CH 9.
Path Integratation

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0 前情回顾

0.1 Field Theory

$$\text{Klei-Gordon: } (\partial^2 + m^2)\phi = 0 \tag{1}$$
$$\text{Dirac: } (i\gamma \cdot \partial - m)\,\Psi = 0 \tag{2}$$
$$\text{Maxwell: } \partial_\mu F^{\mu\nu} = 0 \tag{3}$$

0.2 Lorentz

$$x \to x' = \Lambda x \tag{4}$$
$$\psi \to \psi'(x') = D(\Lambda)\psi(x) \tag{5}$$
$$\psi'(x) = D(\Lambda)\psi(\Lambda^{-1}x) \tag{6}$$

$$D(\Lambda) = \Lambda_0 \quad \text{Scalar Field} \tag{7}$$
$$D(\Lambda) = \Lambda_{\frac{1}{2}} \quad \text{Spinnor Field} \tag{8}$$
$$D(\Lambda) = \Lambda_1 \quad \text{Vector Field} \tag{9}$$

$$\cdots\cdots \tag{10}$$

0.3 经典力学 VS 经典场论

$$S = \int dt\mathcal{L}(q, \dot{q}, t), \qquad \frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{q}}\right) - \frac{\delta\mathcal{L}}{\delta q} = 0 \tag{11}$$

$$S = \int d^4x\mathcal{L}(\phi, \partial_\mu\phi, x), \quad \frac{\partial}{\partial x^\mu}\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right) - \frac{\delta\mathcal{L}}{\delta\phi} = 0 \tag{12}$$

0.4 量子力学 VS 量子场论

QM	QF
x	$\phi(x)$
$[x_i, p_j] = i\hbar\delta_{ij}$	$[\phi(x), \pi(y)] = i\hbar\delta(x-y)$
$\mathcal{L}(t, x_i(t), p_i(t)]$	$\mathcal{L}[x, \phi(x), \pi(x)]$ (Local)

正则量子化**:** 正则坐标, 正则动量, 正则关系

路径积分量子化**:** 路径积分，约束系统

（正则量子化在量子电动力学中的困难）

1 Path Integral in QM

Coordinate Representation

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x), \quad \Phi(x) = \langle x|\Phi\rangle, \langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx} \tag{13}$$

$$H = \frac{p^2}{2m} + V(x) \tag{14}$$

1.1 Transition amplitude

$$\begin{aligned} T &\equiv \langle f,t|i,t_0\rangle && (15)\\ &\equiv \langle f|e^{-i\hat{H}(t-t_0)}|i\rangle && (16)\\ &= \langle f|\left(\int dx_a|x_a\rangle\langle x_a|\right)e^{-i\hat{H}(t-t_0)}\left(\int dx_b|x_b\rangle\langle x_b|\right)|i\rangle && (17)\\ &= \int dx_a dx_b\langle f|x_a\rangle\langle x_a|e^{-i\hat{H}(t-t_0)}|x_b\rangle\langle x_b|i\rangle && (18)\end{aligned}$$

- which Picture used above?

Schrödinger	Heisenberg	
$\|\phi, t\rangle$	$\|\phi\rangle$	(19) (20)

- Initial and final state Wave Function: $\Phi_i(x) = \langle x|i\rangle, \Phi_f^\dagger(x) = \langle f|x\rangle$
- How to calculate the amplitude $U(x_f, t; x_i, t_0) \equiv \langle x_f|e^{-i\hat{H}(t-t_0)}|x_i\rangle$?

1. $\delta t = t - t_0 \to 0$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]/2} \tag{21}$$

$$e^{-i\hat{H}\delta t} = e^{-i(\frac{\hat{p}^2}{2m}+V(\hat{x}))\delta t} \tag{22}$$

$$= e^{-i\frac{\hat{p}^2}{2m}\delta t} \times e^{-iV(\hat{x})\delta t} + \mathcal{O}(\delta t^2) \tag{23}$$

$$= e^{-iV(\hat{x})\delta t} \times e^{-i\delta t\frac{\hat{p}^2}{2m}} + \mathcal{O}(\delta t^2) \tag{24}$$

$$\langle y|e^{-i\hat{H}\delta t}|x\rangle = e^{-iV(x)\delta t}\langle y|e^{-i\frac{\hat{p}^2}{2m}\delta t}|x\rangle \tag{25}$$

$$= e^{-iV(x)\delta t}\langle y|e^{-i\frac{p^2}{2m}\delta t}\left(\int dp|p\rangle\langle p|\right)|x\rangle \tag{26}$$

$$= e^{-iV(x)\delta t}\int dp e^{-i\frac{p^2}{2m}\delta t}\langle y|p\rangle\langle p|x\rangle \tag{27}$$

$$= e^{-iV(x)\delta t}\int \frac{dp}{2\pi}e^{-i\frac{p^2}{2m}\delta t}e^{ip(y-x)} \tag{28}$$

$$= e^{-iV(x)\delta t}\int \frac{dp}{2\pi}e^{-\frac{i\delta t}{2m}\left(p-\frac{m(y-x)}{\delta t}\right)^2}\times e^{i\frac{m}{2}\left(\frac{y-x}{\delta t}\right)^2\delta t} \tag{29}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2m\pi}{i\delta t}}\times e^{i\left(\frac{m}{2}\left(\frac{y-x}{\delta t}\right)^2-V(x)\right)\delta t} \tag{30}$$

$$= \sqrt{\frac{m}{i2\pi\delta t}}e^{i\mathcal{L}(x)\delta t} \tag{31}$$

No operator any more!!

HOMEWORK: *Get the following results*

$$1.\ \text{Eq.}(26) = \int \frac{dp}{2\pi}\exp(i\delta t(p\dot{q}-L)) \tag{32}$$

2. $\int_{-\infty}^{\infty} dx e^{-a(x-b)^2} = \sqrt{\pi/a}$, (33)

3. $\int_{-\infty}^{\infty} dx f(x-b) = \int_{-\infty}^{\infty} dx f(x)$? (34)

2. $\delta t = \frac{t-t_0}{N+1}, N \to \infty$.

$$t_i = t_0 + i\delta t, t - t_0 = t - t_N + t_N \cdots - t_i + t_i \cdots - t_1 + t_1 - t_0, \tag{35}$$

$$\begin{aligned}
\langle x_f, t | x_i, t_0 \rangle &= \langle x_f | e^{-i\hat{H}(t-t_0)} | x_i \rangle && (36)\\
&= \langle x_f | e^{-i\hat{H}(t-t_N)} e^{-i\hat{H}(t_N - t_{N-1})} \cdots e^{-i\hat{H}(t_1-t_0)} | x_i \rangle && (37)\\
&= \int \langle x_f | e^{-i\hat{H}\delta t} | x_N \rangle \langle x_N | \cdots | x_1 \rangle \langle x_1 | e^{-i\hat{H}\delta t} | x_i \rangle && (38)\\
&= A \int dx_1 \cdots dx_N \prod_m e^{i\mathcal{L}(x_m, t_m)\delta t} && (39)\\
&= A \int [dx] e^{i\sum \mathcal{L}_m \delta t} = A \int [dx] e^{i\int dt \mathcal{L}} && (40)
\end{aligned}$$

3. In summary,

$$T \equiv \langle f | e^{-i\hat{H}t} | i \rangle \tag{41}$$

$$= \int dx_i dx_f \phi_f^\dagger(x_f)\phi_i(x_i)\langle x_f|e^{i\hat{H}t}|x_i\rangle \tag{42}$$

$$= A\int dx_i dx_f \phi_f^\dagger(x_f)\phi_i(x_i)U(x_f,t_f;x_i,t_i) \tag{43}$$

$$U(x_f,t_f;x_i,t_i) = A\int [dx]e^{i\int \mathcal{L}dt} = A\int [dx]e^{iS[x(t)]} \tag{44}$$

In Quantum Theory, the total amplitude is the coherent sum of contributions from the all possible paths $x(t)$ with the phase(weight) $\exp(iS[x(t)])$ (See Fig.9.1). The classical Path $x_{\rm cl}(t)$ is determined by the *least action principle*

$$\Big(\delta S[x(t)]\Big)\Big|_{x_{\rm cl}(t)} = 0, \quad \text{or} \quad \left(\frac{d}{dt}\frac{\delta}{\delta\dot{x}} - \frac{\delta}{\delta x}\right) L[x,\dot{x},t)\Big|_{x_{\rm cl}} = 0 \tag{45}$$

