CH 12. Renormalization Group

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1 Renormalization and Effective Field Theory

- The divergence can be canceled with renormalization or shift of few parameters in the theory, that has been shown in last section.
- Physically, the fact that high-momentum virtual quanta has little effect, should be understood at deep level.

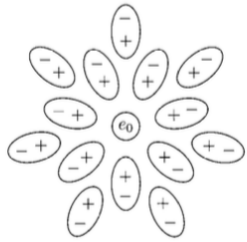

Figure 7.8. Virtual e^+e^- pairs are effectively dipoles of length $\sim 1/m$, which screen the bare charge of the electron.

– High-momentum means the small distance, this can be understood as *effective charge* or *effective theory*.

– The effective theory can be illustrated with multi-pole expansion.

- Practically, the subtraction point is arbitrary.

$$\sigma(\alpha(\mu), m(\mu), \mu) \tag{1}$$

2 Callan-Symanzik Equation

2.1 Off-shell Renormalization-Yukawa

Consider the scalar self-energy diagram in Yukawa theory,

$$-i\Sigma(p^2) = \frac{4ig^2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(1-d/2)}{(-x(1-x)p^2)^{1-d/2}} + i(p^2\delta_Z - \delta_m) \tag{2}$$

- On-shell scheme does not work
- Off-shell scheme: subtract at any unphysical scale $p^2 = -\mu^2$

$$\Sigma(-\mu^2) = 0 \tag{3}$$
$$\Sigma'(-\mu^2) = 0 \tag{4}$$
$$\Gamma(-\mu^2) = i\lambda \tag{5}$$

$$\delta_m = 0, \tag{6}$$
$$\delta_Z = \frac{4g^2}{(4\pi)^{d/2}} \int dx \frac{x(1-x)\Gamma(2-d/2)}{(x(1-x)\mu^2)^{2-d/2}} \tag{7}$$

$$\sim \frac{\Gamma(2-d/2)}{\mu^{4-d}} = \frac{1}{2-d/2} - \ln \mu^2 + \cdots \tag{8}$$

2.2 Callan-Symanzik Equation

The choice of the renormalization scale is arbitrary. Is there connections between different scales?

$$\tilde{G}^n = \langle \phi(x_1) \cdots \phi(x_n) \rangle = Z^{-n/2} \langle \phi_0(x_1) \cdots \phi_0(x_n) \rangle \tag{9}$$

Conside two different scale μ_1^2 and μ_2^2

$$Z_1^{n/2} \tilde{G}_1^{(n)} = Z_2^{n/2} \tilde{G}_2^{(n)} \tag{10}$$

Considering a small variation of renormalization scale

$$\mu_2 = \mu_1 + \delta\mu \tag{11}$$

$$\lambda_2 = \lambda_1 + \delta\lambda \tag{12}$$

$$\phi_2 = (1+\delta\eta)\phi_1 \tag{13}$$

$$\tilde{G}_2^n = (1+n\delta\eta)\tilde{G}_1^n \tag{14}$$

$$\mu \frac{d\tilde{G}^{(n)}}{d\mu} = \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{d\lambda}{d\mu} \frac{\partial}{\partial \lambda} \right) \tilde{G}^{(n)} = n\mu \frac{d\eta}{d\mu} \tilde{G}^{(n)} \tag{15}$$

$$\beta \equiv \frac{d\lambda}{d\ln\mu} = \mu\frac{d\lambda}{d\mu} \tag{16}$$

$$\gamma \equiv -\frac{d\eta}{d\ln\mu} = -\mu\frac{d\eta}{d\mu} \tag{17}$$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma\right) G^{(n)}(x_1, x_2, \ldots, x_n; \mu, \lambda) = 0 \tag{18}$$

- The Callan-Symanzik equation was discovered independently by Curtis Callan and Kurt Symanzik in 1970. Later it was used to understand asymptotic freedom.
- $\beta(\lambda)$ is called beta-function.

  $$\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu} \tag{19}$$

  describing the running of *renormalized* coupling.
- γ is called *anomalous dimension*

  $$\phi_0 = \sqrt{Z}\phi, \delta\eta = \frac{\delta\phi}{\phi} = \frac{\delta Z}{2Z}, \tag{20}$$

$$\gamma = \frac{\partial \ln Z}{\partial \ln \mu^2} = \frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial \mu} \tag{21}$$

$$Z = 1 + \delta_Z, \gamma = \frac{1}{2}\frac{\mu}{\delta_Z + 1}\frac{\partial \delta_Z}{\partial \mu} = \frac{1}{2}\mu\frac{\partial \delta_Z}{\partial \mu} + \text{higher-order} \tag{22}$$

Although Z or δZ is divergent, γ is conserved (Please check the result in §10).

