CH.17 Quantum Chromodynamics

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1 From Quark to QCD

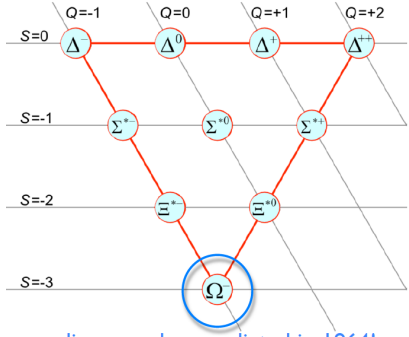
1.1 History

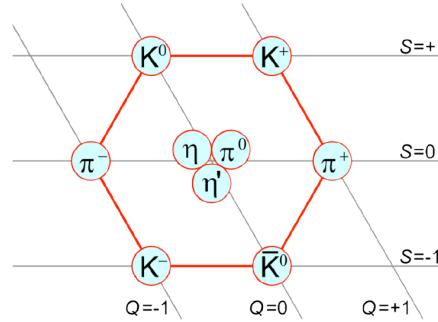
- 1954, Yang-Mills Theory *Phy. Rev.,96(1954),191*
- 1955, Gell-Mann-Nishijima Formula, *Prog. Theor. Phys.,13(1955)*
- 1956, Sakata Model *Prog. Theor. Phys., 16(1956), 686*
- 1959, Ogawa, SU(3) symmetry *Prog. Theor. Phys.,21(1959), 209−211*
- 1961, Neeman, Gell-Mann, SU(3) octet, *Nucl. Phys., 26(1961)*
- 1961, Glashow, $SU(2) \times U(1)$ Electro-Weak Model, *Nucl. Phys., 22(1961), 579*
- 1962, Okubo, Gell-Man, Gell-Man-Okubo Mass formula, *Prog. Theor. Phys. 27(1962), 949*
- 1964, Gell-Man *et al*, Quark Model
- **1964**, Greenberg, New freedom of quark(color), *Phys. Rev. Lett., 13(1964), 598*
- 1964, Higgs *et al*, Higgs Mechanism, *Phys. Lett., 12(1964),132*
- 1967, Weinberg, Salam, Weinberg-Salam Model, *Phys. Rev. Lett., 19(1967), 1264*

- 1969, Feynman, Parton Model, *Phys. Rev. Lett., 23(1969)*
- 1971, t'Hooft, Renormalization of Yang-Mill Theory and Weinberg-Salam Model, *Nucl, Phys., B33(1971),173; B35(1971), 167*
- 1972, Gell-mann, color number is introduced
- **1973**, Gross *et al*, Asymptotic freedom, QCD was constructed.
- 1973, Weak Neutral current was observed. The GWS Model was confirmed.
- **1975**, SLAC, R_Q was observed and the color freedom was confirmed
- **1979**, 3-jets events was observed, gluon was confirmed

Quark Model is successful

- $SU(3) \supset SU(2)$, consistent with iso-spin symmetry
- hadron spectrum





But:

- Δ^{++}: **Spin-3/2, Charge-2**

$$u^{\uparrow}u^{\uparrow}u^{\uparrow} \rightarrow \epsilon_{rgb}(r,g,b) \otimes u^{\uparrow}u^{\uparrow}u^{\uparrow} \tag{1}$$

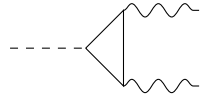
- **Why no free quark: confinement.**

QCD is Successful

- Asymptotic freedom

$$\frac{d\alpha_s}{d\mu} < 0 \tag{2}$$

- $\pi^0 \to \gamma\gamma$

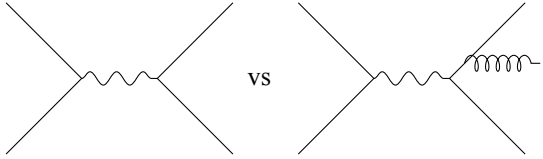


$$\Gamma(\pi^0 \to \gamma\gamma) = N_c^2 (Q_u^2 - Q_d^2) \frac{\alpha_e^2 m_\pi^2}{64\pi^2 f_\pi^2} \tag{3}$$

- R ratio

$$R(Q) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum Q_q^2 \tag{4}$$

- 3-jet events: The discovery of gluons.



(5)

$$R = \frac{\sigma(e^+e^- \to 3-jet)}{\sigma(e^+e^- \to 2-jet)} \sim \alpha_s \tag{6}$$

But

- a) non-perturbative:
  - Factorization and Parametrization: Form Factor, Structure Function, Parton Distribution Function ...
  - Lattice QCD
- b) Exotic sates: glue-ball, tetraquark states($qq\bar{q}\bar{q}$), pentaquark states($qqqq\bar{q}$)

1.2 Foundation of QCD

The generators of SU(3) $(t^a)_{ij} = \lambda^a_{ij}/2\,(a = 1, \cdots 8; \quad i, j = 1, 2, 3 \text{ or } r, g, b)$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{7}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \tag{8}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}. \tag{9}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{10}$$

$$D_\mu = \partial_\mu - igt^a A^a_\mu \tag{11}$$

1.2.1 The Lagrangian of QCD

- Gauge Field

$$\mathcal{L}_G = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{12}$$

$$= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \tag{13}$$

$$+\frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} \quad \text{(3-gluon coupling)} \tag{14}$$

$$-\frac{g^2}{4} f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu \quad \text{(4-gluon coupling)} \tag{15}$$

- Fermion

$$\mathcal{L}_F = \bar{\Psi}(i\partial\!\!\!/ - m)\psi + g\bar{\psi}A\!\!\!/^a t^a \psi. \tag{16}$$

- Gauge Fixing $\partial \cdot A^a = 0$

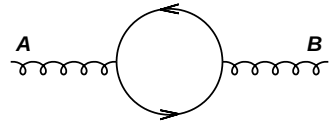
$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A^a)^2. \tag{17}$$

- Ghost Lagrangian

$$\mathcal{L}_{FP} = -\bar{c}^a \partial^\mu D_\mu^{ab} c_b = \partial^\mu \bar{c}^a \partial_\mu c^a - g f^{abc} \partial^\mu \bar{c}^a A_\mu^b c^c. \tag{18}$$

