CH.16. Quantization of Non-Abelian Gauge Theories

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1 Interaction of non-Abelian Gauge Fields

1.1 What we can expect at First Sight

$$\mathcal{L} = -\frac{1}{4}\sum_a (F^a_{\mu\nu})^2 + \bar{\psi} i \partial\!\!\!/ \psi + g\bar{\psi}\gamma^\mu t^a \psi A^a_\mu,\ (a = 1, \cdots n) = \mathcal{L}_0 + \mathcal{L}_{int} \tag{1}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \bar{\psi} i \partial\!\!\!/ \psi \tag{3}$$

$$\mathcal{L}_{int} = g A^a_\mu \bar{\psi}\gamma^\mu t^a \psi - g f^{abc}(\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} \tag{4}$$

$$-\frac{1}{4} g^2 (f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{c\mu} A^{d\nu}) \tag{5}$$

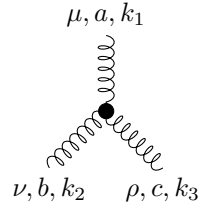
We try to generalize the QED to non-Abelian gauge fields.

Propagator :

$$\langle \psi \bar{\psi} \rangle = \frac{i}{\not{k} - m} \tag{6}$$

$$\langle A^a_\mu A^b_\nu \rangle = \frac{-i g_{\mu\nu}}{k^2} \tag{7}$$

Vertex :



$$: \quad gf^{abc}\left[g_{\mu\nu}(k_1-k_2)_\rho + g_{\nu\rho}(k_2-k_3)_\mu + g_{\mu\rho}(k_3-k_1)_\nu\right] \tag{8}$$

It seems there are n kinds of photons with some additional interactions beyond QED.

But it's not true due to the self-interaction.

1.2 What's the problem of our first sight

1. **WTI: QED vs Yang-Mills**

$$\epsilon_\mu(k)\mathcal{M}^\mu \to k_\mu \mathcal{M}^\mu = 0 \tag{9}$$

Consider fermion-anti-fermion annihilate to two gauge boson in QED and Yang-Mills Theory

$$\text{QED}: \quad k_1^\mu \epsilon_2^\nu \mathcal{M}_{\mu\nu} = k_1^\mu \epsilon_2^\nu (\mathcal{M}_{1\mu\nu} + \mathcal{M}_{2\mu\nu}) = 0 \tag{10}$$

$$\text{YM}: \quad k_1^\mu \epsilon_2^\nu (\mathcal{M}_{1\mu\nu} + \mathcal{M}_{2\mu\nu}) = -g^2 f^{abc} \epsilon_\mu \bar{v} \gamma^\mu t^c u \tag{11}$$

$$k_1^\mu \epsilon_2^\nu \mathcal{M}_{3\mu\nu} = g^2 f^{abc} \epsilon_\mu \bar{v} \gamma^\mu t^c u \tag{12}$$

In the case of QED, the results of exchanging the final two photon cancel each other to keep WTI. In the non-Abelian case, the residual term in nonzero, but cancels the contribution from the gauge boson self-interaction. Eventually, the WTI is kept in Yang-Mills Theory. It also can be checked with gauge boson scattering. *In conclusion, the unphysical polarization states will not appear in final states.*

2. **Optical Theorem**
The two physical polarization states ϵ^T and two unphysical polarization

$$\epsilon_\mu^\pm = \frac{1}{\sqrt{2}k}(k, \pm\vec{k}), \ \sum \epsilon_\mu^T \epsilon_\nu^{T*} = -g_{\mu\nu} + \epsilon_\mu^+ \epsilon_\nu^{-*} + \epsilon_\mu^- \epsilon_\nu^{+*} \tag{13}$$

We have seen that the unphysical polarization does not appear in the final states(or cancels) $\sum \epsilon_\mu^T \epsilon_\nu^{T*} \to -g_{\mu\nu}$. However when consider the fermion pare annihilate to two gauge boson at one loop level, the unphysical states' contribution can not cancel as shown in Fig.(16.4). $\sum \epsilon_\mu^T \epsilon_\nu^{T*} \nrightarrow -g_{\mu\nu}$. A paradox is obtained when the *optical theorem* is considered as shown in Fig.(16.2)

$$\text{Left}: \quad -g_{\mu\nu} = \sum \epsilon_\mu^T \epsilon_\nu^{T*} - \epsilon_\mu^- \epsilon_\nu^{+*} \tag{14}$$

$$\text{Right}: \quad -g_{\mu\nu} = \sum \epsilon^T_\mu \epsilon^{T*}_\nu \tag{15}$$

