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STRING THEORY DYNAMICS IN VARIOUS DIMENSIONS

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The strong coupling dynamics of string theories in dimension $d \geq 4$ are studied. It is argued, among other things, that eleven-dimensional supergravity arises as a low energy limit of the ten-dimensional Type IIA superstring, and that a recently conjectured duality between the heterotic string and Type IIA superstrings controls the strong coupling dynamics of the heterotic string in five, six, and seven dimensions and implies S duality for both heterotic and Type II strings.

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1. Introduction

Understanding in what terms string theories should really be formulated is one of the basic needs and goals in the subject. Knowing some of the phenomena that can occur for strong coupling – if one can know them without already knowing the good formulation! – may be a clue in this direction. Indeed, S-duality between weak and strong coupling for the heterotic string in four dimensions (for instance, see [1,2]) really ought to be a clue for a new formulation of string theory.

At present there is very strong evidence for S-duality in supersymmetric field theories, but the evidence for S-duality in string theory is much less extensive. One motivation for the present work was to improve this situation.

Another motivation was to try to relate four-dimensional S-duality to statements or phemonena in more than four dimensions. At first sight, this looks well-nigh implausible since S-duality between electric and magnetic charge seems to be very special to four dimensions. So we are bound to learn something if we succeed.

Whether or not a version of S-duality plays a role, one would like to determine the strong coupling behavior of string theories above four dimensions, just as S-duality – and its conjectured Type II analog, which has been called U-duality [3] – determines the strong coupling limit after toroidal compactification to four dimensions.[1] One is curious about the phenomena that may arise, and in addition if there is any non-perturbative inconsistency in the higher dimensional string theories (perhaps ultimately leading to an explanation of why we live in four dimensions) it might show up naturally in thinking about the strong coupling behavior.

In fact, in this paper, we will analyze the strong coupling limit of certain string theories in certain dimensions. Many of the phenomena are indeed novel, and many of them are indeed related to dualities. For instance, we will argue in section two that the strong coupling limit of Type IIA supergravity in ten dimensions is eleven-dimensional supergravity! In a sense, this statement gives a rationale for "why" eleven-dimensional

[1] By "strong coupling limit" I mean the limit as the string coupling constant goes to infinity keeping fixed (in the sigma model sense) the parameters of the compactification. Compactifications that are not explicitly described or clear from the context will be toroidal.

supergravity exists, much as the interpretation of supergravity theories in $d \leq 10$ as low energy limits of string theories explains "why" these remarkable theories exist. How eleven-dimensional supergravity fits into the scheme of things has been a puzzle since the theory was first predicted [5] and constructed [6].

Upon toroidal compactification, one can study the strong coupling behavior of the Type II theory in $d < 10$ using U duality, as we will do in section three. One can obtain a fairly complete picture, with eleven-dimensional supergravity as the only "surprise."

Likewise, we will argue in section four that the strong coupling limit of five-dimensional heterotic string theory is Type IIB in six dimensions, while the strong coupling limit of six-dimensional heterotic string theory is Type IIA in six dimensions (in each case with four dimensions as a K3), and the strong coupling limit in seven dimensions involves eleven-dimensional supergravity. These results are based on a relation between the heterotic string and the Type IIA superstring in six dimensions that has been proposed before [3,4]. The novelty in the present paper is to show, for instance, that vexing puzzles about the strong coupling behavior of the heterotic string in five dimensions disappear if one assumes the conjectured relation of the heterotic string to Type IIA in six dimensions. Also we will see – using a mechanism proposed previously in a more abstract setting [7] – that the "string-string duality" between heterotic and Type IIA strings in six dimensions implies S-duality in four dimensions, so the usual evidence for S-duality can be cited as evidence for string-string duality.

There remains the question of determining the strong coupling dynamics of the heterotic string above seven dimensions. In this context, there is a curious speculation[2] that the heterotic string in ten dimensions with $SO(32)$ gauge group might have for its strong coupling limit the $SO(32)$ Type I theory. In section five, we show that this relation, if valid, straightforwardly determines the strong coupling behavior of the heterotic string in nine and eight dimensions as well as ten, conjecturally completing the description of strong coupling dynamics except for $E_8 \times E_8$ in ten dimensions.

The possible relations between different theories discussed in this paper should be

[2] This idea was considered many years ago by M. B. Green, the present author, and probably others, but not in print as far as I know.

taken together with other, better established relations between different string theories. It follows from T duality that below ten dimensions the $E_8 \times E_8$ heterotic string is equivalent to the $SO(32)$ heterotic string [8,9], and Type IIA is equivalent to Type IIB [10,11]. Combining these statements with the much shakier relations discussed in the present paper, one would have a web of connections between the five string theories and eleven-dimensional supergravity.

After this paper was written and circulated, I learned of a paper [12] that has some overlap with the contents of section two of this paper.

2. Type II Superstrings In Ten Dimensions

2.1. Type IIB In Ten Dimensions

In this section, we will study the strong coupling dynamics of Type II superstrings in ten dimensions. We start with the easy case, Type IIB. A natural conjecture has already been made by Hull and Townsend [3]. Type IIB supergravity in ten dimensions has an $SL(2,\mathbf{R})$ symmetry; the conjecture is that an $SL(2,\mathbf{Z})$ subgroup of this is an exact symmetry of the string theory.[3] This then would relate the strong and weak coupling limits just as S-duality relates the strong and weak coupling limits of the heterotic string in four dimensions.

This $SL(2,\mathbf{Z})$ symmetry in ten dimensions, if valid, has powerful implications below ten dimensions. The reason is that in $d < 10$ dimensions, the Type II theory (Type IIA and Type IIB are equivalent below ten dimensions) is known to have a T-duality symmetry $SO(10-d,10-d;\mathbf{Z})$. This T-duality group does not commute with the $SL(2,\mathbf{Z})$ that is already present in ten dimensions, and together they generate the discrete subgroup of the supergravity symmetry group that has been called U-duality. [4] Thus, U-duality is true in

[3] For earlier work on the possible role of the non-compact supergravity symmetries in string and membrane theory, see [13].

[4] For instance, in five dimensions, T-duality is $SO(5,5)$ and U-duality is E_6. A proper subgroup of E_6 that contains $SO(5,5)$ would have to be $SO(5,5)$ itself or $SO(5,5) \times \mathbf{R}^*$ ($\mathbf{R}^*$ is the non-compact form of $U(1)$), so when one tries to adjoin to $SO(5,5)$ the $SL(2)$ that was already present in ten dimensions (and contains two generators that map NS-NS states to RR states and so are

every dimension below ten if the $SL(2,\mathbf{Z})$ of the Type IIB theory holds in ten dimensions.

In the next section we will see that U-duality controls Type II dynamics below ten dimensions. As $SL(2,\mathbf{Z})$ also controls Type IIB dynamics in ten dimensions, this fundamental duality between strong and weak coupling controls all Type II dynamics in all dimensions except for the odd case of Type IIA in ten dimensions. But that case will not prove to be a purely isolated exception: the basic phenomenon that we will find in Type IIA in ten dimensions is highly relevant to Type II dynamics below ten dimensions, as we will see in section three. In a way ten-dimensional Type IIA proves to exhibit the essential new phenomenon in the simplest context.

To compare to $N = 1$ supersymmetric dynamics in four dimensions [14], ten-dimensional Type IIA is somewhat analogous to supersymmetric QCD with $3N_c/2 > N_f > N_c + 1$, whose dynamics is controlled by an effective infrared theory that does not make sense at all length scales. The other cases are analogous to the same theory with $3N_c > N_f > 3N_c/2$, whose dynamics is controlled by an exact equivalence of theories – conformal fixed points – that make sense at all length scales.

2.2. Ramond-Ramond Charges In Ten-Dimensional Type IIA

It is a familiar story to string theorists that the string coupling constant is really the expectation of a field – the dilaton field ϕ. Thus, it can be scaled out of the low energy effective action by shifting the value of the dilaton.

After scaling other fields properly, this idea can be implemented in closed string theories by writing the effective action as $e^{-2\phi}$ times a function that is invariant under $\phi \to \phi +$ constant. There is, however, an important subtlety here that affects the Type IIA and Type IIB (and Type I) theories. These theories have massless antisymmetric tensor fields that originate in the Ramond-Ramond (RR) sector. If A_p is such a p-form field, the natural gauge invariance is $\delta A_p = d\lambda_{p-1}$, with λ_{p-1} a $p-1$-form – and no dilaton in the transformation laws. If one scales A_p by a power of e^{ϕ}, the gauge transformation law becomes more complicated and less natural.

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not in $SO(5,5)$) one automatically generates all of E_6.

Let us, then, consider the Type IIA theory with the fields normalized in a way that makes the gauge invariance natural. The massless bosonic fields from the $(\text{Neveu}-\text{Schwarz})^2$ or NS-NS sector are the dilaton, the metric tensor g_{mn}, and the antisymmetric tensor B_{mn}. From the RR sector, one has a one-form A and a three form A_3. We will write the field strengths as $H = dB$, $F = dA$, and $F_4 = dA_3$; one alse needs $F_4' = dA_3 + A \wedge H$. The bosonic part of the low energy effective action can be written $I = I_{NS} + I_R$ where I_{NS} is the part containing NS-NS fields only and I_R is bilinear in RR fields. One has (in units with $\alpha' = 1$)

$$I_{NS} = \frac{1}{2}\int d^{10}x\sqrt{g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2\right) \tag{2.1}$$

and

$$I_R = -\int d^{10}x\sqrt{g}\left(\frac{1}{2\cdot 2!}F^2 + \frac{1}{2\cdot 4!}F_4'^2\right) - \frac{1}{4}\int F_4 \wedge F_4 \wedge B. \tag{2.2}$$

With this way of writing the Lagrangian, the gauge transformation laws of A, B, and A_3 all have the standard, dilaton-independent form $\delta X = d\Lambda$, but it is not true that the classical Lagrangian scales with the dilaton like an overall factor of $e^{-2\phi}$.

Our interest will focus on the presence of the abelian gauge field A in the Type IIA theory. The charge W of this gauge field has the following significance. The Type IIA theory has two supersymmetries in ten dimensions, one of each chirality; call them Q_α and $Q'_{\dot\alpha}$. The space-time momentum P appears in the anticommutators $\{Q, Q\} \sim \{Q', Q'\} \sim P$. In the anticommutator of Q with Q' it is possible to have a Lorentz-invariant central charge

$$\{Q_\alpha, Q'_{\dot\alpha}\} \sim \delta_{\alpha\dot\alpha}W. \tag{2.3}$$

To see that such a term does arise, it is enough to consider the interpretation of the Type IIA theory as the low energy limit of eleven-dimensional supergravity, compactified on $\mathbf{R}^{10} \times \mathbf{S}^1$. From that point of view, the gauge field A arises from the components $g_{m,11}$ of the eleven-dimensional metric tensor, W is simply the eleventh component of the momentum, and (2.3) is part of the eleven-dimensional supersymmetry algebra. [5]

[5] The relation of the supersymmetry algebra to eleven dimensions leads to the fact that both for the lowest level and even for the first excited level of the Type IIA theory, the states can be

In the usual fashion [17], the central charge (2.3) leads to an inequality between the mass M of a particle and the value of W:

$$M \geq c_0 |W|, \tag{2.4}$$

with c_0 a "constant," that is a function only of the string coupling constant $\lambda = e^{\phi}$, and independent of which particle is considered. The precise constant with which W appears in (2.3) or (2.4) can be worked out using the low energy supergravity (there is no need to worry about stringy corrections as the discussion is controlled by the leading terms in the low energy effective action, and these are uniquely determined by supersymmetry). We will work this out at the end of this section by a simple scaling argument starting with eleven-dimensional supergravity. For now, suffice it to say that the λ dependence of the inequality is actually

$$M \geq \frac{c_1}{\lambda} |W| \tag{2.5}$$

with c_1 an absolute constant. States for which the inequality is saturated – we will call them BPS-saturated states by analogy with certain magnetic monopoles in four dimensions – are in "small" supermultiplets with 2^8 states, while generic supermultiplets have 2^{16} states.

In the elementary string spectrum, W is identically zero. Indeed, as A originates in the RR sector, W would have had to be a rather exotic charge mapping NS-NS to RR states. However, there is no problem in finding classical black hole solutions carrying the W charge (or any other gauge charge, in any dimension). It was proposed by Hull and Townsend [3] that quantum particles carrying RR charges arise by quantization of such black holes. Recall that, in any dimension, charged black holes obey an inequality $GM^2 \geq \text{const} \cdot W^2$ (G, M, and W are Newton's constant and the black hole mass and charge); with $G \sim \lambda^2$, this inequality has the same structure as (2.5). These two inequalities actually correspond in the sense that an extreme black hole, with the minimum mass for given charge, is invariant under some supersymmetry [18] and so should correspond upon quantization to a "small" supermultiplet saturating the inequality (2.5).

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arranged in eleven-dimensional Lorentz multiplets [15]. If this would persist at higher levels, it might be related to the idea that will be developed below. It would also be interesting to look for possible eleven-dimensional traces in the superspace formulation [16].

To proceed, then, I will assume that there are in the theory BPS-saturated particles with $W \neq 0$. This assumption can be justified as follows. Hull and Townsend actually showed that upon toroidally compactifying to less than ten dimensions, the assumption follows from U-duality. In toroidal compactification, the radii of the circles upon which one compactifies can be arbitrarily big. That being so, it is implausible to have BPS-saturated states of $W \neq 0$ below ten dimensions unless they exist in ten dimensions; that is, if the smallest mass of a W-bearing state in ten dimensions were strictly bigger than $c|W|/\lambda$, then this would remain true after compactification on a sufficiently big torus.

If the ten-dimensional theory has BPS-saturated states of $W \neq 0$, then what values of W occur? A continuum of values of W would seem pathological. A discrete spectrum is more reasonable. If so, the quantum of W must be independent of the string coupling "constant" λ. The reason is that λ is not really a "constant" but the expectation value of the dilaton field ϕ. If the quantum of W were to depend on the value of ϕ, then the value of the electric charge W of a particle would change in a process in which ϕ changes (that is, a process in which ϕ changes in a large region of space containing the given particle); this would violate conservation of W.