1.2 Correlation Function

$$\begin{aligned}\langle x_f,t_f|\hat{x}(t_2)\hat{x}(t_1)|x_i,t_i\rangle &= \int dx_1dx_2\langle x_f,t_f|\hat{x}(t_2)|x_2,t_2\rangle\langle x_2,t_2|\hat{x}(t_1)|x_1,t_1\rangle\langle x_1,t_1|x_i,t_i\rangle \\ &= A\int dx_1dx_2\, x_1x_2\langle x_f,t_f|x_2,t_2\rangle\langle x_2,t_2|x_1,t_1\rangle\langle x_1,t_1|x_i,t_i\rangle \\ &= A\int dx_1dx_2x_1x_2\int [dx]_{2\to f}e^{i\int_{t_2}^{t_f}\mathcal{L}}\int [dx]_{1\to 2}e^{i\int_{t_1}^{t_2}\mathcal{L}}\end{aligned} \tag{46}$$

$$\times \int [dx]_{i\to 1} e^{i\int_{t_i}^{t_1}\mathcal{L}} \tag{47}$$

$$= \int [dx] x_1 x_2 e^{i\int \mathcal{L}dt} \tag{48}$$

Final one has

$$\langle x_f, t_f | \hat{O}(x_n)\cdots\hat{O}(x_1) | x_i, t_i\rangle \quad = \quad A\int [dx] O(x_1)\cdots O(x_n) e^{i\int \mathcal{L}dt} \tag{49}$$

If $O(x) = e^{i\int dt J\cdot x}$

$$\langle x_f, t_f | x_i, t_i\rangle^J \quad \equiv \quad \langle x_f, t_f | \exp\left(i\int \hat{J}\cdot\hat{x}dt\right) | x_i, t_i\rangle \tag{50}$$

$$= \quad A\int [dx] e^{i\int Jxdt} e^{i\int \mathcal{L}dt} \tag{51}$$

$$= \quad A\int [dx] e^{i\int (\mathcal{L}+Jx)dt} \tag{52}$$

$$\equiv \quad A\int [dx] e^{iS[x,J]} \tag{53}$$

Scattering, *local* interaction

$$|x,t\rangle = e^{-iHt}|x\rangle \tag{54}$$
$$= \sum_n e^{-iHt}|n\rangle\langle n|x\rangle \tag{55}$$
$$= \sum_n e^{-iE_n t}|n\rangle\langle n|x\rangle \tag{56}$$

$$\lim_{t\to\infty(1-i\epsilon)} |x,t\rangle = e^{-iHt}|0\rangle\langle 0|x\rangle \tag{57}$$

$$\lim_{\substack{t_f\to\infty \\ t_i\to-\infty}} \langle x_f,t_f|x_i,t_i\rangle^J = \langle x_f|0\rangle\langle 0,\infty|e^{i\int jxdt}|0,-\infty\rangle\langle 0|x_i\rangle \tag{58}$$

$$\tag{59}$$

Generating Function

$$\langle 0|0\rangle^J = A\int [dx]e^{iS[x,J]} = \frac{\int [dx]e^{iS[x,J]}}{\int [dx]e^{iS[x]}} \equiv Z[J] \tag{60}$$

$$\langle 0|\hat{O}(x)|0\rangle = \frac{\int O(x)e^{i\int \mathcal{L}}}{\int e^{i\int \mathcal{L}}} = O\left(\frac{\delta}{i\delta J}\right) Z[J]\Big|_{J=0} \tag{61}$$

1.3 In summary

$$GeneratingFunction \qquad Z[J] \equiv \frac{\int [dx] e^{iS[x,J)}}{\int [dx] e^{iS[x]}} \tag{62}$$

$$CorrelationFunction \qquad \langle T(x_1 \cdots x_n) \rangle = \frac{(-i)^n \delta^n}{\delta J_1 \cdots \delta J_n} \frac{\int [dx] e^{iS[x,J)}}{\int [dx] e^{iS[x]}} \Bigg|_{J=0} \tag{63}$$

$$= \frac{1}{Z[J]} \frac{(-i)^n \delta^n}{\delta J_1 \cdots \delta J_n} Z[J] \Bigg|_{J=0} \tag{64}$$

The advantage of the PI:

- Compact and systermatic
- Programmatic
- Can be applied in Gauge theory

2 Functional Quantization of QFT

QM		QF
$x_i(t)$	$\rightarrow$	$\Phi(\vec{x},t)$
$S[x]=\int \mathcal{L}dt$		$S[\Phi]=\int d^4x\mathcal{L}$
$U=A\int[dx]e^{i\int dt\mathcal{L}}$		$U=A\int e^{i\int d^4x\mathcal{L}}$
$Z[J]=A\int[dx]e^{i\int(\mathcal{L}+jx)dt}$		$Z[J]=A\int[D\Phi]e^{i\int d^4x(\mathcal{L}+\Phi J)}$

- Generating Function

$$Z[J]=\frac{1}{Z[0]}\int[D\Phi]e^{i\int d^4x(\mathcal{L}+\Phi J)} \tag{65}$$

- Green Functions $G^{(n)}\equiv\langle\Omega|T\{\Phi(x_1)\cdots\Phi(x_n)\}|\Omega\rangle$

$$\Phi(x)=-i\frac{\delta}{\delta J(x)}e^{i\int d^4xJ(x)\Phi(x)}\Big|_{J=0} \tag{66}$$

$$\langle\Omega|\hat{\Phi}(x)|\Omega\rangle=-i\frac{\delta}{\delta J(x)}\langle\Omega|e^{i\int d^4x\Phi J}|\Omega\rangle\Big|_{J=0} \tag{67}$$

$$\langle\Omega|\hat{O}[\Phi(x)]|\Omega\rangle=A\int[d\Phi]O[\Phi]e^{i\int d^4x\mathcal{L}}=O\left(-i\frac{\delta}{\delta J(x)}\right)Z[J]\Big|_{J=0} \tag{68}$$

$$G^{(n)} = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0} \tag{69}$$

HOMEWORK: *Prove the following expression*

$$Z[J] = \sum_n \frac{i^n}{n!} \prod_i^n \int d^4x_i J(x_i) G^{(n)}(x_1, x_2, \cdots, x_n) \tag{70}$$

3 Feynman Rules of Scalar Field

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4 \equiv \mathcal{L}_0 + \mathcal{L}_I \tag{71}$$

Klein-Gordon Equation: the Equation of motion for Free Scalar Field

$$\left(\partial_\mu \frac{\delta}{\delta\partial_\mu\Phi} - \frac{\delta}{\delta\Phi}\right)\mathcal{L}_0 = 0, \quad (\partial^2 + m^2)\Phi = 0 \tag{72}$$

The Generating Function

$$Z_0[J] = \frac{1}{Z[0]} \int [d\Phi] e^{i \int d^4x \mathcal{L}_0 + J\Phi} \tag{73}$$

$$Z[J] = \frac{1}{Z[0]} \int [d\Phi] e^{i \int d^4x \mathcal{L}_0 + \mathcal{L}_I + J\Phi} \tag{74}$$

$$= e^{i\mathcal{L}_I(\Phi = \frac{\delta}{i\delta J})} Z_0[J] \tag{75}$$

- Functional integral of a Real scalar field

$$[d\Phi] = \prod_i d\Phi(x_i) \tag{76}$$

$$\int \prod_i d\xi_i \exp[-\xi_i B_{ij} \xi_j] = \int \prod_i dx_i \exp[-b_i x_i^2] \tag{77}$$

$$= \frac{\text{const}}{\sqrt{\prod_i b_i}} = \text{const} \times \det[B]^{-1/2} \tag{78}$$

- Functional derivative ,$\delta/\delta J(x)$ .vs. Partial derivative $\partial/\partial x_i$

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}, \qquad \frac{\partial}{\partial x_i}(\sum_j x_j^2) = 2x_i, \qquad \frac{\partial}{\partial x_i}(\sum x_j k_j) = k_i \tag{79}$$

$$\frac{\delta J(y)}{\delta J(x)} = \delta^4(x-y), \quad \frac{\delta}{\delta J(x)} \int d^4y J^2(y) = 2J(x), \quad \frac{\delta}{\delta J(x)} \int d^4y J(y)\Phi(y) = \Phi(x) \tag{80}$$

Our next task is to derive Feynman Rules for a scalar field theory. One can skip Eq.(9.19-9.30) and start from Eq.(9.31). Following Eq.(9.34-9.41), we compute the correlation functions formally based on generating functional.

$$Z[J] = A \int [d\Phi] \exp\left(i \int d^4x (\mathcal{L}_0 + \mathcal{L}_I + \Phi J) \right) \tag{81}$$

3.1 $Z_0[J]$

We first get the expression of $Z_0[J]$ and then the perturbative expression of $Z[J]$.