The paradox tells us the non-Abelian gauge theory is not trivial as QED. The paradox originated in the unphysical freedoms of gauge bosons and the unphysical freedoms comes from the gauge symmetry. The solution to the paradox is to fix the gauge and remove(cancel) the unphysical freedoms. The ghost field will be introduced to cancel the unphysical freedoms to hold the optical theorem. *Question: Why is ghost not introduced in QED.*

2 The Faddeev-Popov Lagrangian

2.1 QED

$$\begin{aligned}
Z[J] &= \int [d\tilde{A}] e^{iS[\tilde{A},J]} && (16)\\
&= \int [dA] e^{iS} \int [d\alpha] \delta(\partial A - \omega) \det(\frac{1}{e}\partial^2) && (17)\\
&= \mathcal{N} \det(\frac{1}{e}\partial^2) \int [d\omega] e^{-i\int \omega^2/2\xi} \int [d\alpha] \int [dA] e^{iS} \delta(\partial A - \omega) && (18)\\
&= \mathcal{N} \det(\frac{1}{e}\partial^2) \int [d\alpha] \int [dA] e^{iS[A,J]} e^{-i\int (\partial A)^2/2\xi} && (19)
\end{aligned}$$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial \cdot A)^2 \tag{20}$$

The propagator is

$$\frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \tag{21}$$

2.2 Yang-Mills

Step 1 : Original generating functional

$$Z[J] = \int [d\hat{A}] e^{iS[A,J]} \tag{22}$$

$\hat{A}$ is gauge-fixed field without unphysical freedoms.

Step 2 : Gauge Condition

The gauge fixing condition can be expressed as

$$G^a[A_\mu]\big|_{\hat{A}} = 0, \tag{23}$$

$$A^a_\mu(\alpha) = g(\alpha)\hat{A}_\mu \tag{24}$$

$$= \quad \hat{A}^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c = \hat{A}^a_\mu + \frac{1}{g} D^{ac}_\mu \alpha^c \tag{25}$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - g f^{cab} A^c_\mu = \partial_\mu \delta^{ab} - i g T^c_{ab} A^c_\mu, \quad T^c_{ab}(G) = -i f^{cab} \tag{26}$$

is the covariant derive for adjoint representation.

$$1 = \int d\alpha^i \delta(G^a[A^\alpha]) \det\left(\frac{\delta G^a}{\delta \alpha^i}\right) \tag{27}$$

$$Z[J] \quad = \quad \int [d\alpha] \int [dA] e^{iS} \delta(G[A]) \det\left(\frac{\delta G^a}{\delta \alpha^i}\right) \tag{28}$$

Step 3 : We choose the generalized Lorentz gauge condition

$$G^a[A] = \partial^\mu A^a_\mu - \omega^a \tag{29}$$

$$A^a_\mu(\alpha) = \hat{A}^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} \hat{A}^b_\mu \alpha^c = \hat{A}^a_\mu + \frac{1}{g} D^{ab}_\mu \alpha^b \tag{30}$$

$$\frac{\delta G^a}{\delta \alpha^b} = \frac{1}{g}\left(\partial^\mu D_\mu\right)^{ab} \tag{31}$$

$$\det \frac{1}{g}\partial^\mu D_\mu^{ab} = \int [dc][d\bar{c}] \exp(-i \int d^4x \bar{c}_a \partial^\mu D_\mu^{ab} c_b) \tag{32}$$

$$\equiv \int [dc][d\bar{c}] \exp(i \int d^4x \mathcal{L}_{\text{ghost}}) \tag{33}$$

$c^a, \bar{c}^a$ are Grassman scalar fields in adjoint representation. The factor $1/g$ is absorbed into the fields and final removed to the normalzation factor. The Grassman integral is introduced in Eq.(9.69).

$$Z[J] = \int [d\alpha] \int [dA][dc][d\bar{c}] \exp(i \int \mathcal{L}_g[A] - \bar{c}\partial^\mu D_\mu c)\delta(\partial^\mu A_\mu^a - \omega^a) \tag{34}$$

