CH 10. Renormalization

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1 Power countering–Why

For any Feynman diagram in $d-$dimension, one has

- $D =$ Divergence degree
- $d =$ Dimension
- $L =$ Number of Loops
- $I_F =$ Number of Internal Fermion line
- $I_B =$ Number of Internal Boson line
- $\delta_i =$ Power of momentum in vertex

$$\sim \int \frac{d^d k_1 \cdots d^d k_L (k_n)^{\delta_n} \cdots}{(\not{k}_i - m) \cdots (k_j^2)} \tag{1}$$

The *Divergence degree* D is expressed as

$$D = dL + \sum \delta_n - I_F - 2I_B. \tag{2}$$

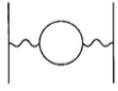
Now with the above expression, we can get the following statements.

1. $L \leq 1$ in $d = 4$

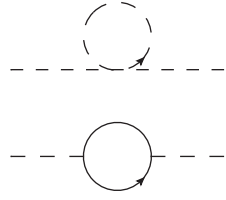

$D = 0$ finite


$D = 0$ $\sim \log \Lambda$


$D = 2$ $\sim \log \Lambda$


$D = -2$ $\sim \log \Lambda$


$D = -2$ finite



The third diagram has $D = 2$ but only a logarithmic divergence, since the photon has no mass(Ward indentity).

$$\Pi_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\gamma_\mu(\not{k} + m)\gamma_\nu(\not{k} + \not{q} + m)\right]}{(k^2 - m^2)((k+q)^2 - m^2)} \tag{3}$$

$$= \quad (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2) \neq \Lambda^2 + \cdots \tag{4}$$

The real divergence degree is worse than superficial DD.

2. D is *superficial degree of divergence* which sums the power of all k_i at the limit $k_i \to \infty$. Problem of the defination? (Divergent subdiagram)
   The statement on P316 after Eq.(10.1):

   - When a diagram contains a divergent sub-diagram, its actual divergence may be worse than that indicated by D. (If the a diagram has no divergent sub-diagram, its actual divergence is not worse than that indicated by D)
   - When symmetries (Such as the Ward Identity) cause certain terms to cancel, the divergence of a diagram maybe reduced or even eliminated. (If a diagram has no sub-diagram, its divergence can not be worse than superficial divergence)

   表观发散度的意义：只有当 Feynman 图及其所有子图的表观发散度都 < 0 时才是真正收敛的 —- Weinerg 定理

3. The divergence is related with dimension of space-time d: Dimension Regularization

4. Topological relation

$$L = I_F + I_B - V + 1 \tag{5}$$

- Total number of internal momentum $I_F + I_B$,
- The number of momentum constrain $V - 1$
- The number of free internal momentum $L = I_F + I_B - V + 1$

5. Consider a specific theory

$$\begin{aligned}
\mathcal{L} &\sim g(i\partial)^{\delta}\Phi^{n_b}\Psi^{n_f} && (6)\\
[\mathcal{L}] &= [g] + \delta + [\Phi]n_b + [\Psi]n_f && (7)\\
[\Psi] &= \frac{d-1}{2} && (8)\\
[\Phi] &= \frac{d-2}{2} && (9)\\
[g] &= d - \delta - \frac{d-2}{2}n_b - \frac{d-1}{2}n_f \equiv r && (10)
\end{aligned}$$

Considering a Feynman diagram with L loops, V vertices, E_B boson and E_F Fermion external lines, I_B boson and E_F fermion internal lines, one has

$$\begin{aligned}
Vn_b &= E_B + 2I_B && (11)\\
Vn_f &= E_F + 2I_F && (12)\\
L &= I_F + I_B - V + 1 && (13)
\end{aligned}$$

$$
\begin{aligned}
D &= dL + V\delta - I_F - 2I_B /. L = I_F + I_B - V + 1 && (14)\\
&= d + (d-1)I_F + (d-2)I_B + V(\delta - d) /. \frac{I_F = (Vn_f - E_F)/2}{I_B = (Vn_b - E_B)/2} && (15)\\
&= d - \frac{d-2}{2}E_B - \frac{d-1}{2}E_F - V\left(d - \delta - \frac{d-2}{2}n_b - \frac{d-1}{2}n_f\right) && (16)\\
&= d - \frac{d-2}{2}E_B - \frac{d-1}{2}E_F - rV && (17)\\
& && (18)
\end{aligned}
$$