The argument just stated involves a hidden assumption that will now be made explicit. The canonical action for a Maxwell field is

$$\frac{1}{4e^2}\int d^n x\sqrt{g}F^2. \tag{2.6}$$

Comparing to (2.2), we see that in the case under discussion the effective value of e is independent of ϕ, and this is why the charge of a hypothetical charged particle is independent of ϕ. If the action were

$$\frac{1}{4}\int d^n x\sqrt{g}e^{\gamma\phi}F^2 \tag{2.7}$$

for some non-zero γ, then the current density would equal (from the equations of motion of A) $J_m = \partial^n(e^{\gamma\phi}F_{mn})$. In a process in which ϕ changes in a large region of space containing a charge, there could be a current inflow proportional to $\nabla\phi\cdot F$, and the charge would in fact change. Thus, it is really the ϕ-independence of the kinetic energy of the RR

fields that leads to the statement that the values of W must be independent of the string coupling constant and that the masses of charged fields scale as λ^{-1}.

Since the classical extreme black hole solution has arbitrary charge W (which can be scaled out of the solution in an elementary fashion), one would expect, if BPS-saturated charged particles do arise from quantization of extreme black holes, that they should possess every allowed charge. Thus, we expect BPS-saturated extreme black holes of mass

$$M = \frac{c|n|}{\lambda}, \tag{2.8}$$

where n is an arbitrary integer, and, because of the unknown value of the quantum of electric charge, c may differ from c_1 in (2.5).

Apart from anything else that follows, the existence of particles with masses of order $1/\lambda$, as opposed to the more usual $1/\lambda^2$ for solitons, is important in itself. It almost certainly means that the string perturbation expansion – which is an expansion in powers of λ^2 – will have non-perturbative corrections of order $\exp(-1/\lambda)$, in contrast to the more usual $\exp(-1/\lambda^2)$ [6]. The occurrence of such terms has been guessed by analogy with matrix models [20].

The fact that the masses of RR charges diverge as $\lambda \to 0$ – though only as $1/\lambda$ – is important for self-consistency. It means that these states disappear from the spectrum as $\lambda \to 0$, which is why one does not see them as elementary string states.

2.3. Consequences For Dynamics

Now we will explore the consequences for dynamics of the existence of these charged particles.

The mass formula (2.8) shows that, when the string theory is weakly coupled, the RR charges are very heavy. But if we are bold enough to follow the formula into strong coupling, then for $\lambda \to \infty$, these particles go to zero mass. This may seem daring, but the familiar argument based on the "smallness" of the multiplets would appear to show that the formula (2.8) is exact and therefore can be used even for strong coupling. In

[6] If there are particles of mass $1/\lambda$, then loops of those particles should give effects of order $e^{-1/\lambda}$, while loops of conventional solitons, with masses $1/\lambda^2$, would be of order $\exp(-1/\lambda^2)$.

four dimensions, extrapolation of analogous mass formulas to strong coupling has been extremely successful, starting with the original work of Montonen and Olive that led to the idea of S-duality. (In four-dimensional $N = 2$ theories, such mass formulas generally fail to be exact [21] because of quantum corrections to the low energy effective action. For $N = 4$ in four dimensions, or for Type IIA supergravity in ten dimensions, the relevant, leading terms in the low energy action are uniquely determined by supersymmetry.)

So for strong coupling, we imagine a world in which there are supermultiplets of mass $M = c|n|/\lambda$ for every λ. These multiplets necessarily contain particles of spin at least two, as every supermultiplet in Type IIA supergravity in ten dimensions has such states. (Multiplets that do not saturate the mass inequality contain states of spin ≥ 4.) Rotation-invariance of the classical extreme black hole solution suggests[7] (as does U-duality) that the BPS-saturated multiplets are indeed in this multiplet of minimum spin.

Thus, for $\lambda \to \infty$ we have light, charged fields of spin two. (That is, they are charged with respect to the ten-dimensional gauge field A.) Moreover, there are infinitely many of these. This certainly does not correspond to a local field theory in ten dimensions. What kind of theory will reproduce this spectrum of low-lying states? One is tempted to think of a string theory or Kaluza-Klein theory that has an infinite tower of excitations. The only other option, really, is to assume that the strong coupling limit is a sort of theory that we do not know about at all at present.

One can contemplate the possibility that the strong coupling limit is some sort of a string theory with the dual string scale being of order $1/\lambda$, so that the charged multiplets under discussion are some of the elementary string states. There are two reasons that this approach does not seem promising: (i) there is no known string theory with the right properties (one needs Type IIA supersymmetry in ten dimensions, with charged string states coupling to the abelian gauge field in the gravitational multiplet); (ii) we do not have evidence for a stringy exponential proliferation of light states as $\lambda \to \infty$, but only for

[7] Were the classical solution not rotationally invariant, then upon quantizing it one would obtain a band of states of states of varying angular momentum. One would then not expect to saturate the mass inequality of an extreme black hole without taking into account the angular momentum.

a single supermultiplet for each integer n, with mass $\sim |n|$.

Though meager compared to a string spectrum, the spectrum we want to reproduce is just about right for a Kaluza-Klein theory. Suppose that in the region of large λ, one should think of the theory not as a theory on $\mathbf{R}^{10}$ but as a theory on $\mathbf{R}^{10} \times \mathbf{S}^1$. Such a theory will have a "charge" coming from the rotations of $\mathbf{S}^1$. Suppose that the radius $r(\lambda)$ of the $\mathbf{S}^1$ scales as $1/\lambda$ (provided that distances are measured using the "string" metric that appears in (2.1) – one could always make a Weyl rescaling). Then for large λ, each massless field in the eleven-dimensional theory will give, in ten dimensions, for each integer n a single field of charge n and mass $\sim |n|\lambda$. This is precisely the sort of spectrum that we want.

So we need an eleven-dimensional field theory whose fields are in one-to-one correspondence with the fields of the Type IIA theory in ten dimensions. Happily, there is one: eleven-dimensional supergravity! So we are led to the strange idea that eleven-dimensional supergravity may govern the strong coupling behavior of the Type IIA superstring in ten dimensions.

Let us discuss a little more precisely how this would work. The dimensional reduction of eleven-dimensional supergravity to ten dimensions including the massive states has been discussed in some detail (for example, see [22]). Here we will be very schematic, just to touch on the points that are most essential. The bosonic fields in eleven-dimensional supergravity are the metric G_{MN} and a three-form A_3. The bosonic part of the action is

$$I = \frac{1}{2}\int d^{11}x\sqrt{G}\left(R + |dA_3|^2\right) + \int A_3 \wedge dA_3 \wedge dA_3. \tag{2.9}$$

Now we reduce to 10 dimensions, taking the eleventh dimensions to be a circle of radius e^γ. That is, we take the eleven-dimensional metric to be $ds^2 = G^{10}_{mn}dx^m\,dx^n + e^{2\gamma}(dx^{11} - A_m dx^m)^2$ to describe a ten-dimensional metric G^{10} along with a vector A and scalar γ; meanwhile A_3 reduces to a three-form which we still call A_3, and a two-form B (the part of the original A_3 with one index equal to 11). Just for the massless fields, the bosonic part of the action becomes roughly

$$I = \frac{1}{2}\int d^{10}x\sqrt{G^{10}}\left(e^\gamma\left(R + |\nabla\gamma|^2 + |dA_3|^2\right) + e^{3\gamma}|dA|^2 + e^{-\gamma}|dB|^2\right) + \dots. \tag{2.10}$$

This formula, like others below, is very rough and is only intended to exhibit the powers of e^γ. The point in its derivation is that, for example, the part of A_3 that does not have an index equal to "11" has a kinetic energy proportional to e^γ, while the part with such an index has a kinetic energy proportional to $e^{-\gamma}$.

The powers of e^γ in (2.10) do not, at first sight, appear to agree with those in (2.1). To bring them in agreement, we make a Weyl rescaling by writing $G^{10} = e^{-\gamma} g$. Then in terms of the new ten-dimensional metric g, we have

$$I = \frac{1}{2}\int d^{10}x\sqrt{g}\left(e^{-3\gamma}\left(R + |\nabla\gamma|^2 + |dB|^2\right) + |dA|^2 + |dA_3|^2 + \ldots\right). \tag{2.11}$$

We see that (2.11) now does agree with (2.1) if

$$e^{-2\phi} = e^{-3\gamma}. \tag{2.12}$$

In the original eleven-dimensional metric, the radius of the circle is $r(\lambda) = e^\gamma$, but now, relating γ to the dilaton string coupling constant via (2.12), we can write

$$r(\lambda) = e^{\frac{2\phi}{3}} = \lambda^{2/3}. \tag{2.13}$$

The masses of Kaluza-Klein modes of the eleven-dimensional theory are of order $1/r(\lambda)$ when measured in the metric G^{10}, but in the metric g they are of order

$$\frac{e^{-\gamma/2}}{r(\lambda)} \sim \lambda^{-1}. \tag{2.14}$$

Manipulations similar to what we have just seen will be made many times in this paper.

Here are the salient points:

(1) The radius of the circle *grows* by the formula (2.13) as $\lambda \to \infty$. This is important for self-consistency; it means that when λ is large the eleven-dimensional theory is weakly coupled at its compactification scale. Otherwise the discussion in terms of eleven-dimensional field theory would not make sense, and we would not know how to improve on it. As it is, our proposal reduces the strongly coupled Type IIA superstring to a field theory that is weakly coupled at the scale of the low-lying excitations, so we get an effective determination of the strong coupling behavior.

(2) The mass of a particle of charge n, measured in the string metric g in the effective ten-dimensional world, is of order $|n|/\lambda$ from (2.14). This is the dependence on λ claimed in (2.5), which we have now in essence derived: the dependence of the central charge on ϕ is uniquely determined by the low energy supersymmetry, so by deriving this dependence in a Type IIA supergravity theory that comes by Kaluza-Klein reduction from eleven dimensions, we have derived it in general.

So far, the case for relating the strong coupling limit of Type IIA superstrings to eleven-dimensional supergravity consists of the fact that this enables us to make sense of the otherwise puzzling dynamics of the BPS-saturated states and that point (1) above worked out correctly, which was not obvious *a priori*. The case will hopefully get much stronger in the next section when we extend the analysis to work below ten dimensions and incorporate U-duality, and in section four when we look at the heterotic string in seven dimensions. In fact, the most startling aspect of relating strong coupling string dynamics to eleven-dimensional supergravity is the Lorentz invariance that this implies between the eleventh dimension and the original ten. Both in section three and in section four, we will see remnants of this underlying Lorentz invariance.

3. Type II Dynamics Below Ten Dimensions

3.1. U-Duality And Dynamics

In this section, we consider Type II superstrings toroidally compactified to $d < 10$ dimensions, with the aim of understanding the strong coupling dynamics, that is the behavior when some parameters, possibly including the string coupling constant, are taken to extreme values.

The strong coupling behaviors of Type IIA and Type IIB seem to be completely different in ten dimensions, as we have seen. Upon toroidal compactification below ten dimensions, the two theories are equivalent under T-duality [10,11], and so can be considered together. We will call the low energy supergravity theory arising from this compactification Type II supergravity in d dimensions.

The basic tool in the analysis is U-duality. Type II supergravity in d dimensions

has a moduli space of vacua of the form G/K, where G is a non-compact connected Lie group (which depends on d) and K is a compact subgroup, generally a maximal compact subgroup of G. G is an exact symmetry of the supergravity theory. There are also $U(1)$ gauge bosons, whose charges transform as a representation of G. [8] The structure was originally found by dimensional reduction from eleven dimensions [23].

In the string theory realization, the moduli space of vacua remains G/K since this is forced by the low energy supergravity. Some of the Goldstone bosons parametrizing G/K come from the NS-NS sector and some from the RR sector. The same is true of the gauge bosons. In string theory, the gauge bosons that come from the NS-NS sector couple to charged states in the elementary string spectrum. It is therefore impossible for G to be an exact symmetry of the string theory – it would not preserve the lattice of charges. The U-duality conjecture says that an integral form of G, call it $G(\mathbf{Z})$, is a symmetry of the string theory. If so, then as the NS-NS gauge bosons couple to BPS-saturated charges, the same must be true of the RR gauge bosons – though the charges in question do not appear in the elementary string spectrum. The existence of such RR charges was our main assumption in the last section; we see that this assumption is essentially a consequence of U-duality.

The BPS saturated states are governed by an exact mass formula – which will be described later in some detail – which shows how some of them become massless when one approaches various limits in the moduli space of vacua. Our main dynamical assumption is that the smallest mass scale appearing in the mass formula is always the smallest mass scale in the theory.

We assume that at a generic point in G/K, the only massless states are those in the supergravity multiplet. There is then nothing to say about the dynamics: the infrared behavior is that of d dimensional Type II supergravity. There remains the question of

[8] To make a G-invariant theory on G/K, the matter fields in general must be in representations of the unbroken symmetry group K. Matter fields that are in representations of K that do not extend to representations of G are sections of some homogeneous vector bundles over G/K with non-zero curvature. The potential existence of an integer lattice of charges forces the gauge bosons to be sections instead of a flat bundle, and that is why they are in a representation of G and not only of K.

what happens when one takes various limits in G/K – for instance, limits that correspond to weak coupling or large radius or (more mysteriously) strong coupling or very strong excitation of RR scalars. We will take the signal that something interesting is happening to be that the mass formula predicts that some states are going to zero mass. When this occurs, we will try to determine what is the dynamics of the light states, in whatever limit is under discussion.

We will get a complete answer, in the sense that for every degeneration of the Type II superstring in d dimensions, there is a natural candidate for the dynamics. In fact, there are basically only two kinds of degeneration; one involves weakly coupled string theory, and the other involves weakly coupled eleven-dimensional supergravity. In one kind of degeneration, one sees toroidal compactification of a Type II superstring from ten to d dimensions; the degeneration consists of the fact that the string coupling constant is going to zero. (The parameters of the torus are remaining fixed.) In the other degeneration one sees toroidal compactification of eleven-dimensional supergravity from eleven to d dimensions; the degeneration consists of the fact that the radius of the torus is going to infinity so that again the coupling constant at the compactification scale is going to zero.[9] (These are actually the degenerations that produce maximal sets of massless particles; others correspond to perturbations of these.)

