15. Non-Abelian Gauge Invariance

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1 Symmetries

The Lagrangian is invariant under some transformations.

$$\begin{cases} \text{Discrete Symmetries:} & C, P, T, R, Z_n \cdots \\ \text{Continuous Symmetries:} & Lorentz, Isospin, U(1)(\text{charge, lepton/baryon number}) \cdots \end{cases}$$

$$\begin{cases} \text{Abelian:} & C, P, T, Z_2, U(1) \cdots \\ \text{non-Abelian:} & Z_n, Lorentz, Isospin \cdots \end{cases}$$

$$\begin{cases} \text{Global:} & C, P, T, R, Z_n, Lorentz, Isospin, U(1)(\text{charge, lepton/baryon number}) \cdots \\ \text{Local:} & GaugeSymmetries : U(1), SU(2), SU(3) \cdots \end{cases}$$

$$U(1) \begin{cases} \text{Global:} & \psi(x) \to e^{i\alpha}\psi(x) \\ \text{Local:} & \psi(x) \to e^{i\alpha(x)}\psi(x) \end{cases}$$

2 Histroy of Gauge Theory

3 Gauge invariance–QED

3.1 Global symmetry

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1}$$

- is invariant under the $U_V(1)$ transformation $\psi \to e^{i\alpha}\psi$, and the conserved current is $j^\mu = \bar{\psi}\gamma^\mu\psi$.
- $m = 0$: is also invariant under the $U_A(1)$ transformation $\psi \to e^{i\beta\gamma^5}\psi$, and the Nother current is $j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$

$$U_V(1): \qquad \psi_{L,R} \to e^{i\alpha}\psi_{L,R} \tag{2}$$
$$U_A(1): \qquad \psi_{L,R} \to e^{\pm i\beta}\psi_{L,R} \tag{3}$$
$$\alpha_{1,2} = \alpha \pm \beta \qquad \psi_L \to e^{i\alpha_1}\psi_L, \psi_R \to e^{i\alpha_2}\psi_R \tag{4}$$

$U_L(1) \times U_R(1) \to U_V(1) \times U_A(1)$. α, β are NOT dependent on x.

3.2 Local symmetry

Considering a local $U(1)$ symmetry

$$\psi(x) \to \psi' = e^{i\alpha(x)}\psi(x), \quad \bar{\psi}' = \bar{\psi}e^{-i\alpha(x)} \tag{5}$$

$$m\bar{\psi}\psi \rightarrow m\bar{\psi}'\psi' = m\bar{\psi}\psi \tag{6}$$

$$\bar{\psi}i\partial\!\!\!/\psi \rightarrow \bar{\psi}'i\partial\!\!\!/\psi' = \bar{\psi}i\partial\!\!\!/\psi - \bar{\psi}\gamma^\mu\psi\partial_\mu\alpha \tag{7}$$

The Gauge Field A_μ is introduced, and transforms under the U(1) transformation as

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{8}$$

The interaction terms

$$e\bar{\psi}\gamma^\mu\psi A_\mu \rightarrow e\bar{\psi}'\gamma^\mu\psi' A'_\mu = e\bar{\psi}\gamma^\mu\psi A_\mu - \bar{\psi}\gamma^\mu\psi\partial_\mu\alpha(x) \tag{9}$$

The $U(1)$ gauge invariant Lagrangian is

$$\mathcal{L} = \bar{\psi}\gamma^\mu(i\partial_\mu - m - eA_\mu)\psi = \bar{\psi}\gamma^\mu i(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi \tag{10}$$

The covariant derivative $D_\mu = \partial_\mu + ieA_\mu$: $D'_\mu = e^{i\alpha}D_\mu e^{-i\alpha}$.