- The divergence degree is related with *dimension*.
- $r = 0, d = 4$, $D = 4 - E_B - \frac{3}{2}E_F$, Finite number of *amplitudes* are divergent; Infinite number of *diagrams* are divergent.
- $r > 0$, The number of divergent diagrams is finite. Some amplitudes diverge at lower order and converge at high order.
- $r < 0$, The divergence become worse with more vertices. $\mathcal{L}_I = G_F(\bar{e}\nu)_{V-A}(\bar{\nu}e)_{V-A}$

1.1 QED

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{\Psi}A\!\!\!/\Psi \tag{19}$$

$$d=4, r=0, n_f=2, n_b=1, \quad D=4-E_\gamma-\frac{3}{2}E_e \tag{20}$$

HOMEWORK: *The divergent degree in d-dimension. Eq.(10.11)*
All the divergent diagrams are $E_\gamma \leq 4, E_e \leq 4$

- (a) $(E_\gamma, E_e) = (0,0), D = 4$, divergent. Vacuum energy which is *unobservable*. Quantum Gravity
- (b) $(1,0), D=3$. zero

$$I_\mu = \int \frac{d^4k}{(2\pi)^4} \frac{Tr(\gamma_\mu(\not{k}+m))}{k^2-m^2} \tag{21}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{4k_\mu}{k^2-m^2} = 0 \tag{22}$$

$$C|0\rangle = |0\rangle, Cj_\mu(x)C^\dagger = -j_\mu(x) \tag{23}$$

$$\langle T(j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n))\rangle = \langle C^\dagger C T(j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n))C^\dagger C\rangle \tag{24}$$

$$= (-1)^n \langle T(j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n))\rangle \tag{25}$$

Furry Theorem: The vacuum expectation value of any odd number of electromagnetic currents must vanish.

- (c) $(2,0), D=2$, Logarithm. Eq.(7.90) with Dimension Regularization

$$\Pi_{\mu\nu} = (g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2), \Pi(q^2) = -\frac{2\alpha}{\pi}\int dx(1-x)\left(\frac{2}{\epsilon} - \ln\Delta - \gamma\right) \tag{26}$$

- (d) $(3,0), D=1$, Zero Furry theorem
- (e) $(4,0), D=0$, Finite
- (f) $(0,2), D=1$, Logarithm. Eq.(7.19) with Pauli-Villars Regularization

$$-i\Sigma_2 \tag{27}$$

- (g) $(1,2), D=0$, Logarithm. Eq.(6.56-6.57) with Pauli-Villars Regularization

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) \tag{28}$$

In summary:

- There are three type divergent diagrams in QED: self-energy diagrams of fermion and photon, interaction of photon with fermions.
- All the divergences are Logarithm.
- The three divergences can be absorbed into Photon Field, Fermion Field and the coupling α_e.

1.2 $\lambda\Phi^n$

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{n!}\Phi^n \tag{29}$$

$$\delta = 0, n_b = n, n_f = 0 \tag{30}$$

$$r = d - \frac{d-2}{2}n \tag{31}$$

$$D = d - \left(d - \frac{d-2}{2}n\right)V - \frac{d-2}{2}E_\Phi \tag{32}$$

$$r > 0 \quad : \quad \text{super} - \text{Renormalizable} \tag{33}$$

$$r = 0 \quad : \quad \text{Renormalizable} \tag{34}$$

$$r < 0 \quad ; \quad \text{non} - \text{Renormalizable} \tag{35}$$

Considering the a theory with $n = 4$ in $d = 4$, one can get

$$r = 0, \text{and } D = 4 - E_\Phi \tag{36}$$

- $E_\Phi = 0, D = 4$, vacuum energy shift, meaningless in QFT.
- $E_\Phi = 1, D = 3$, zero. Z_2-symmetry
- $E_\phi = 2, D = 2$, Quadratic Divergent
- $E_\Phi = 3, D = 1$, Zero, Z_2-symmetry
- $E_\Phi = 4, D = 0$, Logarithm divergent

2 Renormalized Perturbative Theory– $\lambda\Phi^4$

There are only two type divergences in $\lambda\Phi^4$ theory, two- and four- point diagrams.

2.1 Loop Calculation–Regualarization

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{37}$$

(38)

2.1.1 Dimension-Regularization

In $d = 4 - \epsilon$ dimension space, the Lagrangian can be expressed as

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\mu^\epsilon\lambda}{4!}\phi^4, \tag{39}$$

where μ is a arbitrary scale parameter.