Thus, with our hypotheses, one gets a complete control on the dynamics, including strong coupling. Every limit which one might have been tempted to describe as "strong coupling" actually has a weakly coupled description in the appropriate variables. The ability to get this consistent picture can be taken as evidence that the hypotheses are true, that U-duality is valid, and that eleven-dimensional supergravity plays the role in the dynamics that was claimed in section two.

It may seem unexpected that weakly coupled string theory appears in this analysis as a "degeneration," where some particles go to zero mass, so let me explain this situation. For $d < 9$, G is semi-simple, and the dilaton is unified with other scalars. The "string" version

[9] It is only in the eleven-dimensional description that the radius is going to infinity. In the ten-dimensional string theory description, the radius is fixed but the string coupling constant is going to infinity.

of the low energy effective action, in which the dilaton is singled out in the gravitational kinetic energy

$$\int d^d x \sqrt{g} e^{-2\phi} R \tag{3.1}$$

is unnatural for exhibiting such a symmetry. The G-invariant metric is the one obtained by a Weyl transformation that removes the $e^{-2\phi}$ from the gravitational kinetic energy. The transformation in question is of course the change of variables $g = e^{4\phi/(d-2)} g'$, with g' the new metric. This transformation multiplies masses by $e^{2\phi/(d-2)}$, that is, by

$$w_d = \lambda^{2/(d-2)} \tag{3.2}$$

(with λ the string coupling constant). Thus, while elementary string states have masses of order one with respect to the string metric, their masses are of order $\lambda^{2/(d-2)}$ in the natural units for discussions of U-duality. So, from this point of view, the region of weakly coupled string theory is a "degeneration" in which some masses go to zero.

It is amusing to consider that, in a world in which supergravity was known and string theory unknown, the following discussion might have been carried out, with a view to determining the strong coupling limit of a hypothetical consistent theory related to Type II supergravity. The string theory degeneration might then have been found, giving a clue to the existence of this theory. Similarly, the strong coupling analysis that we are about to perform might *a priori* have uncovered new theories beyond string theory and eleven-dimensional supergravity, but this will not be the case.

3.2. The Nature Of Infinity

It is useful to first explain – without specific computations – why NS-NS (rather than RR) moduli play the primary role.

We are interested in understanding what particles become light – and how they interact – when one goes to infinity in the moduli space $G(\mathbf{Z})\backslash G/K$. The discussion is simplified by the fact that the groups G that arise in supergravity are the maximally split forms of the corresponding Lie groups. This simply means that they contain a maximal abelian subgroup A which is a product of copies of $\mathbf{R}^*$ (rather than $U(1)$). [10]

[10] Algebraists call A a "maximal torus," and T would be the standard name, but I will avoid

For instance, in six dimensions $G = SO(5,5)$, with rank 5. One can think of G as the orthogonal group acting on the sum of five copies of a two dimensional real vector space H endowed with quadratic form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

Then a maximal abelian subgroup of G is the space of matrices looking like a sum of five 2×2 blocks, of the form

$$\begin{pmatrix} e^{\lambda_i} & 0 \\ 0 & e^{-\lambda_i} \end{pmatrix} \tag{3.4}$$

for some λ_i. This group is of the form $(\mathbf{R}^*)^5$. Likewise, the integral forms arising in T and U duality are the maximally split forms over $\mathbf{Z}$; for instance the T-duality group upon compactification to $10-d$ dimensions is the group of integral matrices preserving a quadratic form which is the sum of d copies of (3.3). This group is sometimes called $SO(d,d;\mathbf{Z})$.

With the understanding that G and $G(\mathbf{Z})$ are the maximally split forms, the structure of infinity in $G(\mathbf{Z})\backslash G/K$ is particularly simple. A fundamental domain in $G(\mathbf{Z})\backslash G/K$ consists of group elements of the form $g = tu$, where the notation is as follows. u runs over a *compact* subset U of the space of generalized upper triangular matrices; compactness of U means that motion in U is irrelevant in classifying the possible ways to "go to infinity." t runs over A/W where A was described above, and W is the Weyl group.

Thus, one can really only go to infinity in the A direction, and moreover, because of dividing by W, one only has to consider going to infinity in a "positive" direction.

Actually, A has a very simple physical interpretation. Consider the special case of compactification from 10 to $10-d$ dimensions on an orthogonal product of circles $\mathbf{S}^1_i$ of radius r_i. Then G has rank $d+1$, so A is a product of $d+1$ $\mathbf{R}^*$'s. d copies of $\mathbf{R}^*$ act by rescaling the r_i (making up a maximal abelian subgroup of the T-duality group $SO(d,d)$), and the last one rescales the string coupling constant. So in particular, with this choice of A, if one starts at a point in moduli space at which the RR fields are all zero, they remain zero under the action of A.

this terminology because (i) calling $(\mathbf{R}^*)^n$ a "torus" might be confusing, especially in the present context in which there are so many other tori; (ii) in the present problem the letter T is reserved for the T-duality group.

Thus, one can probe all possible directions at infinity without exciting the RR fields; directions in which some RR fields go to infinity are equivalent to directions in which one only goes to infinity via NS-NS fields. Moreover, by the description of A just given, going to infinity in NS-NS directions can be understood to mean just taking the string coupling constant and the radial parameters of the compactification to zero or infinity.

3.3. The Central Charges And Their Role

Let us now review precisely why it is possible to predict particle masses from U-duality. The unbroken subgroup K of the supergravity symmetry group G is realized in Type II supergravity as an R-symmetry group; that is, it acts non-trivially on the supersymmetries. K therefore acts on the central charges in the supersymmetry algebra. The scalar fields parametrizing the coset space G/K enable one to write a G-invariant formula for the central charges (which are a representation of K) of the gauge bosons (which are a representation of G). For most values of d, the formula is uniquely determined, up to a multiplicative constant, by G-invariance, so the analysis does not require many details of supergravity. That is fortunate as not all the details we need have been worked out in the literature, though many can be found in [24].

For example, let us recall (following [3]) the situation in $d = 4$. The T-duality group is $SO(6,6)$, and S-duality would be $SL(2)$ (acting on the axion-dilaton system and exchanging electric and magnetic charge). $SO(6,6) \times SL(2)$ is a maximal subgroup of the U-duality group which is $G = E_7$ (in its non-compact, maximally split form) and has $K = SU(8)$ as a maximal compact subgroup.

Toroidal compactification from ten to four dimensions produces in the NS-NS sector twelve gauge bosons coupling to string momentum and winding states, and transforming in the twelve-dimensional representation of $SO(6,6)$. The electric and magnetic charges coupling to any one of these gauge bosons transform as a doublet of $SL(2)$, so altogether the NS-NS sector generates a total of 24 gauge charges, transforming as $(\mathbf{12}, \mathbf{2})$ of $SO(6,6) \times SL(2)$.

From the RR sector, meanwhile, one gets 16 vectors. (For instance, in Type IIA, the vector of the ten-dimensional RR sector gives 1 vector in four dimensions, and the

three-form gives $6 \cdot 5/2 = 15$.) These 16 states give a total of $16 \cdot 2 = 32$ electric and magnetic charges, which can be argued to transform in an irreducible spinor representation of $SO(6,6)$ (of positive or negative chirality for Type IIA or Type IIB), while being $SL(2)$ singlet. The fact that these states are $SL(2)$ singlets means that there is no natural way to say which of the RR charges are electric and which are magnetic. Altogether, there are $24 + 32 = 56$ gauge charges, transforming as

$$(\mathbf{12}, \mathbf{2}) \oplus (\mathbf{32}, \mathbf{1}) \tag{3.5}$$

under $SO(6,6) \times SL(2)$; this is the decomposition of the irreducible **56** of E_7. Let us call the space of these charges V.

The four-dimensional theory has $N = 8$ supersymmetry; thus there are eight positive-chirality supercharges Q^i_α, $i = 1 \ldots 8$, transforming in the **8** of $K = SU(8)$. The central charges, arising in the formula

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}, \tag{3.6}$$

therefore transform as the second rank antisymmetric tensor of $SU(8)$, the **28**: this representation has complex dimension 28 or real dimension 56. Denote the space of Z^{ij}'s as W.

Indeed, the **56** of E_7, when restricted to $SU(8)$, coincides with the **28**, regarded as a 56-dimensional real representation. (Equivalently, the **56** of E_7 when complexified decomposes as $\mathbf{28} \oplus \overline{\mathbf{28}}$ of $SU(8)$.) There is of course a natural, $SU(8)$-invariant metric on W. As the **56** is a pseudoreal rather than real representation of E_7, there is no E_7-invariant metric on V. However, as V and W coincide when regarded as representations of $SU(8)$, one can pick an embedding of $SU(8)$ in E_7 and then define an $SU(8)$-covariant map $T : V \to W$ which determines a metric on V.

There is no reason to pick one embedding rather than another, and indeed the space of vacua $E_7/SU(8)$ of the low energy supergravity theory can be interpreted as the space of all $SU(8)$ subgroups of E_7. Given $g \in E_7$, we can replace $T : V \to W$ by

$$T_g = T g^{-1}. \tag{3.7}$$

This is not invariant under $g \to gk$, with $k \in SU(8)$, but it is so invariant up to an $SU(8)$ transformation of W. So let $\psi \in V$ be a vector of gauge charges of some string state. Then

$$\psi \to Z(\psi) = T_g \psi \tag{3.8}$$

gives a vector in W, representing the central charges of ψ. The map from "states" ψ to central charges $Z(\psi)$ is manifestly E_7-invariant, that is invariant under

$$\begin{aligned} \psi &\to g'\psi \\ g &\to g'g. \end{aligned} \tag{3.9}$$

Also, under $g \to gk$, with $k \in SU(8)$, Z transforms to $Tk^{-1}T^{-1}Z$, that is, it transforms by a "local $SU(8)$ transformation" that does not affect the norm of the central charge. The formula (3.8) is, up to a constant multiple, the only formula with these properties, so it is the one that must come from the supergravity or superstring theory.

In supersymmetric theories with central charges, there is an inequality between the mass of a state and the central charge. For elementary string winding states and their partners under U-duality, the inequality is $M \geq |Z|$. (More generally, the inequality is roughly that M is equal to or greater than the largest eigenvalue of Z; for a description of stringy black holes with more than one eigenvalue, see [19]. Elementary string states have only one eigenvalue.)

So far, we have not mentioned the integrality of the gauge charges. Actually, states carrying the 56 gauge charges only populate a lattice $V_{\mathbf{Z}} \subset V$. If U-duality is true, then each lattice point related by U-duality to the gauge charges of an elementary string state represents the charges of a supermultiplet of mass $|Z(\psi)|$.

As an example of the use of this formalism, let us keep a promise made in section two and give an alternative deduction, assuming U-duality, of the important statement that the masses of states carrying RR charges are (in string units) of order $1/\lambda$.[11] Starting from any given vacuum, consider the one-parameter family of vacua determined by the following one-parameter subgroup of $SO(6,6) \times SL(2)$: we take the identity in $SO(6,6)$

[11] The following argument was pointed out in parallel by C. Hull.

(so that the parameters of the toroidal compactification are constant) times

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \tag{3.10}$$

in $SL(2)$ (so as to vary the string coupling constant). We work here in a basis in which the "top" component is electric and the "bottom" component is magnetic.

Using the mass formula $M(\psi) = |Z(\psi)| = |Tg^{-1}\psi|$, the t dependence of the mass of a state comes entirely from the g action on the state. The NS-NS states, as they are in a doublet of $SL(2)$, have "electric" components whose masses scale as e^{-t} and "magnetic" components with masses of e^t. On the other hand, as the RR states are $SL(2)$ singlets, the mass formula immediately implies that their masses are independent of t.

These are really the masses in the U-dual "Einstein" metric. Making a Weyl transformation to the "string" basis in which the electric NS-NS states (which are elementary string states) have masses of order one, the masses are as follows: electric NS-NS, $M \sim 1$; magnetic NS-NS, $M \sim e^{2t}$; RR, $M \sim e^t$. But since we know that the magnetic NS-NS states (being fairly conventional solitons) have masses of order $1/\lambda^2$, we identify $e^t = 1/\lambda$ (a formula one could also get from the low energy supergravity); hence the RR masses are of order $1/\lambda$ as claimed. [12]

The basic properties described above hold in any dimension above three. (In nine dimensions, some extra care is needed because the U-duality group is not semi-simple.) In three dimensions, new phenomena, which we will not try to unravel, appear because vectors are dual to scalars and charges are confined (for some of the relevant material, see [25]).

3.4. Analysis Of Dynamics

We now want to justify the claims made at the beginning of this section about the strong coupling dynamics.

[12] We made this deduction here in four dimensions, but it could be made, using U-duality, in other dimensions as well. Outside of four dimensions, instead of using the known mass scale of magnetic monopoles to fix the relation between t and λ, one could use the known Weyl transformation (3.2) between the string and U-dual mass scales.

To do this, we will analyze limits of the theory in which some of the BPS-saturated particles go to zero mass. Actually, for each way of going to infinity, we will look only at the particles whose masses goes to zero as fast as possible. We will loosely call these the particles that are massless at infinity.

Also, we really want to find the "maximal" degenerations, which produce maximal sets of such massless particles; a set of massless particles, produced by going to infinity in some direction, is maximal if there would be no way of going to infinity such that those particles would become massless together with others. A degeneration (i.e., a path to infinity) that produces a non-maximal set of massless particles should be understood as a perturbation of a maximal degeneration. (In field theory, such perturbations, which partly lift the degeneracy of the massless particles, are called perturbations by relevant operators.) We will actually also check a few non-maximal degenerations, just to make sure that we understand their physical interpretation.

To justify our claims, we should show that in any dimension d, there are only two maximal degenerations, which correspond to toroidal compactification of weakly coupled ten-dimensional string theory and to toroidal compactification of eleven-dimensional supergravity, respectively. The analysis is in fact very similar in spirit for any d, but the details of the group theory are easier for some values of d than others. I will first explain a very explicit analysis for $d = 7$, chosen as a relatively easy case, and then explain an efficient approach for arbitrary d.