To complete the construction of the Lagrangian with Local Symmetry, we must write down the kinetic energy for the gauge field A_μ with derivative (momentum) term $(\partial_\mu A_\nu)^2$.

$$\partial_\mu A_\nu \rightarrow \partial_\mu A'_\nu = \partial_\mu A_\nu - \frac{1}{e}\partial_\mu\partial_\nu\alpha(x), \tag{11}$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu A'_\nu - \partial_\nu A'_\mu \equiv F_{\mu\nu} = \frac{1}{ie}[D_\mu, D_\nu] \tag{12}$$

therefore $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i(\partial_\mu + ieA_\mu)\gamma^\mu\psi - m\bar{\psi}\psi.$ (13)

1. **Why $-1/4$ Factor?**
   The Factor is arbatrary and can be changed by rescale of field strength A_μ and coupling e. $\frac{1}{4}$ is chosen to match the classical Maxwell Equation

$$\frac{\delta\mathcal{L}}{\delta A_\mu} - \partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\nu A_\mu} = 0, \quad \partial_\mu F^{\mu\nu} = ej^\nu \tag{14}$$

2. **What do A_μ and $F_{\mu\nu}$ Mean:**

   - Connection and Covariant Derivative

$$\partial_\mu\psi(x) = \lim_{\delta x^\mu\to 0}\frac{\psi(x^\mu + \delta x^\mu) - \psi(x^\mu)}{\delta x^\mu} \tag{15}$$

   We need map(translate) the ψ at x^μ to ψ' at $x^\mu + \delta x^\mu$ with connection,

$$\psi'(x) = U(x^\mu + \delta x^\mu, x)\psi = \psi(x) - ieA_\mu\delta x^\mu\psi(x) \tag{16}$$

   covariant derivative

$$D_\mu\psi \quad = \quad \lim_{\delta x^\mu\to 0}\frac{\psi(x^\mu + \delta x^\mu) - \psi'(x^\mu)}{\delta x^\mu} \tag{17}$$

$$= \lim_{\delta x^\mu \to 0} \frac{\psi(x^\mu + \delta x^\mu) - \psi(x^\mu) + ieA_\mu \delta x^\mu \psi(x)}{\delta x^\mu} \tag{18}$$

$$= (\partial_\mu + ieA_\mu)\psi(x) \tag{19}$$

- Wilson Line and Wilson Loop:

$$U(x+\delta x, x) = 1 - ieA_\mu \delta x^\mu = \exp\left[-ie\int_x^{x+\delta x} A_\mu dx^\mu\right] \tag{20}$$

Comarator $U(y,x)$: Nonlocal and Gauge invariant

$$U(y,x) = U(y,x;S] = e^{-ie\int_S A_\mu dx^\mu} \tag{21}$$

$$U(x,x) = U(x,x;L] = e^{-ie\oint_L A_\mu dx^\mu} = e^{-i\int_L F_{\mu\nu} d\sigma^{\mu\nu}} \tag{22}$$

The phase shift in the EM field.

- Covariant Curvature and Field strength

$$U(x,x) = 1 - ieF_{\mu\nu}\delta x^\mu \wedge \delta x^\nu + \cdots \tag{23}$$

The covariant $Curvature$ tensor is

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \tag{24}$$

3. **Why** *covariant***?**

$$covariant \rightarrow invariant. \tag{25}$$

4. **Other terms:**

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\Psi - m\bar{\Psi}\Psi + ??? \tag{26}$$

It's Lorentz invariant, Gauge invariant and Renormalizable.

- $\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$: violate CP, anormally
- Mass-term $A_{\mu}A^{\mu}$: not gauge invariant, not renormalizable
- Dim-5: $\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$: not renormalizable, appears in EFT
- Dim-6: $(\bar{\psi}\psi)^2_{S,P,V,A}$: not renormlizable, appears in EFT

5. **Scalar QED:**

$$\mathcal{L} = |\partial\phi|^2 - m^2 - \lambda|\phi|^4, \rightarrow |D\phi|^2 - m^2 - \lambda|\phi|^4 \tag{27}$$

The Ψ and ϕ have the same coupling with A_{μ}.

6. **Massive QED:**
The mass term is not gauge invariant,

$$A_\mu A^\mu \to \left(A_\mu - \frac{1}{e}\partial_\mu \alpha\right)\left(A^\mu - \frac{1}{e}\partial^\mu \alpha\right). \tag{28}$$

The unrenormalizable Lagrangian of massive Vetor Field (Proca Field) is

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}m^2 A_\mu A^\mu && (29)\\
&= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu && (30)\\
&= \frac{1}{2}A_\mu\left(-(i\partial)^2 g^{\mu\nu} + m^2 g^{\mu\nu} + (i\partial^\mu)(i\partial^\nu)\right)A_\nu && (31)\\
&= -\frac{1}{2}\int dy A_\mu(y) K^{\mu\nu}(y,x) A_\nu(x) && (32)
\end{aligned}$$