The divergence degrees are

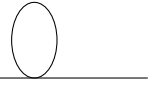
$$4 \to d = 4 - \epsilon \tag{40}$$

$$E_\Phi = 2: \quad 2 \to 2 - \epsilon \tag{41}$$

$$E_\Phi = 4: \quad 0 \to -\epsilon \tag{42}$$

- self-energy correction

$$-i\Sigma(p^2) = \tag{43}$$



$$= \frac{-i\mu^\epsilon\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2 + i\epsilon} \tag{44}$$

$$\overset{\text{(Wick R)}}{=} -\frac{\mu^\epsilon\lambda}{2(2\pi)^d}\int d^dl\frac{i}{l^2 + m^2} \tag{45}$$

$$= -\frac{i\mu^\epsilon\lambda}{2(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{m^{2-d}} \tag{46}$$

$$= -\frac{i}{2}\frac{\lambda m^2}{16\pi^2}(\frac{4\pi\mu^2}{m^2})^{\epsilon/2}\Gamma(-1+\epsilon/2) \tag{47}$$

$$= \frac{i}{2}\frac{\lambda}{16\pi^2}m^2\left(\frac{2}{\epsilon}-\gamma+\ln 4\pi+1+\ln\mu^2/m^2+\mathcal{O}(\epsilon)\right) \tag{48}$$

$$= \frac{i}{2}\frac{\lambda}{16\pi^2}m^2\left(\frac{2}{\epsilon}+1+\ln\hat{\mu}^2/m^2+\mathcal{O}(\epsilon)\right) \tag{49}$$

Wick rotation:

$$\frac{1}{k^2-m^2+i\epsilon} = \frac{1}{(k^0-\sqrt{\vec{k}^2+m^2}+i\epsilon)(k^0+\sqrt{\vec{k}^2+m^2}-i\epsilon)}, \tag{50}$$

$$k^0 = il_E, \quad d^dk = id^dl, \quad k^2 = -l^2 \tag{51}$$

$\epsilon-$expansion

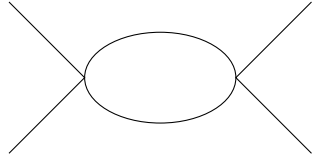
$$x^\epsilon = 1+\epsilon\ln x+\mathcal{O}(\epsilon^2), \quad \Gamma(x) = \frac{(-1)^n}{n!}(\frac{1}{x+n}-\gamma+1+\cdots+\frac{1}{n}+\mathcal{O}(x+n)) \tag{52}$$

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = 1+\frac{\epsilon}{2}\ln\frac{4\pi\mu^2}{m^2} \tag{53}$$

$$\Gamma(-1+\epsilon/2) = -\left(\frac{2}{\epsilon} - \gamma_E + 1 + \mathcal{O}(\epsilon)\right) \tag{54}$$

- Interaction

$$i\Gamma(p^2) = \tag{55}$$



$$= \frac{(-i\lambda\mu^\epsilon)^2}{2}\int \frac{i}{(k-p)^2-m^2}\frac{i}{k^2-m^2} \tag{56}$$

(Feyn Par)
$$= -\frac{(-i\lambda\mu^\epsilon)^2}{2}\int \frac{d^dk}{(2\pi)^d}\int dx \frac{1}{((k-xp)^2-(m^2-x(1-x)p^2))^2} \tag{57}$$

(Shift & Wick R)
$$= -\frac{(-i\lambda\mu^\epsilon)^2}{2}\int dx \int \frac{d^dl}{(2\pi)^d}\frac{i}{(l^2+\Delta^2)^2} \tag{58}$$

(Mom Int)
$$= -\frac{(-i\lambda\mu^\epsilon)^2}{2}\frac{i}{(4\pi)^{d/2}}\int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \tag{59}$$

(ϵ-Expand)
$$= -i\lambda\mu^\epsilon \times \frac{-\lambda}{16\pi^2}\int dx \left(\frac{1}{\epsilon} - \frac{1}{2}\gamma + \ln 4\pi + \frac{1}{2}\ln\frac{\mu^2}{\Delta} + \mathcal{O}(\epsilon)\right) \tag{60}$$

The difference between the result of Eq.(10.23) on P327 and Eq.(11.77) on P377 is $\ln \mu^2$.

$$
\begin{aligned}
i\Gamma(s,t,u) &= i\Gamma(s) + i\Gamma(t) + i\Gamma(u) && (61)\\
&= (-i\lambda\mu^{\epsilon}) \times \frac{-\lambda}{16\pi^2}\left(\frac{3}{\epsilon} - \frac{3}{2}\gamma + 3\ln 4\pi + \frac{1}{2}\int dx \ln\frac{\mu^2}{\Delta_s} + \ln\frac{\mu^2}{\Delta_t} + \ln\frac{\mu^2}{\Delta_u}\right) && (62)
\end{aligned}
$$

$\Delta = m^2 - x(1-x)p^2$ and $s = (p_1+p_2)^2, t = (p_1-p_3)^2, u = (p_1-p_4)^2$.