Step 4 : The physical results can not depend on $\omega^a(x)$, thus we integrate over all $\omega^a(x)$ with an Gaussian weighting function,

$$Z[J] = \int [d\alpha] \int [dA][dc][d\bar{c}] \exp(i \int \mathcal{L}_g[A, J] + \mathcal{L}_{\text{ghost}}[c, \bar{c}]) \tag{35}$$

$$\times \int [d\omega^a]\delta(\partial^\mu A_\mu^a - \omega^a) \exp(-i \int d^4x \frac{(\omega^a)^2}{2\xi}) \tag{36}$$

$$= \int [d\alpha] \int [dA][dc][d\bar{c}] \exp(i \int \mathcal{L}_g[A] + \mathcal{L}_{\text{ghost}}[c,\bar{c}] - \frac{1}{2\xi}(\partial A^a)^2) \tag{37}$$

$$= \mathcal{N} \int [dA][dc][d\bar{c}] \exp(i \int \mathcal{L}_g[A] + \mathcal{L}_{\text{gh}}[c,\bar{c}] + \mathcal{L}_{\text{gf}}) \tag{38}$$

The unphysical function $\omega(x)$ is integrated out, but an arbitrary parameter ξ is left. The physical results does depend on it. In calculation, we can choose any numerical value for it.

$\xi = 1$	t'Hooft-Feynman Gauge
$\xi = 0$	Landau Gauge
$\xi = \infty$	Physical(or Unitary) Gauge

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{gh} + \mathcal{L}_{gf} \tag{39}$$

$$= -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{c}(-\partial_\mu D^\mu)c - \frac{1}{2\xi}(\partial A^a)^2 \tag{40}$$

$$= \frac{1}{2}A^a_\mu \left(\partial^2 g^{\mu\nu} - (1 - \frac{1}{\xi})\partial^\mu \partial^\nu \right) \delta^{ab} A^b_\nu \tag{41}$$

$$+\bar{c}^a(-\partial^2)\delta^{ab}c^b + \mathcal{L}_{int} \tag{42}$$

- **Propagator**

$$\langle A^a_\mu A^b_\nu \rangle = \frac{(-i)^2\delta^2 Z[J]}{\delta J^\mu_a(x)\delta J^\nu_b(y)} = \int \frac{d^4k}{(2\pi)^4}\frac{-i\delta^{ab}}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)e^{-ik(x-y)} \quad (43)$$

$$\langle c^a \bar{c}^b \rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\delta^{ab}e^{-ik(x-y)} \quad (44)$$

- **Vertex**
  HOMEWORK: *Write down the $\mathcal{L}_{int}$ and get the Feynman Rules.*

3 BRST

In the last section, we focus on the gauge fixing in order to proceed the path-integral quantization of non-Abelian gauge field theories. We obtained a Lagrangian with ghost and gauge fixing terms which is not gauge invariant any more. However we will show that the Lagrangian has an extended symmetry which is introduced by Becchi, Rouet, Stora and Tyutin. This symmetry plays a crucial role in proving the renormalizability of Yang-Mills theories.

The Faddeev-Popov Lagrangian is rewritten with auxiliary scalar fields B^a,

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi - \bar{c}\partial^\mu D_\mu c + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu \quad (45)$$

EOM of B^a: $B^a = -\frac{1}{\xi}\partial \cdot A^a$.

Now let ϵ be an infinitesimal Grassman parameter, independent on x. BRST transformation is expressed as

$$\begin{aligned}
\delta A^a_\mu &= \epsilon D^{ab}_\mu c^b \equiv \epsilon \mathcal{Q} A^a_\mu && (46)\\
\delta \psi &= ig\epsilon c^a t^a \psi \equiv \epsilon \mathcal{Q}\psi && (47)\\
\delta \bar{\psi} &= -ig\epsilon c^a \bar{\psi} t^a \equiv \epsilon \mathcal{Q}\bar{\psi} && (48)\\
\delta c^a &= -\frac{1}{2} g\epsilon f^{abc} c^b c^c \equiv \epsilon \mathcal{Q} c^a && (49)\\
\delta \bar{c}^a &= \epsilon B^a \equiv \epsilon \mathcal{Q}\bar{c}^a && (50)\\
\delta B^a &= 0 \equiv \epsilon \mathcal{Q} B^a && (51)
\end{aligned}$$