The Propagator $\Delta(k)$ is

$$K^{\mu\nu}(k) = (k^2 - m^2)g^{\mu\nu} - k^\mu k^\nu \tag{33}$$

$$\Delta_{\mu\nu}(k) = -i\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2} \tag{34}$$

$$\Delta_{\mu\alpha} g^{\alpha\beta} K_{\beta\nu} = -i g_{\mu\nu} \tag{35}$$

At the limit $k \to \Lambda \to \infty$, the propagator's behavior is $\Delta_{\mu\nu}(k) \to \frac{g_{\mu\nu}}{m^2}$ not $\Delta_{\mu\nu}(k) \to \frac{g_{\mu\nu}}{\Lambda^2}$. The more the internal lines, the worse the divergence. *The Proca Lagrangian is not renormalizable.*

The EOM can be gotten with Euler-Lagrange Equation

$$(\partial^2 + m^2) A_\mu - \partial_\mu \partial \cdot A = 0 \tag{36}$$

To construct a renormalizable massive QED Lagrangian, a way is to introduce another field to cancel the extra terms which is called Stúckelberg field

$$\phi(x) \to \phi(x) - \frac{1}{e}\alpha(x), \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}(m A_\mu - \partial_\mu \phi)^2 \tag{37}$$

4 Yang-Mills Theory: non-Abelian Gauge Invariant Lagrangian

Proton and Neutron are Isospin doublet, and they are degenerate except eletronmagnetic interaction(Charge is different). In 1954, Yang and Mills introduced a Local $SU(2)$ symmetry instead the Global Isospin symmetry.

$$\Psi \to \Psi' = V(x)\Psi = e^{i\alpha_i(x)\tau^i}\Psi, \quad \tau^i = \sigma_i/2, \quad i = 1, 2, 3. \tag{38}$$

$$\bar{\Psi}' = \bar{\Psi} V^\dagger = \bar{\Psi} e^{-i\alpha_i \tau^i} \tag{39}$$

The Lagrangian with global $SU(2)$ is

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi, \quad \Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \tag{40}$$

It's not invariant under local $SU(2)$ transfomation,

$$\bar{\Psi}(i\partial\!\!\!/)\Psi \to \bar{\Psi}V^\dagger(i\partial\!\!\!/)V\Psi = \bar{\Psi}(i\partial\!\!\!/)\Psi + \bar{\Psi}(iV^\dagger\partial\!\!\!/V)\Psi \tag{41}$$

Similar with QED, three Gauge Fields $A^i_\mu\tau^i$ are introduced to make the derivative covariant,

$$D_\mu = \partial_\mu - igA^i_\mu\tau^i = \partial_\mu - ig\mathcal{A}_\mu \tag{42}$$
$$D_\mu\Psi \to D'_\mu\Psi' = VD_\mu\Psi \tag{43}$$
$$D_\mu \to D'_\mu = V(x)D_\mu V^\dagger(x) \tag{44}$$

Now we try to obtain the behavior of the Gauge Fields under the local transformations,

$$D'_\mu\Psi' = VD_\mu\Psi \tag{45}$$
$$(\partial_\mu - ig\mathcal{A}'_\mu)V\Psi = V(\partial_\mu - ig\mathcal{A}_\mu)\Psi \tag{46}$$
$$V\partial_\mu\Psi + (\partial_\mu V - ig\mathcal{A}'V)\Psi = V\partial_\mu\Psi - igV\mathcal{A}_\mu\Psi \tag{47}$$

$$\mathcal{A}'_\mu = V\mathcal{A}_\mu V^\dagger + \frac{1}{ig}(\partial_\mu V)V^\dagger = V\mathcal{A}_\mu V^\dagger + \frac{i}{g}V(\partial_\mu V^\dagger) \tag{48}$$

In the same way as in QED, the Field strength $F^i_{\mu\nu}$ can be defined with the commutator

$$[D_\mu, D_\nu] = -igF^i_{\mu\nu}\tau^i = -ig\mathcal{F}_{\mu\nu}, \tag{49}$$

$$\mathcal{F}_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A^i_\nu \tau^i - \partial_\nu A^i_\mu \tau^i + g\epsilon^{ijk} A^i_\mu A^j_\nu \tau^k \tag{50}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu \tag{51}$$

$$\mathcal{L} \supset -\frac{1}{2} Tr(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{4}(F^i_{\mu\nu} F^{i\mu\nu}) \tag{52}$$