1.2.2 Color Factors

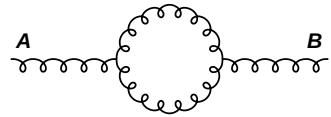
$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2} \tag{19}$$



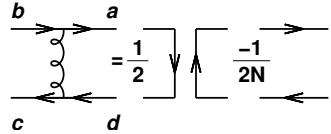
$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \quad C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3} \tag{20}$$

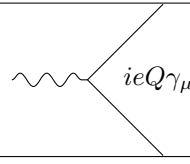
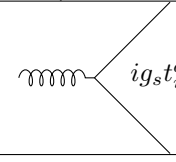


$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \quad C_A = N_C = 3 \tag{21}$$



$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_C}\delta_{ab}\delta_{cd} \text{ (Fierz)} \tag{22}$$



	QED	QCD
Fermion	Q	r, g, b
Gauge Fields	A_μ 0	$G^a_\mu (a = 1, \cdots, 8)$ $r\bar{g}, r\bar{b}, g\bar{r}, g\bar{b}, b\bar{r}, b\bar{g}$ $\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$
Coupling	α_e	α_s
Propagator	$\frac{-ig_{\mu\nu}}{k^2+i\epsilon}$	$\frac{-i\delta^{ab}}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$
Vertex	 $ieQ\gamma_\mu$	 $ig_s t^a_{ij}\gamma_\mu$

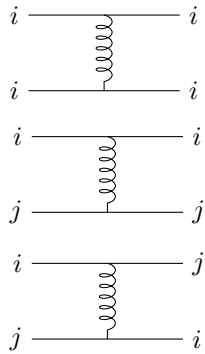
1.2.3 Attractive or Repulsive

Quark-Quark $q - q$

$$\text{QED}: \quad ieQ \ : \ V = \frac{\alpha Q_1 Q_2}{r} \tag{23}$$

$$\text{QCD}: \quad igT^a : V = \frac{\alpha}{r}\sum_a T^a_{ij}T^a_{mn} \tag{24}$$

$$\text{Fierz}: \quad \sum_a T^a_{ij}T^a_{mn} = \frac{1}{2}\delta_{in}\delta_{jm} - \frac{1}{2N}\delta_{ij}\delta_{mn} : \quad > 0(\text{Repulsive}); \quad < 0(\text{Attractive}) \tag{25}$$



$\frac{1}{2} - \frac{1}{6} > 0$ Same Color: Repulsive (26)

$-\frac{1}{6} < 0$ Different Color: Attractive (27)

$\frac{1}{2} > 0$ Exchange Color: Repulsive (28)

(29)

Quark-Antiquark $q - \bar{q}$

$$\text{QED:} \quad ieQ \,:\, V = -\frac{\alpha Q_1 Q_2}{r} \tag{30}$$

$$\text{QCD:} \quad igT^a \,:\, V = -\frac{\alpha}{r}\sum_a T^a_{ij} T^a_{mn} \tag{31}$$

$$\text{Fierz:} \quad \sum_a T^a_{ij} T^a_{mn} = \frac{1}{2}\delta_{in}\delta_{jm} - \frac{1}{2N}\delta_{ij}\delta_{mn} : \quad > 0(\text{Attractive}); \quad < 0(\text{Repulsive}) \tag{32}$$

$$i\bar{i} \to i\bar{i} : \quad +\frac{1}{3} : \text{Attractive} \tag{33}$$

$$i\bar{i} \to j\bar{j} : \quad +\frac{1}{2} : \text{Attractive} \tag{34}$$

$$i\bar{j} \to i\bar{j} : \quad -\frac{1}{6} : \text{Repulsive} \tag{35}$$

The mesons are made of $q\bar{q}$ not qq or $\bar{q}\bar{q}$.

1.3 Renormalization

$$\mathcal{L} \;=\; \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{a\mu\nu}F^{a\mu\nu} \tag{36}$$

$$-\frac{1}{2\xi}(\partial A^a)^2 - \bar{c}^a(\partial^2\delta_{ab} - gC_{abc}\partial A^b)c^c \tag{37}$$

We introduce the renormalised fields $A_R^{a\mu}, \Psi_R, c_R^a$ and coupling g_R in $D = 4 - 2\epsilon$ dimension

$$A^{a\mu} = Z_A^{1/2}A_R^{a\mu}, \quad \Psi = Z_\psi^{1/2}\Psi_R, \quad c^a = Z_c^{1/2}c^a, \quad g = \mu^\epsilon Z_g g_R. \tag{38}$$

The Lagrangian can be rewriten as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}Z_A A_R^{a\mu}(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A_R^{a\nu} - \frac{1}{2\xi}(\partial_\mu A_R^{a\mu})^2 && (39)\\
&\quad + iZ_\Psi\bar{\Psi}_R\partial\!\!\!/\Psi_R + Z_c\partial_\mu c_R^a\partial^\mu c_R^a && (40)\\
&\quad - Z_1 g_R\mu^\epsilon A\!\!\!/_R^a t^a\Psi && (41)\\
&\quad + \frac{Z_3}{2}g\mu^\epsilon f^{abc}A^{a\mu}A^{b\nu}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\cdots && (42)\\
&\quad - \frac{Z_4}{g}^2\mu^{2\epsilon}f^{abe}f^{cde}A^{a\mu}A^{b\nu}A_\mu^c A_\nu^d && (43)\\
&\quad + \tilde{Z}_1 g\mu^\epsilon f^{abc}\bar{c}^a(\partial A^b)c^c && (44)
\end{aligned}$$

$$Z_1 = Z_g Z_\psi Z_A^{1/2}, \quad Z_3 = Z_g Z_A^{3/2}, \quad Z_4 = Z_g^2 Z_A^2, \quad \tilde{Z}_1 = Z_g Z_c Z_A^{1/2} \tag{45}$$

Gauge invariant leads to the Ward-Takahashi identities

$$Z_g Z_A^{1/2}: \quad \frac{Z_1}{Z_\psi} = \frac{Z_3}{Z_A} = \left(\frac{Z_4}{Z_A}\right)^{1/2} = \frac{\tilde{Z}_1}{Z_c}. \tag{46}$$

The re-normalized coupling is universal.

