the same algebra can have different representations.

As we have mentioned, we will be interested in relativistically invariant theories, i.e. theories with an action of the Poincaré group on A, so that the subgroup of time translations acts by $a \to e^{-\imath tH/\hbar} a e^{\imath tH/\hbar}$. When we talk about realizations of such a theory, we will assume that $\mathbf{P}$ acts in $\mathcal{H}$ by unitary operators, with the group of time translations acting by $e^{-\imath tH/\hbar}$.

By a *quantum ground state* of a quantum theory (A, H) in a realization $\mathcal{H}$ we mean a vector $\Omega \in \mathcal{H}$ which is an eigenstate of H with the smallest possible eigenvalue. In a relativistically invariant quantum field theory in dimension ≥ 2, a ground state is called a vacuum state and is the same thing as a $\mathbf{P}$-invariant vector; in this case the energy of the ground state is (by $\mathbf{P}$-invariance) zero.

Remark. In general, an irreducible realization of a quantum theory theory can have many ground states. For example, for the theory of Dirac operator on a compact manifold, ground states are harmonic spinors; the space of harmonic spinors can be more than one-dimensional. However, one can show that for a Minkowski space field theory in two or more dimensions which satisfies Wightman axioms, any realization has exactly one vacuum state. This is also satisfied in many cases when Wightman axioms fail. This is why, in quantum field theory, the words "realization" and " vacuum state" can be used as synonyms.

Suppose that we have a classical theory (A_0, H_0) which has been quantized, and the corresponding $\hbar$-dependent family of quantum theories is (A, H) (here by an $\hbar$-dependent family we mean a family depending on the dimensionless parameter $\hbar/S_0$, where S_0 is a characteristic scale of action). This means that we have a quantization map – some linear map $A_0 \to A$, given by $a \to \widehat{a}$, such that $\widehat{H}_0 = H$, and $[\widehat{a}, \widehat{b}] = i\hbar\widehat{\{a, b\}} + o(\hbar^2)$, $\hbar \to 0$. In this case, we will say that a state $v \in \mathcal{H}$ of norm 1 is localized near a classical state $x \in X = \mathrm{Spec} A_0$ if for any $a \in A_0$ $\langle v, \widehat{a}v\rangle \to a(x)$, as $\hbar \to 0$.

Let us now explain the connection between the classical and the quantum notions of a ground state. Suppose we have a quantum ground state Ω of norm 1 which is localized near a classical state x. In this case x is a classical ground state. Indeed, $H_0(x) = \lim_{\hbar\to 0}\langle\Omega, H\Omega\rangle = 0$, and for any $F \in A_0$, $\{F, H_0\}(x) = \lim_{\hbar\to 0} \frac{1}{i\hbar}\langle\Omega, [\widehat{F}, H]\Omega\rangle = 0$, so x is a stationary point of H_0.

Remark. Not every quantum ground state is localized near a classical ground state. Sometimes a quantum vacuum state is "spread" with some density over the set of classical ground states or even farther afield. We will see examples of this in this lecture and the rest of the term.

Given a quantum theory (A, H), it is convenient to represent its realizations by correlation functions. Namely, given a realization $\mathcal{H}$ of this system, and a ground state $\Omega \in \mathcal{H}$, we can define correlation functions $\langle\Omega, L_1 ... L_n\Omega\rangle$, where $L_i \in A$. Since the action of A in $\mathcal{H}$ is irreducible, the realization $\mathcal{H}$ can be completely reconstructed from these correlation functions.

Remark. The irreducibility condition is not always satisfied in physically interesting examples in dimension ≥ 2. A typical counterexample is $U(1)$ gauge theory, where the operator of electric charge commutes with all local operators. But here for simplicity we will assume that this condition is satisfied.

Sometimes a quantum system can be defined by a Lagrangian. As in the classical case, this is not always possible, and if possible, not in a unique way.

However, such a presentation is very convenient for understanding the behavior of the system.[1] So let us explain (on examples) how to pass from a Lagrangian to the Hamiltonian and the operator algebra.

We will start with the case of quantum mechanics, when the spacetime is just the time line. Consider a Lagrangian for one boson:

$$\mathcal{L} = \int \left[\frac{(\phi')^2}{2} - U(\phi) \right] dt.$$

Then, by definition, the operator algebra is generated by operators ϕ_0, ϕ_0', H, with the canonical commutation relations

$$[\phi_0, \phi_0'] = i\hbar, \ [H, \phi_0] = -i\hbar\phi_0', \ [H, \phi_0'] = i\hbar U'(\phi_0).$$

Define the local operators $\phi(t) = e^{iHt/\hbar}\phi_0 e^{-iHt/\hbar}$, $\phi'(t) = e^{iHt/\hbar}\phi_0' e^{-iHt/\hbar}$. It follows from the above definition that $d\phi/dt = \phi'$, and ϕ satisfies the Newton's differential equation

$$\phi''(t) = -U'(\phi(t)).$$

The operators of the form $F(\phi(t), \phi'(t))$, where F is a polynomial, are called local operators at t (we order products in such a way that ϕ' stands to the right of ϕ).

The operator algebra is spanned (topologically) by operators $\phi(t_1)...\phi(t_n)$. Thus, a realization of the theory is defined by prescribing expectation values of these operators – the correlation functions.

In any realization of the theory, the Hamiltonian is given by the following explicit formula:

$$H = \frac{1}{2}(\phi')^2 + U(\phi) + C.$$

Indeed, the difference $H - \frac{1}{2}(\phi')^2 + U(\phi)$ commutes with ϕ and ϕ', so by irreducibility it acts by a scalar.

Now consider quantum field theory. We first consider the theory in a spacetime $V = L \times \mathbb{R}$, where L is a lattice (finite or infinite). Let ∇_L be the discrete gradient operator on the lattice. Consider a Lagrangian

$$\mathcal{L} = \sum_{x \in L} \int dt \left[\frac{1}{2}(\phi_t^2(x,t) - (\nabla_L \phi(x,t))^2) - U(\phi(x,t)) \right]$$

In this case the operator algebra is generated by operators $\phi(x,0), \phi_t(x,0)$, $x \in L$, and H, satisfying the commutation relations

$$[\phi(x,0), \phi_t(x,0)] = i\hbar, \ [H, \phi(x,0)] = -i\hbar\phi_t(x,0),$$
$$[H, \phi_t(x,0)] = i\hbar[-\Delta_L \phi(x,0) + U'(\phi(x,0))],$$

(where Δ_L is the lattice Laplacian) and such that $\phi(x,0), \phi'(x,0)$ commute with $\phi(y,0), \phi'(y,0)$ if $x \neq y$ (causality). The local operators $\phi(x,t), \phi'(x,t)$ are defined as above. (Observe that since H does not commute with ϕ, for $t_1 \neq t_2$ the operators $\phi(x_1,t_1)$, $\phi(x_2,t_2)$, in general, do not commute).

[1] And in practice, essentially every example we will study this spring can be derived from a Lagrangian in a rather natural way.

As before, the operator algebra is spanned by the operators $\phi(x_1,t_1)...\phi(x_n,t_n)$. Thus, a realization of the theory is determined by expectation values of these operators – the correlation functions.

As in the case of quantum mechanics, in any realization we can compute the Hamiltonian explicitly. Namely, the Hamiltonian is of the form $H = \sum_{x\in L} H_x$, where $H_x = \frac{1}{2}(\phi_t^2(x,0) + (\nabla_L\phi(x,0))^2) + U(\phi(x,0)) + C_x$.

Remark. Of course, if the lattice L is infinite, the definition of H can be problematic, since the sum over L may be divergent. However, since commutators of H with other operators are well defined, one may hope that the constants C_x can in fact be adjusted in such a way that the sum converges. This is indeed true in many situations.

Now let us consider field theory in continuous spacetime. In this case the operator algebra and the Hamiltonian are defined similarly to the case of discrete space, which was considered above. Namely, the operator algebra will be generated by the operators $\phi(x,0)$, $\phi_t(x,0)$, and also H_b, $b \in \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra of the Poincaré group $\mathbf{P}$, with the commutation relations between ϕ, ϕ_t and H_b similar to the above.

However, we will face an additional problem – now expressions like $\phi^2(x,t)$ may not be well defined, because of ultraviolet divergences. This problem can be cured by the ultraviolet renormalization theory, which we discussed last semester, if the Lagrangian we started with was renormalizable. In this case, the algebra of local operators is not quite an algebra, but an OPE algebra (an algebra with operator product expansion). It is almost never possible to compute the structure constants of this algebra exactly (rational conformal field theory in 2 dimensions is the main exception), but it is possible to compute them in perturbation expansion to any finite order. In general, for continuous space we have additional analytic difficulties (compared to the case of discrete space), but they will not be very important in the present lecture, so we will not discuss them here. We will just need the rough general picture, which has been outlined in this introduction.

§1.1. What is symmetry breaking, and why it does not happen in quantum mechanics

Suppose we have some classical physical theory (A,H), which has a symmetry group G. Let us ask the following question: does this theory have a G-invariant ground state?

If we have a quantum theory (A,H), which has a symmetry group G, then, in cases when there is no difference between realizations and vacuum states, the correct analogue of this question is: does this theory have a G-invariant realization?

If the answer is no, one says that the symmetry is broken. If the answer is yes, one says that the symmetry is preserved.

In classical mechanics and classical field theory symmetry breaking can easily happen. For example, consider a classical particle of mass 1 on the line whose potential energy is $U(x) = g(x^2-a^2)^2/4!$, where $a > 0, g > 0$. The space of states of this particle is the plane with coordinates x,p, and its Hamiltonian is $\frac{p^2}{2} + U(x)$. There is an action of the group $G = \mathbb{Z}/2\mathbb{Z}$ on the space of states, by $(x,p) \to (-x,-p)$, which preserves the equations of motion, and there are two

lowest energy states: $s_+ = (a, 0), s_- = (-a, 0)$, which are permuted by G. But there is no lowest energy state which is G-invariant.

In quantum mechanics, symmetry breaking does not occur. Let us prove this for a theory of bosons with the operator algebra generated by operators x_i, p_i, $i = 1, ..., n$, satisfying the Heisenberg algebra relations, and with Hamiltonian $H = \frac{p^2}{2} + U(x)$. There is a realization of the theory in $\mathcal{H} = L^2(\mathbb{R}^n)$, with x_i acting by multiplication by the coordinate functions, and $p_i = -i\hbar\frac{\partial}{\partial x_i}$. It is well known that this representation is irreducible. Any symmetry group G of the potential U also acts (projectively) in $\mathcal{H}$. Thus, symmetry breaking does not occur.

Remark. In fact, according to the Stone - von Neumann theorem, $\mathcal{H}$ is the unique realization of this theory.

In the case of bosons with real Lagrangian we can in fact make a stronger statement. Namely, not only is the realization unique, but the ground state is also unique (and therefore invariant under any symmetry group of U). For simplicity we will show this in the case of only one boson on the line, but the argument we will give generalizes to any number of bosons in a space of any dimension.

We will consider a single boson on the line, in a field with potential $U(x)$ as above.

Theorem 1.1. *Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + U(x)$ be any Schrödinger operator, such that the potential $U(x)$ tends to $+\infty$ at infinity (so that H has discrete spectrum). Let E_0 be the smallest eigenvalue of H. Then there exists a unique, up to a factor, function $\psi \in L^2(\mathbb{R})$ (called the ground state wave function) such that $H\psi = E_0\psi$.*

Proof. For any $f \in L^2(\mathbb{R})$ we have

$$(f, Hf) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|f'(x)|^2 + U(x)|f(x)|^2\right) dx \tag{1.1}$$

Thus, ψ is defined by the condition that it is a global minimum point for the energy functional $E(f) := (f, Hf)$ on the sphere $||f|| = 1$. The proof of uniqueness of ψ rests on the following Lemma.

Lemma. *If ψ is a real global minimum point of (1.1) than ψ has constant sign.*

Proof of the Lemma. Let $E(\psi) = E_0$. Suppose that ψ changes sign at the point x_0. Since ψ satisfies the Euler-Lagrange (=Schrödinger) equation $H\psi = E_0\psi$, we have $\psi'(x_0) \neq 0$. Consider the function $|\psi|$. It is clear that $E(|\psi|) = E(\psi) = E_0$, but $|\psi|$ is not smooth, so it does not satisfy the Euler-Lagrange equation $Hf = E_0 f$, and thus cannot be the global minimum of E. Thus, the smallest value of E on the sphere is less than E_0 – a contradiction.

Now it is easy to prove the theorem. If the space of solutions of $H\psi = E_0\psi$ is more than one-dimensional, then there exist two linearly independent, orthogonal real solutions ψ_1, ψ_2. On the other hand, both of them have to be of constant sign, so $(\psi_1, \psi_2) \neq 0$ – a contradiction. □

It is useful to consider how symmetry breaking, which is absent in the quantum theory, arises in the quasiclassical limit. For this purpose, we should introduce Planck's constant $\hbar$, and consider the $\hbar$-dependent Hamiltonian

$$H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + U(x), \tag{1.2}$$

where $U(x) = g(x^2-a^2)^2/4!$. Let E_0, E_1 be the lowest eigenvalues of H in the space of even and odd functions, respectively, and ψ_0, ψ_1 the corresponding eigenvectors. We assume that ψ_0, ψ_1 have unit norm, and are normalized in such a way that $\psi_0(0) > 0$, $\psi_1'(0) > 0$. It is easy to show in the same way as above that ψ_1 does not change sign in the regions $x > 0$, $x < 0$. Define $\psi_+ = \frac{1}{\sqrt{2}}(\psi_0 + \psi_1)$, $\psi_- = \frac{1}{\sqrt{2}}(\psi_0 - \psi_1)$.

Then it is possible to prove the following.

Theorem 1.2. (i) *As* $\hbar \to 0$, $E_0, E_1 \sim a\hbar\sqrt{g/4!}$, *and* $E_1 - E_0 \sim Ce^{-S_0/\hbar}$, *where* S_0 *is a positive constant.*

(ii) *In the sense of distributions,*

$$\lim_{\hbar\to 0} |\psi_\pm|^2 = \delta(x \mp a). \tag{1.3}$$

This theorem shows that for very small $\hbar$, although there is only one lowest energy state ψ_0 with energy E_0, there is another stationary state ψ_1 with energy E_1 almost indistinguishable from the lowest one, and in the two-dimensional space spanned by ψ_0, ψ_1, there are two orthogonal states ψ_+, ψ_- localized near the classical equilibrium states $a, -a$. The states ψ_+, ψ_- are not stationary (i.e. are not eigenvectors of H), but their failure to be stationary is indistinguishable to any finite order in $\hbar$ (in fact, the angle between $H\psi_\pm$ and $\psi_\pm$ is dominated by const $\cdot\, e^{-S_0/\hbar}$). In particular, symmetry restoration in quantum theory is not seen at the perturbation theory level.

Thus, we have seen how symmetry is lost in the quasiclassical limit. One can consider this effect from a slightly different prospective, by looking at how symmetry appears in the process of quantization.

Recall that we have two classical lowest-energy states $a, -a$, near which the operator H looks approximately like a harmonic oscillator. Therefore, we can look for eigenfunctions of H perturbatively, in the form

$$\widetilde{\psi}_\pm(x) = f_\pm\left(\frac{x \mp a}{\sqrt{\hbar}}\right),\ f_\pm(z) = (\pi\hbar)^{-1/4} g^{1/8} e^{-\sqrt{g}z^2/2}(1 + u_1(z)\hbar^{1/2} + u_2(z)\hbar + ...) \tag{1.4}$$

(the 0-th term of this expansion is the lowest eigenfunction of the harmonic oscillator). It is easy to see that the real "eigenfunction" of the form (1.4), normalized to have unit norm, is unique for each sign.

Now, one can see the following.

(i) The formal series $f_\pm(z)$ do not converge. However, they represent asymptotic expansions of actual functions $\psi_\pm(z\sqrt{\hbar} \pm a)$ (where $\psi_\pm$ are as above), which are smooth from the right in $\hbar$ on $[0, \infty)$, but not analytic.

(ii) The formal series $\widetilde{\psi}_\pm(x)$ are eigenfunctions of H with the same eigenvalue. On the other hand, the actual functions $\psi_\pm(x)$ are not eigenvectors of H, although their failure to be ones is exponentially small. The actual lowest eigenvector of H is unique up to a factor, and equals $\psi_0 = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$. In particular, it is G-invariant, unlike ψ_+, ψ_-.

(iii) Let $W_+^n(t_1, ..., t_n)$, $W_-^n(t_1, ..., t_n)$ be the correlation functions computed perturbatively (by Feynman calculus) using the formal vacua $\widetilde{\psi}_+$, $\widetilde{\psi}_-$. Then W_+^n, W_-^n

do not serve as small $\hbar$ asymptotic expansions of the correlation functions of any realization of our quantum theory. However, the averages $\frac{1}{2}(W_+^n + W_-^n)$ do serve as such asymptotic expansions.

This shows how symmetry appears in quantization, when one goes from the perturbative to the nonperturbative setting.

§1.2. Still no symmetry breaking in quantum field theory in finite volume

Before going over to quantum field theory, we will give one more argument, at the physical level of rigor, which shows why there is no symmetry breaking in quantum mechanics (this argument is more of an explanation than a proof). It is based on the path integral approach. It does not use representation theory of the Heisenberg algebra, nor the positivity of the ground state wave function, and has the advantage that it also works for quantum field theory on a spacetime with a "space" part of finite volume.

As before, we will consider the $\mathbb{Z}/2\mathbb{Z}$-symmetric quartic potential $U(x)$. Suppose that symmetry breaking were the case. Then the two perturbative vacua $\widetilde{\psi}_+, \widetilde{\psi}_-$ do indeed exist nonperturbatively, i..e serve as asymptotic expansions of actual lowest eigenstates ψ_+, ψ_- of the Hamiltonian, in two different realizations of the theory, $\mathcal{H}_+, \mathcal{H}_-$. Consider the space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The vectors $\psi_+, \psi_- \in \mathcal{H}$ are "localized" near $a, -a$, and are orthogonal to each other. Thus, the inner product $(\psi_+, e^{-Ht/\hbar}\psi_-)$ must vanish. Let us now compute the same inner product using path integrals.

Recall the Feynman-Kac formula:

$$(\delta_{x_1}, e^{-Ht/\hbar}\delta_{x_2}) = \int_{\phi:[0,t]\to\mathbb{R},\phi(0)=x_1,\phi(t)=x_2} e^{-S(\phi)/\hbar} D\phi, \tag{1.5}$$

where

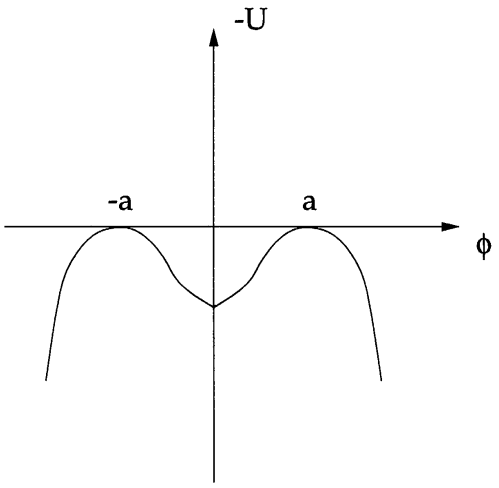
$$S(\phi) = \int_0^t \left[\frac{1}{2}(\phi')^2 + U(\phi)\right] ds. \tag{1.6}$$

The states ψ_+, ψ_- are "localized" near $a, -a$ (i.e. are close to delta-functions at $a, -a$ after a suitable normalization). So, if we believe the Feynman-Kac formula in this situation, we can substitute $\delta(x-a)$, $\delta(x+a)$ instead of them, and apply the Feynman-Kac formula. Using the small $\hbar$ stationary phase estimate on the right hand side of (1.5), we will get

$$(\psi_+, e^{-tH/\hbar}\psi_-) \sim Ce^{-S_*(t)/\hbar}, \tag{1.7}$$

where $S_*(t)$ is the least possible action of a path $\phi : [0,t] \to \mathbb{R}$ such that $\phi(0) = -a, \phi(t) = a$.

For large t, the least action $S_*(t)$ is attained at a classical trajectory $\phi = \phi_*(\tau)$, which is a solution of the Euler-Lagrange differential equation, i.e. Newton's equation $\phi'' = \frac{g}{6}\phi(\phi^2 - a^2)$ with boundary conditions $\phi(0) = -a$, $\phi(t) = a$. (Such a solution exists and is unique.)



Remark 1. The function $x = \phi_*(\tau)$ describes the motion of a ball in the potential field with potential $-U(x)$ (which looks like a camel's back, as in the figure), from one hump to the other. The initial velocity of the ball is such that the time needed to go from the top of one hump to the top of the other is t. The reason that the potential $U(x)$ is replaced here by $-U(x)$ is that we are doing a Euclidean path integral, which means that we performed a Wick rotation $t \to it$. This rotation transforms Newton's equation for the potential U into Newton's equation for the potential $-U$.

Remark 2. The solution $\phi^*(\tau)$ has a limit as $t \to \infty$; this limiting solution is unique up to time translations. The initial velocity of the solution $\phi^*(\tau)$ converges to zero as $t \to \infty$. Since in probing the low-energy states of the Hamiltonian, one is interested in large t, it is usual to express the analysis in terms of this limiting solution, which is called the instanton.

Formula (1.7) contradicts the fact that $(\psi_+, e^{-Ht/\hbar}\psi_-)$ vanishes. So our assumption that there are two vacua was false.

Remark. Formula (1.7) actually gives the correct estimate of the inner product $(\psi_+, e^{-tH}\psi_-)$. In particular, $S_0 = S_*(\infty)$. This estimate can be confirmed by rigorous methods.

In this argument, we have never used the fact that ψ_+, ψ_- are functions on the real line. All we used is that they are "localized" near classical equilibrium states $a, -a$, i.e. that for any local observable A the expectation value of A on $\psi_\pm$ is close to the value of the corresponding classical observable at the point $(\pm a, 0)$ in the phase space. Thus, our argument is independent of the realization of the space of states as $L^2(\mathbb{R})$. This makes it easy to generalize this argument to the case of field theory.

Consider a spacetime $M \times \mathbb{R}$, where M is the "space". We will assume that we have already performed a Wick rotation, so that the metric on the spacetime is a Riemannian product metric. We will assume that the volume of M is finite and equals V.

Consider the field theory on $M \times \mathbb{R}$ with one scalar Bose field ϕ, described by the Euclidean Lagrangian

$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 + U(\phi) \right) dx, \tag{1.8}$$

where U is the quartic potential as above. As before, we have two equilibrium states $a, -a$.

Now consider this theory quantum mechanically. Then we can see that there is still no symmetry breaking. The simplest way to see it is to first consider a discrete space M. In this case, the operator algebra is a finite tensor product of Heisenberg algebras corresponding to points of M, so it is a Heisenberg algebra itself, and the representation-theoretic argument that we used in the case of quantum mechanics shows that symmetry breaking does not occur.

However, it is more instructive to use the path integral argument. As before, assume that symmetry breaking occurs. Then we have two realizations $\mathcal{H}_+$, $\mathcal{H}_-$, and two orthogonal quantum vacua $\Omega_\pm \in \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ which are "localized" near the equilibrium points $a, -a$ for small $\hbar$, in the sense that $\langle \Omega_\pm, \phi(x_1)...\phi(x_n)\Omega_\pm \rangle \to (\pm a)^n$, $\hbar \to 0$.

As in the quantum mechanics case, the inner product $(\Omega_+, e^{-tH/\hbar}\Omega_-)$ vanishes. On the other hand, computing it using the Feynman-Kac formula, we will get

$$(\Omega_+, e^{-tH/\hbar}\Omega_-) \sim e^{-S_*(t)V/\hbar}, \tag{1.9}$$

where $S_*(t)$ is as above and V is the volume of the space. The reason is that the least action is attained on the space-independent classical solution $\phi_*^M(\mu, \tau) = \phi_*(\tau)$, $\mu \in M$, $\tau \in \mathbb{R}$.

Since (1.9) is nonzero (here it is essential that the volume V is finite), we get a contradiction.

As in quantum mechanics, we can define two sets of correlation functions W_+^n, W_-^n, evaluated by using perturbation theory near the lowest energy points $a, -a$ (of course, we can only define them in the renormalizable case, i.e. in 4 dimensions and below; also, one should remember that renormalization is not uniquely determined). Our reasonings show that W_+^n, W_-^n are not small $\hbar$ expansions of the correlation functions of a realization of our quantum theory. On the other hand, the functions $W_0^n = \frac{1}{2}(W_+^n + W_-^n)$ have a chance to be asymptotic expansions of the actual (nonperturbative) correlation functions.

For $\dim(M) = 1, 2$ the existence of the quantum theory $\mathcal{H}$ has been established in constructive field theory. In this case, it is possible to show that (in finite volume) there exists a unique ground state Ω_0, which is G-invariant, and that the correlation functions of the theory with respect to Ω_0 indeed have the asymptotic expansion given by W_0^n.

§1.3. Symmetry breaking in quantum field theory in infinite volume

When the volume V of the space M becomes infinite, the arguments of the previous section fail. The representation-theoretic argument fails (even in the presence of an ultraviolet cutoff), because the canonical representation of the operator algebra, which we used in the case of finite volume, is now an infinite tensor product of spaces corresponding to points; so we have to make sense of it, and there may be no G-invariant way of doing so. The path integral argument also fails. Indeed, the right hand side of (1.9) vanishes, so both computations of $(\Omega_+, e^{-tH/\hbar}\Omega_-)$ give the same answer, and we can derive no contradiction. Moreover, if we assume that we have a representation $\mathcal{H}$ of the operator algebra with two vacua Ω_+, Ω_-, then

using the formula

$$\langle \Omega_+, \phi(x_1^*)....\phi(x_n^*)\Omega_- \rangle = \int_{\phi: X \to \mathbb{R}, \phi \to \pm a, t \to \pm\infty} \phi(x_1)...\phi(x_n) e^{-S(\phi)/\hbar} D\phi, \tag{1.10}$$

where $x^* = (\mu, it)$ for $x = (\mu, t)$, we infer that the inner product (1.10) vanishes for any $x_1, ..., x_n$. This shows that the space $\mathcal{H}$ splits into an orthogonal direct sum: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where the spaces $\mathcal{H}_\pm$ are the Hilbert spaces of separate realizations generated by the vacua Ω_+, Ω_-. This means that symmetry breaking could occur, like in the classical theory: quantum effects are not necessarily strong enough to restore symmetry.

This is what in fact happens in quantum field theory. More precisely, the situation is the following.

(i) If the symmetry group G is a finite group, symmetry can be broken in infinite volume starting with spacetime dimension 2. For example, in the example we considered symmetry breaking does occur.

(ii) If the symmetry group G is a connected Lie group, symmetry breaking can occur starting with spacetime dimension 3. The fact that it does not occur in dimension 2 is a remarkable and a very important fact, which we will discuss in §1.5. This fact was proved by S. Coleman in 1973 (the reference is at the end of this lecture).

Thus, symmetry breaking is an "infrared" effect, associated with the behavior of the theory at large distances and very sensitive to the spacetime dimension.

Remark. There is a sense in which in dimension 2, once we consider solitons that interpolate between the different vacua, all vacua become part of one more unified story. This is explained in an example at the end of Lecture 5; solitons are discussed more systematically in Lecture 9.

§1.4. Infinite volume asymptotics of correlation functions

Let us see how symmetry is broken in the infinite volume limit of finite volume quantum field theories.

Consider *any* way of approaching the infinite volume limit starting from finite volume. For example, we might take spacetime to be a compact d-manifold M, and scale up the metric g of M by $g \to tg$, with $t \to \infty$. In the process, any local region of M converges to $\mathbb{R}^d$. We consider $\mathcal{O}_i(x_i)$ (with $\mathcal{O}_i$ an arbitrary set of local operators, not necessarily the elementary fields) inserted at points x_i that converge, as $t \to \infty$, to specified points in $\mathbb{R}^d$.[2] Then we try to analyze correlation functions

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_t \tag{1.11}$$

– the correlation function of the given product for a given value of t – in the limit of large t.

Another example of much the same thing is that one could consider M to be a ball in $\mathbb{R}^d$ of radius t, with some boundary conditions on the boundary of the

[2]In an application to Donaldson theory that we will make in lecture 19, one must consider as well as operators localized on points, also operators associated with surfaces Σ_α. But those operators are simply integrals over the Σ_α of ordinary local operators; in applying the discussion we are about to give, one can take the $\mathcal{O}_i$ to be those latter local operators.

ball. (More generally, M could be any region in $\mathbb{R}^d$, whose size is being scaled up by a factor t.) Then one considers correlation functions that are defined doing a path integral over fields ϕ that obey the boundary conditions. We shall use the symbol (1.11) in this case to describe correlation functions of fixed operators at points $x_i \in \mathbb{R}^d$ that are kept fixed as $t \to \infty$.

We claim first of all that if multiplied by a suitable function $C(t)$, the correlation functions in (1.11) have a limit as $t \to \infty$. The function $C(t)$ depends on the details of the choice of M with its metric in the first case, or the shape of the region and the boundary conditions in the second. Once the correlation functions are multiplied by $C(t)$ so that they have a limit, we claim the following for the limit. Let $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{(i)}$ be the corresponding correlation functions in the ith vacuum of the theory. Then the claim is that

$$\lim_{t\to\infty} C(t) \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_t = \sum_i p_i \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{(i)} \tag{1.12}$$

for some complex numbers p_i.

Intuitively, the reason for this important formula is that as $t \to \infty$, to avoid having too large an action, one is almost everywhere in space very close to one vacuum or another. But this space-pervading vacuum can be an arbitrary one, the ith vacuum for any i. So the correlation functions are written as the sum over i of the correlation functions in the ith vacuum. Different constants p_i appear for different vacua because the behavior for $t \to \infty$ of the function $C(t)$ was naturally determined only up to multiplication by a complex constant. There is no reason that the "same" constant will arise naturally for each i, so we must weight the different vacua by different constants. (Some of the p_i may vanish, if contributions of some vacua are negligible compared to contributions of others in the limit $t \to \infty$.)

In spirit – though perhaps hard to make rigorous – an attractive explanation for (1.12) is as follows. The correlation functions $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$ of this theory obey a multitude of linear relations that follow from the operator product expansion. Since the relations are linear, the space of solutions is a vector space. One can reasonably hope that a basis of solutions of this vector space is given by the correlation functions $\langle \mathcal{O}_1(x_i) \dots \mathcal{O}_n(x_n) \rangle_{(i)}$ in the various vacua. The left hand side of (1.12) defines an element in this vector space (since the correlators on the left hand side certainly obey all relations that follow from the operator product expansion). So it must have an expansion of the sort written in (1.12).

Note that if more than one of the p_i is nonzero, the correlation functions obtained by starting on the compact manifold M (with or without boundary) and taking $t \to \infty$ do not obey cluster decomposition. They otherwise obey all axioms of field theory. From an axiomatic point of view (recall Lecture 2 of [I-**Axiomatics**]) one can hope to prove that functions obeying all axioms of correlation functions except cluster decomposition can be expanded as in (1.12) in terms of symbols that do obey cluster decomposition.

Now we mention some simple cases where the p_i can be determined. If M is a torus with flat metric, then in most simple cases, the p_i are equal because the path integral on M has the interpretation of a trace, to which all states of given energy contribute equally. (This equality can be avoided under certain conditions, for instance if there is an unbroken discrete gauge symmetry, but that is something I will not talk about today.) For another type of example, suppose that M is a

ball in $\mathbb{R}^d$ with some specified boundary conditions on the boundary of the ball. Suppose we have a weakly coupled theory of a scalar field ϕ with symmetry $\phi \to -\phi$ and two classical ground states at $\phi = a$ and $-a$. We label the corresponding two quantum vacua as $+$ and $-$. If we adopt the boundary conditions that $\phi = a$ on the boundary, then as $t \to \infty$ it becomes harder and harder to see anything but the $+$ vacuum, so we get $p_+ \neq 0$, $p_- = 0$. (One can take $p_+ = 1$ by suitable choice of $C(t)$.) Conversely, if $\phi = -a$ on the boundary, we get $p_+ = 0$, $p_- \neq 0$. Finally, we can take $\phi = 0$ on the boundary. This preserves the symmetry that exchanges the two vacua, so $p_+ = p_-$.

§1.5. Continuous symmetry breaking

Now we will consider symmetry breaking in quantum field theory, when the symmetry group is a connected Lie group. The typical example is a complex valued Bose field ϕ, and the Lagrangian

$$\mathcal{L}(\phi) = \int d^d x \left(\frac{1}{2} |\nabla \phi|^2 + (|\phi|^2 - a^2)^2 \right). \tag{1.13}$$

This Lagrangian has a $U(1)$-symmetry, acting by $\phi \to e^{i\theta}\phi$. In the classical theory, we have lowest energy states $\phi = ae^{i\theta}$, $\theta \in [0, 2\pi)$, so we have symmetry breaking. We will try to find out whether symmetry breaking exists also in the quantum theory.

Above we showed that symmetry breaking does not occur in the one-dimensional case (quantum mechanics). We did it for the case of a real boson ϕ, but for the complex boson the argument (with path integral) works even better. For instance, for Lagrangian (1.13) the set of classical minima of the energy is the circle $|\phi| = a$, which is connected. Therefore, to go from one classical minimum, $ae^{i\theta_1}$, to another, $ae^{i\theta_2}$, one does not need to go over the "hump" of the potential, and can go along the circle of minima, so one can do it with even less action than before. Therefore, the path integral computation described above would show that in the complex case there is even more linking between the states ψ_+, ψ_- than in the real case. This effect might make us think that perhaps in field theory, continuous symmetry breaking does not happen as easily as discrete symmetry breaking. And indeed, it turns out that continuous symmetry breaking cannot happen in two dimensions, and can happen only in dimensions ≥ 3.

Unfortunately, such a simple path integral argument is too crude to show that continuous symmetry breaking does not occur in 2 dimensions. Indeed, it is easy to see that all paths in the integral which computes $(\Omega_+, e^{-tH}\Omega_-)$ have infinite action in infinite volume, so we can derive no contradiction. So let us demonstrate why symmetry is preserved in 2 dimensions and broken above 2 dimensions by considering the simplest example.

The simplest example is the theory of a free massless real scalar Bose field in d dimensions, defined by the Lagrangian

$$\mathcal{L}_0(\phi) = \int d^d x \left(\frac{1}{2} (\nabla \phi)^2 \right). \tag{1.14}$$

This theory has a translation symmetry $\phi \to \phi + c$. Let us show that this symmetry is broken for $d > 2$, and preserved for $d = 2$.

For $d > 2$, symmetry breaking is obvious. Indeed, in this case we have a Wightman field theory generated by an elementary field ϕ, satisfying Wightman axioms (Lecture 1 of [I-**Axiomatics**]). In this theory, the 1-point function of the operator ϕ is zero, while the 1-point function of the operator $\phi + c$ is c. Therefore, the transformation $\phi \to \phi + c$ does not preserve the 1-point function. This shows, that in the case $d > 2$ we in fact have not a single realization, but a family of realizations $\mathcal{H}_c$ parametrized by points c of the line. The theory $\mathcal{H}_c$ is defined by the condition that $\langle \Omega_c, \phi\Omega_c \rangle = c$, where Ω_c is the vacuum of $\mathcal{H}_c$. The vacuum Ω_c is "localized" near the classical equilibrium state c. The map $\phi \to \phi + c$ transforms $\mathcal{H}_b$ to $\mathcal{H}_{b+c}$.

For $d = 2$, the situation is not the same. The problem is that the operator ϕ is not defined in 2 dimensions, although its derivatives are. Indeed, in 2 dimensions, we have

$$\langle \partial\phi(x)\partial\phi(y) \rangle = -\partial_x \partial_y \ln|x - y|, \tag{1.15}$$

so if the operator ϕ was defined in some way, we would have

$$\langle \phi(x)\phi(y) \rangle = -\ln|x - y| + C, \tag{1.16}$$

which contradicts positivity (the function on the RHS of (1.16) is not positive).

Let us say this more precisely. What we have in 2 dimensions is a quantum field theory generated by a Wightman map ϕ from Schwartz functions to operators, which is defined not on the whole space of Schwartz functions $S(V)$ but only on the space $S_0(V)$ of Schwartz functions on V with integral zero. Then derivatives of ϕ can be defined on all Schwartz functions by

$$\partial_i \phi(f) = -\phi(\partial_i f)$$

(the right hand side makes sense since $\int \partial_i f = 0$).

However, in the theory defined by such ϕ the question of symmetry breaking does not arise, since there is no symmetry to begin with: on the space $S_0(V)$, the maps ϕ and $\phi + c$ are the same. So, in order to raise the question about symmetry breaking, we should consider an extension of our operator algebra, which will have a nontrivial action of symmetry. The most reasonable way of doing so is the following.

Instead of considering the theory of an $\mathbb{R}$-valued massless scalar ϕ, we will consider the theory of a circle-valued field ϕ with the same Lagrangian. We take the circle to be $S_\lambda = \mathbb{R}/2\pi\lambda\mathbb{Z}$. The Lagrangian is the same as before. In this case, the local functional ϕ is not defined, but instead we have local functionals $e^{ik\phi/\lambda}$, $k \in \mathbb{Z}$. The local functionals in this theory are Laurent polynomials in $e^{i\phi/\lambda}$ whose coefficients are differential polynomials in the derivatives of the field ϕ, but not in ϕ itself. The translational symmetry in this theory is the $U(1)$-symmetry: for a complex number z with $|z| = 1$, $z \circ e^{ik\phi/\lambda} = z^k e^{ik\phi/\lambda}$, and z acts trivially on the derivatives of ϕ. Classically, this theory has a family of vacua (equilibrium states), given by $\phi = c$, $c \in S_\lambda$.

Let us show that in this system symmetry is not broken quantum-mechanically. It is enough to show that the 1-point function $\langle \Omega, \mathcal{O}(0)\Omega \rangle$ vanishes for any local operator $\mathcal{O}$ of the form $\mathcal{O} = P(\phi)e^{ik\phi/\lambda}$ for $k \neq 0$.[3] Let us show this in the case

[3]All local operators are of this form; and the operator product expansion can be used to show that if there is an n-point function that is not invariant, then there is such a 1-point function.

$P = 1$ (in general, the proof is analogous). Proof: From the OPE in the free theory (cf. Lecture 3 of [I-**Perturbative QFT**]) we get

$$\langle \Omega, \mathcal{O}(x)\mathcal{O}^*(0)\Omega \rangle = |x|^{-k^2/\lambda^2},$$

so this 2-point function vanishes at infinity. But by clustering, the limit of this function at infinity is $|\langle \Omega, \mathcal{O}(0)\Omega \rangle|^2$. So, $\langle \Omega, \mathcal{O}(0)\Omega \rangle = 0$.

Another way to see that there is no symmetry breaking is as follows. The Hilbert space of the quantized theory is of the form

$$\mathcal{H} = F \otimes F^* \otimes l_2(\lambda\mathbb{Z}) = \oplus_{k\in\mathbb{Z}}(F \otimes F^*)_k,$$

where F is the Fock space. The operators corresponding to derivatives of ϕ respect this decomposition, while the operator $e^{ik\phi/\lambda}$ maps the space $(F \otimes F_*)_{k'}$ to the space $(F \otimes F^*)_{k+k'}$. The vacuum vector Ω belongs to the zero component $(F \otimes F^*)_0$. This implies that all correlation functions of operators in this theory are invariant under the action of $U(1)$. In particular, $\langle e^{ik\phi/\lambda} \rangle = \delta_{0k}$, which shows that the vacuum Ω is not "localized" near any classical ground state but is "spread" uniformly over the space S_λ of classical ground states. Thus, symmetry under $U(1)$ is not broken.

Remark 1. It is instructive to see why our argument that there is symmetry breaking for $d > 2$ fails in finite volume, where, as we know, there should be no symmetry breaking. The problem is that in finite volume the field ϕ is not defined, although its derivatives are. Indeed, since the spacetime is the product of a time line with a compact space, at large distances it looks simply like a line; the 2-point function of ϕ (which is the Green's function of the spacetime) at large $|t|$ looks like the one-dimensional Green's function, i.e. $-|t| + C + o(1)$. This function is not bounded from below, so it violates positivity. Moreover, as follows from considering the case $d = 2$, the situation is the same if all spatial directions but one are compactified.

Remark 2. Above we have considered the theory of a free massless scalar ϕ in dimension $d > 2$ and found that its space of quantum vacua is the space of values of ϕ (i.e. the target space $\mathbb{R}$). More generally, if one considers the sigma model with spacetime $\mathbb{R}^d$, $d \geq 2$, and target space M (a Riemannian manifold), the space of quantum vacua, as well as the space of classical ground states, will be M. (Here you should forget for a moment that this sigma model for nonlinear M is not renormalizable. We will in part 8 of today's lecture understand this sigma model as an infrared effective theory.) In particular, M as a Riemannian manifold can be recovered from the quantum theory as the moduli space of quantum vacua (see Remark 3 below).

On the other hand, we saw that for $d = 2$ the theory of a circle-valued field ϕ (which is the same as the two-dimensional sigma model with target space S_λ) has only one vacuum Ω. This is the case for two-dimensional sigma models in general: its moduli space of quantum vacua is generally very small, and does not coincide with the space of classical ground states (=the target space). In particular, the target space cannot be recovered intrinsically from the quantum theory. For example, the circle S_λ cannot be recovered intrinsically from the theory of maps into S_λ considered above. This, in fact, happens for a good reason – one can show that the theories attached to the circles S_λ and $S_{1/\lambda}$ are equivalent (for example, their partition functions coincide – see Lecture 1 of [II-**CFT**]). This is the starting point for the theory of mirror symmetry.

Remark 3. Consider a field theory with the Lagrangian

$$\mathcal{L} = \int d^d x (\frac{1}{2}(\nabla\phi)^2 + U(\phi)),$$

where ϕ takes values in some Riemannian manifold M, and U is a potential function on M ($U \geq 0$). Let $M(0)$ be the set of zeros of U. Assume that $M(0)$ is nonempty and smooth, and that the Hessian matrix of second derivatives of U is nondegenerate on $T_x M/T_x M(0)$ for $x \in M(0)$. Suppose that there is a Lie group G which acts by isometries on M, fixes U, and acts transitively on M_0. In this case, one can show (at the physical level of rigor) that the "infrared behavior" of the theory described by $\mathcal{L}$ is the same as the "infrared behavior" of the sigma model with target space being the space $M(0)$ of classical vacua. The precise meaning of this statement is explained in Section 1.7. (We can ignore nonrenormalizability problems by defining the theories by a cutoff path integral, where integration is taken over fields defined on a lattice with step Λ^{-1}, or over fields having only Fourier modes with $|k| < \Lambda$, with respect to some coordinate system; in this setting, the cutoff Λ is not sent to infinity). This fact can be explained heuristically: if a function ϕ is to have low energy, it has to oscillate slowly and also stay close to the minimum locus $M(0)$, i.e. it has to be close to a map into $M(0)$. So the low energy physics has to do with slowly varying maps to $M(0)$.

Thus, continuous symmetry breaking, being an infrared effect, will occur in the theory described by $\mathcal{L}$ iff it occurs in the corresponding sigma model. So, as follows from remark 2, symmetry breaking tends not to occur in dimension 2, but tends to occur in dimension > 2.

These are, however, mostly heuristic arguments. In the next section we will treat the issue of continuous symmetry breaking in a more systematic way, using Goldstone's theorem.

§1.6. Goldstone's theorem

Recall the standard formalism of Noether's theorem and currents in classical field theory. Suppose we have a Lagrangian $\mathcal{L} = \mathcal{L}(\phi)$ in a d-dimensional spacetime V. Denote the space of solutions of the corresponding Euler-Lagrange equations by X.

Let G^s be a 1-parameter symmetry group of this Lagrangian. Let $D_\phi := \frac{d}{ds}|_{s=0} G^s\phi$. We assume that D_ϕ is a local functional of ϕ.

Let $\eta \in \Omega^1(X, \Omega^{d-1}(V))$ be the canonical one-form on the space of solutions that was discussed in [I-**Classical Fields**] and [I-**Homework**] (the canonical two-form on X was defined as $\int_C d_X\eta$, where C is a $d-1$-dimensional cycle). For instance, the formula for η for the free theory of a massless scalar is

$$\eta(\delta\phi)(x) = \delta\phi(x) * d\phi(x). \tag{1.17}$$

Let $J \in \Omega^0(X, \Omega^{d-1}(V))$ be defined by the formula $J = \eta(D_\phi)$. Since D_ϕ is local, so is J. Thus, J is a local functional on X with values in $\Omega^{d-1}(V)$. For instance, if $\mathcal{L}$ is the Lagrangian of the theory of a free massless scalar, and $G^s\phi = \phi + s$, then $D_\phi = 1$, so $J(x) = *d\phi(x)$. The local functional J is called the *current* corresponding to the symmetry G^s. The main property of the current is that it is *conserved*, i.e. $d_V J = 0$. This follows form the fact that $d_V\eta = 0$.

In finite volume it is useful to define the charge functional $Q = \int_C J(x)$, where C is some spacelike cycle (e.g. $t = \text{const}$). Since the current is conserved, this quantity is independent of the choice of the cycle, as long as it represents the fundamental homology class of "space". If $\Omega = \int_C d_X \eta$ is a nondegenerate two-form on X, then Q is a Hamiltonian which defines the symmetry group G^s, in the sense that $\frac{d}{ds}|_{s=0}(G^s)^* F = \{F, Q\}$, where $\{,\}$ is the Poisson bracket on S, and F any local functional on X.

In infinite volume, the functional Q is not necessarily defined, since the integral does not converge. In this case, it is convenient to pick a time coordinate t and to let $C = C_0 = \{t = 0\}$ be an initial value surface, Then we define the "cutoff charge functional"

$$Q_f = \int_{C_0} f(x) J(x), \tag{1.18}$$

where $f : C_0 \to \mathbb{R}$ is a Schwartz function. The limit $\lim_{f \to 1} Q_f$ in this case does not exist, but for any local functional F

$$\lim_{f \to 1} \{F, Q_f\} = \frac{d}{ds}\bigg|_{s=0} (G^s)^* F. \tag{1.19}$$

Remark. By $f \to 1$ we mean that f converges to 1 uniformly on any compact set, and all derivatives of f go to zero uniformly on the whole space.

In quantum theory, the story is the same, except that (local) functionals are replaced with local operators, and Poisson bracket with commutator times i. That is, to any one-parameter symmetry G^s there corresponds a $(d-1)$-form-valued local operator $J(x)$, which is conserved, and the charge operator $Q = \int_C J(x)$ (in finite volume) has the property

$$[F, Q] = -i \frac{d}{ds}\bigg|_{s=0} (G^s)^* F. \tag{1.20}$$

The case of infinite volume is dealt with in the same way as in the classical theory, by considering cutoff operators Q_f.

Now we will discuss Goldstone's theorem. Suppose that the symmetry G^s in the theory defined by $\mathcal{L}$ is broken not only classically but also quantum mechanically. In this case, if we have a solution of the theory $\mathcal{H}$ (a Hilbert space with an action of the operator algebra), then there exists a scalar local operator ϕ whose 1-point function is not invariant under symmetry.

Remark. Strictly speaking, we only know that some n-point function is not invariant; but in all known situations with symmetry breaking there is also a non-invariant 1-point function. (By use of the operator product expansion, one can argue that if there is a non-invariant n-point function, then there is a non-invariant 1-point function.)

Let Q_f be the cutoff charge operator for the symmetry G^s. We have

$$\langle \Omega | [Q_f, \phi(0)] | \Omega \rangle \neq 0, \tag{1.21}$$

for f sufficiently close to 1. Thus, $\langle \Omega | [J(x)\phi(0)] | \Omega \rangle \neq 0$ for some x.

Consider the 2-point functions

$$M_+(x) := \langle \Omega | J(x)\phi(0)|\Omega\rangle, M_-(x) := \langle \Omega|\phi(0)J(x)|\Omega\rangle \tag{1.22}$$

(these are $(d-1)$-forms on V). Since the symmetry is broken, $M_+ \neq M_-$, although by space-like separation, they coincide if x is spacelike.

Let $\mathcal{H} = \int_{p\in V_+} \mathcal{H}_p$ be the spectral decomposition of $\mathcal{H}$ with respect to the action of the translation group (here V_+ is the positive part of the full light cone). Using the decomposition of the inner product into a sum over intermediate states, $(\langle J(x)\Omega, \phi(0)\Omega\rangle = \sum_n \langle J(x)\Omega, n\rangle\langle n, \phi(0)\Omega\rangle)$, we get

$$M_\pm(x) = \int_{V_+} K_\pm(x,p)dp, \tag{1.23}$$

where $K_+(x,p), K_-(x,p)$ are the contributions to M_+, M_- from intermediate states of 4-momentum p ($K_\pm$ are vector-valued distributions in p).

Because of Lorentz invariance, the distributions K_+, K_- look like

$$K_\pm = pe^{\pm ipx}\rho_\pm(-p^2), \tag{1.24}$$

where $\rho_\pm(s)$ are distributions on the half-line $s \geq 0$.

This yields

$$M_\pm(x) = i * d_V \int_0^\infty \rho_\pm(m^2)W_m(\pm x)dm^2, \tag{1.25}$$

where $W_m(x) = \int_{\mathcal{O}_m^+} e^{ipx}dp$ is the Klein-Gordon propagator with mass m defined in Lecture 1 of [I-**Perturbative QFT**] ($\mathcal{O}_m^+$ is the upper sheet of the hyperboloid $p^2 = -m^2$).

Let $M(x) = M_+(x) - M_-(x)$. Since $W_m(x) = W_m(-x)$ when x is spacelike, for spacelike x (1.25) yields

$$M(x) = i * d_V \int_0^\infty (\rho_+(m^2) - \rho_-(m^2))W_m(x)dm^2. \tag{1.26}$$

However, as we have mentioned, by spacelike separation $M(x)$ vanishes for spacelike x. This implies that $\rho_+ = \rho_- = \rho$, and so (1.25) yields

$$M(x) = i * d_V \int_0^\infty \rho(m^2)(W_m(x) - W_m(-x))dm^2. \tag{1.27}$$

Differentiating both sides of (1.27), at a point x such that $x^2 > 0$, and using the conservation of the current and the Klein-Gordon equation $\nabla^2 W_m = -m^2 W_m$, we get

$$\int_0^\infty m^2\rho(m^2)(W_m(x) - W_m(-x))dm^2 = 0. \tag{1.28}$$

Taking the Fourier transform, we get

$$p^2\rho(-p^2) = 0. \tag{1.29}$$

Thus, $\rho(m^2) = c\delta(m^2)$, where c is a constant. The constant c cannot vanish, otherwise we would prove that $[J(x), \phi(0)]$ has a zero expectation value at the vacuum, which contradicts the assumption of symmetry breaking.

This argument shows that all contributions to the 2-point function $\langle \Omega, J(x)\phi(0)\Omega \rangle$ come from intermediate states of zero mass. This implies that $L^2(\mathcal{O}_0^+)$ is contained in the discrete spectrum of $\mathcal{H}$ (i.e. as an honest subrepresentation of the Poincaré group). Such a subrepresentation is interpreted in quantum field theory as a massless particle of zero spin. Thus, we have proved the following statement, which goes under the name of Goldstone's theorem:

Theorem. *In the Hilbert space of a realization of a field theory with continuous symmetry breaking, there is a massless scalar (i.e. a subrepresentation of the Poincaré group isomorphic to $L^2(\mathcal{O}_0^+)$) which is created by the current of the broken symmetry.*

Remark. "Created" means that $J(x)\Omega$ is not orthogonal to the subrepresentation.

Definition. The massless scalar which we found is called the Goldstone boson corresponding to the broken symmetry G^s.

Remark 1. If the symmetry with respect to G^s was not broken, then $\lim_{f \to 1} Q_f \Omega$ would be zero. On the other hand, when symmetry breaking occurs, Q_f creates Goldstone bosons from the vacuum. Thus, Goldstone bosons "measure" the failure of symmetry.

Remark 2. The Goldstone boson can be created not only by the current operator of the symmetry, but also by other local operators. In fact, as we saw in the proof of Goldstone's theorem, it will be created by any scalar local operator whose 1-point function is not invariant under the symmetry.

Remark 3. There is no claim in Goldstone's theorem that the Goldstone boson is free, i.e. that it can be created by a free field $\phi(x)$. In fact, as we will see, this is often not the case.

Remark 4. If continuous symmetry breaking occurs classically, the Goldstone boson can already be seen in perturbation theory. As an example consider the Lagrangian (1.13). Consider the classical ground state $\phi = a$. This state is degenerate. Therefore, if we introduce real variables $\phi_1 = \mathrm{Re}\phi - a$, $\phi_2 = \mathrm{Im}\phi$, and rewrite the Lagrangian in terms of these variable, then because of the degeneracy of the minimum the field ϕ_2 will be classically massless. Therefore, if we compute the 2-point function of ϕ_2 it will have a pole at $k^2 = 0$ (modulo the perturbation parameters). Of course, in the absence of a symmetry, loop terms might shift this pole. In other words, the classically massless ϕ_2 may get nonzero mass quantum-mechanically. What Goldstone's theorem tells us is that this will not happen if there is a spontaneously broken symmetry under which the condition $\phi_2 = 0$ is not invariant.

Corollary. *Continuous symmetry breaking does not happen in 2 dimensions.*

Proof. Otherwise, by Goldstone's theorem Goldstone bosons would have to exist. But in a two-dimensional quantum field theory, there can be no massless particles created by a local operator. Indeed, the 2-point function of this operator in momentum space equals $w(k^2) = \int_0^\infty \frac{d\mu(m^2)}{k^2+m^2}$, where μ is the spectral measure. If there

is a massless particle, this measure will have an atom at $m = 0$. But in this case the 2-point function $W(x)$ in position space will behave like $-C \ln x^2$ at infinity, i.e. would violate the positivity axiom. □

Now assume that we have a Lagrangian which has a Lie group G of symmetries. Assume that $\mathcal{H}$ is a realization of the quantum field theory defined by this Lagrangian, whose stabilizer is $H \subset G$. In this case one says that in the realization $\mathcal{H}$, the G-symmetry is spontaneously broken to H. (By the above corollary, the spacetime dimension is necessarily greater than two.)

Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H. Goldstone's theorem implies

Corollary. *$\mathcal{H}$ contains in its discrete spectrum a subrepresentation isomorphic to $L^2(\mathcal{O}_0^+) \otimes (\mathfrak{g}/\mathfrak{h})$.*

Proof. The proof is clear: if this is not so than there exists an element $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$, such that $Q_f^Y \Omega \to 0$ (weakly) when $f \to 1$. This means that the symmetry with respect to Y is not broken – a contradiction.

The corollary means that Goldstone bosons corresponding to linearly independent broken infinitesimal symmetries are also linearly independent.

Example. Let $\phi : \mathbb{R}^d \to \mathbb{R}^N$ be a scalar field, and consider the Lagrangian

$$\mathcal{L}(\phi) = \int d^d x \left(\frac{1}{2} (\nabla \phi)^2 + \frac{g}{4!} (\phi^2 - a^2)^2 \right). \tag{1.30}$$

This Lagrangian has an $SO(N)$-symmetry, and the space of its classical ground states is S^{N-1}. Therefore, classically the $SO(N)$-symmetry is broken to $SO(N-1)$. As we know, if $d > 2$, the same will happen quantum mechanically (for a weakly coupled theory), and so any realization (solution) of the theory has $N - 1$ independent Goldstone bosons.

Let P be the classical minimum with coordinates $(0, 0, ..., 0, a)$. Consider the realization $\mathcal{H}_P$ of the theory, where the vacuum Ω is localized near P. Let $Y_i \in \mathfrak{so}_N$, $i = 1, ..., N-1$, be the infinitesimal rotations in the planes generated by the basis vectors e_i, e_N of $\mathbb{R}^N$. The 1-point function of the operator ϕ_i is not invariant under Y_i, so ϕ_i creates the Goldstone boson corresponding to Y_i.

We can construct low energy (non-vacuum) states that are not just localized near a single classical ground state P. Indeed, let P' be another classical ground state, and let $Y \in \mathfrak{so}(N)$ be an element such that $e^Y P = P'$. Let J_Y be the current corresponding to Y, and Q_f^Y the corresponding cutoff charge. Then the state $e^{iQ_f^Y} \Omega$ is a low energy state localized near P'.

At long distances the theory will behave as a sigma model into the space of classical ground states. This is an interesting statement for $N \geq 3$, because in this case the target (S^{N-1}) is not flat, so the sigma model is not free. More precisely, at low energies (or long distances), the theory of bosons ϕ_i, $i = 1, ..., N-1$, will be free in the zeroth approximation (where one considers only the zero momentum limit of scattering amplitudes), but in the first approximation (where one considers the first nonzero terms for small momenta) it will not be free but will be described by the Lagrangian of the sigma model into the sphere.

§1.7. Infrared behavior of purely non-renormalizable field theories

In this section we will clarify the meaning of and basis for the statement that, for $d > 2$, a quantum field theory in which all massless particles are Goldstone bosons behaves in the infrared limit as a sigma model into the space of vacuum states.

Suppose we have a purely nonrenormalizable field theory described by a Lagrangian $\mathcal{L}$. We will call a Lagrangian purely nonrenormalizable if all its couplings have negative dimension. An example of such a Lagrangian is the Lagrangian of a sigma model for $d > 2$. Such Lagrangians are not good for perturbative renormalization in the UV limit, but create no problem in the IR limit, since all their interactions are IR irrelevant from the point of view of the Wilsonian renormalization group flow. Namely, if we introduce an UV momentum cutoff Λ (which is now not being sent to infinity), we can define correlation functions of $\mathcal{L}$ perturbatively: the correlation function is the sum of amplitudes of all Feynman diagrams, which are evaluated as usual, with integration carried out with cutoff $|q| < \Lambda$. Because $\mathcal{L}$ has no mass terms, there may be some IR divergences, but they are rather tame when all couplings have negative dimensions; their meaning is just that the S-matrix is not analytic in momenta near zero momentum. (We give an example of this later; see the figure at the end of the lecture.)

Now suppose that we want to compute the asymptotic expansion of the n-point function in momentum representation, around the point $k_i = 0$. We will have (for $\sum k_i = 0$):

$$G_n(k_1, \dots, k_n) = \frac{G_n^0(k_1, \dots, k_n)}{k_1^2 \dots k_n^2}, \tag{1.31}$$

where according to the LSZ reduction formula that we learned about in Kazhdan's lectures, G_n^0 is the S-matrix element for scattering of n Goldstone bosons. G_n^0 is not analytic near $k_i = 0$, so we cannot expand it in a power series, but we can look at how it behaves under an overall scaling $k_i \to \lambda k_i$.

The key property of this series, which follows from pure nonrenormalizability, is that modulo terms that scale as any given power of λ, G_n^0 is determined by finitely many Feynman graphs. In other words, pure unrenormalizability plus dimensional analysis imply that terms of higher and higher order scale with increasing powers of λ. Thus, we can obtain the IR asymptotics of the correlation functions to any order in k_i just from finitely many terms in perturbation theory.

The loop diagrams constructed in the purely unrenormalizable theory also have ultraviolet divergences, of course. We are viewing the purely unrenormalizable theory as a low energy effective field theory, and from that point of view, we simply regularize the divergences in an arbitrary fashion and regard the coefficients of all possible terms in the low energy effective Lagrangian as phenomenological parameters. Modulo any power of λ, only a finite number of such terms are relevant (since, above $d = 2$, there are only finitely many operators with dimension below a prescribed constant).

Now suppose we have an actual quantum field theory, given by some renormalizable Lagrangian $\mathcal{L}'$. When we say that the theory defined by $\mathcal{L}'$ is described in the IR limit, to a specified accuracy, by a purely nonrenormalizable Lagrangian $\mathcal{L}$ (on the same fields), we mean that up to terms that scale (under $k_i \to \lambda k_i$) as a given power of λ, the functions G_n^0 given by (1.31) are the same for $\mathcal{L}$ as for $\mathcal{L}'$.

For instance, when at the end of the previous section we said that at low energies (or long distances), the theory of bosons ϕ_i, $i = 1, ..., N-1$, is free in the zeroth approximation, and described by the Lagrangian of the sigma model into a sphere whose coordinates are ϕ_i, in the first approximation, we meant that the functions $G_n^0(k_1, ..., k_n)$ for ϕ_i are the same as in the free theory modulo $o(1)$, and the same as in the sigma model modulo $o(k^2)$.

Computing higher terms of the k-expansion, one can construct a purely non-renormalizable low energy effective theory, which will describe our theory in the infrared to any required accuracy.

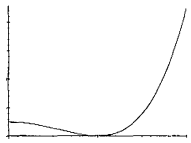
Remark. Since the restriction of the function $G_0^n(k_1, \ldots, k_n)$ to the locus $k_i^2 = 0$ (with k_i not necessarily zero) is the S-matrix element, the statement is that the scattering matrix of n Goldstone bosons in model (1.30) vanishes in order k^0 as in the free theory, and that the term of order k^2 is determined by the curvature of the space of vacua and in particular is nonzero if $N > 2$.

§1.8. Effective action for Goldstone bosons

Let us implement the general discussion of the previous section, and work out the effective action for a Goldstone boson in a particular example. Consider a theory which breaks $SO(3)$ to $SO(2) = U(1)$. We have three real scalars which can be combined to a 3-component object $\vec{\phi}$ which transforms in the 3-dimensional representation of $SO(3)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{\lambda} \int d^n x \left[\frac{1}{2} (\partial_\mu \vec{\phi})^2 + V(|\vec{\phi}|) \right],$$

where, letting $\rho = |\vec{\phi}|$, $V(\rho)$ is a potential which has a nondegenerate minimum away from the origin.



The simplest such potential is $V(\rho) = \frac{1}{8}(\rho^2 - \rho_0^2)^2$, with a minimum at ρ_0. If we let $\Omega = \phi/\rho \in S^2$, we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{\lambda} \int d^n x \left[\frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} \rho^2 (\partial_\mu \Omega)^2 + V(\rho) \right].$$

The term $\frac{1}{2}\rho^2(\partial_\mu \Omega)^2$ represents the round metric on S^2.

When λ is small, we can hope that the classical approximation will be good. Since ρ is a massive field, we can integrate it out of the theory by setting it equal to its minimum value ρ_0 and studying fluctuations

$$\rho = \rho_0 + w,$$

where w is now the quantum field which will appear in our path integrals. To first approximation, we would have an effective Lagrangian

$$\mathcal{L}_{\text{eff}}(\Omega) = \frac{\rho_0^2}{2\lambda} \int d^n x \, (\partial \Omega)^2,$$

describing a nonlinear sigma model of maps to G/H, which in the present case is S^2. The massless fields are described by Ω.

Can this be an answer? In other words, could *any* quantum field theory flow to this sigma model in the infrared? In fact, it *is* a possible answer for spacetime dimension $n > 2$, because the nonlinear sigma model is non-renormalizable above two dimensions, so the interaction we get is irrelevant. That is, if a_i are Riemann normal coordinates near a point $P \in S^2$, we can expand schematically near P

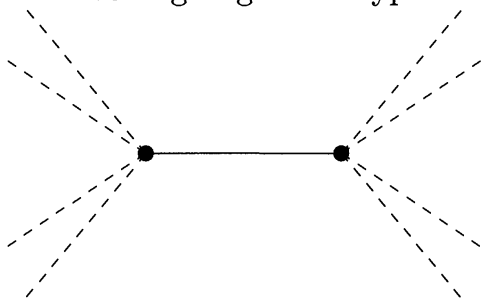
$$(\partial\Omega)^2 = (da)^2 + Ra^2(da)^2 + \dots$$

with R being the Riemann tensor of S^2. From this we see that the interaction is irrelevant above two dimensions. In fact, to give $(da)^2$ dimension n, we must assign dimension $(n-2)/2$ to a, whence the interaction has dimension $2n-2$, which exceeds n for $n > 2$. The fact that the sigma model is a possible answer for $n > 2$ but not for $n \leq 2$ is an aspect of the fact, discussed earlier, that spontaneous breaking of a continuous symmetry is possible for $n > 2$ but not for $n \leq 2$. This expansion would also serve as the starting point for an explicit computation of the Goldstone boson S-matrix from the sigma model.

Is the sigma model the *correct* answer for the infrared behavior of our particular problem, at least for sufficiently small λ? We claim that it is. To study this point, let us treat the effects of w perturbatively. The interactions of w with the Goldstone boson come from the interaction term in the Lagrangian:

$$\frac{1}{2\lambda}\rho^2(\partial\Omega)^2 = \frac{1}{2\lambda}(\rho_0 + w)^2(\partial\Omega)^2 = \frac{1}{2\lambda}(\rho_0^2 + 2\rho_0 w + w^2)(\partial\Omega)^2.$$

The operator $\partial\Omega$ is highly nonlinear, and can be thought of as emitting an arbitrary number of Goldstone bosons. We need to calculate Feynman diagrams involving w's, in order to find the effective Lagrangian. A typical diagram is



Here we are representing w with a solid line and a's with dotted lines. The w-propagator is $\frac{1}{k^2+m_w^2}$; since we are interested in small momenta k, we expand in powers of k. The leading term is

$$\text{const} \cdot (\partial\Omega)^2 \frac{1}{m_w^2}(\partial\Omega)^2.$$

This has $SO(3)$ symmetry and is an irrelevant interaction. It is even more irrelevant than the terms, sketched above, that come by expanding $(\partial\Omega)^2$ in powers of a.

In fact, while it is instructive to study these diagrams, just to show that the sigma model is infrared-stable, we do not need the details of the diagrams. All we need to know is that the effective action has $SO(3)$ symmetry. Given that symmetry, what possible terms could be generated in the effective action?

- A potential $V(\Omega)$ is not possible, because there is no $SO(3)$-invariant function on S^2. This is the basic reason that, given the $SO(3)$ symmetry, the sigma model is infrared stable. A potential function would change the picture completely. For instance, a generic potential would have an isolated, nondegenerate minimum, giving us a unique vacuum with an infrared-trivial massive theory, in contrast to a continuous family of vacua associated with spontaneously broken $SO(3)$.
- The only term with only two derivatives that respects all the symmetries of the problem is $(\partial\Omega)^2$ itself. (That is essentially because any $SO(3)$ invariant metric on S^2 is a multiple of the standard one.) So quantum corrections due to diagrams with w fields should definitely be expected to modify the coefficient of this term.
- Other possible terms like $(\partial\Omega^2)^2$ and $(\partial\Omega)\,\nabla^2(\partial\Omega)$ have more than two derivatives and are more and more irrelevant.

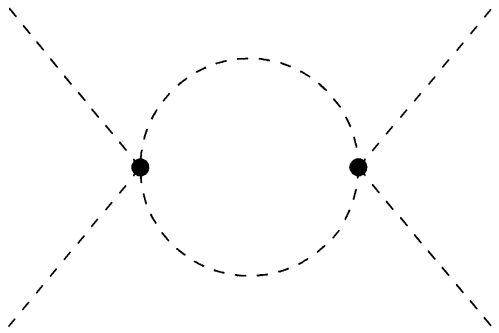
So the leading infrared behavior is determined by an effective action of the form $\frac{1}{f^2}(\partial\Omega)^2$ with

$$\frac{1}{f^2} = \frac{\rho_0^2}{2\lambda} + \text{ loop corrections.}$$

When this is expanded in Riemann normal coordinates about a given vacuum, that is, a given point $P \in G/H$, one gets interactions (of which the first was sketched above) that involve the Riemann tensor of G/H and its covariant derivatives. If one uses these interactions to compute scattering amplitudes involving Goldstone bosons with small momenta of order k, the tree level amplitudes are all proportional to k^2 for k near zero, as the interaction terms all contain precisely two derivatives. Loop contributions are smaller for $k \to 0$, since the interactions are irrelevant in the infrared. To be more precise, loop amplitudes all either (i) renormalize the constant f in the $SO(3)$-invariant Lagrangian, or (ii) give corrections to the scattering amplitudes that vanish faster than k^2 for $k \to 0$.

Hence, the terms of order k^2 in the Goldstone boson scattering are all completely determined by the one constant f (or more generally by the choice of a G-invariant metric on the homogeneous space G/H). In the 1960's, it was discovered that the low energy scattering of pions beautifully fits such a description, with $G = SU(2) \times SU(2)$ and H a diagonal $SU(2)$. This is how it was discovered that the strong interactions have a spontaneously broken approximate chiral symmetry; the discovery played a very major role in the subsequent development of physics.

What happens if one wants to compute terms in the Goldstone boson scattering of higher order than k^2? It is clear that in order k^4, new constants will enter that can only be determined from microscopic calculations (or experiment), since there are G-invariant interactions with four derivatives (such as the $((\partial\Omega)^2)^2$ term found above from the explicit tree diagram considered). However, interestingly, in four spacetime dimensions, the lowest order correction to the k^2 amplitude for Goldstone bosons is not of order k^4 but of order $k^4 \ln k$. It comes from a loop diagram



with vertices drawn from the two-derivative part of the Lagrangian, and hence is uniquely determined in terms of the same constant f that controls the k^2 terms in the scattering amplitudes.

The analysis of low-energy Goldstone boson interactions via the ideas I have explained is known as "current algebra." In particular, via current algebra relations, one can deduce from experiment what is the broken symmetry group G, and many of the parameters in the G-invariant effective Lagrangian.

One final comment about symmetry-breaking examples such as this one: if we begin with a G-invariant microscopic Lagrangian $\mathcal{L}_{\text{micro}}$ which we perturb to

$$\mathcal{L}_{\text{micro}} + \varepsilon(\delta\mathcal{L})$$

with ε a small parameter, and the term $\delta\mathcal{L}$ not being G-invariant, then in the infrared we will get

$$\frac{1}{f^2}(\partial\Omega)^2 + \varepsilon V(\Omega) + O(\varepsilon^2),$$

with $V(\Omega)$ being a non-G-invariant operator – of which the most relevant part is of course a potential with no derivatives, as suggested in the notation V. $V(\Omega)$ is highly constrained by the fact that it must transform under G the same way that $\delta\mathcal{L}$ does. For example, in the case of strong interactions, a small $\delta\mathcal{L}$ term, breaking $SU(2) \times SU(2)$ to a diagonal $SU(2)$, is actually present; it selects a unique vacuum from what would otherwise be a continuous family, and gives small masses to the pions. (See problem 1 in the homework for more on this.) Since the pion actually has a mass, one cannot expand its four-momentum near zero; in current algebra studies of pions, one really takes the momentum k to be of order the pion mass.

References and cross references

More about generalities of field theory, in particular the notion of a symmetry, is in [I-**Classical Fields**] (classical) and [I-**Axiomatics**] (quantum). The nonperturbative estimates of §§1.1, 1.2 are examples of instanton computations which are discussed in more detail in Lecture 7. The theory of of an angle-valued free field in 2 dimensions is discussed in Lecture 1 of [II-**CFT**] and in Lecture 6 of this series. A discussion of Goldstone bosons as the only spin 0 massless particles in 4 dimensions is in Lecture 2. A general discussion of infrared behavior of field theories is in Lecture 2, with a lot of examples in later lectures. Problem 1 at the end of these lectures deals with symmetry breaking.

Coleman's theorem on the absence of spontaneous symmetry breaking in two spacetime dimensions can be found in

S. Coleman, *There Are No Goldstone Bosons In Two Dimensions*, Commun. Math. Phys. **31** (1973), 259.

For an introduction to spontaneously broken global symmetry and its implications, see

S. Coleman, *Aspects Of Symmetry*, Cambridge University Press, 1985.
S. Weinberg, *The Quantum Theory Of Fields*, chapter 19, vol. 2, Cambridge University Press, 1996.

LECTURE 2
Gauge Symmetry Breaking and More on Infrared Behavior

§2.1. Gauge symmetry

Recall what gauge symmetry is. We have a spacetime $X = \mathbb{R}_{time} \times X_0$. We have a compact gauge group G. We have a field theory where a field configuration is a connection in some principal bundle over $\mathbb{R}^d$ and possibly some matter fields.

Recall the Hamiltonian approach to gauge theory. Let $\widetilde{M_0}$ be the space of solutions to the classical equations of motion. On $\widetilde{M_0}$ we have an action of the group $\widehat{G}$ of gauge transformations. Let $M_0 \subset \widetilde{M_0}$ be the space of all solutions where the G-bundle is trivialized in the time direction, and the connection is trivial in that direction. Such solutions as usual are completely determined by the pair $A(t_0), \frac{dA}{dt}(t_0)$ (where $t = t_0$ is a spatial cycle), together with initial data for the matter fields. It is clear that any element of $\widetilde{M_0}$ can be brought to M_0 by a gauge transformation, so M_0 still contains all solutions up to gauge transformations.

Suppose that $X_0 = \mathbb{R}^{d-1}$. In this case we may consider only trivial bundles, and (if the energy is finite) connections which vanish at spatial infinity. In other words, if $\mathcal{A}$ is the space of connections A on the trivial G-bundle over $\mathbb{R}^{d-1}$ which vanish at ∞ then M_0 for pure gauge theory is $T^*\mathcal{A}$. If matter fields are present, then M_0 is a product of $T^*\mathcal{A}$ with some other space.

Define $\widetilde{G}$ to be the group of elements of $\mathrm{Maps}(\mathbb{R}^{d-1}, G)$ which have a limit at infinity, and $\widetilde{G}_0$ to be the subgroup of $\widetilde{G}$ consisting of functions which tend to 1 at ∞. We have $\widetilde{G}/\widetilde{G}_0 = G$. This quotient group is called the group of constant gauge transformations at ∞ and called G_∞.

The group $\widetilde{G}$ acts symplectically on M_0. The physical phase space in gauge theory is the symplectic quotient $M = M_0//\widetilde{G}_0$. Note that we only divide by $\widetilde{G}_0$ and not by the whole group $\widetilde{G}$, so that we inherit an action of the quotient G_∞ on M. It is this symmetry group whose breaking we will discuss.

§2.2. Breaking of gauge symmetry and charges at infinity

Definition. Suppose we have a (classical) gauge theory, and let $s \in M$ be its vacuum state. Let $H \subset G = G_\infty$ be the stabilizer of s. In this case we will say that at the ground state s the gauge symmetry is broken from G to H.

Notes by Pavel Etingof, David Kazhdan, and David Morrison

Thus, by symmetry breaking we mean roughly the same thing as for global symmetry: there is a symmetry of the Poisson algebra of functions on M which does not fix a particular ground state.

Important remark. The above expression "roughly the same thing " should be taken with great care. There are some fundamental differences between the two situations, which will become clear below. They come from the fact that in the situation we are considering here (unlike that in Lecture 1), the physical observables, being gauge invariant by definition, automatically commute with G and therefore do not separate points on M; i.e. not every function on M is "observable." In other words, the action of G on the "theory" (in the sense of Lecture 1) is trivial from the beginning. Related to this, our formulation of spontaneous symmetry breaking for global symmetries in Lecture 1) made sense abstractly, but the notion of spontaneous breaking of gauge symmetry, as we have formulated it here, is tied to an underlying classical description.

A variant of the above remark is that the phrase "gauge symmetry breaking" doesn't really refer to a phenomenon that can be described abstractly and axiomatically. It refers to a specific important phenomenon, called the Higgs mechanism, that occurs in certain weakly coupled gauge theories. We explain this mechanism in the example below.

Let us now compute the action of G (classically). First of all, we have a moment map $\mu : M_0 \to \widetilde{\mathfrak{g}}_0^*$, where $\widetilde{\mathfrak{g}}_0$ is the Lie algebra of $\widetilde{G}_0$ – the algebra of functions from $\mathbb{R}^{n-1}$ to the Lie algebra $\mathfrak{g}$ of G which vanish at infinity. Thus, for any $\varepsilon \in \widetilde{\mathfrak{g}}_0^*$ we have a Hamiltonian $Q(\varepsilon) \in C^\infty(M_0)$ defined by $Q(\varepsilon)(X) = \mu(X)(\varepsilon)$.

In fact, it is easy to compute $Q(\varepsilon)$ using the Noether formalism. Namely,

$$Q(\varepsilon) = \int_{\mathbb{R}^{d-1}} \mathrm{Tr}\left(\frac{dA}{dt}\nabla_A\varepsilon\right) d^{d-1}x + \text{ matter terms }, \tag{2.1}$$

On M, $Q(\varepsilon) = 0$ if ε vanishes at infinity. Thus, on M we have $\nabla_A^* \frac{dA}{dt} =$ matter terms . In particular, in pure gauge theory $\nabla_A^* \frac{dA}{dt} = 0$.

Taking this into account, we see that on M

$$Q(\varepsilon) = \int_{\mathbb{R}^{d-1}} \mathrm{Tr}\left(\nabla_A(\varepsilon\frac{dA}{dt})\right) d^{d-1}x. \tag{2.2}$$

Using Stokes' formula, we can rewrite (2.2) as

$$Q(\varepsilon) = \lim_{r\to\infty} \int_{S^{n-2}(r)} *_{d-1}\mathrm{Tr}\left(\varepsilon\frac{dA}{dt}\right) = \lim_{r\to\infty} \int_{S^{n-2}(r)} *_d\mathrm{Tr}\,(\varepsilon F)\,, \tag{2.3}$$

where F is the curvature of the spacetime connection corresponding to the given point of M_0, and $S^k(r)$ is the k-sphere of radius r. This formula defines the Hamiltonians for the action of $G = G_\infty$ on M.

This formula shows that $Q(\varepsilon)$ vanishes for all gauge transformations (not necessarily vanishing at infinity) on a particular state if $F = o(r^{2-n})$, $r \to \infty$ on that state. However, if this is not the case, then $Q(\varepsilon)$ may be nonzero for a constant ε.

Example. Consider a $U(1)$ gauge theory with a charged complex scalar. The fields are a connection A on a Hermitian line bundle and a section ϕ of this bundle. The Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2}\int F \wedge *F + \int |D_A\phi|^2 d^4x + \int \frac{\lambda}{8}(|\phi|^2 + v^2)^2 d^4x.$$

This is the most general renormalizable Lagrangian in these fields in four dimensions, which is symmetric under $\phi \to -\phi$. Here e, λ, v are parameters and e^2, λ are positive while v^2 can be positive or negative. For simplicity we assume first that $v^2 \neq 0$.

In four dimensions, this theory is not believed to exist in the UV, but we will regard it as an effective theory for some more fundamental theory (or we could consider it in fewer than four dimensions).

Classically (and quantum mechanically for e^2, λ sufficiently small) we have two cases.

1. $v^2 > 0$; the potential has a single minimum.
2. $v^2 < 0$; the potential has a circle of minima.

Let us consider how in these two cases the theory behaves in the infrared.

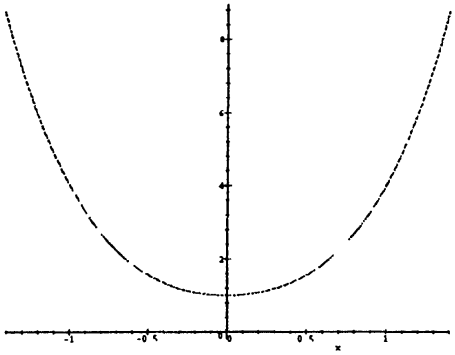


Figure 1. The potential for $v^2 > 0$.

Case 1. $v^2 > 0$. In this case the minimum of energy is attained when $\phi = 0$. First consider the case when the gauge coupling vanishes: $e^2 = 0$. In this case our theory is a direct product of a pure (free) abelian gauge theory and the ϕ^4 theory. Therefore, it has a unique vacuum, and the particles which occur at the lower part of the spectrum are a massless vector, or gauge boson (coming from gauge theory) and two massive real scalars (coming from ϕ^4 theory).

If we turn on small e^2 the situation should remain the same. Indeed, certainly nothing can happen to the massive scalars (the subspace of the Hilbert space consisting of states – like these scalars – with nonzero charge, has a mass gap in the lowest order approximation, and massiveness is an open condition); moreover, their masses must be equal since there is a $U(1)$ symmetry at infinity (the $Q(\varepsilon)$ for constant ε) which prohibits the masses to differ. The fact that $Q(\varepsilon) \neq 0$ is clear since this is so at $e^2 = 0$, when $Q(\varepsilon)$ represents the $U(1)$ global symmetry. The operator $Q(\varepsilon)$ for ε the unit tangent vector is called the operator of electric charge.

Furthermore, the massless vectors cannot become massive. Indeed, recall that a massless vector means an irreducible representation of $SO(3,1)$ with $p^2 = 0$ and spin 1, i.e. the space of sections of a two-dimensional equivariant vector bundle over the light cone. This vector bundle cannot be deformed to an equivariant

vector bundle over the hyperboloid, since the stabilizer group $SO(3)$ of a point on the hyperboloid does not have an irreducible two-dimensional representation. Thus, the quantum theory for small coupling will have the same particles – two massive scalars (the real and imaginary part of ϕ) and a massless vector (the gauge boson).

Remark. The above argument on non-deformability of a massless vector fails in two or three dimensions. For example, in three dimensions, the massless vector is just the space of functions over the cone, which can be successfully deformed into the space of functions over a hyperboloid. This actually happens when in pure $U(1)$ gauge theory one introduces a Chern-Simons term $c\int A\wedge dA$. The theory remains free but becomes massive, yielding one massive scalar. In the theory we are considering (for three dimensions), this cannot happen dynamically since the Chern-Simons term is odd under change of orientation, but in other theories this could happen.

In fact, quantum mechanically the operator $Q(\varepsilon)$ (for a suitable normalization of ε) has integer eigenvalues, and thus defines (in quantum theory) a $\mathbb{Z}$-grading of the corresponding Hilbert space. In particular, since $Q(\varepsilon)\neq 0$, but $Q(\varepsilon)$ commutes with all local operators, there are sectors of the Hilbert space (namely those with nonzero $Q(\varepsilon)$) which cannot be reached from the vacuum by applying local operators. This shows that we have a fundamental violation of the Wightman axioms: the representation of the operator algebra in the physical Hilbert space is not irreducible. However, the theory still has one vacuum only: the minimal energy in the sectors with nonzero charge $Q(\varepsilon)$ is positive.

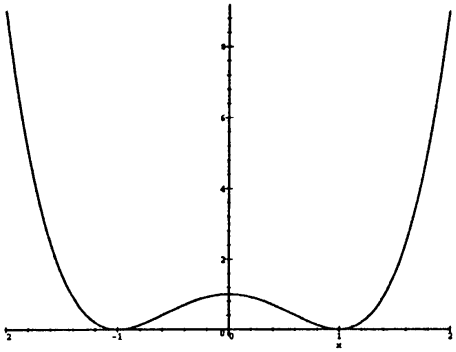


Figure 2. The potential for $v^2<0$.

Case 2. $v^2<0$. Let $v^2=-b^2$. Then classically we have a minimum of energy on the circle $|\phi|=b$. This implies that any finite energy configuration has the property $\phi=be^{i\theta_0}$ at infinity, where θ_0 is a constant. Therefore, by a gauge transformation which has a finite limit at infinity, we can arrange that ϕ is real and positive: $\phi=b+w$, where w is a new real variable. Writing the Lagrangian in terms of the new variables, we will get something with the following quadratic part:

$$L_{quadratic}=\frac{1}{4e^2}\int F^2+\int d^4x((dw)^2+M^2w^2)+\int d^4xb^2A^2. \tag{2.4}$$

It is seen from (2.4) that now all fields are massive. Of course, the Lagrangian (2.4) is not gauge invariant for A, since we have already "spent" the gauge symmetry on making ϕ real.

Thus, the infrared limit of the corresponding quantum theory is trivial for small values of the couplings. In particular, there are no massless gauge bosons: they

have been "eaten" by the ϕ-field. This situation is called the Higgs phenomenon, or spontaneous breaking of gauge symmetry.

Note that in spite of the presence of what looks like a circle of zero energy states, our theory has only one vacuum. In other words, all points of the circle are regarded as the same state, on the grounds that they are gauge equivalent to each other and therefore define equivalent realizations (i.e. give the same expectation values of gauge invariant local operators) This is a fundamental difference between gauge and global symmetry breaking. In global symmetry breaking, the points of the circle represent different vacua (realizations) of the theory, since there exist non-symmetric operators which have different expectation values at different points of the circle.

To summarize, the phrase "gauge symmetry breaking" refers mainly to the Higgs mechanism, or Higgs phenomenon, whereby in a model such as this one with $v^2 < 0$, the gauge field gets a mass even though the coupling is weak. Under much more restrictive conditions (see Lecture 10) one can give an axiomatic explanation of what one means by a Higgs phase. But as the term is most commonly used, the "Higgs mechanism" is just a way to get a massive spectrum (or at least, to make the number of massless spin one fields smaller than the dimension of the gauge group) even for weak coupling. It is important because it is one of the simplest things a gauge theory can do, and because it occurs in nature in the weak interactions.

Remark. If one tries to compute $Q(\varepsilon)$ in Case 2 (when the symmetry is broken) using formula (2.3), the answer will be zero since the integrand dies rapidly at infinity. When a Higgs mechanism occurs, the conserved charge associated with the gauge symmetry is zero.

The particles which are found in the infrared in the situation of Case 2 are, according to (2.4), a massive vector (A) and a massive scalar (ϕ). There is only one scalar since ϕ is now real. Thus, at the level of representation theory the Higgs phenomenon arising in Case 2 boils down to a deformation of representations of the Poincaré group: a massless vector plus a massless scalar is deformed to a massive vector. Recall for comparison that a massless vector separately cannot be deformed into a massive representation.

Finally, consider the special case $v^2 = 0$. In this case classically we have no symmetry breaking as for $v^2 > 0$, and the particles are a massless vector and two massless scalars. However, it is not expected to be the quantum answer, since this configuration is not stable under perturbations.

Remark. If the Lagrangian we start with is not IR free (say, it is the Lagrangian of an asymptotically free gauge theory) then the classical analysis we discussed above does not apply in quantum theory. In this case the infrared behavior of the theory is in general difficult to determine. All kinds of things can happen, some of which we will hear about later this term in my lectures or Seiberg's. Among other things, one can get a massive spectrum in ways that are far more exotic and difficult to understand the the Higgs mechanism; or one can generate in the infrared an unbroken gauge symmetry that is not a subgroup of the original classical gauge symmetry.

§2.3. Symmetry breaking and gauging

In conclusion, let us discuss the connection between global symmetry and gauge symmetry breaking. Suppose we have a Lagrangian L of a field theory (say in 4 dimensions) which has a global $U(1)$ symmetry. A typical example is when the theory contains some scalar fields ϕ_j which are sections of Hermitian vector bundles, and $U(1)$ acts by multiplication of these sections by $e^{in_j\theta}$. This $U(1)$ symmetry can be gauged, by introducing a $U(1)$ gauge field A and new Lagrangian

$$L_{gauged} = \frac{1}{4e^2}\int F_A \wedge *F_A + L_A, \tag{2.5}$$

where L_A is obtained from the original matter Lagrangian L by replacing all derivatives of ϕ_j by covariant derivatives.

The statement is that if L is infrared free, then for small gauge couplings above three dimensions the symmetry breaking behavior of the theories defined by L and L_{gauged} is the same. (The discussion in two or three dimensions needs some modification because of "confinement" in $U(1)$ gauge theory, which we study later in the term, and for certain other reasons that will be discussed later today and in later lectures.) Namely, if there is breaking of global symmetry for L then there is a Higgs mechanism for L_{gauged} and vice versa.

Indeed, let us consider both cases.

Case 1. No global symmetry breaking. We suppose that the theory is sufficiently generic that there are no massless charged scalars. (This corresponds in our above example to avoiding the borderline case $v = 0$.) In the small e limit, the Hilbert space is the tensor product of the Hilbert space $\mathcal{H}_L$ made by quantizing L and the Hilbert space $\mathcal{H}'$ of a free massless spin one field. $U(1)$ acts nontrivially, by its action in $\mathcal{H}_L$. In the absence of charged massless scalars, this spectrum with a massless spin one field is stable; nothing can happen to it for sufficiently weak coupling.

Case 2. Global symmetry breaking. In this case, at the minimum of the energy the charged field ϕ_j is not zero. There is no invariant classical ground state, and there is a Goldstone boson corresponding to this symmetry breaking. In this case, pick a vacuum state and a component $\phi_j^{(n)}$ which is not zero at this vacuum. In the gauged theory, we can perform a gauge transformation which will make this component real and positive. This shows that if there are no other massless particles (in particular, no other broken global symmetries), all fields in the theory will become massive. Thus, we have breaking of gauge symmetry. In describing this case, we have used a classical language, assuming that L is weakly coupled; but the discussion can be made abstractly, only assuming that L describes a theory in which the $U(1)$ symmetry is spontaneously broken.

These considerations are valid only if the gauge coupling is small. (If we take a nonabelian gauge group in four dimensions, then in Case 1, there is a further restriction: this analysis is adequate only if the beta function is positive so that the gauge coupling flows to zero in the infrared.)

§2.4. No massless particles of higher spin

Now we consider in a general way infrared-free theories in 4 dimensions. (The considerations that follow generalize above 4 dimensions but become trivial below

dimension 4). The general discussion of representations of the Poincaré algebra and free fields of all spins seems to suggest that infrared-free theories might have massless particles of any spin. But in practice, in all interesting examples I am familiar with, one can argue *a priori* that any massless particles will have spins 0, 1/2 or 1.

Most theories of interest can be formulated not just on flat $\mathbb{R}^4$, but on a more general curved 4-manifold M^4 with a general metric g. In fact, any theory that is part of the description of nature has this property, since general relativity is part of nature and in general relativity, spacetime is curved! In quantum field theory, the ability to work on a curved spacetime implies the existence of a very special operator, called the stress tensor or energy-momentum tensor $T_{\mu\nu}(x)$. It measures the response to a change in the metric tensor g. We suppose that a theory is formulated with a general g by a Lagrangian $\mathcal{L}(\phi_i; g)$, which is invariant under diffeomorphisms acting both on the ϕ_i and on g. g is not one of the fields of the theory – it is arbitrary but is held fixed in studying the classical or quantum dynamics of the ϕ_i – and this diffeomorphism invariance means that the theory, if formulated in a spacetime (M, g), really depends on g only up to diffeomorphism. In this setup, the stress tensor is defined as

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}.$$

This implies obviously that T is a symmetric tensor

$$T^{\mu\nu} = T^{\nu\mu}.$$

T can also be shown to obey

$$D_\mu T^{\mu\nu} = 0$$

by virtue of diffeomorphism invariance. If our theory is actually *conformally invariant*, then T is traceless, that is, $g^{\mu\nu}T_{\mu\nu} = 0$.

Having such a stress tensor leads to powerful statements even if one specializes to the case that M is flat Euclidean space. For instance, last fall, when we axiomatized quantum field theory, we required Poincaré invariance, with conserved charges $Q(K)$ for every Killing vector field K in spacetime. The existence of a conserved, symmetric stress tensor is a local statement that leads to Poincaré invariance globally. Given any Killing vector field K, one uses the Killing vector equation (which reads $D_\mu K_\nu + D_\nu K_\mu = 0$) plus symmetry and conservation of T to prove that the current

$$J^\nu(K) = K_\mu T^{\mu\nu}$$

is conserved. Then one obtains the conserved charge

$$Q(K) = \int_{\Sigma^{n-1}} * J(K)$$

where the integral is taken over an initial value hypersurface (such as time zero).

I have presented this as if one needs to have a Lagrangian so as to deduce the existence of a stress tensor $T = \delta\mathcal{L}/\delta g$. Though this is a powerful approach, one can also argue more abstractly. Consider any theory which can be formulated for any metric on M. To define an operator $T(y)$, we must define, for any specified

n points $x_1, \ldots, x_n$ on M^4 distinct from each other and from y and operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$, the correlation function $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) T(y) \rangle_g$ (here the subscript serves to emphasize that the correlation function depends on a metric g). We define this as the derivative of $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g$ with respect to g:

$$\frac{\delta}{\delta g(y)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) T(y) \rangle_g.$$

The reader should verify that this definition agrees with the previous one in case a Lagrangian exists. T as defined in this way is obviously symmetric; it is conserved if the theory depends only on the diffeomorphism class of the metric g. Many of the properties of a local quantum field operator follow readily from this definition of T, and it is plausible to believe that they all do in general.

In any event,[4] one normally considers in practice theories that have a local, conserved, symmetric (and of course gauge-invariant) stress tensor. As I have essentially already noted, any theory that appears in nature has this property, since T appears directly in the Einstein equations!

Existence of T leads[5] to sharp restrictions on possible massless particles: as we will see, the possible spins of a massless particle in a theory with a stress tensor are $0, 1/2$, and 1. A further and analogous restriction is the following. Let J^μ be a conserved current associated with a "global symmetry." Thus, J transforms as a vector under Poincaré, and the conserved charge

$$Q(J) = \int *J$$

is Poincaré invariant. Then we will see that Q annihilates any massless particle of spin 1.

The proofs are so similar that we consider the two cases together. In four dimensions, denote by $|p, j\rangle$, a massless one-particle state of momentum p and spin j (in general $j \in \mathbb{Z}/2$). And let $|p', j\rangle$ be a state of different momentum in the same Poincaré representation. There is no essential loss of generality in assuming that $|p, j\rangle$ is an eigenstate of Q with some eigenvalue q. Consider the matrix elements $\langle p', j | J^\nu | j, p \rangle$ and $\langle p', j | T^{\mu\nu} | j, p \rangle$. The latter cannot vanish at all, and the former cannot vanish unless $q = 0$. For in the limit that $p' \to p$, we have by Lorentz invariance

1. $\langle p', j | J^\nu | j, p \rangle \sim p^\nu$, and
2. $\langle p', j | T^{\mu\nu} | j, p \rangle \sim p^\mu p^\nu$.

The proportionality constant is q in the first case and 1 in the second, since (as Q and the momentum operators are obtained by integration of J or T) matrix elements of J or T with identical initial and final states measure the charge or momentum of the state.

On the other hand, one can prove using Lorentz invariance that for all $p' \neq p$, these matrix elements vanish in the first case for spin greater than $1/2$, and in the second case for spin greater than 1. The proof goes by noting first that for $p' \neq p$, the subgroup of the Lorentz group that leaves fixed both p and p' is a

[4]Except in studying quantum gravity, which has a very different flavor from quantum field theory in a fixed spacetime, which is the subject of our lectures this spring.

[5]S. Weinberg and E. Witten, *Limits on massless particles*, Phys. Lett. B **96** (1980), 59–62.

copy of $SO(2)$ (or $SO(n-2)$ if we are in n spacetime dimensions; the present considerations degenerate below four dimensions as $SO(n-2)$ is then trivial). One simply shows that $SO(2)$ invariance of $\langle p', j|J^\nu|j,p\rangle$ and $\langle p', j|T^{\mu\nu}|j,p\rangle$, assuming that these matrix elements are nonzero, implies that the spin j is in absolute value $\leq 1/2$ or ≤ 1, in the two cases. A convenient way to perform this computation is to go to a Lorentz frame in which (writing the time coordinate first), $p = (1,1,0,0)$ and $p' = (1,-1,0,0)$. The $SO(2)$ that leaves fixed both p and p' is the rotation of the last two coordinates. Under the generator of this $SO(2)$, the states $|j,p\rangle$ and $|j,p'\rangle$ have respectively eigenvalues j and $-j$. The minus sign for the $|j,p'\rangle$ state, which is crucial, arises because it describes a particle moving in the opposite direction from $|j,p\rangle$; they each have the same spin relative to their own directions of motion, but opposite spins if referred to a fixed axis. So $\langle p', j|J^\nu|j,p\rangle$ or $\langle p', j|T^{\mu\nu}|j,p\rangle$ can be nonzero only if, in the $SO(2)$ action on J or T, there is a term with spin or eigenvalue $-2j$. As the components of J transform under $SO(2)$ with spin ≤ 1 in absolute value, while for T one has components of spin ≤ 2, we get $|j| \leq 1/2$ or $|j| \leq 1$ in the two cases, as was claimed above.

We can actually be somewhat more precise about this result. We have so far used only representation theory, but in quantum field theory one has also a CPT theorem, which implies in four dimensions that every massless particle of spin j is accompanied by one of spin $-j$. So spins $\pm 1/2$ will go together, and likewise spins ± 1.

In dimension $n > 4$, similar reasoning gives the following result. The spin of a massless particle is classified by a representation of the "little group" $SO(n-2)$. If a stress tensor exists, the allowed representations for massless particles are the spinor representation(s),[6] and the exterior powers of the fundamental $(n-2)$-dimensional representation (including the trivial representation).[7] Global charges vanish except for massless particles transforming in the trivial or spinor representation. This n-dimensional formulation is related to the description in four dimensions (in terms of $j \in \mathbb{Z}/2$) as follows: $j = \pm 1/2$ correspond to the two spinor representations, while $j = 0$ and $j = \pm 1$ come from the exterior powers of the fundamental representation.

§2.5. Infrared limits

Given a quantum field theory X, we want to solve it, that is, to learn the most interesting things about it. A big piece of "solving" a theory is determining what it flows to in the infrared. Fairly often, the answer is "nothing," that is, X flows to a trivial theory. This happens precisely when X has a mass gap, for then all (Euclidean) correlation functions decay exponentially. Showing that a given theory flows to a trivial theory may, however, be a rather deep result.

Very often, the infrared limit is not trivial but is a free theory of massless particles, together with an irrelevant interaction which goes to zero in the infrared. In fact, this happens in most of the simplest examples that we will meet. Note that an irrelevant interaction would, in the ultraviolet, be considered "unrenormalizable"; the perturbations that are ill-behaved in the ultraviolet are just the ones that vanish as one flows to the infrared limit.

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[6] We recall that there are two of these if n is even and one if n is odd.

[7] For $n-2$ divisible by four, the middle exterior power can be decomposed into self-dual or anti-self-dual pieces which are each *real*; one can be present without the other.

When a theory is free in the infrared, the question then becomes: *which* massless particles is it a free theory of? They might not be the same as the ones in the original Lagrangian. (We saw a relatively trivial example of this in §1.8.) In fact, as we saw earlier in this lecture, the answer to this question may depend on the vacuum state we are in.

For the infrared limit to be trivial is a special case of the infrared limit being free; it is the case that there are no massless particles at all in the physical spectrum.

If the couplings in a theory are generic, massless particles must be massless for a reason. One possible reason is supersymmetry, but we won't discuss that now. Other possible reasons are as follows.

Spin 0 particles are massless when they are Goldstone bosons, that is, when there is a broken symmetry. Spin $1/2$ particles in four dimensions are massless when they are chiral fermions; their masslessness is due to an *unbroken* chiral symmetry. This means simply the following: the CPT theorem says that if one has n massless particles of spin $1/2$, one also has n such particles of spin $-1/2$. If there is an unbroken symmetry group G and the massless particles of spin $1/2$ transform in a representation R of G, then the massless particles of spin $-1/2$ transform, by the CPT theorem, in the representation $\overline{R}$ (the complex conjugate of R). If R and $\overline{R}$ are distinct, then this spectrum cannot be perturbed in a G-invariant way to a give masses to the fermions.

The reasons just mentioned are really the only known reasons in four dimensions to have massless particles of spin 0 or $1/2$ without supersymmetry and without adjusting some parameters to make particles massless. For spin 1 the situation is somewhat different. The Poincaré representation of a massless spin 1 particle, in four (or more) dimensions, simply cannot be perturbed to give a mass to the particle, unless there is a massless spin 0 particle that can combine with it in a Higgs mechanism, as described in part 2 of this lecture. If our massless spin 0 particles are Goldstone bosons, then the broken symmetries that shift them ensure that they cannot participate and disappear from the massless spectrum in a Higgs mechanism. So as long as the massless spin 0 particles are Goldstone bosons, and *as long as the theory is infrared-free*, the existence of massless spin 1 particles is stable just from Poincaré symmetry.

This depends heavily on the theory being infrared-free since we applied group theory to one-particle states. If interactions are important even at low energies, the states with different numbers of particles can "mix," and we cannot draw a conclusion just by applying Poincaré invariance to the one-particle states.

What interactions can massless particles of these types have? In the case of spin 0 particles, which we assume to be Goldstone bosons, there are no relevant interactions. We have already seen at the beginning of this lecture that there are no relevant or marginal interactions of Goldstone bosons only. There are likewise no relevant or marginal interactions of fermions only above two dimensions – we explored such questions in the fall term – and with similar arguments and a little more care, one can show that there are no relevant or marginal couplings of Goldstone bosons to fermions.

Spin 1 particles are of a different nature, since they correspond to gauge fields, and gauge fields *can* have relevant interactions in the infrared. If G is the gauge group and A is the connection, the Lagrangian

$$\frac{1}{4g^2}\int |F_A|^2 d^4x$$

is nonlinear. It contains couplings that in four dimensions are marginal classically. Whether the interactions are relevant or irrelevant quantum mechanically depends on the behavior of the β-function. An irrelevant nonlinearity in the infrared will allow the gauge theory to function as an effective description. On the other hand, if the nonlinearity is relevant in the infrared, then the gauge theory is *not* the answer. (In the intermediate case when $\beta = 0$ we would get a non-free theory in the infrared.)

Goldstone bosons are always invariant under gauge symmetries – a gauge group acting on them would violate the global symmetry that leads to having Goldstone bosons in the first place. Hence they do not contribute to the β function. So if we see massless spin 1 fields in the infrared without adjusting parameters to make it so, we should expect that either G must be abelian, or else there must be enough fermions in large enough representations of G so that $\beta > 0$. To explain their masslessness, the fermions are chiral (that is, the states of $j = 1/2$ and $j = -1/2$ transform differently) either under the gauge group itself, or under some unbroken global symmetry group that commutes with the gauge group.

One of the major outstanding problems in physics is that, because of something called the "gauge hierarchy problem," nature is not quite as generic as the last paragraph suggests it should be. This is actually one of the things that have suggested that supersymmetry may be important in nature.

References and cross references

About classical gauge theory, see Chapter 4 of [I-**Classical Fields**]. BRST quantization of gauge theories is described in Lecture 3 of this series and in Lecture 4 of [II-**Strings**]. Problems about the Higgs mechanism and infrared limits are in the Exercises at the end of the lecture series, in Chapter 2 of [I-**Classical Fields**], and also in the addendum to the superhomework in [I-**Homework**]. Global symmetry breaking is discussed in Lecture 1.

For an introduction to the Higgs mechanism in weakly coupled gauge theory, see

S. Coleman, *Aspects Of Symmetry*, Cambridge University Press, 1985.

S. Weinberg, *The Quantum Theory Of Fields*, chapter 21, vol. 2, Cambridge University Press, 1996.

LECTURE 3
BRST Quantization of Gauge Theories

In this lecture we will discuss quantization of gauge theory by using BRST cohomology.[8] This approach is an improvement of the original Faddeev-Popov approach. An advantage of the BRST approach as opposed to the Faddeev-Popov method is that BRST makes explicit the independence of quantization of the choice of the gauge fixing procedure.

A similar approach can be used in gravity (see [II-**Strings**]).

§3.1. The general setup

We start with a general setup, and then consider examples. In the general setup, we have a compact gauge group G with Lie algebra $\mathfrak{g}$, and the group $\widehat{G}$, which is the group of gauge transformations of a principal G-bundle E over a spacetime M. Formally, we want to compute the path integral

$$\frac{1}{Vol(\widehat{G})}\int DAD\phi e^{-L(A,\phi)}, \tag{3.1}$$

where $L(A,\phi)$ is a gauge invariant Lagrangian with a gauge field A and matter fields ϕ.

The difficulty with a perturbative treatment of this path integral is that its kinetic term F_A^2 for the gauge field is degenerate along orbits of $\widehat{G}$. One way to deal with this difficulty is to replace the integrand in (3.1) by some expression that integrates to 1 on orbits of $\widehat{G}$ – then (3.1) would be equal to the integral of this expression (at least if everything were finite dimensional). For example, this expression could be the delta-function of some gauge condition, i.e. of some submanifold in the space of connections and matter fields which is a cross-section for the $\widehat{G}$-action (this procedure is called gauge fixing). As we know (see Lecture 5 of [I-**Faddeev**],Lecture 3 of [II-**Strings**]), this introduces a determinant under the integral (the Faddeev-Popov determinant). The determinant is a nonlocal expression, so in order to work only with local expressions, one should replace this determinant with a Gaussian integral over the space of fields times two copies of the odd space

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Notes by Pavel Etingof and David Kazhdan

[8]The reader who is mainly interested in dynamics of quantum field theory should be alerted that many technical details that we will discuss today are not strictly needed for following the subsequent lectures.

of sections of the coadjoint bundle of E. Thus, one has to introduce additional fermionic fields $c, \bar{c}$ with values in $ad(E)$. These fermions are called ghosts, since they do not correspond to any physical particles and violate spin-statistics. After the introduction of ghosts the path integral can be treated as usual, e.g. by perturbation theory techniques.

§3.2. The BRST differential

Of course, any gauge fixing procedure by definition destroys gauge invariance. Therefore, in order to obtain a sensible quantization, we must make sure that in the final result the gauge symmetry is restored. In particular, we must explain what replaces the gauge symmetry in the ghost setting of the previous section.

It turns out that what replaces the gauge symmetry is a certain odd derivation of the algebra of local functionals, which we will now construct.

We will first consider the classical theory. Let us look at what fields our theory has after the introduction of ghosts. The basic fields are the connection A, the matter fields ϕ, and the ghosts $c, \bar{c}$, which are sections of the adjoint bundle to E. We will add an auxiliary scalar bosonic field h, whose significance will be seen below.

Let R be the algebra of local functionals. We want to define an odd derivation $\delta : R \to R$ such that $\delta^2 = 0$ (the BRST differential).

Recall that the algebra of local functionals R is the quotient $\widetilde{R}/I$, where $\widetilde{R}$ is the algebra of local expressions in the fields and I is the differential ideal generated by field equations.

We first define a derivation $\delta : \widetilde{R} \to \widetilde{R}$, and then make sure that the field equations are respected.

Define δ on generators by

$$\delta c = \frac{1}{2}[c, c], \delta\phi = \delta_c\phi, \delta A = -d_A c, \delta\bar{c} = h, \delta h = 0, \tag{3.2}$$

where $\delta_c\phi$ means the variation of ϕ along the infinitesimal gauge transformation c. It is easy to check that $\delta^2 = 0$.

Recall from Lecture 4 of [II-**Strings**] that the Lagrangian with ghosts for our theory is

$$\widetilde{L} = L(A, \phi) - \delta(\bar{c}(h/2 + \Lambda)) = L(A, \phi) - (h\Lambda + \frac{h^2}{2} - \bar{c}\delta\Lambda). \tag{3.3}$$

where $\Lambda = \Lambda(A, \phi)$ is a non-gauge invariant local expression (the gauge fixing function). This Lagrangian is of course equivalent to

$$\widehat{L} = L(A, \phi) + (\frac{1}{2}\Lambda^2 + \bar{c}\delta\Lambda).$$

by eliminating h.

We want the Lagrangian to be invariant under the derivation δ. Since $L(A, \phi)$ is already invariant (because it is gauge invariant), it is enough to check that the expression $h\Lambda + \frac{h^2}{2} - \bar{c}\delta\Lambda$ is closed under δ, which follows from $\delta^2 = 0$, since as seen in equation (3.3), this expression is of the form $\delta(\dots)$.

Since δ preserves the Lagrangian, it preserves the set of its critical points and hence indeed defines a derivation of the algebra of local functionals.

As an example consider the case when E is the trivial bundle (and there are no matter fields). One of the possible gauge fixing conditions is the Coloumb gauge condition $d^*A = 0$. So we take $\Lambda = d^*A$. Then one has $\delta\Lambda = -d^*d_Ac$, so after elimination of h the Lagrangian for pure gauge theory is

$$L = \frac{1}{4e^2}\int F \wedge *F + ((d^*A)^2 - \bar{c}d^*d_Ac). \tag{3.4}$$

We see that this Lagrangian is nondegenerate, so one can do perturbation theory with it as usual. The derivation δ on R in this case is defined by

$$\delta c = \frac{1}{2}[c,c],\ \ \delta A = -d_Ac, \delta\bar{c} = -d^*A. \tag{3.5}$$

§3.3. The properties of the BRST derivation

Thus, we have constructed a derivation δ. The main properties of δ are:

1. $\delta^2 = 0$.
2. δ is defined on $\widetilde{R}$ a priori, without the use of the Lagrangian $L(A,\phi)$ and the gauge fixing term Λ. It preserves the Lagrangian with ghosts $\widetilde{L}$ and therefore descends to R.

Now let us turn to quantum theory. In this case local functionals are replaced by local operators. It can be shown that there exists a renormalization procedure under which δ can be defined as above, and properties 1 and 2 hold. This is discussed below.

However, in order to use the BRST method for quantization, we will need another, purely quantum, property of δ. Namely, denote by $\mathcal{L}_{\text{eff}}$ the effective Lagrangian, i.e. the Lagrangian for which the classical theory is equivalent to the quantum theory for $\mathcal{L}$. This Lagrangian is of course nonlocal. The third property of δ that we need is

3. δ preserves the effective action $\mathcal{L}_{\text{eff}}$. That is, $\delta\mathcal{L}_{\text{eff}} = 0$.

This property may fail, and if it fails then one says that the theory is anomalous.

Remark. Although $\mathcal{L}_{\text{eff}}$ is nonlocal, it can be shown that $\delta\mathcal{L}_{\text{eff}}$ is always the integral of a local expression.[9] A term in $\delta\mathcal{L}_{\text{eff}}$ is only interesting if it cannot be removed by adding a (non-gauge-invariant) local term to $\mathcal{L}_{\text{eff}}$; if we can cancel $\delta\mathcal{L}_{\text{eff}}$ by adding such a term, we do so and eliminate the anomaly. It can be shown that the obstruction to making $\delta\mathcal{L}_{\text{eff}}$ zero by adding such a local counterterm is given in gauge theories by a 1-loop calculation. Thus, anomalies arise in the 1-loop order of perturbation theory, and don't have higher order corrections. We will see this at the end of the lecture.

[9]For example, later we make a regularization by introducing Pauli-Villars regulator fields of mass m and taking $m \to \infty$. In the limit of $m \to \infty$, all effects of the regulator fields are local. On the other hand, gauge invariance is violated only by the couplings of the regulator fields, so the violation of gauge invariance is local.

§3.4. Operators in gauge theory and BRST cohomology

Assume properties 1-3 hold. Consider the path integral Z given by the Lagrangian $\widetilde{L}$:

$$Z = \int DADcD\bar{c}DhD\phi e^{-\widetilde{L}} \tag{3.6}$$

(possibly with some gauge-invariant insertions). Properties 1-3 imply that Z is independent of the gauge fixing condition Λ. Indeed, for any local expression X we have

$$\int DADcD\bar{c}DhD\phi e^{-\widetilde{L}}\delta X = \delta \int DADcD\bar{c}DhD\phi e^{-\widetilde{L}}X = 0, \tag{3.7}$$

which implies the independence of Z on Λ.

Remark. In (3.7), we used that δ preserves the measure of integration $DADcD\bar{c}DhD\phi$. It is easy to see that this is equivalent to Property 3 (absence of anomalies).

The statement that Z is independent of Λ holds for operators (insertions) which are annihilated by δ; for example, for any gauge-invariant insertions into Z, depending only on A, ϕ. On the other hand, if $\mathcal{O} = \delta\mathcal{O}'$ then the integral of $\mathcal{O}$ is zero by (3.7). Thus, the space of "physical" quantum operators in our theory is the cohomology of δ. This cohomology is called the BRST cohomology.

The BRST cohomology comes with a natural $\mathbb{Z}$-grading. Namely, we have a grading in the space of local operators, in which gauge and matter fields have degree 0, c has degree 1 and $\bar{c}$ degree -1. This degree is called the ghost number. It is easy to see that δ raises the degree by 1. This allows us to introduce a grading in cohomology: we denote by H^q_δ the cohomology in degree q.

The properties of this cohomology, which usually hold in this situation are:

1. H^q_δ vanishes for $q < 0$.
2. H^0_δ is the space of gauge invariant local operators depending only on A and ϕ.

This shows that δ plays the role of the gauge symmetry which was broken when ghosts were introduced. Thus, we have established a setting for gauge theory which works well in perturbation theory and in which the gauge symmetry does not die but rather appears in the form of δ.

§3.5. Renormalization and BRST differential

Now let us discuss the renormalizability and renormalization group equation in the BRST approach. We will restrict ourselves to 4 dimensions and pure nonabelian gauge theory.

Recall that in order for a theory to be renormalizable, all interactions have to have nonnegative dimension. To find out whether this is so for the Lagrangian with ghosts, we will compute the dimensions of fields (assuming that δ preserves the scaling dimensions). It is easy to see that the dimensions are as follows: $[c] = 0, [\bar{c}] = 2, [A] = 1$. This shows that all interactions in the Lagrangian with ghosts are renormalizable.

Remark. In this theory, the dimensions of c and $\bar{c}$ are not uniquely determined; the only thing that is determined canonically is $[\bar{c}c]$, which equals 2. This does not lead to a contradiction, since all operators of nonzero ghost number have zero expectation value, and so the ghost scaling dimension has no intrinsic meaning. This is why we needed to make an additional assumption that δ preserves dimensions to fix precise values of the dimensions. If we had assumed that δ raises dimension by k we would get a different answer, which would be equally good for our purposes.

Now let us look for critical couplings which will be renormalized. In the setting without ghosts, the usual thing to do is to write down renormalizable (non-gauge invariant) operators of dimension 4, which correspond to critical couplings: $[A,A]^2, [A,A]dA, (dA)^2$, and then argue that there is only one gauge invariant combination of these operators, so that the only coupling which is to be renormalized is the charge e. However, in the setting with ghosts, we also have to include operators of degree 4 involving ghosts: $\bar{c}cA^2, \bar{c}d^*dc, \ldots$. The gauge invariance condition is now replaced by the condition that δ is a symmetry, so we need to renormalize only delta-invariant interactions. This cuts down the number of operators to be renormalized, but still leaves us with two renormalizable couplings: the charge e and the gauge fixing parameter a, corresponding to the scaling of the gauge fixing term $\frac{\Lambda^2}{2} + \bar{c}\delta\Lambda$. Thus the renormalization group vector field looks like

$$W = \mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + \widetilde{\beta}(e,a)\frac{\partial}{\partial a}.$$

Here β is the beta-function of the theory, and $\widetilde{\beta}$ is the ghost beta-function. The beta-function of the theory depends only on e and is physically meaningful; for example, the negativity of its leading term ensures asymptotic freedom. However, the ghost beta-function $\widetilde{\beta}$ has no physical meaning: it only matters for renormalization of operators and correlators containing c and $\bar{c}$, which don't make sense physically.

§3.6. The Hamiltonian approach

So far we have considered the BRST method from the Lagrangian point of view. Now let us consider the connection of the BRST method with the Hamiltonian formalism.

Since ghosts violate spin-statistics (being scalar fermions), the "Hilbert space" of the theory with ghosts cannot be an actual Hilbert space. It is possible to construct a certain space $\widetilde{\mathcal{H}}$ with a Hermitian form, which is analogous to the Hilbert space in actual physical theories, but the form will not be positive definite. However, on this space we have local quantum operators, obtained by quantization of classical operators in the usual way. In particular, we have the global charges – the Hamiltonian H as well as the BRST charge Q, obtained from ghosts as explained in Lecture 4 of [II-**Strings**]. We also have a grading of $\widetilde{\mathcal{H}}$ by ghost number, obtained naturally from the quantization procedure.

The operator Q has the property $[Q,\mathcal{O}] = \delta\mathcal{O}$ for any operator $\mathcal{O}$ (not necessarily δ-closed) in the theory with ghosts. Also, $Q\Omega = 0$, where Ω is the vacuum, and the ghost number of Q is 1.

The operator Q has properties analogous to those of δ:

1. $Q^2 = 0$. This can be confirmed by a direct computation.

2. Q is defined a priori, without the use of L and Λ (by an explicit formula as in Lecture 4 of [II-**Strings**],formula (4.52)). In particular, if $\widetilde{\mathcal{H}}$ is an irreducible representation of the operator algebra, then Q is completely determined by δ and the properties $[Q, \mathcal{O}] = \delta\mathcal{O}$, $Q\Omega = 0$.
3. If there are no anomalies, the element Q commutes with the Hamiltonian and with all gauge invariant local operators which involve no ghosts.

Remark. As in the Lagrangian setting, here the explicit expression for Q is independent of Λ only if one uses the operator h corresponding to the auxiliary field in the Lagrangian. This operator can be expressed via other operators in the theory, in a way which depends on Λ: $h = -\Lambda$. If one makes this substitution, the obtained formula for Q will involve Λ. Thus, property 2 should be understood as follows: there exists a formula for Q in terms of the fields (including h!) which is independent of $\mathcal{L}$ and Λ but depends only on the field configuration.

Let H_Q^q be the cohomology of Q on $\widetilde{\mathcal{H}}$, graded by ghost number.
The properties of the Q-cohomology which usually hold are

1. H_Q^q vanishes if $q < 0$.
2. The Hermitian form is degenerate on the kernel of Q in $\widetilde{\mathcal{H}}^0$ (operators of ghost number zero); the kernel of this form is the image of Q. The induced form on H_Q^0 is positive definite.

The space H_Q^0 plays the role of the physical Hilbert space of the theory, so we denote it by H_{phys}.

In the space H_{phys}, we have an action of the Hamiltonian H and "physical" local operators $\mathcal{O} \in H_\delta^0$. These operators no longer involve ghosts and correspond to actual observables of the theory.

Let us now compare the BRST and the "traditional" approaches to quantization of gauge theory. For simplicity, we consider pure gauge theory. Traditionally, a scheme of quantization would be as follows. Suppose that the space part of the spacetime is compact. In this case we have seen that classically the space of solutions to the equations of motion can be realized as $T^*\mathcal{A}$, where $\mathcal{A}$ is the space of connections on a spatial cycle modulo gauge transformations. Therefore we would define the Hilbert space as $L^2(\mathcal{A})$ (with respect to some measure). We call this Hilbert space the traditional Hilbert space.

We claim that these approaches give the same result, i.e. H_{phys} is isomorphic to H_{trad} as a representation of the operator algebra.

First of all, H_{phys} does not depend on the gauge fixing term Λ, and the Hamiltonian and the quantum operators in H_{phys} don't depend on it either. This follows from the fact that when Λ is varied, operators in the pseudo-Hilbert space $\widetilde{\mathcal{H}}$ are changed by adding a δ-exact expression, so their action on δ-closed vectors is changed by a δ-exact expression.

To identify H_{phys} with H_{trad} we can use a convenient gauge fixing term Λ. It is enough to do it for one such term, but we will do it for two – just for fun.

Set $\Lambda = ud^*A + vA_0$, where A_0 is the time component of the connection (this uses the splitting of spacetime into space and time). Then we get a sensible theory unless both $u = 0$ and $v = 0$. Even $u = 0, v \neq 0$ gives a nice theory – this gauge fixing term is called "the temporal gauge".

We first consider the case $u = 0$. Then the Lagrangian is

$$\widehat{L} = \int d^4x \left(\frac{1}{4e^2} F^2 + \frac{1}{2} v^2 A_0^2 - v\bar{c} \frac{Dc}{Dt} \right),$$

where D denotes covariant derivative. Replacing $\bar{c}$ with $-v\bar{c}$, and letting v tend to infinity (using the fact that nothing depends on v), we see that the path integral is localized to the hyperplane $A_0 = 0$, and in the limit we get a Lagrangian

$$\widehat{L} = \int d^4x \left(\frac{1}{4e^2} F^2 + \bar{c} \frac{dc}{dt} \right), \tag{3.8}$$

Since A_0 is now zero, we get usual quantum mechanics where the dynamical variables are a spatial connection, its time derivative, and the ghosts. Thus, $\widetilde{\mathcal{H}}$ has the form $\mathcal{O}(\widetilde{\mathcal{A}}) \otimes \Lambda \widetilde{\mathfrak{g}}^*$, where $\widetilde{\mathcal{A}}$ is the space of connections on the spatial cycle, $\mathcal{O}(\widetilde{\mathcal{A}})$ is the space of functions on $\widetilde{\mathcal{A}}$, $\widetilde{\mathfrak{g}}$ is the Lie algebra of the group of gauge transformations on the spatial cycle, and $\Lambda \widetilde{\mathfrak{g}}^*$ is the space of functions of $c \in \Pi \widetilde{\mathfrak{g}}$. Moreover, the Hamiltonian for the ghosts vanishes since there is no nontrivial evolution on the space of classical solutions (the Euler-Lagrange equations for ghosts are simply $\frac{dc}{dt} = \frac{d\bar{c}}{dt} = 0$). Thus the Hamiltonian in our theory is the usual gauge theory Hamiltonian

$$H = \int d^3x \left(\frac{1}{2e^2} F_A^2 + \frac{e^2}{2} \nabla_A^2 \right), A \in \widetilde{\mathcal{A}} \tag{3.9}$$

acting on the first component of the tensor product.

It is easy to see that the space $\widetilde{\mathcal{H}}$ with the grading by ghost number is nothing but the space of the standard complex of the Lie algebra $\widetilde{\mathfrak{g}}$ with coefficients in the module $\mathcal{O}(\widetilde{\mathcal{A}})$. Moreover, from the explicit formula for Q one gets that in this case Q is exactly the differential in the standard complex. Thus, the physical Hilbert space H_{phys} which is by definition the 0-th cohomology of Q, is the 0-th cohomology of $\widetilde{\mathfrak{g}}$ with coefficients in $\mathcal{O}(\widetilde{\mathcal{A}})$. This is just the space of $\widetilde{\mathfrak{g}}$-invariants in $\mathcal{O}(\widetilde{\mathcal{A}})$, i.e. the space of functions on $\widetilde{\mathcal{A}}/G$, which is by definition the traditional Hilbert space H_{trad}.

This shows that BRST cohomology is an infinite dimensional generalization of Lie algebra cohomology.

Now consider another gauge obtained by setting $v = 0, u = 1$: $\Lambda = d^*A$. Let us try to see the isomorphism between H_{phys} and H_{trad} using this gauge. We have,

$$\widehat{L} = \int d^4x \left(\frac{1}{4e^2} F^2 + \frac{1}{2} (d^*A)^2 - \bar{c} D^* d_A c \right). \tag{3.10}$$

In this case the equations of motion for ghosts are nontrivial and of second order, so the Hilbert space $\widetilde{\mathcal{H}}$ consists of functions of $A_s, c, \bar{c}, A_0$, where A_s is a connection on the spatial cycle. In this case, one finds

$$Q = Q_{\text{Lie algebra cohomology}} + Q', \tag{3.11}$$

where $Q' = \int \pi_{\bar{c}} \pi_{A_0}$, and $\pi_{\bar{c}}$, π_{A_0} are the momentum operators for $\bar{c}$, A_0.

It is easy to check directly that the two summands in (3.11) anticommute, and that $(Q')^2 = 0$. It can also be checked that Q' is acyclic except in 0-th degree, where it has a one-dimensional cohomology. Thus, by the Kunneth formula, we again get $H_{\text{phys}} = H_{\text{trad}}$.

§3.7. Anomalies

Now let us recall conditions 1,2,3 which were necessary for the BRST construction, and analyze when they are satisfied. These conditions are

1. Lagrangian: $\delta^2 = 0$.
   Hamiltonian: $Q^2 = 0$.
2. Lagrangian: δ is independent of Λ and $\widetilde{\mathcal{L}}$.
   Hamiltonian: Q is independent of $\Lambda, \widetilde{\mathcal{L}}$.
3. Lagrangian: $\delta\mathcal{L}_{\text{eff}} = 0$.
   Hamiltonian: $[Q, H] = 0$.

As we mentioned, properties 1 and 2 can always be attained.

However, as we also mentioned, Property 3 may fail if anomalies are present. So let us consider anomalies more closely.

Consider a four-dimensional gauge theory with a chiral spinor ψ with values in a complex representation ρ of the gauge group G and antichiral spinor $\overline{\psi}$ with values in the dual representation $\overline{\rho}$. The basic gauge-invariant Lagrangian for such fields is

$$\mathcal{L} = \frac{1}{4e^2}\int F^2 + \int \overline{\psi} D_A \psi. \tag{3.12}$$

In quantum theory we are interested in the path integral $\int e^{-\mathcal{L}}$. Integrating out ψ in this path integral, we get

$$\int \det(D_A) e^{-L(A)} DA, \tag{3.13}$$

where $D_A : \Gamma(S_+) \to \Gamma(S_-)$ is the covariant Dirac operator and $L(A)$ the Lagrangian of the pure gauge theory.

Integral (3.13) may not have a gauge invariant regularization. What is worse, it may not even have a non-gauge-invariant regularization for which the gauge invariance is restored as the cutoff goes to infinity. In this case the gauge theory we are considering does not make sense quantum mechanically, even in perturbation theory, because gauge symmetry cannot be respected in the quantum theory. This phenomenon is called an anomaly.

A geometric reason for an anomaly is that although the operator D_A is gauge invariant, its determinant $\det(D_A)$, in general, cannot be defined as a gauge invariant function of A. In the usual mathematical treatment, this determinant is not a function on the space of gauge classes of connections but rather a section of some line bundle over this space, called the determinant line bundle; this bundle comes with a canonical connection. If this canonical connection does not trivialize the bundle, the determinant is not a function but a section of a nontrivial bundle, and cannot sensibly be integrated.

It is useful to distinguish two types of anomalies.

1. Local anomaly. The canonical connection has nonzero curvature. In this case, for suitable spacetime manifolds this curvature may represent a nontrivial second cohomology class, so that the determinant bundle is not trivial topologically.

2. Global anomaly. The canonical connection is flat but has nontrivial monodromy (and possibly the bundle is not trivial).

Thus, both local and global anomalies can produce topological anomalies, but only the first one can be seen in perturbation theory (by computing the curvature).

Here we will consider only local anomalies.

Remark 1. To analyze when we can expect local anomalies, one may consider the situation from the topological point of view. We assume that our spacetime M is compact and orientable (e.g. S^d), with a specified point ∞, and will consider bundles, connections, and gauge transformations which are trivial at infinity. In this case the space of gauge classes of connections can be regarded as a classifying space $B\widehat{G}$ for the group of gauge transformations $\widehat{G}$. Nontrivial line bundles on $B\widehat{G}$ are classified by $H^2(B\widehat{G})$.

Now, if M is compact and orientable, we have the transgression map $\tau : H_2(B\widehat{G}) \to H_{d+2}(BG)$ defined as follows: given a two-dimensional homology class, pick a surface S in $B\widehat{G}$ which represents it, and take the corresponding principal $\widehat{G}$-bundle on S. Its transition functions can be considered as transition functions of a G-bundle on the $(d+2)$-dimensional manifold $S \times M$, which defines an element $\tau([S])$ of $H_{d+2}(BG)$. Consider the dual map $\tau^* : H^{d+2}(BG) \to H^2(B\widehat{G})$. It can be shown that the Chern class of our line bundle is $\tau^*(C)$, where C is a fixed $d+2$-dimensional characteristic class which does not depend on M, and is computed locally from the curvature. This class is exactly the local anomaly.

Thus for $d = 4$ local anomalies live in $H^6(BG)$, or $(S^3\mathfrak{g})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of G.

For example, in the standard model of particle physics, the gauge group is $SU(3) \times SU(2) \times U(1)$, and thus the space of anomalies $(S^3\mathfrak{g})^{\mathfrak{g}}$ is four-dimensional: it equals the sum of four subspaces $S^3_{\mathrm{inv}}(\mathfrak{su}(3))$, $S^3_{\mathrm{inv}}(\mathfrak{u}(1))$, $S^2_{\mathrm{inv}}(\mathfrak{su}(2)) \otimes \mathfrak{u}(1)$, $S^2_{\mathrm{inv}}(\mathfrak{su}(3)) \otimes \mathfrak{u}(1)$, which are one-dimensional (here "inv" denotes that we are taking invariant symmetric polynomials).

This discussion illustrates why anomalies don't arise in the case when all fermions are in a real representation of the gauge group. Indeed, in this case, the determinant bundle is real, and thus its Chern class must be zero.

Remark 2. Although the local anomaly can be considered from the above topological point of view, one should remember that it has a purely local nature, and has nothing to do with the macrostructure of the spacetime. If there is a local anomaly, the quantum theory will not make sense on any spacetime, even on $\mathbb{R}^d$. The problem is that even if the determinant bundle is topologically trivial, it will not have a flat connection defined in a local way: otherwise this flat connection would have been good for any simply connected spacetime, and no topological anomaly would arise. Thus, the path integral (3.13) is not sensible even on $\mathbb{R}^d$.

Remark 3. As we noted above in Remark 1, the standard model has a gauge group that admits possible anomalies. However, one can check that the anomalies coming from the different matter fields of the standard model miraculously cancel. Actually, the representation of the gauge group in the standard model fermions extends (after adding some insignificant real summands) to a spinor representation of Spin_{10}; this is a group for which $H^6(BG)$ vanishes so that there are no possible anomalies. In "grand unified theories," one interprets the standard model as arising by gauge symmetry breaking from a more unified gauge group such as Spin_{10}.

Let us show how to analyze the anomaly in perturbation theory. Our goal is to explain why, after possibly enlarging the space of fields, properties 1. and

2. of section 3.3 can always be assumed to hold (that is, $\delta^2 = 0$ and δ is defined independently of the choice of a particular Lagrangian) but one cannot assume that the effective Lagrangian is δ-invariant.

First, let us just try to make sense of integral (3.13) perturbatively. When we write down Feynman diagrams, we will find divergences in the 1-loop order which we cannot remove in a gauge invariant fashion. To fix the 1-loop order, we will regularize the path integral by adding another, very heavy matter field χ such that its determinant bundle is inverse to that for ψ. In favorable cases, our original theory should be recovered from this theory in the limit when the mass m of χ goes to infinity. In other cases, the procedure will exhibit why there is an anomaly.

To satisfy this condition, the matter field χ can be taken to be a bosonic field (χ_+, χ_-) with values in $(S_+ \oplus S_-) \otimes \rho$. In this case the complex conjugate field $\overline{\chi}$ takes values in $(S_+ \oplus S_-) \otimes \overline{\rho}$, where S_+, S_- are the spin representations of the Poincaré (recall that both S_+ and S_- are self-dual and self-complex-conjugate in Euclidean signature). Needless to say, these fields violate spin-statistics and therefore, like ghosts, don't make physical sense.

The natural Lagrangian term for the fields $\chi_\pm$ would be

$$(3.14)\quad L'(A, \chi_\pm) = \int d^4x((\overline{\chi}_+, D_A\chi_-) + (\overline{\chi}_-, D_A\chi_+) + m(\overline{\chi}_+, \chi_+) + m(\overline{\chi}_-, \chi_-))$$

(Here the Dirac operator is skew-selfadjoint).

Remark. The χ's are called Pauli-Villars regulator fields.

If we add expression (3.14) to the Lagrangian, we will get the squared absolute value of the determinant rather than the determinant itself, and will not fix the anomaly. Thus, we modify (3.14) in a way that breaks the gauge invariance: we let A_0 be a fixed connection and set

$$(3.15)\quad L''(A, \chi_\pm) = \int d^4x((\overline{\chi}_+, D_A\chi_-) + (\overline{\chi}_-, D_{A_0}\chi_+) + m(\overline{\chi}_+, \chi_+) + m(\overline{\chi}_-, \chi_-))$$

Now consider the theory with the Lagrangian $\mathcal{L} + L''$. Integrating out the χ fields, we will get a factor $\det(D_A D_{A_0} - m^2)^{-1}$. For $m = 0$ this factor is gauge invariant up to a multiplicative constant, and cancels the determinant in the numerator, which is caused by anomaly. This shows that in this theory, we don't have a topological anomaly for any finite m (i.e. the appropriate determinant bundle is trivial). However, for $m > 0$, the gauge invariance fails. So we have to study the limit $m \to \infty$ (which is supposed to give back our original theory) and see whether the gauge symmetry reappears.

Differentiating the determinant ratio $\det(D_A D_{A_0} - m^2)/\det(D_A)$ in the direction of a gauge transformation $t \in \widehat{\mathfrak{g}}$, we obtain (using the path integral interpretation) that it is equal to $m\langle \int [(\overline{\chi}_+, t\chi_+) + (\overline{\chi}_-, t\chi_-)] \rangle$, where $\langle X \rangle$ denotes the expectation value of X. This expectation value has a decomposition in powers of $1/m$.

To see whether the failure of gauge invariance persists for $m \to \infty$, let us consider the two-point function of the curvature operator F. It is easy to see that the leading contribution (in $1/m$) to the derivative of this function in the direction of t is from the 1-loop diagram with a χ loop having the t operator inside and two outgoing F-edges. This contribution is of the 0-th order in $1/m$, and has the form $\sum d_{abc} t^a F^b F^c$, where d_{abc} is some tensor. So if $d_{abc} \neq 0$, the gauge-noninvariance remains in the limit.

Remark. In case the original fermions were in a real representation, d_{abc} is zero and the regularization in (3.14) is completely satisfactory. The problem arises when the original representation is complex. Then a regularization as in (3.14) doesn't work unless one gives up gauge invariance. This is so even if $d_{abc} = 0$. The standard model is an example with fermions in a complex representation, $d_{abc} = 0$, and no gauge-invariant regularization along the above lines.

Remark. When $d_{abc} = 0$, there are no higher-loop (or nonperturbative) violations of gauge invariance.

Now let us consider anomalies from the BRST viewpoint. If a local anomaly is present, we will have $U = \delta\mathcal{L}_{\text{eff}}(A) \neq 0$ (here $\mathcal{L}_{\text{eff}}(A)$ is the effective Lagrangian for A, with the ghosts integrated out). However, since the anomaly is local, U must be the integral of a local expression of A and c which is linear in c. It is also clear that $\delta U = 0$.

On the other hand, if U is δN where N is the integral of some local expression of A then we can arrange $\delta\mathcal{L}_{\text{eff}} = 0$ by redefining the Lagrangian as $\mathcal{L} \to \mathcal{L} + N$. Hence, we should define the cohomology of δ acting on local functionals of the gauge fields, matter fields (if any), and ghosts, modulo total derivatives. Anomalies lie in the degree 1 (that is, ghost number 1) cohomology of δ.

Now let us show that invariant symmetric tensors on the Lie algebra $\mathfrak{g}$ (or equivalently, cohomology classes of BG) can be used to make 1-cocycles in this cohomology. The converse (these classes generate the degree one cohomology) is true, but we will not prove it.

Let C be a G-invariant element in $S^{n+1}\mathfrak{g}$. To this element there corresponds a $(2n+2)$-dimensional characteristic class of G-bundles, namely $C(F^{n+1})$, where F is the curvature. The Chern-Simons form $CS_C(A)$ corresponding to C is the local $(2n+1)$-form such that $\frac{\widetilde{\delta}C(F^{n+1})}{\widetilde{\delta}A} = CS_C(A) \wedge \widetilde{\delta}A$ modulo differentials of local forms (here $\widetilde{\delta}$ denotes the variation, to distinguish from the BRST differential δ).

The main property of the Chern-Simons form is the following. Although this form is not gauge invariant, its Lie derivative along an infinitesimal gauge transformation is a differential of a local form.

Now let M^{2n} be our spacetime. Let A be a connection on M^{2n}. We want to define a functional of the form $U(A,c) = \int W(A,c)$, where $W(A,c)$ is a $2n$-form on M^{2n} which is local in A and c but not gauge invariant, and such that $\delta W = 0$. W will be linear in c.

Let X^{2n+1} is a smooth manifold with boundary equal to M^{2n}. Choose an extension of the connection A to X^{2n+1} in any way (for simplicity we assume that there is no topological obstruction to the choices of X and the extension of A; this assumption is in fact inessential). Now set $V(A) = \int_X CS_C(A)$. This functional depends on the extension of A to X. However, by the main property of CS, the functional $\delta V(A) = W(A,c)$ (where δ is the BRST differential) does not depend on the extension and therefore is a local functional in A and c linear in c. Hence $U = \int W(A,c)$ represents a cohomology class in the δ cohomology of local functionals; one can show that it is nonzero. Thus, we get an injective (and in fact isomorphic) map $S^{n+1}(\mathfrak{g})^{\mathfrak{g}} \to H_{\delta}^{1,local}$. For four-dimensional theories $n = 2$ and the cocycles come from $(S^3\mathfrak{g})^{\mathfrak{g}} = H^6(BG)$.

References and cross references

Another source for BRST quantization of gauge theories is Lecture 4 of [II-**Strings**] and Lecture 5 of [I-**Faddeev**]. Classical gauge theory is described in Chapter 4 of [I-**Classical Fields**].

For an introduction to BRST quantization, see, for example,

S. Weinberg, *The Quantum Theory Of Fields*, chapter 15, vol. 2, Cambridge University Press, 1996.

An introduction to anomalies can also be found in the same reference.

LECTURE 4
Infrared Behavior and the S-matrix of the Two-Dimensional Sigma Model with Target Space S^{N-1}

§4.1. Infrared behavior of certain two-dimensional sigma models

Consider the theory of N scalar bosons, with the Lagrangian

$$\mathcal{L} = \int d^d x \left(\frac{1}{2} (\nabla \phi)^2 + \frac{g}{4!} (\phi^2 - a^2)^2 \right), \tag{4.1}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}^N$. In Lecture 1 we saw that above $d = 2$ and for sufficiently small g this theory has symmetry breaking, from $SO(N)$ to $SO(N-1)$. In particular, for any point y on the sphere $\phi^2 = a^2$ there exists a realization $\mathcal{H}_y$ of this theory, with $\langle \phi \rangle = y$. We also saw that the low energy effective theory for (2.1) is the sigma model with target space S^{N-1}.

In 2 dimensions the first statement fails: there is no symmetry breaking, and there is only one realization, with an action of $SO(N)$. However, the second statement remains valid: the low energy effective theory is the sigma model.