In $d = 7$, the T-duality group is $SO(3,3)$, which is the same as $SL(4)$; U-duality extends this to $G = SL(5)$. A maximal compact subgroup is $K = SO(5)$.

In the NS-NS sector, there are six $U(1)$ gauge fields that come from the compactification on a three-torus; they transform as a vector of $SO(3,3)$ or second rank antisymmetric tensor of $SL(4)$. In addition; four more $U(1)$'s, transforming as a spinor of $SO(3,3)$ or a **4** of $SL(4)$, come from the RR sector. These states combine with the six from the NS-NS sector to make the second rank antisymmetric tensor, the **10** of $SL(5)$.

In Type II supergravity in seven dimensions, the maximal possible R-symmetry is $K = SO(5)$ or $Sp(4)$. The supercharges make up in fact four pseudo-real spinors Q^i_α, $i = 1 \ldots 4$, of the seven-dimensional Lorentz group $SO(1,6)$, transforming as the **4** of

$Sp(4)$. The central charges transform in the symmetric part of $\mathbf{4} \times \mathbf{4}$, which is the $\mathbf{10}$ or antisymmetric tensor of $SO(5)$. Thus, we are in a situation similar to what was described earlier in four dimensions: the gauge charges transform as the $\mathbf{10}$ of $SL(5)$, the central charges transform in the $\mathbf{10}$ of $SO(5)$, and a choice of vacuum in $G/K = SL(5)/SO(5)$ selects an $SO(5)$ subgroup of $SL(5)$, enabling one to identify these representations and map gauge charges to central charges.

A maximal abelian subgroup A of $SL(5)$ is given by the diagonal matrices. A one-parameter subgroup of A consists of matrices of the form

$$g_t = \begin{pmatrix} e^{a_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{a_2 t} & 0 & 0 & 0 \\ 0 & 0 & e^{a_3 t} & 0 & 0 \\ 0 & 0 & 0 & e^{a_4 t} & 0 \\ 0 & 0 & 0 & 0 & e^{a_5 t} \end{pmatrix} \tag{3.11}$$

where the a_i are constants, not all zero, with $\sum_i a_i = 0$. We want to consider the behavior of the spectrum as $t \to +\infty$. By a Weyl transformation, we can limit ourselves to the case that

$$a_1 \geq a_2 \geq \ldots \geq a_5. \tag{3.12}$$

Let ψ_{ij}, $i < j$ be a vector in the $\mathbf{10}$ of $SL(5)$ whose components are zero except for the ij component, which is 1 (and the ji component, which is -1). We will also use the name ψ_{ij} for a particle with those gauge charges. The mass formula $M(\psi) = |Tg^{-1}\psi|$ says that the mass of ψ_{ij} scales with t as

$$M(\psi_{ij}) \sim e^{-t(a_i + a_j)}. \tag{3.13}$$

By virtue of (3.12), the lightest type of particle is ψ_{12}. For generic values of the a_i, this is the unique particle whose mass scales to zero fastest, but if $a_2 = a_3$ then ψ_{12} is degenerate with other particles. To get a maximal set of particles degenerate with ψ_{12}, we need a maximal set of a_i equal to a_2 and a_3. We cannot set all a_i equal (then they have to vanish, as $\sum_i a_i = 0$), so by virtue of (3.12), there are two maximal cases, with $a_1 = a_2 = a_3 = a_4$, or $a_2 = a_3 = a_4 = a_5$. So the maximal degenerations correspond to

one-parameter subgroups

$$g_t = \begin{pmatrix} e^t & 0 & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 & 0 \\ 0 & 0 & e^t & 0 & 0 \\ 0 & 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & 0 & e^{-4t} \end{pmatrix} \tag{3.14}$$

or

$$g_t = \begin{pmatrix} e^{4t} & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^{-t} \end{pmatrix} \tag{3.15}$$

with $t \to +\infty$. As we will see, the first corresponds to weakly coupled string theory, and the second to eleven-dimensional supergravity.

In (3.14), the particles whose masses vanish for $t \to +\infty$ are the ψ_{ij} with $1 \leq i < j \leq 4$. There are six of these, the correct number of light elementary string states of string theory compactified from ten to seven dimensions. Moreover, in (3.14), g_t commutes with a copy of $SL(4)$ that acts on indices $1-2-3-4$. This part of the seven-dimensional symmetry group $SL(5)$ is unbroken by going to infinity in the direction (3.14), and hence would be observed as a symmetry of the low energy physics at "infinity" (though most of the symmetry is spontaneously broken in any given vacuum near infinity). Indeed, $SL(4)$ with six gauge charges in the antisymmetric tensor representation is the correct T-duality group of weakly coupled string theory in seven dimensions.

There is a point here that may be puzzling at first sight. The full subgroup of $SL(5)$ that commutes with g_t is actually not $SL(4)$ but $SL(4) \times \mathbf{R}^*$, where $\mathbf{R}^*$ is the one-parameter subgroup containing g_t. What happens to the $\mathbf{R}^*$? When one restricts to the *integral* points in $SL(5)$, which are the true string symmetries, this $\mathbf{R}^*$ does not contribute, so the symmetry group at infinity is just the integral form of $SL(4)$. A similar comment applies at several points below and will not be repeated.

Moving on now to the second case, in (3.15), the particles whose masses vanish for $t \to +\infty$ are the ψ_{1i}, $i > 1$. There are four of these, the correct number for compactification of eleven-dimensional supergravity on a four-torus $\mathbf{T}^4$ whose dimensions are growing with t. The gauge charges of light states are simply the components of the momentum along $\mathbf{T}^4$.

The symmetry group at infinity is again $SL(4)$. This $SL(4)$ has a natural interpretation as a group of linear automorphisms of $\mathbf{T}^4$. [13] In fact, the gauge charges carried by the light states in (3.15) transform in the **4** of $SL(4)$, which agrees with the supergravity description as that is how the momentum components along $\mathbf{T}^4$ transform under $SL(4)$. As this $SL(4)$ mixes three of the "original" ten dimensions with the eleventh dimension that is associated with strong coupling, we have our first evidence for the underlying eleven-dimensional Lorentz invariance.

Finally, let us consider a few non-maximal degenerations, to make sure we understand how to interpret them. [14] Degeneration in the direction

$$g_t = \begin{pmatrix} e^{3t} & 0 & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & 0 & e^{-2t} \end{pmatrix} \tag{3.16}$$

leaves as $t \to \infty$ the unique lightest state ψ_{12}. I interpret this as coming from partial decompactification to eight dimensions – taking one circle much larger than the others so that the elementary string states with momentum in that one direction are the lightest. This family has the symmetry group $SL(3) \times SL(2)$, which is indeed the U-duality group in eight dimensions, as it should be.

The family

$$g_t = \begin{pmatrix} e^{2t} & 0 & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 0 & e^{-3t} & 0 \\ 0 & 0 & 0 & 0 & e^{-3t} \end{pmatrix} \tag{3.17}$$

gives three massless states ψ_{ij}, $1 \le i < j \le 3$, transforming as $(\mathbf{3}, \mathbf{1})$ of the symmetry group $SL(3) \times SL(2)$. I interpret this as decompactification to the Type IIB theory in

[13] That is, if $\mathbf{T}^4$ is understood as the space of real variables y^i, $i = 1 \ldots 4$, modulo $y^i \to y^i + n^i$, with $n^i \in \mathbf{Z}$, then $SL(4)$ acts by $y^i \to w^i{}_j y^j$. For this to be a diffeomorphism and preserve the orientation, the determinant of w must be one, so one is in $SL(4)$. Given an n-torus $\mathbf{T}^n$, we will subsequently use the phrase "mapping class group" to refer to the $SL(n)$ that acts linearly in this sense on $\mathbf{T}^n$.

[14] We will see in the next section that when the U-duality group has rank r, there are r naturally distinguished one-parameter subgroups. For $SL(5)$, these are (3.14), (3.15), and the two introduced below.

ten dimensions – taking all three circles to be very large. The three light charges are the momenta around the three circles; $SL(3)$ is the mapping class group of the large three-torus, and $SL(2)$ is the U-duality group of the Type IIB theory in ten dimensions.

Partially Saturated States

I will now justify an assumption made above and also make a further test of the interpretation that we have proposed.

First of all, we identified BPS-saturated elementary string states with charge tensors ψ_{ij} with (in the right basis) only one non-zero entry. Why was this valid?

We may as well consider NS-NS states; then we can restrict ourselves to the T-duality group $SO(3,3)$. The gauge charges transform in the vector representation of $SO(3,3)$. Given such a vector v_a, one can define the quadratic invariant $(v,v) = \sum_{a,b} \eta^{ab} v_a v_b$.

On the other hand, $SO(3,3)$ is the same as $SL(4)$, and v is equivalent to a second rank antisymmetric tensor ψ of $SL(4)$. In terms of ψ, the quadratic invariant is $(\psi,\psi) = \frac{1}{4}\epsilon^{ijkl}\psi_{ij}\psi_{kl}$. By an $SL(4)$ transformation, one can bring ψ to a normal form in which the independent non-zero entries are ψ_{12} and ψ_{34} only. Then

$$(\psi,\psi) = 2\psi_{12}\psi_{34}. \tag{3.18}$$

So the condition that the particle carries only one type of charge, that is, that only ψ_{12} or ψ_{34} is non-zero, is that $(\psi,\psi) = 0$.

Now let us consider the elementary string states. Such a state has in the toroidal directions left- and right-moving momenta p_L and p_R. p_L and p_R together form a vector of $SO(3,3)$, and the quadratic invariant is [8]

$$(p,p) = |p_L|^2 - |p_R|^2. \tag{3.19}$$

BPS-saturated states have no oscillator excitations for left- or right-movers, and the mass shell condition requires that they obey $|p_L|^2 - |p_R|^2 = 0$, that is, that the momentum or charge vector p is light-like. This implies, according to the discussion in the last paragraph, that in the right basis, the charge tensor ψ has only one entry. That is the assumption we made.

Now, however, we can do somewhat better and consider elementary string states of Type II that are BPS-saturated for left-movers only (or equivalently, for right-movers only). Such states are in "middle-sized" supermultiplets, of dimension 2^{12} (as opposed to generic supermultiplets of dimension 2^{16} and BPS-saturated multiplets of dimension 2^8). To achieve BPS saturation for the left-movers only, one puts the left-moving oscillators in their ground state, but one permits right-moving oscillator excitations; as those excitations are arbitrary, one gets an exponential spectrum of these half-saturated states (analogous to the exponential spectrum of BPS-saturated states in the heterotic string [26]). With oscillator excitations for right-movers only, the mass shell condition implies that $|p_L|^2 > |p_R|^2$, and hence the charge vector is not lightlike. The charge tensor ψ therefore in its normal form has both ψ_{12} and ψ_{34} non-zero. For such states, the mass inequality says that the mass is bounded below by the largest eigenvalue of $Tg^{-1}\psi$, with equality for the "middle-sized" multiplets.

With this in mind, let us consider the behavior of such half-saturated states in the various degenerations. In the "stringy" degeneration (3.14), a state with non-zero ψ_{12} and ψ_{34} has a mass of the same order of magnitude as a state with only ψ_{12} non-zero. This is as we would expect from weakly coupled string theory with toroidal radii of order one: the half-saturated states have masses of the same order of magnitude as the BPS-saturated massive modes. To this extent, string excitations show up in the strong coupling analysis.

What about the "eleven-dimensional" degeneration (3.15)? In this case, while the particles with only one type of charge have masses that vanish as e^{-3t} for $t \to \infty$, the particles with two kinds of charge have masses that *grow* as e^{+t}. The only light states that we can see with this formalism in this degeneration are the Kaluza-Klein modes of eleven-dimensional supergravity. There is, for instance, no evidence for membrane excitations; such evidence might well have appeared if a consistent membrane theory with eleven-dimensional supergravity as its low energy limit really does exist.

3.5. Framework For General Analysis

It would be tiresome to repeat this analysis "by hand" in other values of the dimension.

Instead, I will now[15] explain a bit of group theory that makes the analysis easy. One of the main points is to incorporate the action of the Weyl group. This was done above by choosing $a_1 \geq a_2 \geq \ldots \geq a_5$, but to exploit the analogous condition in E_7, for instance, a little machinery is useful.

In d dimensions, the U-duality group G has rank $r = 11-d$. Given any one-parameter subgroup F of a maximal abelian subgroup A, one can pick a set of simple positive roots x_i such that the action of F on the x_i is

$$x_i \to e^{c_i t} x_i \tag{3.20}$$

with c_i *non-negative*. In this restriction on the c_i, we have used the Weyl action. Conversely, for every set of non-negative c_i (not all zero), there is a one-parameter subgroup F that acts as (3.20).

The gauge charges are in some representation R of G; that is, for each weight in R there is a corresponding gauge charge. [16] Let $\rho = \sum_i e_i x_i$ be the highest weight in R. The e_i are positive integers. A particle whose only gauge charge is the one that corresponds to ρ has a mass that vanishes for $t \to +\infty$ as

$$M_\rho \sim \exp\left(-\sum_i c_i e_i t\right). \tag{3.21}$$

Any other weight in R is of the form $\rho' = \sum_i f_i x_i$, with $f_i \leq e_i$. A particle carrying the ρ' charge has mass of order

$$M_{\rho'} \sim \exp\left(-\sum_i c_i f_i t\right). \tag{3.22}$$

Thus $M_{\rho'} \geq M_\rho$ – the particle with only charge ρ always goes to zero mass at least as fast as any other – and $M_{\rho'} = M_\rho$ if and only if

$$c_i = 0 \text{ whenever } f_i < e_i. \tag{3.23}$$

[15] With some assistance from A. Borel.

[16] The particular representations R that actually arise in Type II string theory in $d \geq 4$ have the property (unusual among representations of Lie groups) that the non-zero weight spaces are all one-dimensional. It therefore makes sense to label the gauge charges by weights. (These representations are actually "minuscule" – the Weyl group acts transitively on the weights.) $d \leq 3$ would have some new features, as already mentioned above.