The WT identity implies that the correction of gluon self-energy is proportional to $q^2 g_{\mu\nu} - q_\mu q_\nu$

$$\Pi_{\mu\nu} = \Pi_1 q^2 g_{\mu\nu} - \Pi_2 q_\mu q_\nu = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_2 + (\Pi_2 - \Pi_1) q^2 g_{\mu\nu} \tag{47}$$

$\Pi_1 = \Pi_2$ gives no correction to the gauge fixing term. $Z_A/\xi = 1/\xi_R$

2 e^+e^- Annilation into Hadrons

2.1 $e^+e^- \to \mu^+\mu^-$

$$i\mathcal{M} = (ie)^2 \bar{v}(k_1)\gamma_\mu u(k_2) \frac{-i}{s+i\epsilon} \bar{u}(p_1)\gamma^\mu v(p_2) \tag{48}$$

$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{4s^2} \mathrm{Tr}\,(\not{k}_1\gamma_\mu \not{k}_2\gamma_\nu)\,\mathrm{Tr}\,(\not{p}_1\gamma^\mu \not{p}_2\gamma^\nu) \tag{49}$$

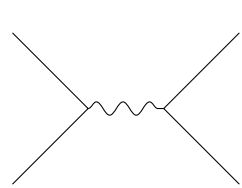


Figure 1: The Feynman diagram of $e^+e^- \to \mu^+\mu^-$.

$$\mathrm{Tr}\left(\not{k}_1\gamma^\mu\not{k}_2\gamma^\nu\right) = 4\left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1\cdot k_2 g^{\mu\nu}\right) \tag{50}$$

$$s = 2k_1\cdot k_2 = 2p_1\cdot p_2 \tag{51}$$

$$t = -2k_1\cdot k_2 = -\frac{s}{2}(1-\cos\theta) \tag{52}$$

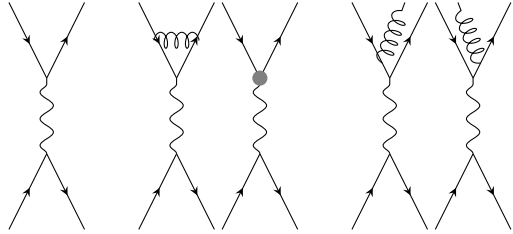
$$u = -2k_1\cdot p_1 = -\frac{s}{2}(1+\cos\theta) \tag{53}$$

$$\sigma = \frac{1}{2s}\int \frac{d\cos\theta}{16\pi}\frac{2(u^2+t^2)}{s^2} \tag{54}$$

$$= \frac{e^4}{32\pi s}\int d\cos\theta(1+\cos^2\theta) \tag{55}$$

$$= \frac{4\pi\alpha_e^2}{3s} \tag{56}$$

2.2 Total X-section of $e^+e^- \to h$



2.2.1 $e^+e^- \to q\bar{q}$ at LO

$$i\mathcal{M} = (ie)^2 Q \bar{v}(k_1)\gamma_\mu u(k_2)\frac{-i}{s+i\epsilon}\bar{u}^a(p_1)\gamma^\mu v^a(p_2) \tag{57}$$

$$|\overline{\mathcal{M}}|^2 = Q^2 N_c |\overline{\mathcal{M}}(\mu)|^2 \tag{58}$$

$$\sigma = N_C Q^2 \frac{4\pi\alpha_e^2}{3s} \tag{59}$$

$$R = \frac{\sigma(e^+e^- \to h)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_C \sum_f Q_f^2. \tag{60}$$

Keeping the final quark mass, the results are

$$\sigma = N_C e_q^2 \frac{4\pi\alpha_e^2}{3s}\left(1+\frac{2M_q^2}{s}\right)\sqrt{1-\frac{4M_q^2}{s}}, \tag{61}$$

$$R = \sum_q N_C Q_q^2 \left(1+\frac{2M_q^2}{s}\right)\sqrt{1-\frac{4M_q^2}{s}}. \tag{62}$$

2.3 $e^+e^- \to q\bar{q}$ at NLO

Following P189-196 in QED:

1. Write down the Amplitude and Loop Integral

$$\begin{aligned}
&\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p) \\
&= 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}'+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p)}{((k-p)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}.
\end{aligned} \tag{6.38}$$

The external line: $u(p) \to u^i(p)$; The vertex : $-ie \to ig_s T^a$; The propagator $-ig_{\nu\rho} \to -ig_{\nu\rho}\delta^{ab}$. So the color factor is

$$T^a_{ij}T^a_{jk} = C_F\delta_{ik} \tag{63}$$

2. Feynman parametrize the Loop Integral

$$\frac{1}{((k-p)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} = \int_0 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{D^3},$$

3. Shift the integral momentum

$$l = k + yq - xp \tag{64}$$

$$\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{D^3}$$
$$\times\,\bar{u}(p')\Big[\gamma^\mu\cdot\big(-\tfrac{1}{2}\ell^2+(1-x)(1-y)q^2+(1-4z+z^2)m^2\big)$$
$$+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\big(2m^2z(1-z)\big)\Big]u(p), \tag{6.47}$$

4. Wick Rotation

   $\ell^0 \equiv i\ell_E^0; \qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E.$

5. Integrate out the Loop momentum and get the UV pole.
   The result in textbook is got with Pauli-Villars regularization.

$$\frac{1}{k^2+i\epsilon} \to \frac{1}{k^2} - \frac{1}{k^2-\Lambda^2} = \frac{\Lambda^2}{k^2(k^2-\Lambda^2)} \sim \frac{1}{k^4} \tag{65}$$



$$= \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\times\,\bar{u}(p')\Big(\gamma^\mu\Big[\log\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\Big((1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big)\Big]$$
$$+\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big[\frac{1}{\Delta}2m^2z(1-z)\Big]\Big)u(p). \tag{6.54}$$

$$\Delta = \tag{66}$$

6. Subtract the UV pole with Counter-terms

UV pole $\ln \Lambda^2$ or $\frac{1}{\epsilon}$.