In $d > 2$, we know that the sigma model is infrared free (in the weak coupling approximation); more precisely, it converges in the infrared limit to a free theory of $N-1$ massless scalars (Goldstone bosons). This is not the case in 2 dimensions. Indeed, in 2 dimensions the sigma model is renormalizable, so it has interactions that classically are marginal (the coupling is dimensionless). On the other hand, it was shown in Lecture 3 of [II-**CFT**] that the β-function of the two-dimensional sigma model into a space of positive curvature is negative (the model is asymptotically free in the UV limit). Therefore, the classically marginal interactions are relevant in the infrared, and we cannot conclude that the model is infrared free. In fact, what happens (as we will see today) is that instead of $N-1$ massless particles that we had in $d > 2$, we will have N massive particles, so the infrared limit is trivial (the correlation functions are analytic at the origin in the momentum space, and decay exponentially at infinity in position space), and rather different from what one sees in perturbation theory.

In the first half of the lecture, we will show that the two-dimensional sigma model is indeed infrared trivial for large N. We will also develop a procedure for computing correlation functions in a series in $1/N$.

Notes by Pavel Etingof and David Kazhdan

Remark 1. Since the β-function of the two-dimensional sigma model into the sphere is negative, it is believed that this theory actually exists. The ability to construct the theory in the $1/N$ expansion, which provides information quite different from what one sees in standard perturbation theory and gives results that agree with standard theory about the renormalization group, gives strong support for this belief.

Remark 2. Since the theory defined by the quartic Lagrangian (4.1) exists rigorously in $d = 2$, it can be regarded as an ultraviolet cutoff of the sigma model. The characteristic momentum scale Λ of this cutoff is the mass of the radial component of ϕ. This cutoff model is called the linear sigma model, as opposed to the nonlinear sigma model, which is the model of maps to the sphere.

§4.2. Computation of the infrared behavior in the $N \to \infty$ limit

We will compute the infrared behavior of the sigma model in the limit $N \to \infty$ by putting the integral in a form in which N only appears as a parameter and then using a saddle point approximation in the path integral. We will operate with path integrals formally, without worrying whether they exist or not.

The path integral which defines the generating series for the Euclidean correlation functions of the sigma model is given by the formula

$$Z(J) = \int_{\phi:\phi^2=1} D\phi \exp\left(-\frac{1}{2\lambda}\int d^2x|d\phi|^2 + Z_J\int d^2xJ\phi\right), \tag{4.2}$$

where $\phi, J : \mathbb{R}^2 \to \mathbb{R}^N$, $J\phi$ is short for (J,ϕ), and λ is a coupling constant. The factor Z_J is added because the operator ϕ has anomalous dimension (see Lecture 3 of [I-**Renorm Groups**]) and requires multiplicative renormalization. (As we know, Z_J is cutoff-dependent, and diverges as the cutoff goes to infinity.)

It is more convenient to integrate over maps to a linear space than over maps to a curved manifold. Therefore, it is useful to rewrite integral (4.2) in the form

$$Z(J) = \int_{\phi:\mathbb{R}^2\to\mathbb{R}^N} D\phi \prod_{x\in\mathbb{R}^2}\delta(\phi^2(x)-1)\exp\left(-\frac{1}{2\lambda}\int d^2x|d\phi|^2 + Z_J\int d^2xJ\phi\right). \tag{4.3}$$

We will now use the formula

$$\delta(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ia\sigma}d\sigma. \tag{4.4}$$

Substituting (4.4) into (4.3), we get (up to a factor independent of J):

$$Z(J) = \int D\phi\int D\sigma\exp\left(i\int d^2x(\frac{\sigma}{2}(\phi^2-1)) - \frac{1}{2\lambda}\int d^2x|d\phi|^2 + Z_J\int d^2xJ\phi\right). \tag{4.5}$$

Thus, the use of formula (4.4) led to the introduction of an auxiliary scalar field σ.

Now we will change the order of integration and integrate first with respect to ϕ. Our goal is to reduce the integral to a form in which N (the dimension of the target space) enters only as a parameter. Happily, the sphere is defined by a

quadratic equation, so in the form just written, the integral with respect to ϕ is Gaussian. So we can compute it explicitly, and get

(4.6)
$$Z(J) = \int D\sigma \det\left(-\frac{\Delta}{\lambda} - i\sigma\right)^{-N/2} \exp\left(-i\int d^2x \frac{\sigma}{2} - \frac{1}{2}Z_J^2 \int d^2x J\cdot(\frac{\Delta}{\lambda}+i\sigma)^{-1}J\right)$$

(the power $-N/2$ of the determinant appears because we are doing N independent Gaussian integrals in the components ϕ_i of ϕ). It is useful to raise the determinant into the exponential:

(4.7)
$$Z(J) = \int D\sigma \exp\left(-\frac{N}{2}\mathrm{Tr}\ln(-\frac{\Delta}{\lambda} - i\sigma) - i\int d^2x\frac{\sigma}{2} - \frac{1}{2}Z_J^2\int d^2x J\cdot(\frac{\Delta}{\lambda}+i\sigma)^{-1}J\right).$$

Now we want to bring the integral (4.6) to the form where there is a factor of N in front of all terms in the exponential (except for the J-term, which serves to expand the answer in the formal series in J; we keep J of order 1 for $N\to\infty$ so that this term is a small perturbation for large N and does not affect the saddle point). We will then evaluate the integral using the saddle-point approximation. For this purpose we have to make the change of variable $\rho = \sigma/N$ and rescale the coupling constant by $\widetilde{\lambda} = \lambda N$. After these changes the integral, up to a factor, looks like

(4.8)
$$Z(J) = \int D\rho \exp\left(-\frac{N}{2}\left[\mathrm{Tr}\ln(-\Delta - i\widetilde{\lambda}\rho) + i\int d^2x\rho\right] - \frac{\widetilde{\lambda}}{2N}Z_J^2\int d^2x J\cdot(\Delta + i\widetilde{\lambda}\rho)^{-1}J\right).$$

In this form, we can already apply the saddle point approximation. (This is really saddle point and not stationary phase, as the function in the exponential is complex).

It is natural to look for a Poincaré invariant saddle point, since the integral we are considering is Poincaré-invariant. Thus, we should look for a saddle point $\rho(x)$ which is a constant, $\rho(x) = \rho$. The value of ρ is found from the saddle point equation

(4.9)
$$\frac{d}{d\rho}\left[\mathrm{Tr}\ln(-\Delta - i\widetilde{\lambda}\rho) + i\rho\int d^2x\right] = 0.$$

(At this point, we temporarily introduce an IR cutoff L, i.e. assume that the spacetime is a torus of size $L\times L$. This makes the total action finite.) Let us rewrite (4.9) using the momentum space representation of the operator $-\Delta - i\widetilde{\lambda}\rho$, and send L to infinity. Then we get

(4.10)
$$\widetilde{\lambda}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - i\widetilde{\lambda}\rho} = 1.$$

The integral on the left hand side is logarithmically UV divergent. So we should introduce an UV cutoff Λ in momentum space (i.e. consider the integral over the ball $|k|<\Lambda$). Then (4.10) becomes

(4.11)
$$\frac{\widetilde{\lambda}}{4\pi}\ln\frac{\Lambda^2}{-i\widetilde{\lambda}\rho} = 1.$$

This shows that

$$\rho = \frac{i\Lambda^2}{\widetilde{\lambda}} \exp\left(-\frac{4\pi}{\widetilde{\lambda}}\right). \tag{4.12}$$

In Lecture 3 of [I-**Renorm Groups**] it was shown that in an asymptotically free theory, for the physics to have a limit as $\Lambda \to \infty$, the coupling must be adjusted according to the formula

$$\widetilde{\lambda}(\Lambda) = \frac{1}{A \ln \Lambda} + o(\frac{1}{\ln \Lambda}), \Lambda \to \infty, \tag{4.13}$$

where A is minus the coefficient of the 1-loop beta-function for $\widetilde{\lambda}$. In the sigma model it turns out that $A = (2\pi)^{-1}\frac{N-2}{N}$, so it equals $(2\pi)^{-1}$ modulo $1/N$ (see the end of Lecture 3 of [II-**CFT**]). The formula for ρ that we have just derived shows that, modulo $1/N$, the beta-function equals its 1-loop approximation. For if and only if this is the beta function modulo $1/N$, ρ as given in equation (4.12) obeys the renormalization group equation, as of course it should.[10] Renormalization is carried out by taking Λ to infinity and $\widetilde{\lambda}$ to zero keeping fixed the renormalization group invariant quantity $M^2 = \lim_{\Lambda\to\infty}(-i\widetilde{\lambda}\rho)$; M will turn out to be the particle mass.

Remark. Thus, in the quantum theory we have a characteristic momentum scale defined by M, despite the fact that the classical theory is conformally invariant, and the Lagrangian has only a dimensionless coupling λ and defines no particular momentum (or length) scale. This phenomenon is called "dimensional transmutation". It gives a very vivid demonstration of the fact that quantization of a theory with a scale-invariant Lagrangian in general does not preserve scale invariance and is thus not completely canonical.

Recalling the form of the J-term in the integral $Z(J)$, we conclude that the J^2 term of the expansion of $Z(J)$ has order $1/N$. Thus, we should rescale the fields ϕ, $\phi \to N^{1/2}\phi$, to get a nonzero limit of correlation functions as $N \to \infty$. After this change it turns out that the 2-point functions of ϕ_i have a nonzero limit, while the connected $4, 6, \ldots$-point functions vanish modulo $1/N$. Thus, in the first order approximation in $1/N$, our theory is free. Moreover, the propagator of this first order theory is $\frac{1}{k^2+M^2}$, where M is defined as above. Thus, the limiting theory is the theory of N independent free massive scalars of mass M.

The connected 4-point function is of order $1/N$. Therefore, the effective coupling constant between the massive scalars is of order $1/N$.

The fact that the fields ϕ_i decouple in the large N limit and become massive scalar fields has the following probabilistic explanation.

Consider the theory of N free massive bosons a_i, described by the path integral

$$Z = \int \left(\prod Da_i\right) \exp\left(-N\sum_i (|da_i|^2 + M^2 a_i^2)\right). \tag{4.14}$$

[10] One can also show, more directly, via Feynman diagrams, that in the large N limit, the beta function has only the one-loop contribution. To show this only involves drawing pictures – no computation is required. Thinking it through is recommended as an exercise.

Consider the expectation value $\langle a_i^2 \rangle$ of the operator a_i^2 in this theory for a fixed i. This one-point function requires renormalization: it is given by the integral

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{N(k^2+M^2)}, \tag{4.15}$$

which is UV divergent. So we should introduce an UV cutoff Λ. Then we have $\langle a_i^2 \rangle = \frac{1}{4\pi N} \ln \Lambda^2/M^2$. Therefore, $\langle \sum_i a_i^2 \rangle = \frac{1}{4\pi} \ln \Lambda^2/M^2$. On the other hand, by the law of large numbers, if d is the dispersion of a_i^2, then the dispersion of $\sum a_i^2$ is $d/\sqrt{N}$, so for large N the vector $(a_1, ..., a_N)$ stays close to the the sphere $\sum x_i^2 = \frac{1}{4\pi} \ln \Lambda^2/M^2$. Thus, the theory of N independent massive fields for large N behaves like the sigma model for the sphere, whose radius varies logarithmically with the cutoff. Conversely, the sigma model of the sphere is the free theory of N massive bosons, modulo corrections of order $1/N$.

§4.3. Computation of the S-matrix

Now we will study the sigma model into the sphere in a totally different way. Namely we will show that this model is integrable even for finite N, in the sense that its S-matrix can be computed explicitly.

Recall the Coleman-Mandula theorem: in a field theory of dimension $d > 2$ with a mass gap and a nondegenerate S-matrix, any even infinitesimal symmetry of the S-matrix is a linear combination of an element of the Poincaré Lie algebra and a symmetry which commutes with the Poincaré Lie algebra. In other words, if there exists a "forbidden" symmetry (not having such a decomposition), then the S-matrix, under some technical conditions, equals 1, i.e. the theory is free.

In two dimensions, the Coleman-Mandula theorem is false. That is, there exist two-dimensional quantum field theories with $S \neq 1$ which have an infinitesimal symmetry not having a decomposition of the type described in the Coleman-Mandula theorem. In fact, we will see that the two-dimensional sigma model with target space a sphere is an example.

Thus, in two dimensions the information that there exists a "forbidden" symmetry is not enough to conclude that $S = 1$. However, this information still allows to compute the S-matrix. In this section we will show how to do it for the sigma model into S^{N-1}.

We will work on a flat Minkowski spacetime with coordinates x_+, x_- and the metric $dx_+ dx_-$. Let

$$\mathcal{L} = \frac{1}{2\lambda} \int d^2x (\partial_+ \phi \partial_- \phi), \phi^2 = 1. \tag{4.16}$$

be the Lagrangian of the sigma model with target S^{N-1} ($\phi \in \mathbb{R}^N$). Since this Lagrangian can be written naturally on a curved spacetime, it has a stress-energy tensor $T = \frac{\delta \mathcal{L}}{\delta g}$. This is a symmetric, rank 2 tensor, so it has 3 components in coordinates x_+, x_-: T_{++}, T_{--}, T_{+-}.

Let us consider some properties of the stress-energy tensor. Classically, we have

$$T_{++} = (\partial_+ \phi)^2, T_{--} = (\partial_- \phi)^2. \tag{4.17}$$

(Expressions like $(\partial_+ \phi)^2$ are abbreviations for $(\partial_+ \phi, \partial_+ \phi)$, with the inner product induced from the metric on $\mathbb{R}^N$.) Since the sigma model is classically conformally

invariant, we have $T_{+-} = 0$. However, we should expect that quantum mechanically conformal invariance is broken (in fact, we know this from the large N limit), so quantum mechanically, $T_{+-} \neq 0$; in fact, we will see that T_{+-} is proportional to $\partial_+\phi\partial_-\phi$ (see below). Still, the stress tensor conservation equations

$$\partial_- T_{++} + \partial_+ T_{+-} = 0, \partial_+ T_{--} + \partial_- T_{+-} = 0, \tag{4.18}$$

(which classically reduce to $\partial_- T_{++} = 0, \partial_+ T_{--} = 0$, because of vanishing of T_{+-}) are still satisfied quantum mechanically, if the operator T_{+-} is suitably defined. Such a conserved stress tensor leads, as we know, to Poincaré invariance.

Let us give a direct demonstration of why the equation

$$\partial_- T_{++} + \partial_+ T_{+-} = 0. \tag{4.19}$$

is satisfied with a proper choice of of T_{+-}. For this purpose recall that classically operators in our theory have a bigrading (d_+, d_-) (with respect to dilations of x_+, x_-). In quantum theory, since conformal invariance is broken, the antidiagonal grading $d_+ - d_-$ survives, while the sum $d_+ + d_-$ becomes a filtration. So the bigrading becomes a (bi)filtration.

Let us classify $SO(N)$-invariant operators of various bidegrees (to allow for the fact that the structure is only a filtration, for given (d_+, d_-) we will list operators of bidegree $d_+ - n, d_- - n$, $n \geq 0$).

(0,0): 1
(0,1): none
(1,0): none
(2,0): $(\partial_+\phi)^2 = T_{++}$
(0,2): $(\partial_-\phi)^2 = T_{--}$
(1,1): $\partial_+\phi\partial_-\phi, 1$
(2,1): $\partial_+^2\phi\partial_-\phi$

This table implies that (4.19) has to be satisfied, under a suitable renormalization of T_{ij}, for dimensional reasons. In fact, $\partial_- T_{++}$ is an operator of bidegree (2,1), as is $\partial_+(\partial_+\phi\partial_-\phi)$, so there must be a linear relation between these operators. The linear relation is not $\partial_+(\partial_+\phi\partial_-\phi) = 0$ (since this operator does not vanish classically), so the relation has a nonzero coefficient of $\partial_- T_{++}$ and takes the form of (4.19) with T_{+-} a suitable multiple of $\partial_+\phi\partial_-\phi$.

Remark. In fact, it can be shown that $T_{+-} = \beta\partial_+\phi\partial_-\phi$, where β is the beta-function. Thus, the beta-function measures the trace of the stress tensor and the failure of conformal invariance.

We will now construct a symmetry forbidden by the Coleman-Mandula theorem using the (quantum) stress-energy tensor. The idea of constructing this symmetry is the following. Suppose that we found two local operators X_+, X_- such that $\partial_+ X_- = -\partial_- X_+$, but there is no Y for which $X_- = -\partial_- Y$, $X_+ = \partial_+ Y$. In this case the operator-valued form $J = X_+ dx_+ - X_- dx_-$ is a nontrivial conserved current (it is not of the form dY for any Y), and the charge operator $Q = \int_C J$, where C is a spacelike cycle, is conserved and generates a symmetry. So we should construct X_+, X_- with the stated properties.

Classically, $X_+ = T_{++}^2$ satisfies the above condition, as $\partial_-(T_{++})^2 = 0$. Quantum mechanically, however, this equation is not satisfied. So we will again use dimension counting to demonstrate the existence of a conservation law.

We want to show that there exist operators X_+, X_- of degrees (4,0) and (3,1), such that $\partial_- X_+ + \partial_+ X_- = 0$, and $X_+ \neq \partial_+ Y$. To do this, let us extend the above table of operators:

(3,0): $\partial_+\phi\partial_+^2\phi$

(3,1): $(\partial_-\phi\partial_+\phi)(\partial_+\phi)^2$, $\partial_-\phi\partial_+^3\phi$.

(4,0): $((\partial_+\phi)^2)^2$, $(\partial_+^2\phi)^2$, $\partial_+^3\phi\partial_+\phi$

(4,1): $\partial_-\phi\partial_+^4\phi$, $(\partial_-\phi\partial_+\phi)(\partial_+\phi\partial_+^2\phi)$, $(\partial_-\phi\partial_+^2\phi)(\partial_+\phi)^2$.

Let H_{d_+,d_-} be the space of operators of bidegree $d_+ - n, d_- - n$ for all $n \geq 0$. We have: $\dim H_{3,0} = 1$, $\dim H_{3,1} = 2$, $\dim H_{4,0} = 3$, $\dim H_{4,1} = 3$. We have two maps $\partial_+ : H_{3,1} \to H_{4,1}$, $\partial_- : H_{4,0} \to H_{4,1}$. For dimensional reasons, there is a two-dimensional subspace Z in $H_{4,0}$ such that for $X \in Z$ one has $\partial_- X \in \mathrm{Im}\partial_+$. On the other hand, the image B of ∂_+ in $H_{4,0}$ is one-dimensional. Therefore, the "cohomology" group $H = Z/B$ is not zero (it is one-dimensional). This "cohomology" group represents the conservation law that we are looking for.

Remark. It is easy to define the complex for which Z are cocycles, B are coboundaries, and H is the cohomology.

Denote the charge operator corresponding to this conservation law by Q. Let $P_\pm$ be the generators of translations of x_+, x_- and K the infinitesimal boost operator. It is easy to show from the definition of Q that $[Q, P_\pm] = 0$, and $[K, Q] = 3Q$.

Thus, Q is an infinitesimal symmetry violating the conclusion of the Coleman-Mandula theorem. The S-matrix of this theory is however not equal to 1; in fact, $S - 1$ is of order $1/N$ and can be computed using the $1/N$ expansion explained in the first part of the lecture.

Remark. Of course, by interchanging $+$ and $-$ in the definition of Q, we can obtain another "forbidden" conserved charge Q', with $[K, Q'] = -3Q'$.

Now let us see how we can use Q. Let $H_m \subset \mathcal{H}$ be a subrepresentation of the Poincaré group which represents a particle of mass m ($H = L^2(\mathcal{O}_m^+)$). Let $|k\rangle$ denote the state of this particle in which its momentum is $k = (k_+, k_-)$, where $k_+k_- = m^2$. (Of course, $|k\rangle \notin H_m$, but for an L^2-function f on $\mathcal{O}_m^+$, $\int f(k)|k\rangle dk \in H_m^+$.) We have $P_\pm|k\rangle = k_\pm|k\rangle$, and $Q|k\rangle = ck_+^3|k\rangle$, where c is a k-independent constant. That c is not zero can be computed explicitly in the $1/N$ expansion.

Let λ be the parameter on the hyperbola $k_+k_- = m^2$ defined by the equations $k_\pm = me^{\pm\lambda}$. As we know, a 1-particle state is represented by a wave function, which satisfies the Klein-Gordon equation:

$$\psi(x_+, x_-) = \int_{-\infty}^{\infty} \widehat{\psi}(\lambda) e^{i(k_+x_+ + k_-x_-)} d\lambda, \tag{4.20}$$

Since $k_\pm = me^{\pm\lambda}$, the exponent in (4.20) is a function of λ. Suppose that we consider a wavepacket with a reasonable amount of localization in both position and momentum, for instance $\widehat{\psi}(\lambda) = \exp(-(\lambda - \lambda_0)^2/\varepsilon)e^{if\lambda}$, with a small constant ε and constant f, is a suitable choice. The wavepacket in (4.20) is approximately

localized on a classical trajectory that can be found as follows. The integral gets its main contribution from λ near λ_0, and is largest if $x_\pm$ are such that the argument of the integral is stationary at $\lambda = \lambda_0$. I leave it to the reader to verify that these conditions determine a straight line with λ_0-dependent velocity and f-dependent position.

Let us now compute how the "forbidden" symmetry $e^{i\delta Q}$ acts on the wave function:

$$e^{i\delta Q}\psi(x_+, x_-) = \int_{-\infty}^{\infty} \widehat{\psi}(\lambda) e^{i(\delta c k_+^3 + k_+ x_+ + k_- x_-)} d\lambda \tag{4.21}$$

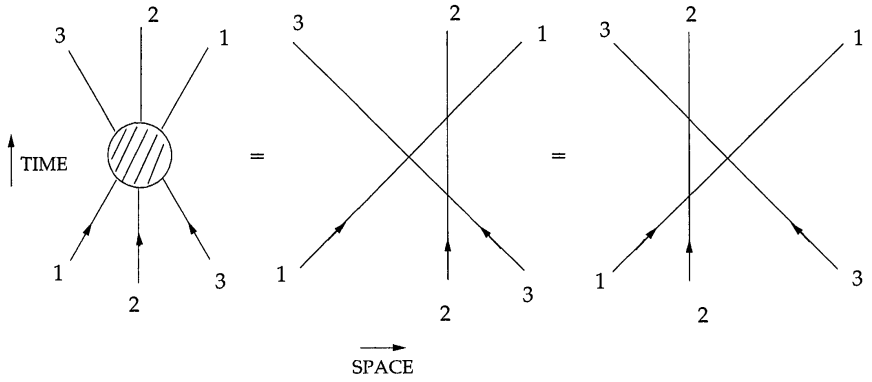
One can again determine the effective classical trajectory by a stationary phase argument. If one does so, one finds that the velocity is unaffected by $e^{i\delta Q}$, but the value of x_+ is shifted by the amount $\gamma_{\text{eff}} = \delta c k_+^2$, where now $k_+ = me^{\lambda_0}$. Since the velocity of the classical trajectory is determined by λ_0, *the shift in position depends on the velocity of the particle.* This is the crucial fact that will make it possible to determine the S-matrix; it is the characteristic consequence of bosonic symmetries that do not have the decomposition described in the Coleman-Mandula theorem.

The presence of the symmetry Q also implies that in scattering with two initial particles, there must be precisely two particles in the final state, and that if the particles are correctly labeled, the momenta (and hence masses) of the incoming particles are equal to those of the outgoing particles. Indeed, let us temporarily assume that there are precisely two particles in the final state. Let $1, 2$ label incoming particles, and $3, 4$ label outgoing particles. The charge Q acts on the state of the i-th particle by by multiplication by $C_i(k_i)_+^3$. So by charge conservation $C_1(k_1)_+^3 + C_2(k_2)_+^3 = C_3(k_3)_+^3 + C_4(k_4)_+^3$. On the other hand, by momentum conservation $(k_1)_+ + (k_2)_+ = (k_3)_+ + (k_4)_+$. So if k_1, k_2 are known, (k_3, k_4) can take only a discrete set of values. This implies, given the analytic properties of S, that $(k_3, k_4) = (k_1, k_2)$ or $(k_3, k_4) = (k_2, k_1)$. The same type of argument implies that there cannot be more than two particles in the final state: conservation of Q gives a relation that cannot be obeyed by an analytic S-matrix if there are different numbers of particles in the initial and final state.

The use of analyticity of S in these arguments is essentially as follows. Analyticity of S does not allow any delta function in S except energy-momentum conservation. In scattering with two initial particles and two final particles, if (after labeling the particles correctly), the initial and final masses are equal, then energy-momentum conservation implies that the initial and final momenta are equal. Thus the relation following from conservation of Q is an identity in this case (if the C_i are suitably restricted), and is impossible otherwise.

Now it is easy to explain the idea of proof of the Coleman-Mandula theorem, and why it fails in two dimensions.

In a theory with a "forbidden" symmetry, if we have two particles heading for a collision (i.e. their worldlines intersect; and in particular their velocities are different), we can apply the "forbidden" symmetry, shifting the positions of the particles by velocity-dependent and hence different amounts. In dimension $d > 2$, two lines generically do not intersect. Application of a "forbidden" symmetry above two dimensions will generically produce two world-lines that do not intersect, and can be made arbitrarily far from each other. Since the S-matrix is invariant under the symmetry, the ability to reduce a computation of the S-matrix to a case of non-intersecting worldlines shows that S equals 1.



In two dimensions, however, two lines generically do intersect, so we cannot conclude from the above argument that the S-matrix is 1. Still, since three lines generically do not intersect at the same point, the above argument shows that scattering reduces, as in the figure, to three successive two-particle collisions, in each of which two particles come in and two particles go out. There are two possibilities for the time sequence of the three collisions, as shown in the picture, and the "forbidden" symmetry implies that the S-matrix for the succession of three collisions is the same with either time sequence.

Now let us return to the sigma model with target space S^{N-1}. In this case, as we know for sufficiently large N (and believe for all $N \geq 3$), we have N massive particles of the same mass m, which form an N-dimensional representation V of $SO(N)$ (i.e. the space of 1-particle states is $V \otimes L^2(\mathcal{O}_m^+)$). The above arguments allow us to make the following conclusions about the scattering matrix.

(i) The S-matrix is completely determined by the "elastic two-body scattering," that is by the matrix elements of S with two initial and two final particles. In this type of scattering, there is no change in the momentum, so the S-matrix reduces to an ordinary matrix-valued function, that is a function of the form $S(\lambda_1 - \lambda_2)$ with values in $\mathrm{End}_{SO(N)}(V \otimes V)$, where λ_i are defined by $k_i = (me^{\lambda_i}, me^{-\lambda_i})$, and k_i are the momenta of the incoming particles. (Lorentz invariance implies that S only depends on the difference $\lambda_1 - \lambda_2$.) Thus, the S-matrix is described explicitly by a matrix-valued function of the form

$$S(\theta)_{ij}^{kl} = F(\theta)\delta_i^k\delta_j^l + G(\theta)\delta_i^l\delta_j^k + H(\theta)\delta_{ij}\delta^{kl}, \tag{4.22}$$

where F, G, H are complex-valued functions.

(ii) $S(\lambda)$ satisfies the quantum Yang-Baxter equation:

$$S_{12}(\lambda_1-\lambda_2)S_{13}(\lambda_1-\lambda_3)S_{23}(\lambda_2-\lambda_3) = S_{23}(\lambda_2-\lambda_3)S_{13}(\lambda_1-\lambda_3)S_{12}(\lambda_1-\lambda_2). \tag{4.23}$$

(iii) $S(\lambda)$ for any real λ is a unitary matrix.

(iv) Crossing symmetry: S continues meromorphically to the complex plane, and $S_{ij}^{kl}(\lambda) = S_{li}^{jk}(\pi i - \lambda)$.

Property (i) we have essentially already explained; it is the statement that any scattering process is determined by a succession of $2 \to 2$ processes, plus $SO(N)$ symmetry to restrict the $2 \to 2$ matrix elements. Property (ii) follows from the fact that the $3 \to 3$ scattering matrix factors as a product of three $2 \to 2$ scattering

matrices, and the order in which these three $2 \to 2$ collisions occur can be reversed, as in the figure, by the "forbidden" symmetry. Property (iii) is satisfied for the S-matrix of any field theory. Property (iv) is called crossing symmetry and is one of the characteristic features of relativistic field theory that do not have analogs in nonrelativistic quantum mechanics. Roughly, it follows from the fact that the $2 \to 2$ S-matrix is the residue of the pole of the 4-point function, but the 4-point function extends to complex values of momenta. So (since the complexification of the Poincaré group is connected) the S-matrix has to be invariant under the disconnected component of the Poincaré group, which permutes past and future and so exchanges initial and final particles. (See the axiomatic lectures from last fall for more detail.)

A. Zamolodchikov and Al. Zamolodchikov showed in 1979 that the S-matrix is uniquely determined by these conditions for $N \geq 3$, up to a scalar θ-dependent factor, provided it is nontrivial (which we can get from the large N expansion). Up to multiplication by a scalar function f that is common for F, G, and H, the functions F, G, H for this S-matrix are given by an easy explicit formula:

$$F(\theta) = 1, G(\theta) = -\frac{2\pi i}{N-2}\theta, H(\theta) = -\frac{2\pi i}{(N-2)(\pi i - \theta)}. \tag{4.24}$$

This solution of the quantum Yang-Baxter equation was one of the main examples from which the theory of quantum groups has originated. The quantum group corresponding to it is now called "the Yangian of $\mathfrak{so}(N)$" (see Drinfeld's talk "Quantum groups" at the International Congress of Mathematicians at Berkeley).

As we indicated, the S-matrix is defined by (4.24) up to multiplication by a scalar function $f(\theta)$. Moreover, this function satisfies the conditions $f(\theta)f(-\theta) = 1$, and $f(\pi i - \theta) = f(\theta)$. Thus, the function f is $2\pi i$-periodic. Which periodic function corresponds to the actual S-matrix of the theory is a nontrivial question. A perturbative calculation as well as a general notion of simplicity strongly suggests that this function should be chosen in such a way that the S-matrix has the smallest possible number of singularities. There are two such choices, and one of them is believed to be realized in the sigma model. The other choice is believed to be realized similarly in the Gross-Neveu model. It is interesting to note that f does not affect the probabilities for different reactions (as it does not affect the absolute value of F, G, and H for real λ), but nonetheless f is important as it affects the "time delays" in the scattering process.

Thus, in this way one determines the S-matrix of the sigma model into S^{N-1}. One can also use such arguments to determine the matrix elements of operators among physical states in this theory (by writing equations that are cousins of the Yang-Baxter equation and presumably have an interpretation in the quantum group language), but I will not discuss that.

One can show in a similar way the the Gross-Neveu model introduced in Lecture 4 of [I-**Renorm Groups**] is also integrable, in the same sense as the sigma model. As we have already remarked, its S-matrix is obtained using the other minimal choice of f.

But the sigma model to a sphere and the Gross-Neveu model are exceptions. Already in the next lecture we will consider a slightly more complicated sigma model – with target space $\mathbb{C}P^N$, which is not integrable and for which the explicit form of the S-matrix is unknown and presumably, in a sense, unknowable.

In a problem in the spring term exercises, one is asked to show that the supersymmetric sigma model with target a sphere is integrable, like its bosonic cousin. You will also see in that problem that the supersymmetric sigma model in a sense combines the ordinary sigma model and the Gross-Neveu model. Incidentally, the supersymmetric model with target $\mathbb{C}P^N$ is also integrable, even though the bosonic sigma model with that target is not.

References and cross references

Axiomatics of scattering in quantum field theory is discussed in Lecture 4 of [I-**Axiomatics**]. Two-dimensional sigma models, in particular negativity of the beta-function of the two-dimensional sigma model into a space of positive curvature, is discussed in Lecture 3 of [II-**CFT**]. The Gross-Neveu model, which is closely related to the nonlinear sigma model discussed in this lecture and has a similar $1/N$ expansion, is discussed in Lecture 4 of [I-**Renorm Groups**]. Problems 3 and 4 at the end of these lectures concern supersymmetric extensions of the ideas presented here. Problem 5 also concerns the large N limit of the sigma model into a sphere.

For the construction of a conserved charge of the sigma model of high spin, see

Y. Y. Goldschmidt, E. Witten, *Conservation laws in some two-dimensional models*, Phys. Lett. **91B** (1980), 392.

For the computation of the S-matrix of the sigma model, see

A. Zamolodchikov, Al. Zamolodchikov, *Factorized S-matrices in two-dimensions as the exact solutions of certain relativistic quantum field models*, Annals Phys. **120** (1979), 253–291.

For an intuitive discussion of the failure of the Coleman-Mandula theorem in two dimensions, and some additional constructions of exact S-matrices, see

R. Shankar, E. Witten, *S matrix of the supersymmetric nonlinear σ model*, Phys. Rev. **D17** (1978), 2134.

LECTURE 5
The Large N Limit of the σ-model into Grassmannians

In this lecture we discuss the large N behavior of the two dimensional σ-model into the Grassmannian $Gr(k,N)$ of k dimensional subspaces of $\mathbb{C}^N$. Here k is fixed as $N \to \infty$. We also consider the real Grassmannian. Since Grassmannians have positive Ricci curvature, these field theories are asymptotically free, but in any case our task is to investigate the infrared behavior.

The Euclidean action of the σ-model is

$$S[\phi] = \frac{1}{g^2}\int_\Sigma d^2x\,|d\phi|^2 - i\theta\int_\Sigma \phi^*(\alpha), \tag{5.1}$$

where Σ is a Riemann surface, $\phi\colon \Sigma \to Gr(k,N)$ a map into the Grassmannian, $\alpha \in H^2\big(Gr(k,N),\mathbb{Z}\big)$ a generator of the cohomology, and g,θ are parameters of the theory. We specify the metric on $Gr(k,N)$ shortly. The second term is a topological term;[11] for Σ closed the integral is integer-valued. Thus a shift $\theta \to \theta + 2\pi$ does not affect the model. The parameter θ is also a parameter of the quantum theory, but renormalization exchanges the dimensionless coupling constant g with a mass parameter μ.

The rescaled coupling constant $\widetilde{g}$, defined by

$$g^2 = \widetilde{g}^2/N, \tag{5.2}$$

is more natural in the large N limit, as we will see.

§5.1. The questions

We ask specific questions about the behavior of the model.

1. Is there a mass gap?
2. What is the θ dependence of the partition function?

Remark. For Σ small the partition function

$$Z(\theta) = \int D\phi\ \exp\left(-\frac{1}{g_{\text{eff}}^2}\int_\Sigma d^2x\,|d\phi|^2 + i\theta\int_\Sigma \phi^*(\alpha)\right)$$

Notes by Daniel Freed

[11] The factor of i is present in the *Euclidean* action so that the action conjugates under orientation reversal. In this way its continuation to Minkowski space is real. See [I-**Classical Fields**, §7.3].

can be studied using perturbation theory. Here g_{eff} is the effective coupling, which varies with the distance scale in the theory set by the size of the surface Σ. (Classically the model is conformally invariant, so does not depend on the size of Σ, but quantum mechanically this is no longer true.) By asymptotic freedom this coupling is small at small distances, hence the assertion that perturbation theory applies in this regime. Now the classical solutions to (5.1) are harmonic maps $\Sigma \to Gr(k, N)$. Note that the only θ dependence is through the degree of ϕ, and so we split the integral as a sum over maps ϕ of varying degrees. In degree 0 we obtain constant maps and in general for degree n some moduli space $\mathcal{M}_n$ of harmonic maps, which are the *instantons* of this model. Degree 1 instantons are (anti)holomorphic maps. The perturbation expansion around these solutions has the rough form

$$Z(\theta) \sim \frac{\operatorname{Vol}\big(Gr(k,N)\big)}{\sqrt{\det(\cdot)}}(1+\cdots) + \sum_{\pm} e^{\pm i\theta} e^{-cN/g_{\text{eff}}^2} \frac{\operatorname{Vol}\mathcal{M}_1}{\sqrt{\det(\cdot)}}(1+\cdots) + \text{higher instantons} \tag{5.3}$$

The constant c in the exponential is the action of a 1-instanton, which is independent of N. The factor of N comes from (5.2). If $g_{\text{eff}} << 1$ we see that the θ dependent term of $Z(\theta)$ vanishes exponentially as $N \to \infty$. As the area of Σ increases the effective coupling g_{eff} also increases. Equation (5.3) is the answer for Σ compact and of small area, but we will find a vastly different result for $\Sigma = \mathbb{R}^2$.

3. *Symmetry Breaking.* Symmetry breaking in two dimensions is possible for a discrete symmetry, and in this model we have the parity symmetry

$$\begin{aligned} P\colon \Sigma &\longmapsto \overline{\Sigma} \qquad (\overline{\Sigma} \text{ is } \Sigma \text{ with the opposite orientation}) \\ \theta &\longmapsto -\theta \end{aligned} \tag{5.4}$$

For $\Sigma = \mathbb{R}^2$ we implement the orientation reversal by an orientation-reversing isometry, i.e., a reflection. Then for $\theta = 0$ and $\theta = \pi$ the parity symmetry P acts on a fixed theory and we ask if it is broken in the quantum theory.

4. The group $PSU(N)$ of isometries of the Grassmannian acts in the classical theory. Since continuous symmetry groups are unbroken in two dimensions, this symmetry acts in the quantum theory as well. However, it is possible that a realization of the theory has a symmetry group which is a cover of $PSU(N)$, the latter being the group which acts on the operator algebra. Does that happen here?

§5.2. An equivalent formulation

To study the model we rewrite it, that is, we construct an action with the same classical and quantum physics. As a preliminary we recall some basic geometry of the Grassmannian. Over $Gr(k, N)$ lies a canonical sequence of vector bundles

$$0 \to S \xrightarrow{s} Gr(k,N) \times \mathbb{C}^N \to Q \to 0, \tag{5.5}$$

where the fiber of S at a k-plane π is simply π viewed as a subspace of $\mathbb{C}^N$. Fix the standard metric on $\mathbb{C}^N$. It induces a metric on S and identifies $Q \cong S^{\perp}$. There is

a canonical connection ∇ on S, obtained by projecting the natural connection on the trivial bundle $Gr(k,N)\times\mathbb{C}^N$. We easily compute

$$\nabla = d - s^* ds, \tag{5.6}$$

where s is the inclusion $S \xrightarrow{s} Gr(k,N)\times\mathbb{C}^N$. Then

$$\nabla s\colon T\big(Gr(k,N)\big) \longrightarrow \mathrm{Hom}(S, S^\perp) \tag{5.7}$$

is an isomorphism. We use it to induce a metric on $Gr(k,N)$, the metric needed to write down the σ-model action (5.1).

Now if $\phi\colon \Sigma \to Gr(k,N)$ we pullback (5.5) to obtain a sequence of bundles over Σ, and by (5.7) the Lagrangian density of ϕ is

$$|d\phi|^2 = |(\phi^*\nabla)\phi^* s|^2. \tag{5.8}$$

Note $\phi^* s\colon \phi^* S \to \Sigma\times\mathbb{C}^N$ determines ϕ. The idea is to replace ϕ by such a bundle map, and so first to replace $\phi^* S$ by a fixed bundle. Note $\deg(\phi^* S) = \deg(\phi)$ so that the topology of $\phi^* S$ determines the cohomology class $\phi^*(\alpha)$, which appears in the second term of the action (5.1). Hence fix a vector bundle $E \to \Sigma$ of rank k and degree d. Also fix a Hermitian metric on E. We introduce a new field

$$\widehat{\phi}\colon E \longrightarrow \Sigma\times\mathbb{C}^N$$

which we constrain to be an isometric immersion:

$$\widehat{\phi}^*\widehat{\phi} = \mathrm{id}_E\,. \tag{5.9}$$

The image of $\widehat{\phi}$ determines a map $\phi\colon \Sigma\to Gr(k,N)$ which is unchanged if we shift $\widehat{\phi}$ by a unitary gauge transformation of E. To rewrite (5.8) in terms of $\widehat{\phi}$ we need a connection on E, and as there is no natural choice we introduce a variable unitary connection A. Using $\widehat{\phi}$ we identify E with a subbundle of the trivial bundle $\Sigma\times\mathbb{C}^N$, so can differentiate $\widehat{\phi}$ using the usual derivative d. Writing A as a one-form plus this trivial connection we find

$$\begin{aligned} |d_A\widehat{\phi}|^2 &= |d\widehat{\phi} + \widehat{\phi}A|^2 \\ &= |d\widehat{\phi}|^2 + 2\Re(d\widehat{\phi}, \widehat{\phi}A) + |A|^2, \end{aligned} \tag{5.10}$$

since $\widehat{\phi}^*\widehat{\phi} = \mathrm{id}_E$. This expression is quadratic in A, so if (5.10) is a classical Lagrangian for A we can use the equations of motion to obtain

$$A_0 = -\widehat{\phi}^* d\widehat{\phi}. \tag{5.11}$$

Comparing with (5.6) we see that A_0 is the pullback of the canonical connection on S, and so by (5.8)

$$|d\phi|^2 = |d_{A_0}\widehat{\phi}|^2.$$

In other words, the Lagrangian (5.10) is equivalent to $|d\phi|^2$ for fields which satisfy the constraint (5.9). We impose the constraint via a Lagrange multiplier field

$$\sigma\colon \Sigma \longrightarrow \mathrm{HermitianEnd}(E).$$

The θ term in the original action (5.1) can be computed using the (skew-Hermitian) curvature F_A via Chern-Weil theory. Altogether we obtain for our new action[12]

$$S[\widehat{\phi}, A, \sigma] = \frac{1}{g^2}\int_\Sigma |d_A\widehat{\phi}^2| - i\int_\Sigma \operatorname{Tr}\sigma(\widehat{\phi}^*\widehat{\phi} - \mathrm{id}_E) + \frac{\theta}{2\pi}\int_\Sigma \operatorname{Tr} F_A. \tag{5.12}$$

As we have explained, the classical equations of motion (and other classical constructs) computed from (5.12) are equivalent to those computed from the original action (5.1). The classical computation which led to (5.11) is valid quantum mechanically since the dependence of (5.12) on A is quadratic and the Hessian is the identity operator (see (5.10)). (We ignore the constant determinant factor which we obtain from the A integral.) Thus the quantum physics is the same as well.

§5.3. The large N effective theory

The argument here is almost identical to that for the large N σ-model into a sphere (see Lecture 4). So we will be brief.

First, the $\widehat{\phi}$ integral is Gaussian, so the partition function is

$$\begin{aligned} Z &= \sum_E \int \frac{DA}{\mathrm{vol}} D\sigma D\widehat{\phi}\, e^{-S[\widehat{\phi},A,\sigma]} \\ &= \sum_E \int \frac{DA}{\mathrm{vol}} D\sigma \exp - \left[\operatorname{Tr}\ln\left(\frac{d_A^* d_A}{g^2} - i\sigma\right) + i\int_\Sigma \operatorname{Tr}\sigma + \frac{\theta}{2\pi}\int_\Sigma \operatorname{Tr} F_A\right]. \end{aligned} \tag{5.13}$$

Here $\sum_E$ is the sum over bundles E. We take $\Sigma = \mathbb{R}^2$. Then the sum over E is irrelevant, as is the topological term. In a first approach to this problem, the topology of the bundle will not be important, since it is spread over an infinite volume. Once we get a basic understanding of what the quantum vacuum looks like, it will not be hard to go back and see how the topology enters. Just like last week, we evaluate the leading behavior in large N by evaluating at a stationary point of the exponential in the integrand, i.e., a classical solution of the effective action.

The only Poincaré invariant (that is, Euclidean invariant) possibility for the gauge field is $A = 0$.[13] So it is natural to look for a stationary point with $A = 0$.

---

[12]Some explanation about the Lagrange multiplier term is in order. It is based on the general formula

$$\int e^{i(\sigma,x)} d\sigma = \delta(x),$$

where σ lies in the dual space to x, the measure $d\sigma$ is suitable normalized, and $\delta(x)$ is the δ-distribution supported at the origin. In the Minkowski space Lagrangian the Lagrange multiplier term is

$$\int_\Sigma \operatorname{Tr}\sigma(\widehat{\phi}^*\widehat{\phi} - \mathrm{id}_E),$$

and σ should be interpreted as a two-form (or density); it lies in the dual space to the function space in which $\widehat{\phi}^*\widehat{\phi} - \mathrm{id}_E$ lives. Rotation to Euclidean space yields the second term of (5.12). Or we could of course introduce the Lagrange multiplier term directly in Euclidean space, with the same reasoning.

[13]A nonzero constant expectation value of the curvature $F = dA$ would not be invariant under spatial reflections – the disconnected part of the Euclidean group. For $\theta \neq 0$ (which violates reflection symmetry) we will eventually find that there is indeed a nonzero average value of F, but too small to affect the large N saddle point.

In that case we rewrite minus the integrand in the last line of (5.13) as

$$S_{\text{eff}}[A,\sigma] = N \operatorname{Tr} \ln \left(\frac{d^* d}{g^2} - i\sigma \right) + i \int \operatorname{Tr} \sigma,$$

where the operator in the first term acts on sections of E^*. Now the only Poincaré invariant possibility is σ constant, and so we diagonalize σ and pass to k one-dimensional problems for an eigenvalue. At this point we rescale

$$\begin{aligned} \sigma &= N\widetilde{\sigma} \\ g^2 &= \widetilde{g}^2/N, \end{aligned} \tag{5.14}$$

and so for each eigenvalue we have exactly the problem we had for the large N σ-model into a sphere. Thus the solution $\widetilde{\sigma}_0$ has all eigenvalues equal and is specified by

$$-i\widetilde{g}^2\widetilde{\sigma}_0 = \Lambda^2 e^{-4\pi/\widetilde{g}^2} = M^2$$

for Λ an ultraviolet cutoff and now $\widetilde{g}^2$ the running coupling constant. We define M^2 to be this dynamically generated mass squared.

So in the large N effective action $\widetilde{\sigma}$ acquires a mass. Note that the A field has no transverse degrees of freedom—it is a gauge field in two dimensions—so does not enlarge the spectrum of the model (though, of course, it affects the Hamiltonian as we will see, and in fact diminishes the spectrum). Thus the large N limit has a mass gap. This is the answer to the first in our list of questions.

To answer the other questions we need to compute something more precise, namely the leading approximation to the large N effective action. This means that we do perturbation theory for the action (5.12) about the point $\widehat{\phi}_0 = 0$, $A_0 = 0$, $\widetilde{\sigma}_0 = iM^2/\widetilde{g}^2$. So shifting $\widetilde{\sigma}$ by $\widetilde{\sigma}_0$ and rescaling $\widehat{\phi}$ we have from (5.12)

$$S'[\widehat{\phi}, A, \widetilde{\sigma}] = \int_\Sigma |d_A \widehat{\phi}|^2 + M^2 \int_\Sigma |\widehat{\phi}|^2 - i\widetilde{g}^2 \int_\Sigma \operatorname{Tr} \widetilde{\sigma} \widehat{\phi}^* \widehat{\phi} + \frac{\theta}{2\pi} \int_\Sigma \operatorname{Tr} F_A. \tag{5.15}$$

Here we omit a constant term (which only shifts the partition function by a constant) and a linear term in $\widetilde{\sigma}$ (since we are expanding around a solution of the effective action).

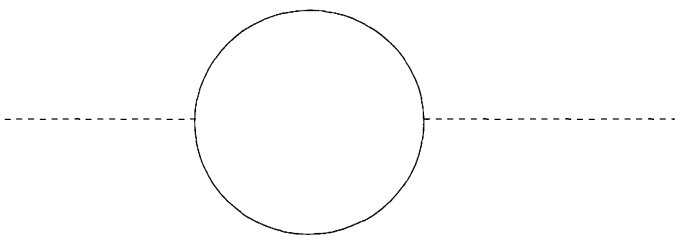


Figure 5.1: The inverse $\widetilde{\sigma}$ propagator

The effective action is computed in perturbation theory using one particle irreducible Feynman diagrams with external lines for A and σ and with internal $\widehat{\phi}$ lines. So the inverse propagator for $\widetilde{\sigma}$ is computed from the diagram shown in

Figure 5.1, where the solid line represents $\widehat{\phi}$ and the dotted line represents σ. (Note that (5.15) has no quadratic term in $\widetilde{\sigma}$, else Figure 5.1 would be a correction to such a term.) We evaluate the diagram in momentum space as

(5.16)
$$\begin{aligned} -\widetilde{g}^4 Nk \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+M^2} \frac{1}{(p-q)^2+M^2} \\ &= -\frac{\widetilde{g}^4 Nk}{4\pi^2} \int_0^1 d\alpha \int d^2q \frac{1}{[q^2+(M^2+\alpha(1-\alpha)p^2)]^2} \\ &= -\frac{\widetilde{g}^4 Nk}{\pi} \int_0^1 \frac{d\alpha}{M^2+\alpha(1-\alpha)p^2} \\ &= -\frac{\widetilde{g}^4 Nk}{\pi M^2} + O(p^2) \qquad \text{as } p \to 0. \end{aligned}$$

This corresponds to a term
$$C|\widetilde{\sigma}|^2$$
in the effective action, with $C > 0$. (The minus sign enters since in the Euclidean theory diagrams compute negative contributions to the effective action.) This is the dominant term in the infrared, which means that $\widetilde{\sigma}$ is massive and does not affect the long range behavior of the theory.

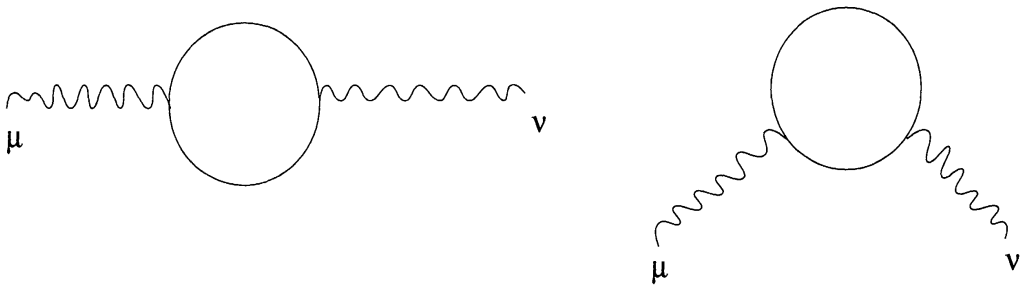


Figure 5.2: The inverse A propagator

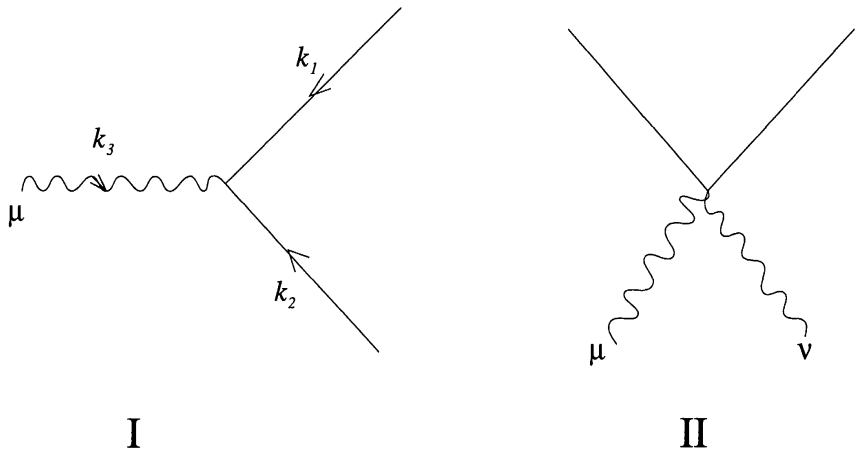


Figure 5.3: The interaction vertices for the A field

The inverse propagator for A is computed by the diagrams in Figure 5.2, which come from the first term in (5.15). Here the wavy line represents A. To compute these diagrams we need the Feynman rules for the vertices indicated in Figure 5.3, which correspond to the terms

$$2\Re(d\widehat{\phi}, \widehat{\phi}A)$$

$$|\widehat{\phi}A|^2$$

in the action (5.15). The indices refer to a standard orthonormal basis for $\mathbb{R}^2$. The Feynman rule (in momentum space) for the second vertex is easy:

$$II = -\delta_{\mu\nu}.$$

(There is a minus sign since the Euclidean functional integral involves e^{-S}.) For the vertex I we must remember that $\widehat{\phi}$ is complex and that A is skew-Hermitian:

$$2\Re(\partial_\mu\widehat{\phi}, \widehat{\phi}A_\mu) = -\partial_\mu\widehat{\phi}A_\mu\widehat{\phi}^* + \widehat{\phi}A_\mu\partial_\mu\widehat{\phi}^*.$$

Thus the vertex is

$$I = -i(k_1 - k_2)_\mu.$$

Note that one of the solid lines in the vertex represents $\widehat{\phi}$ and the other solid line represents $\widehat{\phi}^*$. So the sum of the diagrams in Figure 5.2 is

$$\int \frac{d^2q}{(2\pi)^2} \frac{(p-2q)_\mu(p-2q)_\nu}{(q^2+M^2)((p-q)^2+M^2)} - 2\delta_{\mu\nu}\int \frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+M^2)}. \tag{5.17}$$

The factor of 2 in the second term is from the two ways of attaching the A lines to the external vertices; the corresponding factor of 2 in the first term is canceled by the symmetry which exchanges the two internal vertices in the first diagram. (That is, there is a factor of $1/2$ from the expansion of the exponential, since we have two triple vertices.) Each term in (5.17) is divergent, but the divergences cancel in the difference, and after some computation similar to (5.16) the answer to leading order in p is

$$\frac{N}{12\pi M^2}(p_\mu p_\nu - p^2\delta_{\mu\nu}). \tag{5.18}$$

This corresponds to a term

$$\frac{N}{24\pi M^2}|F_A|^2 \tag{5.19}$$

in the effective action. (Again we must recall that diagrams contribute negatively to the effective action.) In fact, (5.18) and (5.19) correspond precisely if $k = 1$ (the σ-model into projective space), since then A is an abelian connection. In the nonabelian case there are cubic and quartic terms, but by gauge invariance their leading contribution must be as in (5.19). We remark that (5.19) is the lowest order gauge invariant term we can write, and so its appearance can be predicted from gauge invariance alone, but of course we must do a computation to determine the coefficient.

So, finally, the long distance behavior of the two dimensional σ-model into the Grassmannian $Gr(k,N)$, in the large N limit, is equivalent to the long distance behavior of a two dimensional gauge theory with gauge group $U(k)$ and charged massive scalars. A simplified form of the action, keeping the terms of lowest dimension, is

$$\frac{N}{2e^2}\int_\Sigma |F_A|^2 + \frac{\theta}{2\pi}\int_\Sigma \operatorname{Tr} F_A + \int_\Sigma |d_A\phi|^2 + M^2\int_\Sigma |\phi|^2. \tag{5.20}$$

The first term in (5.20) was just computed, where the coupling e^2 summarizes the numerical factor in (5.19). The second term is the θ term in (5.15). The last two terms are the first two terms in (5.15), except we now drop the $\sim$ for convenience. So ϕ is a section of $(E^*)^{\oplus N}$.

§5.4. Real Grassmannians

Let us go back to the beginning and consider the same problem but with a real Grassmannian instead of a complex Grassmannian. In that case the bundle E is real, $\widehat{\phi}\colon E \to \Sigma \times \mathbb{R}^N$, and the gauge group is orthogonal instead of unitary. The bulk of the above derivation goes through with relatively obvious changes, but one point that needs to be reconsidered is the representation of the θ term. For simplicity we consider the σ-model into the Grassmannian $Gr^0_{\mathbb{R}}(k,N)$ of *oriented* k-planes in $\mathbb{R}^N$. (It is a double cover of the Grassmannian of unoriented k-planes.) Now $H^2\big(Gr^0_{\mathbb{R}}(k,N);\mathbb{Z}\big) \cong \mathbb{Z}/2\mathbb{Z}$ and the θ term is meant to detect this class. Thus the second term of (5.1) is replaced by

$$i\theta \deg_2 \alpha,$$

where $\deg_2$ is the mod 2 degree (0 or 1) and $\theta = 0$ or $\theta = \pi$. In the reformulation of the problem E is an oriented real k-plane bundle over Σ, and the topological term in (5.12) and subsequent formulas is

$$i\theta w_2(E)[\Sigma], \tag{5.21}$$

where w_2 is the second Stiefel-Whitney class. For $\Sigma = \mathbb{R}^2$ we can rewrite (5.21) in terms of a *Wilson line* operator. Namely, a field configuration with finite action (5.12) has A essentially flat at infinity. Since $\mathbb{R}^2$ is contractible we can lift the $SO(k)$ connection A to a $\mathrm{Spin}(k)$ connection $\widetilde{A}$, and the holonomy on a large loop $C \subset \mathbb{R}^2$ is approximately ± 1 depending on the Stiefel-Whitney class of the induced bundle on S^2. (We let '-1' denote the nontrivial element of $\mathrm{Spin}(k)$ covering $1 \in SO(k)$.) Choose a representation R of $\mathrm{Spin}(k)$ and consider

$$W_R(C) = \frac{\operatorname{Tr}_R \operatorname{hol}_C(A)}{\dim R}. \tag{5.22}$$

In the limit where the loop C becomes large, this computes the exponential of (5.21), where $\theta = 0$ if R is a representation of $SO(k)$ and $\theta = \pi$ if R is a representation of $\mathrm{Spin}(k)$ but not of $SO(k)$.

Though the application to Grassmannians is special, the notion of representing the theta term by a Wilson line at infinity makes sense for an arbitrary connected compact Lie group G. A G bundle over Σ has a characteristic class in $H^2(\Sigma, \pi_1 G)$. It pairs with a homomorphism $e^{i\theta}\colon \pi_1 G \to \mathbb{T}$ to give a term in the exponentiated action. Here $\mathbb{T} = U(1)$ is the circle group. On the other hand a representation of the simply connected covering group induces a homomorphism $e^{i\theta}\colon \pi_1 G \to \mathbb{T}$, so we can use the Wilson operator (5.22) to write the θ term on $\mathbb{R}^2$.

§5.5. Pure gauge theory

We still must determine the quantum behavior of the theory with effective action (5.20). For this, we will first practice by analyzing the pure gauge theory in two dimensions, whose action is the sum of the first two terms of (5.20). This is a problem we will come back to from many points of view in later lectures, such as Lecture 11.

We may as well consider an arbitrary connected compact Lie group G. The norm in the first term of (5.20) is defined via a bi-invariant metric on G. We quantize the theory on the circle S^1_V of circumference (=volume) V. In general, to quantize a gauge theory in n dimensions on an n-manifold Y, we consider connections on $\mathbb{R} \times Y$ in *temporal gauge* and take solutions to the equations of motion up to time-independent gauge transformations. For pure gauge theory without the θ term the resulting space is the (co)tangent bundle of the space of connections on Y modulo gauge transformations. For $Y = S^1_V$ we first fix a basepoint; then a connection up to gauge equivalence is specified by the holonomy, an element of G. A change of basepoint conjugates the holonomy, and so the quantum Hilbert space is

$$\mathcal{H}_{\text{gauge}}(\theta{=}0) \;=\; L^2(G)^G,$$

the space of conjugacy invariant functions on G. A basis for this space is the set of characters of irreducible representations.

Next we compute the Hamiltonian. Let $A_t = A_t(x)dx$ be a connection on $\mathbb{R}\times S^1_V$ (with coordinates t, x) in temporal gauge, relative to some trivialization, and let $g_t \in G$ be the corresponding path of holonomies. In a gauge where A_t is constant, we have $g_t = e^{VA_t}$. Then the first term in the action (5.20) is

$$\frac{N}{2e^2}\iint \left|\frac{dA}{dt}\right|^2 dtdx = \frac{N}{2e^2V}\int |\dot{g}|^2\, dt. \tag{5.23}$$

This is the Lagrangian for a classical particle of mass N/e^2V moving on the Riemannian manifold G; the corresponding quantum Hamiltonian is

$$H_{\text{gauge}} = \frac{e^2V}{2N}\Delta_G, \tag{5.24}$$

where Δ_G is the Laplacian on G. The eigenfunctions are the characters of the irreducible representations with eigenvalues proportional to the Casimir.

Now consider the θ term in (5.20). For $G = U(k)$ we have a natural closed imaginary one-form $\alpha \in i\Omega^1_G$ which is the trace of the Maurer-Cartan form. Then in terms of the path g_t of holonomies the θ term in the action is

$$\frac{\theta}{2\pi}\int g^*\alpha. \tag{5.25}$$

This is a topological term—it is invariant under reparametrizations of the path g_t. More geometrically, $\theta\alpha$ is a flat connection on a topologically trivial Hermitian line bundle L_θ over G, and up to equivalence it is given by an element in $H^1(G;\mathbb{T})$. Then (5.25) is parallel transport in this flat bundle.[14] More generally, we can "twist" our

[14]In general, the classical field theory action on a manifold with boundary—here the interval—is not a number, but we do not stress that point here.

mechanical system by any Hermitian line bundle L with connection. The quantum Hilbert space is the the space of sections of L with Hamiltonian the Laplacian for such sections. In our case we obtain the space

$$\mathcal{H}_{\text{gauge}}(\theta) = L^2(G, L_\theta)^G$$

of invariant sections with Hamiltonian (5.24). For arbitrary G an element $e^{i\theta} \in H^1(G;\mathbb{T})$ corresponds to a homomorphism $e^{i\theta}\colon \pi_1 G \to \mathbb{T}$. Recall that $\pi_1 G$ is a subgroup of the center of G. Then the eigenfunctions of the Laplacian acting on $\mathcal{H}_{\text{gauge}}(\theta)$ are the characters of representations of the simply connected cover of G whose restriction to $\pi_1 G$ is $e^{i\theta}$; the eigenvalue is again proportional to the Casimir.

For the unitary group $G = U(k)$ we identify $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ as before. The smallest Casimir occurs for a representation $\det^{\widetilde{\theta}/2\pi}$, where $\widetilde{\theta} \in \mathbb{R}$ is a representative of θ of smallest absolute value. If $\theta \neq \pi$ there is a unique such $\widetilde{\theta}$, but for $\theta = \pi$ there are two possibilities: $\widetilde{\theta} = \pi$ and $\widetilde{\theta} = -\pi$. For $G = SO(k)$ the simply connected cover is $\widetilde{G} = \text{Spin}(k)$. For $\theta = 0$ there is a unique lowest representation, the trivial representation of $SO(k)$. For $\theta = \pi$ the lowest representation is the spin representation if k is odd, and the two half spin representations if k is even. In the large volume limit only the lowest eigenvalue survives, so we have two vacua for $\theta = \pi$ in the complex $\big(U(k)\big)$ case and for k even in the real $\big(SO(k)\big)$ case. Observe that the two vacua correspond under the involution $g \mapsto -1 \cdot g$ in these cases. (Notice also that the bundle $L_{\theta=0}$ is real.) We will see that this vacuum structure persists when we add matter. Thus parity symmetry (5.4) is spontaneously broken at $\theta = \pi$ for $G = U(k)$ and $G = SO(2\ell)$. This answers the third of our questions.

As the volume $V \to \infty$ the eigenvalues of (5.24) become widely separated. In particular, in the infinite volume limit there is no state of finite energy above the vacuum (or the vacua in a few special cases we will find with two degenerate vacua). So the physical Hilbert space in infinite volume consists only of the vacuum (or vacua)—pure gauge theory on $\mathbb{R}^2$ is trivial.

Specialize to $G = U(1)$. Then the *vacuum energy density*, which is the minimum eigenvalue of the Hamiltonian (5.24) divided by the volume V, is

$$E_{\text{vac}}(\theta) = \frac{e^2}{2N} \min_{n\in\mathbb{Z}} (n - \frac{\theta}{2\pi})^2. \tag{5.26}$$

Notice that E_{vac} is smooth except at $\theta = \pi$, where its derivative has a discontinuity.

For $G = U(k)$ formula (5.26) is simply multiplied by k.

The partition function $Z_\Sigma(\theta)$ of the pure gauge theory for $\Sigma = [0,T] \times S^1_V$ has the Hilbert space interpretation

$$Z_\Sigma(\theta) \sim \langle \Omega | e^{-TH(\theta)} | \Omega \rangle \quad \text{as } T \to \infty,$$

where $H(\theta)$ is the Hamiltonian and Ω the vacuum. For the pure gauge theory, we obtain from (5.26) for $G = U(1)$

$$Z_\Sigma(\theta) \sim \exp\big(cTV - \frac{e^2 TV}{2N} \min_{n\in\mathbb{Z}} (n - \frac{\theta}{2\pi})^2\big) \quad \text{as } T \to \infty. \tag{5.27}$$

Here c is a constant which represents the indeterminacy of the path integral, or equivalently the fact that we are free to add a constant (independent of θ) to the

Hamiltonian. This is quite different than the prediction (5.3) from the instanton sum. Note that due to the classical conformal invariance of the σ-model, the instantons used in deriving (5.3) do not have a definite size; they can be rescaled. For that reason the instanton sum is not reliable at large distances. In any case (5.27) answers question 2 for pure gauge theory. It also gives the answer for the σ-model to the Grassmannian, because this model looks much like pure gauge theory at long distances.

§5.6. Classical electromagnetism in two dimensions

Before analyzing the quantum gauge theory with bosonic matter—the ϕ field in the action (5.20)—we discuss the classical physics. For the classical analysis we work in the $G = U(1)$ theory. The classical equations—Maxwell's equations—for pure gauge theory are

$$\frac{N}{e^2} d_A^* F_A = 0. \tag{5.28}$$

Since we are in two spacetime dimensions, this implies that the electric field $f = f_A = *F_A$ is a constant. Let 'x' denote the coordinate on space, which is simply a copy of $\mathbb{R}$. Add a point charge of charge 1 at $x = x_0$. Then the electric field f as a function of x satisfies (5.28) with a right hand side due to the charge:

$$\frac{N}{e^2} \frac{df}{dx} = -\delta(x - x_0).$$

Thus the value of the electric field jumps by $-e^2/N$ across a charge. (See Figure 5.4.) Allowing for many charges, and assuming all of them are multiples of the basic charge (which is a conclusion of the quantum theory), we see that

$$\theta = \frac{2\pi N f}{e^2} \in \mathbb{R}/2\pi\mathbb{Z} \tag{5.29}$$

is a constant. So there is an angle in the classical theory (assuming charge quantization).

$$f \uparrow f - \frac{e^2}{N} \qquad\qquad f \downarrow f + \frac{e^2}{N}$$

Figure 5.4: Change in electric field across a positive or negative charge. An arrow pointing up or down is meant to symbolize a positive or negative charge. On the left, for example, we consider a positive charge; the electric field is f to the left of the charge and $f - e^2/N$ to the right.

The classical energy density of an electric field f is computed from the action (5.23) (where $f = dA/dt$) to be $\frac{N}{2e^2} f^2$. As in the quantum theory, for a fixed value of θ in (5.29) there is a unique minimum obtained at some $f = f_0$ as long as

$\theta \neq \pi$. Any valid configuration must have finite energy compared to the vacuum energy. Thus if $\theta \neq \pi$ we must have $f(x) \to f_0$ as $x \to \pm\infty$. Using the formula above for the jump of the electric field across a charge, we see that the total charge of a finite energy configuration is zero. This means that there is *confinement*—it is impossible to have a single charged particle or any other isolated collection of charges with nonzero total charge. On the other hand, for $\theta = \pi$ we have minimum energy density at $f_0 = \pm e^2/2N$, and so for any finite energy configuration $f(x) \to \pm e^2/2N$ as $x \to \pm\infty$. Thus there is a finite energy configuration with a single particle: the electric field satisfies $f(-\infty) = e^2/2N$, $f(\infty) = -e^2/2N$. (See Figure 5.5.) In this case there is no confinement. Also, in this case there are four components of finite energy configurations, depending on the value of f at $\pm\infty$.

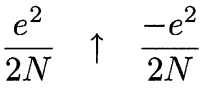


Figure 5.5: A one-particle state for $\theta = \pi$. The electric field has the same absolute value on the two sides.

Notice that whereas in three space dimensions the Coulomb potential between point charges separated by distance r is proportional to $1/r$, the Coulomb potential in one space dimension is proportional to r. This means the potential energy grows as the charges separate, which is another way of understanding that confinement occurs.

§5.7. Quantum theory with matter

Now we want to show that there is confinement in the quantum theory of the Lagrangian (5.20) as long as $\theta \neq \pi$. This is the assertion that every finite energy configuration in the quantum theory has total charge zero.

As a preliminary, consider the theory of matter only (no gauge field). In the simplest case $k = N = 1$ there is a single free complex scalar field ϕ with mass M. The Hilbert space $\mathcal{H}_\phi$ of this theory is the completed symmetric algebra of $W \oplus \overline{W}$, where W is the scalar representation of Poincaré with mass M. There is a global $U(1)$ symmetry which rotates ϕ. The corresponding quantum operator Q_ϕ, the Noether charge, has value 1 on W and -1 on $\overline{W}$, so value $p-q$ on $\mathrm{Sym}^p W \otimes \mathrm{Sym}^q \overline{W}$. A state in this subspace represents p positively charged particles and q negatively charged particles. For arbitrary k, N we have kN copies of this picture. In particular, for $k = 1$ there is a global $SU(N)$ symmetry (beyond the $U(1)$ symmetry already discussed.)

Now add the gauge field. For $k = N = 1$ we have a $U(1)$ gauge theory with a single charged scalar field. The global $U(1)$ symmetry of the preceding paragraph is now a local symmetry. Consider first the case $\theta = 0$. For small coupling e^2 we construct the quantum Hilbert space of the joint system by quantizing the symplectic manifold of classical solutions to (5.20). Thus take Σ to be the cylinder $\mathbb{R} \times S^1_V$. Then the space of classical solutions is a vector bundle over the cotangent bundle $T^*\mathcal{A}$ to the space $\mathcal{A}$ of connections on S^1_V; the fiber of this vector bundle at the trivial connection $A = 0$ is the real symplectic vector space underlying $W \oplus \overline{W}$.

(Note we quantize by a complex polarization, which leads to the Hilbert space $\mathcal{H}_\phi$ described above.) To implement gauge symmetry we must take the symplectic quotient by the group $\mathcal{G}$ of gauge transformations. Fix a basepoint (infinity) on S^1_V. Then the subgroup $\mathcal{G}_0$ of gauge transformations which equal the identity at the basepoint acts freely on $\mathcal{A}$, and the quotient is identified with $U(1)$ via holonomy. In the pure gauge theory the symplectic quotient of $T^*\mathcal{A}$ by $\mathcal{G}_0$ is diffeomorphic to $T^*U(1)$; its quantization $L^2\big(U(1)\big)$ was discussed previously. As the length $V \to \infty$ recall that the only state which remains of finite energy is the vacuum state. Classically, the vacuum corresponds to the zero section, a Lagrangian submanifold of $T^*U(1)$. The subgroup $U(1) \subset \mathcal{G}$ of constant gauge transformations acts trivially on $\mathcal{A}$—hence trivially on $T^*\mathcal{A}$—so does not enter into pure gauge theory. In the theory with matter the symplectic quotient by $\mathcal{G}_0$ is a vector bundle over $T^*U(1)$ with fiber $W \oplus \overline{W}$ at the identity. Now the constant gauge transformations act nontrivially in the fibers by scalar multiplication. To implement the symmetry we have two choices: we can quantize the symplectic quotient or we can consider the subspace of the Hilbert space annihilated by the corresponding quantum operator N_e. Pursuing first the latter, we see that in the infinite volume limit the quantum Hilbert space before implementing the $U(1)$ is simply $\mathcal{H}_\phi$, since the gauge field contributes only the vacuum state. The operator N_e is simply equal to Q_ϕ. Thus the Hilbert space of the theory is the subspace of states with total charge $Q_\phi = 0$. Therefore, just as in the classical theory we have confinement. There is a mass gap, and the smallest mass is $2M$. In the theory with small coupling, the qualitative picture is the same.

If instead we take the symplectic quotient of $W \oplus \overline{W}$ by $U(1)$, we are led to a singular space. The moment map is $\mu(w, \overline{w}') = |w|^2 - |\overline{w}'|^2$, and $\mu^{-1}(0)$ is singular at $(0,0)$. In the quantization this singular point corresponds to the vacuum, and it is not hard to see heuristically that we are led to the same description as before.

Now allow $\theta \neq 0$. In the language of geometric quantization the prequantum line bundle over $T^*U(1)$ is now twisted by the pullback of $L_\theta \to U(1)$. Quantum states correspond to "Bohr-Sommerfeld" leaves of the given polarization, which are equally spaced parallel circles in the cylinder $T^*U(1)$. More precisely, if p is the (momentum) coordinate in the cotangent space, the circles occur at $p = n - \theta/2\pi$ for integral n. The energy of the corresponding quantum state (in pure gauge theory) is given in (5.26), and as $V \to \infty$ we only keep the closest circle(s) to $p = 0$, which corresponds to the vacuum state. If $\theta \neq \pi$ there is a unique such circle, and we have the same picture of confinement as above. For $\theta = \pi$ there are two such circles, corresponding to the two vacuum states. Thus in the theory with matter, before dividing out by the constant gauge transformations, we must quantize two disjoint copies of $W \oplus \overline{W}$ to obtain $\mathcal{H}_\phi \oplus \mathcal{H}_\phi$. If, as before, we assign zero charge to each of the vacuum states, then the operator N_e which corresponds to the $U(1)$ symmetry is $Q_\phi \oplus Q_\phi$. The subspace of $\mathcal{H}_\phi \oplus \mathcal{H}_\phi$ annihilated by $Q_\phi \oplus Q_\phi$ is simply two copies of the system seen previously, each copy with a vacuum. These are two "realizations" of the quantum theory, each with a mass gap of size close to $2M$. We can also construct states containing a single charge, with an energy approximately M, in which for $x \to -\infty$ one sees one realization or vacuum of the theory, and for $x \to +\infty$ one sees the other. These states are "solitons" that interpolate between the two vacua; we discuss such states (mainly in situations in which they can be seen using a different kind of topological argument) in Lecture 9.

To justify these heuristic pictures we compute. From (5.20) we see that the Hamiltonian is

$$H = \int_{-\infty}^{\infty} dx \, \left(\frac{1}{2}|\pi_\phi|^2 + |d_A\phi|^2 + M^2|\phi|^2 + \frac{N}{2e^2}|f_A|^2\right), \tag{5.30}$$

where ϕ is a field on $\mathbb{R}$. Here π_ϕ is the conjugate momentum to ϕ, and $f = f_A$ is the Hodge star of the curvature as before. We derive an effective Hamiltonian for ϕ by plugging in the equation of motion of A. The latter is obtained by varying the Lagrangian (5.20) with respect to A:

$$\frac{N}{e^2}\frac{df}{dx} = j, \tag{5.31}$$

where the current is

$$j = \phi\pi_\phi - \overline{\phi\pi_\phi}.$$

We integrate (5.31) to obtain

$$f(x) = \frac{e^2}{N}\left\{\int_{-\infty}^{\infty} dy \, G(x-y)j(y) + c\right\} \tag{5.32}$$

for some constant c, where

$$G(x-y) = \begin{cases} 1, & x \geq y; \\ 0, & x < y. \end{cases}$$

Note that

$$\lim_{x\to-\infty} f(x) = \frac{e^2 c}{N}. \tag{5.33}$$

Plugging into (5.30) we see

$$H = H_0 + \Delta H,$$

where H_0 is the Hamiltonian for the free scalar particle (if we compute at $A = 0$), and

$$\Delta H = \frac{e^2}{2N}\int_{-\infty}^{\infty} dx \, \left\{\int_{-\infty}^{\infty} dy \, G(x-y)j(y) + c\right\}^2.$$

The perturbation term ΔH is nonlocal and singular.

Now we must determine the subspace of states Ψ on which ΔH is finite. Consider first $c = 0$. Then $\langle\Psi|\Delta H|\Psi\rangle$ is finite if and only if

$$\lim_{x\to\infty}\langle\Psi|f(x)|\Psi\rangle = 0,$$

where $f(x)$ is defined by (5.32). Now

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} dy \, G(x-y)j(y) = \int_{-\infty}^{\infty} dy \, j(y) = Q_\phi$$

is the charge operator in the Hilbert space $\mathcal{H}_\phi$. So if $c = 0$ we have confinement:

$$\langle\Psi|Q_\phi|\Psi\rangle = 0.$$

In general, c is related to θ through (5.33) and (5.29):

$$|c| = \min_n \left(\frac{\theta}{2\pi} + n \right).$$

This is the value of the electric field at $-\infty$. Also, we subtract an (infinite) constant from ΔH to account for the nonzero energy at ∞:

$$(\Delta H)_{\text{normalized}} = \frac{e^2}{2N} \int_{-\infty}^{\infty} dx \left[\left\{ \int_{-\infty}^{\infty} dy\, G(x-y) j(y) + c \right\}^2 - c^2 \right].$$

This is finite on states Ψ which satisfy

$$Q_\phi (Q_\phi + 2c) |\Psi\rangle = 0, \tag{5.34}$$

i.e., $Q_\phi|\Psi\rangle = 0$ or $Q_\phi|\Psi\rangle = -2c$. For $\theta \neq \pi$ the only possibility is $Q_\phi|\Psi\rangle = 0$ and we have confinement.

For $\theta = \pi$ we have $c = \pm 1/2$, and (5.34) is satisfied by states with $Q_\phi|\Psi\rangle = 0$ or $Q_\phi|\Psi\rangle = \mp 1$. As in the classical theory, this gives four sectors. We denote them $\mathcal{H}_{++}, \mathcal{H}_{+-}, \mathcal{H}_{-+}, \mathcal{H}_{--}$ according to the value of the electric field at $-\infty$ and $+\infty$. Note that an example of a state in $\mathcal{H}_{++}$ is a widely separated charge-anticharge pair, each of mass approximately M. In the limit as the charges separate, there is no increase in energy, so one certainly wants to say that this theory is not confining. If we take a limit with one of the two charges at a fixed position and the other going off to infinity, then all local measurements at fixed distance from the first charge converge to measurements in a Hilbert space $\mathcal{H}_{+-}$ (or $\mathcal{H}_{-+}$, depending on the sign of the charge that is taken to infinity and in which direction it is taken). Similarly, by taking two charges to infinity, one can describe in $\mathcal{H}_{++}$ a sequence of measurements that converge to a measurement in $\mathcal{H}_{--}$. In that sense, all four Hilbert spaces should be regarded as part of one story. This is a general aspect of two-dimensional theories with several vacua; it does not have an analog above two dimensions.

In the confining cases the symmetry group $PSU(N)$ acts. In the sectors $\mathcal{H}_{\pm\mp}$ of the $\theta = \pi$ theory, the covering group $SU(N)$ acts, as the one-particle states transform in the fundamental representation of $SU(N)$ and its dual. So the symmetry group is $PSU(N)$, except for $\theta = \pi$, where it is $SU(N)$.

The story for $k > 1$ is similar.

References and cross references

Large N limits are discussed in Lecture 4 of [I-**Renorm Groups**], Lecture 4 of this series, and Problem FE4 of [I-**Homework**]. Symmetry breaking is discussed in Lecture 1 of this series.

The large N limit of the sigma model with target a complex projective space is analyzed in

A. D'Adda, M. Luscher, P. Di Vecchia, *A $1/N$ expandable series of nonlinear sigma models with instantons*, Nucl. Phys. **B146** (1978), 63–76..

E. Witten, *Instantons, the quark model, and the $1/N$ expansion*, Nucl. Phys. **B149** (1979), 285–320.

For a discussion of weakly coupled two-dimensional gauge theory with massive charges, see

S. Coleman, *More about the massive Schwinger model*, Annals of Physics **101** (1976), 239–267.

LECTURE 6
The Bose-Fermi Correspondence and its Applications

§6.1. Two-dimensional gauge theories with fermions

Today we will consider two-dimensional gauge theories with fermions. We will work with Euclidean Lagrangians.

In 2 dimensions, the two spinor representations S_+, S_- are one-dimensional and complex. They are defined by the formula $\theta \to e^{\pm \imath\theta/2}$, $\theta \in [0, 4\pi)$, where θ is the parameter on $Spin(2)$.

Let Σ be a Riemann surface with a specified Spin structure. Then S_+, S_- define holomorphic line bundles on Σ, which are called the Spin bundles, and denoted by the same letter. We have $S_+ \otimes S_- = \mathbb{C}$, and $S_+^2 = T\Sigma, S_-^2 = T^*\Sigma$. Sections of ΠS_- (respectively, ΠS_+) will be called left (respectively, right) moving fermions, by analogy with the Minkowski picture.

We denote by $D_+ : S_- \to S_+$, $D_- : S_+ \to S_-$ the corresponding Dirac operators in Spin bundles.

We will do gauge theory with gauge group G (a compact Lie group). Let E_L, E_R be orthogonal, unimodular representations of G. Let P be a principal G-bundle on Σ, and let E_R, E_L denote the orthogonal vector bundles associated to the representations E_R, E_L. Let $\overrightarrow{\psi_+}, \overrightarrow{\psi_-}$ be sections of the bundles $E_R \otimes S_+$, $E_L \otimes S_-$.

The Lagrangian of a two-dimensional gauge theory with fermions is

$$\mathcal{L} = \int d^2x \left(\frac{1}{4e^2} | * F_A|^2 + \frac{1}{4\pi} \overrightarrow{\psi_+} D_-^A \overrightarrow{\psi_+} + \frac{1}{4\pi} \overrightarrow{\psi_-} D_+^A \overrightarrow{\psi_-} \right), \tag{6.1}$$

where A is a connection in P, and $D_\pm^A$ are the corresponding Dirac operators.

In order for this theory to make sense quantum mechanically, the representations E_L, E_R have to satisfy an additional condition. To derive this condition, recall that the partition function of (6.1) is given by

$$Z = \int DA \int D\overrightarrow{\psi_+} D\overrightarrow{\psi_-} e^{-\mathcal{L}}. \tag{6.2}$$

(in this integral, we sum over all topological types of principal bundles). The fermion integral is easy to compute: it equals $I_A = \mathrm{Pf}(D_-^A|_{E_R})\mathrm{Pf}(D_+^A|_{E_L})$, where

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Pf denotes the Pfaffian. The expression I_A is a section of the Pfaffian line bundle $B = PF(D_-^A|_{E_R})PF(D_+^A|_{E_L})$ on the space of gauge classes of connections. In order for the A-integral to make sense, this expression should be a function, i.e. the bundle B has to be trivial. One can show (using standard mathematical formulas for the curvature of the connection on the Pfaffian line bundle) that this boils down to the condition

$$\mathrm{Tr}(\rho_L(t)\rho_L(t')) = \mathrm{Tr}(\rho_R(t)\rho_R(t')), \tag{6.3}$$

where $t, t' \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Gauge group G, and $\rho_{L,R} : \mathfrak{g} \to SO(E_{L,R})$ are the representation maps. If G is simple, and E_L, E_R are irreducible, this condition means that the Casimirs of E_L, E_R are the same. Equation (6.3) is called the condition of anomaly cancellation. From a different point of view it was described (in the four-dimensional case) in Lecture 3.7.

Today we will consider a simple case: $G = U(1)$, and E_L, E_R are irreducible two-dimensional real representations. In this case (6.3) says that E_L, E_R are the same: $E_L = E_R = E$. We will take $E = \mathbb{C}$ with metric $|z|^2$, and $U(1)$ acting by multiplication (but remember that tensor products $S_\pm \otimes E$ are over $\mathbb{R}$). We decompose $S_\pm \otimes E$ in a direct sum of two two-dimensional representations. Using this decomposition, we will write $\overrightarrow{\psi}_\pm = (\psi_\pm, \overline{\psi}_\pm)$, where for $z \in U(1)$ one has $z(\psi, \overline{\psi}) = (z\psi, \overline{z}\overline{\psi})$.

In the new notation, the Lagrangian (6.1) has the form

$$\mathcal{L} = \int d^2x \left(\frac{1}{4e^2} | * F_A|^2 + \frac{1}{2\pi} \overline{\psi}_+ D_-^A \psi_+ + \frac{1}{2\pi} \overline{\psi}_- D_+^A \psi_- \right), \tag{6.4}$$

We can also add to this Lagrangian a topological term:

$$\mathcal{L}_\theta = \mathcal{L} - \frac{i\theta}{2\pi} \int F, \tag{6.5}$$

and a mass term:

$$\mathcal{L}_{\theta,m} = \mathcal{L}_\theta + \frac{m}{2\pi} \overline{\psi}_- \psi_+ + \frac{\overline{m}}{2\pi} \overline{\psi}_+ \psi_-, \tag{6.6}$$

(here m is complex).

§6.2. Chiral symmetry

Chiral symmetry is a $U(1)$-symmetry of the classical Lagrangian (6.4) or (6.5) given by

$$\psi_+ \to \psi_+, \psi_- \to e^{i\gamma} \psi_-. \tag{6.7}$$

This symmetry is violated by the mass term. However, even if there is no mass term, this symmetry may be violated quantum-mechanically, for the following topological reason.

Let the spacetime be a closed Riemann surface. Then under the chiral symmetry, the "measure" $\mu = D\psi_- D\overline{\psi}_-$ transforms as $\mu \to e^{i\gamma I}\mu$, where I is the index of the operator $D_+^A : S_- \otimes_{\mathbb{C}} E \to S_+ \otimes_{\mathbb{C}} E$. "Proof": Let $\{\psi_-\}$ denote the space

of sections of S_- of charge 1, and let $\{\overline{\psi}_-\}$ be the analogous space for charge -1. Then formally $I = dim(\{\psi_-\}) - dim(\{\overline{\psi}_-\})$; since $(\{\overline{\psi}_-\}) = (\{\psi_-\})^*$, we have $I = dim(\{\psi_-\}) - dim(\{\psi_+\}) = ind(D_+^A|_{S_+\otimes_{\mathbb{C}}E})$. Here *ind* denotes the index of the stated operator.

It is known that the index I equals $c_1(P) = \int \frac{F_A}{2\pi}$. Thus, since the bundle P may be nontrivial topologically, chiral symmetry is violated in the quantum theory.

This effect is called an anomaly: a classical symmetry does not hold quantum mechanically, because the measure is not invariant. The difference from spontaneous symmetry breaking, discussed in Lecture 1, is that in the case of spontaneous symmetry breaking, the classical symmetry does exist in the quantum theory, but is not realized. An anomaly in a symmetry means that the symmetry simply is not a symmetry of the quantum theory. To say it again – since it is a very important though elementary point – an anomalous symmetry is classical symmetry that is not a symmetry of the quantum theory, while a spontaneously broken symmetry is a symmetry of the quantum theory but does not leave invariant the vacuum. (Unlike spontaneous symmetry breaking, anomalies can occur even in one spacetime dimension.)

Let us discuss the mechanism of chiral symmetry breaking in the language of currents. Classically, chiral symmetry is generated by the current

$$J_A = \overline{\psi}_-\psi_- \tag{6.8}$$

(this is a one-form of type (0,1)), which is of course conserved: $dJ_A = 0$. Quantum mechanically, this differential equation may be deformed: $dJ_A = \mathcal{O}$, where $\mathcal{O}$ is some operator with values in two-forms on the surface. It is not hard to show by listing all possible operators that there is only one operator (up to a factor) that can arise: it is the curvature operator F. Thus, $dJ_A = \alpha F$, where α is a constant, and the previous topological computation shows that $\alpha = 1/2\pi$. One can also establish the relation $dJ_A = F/2\pi$ by studying Feynman diagrams; this is how it was first done.

When there is an anomalous symmetry, the correlation functions depend on θ in a very simple way. We will illustrate this statement in the context of the Lagrangian (6.5). Let $\mathcal{O}_i$ be any operators such that $\prod_i \mathcal{O}_i \to e^{in\gamma}\prod_i \mathcal{O}_i$ under chiral symmetry. Then the above index computation implies that all nontrivial contributions to the correlator $\langle\prod \mathcal{O}_i\rangle$ are from bundles P with $c_1 = n$. (This is the condition that the measure times $\prod_i \mathcal{O}_i$ is invariant under the symmetry.) So this correlator has the form $e^{in\theta}\langle\prod O_i\rangle_0$, where $\langle\prod O_i\rangle_0$ is θ-independent. For example, $\langle\psi_-\psi_+^*\rangle = Ce^{i\theta}$, where C is theta-independent.

If instead of (6.5) we consider the Lagrangian (6.6) with the mass term, then this argument shows that

$$\left\langle\prod \mathcal{O}_i\right\rangle(m,\theta) = e^{in\theta}\left\langle\prod \mathcal{O}_i\right\rangle_0(\widetilde{m}), \tag{6.9}$$

where $\widetilde{m} = me^{-i\theta}$. Thus, the really important parameter of the theory is $\widetilde{m}$, which we will write as $m_*e^{-i\theta_*}$, $m_* \geq 0$. Our goal in this lecture to study this theory as a function of $\widetilde{m}$.

§6.3. Behavior of two-dimensional gauge theory with massive fermions

Now we will describe the behavior of the theory defined by the Lagrangian (6.6), and later will justify our claims.

First of all, we will see that for large m_* (i.e. $m_* \gg e$) the theory is similar to the two-dimensional gauge theory with massive bosons, considered in the previous lecture. In particular, it has a mass gap. It also has a unique vacuum for $\theta \neq \pi$, and two of them for $\theta = \pi$. The discrete symmetry of parity or reversal of space orientation ($t \to t, x \to -x$), which acts by $m_* \to m_*$, $\theta_* \to -\theta_*$, is a symmetry for $\theta_* = 0$ or π. For large m_*, the parity symmetry is unbroken if $\theta_* = 0$ and broken if $\theta^* = \pi$.

Next, we will show that for $\widetilde{m} = 0$ the theory is in fact free, in fact it becomes a free massive theory after a change of variables. Thus the theory has a unique vacuum and a mass gap for small m_* (i.e. for $m_* << e$). Parity is unbroken for small m_* whether θ is 0 or π.

At $\theta = \pi$ and a critical value of m_* of order e, it is believed that the theory loses the mass gap; in fact it is believed that the theory at the critical value $\widetilde{m} = -m_c$ is in fact conformal, and isomorphic to the theory of a free real fermion. But the arguments for these assertions are not so precise.

§6.4. Heavy fermions

In this section we will study the case $m_* >> e$. In this case we can regard our theory as a perturbation of a theory with $e = 0$ with perturbation parameter $\lambda = e/m_*$. At $\lambda = 0$, we have a direct product of a two-dimensional pure gauge theory (which is free), and a free theory of massive fermions (to be safe here, we should introduce $B = A/e$; this makes sense, as for $e = 0$, only the trivial $U(1)$-bundle contributes to the path integral).

The situation here is similar to the bosonic case that we studied last week, and can be analyzed by similar arguments. Namely, the small e perturbation of the free theory for $e = 0$ is singular. The space of states of the deformed theory is actually smaller than that of the undeformed theory. More precisely, confinement of fermions takes place: the only allowed states (for $\theta \neq \pi$) are states of total charge 0 (here the charge of $\psi_\pm$ is 1 and of $\overline{\psi}_\pm$ is -1, and $\theta \in [0, 2\pi)$). In particular, the fermions can only exist in pairs, quadruples, and so on, and a single fermion cannot exist.

Like in the bosonic case, the theory has a mass gap by a deformation argument (the fact that the deformation is singular does not invalidate this argument, since the Hilbert space does not increase but only becomes smaller). More precisely, we have one realization for $\theta \neq \pi$, and two of them for $\theta = \pi$ (as in the pure gauge theory), and any realization has a mass gap.

§6.5. Bose-Fermi correspondence

Before studying the case of light fermions, we will consider the Bose-Fermi correspondence, which will be useful in studying the case of small m_*.

Consider two two-dimensional free field theories

1. The fermionic theory defined by the Lagrangian

$$\mathcal{L}_f = \frac{1}{2\pi}\int d^2x(\overline{\psi}_-\partial_+\psi_- + \overline{\psi}_+\partial_-\psi_+). \tag{6.10}$$

2. The bosonic theory defined by the Lagrangian

$$\mathcal{L}_b = \frac{R^2}{4\pi}\int d^2x|d\phi|^2, \tag{6.11}$$

These two theories are conformal (both classically and quantum-mechanically), and have Virasoro central charge 1. So we can suspect there may be some connection between them. And indeed, such a connection exists; it is called the Bose-Fermi correspondence.

Remark. Conformal field theories with small central charge are very scarce. For instance, for $c < 1$ they are completely classified by Friedan, Qiu, Shenker (Phys. Rev. Lett, vol 52, 1984, p. 1575). The answer is that there are no continuous parameters, and the theory is almost completely determined by c, which can take only a discrete sequence of values. For $c = 1$, such a classification is unavailable, but very few examples are known, and all of them have a construction in terms of a free Bose field.

Let us now take a look at the bosonic theory (6.11). As we remember from Lecture 1, in order for this theory to make sense, ϕ has to be angle-valued, i.e. take values in the circle $\mathbb{R}/2\pi\mathbb{Z}$. The constant R in the Lagrangian has the meaning of the radius of this circle.

Remark. We always model the target circle as $\mathbb{R}/2\pi\mathbb{Z}$, but consider various Riemannian metrics on it, which are parametrized by values of the radius R.

Recall from Lecture 1 that the Hilbert space of this theory (in its unique realization) is of the form $\mathcal{H}_b = \oplus_{k\in\mathbb{Z}}(F_b \otimes F_b^*)_k$, where F_b is the bosonic Fock space. The operator algebra A_b of the theory is generated by the derivatives of ϕ (but not ϕ itself), and $:e^{in\phi}:$ (for brevity presently we will drop the colons). The operator product expansion is given by formula (3.9) of Lecture 3 of [**I-Perturbative QFT**], where $D(x-y) = -R^{-2}\ln|x-y|$. The action of A_b in $\mathcal{H}_b$: Derivatives of ϕ do not change k, and $:e^{in\phi}:$ maps $(F_b \otimes F_b^*)_k$ to $(F_b \otimes F_b^*)_{k+n}$.

Now consider the theory (6.10). The Hilbert space of this theory is $\mathcal{H}_f = F_f \otimes F_f^*$, where F_f is generated from the vacuum by holomorphic operators $\psi_+, \overline{\psi}_+$, and F_f^* is generated from the vacuum by $\psi_-, \overline{\psi}_-$. The operator algebra A_f is generated by $\psi_+, \overline{\psi}_+, \psi_-, \overline{\psi}_-$, with the standard OPE of the free theory.

Consider more closely the operator $:e^{in\phi}:$ for $n \in \mathbb{Z}$. As we know (Lecture 3 of [**I-Perturbative QFT**] and Lecture 1; the result follows from exponentiating the formula we just wrote for $D(x-y)$), this operator has holomorphic dimension and antiholomorphic dimension $n^2/4R^2$ (the sum of left and right dimensions is $n^2R^2/2$, as we saw in Lecture 3 of [**I-Perturbative QFT**]).

Classically, the operator $e^{in\phi}$ locally factors as a product of a holomorphic one and an antiholomorphic one: $e^{in\phi} = e^{in\phi_+}e^{in\phi_-}$, where $\phi = \phi_+ + \phi_-$, and $\partial_-\phi_+ = \partial_+\phi_- = 0$. (Of course, here ϕ_+, ϕ_- are defined only up to adding a constant). It is not obvious that one can define operators $e^{in\phi_\pm}$, and such operators

are not present in the algebra A_b. But imagine for a second that the operators $e^{in\phi_\pm}$ make sense. Then we will find from the OPE for $e^{i\phi}$ that $e^{i\phi_+}$ has holomorphic dimension $1/4R^2$ and antiholomorphic dimension 0, and

$$\langle e^{i\phi_+(x_1)}\dots e^{i\phi_+(x_n)}e^{-i\phi_+(y_1)}\dots e^{-i\phi_+(y_n)}\rangle = \frac{\prod_{1\le i<j\le n}(x_i-x_j)^{1/2R^2}\prod_{1\le i<j\le n}(y_i-y_j)^{1/2R^2}}{\prod_{1\le i,j\le n}(x_i-y_j)^{1/2R^2}}. \tag{6.12}$$

where x_i, y_j are viewed as complex numbers. From this formula it is clear that in order for $e^{i\phi_+}$ to make any sense, we need $1/2R^2 \in \mathbb{Z}$, so that the function on the R.H.S. of (6.12) is single-valued. Similarly, in order for $e^{in\phi_+}$ to make sense, $n^2/2R^2$ has to be an integer.

Let us consider the first case where $e^{in\phi_+}$ can make sense, i.e. $R = 1/\sqrt{2}$ and $n = 1$. In this case, we have

$$\langle e^{i\phi_+(x_1)}\dots e^{i\phi_+(x_n)}e^{-i\phi_+(y_1)}\dots e^{-i\phi_+(y_n)}\rangle = \frac{\prod_{1\le i<j\le n}(x_i-x_j)\prod_{1\le i<j\le n}(y_i-y_j)}{\prod_{1\le i,j\le n}(x_i-y_j)}. \tag{6.13}$$

It is clear from (6.13) that $e^{i\phi_+}$ behaves like a fermion; more generally, $e^{in\phi_+}$ behaves like a fermion when n is odd, and like a boson when n is even. This makes us hope that at $R = 1/\sqrt{2}$ our theory is related to the theory of fermions.

Let us see why this is indeed the case. Let us compute the fermionic correlation function $\langle\psi_+(x_1)\dots\psi_+(x_n)\overline{\psi}_+(y_1)\dots\overline{\psi}_+(y_n)\rangle$. Using Wick's formula, it is easy to find that

$$\langle\psi_+(x_1)\dots\psi_+(x_n)\overline{\psi}_+(y_1)\dots\overline{\psi}_+(y_n)\rangle = \det\left(\frac{1}{x_i-y_j}\right). \tag{6.14}$$

The coincidence of the right hand sides of (6.13),(6.14) is a famous combinatorial identity, which follows from comparison of zeros and poles, and asymptotics at infinity.

Remark. In fact, the explicit form of (6.13),(6.14) is not relevant to the proof of the fact that they are equal. What is relevant is only the structure of zeros and poles, and the asymptotics at infinity, which in both cases are obvious from the OPE.

Let $\widehat{A}_b$ be the operator algebra generated by A_b and $e^{in\phi_\pm}$. Equalities (6.13), (6.14) show that we have a homomorphism $\xi : A_f \to \widehat{A}_b$ defined by $\xi(\psi_\pm) = e^{\pm i\phi_\pm}$, $\xi(\overline{\psi}_\pm) = e^{\mp i\phi_\pm}$, which preserves expectation values.

Using the OPE, it is easy to find

$$\xi(:\overline{\psi}_+\psi_+:) = \lim_{\varepsilon\to 0}\xi(\overline{\psi}_+(x+\varepsilon)\psi_+(x) - \frac{1}{\varepsilon}) = \lim_{\varepsilon\to 0}(e^{-i\phi_+}(x+\varepsilon)e^{i\phi_+}(x) - \frac{1}{\varepsilon}) = -i\partial_+\phi. \tag{6.15}$$

Likewise, $\xi(:\overline{\psi}_-\psi_-:) = i\partial_-\phi$. Also, $\xi(\psi_+\overline{\psi}_-) = e^{i\phi}$, $\xi(\psi_-\overline{\psi}_+) = e^{-i\phi}$. In fact, it is not difficult to see that this homomorphism is an isomorphism.

At the level of Hilbert spaces, ξ induces a bigraded isomorphism $F_f \to F_b \otimes l_2(\mathbb{Z})$ (bidegree=(quantum scaling dimension, charge)), where the charge of $e^{i\phi_+}$ and the charge of ψ_+ equal 1. Writing the corresponding character formula, we obtain

$$\frac{\sum_{n\in\mathbb{Z}} q^{n^2/2} z^n}{\prod_{n\geq 1}(1-q^n)} = \prod_{n\geq 1}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}), \tag{6.16}$$

which is the famous Jacobi triple product identity. (Here the left hand side is the partition function of the left-moving bosons, and the right hand side is the analogous partition function for fermions, using the bounding spin structure on the circle. A more careful analysis than we have given is needed to show *a priori* that that is the right spin structure to use in comparing to the bosons.)

The correspondence ξ is called the Bose-Fermi correspondence.

Remark 1. If the radius of the circle is not $1/\sqrt{2}$, but 1, then the operator $e^{i\phi_+}$ is meaningless, as its 2-point function would be $(x-y)^{-1/2}$, which is not single-valued. However, the operator $e^{2i\phi}$ is defined, and its 2-point function is $\frac{1}{(x-y)^2}$. This indicates that $e^{2i\phi}$ behaves like a current of some symmetry. And indeed, it turns out that the corresponding model is equivalent to the $\widehat{SU(2)}$-WZW model with Kac-Moody central charge 1, so it has an $SU(2)$-symmetry. In fact, the Fourier components of the operators $e^{\pm 2i\phi_+}$ and $\partial_+\phi$ generate the left-moving $\widehat{SU(2)}$, and the Fourier components of the operators $e^{\pm 2i\phi_-}$ and $\partial_-\phi$ generate the right-moving $\widehat{SU(2)}$.

Remark 2. In fact, the Bose-Fermi correspondence is true not only locally (at the level of operator product expansions), but also globally (at the level of the path integral on an arbitrary Riemann surface). An elegant global version of the Bose-Fermi correspondence is the following. For any closed Riemann surface Σ one has the identity of partition functions

$$\int D\phi\, e^{-\frac{1}{8\pi}\int |d\phi|^2} = \sum_{\varepsilon} \int D\psi_+ D\overline{\psi}_+ D\psi_- D\overline{\psi}_- e^{-\mathcal{L}_f(\psi)}, \tag{6.17}$$

where ϕ is understood as a map of Σ to the circle, $\mathcal{L}_f$ is the Lagrangian given by (6.10), and ε runs over spin structures on Σ (the same spin structure is taken for ψ_+ and ψ_-). There is a similar identity for correlation functions of operators, if the correspondence between operators is made as explained above.

§6.6. Bose-Fermi correspondence for nonlinear theories

We have established a correspondence between two free theories – the theory of a boson and the theory of fermions. Since we know how to identify operators on one side with operators on the other side, we can compare non-free theories made by perturbing the bosonic theory by adding a given operator to the Lagrangian to non-free theories made by perturbing the fermionic theory by the corresponding operator. This leads to many striking results. We consider several examples.

1. Recall that under our correspondence $\mathcal{L}_f \to \mathcal{L}_b(1/\sqrt{2})$, where $\mathcal{L}_b(R)$ is given by (6.11). Since $\xi(\overline{\psi}_\pm\psi_\pm) = \mp i\partial_\pm\phi$, we find

$$\mathcal{L}_f + \frac{1}{2\pi}\int d^2x (g\overline{\psi}_+\psi_+\overline{\psi}_-\psi_-) \to \mathcal{L}_b\left(1/\sqrt{2}(1+g)^{-1/2}\right). \tag{6.18}$$

This shows that, to our surprise, the theory with the Lagrangian $\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(g\overline{\psi}_+\psi_+\overline{\psi}_-\psi_-)$ is free. In particular, its β-function is zero. This is obvious when the theory is described in Bose variables, but not obvious in Fermi variables.

2. On the other hand, consider the theory of free massive fermions, with the Lagrangian $\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(m\overline{\psi}_-\psi_+ + \overline{m}\overline{\psi}_+\psi_-)$. Using the fact that $\xi(\overline{\psi}_\pm\psi_\mp) = -e^{\mp\imath\phi}$, we get that under ξ, this free Lagrangian goes to

$$\mathcal{L}_b\left(1/\sqrt{2}\right) - \frac{1}{2\pi}\int d^2x(me^{i\phi} + \overline{m}e^{-i\phi}). \tag{6.19}$$

So we get another surprising fact that the nonlinear theory defined by (6.19) is in fact free.

Remark. It may appear that the second term in (6.19) has a wrong scaling dimension. This is not the case, because the operator $e^{\pm\imath\phi}$ has anomalous dimension 1. More precisely, (6.19) does not fix a theory but fixes a family of theories depending on a scale μ of momenta, which is introduced when the operators $e^{\imath\phi}$ are renormalized. This scale enters in front of the corresponding term in the Lagrangian and cancels the discrepancy in dimensions.

Using the symmetry $\phi \to \phi + \theta$ we can reduce (6.19) to the case of real m. In this case, (6.19) looks like

$$\mathcal{L}_b\left(\sqrt{2}\right) - \frac{m}{\pi}\int d^2x\cos\phi. \tag{6.20}$$

Thus, the classical equation of motion is $\Delta\phi = -4m\sin\phi$. This equation is called the sine-Gordon equation, and it is a well-known completely integrable soliton equation. The quantum theory is also integrable (in the sense we learned about in Lecture 4). Of course, at the value of the coupling where the quantum theory is equivalent to the theory of a free massive fermion, it is not just integrable but free.

Now consider the Lagrangian

$$\mathcal{L}_b(R) - \frac{m}{\pi}\int d^2x\cos\phi. \tag{6.21}$$

This Lagrangian is proportional to (6.20), with $m' = m/2R^2$, so classically the two Lagrangians are equivalent. However, quantum mechanically, this is not the case, as multiplying the Lagrangian by a constant definitely affects the quantum theory. In fact, the theory now essentially depends on R. If $R = 1/\sqrt{2}$, the theory is free. For a general R it is not free. The map ξ shows that for a general R the theory is equivalent to the fermionic theory with the Lagrangian

$$\mathcal{L}_f + \int d^2x(m\overline{\psi}_-\psi_+ + m\overline{\psi}_+\psi_- + g\overline{\psi}_+\psi_+\overline{\psi}_-\psi_-), \tag{6.22}$$

where $R = 1/\sqrt{2}(1+g)^{-1/2}$.

As we mentioned, the theory described by the Lagrangian (6.21) is not free for $R \neq 1/\sqrt{2}$, but is integrable. Its S-matrix can be computed using methods we discussed in Lecture 4. Solvability of this theory for large R (i.e. in the classical

limit) is related to the complete integrability of the sine-Gordon equation at the classical level.

Now we want to apply the Bose-Fermi correspondence to gauge theory. This will give results that are rigorous and precise if m is small or zero. Consider the Lagrangian $\mathcal{L}_{\theta,m}$ given by (6.6). Define

$$\mathcal{L}^f_{\theta,m} = \mathcal{L}_{\theta,m} + \frac{1}{2\pi}\int d^2x g\overline{\psi}_+\psi_+\overline{\psi}_-\psi_-. \tag{6.23}$$

Let us rewrite it in Bose variables:

$$\mathcal{L}^b_{\theta,m} = \int d^2x\Big(\frac{1+g}{8\pi}|d\phi|^2 - \frac{m}{2\pi}e^{i\phi} - \frac{\overline{m}}{2\pi}e^{-\imath\phi} + A_+\Big(\frac{i\partial_-\phi}{2\pi}\Big) + A_-\Big(\frac{-i\partial_+\phi}{2\pi}\Big) + \frac{|*F|^2}{4e^2} - \frac{i\theta}{2\pi}F\Big). \tag{6.24}$$

We integrate by parts the term $\int(A_-\partial_+\phi - A_+\partial_-\phi)$, and it equals $\int\phi(\partial_-A_+ - \partial_+A_-)$. The expression $\partial_-A_+ - \partial_+A_-$ equals the curvature F, so (6.24) is simplified:

$$\mathcal{L}^b_{\theta,m} = \int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2 + \frac{m}{2\pi}e^{\imath\phi} + \frac{\overline{m}}{2\pi}e^{-\imath\phi} + \frac{|*F|^2}{4e^2} - \frac{i(\phi+\theta)}{2\pi}F\right). \tag{6.25}$$

The ϕF term globally must really be defined in the sense of a Chern-Simons functional (as ϕ is a map to a circle rather than being real-valued), but it will not be necessary for us to be careful with that.

From this equation it is clear that when $m = 0$, the theory is free (as the Lagrangian is quadratic), and there is no essential θ-dependence (θ can be eliminated by shifting ϕ by $-\theta$). We can also now see that the theory at (or near) $m = 0$ has a mass gap. For this, we need only solve the classical equations. For a theory such as this one with a quadratic Lagrangian, analysis of the classical equations suffices to determine what happens quantum mechanically. I leave it to the reader to study the classical equations derived from the above Lagrangian and to show that ϕ obeys a massive wave equation.

For $m \neq 0$, by changing ϕ to $\phi+\theta$ we find that the theta-angle can be absorbed in m. This was argued based on index theory and anomalies at the beginning of the lecture. Let us now look at the dynamics for $m \neq 0$ and small. It follows from Lecture 5 that for an external field $\phi(x)$ that is approximately constant

$$\int DA\, e^{\frac{\imath}{2\pi}\int\phi F_A - \frac{1}{4e^2}\int|*F_A|^2} = Ce^{-\frac{e^2}{2\pi}\int d^2x \min_n\left(n-\frac{\phi(x)}{2\pi}\right)^2}, \tag{6.26}$$

Therefore, the effective Lagrangian for ϕ for the Lagrangian (6.25) is

$$\mathcal{L}^{\theta_*,m_*}_{\text{eff}} = \int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2 - \frac{m_*}{\pi}\cos(\phi-\theta_*) + \frac{e^2}{2\pi}\min_n\left(n-\frac{\phi}{2\pi}\right)^2\right). \tag{6.27}$$

From this formula, we can analyze the θ dependence for small m_*. The potential

$$U(\phi) = -\frac{m_*}{\pi}\cos(\phi-\theta_*) + \frac{e^2}{2\pi}\min_n\left(n-\frac{\phi}{2\pi}\right)^2 \tag{6.28}$$

has a unique global minimum (modulo 2π) with positive second derivative for all θ if m_* is sufficiently small, and in this regime, the physics is smooth as a function of θ.

By contrast, in the large m_* analysis of the original gauge theory, we found a discontinuity at $\theta = \pi$. We cannot really compare this result to what we learn from (6.28), because the description by the ϕ field becomes strongly coupled if m_* is not small compared to e, so we really can only make firm statements based on this description if m_* is small.

Nevertheless, if we boldly, and without firm justification, assume that the quantum theory can be treated using the effective potential $U(\phi)$ even if m_* is large, then we get an attractive picture. For large m_*, the minimization of $U(\phi)$ gives a discontinuity at $\theta = \pi$, as found from direct study of the gauge theory. At a critical value of m_*, the discontinuity vanishes. At this point, the minimum of $U(\phi)$ is degenerate (the second derivative is zero). Such a degenerate critical point of a classical potential would be expected to correspond in quantum theory to a theory that flows to a nontrivial fixed point in the infrared. In fact, it is believed that the generic quantum behavior in two dimensions in varying one parameter to get such a critical point beyond which a $\mathbb{Z}/2\mathbb{Z}$ symmetry is spontaneously broken is that of the "Ising model," which is equivalent to the theory of a massless free real fermion. This is the basis for a conjecture that was stated at the end of §6.3.

References and cross references

Two-dimensional gauge theory is discussed in Lecture 4. The free theory of an angle-valued field in 2 dimensions is discussed in Lecture 1 of [II-**CFT**] and Lecture 1 of this series. The WZW model is discussed in Lecture 4 of [II-**CFT**].

For more on the model studied in this lecture, see

S. Coleman, *More about the massive Schwinger model*, Annals of Physics **101** (1976), 239–267.

See also, for example, the books of Coleman and Weinberg, cited in the references to earlier lectures, for more about θ dependence and anomalous symmetries.

LECTURE 7
Two-Dimensional Gauge Theory of bosons, the Wilson Line Operator, and Confinement

In the last lecture we considered two-dimensional gauge theories with fermions. Today we will return to two-dimensional gauge theories with bosons, which we learned something about in Lecture 5. As before, we will work with Euclidean Lagrangians.

§7.1. Infrared behavior of $U(1)$ gauge theories with bosons in two dimensions

We consider a $U(1)$ gauge theory with bosons in 2 dimensions.

Our spacetime is a Riemann surface Σ. Our fields are A – a $U(1)$ connection in some line bundle L – and $\phi_1, \dots, \phi_N$ – complex scalar fields, which are sections of L.

Our Lagrangian is

$$\int d^2x \left(\frac{|*F|^2}{4e^2} + \frac{1}{2}|d_A\phi|^2 + \frac{\lambda}{4}(|\phi|^2 - a^2)^2 \right) - \frac{i\theta}{2\pi}\int F, \tag{7.1}$$

where $|\phi|^2 = \sum |\phi_i|^2$, $|d_A\phi|^2 = \sum |d_A\phi_i|^2$.

We want to understand the infrared behavior of this theory.

Classically, the "mass" of the bosons obtained by quantizing the ϕ_i is $m^2 = -\lambda a^2$. The parameters of the theory are m, θ, λ, e (with $\lambda > 0$). We will now study the behavior of the system for different ranges of parameters.

First consider the situation $m^2 \to +\infty$ with fixed λ, e. In this situation the quartic term in the Lagrangian becomes unimportant, and the model essentially reduces to one studied in Lecture 5, with massive charged bosons that interact only via the gauge fields, to which they are weakly coupled. The theory has a mass gap, and the interaction between bosons is approximated by the one-dimensional Coloumb potential, which causes their confinement: all states in the theory are of total charge zero. Also, the space reversal symmetry is preserved for $\theta = 0$, but broken for $\theta = \pi$, thus producing two vacua. The vacuum, as a function of m and θ, has a "cut" or discontinuity at $\theta = \pi$. For example, the vacuum energy density, which is a 2π-periodic function of θ given for $-\pi \le \theta \le \pi$ by formula (7.2) below, has a discontinuous derivative at $\theta = \pi$.

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Now consider the situation when $m^2 \to -\infty$ (again with fixed λ, e). In this case, the situation is totally different: because of the presence of the quartic term, the classical potential for bosons has a minimum at $|\phi|^2 = a^2$, so the space of minima is S^{2N-1}. The gauge group $U(1)$ acts freely on this space, so the space of classical ground states is $\mathbb{C}P^{N-1} = S^{2N-1}/U(1)$. Thus, the low energy effective theory is the theory of maps from Σ to $\mathbb{C}P^{N-1}$ with the round metric (the sigma model).

Remark. In Lecture 4 we saw that the theory with the Lagrangian density $\frac{1}{2}(d\phi)^2 + \lambda(\phi^2 - a^2)^2$ flows in the infrared to the sigma model of maps to the sphere. We remarked that this means that this (superrenormalizable) theory is a good cutoff for the sigma model to the sphere. Similarly, the superrenormalizable (and hence rigorously existing) theory defined by the Lagrangian (7.1) is a good UV cutoff for the theory of maps to $\mathbb{C}P^{N-1}$.

We have already studied the $\mathbb{C}P^{N-1}$ model for large N in Lecture 5; we found that it behaves in the infrared as the gauge theory of N massive bosons. Today we will consider in more detail the opposite extreme, in fact the case $N = 1$. In this case $\mathbb{C}P^{N-1}$ is one point, so the theory defined by (7.1) has a mass gap and a unique vacuum. The question is to understand the properties of physics in that vacuum. Though some of our usual questions are trivial for $N = 1$, we will encounter completely new questions and methods.

§7.2. The vacuum energy density

Recall that for $m^2 \gg e^2$ in Lecture 5 we found the following value of the vacuum energy density:

$$E_{\text{vac}}(\theta) = \frac{e^2}{2}\left(\frac{\theta}{2\pi}\right)^2. \tag{7.2}$$

So the θ-derivative of the energy density is proportional to the first power of the coupling e^2.

In the case $m^2 \ll -e^2$, the situation is totally different. Namely, now the contributions to the path integral from nontrivial line bundles (relative to infinity) is exponentially small in e^2, since any section of such a bundle has to vanish somewhere; this vanishing implies that the action is big due to the presence of the quartic potential. Thus, to any finite order of perturbation theory (in e^2), the path integral, and hence the vacuum energy density, are independent of θ. More precisely, the theta-dependence is exponentially small, and comes from the sum over *instantons* – lowest action configurations of nontrivial first Chern class.

§7.3. Instantons

Let us compute the coefficient of $e^{i\theta}$ in the Fourier expansion of the path integral with Lagrangian (7.1). This is equivalent to taking the path integral over connections in line bundles with $c_1 = 1$.

As usual, the main contribution to the path integral comes from field configurations which are close to the configuration minimizing the action. Such a configuration (ϕ, A) is called an instanton.

Let (ϕ, A) be an instanton. In order for its action to be finite, we must have

$$|\phi(x)| \to a, F_A(x) \to 0 \text{ as } x \to \infty, \tag{7.3}$$

where F_A is the curvature of A. Moreover, since ϕ is classically massive, this convergence is exponentially fast. Thus, an instanton in our problem has to be a highly localized field configuration.

In our further discussion of instantons, we will assume that $\Sigma = \mathbb{R}^2$, and c_1 is measured with respect to a trivialization at infinity. In this case, we expect that instantons are rotationally symmetric, with respect to some center of rotation x_0. Without loss of generality we can assume that $x_0 = 0$, so that the instanton has the form $\phi = ae^{i\alpha}f(r)$, $A = g(r)d\alpha$, where $f(r)$, $g(r)$ are some functions of the radius r (where r, α are polar coordinates on $\mathbb{R}^2$). It is easy to show that in this case one must have $f(0) = 0$, $f(\infty) = 1$, and $g(0) = 0$, $g(\infty) = 1$. Thus, we get a boundary value problem for ordinary differential equations. It can be proved by considering the corresponding ODE that this problem has a unique solution. Thus, the instanton is unique, up to translations. In particular, it has a definite size, and its action is a well-defined positive constant I.

Remark. In this respect, our instanton is different from the instantons of the $\mathbb{C}P^{N-1}$ model, which could be transformed by any conformal automorphism, and therefore had no definite size.

By dimensional analysis (rescaling of space, of ϕ, and of I) it is possible to show that I is of the form $e^{-2}m^2h(\lambda/e^2)$. Hence, if e is taken to zero with fixed m^2 and λ/e^2, the action is of order $1/e^2$. In this limit, instanton contributions are exponentially small.

§7.4. Instanton gas

Now we want to understand how to compute the contribution of the instantons to the path integral, and when such a computation gives a good approximation.

First of all consider line bundles with $c_1 > 1$. It can be shown that there is no instanton in such a topological class if e^2/λ is sufficiently large (but there is one if e^2/λ is sufficiently small). We will assume that there is no instanton for $c_1 > 1$, but what we will say can be generalized to the case when there is one (one simply has additional objects that must be included in the dilute instanton gas that we will describe). In the case that there is no instanton with $c_1 > 1$, the problem of minimization of the action in the case $c_1 = 2$ has no global minimum (the infimum is not attained). However, we can consider approximate instantons, whose action approaches the infimum arbitrarily closely. More precisely, if we take $A(x) = A_*(x - x_1) + A_*(x - x_2), \phi(x) = \phi_*(x - x_1)\phi_*(x - x_2)/a$, where (A_*, ϕ_*) is the instanton centered at 0, then the action of (A, ϕ) equals $2I$, plus a correction which is of order $e^{-c|x_1 - x_2|}$, where c is a positive constant. Thus, the infimum of the actions for $c_1 = 2$ is $2I$ if there is no $c_1 = 2$ instanton.

Similarly, the infimum of the actions for $c_1 = n$ is nI, for any $n \in \mathbb{Z}_+$. This follows from the fact that the action of the field configuration

$$\left(\sum_{i=1}^{n} A_*(x - x_i), a\prod_{i=1}^{n} \frac{\phi_*(x - x_i)}{a}\right)$$

has action which is exponentially close to nI when $|x_i - x_j|$ are big.

Now consider the situation when $c_1 < 0$. It is easy to see that $(-A_*, \overline{\phi}_*)$ is the instanton for $c_1 = -1$. It is called the anti-instanton. Thus, the situation for $n < 0$ is symmetric to the situation for $n \in \mathbb{Z}_+$: the field configuration

$$\left(-\sum_{i=1}^{|n|} A_*(x - y_i), a \prod_{i=1}^{|n|} \frac{\overline{\phi}_*(x - y_i)}{a} \right)$$

has action exponentially close to $|n|I$ when $|y_i - y_j|$ is big.

More generally, one can consider field configurations

$$\left(\sum_{i=1}^{n} A_*(x - x_i) - \sum_{j=1}^{m} A_*(x - y_j), a \prod_{i=1}^{n} \frac{\phi_*(x - x_i)}{a} \prod_{j=1}^{m} \frac{\overline{\phi}_*(x - y_j)}{a} \right),$$

with $c_1 = n - m$ which have action exponentially close to $(n + m)I$ when x_i, y_j are distant from each other. Such a configuration consists roughly of n instantons at points x_i, and m anti-instantons at points y_j.

Remark. Such field configurations make up a so-called "instanton gas." Next week, in a different kind of example (the Polyakov model in $2 + 1$ dimensions), we will meet an "instanton gas" with long range Coulomb forces between the particles, analogous to the situation in a plasma. In the present example, the interactions between instantons are exponentially small at long range; this is more like an ordinary gas of atoms, for instance the air in the atmosphere. It is an almost ideal gas; an ideal gas is the case in which the forces are exactly zero. Any real gas (hydrogen, oxygen) behaves as an almost ideal gas if the density is small enough, because then the particles are generally at long distances where the interactions are small. That is the case for the instantons in this model because the instanton action is big (and the instantons have a definite size). We will presently see how to estimate the density in the instanton gas; it will be exponentially small for small e (with fixed $m^2, \lambda/e^2$).

§7.5. Summing over instantons

As we remarked, the perturbation series with respect to powers of e^2 for the path integral with Lagrangian (7.1) does not involve contributions from instantons, as they are exponentially small. Thus, in perturbation theory there is no dependence at all on θ. Let us introduce a refined perturbation series, which will take instantons into account. For this purpose we will work on a two-dimensional spacetime of volume V, and introduce a new perturbation parameter $W = Ve^{-I}$ that will be characteristic of θ-dependent terms. In this refined perturbation expansion, in addition to constructing perturbation theory around the trivial solution ($A = 0, \phi =$ constant) we also expand around configurations of the instanton gas.

In ordinary perturbation theory, one constructs a partition function that has a form e^{VP_0}, where P_0 is a function of the coupling that can be computed in perturbation theory. The exponential dependence on the volume has a particularly simple intuitive explanation in a theory such as this one that has a mass gap. Consider evaluating Feynman diagrams in *position space* rather than, as is more

usual, in momentum space. Evaluating a Feynman diagram in position space means that one integrates over the positions in space of the vertices, and one evaluates the propagators in position space. This is the representation of Feynman diagrams that appears most directly when Feynman diagrams are derived from a path integral (though to calculate Feynman diagrams one usually works in momentum space).

Given a mass gap, all propagators vanish exponentially at long distances. So in a connected Feynman diagram evaluated in position space, all vertices will be in a bounded region of space (of order $1/m'$ with m' the smallest mass), apart from exponentially small effects. The "center of mass" of the connected Feynman diagram can be anywhere in space, and integrating over its position gives a factor of V for every connected Feynman diagram (with corrections that arise, for example, if the center of mass is very near the boundary of space). In the usual way, summing over possibly disconnected diagrams gives the exponential of the connected contribution, so the partition function behaves as e^{VP_0} where P_0 is the contribution of connected diagrams.

Now, consider the case that an instanton is present. Again we evaluate the diagrams in position space. If the center of mass of the vertices in a Feynman diagram is not within a distance of order $1/m'$ of the instanton, the presence of the instanton does not modify the evaluation of the diagram. If it is within a distance of order $1/m'$, the perturbation expansion is completely different from what it would be in the absence of the instanton. So we get for the perturbation series in the presence of the instanton the "vacuum" result (without the instanton) times a factor, which we will call P_+, which depends on the modification of the perturbation theory near the instanton. The partition function with a single instanton thus behaves as

$$Z_1 = e^{VP_0} e^{-I} V P_+ e^{i\theta},$$

where P_0 and P_+ are to be computed in perturbation theory, I is the instanton action, the factor of V multiplying P_+ comes from integrating over the position of the instanton, and we have included the θ-dependence. Note the appearance of $W = Ve^{-I}$.[15] The one-instanton contribution is negligibly small compared to the "vacuum" contribution if $W << 1$. If one asks, "how large must V be so that there is a reasonable probability to see an instanton?" the order of magnitude is $V \sim e^I$, so that $W \sim 1$ and the one-instanton contribution is comparable to the partition function computed without instantons. The density of instantons per unit volume is thus of order e^{-I}, and is exponentially small for $e \to 0$.

In a similar way, the path integral with a single anti-instanton behaves as $e^{VP_0} e^{-I} V P_- e^{-i\theta}$, where P_- reflects the modification of the perturbation expansion near the anti-instanton. In the present theory, because invariance under reversal of orientation is broken only by the θ term, one has $P_+ = P_-$. (In general one has only $P_- = \overline{P}_+$.) But we will for a time carry P_+ and P_- as separate constants.

Now consider a configuration of m instantons at points x_i and n anti-instantons at points y_j. We will suppose that x_i and y_j are all very far apart from each other compared to $1/m'$. This is a reasonable assumption, because we have seen that

[15]The correction to the perturbation series due to the presence of an instanton is additive for a connected Feynman diagram, while for not necessarily connected diagrams it is multiplicative. In the formula for Z_1, P_+ is computed by evaluating the components of the diagram that are close to the instanton, and the multiplicative factor e^{VP_0} arises by exponentiating the contribution of components that are not close to the instanton.

the density of instantons is exponentially small. Then, the vacuum path integral is modified by a factor of P_+ for each instanton and P_- for each anti-instanton; the fact that the Feynman diagrams are essentially local in space and the instantons and anti-instantons are all far apart means that their contributions can be calculated independently. After integrating over the positions, we get a factor of $W = Ve^{-I}$ for each instanton or anti-instanton, together with a factor of $e^{i\theta}$ for instantons or $e^{-i\theta}$ for anti-instantons.

Thus, the approximation to the partition function obtained this way has the form

$$Z(e^2,\theta,W) = e^{VP_0}\sum_{n,m=0}^{\infty}\frac{W^{m+n}P_+^m P_-^n e^{i\theta(n-m)}}{m!n!}, \tag{7.4}$$

(We divide by $m!n!$ because the m instantons and the n anti-instantons are identical, so in integrating over their positions we do not quite get an independent factor of V for each instanton or anti-instanton.)

Summing (7.4) by ordinary calculus, we get

$$Z(e^2,\theta,W) = e^{VP_0+W(P_+e^{i\theta}+P_-e^{-i\theta})}. \tag{7.5}$$

Setting $P_+ = P_- = P$, we get

$$Z(e^2,\theta,W) = e^{VP_0+2WP\cos\theta}. \tag{7.6}$$

Now we can compute the energy density of the vacuum in this approximation. This can be done from the equality $Z = e^{-VE_{\rm vac}}$, where $E_{\rm vac}$ is the vacuum energy density. Thus,

$$E_{\rm vac} = -P_0 - 2Pe^{-I}\cos\theta. \tag{7.7}$$

As we expected, the θ dependence of the vacuum energy is exponentially suppressed, but we have been able to compute the main term of this dependence. Corrections coming from effects we have ignored such as "interactions" between the instantons (the fact that their separations will not always be large compared to $1/m'$, for instance) give even smaller contributions to the θ dependence of $E_{\rm vac}$ than what we have just computed.

To what extent can one improve upon this estimate of the θ dependence of $E_{\rm vac}$? There is no difficulty, in principle, in computing P in a perturbation series in e^2 to any required accuracy. However, to compute terms in the vacuum energy proportional to $\cos(2\theta)$, for example, one must do an accurate computation for $c_1 = 2$. There is no difficulty with this if there is an instanton with $c_1 = 2$ and action I' such that $I' < 2I$. Then $E_{\rm vac}$ has a term $\cos(2\theta)e^{-I'}P'$ that can be analyzed just as above. If, however, there is no such $c_1 = 2$ instanton, then $E_{\rm vac}$ has a term roughly $\cos(2\theta)e^{-2I}$ that is rather hard to calculate, coming from interactions between two instantons and in particular from the fact that their separation may not be much greater than $1/m'$.

We recall that for $m^2 >> 0$, we had approximately $E_{\rm vac} = e^2\theta^2/8\pi^2$ for $|\theta| \leq \pi$, and in particular, $E_{\rm vac}$ was a non-smooth function of θ (with a cut at $\theta = \pm\pi$). By contrast, in the case $m^2 << 0$ the function $E_{\rm vac}(\theta) = -P_0 - 2Pe^{-I}\cos\theta$ is smooth.

(It is true that we have only computed $E_{\text{vac}}(\theta)$ in a leading approximation, but the weakness of the interactions, the diluteness of the instanton gas and the fixed size of the instantons will make the exact function just as smooth as the approximate function that we calculated.) Thus, for $m^2 << 0$ there is no cut at $\theta = \pm\pi$, and there is a mass gap and a unique vacuum for any θ.

Thus, the cut at $\theta = \pm\pi$ has to end at some point $m^2 = m_c^2 \in \mathbb{R}$. At this point, it is plausible that the theory can be described by an effective potential with a quartic critical point, so that the theory has no mass gap. Such issues were discussed at the end of Lecture 6; it is plausible that an effective potential with the qualitative properties obtained at the end of Lecture 6, for some appropriate effective low energy degree of freedom (which would not coincide with any of the elementary fields in our Lagrangian), governs the system we are studying today near the critical point.

§7.6. The Wilson line operator

In this and subsequent sections we will define the Wilson line operator, and try to understand its physical and formal properties.

Suppose we have a gauge theory with gauge group G on a spacetime M. Let A denote the corresponding gauge field with values in the Lie algebra $\mathfrak{g}$ of G. Let C be a closed loop in M, and let $\text{Hol}(A, C) \in G$ be the holonomy of the connection A along C (it is only defined up to conjugation). Let R be a finite-dimensional representation of G. Define the classical Wilson line (or Wilson loop) functional to be

$$W_R(C)(A) = \text{Tr}|_R(\text{Hol}(A, C)). \tag{7.8}$$

It is clear that this functional is gauge invariant.

If C is a union of disconnected loops C_i labeled with representations $R = (R_i)$, then by definition $W_R(C) = \prod W_{R_i}(C_i)$.

If G is abelian, and C is contractible, then $W_R(C) = e^{i\int_D F}$, where F is the curvature of A, and D is a disk such that $\partial D = C$.

An important generalization of this is the following. Let $\widehat{G}$ be the simply-connected cover of G, and R a representation of $\widehat{G}$. In this case it is easy to see that $W_R(C)$, defined in the same way, is still well-defined when C bounds a disk D. (In the abelian case, this follows from the above integral representation of W).

Now we want to define an analogue of $W_R(C)$ in quantum theory. For this purpose we need to renormalize the classical functional $W_R(C)$. This can be done by expanding $W_R(C)$ in powers of A (like the exponential is expanded in Taylor series):

$$W_R(C) = \dim(R) + \int dl \text{Tr}|_R(A(l)) + \frac{1}{2}\int dl dl' \text{Tr}|_R(A(l)A(l')) + \cdots, \tag{7.9}$$

where l is some parameter on C, and $A(l)$ is the evaluation of the form A on the tangent vector $\frac{d}{dl}$ to C at the point l, with respect to this parametrization. Each term of this expansion is polynomial and can be renormalized as usual.

In fact, one can show in most cases that the operator $W_R(C)$ has only multiplicative renormalization. Roughly speaking, this follows from the fact that classically, $W_R(C)$ is the trace of the monodromy of the differential equation $x' = Ax$

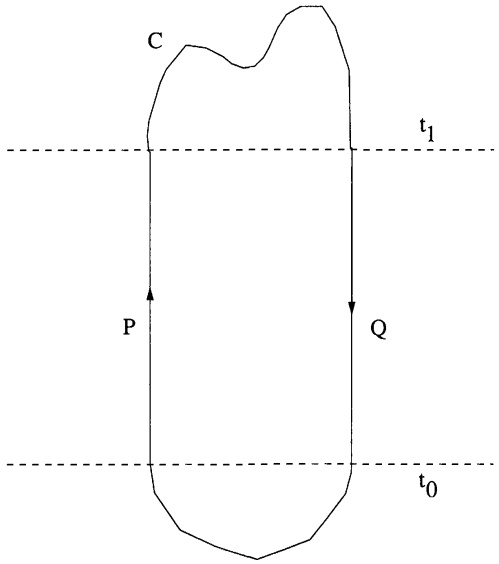
along the loop. This equation is renormalized to $x' = (A + c)x$, where c is an operator in the theory, invariant under the same symmetries as A, in the adjoint representation of the gauge group, of dimension the same and lower than A. If there is no such operator except for constants then this equation will change to $x' = (A + c)x$ under renormalization, where c is a scalar operator. This shows that $W_R(C)$ will have multiplicative renormalization; to renormalize it one must introduce a cutoff and multiply $W_R(C)$ by a function $e^{c'L(C)}$ with $L(C)$ the length (or circumference) of C, and c' a function of the cutoff and coupling.

More precisely, one can show that in critical dimension 4, the divergent renormalization factor has the form $e^{L(C)\Lambda f(e^2)+o(\Lambda)}$, where Λ is the cutoff, and $f(e^2)$ can be computed in perturbation theory (but depends on a precise choice of cutoff). In the superrenormalizable case (in less than 4 dimensions), the divergent factor grows as a smaller power of Λ or is absent.

A possibly more physical way of thinking of the Wilson loop operator is to identify it as an operator on Hilbert space. For this, we will use the Hamiltonian picture. Thus, we take spacetime to be $M = M_s \times \mathbb{R}$, where M_s is the space manifold and $\mathbb{R}$ represents "time." The Hilbert space of the theory $\mathcal{H}$ is then the space of functions on $\mathcal{A}/\widetilde{G}$, where $\mathcal{A}/\widetilde{G}$, $\mathcal{A}$ is the space of all gauge fields in M_s, $\widetilde{G}$ is the group of gauge transformations on M_s, and so $\mathcal{A}/\widetilde{G}$ is the space of gauge classes of connections on M_s. The space $\mathcal{H}$ has the form $\mathcal{H} = (\mathcal{H}_0)^{\widetilde{G}}$, where $\mathcal{H}_0$ is the space of functions on $\mathcal{A}$. (In this description of the Hilbert space, we are really considering "pure gauge theory" without matter fields; matter fields can be included in the statements in a fairly obvious way.)

Now let C be a loop in M. Suppose first that the loop C is in the submanifold $t = 0$. In this case, $W_R(C)$ is a function on $\mathcal{A}/\widetilde{G}$, and as an operator on Hilbert space, it should be understood as multiplication by that function.

Now consider the situation when C is not in a horizontal section of the spacetime, but a general curve. In this case, $W_R(C)$ is no longer multiplication by a function. Let us consider the opposite case in which C intersects the slice $t = 0$ transversely. In fact, let us take C to consist near $t = 0$ of $P \times \mathbb{R} - Q \times \mathbb{R}$, where P and Q are two points in M_s and the minus sign keeps track of the orientation of the intersection of C with M_s. We still have a time-independent system near $t = 0$, but now it does not consist only of gauge fields (and possible other fields) on M_s; there are also two distinguished points on M_s, namely P and Q, at which the physics is different because of the factor $W_R(C)$ in the path integral. We can introduce a Hilbert space $\mathcal{H}_{P,Q}$ and a Hamiltonian $H_{P,Q}$ governing this situation. But they are not the usual Hilbert space and Hamiltonian.



As in the figure, we suppose that to the past of some time t_0 and to the future of some time t_1, C does not just look like $P \times \mathbb{R} - Q \times R$ but is somehow "closed off." The path integral to the past of t_0 and to the future of t_1 computes some initial and final states, say $|\psi_i\rangle$ and $|\psi_f\rangle$, in $\mathcal{H}_{P,Q}$. (Of course, we do not know yet what $\mathcal{H}_{P,Q}$ is.) Propagation between time t_0 and t_1 gives a factor of $e^{-iH_{P,Q}(t_1-t_0)}$ (or $e^{-H_{P,Q}(t_1-t_0)}$ if we are working with Euclidean signature). So the expectation value of the Wilson line in the figure can be computed as

$$\langle\psi_f|e^{-H_{P,Q}(t_1-t_0)}|\psi_i\rangle.$$

But what are $\mathcal{H}_{P,Q}$ and $H_{P,Q}$?

First of all, if for a fixed connection (which we will assume to be time-independent between t_0 and t_1) one wants to compute $W_R(C)$ by a formula along the lines of the one just written, how would one do this? This is just a question in classical mathematics; it does not really involve quantum field theory. One might naturally define $\mathcal{H}_{P,Q}$ to consist of $R \otimes R^*$ (with R^* the dual of R), where we consider R to "live" at P and R^* at Q.[16] Parallel transport from Q to P in the "past" of $t = t_0$ determines a vector $|\psi_i\rangle$ in $R \otimes R^*$. Parallel transport from P back to Q in the future of t_1 likewise gives a vector $|\psi_f\rangle$ in the same space. Now we suppose that between times t_0 and t_1 the connection is time-independent, in the sense that it can be represented relative to some trivialization by a time-independent one-form. Then we can write

$$W_R(C) = \langle\psi_f|e^{-iH_{P,Q}(t_1-t_0)}|\psi_i\rangle$$

where $|\psi_i\rangle$ and $|\psi_f\rangle$ are the vectors just described in $R \otimes R^*$, and $H_{P,Q}$ (which is the sum of a term acting in R and a term acting in R^*) depends on the particular time-independent connection for $t_0 < t < t_1$. (In fact, $H_{P,Q}$ is determined by the component of this connection in the "time" direction.)

So, if the gauge field is fixed, we can get a representation of the desired form if we take $\mathcal{H}_{P,Q} = R \otimes R^*$. Now, what if the gauge field is treated quantum mechanically?

[16]More precisely, the gauge field A is a connection on a principal bundle $\mathcal{P}$ whose fiber at P or Q we call $\mathcal{P}_P$ or $\mathcal{P}_Q$. We write R for $\mathcal{P}_P \times_G R$, and R^* for $\mathcal{P}_Q \times_G R^*$.

We recall what to do if the gauge field is treated quantum mechanically but the Wilson line is absent. In that case, as we noted above, we let $\mathcal{A}$ be the space of all connections, and $\mathcal{H}_0$ the space of $\mathbf{L}^2$ functions on $\mathcal{A}$. Then we consider the action of the gauge group $\widetilde{G}$ on $\mathcal{H}_0$. The physical Hilbert space is the space $\mathcal{H}$ of invariants, that is it is $\mathcal{H}_0^{\widetilde{G}}$.