Now, our problem is to pick the subgroup F, that is, the c_i, so that a maximal set of $M_{\rho'}$ are equal to M_ρ. If the c_i are all non-zero, then (as the highest weight state is unique) (3.23) implies that $\rho' = \rho$ and only one gauge charge is carried by the lightest particles. The condition in (3.23) becomes less restrictive only when one of the c_i becomes zero, and to get a maximal set of $M_{\rho'}$ degenerate with M_ρ, we must set as many of the c_i as possible to zero. As the c_i may not all vanish, the best we can do is to set $r-1$ of them to zero. There are therefore precisely r one-parameter subgroups F_i to consider, labeled by which of the c_i is non-zero.

The x_i are labeled by the vertices in the Dynkin diagram of G, so each of the F_i is associated with a particular vertex P_i. Deleting P_i from the Dynkin diagram of G leaves the Dynkin diagram of a rank $r-1$ subgroup H_i of G. It is the unbroken subgroup when one goes to infinity in the F_i direction.

3.6. Analysis In $d = 4$

With this machinery, it is straightforward to analyze the dynamics in each dimension d. As the rank is $r = 11-d$, there are $11-d$ distinguished one-parameter subgroups to check. It turns out that one of them corresponds to weakly coupled string theory in d dimensions, one to toroidal compactification of eleven-dimensional supergravity to d dimensions, and the others to partial (or complete) decompactifications. In each case, the symmetry group when one goes to infinity is the expected one: the T-duality group $SO(10-d, 10-d)$ for the string degeneration; the mapping class group $SL(11-d)$ for supergravity; or for partial decompactification to d' dimensions, the product of the mapping class group $SL(d'-d)$ of a $d'-d$-torus and the U-duality group in d' dimensions.

I will illustrate all this in $d = 4$, where the U-duality group is E_7. Going to infinity in a direction F_i associated with one of the seven points in the Dynkin diagram leaves as unbroken subgroup H_i one of the following:

(1) $SO(6,6)$: this is the T-duality group for string theory toroidally compactified from ten to four dimensions. This is a maximal degeneration, with (as we will see) 12 massless states transforming in the **12** of $SO(6,6)$.

(2) $SL(7)$: this is associated with eleven-dimensional supergravity compactified to

four dimensions on a seven-torus whose mapping class group is $SL(7)$. This is the other maximal degeneration; there are the expected seven massless states in the **7** of $SL(7)$.

(3) E_6: this and the other cases are non-maximal degenerations corresponding to partial decompactification. This case corresponds to partial decompactification to five dimensions by taking one circle to be much larger than the others; there is only one massless state, corresponding to a state with momentum around the large circle. E_6 arises as the U-duality group in five dimensions.

(4) $SL_2 \times SO(5,5)$: this is associated with partial decompactification to six dimensions. There are two light states, corresponding to momenta around the two large circles; they transform as $(\mathbf{2},\mathbf{1})$ under $SL_2 \times SO(5,5)$. SL_2 acts on the two large circles and $SO(5,5)$ is the U-duality group in six dimensions.

(5) $SL_3 \times SL(5)$: this is associated with partial decompactification to seven dimensions. $SL(3)$ acts on the three large circles (and the three light charges), and $SL(5)$ is the U-duality in seven dimensions.

(6) $SL_4 \times SL(3) \times SL(2)$: this is associated with partial decompactification to eight dimensions. $SL(4)$ acts on the four large circles and light charges, and $SL(3) \times SL(2)$ is U-duality in eight dimensions.

(7) $SL_6 \times SL_2$: this is associated with decompactification to Type IIB in ten dimensions. SL_6 acts on the six large circles and light charges, and $SL(2)$ is the U-duality in ten dimensions.

In what follows, I will just check the assertions about the light spectrum for the first two cases, which are the important ones, and the third, which is representative of the others.

(1) F_1 can be described as follows. E_7 contains a maximal subgroup $SO(6,6) \times SL(2)$. F_1 can be taken as the subgroup of $SL(2)$ consisting of matrices of the form

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{3.24}$$

The gauge charges are in the **56** of E_7, which decomposes under L_1 as $(\mathbf{12},\mathbf{2}) \oplus (\mathbf{32},\mathbf{1})$. The lightest states come from the part of the $(\mathbf{12},\mathbf{2})$ that transforms as e^t under (3.24); these are the expected twelve states in the **12** of $SO(6,6)$.

(2) E_7 contains a maximal subgroup $SL(8)$. F_2 can be taken as the subgroup of $SL(8)$ consisting of group elements $g_t = \mathrm{diag}(e^t, e^t, \ldots, e^t, e^{-7t})$. The **56** of E_7 decomposes as $\mathbf{28} \oplus \mathbf{28'}$ – the antisymmetric tensor plus its dual. The states of highest eigenvalue (namely e^{8t}) are seven states in the **28** transforming in the expected **7** of the unbroken $SL(7)$.

(3) E_7 has a maximal subgroup $E_6 \times \mathbf{R}^*$, and F_3 is just the $\mathbf{R}^*$. The **56** of E_7 decomposes as $\mathbf{27}^1 \oplus \mathbf{27'}^{-1} \oplus \mathbf{1}^3 \oplus \mathbf{1}^{-3}$, where the E_6 representation is shown in boldface and the $\mathbf{R}^*$ charge (with some normalization) by the exponent. Thus in the F_3 degeneration, there is a unique lightest state, the $\mathbf{1}^3$.

The reader can similarly analyze the light spectrum for the other F_i, or the analogous subgroups in $d \neq 4$.

4. Heterotic String Dynamics Above Four Dimensions

4.1. A Puzzle In Five Dimensions

S-duality gives an attractive proposal for the strong coupling dynamics of the heterotic string after toroidal compactification to four dimensions: it is equivalent to the same theory at weak coupling. In the remainder of this paper, we will try to guess the behavior above four dimensions. This process will also yield some new insight about S-duality in four dimensions.

Toroidal compactification of the heterotic string from 10 to d dimensions gives $2(10-d)$ vectors that arise from dimensional reduction of the metric and antisymmetric tensor. Some of the elementary string states are electrically charged with respect to these vectors.

Precisely in five dimensions, one more vector arises. This is so because in five dimensions a two-form B_{mn} is dual to a vector A_m, roughly by $dB = *dA$. In the elementary string spectrum, there are no particles that are electrically charged with respect to A, roughly because A can be defined (as a vector) only in five dimensions. But it is easy to see where to find such electric charges. Letting H be the field strength of B (including the Chern-Simons terms) the anomaly equation

$$dH = \mathrm{tr} F \wedge F - \mathrm{tr} R \wedge R \tag{4.1}$$

(F is the $E_8 \times E_8$ or $SO(32)$ field strength and R the Riemann tensor) implies that the electric current of A is

$$J = *\mathrm{tr}F \wedge F - *\mathrm{tr}R \wedge R. \tag{4.2}$$

Thus, with $G = dA$, (4.1) becomes

$$D^m G_{mn} = J_n, \tag{4.3}$$

showing that J_n is the electric current. So the charge density J_0 is the instanton density, and a Yang-Mills instanton, regarded as a soliton in $4+1$ dimensions, is electrically charged with respect to A.

Instantons (and their generalizations to include the supergravity multiplet [27,28]) are invariant under one half of the supersymmetries. One would therefore suspect that quantization of the instanton would give BPS-saturated multiplets, with masses given by the instanton action:

$$M = \frac{16\pi^2|n|}{\lambda^2}. \tag{4.4}$$

Here n is the instanton number or electric charge and λ is the string coupling.

To really prove existence of these multiplets, one would need to understand and quantize the collective coordinates of the stringy instanton. In doing this, one needs to pick a particular vacuum to work in. In the generic toroidal vacuum, the unbroken gauge group is just a product of $U(1)$'s. Then the instantons, which require a non-abelian structure, tend to shrink to zero size, where stringy effects are strong and the analysis is difficult. Alternatively, one can consider a special vacuum with an unbroken non-abelian group, but this merely adds infrared problems to the stringy problems. The situation is analogous to the study [29] of H-monopoles after toroidal compactification to four dimensions; indeed, the present paper originated with an effort to resolve the problems concerning H-monopoles. (The connection between instantons and H-monopoles is simply that upon compactification of one of the spatial directions on a circle, the instantons become what have been called H-monopoles.)

Despite the difficulty in the collective coordinate analysis, there are two good reasons to believe that BPS-saturated multiplets in this sector do exist. One, already mentioned,

is the invariance of the classical solution under half the supersymmetries. The second reason is that if in five dimensions, the electrically charged states had masses bounded strictly above the BPS value in (4.4), the same would be true after compactification on a sufficiently big circle, and then the BPS-saturated H-monopoles required for S-duality could not exist.

Accepting this assumption, we are in a similar situation to that encountered earlier for the Type IIA string in ten dimensions: there is a massless vector, which couples to electric charges whose mass diverges for weak coupling. (The mass is here proportional to $1/\lambda^2$ in contrast to $1/\lambda$ in the other case.) Just as in the previous situation, we have a severe puzzle if we take the formula seriously for strong coupling, when these particles seem to go to zero mass.

If we are willing to take (4.4) seriously for strong coupling, then we have for each integer n a supermultiplet of states of charge n and mass proportional to $|n|$, going to zero mass as $\lambda \to \infty$. It is very hard to interpret such a spectrum in terms of local field theory in five dimensions. But from our previous experience, we know what to do: interpret these states as Kaluza-Klein states on $\mathbf{R}^5 \times \mathbf{S}^1$.

The $\mathbf{S}^1$ here will have to be a "new" circle, not to be confused with the five-torus $\mathbf{T}^5$ in the original toroidal compactification to five dimensions. (For instance, the T-duality group $SO(21,5)$ acts on $\mathbf{T}^5$ but not on the new circle.) So altogether, we seem to have eleven dimensions, $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{T}^5$, and hence we seem to be in need of an eleven-dimensional supersymmetric theory.

In section two, eleven-dimensional supergravity made a handy appearance at this stage, but here we seem to be in a quandary. There is no obvious way to introduce an eleventh dimension relevant to the heterotic string. Have we reached a dead end?

4.2. The Heterotic String In Six Dimensions

Luckily, there is a conjectured relation between the heterotic string and Type II superstrings [3,4] which has just the right properties to solve our problem (though not by leading us immediately back to eleven dimensions). The conjecture is that the heterotic string toroidally compactified to *six* dimensions is equivalent to the Type IIA superstring

compactified to six dimensions on a K3 surface.

The evidence for this conjecture has been that both models have the same supersymmetry and low energy spectrum in six dimensions and the same moduli space of vacua, namely $SO(20,4;\mathbf{Z})\backslash SO(20,4;\mathbf{R})/(SO(20)\times SO(4))$. For the toroidally compactified heterotic string, this structure for the moduli space of vacua is due to Narain [8]; for Type II, the structure was determined locally by Seiberg [30] and globally by Aspinwall and Morrison [31].

In what follows, I will give several new arguments for this "string-string duality" between the heterotic string and Type IIA superstrings:

(1) When one examines more precisely how the low energy effective actions match up, one finds that weak coupling of one theory corresponds to strong coupling of the other theory. This is a necessary condition for the duality to make sense, since we certainly know that the heterotic string for weak coupling is not equivalent to the Type IIA superstring for weak coupling.

(2) Assuming string-string duality in six dimensions, we will be able to resolve the puzzle about the strong coupling dynamics of the heterotic string in five dimensions. The strongly coupled heterotic string on $\mathbf{R}^5$ (times a five-torus whose parameters are kept fixed) is equivalent to a Type IIB superstring on $\mathbf{R}^5\times\mathbf{S}^1$ (times a K3 whose parameters are kept fixed). The effective six-dimensional Type IIB theory is weakly coupled at its compactification scale, so this is an effective solution of the problem of strong coupling for the heterotic string in five dimensions.

(3) We will also see that – as anticipated by Duff in a more abstract discussion [7] – string-string duality in six dimensions implies S-duality of the heterotic string in four dimensions. Thus, all evidence for S-duality can be interpreted as evidence for string-string duality, and one gets at least a six-dimensional answer to the question "what higher dimensional statement leads to S-duality in four dimensions?"

(4) The K3 becomes singular whenever the heterotic string gets an enhanced symmetry group; the singularities have an $A-D-E$ classification, just like the enhanced symmetries.

(5) Finally, six-dimensional string-string duality also leads to an attractive picture for heterotic string dynamics in seven dimensions. (Above seven dimensions the analysis

would be more complicated.)

I would like to stress that some of these arguments test more than a long distance relation between the heterotic string and strongly coupled Type IIA. For instance, in working out the five-dimensional dynamics via string-string duality, we will be led to a Type IIA theory with a *small* length scale, and to get a semi-classical description will require a T-duality transformation, leading to Type IIB. The validity of the discussion requires that six-dimensional string-string duality should be an exact equivalence, like the $SL(2,\mathbf{Z})$ symmetry for Type IIB in ten dimensions and unlike the relation of Type II to eleven-dimensional supergravity.

4.3. Low Energy Actions

Let us start by writing a few terms in the low energy effective action of the heterotic string, toroidally compactified to six dimensions. We consider the metric g, dilaton ϕ, and antisymmetric tensor field B, and we let C denote a generic abelian gauge field arising from the toroidal compactification. We are only interested in keeping track of how the various terms scale with ϕ. For the heterotic string, the whole classical action scales as $e^{-2\phi} \sim \lambda^{-2}$, so one has very roughly

$$I = \int d^6x \sqrt{g} e^{-2\phi} \left(R + |\nabla\phi|^2 + |dB|^2 + |dC|^2 \right). \tag{4.5}$$

On the other hand, consider the Type IIA superstring in six dimensions. The low energy particle content is the same as for the toroidally compactified heterotic string, at least at a generic point in the moduli space of the latter where the unbroken gauge group is abelian. Everything is determined by $N = 4$ supersymmetry except the number of $U(1)$'s in the gauge group and the number of antisymmetric tensor fields; requiring that these match with the heterotic string leads one to use Type IIA rather than Type IIB. So in particular, the low energy theory derived from Type IIA has a dilaton ϕ', a metric g', an antisymmetric tensor field B', and gauge fields C'. [17] Here ϕ', g', and B' come from the

[17] We normalize B' and C' to have standard gauge transformation laws. Their gauge transformations would look different if one scaled the fields by powers of e^{ϕ}. This point was discussed in section two.