7. Integrate out the Feynman parameters

$$\Delta = -xyq^2 + (1-z)^2 m^2 \to 0 \quad \text{when} \quad z \to 1 \tag{67}$$

- How to deal the IR pole? Introduce gluon mass $\frac{1}{k^2} \to \frac{1}{k^2 - m_g^2}$.

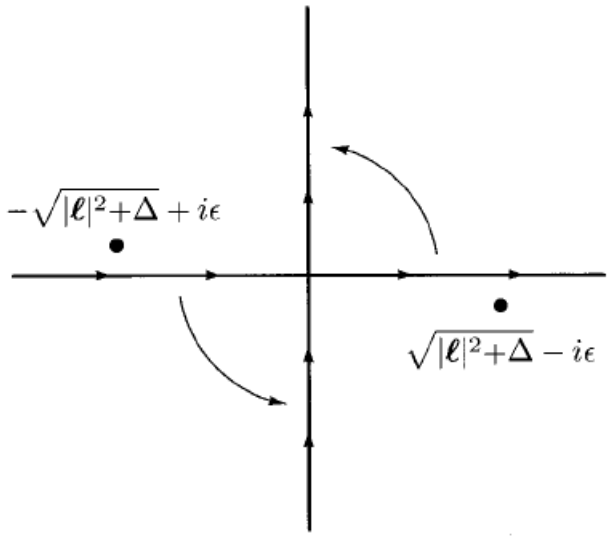
$$\sigma_{MG}(\text{virtual}) = \frac{2\alpha_s}{3\pi}\sigma_0 \left\{ -\log^2(\beta) - 3\log(\beta) - \frac{7}{2} - \frac{2\pi^2}{3} + \pi^2 \right\}_{\text{timelike}}, \tag{2.7.4}$$

with

$$\beta = m_g^2/Q^2 \tag{2.7.5}$$

- Why is there IR pole?

$$\int d^4k \cdots \frac{1}{k^2 - m^2 + i\epsilon} \cdots \to \int \frac{1}{k^2 - (m - i\epsilon)^2} \quad \text{VS} \quad \int d^4k \cdots \frac{1}{k^2 + i\epsilon} \cdots \to \int \frac{1}{k^2 - (0 - i\epsilon)^2} \tag{68}$$



$$I = \int_0^1 dx \ln(a-x) \quad \text{pole at } x = a \tag{69}$$

$$= \int_0^a dx \ln(a-x) + \int_a^1 \ln(a-x) \tag{70}$$

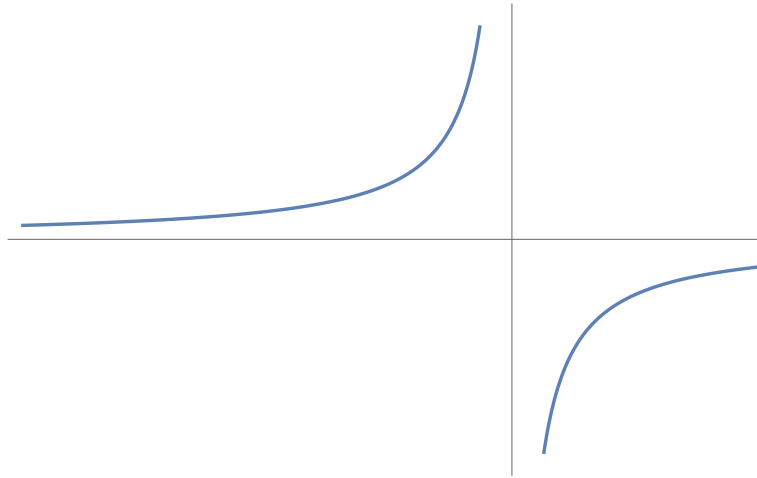
$$= \int_0^a dx \ln(a-x) + i\pi + \int_a^1 \ln(x-a) \tag{71}$$

$$= \lim_{\epsilon\to 0} \left(\int_0^{a-\epsilon} dx \ln(a-x) + \int_{a+\epsilon}^1 \ln(x-a) \right) \tag{72}$$

$$= \lim_{\epsilon\to 0} \left((a\ln a + \epsilon \ln\epsilon) + ((1-a)\ln(1-a) - 1 - \epsilon\ln\epsilon) \right) + i\pi \tag{73}$$

$$= -1 + \ln(1-a) + a\ln\frac{a}{1-a} + i\pi \tag{74}$$

– Similar to the Integration of $1/(x-a)$.



- What happens when $a = 1$
- What happens when $a = 0$
- What happens when $a = 4m^2/s$

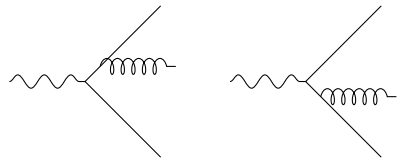


Figure 2: The (part) Feynman diagrams for $q\bar{q}g$.

2.3.1 $e^+e^- \to q\bar{q}g$

The fermion propagators

$$\frac{1}{(k_i+k)^2-m_b^2} = \frac{1}{2k_i\cdot k} = \frac{1}{2E_iE_g(1-\cos\theta_i)} \to \infty \text{ when } E_g \to 0(soft) \text{ or } \cos\theta_i \to 1(collinear) \tag{75}$$

$$\cos\theta_1 \to 0, x_2 = \frac{2E_2}{\sqrt{s}} \to 1 \tag{76}$$

$$\cos\theta_2 \to 0, x_1 \to 1 \tag{77}$$

$$E_g \to 0, x_1 \to x_2 \to 1 \tag{78}$$

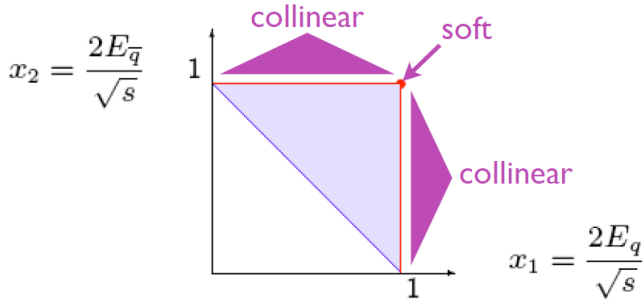


Figure 3: The sof-collinear divergences.