It can be checked that the Lagrangian is invariant under the transformation. With the operator $\mathcal{Q}$, the BRST invariant can be seen more easily:

- $\epsilon\mathcal{Q} = \sum \delta\phi \frac{\delta}{\delta\phi}$

$$\mathcal{Q} = D^{ab}_\mu c^b \frac{\delta}{\delta A^a_\mu} + igc^a t^a \psi \frac{\delta}{\delta\psi} - igc^a \bar{\psi} t^a \frac{\delta}{\delta\bar{\psi}} - \frac{1}{2} g f^{abc} c^b c^c \frac{\delta}{\delta c^a} + B^a \frac{\delta}{\delta c^a} + B^a \frac{\delta}{\delta \bar{c}^a} \quad (52)$$

$\mathcal{Q}$ is an *nilpotent* operator, $\mathcal{Q}^2 = 0$.

- The Lagrangian can be rewritten as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi - \bar{c}\partial^\mu D_\mu c + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu \quad (53)\\ &= \mathcal{L}_{\text{YM}}[A,\psi] + \mathcal{Q}\mathcal{L}_{\text{BRTS}}, \quad (54)\\ \mathcal{L}_{\text{BRTS}} &= \bar{c}^a\partial^\mu A^a_\mu + \frac{\xi}{2}\bar{c}^a B^a \quad (55)\end{aligned}$$

  For general gauge fixing $G^a[A_\mu] = f^a[A_\mu] - \omega^a$, ones

$$\begin{aligned}\mathcal{L}_{\text{BRST}} &= \bar{c}^a f^a[A] + \frac{\xi}{2}\bar{c}^a B^a \quad (56)\\ Q\mathcal{L}_{\text{BRST}} &= B^a f^a[A] - \bar{c}^a\frac{\delta f^a[A]}{\delta A^b}D^{bc}c^c + \frac{\xi}{2}(B^a)^2 \quad (57)\end{aligned}$$

- The BRST transformation of A and ψ is special gauge transformation $\alpha^a = g\epsilon c^a(x)$,

$$\delta\mathcal{L}_{\text{YM}} = \epsilon\mathcal{Q}\mathcal{L}_{\text{YM}} = 0 \quad (58)$$

- $\mathcal{Q}^2 = 0$

$$\delta(\mathcal{Q}\mathcal{L}_{\text{BRST}}) = \epsilon\mathcal{Q}^2\mathcal{L}_{\text{BRST}} = 0 \quad (59)$$

3.1 BRST and Unitarity

Due to the nilpotency of the $\mathcal{Q}$, the states can be divided into three subspaces:

$$\mathcal{H}_1 \quad : \quad \mathcal{H}_1 = \{\phi_1 | Q\phi_1 \neq 0\} \tag{60}$$

$$\mathcal{H}_2 \quad : \quad \mathcal{H}_2 = \{\phi_2 | \phi_2 = Q\phi_1, \phi_1 \in H_1\} \tag{61}$$

$$\mathcal{H}_0 \quad : \quad \mathcal{H}_0 = \{\phi_0 | \phi_0 \notin H_1, \phi_0 \notin H_2\} \tag{62}$$

$$\langle \psi_{2a} | \psi_{2b} \rangle = \langle \psi_{2a} | \mathcal{Q} | \psi_{1b} \rangle = 0, \quad \langle \psi_0 | \psi_2 \rangle = \langle \psi_0 | \mathcal{Q} | \psi_1 \rangle = 0 \tag{63}$$

At the limit $g \to 0$

- $A_\mu^+ \in \mathcal{H}_1, c \in \mathcal{H}_2$

$$\delta A_\mu^a = \epsilon D_\mu^{ab} c^b \to k^\mu c^b : \text{or } Q A_\mu^+ \sim c(x) \tag{64}$$

- $\bar{c} \in \mathcal{H}_1, A_\mu^{a-} \in \mathcal{H}_2$

$$\delta \bar{c} = \epsilon B^a, Q\bar{c} = B^a = \partial^\mu A_\mu^a = \partial^\mu A_\mu^{a-} \tag{65}$$

- $\psi, \bar{\psi}, A_\mu^{aT} \in H_0$ are physical fields.