NS-NS sector, but C' comes from the RR sector, so as we noted in section two, the kinetic energy of ϕ', g', and B' scales with the dilaton just like that in (4.5), but the kinetic energy of C' has no coupling to the dilaton. So we have schematically

$$I' = \int d^6x\sqrt{g'}\left(e^{-2\phi'}\left(R' + |\nabla\phi'|^2 + |dB'|^2\right) + |dC'|^2\right). \tag{4.6}$$

We need the change of variables that turns (4.5) into (4.6). In (4.5), the same power of e^{ϕ} multiplies R and $|dC'|^2$. We can achieve that result in (4.6) by the change of variables $g' = g''e^{2\phi'}$. Then (4.6) becomes

$$I' = \int d^6x\sqrt{g''}\left(e^{2\phi'}(R'' + |\nabla\phi'|^2) + e^{-2\phi'}|dB'|^2 + e^{2\phi'}|dC'|^2\right). \tag{4.7}$$

Now the coefficient of the kinetic energy of B' is the opposite of what we want, but this can be reversed by a duality transformation. The field equations of B' say that $d*(e^{-2\phi'}dB') = 0$, so the duality transformation is

$$e^{-2\phi'}dB' = *dB''. \tag{4.8}$$

Then (4.7) becomes

$$I' = \int d^6x\sqrt{g''}e^{2\phi'}\left(R'' + |\nabla\phi'|^2 + |dB''|^2 + |dC'|^2\right). \tag{4.9}$$

This agrees with (4.5) if we identify $\phi = -\phi'$. Putting everything together, the change of variables by which one can identify the low energy limits of the two theories is

$$\begin{aligned} \phi &= -\phi' \\ g &= e^{2\phi}g' = e^{-2\phi'}g' \\ dB &= e^{-2\phi'} * dB' \\ C &= C'. \end{aligned} \tag{4.10}$$

Unprimed and primed variables are fields of the heterotic string and Type IIA, respectively.

In particular, the first equation implies that weak coupling of one theory is equivalent to strong coupling of the other. This makes it possible for the two theories to be equivalent without the equivalence being obvious in perturbation theory.

4.4. Dynamics In Five Dimensions

Having such a (conjectured) exact statement in six dimensions, one can try to deduce the dynamics below six dimensions. The ability to do this is not automatic because (just as in field theory) the dimensional reduction might lead to new dynamical problems at long distances. But we will see that in this particular case, the string-string duality in six dimensions does determine what happens in five and four dimensions.

We first compactify the heterotic string from ten to six dimensions on a torus (which will be kept fixed and not explicitly mentioned), and then take the six-dimensional world to be $\mathbf{R}^5 \times \mathbf{S}^1_r$, where $\mathbf{S}^1_r$ will denote a circle of radius r. We want to keep r fixed and take $\lambda = e^\phi$ to infinity. According to (4.10), the theory in this limit is equivalent to the Type IIA superstring on $\mathbf{R}^5 \times \mathbf{S}^1_{r'}$ times a K3 surface (of fixed moduli), with string coupling and radius λ' and r' given by

$$\begin{aligned} \lambda' &= \lambda^{-1} \\ r' &= \lambda^{-1} r. \end{aligned} \tag{4.11}$$

In particular, the coupling λ' goes to zero in the limit for $\lambda \to \infty$. However, the radius r' in the dual theory is also going to zero. The physical interpretation is much clearer if one makes a T-duality transformation, replacing r' by

$$r'' = \frac{1}{r'} = \frac{\lambda}{r}. \tag{4.12}$$

The T-duality transformation also acts on the string coupling constant. This can be worked out most easily by noting that the effective five-dimensional gravitational constant, which is λ^2/r, must be invariant under the T-duality. So under $r' \to 1/r'$, the string coupling λ' is replaced by

$$\lambda'' = \frac{\lambda'}{r'} \tag{4.13}$$

so that

$$\frac{r'}{(\lambda')^2} = \frac{r''}{(\lambda'')^2}. \tag{4.14}$$

Combining this with (4.11), we learn that the heterotic string on $\mathbf{R}^5 \times \mathbf{S}^1_r$ and string coupling λ is equivalent to a Type II superstring with coupling and radius

$$\begin{aligned} \lambda'' &= r^{-1} \\ r'' &= \frac{\lambda}{r}. \end{aligned} \tag{4.15}$$

This is actually a Type IIB superstring, since the T-duality transformation turns the Type IIA model that appears in the string-string duality conjecture in six dimensions into a Type IIB superstring.

(4.15) shows that the string coupling constant of the effective Type IIB theory remains fixed as $\lambda \to \infty$ with fixed r, so the dual theory is not weakly coupled at all length scales. However, (4.15) also shows that $r'' \to \infty$ in this limit, and this means that at the length scale of the compactification, the effective coupling is weak. (The situation is similar to the discussion of the strongly coupled ten-dimensional Type IIA superstring in section two.) All we need to assume is that the six-dimensional Type II superstring theory, even with a coupling of order one, is equivalent at long distances to weakly coupled Type II supergravity. If that is so, then when compactified on a very large circle, it can be described at and above the compactification length by the weakly coupled supergravity, which describes the dynamics of the light degrees of freedom.

Moduli Space Of Type IIB Vacua

The following remarks will aim to give a more fundamental explanation of (4.15) and a further check on the discussion.

Consider the compactification of Type IIB superstring theory on $\mathbf{R}^6 \times \mathrm{K3}$. This gives a chiral $N = 4$ supergravity theory in six dimensions, with five self-dual two-forms (that is, two-forms with self-dual field strength) and twenty-one anti-self-dual two-forms (that is, two-forms with anti-self-dual field strength). The moduli space of vacua of the low energy supergravity theory is therefore [32] G/K with $G = SO(21, 5)$ and K the maximal subgroup $SO(21) \times SO(5)$.

The coset space G/K has dimension $21 \times 5 = 105$. The interpretation of this number is as follows. There are 80 NS-NS moduli in the conformal field theory on K3 (that, the moduli space of $(4, 4)$ conformal field theories on K3 is 80-dimensional). There are 24 zero modes of RR fields on K3. Finally, the expectation value of the dilaton – the string coupling constant – gives one more modulus. In all, one has $80 + 24 + 1 = 105$ states. In particular, the string coupling constant is unified with the others.

It would be in the spirit of U-duality to suppose that the Type IIB theory on $\mathbf{R}^6 \times K3$

has the discrete symmetry group $SO(21,5;\mathbf{Z})$. In fact, that follows from the assumption of $SL(2,\mathbf{Z})$ symmetry of Type IIB in ten dimensions [3] together with the demonstration in [31] of a discrete symmetry $SO(20,4;\mathbf{Z})$ for $(4,4)$ conformal field theories on K3. For the $SO(20,4;\mathbf{Z})$ and $SL(2,\mathbf{Z})$ do not commute and together generate $SO(21,5;\mathbf{Z})$. The moduli space of Type IIB vacua on $\mathbf{R}^6 \times$ K3 is hence

$$\mathcal{N} = SO(21,5;\mathbf{Z})\backslash SO(21,5;\mathbf{R})/(SO(21)\times SO(5)). \tag{4.16}$$

Now consider the Type IIB theory on $\mathbf{R}^5 \times \mathbf{S}^1 \times$ K3. One gets one new modulus from the radius of the $\mathbf{S}^1$. No other new moduli appear (the Type IIB theory on $\mathbf{R}^6 \times$ K3 has no gauge fields so one does not get additional moduli from Wilson lines). So the moduli space of Type IIB vacua on $\mathbf{R}^5 \times \mathbf{S}^1 \times$ K3 is

$$\mathcal{M} = \mathcal{N} \times \mathbf{R}^+, \tag{4.17}$$

where $\mathbf{R}^+$ (the space of positive real numbers) parametrizes the radius of the circle.

What about the heterotic string on $\mathbf{R}^5 \times \mathbf{T}^5$? The T-duality moduli space of the toroidal vacua is precisely $\mathcal{N} = SO(21,5;\mathbf{Z})\backslash SO(21,5;\mathbf{R})/(SO(21)\times SO(5))$. There is one more modulus, the string coupling constant. So the moduli space of heterotic string vacua on $\mathbf{R}^5 \times \mathbf{T}^5$ is once again $\mathcal{M} = \mathcal{N} \times \mathbf{R}^+$. Now the $\mathbf{R}^+$ parametrizes the string coupling constant.

So the moduli space of toroidal heterotic string vacua on $\mathbf{R}^5 \times \mathbf{T}^5$ is the same as the moduli space of Type IIB vacua on $\mathbf{R}^5 \times$ K3, suggesting that these theories may be equivalent. The map between them turns the string coupling constant of the heterotic string into the radius of the circle in the Type IIB description. This is the relation that we have seen in (4.15) (so, in particular, strong coupling of the heterotic string goes to large radius in Type IIB).

To summarize the discussion, we have seen that an attractive conjecture – the equivalence of the heterotic string in six dimensions to a certain Type IIA theory – implies another attractive conjecture – the equivalence of the heterotic string in five dimensions to a certain Type IIB theory. The link from one conjecture to the other depended on a

T-duality transformation, giving evidence that these phenomena must be understood in terms of string theory, not just in terms of relations among low energy field theories.

Detailed Matching Of States

Before leaving this subject, perhaps it would be helpful to be more explicit about how the heterotic and Type II spectra match up in five dimensions.

Compactification of the six-dimensional heterotic string theory on $\mathbf{R}^5 \times \mathbf{S}^1$ generates in the effective five-dimensional theory three $U(1)$ gauge fields that were not present in six dimensions. There is the component g_{m6} of the metric, the component B_{m6} of the antisymmetric tensor field, and the vector A_m that is dual to the spatial components B_{mn} of the antisymmetric tensor field. Each of these couples to charged states: g_{m6} couples to elementary string states with momentum around the circle, B_{m6} to states that wind around the circle, and A_m to states that arise as instantons in four spatial dimensions, invariant under rotations about the compactified circle. The mass of these "instantons" is r/λ^2, with the factor of r coming from integrating over the circle and $1/\lambda^2$ the instanton action in four dimensions. The masses of these three classes of states are hence of order $1/r$, r, and r/λ^2, respectively, if measured with respect to the string metric. To compare to Type II, we should remember (4.10) that a Weyl transformation $g = \lambda^2 g'$ is made in going to the sigma model metric of the Type IIA description. This multiplies masses by a factor of λ, so the masses computed in the heterotic string theory but measured in the string units of Type IIA are

$$\begin{aligned} g_{m6}: &\ \frac{\lambda}{r} \\ B_{m6}: &\ \lambda r \\ A_m: &\ \frac{r}{\lambda}. \end{aligned} \tag{4.18}$$

Likewise, compactification of the six-dimensional Type IIA superstring on $\mathbf{R}^5 \times \mathbf{S}^1$ gives rise to three vectors g'_{m6}, B'_{m6}, and A'. The first two couple to elementary string states. The last presumably couples to some sort of soliton, perhaps the classical solution that has been called the symmetric five-brane [28]. Its mass would be of order $r'/(\lambda')^2$ in string units for the same reasons as before. The masses of particles coupling to the three

vectors are thus in string units:

$$\begin{aligned} g'_{m6}: &\ \frac{1}{r'} \\ B'_{m6}: &\ r' \\ A'_m: &\ \frac{r'}{(\lambda')^2}. \end{aligned} \tag{4.19}$$

Now, (4.18) agrees with (4.19) under the expected transformation $\lambda = 1/\lambda'$, $r = \lambda r'$ provided that one identifies g_{m6} with g'_{m6}; B_{m6} with A'_m; and A_m with B'_{m6}. The interesting point is of course that B_{m6} and A_m switch places. But this was to be expected from the duality transformation $dB \sim *dB'$ that enters in comparing the two theories.

So under string-string duality the "instanton," which couples to A_m, is turned into the string winding state, which couples to B'_{m6}, and the string winding state that couples to B_{m6} is turned into a soliton that couples to A'_m.

4.5. Relation To S-Duality

Now we would like to use six-dimensional string-string duality to determine the strong coupling dynamics of the heterotic string in four dimensions. Once again, we start with a preliminary toroidal compactification from ten to six dimensions on a fixed torus that will not be mentioned further. Then we take the six-dimensional space to be a product $\mathbf{R}^4 \times \mathbf{T}^2$, with $\mathbf{T}^2$ a two-torus. String-string duality says that this is equivalent to a six-dimensional Type IIA theory on $\mathbf{R}^4 \times \mathbf{T}^2$ (with four extra dimensions in the form of a fixed K3).

One might now hope, as in six and five dimensions, to take the strong coupling limit and get a useful description of strongly coupled four-dimensional heterotic string theory in terms of Type II. This fails for the following reason. In six dimensions, the duality related strong coupling of the heterotic string to weak coupling of Type IIA. In five dimensions, it related weak coupling of the heterotic string to coupling of order one of Type IIB (see (4.15)). Despite the coupling of order one, this was a useful description because the radius of the sixth dimension was large, so (very plausibly) the effective coupling at the compactification scale is small. A similar scaling in four dimensions, however, will show that the strong coupling limit of the heterotic string in four dimensions is related to a

strongly coupled four-dimensional Type II superstring theory, and now one has no idea what to expect.

It is remarkable, then, that there is another method to use six-dimensional string-string duality to determine the strong coupling behavior of the heterotic string in four dimensions. This was forecast and explained by Duff [7] without reference to any particular example. The reasoning goes as follows.

Recall (such matters are reviewed in [33]) that the T-duality group of a two-torus is $SO(2,2)$ which is essentially the same as $SL(2) \times SL(2)$. Here the two $SL(2)$'s are as follows. One of them, sometimes called $SL(2)_U$, acts on the complex structure of the torus. The other, sometimes called $SL(2)_T$, acts on the combination of the area ρ of the torus and a scalar $b = B_{56}$ that arises in compactification of the antisymmetric tensor field B.