Obviously, we must "mix" the two constructions. The following natural way to mix them can be justified by reexamining the arguments by which path integrals and Hamiltonians are related in the first place. (Alternatively, I sketch in §7.7 the beginning of another approach to justifying the following construction.) We replace $\mathcal{H}_0$ by $\mathcal{H}_0 \otimes R \otimes R^*$, and define $\mathcal{H}_{P,Q} = (\mathcal{H}_0 \otimes R \otimes R^*)^{\widetilde{G}}$. As for the Hamiltonian $H_{P,Q}$, it is given formally by the same expression $\frac{e^2}{2}\int d^3x \mathrm{Tr}(-\delta^2/\delta A(x)^2) + \frac{2}{e^2}\int d^3x \mathrm{Tr} F^2$, but now evaluated in $\mathcal{H}_{P,Q}$.

More generally, given any set of points $P_i \in M_s$, $i = 1, \dots, n$, and associated representations R_i of G, we set $R_i(P) = \mathcal{P}_{P_i} \times_G R_i$. Then we define a Hilbert space $\mathcal{H}_{P_1,R_1;P_2,R_2;\dots;P_n,R_n} = (\mathcal{H}_0 \otimes_{i=1}^n R_i(P))^{\widetilde{G}}$. We say that this is "the Hilbert space of the theory with external charges in representations R_i at points P_i."

We can generalize this to the case that the positions of the P_i are time-dependent. For instance, let us go back to the case of a Wilson loop associated with a circle C. We continue to assume that C intersects a $t = 0$ slice transversely at the points P, Q, but we no longer assume that C is time-independent near $t = 0$. Let $P(t)$, $Q(t)$ be the time-dependent intersection points of C with a slice of given time. We then must work in the time-dependent family of Hilbert spaces $\mathcal{H}(t) = (\mathcal{H}_0 \otimes R(P(t)) \otimes R^*(Q(t)))^{\widetilde{G}}$, which is endowed with a time-dependent Hamiltonian, and hopefully with a natural connection.

§7.7. The path integral representation of the Wilson line operator

It is convenient to represent the classical Wilson line functional as a one-dimensional path integral. Such a representation will be deduced in this section. Among other things, this representation can be used to put in a completely standard form the justification of the above description of $\mathcal{H}_{P,Q}$.

Let R be an irreducible, finite-dimensional representation of a simple compact Lie group G. By Borel-Weil theory, we can think of R as $H^0(G/T, \mathcal{L}_R)$, where $\mathcal{L}_R$ is a suitable holomorphic line bundle. The bundle $\mathcal{L}_R$ has a natural Hermitian metric, and therefore a natural connection B. For any map $\phi : S^1 \to G/T$, define the action $I_R(\phi) = -\ln \mathrm{Hol}(\phi^*(B))$ (Hol is for holonomy). The number I_R is only defined up to $2\pi i n$, but $e^{-I_R(\phi)} = \mathrm{Hol}(\phi^*(B))$ is well defined.

Consider the path integral

$$\int D\phi e^{-I_R(\phi)}. \tag{7.10}$$

It can be represented in the form

$$\int D\phi e^{i\int_D \phi^*(F_B)}, \tag{7.11}$$

where F_B is the curvature of B, and D is a disk whose boundary is the image of S^1. The quantum theory defined by this path integral is the Hilbert space R with the zero Hamiltonian (as the path integral is invariant under diffeomorphisms).

Now let P be the total space of a principal G-bundle over the circle, with a connection A, and let P/T be the associated G/T bundle. Since $\mathcal{L}$ is a G-equivariant line bundle on G/T, it defines a complex line bundle on P, which we denote by $\mathcal{L}_P$. The Hermitian connection B on $\mathcal{L}$ defines a connection on $\mathcal{L}_P$ in the vertical direction (along fibers of P). This connection can be naturally extended to a connection B_P on $\mathcal{L}_P$ using the connection A: in a local gauge where P is trivial and $A = 0$, B is extended by the condition that constant paths are horizontal paths.

For any smooth section ϕ of P/T, define the action

$$I(\phi, A) = -\ln \mathrm{Hol}(\phi^*(B_P)). \tag{7.12}$$

As before, I is defined up to $2\pi i n$.

Let $W_R(A)$ denote the trace in R of the holonomy of A around the circle. It is now easy to see that the functional $W_R(A)$ has the following integral representation:

$$W_R(A) = \int_{\phi \in \Gamma(P/T)} e^{-I(\phi,A)} D\phi. \tag{7.13}$$

Now let C be a loop in M, and let A be a connection in a principal G-bundle P over M. Let $f : S^1 \to C$ be a parametrization. Then we have

$$W_R(C)(A) = \int_{\phi \in \Gamma(f^*P/T)} e^{-I(\phi, f^*A)} D\phi. \tag{7.14}$$

This gives an integral representation of the Wilson line operator. This representation allows us to represent the theory in the presence of C in terms of a path integral that is completely of a standard type except that some fields live throughout spacetime, while ϕ is only defined on C. For instance, in the case of pure gauge theory, we would write

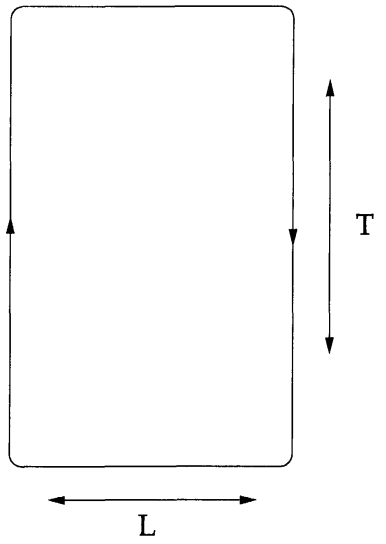
$$\int DAD\phi \exp\left(-\frac{1}{4e^2}\int \mathrm{Tr} F \wedge *F - I(\phi, A|_C)\right). \tag{7.15}$$

Once this is done, if we consider again the case that C is time-independent for $t_0 < t < t_1$, one can introduce a Hilbert space and Hamiltonian using completely standard techniques. By going down that road, one can recover the description of $\mathcal{H}_{P,Q}$ that I stated earlier. The reader is urged to think about this as an exercise. Basically, R and R^* come from quantizing ϕ, while $\mathcal{H}_0$ comes from quantizing A, and the restriction to $\mathcal{H}_{P,Q} = (\mathcal{H}_0 \otimes R \otimes R^*)^{\widetilde{G}}$ is the usual restriction to gauge-invariant states in gauge theories.

Remark. Note that the integrand of (7.13),(7.14) is the holonomy of an abelian connection, so it can be explicitly computed as the exponential of the integral of curvature. This, we have represented the monodromy of an arbitrary linear differential equation on the circle as an explicit integral over infinitely many auxiliary variables. It is well known in classical analysis that such a representation with finitely many auxiliary variables is, in general, impossible: it is only possible for some special equations of "hypergeometric" type.

§7.8. The Higgs and the confinement regimes

We work now in Euclidean space. Consider a Wilson loop which is approximately a rectangle: two charges are born and separated from each other a distance L, stay at that distance from each other for a time $T >> L$, and then annihilate each other. (See the figure.)



Then the expectation value $\langle W_R(C)\rangle$ of the Wilson loop operator (in the Euclidean setting) has the following meaning: it is approximately equal to $Ce^{-TV(L)}$, where $V(L)$ is the energy of interaction of the charges at distance L, and C is a constant that depends on the detailed form of C near the initial and final times. $V(L)$, to be more precise, is the difference between the ground state energy in $\mathcal{H}_{P,Q}$ and the ground state energy in the usual Hilbert space $\mathcal{H}$. The claim that the expectation value of the Wilson loop depends on T as $Ce^{-TV(L)}$ for large T follows from its Hamiltonian representation

$$W_R(C) = \langle \psi_f | e^{-HT} | \psi_i \rangle$$

that was discussed at length above.

So the asymptotics of $\langle W_R(C)\rangle$ depends on the asymptotics of $V(L)$ as $L \to \infty$.

Physicists believe that, in gauge theories with a mass gap above two dimensions (in two dimensions there are some peculiarities, some of which we will see later), there are two possibilities:

1. Higgs regime: $V(L) \to \text{const}, L \to \infty$.

In this regime, charges (of a type specified in the remark below) can separate from each other with finite cost in energy.

2. Confinement regime: $V(L) \sim \text{const} \cdot L$ as $L \to \infty$, where the constant is positive.

In this regime charges are confined, and cannot separate without spending an arbitrarily large amount of energy.

Now let us consider a Wilson loop C of any shape, with circumference S and minimal area of the spanning surface A. There is a general conjecture that, given a mass gap, patterns 1 and 2, if they hold for $T >> L >> 0$, hold for an arbitrary C (large in all directions) in the following form: the Higgs regime corresponds to the asymptotics $\langle W_R(C)\rangle \sim e^{-wS}$ (the circumference law), and the confinement regime

corresponds to the asymptotics $\langle W_R(C)\rangle \sim e^{-wA}$ (the area law). In each case, w is a positive constant.

The intuition behind this conjecture is that, with a mass gap, the response to the Wilson line is more or less local – either localized along C itself, in the Higgs case, or localized along a minimal area surface spanned by C, in the confining case. How the minimal area surface comes in will become clearer next week when we discuss the Polyakov model and subsequently when we discuss 't Hooft's picture of confinement in Lecture 10.

Remark 1. Actually, confinement can only occur if R is a representation of the universal cover $\widehat{G}$ of G and not of G itself. Indeed, if R is a representation of G, there are physical processes contributing to $\langle W_R(C)\rangle$ in which large portions of the Wilson line have zero charge (i.e. carry the trivial representation of G) because some particles in the theory have annihilated the charges on the Wilson line. These processes have an amplitude which is bigger than that predicted by the area law. It is important that the "charges" discussed in patterns 1 and 2 above are external charges by which the theory can be probed – at the cost of replacing the gauge group G by a cover of itself. They are not charges described by the "original" theory; they do not even (in the interesting cases) transform like any charges in the original theory. We will illustrate this remark in our toy example below. I expect this remark to be difficult to understand and do not expect the reader to fully understand it on the first try.

Remark 2. The above criteria 1 and 2 for Higgs regime and confinement are perfectly rigorous within the class of theories in which they are defined. In particular, we can take the perimeter law to be a precise statement of what it means for a gauge theory with mass gap to be in a Higgs phase. This is closely related to, but does not quite always coincide with, the Higgs mechanism as we introduced it in Lecture 2. The perturbative Higgs mechanism of gauge symmetry breaking is a method of calculation that is useful in certain weakly coupled theories. We gave it no meaning outside of weak coupling (in contrast to spontaneous breaking of global symmetry, for which we gave axiomatics definitions in Lecture 1), and it makes perfect sense even when $\pi_1(G) = 0$ (in which case our present more rigorous discussion is vacuous by Remark 1). The perimeter law for a Higgs phase is by contrast an axiomatic criterion governing certain theories with mass gap; the statement of the criterion has nothing to do with whether the theory has a weakly coupled description or with any specific method of calculation. A theory that is in a Higgs phase by the axiomatic criterion cannot necessarily be described by a semiclassical Higgs mechanism (though often it can be so described in some limit of the parameters), and a theory governed by a semiclassical Higgs mechanism is not always in a Higgs phase by the axiomatic criterion (though in practice, when $\pi_1(G)$ is nontrivial, usually it is; in fact the main counterexample is the one we are about to describe, in two dimensions). If the assertions just made seem confusing now, the reader should reexamine them after further study of quantum field theory.

We really already discussed an example of confinement in Lecture 5. For a somewhat different toy example, let us consider the two-dimensional theory with Lagrangian (7.1), which we studied in the first part of the lecture. In this case, $G = U(1)$ and its universal cover is $\widehat{G} = \mathbb{R}$. Let R be the representation of $\widehat{G}$ defined by $\lambda \in \mathbb{R}$: $x \to e^{i\lambda x}$. Then, classically, $W_R(C) = e^{i\lambda \int_D F_A}$, where $\partial D = C$.

Quantum mechanically we can write

$$\langle W_R(C)\rangle = \int DAD\phi D\bar{\phi} e^{-\mathcal{L}} e^{i\lambda \int_D F}. \tag{7.16}$$

Let V be the volume of spacetime, and A_C be the area enclosed by C. We will split the path integral (7.16) into a product of two pieces – the integral over values of fields inside the loop C and over the values outside. Because of the mass gap, the partition function has an approximate factorization as a product of an "inside" partition function and an "outside" one. (There is also a perimeter correction, arising near C; it does not affect the discussion of the area law.) Observe that the last factor in (7.16) (the holonomy factor) is of the same type as the topological term in (7.1). Therefore, (7.16) yields

$$\langle W_R(C)\rangle = \frac{1}{Z} \exp\left(-(V - A_C)E(\theta) - A_C E(\theta + 2\pi\lambda)\right), \tag{7.17}$$

up to boundary terms, which we will neglect here. Here $E(\theta)$ is the vacuum energy density as a function of θ; it can be computed approximately by methods in the first part of the lecture. Since in the absence of the Wilson loop, the partition function is $Z = e^{-VE(\theta)}$, we get

$$\langle W_R(C)\rangle = \exp\left(A_C(E(\theta) - E(\theta + 2\pi\lambda))\right). \tag{7.18}$$

According to our calculations, for $\theta = 0$, $E(\theta + \lambda) - E(\theta)$ reduces to

$$E(\lambda) - E(0) = 2(1 - \cos 2\pi\lambda)e^{-I/e^2} P. \tag{7.19}$$

This is always positive when λ is not an integer. Thus, theory (7.1) for $\theta = 0$ exhibits the confinement regime for all external charges such that λ is not an integer. The condition for λ not to be an integer simply means that the representation we started with is a representation of the universal cover $\mathbb{R}$ of the gauge group, but not a representation of the original gauge group $G = U(1)$. This is in keeping with Remark 1 above: confinement by the criterion we have stated only arises for external charges that represent a cover of the gauge group but not the gauge group itself.

Thus, even though this theory is governed by the perturbative Higgs mechanism, it is in a confining phase at $\theta = 0$, according to the axiomatic criterion above. Thus, in this particular case the perturbative Higgs mechanism is quite different from the abstract criterion for a Higgs phase. In more than two dimensions, one expects (and one proves in three dimensions) that this theory is in a Higgs phase by the abstract criterion as well as exhibiting a perturbative Higgs mechanism.

Going back to two dimensions, one also sees from the above that if $\theta \neq 0$, the coefficient in the area law can be either positive or negative. This is an oddity of two dimensions: a negative coefficient is certainly not expected above two dimensions.

§7.9. The confinement conjecture

The following conjecture is central in quantum field theory.

Conjecture. *Let G be a simple compact Lie group, and let R be a representation of G which is not a representation of the adjoint group G_{ad}. In 3 and 4 dimensions, the pure gauge theory with gauge group G and Lagrangian $\int \mathrm{Tr}(F \wedge *F)$ exhibits confinement for charges with values in R. In 4 dimensions, this remains true for any value of the θ angle.*

The case of this conjecture that is relevant to the strong interactions observed in nature (they govern nuclear physics, among other things) is $G = SU(3)$, $R = \mathbb{C}^3$. This special case of the conjecture amounts to confinement of quarks.

In two dimensions, the analogous statement about pure gauge theory is true at $\theta = 0$; this follows from our discussion in Lecture 5. For $\theta \neq 0$, the situation in two dimensions is more complicated, even in the abelian case, as we have just seen.

References and cross references

Two-dimensional gauge theory is discussed in Lectures 5 and 6. The Wilson line operator and confinement in 3 and 4 dimensions is discussed in later lectures, in particular Lectures 8 and 10.

LECTURE 8
Abelian Duality

§8.1. Introduction

Today, we will discuss abelian duality in two, three and four dimensions.

In two dimensions, abelian duality inverts the radius R of a circle and is often referred to as the "R goes to $1/R$" equivalence. In a certain supersymmetric version, it leads to a linear version of mirror symmetry.

In three dimensions, after studying the duality we will give an application to Polyakov's model of confinement.

Abelian duality in four dimensions is related to an $SL(2,\mathbb{Z})$ symmetry, and we will eventually use it in understanding Donaldson theory, that is, to $N = 2$ supersymmetric theories in dimension 4. In the process we will find another model of confinement. In fact, these two applications of duality give the most concrete models of the phenomenon of confinement which are known, above two dimensions.

We begin with the classical statements of duality. In two dimensions, consider a theory which involves fields ϕ, σ, both obeying the Laplace equation

$$\nabla^2\phi = 0; \qquad \nabla^2\sigma = 0, \tag{8.1}$$

and related by

$$d\phi = *\, d\sigma. \tag{8.2}$$

Classically, either of these fields can be taken as the fundamental field of the theory. For example, if we begin with σ such that $\nabla^2\sigma = *\,d*\,d\sigma = 0$ then $d*\,d\sigma = 0$ so locally we can write $*\,d\sigma = d\phi$ for some field ϕ, i.e., σ determines ϕ (locally and up to an additive constant).

Likewise in three dimensions, consider a theory with a field ϕ obeying the Laplace equation, as well as a connection A on some line bundle $\mathscr{L}$ with curvature $F = dA$ obeying Maxwell's equations

$$dF = d*F = 0, \tag{8.3}$$

whose duality relationship is

$$*\,d\phi = F. \tag{8.4}$$

Again, as above, either of these fields can be taken as fundamental.

Notes by David Morrison

In four dimensions, the analogue is two connections A and B on line bundles $\mathscr{L}_A$ and $\mathscr{L}_B$, each satisfying Maxwell's equations, and related by

$$F_A = *F_B. \tag{8.5}$$

For abelian duality, we could keep going to higher dimensions if we wish, but to find interesting applications we would have to go to string theory.

§8.2. Duality in two dimensions

We wish to study quantum versions of these classical statements. We begin with the two-dimensional case. We identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, and use additive coordinates on the circle. We take ϕ to be a map $\phi : \Sigma \to S^1$ where Σ is a compact oriented surface equipped with a Riemannian metric $g_{\alpha\beta}$. (We will consider a variant later on, in which ϕ is not required to be defined at some specified points P_i, that is, ϕ will map $\Sigma - \{P_i\}$ to S^1.) Our Lagrangian[1] is

$$\mathcal{L}(\phi) = \frac{R^2}{4\pi}\int d^2x\,\sqrt{g}\,\partial_\alpha\phi\,\partial^\alpha\phi = \frac{R^2}{4\pi}\int d\phi\wedge *d\phi. \tag{8.6}$$

The equations of motion $dd\phi = d*d\phi = 0$ reproduce the classical theory discussed above.

We will study this theory in various ways. The usual trick is to introduce new variables with the property that integrating them out would lead back to the original theory, and then integrate out the old variables instead of the new ones in order to produce a dual formulation of the theory.

So we actually wish to study a different theory, one which will contain fields ϕ and A, with ϕ a section of a trivial S^1-bundle $\mathscr{S}$ and A a connection on $\mathscr{S}$. Choosing a trivialization ϕ_0 of $\mathscr{S}$ (so that $A = \phi_0^*\tilde{A}$ for some 1-form $\tilde{A}$ on the total space of the bundle), we can define a covariant derivative

$$D_A\phi = d\phi + A, \tag{8.7}$$

and introduce a new Lagrangian

$$\mathcal{L}(\phi, A) = \frac{R^2}{4\pi}\int d^2x\,\sqrt{g}(\partial_\alpha\phi + A_\alpha)(\partial^\alpha\phi + A^\alpha) = \frac{R^2}{4\pi}\int D_A\phi\wedge *D_A\phi. \tag{8.8}$$

While we can recover the old Lagrangian by setting A to zero, there is no mechanism which enforces this, unless we introduce yet a third field σ which plays the role of a Lagrange multiplier. We take σ to be a map from Σ to S^1, and write a Lagrangian

$$\mathcal{L}(\phi, A, \sigma) = \frac{R^2}{4\pi}\int D_A\phi\wedge *D_A\phi - \frac{i}{2\pi}\int \sigma\wedge F_A, \tag{8.9}$$

where F_A is the curvature of A.

This last term requires some comment. Since we are assuming that the circle bundle $\mathscr{S}$ is trivial, we can globally write $F_A = dA$ and the last term should be interpreted as $\frac{i}{2\pi}\int d\sigma\wedge A$ (after integration by parts); this step is needed because σ is not single-valued. More generally, to define such a term even when $\mathscr{S}$ is non-trivial, one can use a bit of topology to define $\exp(\frac{i}{2\pi}\int\sigma\wedge F_A)$, similar to defining a Chern-Simons form.

The point of writing this Lagrangian is that $\mathcal{L}(\phi, A, \sigma)$ is equivalent to $\mathcal{L}(\phi)$, as we will now show. A naïve analysis goes as follows: σ appears without derivatives,

[1] All Lagrangians in this lecture are written in Euclidean signature.

and its equation of motion is $F_A = 0$; imposing this, we can then go to a gauge where $A = 0$ and recover the original theory.

More globally, we consider the path integral

$$Z = \frac{1}{\mathrm{vol}(G)} \int \mathcal{D}\phi\, \mathcal{D}A\, \mathcal{D}\sigma \exp\left[-\frac{R^2}{4\pi} \int D_A\phi \wedge *D_A\phi + \frac{i}{2\pi} \int \sigma \wedge F_A \right]. \tag{8.10}$$

(We focus on the partition function for now, but path integrals with operator insertions can also be treated this way, as we will discuss later.) In light of the standard formula

$$\int \frac{dx}{2\pi} e^{\imath xy} = \delta(y), \tag{8.11}$$

we would like to say that doing the σ-integral will set F_A to zero. We should treat this statement with care, since we are studying circle-valued functions.

Bearing in mind our identification of S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, our map $\sigma : \Sigma \to S^1$ can be locally written as a real-valued function, but it might not be globally single-valued; however, $d\sigma$ will be a (single-valued) real 1-form on Σ. Choose a circle-valued function $\sigma_h : \Sigma \to S^1$ such that $d\sigma_h$ is the *harmonic* representative in the de Rham cohomology class of $d\sigma$. (We normalize the choice of σ_h by picking some point $P \in \Sigma$ and demanding that $\sigma_h(P) \in 2\pi\mathbb{Z}$.) Then we can write $\sigma = \sigma_h + \sigma_{\mathbb{R}}$, with $\sigma_{\mathbb{R}}$ a single-valued real function.

Notice that $\frac{1}{2\pi} d\sigma$, or equivalently $\frac{1}{2\pi} d\sigma_h$, must have integral periods. In particular, if we choose a basis λ_j of integral harmonic 1-forms, and write $d\sigma_h = \sum 2\pi m_j \lambda_j$, then $m_j \in \mathbb{Z}$.

Now we compute:

$$\int \mathcal{D}\sigma\, e^{\frac{\imath}{2\pi}\int \sigma \wedge F_A} = \int \mathcal{D}\sigma_{\mathbb{R}}\, e^{\frac{\imath}{2\pi}\int \sigma_{\mathbb{R}} \wedge F_A} \sum_{d\sigma_h \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-\frac{\imath}{2\pi} d\sigma_h \wedge A} \tag{8.12}$$

$$= \delta(F_A) \prod_j \left(\sum_{m_j \in \mathbb{Z}} e^{-\imath m_j \int \lambda_j \wedge A} \right). \tag{8.13}$$

The first factor tells us that A is a flat connection. Moreover, among flat connections the gauge equivalence classes $[A]$ are labeled by the holonomies, or equivalently by the quantities $\int \lambda_j \wedge A$. Since the remaining part of the integrand is gauge invariant, we can gauge fix (omitting the factor of $(\mathrm{vol}(G))^{-1}$) and integrate over the space of gauge equivalence classes (a finite dimensional integral):

$$Z = \int \mathcal{D}\phi\, \mathcal{D}[A]\, e^{-\frac{R^2}{4\pi}\int D_A\phi \wedge *D_A\phi} \delta(F_A) \prod_j \left(\sum_{m_j \in \mathbb{Z}} e^{-im_j \int \lambda_j \wedge A} \right) \tag{8.14}$$

$$= \int \mathcal{D}\phi\, e^{-\frac{R^2}{4\pi}\int D_A\phi \wedge *D_A\phi} \delta(F_A)\, \delta\Big(\int \lambda_i \wedge A = 0 \mod 2\pi\mathbb{Z} \Big). \tag{8.15}$$

So there are delta functions setting the holonomies as well as the curvature to zero. So A is zero modulo gauge transformations, and our new theory is indeed equivalent to the original theory, with partition function

$$Z = \int \mathcal{D}\phi \exp\left(-\frac{R^2}{4\pi} \int d\phi \wedge *d\phi \right). \tag{8.16}$$

Now let us integrate out in the opposite order, integrating out ϕ and A but keeping σ. To integrate out ϕ, we fix the gauge in such a way that $\phi = 0$, and

suppress the factor of $(\mathrm{vol}(G))^{-1}$ from the path integral. Then the path integral (8.10) reduces to

$$\int \mathcal{D}A\,\mathcal{D}\sigma\,\exp\left[-\frac{R^2}{4\pi}\int A\wedge *A+\frac{i}{2\pi}\int \sigma\wedge F_A\right]. \tag{8.17}$$

An exercise you might enjoy is verifying that the Faddeev-Popov determinant associated with this gauge fixing is

$$\int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left[-\frac{R^2}{4\pi}\int d^2x\,\bar{c}c\right]. \tag{8.18}$$

To do the integral over A, we need to complete the square, thinking of the second term in the exponent as $-\frac{i}{2\pi}\int A\wedge d\sigma$, i.e., it is the term linear in A. We make a change of variables $A'=A+\frac{i}{R^2}*d\sigma$; the result, including the Faddeev-Popov integral, is

$$\left(\int \mathcal{D}c\,\mathcal{D}\bar{c}\,e^{-\frac{R^2}{4\pi}\int d^2x\,\bar{c}c}\right)\left(\int \mathcal{D}A'e^{-\frac{R^2}{4\pi}\int A'\wedge *A'}\right)\left(\int \mathcal{D}\sigma e^{-\frac{1}{4\pi R^2}\int d\sigma\wedge *d\sigma}\right). \tag{8.19}$$

(The Gaussian integral over A', like the Faddeev-Popov determinant, can be thought of as a normalization factor.)

The Faddeev-Popov integral gives a factor of $(\int dt\,e^{-tR^2/4\pi})^{-1}$ for each 0-form and each 2-form on Σ[2]; the integral over A' gives a factor of $\int dt\,e^{-tR^2/4\pi}$ for each 1-form on Σ. Thus, if n_j denotes the "number of j-forms on Σ", those terms combine to give an overall factor of

$$(\sqrt{\pi/(R^2/4\pi)})^{(-n_0+n_1-n_2)}=(R/2\pi)^{\chi(\Sigma)}. \tag{8.20}$$

That is, the transformation rule which relates these dual formulations is

$$\int \mathcal{D}\phi\,\exp\left[-\frac{R^2}{4\pi}\int d\phi\wedge *d\phi\right]=(R/2\pi)^{\chi(\Sigma)}\int \mathcal{D}\sigma\,\exp\left[-\frac{1}{4\pi R^2}\int d\sigma\wedge *d\sigma\right]. \tag{8.21}$$

(The factor of $(R/2\pi)^{\chi(\Sigma)}$ did not show up in previous explicit calculations by other lecturers this year because they were done in genus 1.) A coupling of R in the first theory maps to a coupling of $1/R$ in the second theory, which is why this duality is sometimes referred to as "R goes to $1/R$."

Now we would like to follow the operators in this theory through the duality transformation, and determine the effect on correlation functions as well as on the partition function. An easy case is $d\phi$, which we expect to map to $*\,d\sigma$, as in the classical theory (actually it will map to a multiple of $*\,d\sigma$, when normalizations are taken into account). We need to repeat the above calculation with an insertion of operators $\prod \mathcal{O}_i(\phi)$, one of which is $d\phi$. In order to do this, we need a gauge-invariant extension of the operator $d\phi$, which is provided by the covariant derivative:

$$d\phi(x)\text{ in first theory}\longrightarrow D_A\phi(x)\text{ in big theory.} \tag{8.22}$$

One then easily checks that this maps back to $d\phi$ when we integrate out σ (and gauge fix A to zero).

The difficult thing about mapping operators in general will be finding the appropriate extension of a given operator to an operator in the larger theory, which

[2]For this purpose, think of c as a zero-form and $\bar{c}$ as a two-form.

reduces to the original operator when σ has been integrated out. For $d\phi$ it is not so hard, as we have just seen.

How will our calculation be modified when insertions of $D_A\phi$ are included? Earlier, when we completed the square, we made a change of variables $A' = A + \frac{i}{R^2} * d\sigma$. If $D_A\phi$ has been inserted in the path integral, it becomes after this change of variables $D_{A'}\phi - \frac{i}{R^2} * d\sigma$. In the rest of the derivatiion, when ϕ is gauge fixed to zero and A' is integrated out, we are left with an insertion of $-\frac{i}{R^2} * d\sigma$ in the dual theory. This is the image, therefore, of $d\phi$ in the original theory.

Some care must be used in manipulating these mappings between operator insertions. For example, even though $d\phi$ maps to $-\frac{i}{R^2} * d\sigma$, the local operators $(d\phi)^2$ will *not* map to $(\frac{i}{R^2} * d\sigma)^2$ in the dual theory, due to nonlinearities introduced when we complete the square.

What made this case easy was that the covariant derivative is a natural gauge-invariant extension of the ordinary derivative.

A harder operator insertion to deal with is $e^{i\phi}$. There is no gauge-invariant version of this. On the other hand, we don't really need it, because $\langle e^{i\phi}(P)\rangle = 0$. But the two-point correlators $\langle e^{i\phi}(P)\, e^{-i\phi}(Q)\rangle$ are not zero in general, so we should try to dualize such insertions, when $P \neq Q$.

We are going to do something rather strange. It is impossible to construct a gauge-invariant extension *locally* for this pair of operator insertions, so we work non-locally during the duality (i.e., in the intermediate theory), introducing a term in the path integral of the form

$$e^{i\phi}(P)e^{-i\phi}(Q)e^{\frac{i}{2\pi}\int \theta\wedge A} \tag{8.23}$$

where θ is a 1-form such that

$$\frac{1}{2\pi}d\theta = \delta_P - \delta_Q \tag{8.24}$$

and all periods of θ are 0 modulo $2\pi\mathbb{Z}$.

To find such a θ, take a path ℓ from P to Q and let $\theta/2\pi$ be Poincaré dual to ℓ (regarded as a distribution). We can rewrite our term (8.23) as

$$e^{i\phi}(P)e^{i\int_\ell A}e^{-i\phi}(Q), \tag{8.25}$$

which is clearly a gauge-invariant expression. (This is an example of a choice of θ, but we allow more general θ's than that.) We will eventually think of θ as a connection form on a trivial S^1-bundle on $\Sigma - P - Q$.

Thus, we have found a gauge-invariant nonlocal object (8.23). Suppose we try to insert this into the path integral. First, if we set A to zero, we recover our original operator insertion $e^{i\phi}(P)e^{-i\phi}(Q)$, so we have correctly extended this pair of operators to the gauge theory.

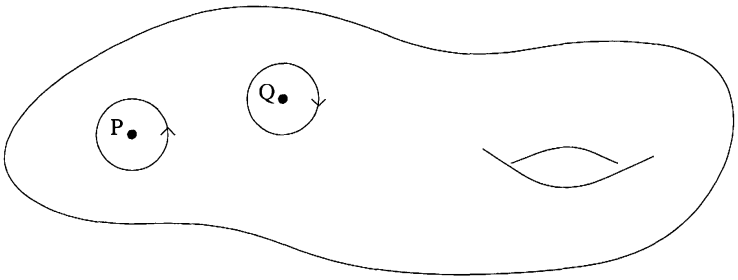
To integrate out the other way, we gauge fix ϕ to zero, which reduces (8.23) to $e^{\frac{i}{2\pi}\int\theta\wedge A}$. This is a new contribution to the term linear in A in the exponent of the path integral, of the form $-\frac{i}{2\pi}\int A\wedge\theta$, so to complete the square we now must make the change of variables $A' = A + \frac{i}{R^2} *(d\sigma + \theta)$, and this makes our dualized path integral take the form

$$\int \mathcal{D}\sigma \exp\left[-\frac{1}{4\pi R^2}\int (d\sigma+\theta)\wedge *(d\sigma+\theta)\right]. \tag{8.26}$$

That is, we have obtained the same theory as before, but now formulated on $\Sigma - P - Q$, and now performing the path integral over maps to S^1 that do *not* extend

over P or Q. That is, we write $\widetilde{\sigma} = \sigma + \alpha$, with $d\alpha = \theta$, α being a map from $\Sigma - P - Q$ to S^1, and then $\widetilde{\sigma}$ is a map of $\Sigma - P - Q$ to S^1 that does not extend over P or Q. In fact, it has winding numbers 1 and -1 around P and Q, respectively.

The point of introducting $\tilde{\sigma}$ is, of course, that in terms of $\tilde{\sigma}$ the Lagrangian is just the usual $(1/4\pi R^2)\int d\tilde{\sigma} \wedge *d\tilde{\sigma}$. Thus, inclusion of the operators $e^{i\phi(P)}e^{-i\phi(Q)}$ gives us the usual Lagrangian for a field $\tilde{\sigma}$ that has winding number 1 or -1 around P or Q.



There are two kinds of operators now, one local in one picture, and the other local in the other picture. The operator $e^{in\phi}(P)$ in one description is mapped to an instruction "delete P from Σ, and perform the path integral over σ's that have winding number n around P" in the other description. Conversely, there is an ordinary local operator $e^{in\sigma}(P)$ in the second description that corresponds to a nonlocal recipe in the first description.

§8.3. Duality in three dimensions

We now turn to three-dimensional theories. First we examine the theory of an S^1-valued function ϕ on a (fixed) compact 3-manifold M (possibly with boundary), governed by the Lagrangian

$$\mathcal{L} = \frac{\Lambda}{4\pi}\int d^3x\,\sqrt{g}\,\partial_\alpha\phi\,\partial^\alpha\phi = \frac{\Lambda}{4\pi}\int d\phi\wedge *d\phi. \tag{8.27}$$

One novel feature of three dimensions is that the prefactor Λ has the dimensions of mass. Since Λ is a dimensionful quantity rather than a constant, it will be impossible for anything special to happen at a numerical value of Λ.

We generalize the theory to the case that ϕ is a section of a trivial S^1-bundle $\mathscr{L}_B$ with a connection B, and write a Lagrangian which includes the covariant derivative $D_B\phi = d\phi + B$:

$$\mathcal{L}(\phi, B) = \frac{\Lambda}{4\pi}\int d^3x\,(\partial_\alpha\phi + B_\alpha)(\partial^\alpha\phi + B^\alpha) = \frac{\Lambda}{4\pi}\int D_B\phi\wedge *D_B\phi \tag{8.28}$$

If we consider this as a theory in its own right, with both B and ϕ as quantum fields,then it is a rather trivial theory; the classical equations of motion tell us that everything is zero up to a gauge transformation, and the quantum theory is equally trivial. To get somewhere, we introduce a line bundle $\mathscr{L}$ with connection A, and the Lagrangian

$$\mathcal{L}(\phi, B, A) = \frac{\Lambda}{4\pi}\int D_B\phi\wedge *D_B\phi - \frac{i}{2\pi}\int A\wedge F_B, \tag{8.29}$$

interpreting the last term as $\frac{-i}{2\pi}\int F_A \wedge B$ (after integrating by parts); this makes sense since $\mathscr{L}_B$ is trivial and, given a trivialization, B is just a one-form. The extension to the case of $\mathscr{L}_B$ not being trivial is a factor in the path integral $e^{\frac{i}{2\pi}\int A\wedge F_B}$, where we interpret the exponent as a Chern-Simons invariant for the connection $A \oplus B$ on the bundle $\mathscr{L} \oplus \mathscr{L}_B$. This factor is well-defined, since the Chern-Simons invariant gives a well-defined map to $\mathbb{R}/2\pi\mathbb{Z}$.

In order to carry out the duality transformation, we need to sum over line bundles $\mathscr{L}$, producing a path integral

(8.30)
$$\frac{1}{\mathrm{vol}(G)\,\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}\phi\,\mathcal{D}A\,\mathcal{D}B\,\exp\left[-\frac{\Lambda}{4\pi}\int D_B\phi\wedge *D_B\phi + \frac{i}{2\pi}\int F_A\wedge B\right].$$

where G and G' are gauge groups for $\mathscr{L}$ and $\mathscr{L}_B$.

We would like to integrate out A. The crude statement is that doing so gives $\delta(F_B)$, which implies that B is a flat connection. As in the computation we did in two dimensions, the complications arise from the possibility of nontrivial holonomies for B. In fact, if $F_B = 0$ we can regard B as an element of $H^1(M, \mathbb{R}/\mathbb{Z})$ and define, for each $x = c_1(\mathscr{L})$, the quantity $e^{i\int x\wedge[B]}$. When this is summed over $[x]$ in doing the path integral over A, the result is $\delta([B])$, showing that B is gauge-equivalent to the trivial flat connection. Thus, we reduce back to the original theory; we've learned that the extended theory with A and $\mathscr{L}$ is equivalent to the original theory.

Now we do the integral in the opposite direction, by doing the ϕ and A integrals. As in the two-dimensional case, we set ϕ to zero by a gauge transformation, removing the normalization factor $(\mathrm{vol}(G))^{-1}$. (Unlike in two dimensions, the Faddeev-Popov determinant does not contribute anything interesting, so we will not explicitly include it.) The square can be completed in the resulting path integral

(8.31)
$$\frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\mathcal{D}B\,\exp\left[-\frac{\Lambda}{4\pi}\int B\wedge *B + \frac{i}{2\pi}\int B\wedge F_A\right]$$

(8.32)
$$= \frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\mathcal{D}B'\,\exp\left[-\frac{\Lambda}{4\pi}\int B'\wedge *B' - \frac{1}{4\pi\Lambda}\int F_A\wedge *F_A\right],$$

where we shifted $B' = B - \frac{i}{\Lambda} * F_A$. The integral in B' is Gaussian, and can be absorbed into the normalization of the path integral, leaving us with

(8.33)
$$\frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\exp\left[-\frac{1}{4\pi\Lambda}\int F_A\wedge *F_A\right],$$

the familiar path integral from abelian gauge theory, with gauge coupling $\mathrm{e} = \sqrt{\Lambda}$.

Now we want to map operators. An analogous calculation to the one we did in two dimensions shows that $d\phi$ extends to the covariant derivative $D_B\phi = d\phi + B$ in the larger theory, and becomes $\frac{i}{\Lambda} * F_A$ in the dual theory. This is the meaning in the quantum theory of the duality between solutions of the Laplace equation and solutions of Maxwell's equation.

What about operators such as $e^{in\phi}(P)$? We will not do this, though I will state the answer. $e^{in\phi}(P)$ is mapped, in the dual gauge theory, to the instruction: delete the point P from the three-manifold M and consider the path integral for connections on a line bundle $\mathscr{L}$ whose first Chern class evaluates to n on a small

sphere around P. The argument leading to this is quite along the lines of what we did in two dimensions.

I will not present that argument (which I recommend as an exercise) and instead discuss what is roughly the reverse. We consider Wilson loop operators in the gauge theory, and try to map those back to the scalar field theory.

Let C be a circle in M which is a boundary, and let $\lambda \in \mathbb{R}$. We have an operator $\exp(i\lambda \oint_C A)$ in the gauge theory. Let us insert this into the path integral of the big theory (the one with Lagrangian $\mathcal{L}(\phi, B, A)$). (More generally, we can do this even if C is not a boundary, but then λ must be restricted to be an integer. C being a boundary, say of a two-surface D, allows us to write $\exp(i\lambda \oint_C A) = \exp(i\lambda \int_D F)$, showing gauge-invariance for any λ.)

When we integrate out A with the extra factor included in the path integral, instead of getting $\delta(F_B = 0)$ we get $\delta(F_B = 2\pi\lambda[C])$ with $[C]$ being the Poincaré dual to C. In other words, performing the A integral determines B to be such that $F_B = 2\pi\lambda[C]$, and the modified Lagrangian after integrating out A is $\int D_B\phi \wedge * D_B\phi$. Here B is a flat $U(1)$-connection with monodromy $e^{2\pi i\lambda}$ around the circle. So the Wilson loop operator for a circle C is dual to a recipe "delete C and interpret ϕ as a section of a flat circle bundle with additive monodromy $2\pi\lambda$ around C." In other words, we modify ϕ to $\widetilde{\phi}$ such that $d\widetilde{\phi} = d\phi + B$, and interpret $\widetilde{\phi}$ as a section of a trivial S^1-bundle on $M - C$ with a flat connection of monodromy $e^{2\pi i\lambda}$ around C. (To get such a trivial S^1-bundle, we either need to assume that C is a boundary, or that λ is an integer. Everything hangs together, since these are also the conditions for the Wilson line we started with to be gauge-invariant.)

§8.4. Application to the Polyakov model

We will describe a model constructed by Polyakov around 1975. It was the first use of duality in a nonlinear relativistic theory. The model exhibits the phenomenon known as confinement.

We work in three dimensions, and study $SO(3)$ gauge theory with a scalar field s in the 3-dimensional representation, governed by the Lagrangian

$$\mathscr{L} = \frac{1}{4\pi e^2}\int d^3x\ \mathrm{Tr}\,|F|^2 + \int d^3x\,(D\vec{s})^2 + \int d^3x\,\lambda(\vec{s}^2 - a^2)^2. \tag{8.34}$$

This model exhibits the Higgs mechanism at tree level.

Classically, the vacuum can be rotated to

$$\langle \vec{s} \rangle = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \tag{8.35}$$

by a gauge transformation. In the vacuum state, the $SO(3)$ symmetry of the Lagrangian is broken to an $SO(2) = U(1)$ symmetry, and the only massless field is the $U(1)$-connection. (This is the so-called Higgs mechanism.) The low-energy theory looks like the $U(1)$ theory in three dimensions.

First Step. This theory looks classically, at low energies, like a $U(1)$ gauge theory with Lagrangian

$$\mathcal{L} = \frac{1}{4e^2}\int F \wedge * F. \tag{8.36}$$

Is that the answer? Is the Gaussian fixed point of the free $U(1)$ theory stable?

One possible source of instability is a Chern-Simons interaction

$$\frac{-in}{2\pi}\int A\wedge F_A \tag{8.37}$$

which could be added to the original Lagrangian $\mathcal{L}$. The free theory (8.36) is conformally invariant if A has dimension $1/2$, so the Chern-Simons interaction has dimension two and thus is a relevant perturbation. (With the Chern-Simons interaction included, the Lagrangian is still quadratic. A straightforward exercise – see homework problem FP4 from the fall term – shows that the Lagrangian with this term included describes a free massive theory.)

There are two obstacles to the Chern-Simons term being a source of instability in our problem. The first is that the coefficient n in the Chern-Simons term must be an integer. This implies that even if you can only approximately calculate the theory, you can determine n if the approximation can be made arbitrarily accurate. One can, in particular, calculate n in perturbation theory, and – as higher loop terms would involve positive powers of e – it could only arise from a one-loop term. So the effective n in *any* three-dimensional gauge theory can be determined by an explicit one-loop computation. In the specific example we are considering here, one can simply notice that the Chern-Simons term is odd under parity (i.e., it depends on a choice of orientation of the 3-manifold) whereas our original theory is not. The parity-invariance is a symmetry of the one-loop determinants (whether or not parity ultimately is spontaneously broken at low energies) and ensures that $n = 0$.

There is no other relevant interaction that can be written in $U(1)$ gauge theory in three dimensions, so anything which could make this theory unstable will be hard to describe in terms of A. However, we have seen that the $U(1)$ gauge theory is equivalent (dual) to a scalar theory

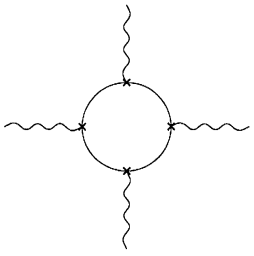
$$\mathcal{L} = \frac{\mathrm{e}^2}{4\pi}\int d^3x\, \partial_\alpha\phi\partial^\alpha\phi \tag{8.38}$$

In this theory, we could add a term

$$g\int d^3x\, \cos n\phi \tag{8.39}$$

for some n. We have chosen the potential to be periodic as ϕ is really a map to a circle. Notice that the theory with $g \neq 0$ has a mass gap, though the theory with $g = 0$ does not. Thus, we have identified something which qualitatively changes the physics, but can only be conveniently interpreted by means of the dual variables. This is strange, because doing the duality requires going to a low energy description to begin with!

Does this effect occur in the original $SO(3)$ theory? First we ask whether it occurs in perturbation theory. Consider the Feynman diagrams of the $SO(3)$ theory. We represent these with massless modes given by wavy lines, massive ones by solid lines. A typical Feynman diagram such as



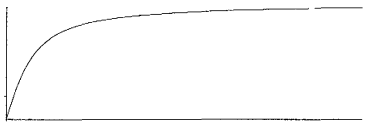
will be completely tame: we have massive propagators, which are analytic in the momenta at low momentum. Such diagrams merely give corrections to the effective action for the $U(1)$ gauge field A which can be described as additional local, gauge-invariant terms in the Lagrangian.

We need something completely different: *instantons*. Let us work on $\mathbb{R}^3$. We have an $SO(3)$ bundle on $\mathbb{R}^3$ whose structure group has been reduced to $U(1)$ at infinity. On the S^2 at infinity, a $U(1)$ bundle $\mathscr{L}$ can have a nontrivial first Chern class, $c_1(\mathscr{L}) \neq 0$.

Since $\mathbb{R}^3$ is contractible, the $SO(3)$ bundle on $\mathbb{R}^3$ is trivial. Pick a trivialization of this $SO(3)$ bundle, so that we can identify the spatial $\mathbb{R}^3$ with the bundle $\mathbb{R}^3$, and simply treat our section s as a map $s : \mathbb{R}^3 \to \mathbb{R}^3$. To construct such a map with nontrivial topology on the S^2 at infinity, we take s of the form

$$s(\vec{x}) = a\frac{\vec{x}}{|x|}f(|x|), \tag{8.40}$$

with $f(|x|)$ an increasing function satisfying $f(0) = 0$, $\lim_{|x|\to\infty} f(|x|) = a$, e.g.,



Such a section is invariant under combined rotations of space and gauge rotations.

We would like the minimum action solution s in this class. It should be spherically symmetric, with asymptotic behavior $Ds \sim \frac{1}{|x|^2}$ for $|x| \to \infty$. This leads to an ODE, which has a unique solution. The action can be written as (I/e^2) for some constant I, so it will diverge as the coupling goes to 0.

Note that since $\vec{s}$ vanishes precisely at the origin, the structure group is reduced to $U(1)$ away from the origin, but this reduction does not extend over the origin. In fact, over a sphere surrounding the origin, the line bundle has a nonzero first Chern class (which actually is 2 if we work in $SO(3)$; that is, the adjoint bundle of $SO(3)$ decomposes over a two-sphere surrounding the origin as $\mathscr{O} \oplus \mathscr{L} \oplus \mathscr{L}^{-1}$ where $\mathscr{L}$ has degree 2).

From this point of view, the difference between a pure $U(1)$ theory and a theory that looks like a $U(1)$ theory only at long distances is that the latter can have bundles that over a large S^2 at infinity have a nonzero first Chern class. The Chern class can be any multiple of the 2 that was found in the explicit solution that we just described.

Note that in abelian gauge theory, it is not possible to use a bundle $\mathscr{L}$ for which $c_1(\mathscr{L}) \neq 0$ over a sphere around the origin, because such a bundle does not extend over $\mathbb{R}^3$. The qualitative difference between an $SO(3)$ theory broken to $U(1)$ and a $U(1)$ theory is that the former admits singularities where the $U(1)$ description breaks down in this way.

If we did a similar thing in $3+1$ dimensions, we would find a time-independent solution of finite energy (rather than finite action). This solution looks like a particle sitting there, and in fact is a magnetic monopole. (The nonzero magnetic charge comes from the fact that $c_1(\mathscr{L}) \neq 0$, which implies that the magnetic field integrates over a sphere at infinity to a nonzero value.) The point of making this analogy is that we can think of the instanton as behaving like a zero-time slice of a magnetic monopole.

What do instantons look like at long distances? A monopole has a field which behaves like

$$F = \text{const}\,\frac{\vec{x}}{|x|^3} = \text{const}\,\frac{\hat{x}}{|x|^2} \tag{8.41}$$

(by Maxwell's equations).

The contribution to the path integral from each instanton is[3] e^{-I/e^2}, which is extremely small for small e.

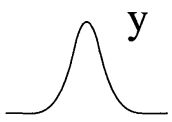
To return to our theory, we now ask how it behaves at long distance. For example, how does a two-point function $\langle F(x)\,F(0)\rangle$ behave as $|x| \to \infty$?

The answer in the free theory, by dimensional analysis, is $1/|x|^3$. (One sees that the curvature F has dimension 3/2 in three-dimensional theories by considering the basic term $\frac{1}{4\pi \mathrm{e}^2}\int F\wedge *F$ in the Lagrangian.) A bit more precisely, we are asserting that

$$\langle F_{ij}(x)\,F_{k\ell}(0)\rangle = \frac{\mathrm{e}^2}{|x|^3}(\delta_{ik}\delta_{j\ell} + \dots). \tag{8.42}$$

The Feynman diagrams don't affect this asymptotic behavior: there is a slight renormalization of e^2 or equivalently of the coefficient of $1/|x|^3$, plus other corrections which are unimportant at big distances.

How does an instanton affect this analysis? Consider an instanton localized near y, and its effect on the two-point function between x and 0.



F(x) • • F(0)

The leading approximation to the path integral in the instanton sector is

$$e^{-I/\mathrm{e}^2}\int d^3y\,\frac{(\vec{x}-\vec{y})}{|x-y|^3}\,\frac{\vec{y}}{|y|^3}. \tag{8.43}$$

Here we integrate over the position y of the instanton; the integrand is simply the classical value of $F(x)F(0)$ in the instanton field. We have denoted the classical action of the instanton as I/e^2 and included a factor of $\exp(-I/\mathrm{e}^2)$ with the integral over instanton position because this is the probability per unit volume for an instanton to be present. (I is independent of e for $\mathrm{e} \to 0$.) The integral scales as $1/|x|$. Thus, the overall behavior, adding the perturbative contribution to the instanton correction, is

$$\langle F(\vec{x})\,F(\vec{0})\rangle \cong \frac{1}{Z}\left(\text{pert. theory} + e^{-I/\mathrm{e}^2}(\text{instanton sector})\right) \tag{8.44}$$

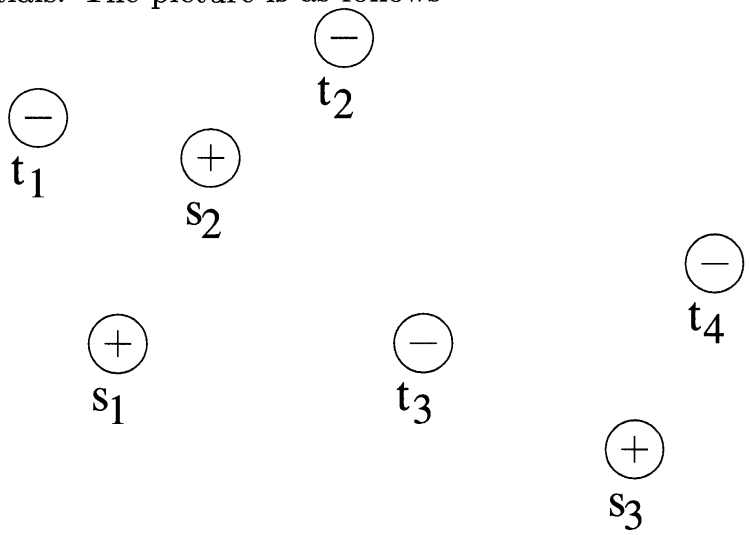
$$\cong \frac{1}{Z}\left((1+\dots)\frac{1}{|x|^3} + e^{-I/\mathrm{e}^2}\left(\frac{1}{|x|}+\dots\right)\right). \tag{8.45}$$

[3]We use 'e' to denote the gauge coupling, as is standard in abelian gauge theory. This should not be confused with 'e', the base of the natural logarithms.

Since the instanton contribution is more important in the infrared than the free theory term ($1/|x|$ compared to $1/|x|^3$), the instanton triggers an instability.

To see in more detail what is happening in the infrared, we first note that instantons are *rare*: the probability to have an instanton in a small volume V_0 is proportional to $V_0 e^{-I/\mathrm{e}^2}$, so the volume of space per instanton is given by $V \sim e^{I/\mathrm{e}^2}$, and the spatial separation between two of them is $R \sim V^{1/3} \sim e^{I/3\mathrm{e}^2}$.

In the infrared, though, if we consider a large enough volume of space, we will get lots of instantons, which we can treat as a gas of particles of definite size. The particles are *charged*, so we can't ignore the interactions between them, given by Coulomb potentials. The picture is as follows



We have labeled the positions of instantons (which are positively charged) with s_i's, and the positions of anti-instantons (which are negatively charged) by t_j's. The sum over all of these takes the form

$$\sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int d^3 s_i \Big|_{i=1}^{n} \int d^3 t_j \Big|_{j=1}^{m} \exp\left(-\frac{I}{\mathrm{e}^2}(n+m)\right) \exp\left(\frac{1}{4\pi\mathrm{e}^2} \sum_{i<j} \left(\frac{1}{|s_i - s_j|} + \frac{1}{|t_i - t_j|}\right) - \frac{1}{4\pi\mathrm{e}^2} \sum_{i,j} \frac{1}{|s_i - t_j|}\right). \tag{8.46}$$

In the integrand, we have written the exponential of the action for a configuration with widely separated instantons at points s_i and anti-instantons at points t_j. The main term in the action is just $(n+m)I$, as if the instantons and anti-instantons do not interact. But there is also an interaction potential

$$\frac{1}{4\pi\mathrm{e}^2} \frac{1}{|s_i - s_j|} \tag{8.47}$$

between instantons at points s_i, s_j, and similar terms, proportional to $1/|t_i - t_j|$ or $-1/|s_i - t_j|$, for two anti-instantons or an instanton and an anti-instanton. These terms amount to the classical "Coulomb potential" between instantons. The instantons behave as "magnetically charged particles" in three dimensions; in three dimensions the potential of interaction between charged particles separated a distance r is proportional to $1/r$, and this leads to the claimed interaction terms. The presence of these long-range interactions between instantons is the difference between the present problem and the dilute instanton gas that we studied in a more elementary example last week.

To compute the desired two-point function of F, we also need an operator insertion, of

$$F(x) = \sum \frac{x - s_i}{|x - s_i|^3} - \sum \frac{x - t_j}{|x - t_j|^3}, \tag{8.48}$$

(where we approximate $F(x)$ by its value in the instanton gas) and similarly for $F(0)$.

The physics involved in performing the sum (8.46) is the classical statistical mechanics of a plasma in space, with chemical potential I/e^2, energy E given by the Coulomb potential between the instantons and anti-instantons, and temperature $T = 4\pi\mathrm{e}^2$.

The phenomenon we need is known as "Debye screening"—a plasma screens external charges. As a result of this screening, the system will have a mass gap.

Here is a quick mathematical derivation of Debye screening (in this context). We go back to our low-energy theory, written in dual variables—as a scalar theory with $F = *\,d\phi$. We will add a term to the Lagrangian to account for the instanton effect: the term we will use (justified by the results of the calculation to come) is $\int e^{-I/\mathrm{e}^2}(e^{2i\phi}+e^{-2i\phi})$. (The 2 is present in the exponent because the basic instanton has first Chern class 2.) The operator insertions of F become insertions of $*\,d\phi$, and the two-point function $\langle F(x)F(0)\rangle$ can be written:

$$\Omega = \int \mathcal{D}\phi\ (*\,d\phi(x))\,(*\,d\phi(0)) \exp\left[-4\pi\mathrm{e}^2 \int |d\phi|^2 + \int e^{-I/\mathrm{e}^2}(e^{2i\phi} + e^{-2i\phi})\right]. \tag{8.49}$$

First we want to show that this is a correct description in the dual variables, then we will analyze this expression.

We will expand Ω in perturbation theory:

$$\Omega = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int \mathcal{D}\phi \exp\left(-\frac{1}{4\pi\mathrm{e}^2}\int d\phi \wedge *\,d\phi\right)(*\,d\phi(x))\,(*\,d\phi(0)) \cdot \cdot \left(\int e^{-I/\mathrm{e}^2} e^{2i\phi(y)} d^3y\right)^n \left(\int e^{-I/\mathrm{e}^2} e^{-2i\phi(z)} d^3z\right)^n. \tag{8.50}$$

We expand further, using the principle that

$$\left(\int dy\, f(y)\right)^n = \int dy_1 \ldots dy_n\, f(y_1)\ldots f(y_n). \tag{8.51}$$

Thus, we can rewrite (8.50) as

$$\Omega = \sum_{n,m=0}^{\infty} \int \mathcal{D}\phi \int d^3s_i|_{i=1}^n d^3t_j|_{j=1}^m\,(*\,d\phi(x))\,(*\,d\phi(0)) \cdot \cdot \exp\left(-\frac{I}{\mathrm{e}^2}(n+m) + 2i\sum_i(\phi(s_i) - \phi(t_i))\right) \exp\left(-4\pi\mathrm{e}^2\int |d\phi|^2\right). \tag{8.52}$$

The ϕ-integral is Gaussian after the square has been completed; that integral will contribute a factor

$$\exp\left(-\frac{1}{4\pi \mathrm{e}^2}\left(\sum_{i<j} G(s_i,s_j)+\sum_{i<j} G(t_i,t_j)-\sum_i\sum_j G(s_i,t_j)\right)\right) \tag{8.53}$$

to the overall answer, where $G(s,t) = 1/|s-t|$ is the Green's function for the Laplacian. We thus recover the previous formula.

Note that we omit from the exponent terms $G(s_i,s_i)$ and $G(t_j,t_j)$. From the point of view of the scalar field theory, omitting these terms is the process of renormalization. From the point of view of the original $SO(3)$ gauge theory, the terms involving the Green's functions all appear in approximating the action of a gas of widely dispersed instantons and anti-instantons. In this approach, diagonal terms do not arise, because the action of a single instanton (or anti-instanton) has already been computed in the constant I/e^2, and an additional divergent correction $G(s_i,s_i)$ or $G(t_j,t_j)$ does not appear. Thus, the low energy theory needs a seemingly arbitrary renormalization whose rationale is clear from the point of view of the high energy theory; this, of course, is a typical situation.

The lesson we have learned is this: our original problem ($SO(3)$ gauge theory) can be described in the infrared, using dual variables, by means of the Lagrangian

$$\frac{1}{4\pi \mathrm{e}^2}\int |d\phi|^2 + 2e^{-I/\mathrm{e}^2}\int d^3x\,\cos 2\phi. \tag{8.54}$$

The second term in unrenormalizable, but is well-behaved in the infrared. Expanding around the minimum of the potential, we see that ϕ has a mass. Thus, what used to be a massless $U(1)$ photon was (i) dualized and reinterpreted as a scalar, and (ii) got a mass that was more easily described in that language.

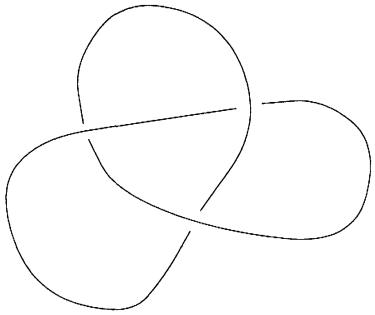
Here is another way to understand the above derivation. Start with the massless scalar field and dualize it, as in section 3 of this lecture, to a massless gauge field A. Now perturb the theory of the massless scalar by adding a weak perturbation $\epsilon\int d^3x(e^{2i\phi}+e^{-2i\phi})$. In terms of the gauge field, the operator $e^{2i\phi(P)}$ becomes, as we noted at the end of section 3, an instruction "delete the point P, and consider bundles with first Chern class 2 on a small sphere around P." In other words, from the point of view of an $SO(3)$ that looks like $U(1)$ at low energies, the instruction is "include an instanton at P." If we take the interaction term $\epsilon\int d^3x(e^{2i\phi}+e^{-2i\phi})$, and expand in powers of ϵ, we simply generate the instanton gas, with each insertion of $e^{2i\phi}$ or $e^{-2i\phi}$ corresponding to an instanton or anti-instanton.

Now we want to study confinement. An important preliminary is to note the symmetries of the problem. In general, in the duality from an abelian gauge theory, ϕ is an angular variable, a map to a circle, so ϕ is equivalent to $\phi+2\pi$. However, because the instanton-induced interaction is a trigonometric function of 2ϕ, there is a symmetry under $\phi\to\phi+\pi$.

Remark. The instanton interaction is a function of 2ϕ because the instanton has $c_1 = 2$. That in turn is because $SO(3)$ has a non-trivial fundamental group $\mathbb{Z}_2$ and hence a nontrivial double cover $SU(2)$. The relation between these facts is as follows. Our theory could be probed by particles in the two-dimensional representation of $SU(2)$. We wrote the $SO(3)$ bundle at infinity as $\mathcal{O}\oplus\mathscr{L}\oplus\mathscr{L}^{-1}$. The associated $SU(2)$ bundle (which exists because $\mathbb{R}^3$ is contractible, so the $SO(3)$

bundle has $w_2 = 0$) is then $\mathscr{L}^{1/2} \oplus \mathscr{L}^{-1/2}$. Existence of a line bundle $\mathscr{L}^{1/2}$ is the topological reason that $c_1(\mathscr{L})$ is even, and hence that the theory of ϕ has the symmetry $\phi \to \phi + \pi$. Now, according to the first Remark in Lecture 7, §8, a precise criterion for confinement in this theory exists only because of the possibility of probing it by charges in the two-dimensional representation of $SU(2)$ (or equivalently, any representation on which the center acts nontrivially). Since this possibility leads directly to the $\phi \to \phi + \pi$ symmetry of the low energy theory, it is fitting that, as we will soon see, that symmetry plays a role in the analysis of confinement.

In the original gauge theory description of this theory, we can consider a curve $C \subset M$ – since we are in three dimensions, it could be knotted, as in the figure!



Knotted Wilson lines are one thing that we do not have outside of three dimensions! For a given C, we can consider the associated Wilson line operator

$$\langle \mathrm{Tr}_R \, \mathrm{Hol}(A, C) \rangle \tag{8.55}$$

where R is the 2-dimensional representation of $SU(2)$ and Hol denotes the holonomy. What does this operator translate to in our infrared description in terms of ϕ? As we have seen at the end of section 3, it translates into a recipe "delete C from spacetime and interpret ϕ as a section of a flat circle bundle with a connection that has monodromy around C." The monodromy is given by the angle $2\pi\lambda$ where λ, introduced in the discussion in section three, is the $U(1)$ charge appearing in the Wilson loop, modulo 1. We have chosen a Wilson line in the two-dimensional representation of $SU(2)$; the $U(1)$ charges are $\pm 1/2$ in units of the weight lattice of $SO(3)$ (a fact exploited in a different way in the above Remark). Hence our example has $\lambda = 1/2$. The monodromy around the circle is thus $\phi \to \phi + \pi$, which, as we noted a moment ago, is a symmetry of the theory even with the instanton-induced interaction.

The problem is now classical: we want to find the minimum of the action

$$\frac{1}{4\pi \mathrm{e}^2} \int |d\phi|^2 - 2e^{-I/\mathrm{e}^2} \int d^3x \, (\cos 2\phi - 1) \tag{8.56}$$

with ϕ a circle-valued function that is required to have monodromy π around C. (We have shifted ϕ so that the minimum energy is at $\phi = 0, \pi$, and subtracted from the action a constant so that the vacuum energy is zero.) Thus, in particular, because of the monodromy, ϕ cannot be constant. In fact, the least action solution will be given approximately in terms of a minimal area surface Σ with $\partial\Sigma = C$; ϕ will be constant far from Σ, and will jump by π in crossing Σ.

The easiest case is when C is a large curve in the plane. Consider the plane in $\mathbf{R}^3$ given by $x = 0$, with x a linear function on $\mathbf{R}^3$. Let C be a very large, essentially infinite, curve in this plane. We want ϕ to jump by π in going from negative x to positive x. In the limit that C is very large, ϕ, if observed somewhere deep in the

interior of C, becomes a function of x only. What needs to be minimized is then

$$\int dx \left(\left(\frac{d\phi}{dx} \right)^2 - 2e^{-I/e^2} (\cos 2\phi - 1) \right) \tag{8.57}$$

with the boundary conditions that $\phi \to 0$ for $x \to -\infty$ and $\phi \to \pi$ for $x \to +\infty$. This variational problem, with the boundary conditions, has a solution that is unique up to translation of x, with ϕ approaching its asymptotic value exponentially fast (because of the mass gap). Let S be the action of this one-dimensional solution.

Now in general, if D is the minimal area surface with boundary C, and $A(D)$ is its area, the minimum action ϕ that is a section of the appropriate flat bundle has the property that ϕ is very near zero or π except near D, jumps by π in crossing D, and looks in profile near D just like the solution of the idealized one-dimensional problem discussed in the last paragraph. Its action is very nearly $A(D)S$, so the expectation value of the Wilson line is approximately $e^{-A(D)S}$, showing the area law and confinement.

§8.5. Duality in four dimensions and $SL(2,\mathbb{Z})$

We next turn to a discussion of abelian duality in four dimensions, and give an application to an $SL(2,\mathbb{Z})$ symmetry of the free $U(1)$ theory in four dimensions. We postpone discussion of $SL(2,\mathbb{Z})$ symmetries of non-free theories to a later lecture, since all known examples of that involve supersymmetry.

We work with a $U(1)$ bundle $\mathscr{L}$ on a 4-manifold M, and a connection A on $\mathscr{L}$, whose curvature is $F{=}F_A$. The gauge theory Lagrangian (in Euclidean signature) including the topological term is

$$\begin{aligned} \mathcal{L}(A) &= \frac{1}{4e^2} \int d^4x \sqrt{g} F_{mn} F^{mn} + \frac{i\theta}{16\pi^2} \int d^4x \sqrt{g} \epsilon_{mnpq} F^{mn} F^{pq} \\ &= \frac{1}{2e^2} \int F_A \wedge *F_A + \frac{i\theta}{4\pi^2} \int F_A \wedge F_A. \end{aligned} \tag{8.58}$$

We have used the standard normalization on the kinetic term, and have normalized the topological term so that replacing θ by $\theta + 2\pi$ does not change the physics. (This property of the topological term derives from the fact that $c_1(\mathscr{L})^2 = \int (F_A/2\pi) \wedge (F_A/2\pi)$ is always an integer. Notice that on a spin manifold, $c_1(\mathscr{L})^2$ is always an *even* integer, and we gain an additional equivalence under replacement of θ by $\theta + \pi$.)

Let $\tau = \frac{\theta}{\pi} + \frac{2\pi i}{e^2} \in \mathfrak{h}$. As we have just observed, $\tau \mapsto \tau + 2$ is a symmetry of this theory, and $\tau \mapsto \tau + 1$ is a symmetry when working on a spin manifold. To extend this to an $SL(2,\mathbb{Z})$ action (in the spin manifold case) we also need a symmetry which maps τ to $-1/\tau$; this will be given by a *duality* transformation $F_A \leftrightarrow *F_C$ (with C being a new "dual" connection).

The computations for this duality transformation are similar to those in §§8.2, 8.3. We begin by defining $F_\pm = \frac{1}{2}(F_A \pm *F_A)$, and rewriting our Lagrangian (8.58) as

$$\begin{aligned} \mathcal{L}(A) &= \frac{i\overline{\tau}}{4\pi} \int F_+ \wedge F_+ + \frac{i\tau}{4\pi} \int F_- \wedge F_- \\ &= \frac{i\overline{\tau}}{4\pi} \int \|F_+\|^2 - \frac{i\tau}{4\pi} \int \|F_-\|^2. \end{aligned} \tag{8.59}$$

Letting $\mathcal{G}$ denote the gauge group associated to A, the partition function for this theory can be written as

$$Z(\tau) = \frac{1}{\mathrm{vol}(\mathcal{G})} \sum_{\mathscr{L}} \int \mathcal{D}A \ \exp\left(-\frac{i\overline{\tau}}{4\pi} \int \|F_+\|^2 + \frac{i\tau}{4\pi} \int \|F_-\|^2 \right). \tag{8.60}$$

Our earlier examples of duality began with a theory of a scalar field ϕ which entered into the Lagrangian only through its derivative $d\phi$ so that the theory had a symmetry under $\phi \mapsto \phi + c$ (with c constant); the first step in the duality transformation was to gauge this symmetry, introducing also an appropriate Lagrange multiplier field.

The present theory is already a gauge theory, being a theory of a connection A which enters into the Lagrangian only through its curvature F_A, so that there is a symmetry under $A \mapsto A + B$ (with B a flat connection). We want to do the analogue of gauging this symmetry, by allowing B to be an arbitary connection on an arbitrary bundle, introducing a kind of "exotic gauge field" G which is a 2-form field, and extending the symmetry to

$$\begin{aligned} A &\to A + B \\ G &\to G + F_B. \end{aligned} \tag{8.61}$$

Then $\mathcal{F} := F_A - G$ plays the role of the "gauge-invariant" quantity, analogous to the covariant derivative of a scalar field. It is to be stressed that two G fields will be considered gauge-equivalent if they differ by $G \to G + F_B$ for F_B the curvature of any connection on any line bundle. In our analysis, we will assume for simplicity that there is no torsion in $H^2(M)$.

We need a "gauge-invariant" extension of our Lagrangian. We might try

$$\mathcal{L}(A, G) = \frac{i\overline{\tau}}{4\pi} \int \|\mathcal{F}_+\|^2 - \frac{i\tau}{4\pi} \int \|\mathcal{F}_-\|^2, \tag{8.62}$$

but this is too simple (because, for example, the classical equations give $\mathcal{F} = 0$). To improve this, we introduce a new connection C on a new line bundle $\mathscr{N}$, with curvature F_C, and consider the Lagrangian

$$\mathcal{L}(A, G, C) = \frac{i\overline{\tau}}{4\pi} \int \|\mathcal{F}_+\|^2 - \frac{i\tau}{4\pi} \int \|\mathcal{F}_-\|^2 - \frac{i}{2\pi} \int F_C \wedge G. \tag{8.63}$$

The partition function for this new theory can be represented as a path integral, which includes sectors associated to all choices of bundles $\mathscr{L}$ and $\mathscr{N}$:

$$\begin{aligned} &\frac{1}{\mathrm{vol}(\widetilde{\mathcal{G}})} \frac{1}{\mathrm{vol}(\mathcal{G})} \frac{1}{\mathrm{vol}(\mathcal{G}_C)} \sum_{\mathscr{L},\mathscr{N}} \int \mathcal{D}A\, \mathcal{D}G\, \mathcal{D}C \\ &\qquad \exp\left(-\frac{i\overline{\tau}}{4\pi} \int \|\mathcal{F}_+\|^2 + \frac{i\tau}{4\pi} \int \|\mathcal{F}_-\|^2 + \frac{i}{2\pi} \int F_C \wedge G \right), \end{aligned} \tag{8.64}$$

where $\mathcal{G}$ and $\mathcal{G}_C$ denote the gauge groups associated to A and C, respectively, and $\widetilde{\mathcal{G}}$ denotes the "exotic" gauge group.

To see that this new theory is equivalent to the original one, we first do the C-integral in (8.64): write $C = C_0 + C'$, for C_0 a fixed connection on the line bundle $\mathscr{N}$. Then the C' integral is

$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C} \int \mathcal{D}C'\, e^{\frac{i}{2\pi}\int C' \wedge dG} = \delta(dG). \tag{8.65}$$

Thus, when we sum over $\mathscr{N}$ we find

(8.66)
$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C}\sum_{\mathscr{N}}\int \mathcal{D}C e^{-\frac{i}{2\pi}\int F_C\wedge G} = \sum_{x\in H^2(M)} e^{i(x,G)}\delta(dG) = \delta\left(\left[\frac{G}{2\pi}\right]\in\mathbb{Z}\right)\delta(dG).$$

The conditions that $dG = 0$ and that $[G/2\pi]$ is an integral class precisely mean that G is of the form F_B for some connection on some line bundle and hence that G can be gauged to zero. After doing this, it follows that the partition function (8.64) coincides with $Z(\tau)$, and we recover the original theory.

Alternatively, we can evaluate the partition function (8.64) by gauging A to 0, using the "exotic" gauge invariance. This leaves the path integral

(8.67)
$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C}\sum_{\mathscr{N}}\int \mathcal{D}G\int \mathcal{D}C\,\exp\left(-\frac{i\overline{\tau}}{4\pi}\int \|G_+\|^2 + \frac{i\tau}{4\pi}\int\|G_-\|^2 + \frac{i}{2\pi}\int F_C\wedge G\right).$$

To evaluate the G integral, we complete the square, bearing in mind that

$$\int F_C\wedge G = \int (F_{C+}\wedge *G_+ - F_{C-}\wedge *G_-). \tag{8.68}$$

In fact, if we define $G' = G - \frac{1}{\overline{\tau}}F_{C+} + \frac{1}{\tau}F_{C-}$, then we can write the exponent from eq. (8.67) as

$$-\frac{i\overline{\tau}}{4\pi}\int\|G'_+\|^2 + \frac{i\tau}{4\pi}\int\|G'_-\|^2 + \frac{i}{4\pi\overline{\tau}}\int\|F_{C+}\|^2 - \frac{i}{4\pi\tau}\int\|F_{C-}\|^2. \tag{8.69}$$

When we carry out the G' integral, the first two terms give a Gaussian integral which contributes to the overall normalization; integrating out G' leaves the path integral

$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C}\sum_{\mathscr{N}}\int\mathcal{D}C\,\exp\left(\frac{i}{4\pi\overline{\tau}}\int\|F_{C+}\|^2 - \frac{i}{4\pi\tau}\int\|F_{C-}\|^2\right). \tag{8.70}$$

This is the same as the original path integral, but with τ replaced by $-1/\tau$, precisely what we wanted to show.

As we did in the case of two dimensions, it is possible to analyze the τ-dependence of the normalization of the path integral, and obtain further interesting results. Some hint of the flavor of the results to be obtained this way is seen if we evaluate the Gaussian integral over G', which yields

$$\left(\frac{2\pi}{\sqrt{i\overline{\tau}}}\right)^{n_{2+}}\left(\frac{2\pi}{\sqrt{-i\tau}}\right)^{n_{2-}}, \tag{8.71}$$

where $n_{2\pm}$ denote the numbers of self-dual and anti-self-dual 2-forms. Of course, these numbers are infinite, so there must be some cancellation against other normalization factors. When this is worked out in detail,[4] the result is found to be

$$Z(\tau) = \tau^{-(\chi+\sigma)/4}\overline{\tau}^{-(\chi-\sigma)/4}Z(-1/\tau), \tag{8.72}$$

where χ and σ are the Euler number and signature of M, respectively. Thus, the partition function $Z(\tau)$ is actually a modular form for $SL(2,\mathbb{Z})$ (or for a subgroup, when the manifold is not spin) of weight $(\frac{\chi+\sigma}{4}, \frac{\chi-\sigma}{4})$.

[4]See my paper cited in the References.

We can also follow certain operator insertions through the duality transformation, as we did in lower dimensions. An insertion of $F_\pm$ in the original theory can be realized by inserting $\mathcal{F}_\pm = F_\pm - G$ in the extended theory, which can be written

$$\mathcal{F}_+ = F_+ - G'_+ - \frac{1}{\overline{\tau}} F_{C+}, \text{ or } \mathcal{F}_- = F_- - G'_- + \frac{1}{\tau} F_{C-}, \tag{8.73}$$

respectively, after making the change of variables to G'. Thus, when we gauge A to zero, and integrate out G', we are left with operator insertions proportional to $F_{C\pm}$, namely:

$$F_+ \mapsto (-1/\overline{\tau}) F_{C+}, \text{ and } F_- \mapsto (1/\tau) F_{C-}. \tag{8.74}$$

Notice that as a consequence of the τ-dependence of these mappings, a correlation function involving insertions of F_+ and F_- will have a different modular weight than that of the partition function.

§8.6. The Hamiltonian formalism

Returning to the case that the gauge group is $U(1)$, let us briefly discuss abelian four-dimensional duality in a Hamiltonian framework. Take a 4-manifold of the form $M_3 \times \mathbb{R}$, where $\mathbb{R}$ is a timelike direction. Note that this is a spin manifold, so we expect full $SL(2,\mathbb{Z})$ symmetry. For simplicity we suppose that there is no torsion in $H_1(M_3)$. Each class $x \in H^2(M_3)$ determines a complex line bundle $\mathscr{L}_x$ on the 3-manifold M_3 (satisfying $c_1(\mathscr{L}_x) = x$). The Hilbert space for our theory on the 3-manifold M_3 can be written in the form

$$\mathcal{H}_\tau(M_3) = \bigoplus_{x \in H^2(M_3,\mathbb{Z})} \mathcal{H}_x, \tag{8.75}$$

where $\mathcal{H}_x$ is the Hilbert space which comes from quantizing connections on $\mathscr{L}_x$. (On the left, we have explicitly indicated the dependence on the coupling constant τ.) To construct $\mathcal{H}_x$, write an arbitrary connection in the form $A = A_0 + \beta$, where A_0 is a harmonic connection (a connection whose curvature is a harmonic two-form) subject to some gauge-fixing condition (the details are immaterial) and β is a 1-form. Let $\mathcal{T}_x$ be the space of harmonic connections on the line bundle $\mathscr{L}_x$ mod gauge transformations. Then the quantization yields

$$\mathcal{H}_x = \mathcal{H}_\beta \otimes L^2(\mathcal{T}_x). \tag{8.76}$$

Here $\mathcal{H}_\beta$ is a Hilbert space obtained by quantizing the space of β's, and $L^2(\mathcal{T}_x)$ is just the space of L^2 functions on $\mathcal{T}_x$.

Note that the factor $\mathcal{H}_\beta$ is independent of x, since the space of one-forms is defined with no reference to x. $\mathcal{H}_\beta$ is in fact the Hilbert space for an abelian gauge theory on M_3 in which the gauge group is not $U(1)$ but its universal cover $\mathbb{R}$. Duality maps $\mathcal{H}_\beta$ to itself while acting separately on $\mathcal{H}' = \bigoplus_x L^2(\mathcal{T}_x)$. The duality action on $\mathcal{H}_\beta$ follows from the operator mapping in (8.74).

The action of duality on $\mathcal{H}'$ can be described as follows. Note that the $\mathcal{T}_x$'s are all principal homogeneous spaces acted on by the torus $H^1(M_3, \mathbb{R}/\mathbb{Z})$, which parametrizes flat line bundles on M_3; the action by an element of $H^1(M_3, \mathbb{R}/\mathbb{Z})$ is defined by tensoring any given line bundle with connection by the corresponding flat line bundle. Let y denote a character of the abelian group $H^1(M_3, \mathbb{R}/\mathbb{Z})$. There is a

decomposition $L^2(\mathcal{T}_x) = \bigoplus_y \mathcal{T}_{x,y}$, where $\mathcal{T}_{x,y}$ is the subspace of $L^2(\mathcal{T}_x)$ transforming in the character y. Each $\mathcal{T}_{x,y}$ is one-dimensional. Hence

$$\mathcal{H}' = \bigoplus_{x,y} \mathcal{T}_{x,y} \tag{8.77}$$

Note that, by Poincaré and Pontryagin duality, the character group of $H^1(M_3, \mathbb{R}/\mathbb{Z})$ is $H^2(M_3, \mathbb{Z})$. Thus, x and y take values in the same space. It is hence possible to exchange them, and this is what the $\tau \to -1/\tau$ transformation does (more precisely, it acts by $(x, y) \to (-y, x)$). Thus duality exchanges a classical notion – the decomposition with respect to x – with a quantum notion – the decomposition with respect to y. The claim about how the duality acts will be justified below where we introduce the operators Q_e and Q_m.

Upon quantization—and suppressing θ for a moment—one writes the four-dimensional curvature as $F_A' + e^2 \pi_A dt$, where F_A' is a two-form on M_3 and π_A—a one-form on M_3—is the momentum conjugate to the connection A. The Hamiltonian becomes

$$H = \int d^3x \left(\frac{1}{2\mathrm{e}^2} F_{A_0}^2 + \frac{\mathrm{e}^2}{2} \nabla_{A_0}^2 \right) + H(\beta). \tag{8.78}$$

Here $H(\beta)$ is the part of the Hamiltonian that acts on $\mathcal{H}_\beta$. The other terms act on $\mathcal{H}'$. The first term is the magnetic energy of the harmonic connection A_0; it comes from the part of the Lagrangian quadratic in F_A' and is a multiple of $\int_{M_3} x \wedge * x$. The second term, which comes from the part of the Lagrangian quadratic in π_A, is the electric energy, the Laplacian ∇^2 on $\mathcal{T}_x$; it is a multiple of $\int_{M_3} y \wedge * y$.

Including the θ term shifts the quantization. In fact, the canonical momentum π_A, which we write as $F_A^\vee = * \pi_A$ for reasons that will appear, can be determined from the original Lagrangian (8.58) and is

$$F_A^\vee = 2\pi i \frac{\delta \mathcal{L}}{\delta F_A} = \frac{2\pi i}{\mathrm{e}^2} * F_A - \frac{\theta}{\pi} F_A \tag{8.79}$$

At non-zero θ, canonical quantization gives not $\mathcal{H}' = \bigoplus_x L^2(\mathcal{T}_x)$ but

$$\mathcal{H}' = \bigoplus_x \Gamma_{L^2}(\mathcal{T}_x, \mathscr{S}_\theta), \tag{8.80}$$

where $\mathscr{S}_\theta$ is a certain flat line bundle over $\mathcal{T}_x$ and Γ_{L^2} is the space of L^2 sections. I leave it as an exercise to the reader to identify $\mathscr{S}_\theta$.[5] Of course, $\mathscr{S}_\theta$ at $\theta = 0$ is the flat bundle with trivial flat connection; and $\mathscr{S}_\theta$ only depends on θ modulo π – we are on a spin manifold so $\theta \to \theta + \pi$ is a symmetry.

Rewriting the formula for $F_A^\vee$ in terms of τ, we can determine how this operator transforms under $\tau \to -1/\tau$. Indeed, under the operator mapping (8.74) one gets

$$F_A^\vee = -\overline{\tau} F_+ + \tau F_- \mapsto F_{C+} + F_{C-} = F_C. \tag{8.81}$$

We will also need the "dual" version of this computation:

$$F_A = F_+ + F_- \mapsto \overline{(-1/\tau)} F_{C+} - (-1/\tau) F_{C-} = -F_C^\vee. \tag{8.82}$$

[5]The starting point in doing the exercise should be to observe that the θ term in the Lagrangian is, in a gauge with the connection trivialized in the time direction, homogeneous and linear in time derivatives; any such term has the effect of introducing a line bundle with connection over the phase space. In the present case, the θ term is a topological invariant and thus a locally constant function on the classical configuration space; this leads to the line bundle $\mathscr{S}_\theta$ being flat.

In the Hamiltonian formalism, for any 2-cycle $\Sigma \subset M_3$ we can define associated "electric" and "magnetic" operators on the Hilbert space $\mathcal{H}_\tau(M_3)$ for any 2-cycle $\Sigma \subset M_3$, by

$$\begin{aligned} Q_e(\Sigma) &= \int_\Sigma \frac{F_A^\vee}{2\pi} = \int_\Sigma \frac{F_C}{2\pi} \\ Q_m(\Sigma) &= \int_\Sigma \frac{F_A}{2\pi} = -\int_\Sigma \frac{F_C^\vee}{2\pi}. \end{aligned} \tag{8.83}$$

(These operators only depend on the class of Σ in $H_2(M_3, \mathbb{Z})$.) Clearly, under $\tau \to -1/\tau$, that is under $A \to C$, one has $Q_e \to Q_m$, $Q_m \to -Q_e$. Since x and y are the eigenvalues of Q_m and Q_e, this means that $\tau \to -1/\tau$ acts by $(x, y) \to (-y, x)$. Moreover, from the explicit formula (8.79) for $F_A^\vee$, one sees that when θ is increased by π, the operator Q_m is unaltered, but the operator Q_e maps to $Q_e + Q_m$. Since we are on a spin manifold, $\theta \to \theta + \pi$ is a symmetry; it corresponds to $\tau \to \tau + 1$ and leaves the Hilbert space unchanged.

The statements made in the last paragraph can be combined to the following: $SL(2, \mathbb{Z})$ acts on $\mathcal{H}'$ via the natural action of $SL(2, \mathbb{Z})$ on the pair (x, y).

References and cross references

The main references for this lecture were

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LECTURE 9
Solitons

§9.1. What is a soliton?

In classical mathematical physics, by a soliton one usually means a "traveling wave" solution of a nonlinear PDE $u_t = F(u, u_x, ...)$, i.e. a solution of the form $u(x,t) = f(x - vt)$. Solitons play a very important role in the theory of integrable systems, where (in certain cases) any solution can be approximated by a superposition of solitons, moving at different velocities. As a result, the theory of integrable systems is sometimes called soliton theory.

Today we will be interested in solitons arising in field theory (as traveling wave solutions of the classical field equations) and primarily in the role they play in quantization of field theories. This is a different point of view from the one in integrable system theory. In particular, no claim is made about nonlinear superposition of solitons, and the models we consider will not, in general, be exactly integrable.

We will consider solitons for Poincaré invariant field theories on Minkowski space. By a soliton for a particular field theory we will mean a traveling wave solution of the field equations (i.e. a solution which depends on $x-vt$), which is localized in space and has finite energy. By Poincaré invariance, we can always assume that $v = 0$, i.e. that the solution is time-independent. We will be mostly interested in solitons which provide the global minimum for the energy in the corresponding homotopy class.

Remark. It is important to distinguish solitons from instantons. Instantons are localized in Euclidean spacetime (i.e. only exist for an instant) and have finite action, while solitons are localized in space (of Minkowski spacetime), exist eternally, and have finite energy. The physical applications are completely different. Today our interest is in solitons.

§9.2. Solitons and components of the space of classical solutions

Classically, existence of solitons is related with existence of different components of the space of classical solutions of finite energy. In a connected component which does not contain a zero energy solution, the minimum of energy is often attained

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at a soliton. As an example of this, you may recall the situation discussed in Lecture 1: a two-dimensional scalar field theory with the potential $U = (\phi^2 - a^2)^2$. In this case, the space of t-independent classical solutions of finite energy has 4 components: $X_{++}, X_{-+}, X_{+-}, X_{--}$, where $X_{+-} = \{\phi : \phi(-\infty) = a, \phi(\infty) = -a\}$, and likewise for other choices of the signs. On two of these components, X_{+-} and X_{-+}, the energy is strictly positive, and its minimum in each case is attained at a soliton solution. These solutions are of the form $\phi = f(x)$ and $\phi = -f(x)$, respectively, where $f(x)$ is the solution of the Newton equation $f'' = U'(f)$ for the potential $-U$, with boundary values $\phi = a$ at $-\infty$ and $\phi = -a$ at ∞. It is elementary to prove that such a solution exists and is unique up to translations of x.

Notice that solitons are not invariant under the Poincaré group. But one often encounters solitons which have rotational symmetry in space, around some "center of mass". In a scalar field theory, this would mean that the group of symmetry of such a soliton is $P_s = \mathcal{E}O(d-1) \times \mathbb{R}$. Since the Poincaré group is $P = SO(d-1,1) \ltimes \mathbb{R}^{d-1,1}$, the P-orbit $\mathcal{O}$ of the soliton in the space of solutions, i.e. the quotient P/P_s, is a $(d-1)$-dimensional vector bundle over the upper part of a 2-sheeted hyperboloid. It is easy to check the following.

1. This bundle is naturally isomorphic to the cotangent bundle of the hyperboloid.
2. The restriction of the symplectic structure on the space of solutions $\mathcal{O}$ is nondegenerate, and thus defines a symplectic structure on $\mathcal{O}$.
3. (normalization of symplectic form) Let m be the mass (i.e. the energy in the center of mass frame) of the soliton. Then there exists a P-equivariant symplectic diffeomorphism $\mathcal{O} \to T^*\mathcal{O}_m^+$, where $\mathcal{O}_m^+$ is the upper part of the hyperboloid $x^2 = -m^2$ in $\mathbb{R}^{d-1,1}$.

§9.3. Solitons and quantization

In a case like the above example in which the space of classical solutions has several disconnected components (in this example the $X_{\pm\pm}$), one expects the quantum Hilbert space to be a direct sum of several subspaces, one for each component. If X is a component of the space of classical solutions that contains a Poincaré-invariant solution of minimum energy , then one may expect that quantization of X will give a Hilbert space containing a Poincaré invariant state. In our above example, X_{++} and X_{--} contain such Poincaré invariant solutions ($\phi = a$ or $-a$). But X_{+-} and X_{-+} do not; in those components, the solution of minimum energy is the soliton or antisoliton. Generally, in a component that does not contain a Poincaré invariant solution of least energy, the solution of least energy is some sort of soliton. So we will call such components solitonic.

Suppose that X_s is a solitonic component. Then by considering all of the Poincaré transforms of the solitonic solution, we get as above a symplectic embedding $T^*\mathcal{O}_m^+ \to X_s$, where m is the mass of the solitonic solution s. Let us assume that the minimum of energy at the image of this map is nondegenerate, in the sense that there is no other solutions of energy m, and the second derivative of the energy in a direction transversal to the image is positive.

In this case, in the weak coupling region we should expect that

1. The component $\mathcal{H}_s$ of the Hilbert space $\mathcal{H}$ of the quantum theory has no vacuum (because classically there is no Poincaré invariant solution).
2. The Hamiltonian H on $\mathcal{H}_s$ satisfies the inequality $H \geq m'$, where m' is some positive mass parameter, such that $m' \to m$ in the weak coupling limit (in general, we should expect m to get quantum corrections). If the theory has a mass gap, we should expect that the states whose mass converges to m as the coupling goes to zero form a space which is a quantization of $T^*\mathcal{O}_m^+$, i.e. the irreducible representation of P of the form $\mathbf{L}^2(\mathcal{O}_m^+)$. We should also expect that the spectrum near m is discrete since the second derivative is positive.

Note that if the action (or energy) functional of the theory is multiplied by a constant C, the mass of the soliton is also multiplied by C, while masses of usual particles do not change. Recall that in such a situation, the coupling scales as C^{-1}. This means that in the semiclassical or weak coupling approximation ($C \to \infty$), solitons are much heavier than usual particles. Therefore, a soliton cannot be seen in perturbation theory: the contribution to the correlation functions of an intermediate state containing a soliton is exponentially small in the weak coupling limit. That is why we have been able to proceed as far as we have without discussing solitons!

§9.4. Solitons in theories with fermions

In theories with fermions, the orbit of a soliton under P is often not the whole space of lowest energy states in the corresponding connected component of the space of solutions. For example, if the model is supersymmetric, it is clear a priori that the orbit is not the whole space of lowest energy solutions: the space of lowest energy states is the orbit of the super-Poincaré group and not just the Poincaré group. Often, fermions play an important role in quantization of solitons even in the absence of supersymmetry.

Let us consider an example of such a situation. Consider the theory in two dimensions with a scalar and a pair of fermions:

$$\mathcal{L} = \frac{1}{2\lambda}\int d^2x(|d\phi|^2 + (\phi^2 - a^2)^2) + i\int d^2x(\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_- - g\phi\psi_+\psi_-). \tag{9.1}$$

Remark 1. For a suitable value of g this model is supersymmetric, but we will not restrict to this value.

Remark 2. This model has a chiral symmetry $\phi \to -\phi$, $\psi_\pm \to \pm\psi_\pm$, which prohibits a mass term $m\psi_+\psi_-$ for the fermions.

Consider the soliton $\phi(x,t) = f(x)$ (interpolating between $\phi = a$ and $\phi = -a$) for the bosonic part of the theory. Fermionic extensions of this solution are functions of finite energy $\psi(x) = \begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix}$ satisfying the Euler-Lagrange equations

$$\left[\partial_x\psi + \begin{pmatrix} 0 & -gf(x)\\ -gf(x) & 0\end{pmatrix}\right]\psi = 0. \tag{9.2}$$

Thus,

$$\psi = \begin{pmatrix}\varepsilon\\ \varepsilon\end{pmatrix}\exp\left(g\int_0^x f(y)dy\right) + \begin{pmatrix}\varepsilon'\\ -\varepsilon'\end{pmatrix}\exp\left(-g\int_0^x f(y)dy\right), \tag{9.3}$$

where $\varepsilon, \varepsilon'$ are odd variables. It is easy to see that for a soliton ($f > 0$ for $x \to \infty$ and $f < 0$ for $x \to -\infty$) precisely the solutions with $\varepsilon = 0$ are in $\mathbf{L}^2$, so the space of solutions of finite energy is one-dimensional. Likewise for an antisoliton, precisely the solutions with $\varepsilon' = 0$ are in $\mathbf{L}^2$. In a vacuum sector (f has the same sign for $x \to \pm\infty$), there are no $\mathbf{L}^2$ solutions.

Considering then the soliton sector, we have thus a single zero energy $\mathbf{L}^2$ solution for the fermions. We could generalize the model slightly to have not a single pair of Fermi fields ψ_+, ψ_-, but n pairs $\psi_{\pm\imath}$, $\imath = 1, \dots n$, each interacting with ϕ via the same coupling as in equation (9.1). The fermionic part of the Lagrangian is thus

$$\sum_{\imath=1}^{n} i \int d^2x (\psi_{+\imath}\partial_-\psi_{+\imath} + \psi_{-\imath}\partial_+\psi_{-\imath} - g\phi\psi_{+\imath}\psi_{-\imath})$$

There is an $SO(n)$ symmetry acting on the fermions in the natural way. In this case, there are n fermionic solutions, so the Poincaré orbit of solutions of lowest energy is not $T^*\mathcal{O}_m^+$, but the supermanifold $T^*\mathcal{O}_m^+ \times \mathbb{R}^{0|n}$.

Therefore, if n is even, in quantum theory the space of lowest energy states in the corresponding component of the Hilbert space is $\mathbf{L}^2(\mathcal{O}_m^+)\otimes S$, where $S = S_+ + S_-$ is the spin representation of $Spin(n)$. For n odd, the description is slightly more subtle and we will not try to analyze that case today. (For an explanation of the essential point, see the end of Lecture 12, part 3.)

§9.5. Solitons in 2+1 and 3+1 dimensions

In spite of the difference between instantons and solitons, there is a connection between them. Namely, often an instanton in a Euclidean field theory in $d-1$ dimensions gives rise to a soliton in the Minkowski version of the same theory in d dimensions. Consider for example the 2+1-dimensional $U(1)$ gauge theory with a complex scalar:

$$\mathcal{L} = \int d^3x \left(\frac{F_A^2}{4e^2} + \frac{1}{\lambda}(|d_A\phi|^2 + (|\phi|^2 - a^2)^2) \right). \tag{9.4}$$

In Lecture 7 we saw that the two-dimensional version of this theory has instantons with Chern classes 1 and -1. In the 2+1-dimensional theory, these instantons become solitons, and govern the lowest energy modes in the corresponding components of the Hilbert space as described above.

Now consider nonabelian gauge theory in 3+1 dimensions. Namely, consider an $SO(3)$ gauge theory with a boson in the three-dimensional representation, and the Lagrangian

$$\mathcal{L} = \int d^4x \left(\frac{\operatorname{Tr} F_A^2}{4e^2} + \frac{1}{2\lambda}(|d_A\phi|^2 + (|\phi|^2 - a^2)^2) \right). \tag{9.5}$$

We can think of A as a connection on a rank three real vector bundle E, and ϕ as a section of E. We considered in Lecture 8 the three-dimensional version of this theory. For convenience, we identified the spacetime $\mathbb{R}^3$ with the Lie algebra of the gauge group and with the space of values of ϕ (the bracket in the Lie algebra is the cross-product). This allows us to write the scalar field ϕ and infinitesimal gauge transformations as vector fields on $\mathbb{R}^3$.

We found that in three dimensions this theory has an instanton in which ϕ is of the form $\vec{\phi} = \frac{\vec{x}}{r} f(r)$, $r = |x|$ (as we have explained, we identify the spacetime and the space of values of ϕ, so $\vec{\phi}$ and the position $\vec{x}$ are both three-vectors). In 4 dimensions this instanton will become a soliton. Such solitons are called magnetic monopoles, because they in fact carry magnetic charge in the classical sense with respect to the low energy unbroken $U(1)$ gauge theory.

To understand this better, recall that any field configuration (A, ϕ) of finite energy defines an integer topological invariant – "the first Chern class at infinity c_1." Indeed, in order for the energy to be finite, we must have $|\phi| = a$ at infinity, so ϕ defines a map from a sphere at infinity in $\mathbb{R}^3$ to the sphere of radius a, and c_1 is twice the degree of this map.

Another definition of c_1 (which may make clearer why the invariant is called c_1 and in what sense c_1 is twice the degree of this map) is as follows: the section ϕ defines at infinity a splitting of our three-dimensional vector bundle E into a direct sum $\mathbb{R}\phi \oplus \phi^\perp$ of a one-dimensional and a two-dimensional vector bundle. (Here $\mathbb{R}\phi$ is the subbundle of E consisting of vectors proportional to ϕ, and $\phi^\perp$ is the orthocomplement.) Thus, ϕ defines a reduction of the structure group to $SO(2) = U(1)$ at infinity. The number c_1 is the first Chern class of the $U(1)$ bundle $\phi^\perp$ restricted to the infinite 2-sphere in $\mathbb{R}^3$. For example, the soliton configuration discussed above has $c_1 = 2$, since in this case the $SO(2)$ bundle is the tangent bundle of the sphere at infinity, and the Euler characteristic of the sphere is 2. In the present model, c_1 measured at infinity is always even, so that 2 is the smallest possible positive value. The reason for the evenness is that, as $\mathbb{R}^3$ is contractible, the second Stiefel-Whitney class of E is zero, so we could pass to the double cover $SU(2)$ of $SO(3)$ and consider the rank two complex vector bundle W associated to E. The Higgs field ϕ splits W at infinity as $\mathcal{T} \oplus \mathcal{T}^{-1}$, with $\mathcal{T}$ some line bundle; the first Chern class of $\mathcal{T}$ is one-half the value of c_1 defined using the $SO(3)$ bundle, and the latter is hence always even. Of course, the first Chern class at infinity was also introduced in Lecture 8, in much the same way.

The soliton solution of this theory is called a magnetic monopole for the following reason. At infinity (or in the vacuum) the gauge symmetry is spontaneously broken from $SO(3)$ to $SO(2) = U(1)$ by the Higgs mechanism that we studied in Lecture 2 this spring. So the low energy theory looks like a $U(1)$ abelian gauge theory. The first Chern class at infinity measures the integral of the "magnetic field" (the curvature $F/2\pi$ of the low energy $U(1)$ theory) over a sphere at infinity; this integral is called the total magnetic charge in classical physics. (We write a formal definition of the magnetic charge in eqn. (9.9) below.) One usually calls $c_1/2$ the magnetic charge of the solution.

Now, let us discuss a consequence for solitons of the fact that the unbroken subgroup of the gauge group at infinity is $U(1)$. This $U(1)$ is a symmetry of the vacuum or of the fields at infinity; but it does not leave invariant the full-fledged soliton or magnetic monopole solution. Acting on a particular solution, it produces a new solution. To understand this action, let us consider the solitons (A, ϕ) with $c_1 = 2$, discussed above. Let us represent the infinitesimal operator of the group $U(1)$ by a spherically symmetric gauge symmetry: $\varepsilon = \frac{x}{r} g(r)$, where g is some function. If the function $g(r)$ satisfies $g(r) \sim c_0 r$ at $r \to 0$ and $g(+\infty) = c$, then this formula defines a smooth element of the Lie algebra of gauge transformations which is "constant" at infinity (with respect to the reduction of structure group

defined by the scalar field ϕ). If we want, we can take $\varepsilon = \phi/a$, a completely natural choice that obeys all the conditions. Then the action of ε is

$$A \to A - d_A\varepsilon, \phi \to \phi. \tag{9.6}$$

It is clear that $d_A\varepsilon$ cannot be identically zero, since the soliton connection A does not have a structure group that reduces globally to $U(1)$; there is thus no covariantly constant ε. So the gauge transformations that lie in the unbroken group at infinity act nontrivially. Now consider making the gauge transformation by $\exp(c\varepsilon)$. If $c \in 2\pi\mathbf{Z}$, this gauge transformation approaches the identity at infinity. In quantizing, we identify two fields that differ by a gauge transformation that approaches the identity at infinity, so $\exp(c\varepsilon)$ should be considered to act trivially for such ε. Modulo gauge transformations that are 1 at infinity, the gauge transformations that leave fixed the fields at infinity generate the unbroken $U(1)$, and this group acts freely on the space of soliton solutions.

Allowing also for spatial translations, this argument shows that the space of time-dependent solitons with magnetic charge 1 is at least $\mathbb{R}^3 \times S^1$ (with no canonical zero on either $\mathbb{R}^3$ or S^1). One can show that in fact it is exactly that. We will denote the circle coordinate on the S^1 factor by α.

By using the Poincaré group transformations, we can generate solutions which are time-dependent and propagate at a constant speed. We can also perform a time dependent gauge transformation, which will make the α-coordinate time dependent, i.e. $\alpha = \alpha_0 + st$, $s \in \mathbb{R}$. As a result, the space of time dependent solitons is the product $T^*\mathcal{O} \times T^*S^1$.

§9.6. The 3+1-dimensional theory with the θ-angle

Consider the 3+1-dimensional theory of the previous section with the θ-angle, i.e. let us add to the (Minkowski) Lagrangian a term

$$-\frac{\theta}{16\pi^2}\int \mathrm{Tr}(F \wedge F). \tag{9.7}$$

We recall that the second Chern class c_2 is given by the integral $\frac{1}{8\pi}\int \mathrm{Tr}(F \wedge F)$. Thus, classically, term (9.7) just counts the second Chern class of the bundle.

Now, consider three operators in our theory:

1. The generator of the unbroken $U(1)$, which we write as $\frac{\partial}{\partial\alpha}$, with α an angular parameter on the circle factor in the space of monopole solutions.
2. The electric charge

$$Q_e = \frac{1}{ae^2}\int_{\mathbb{R}^3} \mathrm{Tr}(d_A\phi \wedge *F_A). \tag{9.8}$$

3. The magnetic charge

$$Q_m = \frac{1}{4\pi a}\int_{\mathbb{R}^3} \mathrm{Tr}(d_A\phi \wedge F_A). \tag{9.9}$$

By integrating by parts, and using the equations of motion and Bianchi identity for F, both Q_e and Q_m can be written as surface integrals at infinity; these surface integrals can be written as integrals over the sphere at infinity of $*F/2\pi$ or $F/2\pi$, where F is the curvature of the unbroken $U(1)$. Thus the definitions reduce in the low energy theory to Q_e and Q_m as introduced in Lecture 8. In particular, Q_m is the magnetic charge or first Chern class at infinity, and so is a locally constant function on the space of classical solutions. For our present purposes, instead of integrating by parts, it will be best to leave these formulas in the form given here.

For ordinary particles that arise by quantizing the fields and enter in perturbation theory, $Q_m = 0$ (as these particles are obtained by quantizing the component of the space of classical solutions that has $c_1 = 0$), and $Q_e = -i\frac{\partial}{\partial\alpha}$. (If this statement is not familiar from previous lectures, a derivation will appear in a moment.) Thus, in perturbation theory Q_e has integer eigenvalues.

Let us see, however, what happens nonperturbatively, i.e. when we take into account field configurations with nonzero c_1.

We compute in the gauge where $A_0 = 0$; we write $\dot{A}$ for dA/dt. Classically we have $\frac{\partial}{\partial\alpha} = a^{-1}\int \mathrm{Tr}(d_A\phi\frac{\delta}{\delta A})$. The reason for this is that as mentioned above we can consider the unbroken $U(1)$ to be generated by the Lie-algebra valued function $\varepsilon = -\phi/a$. A gauge transformation generated by this function leaves ϕ invariant, and transforms A by $\delta A = d_A\phi/a$ in first order. This is the transformation generated by $a^{-1}\int \mathrm{Tr}(d_A\phi\frac{\delta}{\delta A})$, showing that this operator is $\frac{\partial}{\partial\alpha}$.

Therefore, the operator $-i\frac{\partial}{\partial\alpha}$ is the charge for the current $J = \mathrm{Tr}(d_A\phi\wedge\pi_A)$, where π_A is the conjugate (momentum) variable for A.

Now compute π_A:

$$\pi_A = \frac{\delta\mathcal{L}}{\delta\dot{A}} = \frac{*F_A}{e^2} - \frac{\theta}{8\pi^2}F_A. \tag{9.10}$$

Combining this with the formula for $\partial/\partial\alpha$ and the definitions of Q_e and Q_m, we get

$$-i\frac{\partial}{\partial\alpha} = Q_e - \frac{\theta}{2\pi}Q_m. \tag{9.11}$$

In perturbation theory, $Q_m = 0$, so $Q_e = -i\partial/\partial\alpha$, as claimed above.

Since the spectrum of $-i\frac{\partial}{\partial\alpha}$ is integral, we get $Q_e = \frac{\theta}{2\pi}Q_m \bmod \mathbb{Z}$. But Q_m is also an integer, since it is an integer in each component of the fields. Thus, Q_e need not be an integer for soliton states. In other words, there exist states (of very high energy $\sim 1/\hbar$ in the semiclassical limit) on which Q_e is not an integer. So the electric charge is integral in the absence of magnetic monopoles, but not necessarily when monopoles are present. Most physicists suspect that monopoles of non-integral and probably irrational electric charge exist in nature, by essentially this mechanism.

The operators Q_e, Q_m commute since classically Q_m is locally constant. Integrality of both Q_m and $-i\partial/\partial\alpha$ shows, with (9.11), that the joint spectrum of Q_e, Q_m is a lattice in $\mathbb{R}^2$. If we consider the family of theories parametrized by $\theta\in S^1$, the monodromy transformation of this lattice around the circle is given by the matrix $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ in the basis Q_m, Q_e. This assertion is a consequence of (9.11), which shows that in adiabatically increasing θ by 2π, Q_e increases by Q_m.

The computation we just performed, which will be important later when we come to Donaldson theory, is very closely related to one that was performed at the end of Lecture 8 in the abelian theory. Using the explicit formula (8.78), we noted in Lecture 8 that under $\theta \to \theta + 2\pi$, the electric and magnetic fluxes $Q_e(\Sigma)$ and $Q_m(\Sigma)$ through any two-surface Σ are transformed by $Q_e \to Q_e + Q_m$. In the abelian theory, to prevent Q_e and Q_m from being zero, we had to work on a three-manifold with nonzero second Betti number. In the present discussion in the nonabelian case, we could work simply on $\mathbb{R}^3$, taking Σ to be a sphere at infinity. The objects we have here called Q_e and Q_m are simply $Q_e(\Sigma)$ and $Q_m(\Sigma)$, computed in the low energy effective theory and with Σ the sphere at infinity.

In Lecture 8 we also saw (in the free theory) how to extend this monodromy representation under $\theta \to \theta + 2\pi$ to an action of $SL(2,\mathbb{Z})$, i.e. how to construct a family of theories parametrized by a complex parameter $\tau = \frac{\theta}{2\pi} + ir$ modulo modular transformations, such that the monodromy representation is the standard action of $SL(2,\mathbb{Z})$ on a two-dimensional lattice. To accomplish something like that in a nonlinear theory in four dimensions is a more difficult problem, which we will return to only at the end of the term.

References and cross references

For an introduction to solitons, see the chapter on "Lumps" in

S. Coleman, *Aspects Of Symmetry*, Cambridge University Press, 1985.

For the derivation of the θ dependence of the electric charge of a magnetic monopole, see

E. Witten, *Dyons Of Charge* $e\theta/2\pi$, Phys. Lett. B **86** (1979), 283–287.

Solitons in supersymmetric theories (BPS states) are discussed in Lecture 12 and in later lectures of this series. Solitons in 3 and 4 dimensions were discussed in Lectures 6 and 7 (as instantons in 2 and 3 dimensions).

LECTURE 10
Wilson Loops, 't Hooft Loops, and 't Hooft's Picture of Confinement

10.1. 't Hooft loop operator

Let us first recall abelian duality in four dimensions, which we discussed in Lecture 8. Consider a free $U(1)$ gauge theory with a θ-angle. Thus we have two dimensionless couplings e, θ which combine into a single complex coupling $\tau = \frac{2\pi i}{e^2} + \frac{\theta}{\pi}$, and the Lagrangian is

$$\mathcal{L} = \int_M \left(\frac{i\bar{\tau}}{4\pi} F_+^2 - \frac{i\tau}{4\pi} F_-^2 \right),$$

where F is the curvature of a $U(1)$ connection A and F_+, F_- are the selfdual and antiselfdual parts of the curvature. We have seen that if the spacetime M is a spin manifold then this theory is "modular invariant" as a function of τ. One modular symmetry $\tau \to \tau + 1$ is obvious, as it corresponds to shifting the θ-angle by π, which does nothing if M is a spin manifold because c_1^2 for a spin manifold is even. (On a manifold that is not a spin manifold, the symmetry would be only $\tau \to \tau + 2$.) The symmetry under the second generator of the modular group, $\tau \to -1/\tau$, is more interesting and corresponds to electromagnetic duality discovered classically in the nineteenth century. More precisely, this means that the theory of a connection A with coupling constant τ is identical both classically and quantum mechanically to the same theory with coupling $-1/\tau$ and the connection B such that $dA = \text{const} * dB$. Now, like in Lecture 8, we want to see what happens to operators under this duality. In particular, we want to know what happens to the Wilson loop operator.

Recall that the Wilson loop operator has the form $W_\gamma(C) = e^{i\gamma \mathrm{Hol}_C(A)}$, where $Hol_C(A)$ denotes the integral of the connection A along a closed oriented curve C in the spacetime M. This operator is gauge-invariant and well-defined if γ is an integer, or for any real γ if the curve C is homotopically trivial in M. (More generally, there could be several components C_i with real numbers γ_i, and the condition is that $\sum_i \gamma_i C_i$ should be an integral class in $H_1(M)$.) Matrix elements of this operator are computed, as usual, by inserting the above exponential into

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the path integral. Similarly to what we found in similar problems in two and three dimensions, we should get that the dual description of the Wilson loop is a recipe which says that, rather than insert in the path integral an object living on C, we should integrate over connections having a singularity along C.

The precise answer is the following. For any curve C the expectation value $\langle W_\gamma(C)\mathcal{O}_1...\mathcal{O}_n\rangle$ equals $\int e^{-\mathcal{L}(B)}\mathcal{O}_1'...\mathcal{O}_n'DB$, where the integral is taken over connections B on a bundle over $M\setminus C$ such that the integral of the curvature of B over a small 2-sphere S in a normal 3-space to C at any point equals $2\pi\gamma$. (Here $\mathcal{O}_i$ are an arbitrary set of local operators at points disjoint from C; they transform under the duality into local operators $\mathcal{O}_i'$.) Obviously, B is not quite a connection on a $U(1)$ bundle, but there is no difficulty in considering connections of this sort mathematically.

Let us prove the assertion of the last paragraph. We assume that C is a boundary. Let D be a 2-chain whose boundary is C. Recall the calculation from Lecture 8 on abelian duality: our fields are A – the original connection, G – the 2-form, and B – the dual connection. We have

$$\int e^{-L(A)}W_\gamma(C)DA=$$

(10.1)

$$\int DA\,DG\,DB\exp\left(\frac{-i\bar\tau}{4\pi}\int\mathcal{F}_+^2+\frac{i\tau}{4\pi}\int\mathcal{F}_-^2+\frac{i}{2\pi}\int G\wedge F_B+i\gamma\int_D(F_A-G)\right),$$

where $\mathcal{F}=F_A-G$, and the last factor corresponds to the insertion of the Wilson loop (recall from the abelian duality discussion that the Wilson loop classically is $e^{i\gamma\int_D F_A}$, and F_A is to be replaced with F_A-G in the extended theory). Gauging A to 0, we get

$$\int e^{-L(A)}W_\gamma(C)DA=$$

$$\int DG\,DB\exp\left(\frac{-i\bar\tau}{4\pi}\int G_+^2+\frac{i\tau}{4\pi}\int G_-^2+\frac{i}{2\pi}\int G\wedge F_B-i\gamma\int_D G\right)=$$

(10.2)

$$\int DG\,DB\exp\left(\frac{-i\bar\tau}{4\pi}\int G_+^2+\frac{i\tau}{4\pi}\int G_-^2+i\int G\wedge(\frac{1}{2\pi}F_B-\gamma[D])\right),$$

where $[D]$ is the delta-function of D. Now we define $\tilde F_B:=F_B-2\pi\gamma[D]$. Then after integrating out G, (10.2) can be written as

$$\int e^{-L(A)}W_\gamma(C)DA=$$

(10.3)

$$\int DB\exp\left(\frac{i}{4\pi\bar\tau}\int(\tilde F_B)_+^2-\frac{i}{4\pi\tau}\int(\tilde F_B)_-^2\right),$$

Thus, the effect of the insertion $W_\gamma(C)$ is that F_B is replaced in the final answer by $\tilde F_B$. This is equivalent to saying that B is now a connection on a line bundle with singularity along C, as claimed above. Our original description in terms of curvature that integrates over a 2-sphere S to $2\pi\gamma$ refers to the curvature $\tilde F_B$.

Suppose that C is not a boundary. Then we must take γ integral for $W_\gamma(C)$ to be defined. With such a C and γ, $\langle W_\gamma(C)\rangle = 0$ (even with insertion of any number of local operators). Indeed, we have a symmetry $A \to A + d\phi$, and ϕ does not have to be globally defined as a map to a circle; in fact, $d\phi$ can be any closed one-form (or more generally, if torsion is present, $d\phi$ can be any flat connection on a line bundle). So if C is not a boundary then we can choose $d\phi$ in such a way that the operator $W_\gamma(C)$ will multiply by $e^{i\alpha}$ for some $\alpha \notin 2\pi\mathbb{Z}$. Hence its expectation value (even with inclusion of gauge-invariant local operators, which are invariant under $A \to A + d\phi$) vanishes. More generally, the correlator $\langle W_{\gamma_1}(C_1)...W_{\gamma_n}(C_n)\rangle$ (with any local operators) is zero if $\sum \gamma_i C_i \neq 0$ in $H_1(M)$, where M is the spacetime. As noted before, for the product of operators in question to be well-defined, we only need $\sum \gamma_i C_i$ to be an integral class. If the sum is a non-zero integral class, the product of operators is well-defined and its expectation value vanishes.

From our construction so far, for any curve C we have two operators:

1) $W_\gamma(C) = e^{i\gamma \int_C A}$

2) $T_\gamma(C) = e^{i\gamma \int_C B}$.

B is the gauge field of the dual theory. The second operator, which is dual to the Wilson loop, is called the 't Hooft loop operator. What we have described so far is how to express of W_γ in terms of B or T_γ in terms of A.

10.2. Hilbert space interpretation of the 't Hooft loop operator

Now let us consider this picture from the Hamiltonian point of view. Then the spacetime M has the form $M = M^3 \times \mathbb{R}$ with Minkowski metric. Let C, C' be two nonintersecting closed simple curves in M^3. They define operators $W_\gamma(C)$ and $T_{\gamma'}(C')$ on the Hilbert space $\mathcal{H}$. We denote the linking number of C and C' as $l(C, C')$. The following commutation relation for these operators is due to 't Hooft:

$$W_\gamma(C)T_{\gamma'}(C') = \exp(2\pi i\gamma\gamma' l(C, C'))\, T_{\gamma'}(C')W_\gamma(C). \tag{10.4}$$

Let us prove this formula. Let us work in terms of the original connection A. Then the Hilbert space consists of wave functions $\Psi(A)$. In this realization, the Wilson loop operator $W_\gamma(A)$ is simply the operator of multiplication by $e^{i\gamma \int A}$.

However, the 't Hooft loop operator is a bit harder to define. To do this, consider the homomorphism $\pi_1(M^3 \setminus C', x_0) \to \mathbb{Z}$ given by the linking number with C'. Let π_1^0 be the kernel of this homomorphism and X be the $\mathbb{Z}$-cover of $M_3 \setminus C'$ corresponding to π_1^0. Let ϕ be a function $X \to U(1)$ such that the monodromy corresponding to the generator of $\mathbb{Z}$ is $e^{i\gamma}$. Any two such functions differ by a gauge transformation, but ϕ itself is not an honest gauge transformation. Then it is not difficult to check that the 't Hooft loop operator $T_{\gamma'}(C')$ is just the "illegal" gauge transformation by ϕ:

$$(T_{\gamma'}(C')\Psi)(A) = \Psi(A^\phi). \tag{10.5}$$

Note that this is well-defined since any two such ϕ's differ by an honest gauge transformation. To check that (10.5) is true, one uses a path integral on M to represent the operation formulated in Hamiltonian language in (10.5), and then one compares this path integral to the earlier definition of $T_\gamma(C)$ in terms of connections with nontrivial restriction to a 2-sphere S that links C.

Now formula (10.4) is clear since $TWT^{-1} = e^{i\gamma \int_C d\phi} W$, because of the way that T transforms the connection in the definition of W.

Formula (10.4) has a generalization to nonabelian G (with $\pi_1(G) \neq 0$). For this, one needs a definition of $T_\gamma(C)$ in terms of connections with singularity along C; this definition is given in §10.3. The proof of (10.4) is otherwise similar.

10.3. The picture of confinement

Now we discuss a picture of confinement developed by 't Hooft. In general we don't assume that the gauge group is abelian. Recall the definition of confinement. We have a gauge group G and with universal cover $\widehat{G}$. We assume that G is the quotient of $\widehat{G}$ which acts faithfully on all fields in the Lagrangian. We let R be a representation of $\widehat{G}$. As we discussed in Lecture 7, if there is a mass gap, there are two usual patterns of decay of the expectation value $\langle W_R(C)\rangle$ of the Wilson line operator corresponding to the representation R as C gets big:

Pattern 1:

$$\langle W_R(C)\rangle \sim e^{-\lambda \,\mathrm{Length}(C)} \tag{10.8}$$

Pattern 2:

$$\langle W_R(C)\rangle \sim e^{-\lambda \,\mathrm{Area}(C)} \tag{10.9}$$

(here λ is a fixed nonzero number and the area of C means the minimal area of the spanning surface). The first regime is called the Higgs regime (the length law) and the second one is called the confinement regime (the area law).

As we discussed in Lecture 7, one will always find Pattern 1 when R is a representation of G itself. The interesting case is the case that R is a representation of $\widehat{G}$ but not of G. Similarly, two R's can be considered equivalent if the quotient $\widehat{G}/G$ acts on them in the same way. Thus, interesting $W_R(C)$ modulo this equivalence relation are classified by $\pi_1(G)^*$.

Now let us consider the 't Hooft loop operator $T_\gamma(C)$. It is defined for any G by analogy with the definition in the abelian case, as follows. We fix an element $\gamma \in \pi_1(G)$. Recall that G-bundles on a two-sphere S^2 are classified by a characteristic class that takes values in $H^2(S^2, \pi_1(G))$, which is canonically isomorphic to $\pi_1(G)$. The choice of γ therefore canonically determines an isomorphism class of G-bundles on S^2. We can now define the 't Hooft operator: a path integral with insertion of $T_\gamma(C)$ is computed by integrating over connections on $M \setminus C$ which have the property that, when restricted to a small sphere S that links C, the bundle has characteristic class γ.

't Hooft's idea was to consider $T_\gamma(C)$ instead of $W_R(C)$ and find conditions under which there is an area law for its expectation value. This occurs, as he showed, for certain Higgs theories. Then, 't Hooft proposed (following earlier ideas of Nambu, Mandelstam, and others) that confinement would be related to the Higgs mechanism by a duality that maps 't Hooft loop operators into Wilson loop operators. This does not explain confinement, but it reformulates the problem: to reduce the mysterious phenomenon of confinement to the much more easily understood Higgs phenomenon, one must understand the nonlinear duality that exchanges 't Hooft and Wilson loop operators.

To illustrate the area law for the 't Hooft loop in Higgs theories, we consider a familiar example: the $U(1)$ gauge theory with a charged complex scalar ϕ (of charge 1). The Lagrangian is in n dimensions

$$\int d^n x \left(\frac{|F|^2}{4e^2} + |D_A \phi|^2 + V(\phi\bar{\phi}) \right), \tag{10.10}$$

where V is a (quartic) potential. We will study the 't Hooft loop $T_\gamma(C)$, where $\gamma \in \mathbb{Z}$. Thus A is a connection and ϕ is a section of a Hermitian line bundle over $M \setminus C$ such that it has first Chern class γ when restricted to a small sphere linking C. We will compute $\langle T_\gamma(C) \rangle$ for two classes of V:

1) $V = \lambda(\phi\bar{\phi} + a^2)^2$;
2) $V = \lambda(\phi\bar{\phi} - a^2)^2$.

This theory was considered in Lecture 2. Recall the results of this consideration.

Case 1. In the infrared the theory behaves like the product of the theory of a free massive field with a free massless gauge theory. In particular, there is no mass gap. (Hence the classification above in Pattern 1 and Pattern 2 does not apply.) Thus, we can calculate $\langle T_\gamma(C) \rangle$ for large C using the free theory. But in the free theory this expectation value is the same as $\langle W_\gamma(C) \rangle$ in the dual theory. It is easy to see that the expectation values of both W_γ and T_γ behave according to the length law, because of the Coulomb law of charge interaction. A theory behaving in this way is said to be in the "Coulomb phase."

Case 2. In the infrared this theory has breaking of gauge symmetry and a Higgs mechanism. The spectrum, as analyzed in Lecture 8, consists of a massive vector and a real massive scalar. So there is a mass gap. This theory is not believed to exhibit confinement, i.e. it is believed that it exhibits the length law for the Wilson loop. This is certainly what one computes in perturbation theory.

In case 2, we will show that there is an area law for the 't Hooft loop operator, because of the Higgs mechanism. This happens for topological reasons, as explained below. Some of the ideas involved should be familiar from the discussion of the Polyakov model in Lecture 8.4.

As a warmup, consider a closed spacetime M (of dimension $n \geq 2$) and a line bundle $\mathcal{L}$ with a nontrivial c_1. Let us consider the path integral for our theory over sections of this bundle. We suppose that the metric of M is very "large," that is, that all characteristic distances are large compared to $1/m$, where m is the mass gap of the theory. Let R be a characteristic radius of M. It turns out that the action of all field configurations in this integral has to be very large, of order $(mR)^2$ for large R: it is bounded below by a constant (which is independent of M) times the area of the minimal two-surface which represents a cycle Poincaré dual to $c_1(\mathcal{L})$.

Indeed, if ϕ is a section of $\mathcal{L}$ then ϕ has to vanish on a codimension-two cycle Σ which is dual to $c_1(\mathcal{L})$. If we fix Σ, we can look at the configuration of minimal action with such a zero. For this, we can (if the metric of M is scaled up) reduce to the case that $M = \mathbb{R}^n = \mathbb{R}^{n-2} \times \mathbb{R}^2$, with Σ equal to the first factor. We can assume that ϕ and A are invariant under translations of the first factor in M. In the second factor, we want ϕ to vanish at the origin and to approach the vacuum at infinity (up to gauge transformation), such that the first Chern class of the bundle, relative to the trivialization at infinity given by ϕ, equals 1. The same problem appeared in Lecture 7.3 (in the guise of finding an instanton solution

of the two-dimensional version of the same model), and we discussed qualitative properties of the solution. Anyway, let I be the action of this solution in the two-dimensional sense (that is, integrated over just the second factor in M). Then in the n-dimensional sense, I is the action of the solution per unit volume of $\mathbb{R}^{n-2}$. Going back to the global problem on the manifold M, let $\Sigma \subset M$ be a codimension-two cycle of lowest volume representing the first Chern class of $\mathcal{L}$. The minimum action field looks in the normal directions to Σ like the instanton just described; its action is approximately $I \cdot \mathrm{Vol}(\Sigma)$.

Now let us specialize to four dimensions and consider an 't Hooft loop on a circle C in $M = \mathbb{R}^4$.[1] In this case the gauge bundle is defined only over $M \setminus C$. If $\mathcal{L}$ is a line bundle over $M \setminus C$ with Chern class 1 over a two-sphere that links C, and ϕ a section of $\mathcal{L}$, then ϕ must vanish on a two-surface whose boundary is C. The vanishing will occur on a surface D of least area that spans C, and in the normal directions to D the solution will look just as in the last paragraph. Thus, the same argument as above shows that the expectation value of the 't Hooft loop is $\langle T_1(C)\rangle \sim e^{-I\ \mathrm{Area(D)}}$, where area$(D)$ is the smallest area of a surface spanning C. This is the area law which we wanted to demonstrate.

This behavior is characteristic of what is called the Higgs regime, or phase.

Now let us discuss in more detail the relation of the established behavior of the 't Hooft loop operator with confinement. If $\pi_1(G) \neq 0$ we have so far characterized at least three possible phases:

1) Coulomb: no mass gap, massless gauge bosons in the infrared, W and T behave like in the free theory and exhibit the length law.

2) Higgs: mass gap, length law for W, area law for T.

3) Confinement: mass gap, area law for W, length law for T.

(This is not a complete classification of the phases for a variety of reasons.)

As already suggested, 't Hooft's idea was that there should be a nonabelian analogue of duality which interchanges W with T, exchanges the Higgs and the confinement regimes, and maps the Coulomb phase to itself. Thus, the area law for T in a theory implies confinement in the dual theory. This is what happens for some supersymmetric theories; we will find something like it in the theory relevant to Donaldson theory.

In fact, 't Hooft showed that if R and γ are such that $\gamma|_R \neq 1$ then either W_R on T_γ exhibit the area law. Note that the group $\pi_1(G)^* \times \pi_1(G)$ has a natural symplectic form. Wilson loops are classified, modulo the equivalence relation on representations, by the first factor, and 't Hooft loops by the second factor. Given a pair $(c, \gamma) \in \pi_1(G)^* \times \pi_1(G)$, pick a representation R with central character c and consider the operator $W_R(C)T_\gamma(C)$ (which may require some renormalization, but the renormalization does not affect the following comments). 't Hooft showed that the set of (c, γ) such that this operator does not exhibit the area law, is an isotropic subgroup H of $\pi_1(G)^* \times \pi_1(G)$ with respect to the natural symplectic form. The reason for this, very roughly, is the following. If $A(C) = W_{R_1}T_{\gamma_1}(C)$ and $B(C) = W_{R_2}T_{\gamma_2}(C)$ exhibit the length law then, when acting on the vacuum, they produce only effects that are localized along C (or there would be an area law

[1] We could really make this discussion in any dimension, but in general in dimension n, the 't Hooft operator is defined on a codimension three closed submanifold; it is only a loop operator for $n = 4$. However, deeper statements involving duality of 't Hooft and Wilson loops are specail to $n = 4$.

instead of a length law). So

$$\langle A(C)B(C')\rangle = \langle B(C')A(C)\rangle = \langle A(C)\rangle\langle B(C')\rangle(1+o(1)), d \to \infty \tag{10.11}$$

where d is the distance between C and C'. This shows that if $AB = qBA$ where q is a constant then q must be equal to 1. By 't Hooft's formula (10.4) (which is clearly valid in the nonabelian case as well), this implies that H is isotropic. By further physical arguments, one shows that if there is a mass gap, then H is maximal isotropic. This leads to a more refined classification of massive phases than was stated above: associated to each massive phase is a maximal isotropic subgroup of $\pi_1(G)^* \times \pi_1(G)$. All possibilities can arise, in general.

Remark 1. The argument showing that loop operators A and B must commute fails if one of the operators, say A, exhibits the area law. In this case, acting on the vacuum with this operator produces an effect that is not in any way localized near C; it rather has an effect which is localized near a minimal area disk D whose boundary is C. Such a disk will always intersect C' when the linking number of C and C' is not zero. Formula (10.11) is now valid only if d is the distance from C' to D, which is always 0, so the formula does not tell us anything.

Remark 2. The area law for the 't Hooft operator in a Higgs phase has many physical and mathematical applications. For example, with some small adjustments, what we said above in analyzing the behavior of the Higgs phase with a bundle of nonzero first Chern class could serve as an explanation of the Meissner effect, the fact that a superconductor (which is described approximately by the abelian Higgs model that we examined) expels magnetic flux. Perhaps the reader has, at a science fair, seen a demonstration of a magnet floating above a superconductor; this effect has the same origin. Mathematically, C. Taubes's analysis of the Seiberg-Witten invariants of symplectic four-manifolds made use of the same facts: the localization (in a closely analogous system of equations) of the zeroes of ϕ on a surface of smallest area.

References and cross references

Abelian duality is discussed in Lecture 8. The Wilson loop operator is defined in Lecture 7. The theory of Section 10.3 is discussed in Lecture 2. Nonabelian duality in 4 dimensions is discussed in later lectures on supersymmetric field theories in 4 dimensions.

For 't Hooft's picture of confinement, see *inter alia*

G. 't Hooft, *Aspects of quark confinement*, Physica Scripta **24** (1981), 841–846.

G. 't Hooft, *The topological mechanism for permanent quark confinement in a non-abelian gauge theory*, Physica Scripta **25** (1982), 133–142.

LECTURE 11

Quantum Gauge Theories in Two Dimensions and Intersection Theory on Moduli Spaces

§11.1. The partition function in two dimensional Yang-Mills theory

We consider gauge theories in two dimensions with a compact, connected, simple gauge group G. The spacetime is a compact Riemann surface Σ of genus g with no boundary, and we fix a topological type of G-bundle over Σ. We will get certain general results about Yang-Mills theory, and we will also explain why they can sometimes be applied to determine formulas for intersection theory on moduli spaces of holomorphic vector bundles.

The theory we consider is *physical Yang-Mills theory.* The fields are a G-connection A on a G-bundle P on Σ, with curvature F_A; the Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2} \int d^2x \, \mathrm{Tr}(*F_A)^2. \tag{11.1}$$

The partition function is thus

$$Z = \frac{1}{\mathrm{Vol}(\widetilde{G})} \int DA e^{-\mathcal{L}}, \tag{11.2}$$

with $\widetilde{G}$ the group of gauge transformations (or bundle automorphisms). We introduce a scalar field ϕ with values in $\mathrm{ad}(P)$ and rewrite the path integral for the partition function as

$$Z = \frac{1}{\mathrm{Vol}(\widetilde{G})} \int DAD\phi \exp\left\{ i \int \mathrm{Tr}(\phi F) - \frac{e^2}{2} \int d\mu \mathrm{Tr}(\phi^2) \right\} \tag{11.3}$$

with $d\mu = *(1)$ the Riemannian measure. This theory is invariant under area preserving diffeomorphisms since it has no explicit dependence on a Riemannian metric on Σ but only on the *measure* $d\mu$. It is clear that the theory depends on the coupling constant e and the area $\mathbf{a}$ of Σ only through the combination $e^2\mathbf{a}$. The path integral (11.3) is well behaved under taking the limit as $e \to 0$.

We make a further modification to the path integral for the partition function by introducing a fermionic variable ψ which is a one-form in the adjoint representation

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(in other words ψ is an element of of $\Omega^1(\Sigma) \otimes \Gamma(\mathrm{ad}\, P)$). The field ψ should be thought of as lying in the tangent space to the space $\mathcal{A}$ of connections on Σ. The path integral becomes

(11.4)
$$Z = \frac{1}{\mathrm{Vol}(\widetilde{G})} \int DAD\phi D\psi \exp\left\{ i \int \mathrm{Tr}(\phi F) + \frac{1}{2} \int \mathrm{Tr}(\psi \wedge \psi) - \frac{e^2}{2} \int d\mu \mathrm{Tr}(\phi^2) \right\}.$$

Since the Lagrangian in (11.4) contains no terms which involve both ψ and the other fields ϕ and A, we can integrate out ψ and recover the earlier expression (11.3).

We can now define a supersymmetry operation δ on the space of fields:

(11.5)
$$\delta A = i\psi,$$

(11.6)
$$\delta\psi = -d_A\phi$$

(11.7)
$$\delta\phi = 0$$

It follows that $\delta^2 A = -id_A\phi$, in other words $\delta^2 = 0$ up to the action of a gauge transformation. (Here, d_A refers to the de Rham differential d twisted by a connection A on Σ.) We may check the invariance of the action under δ: we have

(11.8)
$$\delta \int \mathrm{Tr}(\phi F) = -i \int \mathrm{Tr}(\phi D_A \psi),$$

while

(11.9)
$$\delta \int \mathrm{Tr}(\psi \wedge \psi) = - \int \mathrm{Tr}(D_A\phi \wedge \psi).$$

After integrating by parts, we find that the action appearing in (11.4) is invariant.

§11.2. A finite dimensional analogue: the Cartan model

Our path integral is a path integral over the infinite dimensional space $\mathcal{A}$ of connections which is acted on by the group $\widetilde{G}$ of gauge transformations with Lie algebra $\mathrm{Lie}(\widetilde{G})$. The *Cartan model* is used to treat a finite dimensional analogue of this situation. The analogue of $\mathcal{A}$ is a finite dimensional manifold M equipped with the action of a compact Lie group H (which is the analogue of $\widetilde{G}$). Functions of A and ψ correspond to differential forms on M (which we denote as $\Omega^*(M)$), while the analogue of functions of A, ψ and ϕ are elements of $\Omega^*(M) \otimes \mathrm{Fun}(\mathbf{h})$. In fact , to get the Cartan model, we restrict to the H-invariant subspace $\Big(\Omega^*(M) \otimes \mathrm{Fun}(\mathbf{h})\Big)^H$ where H acts in the obvious way on $\Omega^*(M)$ and acts on $\mathbf{h}$ (and hence on $\mathrm{Fun}(\mathbf{h})$) via the adjoint action. Here $\mathrm{Fun}(\mathbf{h})$ denotes an appropriate class of functions on $\mathbf{h}$: in the literature one most usually restricts to polynomial functions $\mathrm{Pol}(\mathbf{h}) = S(\mathbf{h}^*)$ (the symmetric algebra on $\mathbf{h}^*$).

The set
$$\Omega^*_H(M) = \Big(\Omega^*(M) \otimes S(\mathbf{h}^*)\Big)^H$$
has a natural grading: it is the differential form grading on $\Omega^*(M)$ plus two times the polynomial grading on $S(\mathbf{h}^*)$ (in other words a linear function on $\mathbf{h}$ is assigned

grading 2). With this grading one sees that both terms in the operator[1]

$$D = d - i\iota_{V(\phi)} \tag{11.10}$$

increase the grading by 1. (Here d is the de Rham differential and $\iota_{V(\phi)}$ is the interior product with the vector field $V(\phi)$ induced on M by the action of $\phi \in \mathbf{h}$.) We may write

$$\iota_{V(\phi)} = \sum_a \phi^a \iota_{V^a},$$

introducing a basis $\{\phi^a\}$ for $\mathbf{h}$. It is easy to check that $D^2 = 0$ (here one uses the fact that we have restricted to H-invariant elements of $\Omega^*(M) \otimes S(\mathbf{h}^*)$), so one may take the cohomology with respect to D. This cohomology is identified with the H-equivariant cohomology $H^*_H(M)$ of M.[2] If H acts freely on M the topological quotient M/H is a manifold (and M is a principal H-bundle over M/H), and the equivariant cohomology is identified with the ordinary cohomology $H^*(M/H)$ of the quotient.

We shall start with classes in the D-cohomology of $\Omega^*_H(M)$. One type of classes come from $(S(\mathbf{h})^*)^H$ (in other words, the polynomials on the Lie algebra $\mathbf{h}$ which are invariant under the adjoint action): this is identified with the H-equivariant cohomology of a point, or in other words with the cohomology $H^*(BH)$ of the classifying space BH. If M is a principal H-bundle over M/H (or, equivalently, if the action of H on M is free) then each invariant polynomial on $\mathbf{h}$ corresponds to a characteristic class of principal bundles with structure group H. Under the isomorphism $H^*_H(M) \cong H^*(M/H)$, the invariant polynomial S on $\mathbf{h}$ is identified with the corresponding characteristic class of the principal H-bundle M. (This is given in Chern-Weil theory as $S(F_A)$, where A is a connection on the bundle $M \to M/H$ and F_A is its curvature, which is a 2-form on M with values in $\mathbf{h}$.)

§11.3. Infinite dimensional Cartan: the descent equations

We shall now outline the analogue of the Cartan model in our infinite dimensional situation. The space M is the infinite dimensional vector space $\mathcal{A}$ of connections on a G-bundle P over a Riemann surface Σ. We start with an Ad-invariant polynomial S on the Lie algebra $\mathfrak{g}$; from this we shall construct an operator $\mathcal{O}_S^{(0)}$ in two dimensional Yang-Mills theory, which corresponds to a cohomology class in the moduli space $\mathcal{M}$. Recall that the field theory contained a field ϕ with values in $\mathfrak{g}$. We define

$$\mathcal{O}_S^{(0)} = S(\phi). \tag{11.11}$$

The object $\mathcal{O}_S^{(0)}$ is thus a function on Σ: we shall see that up to the supersymmetry differential δ, the operator $\mathcal{O}_S^{(0)}(p) = S(\phi(p))$ is independent of the choice of the point $p \in \Sigma$. We decompose the field ϕ into components $\{\phi^b\}$ (where b indexes a basis for $\mathfrak{g}$). Then

$$d\mathcal{O}_S^{(0)} = \sum_b \frac{\partial S}{\partial \phi^b} d_A \phi^b \tag{11.12}$$

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[1] Mathematicians normally use a convention in which the i in (11.10) is omitted.

[2] Throughout this lecture all cohomology groups will be assumed to have complex coefficients.

$$= i\delta\left(\sum_b \frac{\partial S}{\partial \phi^b}\psi^b\right), \tag{11.13}$$

in terms of the supersymmetry operator δ. Thus if we define

$$\mathcal{O}_S^{(1)} = -\sum_b \frac{\partial S}{\partial \phi^b}\psi^b,$$

we have proved

$$d\mathcal{O}_S^{(0)} = -i\delta\mathcal{O}_S^{(1)}. \tag{11.14}$$

Here $\mathcal{O}_S^{(1)}$ should be viewed as a 1-form on Σ.