Now we return to the paradox of optical theorem (why the unphysical states does not appear in the final states)

- Unitary requires all possible state appear in the mediated states.

$$\langle B|S^\dagger S|B\rangle = \sum_{All}\langle B|S^\dagger|C\rangle\langle C|S|B\rangle \geq \sum_{Phy}\langle B|S^\dagger|C\rangle\langle C|S|B\rangle \tag{66}$$

If the states created as collections of transversely polarized bosons in the far past could evolve into states that contained gauge bosons of other polarizations in the far future, the S-matrix projected between transverse gauge boson states would not be unitary.

- The physical states set $\mathcal{H}_0$ is close.

$$\mathcal{Q}\left(S|A;tr\rangle\right) = 0, \qquad S|A;tr\rangle \in \mathcal{H}_0 \oplus \mathcal{H}_2 \tag{67}$$

$$\langle A;tr|S^\dagger S|A;tr\rangle \neq 0 \qquad S|A;tr\rangle \notin \mathcal{H}_2 \tag{68}$$

$$\langle A;tr|S^\dagger S|A;tr\rangle = \sum_C \langle A;tr|S^\dagger|C;tr\rangle\langle C;tr|S|A;tr\rangle \tag{69}$$

HOMEWORK: *Ghost and Unitary §16.3*

3.2 BRST and Renormalization

BRST symmetry leads to a variety of Ward−Takahashi-like identities for various amplitudes. Using these identities one can show that all the UV divergences of the theory can be cancelled by the counter-terms which are present in the bare Lagrangian.

- W do not need counter-terms for the gluon mass, or the ghost mass, or any non-gauge-invariant combinations of the gluon fields.
- BRST symmetry requires that the gauge coupling renormalize in the same way in each term.

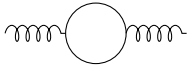
Thus, thanks to the BRST symmetry, the non-abelian gauge theories are renormalizable.

4 One-Loop Divergences of Non-Abelian Gauge Theory

4.1 Divergences and Counter-terms

1. **Gauge boson self-energy**

$$i\Pi^{ab}_{\mu\nu} = \quad (70)$$



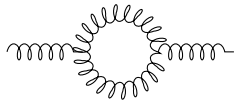
$$= tr[t^a t^b] i(q^2 g^{\mu\nu} - q^\mu q^\nu) \times \frac{-g^2}{(4\pi)^{d/2}} \int dx \frac{8x(1-x)\Gamma(2-d/2)}{\Delta^{2-d/2}} \quad (71)$$

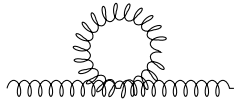
$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\Pi(q^2) \quad (72)$$

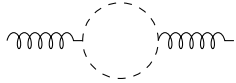
$$\Pi = \frac{-g^2}{(4\pi)^2} C(r) \int dx 8x(1-x) \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln \Delta \right) \quad (73)$$

$$= \quad \frac{-g^2}{(4\pi)^2} C(r) \frac{4}{3} \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \cdots \right) \tag{74}$$

$\Delta = m^2 - x(1-x)q^2$

 $=$ (75)

 $=$ (76)

 $=$ (77)

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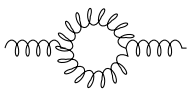
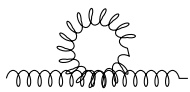
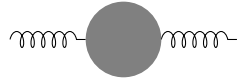
 $+$  $+$  (79)

$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(\frac{-g^2}{(4\pi)^2}\frac{-5}{3}C(G)\Gamma(2-d/2) + \cdots\right) \tag{80}$$



$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\Pi(q^2) \equiv i\Pi^{ab}_{\mu\nu} \tag{81}$$

2. **Fermion self-energy**



$$= \frac{ig^2}{(4\pi)^{d/2}}C_2(r)\not{k}\int_0^1 dx(1-x)(d-2)\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \tag{82}$$

$$= \frac{ig^2}{(4\pi)^2}C_2(r)\not{k}\Gamma(2-d/2) + \cdots \tag{83}$$

$$\equiv i\Sigma_{ab} \tag{84}$$

3. **Interaction**

$$i\Gamma = \tag{85}$$

4. **Counter-terms**
We consider all fields are massless(which approximation is widely applied in QCD), On-shell subtraction scheme is not available. We use off-shell scheme at $p^2 = -\mu^2$.

