Lectures on Supersymmetry

Matteo Bertolini
SISSA

June 9, 2021

Foreword

This is a write-up of a course on *Supersymmetry* I have been giving for several years to first year PhD students attending the curriculum in Theoretical Particle Physics at SISSA, the International School for Advanced Studies of Trieste.

There are several excellent books on supersymmetry and many very good lecture courses are available on the archive. The ambition of this set of notes is not to add anything new in this respect, but to offer a set of hopefully complete and self-consistent lectures, which start from the basics and arrive to some of the more recent and advanced topics. The price to pay is that the material is pretty huge. The advantage is to have all such material in a single, possibly coherent file, and that no prior exposure to supersymmetry is required.

There are many topics I do not address and others I only briefly touch. In particular, I discuss only rigid supersymmetry (mostly focusing on four space-time dimensions), while no reference to supergravity is given. Moreover, this is a theoretical course and phenomenological aspects are only briefly sketched. One only chapter is dedicated to present basic phenomenological ideas, including a bird eyes view on models of gravity and gauge mediation and their properties, but a thorough discussion of phenomenological implications of supersymmetry would require much more.

There is no bibliography at the end of the file. However, each chapter contains its own bibliography where the basic references (mainly books and/or reviews available on-line) I used to prepare the material are reported – including explicit reference to corresponding pages and chapters, so to let the reader have access to the original font (and to let me give proper credit to authors).

I hope this effort can be of some help to as many students as possible!

Disclaimer: I expect the file to contain many typos and errors. Everybody is welcome to let me know them, dialing at `matteo.bertolini@sissa.it`. Your help will be very much appreciated.

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1 Supersymmetry: a bird eyes view

Coming years could represent a new era of unexpected and exciting discoveries in high energy physics, since a long time. For one thing, the CERN Large Hadron Collider (LHC) has been operating for some time, now, and many experimental data have already being collected. So far, the greatest achievement of the LHC has been the discovery of the missing building block of the Standard Model, the Higgs particle (or, at least, a particle which most likely is the Standard Model Higgs particle). On the other hand, no direct evidence of new physics beyond the Standard Model has been found, yet. Still, there are several reasons to believe that new physics should in fact show-up at, or about, the TeV scale.

The most compelling scenario for physics beyond the Standard Model (BSM) is supersymmetry. For this reason, knowing what is supersymmetry is rather important for a high energy physicist, nowadays. Understanding *how* supersymmetry can be realized (and then spontaneously broken) in Nature, is in fact one of the most important challenges theoretical high energy physics has to confront with. This course provides an introduction to such fascinating subject.

Before entering into any detail, in this first lecture we just want to give a brief overview on *what* is supersymmetry and *why* is it interesting to study it. The rest of the course will try to provide (much) more detailed answers to these two basic questions.

Disclaimer: The theory we are going to focus our attention in the (more than) three-hundred pages which follow, can be soon proved to be the correct mathematical framework where to understand high energy physics at the TeV scale, and become a piece of basic knowledge any particle physicist should have. But it can very well be that BSM physics is more subtle and Nature not so kind to make supersymmetry be realized at low enough energy that we can make experiment of. Or, it can also be that all this will eventually turn out to be just a purely academic exercise about a theory that nothing has to do with Nature. An elegant way mankind has worked out to describe in an unique and self-consistent way elementary particle physics, which however is not the one chosen by Nature (but can we ever safely say so?). As I will briefly outline below, and discuss in more detail in the second part of this course, even in such a scenario... studying supersymmetry and its fascinating properties might still be helpful and instructive in many respects.