From $SU(2)$ to $SU(N)$:

	$U(1)$	$SU(2)$	$SU(N)$
Generators	Q	$\tau^i = \frac{\sigma^i}{2}$ $[\tau^i, \tau^j] = i\epsilon^{ijk}\tau^k$	$t^a, a = 1, \cdots, N^2 - 1$ $[t^a, t^b] = if^{abc}t^c$
Gauge Fields	A_μ	A^i_μ	A^a_μ
Derivative	$\partial_\mu - ieQA_\mu$	$\partial_\mu - igA^i_\mu \tau^i$	$\partial_\mu - igA^a_\mu t^a$
Field Strength	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$	$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\mu A^i_\nu + g\epsilon^{ijk} A^j_\mu A^k_\nu$	$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\nu + gf^{abc} A^b_\mu A^c_\nu$

(53)

The elegance of gauge theories comes from the fact that all interactions are given in terms of a single coupling constant, even when quantum corrections are included.

5 Wilson Loop

Wilson Line: comparator between two points.

$$U(x+\delta x,x)=1-ieA_\mu\delta x^\mu,\quad U(y,x)\equiv\exp\left(-ie\int_P A_\mu dx^\mu\right) \tag{54}$$

Wilson Loop

$$W=U(x,x)\equiv\exp\left(-ie\oint A_\mu dx^\mu\right)=\exp\left(-i\frac{e}{2}\int_\Sigma F_{\mu\nu}ds^{\mu\nu}\right) \tag{55}$$

is gauge invariant.

For non-Abelian gauge symmetries,

$$U(y,x)\equiv\exp\left(ig\int_P t^a A^a_\mu dx^\mu\right)=P\left\{\exp\left(ig\int d\tau\frac{dx^\mu}{d\tau}t^a A^a_\mu(x(\tau))\right)\right\} \tag{56}$$

$$U(x,x)\equiv\exp\left(ig\oint_P t^a A^a_\mu dx^\mu\right)=P\left\{\exp\left(ig\oint d\tau\frac{dx^\mu}{d\tau}t^a A^a_\mu(x(\tau))\right)\right\} \tag{57}$$

$$U(y,x) \to V(y)U(y,x)V^\dagger(x), \quad U(x,x) \to V(x)U(x,x)V^\dagger(x) \tag{58}$$

The gauge invariant Wilson Loop is defined as

$$W(x) = \frac{1}{N}\text{tr}\left[P\left\{\exp\left(ig\oint_P t^a A^a_\mu dx^\mu\right)\right\}\right] \tag{59}$$

$$= 1 + \frac{(ig)^2}{32}(F^a_{\mu\nu}\epsilon^{\mu\nu})^2 + \text{Higher Order} \tag{60}$$

6 Homeworks

HOMEWORK: *Try to obtain the relation between $A^{i\prime}_\mu$ and A^i_μ*

$$A^{i\prime}_\mu = A^i_\mu + \frac{1}{g}\partial\alpha^i + \epsilon^{ijk}A^j_\mu\alpha^k + \cdots \tag{61}$$

HOMEWORK: *Try to obtain the Lagrangian of non-Abelian gauge fields*

$$\mathcal{L} \supset -\frac{1}{2}Tr(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{4}(F^i_{\mu\nu}F^{i\mu\nu}) \tag{62}$$

HOMEWORK: *Get the EOM of the non-Abelian Gauge Fields and Fermions*

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}i(\partial\!\!\!/ - igA\!\!\!/^a t^a)\psi - m\bar{\psi}\psi \tag{63}$$

HOMEWORK: The covariant derivative is depend on the transformation of the field(or representation of gauge group)

$$D_\mu = \partial_\mu - igT^aA^a_\mu \tag{64}$$

Construct the gauge invariant Lagrangian for the following field.

1. Φ transforms under $SU(2)$ as

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{65}$$

2. Conjugate of the foundational representation

$$\Phi \to \Phi U^\dagger \tag{66}$$

3. Adjoint representation

$$\Phi \to U \Phi U^\dagger \tag{67}$$