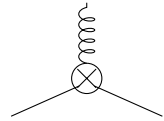
The counter-term for the self-energy for gauge bosons



$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\delta_3 \tag{86}$$



$$= i\not{p}\delta_2 \tag{87}$$



$$= igt^a\gamma^\mu\delta_1 \tag{88}$$

$$\delta_1 = -\frac{\alpha}{4\pi}\left(C_2(r) + C_2(G)\right)\frac{1}{\epsilon} \tag{89}$$

$$\delta_2 = -\frac{\alpha}{4\pi}C_2(r)\frac{1}{\epsilon} \tag{90}$$

$$\delta_3 = \frac{\alpha}{4\pi}(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r))\frac{1}{\epsilon} \tag{91}$$

5. β **function**
Recalling Canllan-Symanzik Equation, one has

$$\beta = -g\frac{\partial}{\partial g}(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3) \tag{92}$$

$$= \frac{-2g^2}{16\pi^2}\left(C_2(r) + C_2(G) - C_2(r) + \frac{1}{2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right)\right) \tag{93}$$

$$= \frac{-g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right) \tag{94}$$

6. **Asymptotic freedom**

$$FermionLoop: \quad C(F)\delta^{ab} = Tr[t^a t^b] \tag{95}$$

$$Gauge/Ghost: \quad C_2(G)\delta^{ab} = f^{acd}f^{bcd} \tag{96}$$

$$Fermion - Gauge: \quad C_2(F)\delta_{ij} = \sum_a t^a_{ik}t^a_{kj} \tag{97}$$

SU(N): Fermions are in fundamental representation, $C(N) = \frac{1}{2}, C_2(N) = \frac{N^2-1}{2N}$; Gauge bosons are in adjoint representation, $C_2(G) = C(G) = N$

$$\beta = \frac{dg}{\ln \mu} = \frac{-g^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right), \frac{d\alpha}{d\ln\mu^2} = -\frac{\alpha^2}{4\pi}\left(\frac{11}{3}N - \frac{2}{3}n_f\right) \equiv -b_0\alpha^2 \tag{98}$$

where $\alpha = g^2/4\pi$. The solution is

$$\alpha(\mu^2) = \frac{\alpha_0}{1 + \alpha_0 b_0 \ln \mu^2/\mu_0^2} \tag{99}$$

$$\alpha(\mu^2) = \frac{4\pi}{b_0 \ln \mu^2/\Lambda^2_{QCD}} \tag{100}$$

HOMEWORK: *Using the QCD beta at NLO, get the running of* $\alpha(\mu^2)$.

$$\beta = -(b_0\alpha^2 + b_1\alpha^3) \tag{101}$$

$$b_0 = (33 - 2n_f)/(12\pi) \tag{102}$$

$$b_1 = (153 - 19n_f)/(24\pi^2) \tag{103}$$

$$\alpha_s(M_Z^2) = 0.118, \tag{104}$$

$$\alpha_s^{(n_f)}(m_f^2) = \alpha_s^{(n_f+1)}(m_f^2) \tag{105}$$

4.2 Relations among Counter-terms–Global view

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{a\mu\nu}F^{a\mu\nu} \tag{106}$$

$$-\frac{1}{2\xi}(\partial A^a)^2 - \bar{c}^a(\partial^2\delta_{ab} - gC_{abc}\partial A^b)c^c \tag{107}$$

1. **Renormalizable**
   The Lagrangian contains no interactions of dimension higher than 4. Therefore the number of divergent diagrams is finite and all divergence can be remove by finite number of counter-terms.
   HOMEWORK: *According the Power Counting, find out all the divergent diagrams* (N_G, N_{FP}, N_F) $(0,0,0), (2,0,0), (0,2,0), (0,0,2), (1,2,0), (1,0,2), (3,0,0), (4,0,0)$

   Except the vacuum energy diagram, there are 7 divergent diagrams need counter-terms which corresponding the 8 terms in Lagrangian(except the renormalization of ξ). The self energy diagram of a fermion need two counter-terms: field strength and mass.