We can iterate this procedure: we find that

$$d\mathcal{O}_S^{(1)} = \sum_{a,b} \frac{\partial^2 S}{\partial\phi^a\partial\phi^b} D_A\psi^a \wedge \psi^b + \sum_a \frac{\partial S}{\partial\phi^a} D_A\psi^a. \tag{11.15}$$

Observing that $D_A\psi = \delta F_A$, we can convert (11.15) into $d\mathcal{O}_S^{(1)} = -i\delta\mathcal{O}_S^{(2)}$, where we have defined

$$\mathcal{O}_S^{(2)} = \frac{1}{2}\sum_{a,b} \frac{\partial^2 S}{\partial\phi^a\partial\phi^b}\psi^a\wedge\psi^b + i\sum_a \frac{\partial S}{\partial\phi^a}F_a. \tag{11.16}$$

These equations may be summarized as follows:

$$(d + i\delta)(\mathcal{O}_S^{(0)} + \mathcal{O}_S^{(1)} + \mathcal{O}_S^{(2)}) = 0. \tag{11.17}$$

(The point is that in an appropriate double complex the total differential is $d + i\delta$ and $\mathcal{O}_S^{(0)} + \mathcal{O}_S^{(1)} + \mathcal{O}_S^{(2)}$ is closed.)

We shall now use this to construct cohomology classes on $\mathcal{A}/\widetilde{G}$. These can be restricted to the moduli space

$$\mathcal{M} = \mathcal{A}^{\text{flat}}/\widetilde{G} \tag{11.18}$$

of flat connections modulo gauge transformations.

If we choose a q-cycle C in Σ, we find (using Stokes' theorem) that

$$Q_S^{(q)}(C) \stackrel{\text{def}}{=} \int_C \mathcal{O}_S^{(q)}$$

satisfies

$$\delta Q_S^{(q)}(C) = 0, \tag{11.19}$$

and if $C = \delta B$ is a boundary, then

$$Q_S^{(q)}(C) = \delta T \text{ for some } T. \tag{11.20}$$

Provided that the group $\widetilde{G}$ of gauge transformations acts freely on $\mathcal{A}$, the $Q_S^{(q)}(C)$ correspond to the generators of the cohomology of $\mathcal{M}$: they are cohomology classes on $\mathcal{A}/\widetilde{G}$ which we will restrict to the moduli space $\mathcal{A}^{\text{flat}}/\widetilde{G}$. These generators are given in a seminal paper of Atiyah and Bott; they are produced by taking the slant product of the characteristic classes of the *universal bundle* over $\mathcal{M}\times\Sigma$ with

classes in the homology of Σ. The quantum field theory will compute a generating functional

$$\int_{\mathcal{M}} \exp\Big\{\alpha Q_S^{(0)}(p) + \sum_{j=1}^{2g} \beta_j Q_{S_j'}^{(1)}(C_j) + \gamma Q_{S''}^{(2)}(\Sigma)\Big\} \tag{11.21}$$

which will encode all intersection numbers in the cohomology of $\mathcal{M}$. Here, p is a point in Σ, the C_j are $2g$ cycles in Σ corresponding to the homology $H_1(\Sigma)$, and the S, S_j' and S'' are arbitrary invariant polynomials on $\mathfrak{g}$ (which are in general distinct). For simplicity we shall mostly treat the case $G = SU(2)$, for which the ring of invariant polynomials on $\mathfrak{g}$ is a polynomial ring on one generator S given by

$$S(\phi) = \mathrm{Tr}(\phi^2).$$

§11.4. Equivariant integration and localization

We now return to the finite-dimensional situation of Section 11.2. We would like to define a map $\mathcal{I} : Z_H^*(M) \to \mathbb{C}$ (where $Z_H^*(M)$ are the D-closed elements in $\Omega_H^*(M)$) by integrating over $M \times \mathbf{h}$. In order to define a convergent integral, we introduce a convergence factor $e^{-\varepsilon \mathrm{Tr}(\phi^2)}$. (In fact, in the mathematical treatment of this convergence factor one may replace the Gaussian $e^{-\varepsilon \mathrm{Tr}(\phi^2)}$ by any collection of rapidly decreasing functions $\{f_\varepsilon\}$ on $\mathbf{h}$ which, as $\varepsilon \to 0$ represent the Dirac delta distribution.) For $\alpha \in Z_H^*(M)$ we define

$$\mathcal{I}(\alpha) = \int_{\phi \in \mathbf{h}} \int_M d\phi_1 \dots d\phi_s e^{-\varepsilon \mathrm{Tr}(\phi^2)} \alpha(\phi). \tag{11.22}$$

In fact, in order to ensure convergence of the integral (11.22) we must place some hypotheses on α. We will allow α to grow at most exponentially in ϕ for large ϕ.

A useful class of equivariantly closed forms are obtained in the special case when M is a symplectic manifold with symplectic form ω, and the action of H on M is Hamiltonian with moment map $\mu : M \to \mathbf{h}^*$: We see readily that

$$\widetilde{\omega} = -i\mathrm{Tr}(\phi\mu) + \omega \tag{11.23}$$

satisfies

$$D\widetilde{\omega} = 0$$

(where D is the Cartan model differential defined by (11.10)). On a symplectic manifold, we will sometimes take α to be the exponential of $\widetilde{\omega}$ times a form that is polynomial in ϕ.

The computation of the integral (11.22) will be governed by a localization principle similar to that which leads to the Duistermaat-Heckman formula for an oscillatory integral over a symplectic manifold with a Hamiltonian torus action. We choose an H-invariant 1-form λ. For such a form,

$$D\lambda = d\lambda - i(V(\phi), \lambda); \tag{11.24}$$

where $V(\phi)$ is the vector field given by the action of ϕ (which of course depends linearly on ϕ). The integral (11.22) is not changed by replacing $\exp(-\varepsilon \mathrm{Tr}(\phi^2))$ by $\exp(-\varepsilon \mathrm{Tr}(\phi^2) + itD\lambda)$ since $\exp(itD\lambda)$ expands as $1 + D(it\lambda + \dots)$. Because $-tD\lambda = it(V(\phi), \lambda) - t(d\lambda)$, as $t \to \infty$ the integral has a wildly oscillatory phase, and localizes around points where $(V(\phi), \lambda) = 0$. One can see the localization directly from a stationary phase argument. Alternatively, one can do the ϕ integral

first, by using the fact that the exponent is quadratic in ϕ. After completing the square and and performing the integral over ϕ, we get a factor

$$\exp\left(-\frac{t^2}{4\varepsilon}\sum_a (V_a,\lambda)^2\right)$$

(and other factors with much weaker dependence on t), so by considering the behavior for $t\to\infty$, we see that the integral localizes around points where $(V_a,\lambda)=0$ for all a.

We now specialize to the case where M is a symplectic manifold equipped with the Hamiltonian action of a compact group H: the moment map will be denoted by μ. We define $I=|\mu|^2: M\to\mathbb{R}$, the norm squared of the moment map. We may choose an almost complex structure $J\in \mathrm{End}(TM)$ compatible with the symplectic structure (in the sense that the symplectic form is of type $(1,1)$ and positive with respect to J), and let $\lambda = J(dI)$ be the invariant one-form defined above. The points where $(V_a,\lambda)=0$ are readily identified with the critical points of I. We see this because

$$dI = 2\sum_a \mu_a d\mu_a,$$

so since $V_a=\omega^{-1}d\mu_a$ (regarding the symplectic form ω as an isomorphism $T_xM\to T_x^*M$ for each $x\in M$, with inverse $\omega^{-1}: T_x^*M\to T_xM$) we have

$$\omega^{-1}dI = 2\sum_a \mu_a V_a.$$

Here $dI\in\Omega^1(M)$, so $\omega^{-1}dI$ should be regarded as an element of $\Gamma(TM)$. Thus $(V_a,\lambda)=0$ implies $(\omega^{-1}dI,\lambda)=0$, in other words

$$(\omega^{-1}dI, J(dI))=0$$

(where both sides of this equation are in $\Gamma(TM)$). This implies $dI=0$ since ω and J combine to give a nondegenerate symmetric bilinear form g on the cotangent space, defined by

$$g(X,Y)=\omega^{-1}(X,JY)$$

(for $X,Y\in\Gamma(T^*M)$). Thus our integral formally localizes on the critical points of I. One obvious source of such critical points is $\mu^{-1}(0)$, but there are other critical points where $I\neq 0$.

We now give a derivation of (one version of) the Duistermaat-Heckman formula. As noted above, the form $\widetilde{\omega}=-i\mathrm{Tr}(\phi\mu)+\omega$ (introduced in (11.23)) is equivariantly closed. We may thus compute

$$\frac{1}{\mathrm{Vol}(H)}\int_{\phi\in\mathbf{h}} d\phi_1\dots d\phi_n\int_M \exp\Big(-i\mathrm{Tr}(\phi\mu)+\omega\Big). \tag{11.25}$$

(Note that if we did not include the integration over ϕ in the formula (11.25), the oscillatory integral over M would correspond to a sum over the components of the fixed point set of T: see for instance Chapter 7 of the book by Berline, Getzler, and Vergne listed in the references.) If however we first integrate over ϕ we obtain a delta function $\delta(\mu)$, so the formula becomes

$$\frac{1}{\mathrm{Vol}(H)}\int_M (\exp\omega)\delta(\mu)=\int_{\mu^{-1}(0)/H} e^{\omega}$$

in other words the symplectic (Liouville) volume of the symplectic quotient $\mu^{-1}(0)/H$.

We wish, however, also to include a factor $\exp(-\varepsilon \mathrm{Tr}(\phi^2))$ in the integrand. Thus we wish to compute

$$Z(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\mathrm{Vol}(H)} \int_{\phi \in \mathbf{h}} d\phi_1 \dots d\phi_n \int_M \alpha \exp\Big(-i\mathrm{Tr}(\phi\mu) + \omega - \varepsilon \mathrm{Tr}(\phi^2)\Big), \tag{11.26}$$

where $\alpha = \alpha(\phi)$ is an equivariantly closed form on M which has *polynomial* dependence on ϕ. We assume that H acts freely on $\mu^{-1}(0)$; near $\mu^{-1}(0)$, the equivariantly closed class $\mathrm{Tr}(\phi^2)$ (which comes from the H-equivariant cohomology of a point in the Cartan model) is the pullback of a class Θ on the symplectic quotient $\mu^{-1}(0)/H$, which is the class which would be evaluated in Chern-Weil theory as $\mathrm{Tr}(F_B^2)$ if B is a connection on the principal H-bundle $\mu^{-1}(0) \to \mu^{-1}(0)/H$ and F_B is its curvature. Similarly, near $\mu^{-1}(0)$, α restricts to the pullback of a class $\widehat{\alpha}$ on $\mu^{-1}(0)/H$.

We find that

$$Z(\varepsilon) = \int_{\mu^{-1}(0)/H} \widehat{\alpha} \exp(\omega - \varepsilon\Theta) + O(e^{-b/\varepsilon}),$$

where b is the smallest nonzero critical value of I. To see heuristically why the nonzero critical value b of I should give a contribution to the integral (11.26) which is of order $e^{-b/\varepsilon}$, we compute (11.26) by first integrating out ϕ (assuming for simplicity that $\alpha = 1$). We compute the integral over ϕ by completing the square in ϕ, which gives

$$Z(\varepsilon) = \int_M \exp\left(\omega - \frac{1}{\varepsilon}\mathrm{Tr}(\mu^2)\right).$$

Thus, naively, a critical point with nonzero μ should contribute an amount of order $\exp(-\mathrm{Tr}(\mu^2)/\varepsilon)$, for small ε.

§11.5. Equivariant integration: the infinite dimensional case

We now pass to the infinite dimensional case and study the path integrals corresponding to equivariant integration. Recall that we had defined the generating functional (11.21). We restrict to $G = SU(2)$ or $SO(3)$; we will pick the invariant polynomials S, S'_j and S'' to all be equal to the generating polynomial (denoted S), namely $S(\phi) = \mathrm{Tr}(\phi^2)$. (We have picked an Ad-invariant inner product on $\mathfrak{g}$, which is denoted Tr; a corresponding measure is defined on G.) We assume $\gamma = 1$ and $\alpha = -e^2/2$ in (11.21), and observe that for $S = \mathrm{Tr}(\phi^2)$ we have

$$Q_S^{(1)}(C_j) = \int_{C_j} \mathrm{Tr}(\phi\psi).$$

Thus the path integral that gives rise to (11.21) becomes (here p is a point in Σ)

$$Z(e, \{\beta_j\}) = \tag{11.27}$$

$$\frac{1}{\mathrm{Vol}(\widetilde{G})} \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\phi \exp\left(-\frac{e^2}{2}\mathrm{Tr}(\phi^2)(p) + \sum_{j=1}^{2g} \beta_j \int_{C_j} \mathrm{Tr}(\phi\psi) + \int_\Sigma Q_S^{(2)}\right).$$

(Note that when $S(\phi) = \mathrm{Tr}(\phi^2)$, the object $\int_\Sigma Q_S^{(2)}$ corresponds to the symplectic form on $\mathcal{A}$, and becomes identified with the standard symplectic form on the moduli space $\mathcal{M}$.)

We can now see how the integral (11.27) can be rewritten in such a way that the generating functional (11.21) involving the $Q_S^{(q)}$ for $q = 0, 1, 2$ is equated with a corresponding generating functional involving only $Q_S^{(q)}$ for $q = 0, 2$. This is accomplished by performing the integral over ψ, which is Gaussian: completing the square to perform this integral we see that we get

$$\frac{1}{\mathrm{Vol}(\widetilde{G})}\int \mathcal{D}A\mathcal{D}\phi \exp\Big\{i\int_\Sigma \mathrm{Tr}(\phi F_A) - \varepsilon \mathrm{Tr}(\phi^2)(p)\Big\}, \tag{11.28}$$

where $\varepsilon = e^2/2 - \sum_{i<j}\beta_i\beta_j\#(C_i \cap C_j)$. In other words we can eliminate the odd-dimensional generators of the cohomology ring of the moduli space corresponding to $Q_S^{(1)}$ (the non-algebraic cycles in the moduli space). This result can be explained by a simple topological argument.

If one generalizes to the case of a Lie group of rank greater than 1, one can still eliminate the odd-dimensional cycles in favor of even-dimensional ones by performing a Gaussian integral over ψ, but in this case a topological explanation for this operation is not yet known.

§11.6. The partition function of Yang-Mills theory

We have now reduced the formula we need to compute to the partition function (11.2) of Yang-Mills theory without the fermionic variable ψ. To compute the partition function we quantize the theory in a Hamiltonian framework and then piece things together to work globally.

To introduce a Hamiltonian framework, we replace the Riemann surface by a cylinder $C \times [0, T]$ (where $C \cong S^1$ should be regarded as space and is assumed to have circumference L, while $[0, T]$ is a time interval of length T). The space of fields at fixed time is equal to the space of connections $\mathcal{A}_C$ on the circle C, so the Hilbert space is

$$\mathcal{H} = C^\infty(\mathcal{A}_C)^{\widetilde{G}},$$

the gauge invariant functions on $\mathcal{A}_C$. Since up to gauge equivalence all connections are classified by their holonomy around the curve C, we find that the Hilbert space is identified with

$$\mathcal{H} = C^\infty(G)^G,$$

the ring of functions on G which are invariant under the adjoint action (or equivalently the functions on the maximal torus T which are invariant under the action of the Weyl group). The Hilbert space is thus identified with the irreducible (complex) representations R of G (or equivalently with their characters, which are denoted χ_R or Tr_R): we take the characters χ_R to form a basis of $\mathcal{H}$, or

$$\mathcal{H} = \oplus_R \mathbb{C}_{(R)}.$$

For every irreducible representation R, the basis element χ_R is identified with a function Ψ_R on $\mathcal{A}_C$, namely

$$\Psi_R(A) = \chi_R(\mathrm{Hol}_C A).$$

The Hamiltonian of the theory is

$$H = -\frac{e^2}{2}\int_C \mathrm{Tr}|F_{01}|^2$$

(where F_{01} is the curvature of the connection), which becomes (identifying the Hamiltonian as an operator on $\mathcal{H}$ in terms of the position variables $A^b(\theta)d\theta$ on $\mathcal{A}_C$ and the corresponding momentum variables $(\partial A^b(\theta)/\partial t)d\theta$ which are identified with $-i\partial/\partial A^b(\theta)$ under canonical quantization)

$$H = \frac{e^2}{2}\int_C \mathrm{Tr}(\frac{\delta}{\delta A})^2.$$

The action of the operator $\delta/\delta A^a(\theta)$ on Ψ_R gives

$$\frac{\delta}{\delta A^a(\theta)}\mathrm{Tr}_R\mathrm{Hol}_C(A) = \mathrm{Tr}_R T^a \mathrm{Hol}_C(A)$$

in terms of a basis T^a for the Lie algebra $\mathfrak{g}$ of G. We thus see that

$$\sum_a \frac{\delta}{\delta A^a(\theta)}\frac{\delta}{\delta A^a(\theta)}\mathrm{Tr}_R\mathrm{Hol}_C(A) = \mathrm{Tr}_R(\sum_a T^aT^a\mathrm{Hol}_C(A)). \tag{11.29}$$

(This actually is a result of Lecture 5, where the same problem was considered.) The element $\sum_a T^aT^a$ is the *quadratic Casimir* of G, whose evaluation in the representation R is denoted

$$\sum_a T^aT^a \stackrel{\mathrm{def}}{=} c_2(R).$$

Thus we see that

$$H\Psi_R = \frac{e^2c_2(R)L}{2}\Psi_R. \tag{11.30}$$

We thus see that

$$< \Psi_R|e^{-HT}|\Psi_R >= \exp(-e^2c_2(R)LT/2) = \exp(-e^2\mathbf{a}c_2(R)/2), \tag{11.31}$$

in terms of the area $\mathbf{a} = LT$ of the cylinder. The answer depends only on the combination $e^2\mathbf{a}$.

Though our interest today is mainly topology, we pause to note a physical interpretation of equation (11.30). It asserts that if R is a nontrivial representation of G, then the energy of the state Ψ_R is proportional to L as $L \to \infty$ (and in fact is exactly proportional to L). This is equivalent to the statement that the pure Yang-Mills theory in two dimensions is confining; it exhibits the area law for all representations R. (Two-dimensional Yang-Mills theory describes no propagating degrees of freedom, so there is no possibility of screening of an external charge by a charge in the theory; that is why to get this result we need not restrict to the case that R is a representation of a cover of G but not of G itself. The need for this restriction when matter fields are present was explained at the end of Lecture 7.)

In fact the effect of renormalization in this quantum field theory is to introduce a factor of the form

$$\exp\left(\alpha\int_\Sigma \mathcal{R}d\mu + (e^2/2)\beta\int_\Sigma d\mu\right) \tag{11.32}$$

into the integrand (where $\mathcal{R}$ is the Riemann curvature and α and β are appropriately chosen real-valued constants). The theory is superrenormalizable, so renormalization only introduces terms of lower dimension than those present in the classical Lagrangian; the terms we have just written are the only possibilities. The effect of the term involving β is to change the quadratic Casimir c_R to a renormalized value $\widetilde{c}_R$ which differs from c_R by addition of a constant. The factor α multiplies the Euler characteristic of Σ: it will eventually be adjusted to ensure that the overall

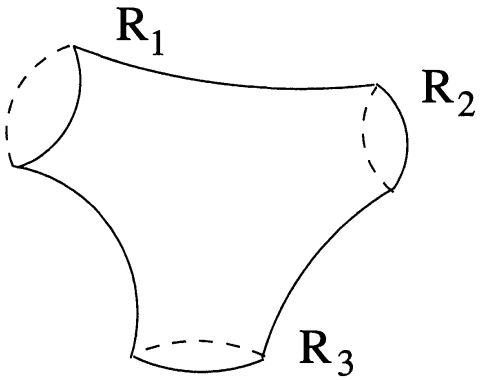

Figure 1

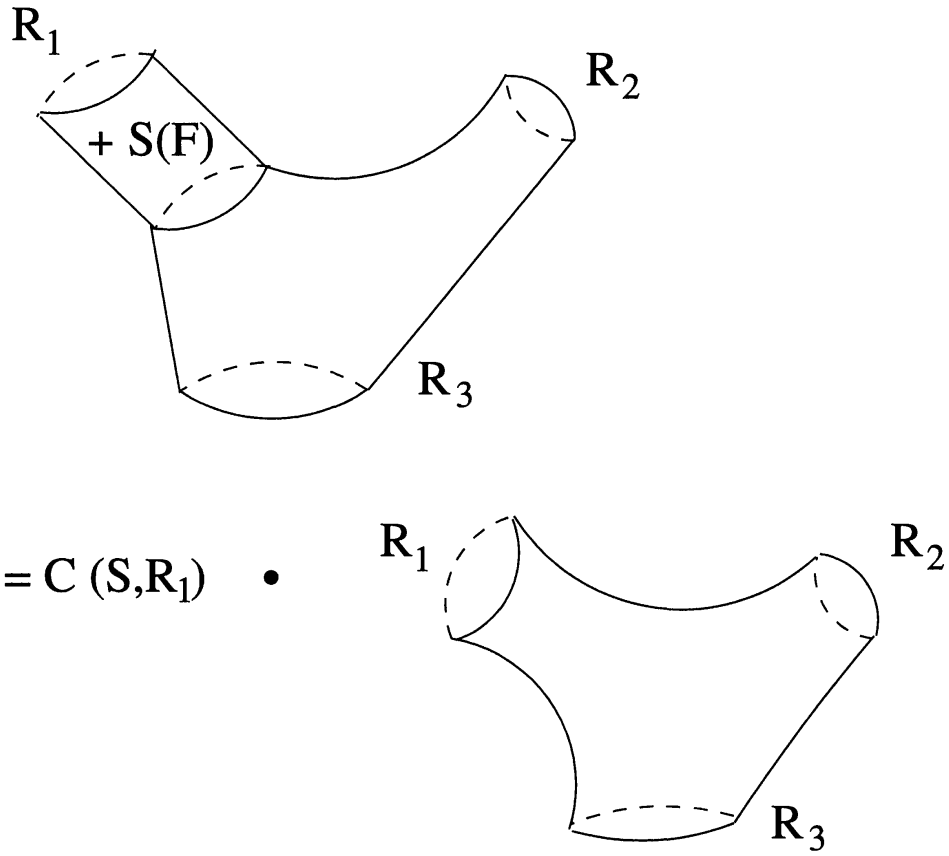

Figure 2

normalization of the partition function and its dependence on the genus g agrees with that found by other methods (for instance by explicit computations in special cases).

To have ingredients from which we can build up any Riemann surface, we must also consider the field theory on the special surface $\mathcal{P}$ which is a sphere with three disks removed (the trinion or pair of pants: see Figure 1). We imagine doing a path integral on $\mathcal{P}$ with the states Ψ_{R_i}, $i = 1, 2, 3$ inserted on the boundary components. This path integral is

$$W_{R_1R_2R_3} = \int DA \exp(-L) \prod_{i=1}^{3} \Psi_{R_i}(A|_{C_i}), \tag{11.33}$$

where A runs over the space of connections on the bundle over $\mathcal{P}$. After computing this path integral for arbitrary R_i, it will be relatively straightforward to piece things together.

The first step is to prove that this path integral $W_{R_1R_2R_3}$ vanishes unless the R_i are all equal. We insert the operator $\widehat{Q_S^{(0)}}(p)$ corresponding to the observable $Q_S^{(0)}(p)$ at some point p in $\mathcal{P}$. We assume that p is near a boundary component labeled by the representation R (where R is one of the R_i). Then we can consider the operator $\widehat{Q_S^{(0)}}(p)$ to act on the state Ψ_R. (This is illustrated in Figure 2, where the operator insertion is labeled $S(F)$ and is considered to act on the state Ψ_{R_1}.) We see using (11.30) and its generalizations to higher order Casimir operators that the state Ψ_R is an eigenstate of $\widehat{Q_S^{(0)}}(p)$ with some eigenvalue $C(S,R)$:

$$\widehat{Q_S^{(0)}}(p)\Psi_R = C(S,R)\Psi_R. \tag{11.34}$$

Here, if S is an invariant polynomial of degree l on $\mathfrak{g}$, $C(S,R)$ is the corresponding l^{th} order Casimir of the representation R. (If S is, for example, the gauge-invariant function $S(\phi) = \mathrm{Tr}(\phi^l)$ with Tr the trace in the adjoint representation, then S becomes upon quantization a multiple of $\mathrm{Tr}(\delta/\delta A)^l$. Ψ_R is an eigenstate of this operator with eigenvalue a multiple of $C(S,R)$.) Since R can equal any one of R_1, R_2, or R_3, we find that this one-point function is equal to $C(S,R_j)W_{R_1R_2R_3}$ (for $j = 1,2,3$) where $W_{R_1R_2R_3}$ is as defined above. The assertion is true for the following reason: since this field theory is invariant under area preserving diffeomorphisms, we can take the operator insertion to be near whichever boundary component we choose, and hence the operator can be taken to act on any one of the external states. It follows that the Casimirs $C(S,R_j)$ are all equal for each j, and since this is true for all invariant polynomials S, we must have $R_1 = R_2 = R_3$ if $W(R_1,R_2,R_3) \neq 0$.

We are thus reduced to computing the partition function of $\mathcal{P}$ with boundary conditions determined by the same external state Ψ_R along all three boundary components. Denote the value of this partition function in the limit of zero area by w_R. Then the area dependence is that if the area is $\mathbf{a}$ one obtains

$$Z_R = w_R \exp\left(-\widetilde{c}_2(R)\frac{\mathbf{a}e^2}{2}\right) \tag{11.35}$$

where we now write Z_R for $W_{R,R,R}$. The reason that the area dependence must be like this is that we can increase the area of the trinion by gluing a cylinder of prescribed area onto any boundary component. This multiplies Z_R by the factor (11.31) that arises in propagating Ψ_R through the given area.

We now consider the problem of computing the partition function for a Riemann surface of genus g with no boundary. Such a surface may be formed by gluing together $2g-2$ copies $\mathcal{P}_j$ of $\mathcal{P}$ along $3g-3$ boundary circles C_γ. We may factor the path integral for the partition function according to the values of the fields restricted to the boundary circles C_γ. If A' denotes the value of a connection on the boundary circles $\coprod_\gamma C_\gamma$ of the $\mathcal{P}_j$, and $\mathcal{A}_{A'} = \{A \in \mathcal{A} : A|_{\coprod_\gamma C_\gamma} = A'\}$ is the

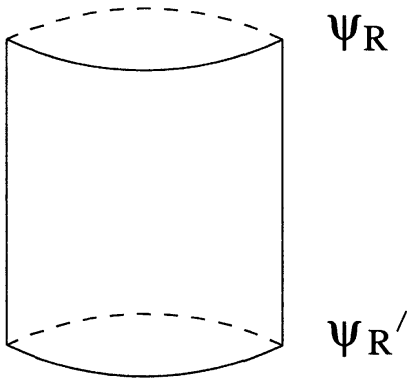

Figure 3

set of all connections which restrict to a given boundary value A', we have[3]

$$\int_{\mathcal{A}} \mathcal{D}A e^{-\mathcal{L}} = \int \mathcal{D}A' \int_{\mathcal{A}_{A'}} DA e^{-\mathcal{L}} \tag{11.36}$$

Here, on the right hand side DA is an integral over A's that restrict to A' on the C_γ's. Once we have fixed the connection A' on the C_γ's, the space $\mathcal{A}_{A'}$ is the product of $2g-2$ copies of the space of connections on $\mathcal{P}$ (with prescribed boundary values). The partition function of $\mathcal{P}$ with prescribed boundary values for the connection and area $\mathbf{a}$ is

$$Z(\mathcal{P}, A|_{\partial\mathcal{P}}) = \sum_R w_R \prod_{\gamma=1}^{3} \Psi_R(A|_{C_\gamma}) \exp\left(-\frac{e^2 \mathbf{a} \widetilde{c}_2(R)}{2}\right). \tag{11.37}$$

(This is equivalent to (11.35), given the orthonormality of the Ψ_R.) To recover the partition function for the closed Riemann surface Σ of genus g, we multiply $2g-2$ copies of $Z(\mathcal{P})$ and integrate over the boundary values A'. Using the orthonormality relations for the group characters which correspond to the states Ψ_R, we learn that a nonzero contribution arises only when all the R's on each of the $2g-2$ trinions must be the same. The sum over R gives

$$Z(\Sigma) = \sum_R w_R^{2g-2} \exp\left(-\frac{e^2 \mathbf{a} \widetilde{c}_2(R)}{2}\right). \tag{11.38}$$

(Now of course $\mathbf{a}$ is the area of the whole surface.)

We now explain how to compute the w_R. We observe that the partition function for a disk D of area $\mathbf{a}$ with an external state Ψ_R on the boundary is

$$Z_R(D) = v_R \exp\left(-\frac{e^2 \mathbf{a} \widetilde{c}_2(R)}{2}\right) \tag{11.39}$$

with some constant v_R. The area dependence is given by the usual factor, since again, the area can be changed by gluing to the boundary a cylinder of prescribed area.

The partition function corresponding to a cylinder with boundary conditions determined by wavefunctions Ψ_R (resp. $\Psi_{R'}$) on the two boundary components is equal to 0 if $R \neq R'$, since Ψ_R is an eigenfunction of the Hamiltonian H. (See

[3]See §4.5 in the first paper of Witten listed in the references at the end of this lecture.

Figure 3.) The partition function of a cylinder $S^1 \times I$ of area $\mathbf{a}$ with external states Ψ_R on both boundary components is

$$Z_R(S^1 \times I) = 1 \cdot \exp\left(-\frac{e^2 \mathbf{a} \widetilde{c}_2(R)}{2}\right) \tag{11.40}$$

(since the Ψ_R are eigenstates of the Hamiltonian H and the path integral on the cylinder computes e^{-HT}; note that because of its Hamiltonian interpretation, this formula, unlike the previous ones, has no unknown multiplicative constant). We may decompose the cylinder as in Figure 4: this yields the partition function of the cylinder with external states Ψ_R on both boundary components as the product of the partition function of $\mathcal{P}$ (with external states Ψ_R on all boundary components) and that of D, so that

$$w_R \cdot v_R = 1. \tag{11.41}$$

To determine v_R, it suffices to consider a disk D of very small area. If we fix the holonomy of a connection around the boundary of D to take the value $U \in G$, the partition function for the disk (restricting to connections on D with boundary holonomy U) is (from (11.39) and using orthogonality of the characters)

$$Z(U) = \sum_R v_R \chi_R(U) \exp\left(-\frac{e^2 \mathbf{a} \widetilde{c}_2(R)}{2}\right) \tag{11.42}$$

(where as above χ_R denotes the character of the representation R). If we instead compute the partition function via the path integral, we write the action as

$$I(U) = \frac{1}{4e^2} \int_D d\mu \mathrm{Tr}(|F|^2), \tag{11.43}$$

and

$$Z(U) = \int_{\mathrm{Hol}(A)=U} DA \exp\left(-\frac{1}{4e^2} \int_D d\mu \mathrm{Tr}(|F|^2)\right), \tag{11.44}$$

(where we restrict to connections for which the boundary holonomy is U). This integral gives a positive function of U, since the integrand is positive. As e^2 (or equivalently the area) converges to zero, the support of $Z(U)$ converges to $U = 1$, since the action of any configuration of fixed $U \neq 1$ diverges as $e \to 0$. So for $e \to 0$, the integral converges to a delta function supported at $U = 1$. This gives

$$Z(U) = \exp(\alpha) \cdot \delta(U - 1), \tag{11.45}$$

Here, α is an as yet unknown constant. We will see that it essentially coincides with the constant in (11.32) multiplying the Euler characteristic of the Riemann surface. Comparing (11.44) to the $e \to 0$ limit of (11.41), we see that

$$\exp(\alpha)\delta(U - 1) = \sum_R v_R \chi_R(U). \tag{11.46}$$

If we multiply both sides of this equation by $\chi_{R'}(U)$ and integrate over U, using the orthogonality relations for group characters, we find

$$v_R = \exp(\alpha) \dim R, \tag{11.47}$$

and hence

$$w_R = \frac{\exp(-\alpha)}{\dim R}. \tag{11.48}$$

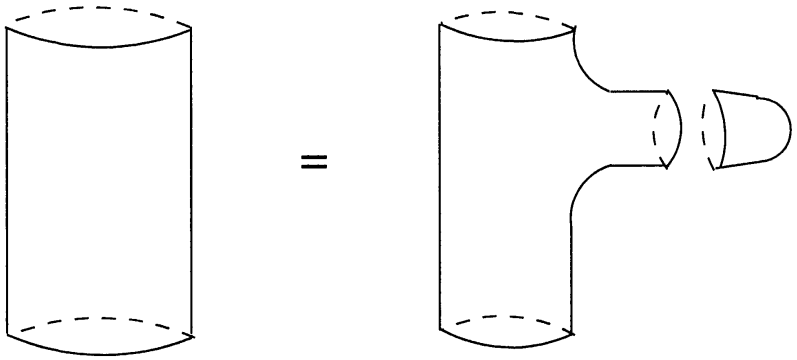

Figure 4

Thus the partition function is

$$Z(\Sigma^g, e^2\mathbf{a}) = \sum_R w_R^{2g-2} \exp\left(-\frac{\widetilde{c}_2(R)e^2\mathbf{a}}{2}\right) \tag{11.49}$$

$$= \sum_R \frac{e^{-\alpha(2g-2)} \exp(-\widetilde{c}_2(R)\varepsilon/2)}{(\dim R)^{2g-2}} \tag{11.50}$$

where we have introduced $\varepsilon = e^2\mathbf{a}$.

The computation we have just performed gives the sum of partition functions corresponding to G-bundles of all possible topological types (recall that if G is not simply connected there will in general be several topological types of bundles over Σ). To study the contribution of one particular topological type, we take G to be simply connected and pick an element $\zeta \in Z(G)$ (the center of G). Choosing a point $p \in \Sigma$ we restrict to connections on $\Sigma - \{p\}$ such that the holonomy around p is equal to ζ. Such a connection will descend to a smooth connection on a quotient bundle with structure group $G/Z(G)$, whose topology is determined by ζ. We are about to make heavy use of this relation between connections with the stated type of singularity on a G-bundle and smooth connections on a $G/Z(G)$-bundle.

To extend the computation to G-bundles with this kind of singularity, we need to calculate the partition function on a small disc D containing the singular point p around which there is prescribed monodromy ζ, as a function of the holonomy U around the boundary of the disc. Then by gluing D into any Riemann surface, we get a result for the partition function on any Riemann surface with the inclusion of the singular point p. In the limit $e \to 0$, the partition function on D is given (via quantization) by

$$\sum_R u_R \chi_R(U) \tag{11.51}$$

for some unknown u_R and (via the path integral) by

$$\exp(\alpha)\delta(U - \zeta). \tag{11.52}$$

The basis for this last assertion is that if the monodromy around p is ζ, but the monodromy around the boundary of the disc is $U \neq \zeta$, then the minimum action in the path integral will diverge as $e \to 0$, and the path integral will vanish; hence, the $e \to 0$ or small area limit of this path integral is a positive multiple of $\delta(U - \zeta)$.

If we multiply by $\chi_{R'}(U)$ and integrate over U we find from the orthogonality of the characters that

$$u_R = \exp(\alpha)\chi_R(\zeta) = \exp(\alpha)(\dim R)\widehat{\chi}_R(\zeta) \tag{11.53}$$

where $\widehat{\chi}_R(\zeta) \in U(1)$ is the normalized character of R evaluated at ζ. If now we build a closed Riemann surface of genus g by gluing together $2g-1$ copies of $\mathcal{P}$ and one copy of the disk D, then for the partition function we get

$$Z(\varepsilon, g, \zeta) = \sum_R w_R^{2g-1} e^{-\widetilde{c}_2(R)e^2\mathbf{a}/2} u_R \tag{11.54}$$

$$= \sum_R \frac{e^{-\alpha(2g-2)} e^{-\varepsilon \widetilde{c}_2(R)/2}}{(\dim R)^{2g-2}} \widehat{\chi_R}(\zeta). \tag{11.55}$$

This is the partition function for G-connections on a Riemann surface of genus g with one puncture and with holonomy ζ around this puncture. Instead we would like the partition function for smooth $G/Z(G)$-connections on the corresponding $G/Z(G)$-bundle with characteristic class of the bundle determined by ζ.

As we have already noted, there is a natural identification between the space of connections on a G-bundle that have monodromy ζ around a puncture p, and the space of connections on a $G/Z(G)$-bundle of topological type determined by ζ. However, the gauge groups – and hence their volumes – are different in the two cases. Let $\widetilde{G}_1$ be the group of gauge transformations on the G-bundle with monodromy ζ around p, and $\widetilde{G}_2$ be the group of gauge transformations on the corresponding $G/Z(G)$-bundle. $\widetilde{G}_2$ has a subgroup of index $(\#Z(G))^{2g}$ (consisting of gauge transformations that in going around loops in Σ are single-valued if lifted from $G/Z(G)$ to G) that is isomorphic to the quotient of $\widetilde{G}_1$ by its center (which is the group of constant gauge transformations by elements of the center of G, and has order $\#Z(G)$). So $\operatorname{Vol}\widetilde{G}_2 = \#Z(G)^{2g-1}\operatorname{Vol}\widetilde{G}_1$, and hence the partition function for the $G/Z(G)$-bundles of topological type determined by ζ is obtained by dividing (11.55) by a factor $(\#Z(G))^{2g-1}$.

So we have obtained the partition function of physical Yang-Mills theory on a $G/Z(G)$-bundle of any desired topological type. The reasoning of §11.4 expresses this integral in terms of intersection theory on the symplectic quotient. In the case at hand, the symplectic quotient is the moduli space of holomorphic stable $G/Z(G)$-bundles of the given topological type. Thus, we should hope to get formulas for intersection theory on that moduli space. If one re-examines the discussion in §11.4 more closely, it will become clear that even in the finite-dimensional case, the identification of the integral with intersection theory on the symplectic quotient holds only when the symplectic quotient is smooth. Otherwise, one gets more elaborate statements (not much studied as of yet) in which the singularities enter. For the application to intersection theory on moduli space, the case that the moduli space is smooth is the case that $G/Z(G) = SU(N)/\mathbb{Z}_n$, ζ is a generator of $\mathbb{Z}_n$, and the genus is bigger than 1; the moduli space is then the moduli space of holomorphic vector bundles of rank n with first Chern class prime to n. Intersection theory on this moduli space is governed by formulas that can be obtained by combining the evaluation above of the partition function with the localization result of §11.4. We will not actually write out the formulas; for precise statements (and rigorous proofs without quantum field theory) see the references. In case $G/Z(G)$ is not $SU(n)/\mathbf{Z}_n$ (or ζ a generator of $\mathbb{Z}_n$), one is in a situation similar to Donaldson theory in four

dimensions. The moduli space of holomorphic $G/Z(G)$-bundles of the given type has singularities, so intersection theory is not uniquely defined. However, in a certain "stable" range, some intersection numbers can be defined while keeping away from the singularities. These should be determined by the above formula for the Yang-Mills partition function; but this, again, has not been investigated mathematically.

We conclude this section by studying the example $G = SU(2)$, $\zeta = -1$, which is the first case where the moduli space of holomorphic bundles is smooth. For each integer $n \geq 1$ there is a unique irreducible representation R_n of dimension n; it has $\widehat{\chi_n}(\zeta) = (-1)^{n+1}$. The quadratic Casimir of the representation R_n is

$$c_2(R_n) = (n^2 - 1)/2;$$

this is renormalized to

$$\widetilde{c}_2(R_n) = n^2/2.$$

The partition function corresponding to an $SO(3)$ bundle of $w_2 \neq 0$ is given (up to an overall multiplicative normalization constant independent of the genus g) by

$$Z(\varepsilon) = e^{-\alpha(2g-2)} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{2g-2}} e^{-\varepsilon n^2/4}. \tag{11.56}$$

Let us now verify that this is the sum of a term which is a polynomial in ε (the polynomial dependence being expected from (11.21) if $Z(\varepsilon)$ does determine intersection theory on the moduli space) and a second term which is exponentially decaying of order $e^{-b/\varepsilon}$ for some positive constant b (and reflects the contribution of higher critical points, which – according to Atiyah and Bott, as cited in the references – correspond to unstable holomorphic bundles). We do this by differentiating the series to find that

$$e^{\alpha(2g-2)} 4^{g-1} \left(\frac{\partial}{\partial \varepsilon} \right)^{g-1} Z(\varepsilon) = F(\varepsilon), \tag{11.57}$$

where we have defined

$$F(\varepsilon) = \sum_{n \geq 1} (-1)^{n+1} e^{-\varepsilon n^2/4}.$$

We have that

$$2F(\varepsilon) - 1 = -\sum_{n \in \mathbb{Z}} (-1)^n e^{-\varepsilon n^2/4},$$

and using the Poisson summation formula this is given by

$$2F(\varepsilon) = 1 - \sqrt{4\pi/\varepsilon} \sum_{m \in \mathbb{Z}} \exp\left(-(2\pi)^2 (m + 1/2)^2/\varepsilon\right). \tag{11.58}$$

(The terms in the sum indexed by m correspond to the contribution of the critical points of the Yang-Mills functional arising from unstable bundles of the form $\mathcal{L}_m \oplus \mathcal{L}_{-m+1}$, where $\mathcal{L}_m$ is a line bundle over Σ of degree m.) We may now apply (11.57) to integrate (11.58), showing that $Z(\varepsilon)$ is the sum of a polynomial in ε (of degree $g - 1$) plus a term exponentially decaying like $e^{-b/\varepsilon}$ for a positive constant b. To consider rank two bundles of trivial first Chern class, we would simply omit the factor $(-1)^{n+1}$ in the partition function. In this case, the same sort of analysis shows that $Z(\varepsilon)$ has an expansion in integral powers of ε up to a certain order, after which one gets a term with half-integral exponent. The failure of $Z(\varepsilon)$ to be of the

form "polynomial plus exponentially small corrections" reflects the singularities of the moduli space of rank two bundles with trivial first Chern class.

References and cross references

Some basic references for this lecture are

A.A. Migdal, Sov. Phys. JETP **42**, 413 (1975) (Zh. Eksp. Teor. Fiz. **69**, 810 (1975)).
E. Witten, On quantum gauge theories in two dimensions, *Commun. Math. Phys.* **141** (1991) 153-209.
E. Witten, Two dimensional gauge theories revisited, hep-th/9204083; *J. Geom. Phys.* **9** (1992) 303-368.

Yang-Mills theory in two dimensions is analyzed in the first two, and the interpretation in terms of intersection theory on the moduli space of holomorphic bundles is developed in the third. A mathematical treatment of localization and equivariant cohomology using the Cartan model may be found in

N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag (Grundlehren vol. 298), 1992.

A mathematical treatment of equivariant integration in the Cartan model may be found in

L. Jeffrey, F. Kirwan, Localization for nonabelian group actions, *Topology* **34** (1995) 291-327.

The moduli space $\mathcal{M}$, in particular its cohomology, is discussed in

M.F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, *Phil. Trans. Roy. Soc. Lond.* **A 308** (1983) 523-615.

A mathematical proof of the intersection number formulas for the case $G = SU(2)$ is given in

M. Thaddeus, Conformal field theory and the cohomology of the moduli space of stable bundles, *J. Diff. Geom.* **35** (1992) 131-149.

This paper also contains a mathematical version of the argument in §11.5. For $G = SU(n)$ (in those cases where the moduli space $\mathcal{M}$ is smooth) a proof of the intersection number formulas is given in

L. Jeffrey, F. Kirwan, Intersection pairings in moduli spaces of vector bundles of arbitrary rank over a Riemann surface, *Ann. Math.* **148**(1) (1998), 109–196.

The supersymmetry transformations in §11.1 and the descent equations of §11.3 will be revisited in a more general context in Lecture 14.

LECTURE 12
Supersymmetric Field Theories

§12.1. Supersymmetric Configurations

Starting from today we study field theories with supersymmetry, i.e. theories whose symmetry group has a nontrivial extension to a supergroup. As usual, by even and odd (infinitesimal) symmetries we mean even, respectively odd, elements of the Lie superalgebra of this supergroup.

It often happens that a solution to the classical field equations has nontrivial odd symmetries. Examples:

gradient flow lines in Morse theory
holomorphic curves
gauge theory instantons
monopoles
Seiberg-Witten solutions
hyperKahler structures
Calabi-Yau metrics
metrics of G_2 and $Spin_7$ holonomy

These examples sound quite different, but they have in common the fact that they are all examples of classical fields invariant under some supersymmetry. They are much studied mathematically, and they also play an important role in understanding the behavior of supersymmetric quantum systems.

§12.2. Supersymmetric solitons (BPS states)

First consider a case in which the gradient flowlines of Morse theory will appear. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a Morse function (i.e. its critical points are nondegenerate and $|\nabla h|$ grows at infinity). A Morse function always has finitely many critical points.

Consider (in Minkowski signature) the theory of maps $\Phi : \mathbb{R}^{2|2} \to \mathbb{R}^n$ with the Lagrangian

$$\mathcal{L} = \int d^2x d^2\theta \left(\frac{1}{2} D_+ \Phi D_- \Phi - h(\Phi) \right), \tag{12.1}$$

Notes by Pavel Etingof and David Kazhdan

where $D_\pm = \frac{\partial}{\partial\theta_\pm} - \theta_\pm\partial_\pm$, $\partial_\pm = \frac{\partial}{\partial x_\pm}$, $x_\pm = \frac{1}{2}(t \pm x)$. This model has an obvious supersymmetry under $Q_\pm = \frac{\partial}{\partial\theta_\pm} + \theta_\pm\partial_\pm$. These supersymmetry operators satisfy the obvious commutation relations $Q_\pm^2 = \partial_\pm$, $\{Q_+, Q_-\} = 0$.

We have $(Q_+ \pm Q_-)^2 = 2\frac{\partial}{\partial t}$. So if we look for classical solutions with time translational symmetry (i.e. for solitons), we may in particular look for those among them which are invariant under one of the supersymmetries, say $Q_+ + Q_-$ or $Q_+ - Q_-$.

We have

$$Q_+ \pm Q_- = \left(\frac{\partial}{\partial\theta_+} \pm \frac{\partial}{\partial\theta_-}\right) + (\theta_+ \pm \theta_-)\frac{\partial}{\partial t} + (\theta_+ \mp \theta_-)\frac{\partial}{\partial x}. \tag{12.2}$$

Thus, the supersymmetry condition for time-independent solutions is

$$\left[\left(\frac{\partial}{\partial\theta_+} \pm \frac{\partial}{\partial\theta_-}\right) + (\theta_+ \mp \theta_-)\frac{d}{dx}\right]\Phi = 0. \tag{12.3}$$

Let us look for even solutions. It is easy to show that such solutions are of the form $\Phi = \phi + \theta_+\theta_-\nabla h(\phi)$. For them, the supersymmetry condition is

$$\frac{d\phi}{dx} \mp \nabla h(\phi) = 0, \tag{12.4}$$

which is the condition for the gradient flowline. Thus, supersymmetric solitons are the flowlines of the gradient flow.

Notice that these first order equations imply the second order equations of motion. Indeed, it is easy to show that the equations of motion are

$$\partial_+\partial_-\phi + \nabla(\nabla h)^2(\phi) = 0, \tag{12.5}$$

or for time-independent solutions

$$\frac{d^2\phi}{dx^2} = \nabla(\nabla h)^2(\phi), \tag{12.6}$$

which can be obtained by differentiation of (12.4) with respect to x.

Another way to see this: the Lagrangian for time-independent fields (i.e the Hamiltonian) is

$$H(\phi) = \int dx \left(\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}(\nabla h(\phi))^2\right). \tag{12.7}$$

Rewriting H, we get

$$H(\phi) = \frac{1}{2}\int dx \left(\frac{d\phi}{dx} \mp \nabla h(\phi)\right)^2 \pm \int_{-\infty}^{\infty} dh(s). \tag{12.8}$$

Since the last term is locally constant on the space of fields of finite energy, a supersymmetric solution provides the global minimum for the energy in each connected component of the space of fields. The value of energy at this minimum is $|S|$ with $S = \int dh$.

Definition. Supersymmetric solitons, that is classical solutions invariant under some but not all supersymmetries, are called *classical BPS states*.

§12.3. The role of BPS states in quantum theory

We have mentioned above and used the fact that the vector fields Q_+, Q_- commute. Since the space of solutions of the classical field equations is a symplectic supermanifold, these vector fields must be (at least locally) generated by some Hamiltonian functions $\widetilde{Q}_+, \widetilde{Q}_-$ (defined up to adding a locally constant function). But these functions need not Poisson commute: their Poisson bracket has to be a locally constant function, not necessarily equal to zero.

In our case, the functions $\widetilde{Q}_+, \widetilde{Q}_-$ are easy to write down: if $\Phi = \phi + \theta_+\psi_+ + \theta_-\psi_- + \theta_+\theta_-F$, then

$$\widetilde{Q}_+ = \int dx\,(\psi_+\partial_+\phi + \psi_-\nabla h(\phi))\,, \widetilde{Q}_- = \int dx\,(\psi_-\partial_-\phi - \psi_+\nabla h(\phi))\,. \tag{12.9}$$

The computation of the Poisson bracket gives $\{\widetilde{Q}_+, \widetilde{Q}_-\} = -2S$, $S = \int dh = h(\phi(\infty)) - h(\phi(-\infty))$.

From this picture it is clear what will happen with operators Q_+, Q_- in quantum theory. We will have $Q_+^2 = P_+, Q_-^2 = P_-$ (where P_+, P_- are the corresponding momentum operators), and $(Q_+ \pm Q_-)^2 = 2(H \mp \widehat{S})$, where $H = (P_+ + P_-)/2$ is the Hamiltonian, and $\widehat{S}$, obtained by quantization of S, commutes with local operators since S is constant on each component of field space.

In quantum theory, to every connected component X_a of the space X of fields of finite energy there corresponds a summand $\mathcal{H}_a$ of the Hilbert space. Under many conditions, to every supersymmetric soliton $\Phi \in X_a$ there corresponds a state Ψ in $\mathcal{H}_a$ which is also supersymmetric: $(Q_+ \pm Q_-)\Psi = 0$. (We will discuss examples in which the existence of such a Ψ can be proved.) Then $(H \mp S_a)\Psi = 0$, where $S_a = \widehat{S}|_{\mathcal{H}_a}$ is a scalar. Therefore, since $H \geq 0$, we have $H\Psi = |S_a|\Psi$. In particular, in this supersymmetric model with just two supercharges, for every connected component X_a of X there is at most one linear combination of the supercharges (determined by S_a) which can annihilate a state in $\mathcal{H}_a$.

In general, on $\mathcal{H}_a$ we have $H \geq |S_a|$.

Now we want to determine whether there is a supersymmetric quantum state, that is a quantum state annihilated by $Q_+ \pm Q_-$, corresponding to the supersymmetric classical state. In the lowest order of perturbation theory one finds no bosonic zero mode except the translations and exactly one fermionic zero mode. Roughly speaking, one now argues that one fermionic zero mode is not enough to represent two supersymmetries, so one linear combination of the supersymmetries must annihilate the lowest energy quantum state of the soliton. Such a quantum state annihilated by a linear combination of supersymmetries is called a quantum BPS state.

It is awkward to make the argument for existence of the quantum BPS state precise in this case because of the fact that the number of fermionic zero modes of the soliton is odd. Solitons in $1+1$ dimensions with an odd number of fermionic zero modes (and no bosonic ones) obey an odd kind of statistics, neither bosonic nor fermionic, that we have not discussed and that I do not wish to enter into in detail.[1] Suffice it to note that if there are n widely separated such solitons (separated by

[1] A system of n identical particles usually gives a representation of the group S_n of permutations of n objects. For bosons, one gets the trivial representation, and for fermions a one-dimensional representation that maps a permutation of parity π to $(-1)^\pi$. $1+1$-dimensional solitons with an

distances much greater than $1/m$, with m the mass gap), for even n, one has n approximate fermion zero modes (one such mode supported near each soliton), and quantization of the Clifford algebra gives $2^{n/2}$ nearly degenerate states of least energy (for fixed positions of the solitons). The number of states per soliton is thus $\sqrt{2}$. By contrast, if the soliton were not annihilated by any linear combination of the supersymmetries, then in studying the one-soliton states we would have to represent the supersymmetry algebra. If $H > |S|$, the states of a single soliton would represent a two-dimensional Clifford algebra $Q_+^2 = Q_-^2 = H$, $\{Q_+, Q_-\} = -2S$ which is *nondegenerate* and has a smallest irreducible representation of dimension 2. So in this case, there would be at least 2, not only $\sqrt{2}$, states per soliton.

The basic idea of proving existence of quantum BPS states is almost always the same: one counts the number of quantum states, and shows that if the states were not annihilated by some supersymmetries, the number would have to be larger. However, the present example is bit bizarre because of the exotic statistics and the non-integral number of quantum states per particle. I do not want to thoroughly analyze this example, since the peculiarities involved will not appear in the remainder of our study of quantum field theory. In our subsequent examples, the details will be more straightforward.

§12.4. $N = 2$ supersymmetry in 2 dimensions

Consider the space $\mathbb{R}^{2|4}$ with coordinates $x_+, x_-, \theta_+, \theta_-, \bar\theta_+, \bar\theta_-$. (See the Supersolutions for much more detail concerning material presented here.) This space admits an action of the N=2 supersymmetry algebra, with supersymmetry generators

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta_+} + \bar\theta_+\frac{\partial}{\partial x_+}, \bar Q_+ = \frac{\partial}{\partial\bar\theta_+} + \theta_+\frac{\partial}{\partial x_+}, \\ Q_- &= \frac{\partial}{\partial\theta_-} + \bar\theta_-\frac{\partial}{\partial x_-}, \bar Q_- = \frac{\partial}{\partial\bar\theta_-} + \theta_-\frac{\partial}{\partial x_-}, \end{aligned} \tag{12.10}$$

Let us also introduce vector fields

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta_+} - \bar\theta_+\frac{\partial}{\partial x_+}, \bar D_+ = \frac{\partial}{\partial\bar\theta_+} - \theta_+\frac{\partial}{\partial x_+}, \\ D_- &= \frac{\partial}{\partial\theta_-} - \bar\theta_-\frac{\partial}{\partial x_-}, \bar D_- = \frac{\partial}{\partial\bar\theta_-} - \theta_-\frac{\partial}{\partial x_-}, \end{aligned} \tag{12.11}$$

which commute with the supersymmetry generators. Therefore, any Lagrangian which is written in terms of the D's is supersymmetric.

Recall that a chiral function (or superfield) on $\mathbb{R}^{2|4}$ is a function satisfying the equations $\bar D_+\Phi = \bar D_-\Phi = 0$. A general solution to these equations has the form

$$\begin{aligned} \Phi = \phi - \theta_+\bar\theta_+\partial_+\phi - \theta_-\bar\theta_-\partial_-\phi + \theta_+\bar\theta_+\theta_-\bar\theta_-\partial_+\partial_-\phi + \theta_+\theta_-F + \\ \theta_+\psi_+ + \theta_-\psi_- - \theta_+\bar\theta_+\theta_-\partial_+\psi_- - \theta_-\bar\theta_-\theta_+\partial_-\psi_+. \end{aligned} \tag{12.12}$$

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odd number of fermion zero modes represent not S_n but a group $\widehat{S}_n$ that is a central extension of S_n by $\mathbf{Z}_2$. For n even, its irreducible representation has dimension $2^{n/2}$. These peculiar solitons also present counterexamples to some of our assumptions in the axiomatic lectures of last term, like the assumption that the states of a quantum field theory reduce asymptotically in the far past or far future to a Fock space of one-particle states. Let me reassure the reader, however, that counterexamples of this kind exist apparently only in $1+1$ dimensions and (with some differences, involving Chern-Simons interactions and quantum groups) in $2+1$ dimensions.

You can read more about chiral functions in the Superhomework and Supersolutions.

Consider the theory of chiral maps Φ of $\mathbb{R}^{2|4}$ into $\mathbb{C}^n$, with the Lagrangian

$$\frac{1}{2}\int d^2x d^4\theta \Phi\bar{\Phi} + \int d^2x d\theta_+ d\theta_- W(\Phi) + \int d^2x d\bar{\theta}_+ d\bar{\theta}_- \bar{W}(\Phi), \tag{12.13}$$

where W is a holomorphic function on $\mathbb{C}^n$, called the superpotential. This Lagrangian is N=2 supersymmetric. In components (for $x_+ = z, x_- = \bar{z}$), it looks like

$$\frac{1}{2}\int d^2x(|d\phi|^2 - |F|^2 + W'(\phi)F + \bar{W}'(\phi)\bar{F} + \text{terms with fermions}). \tag{12.14}$$

Setting all fermions to zero and the "dummy" field F to the stationary point $F = \bar{W}'(\phi)$, we get the bosonic energy functional

$$H = \frac{1}{2}\int d^2x(|d\phi|^2 + |W'(\phi)|^2). \tag{12.15}$$

It is easy to see that this functional coincides with (12.7), for the function $h = \mathrm{Re}(e^{-i\alpha}W)$, where α is any real number. Thus, at the classical level we are doing a special case of the previous problem. However, now we have more supersymmetry and therefore a more interesting theory.

§12.5. $N = 2$ BPS states

The commutation relations for the $N = 2$ supersymmetry algebra are

$$\begin{gathered}\{Q_\pm, \bar{Q}_\pm\} = 2P_\pm, \\ \{Q_+, \bar{Q}_-\} = \{\bar{Q}_+, Q_-\} = 0 \text{ (by R-symmetry)}, \\ \{Q_+, Q_-\} = T,\ \{\bar{Q}_+, \bar{Q}_-\} = \bar{T},\end{gathered} \tag{12.16}$$

where T is analogous to S in Section 12.2 – it is a locally constant function on the space of classical solutions (T is, unlike S, not necessarily real), and for brevity we drop twiddles over Q's. Also, in the above basis, the squares of all the Q's are zero.

In fact, the function T is easy to compute, like the function S in the previous problem. Essentially the same computation as before gives

$$T = W(\phi(\infty)) - W(\phi(-\infty)). \tag{12.17}$$

Now we will look at supersymmetric states. Choose a real number α and look for states which are invariant under two supersymmetries $Q_1(\alpha) = Q_+ + e^{i\alpha}\bar{Q}_-$, $Q_2(\alpha) = \bar{Q}_+ - e^{-i\alpha}Q_-$.

We have

$$\{Q_1, Q_2\} = 2i(H - \mathrm{Re}(e^{-i\alpha}T)). \tag{12.18}$$

This implies that supersymmetric classical states are time-independent. (The symplectic vector fields generated by Q_1, Q_2, and T all vanish for such a state, so the same must be true for H.)

The equation $Q_1\Phi = 0$ for an even function Φ gives (in the time-independent case):

$$\frac{d\phi}{dx} = e^{-i\alpha}\overline{W'(\phi)}, \tag{12.19}$$

and the second equation gives the same result. This implies, in particular, that

$$\bar{T} = \bar{W}(\phi(\infty)) - \bar{W}(\phi(-\infty)) = \int \bar{W}'(\phi)\frac{d\phi}{dx}dx = e^{-i\alpha}\int_{\infty}^{\infty} |W'(\phi)|^2 dx, \tag{12.20}$$

which implies that $\alpha = \arg T$. Thus, from equation (12.18) we get that for a supersymmetric solution of the classical equations we have

$$H = \mathrm{Re}(e^{-i\alpha}T) = |T|. \tag{12.21}$$

For other states in the connected component of this solution we have $H \geq |T|$.

Now, what are the supersymmetric solutions (of finite energy) geometrically? It is clear from equation (12.19) that they are separatrices between critical points of W for the gradient flow of $h = \mathrm{Re}(e^{-i\alpha}W)$.

Now let us turn to quantum theory. From classical considerations we saw that in our theory $H \geq |T|$, and in a nondegenerate case $H = |T|$ only for BPS states. Therefore, we should hope that in quantum theory the same situation takes place, *a priori* with a corrected value of T.

Consider the generic situation when all zero-modes of the Hamiltonian near a classical BPS state arise from the super-Poincaré group. This is the case if for any 3 critical points a, b, c of the potential we have $|T_{ac}| < |T_{ab}| + |T_{bc}|$, where T_{ab} is the value of T on the component of the space of solutions which go from a to b. In other words, this is the case when the gradient flowline between a, b never passes through c. It is thus true for a generic W, but in a family of W's the condition will fail in real codimension one.

In quantum theory, we expect that, as the coupling goes to zero, the lowest eigenvalue of H converges to $|T|$. The multiplicity of the lowest eigenvalue of H (for given momentum) is found by quantizing the zero modes. By our assumption of nondegeneracy, there are no bosonic zero modes except translations (existence of translational zero modes is associated, as we know from Lecture 9, with the fact that the soliton can have any momentum), and there are only two fermion zero modes, coming from the supersymmetries that do not leave the soliton invariant. Quantizing the two fermion zero modes, we get a two-dimensional Clifford algebra with a two-dimensional representation. In the weak coupling limit, the soliton spectrum obtained from this quantization is the space of sections of an equivariant vector bundle on the upper part of the hyperboloid, where the fiber is the standard two-dimensional irreducible representation of the Clifford algebra.

By contrast, if the ground state of the soliton had $H > |T|$, it would have to represent a four-dimensional Clifford algebra generated by all four Q's, and the ground state would have multiplicity four. The states of the soliton, in other words, would be sections of a rank four bundle over the hyperboloid.

So just by counting that there are only two fermion zero modes and hence only two states of the quantum soliton, we can deduce that the ground state of the soliton must be BPS and have mass $H = |T|$. This is of course a more straightforward version of the argument made at the end of section 12.3. It does not require us to leave the world of conventional Bose and Fermi statistics!

§12.6. $N = 1$ Supersymmetry in 4 dimensions

Now consider supersymmetry in 4 dimensions. (See the Supersolutions as well as the Addendum to the Superhomework for much more detail.) We start with $N = 1$ supersymmetry. In this case the odd part of the supersymmetry algebra is similar to the $N = 2$ case in two dimensions. It is generated by $Q_\pm$, $\bar{Q}_\pm$, with relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2P_{\alpha\dot\beta}, \\ \{Q_\alpha, Q_\beta\} &= 0, \\ \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} &= 0\ . \end{aligned} \tag{12.22}$$

where $\alpha, \beta \in \{+, -\}$, and $P_{\alpha\dot\beta}$ are the momentum operators (Because of the isomorphism of Lorentz representations $S_+ \otimes S_- \to V_{\mathbb{C}}$, where $S_\pm$ are the spinor representations, the momentum operators in four dimensions transform in $S_+ \otimes S_-$.) A central extension like in (12.16) cannot arise here because it is prohibited by the Poincaré symmetry (this central extension would have to be in the representation $S^2\mathbb{C}^2$ of $SU(2)$, with $\mathbb{C}^2$ the standard representation; this is the spin 1 representation and contains no invariants).

The determinant of the quadratic form corresponding to the Clifford algebra (12.22) is $(P^2)^2$. The rank of this form if $P^2 = 0$ but $P \neq 0$ is 2, but for $P^2 \neq 0$ it is 4. Therefore, massless representations of the super-Poincaré group correspond to 2-dimensional representations of a rank 2 Clifford algebra, and massive representations correspond to 4-dimensional representations of a rank 4 Clifford algebra.

More precisely, consider a representation W of the super-Poincaré group and the subspace W_p on which $P = p$, where $p \in \mathbb{R}^{1,3}$. Let G_p be the stabilizer of p in the group of rotations, H_p the maximal torus in G_p (always isomorphic to $U(1)$). With respect to H_p, W has a decomposition into a direct sum of representations of integer and half-integer spins. These spins are called helicities of W, and each helicity has a multiplicity.

It is easy to see that the supersymmetry operators $Q_\alpha, \bar{Q}_{\dot\alpha}$ raise helicity by $1/2$ (since they live in the spinor representation of the Lorentz group). Thus, the massless representations have helicities $j, j + 1/2$ with multiplicity 1, and the massive representations have helicities $j, j + 1/2, j + 1$ with multiplicities $1, 2, 1$.

Now let us consider massless particle multiplets which are allowed by $N = 1$ supersymmetry. There are two such basic multiplets.

1. Vector multiplet: a gauge field A and a chiral spinor λ in the adjoint representation. This is the $N = 1$ analogue of the gauge field. In particular, in the $U(1)$ case the minimal theory of such a field is free. It contains a massless vector and a massless spinor, so the helicities are $-1, 1$ (for vector) and $-1/2, 1/2$ (for spinor). In particular, the massless representation of the super-Poincaré group arising in this theory is reducible, and splits into two: the one with helicities $-1, -1/2$ and the one with helicities $1/2, 1$. However, over $\mathbb{R}$ this splitting does not exist and our representation is irreducible. A more physical version of this statement is to say that the helicity $-1, -1/2$ states are related to helicity $1/2, 1$ states by CPT conjugation, so that one pair must be present if the other is. Thus, this representation (or multiplet, as physicists call it) is the minimal supersymmetric one which contains a gauge boson.

2. Chiral multiplet: A massless complex scalar ϕ and a massless chiral spinor χ. In this case the story is analogous. The obtained massless representation has a spinor and a complex scalar, so it has helicities $0, 0$ (for scalar) and $-1/2, 1/2$ (for spinor). Thus this representation is again a sum of two, with helicities $-1/2, 0$ and $0, 1/2$. This decomposition is only valid over $\mathbb{C}$ and not over $\mathbb{R}$; over $\mathbb{R}$, the representation is irreducible, so this combination is the smallest supersymmetric one containing a scalar.

The massive versions of these multiplets are as follows.

1. Massive vector multiplet. The minimal real supersymmetric representation containing a massive vector has helicities $-1, -1/2, 0, 1/2, 1$ with multiplicities $1, 2, 2, 2, 1$. The corresponding eight-dimensional representation of the Clifford algebra is (over $\mathbb{C}$) a sum of two four-dimensional representations: supersymmetry alone would give an irreducible multiplet with helicities $0, 1/2, 1$ of multiplicities $1, 2, 1$; CPT gives the other states. This multiplet contains a massive vector, two massive spinors (chiral and antichiral), and a real massive scalar. One can obtain it by combining a massless vector multiplet and massless chiral multiplet, and perturbing by adding a mass term (which could come from a Higgs mechanism).

2. Massive chiral multiplet. Same as massless chiral multiplet (we could consider the same fields with masses, such that the mass of bosons equals the mass of fermions; the mass term can be represented in superspace by a quadratic term in the superpotential).

§12.7. $N = 2$ Supersymmetry in 4 dimensions

In the $N = 2$ case, we have two copies of operators Q: $Q^{(1)}$ and $Q^{(2)}$. (Again, see the Supersolutions for much further detail on what follows.) The anticommutators of $Q^{(1)}$ with $Q^{(1)}$ or of $Q^{(2)}$ with $Q^{(2)}$ generate the same $N = 1$ supersymmetry algebra that we have just studied. But in the minimal $N = 2$ algebra, $\{Q^{(1)}, Q^{(2)}\} = 0$. However, now there is a possibility for a central extension: it is no longer prohibited by the Poincaré symmetry. The central extension is characterized by the fact that $\{Q^{(1)}, Q^{(2)}\}$ is non-zero (and central).

Consider first the case when the central term is zero. In this case the quadratic form of the Clifford algebra is nondegenerate (in 8 dimensions) in the massive case, and has rank 4 in the massless case. Thus, an irreducible massive representation should be 16-dimensional, and an irreducible massless representation should be 4-dimensional. The helicities for representations are obtained like in the $N = 1$ case. In particular, in a massive representation the helicities are in groups of the form $(j, j + 1/2, j + 1, j + 3/2, j + 2)$ with multiplicities (1,4,6,4,1), and in a massless representation they are in groups $(j, j + 1/2, j + 1)$ with multiplicities (1,2,1).

Now let us consider the simplest $N = 2$ supersymmetric theory. It is obtained by combining fields from a (classically) massless vector multiplet and a massless chiral multiplet in the adjoint representation. Thus the fields are: from vector multiplet – (A, λ), from chiral multiplet – (ϕ, χ). The Lagrangian is the minimal $N = 2$ supersymmetric Lagrangian on these fields. Such a Lagrangian exists and is uniquely determined by the minimality condition. In the $U(1)$ case the theory is free, but in the nonabelian case there are nontrivial interactions.

What does this theory do in the infrared? In the $U(1)$ case, the answer is simple: we should add together the representations for the vector and chiral multiplets. We

get helicities $(-1, -1/2, 0, 1/2, 1)$ with multiplicities (1,2,2,2,1). This is the sum of two complex conjugate massless representations of the $N = 2$ Clifford algebra: the one with helicities (0,1/2,1) and multiplicities (1,2,1) and the one with helicities (-1,-1/2,0) and multiplicities (1,2,1).

But now let us consider the nonabelian case. We assume the gauge group G is simple. The bosonic fields are the gauge field A and a scalar ϕ in the complexified adjoint representation $\mathfrak{g}_{\mathbb{C}}$. The bosonic part of the Lagrangian is

$$L_{bosonic} = \int d^4x \mathrm{Tr}\left(\frac{1}{4e^2}F^2 + (d_A\phi)^2 + [\phi, \bar{\phi}]^2\right). \tag{12.23}$$

Thus the bosonic part of the space of classical ground states is the set of solutions of the equations $[\phi, \bar{\phi}] = 0$ modulo the action of G. This space can be identified with $\mathfrak{t}_{\mathbb{C}}/W$, where $\mathfrak{t}$ is the maximal commutative subalgebra in $\mathfrak{g}$ and W the Weyl group. In the case $G = SU(2)$, this quotient is identified with $\mathbb{C}$ by introducing the global coordinate $u = \mathrm{Tr}(\phi^2)$ (which parametrizes the u-plane of Donaldson theory).

Recall from Lecture 2 that in this situation we have gauge symmetry breaking from G to the centralizer H of ϕ. In particular, for $SU(2)$ with $u \neq 0$ the gauge symmetry is broken classically from $SU(2)$ to $U(1)$ (Higgs mechanism). The components of the gauge field which are charged nontrivially with respect to the surviving $U(1)$ will become massive. By $N = 2$ supersymmetry, the same will happen for the corresponding components of λ, ϕ, ψ. Thus, in the charge 2 and -2 sectors of the Hilbert space the lowest energy states will be in a massive representation with helicities as before: $(-1, -1/2, 0, 1/2, 1)$ with multiplicities (1,2,2,2,1). However, there is no such representation without central extension. Thus, without central extension we get a contradiction, and hence the central extension must appear.

The central extension will show up in the commutation relations between $Q_\alpha^{(1)}$ and $Q_\beta^{(2)}$:

$$\{Q_\alpha^{(1)}, Q_\beta^{(2)}\} = \varepsilon_{\alpha\beta} S, \tag{12.24}$$

where S is an operator which commutes with all local operators (it is called the central charge). In the absence of the central charge, the algebra had a $U(2)$ R-symmetry (action on indices 1 and 2), but the central charge breaks this symmetry down to $SU(2)$. In any event, at the quantum level, the chiral $U(1)$ symmetry is anomalous in our theory, because the fermion measure is not invariant; an anomaly breaks the chiral $U(1)$ symmetry is to $\mathbb{Z}/8\mathbb{Z}$.

In this case the massive particles considered above must have mass exactly $|S|$. This is the value for which the quadratic form of the Clifford algebra becomes degenerate, and just as in the case discussed in section 12.5, if the value of the mass were different, the number of particles would have to be bigger.

This theory also has BPS solitons – magnetic monopoles. If we are at the vacuum $u \in \mathbb{C}$, we have

$$\phi \sim \frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{12.25}$$

where $a = \pm u^{1/2}$. Let $a = \rho e^{i\alpha}$ with real ρ. We should look for classical BPS solutions (i.e. solutions which are time-independent and invariant under some, in

fact half, of the supersymmetry) which satisfy the condition (12.25) at infinity. If $S = |S|e^{i\beta}$ then the equation of being invariant under half of the supersymmetry is

$$F = e^{-i\beta} * d_A\phi, \tag{12.26}$$

where $*$ is in $\mathbb{R}^3$. (the BPS monopole equations). Since F is real, we must have $\alpha = \beta$ modulo π.

It can be shown that such BPS solutions exist; in fact there is an extensive mathematical theory of them. They are characterized by the value of magnetic charge (or the first Chern class at infinity), introduced in Lecture 9. For magnetic charge 1, the space of solutions is as follows. The reduced space is isomorphic to $\mathbb{R}^3 \times S^1$, for reasons explained in Lecture 9. (Recall that $\mathbb{R}^3 \times S^1$ appeared as a homogeneous space for a group consisting of translations of space and $U(1)$ charge rotations at infinity.) The solution is invariant under four linear combinations of supersymmetries; there are precisely four fermion zero modes, generated by the other supersymmetries. Quantizing these fermion zero modes gives a four-dimensional Clifford algebra with a four-dimensional irreducible representation; so the monopole states of charge one are in a four-dimensional representation of supersymmetry. This representation is in fact a massive hypermultiplet – two states of spin zero and two of spin one-half. (The antimonopole, related to the monopole by CPT, likewise has four states, of the same spins.) These states are inevitably BPS, with mass $M = |S|$, since if one had $M > |S|$, one would have to represent an eight-dimensional Clifford algebra, and the number of states would be 16 instead of 4. These BPS states will play a very important role when we analyze the dynamics of this theory (and its applications to Donaldson theory) at the end of the term.

Remark. Computing the commutators of the Q's using currents, one can show that classically

$$S = \int_\Sigma \left(\phi * F + \frac{i}{e^2}\phi F\right), \tag{12.27}$$

where Σ is a distant sphere in the spatial cycle. So it is a linear combination of the electric and magnetic charge for the effective $U(1)$ theory. In fact, S is inevitably a linear combination of the electric and magnetic charge, simply because this theory has no other conserved charge that commutes with the Poincaré group.

References and cross references

The model on $\mathbb{R}^{2|2}$ treated in §12.2 is discussed in [I-**Supersolutions**, §4.5]. Basic formulas for supersymmetry with 4 supercharges appear in [I-**Supersolutions**, §§2.4, 2.6]. The $\mathbb{R}^{2|4}$ model discussed in §12.5 is the dimensional reduction of a model on $\mathbb{R}^{4|4}$. The latter is discussed in Chapter 5 of [I-**Supersolutions**], though the central extension is only seen in the dimensionally reduced model. The various supersymmetric multiplets in four dimensions are the subject of the addendum to the Superhomework in [I-**Homework**]. The quantum behavior of $N = 2$ gauge theory in 4 dimensions is discussed in Lectures 17 and 18 of this series.

For the appearance of central charges in supersymmetry algebras of theories with solitons, see

E. Witten, D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **78B** (1978), 97.

For the mathematical theory of BPS monopoles in the $N = 2$ supersymmetric Yang-Mills in four dimensions, see

M. F. Atiyah, N. Hitchin, *The Geometry And Dynamics Of Magnetic Monopoles*, M. B. Porter Lectures, 1988.

LECTURE 13
$N = 2$ SUSY theories in Dimension Two: Part I

§13.1. Introduction

In this lecture and the next two we consider $N = 2$ supersymmetric field theories in dimension two. Today, after some general introductory remarks about such theories, we consider two-dimensional $U(1)$ gauge theories with complex-valued chiral superfields charged under the gauge group. These theories depend on two functions – a superpotential W and a twisted superpotential $\widetilde{W}$. The superpotential W is a complex polynomial in the superfields. For the class of models that will concern us today, this polynomial must satisfy a certain "weight" condition, but is otherwise free. For gauge group $U(1)$, the twisted superpotential $\widetilde{W}$ of the sort of model we will look at is determined by a single complex parameter $t = -ir + \frac{\theta}{2\pi}$, or more precisely by $\exp(-2\pi i t)$. Today we will keep the superpotential fixed, and we allow the twisted superpotential to vary. Thus, for each (generic) complex polynomial W satisfying the weight conditions, we get a family of theories parameterized by the point $\exp(-2\pi i t)$ in the cylinder $\mathbf{C}/\mathbf{Z}$. We are interested in what happens at the ends of the cylinder $\mathrm{Im}(t) \mapsto +\infty$ and $\mathrm{Im}(t) \mapsto -\infty$. In the cases we consider, the limit as $\mathrm{Im}(t) \mapsto -\infty$ is described in terms of a sigma model on the projective variety X defined by the polynomial W giving the superpotential. In this region, $-\mathrm{Im}(t)$ is the Kähler class of X, and the fact that there is only one parameter in the superpotential is related to the fact that $H^{1,1}(X)$ (or at least the part of it that is pulled back from a certain ambient weighted projective space) is one-dimensional. The limit as $\mathrm{Im}(t) \mapsto +\infty$ is described in terms of an orbifold version of a Landau–Ginzburg model.

When $c_1(X) = 0$, we will find that the beta function vanishes for all t, and we get a family of conformally invariant theories. For $c_1(X) \neq 0$, we get instead a non-trivial renormalization group flow in t. The flow increases $\mathrm{Im}(t)$ when $c_1(X) > 0$ and decreases $\mathrm{Im}(t)$ when $c_1(X) < 0$. (Furthermore, in these latter two cases there are "extra" vacua at the end of the cylinder to which one flows in the infrared.) This allows us to see various Landau–Ginzburg models as infrared limits of sigma models and other sigma models as infrared limits of Landau–Ginzburg models. This is interesting because the descriptions of these types of models are very different

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at the classical level. Most of today's lecture will deal with special cases (the cases that $c_1(X) = 0$, and some of the cases with $c_1(X) > 0$ which flow to a massive infrared theory) that have been extensively studied, but the general picture that will be presented at the end of the lecture is actually new, as far as I know.

§13.2. Generalities on $N = 2$ SUSY theories in dimension two

§13.2.1. The β-function of $N = 2$ SUSY sigma models

The first remark concerns $N = 2$ supersymmetric sigma models with target a compact Kähler manifold X, which we assume has a Kähler-Einstein metric. In [**II-CFT**, §3.8] (Equation 3.42) we have seen that the one-loop β-function is determined by the sign of the Ricci curvature of the metric and in fact, when the Ricci tensor is positive or negative definite, the sign of the one-loop β-function is opposite that of the Ricci curvature. Thus, if $c_1(X) > 0$ then the one-loop β-function is negative. (By $c_1(X)$ we mean c_1 of the tangent bundle of X – so $c_1 > 0$ for complex projective space.) In such a case, at small coupling the β-function is negative so that the sigma model under consideration is a "good" fundamental theory which is asymptotically free. Such a theory is well-defined at the quantum level. The question arises of what it flows to in the infrared. Today we shall see examples of such X for which the infrared theory is massive and trivial, and also examples with flow to a non-trivial infrared fixed point. In the latter case, the limiting superconformal theories that we will get can be described in terms of what are known as supersymmetric Landau–Ginzburg theories. The Landau–Ginzburg models are sometimes explicitly soluble by algebraic methods.

On the other hand, if $c_1(X) < 0$, then the one-loop β-function is positive and hence β is positive at weak coupling. Thus, we have no reason to expect sigma models for such X to be "good" fundamental theories. Rather in this case the sigma model is an effective theory, free in the infrared, and one should ask whether one can describe a concrete, well-defined ultraviolet theory that flows to these sigma models in the infrared. We will solve this problem for this class of sigma models by taking $\mathrm{Im}(t) \to \infty$, where we will find a Landau–Ginzburg theory (which is superconformal), which after a relevant perturbation (to $\mathrm{Im}(t)$ large and positive but not ∞) flows to the sigma model with target space X (plus some extra massive vacua).

The case when $c_1(X) = 0$,. i.e., the case when X is Calabi-Yau, is different. Here, the one loop β-function vanishes and one-loop contributions do not change the Kähler metric. It turns out that there are higher loop corrections to the β-function, so that it is not identically zero, and hence the sigma model on X (with its Ricci-flat metric) is not conformal. The structure is actually as follows. The Kähler metric on X is of course associated with a closed $(1,1)$-form ω, the Kähler form. The renormalization group equation is an equation for the change in ω with respect to the renormalization group scale μ, and takes the general form $d\omega/d\ln\mu = \beta$, where the β "function" is best understood, like ω, as a closed $(1,1)$-form on X. Now, the s-loop contribution to β has the following characteristic: it is a local functional of the Kähler geometry of X, and more specifically, if we define a grading in which we weight derivatives by 1 and the Riemann tensor by 2, then the s-loop part

of the β function is of degree $2s$. [1] For $s = 1$, the Ricci tensor of X has the right properties, and one-loop computation reveals that the one-loop term in β is a negative multiple of the Ricci tensor R. What about $s > 1$? Standard invariant theory shows that any closed $(1,1)$-form that can be written locally and universally on a Kähler manifold and is of degree $2s$ with $s > 1$ is of the form $\bar{\partial}\partial U$, with U a locally constructed function of degree $2s - 2$. Hence, the equation for a zero of the beta function ($\mu d\omega/d\mu = 0$) reduces to

$$R = \bar{\partial}\partial U.$$

At this stage, we cite Yau's theorem about existence of Kähler metrics on X with prescribed Ricci curvature. The theorem asserts that, if $c_1(X) = 0$, then for any fixed function U on X, the equation $R = \bar{\partial}\partial U$ has a unique solution with prescribed complex structure and Kähler class on X. Now, let $\bar{\omega}_0$ be a fixed Kähler class on X, and try to solve $R = \bar{\partial}\partial U$ with Kähler class $p\bar{\omega}_0$, for very large p. In leading approximation, at $p = \infty$, we simply take the solution of $R = 0$. That is consistent because the s-loop contribution to U, for $s \geq 2$, gives a term in the equation of relative order $p^{-(s-1)}$. Having determined the leading order metric, one gets an approximate determination of U, and then one tries to solve the equation $R = \bar{\partial}\partial U$ by successive approximations. By Yau's theorem, a unique Kähler metric obeying the equation exists to all finite orders in $1/p$. Convergence is another story; it is believed that this process converges to give a zero of the beta function if p is sufficiently large, or differently put, if the Kähler class is sufficiently ample and far from the boundaries of the Kähler cone. For today's models these conditions simply mean that $-\mathrm{Im}(t)$ should be large enough.

For very large $-\mathrm{Im}(t)$, a good description of the theory is by the sigma model of maps to X that we started with. What we will be exploring today is really the question of what happens when $-\mathrm{Im}(t)$ is not large.

Today we shall see examples of X with $c_1(X) = 0$ for which there is a family of theories, parameterized by a punctured cylinder, that approaches the sigma model on X as $\mathrm{Im}(t) \to -\infty$ and approaches a Landau–Ginzburg model as $\mathrm{Im}(t) \to +\infty$. To get such a family, the cylinder must be punctured at one point, where the theory has a sort of pole. It is believed that the punctured cylinder parameterizes a smooth family of theories, which flow in the infrared to a family of superconformal field theories. As one check of this belief, we argue that the central charge of the Landau–Ginzburg theory at $+\infty$ and the central charge of the sigma model at $-\infty$ are equal. We will present more evidence for the scenario described today in a later lecture.

§13.2.2. Reasons for considering $N = 2$ SUSY theories

One advantage of studying $N = 2$ supersymmetric theories is that they are much more rigid than theories with less (or no) supersymmetry. For example, if we are considering two-dimensional sigma models into a compact Riemannian manifold X,

[1] β is local because it is computed from Feynman diagrams; all the vertices in the Feynman diagrams are local on X. The statement about the grading of the s-loop contribution to β is proved by examining the weights of the vertices in the diagrams; vertices of higher and higher order arise from higher terms in an expansion of the metric of X in normal coordinates and have higher weights.

then without supersymmetry we can add terms to the Lagrangian of the form

$$\int d^2y\, \varphi^*(h)$$

for a potential function $h\colon X \to \mathbf{R}$. (Here, $\varphi\colon \Sigma \to X$ is the basic field in the Lagrangian.) In $N = 1$ supersymmetric sigma models, such a term is not possible, but there can still be a term

$$\int d^2y\, d^2\theta\, \Phi^*(h)$$

where again h is a real-valued function on X (called the superpotential) and Φ is the basic superfield in the Lagrangian. In both cases this is a relevant perturbation. When we require $N = 2$ supersymmetry, the Riemannian manifold must be Kähler and the superpotential must be of the form $h = \mathrm{Re}(f)$ for f a holomorphic function. Since our Riemannian manifold X is assumed to be compact, this means that f is constant and hence, after performing the θ integrals, this term vanishes. Simple power counting shows that at the classical level this is the only possible relevant perturbation over a 2|4-superspace. This is why sigma models with vanishing potential are natural with $N = 2$ supersymmetry in two dimensions and less natural and harder to study with $N < 2$.

§13.2.3. Relevant and marginal perturbations of the conformal two-dimensional sigma model

Let us study the marginal perturbations of these conformal sigma models. Part of the data of the sigma model is the complex structure on X. Varying this is a marginal perturbation of the theory. Let us examine this perturbation in local coordinates. The metric tensor on X is a $(1,1)$-tensor of the form $h_{i\overline{j}}dz^i \otimes d\overline{z}^j$. A change in the complex structure corresponds to a perturbation of the metric by a term of the form

$$\delta h_{\overline{ij}} + \delta h_{ij}$$

of type $(2,0)+(0,2)$. Of course, this form must be real in order that the new form be a metric tensor, and $\delta h_{\overline{ij}}$ must be $\overline{\partial}$-closed in order that the new almost complex structure is integrable. Let us compute the change in the Lagrangian that goes with this perturbation of the Kähler metric. We write things in local coordinates using bosonic coordinates $y^\alpha, \alpha = 1, 2$, on the Riemann surface and fermionic coordinates $\theta^\pm, \overline{\theta}^\pm$. We write superfields as $X = X(y, \theta, \overline{\theta})$, chiral superfields as $\Phi(y, \theta)$ and anti-chiral superfields as $\overline{\Phi}(y, \overline{\theta})$. The change in the Lagrangian is given by

$$\Delta\mathcal{L} = \left(\int d^2y\, d^2\overline{\theta}\, \delta h_{\overline{ij}} \overline{D}_+ \overline{X}^{\overline{i}} \overline{D}_- \overline{X}^{\overline{j}}\right) + \text{c. c.}$$

It is a nice exercise to verify that the argument of the $\overline{\theta}$ integral in $\Delta\mathcal{L}$ is anti-chiral. (This uses the fact that $\delta h_{\overline{ij}}$ is $\overline{\partial}$-closed.) This is one of the two possible types of marginal perturbations.

The other type of marginal perturbation is to fix the complex structure X and vary the complexified Kähler class of the Kähler metric. One piece of this deformation is an ordinary variation $\delta h^{\mathrm{herm}}_{\overline{ij}} d\overline{z}^{\overline{i}} \wedge dz^j$ by a Hermitian symmetric closed two-form. This ordinary metric perturbation is the real part of a complex perturbation whose imaginary part is called the "B-field" perturbation (which, as we shall see, is analogous to the θ-angle which appears in gauge theories). Thus, we

consider a perturbation $\delta h = \delta h_{i\bar{j}} d\bar{z}^{\bar{i}} \wedge dz^j$ where δh is a closed complex two-form. This leads to a perturbation of the Lagrangian of the form

$$\Delta\mathcal{L} = \left(\int d^2y d\bar{\theta}^+ d\theta^- (\delta h_{i\bar{j}} \overline{D}_+ \overline{X}^{\bar{i}} D_- X^j \right) + \text{c. c.}$$

Other than the marginal deformations of the theory described above, there can only be deformations of the form

$$\int d^2y d^4\theta (\text{something}).$$

For these the θ-integration has weight 2, and hence if the "something" is a function then such terms are marginal (of dimension two) classically. (Notice this count requires four θ's and hence uses the fact that we have $N = 2$ SUSY). But at the quantum level, they have anomalous dimensions (proportional to nonzero eigenvalues of the Laplacian and so strictly positive) and are irrelevant. Such deformations can affect the Kähler metric without changing the complex structure or Kähler class.

These computations show that for the sigma model with $c_1(X) = 0$, the theory has no relevant perturbations and the marginal perturbations are obtained by deforming the complex structure and complexified Kähler metric.

Let us begin with such a sigma model with a Kähler metric that is smooth and has a large radius of curvature close to, but not equal to, the metric that actually gives zero beta function. The renormalization group flow will bring us to a metric that gives conformal invariance, as all other deformations are irrelevant. Were there relevant perturbations, these would dominate in the infrared limit and the limit would be in general impossible to control.

§13.2.4. Central Charge

Let us review some material from [II-**CFT**, §2.4] on central charges in conformal and superconformal field theories in dimension two. Without supersymmetry, we can usefully divide conformal field theories into two broad classes – those with central charge $c \leq 1$, and those with central charge $c > 1$. The $c < 1$ theories are called minimal models, exist only for a discrete set of values of c, and are explicitly described algebraically in terms of representation theory of the Virasoro algebra. $c = 1$ is the onset of the continuous range of c. The $c = 1$ theories can also be explicitly described, in terms of a free boson. When $c \leq 1$, the Virasoro algebras of left- and right- movers act almost irreducibly in the quantum Hilbert space, which is why these theories are known rather explicitly. For $c > 1$, the Virasoro action is very far from being irreducible (the number of highest weight vectors grows exponentially as one goes to higher energies), and conformal field theory has a completely different flavor.

When we go to the $N = 1$ superconformal algebra in dimension two, the central charge again lies either in a discrete set or in a continuous range. The onset of the continuous range of the central charge occurs for a theory that is realized by a free scalar superfield. Being the sum of a free scalar boson and a free scalar fermion,this field, if considered as a representation of the ordinary conformal algebra, has central charge $c = 3/2$. One usually defines $\widetilde{c} = 2c/3$ so that the continuous values for $\widetilde{c}$ begin at $\widetilde{c} = 1$.

The story is similar for the $N = 2$ superconformal algebra in dimension two. Here the lowest value in the continuous part is realized by the free chiral superfield. Since this field consists of a complex scalar boson and a complex fermion, its $N = 0$ central charge is 3. Hence, we set $\widehat{c} = c/3$ so that once again the lowest value in the continuous part is $\widehat{c} = 1$. The discrete set of $N = 2$ theories with $\widehat{c} < 1$ are the "$N = 2$ minimal models" which we will consider in detail later on. Their central charges take the form $\widehat{c} = 1 - \frac{2}{n}$ for $n = 3, 4, 5, \ldots$, and they can be described algebraically and studied in a variety of attractive ways.

A conformal $N = 2$ supersymmetric sigma model with target space a k-dimensional complex manifold X of $c_1(X) = 0$ has central charge $\widehat{c} = k$. This is proved by using the fact that $\widehat{c}$ is constant in a family of conformal field theories, and expanding the metric until X can be approximated, in a local region, to any desired accuracy, by $\mathbf{C}^k$ with a flat metric. Thus, this sigma model has the same $\widehat{c}$ as a sigma model with target a flat $\mathbf{C}^k$. That is a free theory with k chiral superfields, and so has $\widehat{c} = k$, as claimed.

§13.3. The $U(1)$ Theories

Let us turn now to the study of $U(1)$ gauge theory with complex-valued chiral superfields $A_1, \ldots, A_n$. These superfields are charged under the $U(1)$ with charges $q_1, \ldots, q_n \in \mathbf{Z}$. Recall from Chapter 9 of [I-**Supersolutions**] that the connections we use on 2|4-superspace are constrained to be flat in all pairs of odd directions except the pair $\{\overline{\theta}^+, \theta^-\}$ (and its complex conjugate). The basic invariant of such a connection is the nonvanishing part of the curvature in the odd directions:

$$\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\}.$$

Σ is a twisted chiral superfield with values in the complexification of the adjoint bundle over 2|4-superspace. The minimal pure gauge theory Lagrangian that we can write down is

$$\int d^2y d^4\theta \frac{1}{2e^2} \overline{\Sigma}\Sigma.$$

Adding in a kinetic term for the n chiral superfields, the minimal Lagrangian becomes

$$\mathcal{L}_0 = \int d^2y d^4\theta \left(\frac{1}{2e^2} \overline{\Sigma}\Sigma + \sum_{i=1}^{n} \overline{A}_i A_i \right). \tag{13.1}$$

Eventually, we will add a superpotential and a twisted superpotential term to this Lagrangian.

§13.3.1. R-symmetries

Recall from Chapter 9 of [I-**Supersolutions**] that the above $N = 2$ supersymmetric gauge theory in two dimensions (a 2|2 supertheory) comes by dimensionally reducing from $N = 1$ gauge theory in four dimensions (which is a 4|4 supertheory). As such, there is an R-symmetry for the $N = 2$ two-dimensional theory induced by spatial rotation in the remaining two dimensions in four-space. But, in addition, the $N = 1$ supersymmetric version of $U(1)$ gauge theory in four dimensions itself has an R-symmetry. This R-symmetry from dimension four dimensionally reduces to give a second R-symmetry for our 2-dimensional theory.

Independent of their sources, it is easy to see directly in the 2|4-theory what the $U(1) \times U(1)$-group of R-symmetries is. We have J_R, the circle action

$$\theta^+ \mapsto e^{i\alpha}\theta^+$$
$$\overline{\theta}^+ \mapsto e^{-i\alpha}\overline{\theta}^+$$

with θ^- left fixed, and we have J_L, the circle action on θ^- with θ^+ left fixed. The R-symmetry that comes from the R-symmetry of four-dimensional gauge theory is the product of these two (i.e., it acts by the standard representation on both $\theta^{\pm}$). Spatial rotation in the omitted two dimensions induces the circle action which is the product of the first times the inverse of the second of these.

R-symmetries are important, because the $N = 2$ superconformal algebra contains left and right R-symmetries. Hence every $N = 2$ superconformal theory has left and right R-symmetry. Potentially there is an anomaly in one of the R-symmetries of our classical theory. If our theory is to flow in the infrared to a superconformal theory with these R-symmetries in the superconformal algebra, the anomaly must vanish. (It is also possible to flow to a superconformal field theory such that the R-symmetry of the fixed point theory is not a symmetry of the original Lagrangian, but in such cases the Lagrangian description is often less useful.)

Let us make more explicit the earlier remark that the anomaly of the sigma model is proportional to the first Chern class $c_1(X)$. The point is that when we write the Lagrangian out in coordinates we have an expression of the form

$$\mathcal{L} = \int d^2y \left(g_{i\overline{j}} \partial X^i \partial \overline{X^j} + \overline{\psi}_-^{(0,1)} \not{\partial}_+ \overline{\psi}_-^{(1,0)} + \psi_+^{(0,1)} \not{\partial}_- \psi_+^{(1,0)} \right)$$

(plus a four-fermion term that will not affect the present discussion). One of the basic R-symmetries J_L rotates the $\psi_-^{(0,1)}$ and $\psi_-^{(1,0)}$ and the other J_R rotates in the same manner $\psi_+^{(0,1)}$ and $\psi_+^{(1,0)}$. Thus, under a chiral rotation generated by $a_R J_R + a_L J_L$, the measure of integration in the path integral $\mathcal{D}\psi_+ \mathcal{D}\psi_-$ changes by an amount proportional to

$$a_R \operatorname{index}(\not{\partial}_+) + a_L \operatorname{index}(\not{\partial}_-).$$

The operators $\not{\partial}_{\pm}$ are Dirac operators of positive or negative chirality acting on the pullback to the Riemann surface of the holomorphic tangent bundle $T^{(1,0)}(X)$. By the index theorem both of these operators have index given by $c_1(X)$ pulled back and evaluated on the fundamental class of the Riemann surface. Thus, we see that the R-symmetry induced from the R-symmetry in four dimensions is not anomalous, since $\operatorname{index}(\not{\partial}_+) - \operatorname{index}(\not{\partial}_-) = 0$, but that the R-symmetry coming from the spatial rotation can have an anomaly: $\operatorname{index}(\not{\partial}_+) + \operatorname{index}(\not{\partial}_-) = 2 \operatorname{index}(\not{\partial}_+)$. This vanishes for all homotopy classes of maps to X, for all surfaces Σ, if and only if $c_1(X) = 0$.

We have just given a computation for the anomaly for the sigma model. Let us make an analogous computation in $U(1)$ gauge theories with chiral superfields $A_i, 1 \leq i \leq n$. The superfield A_i is a section of the q_i-power of the line bundle associated to the connection on 2|4-superspace. It can be expanded in terms of ordinary fields, both bosons and fermions. The R-symmetries act as chiral symmetries on the fermions. So the same type of anomaly-index computation as above shows that the anomaly of J_R is given by

$$\sum_i q_i \int d^2y \frac{F_A}{2\pi}$$

where F_A is the curvature of the ordinary connection on the Riemann surface. (The anomaly for J_L is the negative of this number.) Thus, in order to get a superconformal field theory we need $\sum_i q_i = 0$.

§13.4. One Example

To make things as simple and concrete as possible let us focus on one special case of our $U(1)$ gauge theory: $n+1$ chiral superfields

$$A_1, \dots, A_n \text{ of weight } q = 1$$
$$A_{n+1} = P \text{ of weight } q = -n.$$

This choice of weights assures us that $\sum_i q_i = 0$, so our R-symmetries are anomaly-free. Now it is time to supplement the minimal Lagrangian $\mathcal{L}_0$ of Equation 13.1 by a superpotential and a twisted superpotential. That is to say, we consider Lagrangians of the form

(13.2)
$$\mathcal{L} = \mathcal{L}_0 + \left(\int d^2y d\theta^+ d\theta^- W(A,P) + \text{c. c.} \right) + \left(\int d^2y d\bar{\theta}^+ d\theta^- \widetilde{W}(\Sigma) + \text{c. c.} \right).$$

Here, W is called the superpotential; it must be a holomorphic function on $\mathbf{C}^{n+1}$. The function $\widetilde{W}(\Sigma)$ is called the twisted chiral superpotential term.

We want to pick W and $\widetilde{W}$ to preserve the R-symmetries. We consider the twisted chiral superpotential term $\widetilde{W}$ first. Here, it is helpful to use the fact that our theory is obtained by dimensionally reducing a gauge theory on 4|4-space. Starting with a connection $A = \sum_{\mu=0}^{3} A_\mu dx^\mu$ on 4|4-space we form

$$\sum_{\alpha=0}^{1} dy^\alpha + \sigma(dx^2 + i dx^3) + \text{c. c.}\ .$$

This shows us how $\Sigma = \sigma + \dots$ transforms under the R-symmetries; it transforms oppositely to $dx^2 + i dx^3$. For the extra term in Equation 13.2 coming from the twisted superpotential to be invariant, it must have the same transformation law under the R-symmetries as Σ does. That is to say, $\widetilde{W}(\Sigma)$ must be a linear function of Σ, i.e., $\widetilde{W}(\Sigma) = t\Sigma$ for some $t \in \mathbf{C}$. Writing

$$t = \frac{\theta}{2\pi} - ir,$$

we have

$$\left(\int d^2y d\bar{\theta}^+ d\theta^- t\Sigma \right) + \text{c. c.} = \theta \int \frac{F}{2\pi} + r \int D.$$

Thus, we see that adding such a linear twisted superpotential produces a Fayet-Iliopulous term $r \int D$ plus a θ-angle term which is θ times an integral topological quantity. The Fayet-Iliopulous term corresponds to adding a constant to the moment map for the $U(1)$-action on $\mathbf{C}^{n+1}$.

Now let us turn to the superpotential $W(A,P)$. For the term

$$\int d^2y d\theta^+ d\theta^- W(A,P)$$

to be invariant, we need $W(A,P)$ to transform with weight two under the R-symmetry induced from the R-symmetry in four dimensions and be invariant under the R-symmetry induced by spatial rotation in four-space. In order to arrange this,

we choose the action of the R-symmetry groups on the line bundles of which the A_i and P are sections as follows: the line bundles L_i with sections A_i are invariant under the R-symmetries and the line bundle L_{n+1} with section P is invariant under the R-symmetry induced by spatial rotation and has weight two under the R-symmetry coming from the R-symmetry of four-space. With this choice of R-symmetry actions any superpotential of the form

$$W = P \cdot F(A_i),$$

for F an arbitrary polynomial of $A_1, \dots, A_n$, is invariant under the R-symmetries. There is of course, one extra condition, namely that W must be gauge invariant. This implies that F must be homogeneous of degree n. Thus, any W of this form will produce a gauge invariant term in the Lagrangian which is invariant under the R-symmetries. This is the most general interaction with these symmetries.

§13.4.1. Classical analysis

Now that we have specified our theory in detail, we want to determine what this theory is, particularly in the limits $r \to \pm\infty$. We begin with a classical analysis. Write each of the superfields appearing in our Lagrangian in components

$$A = a + \theta(\cdots) + \cdots$$
$$P = p + \theta(\cdots) + \cdots$$
$$\Sigma = \sigma + \theta(\cdots) + \cdots$$

The superfield Σ includes auxiliary fields D and F which can be eliminated by means of their equations of motion.

The bosonic potential is a function $V(a, p, \sigma)$ of the bosonic components of these fields. For the Lagrangian we have described, the potential takes the form

$$V(a,p,\sigma) = \frac{1}{2e^2}D^2 + |dW|^2 + |\sigma|^2\left(\sum |a_i|^2 + n^2|p|^2\right). \tag{13.3}$$

The last term in eq. (13.3) arises as follows: in four dimensions, a chiral field $\Phi = \phi + \theta(\cdots) + \cdots$ leads to a term in the potential of the form

$$\sum_{\mu=0}^{3} \left|\frac{D\phi}{Dx^\mu}\right|^2$$

which upon reduction to two dimensions becomes

$$\sum_{\alpha=0}^{1} \left|\frac{D\phi}{Dx^\alpha}\right|^2 + n^2|\sigma|^2|\phi|^2$$

if ϕ has charge n.

In general, the "D-term" (the auxiliary field in Σ), is equal by its equations of motion to $\sum q_i|\phi_i|^2 - r$. This is actually a familiar function mathematically; it is the "moment function" generating the $U(1)$ gauge action on the flat Kähler manifold (a copy of $\mathbf{C}^s$, with s the number of chiral superfields) in which the chiral superfields take their values. In our example, this becomes

$$D = \sum |a_i|^2 - n|p|^2 - r. \tag{13.4}$$

Since $W = P \cdot F(A)$, we also have

$$|dW|^2 = |F|^2 + |p|^2|dF|^2. \tag{13.5}$$

We will focus on the case in which F is *transverse*, i.e., $F = dF = 0$ only at the origin.

We want to find all classical zero-energy solutions, in other words, to solve $V = 0$. Thanks to the form of the potential, this requires that (i) $D = 0$, (ii) $|dW|^2 = 0$ and (iii) either $\sigma = 0$ or $\sum |a_i|^2 + n^2|p|^2 = 0$. We will also need to take the gauge equivalence under $U(1)$ into account.

First, note that thanks to eq. (13.5), setting $|dW|^2 = 0$ when $p \neq 0$ implies $F = dF = 0$, so by our transversality condition, all $a_i = 0$. Thus, there are two cases: $p = 0$ or $a_i = 0$ for all i. On the other hand, if $r \neq 0$, then by eq. (13.4) since $D = 0$ we cannot have both $p = 0$ and $a_i = 0$ for all i. (In fact the sign of r will determine which one of these holds.) Thus, when $r \neq 0$ we have $\sum |a_i|^2 + n^2|p|^2 \neq 0$ and so σ must be 0.

Consider now the case $r > 0$. Our equations for classical vacua become $p = 0$, $\sum |a_i|^2 = r$, and $F = 0$, and we must divide by the action of the gauge group $U(1)$. This gives the hypersurface X defined by the equation $F = 0$ in $\mathbf{CP}^{n-1}$, with Kähler class r. Thus, classically our theory can be described as a nonlinear sigma model whose target space is this hypersurface X, using a metric whose Kähler class has volume proportional to r^{n-2}. As we noted above, such a theory has central charge $\widehat{c} = n - 2$.

This classical description will be a good approximation to the quantum theory when $r \gg 0$, since the nonlinear sigma model is weakly coupled there. In fact, since the nonlinear sigma model is stable (in the sense of having no relevant perturbations), and marginal perturbations merely vary the complexified Kähler and complex structures, we should expect our quantum theory to coincide with one member of this family of sigma models.

On the other hand, when $r < 0$, the space of classical vacua satisfies $a_i = 0$ and $n|p|^2 = -r$. We can use a gauge transformation to fix $p = \sqrt{-r/n}$, leaving a residual gauge invariance of $\mathbf{Z}_n$, i.e., the original $U(1)$ gauge group is broken to $\mathbf{Z}_n$. This will therefore be what is known as an "orbifold" theory, in which the $\mathbf{Z}_n$ shows up in a global analysis of the theory (in a manner which we shall describe later). The local description of the theory (for which we can ignore the orbifolding issue) is this: for $r \ll 0$, the field P has a large mass and can be integrated out, leaving an effective theory of n massless chiral superfields $A_1, \ldots, A_n$ with an effective interaction

$$W_{\text{eff}} = \text{constant} \cdot F(A_1, \ldots, A_n),$$

where F is a homogenous polynomial of degree n.

Such a theory of n massless fields with a polynomial interaction is called a *Landau–Ginzburg theory*. It should apparently flow in the IR to a conformal field theory with $\widehat{c} = n - 2$. We expect this since that was the value for $r \gg 0$. By the Zamolodchikov c-theorem (see the references at the end of the lecture), c is constant in a family of conformal field theories, so if there is flow to a conformal field theory for all t, we should get the same value for $r \to +\infty$ or $r \to -\infty$.

Let's check this prediction. By the Zamolodchikov theorem again, the choice of F should not matter, so we consider a special F of "Fermat type":

$$F(A_1, \ldots, A_n) = \sum_i A_i^n.$$

The Landau–Ginzburg theory with this interaction will (at least locally) factor as a product of n identical theories, each with a single chiral superfield A with an

A^n superpotential. Our prediction implies that each of these theories must have $\widehat{c} = \frac{n-2}{n}$. But that is exactly the central charge of a minimal model! So if we can identify the minimal model with this Landau–Ginzburg theory, we will have established the result. (We will also, at the same time, learn that the Landau–Ginzburg model with Fermat type superpotential is algebraically soluble.) We will return to this issue in Lecture 15. Note that we would have been in difficulties if we had obtained a value of $\widehat{c}$ less than one and not of the form $1 - 2/k$ for some integer k!

§13.4.2. Orbifolding

As we pointed out above, the original gauge symmetry group of $U(1)$ of our theory was broken to $\mathbf{Z}_n$ in the effective theory when $r \ll 0$. Thus, if we consider the path integral for the low energy effective theory on a Riemann surface Σ, the $U(1)$ gauge group is reduced to $\mathbf{Z}_n$, and we need only sum over flat $\mathbf{Z}_n$ bundles. On any surface Σ, a flat $\mathbf{Z}_n$ bundle can be specified by its holonomies $\gamma_1, \ldots, \gamma_k \in \mathbf{Z}_n$ around various loops in Σ. The process of summing over all flat $\mathbf{Z}_n$ bundles in a theory with a $\mathbf{Z}_n$ symmetry is known as *orbifolding* the theory.

In a Hamiltonian approach, formulated on a circle S^1, we need to specify the holonomy $\gamma \in \mathbf{Z}_n$ around the circle as part of the data determining a state. Let $\mathcal{H}_\gamma$ be the Hilbert space of states whose holonomy is γ. (Then Riemann surfaces with boundary on which flat $\mathbf{Z}_n$ bundles have been specified will determine operators mapping among these various Hilbert spaces, depending on the holonomies on the boundary circles.) $\mathbf{Z}_n$ acts on each $\mathcal{H}_\gamma$, since as the group $\mathbf{Z}_n$ is abelian, twisting as one goes around the circle by an element of $\mathbf{Z}_n$ is an operation that commutes with $\mathbf{Z}_n$.

We claim that the Hilbert space for the orbifolded theory is

$$\mathcal{H} = \bigoplus_\gamma (\mathcal{H}_\gamma)^{\mathbf{Z}_n}.$$

To see this, consider a cylinder with holonomy γ on the ends:

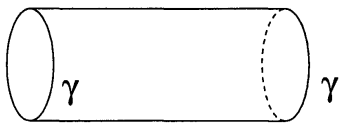

Figure 1.

There is still, as we have noted, an action of $\mathbf{Z}_n$ on each sector $\mathcal{H}_\gamma$; let $\gamma' \in \mathbf{Z}_n$ determine an operator $\gamma' : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$. The trace of this operator can be evaluated as a partition function $Z_{\gamma'\gamma}$ on a torus

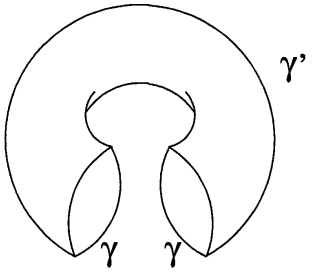

Figure 2.

constructed by using γ' to twist the identification of bundles on the boundary circles of the cylinder, i.e.,

$$Z_{\gamma'\gamma} = \mathrm{Tr}_{\mathcal{H}_\gamma} \gamma' q^H.$$

More invariantly, on this torus there is a twist by γ in one direction and by γ' in the other, as suggested in the figure.

If we sum the operators γ', the result is $n\pi$, where π is the projection onto the $\mathbf{Z}_n$-invariant subspace:

$$\frac{1}{n}\sum_{\gamma'} Z_{\gamma'\gamma} = \mathrm{Tr}_{\mathcal{H}_\gamma} \pi q^H = \mathrm{Tr}_{(\mathcal{H}_\gamma)^{\mathbf{Z}_n}} q^H.$$

Now the partition function for the $\mathbf{Z}_n$ gauge theory is obtained by summing up the genus one partition function with all possible twists in all directions and (for reasons I will describe momentarily) dividing by n. This gives

$$\frac{1}{n}\sum_{\gamma,\gamma'} Z_{\gamma'\gamma} = \mathrm{Tr}_{\oplus(\mathcal{H}_\gamma)^{\mathbf{Z}_n}} q^H,$$

from which we conclude that the Hilbert space for our problem is indeed $\mathcal{H} = \bigoplus_\gamma (\mathcal{H}_\gamma)^{\mathbf{Z}_n}$.

Perhaps I should explain why

$$Z = \frac{1}{n}\sum_{\gamma,\gamma'} Z_{\gamma'\gamma}$$

is the appropriate partition function for a $\mathbf{Z}_n$ gauge theory. In performing the path integral in a gauge theory, one is supposed to sum over connections and to divide by the volume of the gauge group. When the gauge group is a Lie group of positive dimension, the space of connections is infinite-dimensional. The group of gauge transformations is likewise infinite-dimensional, and making sense of "dividing by the volume of the gauge group" requires the Faddeev–Popov construction and introduction of ghosts. In the present case, we are considering a low energy effective theory with gauge group $\mathbf{Z}_n$. The only connections on Σ are the flat connections with values in $\mathbf{Z}_n$. So the sum over γ and γ' is the path integral over connections. The only gauge transformations on Σ are constant transformations by elements of $\mathbf{Z}_n$; the volume of the gauge group is therefore n, the number of elements of the gauge group. So dividing by the volume of the gauge group is accomplished by dividing by n.

§13.4.3. Interpolation from positive to negative r

Our classical analysis can be summarized as follows: we needed to set $D = dW = 0$ and divide by $U(1)$. The step of setting $D = 0$ and dividing by $U(1)$ is the familiar mathematical operation of *symplectic reduction*, in which $D = 0$ defines a level set for the moment map of the $U(1)$ action (with the choice of r specifying the level). There is another mathematical interpretation of this process, as a quotient in the sense of Geometric Invariant Theory (GIT): we complexify the group $U(1)$ to $\mathbf{C}^*$, and consider the action of $\mathbf{C}^*$ on $\mathbf{C}^{n+1}$ with the same weights as before (the A_i's have weight 1 and P has weight $-n$). In this particular example, there are two possible GIT quotients (which actually differ topologically): for $r > 0$ the quotient can be interpreted as the total space of the bundle $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)$ (in which p serves as a fiber coordinate), while for $r < 0$ the quotient is $\mathbf{C}^n/\mathbf{Z}_n$.

In either case we must still impose $dW = 0$: in the $r > 0$ case, $W = PF(A_1, \ldots, A_n)$ is a nondegenerate function (in a generalized sense introduced by Bott) and the the space of critical points in $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)$ is the variety in $\mathbf{P}^{n-1}$ defined by the equation $F = 0$; in the $r < 0$ case we have a homogenous polynomial $F(A_1, \ldots, A_n)$ with a highly degenerate critical point at the origin of $\mathbf{C}^n/\mathbf{Z}_n$.

The natural parameter in our Lagrangian is $t = -ir + \frac{\theta}{2\pi}$. It is the possibility of going to $\theta \neq 0$ that will enable us to interpolate from positive r to negative r without meeting the singularity that one would find in classical geometry.

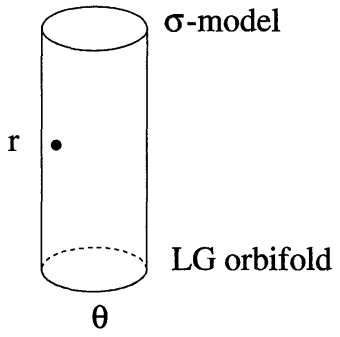

Figure 3.

Classically, there is a singularity at $r = 0$ with arbitrary θ, and interpolation is not possible. Quantum mechanically, we claim that the singularity will be isolated, located at $r = r_0$ (for some r_0) and $\theta = 0$. (Classically, the singularity is at $r = 0$, but quantum mechanically, it is shifted away from $r = 0$ by a one-loop correction that we will discuss.) It is believed that there are no other singularities; we will not prove this claim rigorously, but only explain some of the reasons physicists believe it to be true. The more rigorous part of our analysis will be the demonstration that there *is* a singularity at this location; the less rigorous part will be the argument that this is the only singularity.

To make this argument, we again take space to be a circle S^1 of finite radius. The compactness means that the story is rather similar to ordinary quantum mechanics in zero space dimensions, so let us first recall what happens in ordinary quantum mechanics of a point particle. We suppose that the particle is moving on a manifold X with a potential function V. If X is compact, or if V grows at the "ends" of X, then one can vary the parameters upon which X and V depend without meeting a singularity in the ground state of the quantum mechanics. However, one will get a singularity if one varies in the parameters to a point at which V no longer grows at infinity. For instance, if $X = \mathbf{R}$, and $V = \frac{1}{2}kx^2$ (with x a linear function on $\mathbf{R}$), then the ground state wave is a smooth function of k as long as $k > 0$, but develops a singularity at $k = 0$. The $k = 0$ problem has no normalizable ground state wave function.

Something like that will happen in our problem on a circle. Recall that the form of our potential is

$$V = |P|^2|F|^2 + |dF|^2 + \frac{e^2}{2}\left(\sum |A_i|^2 - n|P|^2 - r\right)^2 + |\sigma|^2\left(\sum |A_i|^2 + n^2|P|^2\right).$$

For generic r, this theory has the property that the potential grows at infinity. This ensures that the theory has a discrete spectrum, and a ground state wave function that varies smoothly with the parameters. However, at $r = 0$, we see that one can go to infinity in σ, at no cost in energy, as long as A_i and P vanish. This suggests

that at $r = 0$ there might be a continuous spectrum and a singularity of the ground state wave function.

To explore this question in more detail, we need to understand how the theory behaves in the dangerous region, that is, very large σ with other fields small. When A and P are very close to zero, and σ is large, A and P have large masses (on the order of $|\sigma|^2$) and they can be integrated out, leaving us with a pure gauge theory with an effective bosonic potential which at the classical level is merely a constant

$$V_{\text{eff}} = \frac{e^2}{2} r^2. \tag{13.6}$$

If this is the correct answer, quantum states decay exponentially in the large σ region if their energy is less than $e^2 r^2/2$. If so, this region could be dangerous for a supersymmetric ground state of zero energy – and could lead to a singularity in the wave function of such a state – only if $r = 0$.

To understand the situation better, we must make the analogous argument quantum mechanically. For this, we consider the superfield to which σ belongs, i.e., we write

$$\Sigma = \sigma + \theta\lambda + \theta\bar{\theta}(F + iD).$$

The action for the supersymmetric gauge theory is

$$\frac{1}{2e^2} \int d^4\theta\, \overline{\Sigma}\Sigma + \left(\int d\bar{\theta}^+ \, d\theta^- \, t\Sigma + \text{c.c.} \right). \tag{13.7}$$

If we perform the θ-integrals in the effective Lagrangian (13.7), we get

$$\frac{1}{e^2} \int d^2x \left(\frac{F^2}{2} + |d\sigma|^2 + \overline{\lambda} \not{\partial} \lambda \right) + \int d^2x \left(\frac{e^2 r^2}{2} \right) + i\theta \int_\Sigma \frac{F}{2\pi},$$

which includes a curvature term (multiplied by a "θ-angle") as well as the bosonic potential previously discussed.

Luckily, we know the θ-dependence of energy in $U(1)$ gauge theories in two dimensions; as computed in §5.5 of Lecture 5 it is:

$$\frac{e^2}{2} \min_{n \in \mathbf{Z}} \left(n - \frac{\theta}{2\pi} \right)^2.$$

So as $\sigma \to \infty$, we have $V(\sigma) \sim \min_n \frac{e^2}{2} |\widetilde{t}|^2$ where $\widetilde{t} \equiv -ir + \frac{\theta}{2\pi} + n$, with $n \in \mathbf{Z}$. In other words, while the classical energy at large σ is $e^2 r^2/2$, quantum mechanically r^2 is replaced by $|\widetilde{t}|^2$. Thus, while classically the singularity is at $r = 0$ and θ is invisible, quantum mechanically the singularity occurs only if θ is zero. For any other value of $\widetilde{t}$, there is a positive energy for $\sigma \to \infty$, and this region is not dangerous for supersymmetric ground states.

There is actually one more quantum effect of relevance: a one loop correction gives a finite renormalization of r, and shifts the position of the singularity from $\widetilde{t} = 0$ to $\widetilde{t} = -ir_0$ for a certain constant r_0. This correction is important for comparing to certain predictions of classical geometry, but not important for what we will say today.

§13.5. Another example: flops

As a second example, we consider a model with gauge group $G = U(1)$, chiral fields A_1 and A_2 of charge 1, chiral fields B_1 and B_2 of charge -1, with *no* superpotential,

and no "P." The Lagrangian takes the form

$$\int d^2x\, d^4\theta \left(\frac{1}{2e^2} \overline{\Sigma}\Sigma + \sum (|A_i|^2 + |B_i|^2) \right) + \left(t \int d\overline{\theta}^+ \, d\theta^- \, \Sigma + \text{c.c.} \right).$$

The D term is

$$D = \sum (|A_i|^2 - |B_i|^2) - r,$$

and the bosonic potential also contains a term

$$|\sigma|^2 \sum (|A_i|^2 + |B_i|^2).$$

As before, we must set $D = 0$ and divide by $U(1)$. When $r > 0$ this yields the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1_A$, whereas when $r < 0$ this yields the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1_B$. Here $\mathbf{P}^1_A$ is a copy of $\mathbf{P}^1$ obtained by requiring A_1, A_2 to be not both zero, and dividing the pair (A_1, A_2) by $\mathbf{C}^*$. Likewise $\mathbf{P}^1_B$ is a copy of $\mathbf{P}^1$ obtained by projectivizing (B_1, B_2).

The transformation from the bundle over $\mathbf{P}^1_A$ to the bundle over $\mathbf{P}^1_B$ is a simple model of a birational transformation known as a "flop." As in the previous example, the singularity only occurs at $r = \theta = 0$ and one can interpolate between these two models. This contrasts with classical geometry, where one passes through a singularity in going from one model of the quotient $\mathbf{C}^4/\mathbf{C}^*$ to the other.

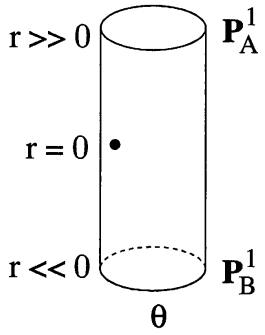

Figure 4.

More generally, consider two Calabi–Yau manifolds X and X' which are birationally equivalent, in such a way that their ample cones meet.

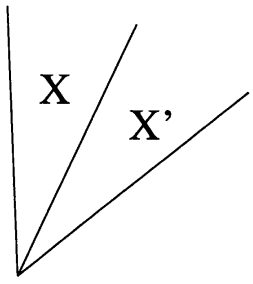

Figure 5.

For simplicity assume $h^{2,0} = 0$, and note that birational transformations won't affect the value of $h^{p,0}$. Classically, to pass from X to X', one goes through a singularity on the wall of the Kähler cone. The singularity is rather similar to the singularity just found in our "flop" example at $r = 0$. In quantum field theory, one can go around the singularity by taking $\theta \neq 0$; thus one can smoothly continue from X to X' by varying the parameters of the conformal field theory.

For example, we might construct such a pair of Calabi–Yau manifolds with $h^{1,1} = 2$ by starting with a gauge group $G = U(1) \times U(1)$, and a superpotential of the form $W = PF(A_i)$. If one sets things up correctly, one of the $U(1)$'s puts us in the world of projective varieties X and X', and by varying the moment map for the other $U(1)$, we can make a "flop" between X and X' that is quite like our above discussion with the four chiral superfields fields A_i and B_j.

§13.6. Cases in which $c_1 \neq 0$

We now wish to consider linear sigma models in which the R-symmetry is anomalous, so that we expect a nontrivial renormalization group flow.

§13.6.1. Negative β-function

We begin with the case of negative β-function, which for nonlinear sigma models corresponds to target spaces X for which $c_1(TX) > 0$. We can build examples of these exactly as before, with $U(1)$ acting on $n+1$ fields A_i, P, but this time we give P charge $-k$ and take a superpotential $W = P \cdot F$ with F homogeneous of degree k. The beta function of the sigma model is negative, zero, or positive for $k < n$, $k = n$, or $k > n$, respectively. For $k \neq n$, even though there is still classically a singularity at $r = 0$, the quantum theory has no singularity on the t-cylinder. (This is shown by analyzing the behavior at large σ, as we do presently.) For $k \neq n$, there is a nontrivial renormalization group flow on the cylinder. The flow is holomorphic and singularity-free, so it is a constant flow "downward" or "upward" on the cylinder, depending on the sign of $n - k$.

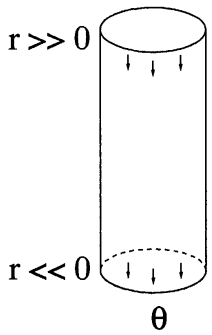

Figure 6.

When we repeat the previous analysis, we find that the R-symmetry which comes from the four-dimensional R-symmetry is anomalous, although the one which comes from rotations in the missing directions is anomaly-free.

The classical analysis can also be repeated: for $r \gg 0$ we find a space of classical vacua described by $\{F = 0\}$ in $\mathbf{P}^{n-1}$, and we expect the nonlinear sigma model on this hypersurface to be a good approximation to our theory. This hypersurface has positive first Chern class if $k < n$, so we expect a well-defined quantum field theory from looking at the nonlinear sigma model. On the other hand, since $k < n$, the sigma model is infrared-unstable. The infrared flow will take us away from the sigma model in the infrared and toward the theory we will find at the $r \to -\infty$ limit. Note that the sigma model has an $(n-2)$-dimensional target space, so for $k < n$ the effective central charge in the ultraviolet limit is $\widehat{c} = n - 2$.

On the other hand, if $r \ll 0$ then on the space of classical vacua $P \neq 0$ but $A_i = \sigma = 0$. Giving a nonzero expectation value to P breaks $U(1)$ to $\mathbf{Z}_k$, and (in

the $r \to -\infty$ limit) we get a Landau–Ginzburg orbifold with an effective interaction

$$W_{\text{eff}}(A_i) = F(A_i).$$

The central charge of this model is

$$\widehat{c} = n(1 - \frac{2}{k}).$$

This is less than $n - 2$ if $k < n$. We will argue shortly that, for $k < n$, the Landau–Ginzburg theory is the infrared limit of the renormalization group flow from the sigma model; and the fact that its central charge is smaller than that of the sigma model illustrates a general theorem by Zamolodchikov, which asserts that the central charge always diminishes along a renormalization group flow:

$$c_{\text{IR}} \leq c_{\text{UV}}.$$

The intuition behind this theorem is that the central charge measures the total number of degrees of freedom of the theory. As one flows toward the infrared, massive degrees of freedom are "integrated out," and no longer contribute to the central charge, which therefore can only diminish.

As we will see, the Landau–Ginzburg model is only one possibility for what the infrared flow can lead to for $k < n$. There are in fact $n - k$ other vacuum states at $r = -\infty$, with a mass gap, which can't be seen classically but will require a 1-loop calculation. Being massive, they have $\widehat{c} = 0$, which is certainly less than $n - 2$.

§13.6.2. The $\mathbf{CP}^{n-1}$ model

An extreme case of the situation we are considering is the case $k = 0$, i.e., the case of no P field and no superpotential. This is a $U(1)$ gauge theory with n free charged superfields $A_1, \ldots, A_n$ (all of charge 1). The D-term takes the form

$$D = \sum |A_i|^2 - r.$$

Setting it to zero and dividing by $U(1)$ we find (for $r >> 0$) the supersymmetric $\mathbf{CP}^{n-1}$ model which we studied in an earlier lecture. It will be very instructive to re-examine this case in detail before going back to the general case.

At first glance, there would appear to be no supersymmetric vacua at all in the $r \to -\infty$ limit of this $\mathbf{CP}^{n-1}$ model, since the moment map is strictly positive. However, it cannot be so that there are no supersymmetric vacua at all in this limit. An obstruction is provided by the supersymmetric index $\text{Tr}\,(-1)^F$. (I discussed this index, with its generalization, the elliptic genus, in lectures last fall.) This index, when computed for large positive r, is seen to coincide with the Euler characteristic of $\mathbf{CP}^{n-1}$, which is n. The supersymmetric index must be invariant under deformations of r, so there must be n supersymmetric vacua even if r is large and negative. We must find them somewhere!

To see what is happening, we must consider the quantum mechanics more carefully. The classical treatment of the $r \to -\infty$ limit is valid in any compact region of field space, so if there are quantum mechanical vacua for arbitrarily negative r that cannot be seen classically, they must disappear from the classical field of view by going off to infinity in field space for $r \to -\infty$. So again we must consider the quantum-mechanical behavior at large σ. For our $\mathbf{CP}^{n-1}$ model, the bosonic potential takes the form

$$V = \frac{e^2}{2} D^2 + |\sigma|^2 \sum |A_i|^2.$$

The theory is weakly coupled for $r \ll 0$.

What happens if $|\sigma| \to \infty$? As earlier, the fields A_i will then be massive and can be integrated out to yield an effective theory which is an abelian gauge theory with Lagrangian

$$\mathcal{L}_{\text{eff}} = \int d^2x\, d^4\theta\, \frac{1}{2e^2}\, \overline{\Sigma}\Sigma + \left(\int d^2x\, d\bar{\theta}^+\, d\theta^-\, \widetilde{W}_{\text{eff}}(\Sigma) + c.c. \right).$$

The effective twisted superpotential depends on the renormalization mass scale μ, and we claim that it takes the novel (R-symmetry violating) form

$$\widetilde{W}_{\text{eff}}(\Sigma) = t\Sigma + \frac{in}{2\pi}\left(\Sigma - \Sigma \ln(\Sigma/\mu)\right) \tag{13.8}$$

which we shall verify by making a 1-loop calculation below. Notice that a change in the choice of logarithm in eq. (13.8) can be compensated for by changing the value of t by an integer; such a change has no effect on the physics. Of course, we must keep Σ away from zero in order to define the logarithm in $\widetilde{W}_{\text{eff}}$, but that is consistent with our assumption that σ is large.

With this new term in the twisted superpotential, the equation for a critical point of $\widetilde{W}_{\text{eff}}$ is

$$0 = t - \frac{in}{2\pi} \ln(\Sigma/\mu),$$

or in components,

$$0 = -ir + \frac{\theta}{2\pi} - \frac{in}{2\pi} \ln(\sigma/\mu) \pmod 1.$$

This has n solutions

$$\sigma = \mu \exp\left(-\frac{2\pi r}{n} - \frac{i\theta}{n} + \frac{2\pi i k}{n} \right), \tag{13.9}$$

$k = 0, \ldots, n-1$, corresponding to n different vacua in the $r \to -\infty$ limit. n is of course, the expected number, the Euler characteristic of $\mathbf{CP}^{n-1}$. Notice that $|\sigma|$ is indeed large when $r \to -\infty$, so the assumptions made in deriving these vacua hold. The fact that the n vacua go off to infinity for $r \to -\infty$, along with the fact that their existence depends on the one-loop quantum correction, explains why one cannot see them classically.

Another very important consequence of the logarithm is that it means that for $\sigma \to \infty$, the energy in the quantum theory grows like $|\ln \sigma|^2$. Hence, in particular, low energy states are limited to a bounded region of field space, and there is no singularity in the vacuum behavior for any value of t on the cylinder. As we will see, the models with $n \neq k$ all have such a logarithm, and hence they are all nonsingular throughout the t cylinder.

§13.6.3. The 1-loop calculation

To understand the origin of the term $\frac{in}{2\pi}\left(\Sigma - \Sigma \ln(\Sigma/\mu)\right)$ in the effective twisted superpotential, note that its presence is equivalent to the effective Lagrangian having a term

$$\int d^2x\, r_{\text{eff}}\, D,$$

where

$$r_{\text{eff}} = r + \frac{n}{2\pi} \ln(|\sigma|/\mu).$$

In this theory, we have $D = \sum \overline{A}_i A_i - r$. Classically, $A_i = 0 = \overline{A}_i A_i$ for large σ, but quantum mechanically $\overline{A}_i A_i$ has an expectation value at large σ; this expectation value can be interpreted in an effective classical description as a shift in $-r$. So we need to calculate the expectation value of $\overline{A}A$ by means of a 1-loop diagram

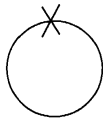

Figure 7.

whose contribution to $\langle \bar{A}_i A_i \rangle$ is

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + m_A^2} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + |\sigma|^2}$$

where we got the mass m_A from the term $|\sigma|^2|A|^2$ in the potential V. Cutting off the divergent integral, we find

$$\int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + |\sigma|^2} = -\frac{1}{2\pi} \ln \frac{|\sigma|}{\mu} + \frac{1}{2\pi} \ln \frac{\Lambda}{\mu}.$$

Here μ is a mass introduced in the renormalization procedure, and renormalization amounts in this example to just dropping the $\ln \frac{\Lambda}{\mu}$ term (which is independent of σ and so can be canceled by a renormalization of r). This formula, after the renormalization, gives a σ-dependent shift in $-r$, and accounts for the claimed formula for $r_{\rm eff}$. There are, by the way, no higher loop corrections to this formula. This can be proved using holomorphy, or by noting that, in this superrenormalizable theory, higher loop diagrams would involve negative powers of σ and would vanish for large σ.

Let us now repeat this for a general case with chiral superfields B_i of charges q_i. (The B_i are to include all of the chiral superfields, including the P field of the sigma model discussion, if it is present.) The D-term is $\sum q_i |B_i|^2$ and for its expectation value at large σ we get the 1-loop correction

$$\sum q_i \int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + q_i^2|\sigma|^2} = -\frac{1}{2\pi} \sum q_i \ln |q_i| - \frac{1}{2\pi} (\sum q_i) \ln \frac{|\sigma|}{\mu},$$

leading to an effective twisted superpotential

$$\widetilde{W}_{\rm eff}(\Sigma) = \left(t - \frac{i}{2\pi} \sum q_i \ln |q_i| \right) \Sigma + \frac{i \sum q_i}{2\pi} (\Sigma - \Sigma \ln(\Sigma/\mu)).$$

In the Calabi-Yau case, $\sum_i q_i = 0$ and the coefficient of the logarithm vanishes. Hence $\widetilde{W}_{\rm eff}$ is linear in Σ, as a result of which its contribution to the potential energy is independent of σ. As a result of this, it is possible for σ to go to infinity at no cost in energy if t has the correct value. At this value of t, one gets a singularity. The correct value is the value at which the coefficient of Σ in the twisted chiral superpotential vanishes; the condition is not $\mathrm{Im}(t) = 0$, as one would expect classically, nor even $t = 0$, but rather $t - (i/2\pi) \sum_i q_i \ln |q_i| = 0$. This is the shift in the position of the singularity that was mentioned in our discussion of the

Calabi-Yau case.[2] In order to compare the present discussion to, for instance, the counting of rational curves on the Calabi-Yau manifold, it is important to know about this shift. When $\sum_i q_i \neq 0$, the logarithm prevents σ from going to infinity, and hence prevents a singularity on the t cylinder, as we have already asserted.

On the other hand, if $\sum q_i \neq 0$, then there are $|\sum q_i|$ extra vacua (as in the $\mathbf{CP}^{n-1}$ model) with σ-values

$$\sigma = \mu \exp\left(-\frac{2\pi r}{\sum q_i} - \frac{i\theta}{\sum q_i} - \frac{\sum q_i \ln q_i}{\sum q_i} + \frac{2\pi i k}{\sum q_i}\right), \tag{13.10}$$

However, the prediction of these vacua is only valid if they occur at large σ. Hence, these vacua are only trustworthy for $r \to -\infty$ if $\sum_i q_i > 0$, or for $r \to +\infty$ if $\sum_i q_i < 0$.

§13.6.4. Hypersurfaces

Let us now return to the situation of a hypersurface defined by a homogeneous polynomial F of degree k in n variables (which for simplicity we take to all be of charge 1). Including also the P field of charge $-k$ (so that the superpotential $PF(A_i)$ is gauge invariant), we have $\sum_i q_i = n - k$. Hence, vacua at large σ are present for $r \to -\infty$ if $n > k$, or at $r \to +\infty$ if $n < k$. The vacua at large σ are the ones that one cannot see classically. In addition to those, one has vacua that one can see classically – a sigma model for $r >> 0$ and a Landau–Ginzburg orbifold for $r << 0$.

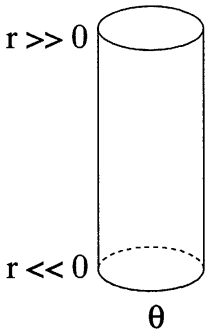

Figure 8.

Let us now consider the two cases in detail. For $k < n$, the sigma model has for its target space a Fano variety, a hypersurface in $\mathbf{CP}^{n-1}$ of degree $k < n$. This is a good description in the ultraviolet, and gives a well-posed quantum field theory whose infrared behavior we would like to determine. The infrared flow brings us to $r \ll 0$, where we find two types of vacua: (1) vacua which are conformal field theories of central charge $\widehat{c} = n(1 - \frac{2}{k})$, and (2) $n - k$ new vacua at large σ. The corresponding effective twisted superpotential for large σ is

$$\widetilde{W}_{\text{eff}} = t\Sigma + \frac{i}{2\pi}(n-k)(\Sigma - \Sigma \ln \Sigma).$$

Both of these types of vacua have central charge less than the value $\widehat{c} = n - 2$ of the UV theory. Thus, in this example, a single quantum field theory in the ultraviolet can flow to quite different possibilities in the infrared. Renormalization group flow

[2] In case $\sum_i q_i = 0$, the $\ln \Lambda$ term in the 1-loop correction cancels by itself, with no need for renormalization. The numerical value of the shift of the position of the singularity is hence not affected by any potentially arbitrary renormalization procedure.

for $N = 2$ sigma models in two dimensions has many invariants (such as $\mathrm{Tr}\,(-1)^F$, the elliptic genus, and some of their cousins). To reproduce the invariants of the UV sigma model with Fano target, one must sum over the infrared vacua of types (1) and (2).

§13.6.5. The case $c_1 < 0$

Consider now the opposite case $k > n$. A hypersurface of degree $k > n$ in $\mathbf{CP}^{n-1}$ is an algebraic variety of general type. In this case $\beta > 0$, and there is not a well-defined quantum field theory of maps from spacetime to this hypersurface. However, the sigma model with this hypersurface as target makes sense as an effective infrared theory. One can ask what well-defined ultraviolet theory can flow to this effective theory in the infrared.

For this question, we will now give an answer. For $k > n$, the ultraviolet stable end of the cylinder is the Landau–Ginzburg end. The Landau–Ginzburg model has central charge $\widehat{c} = n(1 - 2/k)$, which for $k > n$ is greater than the value $\widehat{c} = n - 2$ of the sigma model. Thus the Zamolodchikov theorem permits a renormalization group flow from the LG model to the sigma model. That is precisely what happens, since for $n < k$ the renormalization group flow on the cylinder is "upwards," away from the LG model and towards the sigma model.

If $k > n$, there are no large σ vacua at the LG end of the cylinder. Instead, $k - n$ such vacua accompany the sigma model. Thus, a well-defined ultraviolet fixed point, given by the Landau–Ginzburg model, can flow in the infrared either to the sigma model of the hypersurface, or to a nonclassical massive vacuum at large σ. Just as for $k < n$, to equate renormalization group invariants such as $\mathrm{Tr}\,(-1)^F$ between the ultraviolet and infrared theories, one must sum over both types of vacua in the infrared limit.

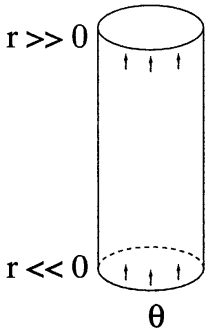

Figure 9.

Note that, regardless of the sign of $n - k$, the nonclassical vacua always appear in the infrared, not the ultraviolet. There is a good intuitive reason for this. In the ultraviolet, where we are defining our theory, we can select what theory we wish to study (from all of the possible UV fixed points), so the UV theory has a vacuum of just one type. Going to the infrared means solving the equations, and at this stage things are out of our hands: a definite quantum field theory may have more than one solution for its infrared behavior. That is what we have found today. In fact, we have seen that a definite UV field theory (based on a Fano variety if $k < n$ or a Landau–Ginzburg model if $k > n$) can flow to quite different possibilities in the infrared.

Both of the $k \neq n$ cases that we have studied today are of considerable methodological interest. For $k < n$, we have seen how the quantum theory with a Fano target gives, in general, an answer which is a sort of mixture of the behavior of a Calabi-Yau manifold and the behavior of complex projective space. For $k > n$, we have found, in a concrete situation, a well-defined UV fixed point with flow to the effective theory based on an algebraic variety of general type.