- The DR does not respect the divergent behavior.

$$D = 2 : \Sigma \to 1/\epsilon \quad (63)$$

$$D = 0 : \Gamma \to 1/\epsilon \quad (64)$$

2.1.2 Cut-Off

$$
\begin{aligned}
-i\Sigma(p^2) &= \frac{-i\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} && (65)\\
&\overset{\text{(Wick R)}}{=} -\frac{i\lambda}{2(2\pi)^4}\int d^4l \frac{1}{l^2+m^2} && (66)
\end{aligned}
$$

$$= -\frac{i\lambda}{2(4\pi)^2}\int_0^{\Lambda^2} dl^2 \frac{l^2}{l^2+m^2} \tag{67}$$

$$= -\frac{i\lambda}{2(4\pi)^2}(\Lambda^2 - m^2 \ln\frac{m^2+\Lambda^2}{m^2}) \tag{68}$$

- $l^2 < \Lambda^2$ is not Lorentz invariant, $l_\mu \in (-\Lambda, \Lambda)$
- Lorentz Cut-Off $p^2 = p_0^2 - p_i^2 < \Lambda^2$: $p_\mu \in (-\infty, +\infty)$.

$$i\Gamma(p^2) = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{(k-p)^2-m^2}\frac{i}{k^2-m^2} \tag{69}$$

(Feyn Par) $$= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\int dx \frac{1}{((k-xp)^2-(m^2-x(1-x)p^2))^2} \tag{70}$$

(Shift & Wick R) $$= \frac{i\lambda^2}{2}\int dx \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+\Delta^2)^2} \tag{71}$$

(Mom Int) $$= \frac{i\lambda^2}{2}\times\frac{1}{16\pi^2}\left(\ln\frac{\Lambda^2+m^2}{m^2} - \frac{\Lambda^2}{\Lambda^2+m^2}\right) \tag{72}$$

Expand $$= \frac{i\lambda^2}{2}\times\frac{1}{16\pi^2}\left(\ln\frac{\Lambda^2}{m^2} - 1 + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)\right) \tag{73}$$

$$i\Gamma(s,t,u) = i\Gamma(s) + i\Gamma(t) + i\Gamma(u) \tag{74}$$

$$= (-i\lambda) \times \frac{-3\lambda}{16\pi^2}\left(\ln\frac{\Lambda^2}{m^2} - 1\right) \tag{75}$$

The divergence behavior is respected but the symmetries are violated.

2.2 Counter terms–Renormalization

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4 \tag{76}$$

$$\mathcal{L}_R = \mathcal{L}_0 - \delta\mathcal{L}, \quad \delta\mathcal{L} = \frac{1}{2}\delta_Z(\partial\phi)^2 - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta_\lambda}{4!}\phi^4 \tag{77}$$

- self-energy: Renormalization of field strengh and mass

$$\Sigma_0 \sim \Lambda^2 + m^2\ln\Lambda + p^2\ln\Lambda + (\text{finite} - \text{terms}) \tag{78}$$

$$\delta\Sigma \sim p^2\delta_Z - \delta_{m^2} \tag{79}$$

$$\delta_Z \sim \ln\Lambda + (\text{fintite} - \text{terms}) \tag{80}$$

$$\delta_{m^2} \sim \Lambda^2 + m^2\ln\Lambda + (\text{fintite} - \text{terms}) \tag{81}$$

- interactions:

$$\Gamma \sim \ln \Lambda + (\text{finite} - \text{terms}) \tag{82}$$
$$\delta\Gamma \sim \delta_\lambda \tag{83}$$
$$\tag{84}$$

- Counter-terms: To generate the Counter-terms $\delta\Sigma \sim p^2\delta_Z - \delta_m$ and δ_λ, one has

$$\begin{aligned}
\delta\mathcal{L} &\sim \frac{1}{2}\delta_Z(\partial\Phi)^2 - \frac{1}{2}\delta_{m^2}\Phi^2 - \frac{\delta_\lambda}{4}\Phi^4 && (85)\\
\mathcal{L}_R &= \mathcal{L}_0 - \delta\mathcal{L} && (86)\\
&= \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda^2}{4!}\Phi^4 && (87)\\
&\quad -\frac{1}{2}\delta_Z(\partial\Phi)^2 + \frac{1}{2}\delta_{m^2}\Phi^2 + \frac{\delta_\lambda}{4!}\Phi^4 && (88)\\
&= \frac{1}{2}(1-\delta_Z)(\partial\Phi)^2 - \frac{1}{2}(m^2 - \delta_{m^2})\Phi^2 - \frac{\lambda - \delta\lambda}{4!}\Phi^4 && (89)\\
&\equiv \frac{1}{2}(\partial\Phi_r)^2 - \frac{1}{2}m_r^2\Phi_r^2 - \frac{\lambda_r^2}{4!}\Phi_r^4 && (90)
\end{aligned}$$