$$\mathcal{M} = \bar{u}(p_1)(ig_s t^a)\not{\epsilon}(k)\frac{-i(\not{p}_1 + \not{k})}{(p_1+k)^2}(ie_q)\gamma_\mu v(p_2) + \bar{u}(p_1)(ie_q)\gamma_\mu \frac{i(\not{p}_2 + \not{k})}{(p_2+k)^2}(ig_s t^a)\not{\epsilon}(k)v(p_2) \quad (79)$$

One can see the result divergence with soft(co-linear) gluon $k \to 0$, i.e. $(p_i + k)^2 - m_i^2 \to 0$. To see the poles, we simplify the above resutl with $\not{A}\not{B} = 2A \cdot B - \not{B}\not{A}$.

$$
\begin{aligned}
a) &= \bar{u}(p_1)(ie_q)t^a \frac{2\epsilon \cdot (p_1 + k) - (\not{p}_1 + \not{k})\not{\epsilon}}{(p_1 + k)^2} \gamma_\mu v(p_2) && (80)\\
&= \bar{u}(p_1)(ie_q)t^a \gamma_\mu v(p_2) \times g_s \frac{2\epsilon \cdot p_1}{2p_1 \cdot k} && (81)\\
b) &= -\bar{u}(p_1)(ie_q)t^a \gamma_\mu v(p_2) \times g_s \frac{2\epsilon \cdot p_2}{2p_2 \cdot k} && (82)\\
\mathcal{M} &= \bar{u}(p_1)(ie_q)t^a \gamma_\mu v(p_2) \times g_s \left(\frac{\epsilon \cdot p_1}{p_1 \cdot k} - \frac{\epsilon \cdot p_2}{p_2 \cdot k} \right) && (83)\\
|\mathcal{M}|^2 &= -|\mathcal{M}_{q\bar{q}}|^2 C_F \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2 && (84)\\
&= |\mathcal{M}_{q\bar{q}}|^2 C_F g_s \frac{2p_1.p_2}{(p_1.k)(p_2.k)} && (85)
\end{aligned}
$$

The result show that in the soft limit, the $|M_{q\bar{q}g}|^2$ squared matrix element can be factorized into two terms: the $|M_{q\bar{q}}|^2$ matrix element and a piece with a rather simple dependence on the gluon momentum. In the soft approximation, we can write this phase space as

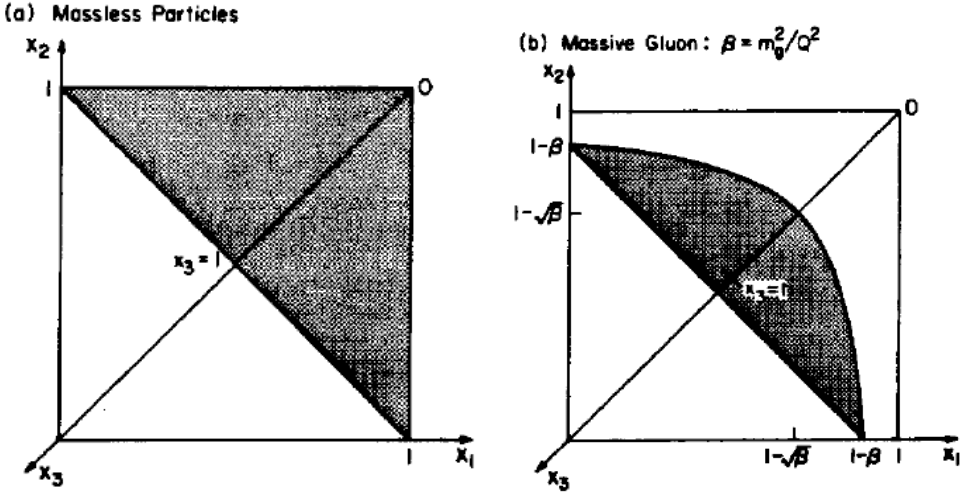
$$
\sigma_{q\bar{q}g} = \frac{1}{2s} \int |\mathcal{M}|^2 d\Phi_3 = \sigma_{q\bar{q}} \times C_F g_s \int \frac{d^3k}{k^0 (2\pi)^3} \frac{2p_1.p_2}{(p_1.k)(p_2.k)} \qquad (86)
$$

$$= \sigma_{q\bar{q}} \times \int \frac{dk^0}{k^0} d\cos\theta \frac{d\phi}{2\pi} \frac{2\alpha_s C_F}{\pi} \frac{1}{(1-\cos\theta)(1+\cos\theta)} \tag{87}$$

The result has two type divergence: 1), called the soft (or infrared) divergence when $k^0 \to 0$ and 2), a collinear divergence, when $\theta \to 0(or\pi)$, i.e., the gluon becomes collinear with the quark (or anti-quark) direction. The complete calculation of $e^+e^- \to q\bar{q}g$ at tree-level is

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \left(3 \sum_f Q_f^2 \right) \times \frac{2\alpha_s}{3\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}. \tag{88}$$

- How to deal with the soft-collinear divergence?



Finally, the x-section of $e^+e^- \to q\bar{q}g$ with gluon mass $m_g \neq 0$

$$\sigma_{MG}(\text{real}) = \frac{2\alpha_s}{3\pi}\sigma_0 \left\{ \log^2(\beta) + 3\log(\beta) - \frac{\pi^2}{3} + 5 \right\}, \qquad (2.5.11)$$

$$\sigma_{MG}(\text{virtual}) = \frac{2\alpha_s}{3\pi}\sigma_0 \left\{ -\log^2(\beta) - 3\log(\beta) - \frac{7}{2} - \frac{2\pi^2}{3} + \pi^2 \right\}_{\text{timelike}}, \tag{2.7.4}$$

with

$$\beta = m_g^2/Q^2 \tag{2.7.5}$$

- How to understand the divergence?

  The quark and anti-quark will undergo further soft interaction with other product to be collected into color-singlet. Thus the presence of one or more soft-collinear gluons has no noticeable effect on the final states, which consists with back-to-back jets.