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} m^2 \Phi^2 \tag{82}$$

$$= \frac{1}{2}\partial_\mu(\Phi\partial^\mu\Phi) - \frac{1}{2}\Phi\partial^2\Phi - \frac{1}{2}m^2\Phi^2 \tag{83}$$

$$= \frac{1}{2}\partial_\mu(\Phi\partial^\mu\Phi) - \frac{1}{2}\Phi(\partial^2 + m^2)\Phi \equiv -\frac{1}{2}\int d^4y\, \Phi(x)K(x,y)\Phi(y) \tag{84}$$

$$K(x,y) = \delta^{(4)}(x-y)(\partial^2 + m^2) \tag{85}$$

$$\tilde{K}(k) = \int d^4z K(x,y)e^{ik(x-y)} = -k^2 + m^2 \tag{86}$$

$$K^{-1}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{1}{-k^2+m^2}e^{-ik(x-y)} = i\Delta(x-y) \tag{87}$$

The i is convenient and we will see later that Δ is the Feynman propagator.

$$\Delta(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik\cdot(x-y)}, \quad \tilde{\Delta}(k) = \frac{i}{k^2-m^2+i\epsilon}, \quad \text{-----} \tag{88}$$

HOMEWORK: *It's easy to check that $i\Delta$ is the inverse of K.*

$$\int d^4z K(x,z)(i\Delta(z,y)) = \int d^4z (i\Delta(x,z))K(z,y) = \delta^{(4)}(x,y) \tag{89}$$

The Fourier transformation is expressed as

$$f(x) = \int \frac{dk}{2\pi}\tilde{f}(k)e^{-ikx} \tag{90}$$

$$\tilde{f}(k) = \int dx f(x) e^{ikx} \tag{91}$$

$$\tag{92}$$

Now the generating functional can be expressed as follows.

$$Z_0[J] = A \int [d\Phi] e^{i \int d^4x d^4y \left(-\frac{1}{2}\Phi(y)K(x,y)\Phi(x) + \delta^{(4)}(x-y)J(y)\Phi(x) \right)} \tag{93}$$

$$= A \int [d\Phi] e^{i \int d^4x d^4y \left(-\frac{1}{2}\Phi(y)K(x,y)\Phi(x) + \delta^{(4)}(x-y)J(y)\Phi(x) \right)} \tag{94}$$

$$= A \int [d\Phi] e^{i \int \left(-\frac{1}{2}(\Phi - K^{-1}J)K(\Phi - K^{-1}J) + \frac{1}{2}JK^{-1}J \right)} \tag{95}$$

$$= A \int [d\Phi] e^{i \int \left(-(\Phi - K^{-1}J)K(\Phi - K^{-1}J) \right)} \times e^{i \int \frac{1}{2} JK^{-1}J} \tag{96}$$

$$= \frac{Z_0[0]}{Z[0]} \times \exp\left(- \int d^4x d^4y \frac{1}{2} J(x) \Delta(x,y) J(y) \right) \tag{97}$$

HOMEWORK: *Free Scalar Feynman Rules*

1. Free 2-point Green Functions

$$G_0^{(2)} = \frac{\delta}{i\delta J(x)} \frac{\delta}{i\delta J(y)} Z_0[J] \Big|_{J=0} = ? \tag{98}$$

2. Free 4-point Green Functions

$$G_0^{(4)} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,3}\Delta_{2,4} + \Delta_{1,4}\Delta_{2,3} \tag{99}$$

3.2 $Z[J]$

$$e^{-\epsilon} = 1 - \epsilon + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{14} + \cdots \tag{100}$$

$$e^{-\frac{i}{\hbar}\int \frac{\lambda}{4!}\Phi^4} = 1 - \frac{i}{\hbar}\frac{\lambda}{4!}\int d^4x \left(\frac{\delta}{i\delta J(x)}\right)^4 \tag{101}$$

$$+ \frac{1}{2}\left(\frac{i}{\hbar}\frac{\lambda}{4!}\right)^2 \int d^4x \left(\frac{\delta}{i\delta J(x)}\right)^4 \int d^4y \left(\frac{\delta}{i\delta J(y)}\right)^4 + \cdots \tag{102}$$

$$Z[J] = A\int [d\Phi] e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_I+J\Phi)} = A\int [d\Phi] e^{\int d^4x \mathcal{L}_I} e^{i\int d^4x(\mathcal{L}_0+J\Phi)} \tag{103}$$

$$= e^{i\int d^4x \mathcal{L}_I(\frac{\delta}{i\delta J})} Z_0[J] \tag{104}$$

$$= A\left(1 - \frac{i}{\hbar}\frac{\lambda}{4!}\int d^4x \left(\frac{\delta}{i\delta J(x)}\right)^4 + \cdots\right) e^{-\frac{1}{2}\int d^4x_1 d^4x_2 J_1 \Delta_{1,2} J_2} \tag{105}$$

$$= Z_0[J] \times \tag{106}$$

$$\left(1 - \frac{i\lambda}{8}\int d^4x\Delta(x,x)\Delta(x,x) + \frac{i\lambda}{4}\int d^4x d^4x_3 d^4x_4 J_3\Delta_{3,x}\Delta_{x,x}\Delta_{x,4}J_4 \right. \tag{107}$$

$$\left. -\frac{i\lambda}{4!}\int d^4x d^4x_{3456}\Delta_{x,3}J_3\Delta_{x,4}J_4\Delta_{x,5}J_5\Delta_{x,6}J_6 + \cdots \right) \tag{108}$$

$$\equiv Z_0[J] \times Z_I[J] \tag{109}$$

Use Sum to instead Int: $\int d^4x_1 d^4x_2 \frac{1}{2}J(x_1)\Delta(x_1,x_2)J(x_2) \equiv \frac{1}{2}J_1\Delta_{1,2}J_2$. Please get the following results

$$\frac{\delta}{\delta J(x)}Z_0[J] = \tag{110}$$

$$\frac{\delta^2}{\delta J^2(x)}Z_0[J] = \tag{111}$$

$$\frac{\delta^3}{\delta J^3(x)}Z_0[J] = \tag{112}$$

$$\frac{\delta^4}{\delta J^4(x)}Z_0[J] = (3\Delta_{xx}\Delta_{x,x} - 6\Delta_{x,x}\Delta_{x,3}J_3\Delta_{x,4}J_4 + \Delta_{x,3}J_3\Delta_{x,4}J_4\Delta_{x,5}J_5\Delta_{x,6}J_6)Z_0[J] \tag{113}$$

3.3 Green Function

$$\langle T\{\Phi_1\Phi_2\cdots\Phi_n\}\rangle = \frac{(-i)^n}{Z[0]}\frac{\delta^n}{\delta J_1\cdots\delta J_n}Z[J]\Big|_{J=0} \tag{114}$$

- One-Point

$$\frac{\delta}{\delta J_1}Z[J] = -\Delta(x_1,x)J_xZ[J] + Z_0[J]\times\left(+\frac{i\lambda}{2}\int\Delta_{1,x}\Delta_{x,x}\Delta_{x,3}J_3\right. \tag{115}$$

$$\left.-\frac{i\lambda}{6}\int\Delta(1,x)\Delta(x,2)J_2\Delta_{x,3}J_3\Delta_{x,4}J_4+\cdots\right) \tag{116}$$

$$\langle\Phi(x_1)\rangle = \frac{-i}{Z[J]}\frac{\delta}{\delta J_1}Z[J]\Big|_{J=0} = 0 \tag{117}$$