References and cross references

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LECTURE 14

$N = 2$ SUSY Theories in Dimension Two, Part II: Chiral Rings and Twisted Theories

This lecture is again concerned with $N = 2$ supersymmetric theories in dimension 2. Using one of the supersymmetries Q, we shall define the chiral ring of local operators and the Q-cohomology of states for theories defined in flat Minkowski or Euclidean space. We then discuss how to twist theories so that the supersymmetry Q has global meaning at least over a large class of $2|4$ supermanifolds. Once Q has been globalized in a theory over Σ we have the notions of the ring of chiral local operators for that theory. We can define what are called descendants of Q-invariant operators, and we show that the integrals of these descendants over cycles in the manifolds lead to correlation functions which have topological meaning. To compute these functions one does integrals over the Q-invariant fields.

We then consider in detail two examples of twisted theories with a global supersymmetry Q. The first is pure $N = 2$ gauge theory in dimension 2 with a simple Lie group G. Here, we see that the ring of chiral functions is identified with the invariant polynomials on the Lie algebra. Classically, the space of Q-invariant states is the usual space of flat connections (plus an extra parallel section of the adjoint bundle in the case of reducible connections). The correlation functions of the Q-invariant operators are identified with the integrals over the moduli space of flat connections of the usual classes derived from the invariant polynomials in the Lie algebra. (There are extra complications from reducible connections.)

The second example is $N = 2$ supersymmetric two-dimensional sigma models into a Kähler manifold. It turns out that there is one way of twisting that leads to a chiral ring which is the topological cohomology of the Kähler manifold. The space of Q-invariant states is the space of holomorphic maps from the Riemann surface into X. In this twisted theory the path integrals which compute the topological correlation functions are the integrals in quantum cohomology. When the Kähler manifold is Calabi-Yau, there is a second, inequivalent way of twisting which leads to a different theory. In this theory the chiral ring is $H^*(X, \wedge^* T^{1,0}(X))$ and the Q-invariant fields are constant maps from the Riemann surface to X. Thus, the path integrals which compute the topological correlation functions are integrals of these objects over X – these integrals are related to variations of Hodge structure on X.

Notes by John Morgan

§14.1. R-symmetry revisited

Most of the $N = 2$ supersymmetric examples we shall consider will have (at least classically) both left- and right-moving R-symmetries – the R-symmetries denoted by J_L and J_R in the last lecture. It is possible to carry out today's discussion with only one of these R-symmetries, but we must have at least one. However, we will suppose that the group of R-symmetries is $U(1) \times U(1)$ so that we have both J_R and J_L. The supermanifold on which the theory is defined in $\mathbf{R}^{2|4}$. We shall also eventually allow a certain class of split $2|4$ super-Riemann surfaces with a Euclidean metric (and $2|4$-supermanifolds with a flat Minkowski metric). The space of odd directions is a copy of the two-dimensional representation for J_L and the two-dimensional representation for J_R. We expect a symmetry between J_L and J_R obtained by reversing the orientation of the worldsheet.

One operation we can attempt to do to a symmetry of a Lagrangian is to gauge it. If the symmetry group to be gauged is H, then the new Lagrangian will involve gauge fields with gauge group H times the gauge group (if any) of the original theory. Intuitively, the process of gauging involves replacing fields with values in a constant space by fields with values in an H-bundle and adding a new field – the H-connection. Gauging is most straightforward when the action or Lagrangian is the integral of a local Lagrangian density. In this case, at the classical level, it is always possible to form a new, gauged Lagrangian. We replace ordinary derivatives in the Lagrangian by covariant derivatives. Quite often there is a natural way to carry out this process, but in any case different choices for gauging the symmetry lead to Lagrangians which differ by gauge invariant terms. There is, however, the issue of whether the gauged theory makes sense as a quantum theory. If the original theory has this property then the necessary and sufficient condition that the gauged theory does as well (if we are in sufficiently low dimension that we do not run into new ultraviolet difficulties as a result of gauging) is that certain anomalies should cancel.

In our case it is never possible to gauge the entire $U(1) \times U(1)$ group of R-symmetries to produce a theory that makes sense as a quantum theory. In fact the only combinations of J_L and J_R that can be gauged in this way are $J_L \pm J_R$. These are referred to as the vector and axial currents

$$J_V = J_L + J_R$$

$$J_A = J_L - J_R.$$

If one of these is gauged, then the other develops anomalies and stops being a symmetry of the quantum theory. To explain this, we need some general observations about anomalies. The statement that a theory has a symmetry, generated by a current J_L, means that correlation functions obey the identity

$$d_x \langle J_L(x) \mathcal{O}(y_1) \dots \rangle = 0 \ (*)$$

as long as x is distinct from the y_i. Here d_x is the exterior derivative in the x variable. If one extends the correlation functions (as distributions) across the diagonal, this identity will pick up extra terms supported on the diagonal. A very important special case is the two-point function of the current itself (or, in $2k$ spacetime dimensions, the $(k+1)$-point function). For this the general structure is

$$d_x \langle J_L(x) J_L(y) \rangle = \frac{k}{2\pi} d_y \delta(x - y) \ (**)$$

with some constant k, which may or may not be zero. The condition that the global symmetry generated by J_L can be gauged is precisely that $k = 0$. To see this, let us try to gauge the symmetry by adding to the theory a $U(1)$ gauge field A. The Lagrangian obtains an extra term $\int A \wedge J_L$. (There may be more terms, higher order in A, if J_L itself contains derivatives of charged fields.) Now, let us try to determine whether the theory with this coupling actually has the symmetry generated by J_L, that is, whether the identity of equation $(*)$ holds for this theory. For this we must look, in the original theory, at

$$d_x \langle J_L(x) \mathcal{O}(y_1) \dots \exp(\int A \wedge J_L) \rangle$$

The exponential is included to express correlation functions of the new theory in terms of the old theory. If we expand this in powers of A, the term linear in A will contribute, upon using $(**)$ and integrating by parts:

$$\langle \mathcal{O}(y_1) \cdot \dots \cdot kF(x) \rangle$$

with $F = dA$ the curvature. This can be expressed as an operator relation

$$dJ_L = \frac{k}{2\pi} F,$$

asserting that after gauging, J_L no longer generates a symmetry.

More generally, if there are several currents $J^1, \dots, J^s$, generating a Lie group G of dimension s, with Lie algebra $\mathcal{G}$, then the two-point functions obey

$$d_x \langle J^a(x) J^b(y) \rangle = \frac{k^{ab}}{2\pi} d_y \delta(x - y)$$

with k^{ab} some invariant quadratic form on $\mathcal{G}$. The quadratic form k has no positivity (since for instance left- and right-moving massless fermions make contributions of opposite signs) but does obey an integrality condition. If we gauge a subgroup H of G, with Lie algebra $\mathcal{H}$, then by a reasoning along the lines of the above, the quantum theory only makes sense if the quadratic form k vanishes when restricted to $\mathcal{H}$. If this is so, then the "global symmetries" of the gauged theory are generated by currents in $\mathcal{H}^\perp/\mathcal{H}$, that is, currents in $\mathcal{G}$ orthogonal to $\mathcal{H}$ modulo those in $\mathcal{H}$.

In the $N = 2$ theories that we will look at, there are two currents J_L, J_R, and there is a symmetry of "parity" (or orientation reversal of the two-dimensional spacetime) that exchanges J_L and J_R. k is odd under orientation reversal, so in such a case the structure of k is that $k^{LL} = -k^{RR} \neq 0$, $k^{LR} = 0$. This quadratic form has two null vectors, $J_L \pm J_R$, so either of those two linear combinations of the R-symmetries can be gauged. If one is gauged, the second becomes "anomalous" – it ceases to be conserved – since the two null vectors are, of course, not orthogonal to each other. "Luckily," these options agree with the choices that we will need below for other reasons.

What we are really interested in is not just gauging but twisting theories by R-symmetries. This process requires first gauging the R-symmetry and then twisting it using a homomorphism from the orthogonal group determined by the metric to the symmetry group. It turns out that there will be two equivalent ways of going about this process. But before we embark on twisting, we must understand the local story. That is the subject of the next section.

§14.2. Q-cohomology of operators

We consider $\mathbf{R}^{2|4}$ with its standard Euclidean or Minkowski metric. Suppose that our $N = 2$ supersymmetric theory in dimension two has the following supersymmetries:

characters of $U(1)_L$	characters of $U(1)_R$
Q_-	Q_+
$\overline{Q}_-$	$\overline{Q}_+$.

Q_- and $\overline{Q}_-$ transform in the standard representation of $U(1)_L$ and its conjugate, and commute with $U(1)_R$, and $Q_+, \overline{Q}_+$ are similarly related to $U(1)_R$, and $U(1)_L$. These supersymmetries obey the relations

$$Q_+^2 = \overline{Q}_+^2 = 0$$

$$\{Q_+, \overline{Q}_+\} = P_+$$

plus symmetric relations with $+$ replaced by $-$. Also,

$$\{Q_+, Q_-\} = \{\overline{Q}_+, Q_-\} = \{Q_+, \overline{Q}_-\} = \{\overline{Q}_+, \overline{Q}_-\} = 0.$$

We will consider at the same time four closely related possibilities. We let Q be one of the following:

$$Q_+ + Q_-, \quad Q_+ + \overline{Q}_-, \quad \overline{Q}_+ + Q_-, \quad \overline{Q}_+ + \overline{Q}_-.$$

(The first two possibilities will give inequivalent twisted theories, and the last two are Hermitian conjugate to the first two and would give equivalent theories.) To fix notation let us choose the generators so that $Q = Q_+ + Q_-$. Letting $\mu = \pm$ we see that P_μ can be written as a commutator

$$P_\mu = \{Q, T_\mu\} \tag{14.1}$$

where T_μ is a linear combination of $Q_\pm, \overline{Q}_\pm$. We are interested in the cohomology of Q acting as a differential (by superbracketing) on the space of local operators (and more generally on products of such operators). Thus, a local operator $\phi(x)$ will define a Q-cohomology class if $\phi(x)$ is Q-closed, i.e., if $\{Q, \phi(x)\} = 0$. A simple computation using Equation 14.1 shows that for a Q-closed operator $\phi(x)$ we have

$$d_\mu \phi = \{Q, [T_\mu, \phi]\}.$$

This equation means that the operator $d\phi$ is Q-exact, and hence that the class of $d\phi$ in the space of Q-cohomology classes of operators is trivial. This means that as we vary the point x in the Riemann surface the Q-cohomology class of the local operator $\phi(x)$ is unchanged. It then follows that in Q-cohomology such Q-closed local operators commute – simply move the points to be space-like separated. That is to say, the space of Q-cohomology of operators is a $\mathbf{Z}/2\mathbf{Z}$-graded commutative ring. We can always represent these operators by 0-form operators. This is an easy consequence of the independence of the Q-cohomology class on the metric and diffeomorphism invariance of the Q-cohomology class. These facts will be established later.

Let us recall a construction that we have seen before in Lecture 11, §11.3. Given a Q-invariant 0-form local operator $\phi = \phi^{(0)}$ we define a 1-form local operator by

$$\phi_\mu^{(1)} = [T_\mu, \phi^{(0)}]$$

and

$$\phi^{(1)} = \phi^{(1)}_\mu dx^\mu.$$

The same computation as we did in the previous paragraph shows that

$$d\phi^{(0)} = \{Q, \phi^{(1)}\}.$$

This means that if Σ_1 is a one-cycle in the super-Riemann surface then

$$\int_{\Sigma_1} \phi^{(1)}$$

is Q-invariant. Let $\phi^{(2)} = [T, \phi^{(1)}]$. Then

$$d\phi^{(1)} = \{Q, \phi^{(2)}\}.$$

Hence $\int_\Sigma \phi^{(2)}$ is Q-invariant. So in this way, beginning with a Q-invariant 0-form local operator $\phi^{(0)}$, we can build a tower of k-form local operators $\phi^{(k)}$, called the descendants of $\phi^{(0)}$, and hence produce more general Q-invariant operators of the form

$$\int_{\Sigma_k} \phi^{(k)},$$

with Σ_k a k-cycle in Σ. Our computations show that for any Q-invariant local operator $\phi^{(0)}$ the operator $\phi^{(0)} + \phi^{(1)} + \phi^{(2)}$ is closed for the total differential $d - Q$.

This "descent" procedure was also followed in Lecture 11, in the context of two-dimensional gauge theory.

§14.3. Twisting the theory to give it global meaning

So far we have been working on $\mathbf{R}^{2|4}$ with fixed coordinates. If we rotate space, then $Q = Q_+ + Q_-$ is not invariant. Our goal is to modify things until Q is invariant under rotations and hence can be defined on more general Euclidean 2|4-manifolds. Let us denote by $SO(2)_E$ the Euclidean rotation group. The entire group of symmetries at our disposal for the Euclidean theory is $SO(2)_E \times U(1)_L \times U(1)_R$. This group has the obvious projection onto $SO(2)_E$ with a splitting back into the first factor. We wish to find another splitting of this projection map, i.e., a group $SO(2)'_E$ inside the product projecting isomorphically onto $SO(2)_E$ but which has the extra property that it commutes with Q. It is easy to see that there exists a unique such splitting and it is given on the Lie algebra level by

$$K' = K \pm \frac{1}{2} J_L \pm \frac{1}{2} J_R$$

where K is the generator for $SO(2)_E$ and the signs depend on which of the four possibilities we have chosen for Q. In the case that we are concentrating on when $Q = Q_+ + Q_-$ we have

$$K' = K + \frac{1}{2} J_L - \frac{1}{2} J_R.$$

The fact that $1/2$ appears in these formulas means that the twisting actually involves the spin form of $SO(2)_E$.

Notice that in order to make such twisting in dimension n there must be nontrivial homomorphisms from $\mathrm{Spin}(n)$ to the R-symmetry group. In dimension two it is possible to do this when the R-symmetry group is $U(1)$, e.g., for $N = 1$ supersymmetry. But for $n = 4$ it is necessary to have $N = 2$ supersymmetry so

that the R-symmetry group is $SU(2)\times U(1)$ which has a non-trivial homomorphism from Spin(4).

With this preliminary discussion in place, we are now ready to globalize our theory of Q-cohomology classes of local operators. Suppose that we have a $2|4$ super-Riemann surface locally modeled on $\mathbf{R}^{2|4}$. Given any ordinary (bosonic) Riemann surface with chosen spin structures, there is a corresponding very simple super-Riemann surface: it is a split $2|4$-manifold complex with the odd directions being

$$\left(\Pi K^{1/2}\oplus\Pi\overline{K}^{1/2}\right)\oplus\left(\Pi\overline{K}^{1/2}\oplus\Pi K^{1/2}\right).$$

Here $\theta^+,\overline{\theta}^+$ are sections of the first two line bundles and $\theta^-,\overline{\theta}^-$ are sections of the last two. Now, we will by "twisting" change the odd part of this supermanifold. We twist the spinor bundles containing $\theta^\pm$ by the square root of the canonical bundle to the power $\pm 1/2$. The result of twisting replaces the above line bundles by

$$(\Pi\mathcal{O}\oplus\Pi K)\oplus\left(\Pi\mathcal{O}\oplus\Pi\overline{K}\right).$$

and makes the supersymmetry

$$Q=\frac{\partial}{\partial\theta^+}+\frac{\partial}{\partial\theta^-}$$

which is a global symmetry. Locally, this symmetry agrees with the symmetry Q on $\mathbf{R}^{2|4}$ that we have been discussing. Any $N=2$ theory that can be formulated on super-Riemann surfaces in general will, when formulated on the particular kind of super-Riemann surface that we have just described, have Q as a symmetry. This ensures that a global version of the twisted theory exists; it can of course be explicitly described in terms of component fields.

Once we have globalized the supersymmetry Q, all the previous discussion globalizes. In particular, for a Q-closed local operator $\mathcal{O}(p)$ the Q-cohomology class of $\mathcal{O}(p)$ is independent of the point p. Thus, the Q-cohomology classes of local operators on this global super-Riemann surface generate a $\mathbf{Z}/2\mathbf{Z}$-graded commutative ring called the *chiral ring* of the theory.

§14.3.1. Certain Correlation Functions and Their Basic Properties

We will consider path integrals of the following types:

$$Z_{(\mathcal{O}_\alpha)}=\int\mathcal{D}\Phi e^{-\mathcal{L}}\prod_{\alpha=1}^{n}\int_{\Sigma_\alpha}\mathcal{O}_\alpha^{(k_\alpha)}$$

where for each α the operator $\mathcal{O}_\alpha$ is a Q-closed local operator, $k_\alpha=0,1,2$ and Σ_α is a k_α-cycle in the super-Riemann surface Σ.

Here are the basic properties of $Z_{(\mathcal{O}_\alpha)}$:

1). It is independent of the metric on the super-Riemann surface.

The reason is that as you vary the metric by δg you get an insertion of $\int_\Sigma\delta g\cdot T$, where T is the stress-energy tensor. But as we have seen before T is Q-exact in the sense that there is a field $S_{\mu\nu}$ with $T_{\mu\nu}=\{Q,S_{\mu\nu}\}$. We then see that this insertion produces a term of the form $\{Q,\cdot\}$. The path integral of such a term is zero since the fields are Q-invariant. (Notice also that it is necessary to assume that Q preserves the measure in the path integral. This will be so if the original theory was supersymmetric and the current $\frac{1}{2}J_L-\frac{1}{2}J_R$ was gaugeable. An important corollary of the fact that the correlation function is metric-independent is that the

zero-form operators $\mathcal{O}_\alpha^{(0)}$ must be scalar operators, of spin zero; objects of higher spin would not be invariant under local rotations, and their insertion in a correlation could not be metric-independent.)

2). It is independent of most of the parameters in $\mathcal{L}$.

First, the correlation function is independent of any term $\int d^2y d^4\theta(\cdots)$. The reason is that to evaluate such a term we differentiate by all four of the Q's so any such term is of the form $\{Q, \dots\}$. The integral with such an insertion vanishes for the same reason as in the last paragraph. For the same reason, only one of the terms

$$\int d^2y d\theta^+ d\theta^- W + \text{c. c.}$$

and

$$\int d^2y d\bar{\theta}^+ d\theta^- \widetilde{W} + \text{c.c.}$$

can be non-trivial depending on which combination of $Q_+, Q_-, \overline{Q}_+, \overline{Q}_-$ we are using. In fact the correlation function is independent of one of these terms and varies holomorphically with the other.

3). We can calculate $Z_{(\mathcal{O}_\alpha)}$ by using a fixed point theorem applied to the action of Q. We consider Q as acting (infinitesimally) on the function space of all the fields in the theory. If this action has no fixed points, then the value of the path integral would be zero. In general, we can reduce the computation to an integral over the fixed point set of Q. These of course are the field configurations which are invariant under the supersymmetry Q.

§14.3.2. Q-cohomology of States

Now we pass from the local operators, path integrals and correlation functions world to the Hamiltonian framework. Our Riemann surface is now a cylinder $S^1 \times \mathbf{R}$. We trivialize all the relevant spin bundles over the S^1. We have the Hilbert space $\mathcal{H}$ of states of the quantum theory over the circle and the Hamiltonian operator H on $\mathcal{H}$ associated with time translation. Since Q is acting on the theory it produces an operator, which we also call Q, on $\mathcal{H}$ commuting with H. Also, from the commutation relations in the superalgebra we see that

$$H = \{Q, \widetilde{Q}\}$$

where $\widetilde{Q}$ is the adjoint of Q. Of course, we still have the relation $Q^2 = 0$, so that we can take the cohomology of the action of Q on $\mathcal{H}$:

$$H^*(Q) = \ker Q / \operatorname{Im} Q.$$

If our theory is sufficiently well-behaved so that H has a discrete spectrum, then $H^*(Q)$ is identified with $\{\psi | H\psi = 0\}$. (This is not the case for the gauge theory example we discuss later today but it is true for the sigma models.) In general $H^*(Q)$ is bigraded by the action of the $U(1) \times U(1)$ R-symmetry group.

§14.3.3. Comparison of Q-cohomology of local operators and Q-cohomology of states

We have seen two types of Q-cohomology – the first involving local operators and yielding the chiral ring and the second involving states on the circle. Let us construct a map between these Q-cohomologies. We consider a disc or hemisphere with a point p, as in the figure.

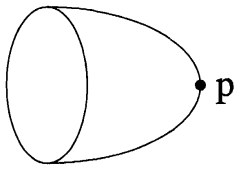

Figure 1.

Let $\mathcal{O}$ be a local operator at p. Doing the path integral over the space of fields on the hemisphere with a given boundary value produces a number, which as the boundary values vary gives a function of the fields on the boundary. This function is an element in the Hilbert space $\mathcal{H}$ obtained by quantizing the fields on the boundary. This then gives a map from local operators at p to $\mathcal{H}$. It commutes with the action of Q since the path integral is Q-invariant and hence induces a map from the Q-cohomology of local operators (the chiral ring) to the Q-cohomology $H^*(Q)$ of states on the circle:

$$\psi\colon (\text{chiral ring}) \to H^*(Q).$$

If the spectrum of Q on $\mathcal{H}$ (at least near zero) is discrete then ψ is an isomorphism.

If J_L and J_R are both R-symmetries classically, then the chiral ring and $\ker H$ are both bigraded by these symmetries. Typically in the twisted theory there is an anomaly in the grading proportional to the Euler characteristic of the Riemann surface (which is 1 for the hemisphere). In such a case the map ψ shifts the bigrading by

$$\psi\colon H^{p,q}(Q) \to \ker H^{p\pm k, q\pm k},$$

where k is a measure of the anomaly. (The signs depend on which of the four twisted theories we have chosen.) Notice that the bigrading need not be by integers, only by rational numbers whose denominators divide the order of the covering of $U(1) \times U(1)$ that actually acts on the theory.

§14.4. A Gauge Theory Example

The first example we take is that of gauge theory of a simple Lie group G. We begin with $N = 1$ supersymmetric gauge theory in four dimensions and dimensionally reduce to an $N = 2$ supersymmetric theory in dimension two. As we saw in the Superhomework, the basic field strength is the field

$$\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\},$$

which is a twisted chiral superfield with values in the adjoint bundle of the principal bundle. We use the expansion

$$\Sigma = \sigma + \cdots + \overline{\theta}^+\theta^-(F + iD).$$

The bosonic fields are the connection A_μ and $\sigma, \overline{\sigma}$. Recall that the connection on four-space is

$$A^{(4)} = \sum_{\mu=0}^{1} A_\mu dy^\mu + \sigma(dy^2 - idy^3) + \overline{\sigma}(dy^2 + idy^3).$$

The two R-symmetries are:

$$J_A = J_L - J_R; \quad J_V = J_L + J_R.$$

The first is induced by rotation in the (y^2, y^3)-plane and the second comes from the chiral $U(1)$ R-symmetry in four dimensions. (This symmetry is anomalous in four dimensions but not in two dimensions – see the Superhomework.) We now twist by J_V, so that $SO(2)'_E$ is generated by $K' = K \pm \frac{1}{2} J_V$. Then A_μ remains a connection one-form and $\sigma, \overline{\sigma}$ become 0-forms. In general, this twist eliminates spin bundles and turns sections of the spin bundles with differential forms. Then the four spinor fields $\lambda_\pm, \overline{\lambda}_\pm$ get twisted into a 0-form $\eta = \psi_{(0)}$, a one-form $\psi = \psi_{(1)}$ and a two-form $\chi = \psi_{(2)}$. (This new theory is of course a topological theory existing on a compact surface. It does not have all the usual properties required of a physical model!)

In this example J_A is twice the rotation of (y^2, y^3)-space, normalized so that $[J_A, Q] = Q$. The J_A degrees of the various propagating fields in the Lagrangian are as follows (cf. the end of [I-**Supersolutions**, §9.2]). (The reference to propagating fields means that we have in mind using the equations of motion to set the auxiliary field $F + iD$ equal to some expression in the other fields; F is set equal to the curvature of A and D is set equal to zero since there are no matter fields.)

J_A-degree	-2	-1	0	1	2
field	$\overline{\sigma}$	$\psi_{(0)}, \psi_{(2)}$	A_μ	$\psi_{(1)}$	σ

Actually, the sign of J_A could be reversed, but with one choice the degrees will be as indicated.

Recall from Lecture 11, §11.1, that the supersymmetry Q lifts to a supersymmetry δ of the space of superconnections[1]. It is not the case that $\delta^2 = 0$, but only that δ^2 is a gauge transformation. In fact, $\delta^2 = [\sigma, \cdot]$. In terms of the coordinate fields it is given by:

$$\delta A = \psi_{(1)}$$
$$\delta \psi_{(1)} = -D_A \sigma$$
$$\delta \sigma = 0.$$

The symbol D_A of course refers to covariant derivative with respect to the connection A.

In Lecture 11, §11.2, using the analogous symmetry led us to the equivariant cohomology of the moduli space of connections, via the standard model of forms. We let $\mathcal{G}$ be the Lie algebra of the gauge group G and M the space on which this group acts. Then we have the complex

$$(\Omega^*(M) \otimes \mathrm{Func}(\mathcal{G}))^G$$

with differential

$$D = d_M + i_{V(\phi)}.$$

[1]In that lecture we had a real field ϕ instead of the complex field σ here, but that changes nothing.

This was found by hand in Lecture 11. In our present approach, starting with $N = 2$ supersymmetry and then twisting, we come upon another closely related model of equivariant cohomology which is not as well known but probably should be. We replace the Lie algebra $\mathcal{G}$ by its complexification $\mathcal{G}_{\mathbf{C}}$ and we form the complex

$$\left(\Omega^*(M) \otimes \Omega^{0,*}(\mathcal{G}_{\mathbf{C}})\right)^G.$$

Here complex linear functions ϕ on $\mathcal{G}_{\mathbf{C}}$ have degree 2, the functions $\overline{\phi}$ have degree -2 and the $(0,1)$-form $d\overline{\phi}$ has degree -1. The differential in this complex is

$$\widetilde{D} = d_M + i_{\widetilde{V}(\phi)} + \overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$$

where $\widetilde{V}$ is a vector field on $M \times \mathcal{G}_{\mathbf{C}}$ giving the action. To see that this complex also computes the equivariant cohomology we view $\widetilde{D}$ as a sum of $(d_M + i_{\widetilde{V}(\phi)})$ and $\overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$. We apply the usual spectral sequence for the double complex, doing $\overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$ first. Since this cohomology vanishes except in degree zero, where it produces the polynomial functions on $\mathcal{G}_{\mathbf{C}}$, we see that one recovers the same answer as for the first complex. (Notice that this second approach leads to a Hodge theory version of equivariant cohomology, using the operator $\widetilde{D}^*\widetilde{D}$ as a "Laplacian"; it would be interesting to understand the resulting $\mathbf{L}^2$ version of equivariant cohomology.)

Let us complete the table describing the action of the supersymmetry δ on the space of propagating fields. The above model of equivariant cohomology translates into the following formulae for the supersymmetry:

$$\begin{aligned}
\delta A &= \psi_{(1)} \\
\delta\psi_{(1)} &= -D_A\sigma \\
\delta\sigma &= 0 \\
\delta\overline{\sigma} &= \psi_{(0)} \\
\delta\psi_{(0)} &= [\sigma, \overline{\sigma}] \\
\delta\psi_{(2)} &= F_A.
\end{aligned}$$

It follows from the last equation that the fixed points of δ will be field configurations where the connection is flat, and from the other equations that the section σ is covariantly constant and commutes with $\overline{\sigma}$. Next we see that the anomaly of J_A after twisting and upon working on a Riemann surface of genus g is $\dim(G) \cdot (-\chi(\Sigma)) = -\chi(\Sigma) \cdot \dim(G)$. [2] The reason for this is that the term that contributes to the anomaly in the measure of integration for the path integral is $\mathcal{D}\psi_{(0)}\mathcal{D}\psi_{(1)}\mathcal{D}\psi_{(2)}$. The symmetry J_A of the Lagrangian acts by -1 on $\psi_{(0)}$ and $\psi_{(2)}$ and by $+1$ on $\psi_{(1)}$. Thus, the index computation for the anomaly becomes the dimension of the adjoint representation times $-\chi(\Sigma)$. By invariance we get

$$\langle \prod_\alpha \int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)} \rangle = 0$$

unless $\sum_\alpha n_\alpha = (2g-2)\dim(G)$ where n_α is the J_A-degree of $\mathcal{O}_\alpha^{(k_\alpha)}$. Let us consider a local operator which is of the form $P(\sigma)$ for some gauge invariant polynomial on the Lie algebra of G. The ring structure on these operators is just the usual ring structure on invariant polynomials. It is easy to see that if P is of degree r, then the operator $\mathcal{O}_P^{(0)}$ has J_A-degree $2r$, and the descendants $\mathcal{O}_P^{(k)}$ have J_A-degree equal to

[2] Upon setting $\chi = 1$, this anomaly produces the shift in the bigrading of the map from operators to states that was noted at the end of §14.3.

$2r - k$. Notice that these degrees agree with the usual dimension of the analogous classes in Donaldson theory.

Now let us calculate $\langle \prod_\alpha \int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)} \rangle$ via fixed point theory. Recall that the Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2} \int d^2y d^4\theta \operatorname{Tr} \overline{\Sigma}\Sigma.$$

We will recover a theoretical formula for this correlation function. (The theoretical formula can be evaluated to get an actual concrete formula using the results of Lecture 11, §11.6.) In coordinates we have

$$\mathcal{L} = \frac{1}{4e^2} \int d^2y \left(\operatorname{Tr} F^2 + \operatorname{Tr} D\sigma D\overline{\sigma} + \operatorname{Tr}([\sigma, \overline{\sigma}]^2) + \cdots \right). \tag{14.2}$$

By the fixed point theorem we need only calculate at points where the curvature is zero and $D_A\sigma = [\sigma, \overline{\sigma}] = 0$. There are two cases. If A is irreducible and $D_A\sigma = 0$ then $\sigma = 0$. These solutions give smooth points in the moduli space of classical solutions to the equations of motion and this part of the moduli space of classical δ-invariant solutions is simply the space of of irreducible representations of $\pi_1(\Sigma) \to G$. The other possibility is that A is reducible and $\sigma \neq 0$. These points produce singular points of the moduli space of classical solutions. Recall that σ takes values in the complexified adjoint bundle. Since $D_A\sigma = 0$, σ is covariantly constant, and it suffices to study its structure over one point of Σ. Since σ and $\overline{\sigma}$ commute, σ can be diagonalized (conjugated into the Cartan subalgebra of the complexification of the Lie algebra of G) by an element of G.

Now let us examine how perturbation theory will compute these topological correlation functions. We will restrict to the smooth case where $\sigma = 0$. Of course, we are taking the coupling constant e in the Lagrangian (14.2) to be small. We are reduced to computing an integral over the moduli space, taken in the supersense, of classical solutions. Let us consider integrating out the odd variables at a point $[\rho]$ of the ordinary moduli space of flat connections. The fermion space is the space of zero modes of the $\psi_{(k)}$. These are the spaces of harmonic forms and hence are identified with the spaces $H^k(\Sigma, \mathrm{ad}(\rho))$. Since we are assuming that ρ is irreducible and that the moduli space is smooth at $[\rho]$ we have that $H^0(\Sigma, \mathrm{ad}(\rho)) = H^2(\Sigma, \mathrm{ad}(\rho)) = 0$ and $H^1(\Sigma, \mathrm{ad}(\rho))$ is the tangent space to the moduli space at $[\rho]$. So in fact the space of odd directions at $[\rho]$ is identified with $\Pi T\mathcal{M}_{[\rho]}$. Thus, at the open subset of irreducible connections, the supermanifold of classical solutions is $\Pi T\mathcal{M}$, the parity reversed tangent bundle over the moduli space of flat connections. As we have seen in Problem FP8 of [I-**Homework**], the Berezinian of such a supermanifold is naturally trivialized so that one integrates functions over this supermanifold. We have also seen that a function on this supermanifold is the same thing as a differential form on $\mathcal{M}$ and integration of a function over $\Pi T\mathcal{M}$ is the same thing as the integral of the corresponding differential form over $\mathcal{M}$. (This requires an orientation of $\mathcal{M}$ which is determined by an orientation of Σ.)

Of course, we wish to compute correlation functions involving the operators $\int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)}$. These operators determine functions on $\Pi T\mathcal{M}$ and the correlation function of a product of them becomes the integral of the product of the functions. In this manner gauge invariant polynomials in σ are mapped to functions on $\Pi T\mathcal{M}$ and hence to differential forms on $\mathcal{M}$. A crucial step in the reduction from polynomials to differential forms is the observation that with fermions present, it is no

longer the case that $\sigma = 0$. Rather the equations of motion give

$$\Box\sigma = -[\psi_{(1)}, \psi_{(1)}].$$

This is important because the right hand side is directly interpreted as a two-form on moduli space (as the zero modes of $\psi_{(1)}$ represent one-forms on the moduli space), so the formula enables us to express the function σ, which is formally of degree two in the quantum field theory formalism, in terms of an ordinary two-form on the moduli space. One evaluates the correlation functions by integrating the differential forms obtained in this way over the moduli space. (Of course, these computations have to be augmented by computations at the components given by reducible connections and a non-zero but parallel field σ.)

In this gauge theory example, because of the presence of reducible connections, the Hamiltonian has a continuous spectrum beginning at zero energy. As a result the map from Q-cohomology of operators to Q-cohomology of states is not very useful.

§14.5. A σ-model example

Let us consider the $N = 2$ supersymmetric sigma model

$$\Phi\colon \Sigma \to X$$

where X is a Kähler manifold and where Σ is a split $2|4$ manifold as in the previous gauge theory example. With this assumption, the fermions of the sigma model lie in

$$K_\Sigma^{\pm 1/2} \otimes T^{\pm}X$$

where K_Σ is the canonical bundle of Σ, and $T^+X = T^{1,0}X$ and $T^-X = T^{0,1}X$ are the holomorphic and anti-holomorphic tangent bundles of X. We denote these fermions by $\chi^{\pm,\pm}$. The first sign refers to whether the fermion is left- or right-moving (i.e., in whether it is a section of K_Σ^+ or K_Σ^-) and the second sign refers to the type of the section (holomorphic or anti-holomorphic) on X. Once again we twist the theory for it to exist globally on Σ, replacing K by

$$K' = K \pm \frac{1}{2}J_L \pm \frac{1}{2}J_R.$$

At this point some fermions are sections of $T^{\pm}X$ and others are sections of $K_\Sigma^{\pm 1} \otimes T^{\pm X}$. Depending on the choice of signs in this twisting we get two possible cases called the A-model (using $-\frac{1}{2}J_L + \frac{1}{2}J_R$) or the B-model (using $\frac{1}{2}J_L + \frac{1}{2}J_R$).

§14.5.1. The A-model

The fermions that transform after twisting as zero-forms on Σ are $\chi^{++} \in T^{1,0}X$ and $\chi^{--} \in T^{0,1}X$. A generic element of the ring of 0-form operators is

$$F(\Phi, \chi^{++}, \chi^{--}).$$

The ring of these operators is the ring of functions on ΠTX, which is identified with the ring of differential forms on the Kähler manifold. Furthermore, with this identification of a differential form F with a corresponding operator $\mathcal{O}_F$, we have

$$Q\mathcal{O}_F = \mathcal{O}_{dF},$$

so that Q becomes the ordinary exterior derivative. The chiral ring is then $H^*(X)$, the usual topological cohomology of X, and the bigrading induced by J_L and J_R

is the usual Hodge bigrading. In the space of maps, the Q-fixed points are the holomorphic curves

$$\phi\colon \Sigma \to X.$$

(To see this one must look at the supersymmetry transformation law of those fermions that transform as $(1,0)$- or $(0,1)$-forms on Σ in the twisted model.) Thus, to compute correlation functions in this model we must do integrals over the spaces of holomorphic curves in X. This will lead, as we shall see in the next lecture, to quantum cohomology.

§14.5.2. The B-model

It is possible to construct this model only in the case X is Calabi-Yau, for this is the only case when the symmetry $(J_L + J_R)$ by which we wish to twist is not anomalous[3]. We fix a non-zero holomorphic n-form ω on X. In this case the zero-forms are $\chi^{+-} \in T^{1,0}X$ and $\chi^{--} \in T^{0,1}X$. The ring of 0-form operators is

$$F(\Phi, \chi^{+-}, \chi^{--})$$

which is the ring

$$\Omega^{0,*}(X) \otimes \Omega^{0,*}(X) = \Omega^{0,*}\left(X, \wedge^* T^{1,0}(X)\right).$$

Under this identification we have

$$Q\mathcal{O}_F = \mathcal{O}_{\overline{\partial} F},$$

so that the chiral ring is

$$\oplus_{q,p} H^q(X; \wedge^p T^{1,0}X).$$

The Q-fixed points in the space of fields are the constant maps $\Sigma \to X$. Thus, the computations of correlation functions in this model will become integrals over X

$$\langle \prod_{i=1}^{s} \mathcal{O}_{F_i} \rangle = \int_X (\wedge_{i=1}^s F_i) \cdot \omega^{\otimes 2}.$$

These integrals lie squarely in classical mathematics; they have to do with variations of Hodge structure under deformation of complex structure.

References and cross references

A reference for much of this material is

E. Witten *Mirror Manifolds And Topological Field Theory*, in Essays on mirror manifolds, 120–158, Internat. Press, Hong Kong, 1992.

Much of the material here is closely related to that contained in Lecture 11. The trivialization of the Berezinian of the supermanifold constructed from ΠTM and the identification of functions on this supermanifold with differential forms is also contained in [I-**Dirac Index**, §1.2]. Basic material on R-symmetry is contained in Chapter 9 of [I-**Supersolutions**]. The relation of the stress-energy tensor and variations of the metric is explained in [I-**Classical Fields**, §2.9] and described in the context of conformal field theory in [II-**CFT**, §2.1] and [II-**Strings**, §2.1]. Two examples of topological correlation functions in topologically twisted quantum field theories are worked out in Problems 10 and 11 at the end of these lectures.

[3]In other words, if $c_1(X)$ is nonzero, then $J_L + J_R$ is simply not a symmetry of the quantum theory, even before gauging. The discussion of the obstruction to gauging in the first part of this lecture assumed that J_L and J_R both generated symmetries prior to gauging, and thus does not apply to this case.

LECTURE 15

The Landau–Ginzburg Description of $N = 2$ Minimal Models; Quantum Cohomology of Kähler Manifolds

§15.1. Landau–Ginzburg models

The topic for the first part of today's lecture is a more detailed discussion of the known evidence for the claimed equivalence between IR limits of certain Landau–Ginzburg models, and the algebraically constructible conformal field theories known as "minimal models."

Consider first the simplest Landau–Ginzburg model: a two-dimensional N=2 theory of a single chiral superfield Φ with Lagrangian

$$\mathcal{L} = \int d^2x\, d^4\theta\, \overline{\Phi}\Phi + \left(\int d^2x\, d\theta^+\, d\theta^-\, \Phi^k + \text{c.c.} \right).$$

Our claim in Lecture 13, §13.6.1, was that this model flows in the IR to a conformal field theory with $\widehat{c} = 1 - \frac{2}{k}$. If true, then the theory it flows to is an algebraically solvable theory, due to the known classification of representations of the N=2 superconformal algebra with $\widehat{c} < 1$. More precisely, fixing $\widehat{c} = 1 - \frac{2}{k}$, there turn out to be a finite number of irreducible representations $\mathcal{H}_\alpha$ of the algebra of central charge $\widehat{c}$ (up to isomorphism). The conformal field theories with $\widehat{c} = 1 - \frac{2}{k}$ are determined by their Hilbert spaces, which are constructed via a sum of tensor products

$$\bigoplus_{\alpha,\beta} (\mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\beta)^{e_{\alpha\beta}}$$

of these representations with a modular-invariant character. These combinations have an *A-D-E* classification. The simplest modular-invariant combination, and the one which will occur here, is the unweighted diagonal sum over all of the representations:

$$\bigoplus_{\alpha} \mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\alpha.$$

(This corresponds to the A series.)

More general Landau–Ginzburg models involve several chiral superfields Φ_1, ..., Φ_n and a quasi-homogeneous polynomial $W(\Phi_1, \ldots, \Phi_n)$ which serves as the superpotential of the theory. We should assume that the polynomial is transverse

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Notes by John Morgan and David Morrison

in the sense that $W = dW = 0$ holds only at the origin; the quasi-homogeneity condition can be expressed as the existence of positive rational numbers α_j such that

$$W = \sum \alpha_j \Phi_j \frac{\partial W}{\partial \Phi_j}.$$

The Lagrangian can be written

$$\mathcal{L} = \int d^2x\, d^4\theta\, \overline{\Phi}_j \Phi_j + \left(\int d^2x\, d\theta^+\, d\theta^-\, W(\Phi_j) + \text{c.c.} \right).$$

Intuitively, the quasi-homogenity is important because it allows the possibility of extending one of the R-symmetries, namely

$$\theta^+ \to e^{i\varepsilon}\theta^+, \qquad \theta^- \to \theta^-,$$

to act on the chiral fields as $\Phi_j \to e^{i\varepsilon\alpha_j}\Phi_j$ so that the superpotential transforms as $W \to e^{i\varepsilon}W$ and hence the Lagrangian is invariant. If we *assume* that this microscopic R-symmetry *becomes* the left-moving R-symmetry of a limiting conformal field theory in the IR, then we can calculate the central charge $\widehat{c}$ of that limiting theory via operator product expansions.

This works as follows. Recall that in Lecture 13 we encountered a constant k which measures the anomaly which arises if one attempts to gauge the R-symmetry J_L. This constant appears in the operator product expansion as

$$J_L(x)\, J_L(x') = \frac{k}{(x^- - x'^-)^2} + \cdots.$$

The interpretation of k as the obstruction to gauging makes it clear that it is a topological invariant which can be reliably measured or computed at any scale. In particular, it can be calculated in the UV, where the superpotential doesn't matter. (Thus, the obstruction to gauging was computed from an index problem and holds whether the Riemann surface on which we work is large or small; we can take it to be so small that the superpotential is irrelevant.) On the other hand, if J_L is the same R-symmetry which appears in the limiting $N = 2$ superconformal field theory, then the operator product expansion of the superconformal algebra asserts that

$$J_L(x)\, J_L(x') = \frac{\widehat{c}}{(x^- - x'^-)^2} + \cdots,$$

where $\widehat{c}$ is the central charge in the $N = 2$ algebra. Thus, $k = \widehat{c}$ and we can measure the IR central charge by computing in the UV.

To compute k in the ultraviolet, one treats the fields as free fields. The obstruction to gauging comes from the fact that the R-symmetries are chiral symmetries for the fermions in the free supermultiplets. According to methods of Lecture 6, §6.2, we have $k = \sum b_i^2 - \sum c_i^2$, where b_i and c_i are the R-charges of left and right chirality fermions. In our problem, $b_i = -1 + \alpha_i$ and $c_i = \alpha_i$, so we get $k = \sum_i (1 - 2\alpha_i)$. Assuming the flow to an $N = 2$ theory in the infrared, with the same R-symmetry appearing in the $N = 2$ algebra, we have $k = \widehat{c}$ and hence $\widehat{c} = \sum_j (1 - 2\alpha_j)$. Notice that in the case of a single field with $W(\Phi) = \Phi^k$, we must have $\alpha = 1/k$ and so we get the desired result $\widehat{c} = 1 - 2/k$ for this theory.

We now want to consider a more sophisticated argument in favor of this equivalence, one which reveals more of the structure of the (limiting) conformal field

theory. Recall the structure of the N=2 algebra:

$$\{Q_\pm, \overline{Q}_\pm\} = P_\pm$$
$$\{Q_\pm, Q_\mp\} = 0.$$

We have $\overline{Q}_+^2 = 0$ so we can consider the cohomology of $\overline{Q}_+$, i.e., consider local operators $\mathcal{O} = \mathcal{O}(x)$ such that $\{\overline{Q}_+, \mathcal{O}\} = 0$. Then

$$\partial_+ \mathcal{O} = \left[\{Q_+, \overline{Q}_+\}, \mathcal{O}\right] = \left[\overline{Q}_+, \{Q_+, \mathcal{O}\}\right],$$

i.e., in Euclidean language (identifying (x^+, x^-) with $(z, \overline{z})$) we would say that the class of $\mathcal{O}(x)$ in $\overline{Q}_+$-cohomology varies holomorphically.

Taking two such operators, using holomorphic language, we write

$$\mathcal{O}(z)\mathcal{O}'(z') = \sum f_k(z - z')\mathcal{O}_k(z') + \{\overline{Q}_+, \cdots\},$$

with the f_k's being holomorphic functions. From this formula, we can see that the $\overline{Q}_+$-cohomology has the general structure of a conformal field theory in which the operators are "purely left-moving," that is, they vary holomorphically.

An important remark is that the $\overline{Q}_+$-cohomology is invariant under Weyl rescaling, and hence it depends only on the complex structure of the Riemann surface (not the metric). To see this invariance, we must show that $T_{z\overline{z}}$ is $\overline{Q}_+$-exact. A natural way to prove this is to modify the usual definition of scale invariance, employing the quasi-homogeneity of our polynomial, so that the superpotential term is preserved under the modified scaling. Then the kinetic term in the action fails to be scale-invariant, but the change in the kinetic term is of the form $\int d^4\theta(\cdots)$ (since the kinetic term itself has this property) and hence is $\overline{Q}_+$-exact.

We have formulated this discussion in terms of $\overline{Q}_+$-cohomology in order to include very general N=2 theories. However, whenever there is a superspace realization of the theory (as is true of the Landau–Ginzburg theories we are studying), it is more convenient to study the $\overline{D}_+$-cohomology rather than the $\overline{Q}_+$-cohomology, where

$$\overline{D}_+ = e^{-\theta^+\overline{\theta}^+\partial_+}\, \overline{Q}_+\, e^{\theta^+\overline{\theta}^+\partial_+},$$

so the $\overline{D}_+$ and $\overline{Q}_+$ cohomology are naturally isomorphic.

In our Landau–Ginzburg theory, the equations of motion are

$$2\overline{D}_+\overline{D}_-\Phi^j = \frac{\partial W}{\partial \Phi^j}.$$

Consider the operator

$$\mathcal{T} = \sum_j \left(\left(\frac{1 - \alpha_j}{2} \right) D_-\Phi^j\, \overline{D}_-\, \overline{\Phi}_j - i\alpha_j \overline{\Phi}_j \partial_- \Phi_j \right);$$

then $\overline{D}_+\mathcal{T} = 0$ so that this defines a $\overline{D}_+$-cohomology class [1].

In the free theory, with $W = 0$, $\mathcal{T}$ is an operator of dimension $(1, 0)$. If we have a superconformal theory, we will be able to write

$$\mathcal{T} = J + \theta^- G + \overline{\theta}^- \overline{G} + \theta^-\overline{\theta}^- T.$$

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[1] We should worry about anomalies here, but in fact there is no quantum anomaly in the statement that $\overline{D}_+\mathcal{T} = 0$, though there are such anomalies in somewhat similar statements in two-dimensional models with $(0, 2)$ supersymmetry. See E. Silverstein and E. Witten, "Global $U(1)$ R Symmetry And Conformal Invariance Of $(0, 2)$ Models," Phys. Lett. **B328** (1994) 307.

If we do arrive at an N=2 conformal field theory in the limit, then we should expect an OPE of the form

$$\mathcal{T}(x)\mathcal{T}(y) = \frac{\widehat{c}}{(x-y)^2} + \frac{\mathcal{T}}{(x-y)} + \text{terms holomorphic in } \mathcal{O}\text{'s.}$$

We can check this structure in the free field theory: the first term comes from a diagram of the form

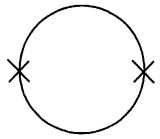

Figure 1.

with two propagators exchanged between the two operators and the second term comes from a diagram of the form

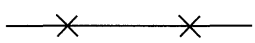

Figure 2.

with exchange of only one propagator.

Note that, in computing these terms in the operator product expansion, we can set $W = 0$, since the superpotential term is too soft (superrenormalizable) to contribute to these singularities. Once this is done, the different superfields are decoupled, so the computation of $\widehat{c}$ is a sum over different superfields. So there is an immediate conclusion that $\widehat{c}$ is *additive* in the α_j's, that is, $\widehat{c} = \sum_j f(\alpha_j)$ for some function f. Computation gives $f(\alpha) = 1 - 2\alpha$, just as in our first approach.

In summary, even in our microscopic Lagrangian, we can see an operator $\mathcal{T}$, which at the level of $\overline{D}_+$ cohomology obeys an $N = 2$ superconformal algebra. So presumably this theory flows in the IR to a conformal field theory whose $\overline{D}_+$-cohomology contains the superconformal algebra.

If we have only one chiral superfield, so $\widehat{c} < 1$, the $N = 2$ algebra acts almost irreducibly in the quantum Hilbert space. Hence one should expect that $\mathcal{T}$ will generate the $\overline{Q}_+$-cohomology. For $\widehat{c} > 1$, the $N = 2$ algebra acts in a way that is far from irreducible, and to get an algebraically soluble model one would need additional generators. It is not known whether such additional generators exist for some Landau–Ginzburg models (apart from the obvious case where the superpotential is a sum of functions of different variables).

§15.2. The elliptic genus

We will describe additional evidence for the equivalence of these theories from a computation of the *elliptic genus*. We work on a Riemann surface of genus 1 with period lattice spanned by 1 and τ.

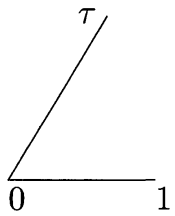

Figure 3.

We let $q = e^{2\pi i\tau}$, and compute $\operatorname{Tr} q^{(H-P)/2}\overline{q}^{(H+P)/2}(-1)^{F_R}$.

We choose periodic boundary conditions for both left and right-moving fermions in the space direction on the torus. The insertion of $(-1)^{F_R}$ means that the right-moving fermions ψ_+, $\overline{\psi}_+$ should be given the periodic boundary conditions in the time direction as well, making the *odd* spin structure (the one periodic in both directions), while the left-moving fermions ψ_-, $\overline{\psi}_-$ should be antiperiodic in the time direction and thus are sections of an *even* spin bundle.

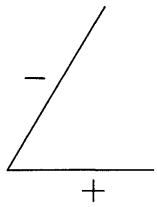

Figure 4.

The path integral with these boundary conditions can be introduced just using $N = 1$ supersymmetry, and as we discussed last fall, it is quite interesting already in that context. It can be interpreted to be an index of the right-moving supercharge of the $N = 1$ theory.

The story is more interesting in the case of N=2 supersymmetry, since there is the possibility of enriching things further by inserting in the trace $e^{i\gamma J_L}$, which commutes with both Q_+ and $\overline{Q}_+$ and so preserves the interpretation as an index.[2] We consider the function

$$F(q,\gamma) = \operatorname{Tr} q^{(H-P)/2}\overline{q}^{(H+P)/2}e^{i\gamma J_L}(-1)^{F_R}.$$

To represent this trace via a path integral, we in the customary fashion build up a genus one surface by gluing together the two ends of a cylinder (the two ends are identified with initial and final times). The factor of $e^{i\gamma J_L}$ in the trace is incorporated in the path integral by identifying the initial and final data with multiplication by $e^{i\gamma J_L}$; and likewise for other factors in the trace.

The first observation is that the quantity $F(q,\gamma)$ is independent of any continuously variable parameters of the theory that preserve the $N = 2$ structure. This is so as it is a linear combination of terms each of which is an index of an operator. For the same reason (or because the trace of the stress tensor of the theory is $\overline{Q}_+$-exact) $F(q,\gamma)$ is invariant under rescaling of the metric on the torus, and only depends on the complex structure.

We want to calculate $F(q,\gamma)$ in two contexts: (i) an algebraic calculation for the known algebraically constructed models in the IR, and (ii) directly for the Landau–Ginzburg models in the UV. The comparison of these two will then provide additional evidence that the Landau–Ginzburg theory indeed flows to the "minimal model" in the IR.

§15.2.1. Algebraic calculation in the IR

The Hilbert space of a "minimal model" (in the A series, relevant to the model considered here) takes the form $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\alpha$, with the left-movers in $\mathcal{H}_\alpha$ and the right-movers in $\overline{\mathcal{H}}_\alpha$. (The sum runs over all unitary irreducible representations of the $N = 2$ superconformal algebra with $\widehat{c} = 1 - 2/k$ in the Ramond sector, that

[2]We use the same symbol J_L for the left R-current and the conserved charge derived from it.

is, with a non-bounding spin structure on the circle.) Thus, the elliptic genus can be written

$$F(q,\gamma) = \sum_\alpha \left(\mathrm{Tr}_{\mathcal{H}_\alpha} q^{(H-P)/2} e^{i\gamma J_L}\right)\left(\mathrm{Tr}_{\overline{\mathcal{H}}_\alpha} \overline{q}^{(H+P)/2}(-1)^{F_R}\right).$$

The right-moving contribution $\mathrm{Tr}_{\overline{\mathcal{H}}_\alpha} \overline{q}^{(H+P)/2}(-1)^{F_R}$ evaluates to 1 if $\overline{\mathcal{H}}_\alpha$ has highest weight with $H+P=0$ and 0 otherwise. This has the effect of selecting a distinguished subset from among the irreducible representations. Let us introduce a restricted sum $\sum'$ to denote the sum over that subset. Then our elliptic genus can be written

$$F(q,\gamma) = \sum_\alpha{}' \mathrm{Tr}_{\mathcal{H}_\alpha} q^{(H-P)/2} e^{i\gamma J_L}.$$

(Note that the eigenvalues of the charge J_L are not integers in general. Rather, the object $\exp(2\pi i J_L)$ commutes with the $N=2$ algebra and is a constant in each representation; the constant depends on the R-charges of the chiral primary fields, which are rational numbers. So $F(q,\gamma)$ is periodic in γ, but the period is not 2π.) Using this formula, $F(q,\gamma)$ can be computed from character formulas for the $N=2$ superconformal algebra.

§15.2.2. Direct calculation in the UV

Going back to the Landau–Ginzburg theory, we can try to directly calculate in the UV. We use the fact that $F(q,\gamma)$ is independent of the area of the torus. We take the area to be very small. Roughly, in the UV the interaction is negligible and we should be able to set $W=0$. The only reason for this to fail has to do with the behavior for large fields. The bosonic part of the action is

$$\int d\mu |d\phi|^2 + \int d\mu \left|\frac{dW}{d\phi}\right|^2.$$

Here $d\mu$ is the Riemannian measure of the torus. In the presence of the W term, the unique minimum of the action is $\phi=0$. If we would set W to zero, the action would be minimized for any constant ϕ. One would have to integrate over the space of constant ϕ's even in a leading approximation to the path integral, and then when ϕ becomes sufficiently large, the W term, if its coefficient is not strictly zero, is important.

The conclusion is, then, that as long as ϕ is a *function* on the torus, the W term cannot quite be ignored. If, however, ϕ were not a function but a section of a nontrivial flat bundle, then even at $W=0$ the $|d\phi|^2$ part of the action would have a unique and nondegenerate minimum. In that situation, it would be straightforward, in the limit of small area on the torus, to set W to zero.

In the path integral evaluation of $F(q,\gamma)$ for generic γ, we get just such a situation in which ϕ is a section of a nontrivial flat bundle. The reason is that J_L acts on ϕ by $\delta\phi = \alpha\phi$. In evaluating the elliptic genus by a path integral, ϕ is a field on the torus that obeys

$$\phi(P+\tau) = e^{i\gamma\alpha}\phi(P).$$

(Here we regard the torus as the quotient of the complex plane by the lattice generated by 1 and τ, and P is any point on the complex plane.) So ϕ is a section of the flat bundle on the torus specified by this gluing. If γ is generic, this flat bundle is non-trivial, and $\phi=0$ is an isolated and nondegenerate minimum of the action

even if we set $W = 0$. The nondegeneracy makes computations straightforward: the calculation of $F(q, \gamma)$ is made by setting $W = 0$ and expressing F as a ratio of determinants. Explaining the details would take us too far afield, but the result is

$$F(q,\gamma) = e^{-i\gamma k\alpha/2} \cdot \frac{1 - e^{i\gamma(k+1)\alpha}}{1 - e^{i\gamma\alpha}} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i\gamma(k+1)\alpha})(1 - q^n e^{-i\gamma(k+1)\alpha})}{(1 - q^n e^{i\gamma\alpha})(1 - q^n e^{-i\gamma\alpha})}.$$

When this is compared with the algebraic calculation in the IR, one can extract formulas for certain characters of the N=2 algebra. These formulas can be verified independently, giving strong support to the claim that the Landau–Ginzburg theory flows to the $N = 2$ minimal model in the infrared.

§15.2.3. Connection to Calabi–Yau models

Consider now the $U(1)$ gauge theory with chiral superfields Φ_j of charge α_j and P of charge -1, such that $\sum \alpha_j = 1$. (We are no longer assuming that all of the α_j's are equal, and we are using a different normalization for the generator of $U(1)$ than in Lecture 13.) Denote the gauge supermultiplet by

$$\Sigma = \sigma + \cdots,$$

let $W(\Phi_1, \ldots, \Phi_n)$ be a transverse, quasi-homogeneous polynomial as above (i.e., satisfying $W = \sum \alpha_j \Phi_j \frac{\partial W}{\partial \Phi_j}$, with $W = dW = 0$ only at the origin), and consider the Lagrangian

$$\mathcal{L} = \int d^2x\, d^4\theta \left(\frac{\overline{\Sigma}\Sigma}{4e^2} + \overline{\Phi}^j \Phi^j \right) + \left(\int d^2x\, d^2\theta\, PW(\Phi) + \text{c.c.} \right) + \left(\int d^2x\, d\theta^+\, d\theta^-\, t\Sigma + \text{c.c.} \right)$$

such that $t = \frac{\theta}{2\pi} - ir$.

The polynomial $PW(\Phi)$ is gauge invariant, and we can repeat our discussion from Lecture 13: for $r \gg 0$ we classically get a Calabi–Yau hypersurface $\{W = 0\}$ in weighted projective space $\mathbf{WCP}^{n-1}_{(\alpha_1,\ldots,\alpha_n)}$ (the hypersurface may inherit some singularities from those of the weighted projective space); for $r \ll 0$ we get a Landau–Ginzburg orbifold whose superpotential is W. The quantum theory is singular only when $r = r_0$, $\theta = 0$. (r_0 is a constant computed as in Lecture 13.) The computation of the central charge in Landau–Ginzburg theory gives

$$\widehat{c} = \sum_j (1 - 2\alpha_j);$$

the computation for the sigma model gives $\widehat{c} = \dim_{\mathbf{C}} X$, and these agree since $\sum \alpha_j = 1$.

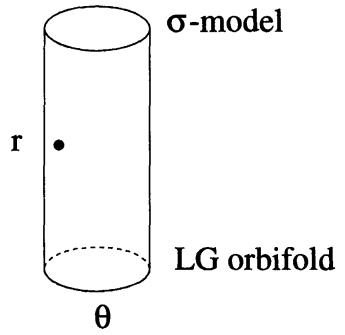

Figure 5.

The elliptic genus should be independent of t since it is a topological invariant; this leads to a formula which relates the elliptic genus of this type of Calabi–Yau manifold to the explicit elliptic genus which can be computed via characters of minimal models on the Landau–Ginzburg side.

Although the elliptic genus is independent of t, one of the other quantities we have discussed for N=2 theories—the $\overline{Q}_t$-cohomology—varies holomorphically with t. The *instantons* in this problem—which are holomorphic curves in the sigma model—will contribute to the path integral with a factor

$$\exp\left(-i\theta\int\frac{F}{2\pi}-r\int d^2x\,D\right)$$

where $D=\sum\alpha_j|\Phi_j|^2-r$. The instanton equation of the $N=2$ model – the condition for the field to be invariant under some supersymmetry – gives $\int d^2xD=\int F/2\pi$, so this factor is $\exp\left(-2\pi it\int_\Sigma\Phi^*(c_1(\mathcal{L}))\right)$. We will explore the instanton sums in the second half of the lecture.

§15.3. Introduction to quantum cohomology

We return now to twisted $N=2$ supersymmetric sigma models in dimension two with target a Kähler manifold X. Recall that there were four possibilities for twisting depending on which supersymmetry $Q\in\{\overline{Q}_++\overline{Q}_-,Q_++\overline{Q}_-,\overline{Q}_++Q_-,Q_++\overline{Q}_-\}$ is preserved. Whichever possibility one selects, the cohomology of Q as a vector space can be identified with the cohomology of X. But there is more structure; if we restrict to the case that the Riemann surface is a fixed two-manifold Σ, then the additional structure is a ring structure on the cohomology of Q. The resulting rings are called the (a,a), (c,a), (a,c), and (c,c) chiral rings.

With one of the choices, the classical version of the chiral ring turns out to be the usual ring structure on cohomology, but quantum mechanically there are corrections, instanton corrections, to this classical answer. For this reason the chiral ring of the sigma model is called the quantum cohomology of the Kähler manifold X. In this half of the lecture, we will discuss quantum cohomology generally and then make some explicit computations in the case $X=\mathbf{CP}^{n-1}$ and the case X is a Fano hypersurface. These computations will be done both mathematically, counting holomorphic rational curves with certain properties, and will be done from the physics point of view by using the renormalization group flow (discussed in Lecture 13 and the first half of this lecture) from these sigma models to Landau–Ginzburg theories with extra vacua.

§15.4. The space of zero-energy states

We wish to compute some of these chiral rings, but before we deal directly with the operators, we compute the zero-energy states in the Hamiltonian framework. The reason for doing this computation is that as we showed in Lecture 14, §14.3.3, there is a map

$$\psi\colon\text{chiral ring}\to\{\text{zero-energy states}\}$$

given by inserting a local operator at a point on the hemisphere. In favorable circumstances this map is an isomorphism.

In the Hamiltonian framework, the bosonic space underlying the space of fields is

$$\mathcal{W} = \mathrm{Maps}(S^1, X).$$

The minimum energy configurations are the constant maps; the space of these is a copy of X. Since there are fermions in two copies of the spin bundle, when we quantize, the resulting Hilbert space is the tensor product of two copies of the spin bundle over X, i.e., the space of differential forms on X. The Hamiltonian is the Laplacian and its space of ground states is then the space of L^2-harmonic forms on X, which is identified with $H^*_{L^2}(X)$. Of course, there are normal directions to the copy of X in $\mathcal{W}$ to consider, but supersymmetry implies that the quadratic form in the normal directions is non-degenerate, and hence these directions produce a tensor product of harmonic oscillators and have a unique ground state. Thus, in the end we find that the ground states for the entire theory are identified with the harmonic forms on X. We shall see that the computation of the chiral rings gives the same answer (additively), so that in the cases we consider today the map ψ is indeed an isomorphism. This is true generally when the classical theory has the property that the energy grows as the fields go to infinity. [3]

§15.5. Generalities on the chiral ring

In order to define a chiral ring we need a global supersymmetry that squares to zero. As we saw in Lecture 14, §14.3, the choice of a twisted form of an $N = 2$ supersymmetric theory selects such a global supersymmetry. Today we shall only consider twists making $\overline{Q}_+ + Q_-$ global. This is what we referred to in Lecture 14 as Case A. It is a twist that can be performed for any Kähler manifold – no assumption of vanishing first Chern class is necessary. We shall consider local operators which are functions of the basic bosonic field $\Phi\colon \Sigma \to X$ (Σ is our Riemann surface) and fermions that are related to Φ by supersymmetry. (If local coordinates are chosen on X, we describe Φ by functions on S^1 that we call ϕ.) Recall that one of the effects of twisting is to make the fermions sections of bundles of integral rather than half-integral spin. That is to say, they are differential forms on Σ (valued in the pullback of the tangent bundle of X) instead of spinors. It will suffice in describing the Q-cohomology of operators to consider only fermions of zero spin. The reason for this is that all of the Q-cohomology classes of operators have representatives that are constructed using only those fermions. To be more precise, the Q-cohomology classes have representatives that depend only on the bosonic fields and spin zero fermions and not their derivatives [4]. Thus, our local operators will be of the form

$$\mathcal{O} = \mathcal{O}(\phi, \psi_L, \psi_R)$$

where, in the untwisted version of the theory, ψ_L is a section of $K^{-1/2} \otimes \phi^* T^{0,1} X$ and ψ_R is a section of $K^{1/2} \otimes T^{1,0} X$. After we twist, these fermion fields lie in $\Phi^* T^{0,1} X$ and $\Phi^* T^{1,0} X$, since the twisting cancels out the line bundle over Σ. In particular,

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[3] For a sigma model with compact target space X, the only sense in which a map $\Phi : S^1 \to X$ can "go to infinity" is that they it can become rapidly varying along S^1. This certainly causes the energy to grow.

[4] This is the operator analog of the fact that, in the Hamiltonian description, the zero energy states are constructed from differential forms on the target space with all oscillators in their ground state. Operators that contain derivatives of fields would correspond to states in which some oscillators are excited.

it is possible to form the sum $\psi = \psi_L + \psi_R$ and consider operators $\mathcal{O} = \mathcal{O}(\phi, \psi)$ where ψ is a zero-form on Σ valued in the pullback of $TX = T^{1,0}X \oplus T^{0,1}X$.

§15.5.1. Local Operators

Each differential form λ on X corresponds to a function $f_\lambda(\phi, \psi)$ (f_λ is a function on ΠTX). So for every λ there is an operator $\mathcal{O}_\lambda(x) = f_\lambda(\phi(x), \psi(x))$. As we remarked in last week's lecture, under the identifications of operators and forms, Q is identified with the usual exterior derivative:

$$\{Q, \mathcal{O}_\lambda\} = \mathcal{O}_{d\lambda}.$$

That is why, additively, the Q-cohomology coincides with the de Rham cohomology of X.

When, as in the case of these $N = 2$ supersymmetric sigma models, the operator-to-state correspondence is an isomorphism, we can compute the ring structure of the chiral ring by computing correlation functions. Our goal here is to compute in more classical topological and geometric terms the correlation function

$$\langle \mathcal{O}_{\lambda_1}(x_1) \cdots \mathcal{O}_{\lambda_s}(x_s) \rangle$$

for closed forms λ_i. As we have already remarked, the correlation function is unchanged if we replace the λ_i by cohomologous forms. Also, to compute this correlation function, which means to compute the path integral

$$\int \mathcal{D}\phi \mathcal{D}\psi e^{-\mathcal{L}} \prod_i \mathcal{O}_{\lambda_i}(x_i)$$

over the space of all fields, it suffices to compute the integral over the subspace of Q-invariant fields. We have already identified the bosonic part of the subspace of Q-fixed points: it is the space of holomorphic maps $\Sigma \to X$. For each i choose a cycle H_i Poincaré dual to $[\lambda_i]$. Then we can choose a representative for λ_i with support in an arbitrarily small neighborhood of H_i.

Now we want to compute the correlation function, by localization of the path integral on Q-invariant classical configurations, that is, on holomorphic maps of the Riemann surface to X. Given the representatives that we picked for the operators, only holomorphic maps that at the points x_i pass through the H_i will contribute. In the "best" case when the moduli space of holomorphic maps passing through these points consists of a finite set of nondegenerate points, one shows that each such map contributes ± 1. In fact, supersymmetry ensures that the Bose and Fermi determinants cancel up to sign. Then the result obtained by summing over all the holomorphic maps equals the product of the exponential of the classical action times the sign coming from the determinants. The exponential of the classical action is, for a holomorphic map, just $\exp\left(-2\pi i t \int_\Sigma \Phi^*(c_1(\mathcal{L}))\right)$.

Mathematically, there is a dimension count which must be satisfied for the problem of "holomorphic maps passing through the H_i at x_i" to produce a zero-dimensional moduli space. To state things more generally, if we interpret the operators $\mathcal{O}_{\lambda_i}(x_i)$ as producing differential forms on the moduli space of all holomorphic maps of Σ to X, then there is a dimensional condition on the H_i for the differential form to be of top dimension; otherwise its integral certainly vanishes. From the physics point of view, one sees the same dimension restriction coming from the anomaly of the R-symmetry. The measure of integration in the path integral has an anomaly under this symmetry, and one must use a set of operators whose product

has precisely the cancelling anomaly for the path integral to have a chance to be non-zero. Not surprisingly, these two conditions are the same: namely, that the sum of the codimensions of the H_i in X must be equal to the virtual dimension of the component one considers of the moduli space of holomorphic maps from Σ to X.

If one wants to see just the classical cohomology ring, rather than the quantum cohomology, then one should take Σ to be of genus zero, and one should consider just the constant maps to X. The constant maps are of course all the holomorphic maps in a certain component of $\mathrm{Maps}(\Sigma, X)$, namely the component consisting of null-homotopic maps. For this homotopy class of maps, the dimension condition is

$$\sum_{i=1}^{s} \text{degree}\, \lambda_i = \dim X$$

and the classical answer is simply

$$\int_X \lambda_1 \wedge \cdots \wedge \lambda_s.$$

That is to say, the classical chiral ring is the cohomology ring of X. This answer gets corrected quantum mechanically by instanton corrections given by the integrals we were describing over the moduli space of non-constant holomorphic maps. Thus, the chiral ring is a quantum correction to the usual cohomology ring of X. For this reason the chiral ring is usually called the quantum cohomology ring of X.

§15.6. More Details on the Ring Structure

Having given the general definition of quantum cohomology as the chiral ring and indicated the sorts of mathematics that go into computing it, it is now time to be more concrete and compute an example. We take the case of $\Sigma = S^2$ mapping into a compact Kähler manifold X. Let us begin with a correlation function with only two operators:

$$\langle \mathcal{O}_{\lambda_1} \mathcal{O}_{\lambda_2} \rangle,$$

with the degree of λ_1 plus the degree of λ_2 equal to the dimension of X. We of course know that the space of constant maps gives a contribution to this correlation function equal to

$$\int_X \lambda_1 \wedge \lambda_2.$$

In this case there is no quantum correction and this is also the answer in the chiral ring. The reason that there is no quantum correction is that the space of non-constant holomorphic maps of $S^2 \to X$ with specified points $x_1, x_2 \in S^2$ mapping respectively to H_1 and H_2 has a free $\mathbf{C}^*$-action on it, induced from the $\mathbf{C}^*$ that acts on S^2 fixing x_1 and x_2. For a component of this space to possibly give a non-zero correction to the correlation function, it must be the case that its formal dimension is zero. If its actual dimension is zero and there is a free $\mathbf{C}^*$-action on it, then it must be empty. (Even for a possible nonempty component of this space, the free $\mathbf{C}^*$ action ensures that the contribution is zero.) Thus, under the identification of the chiral ring with the cohomology of X, the two-point correlation functions, which compute the inner product on the vector space underlying the chiral ring, give the usual inner product on cohomology or equivalently give the intersection form on homology.

Now let us consider the ring structure. Fix a basis $\{\lambda_i\}$ for $H^*(X)$. Let the inner product in this basis be given by η_{ij}:

$$\eta_{ij} = \langle \mathcal{O}_{\lambda_i} \mathcal{O}_{\lambda_j} \rangle.$$

Then the ring structure is determined by structure constants c_{ij}^k defined by

$$\mathcal{O}_{\lambda_i} \mathcal{O}_{\lambda_j} = c_{ij}^k \mathcal{O}_{\lambda_k}.$$

Let ω_{ijk} be the three-point function

$$\langle \mathcal{O}_{\lambda_i} \mathcal{O}_{\lambda_j} \mathcal{O}_{\lambda_k} \rangle.$$

Since this three-point function is also the inner product of the product of the first two local operators with the third, we see

$$\omega_{ijk} = \sum_r c_{ij}^r \eta_{rk}.$$

Since η_{ij} is non-degenerate, this implies that the two- and three-point correlation functions determine the ring structure of the chiral ring.

§15.7. Calculations for $\mathbf{CP}^{n-1}$

We will now consider an example. We do the computation for $\mathbf{CP}^{n-1}$ by using a gauged linear sigma model with $U(1)$ gauge group and linear bosonic fields $\phi_1, \dots, \phi_n$ as in Lecture 13, §13.4 (the fields were called A_i in that lecture). To get all of $\mathbf{CP}^{n-1}$ we set the superpotential W equal to zero. We get a family of theories depending on a complex parameter $t = -ir + \frac{\theta}{2\pi}$, and as we have seen, for $\mathrm{Im}(t) \ll 0$ these are non-linear sigma models on X with some Kähler metric depending on t and Kähler class roughly proportional to $\mathrm{Im}(t)$. The only observable is $\mathcal{O}_H$, where H is the hyperplane section. We call this operator σ. Classically, in $\mathbf{CP}^{n-1}$ we have one relation, namely $\sigma^n = 0$. Let us see what happens quantum mechanically. We need to compute the three-point function

$$\langle \sigma^a \sigma^b \sigma^c \rangle$$

where $a + b + c = n - 1 + dn = \dim \mathcal{M}_d$, the dimension of the moduli space of rational curves of degree d. Clearly, it suffices to consider the case when $a, b, c < n$. This implies that the only possibilities for d are $d = 0, 1$. Of course, $d = 0$ is the moduli space of constant maps and its contribution is the classical ring structure. For $d = 1$, we are considering straight lines in $\mathbf{CP}^{n-1}$. To calculate

$$\langle \sigma^{n-1}(x) \sigma^{n-1}(y) \sigma(z) \rangle$$

we must consider the space of straight lines through two points meeting a fixed hyperplane. There is exactly one such line. Since the instanton action is $e^{-2\pi i t}$, this leads to the relation

$$\langle \sigma^{n-1}(x) \sigma^{n-1}(y) \sigma(z) \rangle = e^{-2\pi i t}.$$

Unraveling the ring structure from this three-point function yields

$$\sigma^n = e^{-2\pi i t}$$

in quantum cohomology.

We can make the same calculation physically. We are considering the sigma model $\mathbf{CP}^1 \to \mathbf{CP}^{n-1}$. In Lecture 13, §13.6.2, we found that in the infrared this flows to n vacua with a mass gap (since this is a case in which $c_1(X) > 0$,

the flow is from the sigma model to the Landau–Ginzburg model). The field σ has different expectation values in each vacuum. Since the fields are massive, we simply set them equal to their expectation values. These vacuum expectation values are $\sigma = \mu e^{2\pi i(-t+k)/n}$, $k = 0, \dots, n-1$. These give the idempotents of the quantum cohomology ring, from which we deduce the same answer as before,[5] namely that the ring is generated by σ modulo the relation $\sigma^n = e^{-2\pi i t}$.

§15.8. Calculations for Fano Hypersurfaces

Now let us generalize these computations to the case of a hypersurface of degree $d < n$ in $\mathbf{CP}^{n-1}$, so-called Fano hypersurfaces.

According to [A. Collino and M. Jinzenji, *On the structure of small quantum cohomology rings for projective hypersurfaces*, preprint hep-th/9611053.] the answer, computed mathematically, is that the hyperplane section σ obeys the following relation:

$$\sigma^{d-1}\left(\sigma^{n-d} - e^{-2\pi i t}\right) = 0. \tag{15.1}$$

Let us think about how this computation fits with the fact that again this sigma model flows in the infrared to a Landau–Ginzburg model plus extra massive vacua. The roots of the equation $\sigma^{d-1}(\sigma^{n-d} - e^{-2\pi i t}) = 0$ are of two types: a root at $\sigma = 0$ of multiplicity $d-1$, and nondegenerate roots (that is, roots of multiplicity one) at nonzero σ. The roots of the equation correspond to the vacua of the quantum field theory. Nondegenerate roots correspond to massive vacua, which give idempotents of the quantum cohomology ring. In analyzing the vacuum structure of this theory in Lecture 13, §13.6.2, we found $n-d$ massive vacua at nonzero σ. These were situated at the $(n-d)^{th}$ roots of $e^{-2\pi i t}$ (times a constant whose interpretation is as in the last footnote) and give the factor $(\sigma^{n-d} - e^{-2\pi i t})$ in the equation for the quantum cohomology. The other factor σ^{d-1} is the contribution of the Landau–Ginzburg vacua at the origin. This factor is present if $d > 1$ (for $d = 1$ we are discussing a projective space of codimension one, and there is no Landau–Ginzburg vacuum), and corresponds to a root at the origin of multiplicity greater than one if $d > 2$. The reason for this restriction is clear from the physics; for $d = 2$, the central charge of the Landau–Ginzburg theory, which in general is $\widehat{c} = n(1 - 2/d)$, vanishes, so in this case the Landau–Ginzburg theory is massive and infrared trivial. For $d > 2$, the Landau–Ginzburg theory has a nontrivial critical point, which can naturally contribute a root of multiplicity greater than one. This root will be at the origin because of scale invariance. How to compute the cohomology of a Landau–Ginzburg orbifold is described in one of the homework problems; with those results, one can compute the multiplicity of the root in the present case.

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[5] Except for a power of μ which has to do with a mismatch between our physical and mathematical conventions. In Lecture 13, σ was defined as a field of dimension one and no effort was made to compare to any topological normalization. In our mathematical discussion today, σ was normalized topologically as a field related to a hyperplane section of projective space, and in particular has dimension zero. To compare the two definitions, one should, in the setting of Lecture 13, integrate out the σ field in the region $r \gg 0$ in favor of the massless fields of the sigma model. In this process, σ will turn into an operator of the low energy theory that can be derived from a two-form on $\mathbf{CP}^{n-1}$. The cohomology class of this operator will be a multiple C of the hyperplane section, and C is the factor by which the physical normalization of Lecture 13 disagrees with the topological normalization that gives $\sigma^n = e^{-2\pi i t}$.

References and cross references

A reference for much of this lecture is

E. Witten, *On the Landau–Ginzburg description of $N = 2$ minimal models*, Int. J. Mod. Phys. A **9** (1994), pp. 4783–4800.

For further computations of elliptic genera of Landau–Ginzburg and related models, see

P. Berglund and M. Henningson, *On The Elliptic Genus And Mirror Symmetry,* in Essays On Mirror Manifolds II, ed. S.-T. Yau (International Press), and references therein.

LECTURE 16
Four-dimensional gauge theories

In this lecture we will survey some of the main known, believed, and conjectured results about four-dimensional gauge theory, with or without supersymmetry.

§16.1. Gauge theory without supersymmetry

We start with no supersymmetry ($N = 0$). We consider a pure nonabelian gauge theory for a simple Lie group G, with Lagrangian

$$L = \frac{1}{4g^2} \int \operatorname{Tr} F \wedge *F.$$

We could add a topological term with the theta angle: $\frac{i\vartheta}{16\pi^2} \int \operatorname{Tr} F \wedge F$. This is a well defined, asymptotically free theory. The problem we are interested in is how this theory behaves in the infrared.

It is believed that (for all ϑ) this theory has a mass gap, and conjectured that it exhibits confinement (the area law for the Wilson loop operator, see Lecture 6). However, to justify the first and especially the second statement is still an important open problem.

Remark. As we will see, the benefit of supersymmetry is that in the supersymmetric gauge theory, both statements can be justified at least partly.

Now let us add matter. In this lecture we work in Euclidean signature. We consider first an example with the group $G = SU(N)$. The take the matter fields to consist of chiral spinors ψ_i, ψ_i^* with values in the vector representation V and the dual representation V^*, and the corresponding antichiral spinors $\overline{\psi}_i$, $\overline{\psi}_i^*$ with values in V^* and V. (Chiral or anti-chiral spinors are sections of the positive or negative spin bundles S_+ or S_- with values in the given representations.) Let the number of spinors of each kind be n_f, n_f^* (the number of flavors). We must have $n_f = n_f^*$ to avoid local anomalies (see the next lecture).

The Lagrangian for the theory with spinors (with zero bare masses) is

$$L = \frac{1}{4g^2} \int \operatorname{Tr} F \wedge *F + \frac{i\vartheta}{16\pi^2} \int \operatorname{Tr} F \wedge F + \sum_i \int (\overline{\psi}_i, D\psi_i) + (\overline{\psi}_i^*, D\psi_i^*). \tag{16.1}$$

Notes by Pavel Etingof and David Kazhdan

Remark. For $G = SU(3)$ and $n_f = 2$ or 3 this is the theory of strong interactions. In these cases we really should add small fermion bare masses if we wish to describe the strong interactions precisely.

This theory is asymptotically free if n_f is not too large. So one is interested in its infrared behavior.

The Lagrangian has a global $U(n_f) \times U(n_f)$ symmetry transforming the flavors. However, quantum mechanically, this theory has a chiral anomaly (because the Dirac operator on chiral spinors may have a nonzero index). Therefore, quantum mechanically the symmetry of the operator algebra is broken to the subgroup $H = \{(A, B) \in U(n_f) \times U(n_f) : \det(A) = \det(B)\}$ (the rest of the group is broken because it does not leave fixed the measure of path integration in the field of an instanton).

It is believed that for $n_f > 1$, the group H is spontaneously broken to the diagonal subgroup H_{diag} of $U(n_f) \times U(n_f)$, leading by Goldstone's theorem to the existence of Goldstone bosons. Note that the diagonal subgroup is the maximal subgroup that would permit bare masses for fermions. If $n_f \leq 1$, H is already diagonal, and we don't get any spontaneous symmetry breaking or Goldstone bosons. In this case it is believed that the theory has a mass gap. For $n_f > 1$, the existence of Goldstone bosons means that there can be no mass gap. In this case, it is believed that the theory is infrared free and has no other massless particles than Goldstone bosons, in which case the low energy effective theory is a sigma model in the space of vacua H/H_{diag}. If small bare masses are added for the fermions to make the model a better description of nature, then, as explored in one of the homework problems, generically a unique vacuum is selected and a mass gap is generated.

Remark. The global symmetry breaking from H to the diagonal subgroup is far more delicate than what we considered in Lecture 1: in the present case this breaking is not seen classically, since the classical space of vacua is a point. It is more similar in that respect to the chiral symmetry breaking in the two-dimensional $(\overline{\psi}\psi)^2$ model, analyzed last fall by Gross in a $1/N$ expansion.

The infrared behavior of this theory is not completely understood, but some aspects are. For example, one can sometimes prove the absence of a mass gap, in the following way. In the following, we generalize the discussion, and consider an arbitrary four-dimensional gauge theory, with any compact gauge group and fermion representation. We let H be the (anomaly-free) global symmetry group of the theory. The currents of H have the following useful properties.

Remark. The following analysis is analogous to one that we made in the two-dimensional case at the beginning of Lecture 14. Both here and in Lecture 14, the goal is to use a short distance calculation to learn something about the infrared. In Lecture 14, we inferred the value of $\widehat{c}$ from a short distance calculation; note that a nonzero value of $\widehat{c}$ implies that there is no mass gap. Here, we simply use the short distance computation of the anomaly to prove that there is no mass gap. Note that in two dimensions we used a two-point function of currents, while in four dimensions we will use a three-point function. An analogous argument in $2n$ dimensions would use an $(n+1)$-point function.

Let J^a, $a = 1, ..., \dim(H)$, be the currents for H. Consider the 3-point function $\langle J^a(x)J^b(y)J^c(z)\rangle$. Since the currents are conserved classically, we get

$$d_x\langle J^a(x)J^b(y)J^c(z)\rangle = 0 \tag{16.2}$$

if x, y, z are distinct. However, when there is an anomaly, the l.h.s. of (16.2) is a nonzero singular distribution supported on the set $x = y = z$. So let us go to momentum space and compute (16.2) to 1-loop order. Let

$$F^{abc}_{\lambda\mu\nu}(k,q,r)\delta(k+q+r) = \langle \widehat{J}^\lambda_a(k)\widehat{J}^\mu_b(q)\widehat{J}^\nu_c(r)\rangle$$

be the 3-point function of J in momentum space. Then we get (to 1-loop accuracy)

$$k_\lambda F^{abc}_{\lambda\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}q^\alpha r^\beta d^{abc}, \tag{16.3}$$

where $k + q + r = 0$ and d^{abc} are the components of an invariant cubic form on the Lie algebra of the gauge group (ε is an antisymmetrization tensor). In fact, $d^{abc} = \frac{1}{2}\mathrm{Tr}\, T^a\{T^b, T^c\}$, where the T^a are the generators of the symmetries associated with J^a, acting on the chiral fermions.

It can be proved that this formula is exact, with no higher order corrections. In fact, we have actually met the tensor d^{abc} in previous lectures (for instance, Lecture 3 and Lecture 14.1 for the two-dimensional analog) in the guise of the obstruction to gauging. Those subgroups of H that can be gauged are precisely the ones such that the d-tensor vanishes when restricted to the Lie algebra of H. The one-loop exactness of our present formula is equivalent to the one-loop exactness of the obstruction to gauging.

For our present purposes, the key point is the following. No function F that is real-analytic near $k = q = r = 0$ can obey (16.3) with nonzero d. Hence if d is not zero, F is singular near zero momentum. This implies that if $d \neq 0$, the theory has massless particles, so there is no mass gap.

Thus, in some cases we can see by a short distance calculation that a four-dimensional gauge theory does not have a mass gap.

We would like to obtain a clearer understanding of the infrared limit in the cases $d^{abc} \neq 0$. We recall from sections 4 and 5 of Lecture 2 that infrared-free critical points in four dimensions are severely constrained. Massless particles have spins 0, 1/2, or 1. In the absence of supersymmetry, one can reasonably expect that spin 0 massless particles are Goldstone bosons, and that spin 1/2 massless particles are massless because of unbroken chiral symmetries.

Let us look at the possibilities in the case when $d^{abc} \neq 0$.

Possibility 1. The theory is not free in the infrared. We will not pursue this possibility, except for a comment below about when it is expected to occur.

Possibility 2. The theory is free in the infrared. Then Green's functions can be calculated using Feynman diagrams of light fields. So there must be diagrams which contribute to the non-analyticity of F. There are two possible cases:

Case 1. There is no Goldstone boson: the field $J(x)$ does not create from the vacuum a state of one massless particle (the two-point function of J has no pole at zero momentum). Then the tensor d^{abc} can come only from loop diagrams. Such diagrams must have massless fermions inside as one-loop diagrams of bosons do not produce an anomaly. The massless fermions may not be the ones in the original Lagrangian – for instance, they may not transform in the same way under H – but they must make the same contribution to d as the fermions in the short-distance Lagrangian. This possibility was pointed out by 't Hooft in the late seventies, but examples were only found in the last few years, using supersymmetry. For more information, consult Seiberg's lectures.

Case 2. There are no massless fermions, and hence no loop contributions to d. In this case, the d tensor must come from a tree diagram. This is possible only if the two point function of J has a pole at zero momentum, corresponding to the fact that J creates from the vacuum a Goldstone boson (see Lecture 1). This state is a massless boson π of spin 0 such that $J = *d\pi$ in the infrared limit. This boson π is the Goldstone boson for the broken global H-symmetry. If there is a Goldstone boson created by J^a, and also a suitable non-zero matrix element $\langle\pi|J^bJ^c|\Omega\rangle$ for the product of two currents to create $|\pi\rangle$ from the vacuum state $|\Omega\rangle$, then one can construct a tree diagram with Goldsone boson exchange that generates the d tensor.

Remark. This has been presented in a somewhat oversimplified way. For example, one could have a hybrid of case 1 and case 2, with H spontaneously broken to a subgroup H' such that d is nonzero when restricted to the Lie algebra of H', and massless fermions in a chiral representation of H'.

Remark. Even in case 1, the two point function need not be analytic at the origin. If there is no mass gap, then the two point function of J is almost always nonanalytic at zero momentum: it has a branch cut beginning at zero momentum.

Thus the conclusion is: if the theory has no massless fermions, the H-symmetry is spontaneously broken to a group $H' \subset H$ such that $d|_{H'} = 0$.

This is what happens in nature. Namely, the vacuum is invariant only under the diagonal subgroup of H, which has $d = 0$ since the representation in which the fermions live is real for this group, and thus all chiral fields are balanced by antichiral ones. This is the most obvious subgroup of H with $d = 0$.

Remark. Case 2 is really quite dramatic in terms of the physics involved. The tree diagram in question describes the decay of the neutral pion to two photons. It was in studying this decay that S. Adler, J. Bell, and R. Jackiw discovered anomalies around 1970.

Now I will say something about the conditions under which possibility 1 – a non-free infrared limit – is expected. For illustrative purposes, go back to the case we started with – $SU(n)$ gauge theory with fermions comprising n_f copies of $V \oplus V^*$. Recall that the condition for asymptotic freedom is $n_f < 11n/3$. If n_f is small compared to $11n/3$, it is believed that the theory is infrared free with the behavior described above (spontaneous breaking of H to an anomaly-free diagonal subgroup; no massless particles except Goldstone bosons). It is also expected that it exhibits confinement for sufficiently small n_f/n. However, if n_f is close to $11n/3$, it is expected that the theory flows to a nontrivial IR stable fixed point. This is

expected because the two-loop correction to the beta function of these theories is positive, even if $n_f = 11n/3$. Hence, if the one-loop beta function is made small by taking the ratio $n_f/(11n/3)$ just slightly less than 1, then the beta function has an infrared-stable zero close to the origin and within reach of perturbation theory. Of course, given that n_f is an integer, to make $n_f/(11n/3)$ just slightly less than one we must simultaneously take n large; so this phenomenon possibly only occurs for sufficiently large n. In the analogous supersymmetric case one knows just how large n and n_f/n must be to get a non-free infrared fixed point; this is explained in Seiberg's lectures.

§16.2. $N = 1$ supersymmetric pure gauge theory

Now consider $N = 1$ supersymmetric theories. We start with "pure gauge theory". In terms of components, this theory has a gauge field A and a chiral spinor λ with values in the adjoint representation. The Lagrangian is

$$L = \frac{1}{4g^2}\int \mathrm{Tr}\, F \wedge *F + \frac{i\vartheta}{16\pi^2}\int \mathrm{Tr}\, F \wedge F + \int \mathrm{Tr}\bar{\lambda} i D\lambda.$$

The symmetry of this theory classically is the chiral $U(1)$ symmetry $\lambda \to e^{i\delta}\lambda$. This chiral symmetry, however, does not exist quantum mechanically since the Dirac operator D on spinors in the adjoint bundle may have a nontrivial index. However, one can show that this index for the sphere S^4 is always divisible by $2h$, where h is the dual Coxeter number of the Lie algebra (it equals $2hk$ for the bundle with second Chern class k). This shows that (if one is on $\mathbb{R}^4$ or S^4) the subgroup of $U(1)$ consisting of symmetries of the measure of integration is $\mathbb{Z}/2h\mathbb{Z}$. This group is a symmetry group of the theory; it acts on the quantum operator algebra.

In the infrared limit, the group $\mathbb{Z}/2h\mathbb{Z}$ could possibly be spontaneously broken to a subgroup. The smallest subgroup it can break down to is $\{-1, 1\}$, since -1 is the central element of the double cover of the Poincaré group.

The standard expectations about this theory are:

1. Mass gap.
2. Confinement.
3. Chiral symmetry breaking: $\mathbb{Z}/2h\mathbb{Z}$ is spontaneously broken to $\{1, -1\}$. There are precisely the h vacua required by this symmetry breaking.

The first statement implies that fermion masses are generated dynamically. This suggests that $\mathbb{Z}/2h\mathbb{Z}$ is broken down to a subgroup that allows bare masses for the elementary fermions. The largest such subgroup is $\mathbb{Z}/2\mathbb{Z}$. So statement 1 suggests statement 3.

It is remarkable that statement 3 about symmetry breaking can actually be checked. In order to do it, it is enough to show that the one point function $\langle \mathrm{Tr}\, \lambda\lambda \rangle$ of the operator $\mathrm{Tr}\, \lambda\lambda$ is not zero (by $\mathrm{Tr}\, \lambda\lambda$ we mean the Lorentz scalar operator obtained by combining the positive chirality components of λ using the skew form on spinors while taking the trace on the Lie algebra).

If there are precisely n vacua permuted by the broken chiral symmetry, then $\langle \mathrm{Tr}\, \lambda\lambda \rangle = Ce^{2\pi ik/h}$ in the k-th vacuum, for some constant C. For the generator of the group $\mathbb{Z}/2h\mathbb{Z}$ sends the k-th vacuum to the $k+1$-st and multiplies $\langle \mathrm{Tr}\, \lambda\lambda \rangle$ by $e^{2\pi\imath/h}$.

Now let us ask ourselves: how does the function $\langle \operatorname{Tr} \lambda\lambda \rangle$ depend on the theta-angle? Let J be the current of the $U(1)$ symmetry. Because of the chiral anomaly, dJ is nonzero and equals a multiple of the Chern-Weil form:

$$dJ = \frac{h}{8\pi^2} \operatorname{Tr} F \wedge F. \tag{16.4}$$

This implies that the rotation of λ by $e^{i\alpha}$ is equivalent to adding $2h\alpha$ to the theta-angle: $\vartheta \to \vartheta + 2h\alpha$ (i.e. these two actions of $U(1)$ on the space of theories coincide). Thus we should expect that $\langle \operatorname{Tr} \lambda\lambda \rangle = Ce^{i\vartheta/h}$, where C is real and independent of the theta-angle. Note that this function is multivalued and has h different values, corresponding to h vacua.

Since $\langle \operatorname{Tr} \lambda\lambda \rangle$ has h different values, it is convenient to consider its h-th power $\langle \operatorname{Tr} \lambda\lambda \rangle^h = C^h e^{i\vartheta}$. This is a single-valued function of ϑ.

The constant C depends only on the gauge coupling g, and is zero in perturbation theory, since perturbation theory is invariant under the full $U(1)$ R-symmetry, and not just the anomaly-free $\mathbb{Z}/2h\mathbb{Z}$. Thus, if $C \neq 0$, it is a nonperturbative phenomenon (like in the Gross-Neveu model, see Gross' lecture 4).

So far all we have done is to explain what will have to happen if the $\mathbb{Z}/2h\mathbb{Z}$ chiral symmetry is spontaneously broken. Now let us compute $\langle \operatorname{Tr} \lambda\lambda \rangle^h$. We have

$$\langle \operatorname{Tr} \lambda\lambda \rangle^h = \lim_{|x_i - x_j| \to \infty} \langle \operatorname{Tr} \lambda\lambda(x_1) \dots \operatorname{Tr} \lambda\lambda(x_n) \rangle \tag{16.5}$$

(cluster decomposition). This statement is true in all the vacua.

We would like to show that $\langle \operatorname{Tr} \lambda\lambda(x_1) ... \operatorname{Tr} \lambda\lambda(x_n) \rangle$ is independent of $x_1, \dots, x_n$. To do this, recall the superspace formulation of our model (see the Superhomework and Supersolutions).

In the superspace formulation, we describe this theory on the super-spacetime $\mathbb{R}^{4,4}$ with coordinates $x_0, \dots, x_3$ and $\theta_+, \theta_-, \overline{\theta}_+, \overline{\theta}_-$. The Lagrangian of our model is equivalent to the Lagrangian

$$L' = \int d^4x (d^2\theta \tau \operatorname{Tr} WW + d^2\overline{\theta}\overline{\tau} \operatorname{Tr} \overline{W}\,\overline{W}),$$

where $\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$, and W is the supercurvature: $W = \lambda + \Theta F_+ + \dots$, $\overline{W} = \overline{\lambda} + \Theta F_- + \dots$. Here Θ is the canonical linear chiral function on $\mathbb{R}^{4,4}$ with values in S_+, F_+ is the self-dual part of the curvature, and $\operatorname{Tr} W^2$ is a scalar obtained by contracting the spinor W using the skew form on spinors. The terms ΘF_+ in the formula for W involves the $SU(2)$-map $\mathbb{C}^2 \otimes \mathbb{C}^3 \to \mathbb{C}^2$.

The supercurvature is a spinor-valued chiral superfield. Namely, let $\overline{D}$ be the spinor valued operator which annihilates chiral functions (we have $[\overline{D}, \overline{D}] = 0$). Then $\overline{D}W = 0$. This implies that $\overline{D}(\operatorname{Tr} WW) = 0$. Since $d = [D, \overline{D}]$, we get $d(\operatorname{Tr} W^2) = \overline{D}D(\operatorname{Tr} W^2)$.

Now recall that for any operator X the 1-point function $\langle \overline{D}X \rangle$ is zero. (This statement, which reflects $\overline{D}$ invariance of the path integral measure, is hopefully familiar from Lectures 14 and 15.) Using $\overline{D}(\operatorname{Tr} W^2) = 0$ and $d(\operatorname{Tr} W^2) = \overline{D}(\dots)$, this implies that $\langle d(\operatorname{Tr} WW(x_1)) \operatorname{Tr} WW(x_1) \dots \operatorname{Tr} WW(x_n) \rangle = 0$. After setting $\Theta = 0$, this implies that $\langle \operatorname{Tr} \lambda\lambda(x_1) \dots \operatorname{Tr} \lambda\lambda(x_n) \rangle$ is independent of $x_1, \dots, x_n$.

This implies that we can compute $\langle(\mathrm{Tr}\,\lambda\lambda)^h\rangle$ at short distances, that is, by assuming that all separations between the x_i are small compared to the length scale introduced in renormalizing the theory (this length scale is $1/\mu$, where μ is the mass scale introduced in renormalization). At short distances, $\langle(\mathrm{Tr}\,\lambda\lambda)^h\rangle$ can be computed using asymptotic freedom. On the other hand, because of (16.5), given that the h-point function of $\mathrm{Tr}\,\lambda\lambda$ is non-zero, a non-zero value at short distances implies that $\langle\mathrm{Tr}\,\lambda\lambda\rangle$ is nonzero. Once one has a single vacuum with that property, there must be at least h of them permuted by $\mathbb{Z}/2h\mathbb{Z}$. This is the structure that we are aiming to establish.

Our parameters are μ (the scale of renormalization), g (the gauge coupling), and ϑ (the theta angle). Thus $\langle\mathrm{Tr}\,\lambda\lambda\rangle = F(\mu, g, \vartheta)$. From dimensional arguments, $F(\mu, g, \vartheta) = \mu^{3h} f(g, \vartheta)$, where f is dimensionless (fermions have scaling dimension $3/2$ in 4 dimensions). Also, we found that $\langle\mathrm{Tr}\,\lambda\lambda\rangle^h$ has the form $Ce^{i\vartheta}$, where C is independent of ϑ. This implies that $f(g, \vartheta) = \rho(g)e^{i\vartheta}$.

Now comes the main point. A key property of our theory, which follows from supersymmetry, is that if X is an operator such that $\overline{D}X = 0$ then $\langle X\rangle$ is a holomorphic function of τ, where τ is the "modular" parameter in the superspace formulation. This follows from the fact that $\frac{\partial L'}{\partial\overline{\tau}} = \int d^4x d^2\overline{\theta}\,\mathrm{Tr}\,\overline{W}^2 = \overline{D}Y$, where $Y = \overline{D}\mathrm{Tr}\,\overline{W}^2$. But for such a Y and for X chiral, $\langle Y\,X\rangle = \langle\overline{D}(\overline{D}\mathrm{Tr}\,\overline{W}^2 \cdot X\rangle = 0$. Therefore, F is holomorphic in τ. This implies that $f(g) = e^{-8\pi^2/g^2}$, i.e.

$$F(\mu, g, \vartheta) = C_0 \exp\left(i\vartheta - \frac{8\pi^2}{g^2}\right). \tag{16.6}$$

where C_0 is a constant.

The constant C_0 is tricky to find. To find C_0, one should notice that the only nontrivial contributions to $\langle\mathrm{Tr}\,\lambda\lambda\rangle^h$ can be from field configurations with instanton number (i.e., second Chern class) equal to 1. The reason that C_0 can come only from instanton number 1 is that $\langle\mathrm{Tr}\,\lambda\lambda\rangle^h$ has charge $2h$ under the R-symmetry, so the index of the Dirac operator must be $2h$. An attractive further check is that $8\pi^2/g^2$ is the instanton action, which explains why the g dependence is $\exp(-8\pi^2/g^2)$. Thus, the constant C_0 can be found by doing perturbation theory in the vicinity of a single instanton. In particular, it can be shown by a somewhat complicated computation that $C_0 \neq 0$ (dynamical symmetry breaking).

Remark 1. Since $\overline{W} = \overline{\lambda} + \Theta F_- + \ldots$, and since instantons are defined by having self-dual curvature and $\lambda = 0$, we have $\overline{W} = 0$ for instantons and hence the Lagrangian is holomorphic in τ at the instantons (and only at them). This is an indication of why the function F should receive its contribution from instantons.

Remark 2. The operator F has zero anomalous dimension, since, for example, one of its components is the density that integrates to the second Chern class of the gauge bundle; this is an integer and not subject to quantum renormalization. Being the expectation value of a gauge-invariant operator, F obeys the renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)F = 0, \tag{16.7}$$

where β is the beta-function. A priori we know only that $\beta(g) = -3hg^3 + O(g^5)$. Formula (16.6) implies that the 1-loop beta function is in fact exact:

$$\beta(g) = -3hg^3. \tag{16.8}$$

This amounts to saying that the parameter τ has only additive renormalization. In perturbation theory, this is in any case evident from holomorphy: renormalization of τ by a non-constant power series in g^2 would certainly violate holomorphy. The additive renormalization of τ takes the form $\tau(\mu) = \tau(\mu_0) + 48\pi^2 h \ln(\mu/\mu_0)$.

§16.3. $N = 1$ theories with chiral superfields

Now let us consider the same theory with an adjoint-valued chiral superfield Φ. The new Lagrangian is

$$L'' = L' + \int d^4x d^4\theta \,\mathrm{Tr}\,\overline{\Phi}\Phi + (\varepsilon \int d^4x d^2\theta \mathrm{Tr}\,\Phi^2 + c.c). \tag{16.9}$$

(c.c. is "complex conjugate".) Here ε is a complex number which is a mass parameter for Φ. If ε is very large, we recover the previous theory as an effective theory after integrating out Φ. If $\varepsilon = 0$ we get $N = 2$ super Yang-Mills, which is intimately related to Donaldson theory.

Remark. It can be deduced from supersymmetry, modulo standard assumptions, that the case of any nonzero ε, no matter how small, is the same as the case of large ε in terms of the qualitative behavior of the theory in the infrared. This is proved by redefining Φ by $\Phi \to \Phi/\sqrt{\varepsilon}$. This operation removes ε from the superpotential. The ε dependence appears now only in the kinetic energy of Φ, which is a term of the form $\int d^4\theta(\dots)$. Such terms are believed to be universally irrelevant in the infrared behavior of a supersymmetric theory, so one expects the qualitative behavior of this theory to be independent of ε.

The remark implies that for $G = SU(2)$, this theory, with $\varepsilon \neq 0$, has two vacua (as in the theory without the superfield Φ). This is shown simply by going to large ε, where Φ is extremely massive and irrelevant. Also, we will show in Lecture 18 that this theory, with small, nonzero ε, has a mass gap and confinement. Hence, by the above remark, these properties hold for $\varepsilon \to \infty$ and thus for the pure $N = 1$ supersymmetric gauge theory.

For $\varepsilon = 0$, the behavior is completely different. There is a one complex parameter family of vacua, parametrized by $u = \mathrm{Tr}\,\Phi^2$, as we will discuss next week.

Now we will consider an example of a very different kind. We specialize to $G = SU(n)$ and discuss $N = 1$ supersymmetric QCD. This means that we omit the adjoint-valued chiral superfield, and consider instead chiral superfields Φ in the representation $R = V^{\oplus n_f} \oplus V^{*\oplus n_f}$, where V is the vector representation of $SU(n)$. (This is the example with which we started the lecture, and as before, the multiplicity of V and V^* must be the same to avoid anomalies.) We write the superfields as $\Phi = (Q_1, \dots, Q_k, Q_1^*, \dots, Q_k^*)$, where $k = n_f$. This theory is asymptotically free if and only if $k < 3n$, since $\beta = -\frac{g^3}{\pi}(3n - k)$.

The dynamical behavior of this theory is very nontrivial. For fuller details see [II-**N=1 Dynamics**]. The following picture is believed to hold:

1. $k = 0$ – discussed above. There is a mass gap.
2. $k = 1, \dots, n-1$ – discussed below. The low energy theory is a supersymmetric sigma model with a nonzero superpotential. There is no stable vacuum at all unless bare masses are added for the chiral multiplets.

3. $k = n, n+1$ – The theory is infrared free and has an interesting vacuum with more, or at least different, massless fields than one can anticipate classically.
4. $n + 2 \leq k < 3n/2$ – The theory is infrared free, and is described in the infrared by a gauge theory, but the IR gauge group is not a subgroup of the UV gauge group.
5. $3n/2 \leq k < 3n$ – the infrared limit is nontrivial and non-free.

Today we only discuss the case $1 \leq k \leq n-1$. We will try to explain why the superpotential is nonzero.

Let us compute the classical moduli space of vacua. According to lecture 2, this moduli space should be the symplectic quotient: $\mathbf{M} = R//G$ (where as before R is the chiral superfield representation). Here the symplectic structure on R is induced by the Hermitian form on R. Since G acts holomorphically, we have $R//G = R/G_{\mathbb{C}}$. So we are down to classical invariant theory.

For $k < n$ all invariants of $SL(n)$ in R are functions of the inner products $M_{ij} = (Q_i, Q_j^*)$. These invariants are independent, and form a coordinate system on $R/G_{\mathbb{C}}$. Moreover, let us compute the stabilizer of a point with $\det(M) \neq 0$. It is easy to see that this stabilizer is isomorphic to $SL(n-k)$. Thus, we have a Higgs phenomenon: the $SU(n)$ gauge symmetry is spontaneously broken to $SU(n-k)$ (see §2.2 of Lecture 2). In particular, for $k = n-1$ it is broken completely. This says that for $k < n-1$ the classical theory has massless gauge fields of $SU(n-k)$ as well as massless chiral multiplets parametrizing the symplectic quotient. Quantum mechanically the story will be more complicated for several reasons.

In particular, for $n - k \geq 2$, since the stabilizer of a point in $\mathbf{M}$ is $SU(n-k)$, we should expect that the low energy effective theory is an $N = 1$ supersymmetric $SU(n-k)$-gauge theory (plus light chiral superfields that are neutral under $SU(n-k)$). As we discussed in section 2 of today's lecture, it is believed that this theory has $n-k$ vacua, so we should expect that there are $n-k$ quantum vacua for every classical vacuum. In other words, the quantum vacuum space will be an $(n-k)$-fold cover of $\mathbf{M}$. Call this cover $\widetilde{\mathbf{M}}$. Thus the low energy effective theory involves a sigma model into $\widetilde{\mathbf{M}}$. Such a supersymmetric sigma model can possibly admit a superpotential. It is shown similarly to the above arguments that this superpotential W must be holomorphic on $\widetilde{\mathbf{M}}$ and as a function of the complexified gauge coupling τ. The superpotential should of course have the same symmetries as the theory itself.

Classically our theory has the $U(k) \times U(k)$ global symmetry acting on the superfield, as well as the R-symmetry, which is the $U(1)$ symmetry rotating λ. Some of these symmetries are anomalous (chiral anomaly). The anomaly free subgroup which acts on the quantum theory is generated by the group $H = \{(A, B) \in U(k) \times U(k); \det(A) = \det(B)\}$ and the group $U(1) = \{(\zeta, \zeta^{-(n-k)/n}, \zeta^{(n-k)/n}) \in U(1) \times U(k) \times U(k)\}$. From these symmetries (and remembering from the first part of the lecture how the R-symmetry acts on the theta angle), we see that the superpotential must equal

$$W(M) = c_{n,k}(\det M)^{\frac{-1}{n-k}}. \tag{16.10}$$

So we do get a function on an $n-k$-fold cover $\widetilde{\mathbf{M}}$ of $\mathbf{M}$, as expected.

Now we need to compute $c_{n,k}$ and in particular to see if it is zero. The key fact for doing this is: If $c_{n,k} \neq 0$ then $c_{n-1,k-1}$ is nonzero, and one can be determined

from the other. Indeed, if in the n, k theory, we take M which has one very large eigenvalue, the gauge group is spontaneously broken from $SU(n)$ to $SU(n-1)$, some chiral superfields get mass, and we are effective reduced to the theory with n, k replaced by $n-1, k-1$.

Thus if $k < n-1$, we can descend from (n, k) to $(n-k, 0)$ and thus reduce the problem to the case of pure $N = 1$ gauge theory. In pure gauge theory, the space $\mathbf{M}$ is a union of $n-k$ points, and the role of the superpotential W is played by the object $\langle \mathrm{Tr}\, \lambda\lambda \rangle$ that we have computed. Thus, $c_{n-k,0} \neq 0$.

Remark. It might seem that the superpotential is defined up to adding a locally constant function. However, it is in fact defined up to adding only a constant function. This is because in the case when the space is compact, there is only one vacuum and so the h points of the moduli space "know" about each other; it is possible to interpolate between them (through a family of physical states that are not vacua) and compare the values of the superpotential for the different vacua.

Now let $k = n-1$. Then the descent procedure will stop at $n = 2, k = 1$. In the case $n = 2$, $k = 1$ one has $W = c_{2,1}(\det(M))^{-1}$. More precisely, by dimension arguments $c_{2,1} = \widetilde{c}_{2,1}\mu^5$, where μ is the renormalization scale of the theory and $\widetilde{c}_{2,1}$ is dimensionless. It is shown as for the discussion of $\langle \mathrm{Tr}\, \lambda\lambda \rangle$ in the pure gauge theory that instantons which contribute to $c_{2,1}$ have instanton number 1. Thus, by holomorphy of W with respect to τ, we have

$$W = \widetilde{c}_{2,1}\mu^5 e^{i\vartheta - 8\pi^2/g^2}/\det(M). \tag{16.11}$$

The computation of $\widetilde{c}_{2,1}$ is tricky. The reason is that there are instantons of all sizes, which makes it more difficult to do perturbation theory around the instantons.

The computation was done by Affleck, Dine, and Seiberg. They showed that $c_{2,1}$ is not zero. Thus $c_{n,n-1}$ is not zero.

I refer to Seiberg's lectures for more on the above and an explanation of what happens if $k \geq n$.

References and cross references

For a general discussion of infrared limits of four dimensional field theories, see Lecture 2. The lagrangian of $N = 1$ gauge theory in four dimensions is discussed in Chapter 8 of [I-**Supersolutions**]. The discussion in §16.3 serves as an introduction to Seiberg's lectures [II-**N=1 Dynamics**], where these theories are explained in detail and more references can be found.

LECTURE 17

$N = 2$ Supersymmetric Yang-Mills Theories in Dimension Four: Part 1

§17.1. Introduction

Today we shall begin our study of the dynamics of $N = 2$ supersymmetric Yang-Mills theories in four dimensions. The underlying manifold on which we shall study these theories, today and next week, will be Minkowski four-space $M^4 = \mathbf{R}^{1,3}$, with coordinates x_0, x_1, x_2, x_3 and metric $dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$. We are mainly interested in the case that the gauge group G is $SU(2)$. Since (after spontaneous symmetry breaking) low energy limits of $SU(2)$ theories can generate $U(1)$ theories, we shall begin by studying $U(1)$ gauge theory. One should note that these $U(1)$ theories are purely unrenormalizable (all terms are unrenormalizable) in dimension four. Thus, they don't make sense as fundamental theories. But they are free in the infrared, so they make sense as infrared limits, and their infrared behavior can be studied semiclassically via an effective lagrangian. Of course, when we get to the $SU(2)$ theory, we will be dealing with an asymptotically free, renormalizable theory which has strong quantum effects. This is then a reasonable fundamental theory, but one whose low energy limit is hard to understand.

In Lecture 19, we shall use the analysis of the low energy behavior of these gauge theories on Minkowski space as a tool in the analysis of their behavior on compact Riemannian four-manifolds. We will define certain correlation functions associated to cycles on the four-manifold. These particular correlation functions can be computed at short distances or long distances because they are invariant under changes in the metric. Computing in the high energy limit, one identifies these correlation functions with certain integrals over the moduli space of classical solutions. These integrals are exactly the ones used to define the Donaldson invariants of the four-manifold. Going to the low energy limit, we get a completely different method of computing the same correlation functions. In this way we obtain totally different expressions for the Donaldson invariants of a four-manifold this time in terms of integrals over the Seiberg-Witten moduli spaces. This then produces the link between the Donaldson polynomial invariants and the Seiberg-Witten $U(1)$

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monopole invariants (and a generalization of the relation between these invariants when $b_2^+ = 1$).

§17.2. Low energy $U(1)$ $N = 2$ super Yang-Mills theories

§17.2.1. The basic fields

Consider pure $N = 2$ super Yang-Mills theory in four dimensions with gauge group $U(1)$. This means that the fields in the Lagrangian will form a single $N = 2$ vector supermultiplet: the gauge field and its $N = 2$ supersymmetric partners. All the fields will be massless at the classical level, prior to possible gauge symmetry breaking. Let us recall from Problem ASH5 of [I-**Homework**] what the $N = 2$ massless vector supermultiplet looks like in dimension 4. Massless particles are associated to representations of the "little" subgroup Spin(2) in the Lorentz group Spin(1, 3) fixing a nontrivial momentum vector of zero norm. The "helicity" of such a particle (i.e., its spin around its direction of motion) is the representation of its stabilizer. These are indexed so that the fundamental representation of Spin(2) is labeled 1/2. The irreducible representations of the Poincaré group that comprise the $N = 2$ vector supermultiplet in dimension four are arrayed as pictured below:

helicity	-1	$-1/2$	0	$\oplus$	0	$1/2$	1
multiplicity of representation	1	2	1		1	2	1.

The action of CPT conjugation on these representations is to interchange the right and left halves of this table. This multiplet has a $U(2)_R$ symmetry, acting so as to preserve helicity. Thus, the $SU(2)_R \subset U(2)_R$ acts trivially except on those particles of helicities $\pm 1/2$. On each of these it acts via the fundamental two-dimensional representation. The $U(1)_R$ acts trivially on the states of helicity ± 1 and acts by (two copies of) the standard representation, and its inverse, on the states of helicity $\pm 1/2$ and by the ± 2 power of the fundamental representation on the states of helicity zero. The CPT involution normalizes the $U(1)_R$ and acts on it by conjugation. For more on all this, see the addendum to the superhomework in [I-**Homework**].

The fields of an $N = 2$ vector supermultiplet on $N = 1$ superspace in dimension four decompose into an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet. The $N = 1$ vector supermultiplet is the gauge $N = 1$ supermultiplet $\mathcal{A}$. The field $\mathcal{A}$ is the superconnection and (after partial gauge fixing to Wess-Zumino gauge) consists of a triple

$$\mathcal{A} = (A, \lambda, D)$$

where A is the usual $U(1)$ gauge field, λ is a fermion field, and D is an auxiliary field. The curvature of $\mathcal{A}$ is a two-form on $N = 1$ superspace. The entire curvature form is determined by certain components W_α which are $(\frac{1}{2}, 0)$ spinor chiral superfields. (Finite-dimensional irreducible representations of $\mathrm{Spin}(1,3) = \mathrm{SL}(2, \mathbf{C})$ are indexed by an ordered pair (a, b) of half-integers. The pair $(1/2, 0)$ indexes the defining two-dimensional complex representation. It is chiral because the second half-integer is zero.) The kinetic energy of the super-connection $\mathcal{A}$ is an expression quadratic in the field W_α. The other fields form an $N = 1$ supermultiplet called Φ. This is a (scalar) chiral supermultiplet. It consists of

$$\Phi = (\phi, \psi, F)$$

where ϕ is a complex-valued scalar field, ψ is a fermion, and F is an auxiliary field. The auxiliary fields D and F enter the action quadratically and without derivatives so that they can be integrated out using the equations of motion. Thus, they play no role in the analysis, and we will ignore them.

§17.2.2. The Lagrangian

Now we are ready to write the Lagrangian for the $N = 2$ pure $U(1)$ gauge theory in four dimensions. Though we impose $N = 2$ supersymmetry, we choose to write the lagrangian first as a lagrangian for fields on $N = 1$ superspace and then study the consequences of the extra supersymmetry. The general $N = 1$ supersymmetric lagrangian that we will consider is of the form:

$$\mathcal{L} = \left(\frac{i}{4\pi} \int d^4x d^2\theta \tau(\Phi) W_\alpha W^\alpha + \text{c.c.} \right) + \frac{1}{4\pi} \int d^4x d^4\theta K(\Phi, \overline{\Phi}). \tag{17.1}$$

As we remarked above, W is a piece of the curvature of the $N = 1$ superconnection $\mathcal{A}$. It is a chiral superfield. The function $\tau(\Phi)$ is a holomorphic function on the complex plane (or more generally the complex curve) where Φ takes its values. Since Φ is a chiral superfield, $\tau(\Phi)$ is also such a chiral superfield. We write

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

with θ and e real. Lastly, $K(\Phi, \overline{\Phi})$ is a Kähler potential on Φ-space.

We have encountered such lagrangians before (in Section 4 of Lecture 13), with one essential difference. That difference is that here we are not assuming that τ is constant. Rather we allow it to be a holomorphic function of Φ. Briefly, the reason that we consider this generalization is that this theory will arise as a low energy limit of some $SU(2)$ theory. That $SU(2)$ theory will have a complex plane of classical vacua (i.e., the space of minima of the potential energy will be $\mathbf{C}$). We find terms in the low energy limit which involve functions on this space of classical vacua. It is in this way that the non-constant holomorphic function τ arises. The main application we have in mind for this study is to the low energy limits of $N = 2$ supersymmetric pure $SU(2)$ gauge theory. Of course, in the low energy limit Φ will be slowly varying, i.e., almost constant. To the extent that Φ is almost constant the same will be true of τ, and we find ourselves expanding around a low energy effective theory with constant τ_{eff}.

We have omitted a superpotential term in writing the $N = 1$ lagrangian. The reason is that while one can have such a term in an $N = 1$ supersymmetric theory on $N = 1$ superspace in four dimensions, the extra conditions imposed by requiring the theory to have $N = 2$ supersymmetry imply that this term must be zero. We also omitted terms with more than the minimum number of derivatives; they are irrelevant in the infrared relative to the interactions we have kept, and will not affect the analysis. We now must specialize the above $N = 1$ supersymmetric lagrangian to the case that one actually has $N = 2$ supersymmetry. $N = 2$ supersymmetry imposes a relation between τ and K. Namely, with an appropriate choice of coordinates on the curve in which Φ takes values (that is after a holomorphic change of variables, $\Phi \to \widetilde{\Phi}(\Phi)$), one has $\text{Im}\,\tau = K_{\phi\bar{\phi}}$. To see this, let us write out the

lagrangian (17.1) in components. We get

$$\mathcal{L} = \frac{1}{4\pi}\int_{M^4} d^4x \Big(i\left[\overline{\tau}(F^+\wedge F^+) - \tau(F^-\wedge F^-)\right] + K_{\phi\overline{\phi}}\partial\phi\overline{\partial\phi} + (\mathrm{Im}\,\tau)\overline{\lambda}i\not{\partial}_A\lambda + K_{\phi\overline{\phi}}\overline{\psi}i\not{\partial}_A\psi\Big) \tag{17.2}$$

One thing that is clear from this formula, and which we have seen before, is that the lagrangian depends not on the Kähler potential K but only on the Kähler metric $K_{\phi\overline{\phi}}\partial\phi\overline{\partial\phi}$. To see the relation of τ and K we ask for $SU(2)_R$-symmetry rotating λ and ψ. Such symmetry will imply $N = 2$ supersymmetry, because commuting the $N = 1$ supersymmetry with $SU(2)_R$ will generate a second supersymmetry. Conversely, it can be shown that $N = 2$ supersymmetry for this class of models implies $SU(2)_R$ symmetry. We will not prove this last assertion here, but note that in any event the $N = 2$ supersymmetric $SU(2)$ gauge theory has $SU(2)_R$ symmetry, so all theories arising as its low energy limits have this property.

Notice that if $K_{\phi\overline{\phi}} = \mathrm{Im}\,\tau$, then λ and ψ have the same kinetic energy. So one can define an $SU(2)_R$ symmetry under which (λ,ψ) transform in the standard representation, while the bosonic fields are invariant. Conversely, if there is an $SU(2)_R$-symmetry, then with an appropriate holomorphic redefinition of the supermultiplet Φ, the fermions (λ,ψ) form a standard multiplet for the $SU(2)_R$ action. Thus, in these coordinates the kinetic terms for λ and ψ must have the same coefficient, and hence with this choice of coordinates we have $\mathrm{Im}\,\tau = K_{\phi\overline{\phi}}$.

Since the kinetic energy of all fields (including the gauge fields once we separate the $F\wedge *F$ and $F\wedge F$ terms in (17.2)) is $\mathrm{Im}\,\tau$, we require $\mathrm{Im}\,\tau > 0$. This seems like a problematic condition. Locally it is not hard to ensure that it holds, but for a global holomorphic function on all of $\mathbf{C}$ it cannot be achieved unless τ is constant. There are two possibilities to contemplate. One is that as τ approaches the real axis, for some reason the physics breaks down. (Another way to say that is that perhaps Φ takes values not in a possibly incomplete algebraic curve, but in something like the upper half plane that admits a holomorphic function τ with positive imaginary part.) One would then be faced with studying what happens at the "bad" limits where $\mathrm{Im}\,\tau \to 0$. Another possibility is that τ is multi-valued. After all, we already know that the transformation $\tau \to \tau + 1$ (i.e., increasing θ by 2π) does not affect the theory. So at least to that extent one must consider τ as multi-valued. Of course, this does not solve the problem, since any non-constant, entire holomorphic function τ to $\mathbf{C}/\{\tau \cong \tau + 1\}$ still cannot have $\mathrm{Im}\,\tau > 0$. To arrange this we need a larger group of indeterminacy for the values of τ. For example, recall that it is easy to construct global holomorphic functions into the upper half-plane if we permit them to be multi-valued, say under the standard action of $SL(2,\mathbf{Z})$. As we shall see, this is exactly what happens in this case.

§17.2.3. Indeterminacy of τ

In order to make the notation consistent with that of the Seiberg-Witten paper (see references at the end of the lecture), we set $\phi = a$. The term in the lagrangian derived from the Kähler potential becomes

$$-\frac{i}{4\pi}\int_{M^4} d^4x(\tau - \overline{\tau})\frac{\partial a}{\partial x^i}\frac{\partial\overline{a}}{\partial\overline{x}^i},$$

which means that the induced Kähler metric on the Φ-plane, now called U, is

$$ds^2 = -\frac{i}{4\pi}(\tau\, da d\bar{a} - da\bar{\tau} d\bar{a}),$$

where a is a local holomorphic coordinate on U. It is convenient to introduce another holomorphic function a_D (called *the dual of* a) locally on an open subset V of U with the property that $da_D = \tau da$. Given this, we can re-write the metric on V as

$$ds^2 = -\frac{i}{4\pi}(da_D d\bar{a} - da d\bar{a}_D).$$

We denote by Ω the Kähler form of this metric:

$$\Omega = \frac{-1}{8\pi}\left(da_D \wedge d\bar{a} - da \wedge d\bar{a}_D\right).$$

We can view this more symmetrically in a_D and a by forming the complex two-space with coordinates (a_D, a). Then there is an embedding of the complex curve $f\colon V \to \mathbf{C}^2$ into this complex two-space so that the image is a solution curve for the differential equation $da_D = \tau da$. This embedding will be the graph of either of the variables as a function of the other. Let $\omega = \frac{-1}{8\pi}\left(da \wedge d\bar{a}_D - da_D \wedge d\bar{a}\right)$. It is a closed, non-positive $(1,1)$-form on complex two-space. The Kähler metric on V is $\Omega = f^*\omega$. Of course, f must be such that $f^*\omega > 0$.

The question is: How much freedom do we have in the representation of the theory in terms of the parameters a_D, a (and implicitly τ)? To what extent can we find other lagrangians of this form with different a_D and a (and hence different τ) which represent the same theory? Just considering the Kähler potential term we see an obvious geometric change of coordinates that brings the lagrangian back into the same form with different values of the parameters. Namely, we can act by an extended version $ISL(2,\mathbf{R})$ of $SL(2,\mathbf{R})$. We think of $SL(2,\mathbf{R})$ as a group of automorphisms of $\mathbf{C}^2 = \mathbf{R}^2 \otimes_{\mathbf{R}} \mathbf{C}$ (here $SL(2,\mathbf{R})$ acts on $\mathbf{R}^2$ in the standard fashion). Then we form the extension

$$0 \to \mathbf{C}^2 \to ISL(2,\mathbf{R}) \to SL(2,\mathbf{R}) \to 0$$

which is a semi-direct product of $SL(2,\mathbf{R})$ with the group of translations of $\mathbf{C}^2$. An element $(M, \binom{c_D}{c})$ in this group (M being an element of $SL(2,\mathbf{R})$) acts by

$$\left(M, \begin{pmatrix} c_D \\ c \end{pmatrix}\right) \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} = M \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} + \begin{pmatrix} c_D \\ c \end{pmatrix}.$$

Since $\tau = \frac{da_D}{da}$, one sees that the action of this group on τ is the usual fractional linear transformation of M on $\tau \in \mathbf{H}$:

$$\begin{pmatrix} r & s \\ u & v \end{pmatrix} \cdot \tau = \frac{r\tau + s}{u\tau + v}.$$

In this way this change of variables brings the $d^4\theta$ term of the lagrangian back into the same form with a different a, a_D, τ.

This computation with the Kähler metric leads us to ask whether the non-uniqueness in our description of the holomorphic function τ should be that we are allowed to compose τ with any $SL(2,\mathbf{R})$ fractional linear transformation. But we must examine the other term in the lagrangian where τ appears. It appears in the term involving the gauge field (or photon). Recall from §5 of Lecture 8 what happens for constant τ. Since $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ and the action is given by $e^{i\mathcal{L}}$, if we

consider strictly upper triangular matrices the only ones that leave invariant this part of the action are those of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

with $b \in \mathbf{Z}$. (This is because a shift of θ is a symmetry only if it is of the form $\theta \to \theta + 2\pi b$ for some integer b.) This action of the strictly upper triangular matrices that are in $SL(2,\mathbf{Z})$ extends to an action of $SL(2,\mathbf{Z})$ on lagrangian descriptions of the theory. That is to say, if τ and τ' in $\mathbf{H}$ differ by the action of $SL(2,\mathbf{Z})$, then the pure $U(1)$ gauge theory with τ as parameter has another representation as a $U(1)$ gauge theory (with different basic fields) where the parameter is τ'. The transformation $\tau \to -1/\tau$ is the duality transformation in $U(1)$ gauge theory interchanging the electric and magnetic charges. The upshot of all this is that the full symmetry group acting on representations of the theory by lagrangians of the given form is

$$ISL(2,\mathbf{Z}) = \mathbf{C}^2 \ltimes SL(2,\mathbf{Z}).$$

The way this group transforms the parameter τ is through composing with the usual fractional linear transformation action of $SL(2,\mathbf{Z})$ on the upper half-plane.

We now see how to describe our low energy effective theory globally. There is a complex curve U of vacua, and on it there is a global holomorphic function $\tilde{\tau}\colon U \to \mathbf{H}/PSL(2,\mathbf{Z})$. Furthermore, for each $u \in U$ there is a neighborhood V of u, a lifting of $\tilde{\tau}$ to a holomorphic function $\tau\colon V \to \mathbf{H}$, and $N = 1$ superfields $\mathcal{A}, \Phi$ such that there is a lagrangian in the form given in Equation 17.1 (or Equation 17.2 in components) which represents the theory in V. As we pass from one of these open subsets to another, the local representation of the low energy effective theory by a lagrangian changes. The function τ changes by the natural action of $SL(2,\mathbf{Z})$ on $\mathbf{H}$. The chiral superfield changes by the action of $ISL(2,\mathbf{Z})$ described above, and the gauge superfield $\mathcal{A}$ changes by the duality action of $SL(2,\mathbf{Z})$. Because there are global holomorphic functions to $\mathbf{H}/SL(2,\mathbf{Z})$, this is a completely consistent picture of a global theory in which $\operatorname{Im}\tau$ never goes to zero.

We are in a new situation now: in the low energy theory on the curve U, no one lagrangian is good everywhere. We must use different lagrangians in different regions of U. The only known way to find a single lagrangian that gives a valid description everywhere is to go back to the original $SU(2)$ theory.

§17.2.4. The family of elliptic curves associated to τ

We have now established all the relevant general features of the low energy effective theories we want to study. We have an $N = 2$ supersymmetric pure $U(1)$ gauge theory with a family of vacua parameterized by a complex curve U, where the scalar field $\phi = a$ takes its values. This curve has an open covering $\{V^\alpha\}$ and on each open subset there is a pair of dual holomorphic coordinates a_D^α, a^α so that $\tau^\alpha = da_D^\alpha/da^\alpha$ is the τ-parameter appearing in the lagrangian. The different local representations of the theory differ by automorphisms in $ISL(2,\mathbf{Z})$. We have just seen that the local holomorphic function τ on U which appears in the lagrangian is in fact a global function of U to $\mathbf{H}/SL(2,\mathbf{Z})$. Thus, for each $u \in U$ it is natural to consider $\tau(u)$ as determining the j-invariant of an elliptic curve $E(u)$. In other words, $\tau(u)$ determines $E(u)$ up to isomorphism. Of course, it is possible that τ goes off to infinity at a discrete set of points $\{p_i\}$ in U, or equivalently that the

elliptic curves $E(u)$ develop a node as $u \to p_i$. At these points the physics changes and there are more massless fields, fields that are massive at nearby $u \in U$.

There is a natural way to fit these elliptic curves together. Namely, there is a complex analytic surface $\mathcal{E}$ together with a proper holomorphic map $\mathcal{E} \to U$ whose fibers form the family of elliptic curves $\{E(u)\}_{u \in U}$. The simplest way to construct such a family is to take the Weierstrass form. That is to say, there are functions A, B on U such that $\mathcal{E}$ is defined by

$$Y^2 = X^3 + AX + B,$$

with an appropriate completion of the missing point at infinity in each elliptic curve. (For globally non-trivial bases U, we will have a line bundle L over U and A and B will be sections of $L^{\otimes 4}$ and $L^{\otimes 6}$, respectively.) This family of elliptic curves has a natural section (the section at infinity in the Weierstrass description), and thus it can be viewed as a family of one-dimensional Abelian varieties. There are, of course, other families of holomorphically varying genus-one curves over the same base with the same j-invariant function but without a section. But given the information of the j-invariant function only, there is no distinguished family except the one with a section.

Suppose that we have a family of elliptic curves $\mathcal{E} \to U$ with a section σ. Furthermore, suppose that we have an open covering $\{V^\alpha\}_\alpha$ of U such that on each V^α there are a pair of dual holomorphic coordinates a^α, a^α_D such that $\tau^\alpha = da^\alpha_D/da^\alpha$ is a function to the upper half-plane. We suppose that these two sets of data are related as above. That is to say, $\mathcal{E}|_{V^\alpha}$ is isomorphic to the family $V^\alpha \times \mathbf{C}/(1, \tau^\alpha(v))$. This allows us to define one-cycles $E^\alpha_1(u), E^\alpha_2(u) \subset E(u)$ for all $u \in V^\alpha$ as the images of the arcs $\{u\} \times [0,1]$ and $\{u\} \times [0, \tau^\alpha(u)]$ in $\{u\} \times \mathbf{C}$. Then, there is a differential of the second kind λ^α (i.e., a meromorphic one-form with trivial residue along its polar locus) on $\mathcal{E}|_{V^\alpha}$ such that

$$\int_{E^\alpha_1(u)} \lambda^\alpha = a^\alpha(u)$$

$$\int_{E^\alpha_2(u)} \lambda^\alpha = a^\alpha_D(u).$$

In general, we cannot fit these differentials λ^α of the second kind together to produce a global object. (We will see the precise conditions when this can be done soon.) But we claim that we can fit together the two-forms $d\lambda^\alpha$. Namely, we claim that we there is a unique global holomorphic two-form η on $\mathcal{E}$ such that on $\mathcal{E}|_{V^\alpha}$ we have

$$\int_{E^\alpha_1} \eta = da^\alpha$$

$$\int_{E^\alpha_2} \eta = da^\alpha_D.$$

Of course, the uniqueness assertion immediately implies that the restriction of any such global two-form η to $\mathcal{E}^\alpha$ is equal to $d\lambda^\alpha$.

Existence and uniqueness of η are seen as follows. Recall that $\mathcal{E}^\alpha$ is identified with $V^\alpha \times \mathbf{C}/(1, \tau^\alpha(u))$. On the universal covering space $V^\alpha \times \mathbf{C}$ we have the holomorphic one-form dz induced from the standard holomorphic coordinate on $\mathbf{C}$. This one-form does not descend to the quotient because τ is not a constant function

of a^α. Nevertheless, the two-form $da^\alpha \wedge dz$ does descend to give a well-defined two-form on $\mathcal{E}^\alpha$. It is clear that the only holomorphic two-form on $\mathcal{E}|_{V^\alpha}$ which satisfies the first integral equation above is

$$\eta^\alpha = -da^\alpha \wedge dz.$$

This two-form η^α also clearly satisfies the second integral equation. Now we need to see that these local two-forms fit together to give a global form η. But this is clear from the uniqueness of the local forms.

Now let us compute $\pi_*(\eta \wedge \overline{\eta})$, where π is the projection from $\mathcal{E} \to U$. Let us restrict attention to $\mathcal{E}|_{V^\alpha}$. From our local description we have

$$\pi_*(\eta \wedge \overline{\eta}) = \int_{\text{fibers}} d\lambda^\alpha \wedge d\overline{\lambda}^\alpha.$$

It follows immediately from the formulas for the integrals of $d\lambda$ over E_1^α and E_2^α and the fact that $E_1^\alpha \cdot E_2^\alpha = 1$ in the homology of $E(u)$ that the result is

$$\pi_*(\eta \wedge \overline{\eta}) = da^\alpha \wedge d\overline{a}^\alpha \overline{\tau}^\alpha + da^\alpha \wedge d\overline{a}^\alpha \tau^\alpha = (\tau^\alpha - \overline{\tau}^\alpha) da^\alpha \wedge d\overline{a}^\alpha = 8\pi\Omega|_{V^\alpha}.$$

Ω is as before the Kähler form of the metric on U.

Since the two-form η is holomorphic, it integrates trivially on any fiber and on the section σ of the elliptic fibration. Thus, its homology class is determined by the integrals of η along two-tori which lie over circles in U.

Now, we will show that the de Rham cohomology class of η determines whether or not the automorphisms required in passing between our local descriptions in terms of various sets of (a_D, a, τ) lie in $SL(2, \mathbf{Z})$ or in the extended group $ISL(2, \mathbf{Z})$. We claim that if the de Rham homology class of η is trivial, then the automorphisms we use lie in $SL(2, \mathbf{Z})$. The point is the following. We already know that the changes in da_D, da are by elements in $SL(2, \mathbf{Z})$. We need to compute the changes in a_D, a. If we go around any loop γ in U with the property that E_1 comes back to itself, then the cycles $E_1(\gamma(t))$ fit together to make a torus and the integral around this torus of η is equal to $\int_\gamma da$. If this integral is trivial, then a can be analytically continued around this loop to a single valued function. Similar arguments apply to a_D. Thus, if all the periods of η are trivial, then a, a_D transform by $SL(2, \mathbf{Z})$.

In general, the de Rham cohomology class of a holomorphic two-form such as η is trivial if and only if there is a global differential of the second kind λ on $\mathcal{E}$ with $d\lambda = \eta$. In the more general case, we can write the holomorphic two-form η as $d\lambda$ for some global differential λ of the third kind on $\mathcal{E}$ (i.e., λ is a meromorphic one-form on $\mathcal{E}$ with constant residues along its polar locus.) In turns out that in our application to pure $SU(2)$ gauge theory the automorphisms between our various descriptions lie in $SL(2, \mathbf{Z})$, so that we shall not need to consider the more general case. The proof of this fact is the subject of the next section.

§17.2.5. The BPS formalism

Our goal here is to show that if the low energy effective $U(1)$ gauge theory that we have been considering is the low energy limit of a pure $N = 2$ supersymmetric $SU(2)$ gauge theory, then the transformations between the various local descriptions in terms of (a_D, a, τ) of the $U(1)$ theory lie in $SL(2, \mathbf{Z})$. We shall establish this by using the BPS formalism. In any $U(1)$ theory we have two conserved charges, the

electric and magnetic charges:

$$Q_e = \frac{1}{4\pi}\int_{S^2} *F$$

$$Q_m = \frac{1}{4\pi}\int_{S^2} F.$$

Here, we are doing the integrals on the sphere at infinity in some time slice in Minkowski four-space. In pure $U(1)$ theory on four-space, these charges are both zero. But in the case that our $U(1)$ theory is the low energy limit of an $SU(2)$ theory these charges can be non-zero. Recall from §5 of Lecture 8 that our $SU(2)$ theory has a $U(1)$ symmetry whose conserved charge at least in the case of constant τ is

$$N_e = Q_e - \frac{\theta Q_m}{2\pi}.$$

Since it generates a $U(1)$ symmetry, this operator has integral eigenvalues. There is another conserved charge, the magnetic charge, given by

$$N_m = Q_m.$$

This is determined by the first Chern class of the line bundle at infinity and hence clearly has integral eigenvalues as well. Having integer eigenvalues, both charges are associated with $U(1)$ symmetries. Of course, as we saw in §5 of Lecture 8 these two charges are interchanged by duality.

The $N = 2$ supersymmetry generators Q^i_α, $i = 1, 2$, satisfy the relations of the super Poincaré group, which include

$$\{Q^i_\alpha, Q^j_{\dot\alpha}\} = \delta^i{}_j P_{\alpha\dot\alpha}$$

where $P_{\alpha\dot\alpha}$ is the generator of a translation in the spatial direction $\{\alpha\dot\alpha\}$. Also, we have

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon^{ij}\epsilon_{\alpha\beta} S$$

where S is the central charge of the theory; in each supermultiplet S is simply multiplication by a scalar.

In general, there is the BPS bound on the mass of any state which says

$$M \geq |S|.$$

One can often gain a good understanding of the BPS-saturated states, i.e., those for which this inequality is an equality.