  $e^+e^- \to q\bar{q}g$ in the soft-collinear region contribute to di-jet events experimentally.

- When the gluon is emitted with substantial transverse momentum with respect to the $q\bar{q}$ axis, it is not possible for subsequent soft exchanges to recall or reverse this transverse momentum. In this case, the $q\bar{q}g$ system will evolves into 3-jet final states.

2.3.2 Total x-section

The inclusive x-section is

$$\sigma_{MG}(\text{real}) = \frac{2\alpha_s}{3\pi}\sigma_0 \left\{ \log^2(\beta) + 3\log(\beta) - \frac{\pi^2}{3} + 5 \right\}, \qquad (2.5.11)$$

$$\sigma_{MG}(\text{virtual}) = \frac{2\alpha_s}{3\pi}\sigma_0 \left\{ -\log^2(\beta) - 3\log(\beta) - \frac{7}{2} - \frac{2\pi^2}{3} + \pi^2 \right\}_{\text{timelike}}, \qquad (2.7.4)$$

with

$$\beta = m_g^2/Q^2 \qquad (2.7.5)$$

$$\sigma(e^+e^- \to h) = \sigma_0 + \sigma_1^V + \sigma_1^R(e^+e^- \to q\bar{q}g) + \cdots \qquad (89)$$

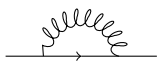
$$= \left| \right|^2 + 2\text{Re}\left(\times \right) + \left| \right|^2 \qquad (90)$$

$$= \left(\sigma_0 + \sigma^V + \sigma^R(E_{g_<}, \Delta R_{i<})\right) + \left(\sigma^R(E_{g_>}, \Delta R_{i>})\right) \qquad (91)$$

$$= \sigma_0 \left(1 + \frac{\alpha_s}{\pi}\right) + O(\alpha_s^2) \qquad (92)$$

2.4 Running of α_s

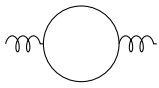
2.4.1 Running of α_s



$$\Sigma(p^2) = \not{p}\frac{g_R^2}{(4\pi)^2}\left(\frac{C_F}{\epsilon} + \text{finite}\right) \tag{93}$$



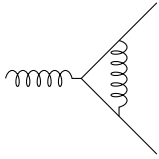
$$Z_2 = 1 - \frac{g_R^2}{(4\pi)^2}\frac{1}{\epsilon}C_F + \text{finite} \tag{94}$$



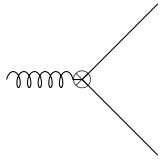
$$\Sigma_3(p^2) \tag{95}$$



$$Z_3 = 1 + \frac{g_R^2}{(4\pi)^2}\frac{1}{\epsilon}\left(\frac{5}{3}C_A - \frac{4}{3}T_R n_f\right) + \text{finite} \tag{96}$$



$$\Gamma \sim \tag{97}$$



$$Z_1 = 1 - \frac{g_R^2}{(4\pi)^2}\frac{1}{\epsilon}\left(C_F + C_A\right) + \text{finite.} \tag{98}$$

$$Z_1 = Z_g Z_2 Z_3^{1/2}, \to Z_g = \frac{Z_1}{Z_2} Z_3^{-1/2} = 1 - \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(C_F + C_A - C_F + \frac{5C_A - 4T_R n_f}{6}\right) + \cdots \tag{99}$$

The bare coupling g_0 is not dependent on μ,

$$g_0 = \mu^\epsilon Z_g(\mu) g_R(\mu), \quad Z_g = 1 + \frac{Z_g^1}{\epsilon} + \cdots \tag{100}$$

$$\beta(g) = -(\beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \cdots) \tag{101}$$

$$\beta_0 = -2\frac{dZ_g^1}{d\alpha_s} = \frac{11C_A - 4T_R n_f}{12\pi} \tag{102}$$

$$= \frac{33 - 2n_f}{12\pi} > 0 \quad (\text{for } n_f < 16) \tag{103}$$

The running of the α_s,

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta_0 \alpha_s^2, \quad \alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + \alpha_s(\mu_0^2)\beta_0 \ln\frac{\mu^2}{\mu_0^2}} = \frac{1}{\beta_0 \ln\frac{\mu^2}{\Lambda_{QCD}^2}} \tag{104}$$

$\beta_0 > 0, \beta < 0$ and we have $\alpha_s(\mu^2 \to \infty) \to 0$, this is called Asymptotic Freedom.

2.4.2 Measurements of α_s

1. Choose the renormalization scheme ($\overline{MS}$): $\alpha_s^{\overline{MS}}(\mu), m^{\overline{MS}}(\mu)$, Counter-terms
2. Calculate the cross-section in the chosen scheme

$$\sigma = \sigma(\mu, \alpha^{\overline{MS}}(\mu)). \tag{105}$$

3. Set the renormalization scale $\mu = M_Z, \sqrt{s}$.
4. Extract the number value from the experimental results.

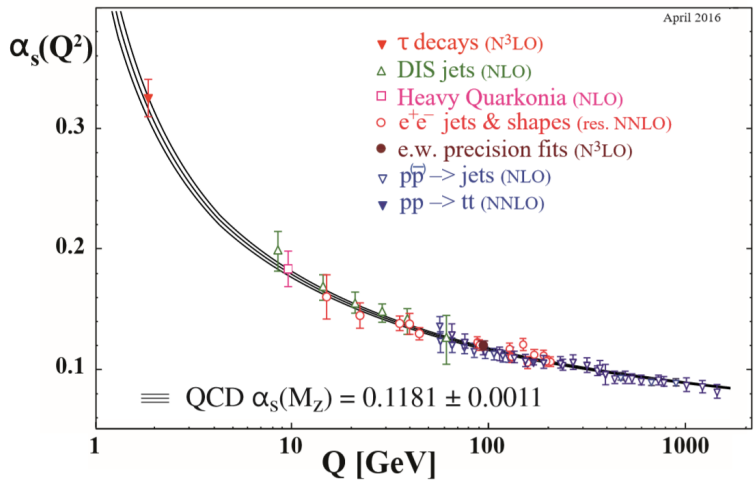


Figure 9.3: Summary of measurements of α_s as a function of the energy scale Q. The respective degree of QCD perturbation theory used in the extraction of α_s is indicated in brackets (NLO: next-to-leading order; NNLO: next-to-next-to leading order; res. NNLO: NNLO matched with resummed next-to-leading logs; N^{3}LO: next-to-NNLO).