- Two-Point

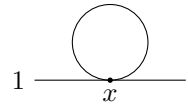
$$\frac{\delta^2}{\delta J_1J_2}Z[J] = -\Delta(x_1,x_2)Z[J] + Z_0[J]\times\left(\frac{i\lambda}{2}\int\Delta_{1,x}\Delta_{xx}\Delta_{x2}-\cdots\right) \tag{118}$$

$$-\Delta(x_1,x)J_x\frac{-i\delta Z[J]}{\delta J_2}+\frac{\delta Z_0[J]}{\delta J_2}\times() \tag{119}$$

$$\langle T\{\Phi(x_1)\Phi(x_2)\}\rangle = \frac{-1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J_1 \delta J_2}\bigg|_{J=0} \tag{120}$$

$$= \Delta_{1,2} + (-i\lambda) \times \frac{1}{2} \times \int \Delta_{1,x}\Delta_{x,x}\Delta_{x,2} \tag{121}$$

$$= \quad 1 \text{ ——————— } 2 \quad + \tag{122}$$



$$= \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x_1-x_2)}}{k^2-m^2} + \frac{(-i\lambda)}{2}\int d^4x \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2} \tag{123}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4}\frac{ie^{-ik_1(x_1-x)}}{k_1^2-m^2}\int \frac{d^4k_2}{(2\pi)^4}\frac{ie^{-ik_2(x-x_2)}}{k_2^2-m^2} \tag{124}$$

$$= \int \frac{d^4k}{(2\pi)^4}e^{-ik(x_1-x_2)} \times \tag{125}$$

$$\left(\frac{i}{k^2-m^2} + \frac{i}{k^2-m^2}\times\frac{-i}{2}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2}\times\frac{i}{k^2-m^2} + (\ldots)\right) \tag{126}$$

- Propagator: $\Delta = \frac{i}{p^2-m^2}$
- vertex: $-i\lambda$

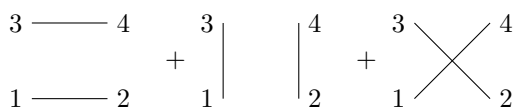
– symmetric factor: $1/2$

- Three-Point [Home Work]

$$\langle T\{\Phi_1\Phi_2\Phi_3\}\rangle = \frac{1}{Z[J]}\frac{(-i)^3\delta^3 Z[J]}{\delta J_1\delta J_2\delta J_3}\bigg|_{J=0} = \tag{127}$$

- Four-Point

$$\begin{aligned}
\langle T\{\Phi_1\Phi_2\Phi_3\Phi_4\}\rangle &= \frac{(-i\hbar)^4}{Z[J]}\frac{\delta^4}{\delta J_1\delta J_2\delta J_3\delta J_4}Z[J] && (128)\\
&= (\Delta_{1,2}\Delta_{3,4} + \Delta_{1,3}\Delta_{2,4} + \Delta_{1,4}\Delta_{2,3}) && (129)\\
&\quad + \frac{-i\lambda}{2}\Delta_{x,x}\left(\Delta_{1,2}\Delta_{3,x}\Delta_{x,4} + (x_2 \leftrightarrow x_3) + (x_2 \leftrightarrow x_4)\right. && (130)\\
&\quad \left. + (x_1 \leftrightarrow x_3) + (x_1 \leftrightarrow x_4) + (x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4)\right) && (131)\\
&\quad - i\lambda\Delta_{1,x}\Delta_{2,x}\Delta_{3,x}\Delta_{4,x} && (132)
\end{aligned}$$

$=$ 

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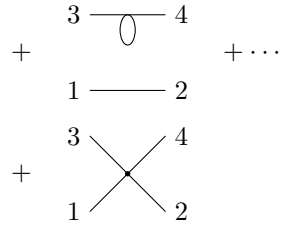
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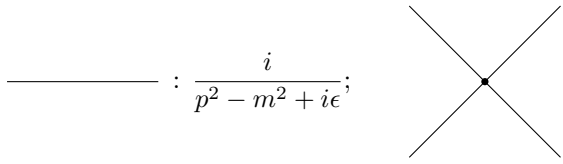
Summary

- We can calculate the Green Functions with Functional Method.
- We get the Feynman Rules for the scale fields

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4 \tag{136}$$



$$: \frac{i}{p^2 - m^2 + i\epsilon}; \qquad : -i\lambda \tag{137}$$

- **HOMEWORK:** *Get the Feynman Rules for the following Lagrangian.*

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{3!}\Phi^3 \tag{138}$$

- **HOMEWORK:** *Treat m^2 as perturbation, get the Feynman Rules for the Lagragian.*

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 \tag{139}$$

3.4 Effective Action

1. Generating function

$$W[J] = -i \ln Z[J] \tag{140}$$

2. Connected Green Functions

$$G_c^{(n)} = \left. \frac{(-i)^n \delta^n i W[J]}{\delta J_1 \cdots \delta J_n} \right|_{J=0} \tag{141}$$

所有的格林函数 $G^{(n)}$ 都可以通过连通格林函数 G_c^m 得到：

$$G^n = \frac{(-i)^n \delta^n}{\delta J_1 \cdots \delta J_n} Z[J] = \frac{(-i)^n \delta^n}{\delta J_1 \cdots \delta J_n} e^{iW[J]} \tag{142}$$

$$= \frac{(-i)^n \delta^n i W}{\delta J_1 \cdots \delta J_n} + \cdots + \frac{\delta W}{\delta J_1} \cdots \frac{\delta W[J]}{\delta J_n} \tag{143}$$

- 1-Point

$$\frac{\delta}{\delta J} W[J] = \frac{-i}{Z[J]} \frac{\delta Z[J]}{\delta J} \equiv \phi_c[J], \quad G^{(1)} = G_c^{(1)} \tag{144}$$

- 2-P

$$G_c^{(2)} = i \left. \frac{(-i)^2 \delta W[J]}{\delta J_1 \delta J_2} \right|_{J=0} \tag{145}$$

$$G^{(2)}(x_1, x_2) = G_c^{(2)}(x_1, x_2) + G_c^{(1)}(x_1) G_c^{(1)}(x_2) \tag{146}$$

- 3-P

$$G_c^{(3)} = i \frac{(-i)^3 \delta^3 W}{\delta J_1 \delta J_2 \delta J_3} \tag{147}$$

- 4-P

$$G_c^{(4)} = i \frac{(-i)^4 \delta^4 W}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \tag{148}$$

3. Effective Action

$$\Gamma(\phi_c) = W[J] - \int d^4 J(x) \phi_c(x), \quad \phi_c = \frac{\delta W}{\delta J} \tag{149}$$

4. Regular Vertex

$$\Gamma^{(n)} = \left. \frac{\delta^n \Gamma(\phi_c)}{\delta \phi_c(x_1) \cdots \delta \phi_c(x_n)} \right|_{\phi_c = 0} \tag{150}$$

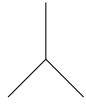
5. Tree-level Vertex

$$\Gamma[\phi_c = S[\phi_c] + \mathcal{O}(\hbar), \quad i\Gamma^{(n)} = \frac{i\delta^n S[\phi_c]}{\delta\phi_c(x_1)\cdots\delta\phi_c(x_n)} \tag{151}$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!} \tag{152}$$

$$i\Gamma^{(3)}_{\text{tree}} = \frac{i\delta^3 \int d^4x \mathcal{L}[\phi_c]}{\delta\phi_c(x_1)\delta\phi_c(x_2)\delta\phi_c(x_3)} \tag{153}$$

$$= -i\lambda \tag{154}$$



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4 Quantization of Electromagnetic Field

The Classical Electromagnetic Field is described by Maxwell equations.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial_\mu F^{\mu\nu} = 0 \tag{156}$$

4.1 Canonical Quantization

The canonical momentum of A_μ is given

$$\Pi_\mu = \frac{\delta \mathcal{L}}{\delta \dot{A}^\mu} = -F_{0\mu}, \quad \dot{A}^\mu = \partial_0 A^\mu \tag{157}$$

Canonical commutation relation would be given as

$$[A_\mu(x), \Pi_\nu(y)] = i g_{\mu\nu} \delta(x-y) \tag{158}$$

However, the commutation relation can not be introduced since

$$\Pi_0 = 0. \tag{159}$$

4.2 Path Integral

$$\begin{aligned} S[A] &= -\int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu} && (160)\\ &= -\frac{1}{2}\int d^4x (\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) && (161)\\ &= \frac{1}{2}\int d^4x A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu && (162) \end{aligned}$$

$$= -\frac{1}{2}\int d^4x d^4y A_\mu(x)\delta(x-y)\left(-\partial^2 g^{\mu\nu}+\partial^\mu\partial^\nu\right)A_\nu(y) \tag{163}$$

$$= -\frac{1}{2}\int d^4x d^4y A_\mu(x)K^{\mu\nu}(x,y)A_\nu(y) \tag{164}$$

$$Z[J] = \int [DA]e^{i(S[A]+\int A_\mu(x)\delta(x-y)J^\mu(y)} \tag{165}$$

$$= \int [DA]\exp\left(-\frac{1}{2}\int (A_\mu - K^{-1}_{\mu\sigma}J^\sigma)iK^{\mu\nu}(A_\nu - K^{-1}_{\nu\rho}J^\rho)\right)$$
$$\times \exp\left(\frac{i}{2}\int JK^{-1}J\right) \tag{166}$$

$$K_{\mu\nu}(x,y) = \delta(x-y)(-\partial^2 g_{\mu\nu}+\partial_\mu\partial_\nu) \tag{167}$$

$$= \int \frac{d^4k}{(2\pi)^4}\left(k^2 g_{\mu\nu}-k_\mu k_\nu\right)e^{-ik\cdot(x-y)} \tag{168}$$