2.3 $\lambda\phi^4$

γ: Field strength and mass renormalization can be done in the same way to that in Yukawa theory.

$$-i\Sigma(p^2) = -i\lambda\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m) \tag{23}$$

$$= \frac{-i\lambda\mu^\epsilon}{2(4\pi)^{d/2}}\frac{\Gamma(1 - d/2)}{m^{2-d}} + i(p^2\delta_Z - \delta_m) \tag{24}$$

The one-loop correction does not depend on p^2, thus the counter terms would not depend on the scale μ^2. $\gamma = 0$. Indeed one has

$$-i\Sigma(p^2) = 0, \quad \text{when } m^2 = 0, \text{ i.e. } \delta_Z = 0, \delta_m = 0. \tag{25}$$

β:

$$\Gamma^{(4)} = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)] - i\delta_\lambda \tag{26}$$

- On-shell renormalization: $s = 4m^2, t = u = 0$
- Off-shell renormalization: $s = t = u = -\mu^2$

$$\Gamma^{(4)}(s = t = u = -\mu^2) = -i\lambda, \tag{27}$$

$$\delta_\lambda = (-i\lambda)^2 3V(-\mu^2) \tag{28}$$

$$= \frac{3\lambda^2}{2(4\pi)^{d/2}} \int dx \frac{\Gamma(2 - d/2)}{(x(1-x)\mu^2)^{(2-d/2)}} \tag{29}$$

where

$$\frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}(x(1-x)\mu^2)^{(2-d/2)}} = \frac{1}{(4\pi)^2}\Big(\frac{1}{2 - d/2} - \ln\mu^2 - \gamma + \ln 4\pi \tag{30}$$

$$- \ln(x(1-x)) + \mathcal{O}(2 - d/2)\Big) \tag{31}$$

$$\delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2}\left(\frac{1}{2 - d/2} - \ln\mu^2 + finite - term\right) \tag{32}$$

The *finite-term* is not really finite and it become divergence at end point $x \to 0, 1$. These divergence does not effect the renormalization and total x-section.

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right) G^{(4)} = 0 \tag{33}$$

$$\mu\frac{\partial}{\partial\mu}G^{(4)} = -i\mu\frac{\partial}{\partial\mu}\delta_\lambda = \frac{3i\lambda^2}{16\pi^2} \tag{34}$$

$$\beta\frac{\partial}{\partial\lambda}G^{(4)} = \beta(-i + \mathcal{O}(\lambda)) \tag{35}$$

$$\gamma = 0 \tag{36}$$

$$\beta = \frac{-\mu}{Z_\lambda}\frac{\partial}{\partial\mu}\delta_\lambda = \frac{3\lambda^2}{16\pi^2} \tag{37}$$

$$\lambda_0 = \mu^\epsilon Z_\lambda \lambda \tag{38}$$

$$0 = \frac{d\lambda_0}{d\mu} = \left(\frac{\epsilon}{\mu}\lambda + \frac{\lambda}{Z_\lambda}\frac{dZ_\lambda}{d\mu} + \frac{1}{\lambda}\frac{d\lambda}{d\mu}\right)\mu^\epsilon Z_\lambda \tag{39}$$

$$\beta = -\frac{\lambda}{Z}\frac{dZ_\lambda}{d\ln\mu} \tag{40}$$

General Green Function

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+n\gamma_\psi+m\gamma_A\right)G^{(n,m)}(\psi_1,\cdots\psi_n,A_1,\cdots A_m,g,\mu)=0 \tag{41}$$

$$\gamma_i = \mu\frac{\partial\delta_{Z_i}}{\partial\mu}, \tag{42}$$

$$\beta = \mu\frac{\partial}{\partial\mu}(-\delta_g+g\sum_i\delta_{Z_i}/2) \tag{43}$$

2.4 QED

$$\mathcal{L}=-\frac{1}{4}(F^{\mu\nu})^2+\bar\psi(i\not{D}-m)\psi-\frac{1}{4}\delta_3(F^{\mu\nu})^2+\bar\psi(i\delta_2\not{\partial}-\delta_m)\psi-e\delta_1\bar\psi\not{A}\psi. \tag{44}$$

WTI: $\delta_2=\delta_1=e^{-1}\delta_e$

$$\delta_2 = \delta_1=-\frac{e^2}{16\pi^2}\frac{\Gamma(2-d/2)}{(\mu^2)^{2-d/2}}+finite=-\frac{e^2}{16\pi^2}\left(\frac{1}{2-d/2}-\ln\mu^2+finite\right) \tag{45}$$

$$\delta_3 = -\frac{e^2}{16\pi^2}\frac{4}{3}\frac{\Gamma(2-d/2)}{(\mu^2)^{2-d/2}}+finite. \tag{46}$$

$$\gamma_2(e) = \frac{\partial \delta_2}{\partial \ln \mu^2} = \frac{e^2}{16\pi^2} \tag{47}$$

$$\gamma_3(e) = \frac{\partial \delta_3}{\partial \ln \mu^2} = \frac{e^2}{12\pi^2} \tag{48}$$

$$\beta(e) = \mu \frac{\partial}{\partial \mu}(-e\delta_1 + e\delta_2 + e\delta_3/2) = \frac{e^3}{12\pi^2} \tag{49}$$

HOMEWORK: *§16.5 One-Loop Divergences of Non-Abelian Gauge Theory. Eq.(16.58-16.85)*