- Practical view (BPHZ scheme)

$$\Phi_0 = Z_\Phi^{1/2}\Phi_r, m_0^2 = Z_m m_r^2, \lambda_0 = Z_\lambda \lambda_r \tag{91}$$

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial\Phi_0)^2 - \frac{1}{2}m_0^2\Phi_0^2 - \frac{\lambda}{4!}\Phi_0^4 && (92)\\
&= \frac{Z_\Phi}{2}(\partial\Phi_r)^2 - \frac{Z_m Z_\Phi}{2}m_r^2\Phi_r - Z_\Phi^2 Z_\lambda\frac{\lambda_r}{4!}\Phi_r^4 && (93)\\
&= \frac{1}{2}(\partial\Phi_r)^2 - \frac{1}{2}m_r^2\Phi_r^2 - \frac{\lambda_r}{4!}\Phi_r^4 && (94)\\
&\quad + \frac{\delta Z_\Phi}{2}(\partial\Phi_r)^2 - \frac{\delta m^2}{2}\Phi_r^2 - \frac{\delta\lambda}{4!}\Phi_r^4 && (95)
\end{aligned}$$

$$\begin{aligned}
&\delta_Z = Z - 1 && (96)\\
&\delta m^2 = (Z_m Z_\Phi - 1)m_r^2 = (\delta Z_m + \delta Z_\phi)m_r^2 && (97)\\
&\delta\lambda = (2\delta Z_\Phi + \delta Z_\lambda)\lambda_r && (98)
\end{aligned}$$

  - m_0, λ_0 are bare parameters which are *unphysical*.
  - m_r, λ_r are *physical* (not really, actually called *renormalized*) parameters which are scheme-dependent.

– From now on, $\Phi_r \to \Phi, m_r \to m, \lambda_r \to \lambda$ and $\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}$ not the bare (or un-renormalized) Lagrangian but the renormalized Lagrangian at Tree-level or Zero-order.

2.3 One-loop renormalization of $\lambda\Phi^4$

In the previous section, we get the relation between renormalized and bare Lagrangian.

To obtain a finite result for an amplitude which involving divergent diagrams, we have used following procedure: 1) Computer the diagrams using renormalized mass m and coupling λ with regularization. 2) Subtract the divergence with counter terms.

In this section we will illustrate how the subtraction works. Now we working from the Lagrangian (Eq10.18,P 324)

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4 \tag{99}$$

$$+ \frac{\delta Z_\Phi}{2}(\partial\Phi)^2 - \frac{\delta m^2}{2}\Phi^2 - \frac{\delta\lambda}{4!}\Phi^4 \tag{100}$$

- Vertex renormalzation

$$i\Gamma^{(4)}(p_i) = -i\lambda + i\Gamma(s,t,u) - i\delta_\lambda \tag{101}$$

$$i\Gamma(s,t,u) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln(m^2 - x(1-x)s)\right) + (s \to t) + (s \to u) \tag{102}$$

We use the amplitude at $\vec{p}_i = 0$ (which can be measured or obtained with continuation) as input,

$$
\begin{aligned}
i\Gamma^{(4)}(\vec{p}_i = 0) &= -i\lambda && (103)\\
\delta_\lambda &= \Gamma(4m^2, 0, 0) && (104)\\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\bigg(\frac{6}{\epsilon} - 3\gamma + 3\ln 4\pi && (105)\\
&\quad - \ln(m^2 - 4m^2x(1-x)) - 2\ln m^2\bigg) && (106)
\end{aligned}
$$

The scattering amplitude

$$
\begin{aligned}
i\mathcal{M}(p_1p_2 \to p_3p_4) &= -i\lambda - \frac{i\lambda^2}{32}\int_0^1 dx\bigg(\ln\frac{m^2 - x(1-x)s}{m^2 - 4m^2x(1-x)} && (107)\\
&\quad + \ln\frac{m^2 - x(1-x)t}{m^2} + \ln\frac{m^2 - x(1-x)u}{m^2}\bigg) && (108)
\end{aligned}
$$