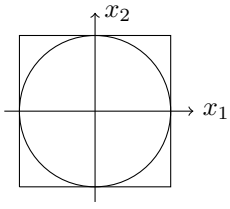
$$K_{\mu\nu}(k) = k^2 g_{\mu\nu}-k_\mu k_\nu = \begin{pmatrix} k^2-k_0^2 & -k_0k_1 & -k_0k_2 & -k_0k_3 \\ -k_0k_1 & -k^2-k_1^2 & -k_1k_2 & -k_1k_3 \\ -k_0k_2 & -k_1k_2 & -k^2-k_2^2 & -k_2k_3 \\ -k_0k_3 & -k_1k_3 & -k_2k_3 & -k^2-k_3^2 \end{pmatrix}, \tag{169}$$

$$\det(K) = 0 \tag{170}$$

The is no inverse of $K^{\mu\nu}(x,y)$, or the following function has no solution.

$$\int d^4 z K^{\mu\alpha}(x,z) K^{-1}_{\nu\beta}(z,y) = g^\mu_\nu \delta(x-y) \tag{171}$$

The reason for that is the un-physical freedom contained in A_μ. Mathematically speaking, the A_0, A_1, A_2, A_3 are not independent, but we take them as dependent function in the path integral.



$$\int [dx] \neq \int [dx_1][dx_2]$$

$$\int [dx] = \int [dx_1][dx_2]\theta(r^2 - x_1^2 - x_2^2)$$

Figure 1: Illustration of the interal

The Lagrangian is invariant under the following tranformation,

$$A_\mu = \tilde{A}_\mu + \frac{1}{e}\partial_\mu \alpha \tag{172}$$

where $\tilde{A}_\mu$ is the physical field.

$$\int [DA] e^{iS[A]} = \int [D\alpha] \int [D\tilde{A}] e^{iS[\tilde{A}]} \tag{173}$$

What we calculated is $\int [DA] e^{iS[A]}$, while what we want is $\int [D\tilde{A}] e^{iS[A]}$. The physical field is the solution of the gauge fixing condition

$$G(A) = 0, \quad \int [D\alpha] \delta(G(A)) \det \left(\frac{\delta G(A)}{\delta \alpha} \right) = 1 \tag{174}$$

The above formula is an extension of the result

$$\int dx \delta(f(x)) \frac{df}{dx} = 1. \tag{175}$$

$$\begin{aligned} Z[J] &= \int [D\tilde{A}] e^{iS[\tilde{A}]} && (176) \\ &= \int [D\alpha] \int [D\tilde{A}] e^{iS[\tilde{A}]} \delta(G(A)) \det \left(\frac{\delta G(A)}{\delta \alpha} \right) && (177) \\ &= \int [DA] e^{iS[A]} \delta(G(A)) \det \left(\frac{\delta G(A)}{\delta \alpha} \right) && (178) \\ & && (179) \end{aligned}$$

- First we deal with the determinant term.

$$G(A) = \partial \cdot A - \omega = \partial \cdot \tilde{A} + \frac{1}{e}\partial^2\alpha - \omega, \qquad \frac{\delta G(A)}{\delta \alpha} = \frac{1}{e}\partial^2 \tag{180}$$

$$\det\left(\frac{1}{e}\partial^2\right) = \int [D\bar{c}][Dc] e^{-\int d^4x \bar{c}(x)\frac{1}{e}\partial^2 c(x)} \tag{181}$$

$$= \int [D\bar{c}][Dc] e^{i\int d^4x \bar{c}(x)\left(\frac{i}{e}\partial^2\right)c(x)} \tag{182}$$

$$\tag{183}$$

$c, \bar{c}$ are the *ghost* fields, and have no physical effect in QED. But in non-Abelian case, there are interactions between gauge bosons, i.e.

$$\hat{D} = \left(\frac{\delta G(A)}{\delta \alpha}\right) \sim A, \quad \det \hat{D} = \int [Dc] \exp\left(-\int \bar{c} D(A) c\right). \tag{184}$$

The ghost will has interaction with gauge bosons and can not be dropped from the quantized Lagrangian.

- Now we deal with the δ-function.

$$\begin{aligned}\int [DA] e^{iS[A]} \delta(G(A)) &= \int [D\omega] e^{-i\int d^4x \frac{\omega^2}{2\xi}} \int [DA] e^{iS[A]} \delta(\partial \cdot A - \omega) && (185)\\ &= \int [DA] e^{i\int d^4x \mathcal{L}[A] - \frac{1}{2\xi}(\partial\cdot A)^2} && (186)\\ & && (187)\end{aligned}$$

- Finally one has

$$\begin{aligned}Z[J] &= \frac{1}{Z[0]} \int [D\bar{c}][Dc][DA] e^{i\int d^4x \mathcal{L}[A] - \frac{1}{2\xi}(\partial\cdot A)^2 + \frac{i}{e}\bar{c}\partial^2 c + A_\mu J^\mu} && (188)\\ &= \frac{1}{Z[0]} \int [DA] e^{i\int d^4x \left(\mathcal{L}[A] - \frac{1}{2\xi}(\partial\cdot A)^2 + A_\mu J^\mu\right)} && (189)\\ & && (190)\end{aligned}$$

The quantized Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot A)^2 \qquad (191)$$

$$Z[J] = \exp\left(-\frac{1}{2}\int dxdy J^{\mu}(x)(-i)K^{-1}_{\mu\nu}(x,y)J^{\nu}(y)\right) \tag{192}$$

$$K_{\mu\nu}(x,y) = \delta(x-y)\left(-\partial^2 + \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right) \tag{193}$$

$$\tilde{K}_{\mu\nu} = k^2 g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu \tag{194}$$

$$\tilde{K}^{-1}_{\mu\nu} = \frac{g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu/k^2}{k^2} \tag{195}$$

$$K^{-1}(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu/k^2}{k^2}e^{-ik\cdot(x-y)}. \tag{196}$$

As a homework, please check the following results.

$$\tilde{K}_{\mu\rho}g^{\rho\sigma}\tilde{K}^{-1}_{\sigma\nu} = g_{\mu\nu} \tag{197}$$

$$\int d^4z K_{\mu\rho}(x,z)g^{\rho\sigma}K^{-1}_{\sigma\nu}(z,y) = \delta(x-y)g_{\mu\nu} \tag{198}$$

The propagator of photon is expressed as (in momentum space)

$$\Delta_{\mu\nu} = -iK^{-1}_{\mu\nu} = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)k_\mu k_\nu/k^2\right) : \quad \mu \sim\!\!\overset{k}{\rightarrow}\!\!\sim \nu \tag{199}$$

5 Quantization of Spinor Fields

- In PI method, there are no operators but only Fields or functions which are (complex) numbers during the calculation of correlation functions.
- The classical fields in path integrals are NOT classical fields.
- Fermions are different with Bosons, can not dealed with normal complex fields. We must represent the classical fields with anti-commuting numbers.

5.1 Grassmann number

Grassmann *number* (also called an anti-commuting number or super-number) is defined in **Algebra**. It is number not operator. The basic feature of Grassmann numbers is that they are anti-commuting.

$$\theta\eta = -\eta\theta, \tag{200}$$

We are familiar with the features of *c-numbers*: product, conjugate, integral.