2. **Renormalization of non-Abelian Gauge theory**

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu})^2 + \bar{\psi}_0(i\not{\partial} - m_0)\psi_0 - \bar{c}^a_0\partial^2 c^a_0 - \frac{1}{2\xi_0}(\partial A^a_0)^2 \tag{108}$$

$$+g_0 A^a_{0\mu}\bar{\psi}_0\gamma^\mu\psi_0 - g_0 f^{abc}(\partial_\mu A^a_{0\nu})A^b_{0\mu}A^c_{0\nu} \tag{109}$$

$$-g^2(f^{eab}A^a_{0\mu}A^b_{0\nu})(f^{ecd}A^c_{0\mu}A^d_{0\nu}) + g\bar{c}^a_0 f^{abc}\partial^\mu A^b_{0\nu}c^c_0 \tag{110}$$

There are 7 kinds divergences which need 7 counter-terms,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}Z_A A^{a\mu}_R(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^{a\nu}_R - \frac{1}{2\xi}(\partial_\mu A^{a\mu}_R)^2 && (111)\\
&\quad + iZ_\Psi \bar{\Psi}_R \partial\!\!\!/ \Psi_R + Z_c \partial_\mu c^a_R \partial^\mu c^a_R && (112)\\
&\quad - Z_1 g_R \mu^\epsilon A\!\!\!/^a_R t^a \Psi + \frac{Z_3}{2} g\mu^\epsilon f^{abc} A^{a\mu}A^{b\nu}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) && (113)\\
&\quad - Z_4 g^2 \mu^{2\epsilon} f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu + \tilde{Z}_1 g\mu^\epsilon f^{abc}\bar{c}^a(\partial A^b)c^c && (114)\\
&= \mathcal{L}_R + \mathcal{L}_{ct} && (115)
\end{aligned}$$

- Gauge invariant leads to the Ward-Takahashi identities which can get with BRST symmetry

$$\frac{Z_1}{Z_\psi} = \frac{Z_3}{Z_A} = \left(\frac{Z_4}{Z_A}\right)^{1/2} = \frac{\tilde{Z}_1}{Z_c} \tag{116}$$

  The gauge coupling is universal.

- All the counter-terms can be absorbed into the renormalization of Fields and coupling.

$$\mathcal{L} = Z_A\left(-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \bar{g}f^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} \right. \tag{117}$$

$$-\frac{1}{4}\bar{g}^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{c\mu}A^{d\nu})\Big) - \frac{1}{2\xi}(\partial A^a)^2 \tag{118}$$

$$+Z_\psi(i\bar{\psi}\partial\!\!\!/\psi + \bar{g}A^a_\mu\bar{\psi}\gamma^\mu t^a\psi) \tag{119}$$

$$+Z_c\left(-\bar{c}^a\partial^2\psi - \bar{g}f^{abc}\bar{c}^a\partial^\mu A^b_\mu c^c\right) \tag{120}$$

$$\tag{121}$$

We introduce the renormalised fields $A^{a\mu}_R, \Psi_R, c^a_R$ and coupling g_R in $D = 4-2\epsilon$ dimension

$$A^{a\mu} = Z_A^{1/2}A^{a\mu}_R, \quad \Psi = Z_\psi^{1/2}\Psi_R, \quad c^a = Z_c^{1/2}c^a, \quad g = \mu^\epsilon Z_g g_R \tag{122}$$

The relation between the renormalization constants and counter-terms are

$$Z_1 = Z_g Z_\psi Z_A^{1/2}, Z_3 = Z_g Z_A^{3/2}, Z_4 = Z_g^2 Z_A^2, \tilde{Z}_1 = Z_g Z_c Z_A^{1/2} \tag{123}$$

The W-T identity is

$$Z_g = \frac{Z_1}{Z_\psi Z_A^{1/2}} = \frac{Z_3}{Z_A^{3/2}} = \left(\frac{Z_4}{Z_A^{3/2}}\right)^{1/2} = \frac{\tilde{Z}_1}{Z_c Z_A^{1/2}} \tag{124}$$

3. **MS Scheme**

The finite term in δ_i do not contribute to β functions. The simplest renormalization scheme is the

Minimal Subtraction Scheme(MS) in which only divergences are subtracted.

$$\delta_1 = -\frac{\alpha}{4\pi}\left(C_2(r)+C_2(G)\right)\frac{1}{\epsilon} \tag{125}$$

$$\delta_\psi = -\frac{\alpha}{4\pi}C_2(r)\frac{1}{\epsilon} \tag{126}$$

$$\delta_A = \frac{\alpha}{4\pi}\left(\frac{5}{3}C_2(G)-\frac{4}{3}n_f C(r)\right)\frac{1}{\epsilon} \tag{127}$$