3 Deep Inelastic Scattering (DIS)

3.1 DIS and Parton Model

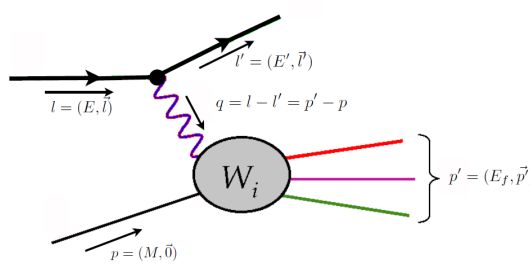


Figure 4: The kinematics of deep inelastic scattering $e^- p^+ \to e^- + X$.

$$i\mathcal{M} = \frac{e^2}{q^2} L_\mu \sum_X \langle X|J^\mu|P\rangle \tag{106}$$

$$|\overline{\mathcal{M}}|^2 = \left(\frac{e^2}{q^2}\right)\frac{1}{2}L_{\mu\nu}\frac{1}{2}\sum_X \langle P|J^\mu|X\rangle\langle X|J^\nu|P\rangle \tag{107}$$

$$\sigma = \frac{1}{4EM}\int\frac{d^3l'}{2E'(2\pi)^3}\left(\frac{e^2}{q^2}\right)^2\frac{1}{2}L_{\mu\nu}\sum_X\frac{d^3P_X}{2E_X(2\pi)^3}(2\pi)^4\delta^{(4)}(P_X-P-q) \tag{108}$$

$$\times\frac{1}{2}\langle P|J^\mu|X\rangle\langle X|J^\nu|P\rangle \tag{109}$$

$$= \frac{1}{4EM}\int\frac{d^3l'}{2E'(2\pi)^3}\left(\frac{e^2}{q^2}\right)^2\frac{1}{2}L_{\mu\nu}\times(4\pi^2M)W^{\mu\nu} \tag{110}$$

$$= \int dE'd\cos\theta\frac{\alpha^2}{Q^4}\frac{E'}{E}L^{\mu\nu}W_{\mu\nu}, \tag{111}$$

where $q^\mu=(l-l')^\mu=(p'-p)^\mu$, and

$$L^{\mu\nu}=\mathrm{Tr}[\not{l}\gamma^\mu\not{l}'\gamma^\nu]=8l^\mu l^\nu-4q^\mu l^\nu-4l^\nu q^\mu+2g^{\mu\nu}q^2. \tag{112}$$

The factor $4\pi^2M$ is included to give the "standard convention" for the normalization of $W_{\mu\nu}$. The $W_{\mu\nu}$ represents the unknown physics at the photon-hadron vertex. And it has the following properties

$$q^\mu W_{\mu\nu}=q^\nu W_{\mu\nu}=0. \tag{113}$$

The most general form is

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right)\frac{W_2}{M^2} \tag{114}$$

$$\frac{d^2\sigma}{dE' d\cos\theta} = E'^2 \frac{8\alpha^2}{Q^4}\left(2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2}\right) \tag{115}$$

$$= \frac{\alpha^2}{4E^2 \sin^4\theta/2}\left(\frac{1}{\nu}F_2 \cos^2\frac{\theta}{2} + \frac{2}{M}F_1 \sin^2\frac{\theta}{2}\right) \tag{116}$$

W_i are *Form Factors* of Proton, which are scalars and dependent on the q^μ and P^μ. Thus one has

$$W_i \sim q^2 = -Q^2, q\cdot P(\text{or } x) \tag{117}$$

where scaling variable is defined

$$x \equiv \frac{-q^2}{2P\cdot q} = \frac{Q^2}{2M\nu}, \quad \nu = \frac{q\cdot P}{M} \tag{118}$$

- The structure function F_is are functions of Q^2 and x (or ν).

  $$F_1(x, Q^2), \quad F_2(x, Q^2). \tag{119}$$

  To determine the $F_i(x, Q^2)$, one need to measure the differential x-section for a given x and Q^2.

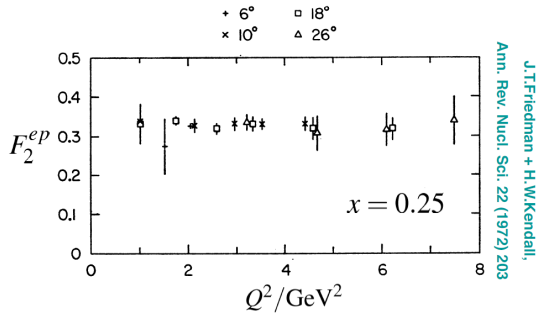
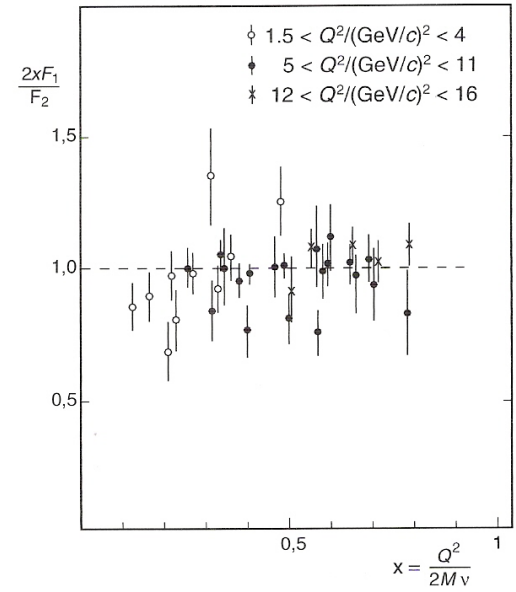


Figure 5: Bjorken Scaling

- Experimentally it is observed that F_is are (almost) independent of Q^2, which is known as *Bjorken Scaling*.