In our case, we wish to compute the central charge S. To do this recall that the lagrangian (for τ constant) is:

$$\mathcal{L} = \frac{1}{4\pi}\int d^4x d^2\theta i\tau W_\alpha W^\alpha + \text{c.c.} + \int d^4x d^4\theta \text{Im}\,\tau \bar\Phi\Phi.$$

From this one knows (from §5 of Lecture 8) that for this value of τ

$$S = aN_e + a_D N_m$$

where $\Phi = a + \ldots$. On dimensional grounds, this formula holds even if τ is not constant; terms involving derivatives of a and a_D are not possible dimensionally. The interpretation is that different lagrangian descriptions of the low energy theory use different values of a_D and a. In each description S depends on the values of a_D and a by the above formula. Recall from §5 of Lecture 8 that (N_m, N_e) transform

under $SL(2,\mathbf{Z})$ as we take different representations of our theory. Since S has to be unchanged, we see that if

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto M \cdot \begin{pmatrix} a_D \\ a \end{pmatrix},$$

then

$$\begin{pmatrix} N_m & N_e \end{pmatrix} \mapsto \begin{pmatrix} N_m & N_e \end{pmatrix} \cdot M^{-1}$$

for any $M \in SL(2,\mathbf{Z})$.

The first important point to notice is that we cannot compensate for the operation of adding a constant to $\begin{pmatrix} a_D \\ a \end{pmatrix}$ by a change in (N_m, N_e), because in this theory there is nothing else (no other conserved charge) that N_m and N_e can transform into. This shows what we claimed at the end of the last section, namely, that in the $SU(2)$ gauge theory with the $N = 2$ vector multiplet only, the automorphisms between the various representations of the $U(1)$ theory in terms of (a_D, a, τ) lie in $SL(2,\mathbf{Z})$. Thus, in this case we can construct a differential λ of the second kind on $\mathcal{E}$ such that for each α we have

$$\int_{E_1} \lambda = a^\alpha$$

$$\int_{E_2} \lambda = a_D^\alpha.$$

on $\mathcal{E}|_{V^\alpha}$.

More generally, if the $SU(2)$ theory has extra symmetries (as might be the case if we add hypermultiplets), then the form of the central charge might become

$$S = aN_e + a_D N_m + W$$

where W is some other conserved charge. In this case it is possible to compensate for the affine translations in (a_D, a). Namely, we can have

$$\begin{pmatrix} a_D \\ a \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} M & \begin{pmatrix} c_D \\ c \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_D \\ a \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} N_m & N_e & W \end{pmatrix} \mapsto \begin{pmatrix} N_m & N_e & W \end{pmatrix} \cdot \begin{pmatrix} M^{-1} & \begin{pmatrix} * \\ * \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix}.$$

In all these transformations no derivatives of τ enter. In fact, what we did was to compute the transformation laws for τ constant and then extend them in the obvious way to the case when τ is holomorphic. The reason that we know that this mechanism works is that, on dimensional grounds alone, no derivatives of τ can enter the formulas.

We shall restrict ourselves mainly to the case that there are no extra symmetries and the transformations between the various representations of our low energy theory lie in $SL(2,\mathbf{Z})$. But we will make a few remarks here and in Lecture 19 on the case with additional hypermultiplets, where one does meet affine translations.

We have already given one derivation of the form of the central charge for the $SU(2)$ theory. Here is another explanation of the same formula (or at least a piece of it). Let us consider $U(1)$ gauge theory with a charged $N = 2$ hypermultiplet. In the $N = 1$ language, this hypermultiplet is a pair of chiral superfields $(T, \tilde{T})$.

Supersymmetry implies that the charges must be opposite; let us denote them by $\pm n_e$. $N = 2$ supersymmetry also requires that, in an $N = 1$ description, apart from a possible bare mass, the $T, \tilde{T}$ dependent part of the superpotential is precisely

$$\tilde{W} = n_e \Phi T \tilde{T}.$$

Since $\Phi = a + \ldots$, we see that the mass of the pair $(T, \tilde{T})$ is an_e. This massive hypermultiplet is BPS (the multiplicity of states is too small for it to be otherwise), so the fact that the mass of this multiplet is an_e means that the central charge must contain a term aN_e. (The T field is an eigenstate of N_e with eigenvalue n_e. It is also an eigenstate of N_m with eigenvalue 0, which is why we cannot detect the $a_D N_m$ term in the central charge this way.)

If we add a bare mass for this multiplet, the relevant part of the superpotential becomes

$$\tilde{W} = (n_e \Phi + m) T \tilde{T},$$

with m a constant. The mass is now $M = n_e a + m$, and the extra constant means that the central charge is now actually $S = an_e + a_D N_m + mW$, where W is a new conserved quantity that acts as 1 or -1 on T and $\tilde{T}$ (and trivially on the vector multiplet). As a result of the appearance of this extra term in S, the monodromies eventually turn out to include affine translations. For details, see the references.

§17.2.6. Jumping of the BPS spectrum

Let us return to our low energy $N = 2$ supersymmetric $U(1)$ gauge theory given locally by a, a_D. In such theories the BPS mass inequality is

$$M \geq |n_e a + n_m a_D|.$$

Since n_e and n_m are integers, this is a good bound in the sense that it implies that there is a positive constant which is a lower bound for M (on states of nonzero n_e, n_m) as long as a and a_D generate a lattice in $\mathbf{C}$.

What happens as the lattice generated by a and a_D degenerates, i.e., when a and a_D become collinear in $\mathbf{C}$? At such points the triangle inequality relating masses of states at different lattice points is no longer strict. This permits states to decay to pairs of lower states that become degenerate with them when a and a_D are collinear. In this way BPS states can disappear (or appear), accounting for the jumping in the BPS spectrum.

Generically, we expect this to happen along a real curve in U. (Notice that the lattice generated by $(1, \tau)$ is not degenerating and nothing is wrong with the elliptic curve. Rather it is the lattice generated by a and a_D which is degenerating.) This phenomenon is similar to that in Vafa's lecture for $N = 2$ supersymmetric theories in dimension two. There the BPS states are solitons (the same solitons that we introduced in Lecture 12.5), and jumping occurred when the values of the central charge for different solitons become collinear. In those two-dimensional models, this occurs when the superpotential W has three collinear critical points in the complex plane. In the four-dimensional example that we are discussing today, we shall obtain explicit information about the curve in U where the BPS spectrum jumps.

§17.2.7. More about the application to $SU(2)$ gauge theory

As we have remarked in passing several times, we will study an $N = 2$ supersymmetric $SU(2)$ gauge theory with no matter hypermultiplets. We shall do this by studying the theory as an $N = 1$ supersymmetric theory with Φ an adjoint-valued chiral superfield. The microscopic (or high energy) lagrangian is given by:

$$\mathcal{L}_{\rm mic} = \frac{1}{4\pi}\int d^4x d^2\theta \tau_0(\Lambda_0)\mathrm{Tr}\, W_\alpha W^\alpha + \text{c.c.} + \frac{1}{4\pi}\int d^4x d^4\theta \mathrm{Im}\,\tau_0(\Lambda_0)\mathrm{Tr}\overline{\Phi}\Phi$$

where W is a piece of the supercurvature of the superconnection $\mathcal{A}$ on a principal $SU(2)$-bundle over Minkowski space, and $\Phi = \phi + \theta\psi + \cdots$ with ϕ being a scalar field with values in the complexification of the adjoint bundle. Λ_0 is the ultraviolet momentum cutoff that we have chosen to regulate the $SU(2)$ theory. Also, τ_0 is a constant (in the fields) which depends on the value of Λ_0. τ_0 is to be adjusted as Λ_0 goes to infinity so that the theory has a limit.

The space of classical vacua is given by configurations consisting of trivial connections and constant (covariantly constant) sections ϕ of $\mathbf{C}\otimes_{\mathbf{R}}\mathrm{su}(2)$ which are zeros for the potential, up to gauge equivalence. The potential for this lagrangian is the norm-squared of the moment map

$$V = \frac{\mathrm{Im}\,\tau_0}{4\pi}\int d^4x \mathrm{Tr}\left([\phi,\overline{\phi}]^2\right).$$

If ϕ is a zero of the potential function, then clearly ϕ and $\overline{\phi}$ commute. Writing $\phi = \gamma + i\delta$ with $\gamma,\delta\in\mathrm{su}(2)$, we see that the condition that ϕ and $\overline{\phi}$ commute is simply that γ and δ commute. This means that γ and δ can be simultaneously diagonalized. That is to say, up to gauge transformation,

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} a & 0\\ 0 & -a\end{pmatrix}$$

for $a\in\mathbf{C}$. There is a further conjugation, namely the action of the Weyl group, which in this case is $\mathbf{Z}/2\mathbf{Z}$ and acts by sending $a\mapsto -a$. Thus, the moduli space of classical vacua is a complex plane parameterized by the complex coordinate

$$u = \mathrm{Tr}\,\phi^2 = a^2.$$

In the classical theory, for $u\neq 0$ the $SU(2)$ gauge symmetry is broken to $U(1)$ by a Higgs mechanism. Many components of the superfields get masses. The low energy theory is just of the sort we have discussed today, with the light fields making up a $U(1)$ gauge multiplet with $N = 2$ supersymmetry. The low energy effective theory at the exceptional point $u = 0$ remains an unbroken $SU(2)$ gauge theory.

Let us examine it in light of our discussion today about low energy effective $U(1)$ theories. As we have just explained, the space of classical vacua is the complex plane – the u-plane. Let us examine the metric on the u-plane. All points of this plane except zero correspond, classically, to a $U(1)$ theory as described in this lecture. The metric on the space of classical vacua is induced from the natural metric on the space of trivial connections and covariantly constant scalars in the complexified Lie algebra. This is simply the ad-invariant metric on the complexified Lie algebra. Restricting to the diagonal matrices as above, we see that the induced metric is $da\wedge d\overline{a}$. Since $u = a^2$, this means that the metric on the u-plane is a multiple of

$$ds^2 = \frac{du d\overline{u}}{|u|^2},$$

which has a singularity at the origin. This singularity reflects the fact that at the origin there are other massless fields so that the low energy effective theory is not a pure $N = 2$ supersymmetric $U(1)$ gauge theory at that point; if one tries to describe it as a pure $U(1)$ theory near $u = 0$, one gets a singularity.

Now we can describe in the classical theory the family of tori over the u-plane punctured at 0, and the holomorphic two-form η on the total space of this family. The function τ is constant, and hence the family of tori has monodromy contained in $\{\pm 1\} \subset SL(2, \mathbf{Z})$. Since the metric on the u-plane minus the origin is given by $\mathrm{Im}\,\tau da \wedge d\bar{a}$ we see that the parameter $a = \sqrt{u}$ is the parameter of the same name in the $U(1)$ theory (i.e., the first of the dual pair of local holomorphic coordinates on the base) and the dual parameter a_D is τa. The formula $u = a^2$ or $a = \sqrt{u}$ means that the monodromy around infinity (or equivalently around zero) in the u-plane sends a to $-a$ and a_D to $-a_D$. This means that the family of tori over the punctured u-plane is obtained from the product family $(\mathbf{C} - \{0\}) \times E$ over the punctured a-plane by dividing out by the involution $(a, z) \mapsto (-a, -z)$. (Here, τ is the complex parameter appearing in the classical $SU(2)$ lagrangian and E is the elliptic curve $\mathbf{C}/L(1, \tau)$, the quotient of the complex plane by the lattice generated by 1 and τ; $z \to -z$ is "multiplication by -1" on the elliptic curve.) The holomorphic two-form η is the image of $da \wedge dz$, where dz is a global holomorphic one-form on E.

This description is of the classical theory. In the next lecture we shall discuss the quantum version of this theory. It turns out that things change a lot in going to the quantum theory, but the changes are in some ways not as drastic as one might imagine. For example the space of quantum vacua will be the same u-plane, though singularities will arise at different points, and they will correspond to a different type of object becoming massless. The monodromy at singularities will accordingly be different.

References and cross references

The basic reference for this lecture and the next two is

N. Seiberg and E. Witten, *Electric-Magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **426** (1994), pp. 19-52.

On jumping of the BPS spectrum, in the two-dimensional case, see

S. Cecotti and C. Vafa, *On Classification Of $N = 2$ Supersymmetric Theories*, Commun. Math. Phys. **158** (1993) 569-644.

A detailed discussion of the $N = 2$ multiplets can be found in Problem ASH5 of [I-**Homework**]. The lagrangian (17.1) is developed in detail in [I-**Supersolutions**, §10.3]. The introductory material on abelian duality in dimension four is contained in Section 5 of Lecture 8. BPS states are discussed in detail in Lecture 12.

LECTURE 18

$N = 2$ Supersymmetric Yang-Mills Theories in Dimension Four: Part 2

§18.1. Review of material from the last lecture

We are considering pure $N = 2$ supersymmetric gauge theory (i.e., without hypermultiplets) in four-dimensional Minkowski space. The gauge group is either $SU(2)$ or the closely related group $SO(3)$. (In fact, working as we are today over Minkowski four-space, which is contractible, we will not see the distinction between these two gauge groups.) We write the theory in the $N = 1$ supersymmetric notation. The fields which form a single $N = 2$ vector multiplet decompose under the $N = 1$ super Poincaré group in four dimensions as a vector multiplet $\mathcal{A}$ (which contains the gauge field) and a chiral superfield

$$\Phi = \phi + \theta\psi + \cdots$$

where ϕ is a scalar field (under the Poincaré group). It is a section of the complexification of the adjoint bundle. The microscopic lagrangian is

$$\mathcal{L} = \int d^4x d^2\theta\, \tau_0(\Lambda_0) \operatorname{Tr} W_\alpha W^\alpha + \text{c.c.} + \int d^4x d^2\theta\, \operatorname{Im} \tau_0(\Lambda_0) \operatorname{Tr} \overline{\Phi}\Phi$$

where W is a certain piece of curvature of the superconnection $\mathcal{A}$. Here Λ_0 is a cut-off used to render finite the Feynman path integrals. We vary τ_0 as a function of Λ_0 in order to renormalize the theory, i.e., render the expectation values in the theory finite, as $\Lambda_0 \mapsto \infty$. In order to decide how to renormalize we must select a so-called "subtraction point" Λ at which to fix the value of one parameter in the theory. Λ is held fixed while $\Lambda_0 \to \infty$, and τ_0 is adjusted so that the theory has a limit. We shall see later a convenient normalization scheme for this particular theory which involves the details of the theory – the u-plane, etc. But for now we leave vague the actual renormalization scheme.) While τ_0 depends on the cutoff, it is independent of the fields.

Our object today is to solve this theory by explicitly understanding the moduli space $\mathcal{M}$ of quantum vacua, and obtaining a description of the theory near each vacuum state. As we established last time, the space of classical vacua of this theory

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Notes by John Morgan

consists of gauge equivalence classes of constant matrices in su(2)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

modulo the action of the Weyl group, which is a group of order two switching a and $-a$. This space is analytically isomorphic to $\mathbf{C}$ and the natural local holomorphic coordinate to use on this plane is $u = \operatorname{Tr}\phi^2 = a^2$. While the analytic space of classical vacua is smooth, there is one point in the moduli space of classical vacua where the Kähler metric becomes singular. This is the origin. The singularity is a consequence of the fact that at this point there are more massless fields (in fact the low energy effective theory is a supersymmetric $SU(2)$ theory). Furthermore, as we discussed last time, away from $u = 0$ the low energy effective theory is a pure $N = 1$ supersymmetric $U(1)$ gauge theory. Thus, away from $u = 0$, the nature of this effective theory is described by a family of tori over the punctured u-plane. In the case of the moduli space of classical vacua, this is a family of tori with constant j invariant given by the choice of τ_0 in the lagrangian. The monodromy around infinity for this family is -1. The Kähler metric on the u-plane is

$$ds^2 = \operatorname{Im}\tau_0 \frac{du d\overline{u}}{4|u|^2}.$$

(Notice that there is a singularity in the metric at $u = 0$ as suggested by the fact that at this vacuum there are extra massless fields classically.)

§18.2. First results about the moduli space $\mathcal{M}$ of quantum vacua

Let $\mathcal{M}$ be the space of quantum vacua of our $N = 2$ supersymmetric pure gauge theory. Using perturbation theory and asymptotic freedom, the low energy spectrum near infinity on $\mathcal{M}$ (i.e., for large u), consists precisely of a single $U(1)$ vector multiplet; we will assume that this is the correct low energy spectrum at a generic point on $\mathcal{M}$. If so, we can apply our reasoning from last week. The space of quantum vacua has a Kähler metric of a rather special kind, and over this space of vacua there is a holomorphic family of tori.

The field u, which is a coordinate on the space of classical vacua, is a chiral field in the theory, and thus its expectation values in the various quantum vacua defines a holomorphic function on $\mathcal{M}$. We denote this function by $u\colon \mathcal{M} \to \mathbf{C}$:

$$u(\Omega) = \langle\Omega|u|\Omega\rangle, \quad \text{for all } \Omega \in \mathcal{M}.$$

The part of $\mathcal{M}$ that is most closely related to the u-plane of classical vacua is the region where u is large. More precisely, let Λ be the renormalization scale of the theory; keep it fixed in the following discussion. Suppose that we are studying the theory near a classical vacuum Ω with $u(\Omega) >> \Lambda^2$. (Notice that since u has dimensions of mass squared it naturally compares to Λ^2.) Because our $SU(2)$ theory is asymptotically free, and the $U(1)$ theory to which it is limiting is free in the infrared, if $u >> \Lambda^2$ then the quantum theory is well approximated near Ω by the classical theory. This means that the low energy quantum theory near Ω is described by a sigma model on the u-plane, perhaps perturbed by some superpotential. But $N = 2$ supersymmetry means that any such superpotential must be a constant, and hence cannot lift the vacuum degeneracy near Ω. (There is also the possibility of a Fayet-Iliopoulos θ-term to consider as well, but the $SU(2)_R$-symmetry rules these out.) What this argument shows is that for large u there is a unique quantum

vacuum for each classical vacuum. That is to say, there is a constant $C(\Lambda) > 0$ and an open subset $\mathcal{U} \subset \mathcal{M}$ with the property that $u\colon \mathcal{U} \to (\{z \in \mathbf{C} | |z| > C\})$ is a holomorphic isomorphism.

In Lecture 16 and in more detail in [II-**N=1 Dynamics**], we have seen other examples of gauge theories where near infinity in the space of classical vacua the gauge group $SU(N_c)$ is broken down to a nonabelian gauge group $SU(N_c - N_f)$. Since the gauge theory for this smaller group is not free in the infrared, we cannot conclude that the space of quantum vacua near infinity can be identified with the subspace of critical points of a superpotential on the space of classical vacua. In fact, in these examples the moduli space of quantum vacua is a finite covering space of the space of classical vacua near infinity. To repeat, this is possible because the low energy theory is not free in the infrared. When it is free, as in the case under discussion now, and when $N = 2$ supersymmetry rules out the existence of superpotentials, the classical and quantum moduli spaces can be identified over a neighborhood of infinity in the space of classical vacua.

One assumption that we shall make without any justification is that the moduli space $\mathcal{M}$ is complete in an appropriate sense so that it corresponds to an open complex algebraic variety.

By considering the classical limit, we have found one component of $\mathcal{M}$ which has an end isomorphic under u to a neighborhood of infinity in $\mathbf{C}$. Two other questions present themselves:

- Are there other "ends" of $\mathcal{M}$, ends that are not seen classically?
- Are there other irreducible components of $\mathcal{M}$ not seen classically?

The answer to the first question is "No." All ends of the quantum moduli space are visible classically, i.e, they are related to ends of the classical moduli space. This is a general claim about quantum field theory; quantum effects are bounded and hence cannot effect a non-proper change in the passage from the space of classical vacua to the space of quantum vacua.

As to the second question, there are no examples known of supersymmetric theories where there are branches of the quantum space of vacua which are not seen classically. Still, in this generality it has not been established that such components do not exist. In our particular case we can give arguments showing that there are no other components. The first thing to note is that, by the answer to the first question, any other component of $\mathcal{M}$ must be compact. To exclude the possibility of other compact branches, we use the fact that the theory has a relevant $N = 1$ supersymmetric perturbation. That is the perturbation

$$\Delta\mathcal{L} = \int d^4x d^2\theta \epsilon u + \text{c.c.}$$

where ϵ is a complex constant. As we saw in §16.2 of Lecture 16 (and in [II-**N=1 Dynamics**, §2.2]), with this perturbation the resulting $SU(2)$ theory has two vacua, each of which is believed to have a mass gap and confinement. This result must somehow be reproduced in the low energy effective description. But what will $\Delta\mathcal{L}$ look like in the low energy description along a hypothetical compact branch that cannot be seen classically? By $N = 1$ supersymmetry, it must look like an effective superpotential (plus terms with derivatives that could not lift a vacuum degeneracy); but, as a holomorphic function on a compact branch of the moduli space will be constant, this superpotential will be trivial. So the vacua of the compact branch would survive even with $\Delta\mathcal{L}$ added to the lagrangian, and the

low energy theory would have too many vacuum states if the compact branch has positive dimension. (This argument still allows for the possibility of a compact branch consisting of isolated points – at most two of them – but we shall see later by counting singularities that this does not occur either.)

We have now argued that $\mathcal{M}$ has a single branch (plus possibly at most two isolated points) and that this branch has a single neighborhood of infinity which is mapped by u isomorphically to a neighborhood of infinity in the u-plane. By our completeness assumption, $\mathcal{M}$ is a possibly singular, irreducible complex curve with one puncture at $u = \infty$. The existence of the holomorphic function u with only one pole, that being the simple pole at infinity of $\mathcal{M}$, implies that $\mathcal{M}$ is actually the complex plane and that u is a global holomorphic parameter on $\mathcal{M}$.

Following the analysis of the last lecture, our goal is to solve the theory by finding the family of elliptic curves (in Weierstrass form) over $\mathcal{M}$

$$y^2 = x^3 + A(u)x + B(u)$$

and a holomorphic two-form η on the total space of this family

$$\eta = f(u)du\frac{dx}{y}.$$

The complex structure of the elliptic curve will determine $\tau(u)$, and the periods of η determine the differentials da and da_D of the dual pair of local holomorphic coordinates. The elliptic curve may become singular, and $\tau(u)$ may diverge, at the singular points in $\mathcal{M}$.

§18.3. The nature of infinity in $\mathcal{M}$

As we have already remarked, we obtain information near infinity in $\mathcal{M}$, i.e., where u is large, using asymptotic freedom of the $SU(2)$ theory. Our goal here is to compute the monodromy at infinity of the family of elliptic curves over the moduli space of quantum vacua. We begin by studying the monodromy at infinity for the space of classical vacua. Recall that the space of classical vacua is the space of matrices

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

up to $a \mapsto -a$. The dual coordinate a_D is given by $a_D = \tau_{\rm cl}a$, where $\tau_{\rm cl}$ is a constant (denoted by τ_0 earlier) with large imaginary part. The global coordinate on the u-plane is $u = \operatorname{Tr}\phi^2 = a^2$, so that $a = \sqrt{u}$. From this it is clear that the monodromy at infinity in this classical family is -1.

The monodromy around infinity for the space $\mathcal{M}$ of quantum vacua is different. We give two arguments calculating it. The first is based on asymptotic freedom and the second is based on the $U(1)_R$-anomaly. To compute $\tau_{\rm eff}(u)$ for large u we consider the 1-loop formula. With an appropriate choice of the mass parameter Λ, the formula is:

$$\operatorname{Im}\tau = \frac{4\pi}{g^2_{\rm eff}} = 4\pi\left(2b_0\ln(\sqrt{|u|/\Lambda})\right), \tag{18.1}$$

where b_0 is the 1-loop β-function coefficient.

One loop computations identify

$$b_0 = \frac{1}{4\pi^2}.$$

Since τ is holomorphic, knowing its imaginary part determines it uniquely. We see that for u large we have

$$\tau_{\text{eff}} = \frac{2i \ln(u/\Lambda^2)}{2\pi}. \tag{18.2}$$

Thus, we see that monodromy around infinity shifts τ by -2 and hence shifts θ by -4π. This implies that the monodromy around infinity in $\mathcal{M}$ is given by

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Said another way we are considering the system

$$\begin{aligned} u^2 &= a \\ \frac{da_D}{da} &= \tau = \frac{2i \ln(u/\Lambda^2)}{2\pi} \end{aligned} \tag{18.3}$$

as we turn once around infinity in the u plane. Clearly, a comes back to $-a$ and τ comes back to $\tau - 2$. The above form for the monodromy then follows easily.

Now let us recompute this monodromy using the anomalous $U(1)_R$ symmetry, just to see how things hang together. Classically, the $U(1)_R$ charges of the component fields in the gauge supermultiplet are given as follows: the charge of A is zero, the charges of λ, ψ are 1 and the charge of a is two. This means that $u = a^2$ has charge 4. (This is related to the fact that in Donaldson theory $\mu(\text{pt})$ is 4-dimensional.) This means that under the $U(1)_R$ symmetry that multiplies the chiral superspace coordinates by $e^{i\beta}$, the variable u transforms by $u \mapsto ue^{\pm 4i\beta}$. In fact, the sign is negative. On the other hand, as we know, there is a quantum anomaly breaking this $U(1)_R$ down to $\mathbf{Z}/8\mathbf{Z}$. This is an index computation which uses the fact that the theory has 2 Fermi fields (λ and ψ in last week's notation) in the adjoint representation, and the Casimir of the adjoint representation of su(2) is 4 times the Casimir of the defining representation. $2 \times 4 = 8$, so the anomaly in an instanton field is always divisible by 8, and the symmetry is $\mathbf{Z}/8\mathbf{Z}$. (Essentially the same computation determines the fact that the Floer groups of three-manifolds are $\mathbf{Z}/8\mathbf{Z}$ graded.) This means that under the $U(1)_R$ symmetry $e^{i\beta}$, the theta angle $\theta = 2\pi \operatorname{Re} \tau$ transforms to $\theta - 8\beta$. Thus, as we transverse a loop around infinity in the u-plane, the argument of u changes by 2π, there is a $U(1)_R$ transformation by $2\pi/4 = \pi/2$ and θ decreases by 4π. This means that $\tau \mapsto \tau - 2$ under the monodromy around infinity.

Of course, these are not independent computations. Supersymmetry imposes a relation between the $U(1)_R$ anomalies and the β-function and ensures that everything hangs together.

Actually, we can say more about the nature of τ at infinity. For $|u| >> \Lambda^2$, we have in perturbation theory a series

$$\tau(u) = \frac{2i \ln(u/\Lambda^2)}{2\pi} + \tau^{\geq 2} \tag{18.4}$$

where the first term is the sum of the classical value and the one-loop correction (as calculated above) and the second term is the sum of the corrections for higher loops. These higher loop corrections produce a power series in g^2 (k-loop corrections produce the power g^{2k-2}). Since there are no loop corrections to $\theta = \operatorname{Re} \tau$, and since τ is a holomorphic function of u, it follows immediately that all the higher loop corrections vanish: $\tau^{\geq 2} = 0$. Thus, eqn. (18.2) is valid to all orders in perturbation theory. There are however non-zero corrections to the perturbation

expansion coming from instantons. Of course, these cannot be seen in perturbation theory. Let's study their nature (using arguments of a type that we have seen in Lecture 16). Holomorphy of τ implies that the corrections to perturbation theory should come from fields invariant under half the supersymmetries, that is, from instantons. A k-instanton field has action $8\pi^2 k/g^2$, so the k-instanton amplitude has a factor $\exp(-8\pi^2 k/g^2)e^{ik\theta} = \exp(2\pi i k\tau)$. The renormalization group implies that this factor must be accompanied by a factor of Λ^{4k}. On dimensional grounds, this must be Λ^{4k}/u^{2k}. As a check, we note, as in similar problems discussed in Lecture 16, that the k-instanton field violates the $U(1)_R$ symmetry by $8k$ units; as u has charge 4, this accounts for the factor of u^{-2k}. Thus, the k-instanton contribution to τ is of the form

$$f_k \exp(-8\pi^2 k/g^2)e^{ik\theta} = f_k \Lambda^{4k} e^{2k\pi i\tau} = f_k \frac{\Lambda^{4k}}{u^{2k}}$$

for a dimensionless constant f_k, and the full expansion of $\tau(u)$ for $|u| >> 1$ is of the form

$$\tau(u) = \frac{2i\ln(u/\Lambda^2)}{2\pi} + \sum_{k\geq 1} f_k \frac{\Lambda^{4k}}{u^{2k}}. \tag{18.5}$$

§18.4. BPS states and singularities in $\mathcal{M}$

Next, we wish to understand the singularities in $\mathcal{M}$. By a *singularity* we mean a point of the moduli space where the physics changes. This is likely to be a point at which there is some singularity in the Kähler metric or other low energy couplings. As a general principle, we expect that the singularities have to do with particles becoming massless. In this case, since the particles are to add singularities to the low energy limit which is a $U(1)$ theory, it is plausible that they should be charged under the $U(1)$. The particles becoming massless will of course form $N = 2$ supermultiplets. More specifically, if the particles becoming massless have spins at most $1/2$, they must consist of a BPS-saturated hypermultiplet, since this is the only type of $N = 2$ multiplet with all spins $\leq 1/2$. We give below reasons to believe that the theory will not gain extra massless particles of spin 1, and as we learned in Lecture 2, it cannot have massless particles of spin greater than 1. So we will try to interpret the singularities in terms of BPS-saturated hypermultiplets becoming massless. BPS-saturated hypermultiplets, if they are not always massless, must carry a non-zero central charge; in the present theory (as we discussed last week), the central charges are the electric and magnetic charge, so the BPS-saturated hypermultiplets of interest must be electrically or magnetically charged.

In pure $SU(2)$ gauge theory with $N = 2$ supersymmetry, we have two types of BPS states near infinity in the u-plane. We have the charged vector multiplets, i.e., the gauge bosons associated with the spontaneously broken part of $SU(2)$. They have $n_e = \pm 1$ and $n_m = 0$.[1] We also have the magnetic monopoles with $n_m = \pm 1$ and any value of n_e. Notice that this BPS spectrum is invariant under the monodromy at infinity, which is $n_e \mapsto -n_e + 2n_m$ and $n_m \mapsto -n_m$. The monopoles transform as BPS-saturated hypermultiplets. (The term dyon is sometimes used for a particle that carries both electric and magnetic charge, so most of these particles are dyons.)

[1] n_e and n_m are of course the eigenvalues of the $U(1)$ generators N_e and N_m.

We claim that this spectrum, though invariant under the monodromy at infinity, cannot be invariant under the entire monodromy of our family. The reason is that the image of the monodromy on all of $\mathcal{M}$ is a nonabelian subgroup of $SL(2, \mathbf{Z})$ (since otherwise $\operatorname{Im} \tau$ could not be positive everywhere); but the above-described BPS spectrum is not invariant under such a non-abelian group. This means that the BPS spectrum cannot be continuous throughout $\mathcal{M}$, that is to say there must be a jumping locus; that is, a real curve in $\mathcal{M}$ along which extra BPS states become massless. As we saw in the last lecture, this happens when a/a_D becomes real.

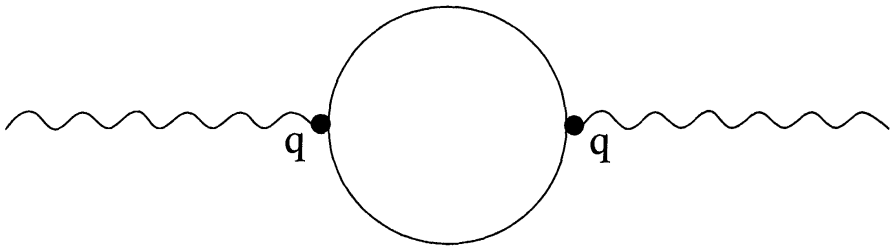
We have identified $\mathcal{M}$ with the complex u-plane. Next, we wish to analyze the singularities of $\mathcal{M}$. There must be singularities in $\mathcal{M}$ since the monodromy at infinity is non-trivial. As we have argued, singularities in $\mathcal{M}$ are associated with a certain supersymmetric multiplet consisting of charged BPS states becoming massless. We have two types of BPS multiplets – those where the maximal spin in the multiplet is 1 (the vector multiplet) and those where the maximal spin is $1/2$ (the hypermultiplet). A vector multiplet becoming massless corresponds to restoring a non-abelian gauge group ($SU(2)$). There are several reasons one does not expect this to happen. First of all, with unbroken $SU(2)$ one has a theory which is unstable in the infrared. This is exactly what happens at the singularity in the classical moduli space. Since the β-function is negative, we have flowed away from this theory, so we don't expect to see it occurring in the low energy limit which we are assuming to be free in the infrared. Of course, if our gauge theory has enough hypermultiplets becoming massless to make $\beta > 0$ then we might see such a low energy limit. But at these points we have multiplets with maximal spins both 1 and $1/2$ becoming massless. So the simplest thing that we can do is to suppose that only a hypermultiplet with maximal spin $1/2$ become massless.[2] Let us analyze this possibility.

Suppose that the singularity in $\mathcal{M}$ is associated to a multiplet whose maximal spin is $1/2$ becoming massless. Recall that a massive $N = 2$ supersymmetric multiplet of spin at most $1/2$ is automatically BPS saturated. Thus, we might expect that what is happening is that one of the BPS states that exist in the theory for $|u| >> \Lambda^2$ is becoming massless at the singularity. Of course, more complicated phenomena could occur. For example, as we come in from infinity in $\mathcal{M}$ we could cross a jumping line of the BPS spectrum before we reach the singularity, and then at the singularity it might be a "new" BPS saturated state that becomes massless, rather than one of the original states that exist near $u = \infty$. But we will eventually see that the state producing the singularity is one of the BPS states at infinity.

Let us assume that a hypermultiplet becomes massless. Denote its charges by (n_m, n_e). We have already argued that this hypermultiplet must be charged, so that $q = \text{g.c.d.}\,(n_m, n_e)$ is a well-defined positive number. After an $SL(2, \mathbf{Z})$ change in the representation of the theory by a lagrangian, we can arrange that $n_e = q$ and $n_m = 0$. Thus, near the singular point $u_0 \in \mathcal{M}$ our theory is a $U(1)$ theory with a charged particle which is becoming massless at $u = u_0$. (But if $n_m \neq 0$ then the $U(1)$ gauge fields in the representation of the theory near $u = u_0$ are different

[2]In two dimensions, one has the Zamolodchikov c-theorem – mentioned in Lecture 13 – which gives a precise sense in which there are fewer massless degrees of freedom in the infrared than in the ultraviolet. It is suspected but not known that a similar theorem holds in any dimension. If one had such a theorem in four dimensions, one might have a sharp *a priori* bound on how many and what massless fields there can be at singularities. Such a bound would probably exclude most of the bizarre hypotheses about how this theory might behave.

from the $U(1)$ gauge fields that we naturally have at infinity.) Unlike the spin-one case, this theory is stable in the infrared, i.e., $\beta > 0$. Since the behavior of the theory near $u = u_0$ is infrared free, we can compute g_{eff} and θ_{eff} in this region using perturbation theory. In general, in a theory whose one-loop beta function is positive, g_{eff} vanishes logarithmically in the infrared, with a coefficient that is determined by the one-loop β-function. The relevant one-loop Feynman diagram is:



(The solid line represents the hypermultiplet, while the wavy line represents the vector multiplet.) This diagram renormalizes the kinetic energy of the gauge field; the renormalization factor is a function of $u - u_0$ (because the hypermultiplet mass is proportional to $|u - u_0|$, and it diverges logarithmically for $u \mapsto u_0$ because of the massless hypermultiplet.

The result is that for u near u_0 we have:

$$\tau(u) = 2q^2 \frac{\ln(u - u_0) + \text{constant}}{2\pi i}.$$

This means that the monodromy of $\tau(u)$ in $PSL(2, \mathbf{Z})$ around this singularity is

$$\pm \begin{pmatrix} 1 & 2q^2 \\ 0 & 1 \end{pmatrix}. \tag{18.6}$$

Of course, the monodromy is actually an element in $SL(2, \mathbf{Z})$ and is only determined up to sign by the monodromy in $PSL(2, \mathbf{Z})$. To pin down the sign we argue as follows. The physics near $u = u_0$ is described by a $U(1)$ gauge theory with gauge fields $(\tilde{A}, \tilde{a})$ and a charged hypermultiplet $(T, \tilde{T})$ which is becoming massless at u_0. Thus, the lagrangian is of the form of the gauge couplings plus a superpotential $q \cdot \tilde{a} T \tilde{T}$. The scalar field $\tilde{a}$ is a local coordinate on the u-plane and in fact is related to a, a_D at infinity by an $ISL(2, \mathbf{Z})$ transformation. (The image of this transformation in $SL(2, \mathbf{Z})$ is determined by the charge (n_e, n_m) of the hypermultiplet $(T, \tilde{T})$ with respect to the $U(1)$ at infinity in the u-plane.) Since the $U(1)$ theory exists throughout this neighborhood even at the singularity, we see that under monodromy around u_0 the field $\tilde{a}$ comes back to itself. This implies that there is an invariant vector under the $SL(2, \mathbf{Z})$-monodromy around u_0 and hence shows that the monodromy is as written in Equation 18.6 with a plus sign.

We can generalize this picture. Suppose that at u_0 there are several hypermultiplets of charge q_i becoming massless. Then the local form of τ would be

$$\tau = 2 \sum_i q_i^2 \frac{\ln(u - u_0) + \text{constant}}{2\pi i}$$

and the monodromy around u_0 would be

$$\begin{pmatrix} 1 & \sum_i 2q_i^2 \\ 0 & 1 \end{pmatrix}.$$

Notice that the monodromy around a singularity where a hypermultiplet is becoming massless has trace 2 whereas the monodromy at infinity has trace -2. This means, in particular, that there cannot be only one singularity in the u-plane if the singularity arises from massless hypermultiplets. (From positivity of $\operatorname{Im}\tau$ away from the singularities, it in any case follows that there must be a nonabelian monodromy group and hence more than one singularity.)

§18.5. The number of singularities in $\mathcal{M}$

To understand the number of singularities in $\mathcal{M}$, let us begin with a microscopic $N = 2$ supersymmetric $SU(2)$ theory and break it to an $N = 1$ supersymmetric theory by adding a perturbation of the form

$$\Delta\mathcal{L} = \int d^4x d^2\theta mu + \text{c.c}, \tag{18.7}$$

with m a complex constant. (Recall that $u = \operatorname{Tr}\phi^2$.) As discussed in Lecture 16 and in [II-**N=1 Dynamics**, §2.2], this theory has two vacua, each with a mass gap and with confinement.

Let us suppose that, at a point in the space of quantum vacua, we have t hypermultiplets becoming massless, say $(T_i, \tilde{T}_i)$ of charge q_i. So near this point the physics is described by a superpotential

$$W = \tilde{a}\left(\sum_i q_i T_i T_i'\right) + mu(\tilde{a}).$$

where, as before $\tilde{a}$ is a local parameter on the u-plane and is related by an $SL(2,\mathbf{Z})$ transformation to a, a_D at infinity. Let us find the space of vacua in the macroscopic theory. That is to say, we solve $dW = 0$ and divide by the $\mathbf{C}^*$-action. Clearly, the equation $dW = 0$ is equivalent to

$$\tilde{a}T_i = \tilde{a}T_i' = 0 \quad \text{for all} \quad i.$$

$$m\frac{du}{d\tilde{a}} + \sum_{i=1}^{t} q_i T_i \tilde{T}_i = 0.$$

If $t > 0$ this gives us a positive dimensional space of vacua. This is impossible since the microscopic theory has only two vacua. This means that at each singularity we have only one hypermultiplet becoming massless. Furthermore, since $\frac{du}{d\tilde{a}} \neq 0$ we see from the second equation that for $m \neq 0$, a vacuum state has $T, \tilde{T} \neq 0$. By the first equation, this means that $\tilde{a} = 0$. Since there is only one hypermultiplet, the equation becomes

$$m\left.\frac{du}{d\tilde{a}}\right|_{\tilde{a}=0} + qT\tilde{T} = 0.$$

This determines $T\tilde{T}$, so it determines the vacuum uniquely up to the action of $\mathbf{C}^*$. Thus, we see that there is a unique vacuum associated to this singularity.

Having one vacuum for each singularity in the macroscopic theory, and having only two vacua in the microscopic theory, it follows that there are at most two singularities in $\mathcal{M}$. Thus, we see

$$N_h \leq 2,$$

where N_h is the number of points at which a hypermultiplet becomes massless.

As we have already remarked, the monodromy around a point where a hypermultiplet is massless and the monodromy at infinity have opposite traces and hence are not conjugate in $SL(2,\mathbf{Z})$. So on these grounds (or because we need nonabelian monodromy to get $\mathrm{Im}\,\tau > 0$), there must be at least two points with massless hypermultiplets; that is, $N_h \geq 2$. We conclude then that $N_h = 2$.

Thus, though we have found that $\mathcal{M}$ is the same as the classical moduli space of vacua, there are important differences. Classically, there is one singular point on the u-plane; a nonabelian gauge group is restored there, but the monodromies are abelian. Quantum mechanically, there are two singular points on the u-plane; the monodromies are nonabelian but one never meets at long distances a nonabelian unbroken gauge symmetry group. The nonabelian monodromy of the quantum theory implies that, unlike the classical case, the family of tori over the space of quantum vacua do not have constant j-invariant. This was already clear from the asymptotic formula (18.2) for the behavior near $u = \infty$.

By now we have found two singularities in the u-plane, each contributing a vacuum when we add the perturbation in Equation 18.7. Since there are at most two vacua in the perturbed theory, it follows that there cannot be isolated points of the space of quantum vacua of the original $N = 2$ theory; these isolated points would also produce quantum vacua of the perturbed theory. This ties up a loose end in our argument in §18.2 to identify $\mathcal{M}$ with the u-plane.

We recall that quantum mechanically, the $U(1)_R$-symmetry of the classical lagrangian is broken down to a $\mathbf{Z}/8\mathbf{Z}$-symmetry. Since u has charge four under $U(1)_R$, this symmetry acts on the u-plane as a $\mathbf{Z}/2\mathbf{Z}$-symmetry. It is the symmetry $u \mapsto -u$. The two singular points in $\mathcal{M}$ must form a $\mathbf{Z}/2\mathbf{Z}$ orbit, so they are the points $\pm u_0$ for some $u_0 \neq 0$. As we have already seen, dimensional analysis tells us that u goes like Λ^2, the square of the mass parameter in the theory. This means that we can make a choice of the mass parameter Λ so that the quantum theory labeled by Λ has singularities at precisely $u = \pm\Lambda^2$. In this way we are adopting the precise definition of the mass scale parameter Λ promised in the introduction. Notice that this choice of Λ is different from the one that we have used until now. The only effect of this change in the choice of Λ simply is to change the constants in Formulas 18.1 through 18.5.

§18.6. The new massless particles

The next step is to identify what is becoming massless. Let (n_e, n_m) be the charge of the hypermultiplet becoming massless at Λ^2 when we express things in a basis in which the monodromy at infinity is upper triangular. If $n_m = 0$, then the monodromy at Λ^2 commutes with the monodromy at infinity. Since the total space of the non-singular part of the family is the thrice punctured sphere, the monodromies at Λ^2 and ∞ generate the entire monodromy image. Thus, if the monodromy at ∞ and Λ^2 are simultaneously upper triangular, then the entire monodromy image is upper triangular and hence commutative. This would mean that $\mathrm{Im}\,\tau$ could not be positive everywhere. Our conclusion is that $n_m \neq 0$. That is to say, the particle

becoming massless at $u = \Lambda^2$ is magnetically charged. Exactly the same analysis holds for the singularity at $-\Lambda^2$. This is a very interesting phenomenon. Particles which started life when $|u| >> \Lambda^2$ as massive solitons are becoming massless as u approaches $\pm\Lambda^2$. Also, it means that our description of the theory near $\pm\Lambda^2$ as a $U(1)$ theory is a different representation from the $U(1)$ representation of the theory near infinity in the u-plane.

Now we wish to calculate the magnetic charge $\pm n_m$ of these particles. Notice that as we come in from infinity $\pm n_m$ is well-defined independent of the path we choose from infinity to $\pm\Lambda^2$ but the electric charge depends on the path – if we wind around the pair of singularities, $\pm n_e$ changes by twice the magnetic charge, since the monodromy around infinity is $n_e \mapsto -n_e + 2n_m$. The magnetic monopoles near $u = \infty$ have $n_m = \pm 1$. This strongly suggests that the hypermultiplets that become massless at the singularities have $n_m = \pm 1$, an important statement that we will argue more precisely in §18.7.

The model we are discussing today has various mathematical applications, but just in terms of physics one of its most striking applications was to give a new model of confinement. We recall that confinement is the statement that there is a linear potential between external electric charges or alternatively that there is area law decay for a Wilson loop in an appropriate representation (in the present example, the two-dimensional representation of $SU(2)$ is a good choice). We recall from Lecture 10.3 that in a Higgs phase, where a charged field gets a vacuum expectation value[3], one can explicitly calculate by a topological argument that the 't Hooft loop gets area law decay. If one could just naively exchange electricity and magnetism, then Wilson loops would be exchanged with 't Hooft loops and we would conclude that confinement will arise if a magnetically charged field gets a vacuum expectation value. This was argued heuristically in the 1970's by 't Hooft, Mandelstam, Nambu and others but with great difficulty in exhibiting actual models, since usually one does not have magnetically charged fields in the formalism and magnetically charged objects (which arise as classical solutions) are heavy in the weakly coupled regime where one understands them concretely. Thus it was generally rather hard to see how one could actually exhibit a phenomenon in which a magnetically charged field gets a vacuum expectation value.

In the present context, this happens. We have already seen that after adding the mu perturbation to the superpotential, which is expected to give confinement, the vacuum near $u = \Lambda^2$ has the hypermultiplet fields $T, \tilde{T}$ nonzero, so in fact in this vacuum (n_m being non-zero) a magnetically charged object has obtained a vacuum expectation value. Thus, as expected, a magnetically charged field obtained a vacuum expectation value just in the confining case. To pursue this with greater precision, it is important that the magnetically charged field in question has n_m odd. Indeed, as the center of $SU(2)$ is $\mathbf{Z}/2\mathbf{Z}$, the explanation of confinement via a dual of the usual Higgs mechanism requires a nonzero expectation value of a field of odd n_m. T and $\tilde{T}$ actually have n_m odd, in fact ± 1, as we argued heuristically above and will argue more precisely in §18.7.

One of the surprising things that have happened is the following. The magnetic monopoles have mass at infinity on the order of $\sqrt{|u|}\ln(|u|)$ whereas there are other BPS states of masses on the order of $\sqrt{|u|}$. Thus, at infinity the monopoles are not

[3]This is a somewhat imprecise description, as we have seen in previous lectures, but is valid in weak coupling.

the least massive of the massive particles. Nevertheless, they are the ones becoming massless at the singular points. In this paragraph we are really jumping ahead of the story and assuming that the particles that become massless at the singularities are the BPS monopoles that one sees semiclassically for large $|u|$.

§18.7. Explicit nature of the family of elliptic curves

We are now ready to put the pieces together and solve the model. We will describe explicitly the family of elliptic curves and the two-form η associated with the space of quantum vacua of this theory. As we have seen, for this theory the moduli space of quantum vacua is $\mathbf{C}$ with parameter $u = \mathrm{Tr}\,\phi^2$. The $SL(2,\mathbf{Z})$-monodromy at infinity for the family is

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

There are in addition two other points $\pm\Lambda^2$ in the u-plane where there are singularities. The $SL(2,\mathbf{Z})$-monodromy $m_{\pm\Lambda^2}$ at each of these points has trace two. We write

$$m_{\Lambda^2}^{-1} = \begin{pmatrix} 1+b & -b^2/a \\ a & 1-b \end{pmatrix},$$

for integers a, b. (This is the general form of a matrix of trace two and determinant one.) Since

$$m_{-\Lambda^2} = m_{\Lambda^2}^{-1} m_\infty$$

has trace two, we conclude that

$$2 = \mathrm{Tr} \begin{pmatrix} 1+b & -b^2/a \\ a & 1-b \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

It follows by a direct computation that $a = 2$. Since $-b^2/a$ is an integer, we see that b is even. Now let us conjugate the matrix for $m_{\Lambda^2}^{-1}$ by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which commutes with the monodromy at infinity. The result is to change b by 2. Thus, after a number of these conjugations we arrive at $b = 0$. This means that in an appropriate basis the monodromy at infinity is

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

and the monodromy at Λ^2 is of the form

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Thus, we have shown that the monodromy representation is uniquely determined up to conjugation in $SL(2,\mathbf{Z})$. Notice also that according to Equation 18.6, in order to get this monodromy, the magnetic charge n_m of the hypermultiplet that is becoming massless at Λ^2 is ± 1. The same of course holds at $-\Lambda^2$. In fact, because the $SL(2,\mathbf{Z})$ representation is uniquely determined up to conjugation, it is invariant, up to conjugation, under the symmetry $u \mapsto -u$.

It is easy to identify the family of elliptic curves over the twice punctured plane with this monodromy. The base of the family can be identified with the modular curve $\mathbf{H}/\Gamma(2)$ given by dividing out the upper half-plane by the (free) action of the

subgroup of $PSL(2, \mathbf{Z})$ consisting of all matrices congruent to the identity modulo two. The family is the universal family of elliptic curves with "level two structure" over this base. Since the singularities are at $u = \pm\Lambda^2$, the Weierstrass equation for the family of elliptic curves associated to this quantum theory is

$$y^2 = (x - u)(x - \Lambda^2)(x + \Lambda^2).$$

Now let us consider the two-form η as described in the last lecture. We know that in the Weierstrass model given above, η has the general form

$$\eta = f(u)du\frac{dx}{y}.$$

Implicitly, in the discussions before we assumed that η was a complex symplectic form without zeros or poles, at least away from the exceptional values of u. Such zeros or poles would produce non-physical singularities. If we take f to be a non-zero constant, then the Kähler form $\pi_*(\eta \wedge \overline{\eta})$ is smooth except at $\infty, \pm\Lambda^2$ where it has the correct singular behavior. Any other f would produce either poles or zeros at other points or the wrong singular behavior at one of $\infty, \pm\Lambda^2$. It follows that $\eta = Cdu\frac{dx}{y}$ for an appropriate constant C, which can be computed by comparing to the formula $a^2 = u$ near $u = \infty$.

Clearly,

$$\eta = d\left(C\frac{udx}{y}\right) = d\left(C\frac{2ydx}{x^2 - \Lambda^4}\right).$$

The form $\lambda = 2Cydx/(x^2 - \Lambda^4)$ is a differential of the second kind on the total space $\mathcal{E}$ of the family of elliptic curves. This means, as we have been asserting all along, that $[\eta] \in H^2(\mathcal{E}; \mathbf{C})$ is trivial and hence that the monodromy of our family of representations of the theory is contained in $SL(2, \mathbf{Z})$, inside the bigger group $ISL(2, \mathbf{Z})$. This was expected since the theory has no conserved charges except n_e, n_m.

§18.8. Description of the BPS spectrum

By further study of the family of elliptic curves that we have found, it can be proved that the curve in the u-plane where a/a_D is real is a simple closed curve that passes through the points $u = \pm\Lambda^2$ and has no other intersection with the axis of real u. The spectrum of BPS states is continuous off this circle. Approaching from $u = \infty$, we see that at $u = \Lambda^2$ we have $a_D = 0$ and $a \neq 0$, whereas at $u = -\Lambda^2$ we have $-a + a_D = 0$ and $a \neq 0$. (Of course, the exact values depend on the path chosen from ∞ to $\pm\Lambda^2$, but if we come in from infinity along the positive or negative real axis then the given equations hold.) Since we can approach the two singularities from infinity without crossing the jumping locus for the BPS states, we see that the BPS states that are becoming massless at $u = \pm\Lambda^2$ are indeed among the original BPS states at infinity. That is to say, these particles are analytic continuations of the original magnetic monopoles at infinity. This confirms what was suggested earlier more informally.

Now let us examine whether, at other points on the jumping locus besides $\pm\Lambda^2$, there are particles forced to become massless. As we move along the jumping line on the open upper arc from Λ^2 to $-\Lambda^2$, we find that for every s with $0 < s < 1$, there is a point where $sa - a_D = 0$. So if there is a BPS state at infinity with charges (n_e, n_m) that satisfy $(n_e, n_m) = t(s, -1)$ for some $t \in \mathbf{R}$, then at some

point along this open arc we find that this state becomes massless. This, and the symmetric argument for the open lower arc of the jumping line, show that all the BPS states at infinity that satisfy of charges (n_e, n_m) with $0 \leq |n_e| \leq |n_m|$ become massless somewhere on the jumping curve (this happens at the points $u = \pm\Lambda^2$ if and only if $n_e = 0$ or $n_e = \pm n_m$). By a monodromy transformation at infinity (which is a symmetry of the spectrum for large u), any BPS state with $n_m \neq 0$ can be mapped to one with $|n_e| \leq |n_m|$. So every BPS state at infinity with $n_m \neq 0$ becomes massless somewhere on the jumping line, if one follows it in from infinity along the appropriate contour.

Since there are no singularities on the jumping line except the ones corresponding to $a_D = 0$ or $a_D - a = 0$, and since those singularities are interpreted in terms of just a single hypermultiplet that becomes massless, we conclude that there are for large u no BPS states with $n_m \neq 0$ except the states with $n_m = \pm 1$ that have been included in our analysis.

This assertion implies the following statement about classical geometry. Let $\mathcal{W}_k$ be the moduli space of k-monopole solutions of this theory, reduced by dividing by translations and global $U(1)$ rotations. It is a hyper-Kähler manifold of dimension $4k - 4$ that has been much studied mathematically. It can be shown that L^2 holomorphic forms (possibly valued in a flat line bundle) on $\mathcal{W}_k$ would give k-monopole bound states, for $k > 1$. Such bound states would create unwanted singularities or would make unwanted contributions to the monodromies at singularities, so there should be no L^2-holomorphic forms on $\mathcal{W}_k$. More generally, an argument by A. Sen using duality of $N = 4$ super Yang-Mills (rather than the $N = 2$ theory that we have considered in this lecture) determines the full L^2-cohomology of the $\mathcal{W}_k$.

References and cross references

As the titles indicate, this lecture builds on Lecture 17; as before the basic reference is

N. Seiberg and E. Witten, *Electric-Magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **426** (1994), pp. 19–52.

A reference for Sen's work is

A. Sen, *Dyon-Monopole Bound States, Selfdual Harmonic Forms On The Multi-Monopole Moduli Space, and $SL(2, \mathbf{Z})$ Invariance In String Theory*, Phys. Lett. **B329** (1994) pp. 217–221.

't Hooft loops and confinement are discussed in Lecture 10. A discussion of one-loop beta function computations occurs in §2 of Lecture 16, and also in Lectures 2 and 3 of [I-**Renorm Groups**], and Lecture 3 of [II-**CFT**]. BPS states are discussed in detail in Lecture 12.

LECTURE 19

$N = 2$ Super-symmetric Yang-Mills Theories in Dimension Four: Part 3, Topological Applications

§19.1. A survey of $N = 2$ supersymmetric gauge theories in dimension four

Let us begin with a brief overview of the range of $N = 2$ supersymmetric gauge theories. We must specify a gauge group G, which will be a compact, connected group. To achieve $N = 2$ supersymmetry, our fields include a vector multiplet consisting of the gauge fields and their super-partner fields with values in the adjoint bundle of the principal G-bundle. We can also have hypermultiplets. Let n be the number of hypermultiplets. The global symmetry group of n free hypermultiplets is the symplectic group $Sp(n)$ (the unitary group of $n \times n$ quaternion matrices). Gauging the theory in an $N = 2$-invariant fashion can be done by gauging any desired subgroup of $Sp(n)$. So in general the hypermultiplets lie in vector bundles associated to the principal G-bundle by a representation $\rho\colon G \to Sp(n)$.

The most important invariant for the qualitative nature of the theory is the β-function, and in particular, the one-loop β-function. The formula for the one-loop β-function is

$$\beta_1(g) = -\frac{g^3}{8\pi^2}(4h - c_2(\rho)),$$

where h is the dual Coxeter number of the group G and $c_2(\rho)$ is the trace of the quadratic Casimir for ρ, normalized so that the defining representation of $SU(2)$ has c_2 equal to one. Of course, $2h$ is simply $c_2(\text{adjoint})$, the quadratic Casimir of the adjoint representation of G on its Lie algebra. There are three possibilites:

- The one-loop β-function vanishes.
- The one-loop β-function is negative.
- The one-loop β-function is positive.

Since

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Notes by John Morgan

we have the expansion

$$\mu\frac{d}{d\mu}(\tau) = \text{constant} + \sum_{\ell\geq 1} C\cdot g^{2\ell} + \sum c_{n,\ell} g^{2\ell}\exp(2\pi i n\tau),$$

where the constant term is computed from the one-loop β-function, the higher powers of g arise from higher loop contributions and the exponential terms come from instanton corrections (which are of course not seen in perturbation theory). By supersymmetry $\mu\frac{d}{d\mu}(\tau)$ is a holomorphic function, and hence the positive powers of g in this expansion vanish identically. Thus, if the one-loop β function vanishes then β vanishes in perturbation theory. In this case it is believed that $\beta \equiv 0$. This has not been shown rigorously, but apparently, when the one-loop β-function vanishes, the sort of low energy effective field theory that we analyzed in the last two lectures does not make sense unless the β-function is identically zero. (By contrast, $N = 2$ theories with a nonzero one-loop β-function do have nonperturbative instanton corrections, as we have seen for the pure $SU(2)$ theory.)

One important special case of an $N = 2$ theory with $c_2(\rho) = 4h$ is the case that the hypermultiplet representation ρ is two copies of the complexification of the adjoint representation.[1] This theory in fact has $N = 4$ supersymmetry. Using this enhanced supersymmetry one can show that the β-function vanishes identically.

When $\beta \equiv 0$, the quantum theory has a well-defined constant τ just like the classical theory. It is believed that in this case there is a subgroup Γ of finite index in $SL(2,\mathbf{Z})$ with the property that the theory is invariant under a faithful action of Γ. Of course, when $\beta \neq 0$ the theory has a mass scale and τ is no longer constant. For these theories one would expect, as we saw for the pure $SU(2)$ theory in the last two lectures, that there is not an exact duality symmetry of the theory, but rather that a duality among possible low energy descriptions is important in understanding the low energy physics.

When the one-loop β-function is negative, then the β-function is negative, at least for sufficiently high energy scales. (The pure $SU(2)$ theory we studied last time is of this type.) In this case, we have an asymptotically free theory which is a good fundamental theory. That is to say we have a well-defined *problem* (to describe the low energy effective version of the theory). We solved this problem in the last lecture for $G = SU(2)$ or $SO(3)$ with ρ being the trivial representation.

When the one-loop β-function is positive, we do not have a well-defined problem, since the theory is not asymptotically free. Rather these theories are candidate *solutions* to the problems posed by the theories with one-loop β-function negative. To be more precise, a theory $T_{\rm mac}$ with positive β-function theory is a candidate solution in the sense that its low energy effective theory is well-defined and infrared free. It is a solution to the problem posed by a theory, $T_{\rm mic}$, with negative β, if $T_{\rm mic}$ flows in the infrared to $T_{\rm mac}$ in the sense that at long distances (or equivalently low energies) the correlation functions of the two theories converge to each other. The general problem of matching questions and answers, of figuring out which $T_{\rm mac}$'s can arise as infrared limits of a given $T_{\rm mic}$, has not been solved, though there are extensive results in many interesting cases.

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[1] Any real representation W of rank n – here W is the adjoint representation and n is the dimension of the group – can be regarded as a representation in $SO(n)$. Two copies of $W\otimes_{\mathbf{R}}\mathbf{C}$ can be regarded as a representation in a symplectic group $Sp(n)$. We will for brevity refer to "two copies of the complexification of W" as "W tensored (over $\mathbf{R}$) with the quaternions," where the quaternions are simply understood as $\mathbf{C}^2$ with a nondegenerate skew form.

The gauge theories we are discussing today are lagrangian models that arise by quantizing classical physics. At least in passing, I should mention that there exist more exotic $N = 2$ theories in four dimensions, some of which come from string theory. Not all of these have known lagrangian formulations. Whether or not they have lagrangians, these $N = 2$ supersymmetric theories should also lead to four-manifold invariants upon twisting. Those without lagrangians could possibly lead to four-manifold invariants which do not have a classical description in terms of the moduli space of solutions to some differential equation.

§19.2. From Minkowski space to a compact Riemannian four-manifold

Let us fix one of the $N = 2$ supersymmetric asymptotically free gauge theories (one with $\beta < 0$). For example, if we wish to study Donaldson theory we will take pure $N = 2$ supersymmetric $SU(2)$ gauge theory. But for a while we can be more general. The theory, as we have discussed it so far, has been on Minkowski four-space. That is to say, we have written down an action which involves the integration over Minkowski space of a lagrangian function of various fields on this space. We wish to pass from this to a theory defined on a compact Riemannian four-manifold. This passage is carried in two steps – i) rotating from Minkowski four-space to Euclidean four-space, and ii) globalizing to a compact Riemannian four-manifold.

Let us recall from what we learned last fall how to Wick rotate the theory to Euclidean four-space. Our theory consists of a family of local operators with an operator product expansion (O.P.E). A realization of the theory gives values to the correlation functions of these operators at various points of Minkowski four-space. But there is an open subset of the complexification of Minkowski four-space where the O.P.E. and the correlation functions are defined by analytic continuation. This open subset contains Euclidean four-space. In this way we Wick rotate to define the theory on Euclidean four-space (see, for example, Lecture 2 of [I-**Axiomatics**]). In our case we are dealing with (supersymmetric) actions

$$S = \int_{M^4} \mathcal{L}$$

which are integrals of lagrangians $\mathcal{L}$ which are analytic expressions involving fields ϕ on the underlying Minkowski space. These expressions extend to holomorphic expressions on complexified Minkowski space and then can be restricted to the Euclidean subspace to produce a Euclidean lagrangian $\mathcal{L}_E$. The Euclidean action is defined by

$$S_E = \int_{\mathbf{R}^4} \mathcal{L}_E.$$

The path integral for a correlation function in Minkowski space is

$$\int \mathcal{D}\phi \mathcal{O}_1(x_1) \cdots \mathcal{O}_t(x_t) e^{iS},$$

and under this process it Wick rotates to the path integral

$$\int \mathcal{D}\phi \mathcal{O}_1(x_1) \cdots \mathcal{O}_t(x_t) e^{-S_E}$$

computing the correlation function in Euclidean space. Even if one is interested in Minkowski space, computations are often done in Euclidean space because of some technical simplifications. For more details on the relation between the two, see Chapter 7 of [I-**Classical Fields**].

§19.2.1. Globalizing by twisting – the case of the supersymmetry algebra

Now let us examine the question of putting the quantum field theory on an oriented Riemannian four-manifold X. Our supersymmetric theory is defined over four-space by an action S which is an integral of a lagrangian function of local fields; these fields transform as scalars, spinors, and vectors under Spin(4). Of course, there are bundles on Minkowski or Euclidean four-space associated to these representations. In order to write the "same" action on a four-manifold X we need to be able to write down the supersymmetric lagrangian on that manifold. This means that all fields that appear in the lagrangian must be globally defined (as sections of bundles) on X. This will not in general be the case for the theory as it has been presented so far because the relevant spin bundles may not exist globally on X. In fact the "physical" theory on a Riemannian four-manifold (which one would use in actually coupling the theory to general relativity) requires that X be spin and generically breaks all supersymmetry. We have seen before (§3 of Lecture 14) how to deal with this problem for topological applications – we twist the theory by an appropriate homomorphism of the spin group Spin(4) to the R-symmetry group. As we shall see, an appropriate twist will both allow us to write down the lagrangian on any oriented Riemannian four-manifold and also allow us to preserve a crucial piece of the supersymmetry.

The decomposition of $\Lambda^2 T^*\mathbf{R}^4$ into self-dual and anti-self-dual components is covered by a decomposition of the group Spin(4) as $SU(2)_+ \times SU(2)_-$. Fortunately, for $N = 2$ the R-symmetry group is $U(2)_R = SU(2)_R \times U(1)_R$. (The $U(1)_R$ is often absent, e.g., if the hypermultiplet has nonzero bare mass. It is also anomalous if $\beta \neq 0$. But even so, we still have $SU(2)_R$ as an R-symmetry group.) With this R-symmetry group twists are possible, since there exist nontrivial homomorphisms from Spin(4) to $SU(2)_R$. (For $N = 1$ the R-symmetry is $U(1)_R$ and, in four dimensions, there is no possibility for twisting.) Recall from Chapter 10 of [I-**Supersolutions**] that the supersymmetric generators are right-invariant vector fields on $\mathbf{R}^{4|8}$, denoted as Q^i_α and $\overline{Q}_{j\dot\alpha}$ where $1 \leq i, j, \alpha \leq 2$. Both of these sets are acted on by $SU(2)_R$ in the standard fashion on the i, j indices. (The α and $\dot\alpha$ are the spinor indices for $SU(2)_+ \times SU(2)_-$.) The actions of the spin group and the $SU(2)_R$ are summarized in the following table.

	Transformation under		
Charges	$SU(2)_+$	$SU(2)_-$	$SU(2)_R$
Q^i_α	$\frac{1}{2}$	0	$\frac{1}{2}$
$\overline{Q}_{j\dot\alpha}$	0	$\frac{1}{2}$	$\frac{1}{2}$

We define $SU(2)'$ as the diagonal subgroup of $SU(2)_+ \times SU(2)_R$ under the obvious identification of these two groups. Working still on $\mathbf{R}^4$, we shall be interested

in an action of the Poincaré group in which the usual Spin(4) group of rotations is replaced by $SU(2)' \times SU(2)_-$. If follows from the information in the table that under $SU(2)' \times SU(2)_-$, the Q^i_α transform in the representation

$$\left(\frac{1}{2} \otimes \frac{1}{2}, 0\right) = (0,0) \oplus (1,0).$$

This means that the twisted form of the vector fields Q^i_α decompose as a complex-valued vector field and a vector field with values in complex self-dual two-forms. Similarly, the $\overline{Q}_{\jmath\dot\alpha}$ transform in the representation

$$\left(\frac{1}{2}, \frac{1}{2}\right)$$

of this same group and thus are tangent vectors, so that the $\overline{Q}_{\jmath\dot\alpha}$ become vector fields with values in the complexified tangent bundle. Thus, both these representations are vector representations of Spin(4) and hence exist globally on any oriented Riemannian 4-manifold. Also, notice that these representations are naturally real representations (even though the original ones were not). This means that on the Riemannian manifold we can take real fields.

In the superspace language, twisting and going to a general four-manifold means replacing the supermanifold $\mathbf{R}^{4|8}$ (which is the supersymmetric extension of $\mathbf{R}^4$) by a split supermanifold whose even part is a four-manifold X and whose odd part is the parity reversed vector bundle $\wedge^0 TX \oplus TX \oplus \wedge^2_+(TX)$. On this supermanifold we have a globally defined real vector field Q defined as follows. Suppose that we have local coordinates $(x^1, \ldots, x^4)$ on an open subset of X. We let $\theta^{\mathbf{R}}$ be the natural coordinate in the odd $\wedge^0 TX$-direction and θ^i be the coordinates in the odd TX-direction determined by the x^i. We denote by $\partial_{\mathbf{R}^{\text{odd}}}$ and $\partial_{i,\text{odd}}$ be the corresponding vector fields in these odd directions. Lastly, $\partial_{i,\text{even}}$ is the usual partial derivative in the x^i-direction. We write

$$Q = \partial_{\mathbf{R}^{\text{odd}}} + \theta^i \partial_{i,\text{even}}.$$

Clearly, as we change local coordinates, this expression is invariant. This gives us a globally defined real vector field Q on the supermanifold which is the globalization of the $(0,0)$ component of the twisted form of the local supersymmetry generators Q^i_α. It is because of the presence of this one global, everywhere nonzero, supersymmetric generator satisfying $Q^2 = 0$ that we can do global topological quantum field theory on X.

We also have a globally defined one-form K on X with values in vector fields on the supermanifold. It is derived from the twisted form of $\overline{Q}_{\jmath,\dot\alpha}$. A formula for it in the same local coordinates is:

$$K = \left(\partial_{i,\text{odd}} + \theta^{\mathbf{R}} \partial_{i,\text{even}}\right) dx^i.$$

One sees directly from the formula that K is invariant under change of coordinates, and hence is a well-defined global object on X.

It is easy to deduce from the super-Poincaré algebra structure, or from the above explicit formulas, that $Q^2 = 0 = \{K, K\}$. Direct computation from the above formulas, or the general supersymmetry algebra structure, also gives

$$\{Q, K\} = d, \tag{19.1}$$

where d is the exterior derivative on X.

In fact, by Noether's theorem there is a local (or unintegrated) form of these equations. In Minkowski space, the translation generators P_μ are obtained by integrating the stress tensor T, and the supersymmetry charges are similarly obtained by integrating the supercurrent $\mathcal{S}$; it a three-form (or by duality a vector field) with values in spinors:

$$\begin{aligned} P_\mu &= \int_\Gamma d^3x T_{\mu 0} \\ \overline{Q}_{\dot\alpha} &= \int_\Gamma d^3x \bar{\mathcal{S}}_{\dot\alpha 0}. \end{aligned}$$

After twisting, the supercurrent decomposes into several pieces, one of which can be regarded as a three-form with one-form values; we denote this simply as $\mathcal{S}$. The second of the above equations becomes in the twisted theory

$$K = \int_\Gamma \mathcal{S}.$$

The unintegrated form of Equation 19.1 is

$$\{Q, \mathcal{S}\} = T.$$

This equation is very important, as it is the basis for a natural proof of topological invariance of the formalism we develop presently.

Like all such relations, the formula $\{Q, \mathcal{S}\} = T$ can be read as a statement about Hamiltonian vector fields or about the transformations they generate. However – for our application to a global four-manifold X – the formula is most powerfully read as follows: $\mathcal{S}$ and T are local operators on X; Q is the globally-defined automorphism of the operator algebra on X that is induced from the odd symmetry $Q = \partial_{\mathbf{R}^{\text{odd}}} + \theta^i \partial_{i,\text{even}}$ of the split supermanifold constructed from X. With that interpretation of the symbols, the statement $\{Q, \mathcal{S}\} = T$ makes sense.

§19.2.2. Topological Quantum Field Theory on X

We shall be considering expectation values of products of Q-closed operators. Since the expectation value of any Q-exact operator is trivial, it follows that the expectation value of any product of Q-closed operators is zero provided that at least one of them is Q-exact.

The fact that $\{Q, T\} = 0$ means that as we vary the metric, the correlation functions change by the insertion of an operator which is in the image of Q. It then follows that the correlation functions we will consider are independent of the metric. The operators that we have in mind are the analogues in this more general gauge theory setting of the Donaldson classes. The independence on the metric of the correlation functions of Q-closed operators is the generalization of the independence of the Donaldson invariants under change of metric.

Let us recall the general scheme of generating operators with values in higher dimensional differential forms out of local operators with values in functions. Here we are generalizing from two dimensions to four dimensions the discussion and results of Lecture 11 and 14.

Begin with a polynomial function P of degree d on the Lie algebra $\mathfrak{g}$ of the gauge group G, invariant under the adjoint representation. Classically, this produces a Chern form $P(F_A)$ of degree $2d$ when applied to any connection A on a principal G-bundle. Applying this to the universal connection on the universal bundle over

the product of the manifold X with the space $\mathcal{B}$ of gauge equivalence classes of configurations yields a closed form of degree $2d$ on $X \times \mathcal{B}$. Recall from the last two lectures that part of the $N = 2$ supersymmetric gauge multiplet is an $N = 1$ chiral multiplet which includes a scalar field ϕ with values in the adjoint bundle. Quantum mechanically, the polynomial function P applied to ϕ produces a Q-closed local operator $\mathcal{O}_P^{(0)}$ in the theory. Applying K repeatedly yields operators $\mathcal{O}_P^{(n)} = K^n \mathcal{O}_P^{(0)}$. These are local operators with values in n-forms on X. For an n-cycle $\Sigma^n \subset X$, the operator

$$\int_{\Sigma^n} \mathcal{O}_P^{(n)}$$

is the analogue of taking the slant product of the Chern form $P(F_A)$ of the universal connection on $X \times \mathcal{B}$ with the cycle Σ^n to produce a closed form $\mu_P(\Sigma)$ of degree $2d - n$ on $\mathcal{B}$. As we shall see, the expectation value

$$\left\langle \prod_{i=1}^{s} \int_{\Sigma_i^{n_i}} \mathcal{O}_P^{(n_i)} \right\rangle$$

computes

$$\sum_E \int_{\mathcal{M}_E} \mu_P(\Sigma_1^{n_1}) \cup \cdots \cup \mu_P(\Sigma_s^{n_s})$$

where E ranges over the topological types of principal G-bundles over X and where $\mathcal{M}_E \subset \mathcal{B}_E$ is the moduli space of classical solutions. This integral of course is only taken when the virtual dimension of $\mathcal{M}_E$ is equal to $\sum_{i=1}^{s}(2d - n_i)$.

From the fact that $\{Q, K\} = d$ and the fact that $\left\{Q, \mathcal{O}^{(0)}\right\} = 0$, we have

$$\left\{Q, \mathcal{O}^{(n)}\right\} = d\mathcal{O}^{(n-1)}.$$

This implies that the integral of $\mathcal{O}^{(n)}$ over an n-cycle is Q-invariant.

§19.2.3. Twisting and globalizing the $N = 2$ vector multiplet

We still have work to do before we can install the supersymmetric theories on a general Riemannian manifold. So far we have studied the effect of twisting on the super-Euclidean group, and seen how to implement that on an oriented Riemannian manifold. We must still examine the effect on the fields in the lagrangian. In this section we study the vector multiplet.

Let us recall the basic configurations of the (untwisted) fields from an $N = 2$ vector multiplet in the $N = 1$ language. First the $N = 2$ vector multiplet decomposes as an $N = 1$ vector multiplet

$$\mathcal{A} = (A, \lambda, D)$$

and an $N = 1$ chiral multiplet

$$\Phi = (\phi, \zeta, F).$$

In the first (vector) multiplet, A is a gauge field, λ is a spinor with values in the adjoint bundle and D is a real auxiliary field with values in the adjoint bundle. In the chiral multiplet, ϕ is section of the complexification of the adjoint bundle, ζ is a spinor with values in the adjoint bundle, and F is a complex-valued auxiliary field with values in the adjoint bundle. Under the $SU(2)_R$-symmetry, the pair (λ, ζ) transform in a two-dimensional representation denoted ψ_α^i and $\vec{D} = (D, \operatorname{Re} F, \operatorname{Im} F)$ transforms under the adjoint representation of $SU(2)_R$.

The following table summarizes the $U(1)_R$-charge of the various fields

The $N = 2$ Vector Multiplet

$U(1)_R$ charge	-2	-1	0	1	2
Field	$\overline{\phi}$	$\overline{\psi}_{\dot{\alpha}j}$	A $\vec{D}$	ψ^i_α	ϕ

Now we twist as described above using $SU(2)_R$. The $SU(2)_R$ acts trivially except on $\vec{D}$ and on the fermions; thus these are the only fields which change character when we twist. After twisting, the ψ^i_α transform in the $(1/2, 1/2)$ representation. This means that after twisting, the ψ^i_α become a one-form with values in the adjoint representation; we denote it as $\psi_{(1)}$. Similarly, after twisting the $\overline{\psi}_{\alpha j}$ transform in $(\frac{1}{2} \otimes \frac{1}{2}, 0)$ and hence decompose as a direct sum of a function $\psi_{(0)}$ and a self-dual two-form $\psi_{(2)}$. The auxiliary field $\vec{D}$ also transforms in $(1, 0)$ and hence becomes a self-dual two-form on X.

We see that, after twisting, all these fields exist globally on X since they have become differential forms. This means that as far as pure gauge theories are concerned, we can write the lagrangian on any closed oriented 4-manifold preserving the supersymmetric charge Q.

Let us examine the action of Q on these fields. The fields $(A, \psi_{(1)}, \phi)$, with Q acting as a differential, form a model of the equivariant cohomology of the gauge group action on the space of connections. To make the notation more suggestive of a differential we let δ denote the action of Q. Then we have

$$\begin{aligned} \delta A &= \psi_{(1)} \\ \delta\psi_{(1)} &= -d_A\phi \\ \delta\phi &= 0. \end{aligned}$$

The Q-closed operators have representatives modulo Q-exact operators using only these fields. We begin with any operator $\mathcal{O}_P^{(0)} = P(\phi)$ given by a gauge-invariant polynomial in ϕ. For example, in the $SU(2)$ case we take $P(\phi) = \operatorname{Tr}\phi^2$. This polynomial has $U(1)_R$-charge four and corresponds to the Donaldson class $\mu(\mathrm{pt})$ in the fourth cohomology. From these local operators we build operators with values in forms by our formula $\mathcal{O}_P^{(j)} = K^j\mathcal{O}_P^{(0)}$.

The action of δ on the other fields is:

$$\begin{aligned} \delta\overline{\phi} &= \psi_{(0)} \\ \delta\psi_{(0)} &= [\phi, \overline{\phi}] \\ \delta\psi_{(2)} &= F_A^+ - \vec{D} \\ \delta(F_A^+ - \vec{D}) &= [\phi, \psi_{(2)}]. \end{aligned}$$

Here we are following the notation that the curvature F_A is decomposed into its self-dual and anti-self-dual components $F_A^+ + F_A^-$. The interpretation is as follows. If one includes just $\overline{\phi}$ and $\psi_{(0)}$, one gets a model of the equivariant cohomology of the gauge

group that was described in Lecture 14.4. Including $\psi_{(2)}$ and $\widetilde{D} = F_A^+ - \overrightarrow{D}$ gives a complex that, formally, computes the equivariant cohomology of the gauge group with values in the parity reversal of the space of adjoint-valued two-forms. Since the space of adjoint-valued two-forms is equivariantly contractible, its inclusion does not affect the equivariant cohomology. Inclusion of $\overline{\phi}$, $\psi_{(0)}$ likewise does not change things, as explained in Lecture 14.4. That is why the classes in Q-cohomology have representatives constructured from $\phi, \psi_{(1)}, A$ only.

§19.2.4. Twisting and Globalizing the $N = 2$ hypermultiplet

Let us turn now to the $N = 2$ hypermultiplet. We have the following table describing the nature of the fields in this $N = 2$ multiplet before twisting.

The $N = 2$ Hypermultiplet

Helicity	$-1/2$	0	$1/2$
$U(1)_R$ – charge	-1	0	1
$SU(2)_R$ – action	trivial	$1/2$	trivial
Field	$\overline{\lambda}_{\dot{\alpha}}$	M	λ_α
Type	$-$chirality spinor	boson	$+$chirality spinor

The boson and the spinors of both chiralities in the above table all take values in the quaternionic bundle over X associated to the representation ρ.

After twisting, the boson M becomes a spinor of type $(1/2, 0)$, that is to say M becomes a plus chirality spinor (a section of the plus spin bundle). The fermions λ and $\overline{\lambda}$ are unchanged under twisting since the $SU(2)_R$ acts trivially on them. Thus, λ remains a spinor of plus chirality and $\overline{\lambda}$ remains a spinor of minus chirality. Thus, in order to write the part of the lagrangian involving the hypermultiplet, we must be able to make sense of the tensor product of the spin bundles $S^{\pm}(X)$ of each chirality with the quaternionic representation ρ. (Of course, it is not necessary that the spin bundles actually exist globally, only that these tensor products exist.)

Assuming that we are in this situation, let us describe the action of the differential Q on these fields. Again denoting it by δ we get

$$\begin{aligned} \delta M &= \lambda \\ \delta\lambda &= \phi(M) \\ \delta\overline{\lambda} &= \not{\partial}_A(M) + \text{fermions}. \end{aligned}$$

In writing $\phi(M)$, we understand that the element ϕ of the Lie algebra acts on the hypermultiplet according to the representation of the gauge group in which the hypermultiplet transforms, whatever that may be.

§19.3. The general form of the high energy computations

At this point, we have succeeded in defining a twisted version of our $N = 2$ supersymmetric gauge theory on any oriented Riemannian four-manifold X for which the

bundles $S^{\pm}(X) \otimes \rho$ are defined. In particular, this includes all pure $N = 2$ gauge theories on any oriented Riemannian four-manifold. Since the ultraviolet behavior of the theory is independent of the metric on the manifold and independent of any global topology, we see that if we began with a fundamental theory (one which has a negative β-function and hence is asymptotically free), then the implementation of the theory on a complete Riemannian four-manifold will also be asymptotically free. Thus, it is a well-defined problem to compute the correlation functions in this theory.

We are interested in the correlation functions of the Q-closed local operators in the theory. There is a localization result to the effect that to compute these in the UV limit one does not need to integrate over the entire infinite dimensional space of fields, but rather only over the fixed points of the action of Q on the space of fields (see §4 of Lecture 14). Another way to think about this is that the correlation functions we are computing are independent of the metric and we can compute in the limit when as the metric shrinks to zero, or equivalently by asymptotic freedom, as the coupling constant goes to zero. In the limit we are doing a classical computation over the minima of the lagrangian – these minima are exactly the fixed points of Q.

Let us examine the bosonic part of the fixed point set of Q. From the equations above, ignoring the fermion components and working modulo the action of the gauge group, we see that the fixed points of Q are given by:

$$\begin{aligned} d_A\phi &= 0 \\ 0 &= F_A^+ - \vec{D} \\ 0 &= \not{\partial}_A(M) \end{aligned}$$

Now using the equations of motion allows us to integrate out the auxiliary field $\vec{D}$ giving the equation

$$\vec{D} = \mu(M)$$

where μ is the hyper-Kähler moment map. Thus, the above system becomes

$$d_A\phi = 0 \tag{19.2}$$

$$F_A^+ = \mu(M) \tag{19.3}$$

$$0 = \not{\partial}_A(M) \tag{19.4}$$

Away from solutions where A is reducible, the first equation implies that $\phi = 0$, and we can ignore ϕ and this equation. At the reducible solutions, however, this equation is important. Inclusion of the ϕ field is very important in correctly treating the contributions of reducible solutions, which of course are the source of many important subtleties in Donaldson theory.

Notice that Equations 19.3 and 19.4 are the Seiberg-Witten equations in the more general setting where the gauge group is not required to be $U(1)$. Also, notice that if we are considering a pure $N = 2$ supersymmetric gauge theory (with no hypermultiplets), then $M = 0$ and these equations become the usual Yang-Mills instanton equations.

Our correlation functions will localize to be integrals over supermanifolds whose underlying geometric manifold is a disjoint union of the moduli spaces

$$\mathcal{M}_E = \left\{ \begin{array}{l} d_A\phi = 0 \\ F_A^+ = \mu(M) \\ 0 = \not{\partial}_A(M) \end{array} \right\} /(\text{Modulo gauge}),$$

where the union is taken over all components of the moduli space for a given E; one may also find it natural to sum over E.

Recall that, assuming that X is compact, there is an index theorem which computes the virtual dimension of $\mathcal{M}_E$. It is

$$\begin{aligned} &\text{virt. dim}(\mathcal{M}_E) \\ &\quad = k\,(4h - c_2(\rho)) - \frac{1}{2}\dim(G)\,(\chi(X) + \sigma(X)) - \frac{\sigma(X)}{4}\text{quaternion dim}(\rho), \end{aligned}$$

where k is the instanton number of the bundle E, $\chi(X)$ is the Euler characteristic of X and $\sigma(X)$ is its signature. If G is simply connected and simple then

$$k = \int_X \frac{\text{Tr}(F \wedge F)}{8\pi^2}.$$

Notice that if $\beta = 0$ then the virtual dimension of $\mathcal{M}_E$ is independent of E, so (if this virtual dimension is non-negative) all values of k can be considered. When $\beta < 0$, the virtual dimension of the $\mathcal{M}_E$ go to infinity as the instanton number of E goes to infinity, so that for large k, correlation functions of seemingly greater and greater complexity can be considered. But when $\beta > 0$, the virtual dimension is negative for $k >> 0$. Since a vanishing theorem shows also that the moduli spaces are empty for $k << 0$, it follows that for $\beta > 0$ there are only finitely many values of k to be considered, and hence only finitely many invariants to be computed. This gives another sense in which theories with $\beta > 0$ are "answers": the topological part of a $\beta > 0$ theory involves only finitely many invariants which hopefully can be computed, in principle (for a given four-manifold) in finitely many steps, in contrast to the seemingly unbounded complexity of the $\beta \leq 0$ theories.

As we have said before, we are interested in computing the expectation values of products of operators of the form

$$\int_{\Sigma_i} \mathcal{O}_{P_i}^{(k_i)}$$

for invariant polynomials P_i on the Lie algebra $\mathfrak{g}$ and for k_i-cycles Σ_i in X. In the case of pure $SU(2)$ gauge theory, these operators correspond to the usual operators in Donaldson theory in the following way. Since the gauge group is $SU(2)$, the space of irreducible polynomials is generated by $\text{Tr}\,\phi^2 = u$. If we use this polynomial invariant to define $\mathcal{O}^{(0)}$, then this operator corresponds to what is usually denoted $\mu(\text{pt})$ in Donaldson theory. That is to say, the operator corresponds to the cohomology class on the moduli space which is minus one quarter the first Pontrjagin class of the $SO(3)$-bundle over the moduli space given by the based moduli space. Furthermore, the derived operators

$$\int_\Sigma \mathcal{O}^{(k)}$$

(for Σ a k-cycle) correspond to the cohomology classes $\mu([\Sigma]) \in H^{4-k}(\mathcal{M}_E)$.

More generally, one can begin with any polynomial function P of degree d on the Lie algebra invariant under the adjoint representation to produce $\mathcal{O}_P^{(0)}$ and the derived operators $\mathcal{O}_P^{(n)}$. In general, we know from topological considerations (the Künneth formula) how to decompose operators derived out of a decomposable invariant polynomial function as a sum of products of operators constructed from indecomposable invariant functions. Thus, in general it suffices to work exclusively with operators derived from indecomposable invariant polynomial functions on the Lie algebra. In the case of su(2) there is only the indecomposable invariant polynomial $\mathrm{Tr}\,\phi^2$. Hence, it suffices to compute operators derived from this one local zero dimensional operator. Restricting to this operator and its "descendants" is standard in Donaldson theory.

In general, to compute a correlation function

$$\left\langle \prod_{i=1}^{s} \int_{\Sigma_i} \mathcal{O}_P^{(b_i)} \right\rangle,$$

with P_i of degree d_i, we must do the integration over all components of the supermanifold of solutions whose underlying geometric manifold, $\mathcal{M}_E$, satisfies

$$\text{virt. dim}(\mathcal{M}_E) = \sum_{i=1}^{s} (2d_i - b_i).$$

Actually, we can do slightly better in the case that G is not simply connected. We can sum over all principal G-bundles E which have a fixed isomorphism class over the two-skeleton of X. [2] This restriction "commutes" with the evaluation of the path integral by expressing it in terms of a low energy effective lagrangian., because the restriction of the bundle to the two-skeleton cannot be changed by changing the data on a ball in X.[3] A more obvious restriction, fixing the value of k, would not be similarly well-behaved, because a sum over all values of the instanton number in every local region of spacetime is one of the ingredients in the construction of the low energy effective lagrangian that we introduced last week.

When the instantons are all smooth and irreducible, the supermanifold of instanton solutions is simply the parity reversal of the tangent bundle of the ordinary instanton moduli space. Integration of functions on that particular type of supermanifold is equivalent to integration of differential forms over the underlying geometric manifold. The dimension equation ensures that we are integrating a top dimensional form over the geometric manifold $\mathcal{M}_E$ underlying the supermanifold of solutions.

§19.3.1. A case when $\beta = 0$

Let us continue with the case when $\beta = 0$. For simplicity, we take the gauge group G to be a connected (compact) semisimple group. Then, since the $\mathcal{M}_E$ all have the same formal dimension, our integration is over all values of the instanton number at once. To write down precisely the correct formula, we need the following subtlety.

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[2]This refinement is vacuous if G is simply connected, for then all G-bundles are trivial when restricted to the two-skeleton.

[3]When the metric of X is scaled up, the ball approximates $\mathbf{R}^4$, and the theory in the ball can be analyzed by knowledge of low energy effective field theory on $\mathbf{R}^4$. Low energy effective field theory sums up the effects of all fields in the ball with given values (of relatively low energy) on the boundary of the ball. All such fields have the same topological restriction to the two-skeleton.

We have not written down the lagrangian explicitly on the curved four-manifold, only on flat four-space. In putting the theory on a curved space we can add terms to the action which involve the metric. Since $\beta = 0$, there is a well-defined coupling constant τ for these theories. There are two such terms that preserve the topological invariance, namely:

$$\int_X (f(\tau)\mathrm{Pf}(R) + g(\tau)L(R))$$

where R is the Riemann curvature tensor of the metric on the four-manifold, Pf denotes the Pfaffian, and L denotes the L-polynomial, and f, g are holomorphic functions of τ. (Since $\overline{\partial}/\partial\overline{\tau}$ acting on the lagrangian gives $\{Q, \cdot\}$, the expectation value of any product of Q-invariant operators is a holomorphic function of τ.) These give topological terms computing universal multiples of $\chi(X)$ and $\sigma(X)$, respectively. Adding in such terms inserts a multiplicative factor in the correlation function. Even if we begin with a classical lagrangian without these extra terms, they would be added in the renormalization process, and the coefficients with which they appear will depend on the explicit nature of that process. Thus, in the end what we are computing is of the form

$$q^{a(\tau)\chi(X)+b(\tau)\sigma(X)} \sum_k \left\langle \prod_{i=1}^{s} \int_{\Sigma_i^{b_i}} \mathcal{O}^{(b_i)} \right\rangle_k q^k$$

for functions a, b derived from f, g. For example, if f, g are linear functions of τ, then a, b are constants. Here, we have fixed the isomorphism class of E over the two-skeleton and indicated by k the instanton number. [4] Also, $q = e^{2\pi i\tau}$ is the usual instanton-counting parameter, and the factor $q^k \cdot q^{a(\tau)\chi(X)+b(\tau)\sigma(X)}$ is the value of $\exp(-S)$ on the component of the moduli space of classical solutions with instanton number k. Because of the proposed invariance of such a theory under a subgroup $\Gamma \subset SL(2, \mathbf{Z})$ of finite index, this result should transform as a modular form under Γ, possibly with poles at the cusps of the quotient of the upper half-plane by Γ.

An important step in understanding this sum is to determine the order of the poles at the cusps. One can effectively determine an integer k_0 (possibly negative in general) such that if $\mathcal{M}_E$ is non-empty then $k \geq k_0$. This gives a bound on the order of the pole at the cusp $q = 0$. By similarly estimating the appropriate k_0 after a duality transformation, one determines the order of the poles at other cusps.

Another important step is to get a simplified method of computation that is valid for generic k. This is done by noting that if $k >> 0$, then the index of $\partial\!\!\!/_A$ is negative, and hence at least for generic A there are no solutions to $\partial\!\!\!/_A(M) = 0$ except the trivial solution $M = 0$. Thus, for sufficiently big k, the computation on $\mathcal{M}_E$ reduces to a computation on the more familiar moduli space $\mathcal{M}_E^{\text{instantons}}$ of solutions to the anti-self-dual equations with $M = 0$. In this case, a topological argument shows that the correlation function is given by

$$\int_{\mathcal{M}_E^{\text{instanton}}} \chi(\mathcal{V}) \prod_i \int_{\Sigma_i^{b_i}} \mathcal{O}^{(b_i)}$$

where $\mathcal{V}$ is the obstruction bundle, which in this case is identified with the cokernel of $\partial\!\!\!/_A \colon S^+ \otimes \rho \to S_- \otimes \rho$, and $\chi(\mathcal{V})$ is its Euler class. (In one of the homework problems, the two-dimensional analog of this statement is treated.)

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[4] Depending on the gauge group, the instanton number may not be an integer but its value modulo $\mathbf{Z}$ is determined by the isomorphism class of the bundles over the two-skeleton.

One case where there is a vanishing theorem for $\not{\partial}_A$ for all instanton numbers is the case when the gauge group is $SU(2)$, $\rho = \mathrm{su}(2) \otimes$ (quaternions), and the manifold is a hyper-Kähler manifold. In this case we have $N = 4$ supersymmetry, and $\mathcal{V}$ is identified with the tangent bundle of the instanton moduli space. In particular, the virtual dimension of $\mathcal{M}_E$ is zero, so the $\int_{\Sigma_i^{b_i}} \mathcal{O}^{(b_i)}$ must all be set to 1. Also, $\chi(\mathcal{V})$ is the Euler class of $\mathcal{M}_E$, and the integral over $\mathcal{M}_E$ computes the topological Euler characteristic of instanton moduli space. That is to say, we are computing

$$q^C \sum_k q^k \chi(\mathcal{M}_{E_k}),$$

for an appropriate constant C. Duality symmetry of the quantum field theory implies that this series in q transforms like a modular form (possibly with poles at cusps), for a subgroup $\Gamma \subset SL(2, \mathbf{Z})$ of finite index. (In fact, Γ is the 2-congruence subgroup $\Gamma_0(2)$.) This prediction can be further sharpened by describing the dependence on $w_2(E)$, and has been compared successfully in some detail to mathematical computations.

§19.3.2. Donaldson Theory

Now we are ready to specialize to Donaldson theory, which is the case of pure $SU(2)$ theory, pure in the sense that the representation ρ is trivial so that there are no hypermultiplets. We change notation slightly from the last lecture; we set the renormalization scale Λ to 1, so the singularities on the u-plane are at $u = \pm 1$. (This choice is provisional and is slightly modified at the end of the lecture.)

As we remarked above, the basic operator is $\mathcal{O}^{(0)} = u = \mathrm{Tr}\,\phi^2$, coming from the quadratic invariant polynomial on su(2), and the derived operators $\mathcal{O}^{(i)}, i = 1, \ldots, 4$ which are to be integrated over i-cycles in X. We denote the various components of the geometric manifold underlying the supermanifold of classical solutions by $\mathcal{M}_k$ where $k \in \mathbf{Z}$ is the instanton number, i.e., the value of the second Chern class of the principal $SU(2)$-bundle. The short distance arguments above show that

$$\left\langle \prod_{i=1}^{s} \int_{\Sigma_i^{b_i}} \mathcal{O}^{(b_i)} \right\rangle_k = \int_{\mathcal{M}_k} \mu(\Sigma_1^{b_1}) \cup \cdots \cup \mu(\Sigma_s^{b_s})$$

where $\mathcal{M}_k$ is the moduli space of instantons on X with second Chern class k, provided that

$$\text{virt. dim}(\mathcal{M}_k) = \sum_{i=1}^{s} (4 - b_i),$$

or equivalently that

$$8k - \frac{3}{2}(\chi(M) - \sigma(M)) = \sum_{i=1}^{s} (4 - b_i).$$

The integral on the right-hand-side is the value of the usual Donaldson polynomial invariant

$$D([\Sigma_1^{b-1}], \ldots, [\Sigma_s^{b_2}]) = \int_{\mathcal{M}_k} \mu(\Sigma_1^{b_1}) \cup \cdots \cup \mu(\Sigma_s^{b_s}).$$

§19.4. Low Energy Computations for Donaldson theory

To get something interesting we must compare this high energy computation, which we have just showed limits to the usual Donaldson polynomial invariants, with a computation using the low energy effective limit of the theory. In terms of metrics, we imagine a one-parameter family of metrics $t^2 g_0$ for some fixed metric g_0. Since the stress tensor is of the form $T = \{Q, \dots\}$ and we are computing correlation functions of Q-cohomology classes of operators, the answers are independent of the metric. As $t \mapsto 0$ we recover the short distance (or high energy) computations we sketched above and we find that the results are the Donaldson polynomial invariants. As $t \mapsto \infty$ we can compute the correlation functions in terms of the space of vacua of the theory.

The basic idea behind this computation was explained in Lecture 1, section 4, for example see eqn. (1.12). As $t \to \infty$ and the metric becomes large in all directions, any computation of correlation functions, in a theory with finitely many vacuum states, will converge to a linear combination of contributions of different vacuum states. In our present example, there are infinitely many vacuum states, parametrized by the complex variable u. The analog of summing over the vacua (in case there are finitely many of them) is to integrate over u. However, we must anticipate that, in this integration, there will be delta-function contributions at the singularities at $u = \pm 1$. The reason is that at $u = \pm 1$, there are low energy configurations (the essential ones being, as we will see, solutions of the Seiberg-Witten equation with nonzero monopole fields) that do not extend nicely to $u \neq \pm 1$. Thus, the general structure will be an integral over the u-plane plus delta-function contributions at $u = \pm 1$. It turns out, however, that if $b_2^+(X) > 1$, the continuous integral over the u-plane vanishes, and the Donaldson invariants are given just by the delta-function contributions at $u = \pm 1$. The reason is briefly sketched in the next paragraph. When $b_2^+(X) = 1$, the u-plane does contribute; the Donaldson invariants are then the sum of three contributions – discrete contributions at $u = \pm 1$ as before and an actual integral of a function over the u-plane. The most interesting topological information is still in the contributions from $u = \pm 1$, even if $b_2^+(X) = 1$, but the integral over the u-plane enters in the formula for the Donaldson invariants.

Everywhere except at $u = \pm 1$, the low energy theory is a pure $U(1)$ theory. All interactions are "purely unrenormalizable" and their effects vanish as a power of t for $t \to \infty$. On the other hand, if the theory near $u = u_0$, with any u_0 except $u_0 = \pm 1$, is treated as a free theory, then the path integral vanishes because there are fermion zero modes. These simple observations are the starting point of a more detailed analysis (see the Moore-Witten paper cited in the references at the end of the lecture) which shows that the contribution of the generic abelian vacua vanishes for $b_2^+(X) > 1$. The number $b_2^+(X)$ enters because it equals the number of zero modes of the Fermi field that was earlier called ψ_2^+; the more of these zero modes there are, the harder it is to avoid vanishing for $t \to \infty$.

But at $u = \pm 1$ the story is completely different. In these vacua, there are massless hypermultiplets that we cannot ignore. At $u = \pm 1$, the classical equations for a field configuration of unbroken supersymmetry are the SW equations

$$\begin{aligned} F_A^+ &= (M\overline{M})^+ \\ \not{\partial}_A(M) &= 0. \end{aligned}$$

Here A is a $U(1)$-connection on a line bundle $\det(S^+ \otimes \mathcal{L})$ and M is a section of $S^+ \otimes \mathcal{L}$. Of course, we must sum over all complex line bundles.

Actually this terminology is imprecise. What we are writing informally as $S^+ \otimes \mathcal{L}$ is usually called mathematically a Spin^c structure on X. Thus, we are really summing over Spin^c-structures on X; the set of such structures forms a principal homogeneous set for the group of isomorphism classes of complex line bundles on X. For a detailed explanation of how the duality transformation from the description at large u to the description at $u = \pm 1$ leads to Spin^c structures, see the Moore-Witten paper cited in the references.

The appearance of M only at $u = \pm 1$ is the reason that there are delta-function contributions to the measure at these points. Solutions of the equations with $M \neq 0$ cannot be sensibly continued away from $u = \pm 1$, since M acquires a mass. We can make this argument in a more formal way by looking back to the very end of section 19.2, where we learn that with hypermultiplets included, one of the conditions for a Q-invariant field is $0 = \phi(M)$. In the present case, in the effective theory near $u = 1$, ϕ is the superpartner a of the photon in a lagrangian description that is good near $u = 1$. The hypermultiplet M has components of charge 1 or -1 in this description; call these components $T, \tilde{T}$. $\phi(M)$ has components aT and $-a\tilde{T}$, so the equation $0 = \phi(M)$ becomes $0 = aT = a\tilde{T}$. Hence there are two distinct branches: $a = 0$, but $T, \tilde{T}$ possibly non-zero, corresponding to a delta function supported at $u = 1$; or $T = \tilde{T} = 0$ but a possibly zero, corresponding to a contribution from a generic point on the u-plane with the hypermultiplets set to zero.

At first sight, one might think that there might not be a clear separation between the two types of contribution, since the two branches might meet at $a = M$. Notice, though, that if $a = M = 0$, the remaining equations reduce to $F^+ = 0$, that is, the instanton equation of $U(1)$ gauge theory. If $b_2^+(X) \geq 1$, then for the generic metric on X there are no $U(1)$ instantons; and if $b_2^+(X) > 1$, then in a generic one-parameter family of metrics, there are no such solutions.[5] Hence, the contributions of the two branches are well-separated if $b_2^+(X) > 0$, and this separation is independent of metric of $b_2^+(X) > 1$. The case $b_2^+(X) = 0$ needs more study (which has not yet been made).

§19.4.1. Operators in the effective low energy theory

In order to compute correlation functions via low energy effective field theory, we need to know the image in the low energy effective theory of operators defined in the fundamental or high energy theory. In defining the low energy effective version of an operator, we must integrate out all the high energy modes. This operation does not commute with taking products. Thus, while we shall see that several of the elementary operators will correspond in the obvious way, funny things will happen to products.

To begin with, in the high energy theory we have the operator $u = \mathrm{Tr}\,\phi^2$. We have seen that this operator flows in the low energy limit to an operator u which we know explicitly. This statement is equivalent to the statement that the exact

[5] One-parameter families of metrics are important because, in showing that a given expression defined using a metric (like a Donaldson invariant or quantum field theory correlation function) is actually metric-independent, one usually looks at a one-parameter family of metrics interpolating between two specified metrics.

moduli space of vacua is the u-plane; we take the expectation value of the operator u to label the vacua.

Of course, local operators at the point x in the high energy theory map to local operators at the same point x in the low energy theory. Hence the derivative of a local operator $\mathcal{O}$ in the high energy theory maps to the derivative of the image $\overline{\mathcal{O}}$ of $\mathcal{O}$ in the low energy theory. Similarly $\int_\Sigma \mathcal{O}$ in the high energy theory flows to $\int_\Sigma \overline{\mathcal{O}}$. Since the low energy theory is still $N = 2$ supersymmetric, it has the operators $\overline{Q}$ and $\overline{K}$, which are the images of the operators Q and K in the high energy theory. Thus, the image in the low energy theory of $K^n u$ is $\overline{K}^n u$. Thus, as long as Σ_1, Σ_2 are disjoint cycles in X, the operation of taking products commutes with integrating over the high energy modes. This means that the high energy operator

$$\int_{\Sigma_1} K^{n_1} u \cdot \int_{\Sigma_2} K^{n_2} u$$

goes in the low energy theory to the product of the images of the individual operators:

$$\int_{\Sigma_1} \overline{K}^{n_1} u \cdot \int_{\Sigma_2} \overline{K}^{n_2} u$$

as long as

$$\Sigma_1 \cap \Sigma_2 = \emptyset.$$

Where products will not be preserved in passing to the low energy effective theory is when the cycles intersect. In this case the general form of the answer is that the product

$$\int_{\Sigma_1} K^{n_1} u \cdot \int_{\Sigma_2} K^{n_2} u$$

of high energy operators flows in the low energy limit to an operator of the form

$$\int_{\Sigma_1} \overline{K}^{n_1} u \cdot \int_{\Sigma_2} \overline{K} n_2 u + \int_{\Sigma_1 \cap \Sigma_2} \overline{K}^{n_1+n_2-4} f(u)$$

for some function $f(u)$. (The dimension of $\Sigma_1 \cap \Sigma_2$, if those cycles intersect transversely, is $n_1 + n_2 - 4$.) The reason for the extra term is that if integrating out massive degrees of freedom gives $\mathcal{O}_i(x) \mapsto \overline{\mathcal{O}}_i(x)$ then to a product, it gives, in general,

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \mapsto \overline{\mathcal{O}}_1(x)\overline{\mathcal{O}}_2(y) + \delta(x-y)\tilde{\mathcal{O}}_{12}(x) + \dots$$

for an appropriate operator $\tilde{\mathcal{O}}(x)$ in the low energy theory (the $\dots$ terms are proportional to derivatives of $\delta(x-y)$). All that this formula asserts is that if x and y are distinct the operation of integrating out massive degrees of freedom can be applied to the two operators separately; and that if one does not require x and y to be distinct, there is a new distributional contribution, supported at $x = y$, about whose details no claim is made. In other words, something new can possibly happen when, and only when, x and y meet, and we make no general claim about what happens there except that new local operators might appear in the formula. In our case if $\mathcal{O}_i = K^{n_i} u$, then it must be the case (because of Q-invariance and modulo $\{Q, \dots\}$) that $\tilde{\mathcal{O}}_{12} = \overline{K}^{n_1+n_2-4} f(u)$ for some holomorphic function $f(u)$. Integrating this result for local operators gives us the formula above for the image in the low energy theory of the product

$$\int_{\Sigma_1} K^{n_1} u \cdot \int_{\Sigma_2} K^{n_2} u.$$

If we consider three or more such operators integrated for example on cycles Σ_1, Σ_2, Σ_3, there will be further new terms, of a similar origin, arising on triple intersections $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$.

Let us simplify the discussion by supposing that $b_1(X) = 0$. Then the only operators that we need to consider are u, K^2u, K^4u. But there is not much to novelty to be gained from K^4u since it is a numerical invariant which is simply the second Chern class, so that inserting it in a correlation function simply multiples the expectation value on the bundle by the instanton number k. From now on we drop this operator from consideration. Among the remaining operators there are no triple intersections assuming, as we always can by topological invariance, that the two-cycles over which we integrate K^2u are in general position. Furthermore, the only double intersections are finite sets of points where a pair of two-cycles meet transversely. In this case we have that the image of the product of the high energy operators

$$\int_{\Sigma_1} K^2u \cdot \int_{\Sigma_2} K^2u$$

in the low energy effective theory will be of the form

$$\int_{\Sigma_1} \overline{K}^2u \cdot \int_{\Sigma_2} \overline{K}^2u + (\Sigma_1 \cap \Sigma_2)T(u),$$

where T is a holomorphic function of u and where $\Sigma_1 \cap \Sigma_2$ denotes the algebraic intersection of Σ_1 and Σ_2. The reason that $T(u)$ is holomorphic is that since Q-closed operators map to $\overline{Q}$-closed operators, we must have that the correction term in the formula is $\overline{Q}$-closed. On the other hand, up to $\overline{Q}$-exact terms any $\overline{Q}$-closed scalar operator is a holomorphic function of u.

§19.4.2. Warm-up with $U(1)$ theory with a hypermultiplet

We shall now do a practice computation that will help in learning to relate the $SU(2)$ and the $U(1)$ theories. But we shall do this practice computation in a simpler theory – the $U(1)$ theory with a hypermultiplet and a minimal lagrangian, minimal in the sense that we will omit certain couplings. We simply take the lagrangian for the vector multiplet and hypermultiplet as we wrote them down in flat space with τ constant. We set $q = \exp(2\pi i\tau)$. The only invariant polynomials for the Lie algebra u(1) are polynomials in a. We are assuming that $H^1(X) = 0$ so that the operators under discussion are the zero-dimensional operator $\mathcal{O}^{(0)} = a$ and the two-dimensional operator $\mathcal{O}^{(2)} = c_1(\mathcal{L})$ (with $\mathcal{L}$ the line bundle on which the gauge field is a connection.) Once again insertion of this latter operator simply multiplies the correlation function by the first Chern class of the line bundle, and hence presents no novelty. That is to say the only operators whose expectation values we need to compute are holomorphic (or polynomial) functions $P(a)$.

Recall that in this theory the bosonic part of the hypermultiplet field M becomes after twisting a section of $S^+ \otimes \mathcal{L}$. We recall that the expression $S^+ \otimes \mathcal{L}$ is to be understood as an informal way to designate a Spin^c structure. We use the name λ to denote either the Spin^c structure or its first Chern class (which, assuming there is no two-torsion in $H^2(X, \mathbf{Z})$, is defined to be one-half the first Chern class of $\lambda = S^+ \otimes \mathcal{L}$).

Let us suppose that we wish to compute $\langle P(a)\rangle_\lambda$ for some polynomial P and some Spin^c-structure λ. The high energy arguments outlined above show that this expectation value concentrates along the geometric manifold $\mathcal{M}_\lambda$ underlying the

moduli space of solutions to the classical equations for the Spin^c-structure λ. Since the value of $\exp(-S)$ on $\mathcal{M}_\lambda$ is q^{λ^2}, we have:

$$\langle P(a)\rangle_\lambda = q^{\lambda^2}\int_{\mathcal{M}_\lambda} P(a).$$

Since a is a two-form on $\mathcal{M}_\lambda$, this integral is equal to

$$q^{\lambda^2} c_{n_\lambda}\int_{\mathcal{M}_\lambda} a^{n_\lambda},$$

where n_λ is one-half the virtual dimension of $\mathcal{M}_\lambda$ and c_{n_λ} is the coefficient of a^{n_λ} in the Taylor series for $P(a)$ at $a = 0$. Of course,

$$SW_X(\lambda) = \int_{\mathcal{M}_\lambda} a^{n_\lambda}$$

is by definition the Seiberg-Witten invariant of the manifold X evaluated on the Spin^c-structure λ. (By convention, all of these integrals are interpreted to be zero if n_λ is not an integer.) Thus, we have evaluated the expectation value $\langle P(a)\rangle_\lambda$ to be the coefficient of a^{n_λ} in the Taylor series for $P(a)$ times $q^{\lambda^2} SW_X(\lambda)$. It is convenient for some of the later computations to rewrite this as

$$\langle P(a)\rangle_\lambda = q^{\lambda^2} SW_X(\lambda)\cdot \mathrm{Res}_{a=0}\left(\frac{da}{a^{1+n_\lambda}} P(a)\right).$$

Summing over all Spin^c-structures yields

$$\langle P(a)\rangle = \sum_\lambda \langle P(a)\rangle_\lambda = \sum_\lambda q^{\lambda^2} SW_X(\lambda)\cdot \mathrm{Res}_{a=0}\left(\frac{da}{a^{1+n_\lambda}} P(a)\right).$$

There is a similar formula when we allow a two-dimensional operator as well. Since this operator just measures the first Chern class, we get

$$\begin{aligned}\left\langle P(a)\cdot exp\left(\alpha\int_\Sigma \mathcal{O}^{(2)}\right)\right\rangle &= \sum_\lambda q^{\lambda^2}\cdot \exp(\alpha\langle c_1(\lambda),\Sigma\rangle)\langle P(a)\rangle_\lambda \\ &= \sum_\lambda q^{\lambda^2}\cdot \exp\left(\alpha\langle c_1(\lambda),\Sigma\rangle\right) SW_X(\lambda)\cdot \mathrm{Res}_{a=0}\left(\frac{da}{a^{1+n_\lambda}} P(a)\right).\end{aligned}$$

Now let us generalize this $U(1)$ gauge theory computation by allowing more general terms that preserve topological invariance. The most general form for the lagrangian on a curved manifold X, preserving topological invariance, is

$$\mathcal{L}(a, A, M) = \{Q, \cdot\} + \int K^4 e(a) + \int g(a)\mathrm{Pf}(R_X) + \int h(a) L(R_X)$$

where R_X is the Riemannian curvature tensor of X. Topological invariance implies that the functions e, g, h must be holomorphic functions. Of course $\int \mathrm{Pf}(R_X)$ gives a multiple of the Euler characteristic whereas $\int L(R_X)$ gives a multiple of the signature of X.

The expectation value is now

$$\langle P(a)\rangle = \sum_\lambda SW(\lambda)\mathrm{Res}_{a=0}\left\{\frac{da}{a^{1+n_\lambda}} P(a)\left(E(a)^{\lambda^2} G(a)^{\chi(X)} H(a)^{\sigma(X)}\right)\right\},$$

where $G = e^{-g}, H = e^{-h}$, and $E = \exp(-\partial^2 e/\partial a^2)$. The reason for the last equation is that since $K(a) = \psi$ and $K(\psi) = F$, we have $K(e(a)) = e'(a)\psi$ and

$$K^2(e(a)) = e''\psi\wedge\psi + e'(a)K(\psi) = e''\psi\wedge\psi + e'(a)F.$$

Continuing the differentiation, we find that

$$K^4(e(a)) = e''(a)F \wedge F + \psi \wedge (\cdots).$$

In our case all the terms involving ψ vanish since $b_1(X) = 0$. The factor $E(a)^{\lambda^2}$ reduces to q^{λ^2}, as written above, if τ is constant.

§19.4.3. The case of $SU(2)$

Now we are ready to return to the pure $SU(2)$ theory and understand the low energy effective computation in terms of the $U(1)$ computations we just did. Suppose that we fix $\alpha \in \mathbf{C}$ and a class $\Sigma \in H^2(X; \mathbf{C})$. We wish to compute

$$\left\langle \exp\left(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\right)\right\rangle_{SU(2)}.$$

The general form of the answer will be

$$\left\langle \exp\left(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\right)\right\rangle_{u=1} + \left\langle \exp\left(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\right)\right\rangle_{u=-1} + \int_{u-\text{plane}} \cdots.$$

The integrand of the u-plane integral (which in fact is a very complicated expression that depends on only elementary topological data) has been denoted just as ....

Let us see what operators we obtain in the low energy effective theory. Using the fact that K commutes with the passage to the low energy limit, plus our above computation of $K^2(e(a))$, applied to the case $e(a) = u$, we have

$$\int_\Sigma \mathcal{O}^{(2)} \mapsto \int_\Sigma K^2 u = \int_\Sigma \frac{\partial u}{\partial a} F + \int_\Sigma \frac{\partial^2 u}{\partial a^2} \psi \wedge \psi.$$

Here, we assume that in a local region of the u-plane, a good choice of a has been made; the formula is then valid in that region with that choice of a. Near $u = \infty$, for instance, we can use the usual a, but near $u = 1$ we have to use a different a. If $b_1(X) = 0$ then we can drop the second term so that we have

$$\int_\Sigma \mathcal{O}^{(2)} \mapsto \int_\Sigma \frac{\partial u}{\partial a} F.$$

Now using the rules for multiplication of the two-dimensional operators worked out above we see

$$\exp\left(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\right) \mapsto \exp\left(\alpha u + \beta \frac{\partial u}{\partial a} \int_\Sigma F + \beta^2 \Sigma^2 T(u)\right).$$

Now let us compute the $u = 1$ contribution to the Donaldson invariant. The a we use here is the one produced by the monopole field that is massless at $u = 1$. We translate this coordinate so that $u = 1$ corresponds to $a = 0$. The effective low energy theory is a $U(1)$ theory and is a special case of the one we discussed in the last section with some unknown universal functions e, g, h (or equivalently unknown functions E, G, H).

So, we have

$$\left\langle \exp\left(\alpha u + \beta\frac{\partial u}{\partial a}(\Sigma,\lambda) + \beta^2\Sigma^2 T(u)\right)\right\rangle$$
$$= 2\sum_\lambda SW(\lambda)\mathrm{Res}_{a=0}\left\{\frac{da}{a^{1+n_\lambda}}\exp\left(\alpha u + \beta\frac{\partial u}{\partial a}(\Sigma,\lambda)\right.\right.$$
$$\left.\left. + \beta^2\Sigma^2 T(u)\right)E^{\lambda^2}H^{\sigma(X)}G^{\chi(X)}\right\}.$$

The multiplicative factor of 2 on the right hand side is introduced because of the following reason. The group of gauge transformations acts by $SU(2)$-bundle automorphisms, but its action on the space of connections is not faithful. There is a central group of order 2 which acts trivially. In performing the path integrals in gauge theory one has to normalize by dividing out by the volume of the group of gauge transformations. The fact that this group is twice as big as the group that acts faithfully introduces a factor of two in the comparison of the path integral with the conventional topological formulas, which are obtained by integrating over the moduli space without dividing by 2.

Let the virtual dimension of the SW moduli space by $2n_\lambda$. By the index theorem, n_λ is a linear function of $\lambda^2, \chi(X), \sigma(X)$; we will not need the precise formula.

Let us suppose that the manifold X is of simple type (in SW theory). By definition this means that all the invariants $SW_X(\lambda)$ vanish except those for which $n_\lambda = 0$. All known simply connected four-manifolds with $b_2^+ > 1$ have this property. Our expression then simplifies to

$$2\sum_\lambda SW_X(\lambda)\mathrm{Res}_{a=0}\frac{da}{a}\exp(\cdots)$$

where the unwritten expression is a regular expression in a. We can evaluate at $u = 1$ to obtain

$$2\sum_\lambda SW_X(\lambda)\exp\left(\alpha u|_{u=1} + (\Sigma,\lambda)\frac{du}{da}\bigg|_{u=1} + \Sigma^2 T(1)\right)\tilde{G}^{\chi(X)}\tilde{H}^{\sigma(X)}.$$

Here $\tilde{G}$ and $\tilde{H}$ are complex numbers derived from E, G, H by setting $u = 1$ and using the index theorem (recall $n_\lambda = 0$) to express λ^2 as a linear combination of $\chi(X), \sigma(X)$. To completely determine the answer we must evaluate

$$u(1), \frac{\partial u}{\partial a}\bigg|_{u=1}, \tilde{G}, \tilde{H}, T(1).$$

Of course, we also have the contribution at $u = -1$ which differs from this one by the involution $u \mapsto -u$.

This type of computation gives a complete evaluation of the Donaldson invariants when $b_2^+(X) > 1$, because in that case there is no continuous integral over the u-plane (away from the singularities). The formula shows that for $b_2^+(X) > 1$, the Donaldson polynomial invariant of X is completely determined by the function $SW_X(\lambda)$ on Spin^c structures. The explicit relation involves five universal constants, which were listed in the last paragraph. Fortunately, as Donaldson invariants have been computed in many cases, these constants can be fixed. In fact, this can be done in many ways, with results that always turn out to be consistent.

The final result is that for manifolds X with $b_2^+(X) > 1$ which are of Seiberg-Witten simple type we have:

$$\left\langle \exp\left(\sum_a \beta_a \int_{\Sigma_a} \mathcal{O}^{(2)} + \alpha \mathcal{O}^{(0)}\right)\right\rangle$$
$$= 2^{1+\frac{1}{4}(7\chi+11\sigma)}\left(\exp\left(\frac{v^2}{2}+2\alpha\right)\sum_\lambda SW_X(\lambda)e^{v\cdot c_1(\lambda)}\right.$$
$$\left. + i^\Delta \exp\left(-\frac{v^2}{2}-2\alpha\right)\sum_\lambda SW_X(\lambda)e^{-iv\cdot c_1(\lambda)}\right),$$

where $v = \sum_a \beta_a[\Sigma_a]$ and where $\Delta = (\chi(X)+\sigma(X))/4$. The first of the two terms on the right-hand-side is the contribution at $u = 1$ and the second is the contribution at $u = -1$. It can be derived from the first by using the involution $u \leftrightarrow -u$.

In the exponent, we have here put in a factor of 2 in front of α that was plucked from thin air; up until now the coefficient of α in such formulas has always been one. The reason for this discrepancy is that we made an implicit assumption throughout this discussion which now has to be corrected in order to compare with Donaldson theory. Recall that in the low energy theory there is a mass scale Λ^2 and that the monopoles become massless at $u = \pm\Lambda^2$. Today we have set $\Lambda = 1$ and, thus, worked in units such that the monopoles become massless at $u = \pm 1$. But there is no reason that this renormalization scheme should be the one that is compatible with topology. So we should do the computations with an extra unknown variable $\pm\Lambda^2$ and then determine Λ by comparison with topological computations in examples. (Or, with sufficiently strong muscles, we could try to determine *a priori* what renormalization scheme or equivalently what value of Λ with our fixed renormalization scheme will coincide with standard topological definitions.) It turns out that we should place the singularities at $u = \pm 2$ in order for the field theory computations to give the usual Donaldson polynomial answers; this is a reflection of the fact that the simple type condition of Kronheimer and Mrowka, with standard topological definitions, reads $u^2 = 4$, an equation whose roots are at $u = \pm 2$. Setting u at the singularities to ± 2 rather than ± 1 explains why in the exponent of the final formula, we get $\pm 2\alpha$ and not $\pm\alpha$.

References and cross references

The basic references for this material are

N. Seiberg and E. Witten, *Electric-Magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **426** (1994), pp. 19-52.

N. Seiberg and E. Witten, *Monopoles, Duality and Chiral Symmetry Breaking In $N = 2$ Supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484.

E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. 1 (1994), pp. 769–796.

Examples of solutions for $N = 2$ theories with hypermultiplets are worked out in the second paper. The application to Donaldson theory is made in the third paper, and it is developed in detail in

G. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. 1 (1997), pp. 298–387.

In particular, in that reference the theory for $b_2^+ = 1$ is developed, and it is shown that if $b_2^+(X) > 1$, then the u-plane does not contribute, except for the singularities.

The comparison of the Euler characteristic of moduli space to mathematical computations is explained in

C. Vafa and E. Witten, *A strong coupling test of S-duality*, Nuclear Physics B, **431** (1994), pp. 3-77.

The relationship of the β-function to asymptotic freedom and infrared freedom is described in [I-**Renorm Groups**, §§2,3]. Twisting theories in order to globalize supersymmetries is described in §3 of Lecture 14.

EXERCISES

Problem 1. The strong interactions are described to good approximation by a theory which has a symmetry group $G = SU(2) \times SU(2)$ spontaneously broken down to a diagonal subgroup $H \cong SU(2)$. There are no massless particles except the Goldstone bosons, which, to the extent G is a valid symmetry, are exactly massless. The theory, to good approximation, has a G-invariant Lagrangian L.

However, G is not an exact symmetry of nature. It is broken by perturbations that transform in the $(2,2)$ representation of G (that is, the tensor product of the two-dimensional representations of the two $SU(2)$'s.) Among the operators of the G-invariant theory are two "multiplets" of operators $\mathcal{O}_i$ and $\widetilde{\mathcal{O}}_j$, $i,j = 1 \ldots 4$, each transforming as $(2,2)$. The Lagrangian is, more accurately,

$$\widetilde{L} = L + \sum_{i=1}^{4} \left(\epsilon_i \mathcal{O}_i + \widetilde{\epsilon}_i \widetilde{O}_i \right). \tag{1}$$

The ϵ_i and $\widetilde{\epsilon}_j$ are, of course, all real. (That there are precisely two possible multiplets $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ of perturbations is explained by the underlying gauge theory, but for the present exercise we simply assume it.)

For a generic choice of the ϵ_i and $\widetilde{\epsilon}_i$ the exact symmetry group of the model is only $H' = U(1)$, and this is, in fact, the situation in nature.

Treat the effective theory to first order in ϵ and $\widetilde{\epsilon}$ (which are in fact in nature small and comparable to each other).

(a) Show that, in first order, you would expect the degeneracy of the vacuum to be lifted, and there to appear a unique vacuum with mass gap. In particular, you expect the H' symmetry to be unbroken. (Abstractly, H' can be embedded in H; the point that is slightly non-trivial is that the perturbation picks a vacuum, of the unperturbed theory, that is invariant under an $H \cong SU(2)$ that actually does contain H'. H could of course have been conjugated to a group that does not contain H'.) In particular, there is no exact symmetry that is spontaneously broken, and no exact Goldstone boson; the former Goldstone bosons all get mass.

(b) More surprisingly, show that to first order in ϵ and $\widetilde{\epsilon}$, the three Goldstone bosons – the one that is invariant under H' and the two that transform in a non-trivial representation of H' – all get *equal* masses. (This is also observed in nature, to high precision, the particles being respectively the neutral and charged pions.)

(c) The three Goldstone bosons will not have *exactly* equal masses. In what order in ϵ and $\widetilde{\epsilon}$ will the degeneracy be lifted and how would you describe this lifting in the effective Lagrangian?

Problem 2. Here we consider (as in the standard model of weak interactions) an $SU(2) \times U(1)$ gauge theory in four dimensions. The $SU(2)$ and $U(1)$ gauge fields will be called A_2 and A_1, the curvatures F_2 and F_1, and the gauge couplings e_2 and e_1. The Lagrangian of the pure gauge theory would be

$$L = \int d^4x \left(\frac{1}{4e_1^2} |F_1|^2 + \frac{1}{4e_2^2} \mathrm{Tr}\, F_2^2 \right) \tag{2}$$

where Tr is the trace in the two-dimensional representation of $SU(2)$.

In addition there are complex scalar fields ϕ which transform in the two-dimensional representation of $SU(2)$ and with "charge 1/2" for the $U(1)$. Their Lagrangian is

$$L_\phi = \int d^4x \left(|D\phi|^2 + \frac{\lambda}{8} \left(|\phi|^2 - v^2 \right)^2 \right). \tag{3}$$

v^2 is positive.

(a) Show that $SU(2) \times U(1)$ is spontaneously broken to a diagonal $U(1)$.

(b) Determine the spectrum of the model: what are the massless and massive scalars and vectors, and how do they transform under the unbroken $U(1)$?

(c) The quantities of the model that are observed to the highest precision are the two gauge couplings and the masses of the charged and neutral massive vectors. This makes four observables. As the model has four parameters (e_1, e_2, λ, and v), one might not expect any relation among these four observables. However, the four observables are independent of λ in the classical approximation which (as the model is weakly coupled) is a good approximation. So in practice the model does predict a relation between the two vector masses (given a measurement of e_1 and e_2). This relation was one of the early successes of the electroweak standard model.

Of course, this relation will not be exact. It will be modified by loops, and the loops will depend on additional fields that are present in nature but were not mentioned so far (and do not affect the classical discussion carried out here). Present experimental accuracy is just about good enough to see the loop effects, and thereby to get an estimate of the parameter λ that does not enter the classical relation but does enter the loops.

Problem 3. Consider the supersymmetric model of maps of $\mathbb{R}^{2,2}$ to S^N. We formulated the Lagrangian in the past for maps $X : \mathbb{R}^{2,2} \to M$ for M an arbitrary Riemannian manifold:

$$L = \int d^2x d^2\theta \; g_{IJ}(X) D_+ X^I D_- X^J. \tag{4}$$

Consider the case that M is a sphere S^N with a round metric.

By imitating the steps that we carried out in class for the bosonic model with the same target, determine the large N behavior of this model.

Suggestion: Include the Lagrange multiplier in a manifestly supersymmetric way in superspace. But when it comes to doing the path integral, first do the theta integrals to reduce to ordinary Lagrangians and path integrals and proceed in a fashion that should by now be familiar. (We haven't developed supergraph techniques for superspace evaluation of Feynman diagrams, and such techniques really don't give much simplification for most problems as simple as this one, anyway.)

This explicit procedure should make it clear that two similar-looking mechanisms that we've studied (in the bosonic sigma model and the Gross-Neveu model) are in fact related to each other by supersymmetry.

Problem 4. Prove that the model considered in the previous problem, though it has a mass gap, has a conserved charge of the sort "not allowed" by the Coleman-Mandula theorem, that is, it is not a linear combination of an element of the Poincaré Lie algebra and an operator commuting with Poincaré.

Problem 5. In the two-dimensional sigma model with target space a sphere S^n, for large n, we introduced a Lagrange multiplier field σ in order to quantize the theory.

(a) By using the equations of motion, identify σ as an operator in terms of the original variables.

(b) What is the dimension of the σ field and therefore what would you expect to be the large q behavior (with Euclidean q) of $\langle\sigma(q)\sigma(-q)\rangle$. Compare to the actual behavior.

Problem 6.[1] Consider in two dimensions a $U(1)$ gauge theory with *two* species of fermion, ψ_i, $i = 1, 2$, in the same representation of $U(1)$, of bare masses m_i. The Lagrangian is thus

$$L_\psi(\psi_1, m_1) + L_\psi(\psi_2, m_2) + \int \left(\frac{F^2}{4e^2} + i\frac{F}{2\pi}\right) \tag{5}$$

where L_ψ is the Lagrangian that we considered in class:

$$L_\psi(\psi, m) = \int \left(\overline{\psi}_+ D_- \psi_+ + \overline{\psi}_- D_+ \psi_- + m\overline{\psi}_- \psi_+ + \overline{m}\overline{\psi}_+ \psi_-\right) d^2x. \tag{6}$$

(a) Show that the θ dependence disappears if either of the m_i vanishes.

(b) Show that in the limit that m_1 becomes very large keeping m_2 fixed (or letting m_2 go to zero while m_1 becomes large) the theory reduces to the one that we studied in class, but with a possibly shifted effective value of θ.

(c) Describe qualitatively the behavior when both of the m_i are large.

(d) Describe the behavior when both of the m_i are small.

(e) Show that if the m_i are equal, the theory has an $SU(2)$ symmetry. Now, by considering the case that the m_i are equal and small, deduce that the two-dimensional sine-Gordon model, that is, the model of a scalar field with potential $V(\phi) = \mu^2 \cdot \cos(\beta\phi)$, has an $SU(2)$ symmetry for a certain value of β which you should identify.

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[1] A reference for most of this problem (except the last paragraph) is S. Coleman, *More about the massive Schwinger model*, Annals of Physics **101** (1976), 239–267.

(f) Taking the limit as μ goes to zero, where the sine-Gordon theory mentioned in (e) becomes a massless conformally invariant free field theory, deduce that the two-dimensional theory of a free massless scalar of the correct radius (or equivalently of fixed radius 2π and the correct coefficient k of the kinetic energy in $L = k\int d^2x\,|d\phi|^2$) has an $SU(2)$ symmetry. Because of the conformal invariance this theory actually represents the affinization of $SU(2)$ (more exactly, two copies of it, one for left-movers and one for right-movers) and not only the finite-dimensional group $SU(2)$. This is called the vertex construction of the affine Lie group of $SU(2)$.

To obtain it, we didn't really need to go via the gauge theory. One could, more generally, simply consider several massless free fermions and, by applying the bose-fermi correspondence to them, get the vertex construction for $SO(2n)$ and (with a slight modification) for $SU(n)$.

Problem 7. Consider in two dimensions an $SO(n)$ gauge theory with scalar fields ϕ consisting of $n-1$ copies of the n-dimensional representation of $SO(n)$ and a potential energy

$$V = \sum_{i=1}^{n-1} \lambda_i \left[(\phi_i,\phi_i) - a_i^2\right]^2 + \sum_{i<j} c_{ij}(\phi_i,\phi_j)^2. \tag{7}$$

The λ_i and the c_{ij} are all positive. Here $(\ ,\ )$ is the invariant quadratic form on the n-dimensional representation of $SO(n)$.

Because $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$, this theory has a two-valued "theta angle." Explain how you would calculate, for weak coupling, the energy difference between the vacua with the two possible values of theta.[2]

Problem 8.[3] Consider in two dimensions the $(\psi_+\psi_-)^2$ model (introduced in Lecture 4 of [I-**Renorm Groups**]) with the fields being two-dimensional fermions ψ_i with Lagrangian

$$L = \int d^2x \left(\sum_i (\psi_{+i}\partial_-\psi_{+i} + \psi_{-i}\partial_+\psi_{-i}) + g\sum_i (\psi_{+i}\psi_{-i})^2\right). \tag{8}$$

(a) Consider the classical conservation law $\partial_- S = 0$ where $S = \psi_{+1}\psi_{+2}\dots\psi_{+n}$. By considering its deformation in quantum mechanics, show that the model is integrable.

In this way you obtain a conserved charge R with $[K,R] = (n/2-1)R$ where K is the generator of the Lorentz group (the subgroup $SO(1,1)$ of the two dimensional Poincaré group.) By exchanging $+$ and $-$, you get another conserved charge R' with $[K,R'] = -(n/2-1)R'$.

(b) Now set $n = 4$, so $(n/2) - 1 = 1$. Hence R and R' have the same Lorentz transformation law as the momentum components P_+ and P_-.

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[2] If you are unsure of how to think about theta for the case that $\pi_1(G)$ is a finite group, you can consult E. Witten, *Theta vacua in two-dimensional quantum chromodynamics*, Nuovo Cimento A **51** (1979), 325–338.

[3] A reference for this problem is E. Witten, *Some properties of the* $(\bar\psi\psi)^2$ *model in two dimensions*, Nucl.Phys.B **142** (1978), 285.

You should be able to sharpen the argument in (a) above to show for $n = 4$ that the theory has two stress tensors T_{mn} and T'_{mn}, where T is the standard stress tensor whose appropriate integrals give the Poincaré generators, and T'_{mn} has the property that its suitable moments are R and R'. But now, by "contracting" T' with the generator of $SO(1,1)$, you get a new "Lorentz generator" K'; show that the usual Poincaré operators together with R, R', and K' generate two commuting copies of the Poincaré group. (To get the commuting copies, you have to take sums and differences of "old" and "new" operators.)

Show that no linear combination of P_+ and R (or P_- and R') vanishes so that each copy of the Poincaré group acts nontrivially. In fact, you should be able to see that the outer automorphism of $SO(4)$ (which acts in this theory by $\psi_1 \to -\psi_1$ with $\psi_i \to \psi_i$ for $i > 1$) exchanges the two copies of Poincaré.

Deduce that for $n = 4$ this theory is "reducible" as the product of two decoupled theories, in one of which one copy of Poincaré acts and in the other the second.

(c) Use the bose-fermi correspondence to verify this and identify the factors. You should be able to draw an inference about the symmetries of the quantum sine-Gordon model at a special value of the coupling. (You drew the same inference last week in a different way.)

(d) Now consider the case $n = 3$. Show that R and R' are supersymmetries, so that the model is supersymmetric for $n = 3$. Can you verify this using the bose-fermi correspondence?

Problem 9. In class we considered in two dimensions a theory with n chiral superfields A_i, $i = 1, \ldots, n$, and superpotential $F(A_1, \ldots, A_n)$ a homogeneous function of degree n. Then we constructed a $\mathbb{Z}_n$ orbifold of this theory in which one divides by the $\mathbb{Z}_n$ group that acts as $A_i \to e^{2\pi i/n} A_i$. The theory has a left and right moving R symmetry group $U(1) \times U(1)$, with which you should be familiar. So the Hilbert space is bigraded.

Formulate this theory on a circle S^1 (with supersymmetric, periodic boundary conditions for all fields). There is a finite-dimensional space of supersymmetric ground states of the theory, in fact for reasons that should be clear from the lecture this space is closely related to the cohomology of the projective hypersurface defined by the equation $F = 0$ in $\mathbb{P}^{n-1}$. Both are bigraded, the bigrading on one side coming from the R symmetries and the bigrading on the other side coming from the Hodge decomposition of the cohomology of the hypersurface; the two bigradings will match (after a shift by $n/2$, that is: what is usually called $H^{p,q}$ maps to states whose R charges are $p', q' = p - n/2, q - n/2$).

Compute the space of supersymmetric ground states of the orbifold theory. There are different pieces:

(1) There are $n - 1$ twisted sectors. How many zero energy states come from each twisted sector? To find the p', q' values of the ground states in the twisted sectors, you can note that (a) perturbation theory is straightforward as there are no zero modes, (b) you can go to the weak coupling limit in computing p' and q', (c) therefore you can use free field theory, (d) the computation is hence extremely similar to computations that appear in superstring theory in comparing Ramond and Neveu-Schwarz sectors, (e) you essentially just have to compute an eta invariant of a rather trivial one-dimensional operator.

(2) In the untwisted sector, perturbation theory is not straightforward. However, in this case the non-zero modes do not make any important contribution (there is no eta invariant as in the previous case) and hence you lose nothing essential if you think of the S^1 as very small, drop the non-zero modes, and reduce to $0+1$ dimensions. We are hence in the realm of classical mathematics. Using whatever knowledge you have of quantum mechanics, L^2 cohomology, singularity theory, or whatever, determine the spectrum of zero energy states of this $0+1$ dimensional problem.

Now combine the answers in (1) and (2) and compare to the cohomology of the projective hypersurface.

The next two exercises are concerned with twisted versions of $N=2$ supersymmetric theories in two dimensions. Recall that in such a twisted theory one considers only operators that commute with a Q such that $Q = Q_+ + Q_-$ or $Q = \overline{Q}_+ + Q_-$ (or the hermitian conjugate of one of these). If one works on a flat $\mathbb{R}^2$, Q commutes with $S+S'$, where S is the generator of rotations of $\mathbb{R}^2$ and S' is a sum or difference of left and right moving R-symmetries.

Problem 10. Consider a theory of chiral superfields Φ_i with some polynomial superpotential $W(\Phi_i)$. We assume that W has only finitely many critical points (perhaps degenerate). The Lagrangian is

$$L = \int d^2x d^4\theta \overline{\Phi}_i \Phi_i + \left(\int d^2x d^2\theta W(\Phi) + \text{c.c.} \right). \tag{9}$$

We recall that this theory can be obtained by dimensional reduction of a four-dimensional theory with coordinates $x^1, x^2, \dots, x^4$; the reduction to two dimensions is made by taking fields to be independent of x^3, x^4. Let S be the generator of rotations of the $1-2$ plane and S' the generator of rotations of the $3-4$ plane. S' is a symmetry for any W. Twist by S' to get a topological field theory in which the Q-invariant operators are chiral superfields (plus their q-form descendants for $q = 1, 2$); this theory can be defined for any W as S' is a symmetry for any W.

Let $\mathcal{O}_i$ be chiral superfields (note that one can take these to be polynomial functions of the Φ_i only, without derivatives) and compute in genus zero the path integral

$$\langle \prod_{i=1}^{s} \mathcal{O}_i(x_i) \rangle. \tag{10}$$

As a preliminary reduction, note that by using the chiral ring you can reduce to the case $s = 1$.

To compute the path integral in genus zero, the main step is to reduce to a finite dimensional integral by throwing away the non-zero modes of all fields. If you don't know how to do or interpret the resulting finite dimensional integral, practice with the case of only one chiral superfield.

Problem 11. Consider first of all an $N = 2$ gauge theory in two dimensions with gauge group G. The curvature is the twisted chiral superfield $\Sigma = \sigma + \dots$.

This theory also can come by dimensional reduction from four dimensions. In four dimensions there is an R symmetry S''. Upon reduction to two dimensions, we made a twisted topological theory in which rotations were generated by $S+S''$ and the topological symmetry is $\overline{Q}_+ + Q_-$; observables in this theory are twisted chiral superfields (and not chiral superfields, as in the previous problem).

We showed the following:

(a) Invariant polynomials in σ (and their q-form descendants) gave the topological observables.

(b) Such a polynomial P corresponds to a cohomology class w_P on the moduli space of flat connections.

(c) Correlation functions are given by integrals over moduli space of a product of such cohomology classes. (It is important to consider the q-form descendants since otherwise everything vanishes, a statement first conjectured by Newstead for $G = SU(2)$.)

Now modify this by including chiral superfields Φ_i in some representation of G. Note that (in contrast to problem (1) where we made a twist that allows an arbitrary superpotential) here we are making the twist by $S + S''$ that forbids a generic superpotential.[4] Because of this, take the superpotential to be zero.