In addition to $SL(2)_U$ and $SL(2)_T$, the heterotic string in four dimensions is conjectured to have a symmetry $SL(2)_S$ that acts on the combination of the four-dimensional string coupling constant

$$\lambda_4 = \lambda \rho^{-1/2} \tag{4.20}$$

and a scalar a that is dual to the space-time components B_{mn} $(m, n = 1 \dots 4)$. We know that the heterotic string has $SL(2)_U$ and $SL(2)_T$ symmetry; we would like to know if it also has $SL(2)_S$ symmetry. If so, the strong coupling behavior in four dimensions is determined.

Likewise, the six-dimensional Type IIA theory, compactified on $\mathbf{R}^4 \times \mathbf{T}^2$, has $SL(2)_{U'} \times SL(2)_{T'}$ symmetry, and one would like to know if it also has $SL(2)_{S'}$ symmetry. Here $SL(2)_{U'}$ acts on the complex structure of the torus, $SL(2)_{T'}$ acts on the area ρ' and scalar b' derived from B'_{56}, and $SL(2)_{S'}$ would conjecturally act on the string coupling constant λ'_4 and the scalar a' that is dual to the $\mathbf{R}^4$ components of B'.

If string-string duality is correct, then the metrics in the equivalent heterotic and Type IIA descriptions differ only by a Weyl transformation, which does not change the complex structure of the torus; hence $SL(2)_U$ can be identified with $SL(2)_{U'}$. More interesting is what happens to S and T. Because the duality between the heterotic string and Type IIA involves $dB \sim *dB'$, it turns a into b' and a' into b. Therefore, it must turn $SL(2)_S$ into $SL(2)_{T'}$ and $SL(2)_{S'}$ into $SL(2)_T$. Hence the known $SL(2)_T$ invariance of the heterotic

and Type IIA theories implies, if string-string duality is true, that these theories must also have $SL(2)_S$ invariance!

It is amusing to check other manifestations of the fact that string-string duality exchanges $SL(2)_S$ and $SL(2)_T$. For example, the four-dimensional string coupling $\lambda_4 = \lambda\rho^{-1/2} = \lambda/r$ (r is a radius of the torus) turns under string-string duality into $1/r' = (\rho')^{-1/2}$. Likewise $\rho = r^2$ is transformed into $\lambda^2(r')^2 = (r'/\lambda')^2 = 1/(\lambda'_4)^2$. So string-string duality exchanges λ_4 with $\rho^{-1/2}$, as it must in order to exchange $SL(2)_S$ and $SL(2)_T$.

Some Other Models With S-Duality

From string-string duality we can not only rederive the familiar S-duality, but attempt to deduce S-duality for new models. For instance, one could consider in the above a particular two-torus $\mathbf{T}^2$ that happens to be invariant under some $SL(2)_U$ transformations, and take the orbifold with respect to that symmetry group of the six-dimensional heterotic string. This orbifold can be regarded as a different compactification of the six-dimensional model, so string-string duality – if true – can be applied to it, relating the six-dimensional heterotic string on this orbifold (and an additional four-torus) to a Type IIA string on the same orbifold (and an additional K3).

Orbifolding by a subgroup of $SL(2)_U$ does not disturb $SL(2)_T$, so the basic structure used above still holds; if six-dimensional string-string duality is valid, then $SL(2)_S$ of the heterotic string on this particular orbifold follows from $SL(2)_T$ of Type IIA on the same orbifold, and vice-versa. This example is of some interest as – unlike previously known examples of S-duality – it involves vacua in which supersymmetry is completely broken. The S-duality of this and possible related examples might have implications in the low energy field theory limit, perhaps related to phenomena such as those recently uncovered by Seiberg [14].

4.6. Enhanced Gauge Groups

Perhaps the most striking phenomenon in toroidal compactification of the heterotic string is that at certain points in moduli space, an enhanced non-abelian gauge symmetry

appears. The enhanced symmetry group is always simply-laced and so a product of A, D, and E groups; in toroidal compactification to six dimensions, one can get any product of A, D, and E groups of total rank ≤ 20.

How can one reproduce this with Type IIA on a K3 surface? [18] It is fairly obvious that one cannot get an enhanced gauge symmetry unless the K3 becomes singular; only then might the field theory analysis showing that the RR charges have mass of order $1/\lambda$ break down.

The only singularities a K3 surface gets are orbifold singularities. (It is possible for the distance scale of the K3 to go to infinity, isotropically or not, but that just makes field theory better.) The orbifold singularities of a K3 surface have an $A-D-E$ classification. Any combination of singularities corresponding to a product of groups with total rank ≤ 20 (actually at the classical level the bound is ≤ 19) can arise.

Whenever the heterotic string on a four-torus gets an enhanced gauge group G, the corresponding K3 gets an orbifold singularity of type G. This assertion must be a key to the still rather surprising and mysterious occurrence of extended gauge groups for Type IIA on K3, so I will attempt to explain it.

The moduli space

$$\mathcal{M} = SO(20,4;\mathbf{Z})\backslash SO(20,4;\mathbf{R})/(SO(20)\times SO(4)) \tag{4.21}$$

of toroidal compactifications of the heterotic string to six dimensions – or K3 compactifications of Type II – can be thought of as follows. Begin with a 24 dimensional real vector space W with a metric of signature $(4,20)$, and containing a self-dual even integral lattice L (necessarily of the same signature). Let V be a four-dimensional subspace of W on which the metric of W is positive definite. Then $\mathcal{M}$ is the space of all such V's, up to automorphisms of L. Each V has a twenty-dimensional orthocomplement $V^\perp$ on which the metric is negative definite.

In the heterotic string description, V is the space of charges carried by right-moving string modes, and $V^\perp$ is the space of charges carried by left-moving string modes. Gener-

[18] This question was very briefly raised in section 4.3 of [34], and has also been raised by other physicists.

ically, neither V nor $V^\perp$ contains any non-zero points in L. When $V^\perp$ contains such a point P, we get a purely left-moving (antiholomorphic) vertex operator $\mathcal{O}_P$ of dimension $d_P = -(P,P)/2$. (Of course, $(P,P) < 0$ as the metric of W is negative definite on $V^\perp$.) d_P is always an integer as the lattice L is even. The gauge symmetry is extended precisely when $V^\perp$ contains some P of $d_P = 1$; the corresponding $\mathcal{O}_P$ generate the extended gauge symmetry.

In the K3 description, W is the real cohomology of K3 (including H^0, H^2, and H^4 together [31]). The lattice L is the lattice of integral points. V is the part of the cohomology generated by self-dual harmonic forms. The interpretation is clearest if we restrict to K3's of large volume, where we can use classical geometry. Then H^0 and H^4 split off, and we can take for W the 22 dimensional space H^2, and for V the three-dimensional space of self-dual harmonic two-forms.

Consider a K3 that is developing an orbifold singularity of type G, with r being the rank of G. In the process, a configuration of r two-spheres S_i (with an intersection matrix given by the Dynkin diagram of G) collapses to a point. These two-spheres are holomorphic (in one of the complex structures on the $K3$), and the corresponding cohomology classes $[S_i]$ have length squared -2. As they collapse, the $[S_i]$ become anti-self-dual and thus – in the limit in which the orbifold singularity develops – they lie in $V^\perp$. (In fact, as S_i is holomorphic, the condition for $[S_i]$ to be anti-self-dual is just that it is orthogonal to the Kahler class and so has zero area; thus the $[S_i]$ lie in $V^\perp$ when and only when the orbifold singularity appears and the S_i shrink to zero.) Conversely, the Riemann-Roch theorem can be used to prove that any point in $V^\perp$ of length squared -2 is associated with a collapsed holomorphic two-sphere.

In sum, precisely when an orbifold singularity of type G appears, there is in $V^\perp$ an integral lattice of rank r, generated by points of length squared -2, namely the S_i; the lattice is the weight lattice of G because of the structure of the intersection matrix of the S_i. This is the same condition on $V^\perp$ as the one that leads to extended symmetry group G for the heterotic string. In the K3 description, one $U(1)$ factor in the gauge group is associated with each collapsed two-sphere. These $U(1)$'s should make up the maximal torus of the extended gauge group.

Despite the happy occurrence of a singularity – and so possible breakdown of field theory – precisely when an extended gauge group should appear, the occurrence of extended gauge symmetry in Type IIA is still rather surprising. It must apparently mean that taking the string coupling to zero (which eliminates the RR charges) does not commute with developing an orbifold singularity (which conjecturally brings them to zero mass), and that conventional orbifold computations in string theory correspond to taking the string coupling to zero first, the opposite of what one might have guessed.

4.7. Dynamics In Seven Dimensions

The reader might be struck by a lack of unity between the two parts of this paper. In sections two and three, we related Type II superstrings to eleven-dimensional supergravity. In the present section, we have presented evidence for the conjectured relation of Type II superstrings to heterotic superstrings. If both are valid, should not eleven-dimensional supergravity somehow enter in understanding heterotic string dynamics?

I will now propose a situation in which this seems to be true: the strong coupling limit of the heterotic string in seven dimensions. I will first propose an answer, and then try to deduce it from six-dimensional string-string duality.

The proposed answer is that the strong coupling limit of the heterotic string on $\mathbf{R}^7 \times \mathbf{T}^3$ gives a theory whose low energy behavior is governed by eleven-dimensional supergravity on $\mathbf{R}^7 \times \mathrm{K3}$! The first point in favor of this is that the moduli spaces coincide. The moduli space of vacua of the heterotic string on $\mathbf{R}^7 \times \mathbf{T}^3$ is

$$\mathcal{M} = \mathcal{M}_1 \times \mathbf{R}^+ \tag{4.22}$$

with

$$\mathcal{M}_1 = SO(19,3;\mathbf{Z})\backslash SO(19,3;\mathbf{R})/SO(19) \times SO(3). \tag{4.23}$$

Here $\mathcal{M}_1$ is the usual Narain moduli space, and $\mathbf{R}^+$ parametrizes the possible values of the string coupling constant. For eleven-dimensional supergravity compactified on $\mathbf{R}^7 \times \mathrm{K3}$, the moduli space of vacua is simply the moduli space of Einstein metrics on K3. This does *not* coincide with the moduli space of $(4,4)$ conformal field theories on K3, because there is no second rank antisymmetric tensor field in eleven-dimensional supergravity.

Rather the moduli space of Einstein metrics of volume 1 on K3 is isomorphic to $\mathcal{M}_1 = SO(19,3;\mathbf{Z})\backslash SO(19,3;\mathbf{R})/SO(19)\times SO(3)$. [19] Allowing the volume to vary gives an extra factor of $\mathbf{R}^+$, so that the moduli space of Einstein metrics on K3 coincides with the moduli space $\mathcal{M}$ of string vacua.

As usual, the next step is to see how the low energy effective theories match up. Relating these two theories only makes sense if large volume of eleven-dimensional supergravity (where perturbation theory is good) corresponds to strong coupling of the heterotic string. We recall that the bosonic fields of eleven-dimensional supergravity are a metric G and three-form A_3 with action

$$I = \frac{1}{2}\int d^{11}x\sqrt{G}\left(R + |dA_3|^2\right) + \int A_3 \wedge dA_3 \wedge dA_3. \tag{4.24}$$

To reduce on $\mathbf{R}^7 \times$ K3, we take the eleven-dimensional line-element to be $ds^2 = \widetilde{g}_{mn}dx^m dx^n + e^{2\gamma}h_{\alpha\beta}dy^\alpha dy^\beta$, with $m,n = 1\ldots 7$, $\alpha,\beta = 1\ldots 4$; here $\widetilde{g}$ is a metric on $\mathbf{R}^7$, h a fixed metric on K3 of volume 1, and e^γ the radius of the K3. The reduction of A_3 on $\mathbf{R}^7 \times$ K3 gives on $\mathbf{R}^7$ a three-form a_3, and 22 one-forms that we will generically call A. The eleven-dimensional Lagrangian becomes very schematically (only keeping track of the scaling with e^γ)

$$\int d^7x\sqrt{\widetilde{g}}\left(e^{4\gamma}(\widetilde{R} + |d\gamma|^2 + |da_3|^2) + |dA|^2\right). \tag{4.25}$$

To match this to the heterotic string in seven dimensions, we write $\widetilde{g} = e^{-4\gamma}g$, with g the heterotic string metric in seven dimensions. We also make a duality transformation $e^{6\gamma}da_3 = *dB$, with B the two-form of the heterotic string. Then (4.25) turns into

$$\int d^7x\sqrt{g}e^{-6\gamma}\left(R + |d\gamma|^2 + |dB|^2 + |dA|^2\right). \tag{4.26}$$

The important point is that the Lagrangian scales with an overall factor of $e^{-6\gamma}$, similar to the overall factor of $\lambda^{-2} = e^{-2\phi}$ in the low energy effective action of the heterotic string. Thus, to match eleven-dimensional supergravity on $\mathbf{R}^7 \times$ K3 with the heterotic string in

[19] This space parametrizes three-dimensional subspaces of positive metric in $H^2(\mathrm{K3},\mathbf{R})$. The subspace corresponding to a given Einstein metric on K3 consists of the part of the cohomology that is self-dual in that metric.

seven dimensions, one takes the radius of the K3 to be

$$e^{\gamma} = e^{\phi/3} = \lambda^{1/3}. \tag{4.27}$$

In particular, as we hoped, for $\lambda \to \infty$, the radius of the K3 goes to infinity, and the eleven-dimensional supergravity theory becomes weakly coupled at the length scale of the light degrees of freedom.

Now, let us try to show that this picture is a consequence of string-string duality in six dimensions. We start with the heterotic string on $\mathbf{R}^6 \times \mathbf{S}^1 \times \mathbf{T}^3$, where $\mathbf{S}^1$ is a circle of radius r_1, and $\mathbf{T}^3$ is a three-torus that will be held fixed throughout the discussion.[20] If λ_7 and λ_6 denote the heterotic string coupling constant in seven and six dimensions, respectively, then

$$\frac{1}{\lambda_6^2} = \frac{r_1}{\lambda_7^2}. \tag{4.28}$$

We want to take r_1 to infinity, keeping λ_7 fixed. That will give a heterotic string in seven dimensions. Then, after taking r_1 to infinity, we consider the behavior for large λ_7, to get a strongly coupled heterotic string in seven dimensions.