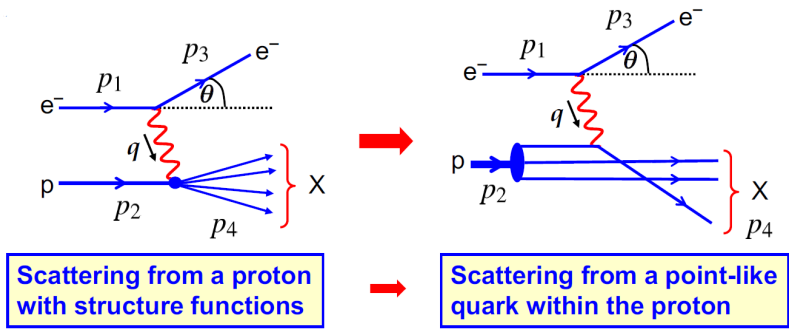
- It is also observed that F_1 and F_2 are not independent but satisfy the *Callan-Gross Relation*

$$F_2(x) = 2xF_1(x) \tag{120}$$



Callan-Gross Relation

3.2 Parton Model



- Any hadron involved in a high energy collision may be seen as constituted by free pointlike particles (partons), each one carrying a fraction x of the hadron longitudinal momentum
- Any hadronic cross section $\sigma_H(P)$ is the incoherent sum of the parton level cross sections $\hat{\sigma}_i$

$$\sigma_H(P) = \sum_i \int_0^1 dx f_{i/H}(x)\hat{\sigma}(xP) \tag{121}$$

- The function $f_{i/H}(x)$ is called as *Parton Distribution Function*(PDF), that is the density of the probability that parton i has momentum xP.

$$\left(\begin{array}{c} \text{probability of finding constituent} f \\ \text{in Hadron } H \text{ with longitudinal fraction} x \end{array} \right) = f_{f/H}(x)dx \tag{122}$$

- Feynman proposed the assumptions of parton even before the quark model was generally accepted.

3.2.1 DIS in Parton Model

$$\sigma(e + P \to e + X) = \sum_q \int dx f_q(x) \hat{\sigma}(e + q \to e + q), \tag{123}$$

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4} \left(1 + \left(1 - \frac{Q^2}{\hat{s}} \right)^2 \right), \tag{124}$$

$$\frac{d^2\sigma}{dQ^2 dx} = \sum_q f_q(x) \frac{d\hat{\sigma}}{dQ^2}, \tag{125}$$

$$= \sum f_q(x) e_q^2 \frac{2\pi\alpha^2}{Q^4} \left(1 + \left(1 - \frac{Q^2}{\hat{s}} \right)^2 \right). \tag{126}$$

- Q^2

$$Q^2 = -q^2 = (k' - k)^2 = 2EE'(1 - \cos\theta) \tag{127}$$

- x

$$\begin{aligned} q_\mu &= (k - k')_\mu = (p' - p)_\mu && (128)\\ p'^2 &= (p + q)^2 = p^2 + q^2 + 2p \cdot q && (129)\\ Q^2 &= 2p \cdot q = 2xP \cdot q && (130)\\ x &= \frac{Q^2}{2P \cdot q} && (131) \end{aligned}$$

- $q = u, d$

$$\sum_q e_q^2 f_q(x) = \frac{4}{9}(f_u + f_{\bar{u}}) + \frac{1}{9}(f_d + f_{\bar{d}}) \tag{132}$$

3.2.2 Bjorken Scaling and Callan-Gross Relationship

Both Bjorken Scaling and the Callan-Gross relationship can be explained by assuming that Deep Inelastic Scattering is dominated by the scattering of a single virtual photon from point-like spin-half constituents

of the proton.

$$F_1(x) = \frac{1}{2} \sum e_i^2 f_i(x) \tag{133}$$

$$F_2(x) = x \sum_i e_i^2 f_i(x) = 2xF_1(x) \tag{134}$$

3.2.3 Deep Inelastic Neutrino Scattering

3.3 The Parton Distribution Functions

$$\int dx(f_u - f_{\bar{u}}) = 2 \tag{135}$$

$$\int dx(f_d - f_{\bar{d}}) = 1 \tag{136}$$

$$\int dxx(f_u + f_d + f_{\bar{u}} + f_{\bar{d}} + f_g(x)) = 1 \tag{137}$$

$SU(2)$-symmetry: $uud \leftrightarrow ddu$

$$f_u^p = f_d^n, \quad f_d^p = f_u^n, \qquad f_{\bar{u}}^p = f_{\bar{d}}^n, \quad f_{\bar{d}}^p = f_{\bar{u}}^n \tag{138}$$

CP-symemry: $uud \leftrightarrow \bar{u}\bar{u}\bar{d}$

$$f_u^p = f_{\bar{u}}^{\bar{p}}, \quad f_d^p = f_{\bar{d}}^{\bar{p}}, \qquad f_{\bar{u}}^p = f_u^{\bar{p}}, \quad f_{\bar{d}}^p = f_d^{\bar{p}}. \tag{139}$$

4 Hard Scattering Processes in Hadron Collisions

$$\sigma\big(P(p_1) + P(p_2) \to Y + X\big) = \int dx_1 dx_2 \sum f_1(x_1) f_2(x_2) \sigma(q_1 + q_2 \to Y) \tag{140}$$

$$= \int dx dy f_1(x) f_2(y/x) \sigma(\hat{s} = yS) \tag{141}$$

The cross-section for the lepton pair production (Drell-Yan process) is obtained as

$$\frac{d^2\sigma}{d\hat{s}dY}(pp \to e^+e^- + X) = \sum_q x_1 f_q(x_1) f_{\bar{q}}(x_2) \frac{Q_q^2}{3} \frac{4\pi\alpha^2}{3\hat{s}^2} \tag{142}$$

$$\hat{s} = 2E_1E_2(1 - \cos\theta), \quad \cosh Y = \frac{E_1 + E_2}{\sqrt{2E_1E_2(1 - \cos\theta)}} \tag{143}$$

$$x_1 = \sqrt{\frac{\hat{s}}{s}} e^Y, \qquad x_2 = \sqrt{\frac{\hat{s}}{s}} e^{-Y} \tag{144}$$

5 Parton Evolution