1.1 What is supersymmetry?

Supersymmetry (SUSY) is a *space-time symmetry* mapping particles and fields of integer spin (bosons) into particles and fields of half integer spin (fermions), and viceversa. The generators Q act as

$$Q\,|Fermion\rangle \;=\; |Boson\rangle \quad \text{and viceversa} \tag{1.1}$$

From its very definition, this operator has two obvious but far-reaching properties that can be summarized as follows:

- It changes the spin of a particle (meaning that Q transforms as a spin-1/2 particle) and hence its space-time properties. This is why supersymmetry is not an internal symmetry but a space-time symmetry.
- In a theory where supersymmetry is realized, each one-particle state has at least a superpartner. Therefore, in a SUSY world, instead of single particle states, one has to deal with (super)multiplets of particle states.

Supersymmetry generators have specific commutation properties with other generators. In particular:

- Q commutes with translations and internal quantum numbers (e.g. gauge and global symmetries), but it does not commute with Lorentz generators

  $$[Q, P_\mu] = 0 \quad , \quad [Q, G] = 0 \quad , \quad [Q, M_{\mu\nu}] \neq 0 \; . \tag{1.2}$$

  This implies that particles belonging to the same supermultiplet have different spin but same mass and same quantum numbers.

A supersymmetric field theory is a set of fields and a Lagrangian which exhibit such a symmetry. As ordinary field theories, supersymmetric theories describe particles and interactions between them: SUSY manifests itself in the specific particle spectrum a theory enjoys, and in the way particles interact between themselves.

A supersymmetric model which is covariant under *general coordinate transformations* is called supergravity (SUGRA) model. In this respect, a non-trivial fact, which again comes from the algebra, in particular from the (anti)commutation relation

$$\{Q, \bar{Q}\} \sim P_\mu \; , \tag{1.3}$$

is that having general coordinate transformations is equivalent to have *local* SUSY, the gauge mediator being a spin 3/2 particle, the gravitino. Hence local supersymmetry and General Relativity are intimately tied together.

One can have theories with different number of SUSY generators Q: $Q^I \quad I = 1, ..., N$. The number of supersymmetry generators, however, cannot be arbitrarily large. The reason is that *any supermultiplet contains particles with spin at least as large as* $\frac{1}{4}N$. Therefore, to describe local and interacting theories, N can be at most as large as 4 for theories with maximal spin 1 (gauge theories) and as large as 8 for theories with maximal spin 2 (gravity). Thus stated, this statement is true in 4 space-time dimensions. Equivalent statements can be made in higher/lower dimensions, where the dimension of the spinor representation of the Lorentz group is larger/smaller (for instance, in 10 dimensions, which is the natural dimension where superstring theory lives, the maximum allowed N is 2). What really matters is the number of single state supersymmetry generators, which is an invariant, dimension-independent statement.

Finally, notice that since supersymmetric theories automatically accomodate both bosons and fermions, SUSY looks like the most natural framework where to formulate a theory able to describe matter and interactions in a unified way.

1.2 What is supersymmetry useful for?

Let us briefly outline a number of reasons why it might be meaningful (and useful) to have such a bizarre and unconventional symmetry actually realized in Nature.

i. Theoretical reasons.

- What are the more general allowed symmetries of the S-matrix? In 1967 Coleman and Mandula proved a theorem which says that in a generic quantum field theory, under a number of (very reasonable and physical) assumptions, like *locality*, *causality*, *positivity of energy*, *finiteness of number of particles*, *etc...*, the only possible continuos symmetries of the S-matrix are those generated by Poincaré group generators, P_μ and $M_{\mu\nu}$, plus some internal symmetry group G (where G is a semi-simple group times abelian factors) commuting with them

$$[G, P_\mu] = [G, M_{\mu\nu}] = 0 \ . \tag{1.4}$$

  In other words, the most general symmetry group enjoyed by the S-matrix is

Poincaré × Internal Symmetries

The Coleman-Mandula theorem can be evaded by weakening one or more of its assumptions. One such assumptions is that the symmetry algebra only involves *commutators*, all generators being *bosonic* generators. This assumption does not have any particular physical reason not to be relaxed. Allowing for *fermionic* generators, which satisfy anti-commutation relations, it turns out that the set of allowed symmetries can be enlarged. More specifically, in 1975 Haag, Lopuszanski and Sohnius showed that supersymmetry (which, as we will see, is a very specific way to add fermionic generators to a symmetry algebra) is the only possible such option. This makes the Poincaré group becoming SuperPoincaré. Therefore, the most general symmetry group the S-matrix can enjoy turns out to be