Now we try to get the renormalization group equation of the gauge coupling.

$$g_0 = Z_g g\mu^\epsilon, \quad Z_g = \frac{Z_1}{Z_\psi Z_A^{1/2}} = 1+\delta_g, \quad \delta_g = \delta_1 - \delta_\psi - \frac{1}{2}\delta_A \tag{128}$$

$g = g(\mu,\epsilon)$ is the renormalized coupling constant in $d = 4-2\epsilon$ dimension. μ is an arbitrary scale to keep the renormalized gauge coupling dimensionless. Since $\mu^\epsilon = 1+\epsilon\ln\mu+\frac{1}{2}\ln^2\mu+\cdots$, it usually add a finite term $\ln\mu$ into the result of an divergent diagram at the limit $\epsilon\to 0$. This term has no effect on the *renormalization group functions* and is useless in On- off- shell subtraction schemes. However in MS or $\overline{\text{MS}}$ scheme which subtraction is not dependent on the scale, the μ is taken as renormalization scale.

$$g_0 = \mu^\epsilon Z_g g(\mu,\epsilon), \quad \beta(g) \equiv \lim_{\epsilon\to 0}\frac{dg(\mu,\epsilon)}{d\ln\mu} = \lim_{\epsilon\to 0}\beta(g,\epsilon) \tag{129}$$

$$\frac{dg_0}{d\ln\mu} = g\frac{dZ_g}{d\ln\mu}\mu^\epsilon + Z_g\frac{dg}{d\ln\mu}\mu^\epsilon + \epsilon Z_g g\mu^\epsilon = 0 \tag{130}$$

$$\beta(g,\epsilon) = \frac{dg(\mu,\epsilon)}{d\ln\mu} = -\epsilon g - \frac{g}{Z_g}\frac{dZ_g}{d\ln\mu} \tag{131}$$

$$\beta(g) = \lim_{\epsilon\to 0}\beta(g,\epsilon) \tag{132}$$

$$= -\frac{g}{Z_g}\frac{dZ_g}{d\ln\mu} \tag{133}$$

The General Form of the renormalization constant can be expressed as

$$Z_g = 1 + \frac{Z_{g,1}}{\epsilon} + \frac{Z_{g,2}}{\epsilon^2} + \cdots = 1 + \sum_n \frac{Z_{g,n}}{\epsilon^n}, \tag{134}$$

We will see $\beta(g)$ only related with $Z_{g,1}$.

$$\beta(g)Z_g = -g\frac{dZ_g}{d\ln\mu} \tag{135}$$

Expand the left and right sides with ϵ

$$-g\frac{dZ_g}{d\ln\mu} = g\frac{dg}{d\ln\mu}\frac{\partial Z_g}{\partial g} \tag{136}$$

$$= -g(\beta(g) - \epsilon g)(\frac{1}{\epsilon}\frac{\partial Z_{g,1}}{\partial g} + \frac{1}{\epsilon^2}\frac{\partial Z_{g,2}}{\partial g} + \cdots) \tag{137}$$

$$= g^2\frac{\partial Z_{g,1}}{\partial g} + \mathcal{O}(1/\epsilon) \tag{138}$$

$$\beta(g)Z_g = \beta(g)(1 + \frac{Z_{g,1}}{\epsilon} + \frac{Z_{g,2}}{\epsilon^2} + \cdots) \tag{139}$$

$$= \beta(g) + \mathcal{O}(1/\epsilon) \tag{140}$$

Similar we can apply the analysis to the mass renormalization, and finally we get the equation,

$$\beta(g) = \frac{dg}{d\ln\mu} = g^2\frac{\partial Z_{g,1}}{\partial g} = g^2\frac{\partial(\epsilon\delta_g)}{\partial g} \tag{141}$$

$$\gamma_m(g) = \frac{1}{Z_m}\frac{dZ_m}{d\ln\mu} = -g\frac{\partial Z_{g,1}}{\partial g} = -g\frac{\partial(\epsilon\delta_m)}{\partial g} \tag{142}$$

In MS($\overline{MS}$) scheme, the renormalization of the theory become simple, since only the coefficient

of $\frac{1}{\epsilon}$ was considered without dealing the Feynman parameter integral.

HOMEWORK: *Get the beta function of QCD with Eq(124-126) and Eq.(140)*