- Field strength and mass renormalization

$$
-i\Sigma(p^2) = -i\lambda\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m) \qquad (109)
$$

$$= \frac{-i\lambda\mu^\epsilon}{2(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{m^{2-d}} + i(p^2\delta_Z - \delta_m) \tag{110}$$

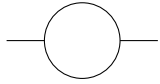
On-shell Renormalization:

$$\Sigma(p^2 = m^2) = 0, \Sigma'(p^2 = m^2) = 0 \tag{111}$$

$$\delta_Z = 0 \tag{112}$$

$$\delta_m = \frac{-\lambda\mu^\epsilon}{2(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{m^{2-d}} \tag{113}$$

In summary the *renormalization conditions* are summarized as



$$= \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2). \quad \text{on-shell subtraction} \tag{114}$$



$$= -i\lambda(\text{at } s = 4m^2, t = u = 0). \quad \text{0-momentum subtraction} \tag{115}$$

In $\overline{MS}$ scheme:

$$\delta_\lambda = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + 3\ln 4\pi\right) \tag{116}$$

$$\delta_Z = 0 \tag{117}$$

$$\delta_m = \frac{-m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{118}$$

2.4 Comments and Summary

Renormalization include:

- Regularization Scheme: Cut-Off (Power Counting), DR, Pauli-Villars, High Dirive$\cdots$

  Cut-Off

$$\int_{-\infty}^{\infty} d^4k = i\int_{-\infty}^{\infty} d^4l = \frac{i}{2}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} l^3 dl d\Omega = i\lim_{\Lambda\to\infty}\int_0^{\Lambda^2} l^2 dl^2 d\Omega \tag{119}$$

  Problem: Lorentz invariant, momentum shift, symmetry(Ward identity, P248 Eq.(7.79)),

  Pauli-Villars

$$\frac{1}{p^2-m^2} \to \lim_{\Lambda\to\infty}\frac{1}{p^2-m^2} - \frac{1}{p^2-\Lambda^2} \tag{120}$$

  The integral is unaffected at small k region, but cuts off smoothly when $k \geq \Lambda$.
  Problem: momentum shift, the regulators are introduced diagram by diagram.

Higher derivative

$$\delta\mathcal{L} = -\frac{1}{2\Lambda^2}\Phi(\partial)^4\Phi \tag{121}$$

$$\mathcal{L} \sim \Phi(-\partial^2 - m^2 - \partial^4/\Lambda^2)\Phi \tag{122}$$

$$\Delta(p) = \frac{i}{p^2 - m^2 - p^4/\Lambda^2} \tag{123}$$

There is no regulator any more. The integral is unaffected at low momentum region and improved $\frac{1}{k^2} \to \frac{1}{k^4}$.

Problem: Higher power terms need if the divergence is worse.

Indeed this defines a completely physical finite theory (Lee-Wick model: Finite QED).

Dimension Regularization $4 \to d$, Eq.(A.41-48). No regulators and no extra spectrum.

Problem: γ^5, SUSY

$$\gamma^5 = \gamma^0\gamma^1\cdots\gamma^{d-1}, \gamma^5\gamma^\mu = (-1)^{d-1}\gamma^\mu\gamma^5 \tag{124}$$

- Subtract condition(Renormalization conditions): The relation between the renormalized parameters and observed values

– on-shell: Physical Mass —- Pole mass $p^2 = m^2$

$$\begin{aligned}
\Sigma(p^2) &= (\Lambda^2 + p^2 \ln\Lambda + m^2 \ln\Lambda + \cdots) + p^2\delta_Z - \delta_m && (125)\\
&= \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \tilde{\Sigma}(p^2) && (126)\\
\langle T\{\Phi(x)\Phi(0)\}\rangle &= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(i\Sigma)\frac{i}{p^2 - m^2} + \cdots && (127)\\
&= \frac{i}{p^2 - m^2 + \Sigma_r(p^2)} && (128)\\
\Sigma(m^2) &= \tilde{\Sigma}(m^2) = 0 \text{ (mass renormalization)} && (129)\\
\Sigma'(m^2) &= 0 \text{ (field Renormalization)} && (130)\\
\langle T\{\Phi(x)\Phi(0)\}\rangle &= \frac{i}{p^2 - m^2 + \tilde{\Sigma}(p^2)} && (131)
\end{aligned}$$

$$\Gamma(m, m, m, m) = -i\lambda \text{ (zero} - \text{momentum scattering)} \quad (132)$$

– Off-Shell, for massless particle $p^2 = -\mu^2$

– MS or $\overline{MS}$

3 QED

1. Write down the bare Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}(F_0^{\mu\nu})^2 + \bar{\Psi}_0(i\partial\!\!\!/ - m_0)\Psi_0 - e_0\bar{\Psi}_0\gamma_\mu\Psi_0 A_0^\mu \tag{133}$$