- **Represent**

$$\text{c-number: } \{1, i\}, \quad c = c_1 + ic_2 \tag{201}$$
$$\text{a-number: } \{\theta_i\}, \quad \theta = c_i\theta_i \tag{202}$$

- **Product 1**

$$\text{c-number: } 1 \times i = i \times 1,\ 1^2 = 1,\ i^2 = -1 \tag{203}$$
$$\text{a-number: } \theta_i\theta_j = -\theta_j\theta_i,\ \theta_i^2 = 0 \tag{204}$$

- **Product 2**

$$c\theta = \theta c \equiv \sum_i (cc_i)\theta_i \tag{205}$$

- **Conjugation**

$$\text{c-number: } 1^* = 1, \quad i^* = -i \tag{206}$$

a-number: $(\theta_i^*)^* = \theta_i$ (207)

$(c\theta_i)^* = c^*\theta^*$ (208)

$(\theta_1 \cdots \theta_n)^* = \theta_n^* \cdots \theta_i^*$ (209)

- **Countable**

c-number: $c, c^*, \cdots, c^m c^{*n}$ (210)

a-number: $\theta, \theta^*, \theta\theta^* = -\theta^*\theta$ (211)

$F(\theta) = f_0 + f_1\theta,$ (212)

$F(\theta, \theta^*) = f_0 + \theta f_1 + \theta^* \bar{f}_1 + \theta^*\theta f_3,$ (213)

If F is an ordinary number, then $f_0(f_3)$ is a ordinary number and $f_1(\bar{f}_1)$ is a Grassmann number which anticommutes with $\theta(\theta^*)$ while F is a Grassmann function in the which case $f_0(f_3)$ is Grassmann number and $f_1(\bar{f}_1)$ is ordinary number.

- **Derivatives**

$$\frac{dx_j}{dx_i} = \delta_{ij}, \qquad \frac{d\theta_j}{d\theta_i} = \delta_{ij} \tag{214}$$

$$\frac{d(ax)}{dx} = x, \qquad \frac{d(c\theta)}{d\theta} = c \tag{215}$$

$$\frac{d(xy)}{dx} = y, \qquad \frac{d\theta\eta}{d\theta} = \eta \tag{216}$$

From the above formulas, one can get the following formulas

$$\frac{d\eta\theta}{d\theta} = -\frac{d\theta\eta}{d\theta} = -\eta \tag{217}$$

$$\frac{dF}{d\theta} = f_1 - \theta^* f_3 \tag{218}$$

$$\frac{dF}{d\theta^*} = \bar{f}_1 + \theta f_3 = \tag{219}$$

$$\frac{d^2F}{d\theta^* d\theta} \equiv \frac{d}{d\theta^*}\frac{dF}{d\theta} = \tag{220}$$

$$\tag{221}$$

- **Integral**
  We do not need general definite integrals of Grassmann number. In last sections, what we need are as follows

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} f(x+a)dx \tag{222}$$

$$\int [d\phi] f[\phi] = \int [d\phi] f[\phi + \phi_0] \tag{223}$$

The Grassmann integral should be also invariant under the shift $\theta \to \theta + \eta$. The only non-vanishing integral of θ is defined as

$$\int d\theta(\theta + \eta) = \int d\theta\ \theta, \quad \to \int d\theta\ \eta = (\int d\theta)\eta = 0, \quad \int d\theta\ \theta = 1 \tag{224}$$

With the two rules, we can get the integral of any functions with θ

$$\int d\theta(A + \theta B) = (\int d\theta\ 1) \times A + (\int d\theta\ \theta)B = B \tag{225}$$

$$\int d\theta d\eta\ \eta\theta \equiv \int d\theta \int d\eta\eta\theta = \int d\theta\theta = +1. \tag{226}$$

- **Gaussian Integral**
  Ordinary Gaussian Integral is as follows

$$\int dcdc^* e^{-bc^* c} = \int dxdy e^{-b(x^2+y^2)/2} = \int rdrd\phi e^{-br^2/2} = \frac{2\pi}{b} \tag{227}$$

$$x = (c^* + c)/\sqrt{2}, \quad y = i(c^* - c)/\sqrt{2} \quad dxdy = \int dc^* dc. \tag{228}$$

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = \int d\theta^* d\theta (1 - \theta^* b\theta) = b \tag{229}$$

$$\int d\theta_1 d\theta_2 e^{-\theta_1 b\theta_2/2} = b \tag{230}$$

The multi-variable Gaussian integral can be processed as follows.

$$\int \exp(-\sum_{ij} \theta_i^* B_{ij} \theta_j) \to \int \exp(-\sum_i \theta_i'^* b_i \theta_i') \tag{231}$$

$$\to \int \exp(-\theta_1'^* b_1 \theta_1') \times \cdots \times \int \exp(-\theta_n'^* b_n \theta_n') \tag{232}$$

$$\to \prod_i b_i \equiv \det B \tag{233}$$

The b_is are the eigenvalues of the B_{ij}. Thus we get the result of Gaussian integral over Grassmann variables.

$$\left(\prod_k \int d\theta_k^* d\theta_k\right) \exp(-\theta_i^* B_{ij} \theta_j) = \prod_i b_i = \det B = e^{Tr[\log B]} \tag{234}$$

HOMEWORK: *Calculate the following integrals*

$$\mathrm{r} - number \int (\prod_k dx_k) e^{-x_i B_{ij} x_j} = \tag{235}$$

$$\mathrm{c} - number \int (\prod_k dc_k dc_k^*) e^{-c_i^* B_{ij} c_j} = \tag{236}$$

$$\mathrm{a} - number \int (\prod_k d\bar{\theta}_k d\theta_k) e^{-\bar{\theta}_i B_{ij} \theta_j} = \tag{237}$$

- **non-Gaussian integral**

$$\int d\theta^* d\theta \; \theta^* \theta \exp(-\theta^* b \theta) = 1 = \frac{b}{b} \tag{238}$$

$$(\prod_k \int d\theta_k^* d\theta_k) \theta_i^* \theta_j \exp(-\theta^* B \theta) = \int d\eta^* d\eta \quad U_{im}^* U_{jn} \eta_m^* \eta_n \exp(-\eta^* b \eta) \tag{239}$$

$$= \sum_{mn} U^*_{im} U_{jn} \int d\eta^* d\eta \quad \eta^*_m \eta_n \exp(-\eta^* b \eta) \tag{240}$$

$$= \sum_{mn} U^*_{im} U_{jn} \left(\int d\eta^*_1 d\eta_1 \exp(-\eta^*_1 b_1 \eta_1) \times \cdots \right. \tag{241}$$

$$\left. \times \ \delta_{mn} \int d\eta^*_m d\eta_m \ \eta^*_m \eta_m \exp(-\eta^*_m b_m \eta_m) \times \cdots \right), \tag{242}$$

$$= \sum_{mn} U^*_{im} U_{jn} \delta_{mn} (b_1 \times \cdots \frac{b_m}{b_m} \times \cdots) \tag{243}$$

$$= \sum_{mn} U^*_{im} U_{jn} \delta_{mn} \frac{1}{b_m} \det B = B^{-1}_{ij} \det B \tag{244}$$

Question:

$$\frac{d(\theta_1 \theta_2 \theta_3)}{d\theta_2} = \tag{245}$$

$$\frac{d^2 F(\theta, \theta^*)}{d\theta^* d\theta} = \tag{246}$$

$$\int d\theta d\eta \theta \eta = \tag{247}$$

$$\int d\theta(A + B\theta) = \tag{248}$$

$$\tag{249}$$

$$\text{Real: } \int \prod_i^n \frac{dx_i}{\sqrt{2\pi}} \exp(-\frac{1}{2} x_i A_{ij} x_j + x_i J_i) = [\det A]^{-\frac{1}{2}} \exp(\frac{1}{2} J_i A_{ij}^{-1} A_j) \tag{250}$$

$$\text{Complex: } \int \prod_i^n \frac{dc_i^* dc_i}{2\pi} \exp(-c_i^* H_{ij} c_j + c_i^* J_i + J_i^* c_i) = [\det A]^{-1} \exp(J_i^* A_{ij}^{-1} J_j) \tag{251}$$

$$\text{Grassmann: } \int \prod_i^n d\theta_i^* d\theta_i \exp(-\theta_i^* H_{ij} \theta_j + \theta_i^* \eta_i + \eta_i^* \theta_i) = [\det H] \exp(\eta_i^* H_{ij}^{-1} \eta_j) \tag{252}$$

$$\tag{253}$$

5.2 Free Fermions

The Lagrangian density for free massive fermions is given as

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{254}$$

With the Lagrangian, we can write the generating function as

$$Z[\bar{\eta},\eta] = A\int[D\bar{\psi}][D\psi]\exp\left[i\int d^4x\,[\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]\right] \tag{255}$$

$$= A\int[D\bar{\psi}][D\psi]\exp\left[-\int(\bar{\psi} - \bar{\eta}K^{-1})iK(\psi - K^{-1}\eta) - \bar{\eta}(-iK^{-1})\eta\right] \tag{256}$$

$$= \exp[-\int d^4x d^4y\bar{\eta}(x)S_F(x-y)\eta(y)] \tag{257}$$

$$K = -(\not{p} - m), S_F = (-i)\times K^{-1} = \frac{i}{\not{p} - m} \tag{258}$$

where the field ψ and source η are Grassmann-valued functions.