SuperPoincaré × Internal Symmetries

From a purely theoretical view point, one could then well expect that Nature might have realized all possible kind of allowed symmetries, given that we already know this is indeed the case (cf. the Standard Model) for all known symmetries, but supersymmetry.

- The history of our understanding of physical laws is an history of unification. The first example is probably Newton's law of universal gravitation, which says that one and the same equation describes the attraction a planet exert on another planet and on... an apple! Maxwell equations unify electromagnetism with special relativity. Quantumelectrodynamics unifies electrodynamics with quantum mechanics. And so on and so forth, till the formulation of the Standard Model which describes in an unified way all known non-gravitational interactions. Supersymmetry (and its local version, supergravity), is the most natural candidate to complete this long journey. It is a way not just to describe in a unified way all known interactions, but in fact to describe matter *and* radiation all together. This sounds compelling, and from this view point it sounds natural studying supersymmetry and its consequences.

- Finally, I cannot resist to add one more reason as to why one could expect that supersymmetry is out there, after all. Supersymmetry is possibly one of the two more definite predictions of String Theory, the other being the existence of extra-dimensions.

Note: all above arguments suggest that supersymmetry maybe realized as a symmetry in Nature. However, none of such arguments gives any obvious indication on the energy scale supersymmetry might show-up. This can be very high, in fact. Below, we will present a few more arguments, more phenomenological in nature, which deal, instead, with such an issue and actually suggest that low energy supersymmetry (as low as TeV scale or slightly higher) would be the preferred option.

ii. Elementary Particle theory point of view.

- *Naturalness and the hierarchy problem.* Three out of four of the fundamental interactions among elementary particles (strong, weak and electromagnetic) are described by the Standard Model (SM). The typical scale of the SM, the electroweak scale, is

$$M_{\rm ew} \sim 250 \text{ GeV} \;\Leftrightarrow L_{\rm ew} \sim 10^{-16} \text{ mm} \; . \tag{1.5}$$

  The SM is very well tested up to such energies. This cannot be the end of the story, though: for one thing, at high enough energies, as high as the Planck scale $M_{\rm pl}$, gravity becomes comparable with other forces and cannot be neglected in elementary particle interactions. At some point, we need a quantum theory of gravity. Actually, the fact that $M_{\rm ew}/M_{\rm pl} << 1$ calls for new physics at a much *lower* scale. One way to see this, is as follows. The Higgs potential reads

$$V(H) \sim \mu^2 |H|^2 + \lambda |H|^4 \quad \text{where} \quad \mu^2 < 0 \; . \tag{1.6}$$

  Experimentally, the minimum of such potential, $\langle H \rangle = \sqrt{-\mu^2/2\lambda}$, is at around 174GeV. This implies that the bare mass of the Higgs particle is roughly around 100 GeV or so, $m_H^2 = -\mu^2 \sim (100\text{GeV})^2$. What about radiative corrections? Scalar masses are subject to quadratic divergences in perturbation theory. The SM fermion coupling $-\lambda_f H \bar{f} f$ induces a one-loop correction to the Higgs mass as

$$\Delta m_H^2 \sim -\, 2\, \lambda_f^2\, \Lambda^2 \tag{1.7}$$

  due to the diagram in Figure 1.1. The UV cut-off Λ should then be naturally around the TeV scale in order to protect the Higgs mass, and the SM should then be seen as an effective theory valid at $E < M_{\rm eff} \sim$ TeV.