There is no change in (a) and (b). However, (c) will be modified. The correlation functions of the operators $P^{(q)}$ (integrated over q-cycles in Σ) in the twisted theory with the chiral superfields will be the integral over the moduli space of flat connections of the same product of cohomology classes that we had before times a new class that depends on the chiral superfields. Can you identify this class?

[4]If one wants to include a superpotential, (i) it must have weight 2 under some $\mathbb{C}^*$ action on the Φ_i, (ii) one must add to S'' the generator of this $\mathbb{C}^*$, thus modifying how the geometrical R symmetry acts on the chiral superfields.

SOLUTIONS TO SELECTED EXERCISES

Solution to Problem 1

Since the only massless particles in the unbroken ($\varepsilon = \tilde{\varepsilon} = 0$) theory are the Goldstone bosons, in the infrared it flows to a σ-model into the homogeneous space G/H. Topologically G/H is a 3-sphere. Now with $\varepsilon, \tilde{\varepsilon}$ nonzero we expect a perturbation of the σ-model lagrangian, and we are asked to contemplate first a perturbation *linear* in $\varepsilon, \tilde{\varepsilon}$. The most relevant perturbation involves no derivative of the field in the σ-model, so is simply a potential function V on G/H. Now G double covers SO_4, and we are given that the operators $\mathcal{O}$, $\widetilde{\mathcal{O}}$ in the problem transform as vectors under SO_4. Thus if we transform $\varepsilon, \tilde{\varepsilon}$ as vectors then the perturbed theory retains the full G invariance. For fixed $\varepsilon, \tilde{\varepsilon}$ the group G is broken to the subgroup which fixes $\varepsilon, \tilde{\varepsilon}$, which for generic vectors $\varepsilon, \tilde{\varepsilon}$ is isomorphic to SO_2. (This is called "H'" in the problem.) Now the potential V must be SO_4-invariant if we transform $\varepsilon, \tilde{\varepsilon}$, and it follows easily that it has the form

$$V_{\varepsilon,\tilde{\varepsilon}}(x) = (\lambda\varepsilon + \tilde{\lambda}\tilde{\varepsilon}) \cdot x.$$

Here we write $x \in G/H \simeq S^3 \subset \mathbb{R}^4$, the '$\cdot$' denotes the dot product in $\mathbb{R}^4$, and $\lambda, \tilde{\lambda}$ are real constants. For fixed $\varepsilon, \tilde{\varepsilon}$ the potential $V_{\varepsilon,\tilde{\varepsilon}}$ is the restriction to S^3 of a linear function on $\mathbb{R}^4$. Thus it has a unique minimum on S^3. This is the unique vacuum of the perturbed theory (question (a)). Furthermore, it is clear that the Hessian of $V_{\varepsilon,\tilde{\varepsilon}}$ at the minimum is scalar; that is, its eigenvalues – the masses of the pseudo-Goldstone bosons – are all equal (question (b)). Observe that the potential is invariant under an SO_3 subgroup of SO_4 (which fixes the vector $\lambda\varepsilon + \tilde{\lambda}\tilde{\varepsilon}$), an accidental symmetry of the infrared theory. (We only expected SO_2 invariance.) Finally, a perturbation quadratic in $\varepsilon, \tilde{\varepsilon}$ — for example $V_{\varepsilon,\tilde{\varepsilon}}(x) = (\varepsilon \cdot x)\,(\tilde{\varepsilon} \cdot x)$ — can break this SO_3 invariance and so destroy the equality of the Goldstone boson masses (question (c)).

Solution to Problem 2

(a) In this problem we work classically. Then the minimum occurs when the gauge fields are zero and $\phi = \phi_0$ is a constant with $|\phi_0|^2 = v^2$. Locally we can identify the

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$SU_2 \times U_1$ in the problem with U_2, which then acts on ϕ via the two-dimensional representation. Fixing the gauge fields and ϕ breaks the gauge symmetry down to the global U_1 gauge transformations which fix ϕ. (A nonzero vector in $\mathbb{C}^2$ is fixed by $U_1 \subset U_2$.)

(b), (c) We expand the action around the minimum up to quadratic terms. Write A, iB for the SU_2 and U_1 connections and set

$$A = \begin{pmatrix} iA_3 & A_1 + iA_2 \\ -A_1 + iA_2 & -iA_3 \end{pmatrix}.$$

Here A_1, A_2, A_3, B are all real. Choose $\phi_0 = \binom{0}{v}$ and write

$$\phi = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Then up to quadratic terms we have

$$\begin{aligned} |F_B|^2 &\sim |dB|^2, \\ |F_A|^2 &\sim |dA_1|^2 + |dA_2|^2 + |dA_3|^2, \\ |D\phi|^2 &\sim \left| \begin{pmatrix} d\phi_1 + (A_1 + iA_2)v \\ d\phi_2 + iBv - iA_3 v \end{pmatrix} \right|^2, \\ &\sim |d\phi_1|^2 + |d\phi_2|^2 + v^2 A_1^2 + v_2 A_2^2 + v^2 (B - A_3)^2, \\ \frac{\lambda}{8}\left(|\phi|^2 - v^2\right) &\sim \frac{\lambda}{8}|\phi_1|^2 + \frac{\lambda}{8}|\phi_2|^2 + 2v \operatorname{Re}\phi_2. \end{aligned}$$

Now rescale $A \to e_2 A$, $B \to e_1 B$ so that after introducing a gauge fixing term (as in Problem FP18 of [I-**Homework**]), the quadratic lagrangian is (putting a coefficient $1/2$ in front of $|D\phi|^2$):

$$\begin{aligned} L_{\text{quad}} = \int d^4x \quad & \left(\frac{1}{2}|d\phi_1|^2 + \frac{\lambda}{8}|\phi_1|^2\right) + \left(\frac{1}{2}|d\phi_2|^2 + 2v \operatorname{Re}\phi_2 + \frac{\lambda}{8}|\phi_2|^2\right) \\ & + \left(\frac{1}{2}(\Delta A_1, A_1) + \frac{1}{2}v^2 e_2^2 A_1^2\right) + \left(\frac{1}{2}(\Delta A_2, A_2) + \frac{1}{2}v^2 e_2^2 A_2^2\right) \\ & + \left(\frac{1}{2}(\Delta A_3, A_3) + \frac{1}{2}(\Delta B, B) + \frac{1}{2}v^2 (e_1 B - e_2 A_3)^2\right). \end{aligned}$$

From this we see that there are two massive complex scalars (ϕ_1, ϕ_2); two massive vectors (A_1, A_2) with mass ve_2; a massless vector and a massive vector with mass $v\sqrt{e_1^2 + e_2^2}$. To prove the last statement observe that in L_{quad} the gauge fields A_3, B have a mass matrix with eigenvalues 0 and $v\sqrt{e_1^2 + e_2^2}$.

The Lie algebra of the unbroken U_1 is the subalgebra where $A_1 = A_2 = 0$ and $e_2 A_3 = e_1 B$. Thus ϕ_1 has charge 2, ϕ_2 is uncharged, A_3 and B are uncharged, and A_1, A_2 have charge 2.

Remark. If we had followed the normalization indicated in the problem, we would have replaced A, B by $\frac{A}{2}$, $\frac{B}{2}$, and then the charge of ϕ_1, A_1, A_2 would be 1 rather than 2. However, we wanted to avoid the denominators in the computation.

Solution to Problem 3

A reference for the supersymmetric σ-model is [I-**Supersolutions**, §§4.1, 4.2]; the formulas there must be dimensionally reduced to two dimensions. We work first on super Minkowski space and only rotate to Euclidean space after we derive the component lagrangian. Following [II-**Dynamics of QFT**, §4.2] we take the fields in the theory to be scalar superfields

$$\Phi\colon M^{2|2} \longrightarrow \mathbb{R}^N,$$
$$\Gamma\colon M^{2|2} \longrightarrow \mathbb{R}.$$

Here Γ is the lagrange multiplier and we have shifted N for convenience. The superspace lagrangian we consider is

$$\mathcal{L} = |d^2x|\, d^2\theta \left\{ \frac{1}{2\lambda}\langle D_+\Phi, D_-\Phi\rangle - \frac{\Gamma}{2}\big(|\Phi|^2 - 1\big)\right\}.$$

Our task is to expand in components. Let $i\colon M^2 \hookrightarrow M^{2|2}$ denote the inclusion of Minkowski space in super Minkowski space. Define the component fields:

$$\begin{aligned} \phi &= i^*\Phi & \gamma &= i^*\Gamma \\ \psi_\pm &= i^*D_\pm\Phi & \tau_\pm &= i^*D_\pm\Gamma \\ F &= i^*D_-D_+\Phi & \sigma &= i^*D_-D_+\Gamma \end{aligned}$$

The component expansion of the first term was already computed in [I-**Supersolutions**, §4.2]. To compute the component expansion of the second term we integrate over the odd variables by applying $i^*D_-D_+$:

$$\begin{aligned} i^*D_-D_+\big\{-\frac{1}{2}\Gamma\langle\Phi,\Phi\rangle + \frac{1}{2}\Gamma\big\} = -\frac{1}{2}\sigma\big(|\phi|^2-1\big) + \tau_+\langle\psi_-,\phi\rangle \\ - \tau_-\langle\psi_+,\phi\rangle - \gamma\langle F,\phi\rangle + \gamma\langle\psi_+,\psi_-\rangle. \end{aligned}$$

So the component lagrangian is

$$\begin{aligned} L = |d^2x|\, \Big\{ \frac{1}{2\lambda}|d\phi|^2 + \frac{1}{2}\langle\psi, \not{D}\psi\rangle + \frac{1}{2}|F|^2 - \frac{1}{2}\sigma\big(|\phi|^2-1\big) + \tau_+\langle\psi_-,\phi\rangle \\ - \tau_-\langle\psi_+,\phi\rangle - \gamma\langle F,\phi\rangle + \gamma\langle\psi_+,\psi_-\rangle.\Big\} \end{aligned}$$

Now we eliminate the auxiliary fields: first solve classically for F (set $F = \gamma\phi$) and then for γ (set $\gamma = \langle\psi_+,\psi_-\rangle^2/2|\phi^4|$). The simplified lagrangian is then

$$\begin{aligned} L = |d^2x|\, \Big\{ \frac{1}{2\lambda}|d\phi|^2 - \frac{\sigma}{2}\big(|\phi|^2-1\big) + \frac{1}{2}\langle\psi, \not{D}\psi\rangle + \frac{1}{2}\frac{\langle\psi_+,\psi_-\rangle^2}{|\phi|^4} \\ + \tau_+\langle\psi_-,\phi\rangle - \tau_-\langle\psi_+,\phi\rangle.\Big\} \end{aligned}$$

We shift σ by $\frac{\langle\psi_+,\psi_-\rangle^2}{|\phi|^2+1}$ to eliminate the $|\phi|^4$ in the denominator of the fourth term. Finally, we rotate to Euclidean space following Chapter 7 of [I-**Classical Fields**]:

$$\begin{aligned} L_e = |d^2x|\, \Big\{ \frac{1}{2\lambda}|d\phi|^2 + \frac{i\sigma}{2}\big(|\phi|^2-1\big) + \frac{1}{2}\langle\psi, \not{D}\psi\rangle - \frac{1}{2}\langle\psi_+,\psi_-\rangle^2 \\ - \tau_+\langle\psi_-,\phi\rangle + \tau_-\langle\psi_+,\phi\rangle.\Big\} \end{aligned}$$

For an explanation of the factor of i in the second term, see the footnote at the end of §5.2 of these lectures.

The sum of the first two terms is the lagrangian considered in §4.2. The sum of the second two terms is the lagrangian of the Gross-Neveu model (see [I-**Renorm Groups**, §4.2]). Since the large N behavior of these models separately is discussed in detail in these cross-references, we content ourselves with the observation that at the saddle point the spinor fields τ_+, τ_- vanish, by Poincaré invariance. Thus to leading order in large N the fields in the two models decouple and the behavior is as explained in the references.

Solution to Problem 4

The argument here is modeled after §4.3. We work in super Minkowski space, as in Problem 3, but now the superfield $\Phi\colon M^{2|2} \to S^N$ is a map to the sphere. We seek a conserved current $\mathfrak{J}$ on superspace, which is an even (bosonic) integral density, or twisted integral form, of cohomological degree $|-1|$. Suppose $\mathfrak{J}$ is of the form

$$\mathfrak{J} = \iota(D_+)|d^2x|\, d^2\theta\, \mathfrak{X}_- + \iota(D_-)|d^2x|\, d^2\theta\, \mathfrak{X}_+,$$

where $\mathfrak{X}_\pm$ are odd (fermionic) scalar operators. Then the conservation law $d\mathfrak{J} = 0$ is

$$D_+\mathfrak{X}_- + D_-\mathfrak{X}_+ = 0.$$

We want to choose $\mathfrak{J}$ which is not exact, which means there does not exist $\mathfrak{Y}$ with $D_-\mathfrak{Y} = \mathfrak{X}_-$ and $D_+\mathfrak{Y} = \mathfrak{X}_+$. The corresponding conserved charge, obtained by integration over a spacelike cycle of dimension $1|2$, is the desired conservation law.

As in §4.3 the quantum operators are bifiltered, only now the filtration is by half-integers. Furthermore, a classical operator of bidegree (d_+, d_-) can mix in the quantum theory only with operators of bidegree $(d_- - n/2, d_+ - n/2)$ for any nonnegative integer n. We work with $SO(N)$-invariant operators only. We list bases for the relevant spaces of operators, where the label (d_+, d_-) denotes operators of bidegree $(d_- - n/2, d_+ - n/2)$ for all n, and the subscripts 'b', 'f' denote 'bosonic', 'fermionic':

$$\begin{aligned}
(3,0)_{\mathrm{b}} &: (\partial_+\Phi, \nabla_+\partial_+\Phi)\\
(3,1/2)_{\mathrm{f}} &: (D_-\Phi, \partial_+\Phi)|\partial_+\Phi|^2,\ (D_-\Phi, D_+\Phi)(D_+\Phi, \partial_+\Phi),\\
&\quad\ (D_+\Phi, \nabla_+\partial_+\Phi),\ (\nabla_+D_+\Phi, \partial_+\Phi)\\
(7/2,0)_{\mathrm{f}} &: (D_+\Phi, \partial_+\Phi)|\partial_+\Phi|^2,\ (D_+\Phi, \nabla_+D_+\Phi)(D_+\Phi, \partial_+\Phi),\ (D_+\Phi, \nabla_+^2\partial_+\Phi)\\
(7/2,1/2)_{\mathrm{b}} &: (D_-\Phi, D_+\Phi)\cdot(3,0)_{\mathrm{b}},\ (D_-\Phi, \partial_+\Phi)\cdot(5/2,0)_{\mathrm{f}},\\
&\quad\ (D_-\Phi, \nabla_+\partial_+\Phi)(D_+\Phi, \partial_+\Phi)\\
(5/2,0)_{\mathrm{f}} &: (D_+\Phi, \nabla_+\partial_+\Phi),\ (\nabla_+D_+\Phi, \partial_+\Phi).
\end{aligned}$$

Here '∇' is the covariant derivative on the sphere. The relevant spaces and their dimensions are:

$$\begin{array}{ccc}
(3,0)_{\mathrm{b}} & \xrightarrow{D_+} & (7/2,0)_{\mathrm{f}} \\
{\scriptstyle D_-}\downarrow & & \downarrow{\scriptstyle D_-} \\
(3,1/2)_{\mathrm{f}} & \xrightarrow{D_+} & (7/2,1/2)_{\mathrm{b}}
\end{array}
\qquad\qquad
\begin{array}{ccc}
1 & \longrightarrow & 3 \\
\downarrow & & \downarrow \\
4 & \longrightarrow & 5
\end{array}$$

It follows by dimension counting, as in §4.3, that we can choose the desired $\mathfrak{X}_+$ of bifiltration $(7/2, 0)$ and $\mathfrak{X}_-$ of bifiltration $(3, 1/2)$. Then computing the weight of $\mathfrak{J}$, and so of the corresponding conserved charge Q, we find $[K, Q] = 3Q$, where K is the Lorentz boost. So Q does not commute with Poincaré, as we wanted.

Solution to Problem 5

(a) This model has two bosonic fields: a map $\phi : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ and a "Lagrange multiplier" $\sigma : \mathbb{R}^2 \to \mathbb{R}$. The action is

$$\mathcal{L} = \frac{1}{2\lambda} |d\phi|^2 - \frac{i\sigma}{2}(|\phi|^2 - 1).$$

It is easy to compute the classical equations of motion

$$|\phi|^2 = 1$$
$$\frac{\Delta}{\lambda}\phi = i\sigma\phi.$$

Our Laplacian 'Δ' is nonnegative. The first equation says that ϕ maps into the n-sphere S^n and the second that the Laplacian of ϕ is perpendicular to S^n. This means that ϕ is a harmonic map into S^n. We express σ in terms of ϕ as

$$\sigma = \frac{1}{i\lambda}(\Delta\phi, \phi). \tag{5.1}$$

Up to a total derivative $\sigma = \frac{1}{i\lambda}|d\phi|^2$.

(b) According to equation (5.1) the σ field has dimension 2, since ϕ is a scalar field in two dimensions and so has dimension 0. We expect the behavior of the 2-point function in position space to be

$$\langle \sigma(x)\sigma(0) \rangle \sim \frac{1}{|x|^4} \text{ as } x \to 0,$$

or equivalently in momentum space to be

$$\langle \sigma(q)\sigma(-q) \rangle \sim q^2 \quad \text{as} \quad q \to \infty, \tag{5.2}$$

with possible logarithmic corrections due to anomalous dimensions. We could integrate out the σ field first, thereby obtaining the σ-model into S^n, and since this theory is asymptotically free the 2-point function of $|d\phi|^2$ does exhibit the behavior (5.2) (up to logarithmic corrections). Instead, we integrate out ϕ first and compute the effective action for σ, at least in the large n limit.

The computation of the effective action appears in §4.2 of these lectures:

$$\Gamma(\tilde{\sigma}) = \frac{n}{2}\{\text{Tr ln}\,(\Delta - i\tilde{\sigma}\tilde{\lambda}) + i\tilde{\sigma}\}. \tag{5.3}$$

where $\tilde{\sigma} = \sigma/n$ and $\tilde{\lambda} = \lambda n$. We know that we should expand around the value $-i\tilde{\sigma}_0\tilde{\lambda} = M^2$ for a positive constant M^2. The $\tilde{\sigma}$ propagator is the inverse of the

second derivative of the action (5.3), which we evaluate at the constant $\tilde{\sigma}_0 = iM^2/\tilde{\lambda}$:

$$\begin{aligned}\left.\frac{\delta^2\Gamma}{\delta\tilde{\sigma}^2}\right|_{\tilde{\sigma}_0} &= \frac{n\tilde{\lambda}^2}{2}\,\mathrm{Tr}\left((\Delta+M^2)^{-1}\delta\tilde{\sigma}(\Delta+M^2)^{-1}\delta\tilde{\sigma}\right)\\ &= \frac{n\tilde{\lambda}^2}{2}\int dxdy\,G(x,y)\,\delta\tilde{\sigma}(y)\,G(y,x)\,\delta\tilde{\sigma}(x)\\ &= \frac{n\tilde{\lambda}^2}{2}\int dxdy\,G(x,y)^2\,\delta\tilde{\sigma}(x)\,\delta\tilde{\sigma}(y),\end{aligned} \tag{5.4}$$

where G is the kernel of the Green's function $(\Delta+M^2)^{-1}$. So the kernel of the inverse $\tilde{\sigma}$-propagator is $\frac{n\tilde{\lambda}^2}{2}G(x,y)^2$. In momentum space the square goes over to convolution, and so the kernel of the inverse $\tilde{\sigma}$-propagator is

$$\begin{aligned}K(q,-q) &= \frac{n\tilde{\lambda}^2}{2}\int\frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+M^2}\,\frac{1}{(q-p)^2+M^2}.\\ &= \frac{n\tilde{\lambda}^2}{8\pi^2}\int_0^1 d\alpha\int d^2p\,\frac{1}{[p^2+M^2+\alpha(1-\alpha)q^2]^2}\\ &= \frac{n\tilde{\lambda}^2}{8\pi}\int_0^1\frac{d\alpha}{M^2+\alpha(1-\alpha)q^2}\\ &= \frac{n\tilde{\lambda}^2}{4\pi}\,\frac{\ln\left(\frac{\sqrt{1+4M^2/q^2}+1}{\sqrt{1+4M^2/q^2}-1}\right)}{q^2\sqrt{1+4M^2/q^2}}\\ &\sim \frac{n\tilde{\lambda}^2}{2\pi}\,\frac{\ln(q^2/M^2)}{q^2}\qquad\text{as}\qquad q\to\infty.\end{aligned} \tag{5.5}$$

So the $\tilde{\sigma}$-propagator behaves as

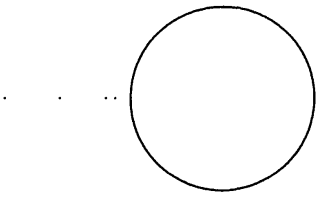
$$\langle\tilde{\sigma}(q)\tilde{\sigma}(-q)\rangle \sim \frac{2\pi}{n\tilde{\lambda}^2}\,\frac{q^2}{\ln(q^2/M^2)}\qquad\text{as}\qquad q\to\infty.$$

Since $\tilde{\sigma}=\sigma/n$, we have finally

$$\langle\sigma(q)\sigma(-q)\rangle \sim \frac{2\pi n}{\tilde{\lambda}^2}\,\frac{q^2}{\ln(q^2/M^2)}\qquad\text{as}\qquad q\to\infty. \tag{5.6}$$

Notice that the behavior in n could have been predicted from (5.1), since the 2-point function of $(\Delta\phi,\,\phi)$ is $n+1$ times the 2-point function of a single scalar field. The large q behavior is as predicted in (5.2), but corrected by a logarithm.

Notice that the inverse propagator (5.5) can be computed by diagrams. The effective action obtained by integrating our ϕ is computed by diagrams with external σ lines and internal ϕ lines. We should expand around $\sigma = iM^2/\lambda$, of course. For the quadratic term there is a single diagram, as shown, where the solid line



represents ϕ and the dotted line represents σ. Note that the ϕ propagator is massive due to the shift in σ. This diagram is evaluated by (5.5).

Solution to Problem 10

We follow the notation and conventions of [I-**Supersolutions**].

A chiral superfield Φ in 2 dimensions has component fields $\phi, \psi_\pm, \bar{\psi}_\pm$. Let K denote the infinitesimal generator of rotation and J_- and J_+ the infinitesimal generators of the R-symmetries, called J_L and J_R in Lecture 14. The topological twist we consider uses $K - \frac{1}{2}J_- + \frac{1}{2}J_+$ as the new infinitesimal generator of Euclidean rotations. The following chart shows the weight of the component fields under these infinitesimal symmetries:

component field	K	J_-	J_+	$K - \frac{1}{2}J_- + \frac{1}{2}J_+$
ϕ	0	0	0	0
ψ_+	1/2	0	1	1
$\bar{\psi}_+$	1/2	0	−1	0
ψ_-	−1/2	1	0	−1
$\bar{\psi}_-$	−1/2	−1	0	0

The supersymmetry $\overline{Q}_+ + \overline{Q}_-$ commutes with the new generator of rotation, so is a supersymmetry of the twisted theory.

Consider a chiral superfield in the twisted theory which maps a super Riemann surface into a Kähler manifold X. The last column of the chart indicates how the component fields transform on the underlying surface Σ. Thus ϕ is a map from the underlying surface Σ to X, the odd fields $\bar{\psi}_\pm$ are scalars on Σ with values in the antiholomorphic tangent bundle $\phi^* T_{0,1}X$, and the odd field ψ_+ (resp. ψ_-) is a $(1,0)$- (resp. $(0,1)$-) form on Σ with values in the holomorphic tangent bundle $\phi^* T_{1,0}X$. It is convenient to define new odd fields $\bar{\chi}$ and $\bar{\chi}'$ by

$$\begin{aligned} \bar{\chi} &= \bar{\psi}_+ + \bar{\psi}_- \\ \bar{\chi}' &= \bar{\psi}_+ - \bar{\psi}_-. \end{aligned} \tag{10.1}$$

We now specialize the target $X = \mathbb{C}^n$. As explained in §14.3.1 of these lectures we reduce the computation of a topological expectation value to an integral over the lowest modes of the fields. When Σ has genus 0 there are no holomorphic (1,0)-forms (or antiholomorphic (0,1)-forms), so we can eliminate $\psi_\pm$. The other fields $\phi, \bar{\chi}, \bar{\chi}'$ are reduced to constants. For these constant fields we compute the supersymmetry transformation laws by dimensionally reducing the last equations in [I-**Supersolutions**, §5.1]. For $\delta = \bar{\eta}(\overline{Q}_+ + \overline{Q}_-)$ we find

$$\begin{aligned} \delta\bar{\phi} &= -\sqrt{2}\bar{\eta}\bar{\chi} \\ \delta\bar{\chi} &= 0 \\ \delta\bar{\chi}' &= 2\sqrt{2}\bar{\eta}\overline{\operatorname{grad} W(\phi)}. \end{aligned}$$

The lagrangian for these fields is computed by dimensionally reducing the general lagrangian in [I-**Supersolutions**, §5.3], restricting to constant fields, and Wick rotating to Euclidean signature:

$$L = \| \operatorname{grad} W(\phi) \|^2 - \frac{1}{2} \operatorname{Hess} \overline{W}(\phi)(\bar{\chi}', \bar{\chi}).$$

As explained in the statement of the problem, we take a single observable which is a polynomial $P(\phi)$. Then the preceding discussion shows that its one-point function E reduces to a finite dimensional integral over $\mathbb{C}^n$ and over the odd variables:

$$E = \int P(\phi) e^{-\| \operatorname{grad} W(\phi)\|^2 + \frac{1}{2} \operatorname{Hess} \overline{W}(\phi)(\bar{\chi}', \bar{\chi})} \frac{d^n \bar{\phi} d^n \phi}{(2\pi i)^n} d^n \bar{\chi} d^n \bar{\chi}'. \tag{10.2}$$

To make sense of this we note that $d^n \bar{\phi} d^n \bar{\chi}$ has a canonical meaning, since $d^n \bar{\phi}$ is a section of $\operatorname{Det} T^*_{0,1}$ and $d^n \bar{\chi}$ is a section of $\operatorname{Det} T_{0,1}$. However, $d^n \phi$ is a section of $\operatorname{Det} T^*_{1,0}$ and $d^n \bar{\chi}'$ is a section of $\operatorname{Det} T_{0,1} \cong \operatorname{Det} T^*_{1,0}$, where the last isomorphism uses the standard hermitian metric on $\mathbb{C}^n$. Thus we must choose an element

$$\omega = d^n \phi d^n \bar{\chi}' \in (\operatorname{Det} T^*_{1,0})^{\otimes 2}$$

to define the integral (10.2) as a number.

As explained in §14.3.1, a topological correlation function is independent of most parameters in the lagrangian. Indeed, in this finite dimensional model L is exact, as we compute from (10.1):

$$\bar{\eta} L = \delta \Big\{ \frac{1}{2\sqrt{2}} \langle \bar{\chi}', \operatorname{grad} W(\phi) \rangle \Big\}.$$

This means that in (10.2) we can insert a constant $1/\epsilon$ in the exponential without changing the answer. (In any case there is a factor of the volume of Σ which enters into the action, and so in the exponential in the integrand in (10.2).)

The fermionic integral is easy to do, and we reduce the computation of E to an ordinary integral over $\mathbb{C}^n$:

$$E = \int_{\mathbb{C}^n} P(\phi)\, \epsilon^{-n} \det \operatorname{Hess} \overline{W}(\phi)\, e^{-\frac{1}{\epsilon} \| \operatorname{grad} W(\phi)\|^2} \frac{d^n \bar{\phi} d^n \phi}{(2\pi i)^n}.$$

Note that we use $\bar{\omega}$ to interpret $\det \operatorname{Hess} \overline{W}(\phi)$ as a number. In the limit $\epsilon \to 0$ the remaining integral localizes to the critical point set of W. Assume for simplicity that W has isolated nondegenerate critical points. Near a critical point we approximate the integral by a Gaussian integral, which gives $\| \det \operatorname{Hess} W(\phi)\|^2$ in the denominator. This cancels the antihomolorphic piece in the integrand and the final result is

$$E = \sum_{\text{crit pts } \phi_i} \frac{P(\phi_i)}{\det \operatorname{Hess} W(\phi_i)}.$$

Solution to Problem 11

Before attacking the path integral, we review[1] a formula for a certain universal curvature. Suppose $P \to \Sigma$ is a principal G-bundle over a Riemannian manifold Σ. Let $\mathcal{A}_P$ denote the space of connections on P and $\mathcal{G}_P$ the group of gauge transformations. Throughout we pass freely from a connection A to the $\bar{\partial}$ operator

[1] A reference for at least part of the computation is S. K. Donaldson, P. Kronheimer, "The geometry of four-manifolds," Oxford University Press, New York, 1990, p. 197.

(complex structure) it defines on associated bundles by its $(1,0)$ part. Now the bundle $\mathcal{A}_P \times P \to \mathcal{A}_P \times X$ has a tautological connection. To descend it to the quotient[2] $\mathcal{P} \to (\mathcal{A}_P/\mathcal{G}_P) \times X$ we need a horizontal distribution on $\mathcal{A}_P \to \mathcal{A}_P/\mathcal{G}_P$, which we obtain by taking the orthogonal complements to orbits. Restrict now to $\mathcal{M}_P \times X$, where $\mathcal{M}_P$ is the moduli space of flat connections modulo gauge transformations. We further restrict to irreducible connections. A tangent vector to $\mathcal{M}_P$ at the equivalence class of A is an element of the cohomology group $H^1_A(\mathfrak{g}^P)$, represented by a 1-form α on Σ with values in the adjoint bundle. Given two such α_1, α_2 the curvature ω_1 of this connection on $\mathcal{P} \to \mathcal{M}_P \times X$ in directions tangent to $\mathcal{M}_P$ is

$$\omega_1(\alpha_1, \alpha_2) = 2\Delta_A^{-1}\langle[\alpha_1 \wedge \alpha_2]\rangle,$$

where $\Delta_A = d_A^* d_A$ is the laplacian on $\Omega^0(\mathfrak{g}^P)$ and the notation $\langle[\alpha_1 \wedge \alpha_2]\rangle$ means, for $\alpha_i = a_i \otimes \zeta_i \in \Omega^1 \otimes \mathfrak{g}$,

$$\langle[a_1 \otimes \zeta_1, a_2 \otimes \zeta_2]\rangle = \langle a_1, a_2\rangle[\zeta_1, \zeta_2]. \tag{11.1}$$

If Σ is a Riemann surface, we rewrite (11.1) in terms of $\Box_A \colon \Omega^0(\mathfrak{g}^P) \to \Omega^2(\mathfrak{g}^P)$ defined by $\Box_A = -d_A * d_A$:

$$\omega_1(\alpha_1, \alpha_2) = 2\Box_A^{-1}[\alpha_1 \wedge *\alpha_2]. \tag{11.2}$$

(Although we make no use of this fact, observe that ω_1 depends only on the conformal structure of Σ.)

Now fix a unitary representation $\rho \colon G \to U(W)$ on a hermitian vector space W, and let $W^P \to \Sigma$ be the associated hermitian vector bundle. The trivial (infinite rank) bundle $\mathcal{A}_P \times \Omega^0(W^P)$ descends under gauge transformation to a bundle $\mathcal{S} \to \mathcal{M}_P$, after restricting to flat connections. This bundle may also be described as the bundle of relative sections of $W^{\mathcal{P}} \to \mathcal{M}_P \times X$. The connection on $\mathcal{P} \to \mathcal{M}_P \times X$ induces a connection on $\mathcal{S} \to \mathcal{M}_P$: given a tangent vector to $\mathcal{M}_P$ we lift it to $\mathcal{M}_P \times X$ trivially and pointwise apply the covariant derivative in $W^{\mathcal{P}}$. This connection respects the L^2 metric on $\mathcal{S} \to \mathcal{M}_P$ obtained by integrating the pointwise inner product on $W^P \to \Sigma$ against the Riemannian measure on Σ. The curvature of this connection is given by acting $\dot{\rho}(\omega)$ pointwise. We are in fact interested in $(0,1)$-forms with values in W, which is the space of sections of the fixed complex line bundle $\Lambda^{0,1}$ tensored with W^P. The connection and curvature do not see this line bundle, so the formula (11.2) is unchanged. Consider now the subbundle of relative harmonic sections of $\Lambda^{0,1} \otimes W^P$. It is a finite dimensional subbundle[3]

$$H^1(W^P) \to \mathcal{M}_P \tag{11.3}$$

of $\mathcal{S} \to \mathcal{M}_P$, and we can project the connection to it using the L^2 metric. The curvature of this projected connection is then the projection of (11.2) acting pointwise on holomorphic sections plus a second fundamental form term. Let π_A denote projection onto the forms in $\Omega^{0,1}(W^P)$ harmonic with respect to the connection A. A bit of computation shows that the second fundamental form term is

$$\begin{aligned}\omega_2(\alpha_1, \alpha_2) = {} & \pi_A \dot{\rho}_{\mathbb{C}}\big(\alpha_2^{(0,1)}\big)\Delta_A^{-1} * \dot{\rho}_{\mathbb{C}}\big(\alpha_1^{(1,0)}\big)\pi_A \\ & - \pi_A \dot{\rho}_{\mathbb{C}}\big(\alpha_1^{(0,1)}\big)\Delta_A^{-1} * \dot{\rho}_{\mathbb{C}}\big(\alpha_2^{(1,0)}\big)\pi_A.\end{aligned} \tag{11.4}$$

[2]We ignore singularities in this account.

[3]We restrict to the space of complex structures for which $H^0(W^P)$ vanishes.

The curvature of (11.3) is the sum

$$\pi_A \omega_1 \pi_A + \omega_2. \tag{11.5}$$

Now to the physics. The twisted chiral superfield has component fields[4] $\sigma, \lambda^{\pm}, \bar{\lambda}^{\pm}$ which take values in the adjoint bundle $\mathfrak{g}^P$. There is also a connection A on P. As in Problem 10, the chiral superfield has component fields $\phi, \psi_{\pm}, \bar{\psi}_{\pm}$ which are sections of the associated hermitian bundle V^P (or its conjugate $\overline{V}^P$.) The topological twist used here is different than that in Problem 10; it commutes with the supersymmetry $\overline{Q}_+ + Q_-$. We summarize in the following chart:

component field	K	J_-	J_+	$K + \frac{1}{2}J_- + \frac{1}{2}J_+$
A	$1 \oplus -1$	0	0	$1 \oplus -1$
σ	0	1	-1	0
$\bar{\sigma}$	0	-1	1	0
λ^+	$-1/2$	0	-1	-1
$\bar{\lambda}^+$	$-1/2$	0	1	0
λ^-	$1/2$	-1	0	0
$\bar{\lambda}^-$	$1/2$	1	0	1
ϕ	0	0	0	0
$\bar{\phi}$	0	0	0	0
ψ_+	$1/2$	0	1	1
$\bar{\psi}_+$	$1/2$	0	-1	0
ψ_-	$-1/2$	1	0	0
$\bar{\psi}_-$	$-1/2$	-1	0	-1

Here K is the infinitesimal generator of rotation and $J_{\pm}$ are the R-symmetries. In the twisted theory on a surface Σ we interpret $\sigma, \bar{\sigma}$ as adjoint-valued complex scalars; $\lambda^-, \bar{\lambda}^+$ as odd adjoint-valued complex scalars; $\bar{\lambda}^-$ as a $(1,0)$-form on Σ with values in the complexified adjoint bundle; λ^+ as a $(0,1)$-form on Σ with values in the complexified adjoint bundle; ϕ as a section of V^P; $\bar{\psi}_+$ (resp. ψ_-) as an odd section of $\overline{V}^P$ (resp. V^P); ψ_+ as a $(1,0)$-form on Σ with values in V^P; and $\bar{\psi}_-$ as a $(0,1)$-form on Σ with values in $\overline{V}^P$.

The difference $J_- - J_+$ commutes with $\overline{Q}_+ + Q_-$, and it is convenient to check the lagrangian and equations below for balance under this symmetry.

The lagrangian is obtained by dimensional reduction from the general $N = 1$ lagrangian in four dimensions. The relevant formulas appear in [I-**Supersolutions**, §6.2] under the "$\mathbb{F} = \mathbb{C}$" case. We take the lagrangian there, dimensionally reduce to two dimensions, then Wick rotate to Euclidean space. Let

[4] In [I-**Supersolutions**, §9.2] we use $\lambda_{\pm}$ which are related to $\lambda^{\pm}$ of [I-**Supersolutions**, §6.2] we use here by

$$\lambda^+ = -\lambda^-. \qquad \lambda^- = \lambda^+.$$

$\dot\rho_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(V)$ denote the complexified infinitesimal action of G on V. We find (see [I-**Supersolutions**, §9.3] for the dimensionally reduced Yang-Mills action):

(11.6)
$$\begin{aligned} L_e = \Big\{ &\frac{1}{2}|F_A|^2 + \langle d_A\bar\sigma, d_A\sigma\rangle + \langle d_A\bar\phi, d_A\phi\rangle + \langle \bar\lambda \not{D}_A \lambda\rangle + \langle \bar\psi \not{D}_A \psi\rangle \\ &+ \frac{1}{2}\|[\sigma,\bar\sigma]\|^2 + 2\|\dot\rho_{\mathbb{C}}(\sigma)\phi\|^2 + 2|\mu(\phi)|^2 \\ &- \sqrt{2}\big(\langle\bar\lambda^-, [\bar\sigma,\lambda^+]\rangle + \langle\bar\lambda^+, [\sigma,\lambda^-]\rangle\big) \\ &+ \sqrt{2}\big(\langle\bar\psi_-, \dot\rho_{\mathbb{C}}(\sigma)\psi_+\rangle + \langle\bar\psi_+, \dot\rho_{\mathbb{C}}(\bar\sigma)\psi_-\rangle\big) \\ &+ \sqrt{2}\big(\langle\dot\rho_{\mathbb{C}}(\lambda^a)^*\phi^*, \psi_a\rangle - \langle\bar\psi_{\dot a}, \dot\rho_{\mathbb{C}}(\bar\lambda^{\dot a})\phi\rangle\big)\Big\}\, |d^2x|. \end{aligned}$$

In the second line $\mu\colon V \to \mathfrak{g}^*$ is the moment map of the G action on V. The sum in the last line is over $\pm$. The notation for the penultimate term is explained in [I-**Supersolutions**, §6.2]; it is simply the complex conjugate of the last term. The terms on the third and fourth lines come from dimensionally reducing the fermion kinetic terms in four dimensions; the second term in the second line comes from dimensionally reducing $\langle d_A\bar\phi, d_A\phi\rangle$ in four dimensions. Similarly, we dimensionally reduce the supersymmetry transformation laws for the fields to two dimensions, Wick rotate, and specialize to the supersymmetry $\delta = \eta(\overline{Q}_+ + Q_-)$ of the twisted theory. The resulting equations are (see [I-**Supersolutions**, §9.2] for the transformation laws of pure gauge theory):

(11.7)
$$\begin{aligned} (\delta A)_{1,0} &= 2\eta\bar\lambda^- & \delta\lambda^+ &= -2\sqrt{2}\,\eta\bar\partial_A\sigma \\ (\delta A)_{0,1} &= 2\eta\lambda^+ & \delta\bar\lambda^+ &= \eta i\big(\tilde f_A + 2\check\mu(\phi) - i[\sigma,\bar\sigma]\big) \\ \delta\sigma &= 0 & \delta\lambda^- &= -\eta i\big(\tilde f_A + 2\check\mu(\phi)\big) + i[\sigma,\bar\sigma]\big) \\ \delta\bar\sigma &= -\sqrt{2}\,\eta(\lambda^- + \bar\lambda^+) & \delta\bar\lambda^- &= 2\sqrt{2}\,\eta\partial_A\sigma \\ \delta\phi &= -\sqrt{2}\,\eta\psi_- & \delta\psi_+ &= -2\sqrt{2}\,\eta\,\partial_A\phi \\ \delta\bar\phi &= -\sqrt{2}\,\eta\bar\psi_+ & \delta\bar\psi_+ &= 2\eta\dot\rho_{\mathbb{C}}(\bar\sigma)\bar\phi \\ & & \delta\psi_- &= 2\eta\dot\rho_{\mathbb{C}}(\sigma)\phi \\ & & \delta\bar\psi_- &= -2\sqrt{2}\,\eta\,\bar\partial_A\bar\phi \end{aligned}$$

Here $\check\mu\colon V \to \mathfrak{g}$ is constructed from the moment map $\mu\colon V \to \mathfrak{g}^*$ using the metric on $\mathfrak{g}$. We have used the standard operator

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial\tau} - i\frac{\partial}{\partial x}\right)$$

and its conjugate (on $\mathbb{R}^2$). The curvature of A is[5]

(11.8)
$$F_A = \tilde f_A\, d\tau \wedge dx.$$

[5]In [I-**Supersolutions**, §9.2] we write the curvature of the gauge field A in Minkowski space M^2 as

$$F_A = -f_A\, dt \wedge dx$$

for standard coordinates t, x on M^2. Setting $\tau = it$ in both the connection and curvature we find (11.8) with $\tilde f_A = i f_A$.

After twisting, these formulas continue to hold on a Riemann surface Σ.

According to the fixed point idea described in §14.3.1, the functional integral for a topological correlation function localizes to the bosonic configurations invariant under the supersymmetry. The bosonic equations are thus obtained by setting to zero the right hand sides of the transformation laws for $\delta(\text{fermion})$. In our problem we find from (11.7) that

$$\begin{aligned} \tilde{f}_A &= -2\check{\mu}(\phi) \\ \partial_A\sigma = \bar{\partial}_A\sigma &= 0 \\ \bar{\partial}_A\bar{\phi} &= 0 \\ [\sigma,\bar{\sigma}] &= 0 \\ \dot{\rho}_{\mathbb{C}}(\sigma)\phi &= 0. \end{aligned} \tag{11.9}$$

Now the moment map for the action of the unitary group $U(V)$ on V is given by the formula

$$\mu(\phi)(T) = -\frac{i}{2}\langle\bar{\phi}, T\phi\rangle, \qquad \phi \in V, \quad T \in \mathfrak{u}(V). \tag{11.10}$$

We pullback by $\dot{\rho}\colon \mathfrak{g} \to \mathfrak{u}(V)$ to obtain a moment map for the action of G on V. If $\mathfrak{g}$ has no abelian factors, then this is the only possibility. In that case, if the representation is irreducible the only solution to $\mu(\phi) = 0$ is $\phi = 0$. So the functional integral is reduced to the zero section of (11.3). We restrict ourselves to this case. We further assume our bundle is chosen so that $H^0_A(W^P) = 0$, where

$$W = \overline{V},$$

or rather we restrict to complex structures for which that is true. Then it follows from the third equation in (11.9) that $\bar{\phi} = 0$. The first equation now implies that A is flat, since by (11.10) the moment map vanishes. We assume that A is irreducible so that the second equation implies $\sigma = 0$. The remaining equations do not add anything new. Summarizing, the equations for a bosonic fixed point are:

$$\begin{aligned} &A \text{ flat} \\ &\phi = \sigma = 0. \end{aligned}$$

Now to the fermions. By the general principles laid out in §14.3 we reduce to integrating over the lowest modes of the fermions. It was explained in §14.4 that if A is irreducible, as we are assuming, then we can set to zero the complex scalars

$$\lambda^- = \bar{\lambda}^+ = 0.$$

Furthermore, under our assumption that $H^0_A(W^P) = 0$ we have

$$\psi_- = \bar{\psi}_+ = 0.$$

The nonzero fermion zero modes are

$$\chi := \psi_+, \qquad \bar{\chi} := \bar{\psi}_-, \qquad \lambda^+, \qquad \bar{\lambda}^-,$$

restricted to be harmonic 1-forms.

We are ready to compute the expectation value of a topological observable. As stated in the problem, and as is evident from (11.7), the observables are polynomials in σ and their descendents, as in the pure gauge theory discussed in §14.4. By general principles of topological field theory (§14.3.1) the path integral reduces to a neighborhood of the supermanifold consisting of the bosonic configurations satisfying (11.9) and the fermionic zero modes. The boson A and the zero modes $\lambda^+, \bar\lambda^-$ comprise $\Pi T\mathcal{M}_P$, and as explained in §14.4 an integral over $\Pi T\mathcal{M}_P$ is equivalent to an integral of a differential form over $\mathcal{M}_P$. The integral over the zero modes of $\chi, \bar\chi$ is a fermionic integral over the vector space $\Pi H^1_A(W^P)$. We reduce to these finite dimensional integrals by integrating out the nonzero modes of these fields, as well as the remaining fields—$\phi, \bar\phi, \sigma, \bar\sigma, A, \bar\lambda^+, \lambda^-, \bar\psi_+, \psi_-$—over the normal bundle to $\Pi\big(T\mathcal{M}_P \times H^1_A(W^P)\big)$ in the quadratic approximation to the action. That integral is Gaussian, so can be done explicitly. Then we evaluate the integral over $\Pi H^1_A(W^P)$ to reduce to an integral of a differential form over $\mathcal{M}_P$. Only at the last stage does the topological observable enter, so the result of our computation will be a differential form which multiplies the form corresponding to the observable. We will recognize that form as the Euler form of (11.3).

First, we do the Gaussian integral over the scalar fields $\phi, \bar\phi, \sigma, \bar\sigma$. In the lagrangian (11.6) we eliminate the potential terms in the second line, since they are quartic in these scalar fields. What remains in the action is an infinite-dimensional integral analogous to the following integral over $\mathbb{C}^N$:

$$\iint_{\mathbb{C}^N} \frac{d\overline{\Phi} d\Phi}{(2\pi i)^N} e^{-(\overline{\Phi}Q\Phi + \overline{\Phi}L + \overline{L}\Phi)} = (\det Q)^{-1} e^{\overline{L}Q^{-1}L}.$$

Here Q is a hermitian operator, so $\overline{\Phi}Q\Phi$ is a real quadratic form, and L a complex linear operator, so $\overline{\Phi}L + \overline{L}\Phi$ is a real linear form. In our problem $\Phi = \begin{pmatrix} \phi \\ \sigma \end{pmatrix}$ and

$$\begin{aligned} Q &= \Box_A \\ L &= \sqrt{2} \begin{pmatrix} \dot\rho_{\mathbb{C}}(\lambda^a)\psi_a \\ \\ [\lambda^+, \bar\lambda^-] + B(\bar\chi, \chi) \end{pmatrix} \end{aligned}$$

for a bilinear form B which satisfies

$$\langle \bar\chi, \dot\rho_{\mathbb{C}}(\bar\sigma)\chi \rangle = \langle \bar\sigma, B(\bar\chi, \chi) \rangle. \tag{11.11}$$

Note that L is quadratic in fermions so $\overline{L}Q^{-1}L$ is quartic in fermions. There is a similar bosonic integral over the nonzero modes of A. The interaction terms of A involving fermions—$\langle \bar\lambda \not{D}_A \lambda \rangle$ and $\langle \bar\psi \not{D}_A \psi \rangle$—pair $\bar\lambda^\pm$ with $\lambda^\pm$ and $\bar\psi_\pm$ with $\psi_\pm$. When we restrict to zero modes such pairs vanish, so although this integral contributes terms in the effective action quartic in fermions, those terms will not play a role. Of course, the integral over the nonzero modes of A also contributes an inverse determinant.

Next, we integrate over the nonzero modes of all fermions. The effective lagrangian for the fermions is of the form

$$\langle \bar\lambda \not{D}_A \lambda \rangle + \langle \bar\psi \not{D}_A \psi \rangle + \overline{\mathcal{L}}\mathcal{Q}^{-1}\mathcal{L}$$

where $\mathcal{Q}, \mathcal{L}$ include Q, L and the corresponding forms for A. In the Gaussian approximation we drop the quartic term $\overline{\mathcal{L}}\mathcal{Q}^{-1}\mathcal{L}$ and are left with a simple Gaussian integral. It evaluates to determinants in the numerator which exactly cancel the bosonic determinants.

We are left with fermionic integrals over the zero modes of $\chi, \bar{\chi}, \lambda^+, \bar{\lambda}^-$ and a bosonic integral over $\mathcal{M}_P$. We do only the integral over $\chi, \bar{\chi}$ to reduce to an integral over $\Pi T\mathcal{M}_P$, as explained earlier. The effective lagrangian now just involves the potential term $\overline{\mathcal{L}}\mathcal{Q}^{-1}\mathcal{L}$, and restricted to the zero modes it is

$$-2\langle \dot{\rho}_{\mathbb{C}}(\bar{\lambda}^-)^*\bar{\chi}, \Box_A^{-1}\dot{\rho}_{\mathbb{C}}(\lambda^+)\chi\rangle + 2\langle B(\chi,\bar{\chi}), \Box_A^{-1}[\lambda^+,\bar{\lambda}^-]\rangle.$$

According to (11.11) we can rewrite the second term as

$$-2\langle \bar{\chi}, \dot{\rho}_{\mathbb{C}}\big(\Box_A^{-1}[\lambda^+,\bar{\lambda}^-]\big)\chi\rangle.$$

(This is a term in the original action (11.6), but with σ substituted by the solution to its equation of motion $\Box_A\sigma = -\sqrt{2}[\lambda^+,\bar{\lambda}^-]$.) So the integral we have is

$$\iint_{H^1_A(W^P)} \frac{D\bar{\chi}D\chi}{(2\pi i)^m}\, e^{\int_\Sigma \langle \bar{\chi},\omega\chi\rangle} = \frac{\det(\omega)}{(2\pi i)^m}, \tag{11.12}$$

where $m = \dim H^1(W)$ and

$$-\omega = 2\dot{\rho}_{\mathbb{C}}\big(\Box_A^{-1}[\lambda^+,\bar{\lambda}^-]\big) + 2\dot{\rho}_{\mathbb{C}}(\bar{\lambda}^-)\Box_A^{-1}\dot{\rho}_{\mathbb{C}}(\lambda^+).$$

In (11.12) ω operates on χ, which is of type $(1,0)$. To compare to (11.5) we need to write the adjoint operator ω^*, which operates on $\bar{\chi}$ of type $(0,1)$. Recalling that $\dot{\rho}$ is skew-Hermitian on real vectors, we see that the adjoint of ω is

$$\omega^* = 2\dot{\rho}_{\mathbb{C}}\big(\Box_A^{-1}[\lambda^+,\bar{\lambda}^-]\big) - 2\dot{\rho}_{\mathbb{C}}(\lambda^+)\Box_A^{-1}\dot{\rho}_{\mathbb{C}}(\bar{\lambda}^-).$$

By (11.5) the differential form which corresponds to ω^* is twice the curvature of $H^1(V^P) \to \mathcal{M}_P$, and so $-\omega^*$ is twice the curvature form of the conjugate bundle $H^1(W^P) \to \mathcal{M}_P$. Therefore, (11.12) is $(i/\pi)^m$ times the determinant of the curvature of $H^1(W^P) \to \mathcal{M}_P$, which is 2^m times[6] the Euler form of this "obstruction bundle".

[6]The arduous reader may find from a more careful computation that this factor is absent. Presumably this is resolved by a careful treatement of the normalization of the integral over $\chi, \bar{\chi}$.

Dynamics of $N = 1$ Supersymmetric Field Theories in Four Dimensions

Nathan Seiberg

Introduction

The exact solutions of supersymmetric field theories have four distinct applications:

1. They teach us about the dynamics of strongly coupled field theories. Here the main lesson is the role of electric-magnetic duality in the dynamics.
2. The ideas from field theory have extensions to string theory, whose dynamics is richer. This has already led to a revolution in our understanding of string theory.
3. Supersymmetry might be discovered experimentally in the next generation of experiments. In this case, the dynamics of supersymmetric field theories is of direct phenomenological interest, especially for the question of how supersymmetry is spontaneously broken, leading to a non-supersymmetric spectrum.
4. Witten's topological field theory is based on supersymmetry. Understanding these theories has applications to mathematics.

In these lectures we will focus on the first application. Other applications were described by other lecturers.

Exact solutions play a crucial role in physics. It is often the case that a simple model exhibits the same phenomena which are also present in more complicated examples. The exact solution of the simple model then teaches us about more generic situations. The main point of these lectures is to show how four dimensional supersymmetric quantum field theories can play a similar role as laboratories and testing grounds for ideas in more generic quantum field theories. This follows from the fact that these theories are more tractable than ordinary, non-supersymmetric theories, and many of their observables can be computed exactly. Yet, it turns out that these theories exhibit explicit examples of various phenomena in quantum field theory. Some of them had been suggested before without an explicit realization and others are completely new.

The generic field theory in four dimensions has no supersymmetry at all. With current techniques its dynamics is difficult to control. With more supersymmetry

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Notes by Siye Wu

the theories are less generic. The most constrained theories have $N = 4$ supersymmetry. A more general class of theories has $N = 2$. An even more general class has the minimal amount of supersymmetry: $N = 1$ supersymmetry. These lectures are devoted to the analysis of this largest class. (The theories with $N = 1$ supersymmetry are also the relevant ones for the third application above—particle physics phenomenology.) Since this class includes more generic theories than the more special $N = 2$ or $N = 4$ theories, the dynamics which is being exhibited is richer; it includes phenomena which are impossible with more supersymmetry. Therefore these theories are more useful for our first application. They are the most generic, yet still amenable to exact analysis.

In Lecture 1 we discuss the simplest theories (Wess-Zumino models) and prove their non-renormalization theorem. We will then turn to a discussion of gauge theories. The key feature is the existence of flat directions of the classical potential, leading to a moduli space of classical ground states. In Lecture 2 we analyze some of the quantum effects in these gauge theories. We show how the moduli space of classical vacua can disappear (by generating a potential) in the quantum theory. Alternatively, the moduli space is deformed. We will also see an example where the space is not deformed, but the massless particles at the singularities are different in the quantum theory than in the short distance classical theory. Finally, in Lecture 3 we complete the analysis of these theories by uncovering a new phenomenon—electric-magnetic duality in $N = 1$ theories. It generalizes previously noticed dualities in $N = 4$ and $N = 2$ theories and explains all the phenomena which we discuss in the earlier lectures.

LECTURE 1
Basic Aspects of $N = 1$ QCD

We will study various supersymmetric theories in 4 dimensions, i.e., the underlying bosonic spacetime is the Minkowski space $\mathbb{R}^{1,3}$. Supersymmetry puts constraints on the geometric setting of the field theory. In fact, $N = 0, 1, 2, 4$ supersymmetry corresponds to having a target space of Riemannian, Kähler, hyper-Kähler (or special Kähler) and flat geometry, respectively.

§1.1. Wess-Zumino model

The Wess-Zumino model is a simple example to demonstrate the principle of holomorphy and the non-renormalization theorem. It is an $N = 1$ supersymmetric theory on $\mathbb{R}^{1,3|4}$ with a multiplet of chiral superfields Φ valued in a complex vector space R. In components, we have

$$\Phi = \phi + \theta\,\psi + \theta^2 F, \tag{1.1}$$

where ϕ, ψ and F are a complex boson, Weyl fermion and the auxiliary field, respectively, all valued in R. The most general Lagrangian (density) with at most two derivatives is

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\, K(\Phi, \bar{\Phi}) + \left(\int d^2\theta\, W(\Phi) + \text{c.c.}\right) \\ &= g(\partial\phi, \bar{\partial}\bar{\phi}) + g(\not{\partial}\psi, \bar{\psi}) + g(F, \bar{F}) + (\text{Hessian}(W)_\phi(\psi, \psi) + \partial W(F) + \text{c.c.}).\end{aligned} \tag{1.2}$$

Here K is a Kähler potential on R, g the corresponding metric, and W a holomorphic function on R called the superpotential. After integrating out the auxiliary field F by its (algebraic) equation of motion, we get a bosonic potential $V = g(\partial W, \overline{\partial W})$. Supersymmetry is unbroken if and only if there exists a solution to the equation $\partial W = 0$.

We consider the following examples. In all the cases considered, let K be a Kähler potential on R such that the metric is flat.

Example 1. Φ is a single chiral superfield valued in $\mathbb{C}$ and $W = \frac{1}{2}m\Phi^2$ $(m \in \mathbb{C})$. This is a free theory in which the boson ϕ and the fermion ψ are of equal mass $|m|$. The free action contains terms $m\psi\psi + |m|^2|\phi|^2$. The phase of $m \in \mathbb{C}$, and more generally that of W, can be rotated away by a $U(1)$ rotation of the fermionic coordinate θ (called the *R-symmetry*). We denote this $U(1)$ group by $U(1)_R$. By (1.1), the $U(1)_R$ rotation produces a redefinition of ψ in component language.

Example 2. Φ is valued in $\mathbb{C}$ and $W = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3$ $(m, \lambda \in \mathbb{C},\ \lambda \neq 0)$. This is the original Wess-Zumino model with the Yukawa coupling $\lambda\phi\psi\psi$ and the quartic interaction $|\lambda|^2|\phi|^4$. The equation $\partial W = m\Phi + \lambda\Phi^2 = 0$ has two solutions: $\Phi = 0$ and $\Phi = -\frac{m}{\lambda}$, corresponding to two classical ground states. The bosonic potential of ϕ is shown in Figure 1.1. Under the transformation $\Phi \mapsto -\frac{m}{\lambda} - \Phi$ which interchanges the two ground states, $W \mapsto -W$ up to an additive constant. The sign of W can be rotated away by a $U(1)_R$ symmetry which maps ψ to $i\psi$. Therefore we have a $\mathbb{Z}_4$ discrete symmetry spontaneously broken to $\mathbb{Z}_2$: the generator of $\mathbb{Z}_4$ relates the two ground states 0 and $-\frac{m}{\lambda}$; its square acts as a 2π rotation in $\mathrm{Spin}(1,3)$ and is therefore unbroken.

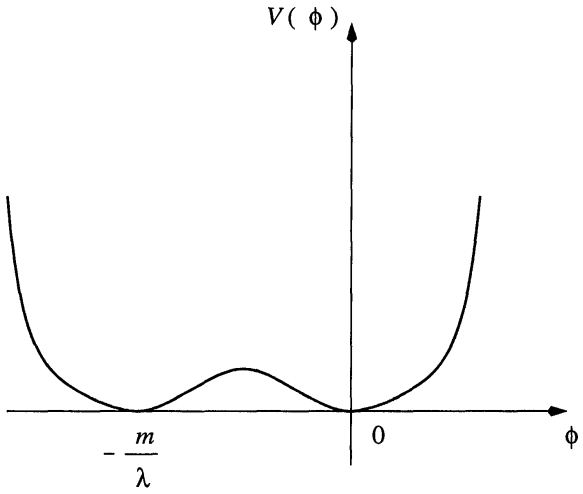

Figure 1.1. The bosonic potential of Example 2, with two minima

Example 3. Φ consists of two complex valued fields L and H (which stand for light and heavy, respectively). The superpotential is $W = \frac{\lambda}{2}LH^2$. The minimum of the bosonic potential is achieved at $H = 0$ and arbitrary L. The field L is massless (light) whereas H has mass $\frac{\lambda}{2}\langle L\rangle$; H is heavy except when $\langle L\rangle = 0$. The moduli space of vacua is shown in Figure 1.2 These (classical) vacua are inequivalent—there are no symmetries that relate them. The metric on the L-space is given by $K_{\text{eff}} = \bar{L}L$, which is the restriction of the original Kähler potential K to the L-space.

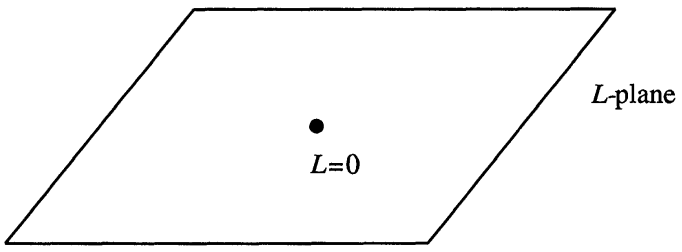

Figure 1.2. The moduli space of vacua of Example 3 with H being massless at $L = 0$

Example 4. Φ consists of three chiral superfields X, Y and Z, and the superpotential is $W = XYZ$. The minimum is achieved at $X = Y = 0$, $Y = Z = 0$ and

$Z = X = 0$. The moduli space of vacua has three branches, as shown in Figure 1.3. Even in the classical theory, there is a singularity $X = Y = Z = 0$, where new massless fields appear.

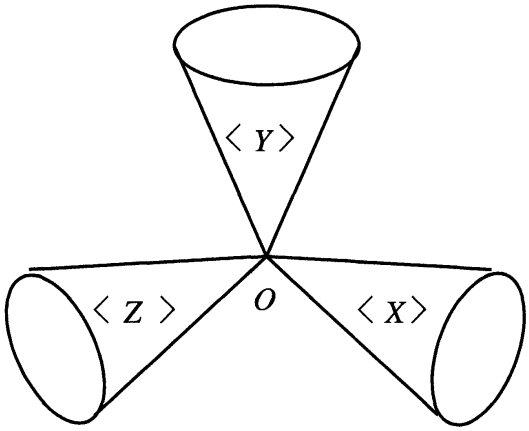

Figure 1.3. The singularity of the moduli space in Example 4

At the quantum level, all theories considered above are renormalizable (but not asymptotically free) and make sense perturbatively in 4 dimensions. The non-renormalization theorem says that the superpotential is not changed by the quantum effects. Although the Wess-Zumino model in 4 dimensions is not an interesting theory non-perturbatively and the standard way to prove the theorem in perturbation theory is to use Feynman diagrams, we are looking for a conceptual and non-perturbative proof which will shed light on other theories.

We need a digression on two notions of effective actions. First, the *1PI* (one-particle irreducible) *effective action* is obtained by adding sources to the (classical) Lagrangian, performing the entire path integral, and doing a Legendre transformation. This procedure fails if the integral diverges in the infrared. On the other hand, in defining the *Wilsonian effective action*, we separate a field, say Φ, into the high energy modes Φ_H and the low energy modes Φ_L in the path integral, i.e., $\int D\Phi = \int D\Phi_H \, D\Phi_L$, do the integration $\int D\Phi_H$ only, and obtain a Lagrangian for Φ_L. Here integrating over Φ_H is putting a cut-off. The infrared difficulty is absent in the Wilsonian effective action. When there are no interacting massless fields, the two notions are identical.

The effective action is subject to constraints of symmetries present in the classical theory. In addition, supersymmetry requires that the superpotential is holomorphic in the effective fields. This is the *principle of holomorphy.* Finally, the effective action is constrained at various limits where the theory is in the perturbative or classical regimes.

A key technique in our proof of the non-renormalization theorem is to think of all coupling constants, such as m, λ in Example 2, as expectation values of some background superfields. (We assume that the extended theory make sense and is still supersymmetric at the quantum level.) As a result of holomorphy, the superpotential is holomorphic not only in the effective fields, but also in the coupling constants.

We consider Example 2, in which the superpotential is $W = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3$ classically. To find the (Wilsonian) effective superpotential, we regard m, λ as background superfields and assign the following $U(1) \times U(1)_R$ charges to the fields:

(1.3)

	$U(1)$	$\times$	$U(1)_R$
Φ	1		1
m	-2		0
λ	-3		-1

$U(1)_R$ is the rotation on the fermionic coordinates in superspace. W is invariant under the first $U(1)$ factor and has $U(1)_R$ charge 2, as required by supersymmetry. By holomorphy, W_{eff} is holomorphic not only in Φ, but also in m, λ. (If supersymmetry was unbroken for $W(\Phi)$ but broken for $W(\Phi, m, \lambda)$, this argument would fail.) So

$$\begin{aligned} W_{\text{eff}}(\Phi, m, \lambda) &= f(m\Phi^2, \lambda\Phi^3) \quad \text{(from the first } U(1) \text{ symmetry)} \\ &= m\Phi^2 h\left(\frac{\lambda\Phi}{m}\right) \quad \text{(from } R\text{-charge of } W \text{ is 2)} \end{aligned} \tag{1.4}$$

for some yet unknown holomorphic functions f and h. We can now forget that m, λ are background fields. Next, we consider the limits. If $|\lambda| << 1$, perturbation theory should be valid. So

$$m\Phi^2 h\left(\frac{\lambda\Phi}{m}\right) = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3 + \sum_{n=2}^{\infty} a_n \frac{\lambda^n \Phi^{n+2}}{m^{n-1}}. \tag{1.5}$$

The n-th term must come from the diagram in Figure 1.4 with n vertices and $n+2$ external legs, with each internal line contributing a factor of m^{-1}. No loop could appear because each loop would increase the power of λ by 1 but not that of Φ. However for $n \geq 2$, the diagram is not 1PI and therefore cannot contribute. So $W_{\text{eff}} = W$, i.e., there is no quantum correction to the superpotential. This is the *non-renormalization theorem.* Notice however there is no claim about the quantum correction to the metric K.

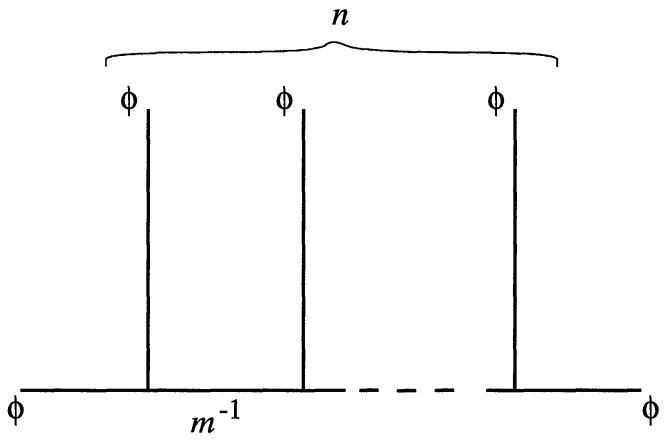

Figure 1.4. The diagram for the n-th term in formula (1.5)

Theories with more fields behave similarly. Consider a variant of Example 3, a theory with two chiral superfields L, H and a superpotential $W = \frac{1}{2}mH^2 + \lambda L^2 H$. Classically, the minimum is reached at $\frac{\partial W}{\partial H} = mH + \lambda L^2 = 0$, $\frac{\partial W}{\partial L} = 2\lambda LH = 0$, i.e.,

$L = H = 0$. So L is a light field. Fixing the value of L, the minimum of W is reached at $H = -\frac{\lambda L^2}{m}$, and we get $W_{\text{eff}} = -\frac{\lambda^2}{2m}L^4$. This procedure is called *integrating out* the heavy field H. The effective superpotential W_{eff} corresponds to the tree diagram in Figure 1.5. The non-renormalization theorem in this case says that W is renormalized by the tree diagram only and not by loop diagrams. In fact, we again regard m, λ as background fields and assign the following $U(1)_H \times U(1)_L \times U(1)_R$ charges to the fields:

(1.6)

	$U(1)_H$	$U(1)_L$	$U(1)_R$
H	1	0	1
L	0	1	1
m	-2	0	0
λ	-1	-2	-1

Invariance under $U(1)_H \times U(1)_L$ and holomorphy imply that

$$W_{\text{eff}}(L, m, \lambda) = h\left(\frac{\lambda^2 L^4}{m}\right) \tag{1.7}$$

for some holomorphic function h. Using the fact that W_{eff} has R-charge 2 or mass dimension 3, we conclude that $W_{\text{eff}} \sim \frac{\lambda^2}{m}L^4$.

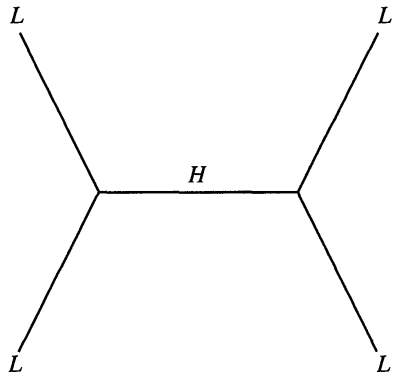

Figure 1.5. The tree diagram that produces the interaction L^4

Several remarks are in order.

1. Supersymmetry is crucial in the above arguments; without holomorphy, we cannot rule out terms such as $e^{-1/\bar{\lambda}\lambda}$.
2. Strictly speaking, we need to show the existence of a regularization scheme that respects all the symmetries. Here we simply assume its existence without discussing the details.
3. These arguments are valid both in perturbation theory and non-perturbatively.
4. If supersymmetry is unbroken classically, it is unbroken quantum mechanically because $W_{\text{eff}} = W$.

5. In usual quantum field theories, we have to include in the Lagrangian all terms compatible with renormalizability and the symmetries. For supersymmetric theories, this is not necessary because renormalization does not generate new terms in the superpotential.
6. Supersymmetry is the only known way to have widely separated scales in a theory which are not ruined by quantum corrections. Therefore the gauge hierarchy problem could be explained if supersymmetry holds in Nature.
7. The kinetic term is usually very complicated. However, in $N = 2$ theories (for example, $N = 2$ super Yang-Mills), it is possible to determine the kinetic term by an extra supersymmetry.
8. We will see examples of supersymmetric gauge theories in which the non-renormalization theorem can be avoided: new terms are generated non-perturbatively. Even so, the constraints of symmetries and holomorphy are strong enough to determine the superpotential completely.

§1.2. Pure supersymmetric gauge theory

Recall that an $N = 1$ gauge field is a connection (with some constraints on the curvature) on a principal G-bundle P over the superspace $\mathbb{R}^{1,3|4}$. In components, a constrained connection on the superspace consists of a gluon field A (a connection on the ordinary spacetime $\mathbb{R}^{1,3}$) and a gluino field λ (a section of the spinor bundle over $\mathbb{R}^{1,3}$ twisted by the adjoint bundle). Let W_α be the superfield strength and F, the usual field strength (curvature of A). The Lagrangian density is

$$\mathcal{L} = \frac{\tau}{4\pi i}\int d^2\theta \operatorname{tr} W_\alpha W^\alpha + \text{c.c.} = \frac{1}{g^2}(\operatorname{tr} F \wedge *F + \lambda i \not{D} \lambda) - \frac{i\theta}{8\pi^2}\operatorname{tr} F \wedge F, \tag{1.8}$$

where g is the coupling constant, θ is the θ-angle, and $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

Classically, there is a $U(1)_R$ symmetry. Its action on the gluino field λ is the chiral rotation $\lambda \mapsto e^{i\alpha\gamma_5}\lambda$ ($e^{i\alpha} \in U(1)_R$). Here γ_5 is ± 1 on the positive (negative) chiral spinors, respectively, and can be regarded as an operator acting on the space of sections of the twisted spinor bundle. The associated current j_5 has an anomaly

$$\partial_\mu j_5^\mu = -\frac{c_2(G)}{32\pi^2} * \operatorname{tr} F \wedge F, \tag{1.9}$$

which perturbatively comes from the Feynman diagram in Figure 1.6. Here $c_2(G)$ is the second Casimir of the gauge group G. Numerically, it is equal to $2h$, h being the dual Coxeter number of G. Consequently, $U(1)_R$ is not a symmetry at the quantum level. This is called the *axial anomaly*. In the path-integral language, a classical symmetry survives at the quantum level if it preserves the path-integral measure in addition to the classical action. Under $e^{i\alpha} \in U(1)_R$, the measure $D\lambda$ transforms as

$$D\lambda \mapsto D\lambda\,(\det e^{i\alpha\gamma_5})^{-1} = D\lambda\, e^{-i\alpha \operatorname{tr}\gamma_5} = D\lambda\, e^{-i\alpha \operatorname{ind} \not{D}_{\text{adj}}}. \tag{1.10}$$

Since the index of the Dirac operator twisted by the adjoint bundle is always an integral multiple of $c_2(G) = 2h$, the measure $D\lambda$ is invariant only under the $\mathbb{Z}_{2h}$ subgroup of $U(1)_R$. Thus the symmetry that remains at the quantum level is $\mathbb{Z}_{2h}$. For $G = SU(N_c)$ (N_c is called the *number of colors*), $h = N_c$ and $c_2 = 2N_c$. Hence the symmetry is $\mathbb{Z}_{2N_c}$.

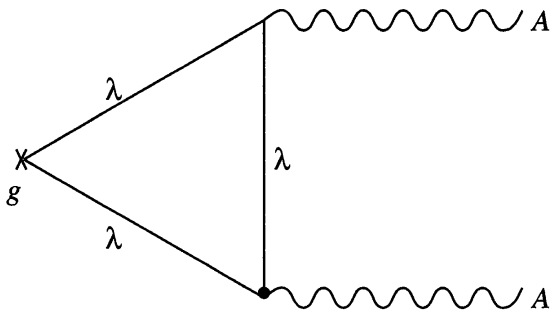

Figure 1.6. A Feynman diagram that causes the axial anomaly

The standard lore is that the theory has a mass gap, that it has confinement, i.e., $\langle W(C)\rangle \sim e^{-\text{Area}}$ for a Wilson loop in the fundamental representation, and that $\langle\lambda\lambda\rangle \neq 0$. The non-zero value of $\langle\lambda\lambda\rangle$ spontaneously breaks the $\mathbb{Z}_{2h}$ symmetry to $\mathbb{Z}_2$. The non-trivial element of $\mathbb{Z}_2$ can be identified as the 2π rotation in Spin$(1,3)$ because its action is equal to $(-1)^F$. Therefore this $\mathbb{Z}_2$ subgroup is unbroken. Consequently the theory has h vacua (N_c if $G = SU(N_c)$). In the k-th vacuum,

$$\langle\lambda\lambda\rangle_k \sim e^{\frac{2\pi i}{h}k}\Lambda^3, \tag{1.11}$$

where Λ is the mass scale generated by renormalization, which we now explain.

Due to the regularization procedure, a quantum field theory depends not only on the classical Lagrangian, but also on a cutoff μ. If we change μ but keep the theory fixed, the coupling constant g then depends on μ. The change of g is governed by

$$\mu\frac{dg}{d\mu} = \beta(g), \tag{1.12}$$

where the $\beta(g)$ is the *β-function*. Up to 1-loop,

$$\beta(g) = -\frac{b_0}{(4\pi)^2}g^3 \tag{1.13}$$

for some number b_0. If $b_0 \neq 0$, a renormalization group invariant mass scale Λ given by the equation

$$\left(\frac{\Lambda}{\mu}\right)^{b_0} = e^{-\frac{8\pi^2}{g(\mu)^2}} \tag{1.14}$$

is generated. This is called *dimensional transmutation*. Notice that Λ is intrinsic to the theory; it is not a cutoff. Because Λ has mass dimension 1, the quantum theory is not scale invariant even if the classical theory is.

In gauge theories, it is convenient to define Λ as a complex number satisfying

$$\left(\frac{\Lambda}{\mu}\right)^{b_0} = e^{-\frac{8\pi^2}{g(\mu)^2}+i\theta} = e^{2\pi i\tau(\mu)}. \tag{1.15}$$

The exponent in (1.15) is simply the value of the classical action for the one-instanton. Moreover, any quantity holomorphic in τ, which appears in the Lagrangian density (1.8), is also holomorphic in Λ. Of course Λ defined by (1.15) is not single valued. If a physical quantity expressed in term of Λ is not single valued, it is often because there are many vacua in the theory. For $N = 1$ pure gauge

theory, $b_0 = \frac{3}{2}c_2(G) = 3h$. Λ^3 is h-valued and there are h vacua in the theory. Fixing a choice of Λ^3, $\langle\lambda\lambda\rangle$ takes values according to (1.11) in the h vacua.

We need a digression on confinement. Recall that the Wilson loop $W_R(C) = \mathrm{tr}_R \mathrm{P} \exp \oint_C A$ is the trace in the representation R of G of the holonomy around a closed curve C. For the adjoint representation, we always have $\langle W_{\mathrm{adj}}(C)\rangle \sim e^{-\mathrm{Perimeter}(C)}$. This is because it is possible to screen the charges by the gluons. So the potential is independent of the distance. Likewise, for a theory with fundamental quarks that transform in a representation where the center of G acts faithfully, we have $\langle W_R(C)\rangle \sim e^{-\mathrm{Perimeter}(C)}$ for any representation R because the quarks can screen any charges. In both cases, the Wilson loop obeys the perimeter law and the charges are not confined. However, in a theory without fundamental quarks, the Wilson loop in fundamental representation may exhibit confinement, in which case we have the area law $\langle W_R(C)\rangle \sim e^{-\mathrm{Area}(D)}$, where D is the disk bounded by C.

§1.3. Supersymmetric QCD

Supersymmetric QCD is a theory of $N = 1$ matter coupled to $N = 1$ gauge theory. It is labeled by a compact Lie group G, a complex representation R of G (where the quarks take value), and a classical superpotential W, which is a G-invariant function on the representation space R. Recall that we have a principal G-bundle $P \to \mathbb{R}^{1,3|4}$ in $N = 1$ gauge theory. Let R be an Hermitian vector space with a unitary representation of G. An $N = 1$ matter (or quark) field Q is a chiral section of the associated bundle $P \times_G R \to \mathbb{R}^{1,3|4}$. The superpotential $W(Q)$ may contain a mass term for Q. It is at most cubic in Q if the theory is to be renormalizable.

Our goal is to find the physics of the theory and its dependence on the bare mass. Does the theory have confinement? Which phase is the theory in? If the theory flows to a non-trivial infrared fixed point (a conformal field theory in four dimensions), what are the anomalous dimensions of the operators?

§1.3.1. Classical theory

We start with the case $W = 0$. Since Q couples to the $N = 1$ gauge theory, the potential for the bosonic part of Q (also denoted by Q) is $V(Q) = |D(Q)|^2$, where D is the moment map of the Hamiltonian G-action on R. The minimum is reached at $Q \in D^{-1}(0)$ (which is non-empty if G is semisimple). Modulo the gauge transformations (the G-action), the moduli space of classical vacua $\mathcal{M}$ is the symplectic quotient $D^{-1}(0)/G$, which is an open dense set in the complex quotient $R/G^{\mathbb{C}}$.

In the lectures, we will consider theories with the gauge group $G = SU(N_c)$. Let C be the fundamental representation (or the defining representation) of G. C is an Hermitian vector space of dimension N_c with a complex volume form v_c preserved by $G = SU(C)$ (and $G^{\mathbb{C}} = SL(C)$ as well). The matter field consists of N_f copies of quarks in the fundamental representation C and N_f copies of anti-quarks in the conjugate representation $\bar{C} \cong C^*$. N_f is called the *number of flavors*. There are altogether $2N_fN_c$ chiral superfields. It is convenient to introduce two Hermitian vector spaces F, F' of complex dimension N_f and let $R = \mathrm{Hom}(F, C) \oplus \mathrm{Hom}(C, F')$. The values of the quark, anti-quark fields are $Q \in \mathrm{Hom}(F, C)$, $Q' \in \mathrm{Hom}(C, F')$, respectively, or $(Q, Q') \in R$.

The action of the group $G = SU(C)$ on R is Hamiltonian and the moment map $D\colon R \to \mathfrak{su}(C)^*$ is

$$\langle D(Q,Q'), X\rangle = \mathrm{tr}_F Q^\dagger X Q - \mathrm{tr}_{F'} Q' X Q'^\dagger \tag{1.16}$$

for any $X \in \mathfrak{su}(C)$. Under the isomorphism $\mathfrak{su}(C)^* \cong \mathfrak{su}(C)$, D takes values in $\mathfrak{su}(C)$ and is given by

$$D(Q,Q') = \text{traceless part of } i(QQ^\dagger - Q'^\dagger Q'). \tag{1.17}$$

So $(Q,Q') \in D^{-1}(0)$ if and only if $QQ^\dagger - Q'^\dagger Q' = rI_C$ for some $r \in \mathbb{R}$. For $N_f < N_c$, we have $r = 0$. There exist unitary bases of C, F, F' such that

$$Q \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{N_f} \end{pmatrix} \sim Q'^\dagger \tag{1.18}$$

for some $a_k \geq 0$ $(1 \leq k \leq N_c)$. Generically, these $a_k > 0$ and the gauge group $SU(N_c)$ is broken to $SU(N_c - N_f)$, the isotropy group of (Q,Q'). For $N_f \geq N_c$, r can be non-zero. There exist unitary bases such that

$$Q \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{N_c} \end{pmatrix}, \quad Q'^\dagger \sim \begin{pmatrix} a'_1 & & \\ & \ddots & \\ & & a'_{N_c} \end{pmatrix} \tag{1.19}$$

for some $a_k \geq 0$, $a'_k \geq 0$ $(1 \leq k \leq N_c)$ satisfying $a_k^2 - {a'_k}^2 = r$ (independent of k). Generically, the gauge group is completely broken.

We now construct gauge invariants from Q and Q'. First, we have the *meson* field $M = Q'Q$, which is clearly invariant under $SL(C)$. If $N_f \geq N_c$, then we have additional invariants $B = Q^* v_c \in \wedge^{N_c} F^*$ and $B' = Q'_* v_c^* \in \wedge^{N_c} F'$, called the *baryon* fields. They are subject to the constraints

$$B, B' \text{ are decomposable} \tag{1.20}$$

and

$$\wedge^{N_c} M = B \otimes B'. \tag{1.21}$$

If $N_f > N_c$, then further constraints

$$M \wedge B = 0, \qquad B' \wedge M = 0 \tag{1.22}$$

shall be satisfied. In particular, if $N_f = N_c$, then (1.20) is vacuous and (1.21) reduces to

$$\det M = B \otimes B'. \tag{1.23}$$

If $N_f = N_c + 1$, then B, B' can be identified as elements of $F \otimes \wedge^{N_f} F^*$, $\wedge^{N_f} F' \otimes F'^*$, respectively. In this case, (1.20) is again vacuous and (1.22) reduces to

$$MB = 0, \qquad B'M = 0. \tag{1.24}$$

The gauge invariant quantities M, B and B' subject to the constraints (1.20), (1.21) and (1.22) provide a good coordinate system on an open dense subset of $R/SL(C)$ and hence of $\mathcal{M}$. This is because, first, there exist Q, Q' such that $M =$

$Q'Q$, $B = Q^* v_c$ and $B' = Q'_* v_c^*$ if M, B, B' satisfy these constraints,[1] and secondly, generically, the gauge invariants M and B, B' (if $N_f \geq N_c$) determine Q, Q' up to a (complexified) gauge transformation.[2] As a consistency check, when $N_f < N_c$, the complex dimension of a generic $G^{\mathbb{C}}$-orbit in R is $\dim_{\mathbb{C}}(SL(N_c)/SL(N_c - N_f)) = 2N_f N_c - N_f^2$. Therefore the quotient $\mathcal{M}$ has complex dimension $\dim_{\mathbb{C}} R - (2N_f N_c - N_f^2) = N_f^2$ and is labeled by N_f^2 gauge invariants M.

§1.3.2. Quantum theory

With an equal number of quarks and anti-quarks in the representation $R = \mathrm{Hom}(F, C) \oplus \mathrm{Hom}(C, F')$, the quantum theory is free from gauge and gravitational anomalies. (We shall come to this point later.) But the quantum theory is different from the classical theory in several aspects.

First, classically the global symmetry is $U(F) \times U(F') \times U(1)_X$, where $U(1)_X$ is the $U(1)$ rotation on the fermionic coordinate θ in the superspace $\mathbb{R}^{1,3|4}$. (The notation $U(1)_R$ is reserved for an anomaly-free group.) The group $U(1)_X$ and the $U(1)$ subgroups of $U(F)$ and $U(F')$ are anomalous at the quantum level. Let $U(1)_A$, $U(1)_B$ be the anti-diagonally and diagonally embedded $U(1)$ subgroups in $U(F) \times U(F')$, respectively. The weights of these $U(1)$ groups on the fermionic fields and on the path-integral measure (in k-instanton background), calculated similarly as in (1.10), are given by

(1.25)

	$U(1)_A$	$U(1)_B$	$U(1)_X$
ψ_Q	-1	-1	-1
$\psi_{Q'}$	-1	1	-1
λ	0	0	1
$D\lambda D\psi_Q D\psi_{Q'}$	$2N_f k$	0	$-2(N_c - N_f)k$

So the group $U(1)_B$ is anomaly-free. Let $U(1)_R$ be a linear combination of $U(1)_A$ and $U(1)_X$ under which the fermionic measure in (1.25) is invariant. The generator is chosen so that it projects to that of $U(1)_X$. Thus a weight of $U(1)_R$ is fractional in general and is related to those of $U(1)_A$ and $U(1)_X$ by

$$R = X - \frac{N_f - N_c}{N_f} A. \tag{1.26}$$

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[1] This is obvious if $N_f < N_c$, when B, B' do not exist and if $N_f = N_c$, when the constraint is (1.23). If $N_f > N_c$, then B (if non-zero) determines a proper subspace $F_0 \subset F$ of dimension N_c. One can choose an isomorphism $Q_0 \colon F_0 \to C$ such that $\wedge^{N_c} Q_0 = B$, let $Q' = M Q_0^{-1}$ and extend Q_0 to Q on F by setting it to be 0 on $F_0^{\perp}$.

[2] When $N_f < N_c$, then generically $\ker Q = 0$, $\mathrm{im}\, Q$ is a proper subspace in C, $\mathrm{im}\, Q' = F'$ and $C = \mathrm{im}\, Q \oplus \ker Q'$. If $Q_1' Q_1 = Q'Q$ for generic variables, then there is an invertible map $U \colon \mathrm{im}\, Q \to \mathrm{im}\, Q_1$ such that $Q_1 = UQ$. Therefore $Q_1' U = Q'$ on $\mathrm{im}\, Q$. Extend U by adding an invertible map $\ker Q' \to \ker Q_1'$ such that $U \in SL(C)$. Then $Q_1' U = Q'$ on C. When $N_f \geq N_c$, then generically $\mathrm{im}\, Q = C$ and $\ker Q' = 0$. If $Q_1' Q_1 = Q'Q$, $Q_1^* v_c = Q^* v_c$, $Q_{1*}' v_c^* = Q'_* v_c^*$ for generic variables, then there exists $U \in GL(C)$ defined as follows: for any $x \in C$, choose $y \in F$ such that $Qy = x$, and set $Ux = Q_1 y$. U is well-defined because $\ker Q = \ker Q_1$ (from $Q_1^* v_c = Q^* v_c$). Since $Q_1^* v_c = \det U\, Q^* v_c$, we conclude that $U \in SL(C)$. It is easy to see $Q_1' = Q' U^{-1}$.

The global symmetry of the quantum theory is $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$. The matter fields Q, Q' (together with their composites M, B, B') transform according to

(1.27)

	$SU(F)$	$\times$ $SU(F')$	$\times$ $U(1)_B$	$\times$ $U(1)_R$
Q	F^*	$\mathbf{1}$	-1	$\frac{N_f - N_c}{N_f}$
Q'	$\mathbf{1}$	F'	1	$\frac{N_f - N_c}{N_f}$
M	F^*	F'	0	$2\frac{N_f - N_c}{N_f}$
B	$\wedge^{N_c} F^*$	$\mathbf{1}$	$-N_c$	$N_c - \frac{N_c^2}{N_f}$
B'	$\mathbf{1}$	$\wedge^{N_c} F'$	N_c	$N_c - \frac{N_c^2}{N_f}$

Here $\mathbf{1}$ is the trivial representation. The representations of the $U(1)$ groups are labeled by their weights.

Secondly, whereas the classical Lagrangian depends only on a dimensionless coupling constant g, we need to introduce a mass scale Λ to define the quantum theory; this is due to dimensional transmutation discussed earlier. The running of the coupling constant is governed by the β-function as in (1.12) and (1.13). For $N = 1$ supersymmetric QCD with gauge group G and matter representation R,

$$b_0 = \tfrac{3}{2} c_2(G) - \tfrac{1}{2} c_2(R), \tag{1.28}$$

where $c_2(R)$ is the value of the Casimir operator in R. In the theories we are considering, $G = SU(C)$ and $R = \mathrm{Hom}(F, C) \oplus \mathrm{Hom}(C, F')$. So $b_0 = 3N_c - N_f$. The theory is asymptotically free if $N_f < 3N_c$. Λ^{b_0} is a scalar under $U(1)_B$. However, since the $U(1)$ subgroups of $U(F)$ and $U(F')$ shift the θ-angle by units of $-N_f$ and N_f, respectively, it is more appropriate to regard Λ^{b_0} in (1.15) as an element in a suitable complex line, i.e.,

$$\Lambda^{b_0} \in \wedge^{N_f} F^* \otimes \wedge^{N_f} F'. \tag{1.29}$$

Thirdly, we have a moduli space of classical vacua $\mathcal{M}$ and want to know whether the vacuum degeneracy will be lifted after quantization. There are several possibilities. The space of vacua could disappear or be modified. If the moduli space remains the same, the metric on it could be changed. We will see examples of each case.

Consider the region in $\mathcal{M}$ where all $a_k >> \Lambda$. The gauge group $SU(N_c)$ is broken to $SU(N_c - N_f)$ if $N_f < N_c - 1$ and is completely broken otherwise. The light fields are the massless fields M and B, B' (if $N_f \geq N_c$) and the $N = 1$ pure $SU(N_c - N_f)$ gauge fields (if $N_f < N_c - 1$), which are not coupled to the former. (Gauge bosons in the directions complementary to $SU(N_c - N_f)$ have acquired masses of the order of a_k through spontaneous symmetry breaking.) Since the $N = 1$ $SU(N_c - N_f)$ gauge theory has $N_c - N_f$ vacua, we obtain an $N_c - N_f$ cover of the space $\mathcal{M}$. The $N = 1$ pure gauge theory has a mass gap. Going to a lower energy, we have only M, and B, B' (if $N_f \geq N_c$), which are combinations of Q, Q'. We wish to find out whether an effective superpotential W_{eff} could be generated for these light fields. The only holomorphic combination of Q, Q' that is invariant under both the gauge and the global $SU(F) \times SU(F') \times U(1)_B$ symmetry is $\det Q'Q$. Since W_{eff} has $U(1)_R$ charge 2, we get $W_{\text{eff}} \sim (\det Q'Q)^{-1/(N_c - N_f)}$.

Since W_{eff} has mass dimension 3 and the only parameter in our theory which has a mass dimension is Λ, we get

$$W_{\text{eff}}(Q, Q') = b_{N_c,N_f} \left(\frac{\Lambda^{3N_c - N_f}}{\det Q'Q} \right)^{\frac{1}{N_c - N_f}} \tag{1.30}$$

for some coefficient b_{N_c,N_f}. (Both $\Lambda^{3N_c-N_f} = \Lambda^{b_0}$ and $\det Q'Q$ are in $\wedge^{N_f} F^* \otimes \wedge^{N_f} F'$. Therefore their ratio is a number.) By holomorphy, (1.30) should also be valid in $\mathcal{M}$ away from infinity. We emphasize here that (1.30) is derived using constraints of global symmetry, holomorphy and the limit far out in the moduli space $\mathcal{M}$. Note that the form of an effective potential need not be constrained by the renormalizability condition.

For $N_f > N_c$, $\det Q'Q = 0$. For $N_f = N_c$, the exponent in (1.30) is singular. So $W_{\text{eff}} = 0$ in both cases. Thus the vacuum degeneracy is not lifted, and we have a moduli space of vacua in the quantum theory. However, the quantum moduli space may be modified away from infinity. For $N_f < N_c$, we have a superpotential (1.30) compatible with all symmetries; this is unlike the Wess-Zumino model. If $b_{N_c,N_f} \neq 0$, then there is no non-renormalization theorem which is valid non-perturbatively for supersymmetric theories with non-Abelian gauge fields.

In $N = 2$ gauge theory, the $N = 1$ supersymmetry already forbids a superpotential. In fact, the pure $N = 2$ gauge theory is an $N = 1$ supersymmetric QCD with matter field Q in the adjoint representation. Since the fermions λ and ψ_Q are both in the adjoint representation, and have opposite charges under the $U(1)_R$ rotation in $N = 1$ superspace, this $U(1)_R$ is anomaly-free. So Q has R-charge 0 and it is not possible to have a superpotential holomorphic in Q that has an R-charge 2.

References and cross references

More about the non-renormalization theorem and holomorphy in supersymmetric field theories can be found in

N. Seiberg, *The power of holomorphy - exact results in 4D SUSY field theories*, PASCOS 1994, pp. 357–369, **hep-th/9408013**;

K. Intriligator and N. Seiberg, *Lectures on supersymmetric theories and electric-magnetic duality*, in: String theory, gauge theory and quantum gravity, Proc. of the Trieste Spring School and Workshop, eds. R. Dijkgraaf et al., Nucl. Phys. B (Proc. Suppl.) 45B, C (1996) pp. 1-28.

and in references therein. Both papers are reviews of supersymmetric gauge theories. Another good expository account of the general ideas is

M. E. Peskin, *Duality in supersymmetric Yang-Mills theory*, in: Fields, strings and duality, Proceedings of the 1996 TASI (Boulder, CO), eds. C. Efthimiou and B. Greene, World Sci. Publishing, River Edge, NJ (1997), pp. 729-809.

In these volumes pure supersymmetric gauge theory is also covered in Lecture 16 of [II-**Dynamics of QFT**]. The classical lagrangian for the Wess-Zumino model is discussed in Chapter 5 of [I-**Supersolutions**] and for $N = 1$ supersymmetric gauge theory in Chapter 8 of [I-**Supersolutions**]. For background on the renormalization group, see [I-**Renorm Groups**].

LECTURE 2
Quantum Behavior of Super QCD: N_f Small

We review some aspects of $N = 1$ supersymmetric QCD with N_c number of colors and N_f number of flavors covered in the last lecture. The gauge group is $G = SU(C)$ where C is an Hermitian vector space of dimension N_c equipped with a complex volume form v_c. The matter (quark) fields Q and Q' take values in $\mathrm{Hom}(F, C)$ and $\mathrm{Hom}(C, F')$, respectively. Here F, F' are Hermitian vector spaces of dimension N_f. Under $SU(C)$, the quarks Q, Q' transform as C and $C^* \cong \bar{C}$, respectively. The (anomaly-free) global symmetry is $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$, where $U(1)_B$ is the diagonally embedded $U(1)$ subgroup in $U(F) \times U(F')$, and $U(1)_R$ is a linear combination of the anti-diagonal $U(1)$ subgroup in $U(F) \times U(F')$ and the $U(1)$ rotation in the superspace. Under this global symmetry group, Q, Q' transform as (1.27).

The moduli space $\mathcal{M}_c$ of classical vacua can be parametrized by gauge invariant quantities $M = Q'Q$ and (if $N_f \geq N_c$) $B = Q^* v_c$, $B' = Q'_* v_c^*$ constructed from Q, Q'. In fact, an open dense subset of $\mathcal{M}_c$ is equivalent to $\{M\}$ if $N_f < N_c$, $\{M, B, B' \mid \det M = B \otimes B'\}$ if $N_f = N_c$, and $\{M, B, B' \mid \wedge^{N_c} M = B \otimes B', MB = B'M = 0\}$ if $N_f = N_c + 1$. This classical moduli space may be modified in the quantum theory. We showed using constraints of global symmetry and holomorphy that for $N_f \geq N_c$, no superpotential can be dynamically generated, whereas for $N_f < N_c$, the only possible form is (1.30). The central question is whether the coefficient b_{N_c,N_f} is non-zero when $N_f < N_c$. We shall give an outline of the computation of b_{N_c,N_f}.

§2.1. $N_f = N_c - 1$

Consider the case $N_f = N_c - 1$. The gauge group $SU(C)$ is completely broken at a generic point of $\mathcal{M}_c$. Having no massless non-Abelian gluons, the theory is not strongly interacting at low energies. Moreover, there is no branch cut in the superpotential (1.30). This suggests that the theory does not have multiple vacua. In this case, the superpotential is generated by instantons for the following three reasons.

Suppose in a field theory with Euclidean action $S(\phi)$ we do perturbation around an instanton ϕ_0, which is a solution of the equation of motion $\frac{\delta S}{\delta \phi}\Big|_{\phi_0} = 0$. Then

$$S(\phi) = S(\phi_0) + \frac{1}{2}\frac{\delta^2 S}{\delta \phi^2}\Big|_{\phi_0} + \cdots . \tag{2.1}$$

The result of any perturbative path integral computation in the instanton background ϕ_0 is of the form

$$e^{-S(\phi_0)}\frac{\det_F}{\det_B}\cdots , \tag{2.2}$$

where $\det_B$, $\det_F$ are the determinants of the operators in $\frac{\delta^2 S}{\delta \phi^2}\Big|_{\phi_0}$ acting on the bosonic, fermionic fields, respectively. In gauge theories, the instanton contribution is non-zero and is proportional to

$$e^{-S(\phi_0)} = e^{-\frac{8\pi^2}{g(\mu)^2}+i\theta} = e^{2\pi i\tau(\mu)}. \tag{2.3}$$

So (2.2) is proportional to $\Lambda^{b_0} = \Lambda^{3N_c-N_f}$ by (1.15); the appropriate power of μ comes from regularizing the determinants and other factors. The superpotential (1.30) has the same power of Λ when $N_f = N_c - 1$. Thus for $N_f = N_c - 1$, the superpotential is instanton generated, and the coefficient b_{N_c,N_c-1} is non-zero. We can choose a renormalization subtraction scheme such that $b_{N_c,N_c-1} = 1$. In other words, the non-zero coefficient is absorbed in the mass scale Λ. Here we see how in a supersymmetric theory, the instanton action, β-function and anomaly are interrelated.

Secondly, in instanton perturbation theory outlined above, because of the zero modes of the Dirac operator, an instanton absorbs $2N_c$ of the λ and N_f each of the ψ_Q and $\psi_{Q'}$. This can be represented graphically by an instanton vertex (Figure 2.1(a)). On the other hand, the Lagrangian contains interactions $\lambda\phi_Q Q$ and $\lambda\psi_{Q'}Q'$ which are represented by the usual vertices (Figure 2.1(b)). In the presence of expectation values $\langle Q\rangle$, $\langle Q'\rangle$, the external Q, Q' lines can be absorbed. When $N_f = N_c - 1$, it is possible to form diagrams with one external line of ψ_Q and $\psi_{Q'}$. (An example is shown in Figure 2.1(c).) The existence of such diagrams is the hallmark of having a superpotential.

Finally, if $N_f = N_c - 1$, instanton computations are reliable and yield finite results. In theories with strong interaction, small instantons correspond to a small coupling constant g (by asymptotic freedom), whereas a large instanton size means strong coupling. When the instanton size becomes large, instantons overlap and have long range interactions. The integration over instanton size diverges. In our theory, at energies much higher than $\langle Q\rangle$, $\langle Q'\rangle$, the gauge group $SU(C)$ is unbroken, so g is small by asymptotic freedom. It turns out that our theory does not reach strong coupling in the infrared as well. This is because when the energy is smaller than $\langle Q\rangle$, $\langle Q'\rangle$, the gauge group is completely broken. So the instantons cannot grow and our theory is free from the usual troubles in the instanton calculations.

Higher order corrections beyond the one-instanton contribution would have a dependence on Λ not allowed by symmetry and therefore are not present. So when $N_f = N_c - 1$, the one-instanton contribution gives the exact answer for the effective superpotential. In this case, we see that there is no non-renormalization theorem because a non-zero superpotential is generated non-perturbatively.

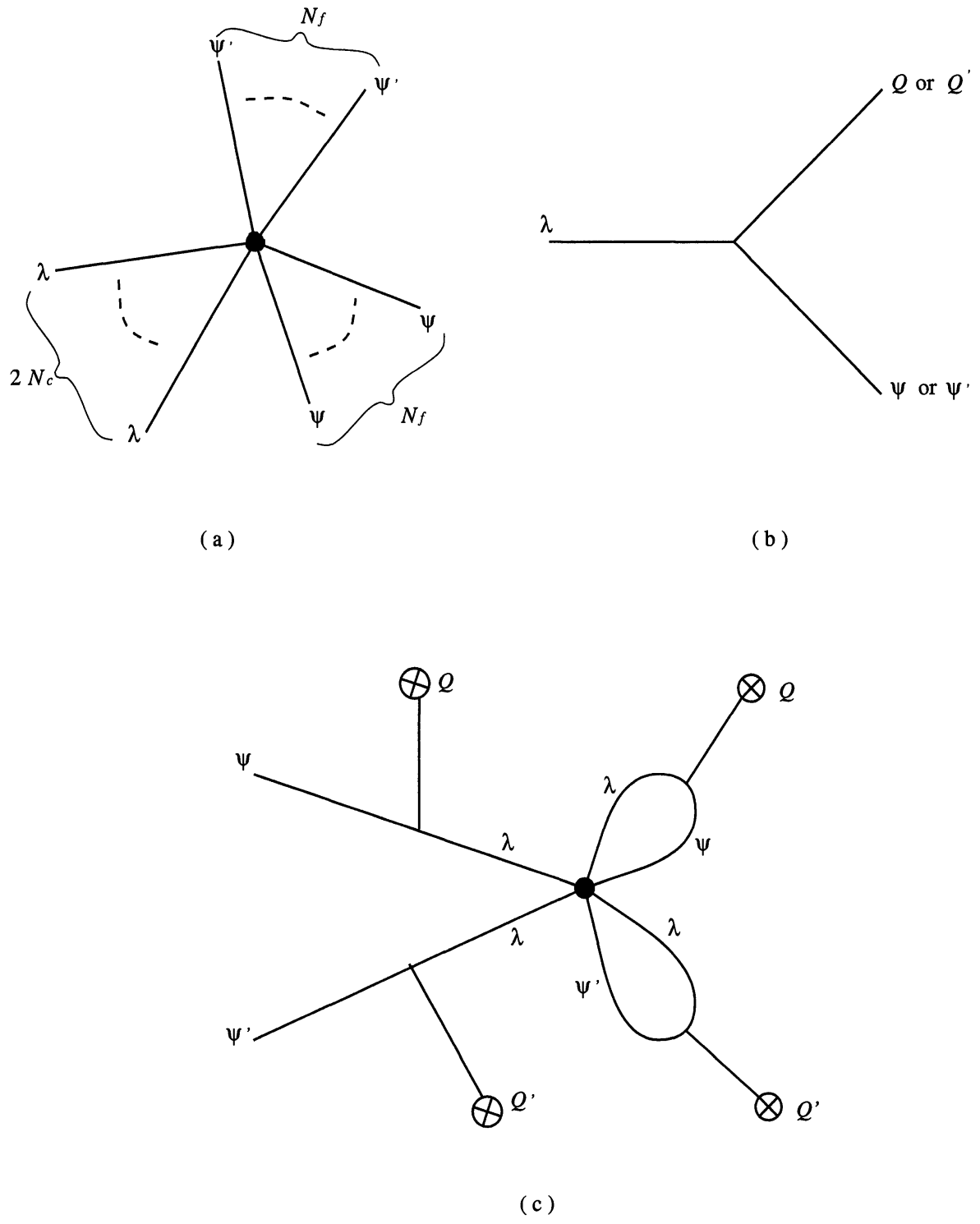

Figure 2.1. (a) The instanton vertex; (b) Yukawa interaction vertex; (c) A diagram with one external line of ψ_Q and $\psi_{Q'}$

§2.2. $N_f < N_c - 1$

We start with the theory with N_c-1 number of flavors, i.e., $\dim F = \dim F' = N_c-1$ and $Q \in \mathrm{Hom}(F, C)$, $Q' \in \mathrm{Hom}(C, F')$. We then add to it a mass term. At short distances (ultraviolet), the mass term is the tree level superpotential

$$W_{\text{tree}} = \operatorname{tr} mQ'Q, \tag{2.4}$$

where $m \in \mathrm{Hom}(F', F)$ is of rank $N_c - N_f - 1$, N_f being the number of light flavors. At long distances (infrared), we shall have an effective superpotential $W_{\mathrm{eff}}(M, \Lambda, m)$ which reduces to (1.30) for $N_c - 1$ flavors when $m = 0$. (Recall that we have chosen a subtraction scheme such that $b_{N_c, N_c-1} = 1$.) Since W_{eff} is holomorphic in m, is invariant under $SU(F) \times SU(F')$, and has R-charge 2 and mass dimension 3, it must be of the form

$$W_{\mathrm{eff}} = \frac{\Lambda^{2N_c+1}}{\det M} + \mathrm{tr}\, mM. \tag{2.5}$$

When the energy drops below m, the effective theory has N_f number of flavors. Take $F = \hat{F} \oplus \hat{F}^{\perp}$ and $F' = \hat{F}' \oplus \hat{F}'^{\perp}$ with $\dim \hat{F} = \dim \hat{F}' = N_f$ such that

$$m = \begin{pmatrix} 0 & 0 \\ 0 & m^{\perp} \end{pmatrix}, \tag{2.6}$$

where $m^{\perp} \in \mathrm{Hom}(\hat{F}^{\perp}, \hat{F}'^{\perp})$ is non-degenerate.

The mass scales $\hat{\Lambda}$ of the infrared theory and Λ of the ultraviolet theory are related by the renormalization group flow. At high energy (large μ), $b_0 = 2N_c + 1$. So the coupling constant $g(\mu)$ runs according to

$$e^{-\frac{8\pi^2}{g(\mu)^2} + i\theta} = \left(\frac{\Lambda}{\mu}\right)^{2N_c+1}. \tag{2.7}$$

At low energy (small μ), $b_0 = 3N_c - N_f$. So $g(\mu)$ flows as

$$e^{-\frac{8\pi^2}{g(\mu)^2} + i\theta} = \left(\frac{\hat{\Lambda}}{\mu}\right)^{3N_c-N_f}. \tag{2.8}$$

Matching $g(\mu)$ at a typical mass scale $\mu = (\det m^{\perp})^{1/(N_c-N_f)}$ (see Figure 2.2), we get in a specific subtraction scheme (called $\overline{DR}$ scheme)

$$\hat{\Lambda}^{3N_c-N_f} = \Lambda^{2N_c+1} \det m^{\perp}. \tag{2.9}$$

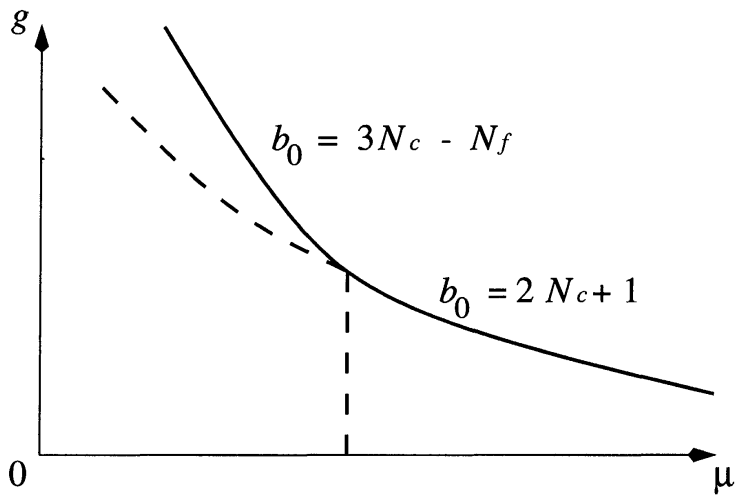

Figure 2.2. Matching of renormalization group flows at high and low energies

Since (2.5) is an effective potential, we can integrate out the heavy quarks classically. The light components of

$$M = \begin{pmatrix} \hat{M} & M_{12} \\ M_{21} & M^{\perp} \end{pmatrix} \tag{2.10}$$

are in $\hat{M} \in \mathrm{Hom}(\hat{F}, \hat{F}')$. We integrate out the heavy components M_{12}, M_{21} and $M^{\perp}$ of M. Fixing $\hat{M}$, the minimum of W_{eff} in (2.5) is reached at $M_{12} = M_{21} = 0$ and

$$M^{\perp} = \left(\frac{\Lambda^{2N_c+1} \det m^{\perp}}{\det \hat{M}} \right)^{\frac{1}{N_c - N_f}} (m^{\perp})^{-1}. \tag{2.11}$$

Substituting these values into (2.5) and using (2.9), we get

$$W_{\text{eff}}(\hat{M}) = (N_c - N_f) \left(\frac{\hat{\Lambda}^{3N_c - N_f}}{\det \hat{M}} \right)^{\frac{1}{N_c - N_f}}. \tag{2.12}$$

This is of the same form as (1.30). Moreover, we conclude that in our renormalization scheme, $b_{N_c,N_f} = N_c - N_f \neq 0$ when $N_f < N_c$.

We also want to find out the physical mechanism that generates the superpotential (1.30) when $N_f < N_c - 1$ without embedding the theory into a larger one. At a generic point in $\mathcal{M}$, expectation values $\langle Q \rangle$, $\langle Q' \rangle$ break the $SU(N_c)$ gauge symmetry to $SU(N_c - N_f)$. The gauge bosons not in the unbroken subgroup acquire a mass classically. At an energy below this mass scale, we have an effective theory whose Lagrangian density is

$$\mathcal{L}_{\text{eff}} = \int d^4\theta \, K(M, M^{\dagger}) + \left(\int d^2\theta \, \frac{\tau(M)}{4\pi i} \operatorname{tr} W_{\alpha} W^{\alpha} + \text{c.c.} \right), \tag{2.13}$$

where K is a Kähler potential on $\mathcal{M}$ which determines the kinetic term of M, W_{α} is the superfield strength of the $SU(N_c - N_f)$ gauge theory, and $\tau(M) = \frac{\theta}{2\pi} + \frac{4\pi i}{g(M)^2}$ is determined by the coupling constant $g(M)$ at scale M. Using the 1-loop β-function, $\tau(M) \sim \log \det M$. The pure $SU(N_c - N_f)$ gauge theory has a mass gap and depends on a scale $\Lambda_{SU(N_c - N_f)}$. Below this mass scale, we have M only. Again, $\Lambda_{SU(N_c - N_f)}$ is related to Λ by matching two renormalization group flows at $\mu = (\det M)^{1/2N_f}$. In fact,

$$\Lambda^{3N_c - N_f} = \Lambda_{SU(N_c - N_f)}^{3(N_c - N_f)} \det M, \tag{2.14}$$

where the same subtraction scheme is used. We claim that the effective superpotential of M is generated by the pure gauge theory. Indeed, the leading order term of $W_{\alpha} W^{\alpha}$ is $\lambda\lambda$. According to (1.11),

$$\langle \lambda\lambda \rangle \sim \Lambda_{SU(N_c - N_f)}^3. \tag{2.15}$$

Using (2.14), we get

$$\langle \lambda\lambda \rangle \sim \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}}. \tag{2.16}$$

Therefore the bosonic potentials from (1.30) and (2.13) agree:

$$\frac{\partial W_{\text{eff}}(M)}{\partial M} \sim \frac{\partial \tau(M)}{\partial M} \langle \lambda\lambda \rangle. \tag{2.17}$$

That $b_{N_c,N_f} \neq 0$ also confirms that $\langle \lambda\lambda \rangle \neq 0$ in the pure gauge theory.

The bosonic potential of Q, Q' is qualitatively illustrated in Figure 2.3(a). We see that this potential pushes the vacuum to infinity. Therefore theories with $N_f < N_c$ do not have a vacuum. We can add a bare mass to the theory and recover the original theory by taking the limit as the mass goes to zero. If $N_f < N_c$, the effective superpotential is

$$W_{\text{eff}} = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}} + \operatorname{tr} mM. \tag{2.18}$$

The bosonic potential now looks like Figure 2.3(b). The minimum is reached at

$$\langle M \rangle = (\Lambda^{3N_c - N_f} \det m)^{\frac{1}{N_c}} m^{-1}. \tag{2.19}$$

The N_c solutions (in the choices of the N_c-th root) correspond to the N_c vacua of the pure gauge theory. Indeed as $m \to 0$, all the eigenvalues of M go to infinity.

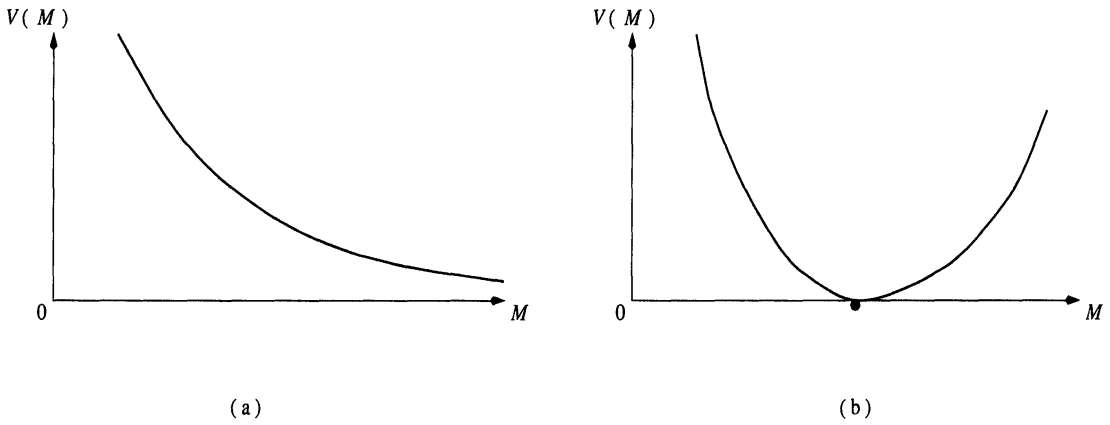

Figure 2.3. The bosonic effective potential in supersymmetric QCD: (a) $m = 0$; (b) $m \neq 0$

The result (2.19) is consistent with all the symmetries. We claim that it is also valid in the range $N_f \geq N_c$, where no superpotential is generated. To show this, we imagine adding a huge mass to some of the quarks, integrating them out, and reducing to a problem of $N_f < N_c$. Since $\langle M \rangle$ is holomorphic in m, (2.19) for $N_f \geq N_c$ follows from the case $N_f < N_c$. For $N_f \geq N_c$, the limit of $\langle M \rangle$ as $m \to 0$ depends on the way the limit is taken. This is not a surprise because there is a moduli space of vacua when $m = 0$. With different ways of taking the limit, the massive theory lands on different points on the moduli space.

§2.3. $N_f = N_c$

As before, we first add a mass term $\operatorname{tr} mM$, then take the limit $m \to 0$. If $m \in \operatorname{Hom}(F', F)$ is invertible then $\langle M \rangle$ is given by (2.19). The invariance under the $U(1)_B$ global symmetry implies that $\langle B \rangle = \langle B' \rangle = 0$. These expectation values lie on the surface

$$\det M - B \otimes B' = \Lambda^{2N_c}. \tag{2.20}$$

Because (2.20) is independent of m, the quantum ground states are not on the classical moduli space $\mathcal{M}_c = \{M, B, B' \mid \det M - B \otimes B' = 0\}$ even in the limit $m \to 0$. We call $\mathcal{M}_q = \{M, B, B' \mid \det M - B \otimes B' = \Lambda^{2N_c}\}$ the *quantum moduli space.*

The constraint (2.20) is consistent with all the symmetries. We postulate that the low energy effective theory consists of fields M, B, B' constrained on $\mathcal{M}_{\rm q}$. Unlike the classical moduli space $\mathcal{M}_{\rm c}$, the quantum moduli space $\mathcal{M}_{\rm q}$ has no singularities near the origin because of the correction Λ^{2N_c}, as shown in Figure 2.4. This suggests that there are no extra massless particles (other than the modes in the flat directions) – a point we shall check later by 't Hooft's anomaly matching condition. If so, the original gluons are not light, even near the origin, the region of strong coupling. Thus the gluons are confined. Far out along the flat directions (when M is big), the quantum correction is small. There the classical moduli space $\mathcal{M}_{\rm c}$ is a good approximation to $\mathcal{M}_{\rm q}$, as expected. Again there are no massless gluons. But in that region, it is more natural to interpret it as a Higgs phenomenon, in which the gauge group is completely broken by $\langle M \rangle$ and all the gauge bosons acquire a mass. Therefore there is a continuous interpolation between the confinement and the Higgs descriptions.

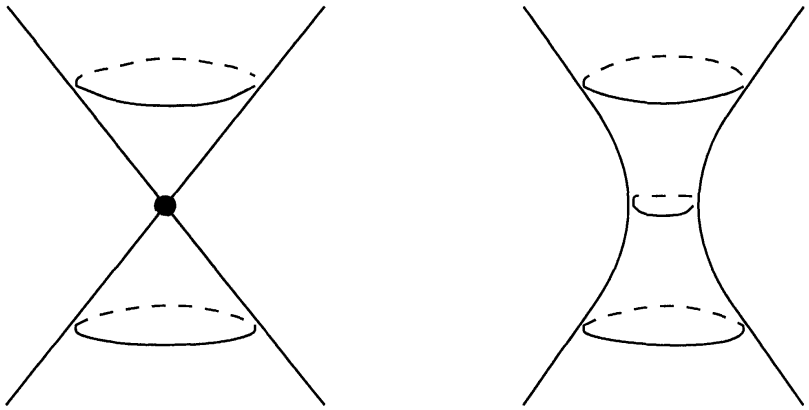

Figure 2.4. (a) The classical moduli space has a singularity at the origin; (b) The quantum moduli space is smooth.

We also remark on chiral symmetry breaking. The global symmetry $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$ is unbroken at the origin of the classical moduli space $\mathcal{M}_{\rm c}$. But at any point on the quantum moduli space $\mathcal{M}_{\rm q}$, the symmetry is always broken to a proper subgroup H, the isotropy subgroup, which varies according to the point. The massless particles live in the tangent space of $\mathcal{M}_{\rm q}$ and form a representation of H. For example, at $M = 0$, $B = B' = \sqrt{-\Lambda^{2N_c}}$, we have $H = SU(F) \times SU(F') \times U(1)_R$. So we have confinement with chiral symmetry breaking.

Since the values of M B, B' in the low energy effective theory are constrained on $\mathcal{M}_{\rm q}$, the theory can be described by

$$W_{\rm eff} = \frac{1}{\Lambda^{2N_c-1}} A\,(B \otimes B' - \det M + \Lambda^{2N_c}), \tag{2.21}$$

where A is a superfield (of mass dimension 2) that imposes the constraint (2.20) Now M, B, B' can take arbitrary values. We check that our theory reduces to supersymmetric QCD with $N_f < N_c$ after integrating out the quarks to which a

bare mass is added. The effective potential with a mass term is

$$W_{\text{eff}} = \frac{1}{\Lambda^{2N_c-1}} A\,(B \otimes B' - \det M + \Lambda^{2N_c}) + \operatorname{tr} mM, \tag{2.22}$$

where $m \in \operatorname{Hom}(F', F)$ is of rank $N_c - N_f$. (N_f is the number of light flavors.) Write $F = \hat{F} \oplus \hat{F}^{\perp}$ and $F' = \hat{F}' \oplus \hat{F}'^{\perp}$ with $\dim \hat{F} = \dim \hat{F}' = N_f$ such that m has the form (2.6). Again the light components in M of the form (2.10) are in $\hat{M} \in \operatorname{Hom}(\hat{F}, \hat{F}')$. Integrating out the rest of M and B, B' in (2.22), we get

$$\hat{W}_{\text{eff}} = (N_c - N_f) \left(\frac{\hat{\Lambda}^{3N_c-N_f}}{\det \hat{M}} \right)^{\frac{1}{N_c-N_f}}. \tag{2.23}$$

This is precisely the superpotential (1.30) with the correct coefficient. Here $\hat{\Lambda}^{3N_c-N_f} = \Lambda^{2N_c} \det m^{\perp}$, which is again consistent with the renormalization group flow.

§2.4. $N_f = N_c + 1$

Again, we perturb the theory by adding a mass term $\operatorname{tr} mM$. Then we have $\langle B \rangle = \langle B' \rangle = 0$ and using (2.19), $\wedge^{N_c}\langle M \rangle = \Lambda^{2N_c-2} \det m\, (\wedge^{N_c} m^{-1}) = \Lambda^{2N_c-2} m$. In the limit $m \to 0$, the expectation values satisfy the constraints (1.21) and (1.24) of the classical configurations. However, for $m \neq 0$, they are not on the classical moduli space $\mathcal{M}_c$. In fact, all the classical values of M with $B = B' = 0$ can be obtained by taking an appropriate limit $m \to 0$. Unlike the case $N_f = N_c$, there is not a constraint which $\langle B \rangle$, $\langle B' \rangle$ and $\langle M \rangle$ satisfy for all values of m. We propose that the long distance effective theory of the original theory without a bare mass is a Wess-Zumino model with the unconstrained fields M, B, B' and a superpotential

$$W_{\text{eff}}(M, B, B') = \frac{1}{\Lambda^{2N_c-1}} (B'MB - \det M). \tag{2.24}$$

The stationary points of W_{eff} are at $\frac{\partial W}{\partial M} = B \otimes B' - \wedge^{N_c} M = 0$, $\frac{\partial W}{\partial B} = B'M = 0$, $\frac{\partial W}{\partial B'} = B'M = 0$; these are precisely the constraints that every classical configuration satisfies. However, the moduli space is interpreted differently: it is embedded into a larger space. As a result, more particles become light in the effective theory. For example, at $M = 0$, B, B' are the new massless fields.

Similar to the theory with $N_f = N_c$, we see confinement at $M = 0$ since there are no massless gluons. There is again a continuum between the confinement description near the origin and the Higgs description in the semiclassical region. However, at $M = B = B' = 0$, the full global symmetry is unbroken. So we can have confinement without chiral symmetry breaking.

We check that by adding a mass term to the superpotential (2.24), i.e,

$$W_{\text{eff}}(M, B, B') = \frac{1}{\Lambda^{2N_c-1}} (B'MB - \det M) + \operatorname{tr} mM, \tag{2.25}$$

the theory with $N_f = N_c$ can be recovered. Here m is of rank 1. Again we write m in the form (2.6). The light components in

$$M = \begin{pmatrix} \hat{M} & * \\ * & A \end{pmatrix}, \quad B = \begin{pmatrix} * \\ \hat{B} \end{pmatrix}, \quad B' = \begin{pmatrix} * , & \hat{B}' \end{pmatrix} \tag{2.26}$$

are $\hat{M}$, A, $\hat{B}$, $\hat{B}'$. Integrating out the rest, we get

$$W_{\text{eff}}(\hat{M}, A, \hat{B}, \hat{B}') = \frac{1}{\Lambda^{2N_c-1}} A\,(\hat{B} \otimes \hat{B}' - \det \hat{M} + \hat{\Lambda}^{2N_c}), \tag{2.27}$$

where $\hat{\Lambda}^{2N_c} = m^{\perp}\Lambda^{2N_c-1}$. This agrees with the superpotential for $N_f = N_c$ after a rescaling of the field A.

We shall further check the postulate on the field content of the low energy theory by 't Hooft's anomaly matching condition.

§2.5. 't Hooft anomaly matching condition

A low energy theory may have massless fermions very different from the fundamental theory. The anomaly matching condition of 't Hooft is a highly non-trivial test on the compatibility of the two theories. We first introduce the condition, then apply it to check some of the claims we made about supersymmetric QCD with $N_f = N_c$ and $N_f = N_c + 1$.

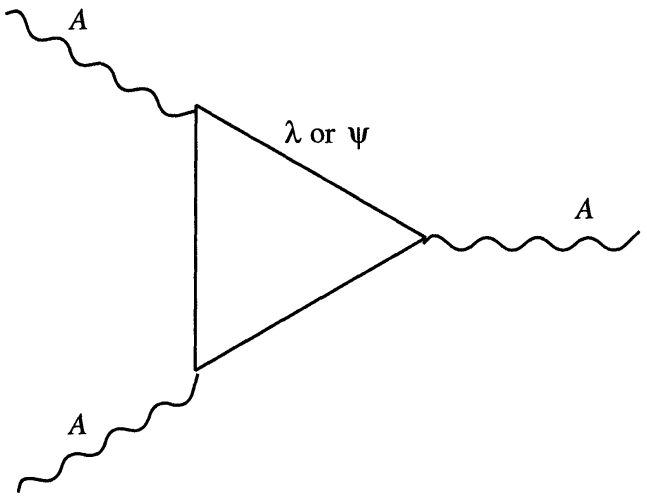

Figure 2.5. The triangular Feynman diagram that causes the gauge anomaly

Any gauge theory with Weyl fermions may suffer from anomalies. If so, the quantum theory is ill-defined. Suppose the gauge group is G and the fermions are in the representation R of G. Perturbatively, the gauge anomaly comes from the triangular Feynman diagram in Figure 2.5. This is proportional to a totally symmetric cubic form d_R on the Lie algebra $\mathfrak{g}$ of G given by

$$d_R(x, y, z) = \mathrm{tr}_R x\{y, z\} \qquad (x, y, z \in \mathfrak{g}). \tag{2.28}$$

$d_R = 0$ if R is a real or quaternionic (pseudo-real) representation. For simple Lie groups, $d_R \neq 0$ only if $\mathfrak{g} = \mathfrak{su}(N_c)$ $(N_c \geq 3)$. In the path-integral language, the fermionic path integral is the determinant of the Dirac operator $\not{D}_R^+$ (twisted by R) on the spacetime 4-manifold X. This determinant is a section of the determinant line bundle Det $\not{D}_R^+$ over $\mathcal{A}/\mathcal{G}$, the space $\mathcal{A}$ of connections modulo the group $\mathcal{G}$ of gauge transformations. By the local family index theorem, the curvature $\mathcal{F}_R$ of Det $\not{D}_R^+$ evaluated on two tangent vectors B, B' at $A \in \mathcal{A}$ is given by

$$\mathcal{F}_R(B, B') = \frac{-i}{24\pi^3} \int_X d_R(F, F, \Delta_A^{-1} * [B, *B']) + d_R(F, B, B'), \tag{2.29}$$

where F is the curvature of $A \in \mathcal{A}$ and Δ_A, is the Laplacian acting on $T_A\mathcal{A}$. The quantum theory makes sense only when the bundle Det $\not{D}_R^+$ is canonically trivial. This requires $\mathcal{F}_R = 0$, hence $d_R = 0$, in agreement with the perturbative argument. Strictly speaking, (2.29) is true only when $\hat{A}(X) = 1$. In general, $\mathcal{F}_R$ contains terms

proportional to the linear form tr_R on $\mathfrak{g}$ (which is non-zero only if G has a $U(1)$ factor). This is the mixed gauge and gravitational anomaly. So the quantum theory is free from gauge and gravitational anomalies if and only if the representation R is arranged so that $d_R = \mathrm{tr}_R = 0$. For example, if $G = SU(C)$, it is easy to check that $d_C = -d_{\bar{C}}$ and $d_{\mathrm{adj}} = 0$. So $N = 1$ supersymmetric QCD with an equal number of quarks and anti-quarks is free from anomalies.

Recall that the global symmetry of our theory is $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$. At a particular point in the moduli space, this symmetry may be broken to a subgroup H. The massless fermions will form a representation R of H. H is not gauged; if we gauge it (adding gauge fields interacting with the fermions), there will be anomalies proportional to d_R and tr_R. But imagine that in the gauged theory we add some massless fermions in another representation R_0 that couple only to the gauge fields of H. The representation R_0 is chosen so that the combined theory is anomaly-free, i.e., $d_R + d_{R_0} = 0$, $\mathrm{tr}_R + \mathrm{tr}_{R_0} = 0$. Suppose our original theory with representation R has a low energy description in which the fermions are in representation R'. Then the low energy description of the combined theory is anomaly-free because it comes from an anomaly-free ultraviolet theory. Since the additional massless fermions stay the same, we have $d_{R'} + d_{R_0} = 0$, $\mathrm{tr}_{R'} + \mathrm{tr}_{R_0} = 0$. Therefore

$$d_R = d_{R'}, \quad \mathrm{tr}_R = \mathrm{tr}_{R'}. \tag{2.30}$$

This is the 't Hooft anomaly matching condition. It is a stringent check on whether two theories with fermions in representations R and R' describe the same low energy physics.

Now we check the claimed solutions of the low energy theories with $N_f = N_c$ and $N_f = N_c + 1$ against 't Hooft's criterion. Using (1.27), the fermions in the ultraviolet theory transform under the global symmetry according to

(2.31)

	$SU(F)$	$\times$ $SU(F')$	$\times$ $U(1)_B$	$\times$ $U(1)_R$
ψ_Q	F^*	$\mathbf{1}$	-1	$-\frac{N_c}{N_f}$
$\psi_{Q'}$	$\mathbf{1}$	F'	1	$-\frac{N_c}{N_f}$
λ	$\mathbf{1}$	$\mathbf{1}$	0	1

In the infrared theory, the fermionic components ψ_M, ψ_B, $\psi_{B'}$ of the composite fields M, B, B' transform as

(2.32)

	$SU(F)$	$\times$ $SU(F')$	$\times$ $U(1)_B$	$\times$ $U(1)_R$
ψ_M	F^*	F'	0	$1 - 2\frac{N_c}{N_f}$
ψ_B	$\wedge^{N_c} F^*$	$\mathbf{1}$	$-N_c$	$N_c - 1 - \frac{N_c^2}{N_f}$
$\psi_{B'}$	$\mathbf{1}$	$\wedge^{N_c} F'$	N_c	$N_c - 1 - \frac{N_c^2}{N_f}$

We can calculate the cubic and the trace forms on the Lie algebra $\mathfrak{su}(F) \oplus \mathfrak{su}(F') \oplus \mathfrak{u}(1)_B \oplus \mathfrak{u}(1)_R$ in both the ultraviolet and the infrared theories. The non-zero combinations are given by

(2.33)

	ultraviolet	infrared
$\mathfrak{su}(F)^3$	$N_c\,(-d_F)$	$N_f\,(-d_F) + (-d_{\wedge^{N_c}(F)})$
$\mathfrak{su}(F')^3$	$N_c\,d_{F'}$	$N_f\,d_{F'} + d_{\wedge^{N_c}F'}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_B$	$N_c\,c_2(F)\cdot(-1)$	$c_2(\wedge^{N_c}F)\cdot(-N_c)$
$\mathfrak{su}(F')^2\mathfrak{u}(1)_B$	$N_c\,c_2(F')\cdot 1$	$c_2(\wedge^{N_c}F')\cdot N_c$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$N_c\,c_2(F)\cdot(-\frac{N_c}{N_f})$	$N_f\,c_2(F)\cdot(1-\frac{2N_c}{N_f}) + c_2(\wedge^{N_c}F)\cdot(N_c-1-\frac{N_c^2}{N_f})$
$\mathfrak{su}(F')^2\mathfrak{u}(1)_R$	$N_c\,c_2(F')\cdot(-\frac{N_c}{N_f})$	$N_f\,c_2(F')\cdot(1-\frac{2N_c}{N_f}) + c_2(\wedge^{N_c}F')\cdot(N_c-1-\frac{N_c^2}{N_f})$
$\mathfrak{u}(1)_B^2\mathfrak{u}(1)_R$	$N_cN_f[(-1)^2+1^2]\cdot(-\frac{N_c}{N_f})$	$2\binom{N_f}{N_c}[(-N_c)^2+N_c^2]\cdot(N_c-1-\frac{N_c^2}{N_f})$
$\mathfrak{u}(1)_R^3$	$2N_cN_f(-\frac{N_c}{N_f})^3 + (N_c^2-1)1^3$	$N_f^2(1-\frac{2N_c}{N_f})^3 + 2\binom{N_f}{N_c}(N_c-1-\frac{N_c^2}{N_f})^3$
$\mathfrak{u}(1)_R$	$2N_cN_f(-\frac{N_c}{N_f}) + (N_c^2-1)1$	$N_f^2(1-\frac{2N_c}{N_f}) + 2\binom{N_f}{N_c}(N_c-1-\frac{N_c^2}{N_f})$

When $N_f = N_c$, the quantum moduli space $\mathcal{M}_{\rm q}$ is defined by the constraint (2.20). At the point $M = 0$, $B = B' = \sqrt{-\Lambda^{2N_c}}$, the global symmetry group is broken to $H = SU(F) \times SU(F') \times U(1)_R$. Using the condition $N_f = N_c$, it is straightforward to check that for $\mathfrak{su}(F)^3$, $\mathfrak{su}(F')^3$, $\mathfrak{su}(F)^2\mathfrak{u}(1)_R$ and $\mathfrak{su}(F')^2\mathfrak{u}(1)_R$, the entries in the left and right columns match. However for $\mathfrak{u}(1)_R^3$ and $\mathfrak{u}(1)_R$, the entry in the left column is $-(N_c^2+1)$ whereas as that in the right column is $-(N_c^2+2)$. The discrepancy is because M, B, B' are constrained on $\mathcal{M}_{\rm q}$. Alternatively, we can introduce a Lagrange multiplier A as in (2.22). A is invariant under $SU(F) \times SU(F') \times U(1)_B$. Since the R-charges of M, B, B' are 0 and that of $W_{\rm eff}$ should be 2, the R-charge of A is 2. So ψ_A has R-charge 1 and contributes 1^3 and 1 to $\mathfrak{u}(1)_R^3$ and $\mathfrak{u}(1)_R$, respectively, in the right column. Thus we have the matching $-(N_c^2+1) = -(N_c^2+2)+1$. This computation confirms that for $N_f = N_c$, the fields M, B, B' are constrained on $\mathcal{M}_{\rm q}$ and that there are no additional light fields.

For $N_f = N_c + 1$, consider the point $M = B = B' = 0$ on the moduli space, where the full global symmetry group is unbroken. Using $N_f = N_c + 1$, it is straightforward to check that in (2.33), all the corresponding entries in the two columns match. This shows that in the low energy theory, M, B, B' are indeed unconstrained and that they are the only light fields.

References and cross references

For more about supersymmetric QCD with gauge group $SU(N_c)$ and $N_f \le N_c - 1$, see

I. Affleck, M. Dine and N. Seiberg, *Dynamical supersymmetric breaking in supersymmetric QCD*, Nucl. Phys. B241 (1984) 493-534.

The theories with $N_f = N_c$ and $N_f = N_c + 1$ were studied in

N. Seiberg, *Exact results on the space of vacua of four-dimensional SUSY gauge theories*, Phys. Rev. D49 (1994) 6857-6863.

The original literature on 't Hooft's anomaly matching condition is

G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, in: Recent developments in gauge theories (Cargèse, 1979), eds. G. 't Hooft et al., Plenum Press, New York (1980).

LECTURE 3
Quantum Behavior of Super QCD: N_f Large

In the previous lectures, we studied $N = 1$ supersymmetric QCD with $N_f \leq N_c + 1$. More concretely, we have an $N = 1$ gauge theory with gauge group $G = SU(C)$ coupled to $N = 1$ matter which consists of the quarks $Q \in \mathrm{Hom}(F, C)$ and the anti-quarks $Q' \in \mathrm{Hom}(C, F')$. $N_c = \dim C$ is called the number of colors and $N_f = \dim F = \dim F'$, the number of flavors. The anomaly-free global symmetry of the theory is $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$, under which Q, Q' transform as (1.27). The moduli space $\mathcal{M}_c$ of classical ground states is parametrized by M, B, B', which are also the fields in the low energy effective theory. Their transformations under the global symmetry are also given by (1.27).

If $N_f < N_c$, a non-zero superpotential (1.30) is dynamically generated. Consequently, the quantum theory has no vacuum. If $N_f = N_c$, the classical moduli space $\mathcal{M}_c$ is modified in the quantum theory. The light fields M, B, B' satisfy the constraint (2.20). If $N_f = N_c + 1$, the quantum moduli space is the same as $\mathcal{M}_c$. But all the components of M, B, B' become physical in the low energy theory. We checked these statements on the low energy field theory using 't Hooft's anomaly matching condition.

In this lecture, we will study the theory for all ranges of N_f. If $N_f \geq 3N_c$, then the β-function $\beta(g) > 0$, so the theory is infrared free. (If $N_f = 3N_c$, the 1-loop β-function vanishes but the 2-loop contribution is positive. See below.) At long distances, we see elementary degrees of freedom: the massless quarks and gluons. We will show that when $\frac{3N_c}{2} < N_f < 3N_c$, the theory has a non-trivial infrared fixed point, which is a superconformal field theory in four dimensions. When $N_c + 2 < N_f < \frac{3N_c}{2}$, the theory is strongly coupled in the infrared. But there is a dual description, the magnetic theory, which is infrared free.

§3.1. $\frac{3N_c}{2} < N_f < 3N_c$: non-trivial infrared fixed points

Up to 2 loops, the β-function is

$$\beta(g) = -\frac{g^3}{16\pi^2}(3N_c - N_f) + \frac{g^5}{128\pi^4}\left(2N_cN_f - 3N_c^2 - \frac{N_f}{N_c}\right) + o(g^7). \tag{3.1}$$

Note that the 2-loop term is positive when $N_f = 3N_c$. Consider the limit $N_c, N_f \to \infty$ while $\frac{N_f}{N_c} = 3 - \epsilon$ is fixed for some $\epsilon > 0$. The β-function is shown in Figure 3.1. If ϵ is sufficiently small, the higher order terms in $\beta(g)$ are negligible. So there is a

solution g_* of $\beta(g) = 0$ with

$$g_*^2 N_c = \frac{8\pi^2}{3}\epsilon + o(\epsilon^2). \tag{3.2}$$

The coupling constant runs and stops at this point in the infrared.

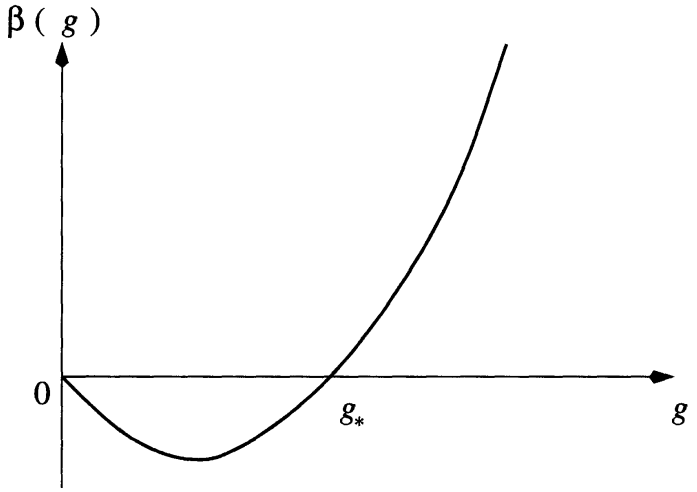

Figure 3.1. The behavior of β-function to two loops

At g_*, the we have a scale invariant and supersymmetric field theory, i.e., a superconformal field theory in four dimensions. It is moreover an non-trivial interacting theory; this will be confirmed by computing the mass dimension of certain fields later. At such a fixed point, the notion of particle is ill-defined. Because of scale invariance, there is no mass gap. The energy spectrum contains a vacuum and a continuum above it. So massless particles can decay arbitrarily since they are not free at long distances. Consequently, there are no well-defined one-particle states in the Hilbert space and there is no S-matrix.