The strategy of the analysis is of course to first dualize the theory, to a ten-dimensional Type II theory, and then see what happens to the dual theory when first r_1 and then λ are taken large. Six-dimensional string-string duality says that for fixed r_1 and λ, the heterotic string on $\mathbf{R}^6 \times \mathbf{S}^1 \times \mathbf{T}^3$ is equivalent to a Type IIA superstring on $\mathbf{R}^6 \times \mathbf{K3}$, with the following change of variables. The six-dimensional string coupling constant λ_6' of the Type IIA description is

$$\lambda_6' = \frac{1}{\lambda_6} = \frac{r_1^{1/2}}{\lambda_7}. \tag{4.29}$$

The metrics g and g' of the heterotic and Type IIA descriptions are related by

$$g = e^{2\phi} g' = \lambda_6^2 g' = \frac{\lambda_7^2}{r_1} g'. \tag{4.30}$$

In addition, the parameters of the K3 depend on r_1 (and the parameters of the $\mathbf{T}^3$, which will be held fixed) in a way that we will now analyze.

[20] Generally, there are also Wilson lines on $\mathbf{T}^3$ breaking the gauge group to a product of $U(1)$'s; these will be included with the parameters of the $\mathbf{T}^3$ that are kept fixed in the discussion.

There is no unique answer, since we could always apply an $SO(20,4;\mathbf{Z})$ transformation to the K3. However, there is a particularly simple answer. The heterotic string compactified on $\mathbf{S}^1 \times \mathbf{T}^3$ has 24 abelian gauge fields. As the radius r_1 of the $\mathbf{S}^1$ goes to infinity, the elementary string states carrying the 24 charges behave as follows. There is one type of charge (the momentum around the $\mathbf{S}^1$) such that the lightest states carrying only that charge go to zero mass, with

$$M \sim \frac{1}{r_1}. \tag{4.31}$$

There is a second charge, the winding number around $\mathbf{S}^1$, such that particles carrying that charge have masses that blow up as r_1. Particles carrying only the other 22 charges have fixed masses in the limit.

Any two ways to reproduce this situation with a K3 will be equivalent up to a T-duality transformation. There is a particularly easy way to do this – take a fixed K3 and scale up the volume V, leaving fixed the "shape." This reproduces the above spectrum with a relation between V and r_1 that we will now determine.

We start with the Type IIA superstring theory in ten dimensions. The bosonic fields include the metric g'_{10}, dilaton ϕ'_{10}, gauge field A, and three-form A_3. The action is schematically

$$\int d^{10}x \sqrt{g'_{10}} \left(e^{-2\phi'_{10}} R'_{10} + |dA|^2 + |dA_3|^2 + \ldots \right). \tag{4.32}$$

Upon compactification on $\mathbf{R}^6 \times$ K3, massless modes coming from A and A_3 are as follows. A gives rise to a six-dimensional vector, which we will call a. A_3 gives rise to 22 vectors – we will call them C_I – and a six-dimensional three-form, which we will call a_3. If V is the volume of the $K3$, the effective action in six dimensions scales schematically as

$$\int d^6x \sqrt{g'} \left(\frac{1}{(\lambda'_6)^2} R' + V|da|^2 + V|da_3|^2 + |dC_I|^2 \right). \tag{4.33}$$

Visible in (4.33) are 23 vectors, namely a and the C_I. However, precisely in six dimensions a three-form is dual to a vector, by $V da_3 = *db$. So we can replace (4.33) with

$$\int d^6x \sqrt{g'} \left(\frac{1}{(\lambda'_6)^2} R' + V|da|^2 + \frac{1}{V}|db|^2 + |dC_I|^2 \right), \tag{4.34}$$

with 24 vectors. As the canonical kinetic energy of a vector is

$$\int d^6x \frac{1}{4e^2_{\rm eff}} |dA|^2, \tag{4.35}$$

with $e_{\rm eff}$ the effective charge, we see that we have one vector with effective charge of order $V^{-1/2}$, one with effective charge of order $V^{1/2}$, and 22 with effective charges of order one.

According to our discussion in section two, the mass of a particle carrying an RR charge is of order $e_{\rm eff}/\lambda'_6$. So for fixed λ'_6 and $V \to \infty$, one type of particle goes to zero mass, one to infinite mass, and 22 remain fixed – just like the behavior of the heterotic string as $r_1 \to \infty$. The lighest charge-bearing particle has a mass of order

$$M' = \frac{1}{V^{1/2}\lambda'_6}. \tag{4.36}$$

To compare this to the mass (4.31) of the lightest particle in the heterotic string description, we must remember the Weyl transformation (4.30) between the two descriptions. Because of this Weyl transformation, the relation between the two masses should be $M = \lambda_6^{-1} M' = \lambda'_6 M'$. So λ'_6 scales out, and the relation between the two descriptions involves the transformation

$$V = r_1^2. \tag{4.37}$$

The reason that the string coupling constant scales out is that it does not enter the map between the moduli space of heterotic string vacua on a four-torus and $(4,4)$ conformal field theories on K3; the relation (4.37) could have been deduced by studying the description of quantum K3 moduli space in [31] instead of using low energy supergravity as we have done.

Since we know from (4.37) and (4.29) how the parameters V and λ'_6 of the Type IIA description are related to the heterotic string parameters, we can identify the ten-dimensional Type IIA string coupling constant λ'_{10}, given by

$$\frac{V}{(\lambda'_{10})^2} = \frac{1}{(\lambda'_6)^2}. \tag{4.38}$$

We get

$$\lambda'_{10} = \frac{r_1^{3/2}}{\lambda_7}. \tag{4.39}$$

Thus, for $r_1 \to \infty$, the Type IIA theory is becoming strongly coupled. At the same time, according to (4.37) one has $V \to \infty$, so the Type IIA theory is becoming decompactified.

In section two, we proposed a candidate for the strong coupling behavior of Type IIA on $\mathbf{R}^{10}$: it is given by eleven-dimensional supergravity on $\mathbf{R}^{10} \times \mathbf{S}^1$. To be more precise, the relation acted as follows on the massless modes. If the line element of the eleven-dimensional theory is $ds^2 = G^{10}_{ij} dx^i dx^j + r_{11}^2 (dx^{11})^2$, $i, j = 1 \dots 10$, with G^{10} a metric on $\mathbf{R}^{10}$ and r_{11} the radius of the circle, then r_{11} is related to the ten-dimensional Type IIA string coupling constant by

$$r_{11} = (\lambda'_{10})^{2/3} = \frac{r_1}{\lambda_7^{2/3}} \tag{4.40}$$

and the Type IIA metric g' is related to G^{10} by

$$g' = (\lambda'_{10})^{2/3} G^{10}. \tag{4.41}$$

As this result holds for any fixed metric g' on $\mathbf{R}^{10}$, it must, physically, hold on any ten-manifold M as long as the dimensions of M are scaled up fast enough compared to the growth of the ten-dimensional string coupling constant. I will assume that with λ'_{10} and V going to infinity as determined above, one is in the regime in which one can use the formulas (4.40), (4.41) that govern the strong coupling behavior on $\mathbf{R}^{10}$.

If this is so, then from (4.41) the volume V_{11} of the K3 using the metric of the eleven-dimensional supergravity is related to the volume V using the string metric of the Type IIA description by

$$V_{11} = (\lambda'_{10})^{-4/3} V = \lambda_7^{4/3} r_1^{-2} V = \lambda_7^{4/3}. \tag{4.42}$$

Now we have the information we need to solve our problem. The heterotic string on $\mathbf{R}^6 \times \mathbf{S}^1 \times \mathbf{T}^3$, with radius r_1 of the $\mathbf{S}^1$ and string coupling constant λ_7, is related to eleven-dimensional supergravity on $\mathbf{R}^6 \times \mathbf{S}^1 \times \mathrm{K3}$, where the radius of the $\mathbf{S}^1$ is given in (4.40) and the volume of the K3 in (4.42). We are supposed to take the limit $r_1 \to \infty$ and then consider the behavior for large λ_7. The key point is that V_{11} is independent of r_1. This enables us to take the limit as $r_1 \to \infty$; all that happens is that $r_{11} \to \infty$, so the $\mathbf{R}^6 \times \mathbf{S}^1 \times \mathrm{K3}$ on which the supergravity theory is formulated becomes $\mathbf{R}^7 \times \mathrm{K3}$. (Thus we see Lorentz invariance between the "eleventh" dimension which came from strong coupling and six of the "original" dimensions.) The dependence on the heterotic string coupling λ_7 is now easy to understand: it is simply that the volume of the K3 is $V_{11} \sim \lambda_7^{4/3}$. That

is of course the behavior of the volume expected from (4.27). So the relation that we have proposed between the heterotic string in seven dimensions and eleven-dimensional supergravity on $\mathbf{R}^7 \times$ K3 fits very nicely with the implications of string-string duality in six dimensions.

5. On Heterotic String Dynamics Above Seven Dimensions

By now we have learned that the strong coupling dynamics of Type II superstrings is, apparently, tractable in any dimension and that the same appears to be true of the heterotic string in dimension ≤ 7. Can we also understand the dynamics of the heterotic string above seven dimensions?

It might be possible to extend the use of six-dimensional string-string duality above seven dimensions (just as we extended it above six dimensions at the end of the last section). This will require more careful analysis of the K3's and probably more subtle degenerations than we have needed so far.

But is there some dual description of the heterotic string above seven dimensions that would give the dynamics more directly? For instance, can we find a dual of the heterotic string directly in ten dimensions?

Once this question is asked, an obvious speculation presents itself, at least in the case of $SO(32)$. (For the $E_8 \times E_8$ theory in ten dimensions, I have no proposal to make.) There is another ten-dimensional string theory with $SO(32)$ gauge group, namely the Type I superstring. Might they in fact be equivalent? [21]

The low energy effective theories certainly match up; this follows just from the low energy supersymmetry. Moreover, they match up in such a way that strong coupling of one theory would turn into weak coupling of the other. This is an essential point in any possible relation between them, since weak coupling of one is certainly not equivalent to weak coupling of the other. In terms of the metric g, dilaton ϕ, two-form B, and gauge field

[21] The $SO(32)$ heterotic string has particles that transform as spinors of $SO(32)$; these are absent in the elementary string spectrum of Type I and would have to arise as some sort of solitons if these two theories are equivalent.

strength F, the heterotic string effective action in ten dimensions scales with the dilaton like

$$\int d^{10}x\sqrt{g}e^{-2\phi}\left(R+|\nabla\phi|^2+F^2+|dB|^2\right). \tag{5.1}$$

If we transform $g=e^{\phi}g'$ and $\phi=-\phi'$, this scales like

$$\int d^{10}x\sqrt{g'}\left(e^{-2\phi'}\left(R'+|\nabla\phi'|^2\right)+e^{-\phi'}F^2+|dB|^2\right). \tag{5.2}$$

This is the correct scaling behavior for the effective action of the Type I superstring. The gauge kinetic energy scales as $e^{-\phi'}$ instead of $e^{-2\phi'}$ because it comes from the disc instead of the sphere. The B kinetic energy scales trivially with ϕ' in Type I because B is an RR field. The fact that $\phi=-\phi'$ means that strong coupling of one theory is weak coupling of the other, as promised.

Though a necessary condition, this is scarcely strong evidence for a new string-string duality between the heterotic string and Type I. However, given that the heterotic and Type II superstrings and eleven-dimensional supergravity all apparently link up, one would be reluctant to overlook a possibility for Type I to also enter the story.

Let us try to use this hypothetical new duality to determine the dynamics of the heterotic string below ten dimensions. (Below ten dimensions, the $SO(32)$ and $E_8\times E_8$ heterotic strings are equivalent [9], so the following discussion applies to both.) We formulate the heterotic string, with ten-dimensional string coupling constant λ, on $\mathbf{R}^d\times\mathbf{T}^{10-d}$ with $\mathbf{T}^{10-d}$ a $(10-d)$-torus of radius r. This would be hypothetically equivalent to a toroidally compactified Type I theory with coupling constant $\lambda'=1/\lambda$ and (in view of the Weyl transformation used to relate the low energy actions) compactification scale $r'=r/\lambda^{1/2}$. Thus, as $\lambda\to\infty$ for fixed r, λ' goes to zero, but r' also goes to zero, making the physical interpretation obscure. It is more helpful to make a T-duality transformation of the Type I theory to one with radius $r''=1/r'$. The T-duality transformation has a very unusual effect for Type I superstrings [11], mapping them to a system that is actually somewhat similar to a Type II orbifold; the relation of this unusual orbifold to the system considered in section four merits further study. The T-duality transformation also changes the ten-dimensional string coupling constant to a new one λ'' which obeys

$$\frac{(r')^{10-d}}{(\lambda')^2}=\frac{(r'')^{10-d}}{(\lambda'')^2} \tag{5.3}$$

so that the d-dimensional effective Newton constant is invariant. Thus

$$\lambda'' = \lambda' \left(\frac{r''}{r'} \right)^{(10-d)/2} = \frac{\lambda^{(8-d)/2}}{r^{10-d}}. \tag{5.4}$$

So for $d = 9$, the strong coupling problem would be completely solved: as $\lambda \to \infty$ with fixed r, $\lambda'' \to 0$ (and $r'' \to \infty$, which gives further simplification). For $d = 8$, we have a story similar to what we have already found in $d = 5$ and 7 (and for Type IIA in $d = 10$): though λ'' is of order 1, the fact that $r'' \to \infty$ means that the coupling is weak at the compactification scale, so that one should have a weakly coupled description of the light degrees of freedom. But below $d = 8$, the transformation maps one strong coupling limit to another.

Of course, once we get down to seven dimensions, we have a conjecture about the heterotic string dynamics from the relation to Type II. Perhaps it is just as well that the speculative relation of the heterotic string to Type I does not give a simple answer below eight dimensions. If there were a dimension in which both approaches could be applied, then by comparing them we would get a relation between (say) a weakly coupled Type II string and a weakly coupled Type I string. Such a relation would very likely be false, so the fact that the speculative string-string duality in ten dimensions does not easily determine the strong coupling behavior below $d = 8$ could be taken as a further (weak) hint in its favor.

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