$$\langle\psi(x)\rangle = \int[D\bar{\psi}][D\psi]\psi(x)\exp\left[i\int d^4x\,[\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]\right] \tag{259}$$

$$= \frac{-i\delta}{\delta\bar{\eta}}Z[\bar{\eta},\eta], \quad \psi \to \frac{-i\delta}{\delta\bar{\eta}} \tag{260}$$

$$\langle\bar{\psi}(x)\rangle = \int[D\bar{\psi}][D\psi]\bar{\psi}(x)\exp\left[i\int d^4x\,[\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]\right] \tag{261}$$

$$= \frac{i\delta}{\delta\eta} Z[\bar{\eta}, \eta], \quad \bar{\psi} \to \frac{(+i)\delta}{\delta\eta} \tag{262}$$

$$\tag{263}$$

Thus the two-point Green function is

$$\langle T\psi_1\bar{\psi}_2\rangle = \frac{-i\delta}{\delta\bar{\eta}_1}\frac{i\delta}{\delta\eta_2} Z[\bar{\eta}, \eta]\Big|_{\eta,\bar{\eta}=0} \tag{264}$$

$$= S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik\cdot(x_1-x_2)}}{\not{k} - m + i\epsilon} \tag{265}$$

5.3 QED & Yukawa

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + Qe\bar{\Psi}\gamma_\mu A^\mu\Psi \tag{266}$$

$$Z_0[J] = \exp\left[-\int \frac{1}{2}J_\mu(x)\Delta^{\mu\nu}(x,y)J_\nu(y) + \bar{\eta}(x)S_F(x,y)\eta(y)\right] \tag{267}$$

$$Z[J] = \left(1 + ie\int \frac{i\delta}{\eta}\gamma_\mu \frac{-i\delta}{\delta J_\mu}\frac{-i\delta}{\delta\bar{\eta}} + \mathcal{O}(e^2)\right) Z_0[J] = Z_0[J] \times Z_I[J] \tag{268}$$

$$= Z_0[J] \times \left(1 + e\gamma_\mu \int J_3^\mu \Delta(3,x)(S_F(x,x) + \bar{\eta}_1 S_F(1,x) S_F(x,2)\eta_2)\right) \tag{269}$$

The Green Functions

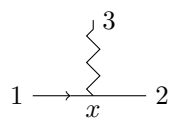
$$\langle T\{\Psi(x)\bar{\Psi}(y)\}\rangle = \frac{1}{Z[J]} \frac{(-i \times i)\delta^2 Z[J]}{\delta\bar{\eta}(x)\delta\eta(y)}\bigg|_0 = S_F(x,y) \tag{270}$$

$$\langle T\{A^\mu(x)A^\nu(y)\}\rangle = \frac{1}{Z[J]} \frac{(-i)^2\delta^2 Z[J]}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_0 = \Delta^{\mu\nu}(x,y) \tag{271}$$

$$\tag{272}$$

The three-point Green function can be obtained as

$$\langle T\{\psi_2\bar{\psi}_1 A^\mu(x_3)\}\rangle = \frac{1}{Z[J]} \frac{(-i)\delta^3 Z[J]}{\delta\bar{\eta}_2\delta\eta_1\delta J_\mu(x_3)}\bigg|_0 =$$



$$(ieQ\gamma^\mu) + \cdots \tag{273}$$

6 Symmetries

6.1 EOM: Schwinger-Dyson equations

Minimal Action Principle (Stationary Action) in Classical Mechanism

$$x \to x + \epsilon, \mathcal{L}(x, \dot{x}) \to \mathcal{L} + \mathcal{O}(\epsilon^2) \tag{274}$$

$$\begin{aligned} \delta\mathcal{L} &= \epsilon \frac{\partial \mathcal{L}}{\partial x} + \dot{\epsilon} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \mathcal{O}(\epsilon^2) && (275) \\ &= \epsilon \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \frac{d}{dt} \left(\epsilon \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \mathcal{O}(\epsilon^2) && (276) \end{aligned}$$

Euler-Lagrange Equation

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \tag{277}$$

In Classical Field Theory, the fields satisfy the Euler-Lagrange Equation (which is called Equation of Motion)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \tag{278}$$

In QFT, the Green Function is *physical*. The Schwinger-Dyson equations can be treated as EOM of Green Functions.

$$Classical \rightarrow Quantum \tag{279}$$

$$O \quad \langle \hat{O} \rangle \tag{280}$$

$$\Phi \quad \langle \Phi \rangle \tag{281}$$

$$\langle T\phi(x_1) \rangle = Z^{-1} \int [D\phi] \phi(x_1) e^{i \int d^4x \mathcal{L}(\phi, \partial\phi)} \tag{282}$$

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \epsilon(x), D\phi' = D\phi, \tag{283}$$

$$\mathcal{L}[\phi] \rightarrow \mathcal{L}[\phi'] = \mathcal{L}[\phi] + \epsilon(x) \left(\frac{\partial}{\partial \phi} - \partial_\mu \frac{\partial}{\partial \partial_\mu \phi} \right) \mathcal{L} + \mathcal{O}(\epsilon^2) \tag{284}$$

$$\int [D\phi'] \phi'(x_1) e^{i \int d^4x \mathcal{L}[\phi']} = \int [D\phi] \phi(x_1) e^{i \int \mathcal{L}} \tag{285}$$

$$+ \int [D\phi] e^{i \int \mathcal{L}} \left(\epsilon(x_1) + i \int d^4x \epsilon(x) \phi(x_1) \frac{\delta \mathcal{L}}{\delta \phi} \right) + \tag{286}$$

$$0 = \int [D\phi] e^{i\int \mathcal{L}} \int d^4x \epsilon(x) \left(\delta^{(4)}(x - x_1) + i\phi(x_1) \frac{\delta \mathcal{L}}{\delta \phi} \right) \quad (287)$$

$$\int [D\phi] e^{i\int \mathcal{L}} \phi(x_1) \frac{\delta \mathcal{L}}{\delta \phi} = i \int [D\phi] e^{i\int \mathcal{L}} \delta^{(4)}(x - x_1) \quad (288)$$

$$\langle \frac{\delta \mathcal{L}}{\delta \phi} \phi(x_1) \rangle = i\delta^{(4)}(x - x_1) \quad (289)$$

$$\frac{\delta \mathcal{L}}{\delta \phi} \equiv (\frac{\partial}{\partial \phi} - \partial_\mu \frac{\partial}{\partial \partial_\mu \phi}) \mathcal{L} = (-\partial^2 - m^2)\phi, \quad \text{for KG Field} \quad (290)$$

$$(\partial^2 + m^2) \langle T(\phi(x)\phi(x_1)) \rangle = -i\delta^{(4)}(x - x_1) \quad (291)$$

$$(\partial^2 + m^2) \langle T(\phi(x)\phi_1 \cdots \phi_n) \rangle = \sum_i \langle T(\phi_1 \cdots (-i\delta^{(4)}(x - x_i)) \cdots \phi_n \rangle \quad (292)$$

For General Case, Eq.(9.88)

6.2 Nother Theorem

6.2.1 Nother Current

$$\phi_a \to \phi'_a = \phi_a + \epsilon \Delta \phi_a \quad (293)$$

$\Delta\phi_a$ can be functions of x and ϕ_a. The Lagrangian is invariant under the transfomation: $\mathcal{L}[\phi'] = \mathcal{L}[\phi] + \epsilon\partial_\mu \mathcal{J}^\mu$

$$\begin{aligned}
\mathcal{L}[\phi] &= \mathcal{L}[\phi + \epsilon\Delta\phi] - \epsilon\partial_\mu \mathcal{J}^\mu && (294)\\
&= \mathcal{L}[\phi] - \epsilon\partial_\mu \mathcal{J}^\mu + \frac{\partial\mathcal{L}}{\partial\phi_a}\epsilon\Delta\phi_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\epsilon\partial_\mu\Delta\phi_a && (295)\\
&= \mathcal{L}[\phi] - \epsilon\partial_\mu \mathcal{J}^\mu + \epsilon(\Delta\phi_a\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\mu\Delta\phi_a) && (296)\\
&= \mathcal{L}[\phi] - \epsilon\partial_\mu \mathcal{J}^\mu + \epsilon\partial_\mu\sum_a\left(\Delta\phi_a\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right) && (297)
\end{aligned}$$