Unlike in two dimensions, the conformal algebra in four dimensions is finite dimensional. Yet it is still possible to extract information from it. We consider radial quantization, i.e., regard the Euclidean spacetime $\mathbb{R}^4 - \{0\}$ as an expanding continuum of S^3. There is a one-to-one correspondence between operators and states in radial quantization. Let $|0\rangle$ be the vacuum. Then an operator $\mathcal{O}$ defines a state $|\mathcal{O}\rangle = \mathcal{O}|0\rangle$ in the Hilbert space. Since the theory is unitary, the operators form a unitary representation of the superconformal algebra.

Recall that the Lie algebra of the conformal group in spacetime of dimension $d \geq 3$ is generated by translations $P_\mu = \partial_\mu$, Lorentz generators $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, dilation $D = -x^\mu\partial_\mu$ and special inversions $K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu$. P_μ, $M_{\mu\nu}$ form the standard Poincaré Lie algebra. The commutation relations of the other generators are

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [K_\mu, P_\nu] = 2(\eta_{\mu\nu}D - M_{\mu\nu}). \tag{3.3}$$

We look for representations in which P_μ are the creation operators and K_μ are the annihilation operators, and $P_\mu^\dagger = K^\mu$. There is a vacuum or highest weight vector $|\mathcal{O}\rangle$ such that it is an eigenvector of D (whose eigenvalue is also denoted by D) and $K_\mu|\mathcal{O}\rangle = 0$. Assume that $\mathcal{O}$ is a scalar operator, i.e., $M_{\mu\nu}|\mathcal{O}\rangle = 0$. Using (3.3), the

square of the norm of $P_\mu|\mathcal{O}\rangle$ is

$$\langle\mathcal{O}|K^\mu P_\mu|\mathcal{O}\rangle = 2dD. \tag{3.4}$$

Therefore unitarity implies that $D \geq 0$ and $D = 0$ if and only if $P_\mu|\mathcal{O}\rangle = 0$, i.e., $\mathcal{O}$ is a constant operator. Furthermore, using (3.3), the square of the norm of $P^\mu P_\mu|\mathcal{O}\rangle$ is

$$\langle\mathcal{O}|K^\nu K_\nu P^\mu P_\mu|\mathcal{O}\rangle = 8dD(D - \tfrac{d-2}{2}). \tag{3.5}$$

For $d = 4$, unitarity implies that $D \geq 1$ and $D = 1$ if and only if $P^\mu P_\mu|\mathcal{O}\rangle = 0$, i.e., $\mathcal{O}$ corresponds to a free field.

When $d = 4$, the superconformal algebra has extra generators Q_α, $\bar{Q}_{\dot\alpha}$ of conformal dimension $\frac{1}{2}$ and S^α, $\bar{S}^{\dot\alpha}$ of conformal dimension $-\frac{1}{2}$. The latter are the supersymmetric partners of $K_m u$, i.e., $[Q_\alpha, K^\mu] = \sigma^\mu_{\alpha\dot\alpha}\bar{S}^{\dot\alpha}$ and $[\bar{Q}_{\dot\alpha}, K^\mu] = \bar\sigma^\mu_{\dot\alpha\alpha}S^\alpha$. Moreover, we have

$$\{Q_\alpha, S^\beta\} = (\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + 2\delta_\alpha{}^\beta(D - \tfrac{3}{2}R), \tag{3.6}$$

where R is the generator of the $U(1)_R$ transformation (whose eigenvalue is also denoted by R). Again, we look for highest weight representations such that $P_\mu^\dagger = K^\mu$ and $Q_\alpha^\dagger = S^\alpha$. An operator $\mathcal{O}$ is called chiral if $\bar{Q}_{\dot\alpha}|\mathcal{O}\rangle = 0$. Again we assume that $\mathcal{O}$ is a scalar operator. If $|\mathcal{O}\rangle$ is also a highest weight vector, i.e., $K_\mu|\mathcal{O}\rangle = 0$ and $S^\alpha|\mathcal{O}\rangle = 0$, then the square of the norm of $Q_\alpha|\mathcal{O}\rangle$ is

$$\langle\mathcal{O}|S^\alpha Q_\alpha|\mathcal{O}\rangle = 4(D - \tfrac{3}{2}R). \tag{3.7}$$

So $D \geq \frac{3}{2}R$ and the equality holds for chiral operators only.

A consequence of the above study is that the chiral operators form a ring. Consider the product of two chiral operators $\Phi_1(x_1)$ and $\Phi_2(x_2)$. When x_1 is close to x_2, we have an expansion

$$\Phi_1(x_1)\Phi_2(x_2) \sim \sum_i (x_1 - x_2)^{\alpha_i}\Phi_i(x_2). \tag{3.8}$$

All the operators Φ_i in the expansion have $R(\Phi_i) = R(\Phi_1) + R(\Phi_2)$. So

$$D(\Phi_i) \geq \tfrac{3}{2}(R(\Phi_1) + R(\Phi_2)) = D(\Phi_1) + D(\Phi_2). \tag{3.9}$$

Therefore all the exponents $\alpha_i \geq 0$. In the limit $x_1 \to x_2$, the surviving terms have $\alpha_i = 0$ and saturate the inequality (3.9). These are the chiral operators. So the chiral operators form a ring.

We return to the study of the non-trivial fixed point g_* with $\beta(g_*) = 0$. The dimension of the chiral fields can be computed from their R-charges. For example, using the R-charges in (1.27), we get, in the infrared limit,

$$D(M) = \frac{3}{2}R(M) = 3\frac{N_f - N_c}{N_c} \tag{3.10}$$

and

$$D(B) = D(B') = \frac{3}{2}\left(N_c - \frac{N_c^2}{N_f}\right). \tag{3.11}$$

The meson M has dimension 2 in the ultraviolet. The difference with (3.10) is called the *anomalous dimension*. Perturbatively, the anomalous dimension of mass is

$$\gamma(g) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + o(g^4). \tag{3.12}$$

At the infrared fixed point g_* given by (3.2), $\gamma(g_*) = -\frac{\epsilon}{3} + o(\epsilon^2)$. In the perturbative expansion, (3.10) is $D(M) = 2 - \frac{\epsilon}{3} + o(\epsilon^2)$. This matches $D(M) = 2 + \gamma(g_*)$.

The requirement $D(M) \geq 1$ implies that $N_f \geq \frac{3}{2}N_c$. For $N_f \geq 3N_c$, the theory is free in the infrared. We claim that for $\frac{3}{2}N_c < N_f < 3N_c$, the theory is an interacting superconformal field theory in the infrared limit. The fixed point is non-trivial because $D(M) > 1$ in this range. If $N_f = \frac{3}{2}N_c$ (for N_c even), then $D(M) = 1$. So M (and perhaps the whole low energy theory) is free. If $N_f < \frac{3}{2}N_c$, (3.10) would imply $D(M) < 1$. Therefore the picture breaks down and new physics should emerge.

§3.2. $N_c + 2 \leq N_f \leq \frac{3N_c}{2}$: infrared free magnetic theory

Associated with the original supersymmetric QCD which we now call the *electric theory*, we introduce a dual theory when $N_f \geq N_c + 2$, called the *magnetic theory*. The dual theory has gauge group $\tilde{G} = SU(\tilde{C})$, where $\tilde{C}$ is an Hermitian vector space of dimension $\tilde{N}_c = N_f - N_c$ and is equipped with a complex volume form $\tilde{v}_c \in \wedge^{\tilde{N}_c}\tilde{C}^*$ invariant under $\tilde{G}$ (and $\tilde{G}^{\mathbb{C}} = SL(\tilde{C})$). The quarks and anti-quarks are $\tilde{Q} \in \mathrm{Hom}(\tilde{F}, \tilde{C})$ and $\tilde{Q}' \in \mathrm{Hom}(\tilde{C}, \tilde{F}')$, respectively, where

$$\tilde{F} = F^* \otimes (\wedge^{N_f} F)^{1/\tilde{N}_c}, \quad \tilde{F}' = F'^* \otimes (\wedge^{N_f} F')^{1/\tilde{N}_c} \tag{3.13}$$

are the spaces of dual flavors (defined up to a root of unity). In addition, the dual theory has a meson $M \in \mathrm{Hom}(F, F')$, which comes from the electric theory but is regarded as an elementary field here. In the electric theory, M has mass dimension 2 in the ultraviolet and acquires an anomalous dimension in the infrared. Therefore, in the magnetic theory, it is more natural to regard the field $\mu^{-1}M \in \mathrm{Hom}(\tilde{F}', \tilde{F})$ of mass dimension 1 as the elementary meson, where

$$\mu \in (\wedge^{N_f} F^* \otimes \wedge^{N_f} F')^{1/\tilde{N}_c} \tag{3.14}$$

is a parameter of mass dimension 1. M is not coupled to the $SU(\tilde{C})$ gauge fields but interacts with the dual quarks through a superpotential

$$W = \frac{1}{\mu}\mathrm{tr}_F\, \tilde{Q}'\tilde{Q}M. \tag{3.15}$$

Here $\tilde{Q}'\tilde{Q} \in \mathrm{Hom}(\tilde{F}, \tilde{F}') \cong \mathrm{Hom}(F', F) \otimes (\wedge^{N_f} F^* \otimes \wedge^{N_f} F')^{1/\tilde{N}_c}$ is the meson of the magnetic theory. W has mass dimension 3.

Under the global symmetry $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$ of the electric theory, the fields $\tilde{Q}$, $\tilde{Q}'$, M transform as, according to (3.13) and (1.26),

(3.16)

	$SU(F)$	$SU(F')$	$U(1)_B$	$U(1)_R$
$\tilde{Q}$	F	$\mathbf{1}$	$-\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{N_f}$
$\tilde{Q}'$	$\mathbf{1}$	F'^*	$\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{N_f}$
M	F^*	F'	0	$2\frac{N_f - N_c}{N_f}$

So the superpotential (3.15) has R-charge $2\frac{N_f - N_c}{N_f} + 2\frac{N_f - \tilde{N}_c}{N_f} = 2$, as required. Their fermionic components $\psi_{\tilde{Q}}$, $\psi_{\tilde{Q}'}$, ψ_M and the gluino field $\tilde{\lambda}$ (of the dual theory) transform as

(3.17)

	$SU(F)$	$\times$ $SU(F')$	$\times$ $U(1)_B$	$\times$ $U(1)_R$
$\psi_{\tilde{Q}}$	F	$\mathbf{1}$	$-\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{N_f}-1$
$\psi_{\tilde{Q}'}$	$\mathbf{1}$	F'	$\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{N_f}-1$
ψ_M	F^*	$\tilde{F}$	0	$1-2\frac{N_c}{N_f}$
$\tilde{\lambda}$	$\mathbf{1}$	$\mathbf{1}$	0	1

It is easy to check that the measure $D\psi_{\tilde{Q}}\, D\psi_{\tilde{Q}'}\, D\psi_{\tilde{\lambda}}$ of the fermions that are coupled to the $SU(\tilde{C})$ gauge fields is invariant under the global symmetry. Therefore the anomaly-free global symmetry of the dual theory is $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$ as well.

We argue that the electric and magnetic theories describe the same physics, at least at long distances. First, we already see that both have the same anomaly-free global symmetry $SU(F) \times SU(F') \times U(1)_B \times U(1)_R$. That they do not have the same gauge group is not a problem. Strictly speaking, gauge symmetry is not a symmetry, but a redundancy in the description of the theory. So gauge symmetry is not fundamental. In fact, the gauge fields in the magnetic theory are composites of the electric degrees of freedom, and vice versa.

Secondly, the electric and magnetic theories have the same gauge invariant operators, though their field contents seem rather different. We construct the mapping of gauge invariant chiral operators. The composite meson $Q'Q$ of the electric theory corresponds to the elementary meson M of the magnetic theory, whereas the composite meson $\tilde{Q}'\tilde{Q}$ of the magnetic theory vanishes by the equation of motion of M. The baryons of the magnetic theory are $\tilde{B} = \tilde{Q}^* \tilde{v}_c \in \wedge^{\tilde{N}_c} \tilde{F}^* \cong \wedge^{N_c} F^*$ and $\tilde{B}' = \tilde{Q}'_* \tilde{v}_c^* \in \wedge^{\tilde{N}_c} \tilde{F}' \cong \wedge^{N_c} F'$. Thus they have the same quantum numbers as B, B' of the electric theory, which we shall identify. In fact, taking into account (1.29), (3.14) as well as their mass dimensions, the only possible relations are

$$B = i^{\tilde{N}_c+1} \Lambda^{b_0/2} \mu^{-\tilde{N}_c/2} \tilde{B}, \quad B' = i^{\tilde{N}_c+1} \Lambda^{b_0/2} \mu^{-\tilde{N}_c/2} \tilde{B}', \tag{3.18}$$

where the numerical factors are chosen to fit future calculations. However, $\tilde{Q}$, $\tilde{Q}'$ are not polynomials of Q, Q' and vice versa because the $U(1)_B$ charges of $\tilde{Q}$, $\tilde{Q}'$ are fractional. Rather, the dual quarks $\tilde{Q}$, $\tilde{Q}'$ should be interpreted as solitons of Q, Q' in the electric theory, and vice versa.

Thirdly, we check the 't Hooft anomaly matching condition. Under the global symmetry, the fermions in the electric theory transform as (2.31). Their contribution to the anomaly is given by the left column of (2.33). In the magnetic theory, the fermions transform as (3.17). Their contribution to the anomaly is given by

(3.19)

	magnetic
$\mathfrak{su}(F)^3$	$\tilde{N}_c d_F + N_f(-d_F)$
$\mathfrak{su}(F')^3$	$\tilde{N}_c(-d_{F'}) + N_f d_{F'}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_B$	$\tilde{N}_c\, c_2(F)\cdot(-\frac{N_c}{\tilde{N}_c})$
$\mathfrak{su}(F')^2\mathfrak{u}(1)_B$	$\tilde{N}_c\, c_2(F')\cdot\frac{N_c}{\tilde{N}_c}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$\tilde{N}_c\, c_2(F)\cdot(\frac{N_c}{N_f}-1) + N_f\, c_2(F)\cdot(1-2\frac{N_c}{N_f})$
$\mathfrak{su}(F')^2\mathfrak{u}(1)_R$	$\tilde{N}_c\, c_2(F')\cdot(\frac{N_c}{N_f}-1) + N_f\, c_2(F')\cdot(1-2\frac{N_c}{N_f})$
$\mathfrak{u}(1)_B^2\mathfrak{u}(1)_R$	$\tilde{N}_c N_f[(-\frac{N_c}{\tilde{N}_c})^2 + (\frac{N_c}{\tilde{N}_c})^2]$
$\mathfrak{u}(1)_R^3$	$2\tilde{N}_c N_f(\frac{N_c}{N_f}-1)^3 + N_f^2(1-2\frac{N_c}{N_f})^3 + (\tilde{N}_c^2-1)1^3$
$\mathfrak{u}(1)_R$	$2\tilde{N}_c N_f(\frac{N_c}{N_f}-1) + N_f^2(1-2\frac{N_c}{N_f}) + (\tilde{N}_c^2-1)1$

It is straightforward to check that these two columns match.

In the range $N_c+2 \le N_f \le \frac{3N_c}{2}$ (not yet studied except for $N_f = \frac{3}{2}N_c$), we have $\tilde{N}_c \ge 2$ and $N_f \ge 3\tilde{N}_c$, therefore the magnetic theory is infrared free. So the magnetic theory is a better description in this range, where the electric theory is hard to study because of the strong coupling in the infrared. In particular, at $N_f = \frac{3N_c}{2}$ (provided that N_c is even), we calculated $D(M) = 1$ in the electric theory. This confirms that M is a free field in the infrared, a consequence of the magnetic description. Generally, as N_f decreases, the coupling in the electric theory gets stronger until the theory confines when $N_f \le N_c + 1$, whereas the coupling of the magnetic theory gets weaker and the theory becomes infrared free when $N_f \le \frac{3N_c}{2}$. The infrared behavior of the theory in various ranges is summarized in the following table.

	electric theory	magnetic theory
$N_c \ge 2$, $N_f \ge 3N_c$ $(\tilde{N}_c+2 \le N_f \le \frac{3\tilde{N}_c}{2})$	infrared free	strongly coupled
$\frac{3N_c}{2} < N_f < 3N_c$ $(\frac{3\tilde{N}_c}{2} < N_f < 3\tilde{N}_c)$	non-trivial infrared fixed point	non-trivial infrared fixed point
$N_c+2 \le N_f \le \frac{3N_c}{2}$ $(\tilde{N}_c \ge 2,\ N_f \ge 3\tilde{N}_c)$	strongly coupled	infrared free
$N_f \le N_c + 1$	analysed in lectures 1, 2	——

§3.3. Further tests of duality

Dynamically, the electric theory has a scale Λ as in (1.29), where $b_0 = 3N_c - N_f$ determines the β-function. The scale $\tilde{\Lambda}$ of the magnetic theory must be related to that of the electric theory. Because Λ, $\tilde{\Lambda}$, μ all have mass dimension 1 and because of

(1.29) (and its magnetic counterpart) and (3.14), the only possible relation among them is, up to a numerical factor,

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3\tilde{N}_c-N_f} = (-1)^{\tilde{N}_c}\mu^{N_f}. \tag{3.20}$$

The phase $(-1)^{\tilde{N}_c}$ will be needed in a number of places below. This, together with (1.15), shows that if the electric theory is strongly coupled, then the magnetic theory is weakly coupled, and vice versa.

§3.3.1. Duality is an involution

We claim that the dual of the magnetic theory is the electric theory itself. The number of colors in the double dual theory is $N_f - \tilde{N}_c = N_c$. Therefore, its color space can be identified with C. Next, the flavor spaces are $\tilde{F}^* \otimes (\wedge^{N_f}\tilde{F})^{1/\tilde{N}_c} \cong F$ and $\tilde{F}'^* \otimes (\wedge^{N_f}\tilde{F}')^{1/\tilde{N}_c} \cong F'$, the same as the electric theory. The mass scale of the double dual theory is again Λ; this follows from (3.20), provided that

$$\tilde{\mu} = -\mu \in (\wedge^{N_F}\tilde{F}^* \otimes \wedge^{N_f}\tilde{F}')^{1/N_c} \cong (\wedge^{N_f}F^* \otimes \wedge^{N_f}F')^{1/\tilde{N}_c}. \tag{3.21}$$

The double dual theory has a new term $\frac{1}{\tilde{\mu}}\mathrm{tr}_{\tilde{F}}Q'Q\tilde{M}$ in the superpotential, where $\tilde{M} \in \mathrm{Hom}(\tilde{F}, \tilde{F}')$ is a new meson field. This, combined with (3.15), gives the total superpotential

$$W = \frac{1}{\mu}\mathrm{tr}_F\,\tilde{M}(M - Q'Q). \tag{3.22}$$

The equation of motion of $\tilde{M}$ implies that $M = Q'Q$. Therefore we recover the electric theory.

§3.3.2. Deformation along the flat directions

So far, we have only studied the theory with $N_f \geq N_c + 2$ at the origin of the moduli space $\mathcal{M}$. Recall that in the electric theory, $\mathcal{M}$ is the space of M, B, B' subject to the classical constraints (1.20), (1.21), (1.22).

We show that the procedure of deforming M in $\mathcal{M}$ commutes with taking the dual. Suppose $\mathrm{rank}\,M = r$. We write

$$F = \hat{F} \oplus \hat{F}^{\perp}, \quad F' = \hat{F}' \oplus \hat{F}'^{\perp} \quad (\dim\hat{F}^{\perp} = \dim\hat{F}'^{\perp} = r) \tag{3.23}$$

such that M has the form

$$M = \begin{pmatrix} 0 & 0 \\ 0 & M^{\perp} \end{pmatrix}. \tag{3.24}$$

At low energies, the gauge group is broken to $SU(N_c - r)$, and there are $N_f - r$ light flavors. The mass scale $\hat{\Lambda}$ of the low energy effective theory is given by

$$\hat{\Lambda}^{3(N_c-r)-(N_f-r)} = \frac{\Lambda^{3N_c-N_f}}{\det M^{\perp}}. \tag{3.25}$$

In the dual description, the fundamental theory has $\tilde{N}_c$ colors and N_f flavors, and has a scale $\tilde{\Lambda}$ related to Λ by (3.20). Now $\mu^{-1}M \in \mathrm{Hom}(\tilde{F}', \tilde{F})$ gives mass to some

components of the dual quarks $\tilde{Q}$, $\tilde{Q}'$. The low energy magnetic theory has $\tilde{N}_c$ colors and $N_f - r$ flavors. Its scale $\hat{\tilde{\Lambda}}$ is given by

$$\hat{\tilde{\Lambda}}^{3\tilde{N}_c-(N_f-r)} = \frac{\tilde{\Lambda}^{3\tilde{N}_c-N_f} \det M^{\perp}}{\mu^r}. \tag{3.26}$$

It follows from (3.20), (3.25), (3.26) that

$$\hat{\Lambda}^{3(N_c-r)-(N_f-r)} \hat{\tilde{\Lambda}}^{3\tilde{N}_c-(N_f-r)} = (-1)^{\tilde{N}_c} \mu^{N_f-r}. \tag{3.27}$$

Thus the low energy magnetic theory is dual to the low energy electric theory.

We further check that the magnetic theory has the same space of ground states $\mathcal{M}$. In the electric theory, the restriction $r = \operatorname{rank} M \leq N_c$ is a classical constraint. In the magnetic theory, the same constraints are enforced by non-perturbative quantum effects. In fact, if $r > N_c$, then the number of flavors in the low energy magnetic theory is $N_f - r < \tilde{N}_c$. In this case, a superpotential of the form (1.30) is generated, and consequently there is no vacuum. So generically, $r = N_c$ from both the electric and the magnetic points of view. In this case, the number of flavors of the low energy magnetic theory is $N_f - r = \tilde{N}_c$, equal to the number of colors. According to (2.20), the light components $\hat{\tilde{M}}$, $\hat{\tilde{B}}$, $\hat{\tilde{B}}'$ of the dual meson and baryons satisfy

$$\det \hat{\tilde{M}} - \hat{\tilde{B}} \otimes \hat{\tilde{B}}' = \hat{\tilde{\Lambda}}^{2\tilde{N}_c}. \tag{3.28}$$

Moreover, $\hat{\tilde{M}} = 0$ here because of the equation of motion of M. Using (3.26) and (3.20) when $r = N_c$, we obtain

$$\hat{\tilde{\Lambda}}^{2\tilde{N}_c} = \frac{\tilde{\Lambda}^{3N_c-N_f} \det M^{\perp}}{\mu^{N_c}} = \frac{(-1)^{\tilde{N}_c} \det M^{\perp}}{\Lambda^{3N_c-N_f} \mu^{-\tilde{N}_c}}. \tag{3.29}$$

By the mapping relation (3.18), we get

$$\hat{B} \otimes \hat{B}' = \det M^{\perp}. \tag{3.30}$$

This is precisely the constraint (1.23) in the electric theory. Other constraints are trivially satisfied.

§3.3.3. Mass deformation

We now check that mass deformation commutes with taking the dual. Suppose a mass term $\operatorname{tr} mM$ with $\operatorname{rank} m = r$ is added to the electric theory. Again, we write (3.23) such that m has the form (2.6). The low energy theory has light quarks $\hat{Q} \in \operatorname{Hom}(\hat{F}, C)$, $\hat{Q}' \in \operatorname{Hom}(C, \hat{F}')$ and has a scale $\hat{\Lambda}$ given by

$$\hat{\Lambda}^{3N_c-N_f+r} = \Lambda^{3N_c-N_f} \det m^{\perp}. \tag{3.31}$$

M has the form (2.10), where $\hat{M}$ contains the light degrees of freedom. Let $\hat{\tilde{F}} = \hat{F}^* \otimes (\wedge^{N_f} F)^{1/\tilde{N}_c}$ and $\hat{\tilde{F}}' = \hat{F}'^* \otimes (\wedge^{N_f} F')^{1/\tilde{N}_c}$. Then the equations of motion of the heavy components of M imply that

$$\tilde{Q}'\tilde{Q} = \begin{pmatrix} \hat{\tilde{M}} & 0 \\ 0 & -\mu m^{\perp} \end{pmatrix} \tag{3.32}$$

for some light dual meson $\hat{\tilde{M}} \in \mathrm{Hom}(\hat{\tilde{F}}, \hat{\tilde{F}}')$. This breaks the $SU(\tilde{N}_c)$ gauge symmetry to $SU(\tilde{N}_c - r)$. We assume that $\tilde{N}_c - r \geq 2$, i.e., $r \leq N_f - (N_c + 2)$. Integrating out the heavy components in M, $\tilde{Q}$, $\tilde{Q}'$, we obtain a low energy effective superpotential

$$\hat{W} = \frac{1}{\mu} \mathrm{tr}\, \hat{\tilde{M}} \hat{M}. \tag{3.33}$$

This is exactly a magnetic theory of $\tilde{N}_c - r$ flavors and $N_f - r$ quarks. Its mass scale $\hat{\tilde{\Lambda}}$ is given by

$$\hat{\tilde{\Lambda}}^{3(\tilde{N}_c - r) - (N_f - r)} = \frac{\tilde{\Lambda}^{3\tilde{N}_c - N_f}}{(-\mu)^r \det m^{\perp}}. \tag{3.34}$$

From (3.31), (3.34) and (3.20), we obtain

$$\hat{\Lambda}^{3N_c - (N_f - r)} \hat{\tilde{\Lambda}}^{3(\tilde{N}_c - r) - (N_f - r)} = (-1)^{\tilde{N}_c - r} \mu^{N_f - r}. \tag{3.35}$$

So the low energy theory of the magnetic theory after mass deformation is dual to that of the electric theory.

If $r = N_f - N_c - 1$, then in the low energy electric theory, the number of flavors is $N_f - r = N_c + 1$ and in the low energy magnetic theory, the gauge symmetry is completely broken since $\tilde{N}_c - r = 1$. The light baryons in the magnetic theory $\hat{\tilde{B}} = \hat{\tilde{Q}}$, $\hat{\tilde{B}}' = \hat{\tilde{Q}}'$ are related to $\hat{B}$ and $\hat{B}'$ by (3.18). The superpotential (3.15) can be written as

$$\frac{1}{\mu} \hat{\tilde{Q}}' \hat{M} \hat{\tilde{Q}} = \frac{1}{\hat{\Lambda}^{2N_c - 1}} \hat{B}' \hat{M} \hat{B}. \tag{3.36}$$

In addition, because the gauge symmetry is completely broken, instanton contributions should be included just as in the case $N_f = N_c - 1$. Now there are more than $\tilde{N}_c - 1$ zero modes of $\psi_{\tilde{Q}}$, $\psi_{\tilde{Q}'}$; these $N_c + 1$ extra zero modes are absorbed by the interaction (3.15), generating a term proportional to the $(N_c + 1)$-th power of $(\mu^{-1} M)_{\mathrm{light}}$ in the effective superpotential. So the instanton contribution is

$$\begin{aligned} \frac{\tilde{\Lambda}^{3\tilde{N}_c - N_f} \det(\mu^{-1} M)_{\mathrm{light}}}{\det(\tilde{Q}' \tilde{Q})_{\mathrm{heavy}}} &= \frac{\tilde{\Lambda}^{3\tilde{N}_c - N_f} \det(\mu^{-1} \hat{M})}{\det(-\mu m^{\perp})} = \frac{\tilde{\Lambda}^{3\tilde{N}_c - N_f} \det \hat{M}}{(-1)^{\tilde{N}_c + 1} \mu^{N_f} \det m^{\perp}} \\ &= -\frac{\det \hat{M}}{\Lambda^{3N_c - N_f} \det m^{\perp}} = -\frac{\det \hat{M}}{\hat{\Lambda}^{2N_c - 1}}, \end{aligned} \tag{3.37}$$

where (3.32), (3.20), (3.31) have been used. Combining (3.36) and (3.37), we get the total effective superpotential

$$W_{\mathrm{eff}} = \frac{1}{\hat{\Lambda}^{2N_c - 1}} (\hat{B}' \hat{M} \hat{B} - \det \hat{M}), \tag{3.38}$$

which is exactly (2.24).

There are many other checks of duality. We remark here that a theory with fewer flavors can be regarded as the deformation of a larger theory by a mass term. This duality is related to various phenomena in $N = 2$ theories and to Langlands duality in $N = 4$ theories.

References and cross references

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EXERCISES

We repeat the discussion of $SU(N_c)$ gauge theories for $Sp(N_c)$ (Problems 1,3,5) and $SO(N_c)$ (Problems 2,4,6).

Problem 1. Consider an $Sp(N_c)$ gauge theory with $2N_f$ chiral superfields in the fundamental ($2N_c$-dimensional) representation.

(a) Find the moduli space of classical ground states and parametrize them by gauge invariants. Write explicitly the constraints when they exist. What is the unbroken gauge group at generic points in the moduli space?

(b) Find the symmetries of the quantum theory. What is the most general invariant superpotential which can be generated? When can we conclude that no superpotential is generated?

(c) What happens if the number of chiral superfields is odd?

Problem 2. Repeat the analysis of Problem 1 for an $SO(N_c)$ gauge theory with N_f chiral superfields in the fundamental (N_c-dimensional) representation.

Problem 3. We continue our study of $Sp(N_c)$ gauge theory with $2N_f$ chiral superfields in the fundamental ($2N_c$-dimensional) representation from Problem 1.

(a) Find when instantons can generate the superpotential, and check the necessary conditions we discussed for $SU(N_c)$. Use the results for $SU(N_c)$ to show that the coefficient is not zero in this case.

(b) Use (a) to derive the superpotential for smaller values of N_f. Show that in the pure gauge $Sp(N_c)$ theory $\langle\lambda\lambda\rangle \neq 0$.

(c) Describe the moduli space of quantum vacua for one more flavor than in (a). Find the most symmetric points on the moduli space, and identify the spectrum of massless particles there. Check the 't Hooft anomaly matching conditions there.

(d) Add one more flavor to (c). What is the moduli space of quantum vacua? What is the low energy field theory? Find the most symmetric point in the moduli space and the spectrum of massless particles there. Check the 't Hooft anomaly matching conditions there.

Problem 4. We continue our study of $SO(N_c)$ gauge theory with N_f chiral superfields in the fundamental (N_c-dimensional) representation from Problem 2. For $N_c = 4$ limit the analysis to equal gauge coupling constants for the two $SU(2)$ subgroups.

(a) Use these theories and the information in $SU(N_c)$ theories to show that $\langle\lambda\lambda\rangle \neq 0$ in the pure gauge $SO(N_c)$ theories. Derive the superpotential for $SO(N_c)$ with $N_f \leq N_c - 5$.

(b) Find the effect of gluino condensation in the unbroken gauge group for $N_f = N_c - 4$. How many branches are there? Find the spectrum of massless particles in all supersymmetric vacua.

(c) Use (b) to find the superpotential for $N_f = N_c - 3$. What is the spectrum of massless particles in the supersymmetric vacua? Check the 't Hooft anomaly matching conditions. Find the physical origin of the various terms in the superpotential.

(d) Use (c) to determine the coefficients of all instanton effects we discussed and the value of $\langle\lambda\lambda\rangle$ in the pure gauge theories using information involving one loop computations only.

(e) Find the moduli space of vacua for $N_f = N_c - 2$. At the generic point on this space the spectrum includes a massless photon. Limit your analysis to $N_c \geq 4$. Find the elliptic curve which determines its coupling constant, τ_{eff}. Note that this problem has only $N = 1$ supersymmetry but τ_{eff} is still holomorphic, and therefore can be determined. Identify the singularities on the moduli space, and find the spectrum of massless particles there. Check the 't Hooft anomaly matching conditions. Show that upon adding a mass to one of the flavors the theory goes over to the one found in (c). Compare with the theory with $N_c = 3$, which has $N = 2$ supersymmetry.

Problem 5. We continue our study of $Sp(N_c)$ gauge theory with $2N_f$ chiral superfields in the fundamental ($2N_c$-dimensional) representation.

(a) Find the dual gauge group. When is it UV free and when is it IR free?

(b) Find the map of the chiral operators. Show that their dimensions satisfy the unitarity bound.

(c) Check the 't Hooft anomaly matching conditions.

(d) Check the flat directions.

(e) Check the mass deformations.

Problem 6. We continue our study of $SO(N_c)$ gauge theory with N_f chiral superfields in the fundamental (N_c-dimensional) representation. For $N_c = 4$, limit the analysis to equal gauge coupling constants for the two $SU(2)$ subgroups. Repeat all the questions from Problem 5 for this case. Note the special peculiarities when the dual gauge group is $SO(3)$.

SOLUTIONS TO EXERCISES (BY SIYE WU)

General Remarks

The classical moduli space

We recall that the quark superfield Q in an $N = 1$ supersymmetric gauge theory is valued in a representation R of the gauge group G. The bosonic potential is $V = |D|^2$, where $D\colon R \to \mathfrak{g}^* \cong \mathfrak{g}$ is the moment map of the Hamiltonian G-action on R. Therefore the space $\mathcal{M}_c$ of classical ground states (modulo gauge transformations), called the moduli space of classical vacua, is the symplectic quotient $D^{-1}(0)/G$.

The space $\mathcal{M}_c = D^{-1}(0)/G$ is also the categorical quotient of R by $G^{\mathbb{C}}$.That is to say, $\mathcal{M}_c$ is a complex space whose coordinate ring is isomorphic to $\mathbb{C}[R]^G$. In fact, for any $Q \in R$, the orbit $G^{\mathbb{C}} \cdot Q$ is closed if and only if $G^{\mathbb{C}} \cdot Q \cap D^{-1}(0) \neq \emptyset$. If $Q \in D^{-1}(0)$, then $G^{\mathbb{C}} \cdot Q \cap D^{-1}(0) = G \cdot Q$. Let the semistable part of R be $R^{ss} = \{Q \in R \,|\, G^{\mathbb{C}} \cdot Q \cap D^{-1}(0) \neq \emptyset\}$. Let $\pi\colon R^{ss} \to R^{ss}/G^{\mathbb{C}}$ be the quotient map. We have
1) the inclusion $D^{-1}(0) \to R^{ss}$ induces a homeomorphism $D^{-1}(0)/G \to R^{ss}/G^{\mathbb{C}}$;
2) for any point $O \in R^{ss}/G^{\mathbb{C}}$, $\pi^{-1}(O)$ contains exactly one closed $G^{\mathbb{C}}$-orbit in R;
3) for any open set $U \subset R^{ss}/G^{\mathbb{C}}$, we have an isomorphism $\mathcal{O}(U) \cong \mathcal{O}(\pi^{-1}(U))^G$.
This justifies the identification of the structure sheaf $\mathcal{O}_{D^{-1}(0)/G}$ with $(\mathcal{O}_R)^G$. (However the invariants in $\mathbb{C}[R]^G$ do not distinguish the $G^{\mathbb{C}}$-orbits in R.)

In what follows we will consider the case $R = \mathrm{Hom}(F, C)$. Here C, called the space of colors, is an irreducible unitary representation of G. F, the space of flavors, is an Hermitian vector space on which G acts trivially. At the classical level, the global symmetry group is $U(F) \times U(1)_X$, where $U(1)_X$ is the rotation on the fermionic coordinates in the $N = 1$ superspace. We write $U(F) = SU(F) \times_{Z(SU(F))} U(1)_A$, where $Z(SU(F))$ is the center of $SU(F)$ and $U(1)_A$ is the scalar multiplication of $U(1)$ on F.

Axial anomaly and the global symmetry

The classical action depends on the gauge group G, the matter representation R, the coupling constant g and the θ-term. In the quantum theory, the coupling constant g depends on the renormalization scale μ: $g(\mu)$ runs according to $\mu \frac{dg}{d\mu} = \beta(g)$. Here

$\beta(g)$ is the β-function; at one loop it is given by

$$\beta(g) = -\frac{b_0}{(4\pi)^2} g^3, \tag{4.1}$$

where $b_0 = \frac{3}{2}c_2(G) - \frac{1}{2}c_2(R)$. The quantum theory thus has a parameter of mass dimension 1: the (complex) scale Λ defined by

$$\Lambda^{b_0} = \mu^{b_0} e^{-\frac{8\pi^2}{g^2(\mu)}} e^{i\theta} = \mu^{b_0} e^{2\pi i \tau(\mu)}, \tag{4.2}$$

where θ is the theta angle and $\tau(\mu) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2(\mu)}$.

Recall that if $R = \mathrm{Hom}(F, C)$, then the classical theory has a global symmetry $U(F) \times U(1)_X$. Upon quantization, both $U(1)_A$ and $U(1)_X$ shift the theta angle θ, and hence Λ^{b_0}, whereas $SU(F)$ does not. Consequently neither of these $U(1)$ groups is a true symmetry of a fixed quantum theory; this is called the axial anomaly. Rather, the group $U(1)_A \times U(1)_X$ acts on a family of quantum theories. The $U(1)_A$ and $U(1)_X$ charges of Q, its fermionic component ψ_Q, the gluino λ and the scale Λ are summarized as follows:

(4.3)

	$U(1)_A$	$U(1)_X$	$U(1)_R$
Q	-1	0	$\frac{c_2(R)-c_2(G)}{c_2(R)}$
ψ_Q	-1	-1	$-\frac{c_2(G)}{c_2(R)}$
λ	0	1	1
Λ^{b_0}	$c_2(R)$	$c_2(R) - c_2(G)$	0

Here and below, representations of $U(1)$ are labeled by their weights.

The anomaly-free subgroup of $U(1)_A \times U(1)_X$ is a product of a $U(1)$ subgroup, denoted by $U(1)_R$, and a finite group containing the center of $SU(F)$. We normalize the generator of the Lie algebra $\mathfrak{u}(1)_R$ so that it projects to that of $\mathfrak{u}(1)_X$. The weights of $U(1)_R$, which are also shown in the above table, are related to those of $U(1)_A$ and $U(1)_X$ by

$$R = X + \frac{c_2(G) - c_2(R)}{c_2(R)} A. \tag{4.4}$$

The true quantum symmetry is $SU(F) \times U(1)_R$ up to a discrete factor.

Global gauge anomalies

Consider a principal G-bundle P over the (Euclidean) space-time. Let $\mathcal{A}$ be the space of connections, $\mathcal{G}$ the group of gauge transformations, and $\not{D}_R^+$ be the Dirac operator acting on Weyl spinors twisted by a representation R. A gauge theory with Weyl fermions has gauge anomalies if the determinant line bundle $\mathrm{Det}\not{D}_R^+$ over $\mathcal{A}/\mathcal{G}$ is not trivial. When the gauge group G is semisimple, the curvature $\mathcal{F}$ of $\mathrm{Det}\not{D}_R^+$ is proportional to a multilinear form d_R on the Lie algebra $\mathfrak{g}$ defined by

$$d_R(x, y, z) = \mathrm{tr}_R x\{y, z\} \qquad (x, y, z \in \mathfrak{g}). \tag{4.5}$$

Local gauge anomalies come from the non-vanishing of $\mathcal{F}$. For $G = Sp(n)$ or $G = SO(n)$, we have $d_R = 0$ for any representation R of G. Consequently the bundle $\mathrm{Det}\not{D}_R^+$ is flat and there are no local gauge anomalies.

However if there is a non-contractible loop in $\mathcal{A}/\mathcal{G}$, the holonomy along it may be non-trivial even if $\mathcal{F} = 0$. This is called the global gauge anomaly. Since the

space $\mathcal{A}$ is contractible, $\pi_1(\mathcal{A}/\mathcal{G}) \cong \pi_0(\mathcal{G})$. When the (Euclidean) space-time is S^4, we have $\pi_0(\mathcal{G}) \cong \pi_4(G)$. When G is a simple Lie group, $\pi_4(G) \neq 1$ only if only if the universal covering group is $Sp(n)$ for some $n \geq 1$, in which case $\pi_4(G) \cong \mathbb{Z}_2$. Thus the holonomy of $\mathrm{Det}\not{D}_R^+$ for any R is ± 1, and consequently, the squared bundle $(\mathrm{Det}\not{D}_R^+)^{\otimes 2}$ is always trivial. In particular, consider $R = \mathrm{Hom}(F, C)$, where G acts on F trivially. When $G = Sp(n)$ and $C \cong \mathbb{C}^{2n}$ is its defining representation, the bundle $\mathrm{Det}\not{D}_R^+ = (\mathrm{Det}\not{D}_C^+)^{\otimes \dim F}$ is trivial if $\dim F$ is even. This condition will guarantee the absence of global gauge anomalies. That the bundle $\mathrm{Det}\not{D}_C^+$ itself is non-trivial is a deeper fact related to Bott periodicity.

For $G = SO(n)$, the curvature $\mathcal{F}$ of $\mathrm{Det}\not{D}_R^+$ vanishes for any representation R. If $n \geq 6$, since $\pi_4(G) = 1$, the bundle $\mathrm{Det}\not{D}_R^+$ is trivial for any representation R (including the defining representation). If G is simple and the representation R is non-trivial and of real type, then up to a finite subgroup of the center, G can be embedded into $SO(d)$, where $d = \dim R$. The G-bundle P induces a principal $SO(d)$-bundle P_0. Let $\mathcal{A}_0$ be the space of connections on P_0, and $\mathcal{G}_0$ the group of gauge transformations. Then there exist maps $\mathcal{A} \to \mathcal{A}_0$, $\mathcal{G} \to \mathcal{G}_0$ and $\mathcal{A}/\mathcal{G} \to \mathcal{A}_0/\mathcal{G}_0$. The determinant line bundle $\mathrm{Det}\not{D}_R^+$ over $\mathcal{A}/\mathcal{G}$ is the pull-back of corresponding bundle over $\mathcal{A}_0/\mathcal{G}_0$. If $d \geq 6$, the latter is trivial. If $2 \leq d \leq 5$, the case can be reduced to $d \geq 6$ since $\mathrm{Det}\not{D}_{R\oplus R\oplus R}^+ = (\mathrm{Det}\not{D}_R^+)^{\otimes 3} \cong \mathrm{Det}\not{D}_R^+$. Therefore $\mathrm{Det}\not{D}_R^+$ is trivial for any real representation R of any compact Lie group G.

We list a few consequences of this result. First, if R and R' are both real or quaternionic representations of G, then the representation $R \otimes R'$ is real. So the bundle $\mathrm{Det}\not{D}_{R\otimes R'}^+$ is trivial. Second, if $G = SO(n)$ for any $n \geq 3$, and $C \cong \mathbb{C}^n$, the defining representation, then the bundle $\mathrm{Det}\not{D}_C^+$ is trivial. Finally, since the adjoint representation is always of real type, the bundle $\mathrm{Det}\not{D}_{\mathrm{adj}}^+$ is trivial. Thus the gluino λ does not contribute to the global gauge anomaly.

References and cross references

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SOLUTIONS TO EXERCISES (BY SIYE WU)
Problems 1–2

Solution to Problem 1

Let $C \cong \mathbb{C}^{2N_c}$ be an Hermitian vector space with a compatible symplectic form ω_c. C is the defining representation of the compact group $G = Sp(C) \cong Sp(N_c)$. We write the complexification $G^{\mathbb{C}} = Sp(C, \mathbb{C})$. The representation on C is quaternionic. We have $c_2(C) = 1$. The dual Coxeter number of G is $h = N_c+1$. Therefore $c_2(G) = 2(N_c + 1)$. Let the flavor space F be of dimension $2N_f$. (It is even to ensure the absence of the global gauge anomaly.) Then $c_2(R) = 2N_f$ and $b_0 = 3(N_c+1) - N_f$.

Classical theory

The moduli space of classical vacua is $\mathcal{M}_c \cong \mathrm{Hom}(F, C)^{ss}/Sp(C, \mathbb{C})$. We define gauge invariant quantities

$$M = Q^* \omega_c \in \wedge^2 F^*. \tag{5.1}$$

Clearly, M satisfies the constraint

$$\wedge^{N_c+1} M = 0. \tag{5.2}$$

It is a classical result of Weyl that the ring of invariants $\mathbb{C}[\mathrm{Hom}(F, C)]^G$ is generated by $M \in \wedge^2 F^*$ with the (only) relations (5.2).So the moduli space $\mathcal{M}_c$ is parametrized by M subject to the constraint (5.2).

Recall that C has a quaternionic structure. We have an orthogonal decomposition of the Lie algebra

$$\mathfrak{u}(C) = \mathfrak{u}(C, \mathbb{H}) \oplus i\{\mathbb{H}\text{-self-adjoint operators on } C\}. \tag{5.3}$$

Since $\mathfrak{u}(C, \mathbb{H}) = \mathfrak{sp}(C)$, the moment map is

$$D(Q) = \text{orthogonal projection of } iQQ^\dagger \text{ onto } \mathfrak{u}(C, \mathbb{H}). \tag{5.4}$$

So $D(Q) = 0$ if and only if $QQ^\dagger$ is $\mathbb{H}$-self-adjoint on C. In this case, there is an $\mathbb{H}$-unitary basis of C such that $QQ^\dagger$ has the form

$$QQ^\dagger \sim \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_r^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.5}$$

where $a_1, \ldots, a_r > 0$, $r = \frac{1}{2}\operatorname{rank} Q$. Hence there is a unitary basis in F such that Q is of the form

$$Q \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.6}$$

In the complex language, any $Q \in \operatorname{Hom}(F, C)^{ss}$ can be put into the standard form (5.6) by a complexified gauge transformation and a global symmetry. The stabilizer of Q in $Sp(C, \mathbb{C})$ is isomorphic to $Sp(N_c - r, \mathbb{C})$ and that in $Sp(C)$ is $Sp(N_c - r)$. At a generic point in $\mathcal{M}_c$, the gauge group is broken to $Sp(N_c - N_f)$ if $N_c > N_f$; it is completely broken if $N_c \leq N_f$.

Quantum theory

Let Λ be the (complex) scale of the quantum theory. Since Λ^{b_0} has $U(1)_A$-charge $c_2(R) = 2N_f$, it is appropriate to write

$$\Lambda^{3(N_c+1)-N_f} \in \wedge^{2N_f} F^*. \tag{5.7}$$

The anomaly-free subgroup in $U(1)_A \times U(1)_X$ is $U(1)_R \times \mathbb{Z}_{2N_f}$, where the R-charge can be computed by, following (4.4),

$$R = X + \frac{N_c + 1 - N_f}{N_f} A. \tag{5.8}$$

Since $\mathbb{Z}_{2N_f} = Z(SU(F))$, the global symmetry of the quantum theory is $SU(F) \times U(1)_R$. The superfields Q and M transform as

	$SU(F)$	$\times$	$U(1)_R$
Q	F^*		$1 - \frac{N_c+1}{N_f}$
M	$\wedge^2 F^*$		$2(1 - \frac{N_c+1}{N_f})$

(5.9)

The vacuum degeneracy in the classical moduli space $\mathcal{M}_c$ may be lifted by a dynamically generated superpotential W for the light fields M. W must be invariant under the gauge and global symmetries. The fields in M are already gauge invariant; the only combination invariant under $SU(F)$ is the Pfaffian $\operatorname{Pf} M$. Also, the superpotential is holomorphic on $\mathcal{M}_c$, hence in M. These restrictions imply that W is of the form

$$W \sim (\Lambda^{b_0})^x (\operatorname{Pf} M)^y. \tag{5.10}$$

The exponents x, y can be fixed as follows:
1) Since $\Lambda^{b_0}, \operatorname{Pf} M \in \wedge^{2N_f} F^*$ and W is a number, we have $x + y = 0$.
2) As a superpotential, W has $U(1)_X$ charge $2(N_c + 1 - N_f)x = 2$.
3) W has mass dimension $b_0 x + 2N_f y = 3$.
Notice that 1) and 2) together imply that W has $U(1)_R$-charge 2. The above three equations for two unknowns x and y are consistent and we get $x = -y = \frac{1}{N_c+1-N_f}$. Hence the superpotential is

$$W = A \left(\frac{\Lambda^{3(N_c+1)-N_f}}{\operatorname{Pf} M} \right)^{\frac{1}{N_c+1-N_f}}, \tag{5.11}$$

where A is a constant that depends only on N_c, N_f and the renormalization scheme. Notice that if $N_f \geq N_c + 1$, then the superpotential W in (5.11) does not exist because $\operatorname{Pf} M = 0$. In this case, the classical vacuum degeneracy is not lifted by quantum effects. When $N_f \leq N_c$, the decay of W at $M \to \infty$ is consistent with asymptotic freedom, which holds when $N_f < 3(N_c + 1)$.

Solution to Problem 2

Let $C \cong \mathbb{C}^{N_c}$ be a complex vector space with a real structure, i.e., C is the complexification of $C_{\mathbb{R}} \cong \mathbb{R}^{N_c}$. Suppose $C_{\mathbb{R}}$ is equipped with a metric g_c and a compatible volume form v_c, which provides a trivialization of $\det C_{\mathbb{R}}^*$. (In the symplectic case, v_c is fixed by the symplectic form ω_c.) C is the defining representation of the compact Lie group $G = SO(C) \cong SO(N_c)$. (The complexification is denoted by $G^{\mathbb{C}} = SO(C, \mathbb{C})$.) Let the flavor space F have dimension N_f. For $N_c \geq 5$, the dual Coxeter number $h = N_c - 2$ and $c_2(C) = 2$. So $c_2(G) = 2(N_c - 2)$, $c_2(R) = 2N_f$ and $b_0 = 3(N_c - 2) - N_f$. For $N_c = 4$, $G = (G_+ \times G_-)/\mathbb{Z}_2$, where $G_\pm \cong SU(2)$. For each factor $G_\pm$, we have $c_2(G_\pm) = 4$, $c_2(C) = 2$, $c_2(R) = 2N_f$. So $b_0 = 6 - N_f$ for the coupling constant of each $G_\pm$. These are formally the continuation of the cases $N_c \geq 5$. For $N_c = 3$, $G \cong SU(2)$ up to a $\mathbb{Z}_2$ factor. We have $c_2(G) = c_2(C) = 4$, $c_2(R) = 4N_f$ and $b_0 = 6 - 2N_f$.

Classical theory

The moduli space of classical vacua is $\mathcal{M}_c \cong \operatorname{Hom}(F, C)^{ss}/SO(C, \mathbb{C})$. The gauge invariants are the mesons $M = Q^* g_c \in S^2 F^*$ and, if $\operatorname{rank} Q = N_c$, the baryons $B = Q^* v_c \in \wedge^{N_c} F^*$. They satisfy the constraints

$$\operatorname{rank} M \leq N_c, \qquad B \otimes B = \wedge^{N_c} M. \tag{5.12}$$

Here $\wedge^{N_c} M \in S^2(\wedge^{N_c} F^*)$ corresponds to the product of non-zero eigenvalues of M. The ring of invariants $\mathbb{C}[\operatorname{Hom}(F, C)]^G$ is generated by $M \in S^2 F^*$ and, if $N_f \geq N_c$, $B \in \wedge^{N_c} F^*$ with the relations (5.12).So M and B subject to the constraints in (5.12) parametrize the moduli space $\mathcal{M}_c$.

Consider the orthogonal decomposition

$$\mathfrak{u}(C) = \mathfrak{so}(C) \oplus i\{\mathbb{R}\text{-self-adjoint operators on } C_{\mathbb{R}}\}. \tag{5.13}$$

Note that $\mathfrak{so}(C)$ is the set of skew-self-adjoint operators on $C_{\mathbb{R}}$. The moment map of the G-action on $\operatorname{Hom}(F, C)$ is

$$D(Q) = \text{orthogonal projection of } iQQ^\dagger \text{ onto } \mathfrak{so}(C). \tag{5.14}$$

So $D(Q) = 0$ if and only if $QQ^\dagger$ is $\mathbb{R}$-self-adjoint on $C_{\mathbb{R}}$. In this case, there is an orthogonal basis of C (i.e., up to a gauge transformation) such that $QQ^\dagger$ has the form

$$QQ^\dagger \sim \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_r^2 \end{pmatrix}, \tag{5.15}$$

where $a_1, \ldots, a_r > 0$ if $r = \operatorname{rank} Q < N_c$ and $a_1, \ldots, a_{N_c - 1} > 0$, $a_{N_c} \neq 0$ if $r = N_c$. In the latter case, $B^{1,\ldots,N_c} = a_1 \ldots a_{N_c}$ is the only non-zero component of B. Hence

up to gauge and global symmetries Q is of the form

$$Q \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}. \tag{5.16}$$

The stabilizer of Q in $SO(C, \mathbb{C})$ is isomorphic to $SO(N_c - r, \mathbb{C})$, and that in $SO(C_{\mathbb{R}})$ is isomorphic to $SO(N_c - r)$. At a generic point in $\mathcal{M}_c$, the gauge group is broken to $SO(N_c - N_f)$ if $N_c \geq N_f + 2$; it is totally broken if $N_c < N_f + 2$.

Quantum theory

Let Λ be the (complex) scale of the quantum theory. We have

$$\Lambda^{3(N_c-2)-N_f} \in (\wedge^{N_f} F^*)^{\otimes 2} \tag{5.17}$$

if $N_c \geq 4$. (For $N_c = 4$, this holds for the scale of each $SU(2)$ theory.) If $N_c = 3$, then

$$\Lambda^{6-2N_f} \in (\wedge^{N_f} F^*)^{\otimes 4}. \tag{5.18}$$

The anomaly-free subgroup of $U(1)_A \times U(1)_X$ is $U(1)_R \times \mathbb{Z}_{2N_f}$ if $N_c \geq 4$ and $U(1)_R \times \mathbb{Z}_{4N_f}$ if $N_c = 3$. Here the R-charges are computed by

$$R = X + \frac{N_c - 2 - N_f}{N_f} A. \tag{5.19}$$

The $\mathbb{Z}_{N_f}$ subgroup in $\mathbb{Z}_{2N_f}$ or $\mathbb{Z}_{4N_f}$ is the center of $SU(F) \cong SU(N_f)$. Therefore the global symmetry of the quantum theory is $(SU(F) \rtimes \mathbb{Z}_2) \times U(1)_R$ if $N_c \geq 4$ and $(SU(F) \rtimes \mathbb{Z}_4) \times U(1)_R$ if $N_c = 3$. Here $SU(F) \rtimes \mathbb{Z}_n = \{U \in U(F) \,|\, U^n \in SU(F)\}$. For all $N_c \geq 3$, the superfields Q and M transform as

(5.20)

	$SU(F)$	$\times$	$U(1)_R$
Q	F^*		$1 - \frac{N_c-2}{N_f}$
M	S^2F^*		$2(1 - \frac{N_c-2}{N_f})$

Invariance under the gauge and global symmetries and holomorphicity imply that the superpotential, if it exists, is of the form

$$W \sim (\Lambda^{b_0})^x (\det M)^y \tag{5.21}$$

for some exponents x, y. When $N_c \geq 4$, considerations exactly the same as in the symplectic case yield three equations $x + y = 0$, $2 = 2(N_c - N_f - 2)x$ and $b_0 x + 2N_f y = 3$ for the two unknowns x and y. Again, they have a solution $x = -y = \frac{1}{N_c - 2 - N_f}$. Hence

$$W = A \left(\frac{\Lambda^{3(N_c-2)-N_f}}{\det M} \right)^{\frac{1}{N_c-2-N_f}}, \tag{5.22}$$

where A is a constant that depends only on N_c, N_f and the renormalization scheme. Clearly W does not exist if $N_f = N_c - 2$. If $N_f > N_c$, then $\operatorname{rank} M < N_f$ and $\det M = 0$. So W does not exist either. If $N_c - 2 < N_f \leq N_c$, $W \to \infty$ as $Q \to \infty$; this is in contradiction with asymptotic freedom ($b_0 > 0$ in this range). So the superpotential is not generated when $N_f \geq N_c - 2$. When $N_c = 4$, (5.22) is the superpotential generated by each of the $SU(2)$ factors. We limit the analysis to the

case of equal scales for the two $SU(2)$ factors. When $N_c = 3$, the three equations $2x + y = 0$, $2 = (4 - 4N_f)x$ and $b_0 x + 2N_f y = 3$ are also consistent and have a solution $2x = -y = \frac{1}{1-N_f}$. So,

$$W = A \left(\frac{\Lambda^{3-N_f}}{\det M} \right)^{\frac{1}{1-N_f}} . \tag{5.23}$$

Similarly, W does not exist if $N_f = 1$ or $N_f > 3$. When $N_f = 2$, the form of W is not consistent with asymptotic freedom ($b_0 = 2 > 0$ in this case). When $N_f = 3$, we have $b_0 = 0$, but the two-loop β-function is positive. The theory is infrared free and does not have a superpotential. In summary, no superpotential can be generated when $N_c = 3$.

References and cross references

Weyl's results on the invariants of symplectic and orthogonal groups are in Sects. VI.1 and II.9 of

H. Weyl, *The classical groups*, Princeton University Press, Princeton (1939).

Supersymmetric QCD with gauge group $Sp(N_c)$ and $SO(N_c)$ were systematically studied in

K. Intriligator and P. Pouliot, *Exact superpotential, quantum vacua and duality in supersymmetric $Sp(N_c)$ gauge theories*, Phys. Lett. B353 (1995) 471-476;
K. Intriligator and N. Seiberg, *Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric $SO(N_c)$ gauge theories*, Nucl. Phys. B444 (1995) 125-160.

These two papers will also be relevant to the solutions of Problems 3-6.

SOLUTIONS TO EXERCISES (BY SIYE WU)
Problems 3–4

Solution to Problem 3

$N_f = N_c$: instanton generated superpotential

Recall that when $N_f \leq N_c$, it is possible to have a dynamically generated superpotential (5.11). This superpotential is generated by instantons when $N_f = N_c$, because the following three requirements are independently met:
1) The result of any calculation in instanton perturbation theory is proportional to $e^{2\pi i \tau(\mu)} \sim \Lambda^{b_0}$. The superpotential W in (5.11) has the same power of Λ when $N_f = N_c$.
2) An instanton vertex absorbs $2(N_c + 1)$ of the λ and $2N_f$ of the ψ_Q. The usual vertex is given by the term $\lambda \psi_Q Q$. When $N_f = N_c$, it is possible to form a diagram with 2 external lines of ψ_Q.
3) The $Sp(N_c)$ gauge symmetry is completely broken when the renormalization scale μ is small compared to $\langle Q \rangle$. This means the coupling constant stays small for all μ. Hence the instanton calculation is trustworthy.
Thus we expect $A \neq 0$ when $N_f = N_c$.

We study how the constant A depends on N_c and N_f using deformations on the moduli space. Suppose Q acquires an expectation value $\langle Q \rangle \in \mathrm{Hom}(F, C)$ with $\frac{1}{2} \operatorname{rank} \langle Q \rangle = r$. Then $C = \hat{C} \oplus \hat{C}^{\perp}$, $F = \hat{F} \oplus \hat{F}^{\perp}$, where $\hat{C} = (\mathrm{im} \langle Q \rangle)^{\perp}$, $\hat{F} = \ker \langle Q \rangle$ have dimensions $2(N_c - r)$, $2(N_f - r)$, respectively. The expectation value $\langle Q \rangle$ breaks the gauge group to $Sp(\hat{C})$. Let $\langle M^{\perp} \rangle \in \wedge^2 \hat{F}^{\perp *}$ be the restriction of $\langle M \rangle$ to $\hat{F}^{\perp}$; $\langle M^{\perp} \rangle$ is nondegenerate. Because of the superpotential (5.11), the excitations of M in the $\hat{F}^{\perp}$ directions become massive. In the low energy theory, M is of the form

$$M = \begin{pmatrix} \hat{M} & \\ & \langle M^{\perp} \rangle \end{pmatrix}, \tag{6.1}$$

where $\langle M^{\perp} \rangle$ is fixed and $\hat{M} \in \wedge^2 \hat{F}^2$ contains the light modes. The space of light flavors is $\hat{F}$.

When the renormalization scale $\mu >> (\mathrm{Pf}\,(\langle M^{\perp} \rangle / 2))^{1/2r}$, the coupling constant $g(\mu)$ runs according to

$$(\Lambda/\mu)^{3(N_c+1)-N_f} = e^{-\frac{8\pi^2}{g^2(\mu)}}. \tag{6.2}$$

When $\mu << (\mathrm{Pf}\,(\langle M^\perp\rangle/2))^{1/2r}$, the low energy theory has $N_c - r$ colors and $N_f - r$ flavors. Let $\hat{\Lambda}$ be its scale. Then

$$(\hat{\Lambda}/\mu)^{3(N_c-r+1)-(N_f-r)} = e^{-\frac{8\pi^2}{g^2(\mu)}}. \tag{6.3}$$

We use the $\overline{\mathrm{DR}}$ subtraction scheme. Matching the coupling constant $g(\mu)$ in (6.2) and (6.3) at $\mu = (\mathrm{Pf}\,(\langle M^\perp\rangle/2))^{1/2r}$, we get

$$\Lambda^{3(N_c+1)-N_f} = \mathrm{Pf}\,(\langle M^\perp\rangle/2)\,\hat{\Lambda}^{3(N_c-r+1)-(N_f-r)}. \tag{6.4}$$

This relation is valid for any N_c, N_f and $r \le N_c, N_f$. Using (6.4), the superpotential of the low energy theory has the same form (5.11) but with a different coefficient $\hat{A}$ related to A by

$$A = 2^{\frac{r}{N_c+1-N_f}}\hat{A}. \tag{6.5}$$

When $N_f = N_c$, if we choose $r = N_c - 1$, then $\frac{1}{2}\dim\hat{C} = \frac{1}{2}\dim\hat{F} = 1$. Due to the accidental isomorphism $Sp(1) \cong SU(2)$ and their defining representations, the low energy theory is identical to the $SU(2)$ supersymmetric gauge theory with a single flavor of quark. We know from the $SU(2)$ theory that $\hat{A} = 1$ under the $\overline{\mathrm{DR}}$ scheme. So $A = 2^{N_c-1}$ when $N_f = N_c$.

$N_f < N_c$ and the pure gauge theory

To derive the superpotential for smaller values of N_f but with a fixed N_c, we need to apply a gauge invariant mass deformation. We perturb the theory by adding at the tree level a superpotential $W_{\mathrm{tree}} = \langle M, m\rangle$, where $m \in \wedge^2 F$. Let $r = \frac{1}{2}\,\mathrm{rank}\,m$. We write $F = \hat{F} \oplus \hat{F}^\perp$ such that $\frac{1}{2}\dim\hat{F} = N_f - r$ and the projection $m^\perp \in \wedge^2\hat{F}^\perp$ of m is nondegenerate. The gauge group of the low energy theory is still $Sp(C)$, but the flavor space is $\hat{F}$. Let $\hat{\Lambda}$ be the scale. When $\mu >> (\mathrm{Pf}\,m^\perp)^{1/r}$, $g(\mu)$ runs according to (6.2). However when $\mu << (\mathrm{Pf}\,m^\perp)^{1/r}$, there are $N_f - r$ flavors of light quarks, and we get

$$(\hat{\Lambda}/\mu)^{3(N_c+1)-(N_f-r)} = e^{-\frac{8\pi^2}{g^2(\mu)}}. \tag{6.6}$$

Matching the coupling constant $g(\mu)$ in (6.2) and (6.6) at $\mu = (\mathrm{Pf}\,m^\perp)^{1/r}$, we get

$$\hat{\Lambda}^{3(N_c+1)-(N_f-r)} = \mathrm{Pf}\,m^\perp\Lambda^{3(N_c+1)-N_f}. \tag{6.7}$$

This relation is valid for any N_c, N_f and $r \le N_f$.

If $N_f \le N_c$ in the ultraviolet theory, the total superpotential after the mass deformation is

$$W = A\left(\frac{\Lambda^{3(N_c+1)-N_f}}{\mathrm{Pf}\,M}\right)^{\frac{1}{N_c+1-N_f}} + \langle M, m\rangle. \tag{6.8}$$

We eliminate the heavy fields in M by minimizing the bosonic potential. At the extremum of the superpotential (6.8), M is of the block-diagonal form (6.1), where

$$\langle M^\perp\rangle = \left(\frac{A}{N_c+1-N_f}\right)^{\frac{N_c+1-N_f}{N_c+1-(N_f-r)}}\left(\frac{\mathrm{Pf}\,m^\perp\Lambda^{3(N_c+1)-N_f}}{\mathrm{Pf}\,\hat{M}}\right)^{\frac{1}{N_c+1-(N_f-r)}}(m^\perp)^{-1}. \tag{6.9}$$

Here $(m^\perp)^{-1} \in \wedge^2\hat{F}^{\perp *}$ is induced by $m^\perp \in \wedge^2\hat{F}^\perp$. We compute the superpotential of the light mesons $\hat{M}$ by substituting in (6.8) the solution of $\langle M^\perp\rangle$. Using

$\langle (m^\perp)^{-1}, m^\perp \rangle = r$, $d\operatorname{Pf} M = \operatorname{Pf} M \langle dM, M^{-1} \rangle$ and (6.7), we deduce that the superpotential has precisely the same form as (5.11), with the coefficient

$$\hat{A} = (N_c + 1 - (N_f - r)) \left(\frac{A}{N_c + 1 - N_f} \right)^{\frac{N_c+1-N_f}{N_c+1-(N_f-r)}}. \tag{6.10}$$

If we have $N_f = N_c$ to begin with, then $A = 2^{N_c-1}$ and $\hat{A} = (N_c + 1 - (N_f - r))2^{\frac{N_c-1}{N_c+1-(N_f-r)}}$. Equivalently, for a theory with $N_f \leq N_c$, the coefficient A in (5.11) is given by

$$A = (N_c + 1 - N_f)2^{\frac{N_c-1}{N_c+1-N_f}}. \tag{6.11}$$

In all cases of $N_f \leq N_c$, the superpotential is nonzero. When $1 \leq N_f \leq N_c$, the classical vacuum degeneracy is lifted by W and the quantum theory has no vacuum.

In the pure $Sp(C) \cong Sp(N_c)$ gauge theory, $N_f = 0$, and the superpotential

$$W = (N_c + 1)2^{\frac{N_c-1}{N_c+1}} \Lambda^3 = (N_c + 1)\mu^3 2^{\frac{N_c-1}{N_c+1}} e^{\frac{2\pi i}{N_c+1}\tau}. \tag{6.12}$$

This is $(N_c + 1)$-valued, corresponding to the fact that there are $N_c + 1$ vacua in quantum theory. In particular, when $N_c = 1$, (6.12) agrees with the result for $SU(2)$ gauge theory. Notice also that the superpotential W is a constant: although nonzero, it does not contribute to the effective Lagrangian. Nevertheless, the result (6.12) implies gaugino condensation. We consider an extended theory in which the parameter τ becomes a superfield and assume that supersymmetry is unbroken after quantization. The partition function

$$Z(\tau) = \int DA\, D\lambda\, e^{iS(A,\lambda,\tau)} = e^{i\int d^4x d^2\theta\, W(\tau)} \tag{6.13}$$

is holomorphic in τ and $W(\tau)$ has the same form as (6.12). Since the Lagrangian density of the pure gauge theory is

$$\mathcal{L} = \frac{1}{4\pi i} \int d^2\theta\, \tau\, \operatorname{tr} W_\alpha W^\alpha + \text{c.c.}, \tag{6.14}$$

where $W_\alpha = \lambda_\alpha + \cdots$ is the superfield strength, we get

$$\begin{aligned} \langle \lambda\lambda \rangle &= 4\pi \frac{\delta}{\delta F_\tau} \log Z(\tau) = 4\pi \frac{\partial}{\partial F_\tau} \int d^2\theta\, W(\tau) \\ &= 4\pi \frac{\partial}{\partial \tau} W(\tau) = 2^{4-\frac{2}{N_c+1}} \pi^2 i \Lambda^3 = 2^{4-\frac{2}{N_c+1}} \pi^2 i \mu^3 e^{\frac{2\pi i}{N_c+1}\tau}. \end{aligned} \tag{6.15}$$

Again, this is $(N_c + 1)$-valued.

$N_f = N_c + 1$

In the above mass deformation of theories with $N_f \leq N_c$, if $r = N_f$, then $\hat{F}^\perp = F$, $m^\perp = m$ and $M^\perp = M$. The extremum of the total superpotential (6.8) is reached at

$$\langle M \rangle = 2^{\frac{N_c-1}{N_c+1}} (\Lambda^{3(N_c+1)-N_f} \operatorname{Pf} m)^{\frac{1}{N_c+1}}\, m^{-1}. \tag{6.16}$$

Just as in the supersymmetric $SU(N_c)$ gauge theories, the above result is valid in the range $N_f \geq N_c + 1$. For $N_f = N_c + 1$, the solution (6.16) satisfies

$$\operatorname{Pf} M = 2^{N_c-1} \Lambda^{2(N_c+1)}. \tag{6.17}$$

This is independent of m, and shall remain valid in the absence of the mass term. We claim that at low energies the quantum theory is described by M subject to the constraint (6.17). To check that the theory reduces to $N_f = N_c$ via mass deformation, we add a mass term $\langle M, m\rangle$ with $\frac{1}{2}\operatorname{rank} m = 1$. Then $\frac{1}{2}\dim \hat{F} = N_c$. Because of the constraint (6.17), we have

$$\langle M, m\rangle = \frac{2^{N_c-1} m^{\perp} \Lambda^{2(N_c+1)}}{\operatorname{Pf} \hat{M}} = 2^{N_c-1} \frac{\hat{\Lambda}^{2N_c+3}}{\operatorname{Pf} \hat{M}}, \tag{6.18}$$

where the last equality follows from (6.7) when $N_f = N_c + 1$, $r = 1$. This is the superpotential (5.11) with the correct coefficient (6.11) when $N_f = N_c$.

The quantum moduli space $\mathcal{M}_{\mathrm{q}}$ is smooth. There are no additional massless fields other than M subject to the constraint (6.17). The absence of massless gluons is known as the confinement of color charges. We check the consistency of these postulates using 't Hooft's anomaly matching conditions. Each point $M_0 \in \Lambda^2 F^*$ on $\mathcal{M}_{\mathrm{q}}$ is a symplectic form on F. Let $Sp(F, \mathbb{C})$ be the (complex) symplectic group preserving M_0. The most symmetric points on $\mathcal{M}_{\mathrm{q}}$ are those compatible with the Hermitian structure on F. There the global symmetry $SU(F) \times U(1)_R$ is broken to $Sp(F) \times U(1)_R$, where the (compact) symplectic group $Sp(F) = Sp(F, \mathbb{C}) \cap U(F)$ depends on the choice of M_0. Notice that when $N_f = N_c + 1$, $U(1)_X = U(1)_R$ is anomaly-free. In the ultraviolet theory, the massless fermions are ψ_Q and λ. In the infrared theory, they are the fermionic partners ψ_M of M in $T_{M_0}\mathcal{M}_{\mathrm{q}}$. Under the unbroken global symmetry, these light fermions transform according to

(6.19)

		$Sp(F)$	$\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*		$-\frac{N_c+1}{N_f} = -1$
	λ	$\mathbf{1}$		1
infrared	ψ_M	$\Lambda^2 F^* \ominus \mathbf{1}$		$1 - 2\frac{N_c+1}{N_f} = -1$

Here and below, $\mathbf{1}$ denotes the trivial representation. The nonzero combinations to 't Hooft's condition are given by

(6.20)

	ultraviolet	infrared
$\mathfrak{sp}(F)^2\mathfrak{u}(1)_R$	$2N_c c_2(F)\cdot(-1)$	$c_2(\Lambda^2 F \ominus \mathbf{1})\cdot(-1)$
$\mathfrak{u}(1)_R$	$4N_c N_f\,(-1) + N_c(2N_c+1)\,1$	$(N_f(2N_f-1)-1)\,(-1)$
$\mathfrak{u}(1)_R^3$	$4N_c N_f\,(-1)^3 + N_c(2N_c+1)\,1^3$	$(N_f(2N_f-1)-1)\,(-1)^3$

To calculate $c_2(\Lambda^2 F \ominus \mathbf{1})$, we note that the root system of $\mathfrak{sp}(N_f)$ is $\{\epsilon_i \pm \epsilon_j, 2\epsilon_i \mid 1 \le i \ne j \le N_f\}$ and the positive roots are $\epsilon_i \pm \epsilon_j$ $(1 \le i \le j \le N_f)$. The highest weights of the representations of F and $\Lambda^2 F \ominus \mathbf{1}$ are $\lambda = \epsilon_1$ and $\lambda_2 = \epsilon_1 + \epsilon_2$, respectively. Therefore $c_2(\Lambda^2 F \ominus \mathbf{1}) = \frac{(\lambda_2+2\rho,\lambda_2)\,\dim(\Lambda^2 F \ominus \mathbf{1})}{(\lambda_1+2\rho,\lambda_1)\,\dim F} c_2(F) = 2(N_f-1)c_2(F)$, where $\rho = \sum_{i=1}^{N_f}(N_f+1-i)\epsilon_i$ is half of the sum of the positive roots. The two columns match when $N_f = N_c + 1$.

Finally, we propose a global anomaly matching condition and show that it is satisfied in the above situation. Consider two theories with the same (anomaly-free) global symmetry group H but with different contents of light fermionic fields, which form representations R and R' of H, respectively. If the two theories describe the same low energy physics, then if H is hypothetically gauged, the determinant line

bundles $\mathrm{Det}\not{D}^+_R$ and $\mathrm{Det}\not{D}^+_{R'}$ of the Dirac operators twisted by R and R' shall be isomorphic. 't Hooft's anomaly matching condition checks whether the curvatures of the two bundles are the same. This is a stringent condition for the two theories to describe the same low energy physics. However, the two bundles can still differ by a flat bundle, which can be nontrivial if $\pi_4(H) \neq 1$. If the representation $R^* \oplus R'$ of H is real, then the determinant line bundle $\mathrm{Det}\not{D}^+_{R^*\oplus R'} \cong (\mathrm{Det}\not{D}^+_R)^{-1} \otimes \mathrm{Det}\not{D}^+_{R'}$ is trivial. Therefore the reality of the representation $R^* \oplus R'$ is a sufficient condition for global anomaly matching. In the above problem, suppose only the $Sp(F)$ factor of the global symmetry is gauged. In the ultraviolet theory, the representation is an even number $(2N_c)$ of copies of F^* and the trivial representations. In the infrared theory, the representation $\wedge^2 F^* \ominus \mathbf{1} \subset F^* \otimes F^*$ is real because F^* is quaternionic. So the global anomalies match.

$N_f = N_c + 2$

When $N_f = N_c + 2$, we obtain from (6.16) that $\wedge^{N_c+1} M \to 0$ as $m \to 0$. So there is no quantum correction to the classical moduli space $\mathcal{M}_c$. We claim that in the low energy theory, all components of M become dynamical and that there is a superpotential

$$W = -\frac{\mathrm{Pf}\, M}{2^{N_c-1}\Lambda^{2N_c+1}}. \tag{6.21}$$

Indeed, varying W with respect to M, we get the constraint (5.2). If we add a mass term $W_{\text{tree}} = \langle M, m\rangle$ with $\frac{1}{2}\operatorname{rank} m = 1$, then $\frac{1}{2}\dim \hat{F} = N_c + 1$. At the extremum of $W + W_{\text{tree}}$, M is of the block-diagonal form (6.1). From varying $M^\perp$, we get

$$\mathrm{Pf}\, \hat{M} = 2^{N_c-1} m^\perp \Lambda^{2N_c+1}. \tag{6.22}$$

By (6.7), this is exactly the constraint (6.17).

The most symmetric point on $\mathcal{M}_c$ is $M = 0$, where the full global symmetry $SU(F) \times U(1)_R$ is unbroken. This is also the most singular point. The singularity is the result of additional massless fields. In fact all the light fields in M are massless at the origin. We check this using 't Hooft's anomaly matching condition. Under the global symmetry, the light fermions in the ultraviolet and the infrared theories transform as

(6.23)

		$SU(F)$	$\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*		$-\frac{N_c+1}{N_f}$
	λ	$\mathbf{1}$		1
infrared	ψ_M	$\wedge^2 F^*$		$1 - 2\frac{N_c+1}{N_f}$

The nonzero combinations to be matched are

(6.24)

	ultraviolet	infrared
$\mathfrak{su}(F)^3$	$2N_c\,(-d_F)$	$-d_{\wedge^2 F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$2N_c c_2(F) \cdot (-\frac{N_c+1}{N_f})$	$c_2(\wedge^2 F) \cdot (1 - 2\frac{N_c+1}{N_f})$
$\mathfrak{u}(1)_R$	$4N_cN_f\,(-\frac{N_c+1}{N_f}) + N_c(2N_c+1)\,1$	$N_f(2N_f-1)\,(1 - 2\frac{N_c+1}{N_f})$
$\mathfrak{u}(1)_R^3$	$4N_cN_f\,(-\frac{N_c+1}{N_f})^3 + N_c(2N_c+1)\,1^3$	$N_f(2N_f-1)\,(1 - 2\frac{N_c+1}{N_f})^3$

We claim that $c_2(\wedge^2 F) = 2(N_f - 1)c_2(F)$ and $d_{\wedge^2 F} = 2(N_f - 2)d_F$. Indeed, let $\epsilon_1, \ldots, \epsilon_{2N_f}$ be the weights of the fundamental representation of $SU(F)$. Since $\sum_{i=1}^{2N_f} \epsilon_i = 0$, we have $\sum_{i<j}(\epsilon_i + \epsilon_j)^2 = 2(N_f - 1)\sum_{i=1}^{2N_f} \epsilon_i^2$ and $\sum_{i<j}(\epsilon_i + \epsilon_j)^3 = 2(N_f - 2)\sum_{i=1}^{2N_f} \epsilon_i^3$. Therefore the two columns match when $N_f = N_c + 2$.

Solution to Problem 4

$N_f \leq N_c - 5$ and pure gauge theory

When $N_f \leq N_c - 5$, it is possible to have a dynamically generated superpotential of the form (5.22). Since most of the analysis is parallel to the $Sp(N_c)$ gauge theories, we only give an outline here.

Suppose M acquires a large expectation value $\langle M \rangle$ of rank r. Let $\hat{F} = \ker\langle M \rangle$. Then $F = \hat{F} \oplus \hat{F}^\perp$ and the restriction $\langle M^\perp \rangle \in S^2\hat{F}^{\perp *}$ of $\langle M \rangle$ to $\hat{F}^\perp$ is nondegenerate. The low energy theory has $N_c - r$ colors and $\dim \hat{F} = N_f - r$ flavors. So $b_0 = 3(N_c - r - 2) - (N_f - r)$. Matching the coupling constant at $\mu = (\det\langle M^\perp \rangle)^{1/2r}$, we obtain

$$\Lambda^{3(N_c-2)-N_f} = \det\langle M^\perp \rangle \hat{\Lambda}^{3(N_c-r-2)-(N_f-r)}. \tag{6.25}$$

In fact, this relation remains valid for all values of N_c, N_f as long as $N_c - r \geq 5$. For future needs, we also consider here the cases $N_c - r \leq 4$. For $N_c - r = 4$, in the low energy $SO(4)$ theory, both scales $\hat{\Lambda}_\pm$ of the $SU(2)_\pm$ factors are given by (6.25). If $N_c - r = 3$, we derive the relation in two stages: first from $N_c \geq 5$ to $N_c = 4$, then from $N_c = 4$ to $N_c = 3$. In the second stage, $N_c = 4$, $r = 1$. The low energy theory has 3 colors and $N_f - 1$ flavors. Above the scale $\hat{\Lambda}$, the gauge group is $SU(2)_+ \times SU(2)_-$; the coupling constants $g_\pm(\mu)$, or $\tau_\pm(\mu)$ run according (4.2) with $b_0 = 6 - N_f$. Since $SU(2)$ is diagonally embedded, its coupling constant is given by $\tau(\mu) = \tau_+(\mu) + \tau_-(\mu)$. Therefore the scale is $(\hat{\Lambda}_+^{6-N_f}\hat{\Lambda}_-^{6-N_f})^{1/2(6-N_f)} = (\hat{\Lambda}_+\hat{\Lambda}_-)^{1/2}$. At low energies, the $SU(2)$ gauge theory has $b_0 = 6 - 2(N_f - 1)$. Matching the coupling constant $g(\mu)$ at $\mu = \frac{1}{2}\langle M^\perp \rangle$, we get

$$4\hat{\Lambda}_+^{6-N_f}\hat{\Lambda}_-^{6-N_f} = \langle M^\perp \rangle^2 \hat{\Lambda}^{6-2(N_f-1)}. \tag{6.26}$$

Combining (6.25) and (6.26), we get, for any $N_c \geq 4$, N_f and $r = N_c - 3$,

$$4(\Lambda^{3(N_c-2)-N_f})^2 = (\det\langle M^\perp \rangle)^2 \hat{\Lambda}^{6-2(N_f-N_c+3)}. \tag{6.27}$$

When $N_f \leq N_c - 5$ and $N_c - r \geq 5$, a comparison of the superpotential (5.22) above and below $\mu = (\det\langle M^\perp \rangle)^{1/2r}$ yields $A = \hat{A}$. If $N_f = N_c - 5$ and $r = N_f$, the low energy theory is a pure $SO(5) \cong Sp(2)/\mathbb{Z}_2$ gauge theory. Using (6.11), we get $\hat{A} = 3 \cdot 2^{\frac{1}{3}}$. So $A = 3 \cdot 2^{\frac{1}{3}}$ when $N_f = N_c - 5$.

Now we add a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m \rangle$ with $\operatorname{rank} m = r$. Again $F = \hat{F} \oplus \hat{F}^\perp$. Here $\dim \hat{F} = N_c - r$ and the projection $m^\perp \in S^2\hat{F}^\perp$ of m is nondegenerate. The low energy theory has the same number of colors and $N_f - r$ flavors. So $b_0 = 3(N_c - 2) - (N_f - r)$. Matching the coupling constant at $\mu = (\det m^\perp)^{1/r}$, we obtain

$$\hat{\Lambda}^{3(N_c-2)-(N_f-r)} = \det m^\perp \Lambda^{3(N_c-2)-N_f}. \tag{6.28}$$

This is valid for $N_c \geq 5$ and all N_f. When $N_c = 4$, the scales $\hat{\Lambda}_\pm$ of the $SU(2)_\pm$ factors are related to the scales $\Lambda_\pm$ of the ultraviolet theory by the same formula.

When $N_c = 3$, (6.28) should be replaced by

$$\hat{\Lambda}^{6-2(N_f-r)} = (\det m^\perp)^2 \Lambda^{6-2N_f}. \tag{6.29}$$

When $N_f \le N_c - 5$, the superpotential W before mass deformation is given by (5.22). Varying the heavy modes in M, the extremum of the total superpotential $W + W_{\text{tree}}$ is reached when M is of the diagonal form (6.1) with

(6.30)

$$\langle M^\perp \rangle = \left(\frac{2A}{N_c - 2 - N_f}\right)^{\frac{N_c-2-N_f}{N_c-2-(N_f-r)}} \left(\frac{\det m^\perp \Lambda^{3(N_c-2)-N_f}}{\det \hat{M}}\right)^{\frac{1}{N_c-2-(N_f-r)}} (m^\perp)^{-1}.$$

In terms of the of the light fields $\hat{M}$, this extremum has the same form as (5.22) with

$$\hat{A} = \frac{N_c - 2 - (N_f - r)}{2} \left(\frac{2A}{N_c - 2 - N_f}\right)^{\frac{N_c-2-N_f}{N_c-2-(N_f-r)}}. \tag{6.31}$$

If we start from $N_f = N_c - 5$, then $A = 3 \cdot 2^{\frac{1}{3}}$ and $\hat{A} = \frac{1}{2}(N_c - 2 - (N_f - r))2^{\frac{4}{N_c-2-(N_f-r)}}$. Equivalently, for a theory with $N_f \le N_c - 5$, the coefficient A in (5.22) is given by

$$A = \tfrac{1}{2}(N_c - 2 - N_f)2^{\frac{4}{N_c-2-N_f}}. \tag{6.32}$$

For pure gauge theories, $N_f = 0$ and $A = \frac{1}{2}(N_c - 2)2^{\frac{4}{N_c-2}}$. As in (6.33), for the pure $SO(N_c)$ gauge theory with $N_c \ge 5$,

$$\langle \lambda\lambda \rangle = 2^{2+\frac{4}{N_c-2}} \pi^2 i \Lambda^3 = 2^{2+\frac{4}{N_c-2}} \pi^2 i \mu^3 e^{\frac{2\pi\imath}{N_c-2}\tau}; \tag{6.33}$$

this is $(N_c - 2)$-valued, corresponding to the $N_c - 2$ vacua. Finally, the result is consistent with another accidental isomorphism $SO(6) \cong SU(4)/\mathbb{Z}_2$. Indeed, when $N_c = 6$, $A = 4$; this agrees with the coefficient of the pure $SU(4)$ gauge theory.

$N_f = N_c - 4$

A generic flat deformation on the moduli space breaks the gauge group to $SO(4) \cong (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$. From (6.25), the scales $\hat{\Lambda}_\pm$ of the two $SU(2)$ gauge theories (without matter) are related to the scale Λ of the ultraviolet theory by

$$\hat{\Lambda}_\pm^6 = \frac{\Lambda^{2(N_c-1)}}{\det M}. \tag{6.34}$$

In the infrared, the superpotential of the two $SU(2)_\pm$ gauge theories are $W_\pm = 2\hat{\Lambda}_\pm^3$, respectively. But both of them are two-valued, corresponding to the two vacua of each $SU(2)$ theory. We fix a square root. Then the superpotential of the original theory is

$$W = 2(\epsilon_+ + \epsilon_-)\left(\frac{\Lambda^{2(N_c-1)}}{\det M}\right)^{1/2}, \tag{6.35}$$

where $\epsilon_+, \epsilon_- = \pm 1$. There are two physically inequivalent branches $\epsilon_+ = \epsilon_-$ and $\epsilon_+ = -\epsilon_-$.

The superpotential (6.35) is compatible with mass deformation. In the branch $\epsilon_+ = \epsilon_-$, it is the continuation to $N_f = N_c - 4$ of (5.22). As before, adding a mass term $\frac{1}{2}\langle M, m \rangle$ of $\operatorname{rank} m = r$ reduces the theory to $N_f = N_c - 4 - r$ with the same superpotential (5.22) in the low energies. In the other branch $\epsilon_+ = -\epsilon_-$, the total

superpotential is $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$. The theory has no vacuum and hence does not contribute to the low energy theory with a smaller N_f.

For $\epsilon_+ = \epsilon_-$, the superpotential (6.35) lifts the classical vacuum degeneracy and the quantum theory has no vacuum. In the branch $\epsilon_+ = -\epsilon_-$, the vacuum degeneracy is not lifted by quantum effects. Classically there is a singularity at $M = 0$, where the full gauge symmetry is restored and all the gluons become massless. We claim that in quantum theory the singularity is smoothed out and the color charges are confined. That is, there are no massless gluons and all the massless fermions are contained in M, even at the origin of $\mathcal{M}$. We check this claim by the 't Hooft and the global anomaly matching conditions. At $M = 0$, the identity component of the unbroken global symmetry is $SU(F) \times U(1)_R$. The light fermions in high and low energy theories transform according to

(6.36)

		$SU(F)$	$U(1)_R$
ultraviolet	ψ_Q	F^*	$-\frac{N_c-2}{N_f}$
	λ	$\mathbf{1}$	1
infrared	ψ_M	S^2F^*	$1-2\frac{N_c-2}{N_f}$

The nonzero combinations to be matched are given by

(6.37)

	ultraviolet	infrared
$\mathfrak{su}(F)^3$	$-N_c d_F$	$-d_{S^2F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$N_c c_2(F)\cdot(-\frac{N_c-2}{N_f})$	$c_2(S^2F)\cdot(1-2\frac{N_c-2}{N_f})$
$\mathfrak{u}(1)_R$	$N_cN_f\,(-\frac{N_c-2}{N_f}) + \frac{1}{2}N_c(N_c-1)\,1$	$\frac{1}{2}N_f(N_f+1)\,(1-2\frac{N_c-2}{N_f})$
$\mathfrak{u}(1)_R^3$	$N_cN_f\,(-\frac{N_c-2}{N_f})^3 + \frac{1}{2}N_c(N_c-1)\,1^3$	$\frac{1}{2}N_f(N_f+1)\,(1-2\frac{N_c-2}{N_f})^3$

Let ϵ_i $(i = 1,\dots,N_f)$ be the weights in the defining representation F. Using $\sum_{i=1}^{N_f}\epsilon_i = 0$, it is easy to check $\sum_{i\le j}(\epsilon_i+\epsilon_j)^2 = (N_f+2)\sum_{i=1}^{N_f}\epsilon_i^2$ and $\sum_{i\le j}(\epsilon_i+\epsilon_j)^3 = (N_f+4)\sum_{i=1}^{N_f}\epsilon_i^3$. So $c_2(S^2F) = (N_f+2)c_2(F)$ and $d_{S^2F} = (N_f+4)d_F$. It is now straightforward to check that the two columns of (6.37) match when $N_f = N_c - 4$, fulfilling 't Hooft's criterion. For $N_f = 2$, $SU(F) \cong Sp(1)$. In the ultraviolet theory, the number of copies of F^* is even ($N_c = 6$). In the infrared, the representation $S^2F^* \subset F^* \otimes F^*$ is real. So the global anomalies also match.

$N_f = N_c - 3$

We first find the superpotential by flat deformations on the moduli space. A generic expectation value of Q breaks the gauge group to $SO(3)$. Let $\hat{F} \subset F$ be a subspace of dimension 1. We consider the limit where a large expectation value $\langle M^\perp\rangle \in S^2\hat{F}^{\perp *}$ with rank $N_f - 1$ breaks the theory to $SO(4) \cong (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$ with one quark $\hat{Q} \in \mathrm{Hom}(\hat{F}, C)$, which further breaks the gauge group to a diagonal subgroup $SU(2)/\mathbb{Z}_2$.[1] In the first stage of deformation, from (6.25), the scales $\hat{\Lambda}_\pm$

[1]The decomposition of $SO(4)$ into $SU(2)_+ \times SU(2)_-$ is canonical. The isotropy group of $\hat{Q} \in \mathrm{Hom}(\hat{F}, C)$ (in $SO(4)$) and its isomorphisms onto $SU(2)_\pm$ depend on $\hat{Q}$.

of the $SU(2)_+ \times SU(2)_-$ theory are given by

$$\hat{\Lambda}_\pm^5 = \frac{\Lambda^{2N_c-3}}{\det\langle M^\perp\rangle}. \tag{6.38}$$

The effective $SU(2)_\pm$ theories have instanton generated superpotentials

$$\hat{W}_\pm = 2\frac{\hat{\Lambda}_\pm^5}{\det \hat{M}} = 2\frac{\Lambda^{2N_c-3}}{\det M}. \tag{6.39}$$

In the second stage, we have the low energy pure $SU(2)$ gauge theory whose scale $\hat{\Lambda}$ is given by (6.27). It has a superpotential

$$\hat{W} = 2\hat{\Lambda}^3 = 4\epsilon\frac{\Lambda^{2N_c-3}}{\det M}, \tag{6.40}$$

where $\epsilon = \pm 1$ comes from taking the square root of (6.27) and labels the two vacua of the pure $SU(2)$ theory. Combining (6.39) and (6.40), we find that the superpotential of the $N_f = N_c - 3$ theory is

$$W = \hat{W}_+ + \hat{W}_- + \hat{W} = 4(1+\epsilon)\frac{\Lambda^{2N_c-3}}{\det M}. \tag{6.41}$$

The first term in (6.41) is due to the instanton effect whereas the second term comes from gaugino condensation in the unbroken $SU(2)$ theory. As in $SU(N_c)$ and $Sp(N_c)$ gauge theories, the superpotential (6.41) is proportional to the exponential of the instanton action, and it is possible to form a diagram with two external fermion lines using the instanton vertex. The novelty here is that the unbroken gauge group $SU(2)$ is nonabelian. However, when the gauge group is broken to the diagonal SU(2), an instanton in one $SU(2)_\pm$ cannot be rotated into the unbroken group. Therefore the instanton computation is still reliable because there is no infrared divergence.

We check that the result (6.41) is consistent with mass deformation. The theory has two branches: $\epsilon = 1$ and $\epsilon = -1$. After adding a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$, the two branches should correspond to those of $N_f = N_c - 4$. In fact, when $\epsilon = 1$, the total superpotential (6.41) is the continuation of (5.22), just as (6.35) when $\epsilon_+ = \epsilon_-$. So the extremum of $W + W_{\text{tree}}$ matches (6.35) in the branch $\epsilon_+ = \epsilon_-$. In particular, from (6.30), the extremum is reached when

$$\langle M^\perp\rangle = 4\left(\frac{\Lambda^{2N_c-3}}{m^\perp \det \hat{M}}\right)^{1/2}. \tag{6.42}$$

The ambiguity in the sign of the square root corresponds to the two possibilities $\epsilon_+ = \epsilon_- = \pm 1$. In the other branch $\epsilon = -1$, we claim that the theory has additional light fields $q \in F \otimes \wedge^{N_f} F$ coupling to M and that the superpotential near $M = 0$ behaves as

$$W \sim \frac{1}{2\mu} M(q,q), \tag{6.43}$$

where $\mu \in (\wedge^{N_f} F)^{\otimes 2}$ is a parameter of mass dimension 1. The R-charges of M, q are $-\frac{2}{N_f}$, $1 + \frac{1}{N_f}$, respectively. This is consistent with requirement that W has R-charge 2. Upon mass deformation, we write $F = \hat{F} \oplus \hat{F}^\perp$ as before. The extremum of $W + W_{\text{tree}}$ is reached at $q^\perp = \pm(-m\mu)^{1/2}$, $\hat{q} = 0$, $M(q,\cdot) = 0$. The light fields are in $M = \hat{M} \in S^2\hat{F}$. This matches the low energy behavior of $N_f = N_c - 4$. The signs of $q^\perp$ correspond to the choices of ϵ_+, ϵ_- in $\epsilon_+ = -\epsilon_- = \pm 1$.

We check the postulate on the existence of q using the 't Hooft and global anomaly matching conditions. In the low energy theory, the global symmetry $SU(F) \times U(1)_R$ is unbroken at $M = 0$. Also, at $M = 0$ the fields q are massless. The R-charge of ψ_q is $\frac{1}{N_f}$. So their contribution to the left column of (6.37) is

(6.44)

	infrared: additional contribution from q
$\mathfrak{su}(F)^3$	d_F
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$c_2(F) \cdot \frac{1}{N_f}$
$\mathfrak{u}(1)_R$	$N_f(\frac{1}{N_f})$
$\mathfrak{u}(1)_R^3$	$N_f(\frac{1}{N_f})^3$

It is straightforward to check that for $N_f = N_c - 3$, each term in the left column of (6.37) is equal to the sum of the corresponding one in the right column and that in (6.44). When $N_f = 2$, the number of copies of F^* in the ultraviolet theory is odd ($N_c = 5$). In the infrared theory, the representation S^2F^* is real, but there is an additional quark q in the representation F. Therefore the global anomalies match.

$N_f = N_c - 2$

Classically, a generic expectation value $\langle M \rangle$ breaks the gauge group to the Abelian group $SO(2) \cong U(1)$. At the singularities, where $\langle M \rangle$ is of a lower rank, the unbroken gauge group is larger and hence there are more massless gauge bosons. At the quantum level, since no superpotential is generated, the vacuum degeneracy is not lifted. In fact when $N_f = N_c - 2$, $U(1)_X = U(1)_R$ is anomaly-free and M is neutral, whereas a superpotential should have R-charge 2. We expect the classical picture to be valid when $\langle M \rangle$ is large. There the theory is in the Higgs phase. Modification is needed near the singularities due to strong coupling effects. In particular, the quantum moduli space may have different singularities.

Over a smooth point M in the moduli space, we have a $U(1)$ gauge theory whose scale is set by M. By $N = 1$ supersymmetry, the effective gauge coupling $\tau_{\text{eff}}(M)$ is holomorphic in M. However, due to the periodicity in the (effective) θ-angle and the electric-magnetic duality, τ_{eff} needs not be single-valued in M; it is defined up to fractional linear transformations of $SL(2,\mathbb{Z})$. At the singularities, τ_{eff} is not defined because there are extra massless fields. Consequently, there are possible nontrivial $SL(2,\mathbb{Z})$ monodromies around the singularities. These information can be summarized by a family of elliptic curves over the smooth part $\mathcal{M}_{\text{smooth}}$ of the moduli space whose fibre over M has modular parameter $\tau_{\text{eff}}(M)$. The j-invariant of the family is a well-defined holomorphic function on $\mathcal{M}_{\text{smooth}}$. In our situation, the j-invariant is a function of $U = \det M$ because of the (anomaly-free) $SU(F)$ symmetry. We construct a family of elliptic curves over the smooth part of $\mathcal{U} = (\wedge^{N_f} F^*)^{\otimes 2}$; the family over $\mathcal{M}_{\text{smooth}}$ is then the pullback by the determinant map.

We consider the range where M is of the form (6.1): the large expectation value $\langle M^\perp \rangle$ of rank $N_c - 3$ breaks the gauge group to $SO(3) \cong SU(2)/\mathbb{Z}_2$, which is further broken to $U(1)$ by $\hat{M}$. The $SO(3)$ theory in the intermediate stage has a single flavor. It is the $N = 2$ theory which gives rise to Donaldson and Seiberg-Witten

invariants. Using (6.27), the scale $\hat{\Lambda}$ is given by

$$\hat{\Lambda}^4 = 4\left(\frac{\Lambda^{2N_c-4}}{\det\langle M^\perp\rangle}\right)^2. \tag{6.45}$$

The family of elliptic curves is

$$y^2 = (x-\hat{\Lambda}^2)(x+\hat{\Lambda}^2)(x-u), \tag{6.46}$$

where the parameter u is related to our theory by $u = \hat{M}/4$. After making a change of variables $x \mapsto x/4\det\langle M^\perp\rangle$, $y \mapsto y/(4\det\langle M^\perp\rangle)^{3/2}$, (6.46) becomes

$$y^2 = (x-8\Lambda^{2N_c-4})(x+8\Lambda^{2N_c-4})(x-U). \tag{6.47}$$

This is a good approximation for large U. When U goes around ∞, the monodromy is PT^{-2}. For large U, we have

$$e^{2\pi i\tau_{\text{eff}}(M)} = \left(\frac{\Lambda}{U^{1/2(N_c-2)}}\right)^{4N_c-8}. \tag{6.48}$$

The exact family of curves which agrees with (6.48) for large U must be of the form

$$y^2 = (x-8\Lambda^{2N_c-4})(x+8\Lambda^{2N_c-4})(x-U+\alpha\Lambda^{2N_c-4}). \tag{6.49}$$

The curve is singular at $U = (\alpha \pm 8)\Lambda^{2N_c-4}$. To determine α, we add a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$. Then the vacuum degeneracy is lifted except at two singularities, which shall correspond to the two branches $\epsilon = \pm 1$ of the $N_f = N_c - 3$ theory. The $\epsilon = 1$ branch has a superpotential

$$\tfrac{1}{2}M^\perp m^\perp = 8\frac{\hat{\Lambda}^{2N_c-3}}{\det\hat{M}}. \tag{6.50}$$

So $U = 16\hat{\Lambda}^{2N_c-3}/m^\perp = 16\Lambda^{2N_c-4}$. The $\epsilon = -1$ branch has $\det\hat{M} = 0$, hence $U = 0$. Therefore we have a unique solution $\alpha = 8$. After making another change of variable $x \mapsto x - 8\Lambda^{2N_c-4}$, we get the family of elliptic curves of our theory

$$y^2 = x(x-16\Lambda^{2N_c-4})(x-U). \tag{6.51}$$

The monodromies when U goes around $16\Lambda^{2N_c-4}$ and 0 are $(T^2S)T(T^2S)^{-1}$ and ST^2S^{-1}, and the fields that become massless at $U = 16\Lambda^{2N_c-4}$ and $U = 0$ are dyons and magnetic monopoles, respectively. Unlike the $N = 2$ gauge theory, the theory here does not have the $\mathbb{Z}_2$ symmetry that relates the two singular components in the moduli space. We will see that the physics in these branches are inequivalent.

Taking M around the surface $\det M = 16\Lambda^{2N_c-4}$ in $\mathcal{M}$, U goes around $16\Lambda^{2N_c-4}$ once, so the monodromy is $(T^2S)T(T^2S)^{-1}$. From the one-loop β-function, we conclude that a pair of dyons $E^\pm$ of charges ± 1 become massless at $U = 16\Lambda^{2N_c-4}$. The effective action when U is near $16\Lambda^{2N_c-4}$ is of the form

$$W \sim (U - 16\Lambda^{2N_c-4})\, E^+ \cdot E^-, \tag{6.52}$$

where $\cdot$ is the pairing between fields of opposite $U(1)$ charges. After adding the mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$, the $M^\perp$ equation of motion implies that $E^+ \cdot E^- = -m^\perp/2\det\hat{M}$. The light fields are in $\hat{M}$ and the superpotential (6.50) agrees with the $\epsilon = 1$ branch of $N_f = N_c - 3$. The $U(1)$ gauge bosons become heavy due to the expectation values of $E^\pm$. But since $E^\pm$ are dyons, this is called oblique confinement.

We check 't Hooft's anomaly condition at a generic point M_0 on the moduli space. The light fermions in the infrared theory are the mesons M, the photino λ_0 and, if $\det M = 16\Lambda^{2N_c-4}$, the dyons $E^{\pm}$. When M_0 is nondegenerate and is compatible with the Hermitian form on F, the global symmetry is broken to $SO(F) \times U(1)_R$. The light fermions in the high and low energy theories transform as

(6.53)

		$SO(F)$	$\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*		$-\frac{N_c-2}{N_f} = -1$
	λ	**1**		1
infrared	ψ_M	S^2F^*		$1 - 2\frac{N_c-2}{N_f} = -1$
	λ_0	**1**		1
	$(E^{\pm})$	**1**		0

With or without $E^{\pm}$, the nonzero combinations to be matched are

(6.54)

	ultraviolet	infrared
$\mathfrak{so}(F)^2\mathfrak{u}(1)_R$	$N_c c_2(F) \cdot (-1)$	$c_2(S^2F) \cdot (-1)$
$\mathfrak{u}(1)_R$	$N_c N_f\ (-1) + \frac{1}{2}N_c(N_c-1)\ 1$	$\frac{1}{2}N_f(N_f+1)\ (-1) + 1$
$\mathfrak{u}(1)_R^3$	$N_c N_f\ (-1)^3 + \frac{1}{2}N_c(N_c-1)\ 1^3$	$\frac{1}{2}N_f(N_f+1)\ (-1)^3 + 1^3$

It is easy to check that the two columns match when $N_f = N_c - 2$. The global anomalies also match because all the representations of $SO(F)$ are of real type here. We compare this and the $Sp(N_c)$ theory with $N_f = N_c + 1$. There, the meason M is constrained on the surface (6.17) and there is no massless photon. In the low energies, the fermion ψ_M satisfies a traceless conditon. In the present theory, M can be deformed away from the singular branch and hence there is no traceless condition on ψ_M.

At the other singularity $U = 0$, a non-trivial loop around a point M_0 with $\operatorname{rank} M_0 = r$ is mapped to one that goes $N_f - r$ times around 0. So the monodromy is $ST^{2(N_f-r)}S^{-1}$. Therefore the number of pairs of monopoles becoming massless at M_0 is $N_f - r$. This suggests a superpotential

$$W \sim \frac{1}{2\mu}\langle M, \tilde{Q}^+ \cdot \tilde{Q}^- \rangle, \tag{6.55}$$

where μ is of mass dimension 1 and $\tilde{Q}^{\pm}$ represent magnetic monopoles that become massless at the singularity. After adding a mass term $W = \frac{1}{2}\langle M, m \rangle$ with $\operatorname{rank} m = 1$, the fermionic content reduces to the $\epsilon = -1$ branch of $N_f = N_c - 3$. Again, the expectation values of $\tilde{Q}^{\perp}$ lift the $U(1)$ gauge boson. Consequently we see confinement in the original description. The light superfield q in the $N_f = N_c - 3$ theory can be constructed by

$$q = \frac{1}{2\sqrt{m^{\perp}\mu}}(\hat{\tilde{Q}}^+ \cdot (\tilde{Q}^-)^{\perp} - (\tilde{Q}^+)^{\perp} \cdot \hat{\tilde{Q}}^-). \tag{6.56}$$

We check 't Hooft's anomaly matching condition at $M = 0$. In addition to M, the fermionic component $\psi_{\tilde{Q}^\pm}$ of $\tilde{Q}^\pm$ and the photino λ_0 transform as

(6.57)

	$SU(F)$	$\times$	$U(1)_R$
$\psi_{\tilde{Q}^\pm}$	F		0
λ_0	$\mathbf{1}$		1

Their contribution to the anomaly is

(6.58)

	infrared: additional contribution from λ_0, $\tilde{Q}^\pm$
$\mathfrak{su}(F)^3$	$2d_F$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	0
$\mathfrak{u}(1)_R$	1
$\mathfrak{u}(1)_R^3$	1^3

It is easy to check that the left column of (6.37) is the sum of the right column and (6.58). The global anomalies also match.

References and cross references

See the papers of K. Intriligator and P. Pouliot and of K. Intriligator and N. Seiberg quoted after the solutions to Problems 1-2. In addition, a general discussion of the Abelian Coulomb phase of $N = 1$ gauge theories, including the family of elliptic curves that determines the coupling constant, can be found in

K. Intriligator and N. Seiberg, *Phases of $N = 1$ supersymmetric gauge theories in four dimensions*, Nucl. Phys. B431 (1994) 551-565.

and the review paper of Intriligator and Seiberg (Sect. 7) quoted after Lecture 1. For $N = 2$ supersymmetric gauge theories, see

N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B426 (1994) 19–52, Erratum, ibid., B430 (1994) 485–486;

N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. B431 (1994) 484-550.

and Lecture 17–19 of [II-**Dynamics of QFT**] in this volume.

SOLUTIONS TO EXERCISES (BY SIYE WU)
Problems 5–6

Solution to Problem 5

The supersymmetric gauge theory with gauge group $G = Sp(C)$ and quarks $Q \in \mathrm{Hom}(F, C)$ is called the electric theory. When $N_f \geq N_c+3$, we define a dual theory, called the magnetic theory, with gauge group $\tilde{G} = Sp(\tilde{C})$. The color space $\tilde{C}$ is an Hermitian vector space of dimension $2\tilde{N}_c = 2(N_f - N_c - 2)$ equipped with a compatible symplectic form $\tilde{\omega}_c$. The flavor space is

$$\tilde{F} = F^* \otimes (\wedge^{2N_f} F)^{\frac{1}{2(\tilde{N}_c+1)}}. \tag{7.1}$$

In addition to the dual quarks $\tilde{Q} \in \mathrm{Hom}(\tilde{F}, \tilde{C})$, the magnetic theory has elementary mesons $M \in \wedge^2 F^*$. The fields $\tilde{Q}$, M, their fermionic components $\psi_{\tilde{Q}}$, ψ_M and the gluino $\tilde{\lambda}$ of the magnetic theory transform under the group $SU(F) \times U(1)_R$ according to

(7.2)

	$SU(F)$	$\times$	$U(1)_R$
$\tilde{Q}$	F		$\frac{N_c+1}{N_f}$
$\psi_{\tilde{Q}}$	F		$\frac{N_c+1}{N_f} - 1$
M	$\wedge^2 F^*$		$2(1 - \frac{N_c+1}{N_f})$
ψ_M	$\wedge^2 F^*$		$1 - 2\frac{N_c+1}{N_f}$
$\tilde{\lambda}$	$\mathbf{1}$		1

The global symmetry $SU(F) \times U(1)_R$ is anomaly-free in the dual theory; this can be checked by a simple calculation $(\frac{N_c+1}{N_f} - 1) \cdot 2N_f c_2(\tilde{C}) + c_2(\tilde{G}) = 0$. The fields $\tilde{Q}$ and M couple through a superpotential

$$W = \frac{1}{2\mu}\langle M, \tilde{Q}^* \tilde{\omega}_c \rangle, \tag{7.3}$$

where

$$\mu \in (\wedge^{2N_f} F^*)^{\frac{1}{\tilde{N}_c+1}} \tag{7.4}$$

has mass dimension 1. The superpotential has R-charge 2 and is invariant under $SU(F)$. Therefore the dual theory has the same global symmetry $SU(F) \times U(1)_R$.

The β-functions of the electric and magnetic theories are given by $b_0 = 3(N_c + 1) - N_f$ and $\tilde{b}_0 = 3(\tilde{N}_c + 1) - N_f = 2N_f - 3(N_c + 1)$, respectively. The scale $\tilde{\Lambda}$ of the dual theory should be related to Λ and μ. Because Λ, $\tilde{\Lambda}$ and μ all have mass dimension 1 and because of (7.4), (5.7) and its magnetic counterpart, the only possible relation among them is

$$\Lambda^{3(N_c+1)-N_f}\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f} = c(-1)^{\tilde{N}_c+1}\mu^{N_f}, \tag{7.5}$$

where c is a constant to be determined later.

We check that duality is an involution. In the double dual theory, the number of colors is $N_f - \tilde{N}_c - 2 = N_c$, and the flavor space is $\tilde{F}^* \otimes (\wedge^{2N_F}\tilde{F})^{\frac{1}{2(N_c+1)}} \cong F$. There are new mesons $\tilde{M} \in \wedge^2 \tilde{F}^*$ and quarks $Q \in \mathrm{Hom}(F, C)$. The total superpotential is

$$W = \frac{1}{2\mu}\langle M, \tilde{M}\rangle + \frac{1}{2\tilde{\mu}}\langle \tilde{M}, Q^*\omega_c\rangle, \tag{7.6}$$

where $\tilde{\mu} \in (\wedge^{2N_f}\tilde{F}^*)^{\frac{1}{N_c+1}} \cong (\wedge^{2N_f}F^*)^{\frac{1}{N_c+1}}$. We propose that $\tilde{\mu} = -\mu$. Then the double dual theory has the same scale Λ as the electric theory, and the equation of motion of $\tilde{M}$ is $M = Q^*\omega_c$. Therefore we recover the electric theory.

The electric theory is infrared free when $b_0 \leq 0$, i.e., when $N_f \geq 3(N_c + 1)$. (When $b_0 = 0$, the two-loop β-function is positive.) It is ultraviolet free when the dual theory is infrared free, i.e., when $\tilde{b}_0 \leq 0$ or $N_f \leq \frac{3}{2}(N_c + 1)$. We claim that for $\frac{3}{2}(N_c+1) < N_f < 3(N_c+1)$, the theory flows to an interacting superconformal field theory in the infrared. The infrared behavior of the $Sp(N_c)$ theories with $N_f \geq N_c + 3$ and their duals is summarized in the following table.

	electric theory	magnetic theory
$N_c \geq 1$, $N_f \geq 3(N_c+1)$ $(\tilde{N}_c + 3 \leq N_f \leq \frac{3}{2}(\tilde{N}_c+1))$	infrared free	strongly coupled
$\frac{3}{2}(N_c+1) < N_f < 3(N_c+1)$ $(\frac{3}{2}(\tilde{N}_c+1) < N_f < 3(\tilde{N}_c+1))$	non-trivial infrared fixed point	non-trivial infrared fixed point
$N_c + 3 \leq N_f \leq \frac{3}{2}(N_c+1)$ $(\tilde{N}_c \geq 1, N_f \geq 3(\tilde{N}_c+1))$	strongly coupled	infrared free

That the electric and the magnetic theories describe the same low energy physics will be established through the following highly nontrivial consistency tests.

Chiral operators and the unitarity bound

Though their field contents seem rather different, the electric and the magnetic theories have the same gauge invariant operators. We construct the mapping of gauge invariant chiral operators. The composite meson $Q^*\omega_c$ of the electric theory corresponds to the elementary meson M of the magnetic theory, whereas the composite meson $\tilde{M} = \tilde{Q}^*\tilde{\omega}_c$ of the magnetic theory vanishes by the equation of motion of M.

At the non-trivial infrared fixed points, the dimensions of the chiral fields can be calculated by their R-charges. In fact,

$$D(M) = \frac{3}{2}R(M) = 3\left(1 - \frac{N_c+1}{N_f}\right). \tag{7.7}$$

The unitarity bound $D(M) \geq 1$ is satisfied if $N_f \geq \frac{3}{2}(N_c + 1)$.

't Hooft anomaly matching condition

Under $SU(F) \times U(1)_R$, the fermions $\psi_{\tilde{Q}}$, ψ_M, $\tilde{\lambda}$ transform as (7.2). At $M = 0$, all matter fields are massless and the whole global symmetry group is unbroken. The non-zero contributions to the 't Hooft condition are

(7.8)

	magnetic
$\mathfrak{su}(F)^3$	$2\tilde{N}_c\, d_F - d_{\wedge^2 F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$2\tilde{N}_c\, c_2(F)\cdot(\frac{N_c+1}{N_f}-1) + c_2(\wedge^2 F)\cdot(1-2\frac{N_c+1}{N_f})$
$\mathfrak{u}(1)_R$	$4\tilde{N}_c N_f\,(\frac{N_c+1}{N_f}-1) + N_f(2N_f-1)\,(1-2\frac{N_c+1}{N_f}) + \tilde{N}_c(2\tilde{N}_c+1)\,1$
$\mathfrak{u}(1)_R^3$	$4\tilde{N}_c N_f\,(\frac{N_c+1}{N_f}-1)^3 + N_f(2N_f-1)\,(1-2\frac{N_c+1}{N_f})^3 + \tilde{N}_c(2\tilde{N}_c+1)\,1^3$

Using $d_{\wedge^2 F} = 2(N_f - 2)d_F$ and $c_2(\wedge^2 F) = 2(N_f - 1)$, the above matches the left column of (6.24).

Deformation along the flat directions

The magnetic theory has the same space of ground states as the electric theory. In the electric theory, the classical moduli space $\mathcal{M}_c$ consists of M satisfying (5.2) as a classical constraint. Since the superpotential is absent, the quantum moduli $\mathcal{M}_q$ space is the same as $\mathcal{M}_c$; we denote both by $\mathcal{M}$. In the magnetic theory, the equations of motion of M set $\tilde{Q} = 0$. The same constraint (5.2) is enforced by non-perturbative quantum effects. In fact, if $\frac{1}{2}\operatorname{rank} M > N_c$, then the number of flavors in the low energy magnetic theory is $N_f - \frac{1}{2}\operatorname{rank} M < \tilde{N}_c + 2$. When $N_f - \frac{1}{2}\operatorname{rank} M \leq \tilde{N}_c$, a superpotential of the form (5.11) is generated, and consequently there is no vacuum. When $N_f - \frac{1}{2}\operatorname{rank} M = \tilde{N}_c + 1$, the light components of $\tilde{M} = \tilde{Q}^* \tilde{\omega}_c$ are constrained by an equation like (6.17) which implies $\tilde{Q} \neq 0$. Therefore the superpotential (7.6) does not have an extremum on the constraint surface and there is no vacuum in this case either. Thus $\frac{1}{2}\operatorname{rank} M \leq N_c$ in the low energy magnetic theory.

We show that the procedure of deforming M in $\mathcal{M}$ commutes with taking the dual. Suppose in the electric theory, M has an expectation value $\langle M \rangle$ with $\frac{1}{2}\operatorname{rank}\langle M \rangle = r$. At low energies, the gauge group is broken to $Sp(N_c - r)$, and there are $N_f - r$ light flavors. Using the previous notations, the mass scale $\hat{\Lambda}$ of the effective theory is given by (6.4). The original magnetic theory has $\tilde{N}_c$ colors and N_f flavors, and the scale $\tilde{\Lambda}$ is related to Λ by (7.5). Now the expectation value $\langle M \rangle$ gives mass to r components of the dual quark $\tilde{Q}$. The low energy magnetic theory has $\tilde{N}_c$ colors and $N_f - r$ flavors. Its scale $\hat{\tilde{\Lambda}}$ is given by

$$\hat{\tilde{\Lambda}}^{3(\tilde{N}_c+1)-(N_f-r)} = \operatorname{Pf}(\langle M^{\perp} \rangle / 2\mu)\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f}. \tag{7.9}$$

It follows from (6.4) and (7.9) that the relation (7.5) is preserved at low energies. Thus the low energy magnetic theory is dual to the low energy electric theory.

Mass deformation

We now check that mass deformation commutes with taking the dual. Suppose a mass term $\frac{1}{2}\langle M, m\rangle$ with $\frac{1}{2}\operatorname{rank} m = r$ is added to the electric theory. We follow the previous notations. The low energy theory has quarks $\hat{Q} \in \operatorname{Hom}(\hat{F}, C)$. Its scale $\hat{\Lambda}$ is given by (6.7). When $r \leq N_f - N_c - 2$, it has a dual description as well. In the magnetic theory dual to the original electric theory, let $\hat{\tilde{F}} = \hat{F} \otimes (\wedge^{2N_f} F)^{\frac{1}{2(\tilde{N}_c+1)}}$. The equations of motion of the heavy components of M imply that

$$\tilde{Q}^* \tilde{\omega}_c = \begin{pmatrix} \hat{\tilde{M}} & \\ & -2\mu m^{\perp} \end{pmatrix}, \tag{7.10}$$

where $\hat{\tilde{M}} \in S^2 \hat{\tilde{F}}$ contains the light components. This breaks the $Sp(\tilde{N}_c)$ gauge symmetry to $Sp(\tilde{N}_c - r)$. Integrating out the heavy components in M, $\tilde{Q}$, we obtain a low energy effective superpotential

$$\hat{W} = \frac{1}{2\mu}\langle \hat{M}, \hat{\tilde{M}}\rangle. \tag{7.11}$$

This is exactly a magnetic theory of $\tilde{N}_c - r$ flavors and $N_f - r$ quarks. Its mass scale $\hat{\tilde{\Lambda}}$ is given by the magnetic counterpart of (6.4), i.e.,

$$\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f} = (-\mu)^r \operatorname{Pf} m^{\perp} \hat{\tilde{\Lambda}}^{3(\tilde{N}_c-r+1)-(N_f-r)}. \tag{7.12}$$

From (6.7) and (7.12), we see that the relation (7.5) is preserved at low energies. So the low energy theory of the magnetic theory after mass deformation is dual to that of the electric theory.

If $r = N_f - N_c - 2$, then in the low energy electric theory the number of flavors is $N_f - r = N_c + 2$. In the low energy magnetic theory, the gauge symmetry is completely broken since $\tilde{N}_c - r = 0$. Therefore, the instanton contribution should be included in the magnetic theory just as in the case $N_f = N_c$ for the electric theory. Now the number of zero modes of $\psi_{\tilde{Q}}$ is $2N_f > 2\tilde{N}_c$; these $2(N_c + 2)$ extra ones are absorbed by the interaction (7.3), generating a term proportional to the $(N_c + 2)$-th power of $(M/2\mu)_{\text{light}}$ in the effective superpotential. So the instanton contribution is

$$\begin{aligned} 2^{\tilde{N}_c - 1} \frac{\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f} \operatorname{Pf}(M/2\mu)_{\text{light}}}{\operatorname{Pf}(\tilde{M})_{\text{heavy}}} &= \frac{(-1)^r \tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f} \operatorname{Pf} \hat{M}}{2^{N_c+3} \mu^{N_f} \operatorname{Pf} m^{\perp}} \\ &= -\frac{c \operatorname{Pf} \hat{M}}{2^{N_c+3} \operatorname{Pf} m^{\perp} \Lambda^{3(N_c+1)-N_f}} = -\frac{c \operatorname{Pf} \hat{M}}{2^{N_c+3} \hat{\Lambda}^{2N_c+1}}, \end{aligned} \tag{7.13}$$

where (7.5), (6.7) have been used. This is exactly (6.21) if we choose $c = 16$.

Solution to Problem 6

Recall that the electric theory has a gauge group $G = SO(C)$ and quarks $Q \in \operatorname{Hom}(F, C)$. The cases with $N_c \geq 4$ and $N_f \leq N_c - 2$ have been treated earlier. When $N_f \geq N_c - 1$, we define a magnetic theory whose gauge group is $\tilde{G} = SO(\tilde{C})$. Here the space of dual colors $\tilde{C}$ is a complex vector space of dimension $\tilde{N}_c = N_f - N_c + 4$. It has a linear metric $\tilde{g}_c$ and a compatible volume form $\tilde{v}_c$. The space

of flavors is

$$\tilde{F} = F^* \otimes (\wedge^{N_f} F)^{\frac{1}{N_c-2}}. \tag{7.14}$$

The matter fields of the magnetic theory are quarks $\tilde{Q} \in \mathrm{Hom}(\tilde{F}, \tilde{C})$ and (elementary) mesons $M \in \wedge^2 F^*$.

We first treat the cases $N_c \geq 4$, $N_f \geq N_c$. We concentrate on the identity component $SU(F) \times U(1)_R$ of the global symmetry. The fields $\tilde{Q}$, M, their fermionic components $\psi_{\tilde{Q}}$, ψ_M and the gluino $\tilde{\lambda}$ transform as

(7.15)

	$SU(F)$	$\times$	$U(1)_R$
$\tilde{Q}$	F		$\frac{N_c-2}{N_f}$
$\psi_{\tilde{Q}}$	F		$\frac{N_c-2}{N_f} - 1$
M	S^2F^*		$2(1 - \frac{N_c-2}{N_f})$
ψ_M	S^2F^*		$1 - 2\frac{N_c-2}{N_f}$
$\tilde{\lambda}$	$\mathbf{1}$		1

It is easy to check that $SU(F) \times U(1)_R$ is anomaly-free in the dual theory. The fields $\tilde{Q}$ and M couple through a superpotential

$$W = \frac{1}{2\mu}\langle M, \tilde{Q}^* \tilde{g}_c \rangle, \tag{7.16}$$

where

$$\mu \in (\wedge^{N_f} F^*)^{\frac{2}{N_c-2}} \tag{7.17}$$

has mass dimension 1. The superpotential has R-charge 2 and is invariant under $SU(F)$.

When $N_f \geq N_c$, the β-functions of the electric and magnetic theories are given by $b_0 = 3(N_c - 2) - N_f$ and $\tilde{b}_0 = 3(\tilde{N}_c - 2) - N_f = 2N_f - 3(N_c - 2)$, respectively. The scale $\tilde{\Lambda}$ of the magnetic theory is given by

$$\Lambda^{3(N_c-2)-N_f} \tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f} = c(-1)^{\tilde{N}_c} \mu^{N_f}, \tag{7.18}$$

where c is a constant to be determined. It is easy to check that duality is an involution just as in the symplectic case. The electric theory is infrared free when $N_f \geq 3(N_c - 2)$. It is ultraviolet free when the dual theory is infrared free, i.e., when $N_f \leq \frac{3}{2}(N_c-2)$. We claim that when $\frac{3}{2}(N_c-2) < N_f < 3(N_c-2)$, the theory flows to an interacting superconformal field theory in the infrared. The infrared behavior of the $SO(N_c)$ theories with $N_f \geq N_c + 3$ and their duals is summarized in the following table.

	electric theory	magnetic theory
$N_c \geq 4$, $N_f \geq 3(N_c - 2)$ $(\tilde{N}_c \leq N_f \leq \frac{3}{2}(\tilde{N}_c - 2))$	infrared free	strongly coupled
$\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$ $(\frac{3}{2}(\tilde{N}_c - 2) < N_f < 3(\tilde{N}_c - 2))$	non-trivial infrared fixed point	non-trivial infrared fixed point
$N_c + 3 \leq N_f \leq \frac{3}{2}(N_c - 2)$ $(\tilde{N}_c \geq 4, N_f \geq 3(\tilde{N}_c - 2))$	strongly coupled	infrared free

Chiral operators and the unitarity bound

We construct the mapping of gauge invariant chiral operators. Obviously, the composite mesons $Q^*\omega_c$ of the electric theory correspond to the elementary meson M of the magnetic theory. Recall the baryons $B = Q^*v_c \in \wedge^{N_c}F^*$ in the electric theory. Since the (fermionic) superfield strength W_α is valued in the Lie algebra $\mathfrak{so}(C) \cong \wedge^2 C$, we can define additional gauge invariant fields $b = Q^*v_c(W_\alpha, W^\alpha, \cdot) \in \wedge^{N_c-4}F^*$ and $w_\alpha = Q^*v_c(W_\alpha, \cdot) \in \wedge^{N_c-2}F^*$. The field q in the $N_f = N_c - 3$ theory is proportional to b.

In the dual theory, the baryons $\tilde{B} \in \wedge^{\tilde{N}_c}\tilde{F}^* \cong \wedge^{N_f-N_c+4}F \otimes (\wedge^{N_f}F^*) \otimes (\wedge^{N_f}F^*)^{\frac{2}{\tilde{N}_c-2}}$ and $\tilde{b} \in \wedge^{\tilde{N}_c-4}\tilde{F}^* \cong \wedge^{N_f-N_c}F \otimes (\wedge^{N_f}F^*) \otimes (\wedge^{N_f}F^*)^{-\frac{2}{\tilde{N}_c-2}}$, $\tilde{w}_\alpha \in \wedge^{\tilde{N}_c-2}\tilde{F}^* \cong \wedge^{N_f-N_c+2}F \otimes (\wedge^{N_f}F^*)$. It is easy to check that $R(B) = R(\tilde{b}) = \frac{N_c(N_f-N_c+2)}{N_f}$, $R(b) = R(\tilde{B}) = \frac{(N_c-2)(N_f-N_c+4)}{N_f}$ and $R(w_\alpha) = R(\tilde{w}_\alpha) = 1 + \frac{(N_c-2)(N_f-N_c+2)}{N_f}$. So B and $\tilde{b}$, b and $\tilde{B}$, w_α and $\tilde{w}_\alpha$ have the same quantum numbers, respectively. The mapping of chiral operators in the electric and magnetic theories are given by $B \mapsto \tilde{b}$, $b \mapsto \tilde{B}$ and $w_\alpha \mapsto \tilde{w}_\alpha$, upon choosing a trivialization of $\wedge^{N_f}F$.

At non-trivial infrared fixed points, the dimension of any chiral field is given by Φ as $D(\Phi) = \frac{3}{2}R(\Phi)$. Since $R(Q) = 1 - \frac{N_c-2}{N_f}$ and $R(W_\alpha) = 1$, we have $D(B), D(b), D(w_\alpha) \geq 1$ and $D(M) = 3(1 - \frac{N_c-2}{N_f}) \geq 1$ if $N_f \geq \frac{3}{2}(N_c - 2)$. The unitarity bounds are satisfied.

't Hooft anomaly matching condition

Under the global symmetry, the light fermions $\psi_{\tilde{Q}}$, ψ_M, $\tilde{\lambda}$ transform according to (7.15). At $M = 0$ in $\mathcal{M}_q$, the whole group is unbroken. The non-zero combinations in the 't Hooft condition are

(7.19)

	magnetic
$\mathfrak{su}(F)^3$	$\tilde{N}_c\, d_F - d_{S^2F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$\tilde{N}_c\, c_2(F)\cdot(\frac{N_c-2}{N_f} - 1) + c_2(S^2F)\cdot(1 - 2\frac{N_c-2}{N_f})$
$\mathfrak{u}(1)_R$	$\tilde{N}_c N_f\,(\frac{N_c-2}{N_f} - 1) + \frac{1}{2}N_f(N_f+1)\,(1 - 2\frac{N_c-2}{N_f}) + \frac{1}{2}\tilde{N}_c(\tilde{N}_c - 1)\,1$
$\mathfrak{u}(1)_R^3$	$\tilde{N}_c N_f\,(\frac{N_c-2}{N_f} - 1)^3 + \frac{1}{2}N_f(N_f+1)\,(1 - 2\frac{N_c-2}{N_f})^3 + \frac{1}{2}\tilde{N}_c(\tilde{N}_c - 1)\,1^3$

Using $d_{S^2F} = (N_f + 4)d_F$ and $c_2(S^2F) = (N_f + 2)c_2(F)$, the above matches the left column of (6.37).

Deformation along the flat directions

First, the dual theory has the same moduli space of vacua. In the electric theory, the condition $\operatorname{rank} M \leq N_c$ is a classical constraint. If $\operatorname{rank} M > N_c$ in the magnetic theory, then the number of light flavors is $N_f - \operatorname{rank} M < \tilde{N}_c - 4$. So a superpotential of the form (5.22) is generated and consequently the quantum theory has no vacuum. If $\operatorname{rank} M = N_c$, then according to (5.12), there are two choices of sign of $B = \pm(\wedge^{N_c}M)^{1/2}$ in the electric theory. In the dual theory, the number of light flavors is $N_f - \operatorname{rank} M = \tilde{N}_c - 4$. So there are two branches. One branch has a superpotential (6.35) with $\epsilon_+ = \epsilon_-$ and hence does not have a vacuum. The

other branch has two vacua, labeled by $\epsilon_+ = -\epsilon_- = \pm 1$. Therefore the magnetic theory also has two vacua for each M with $\operatorname{rank} M = N_c$.

We check that flat deformations on the moduli space commute with taking the dual. If M acquires an expectation value $\langle M \rangle$ with $\operatorname{rank}\langle M \rangle = r$, then the low energy electric theory has N_c colors and $N_f - r$ flavors. In the magnetic theory, the expectation value $\langle M \rangle$ gives mass to some of the dual quarks. The low energy magnetic theory has $\tilde{N}_c - r$ colors and $N_f - r$ flavors. This is dual to the low energy electric theory. Further, using (6.25) and the magnetic analog of (6.28), one can show that the relation (7.18) is preserved in the low energy theories.

Mass deformation

If we add a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m \rangle$ with $\operatorname{rank} m = r$ in the electric theory, then the low energy theory has N_c colors and $N_f - r$ flavors. Its scale $\hat{\Lambda}$ is given by (6.28). When $r \le N_f - N_c + 1$, this low energy theory has a dual theory. The latter has $\tilde{N}_c - r$ colors and $N_f - r$ flavors. Its scale $\hat{\tilde{\Lambda}}$ is given by the magnetic analog of (6.25). It is easy to show that the relation (7.18) is preserved in low energy theories and so is the form of the superpotential (7.16) after integrating out the heavy fields.

When $r = N_f - N_c + 1$, the number of colors in the low energy magnetic theory is $\tilde{N}_c - r = 3$. Just as the electric theory when $N_f = N_c - 3$, we consider an intermediate stage in which the gauge group is broken to $SO(4) \cong (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$. When the groups $SU(2)_\pm$ are further broken, there are instanton generated superpotentials. Now the number of zero modes of $\psi_{\tilde{Q}}$ is $2N_f > 2(\tilde{N}_c - 3)$. The extra $2(N_c - 1)$ ones are absorbed by the interaction (7.16), generating a term proportional to the $(N_c - 1)$-th power of $(\mu^{-1} M)_{\text{light}}$. Thus the superpotential contains an extra term

$$4\frac{\tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f} \det(\mu^{-1}M)_{\text{light}}}{(\det \tilde{M})_{\text{heavy}}} = \frac{4(-1)^r \tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f} \det \hat{M}}{\mu^{N_f} \det m^\perp} = -\frac{4c \det \hat{M}}{\det m^\perp \Lambda^{3(N_c-2)-N_f}} = -\frac{4c \det \hat{M}}{\hat{\Lambda}^{2N_c-5}}. \tag{7.20}$$

Here we have used (7.18) and (6.28). So the total superpotential in the low energy magnetic theory is

$$\hat{\hat{W}} = \frac{1}{2\mu}\langle \hat{M}, \hat{\tilde{Q}}^* \tilde{g}_c \rangle - \frac{4c \det \hat{M}}{\hat{\Lambda}^{2N_c-5}}. \tag{7.21}$$

Finally, we derive the relation between the scales of the low energy theories. The scale $\hat{\Lambda}$ of the low energy electric theory is given by (6.28). The scale $\hat{\tilde{\Lambda}}$ of the low energy magnetic theory is given by the magnetic analog of (6.27), i.e.,

$$4(\tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f})^2 = \det(-\mu m^\perp)^2 \hat{\tilde{\Lambda}}^{6-2(N_f-\tilde{N}_c+3)}. \tag{7.22}$$

Using (6.28) and (7.22), the relation (7.18) reduces at low energies to

$$(\hat{\Lambda}^{2N_c-5})^2 \hat{\tilde{\Lambda}}^{6-2(N_c-1)} = 4c^2 \mu^{2N_c-2}. \tag{7.23}$$

$N_f = N_c - 1$

We consider the theory with $N_f = N_c - 1$ and scale Λ. The electric theory is strongly coupled at low energies. From (7.21), the superpotential of the magnetic

theory is

$$W = \frac{1}{2\mu}\langle M, \tilde{Q}^* \tilde{g}_c \rangle - \frac{4c \det M}{\Lambda^{2N_c-5}}. \tag{7.24}$$

From (7.23), the scale of the dual theory is given by

$$(\Lambda^{2N_c-5})^2 \tilde{\Lambda}^{6-2(N_c-1)} = 4c^2 \mu^{2N_c-2}. \tag{7.25}$$

In fact, up to a constant, (7.25) is the only possible relation consistent with (7.17), (5.17), the magnetic analog of (5.18), and the fact that Λ, $\tilde{\Lambda}$, μ all have mass dimension 1. Under a flat deformation, (7.25) is satisfied by the low energy electric and magnetic theories.

We show that adding the mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m \rangle$ with $\operatorname{rank} m = 1$ reduces the theory to $N_f = N_c - 2$. The total superpotential is $W + W_{\text{tree}}$. Following previous notations, we have $\tilde{F} = \hat{\tilde{F}} \oplus \hat{\tilde{F}}^\perp$ with $\dim \hat{\tilde{F}}^\perp = 1$. The dual quarks have the decomposition $\tilde{Q} = (\hat{\tilde{Q}}, \tilde{Q}^\perp) \in \operatorname{Hom}(\hat{\tilde{F}}, \tilde{C}) \oplus \operatorname{Hom}(\hat{\tilde{F}}^\perp, \tilde{C})$. The $M^\perp$ equations of motion imply

$$(\tilde{Q}^\perp)^* \tilde{g}_c = \frac{8c\mu \det \hat{M}}{\Lambda^{2N_c-5}} - \mu m^\perp. \tag{7.26}$$

In particular, $\operatorname{im}(\hat{\tilde{Q}}^\perp)$ is not isotropic in $\tilde{C}$. We write $\tilde{C} = \hat{\tilde{C}} \oplus \hat{\tilde{C}}^\perp$, where $\hat{\tilde{C}}^\perp = \operatorname{im}(\hat{\tilde{Q}}^\perp)$. The value of $\tilde{Q}^\perp$ breaks the dual gauge group $SO(\tilde{C})$ to $SO(\hat{\tilde{C}}) \cong SO(2)$. The equations of motion imply that M is block-diagonal and that $\tilde{g}_c(\hat{\tilde{Q}}, \hat{\tilde{Q}}^\perp) = 0$, i.e., $\hat{\tilde{Q}} \in \operatorname{Hom}(\hat{\tilde{F}}, \hat{\tilde{C}})$. Let $\hat{\tilde{C}} = \hat{\tilde{C}}^+ \oplus \hat{\tilde{C}}^-$, where the weights of $SO(\hat{\tilde{C}})$ in $\hat{\tilde{C}}^\pm$ are ± 1, respectively. We write $\hat{\tilde{Q}} = (\hat{\tilde{Q}}^+, \hat{\tilde{Q}}^-)$, where $\hat{\tilde{Q}}^\pm \in \operatorname{Hom}(\hat{\tilde{F}}, \hat{\tilde{C}}^\pm)$. Then the superpotential $W + W_{\text{tree}}$ reduces at low energies to

$$\hat{W} \sim \frac{1}{2\mu}\langle \hat{M}, \hat{\tilde{Q}}^+ \cdot \hat{\tilde{Q}}^- \rangle, \tag{7.27}$$

where $\cdot$ is the pairing between $\hat{\tilde{C}}^+$ and $\hat{\tilde{C}}^-$. This matches (6.55). Due to the instanton effects in symmetry breaking, it is only an approximate result near $\hat{M} = 0$. When $\det \hat{M}$ is large, the quarks $\hat{\tilde{Q}}$ become heavy. After integrating out the heavy fields, the effective superpotential is

$$W = \frac{1}{2\mu} M^\perp \left(4u - \frac{8c\mu \det \hat{M}}{\Lambda^{2N_c-5}} + \mu m^\perp \right), \tag{7.28}$$

where $u = \tilde{g}_c(\tilde{Q}^\perp, \tilde{Q}^\perp)/4$. The effective theory, which is the $SO(\tilde{C}) \cong SO(3)$ gauge theory with a single flavor $\tilde{Q}^\perp$, has $N = 2$ supersymmetry. Its scale $\tilde{\Lambda}'$ is given by

$$\tilde{\Lambda}'^4 = \det(\mu^{-1}\hat{M})^2 \tilde{\Lambda}^{6-2(N_c-1)} = \left(\frac{2c\mu \det \hat{M}}{\Lambda^{2N_c-5}} \right)^2. \tag{7.29}$$

The superpotential (7.28) lifts the vacuum degeneracy of the u-plane except at the two singularities $u = \pm \tilde{\Lambda}'^2 = \pm 8c\mu \det \hat{M}/\Lambda^{2N_c-5}$. There is still no vacuum with the plus sign. With the minus sign, there is a vacuum if $\hat{M}$ satisfies

$$\det \hat{M} = \frac{m^\perp \Lambda^{2N_c-5}}{16c} = \frac{\hat{\Lambda}^{2N_c-4}}{16c}. \tag{7.30}$$

This matches the non-zero singular branch of the $N_f = N_c - 2$ theory when $c = 2^{-8}$.

References and cross references

In addition to the papers of K. Intriligator and P. Pouliot and of K. Intriligator and N. Seiberg quoted after the solutions to Problems 1-2, duality in $SO(N_c)$ theories was also studied in

N. Seiberg, *Electric-magnetic duality in supersymmetric non-Abelian gauge theories*, Nucl. Phys. B435 (1995) 129-146 (Sect. 5).

See also the review papers of Intriligator and Seiberg (Sect. 6) and of Peskin (Sect. 8) quoted at the end of Lecture 1.

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Quantum Fields and Strings: A Course for Mathematicians

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