2. Introduce the renormalization constant

$$e_0 = Z_e e_r, m_0 = Z_m m_r, \Psi_0 = \sqrt{Z_2}\Psi_r, A_0^\mu = \sqrt{Z_3}A_r^\mu \tag{134}$$

3. Get the Lagrangian with counter-terms

$$\mathcal{L} = -\frac{1}{4}Z_3(F^{\mu\nu})^2 + Z_2\bar{\Psi}(i\partial\!\!\!/ - Z_m m)\Psi - Z_1 e\bar{\Psi}\gamma_\mu\Psi A^\mu \tag{135}$$

$$= -\frac{1}{4}(F^{\mu\nu})^2 + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - e\bar{\Psi}\gamma_\mu\Psi A^\mu \tag{136}$$

$$-\frac{1}{4}\delta_3(F^{\mu\nu})^2 + \bar{\Psi}(i\delta_2\partial\!\!\!/ - \delta_m)\Psi - \delta_1 e\bar{\Psi}\gamma_\mu\Psi A^\mu \tag{137}$$

where

$$\delta_3 = Z_3 - 1, \delta_2 = Z_2 - 1, \delta_m = (Z_2 Z_m - 1)m, \delta_1 = (Z_1 - 1) \tag{138}$$

The charge renormalization(Please see §7.5) can be gotten with Ward Identity.

$$Z_2 = Z_1 = Z_e Z_2 Z_3^{1/2}, \quad Z_e = Z_3^{-1/2}. \tag{139}$$

4. Get the Feynman Rules and Calculate the loop diagrams with some regularization scheme.

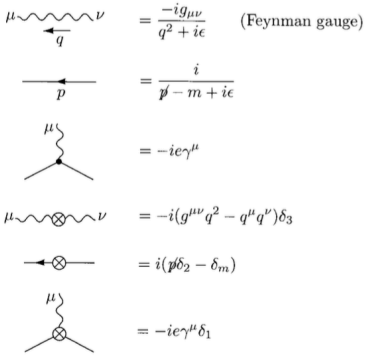
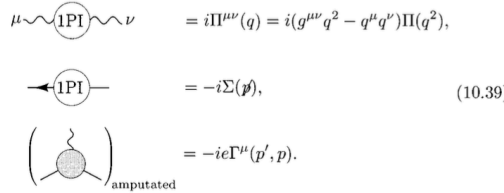


Figure 10.4. Feynman rules for Quantum Electrodynamics in renormalized perturbation theory.



(10.39)

5. Fix the renormalization condition to get the counter terms.

$$\begin{aligned} \Sigma(\not{p} = m) &= 0; \\ \frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=m} &= 0; \\ \Pi(q^2 = 0) &= 0; \\ -ie\Gamma^\mu(p' - p = 0) &= -ic\gamma^\mu. \end{aligned} \tag{10.40}$$

$$\delta_2 = \frac{d}{d\not{p}}\Sigma_2(p = m), \qquad m\delta_2 - \delta_m = \Sigma_2(m) \tag{140}$$

$$\delta_3 = \Pi_2(0), \qquad \delta_1 = -\delta F_1(0) \tag{141}$$

The vertex correction can be expressed with *form factors* F_1, F_2 as in Eq6.33 on P186

$$\Gamma^\mu = \gamma^\mu F_1(q) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q). \tag{142}$$

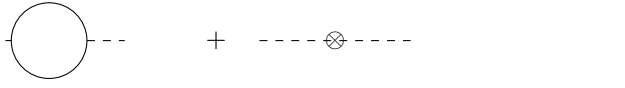
The final results are shown in Eq.(10.41)-Eq.(10.46).

4 Yukawa

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m_f)\Psi + \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - y\bar{\Psi}\Psi\Phi \tag{143}$$

- $r = 0$: $D = 4 - E_\Phi - \frac{3}{2}E_\Psi$
- $(0,0), D = 4$
- $(1,0), D = 3$
- $(2,0), D = 2$
- $(4,0), D = 0$
- $(0,2), D = 1$
- $(1,2), D = 0$

1.Field strength and mass renormalization

$-i\Sigma(p^2) =$  (144)

$$= \frac{4ig^2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + i(p^2\delta_Z - \delta_m) \tag{145}$$

$$\Delta = m_f^2 - x(1-x)p^2 \tag{146}$$

subtract at $p^2 = m^2$

$$\Sigma(p^2 = m^2) = 0 \quad : \quad \delta_m = \frac{4g^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(1-\frac{d}{2})}{(m_f^2 - x(1-x)m^2)^{1-d/2}} \tag{147}$$

$$\Sigma'(p^2 = m^2) = 0 \quad ; \quad \delta_Z = -\frac{4g^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{x(1-x)\Gamma(2-d/2)}{(m_f^2 - x(1-x)m^2)^{2-d/2}} \tag{148}$$