Then one has Nother current which is conserved,

$$\partial_\mu\left(\sum_a \Delta\phi_a\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a} - \mathcal{J}^\mu\right) = 0 \quad (298)$$

- complex scalar

$$\phi \to e^{i\alpha}\phi, \quad j^\mu = i(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \quad (299)$$

- fermion

$$\psi = e^{ie\alpha}\psi, \quad j^\mu = e\bar{\psi}\gamma^\mu\psi \quad (300)$$

- spacetime-translation

$$x_\mu \to x'_\mu = x_\mu - a_\mu \tag{301}$$
$$\phi \to \phi' = \phi + a_\mu \partial^\mu \phi \tag{302}$$
$$\mathcal{L}[\phi, x] \to \mathcal{L}[\phi', x'] \tag{303}$$

$$\begin{aligned} \mathcal{L}[\phi, x] &= \mathcal{L}[\phi', x'] && (304)\\ &= \mathcal{L}[\phi', x] - a_\mu \partial^\mu \mathcal{L} && (305)\\ j^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} a_\nu \partial^\nu \phi - a^\mu \mathcal{L} && (306)\\ &= a_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \right) && (307)\\ &\equiv a_\nu T^{\mu\nu} && (308) \end{aligned}$$

6.2.2 QFT

$$\phi' = \phi + \epsilon \Delta\phi, \quad \mathcal{L}[\phi + \epsilon\Delta\phi] - \mathcal{L}[\phi] = -\epsilon \partial_\mu j^\mu(x) \tag{309}$$

Similar to Schwinger-Dyson Equation, consider 2-Point Green Fucntion as an example,

$$\begin{aligned}\int [D\phi] e^{i\int d^4x \mathcal{L}[\phi]} \phi_1\phi_2 &= \int [D\phi'] e^{i\int d^4x \mathcal{L}[\phi']} \phi_1'\phi_2' \\ 0 &= \epsilon \int [D\phi] e^{i\int d^4x \mathcal{L}[\phi]} \left(\int d^4x (-i)\partial_\mu j^\mu(x)\phi_1\phi_2 + \Delta\phi_1\phi_2 + \phi_1\Delta\phi_2 \right) \\ &= \epsilon \int [D\phi] e^{i\int d^4x \mathcal{L}[\phi]} \int d^4x \left(-i\partial_\mu j^\mu(x)\phi_1\phi_2 + \Delta\phi(x)\delta(x-x_1)\phi_2 + \phi_1\Delta\phi\right.\end{aligned}$$

$$\langle \partial_\mu j^\mu(x)\phi_1\phi_2\rangle = (-i)(\langle \Delta\phi(x)\delta(x-x_1)\phi_2\rangle + \langle \phi_1\Delta\phi(x)\delta(x-x_2)\rangle) \tag{313}$$

6.3 Ward-Takahashi Identity

$$\partial_\mu \langle j^\mu(x)\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle + i\sum_i \langle \phi_{a_1}(x_1)\cdots\Delta\phi_{a_j}(x)\delta(x-x_i)\cdots\phi_{a_n}(x_n)\rangle = 0 \tag{314}$$

$$\Delta\psi \to ie\epsilon\psi, \Delta\bar{\psi} = -ie\epsilon\bar{\psi}, \mathcal{L}[\psi + ie\epsilon\psi] - \mathcal{L}[\phi] = \epsilon\partial_\mu j^\mu(x) \tag{315}$$

$$i\partial_\mu \langle J^\mu\psi(x_1)\bar{\psi}(x_2)\rangle = -ie\delta(x-x_1)\langle\psi(x)\bar{\psi}(x_2)\rangle + ie\delta(x-x_2)\langle\psi(x_1)\bar{\psi}(x)\rangle \tag{316}$$

- These contact terms don't contribute to the transition amplitude, so we specialize to the Ward identity $k_\mu \mathcal{M}^\mu = 0$

$$k \cdot \varepsilon = 0,\ k_\mu = (k,0,0,k),\ \varepsilon_\mu^\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0),\ \mathcal{M} = \varepsilon_\mu \mathcal{M}^\mu, \tag{317}$$

$$\begin{aligned}
\sum |\bar{\mathcal{M}}|^2 &= |\mathcal{M}^\mu \epsilon_\mu^+|^2 + |\mathcal{M}^\mu \epsilon_\mu^-|^2 = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 && (318)\\
&= \sum_\lambda \mathcal{M}^\mu \mathcal{M}^{\nu *} \epsilon_\mu^\lambda \varepsilon_\nu^{\lambda *} = \mathcal{M}^\mu \mathcal{M}^{\nu *}(-g_{\mu\nu}) && (319)\\
&= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2 && (320)
\end{aligned}$$

$$k_\mu \mathcal{M}^\mu = k(\mathcal{M}^0 + \mathcal{M}^3) = 0, \quad |\mathcal{M}^3|^2 = |\mathcal{M}^0|^2 \tag{321}$$

- $Z_2 = Z_1$

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - e \bar{\psi} A\!\!\!/ \psi = Z_2 \bar{\psi} i \partial\!\!\!/ \psi - Z_1 e \bar{\psi} A\!\!\!/ \psi \tag{322}$$

$$i\partial_\mu \langle J^\mu \bar{\psi}(x_1)\bar{\psi}(x_2)\rangle = -ie\delta(x-x_1)\langle \psi(x)\bar{\psi}(x_2)\rangle + ie\delta(x-x_2)\langle \psi(x_1)\bar{\psi}(x)\rangle \tag{323}$$

$$Z_2 i\partial_\mu \langle J^\mu \bar{\psi}(x_1)\bar{\psi}(x_2)\rangle = -iZ_1 e\delta(x-x_1)\langle \psi(x)\bar{\psi}(x_2)\rangle + ie\delta(x-x_2)\langle \psi(x_1)\bar{\psi}(x)\rangle \tag{324}$$

- In $\overline{MS}$ scheme Z_1, Z_2 has same pole, $Z_1 = Z_2$
- in On-shell (Eq.7.69)

$$-ik_\mu Z_2 \tilde{\Gamma}^\mu(p+k,k) = Z_1 \tilde{e}(\tilde{S}^{-1}(p+k) - \tilde{S}^{-1}(p)), \tag{325}$$

at the limit $k \to 0$ and $p^2 = m^2$

$$\tilde{\Gamma}^\mu(k=0) = \tilde{e}\gamma^\mu, \tilde{S}^{-1}(p) = \not{p} - m \tag{326}$$

$$Z_1 = Z_2 \tag{327}$$

$$\mathcal{L} = \bar{\Psi} i(\not{\partial} + ie\not{A})\Psi \equiv \bar{\Psi} i \not{D} \Psi \tag{328}$$

$$\mathcal{L} = Z_2 \bar{\Psi} i \not{\partial} \Psi - Z_1 e \bar{\Psi} \not{A} \Psi = Z_2 \bar{\Psi} i \not{D} \Psi \tag{329}$$

The renormalized Lagrangian is still gauge invariant.

7 Answers

The answers are as follows:

$$\frac{\delta}{\delta J(x)} Z_0[J] = -(\Delta_{3,x} J_3) Z_0[J] \tag{330}$$

$$\frac{\delta^2}{\delta J^2(x)} Z_0[J] = (\Delta_{3,x} J_3 \Delta_{x,4} J_4 - \Delta_{x,x}) Z_0[J] \tag{331}$$

$$\frac{\delta^3}{\delta J^3(x)} Z_0[J] = (3\Delta_{xx}\Delta_{x,3} J_3 - \Delta_{x,3} J_3 \Delta_{x,4} J_4 \Delta_{x,5} J_5) Z_0[J] \tag{332}$$

$$\frac{\delta^4}{\delta J^4(x)} Z_0[J] = (3\Delta_{xx}\Delta_{x,x} - 6\Delta_{x,x}\Delta_{x,3} J_3 \Delta_{x,4} J_4 + \Delta_{x,3} J_3 \Delta_{x,4} J_4 \Delta_{x,5} J_5 \Delta_{x,6} J_6) Z_0[J] \tag{333}$$