For massless particle $m_f^2 = m^2 = 0$

$$-i\Sigma(p^2) = \frac{4ig^2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(1-d/2)}{(-x(1-x)p^2)^{1-d/2}} + i(p^2\delta_Z - \delta_m) \tag{149}$$

- can not subtracte on-shell $p^2 = 0$ any more. subtract off shell.
- The integration of x divergent at $x \to 0$ and $x \to 1$, IR pole.

$$\int d^4p \frac{1}{p^2 - m^2 + i\epsilon} \tag{150}$$

The pole $\pm(m - i\epsilon)$ can be avoid by Wick Rotation, but the $0 \pm i\epsilon$ can not.

2. Interaction Renormalization

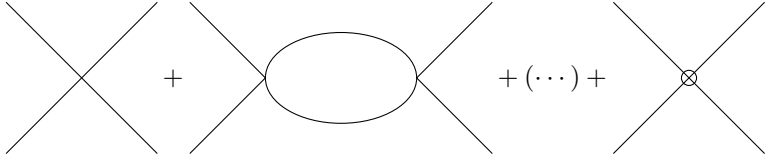
5 Beyond one-loop

1. Complete form of renormalized Lagrangian and Feynman Rules

$$\mathcal{L} = \frac{1}{2}(\partial\Phi_r)^2 - \frac{1}{2}m_r^2\Phi_r^2 - \frac{\lambda_r}{4!}\Phi_r^4 \tag{151}$$

$$+ \frac{\delta Z_\Phi}{2}(\partial\Phi_r)^2 - \frac{\delta m^2}{2}\Phi_r^2 - \frac{\delta\lambda}{4!}\Phi_r^4 \tag{152}$$

2. Renormalization at one-loop order

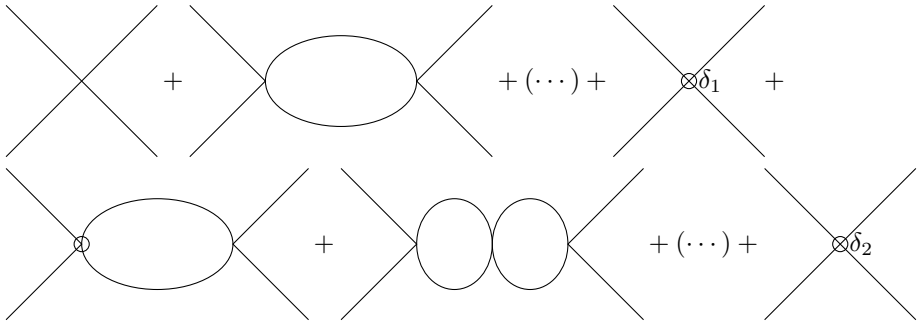


(153)

$$\to -i\lambda, \quad (s = 4m^2, t = u = 0) \tag{154}$$

$$\delta_\lambda = \delta_1(\lambda^2) \tag{155}$$

3. Renormalization at two-loop order



$$\to -i\lambda, \quad (s = 4m^2, t = u = 0) \tag{156}$$

$$\delta_\lambda = \delta_1(\lambda^2) + \delta_2(\lambda^3) \tag{157}$$

4. High order counter-terms can be obtained with the same way.

$$\delta = \lambda^2 + \lambda^3 + \lambda^4 + \cdots \tag{158}$$

6 Lee-Wick

$$D = dL - \delta_B I_B - \delta_F I_F + V\delta \tag{159}$$

$$L = I_F + I_B - V + 1 \tag{160}$$

$$E_F \qquad Vn_i = 2(I_i + E_i) \tag{161}$$

$$= d - \frac{dE_B}{2} - \frac{dE_F}{2} + \frac{\delta_B E_B}{2} + \frac{\delta_F E_F}{2} + n_B V + \frac{n_F V}{2} - \frac{\delta_B n_B V}{2} - \frac{\delta_F n_F V}{2} - rV \tag{162}$$

$$= d - \frac{d-\delta_B}{2} E_B - \frac{d-\delta_F}{2} E_F - Vr - Vn_B \frac{\delta_B - 2}{2} - Vn_F \frac{\delta_F - 1}{2} \tag{163}$$

$$\mathcal{L} = \tag{164}$$