  What can be the new physics beyond such scale and how can such new physics protect the otherwise perturbative divergent Higgs mass? New physics, if any,

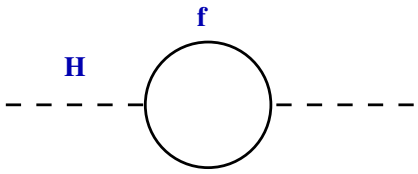


Figure 1.1: One-loop radiative correction to the Higgs mass due to fermion couplings.

may include many new fermionic and bosonic fields, possibly coupling to the SM Higgs. Each of these fields will give radiative contribution to the Higgs mass of the kind above, hence, no matter what new physics will show-up at high energy, the natural mass for the the Higgs field would always be of order the UV cut-off of the theory, generically around $\sim M_{\rm pl}$. We would need a huge fine-tuning to get it stabilized at $\sim$ 100GeV (we now know that the physical Higgs mass is at 125 GeV, in fact)! This is known as the *hierarchy problem*: the experimental value of the Higgs mass is unnaturally smaller than its natural theoretical value.

In principle, there is a very simple way out of this. This resides in the fact that (as you should know from your QFT course!) scalar couplings provide one-loop radiative contributions which are opposite in sign with respect to fermions. Suppose there exist some new scalar, S, with Higgs coupling $-\lambda_S|H|^2|S|^2$. Such coupling would also induce corrections to the Higgs mass via the one-loop diagram in Figure 1.2.

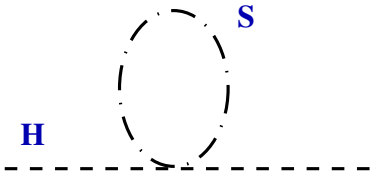


Figure 1.2: One-loop radiative correction to the Higgs mass due to scalar couplings.

Such corrections would have *opposite* sign with respect to those coming from fermion couplings

$$\Delta m_H^2 \sim \ \lambda_S \, \Lambda^2 \ . \tag{1.8}$$

Therefore, if the new physics is such that each quark and lepton of the SM were accompanied by two complex scalars having the same Higgs couplings of the quark and lepton, i.e. $\lambda_S = |\lambda_f|^2$, then all Λ^2 contributions would automatically cancel, and the Higgs mass would be stabilized at its tree level value!

Such conspiracy, however, would be quite ad hoc, and not really solving the fine-tuning problem mentioned above; rather, just rephrasing it. A natural thing to invoke to have such magic cancellations would be to have a symmetry protecting m_H, right in the same way as gauge symmetry protects the masslessness of spin-1 particles. A symmetry imposing to the theory the correct matter content (and couplings) for such cancellations to occur. This is exactly what supersymmetry is: in a supersymmetric theory there are fermions and bosons (and couplings) just in the right way to provide *exact* cancellation between diagrams like the ones above. In summary, supersymmetry is a very natural and economic way (though not the only possible one) to solve the hierarchy problem.

Known fermions and bosons cannot be partners of each other. For one thing, we do not observe any degeneracy in mass in elementary particles that we know. Moreover, and this is possibly a stronger reason, quantum numbers do not match: gauge bosons transform in the adjoint reps of the SM gauge group while quarks and leptons in the fundamental or singlet reps. Hence, in a supersymmetric world, each SM particle should have a (yet not observed!) supersymmetric partner, usually dubbed *sparticle*. Roughly, the spectrum should be as follows

SM particles	**SUSY partners**
gauge bosons	gauginos
quarks, leptons	scalars
Higgs	higgsino

Notice: the (down) Higgs has the same quantum numbers as the scalar partner of neutrino and leptons, sneutrino and sleptons respectively, $(H_d^0, H_d^-) \leftrightarrow (\tilde{\nu}, \tilde{e}_L)$. Hence, one can imagine that the Higgs is in fact a sparticle. This cannot be. In such scenario, there would be phenomenological problems, e.g. lepton number violation and (at least one) neutrino mass in gross violation of experimental bounds.

In summary, the world we already had direct experimental access to, is not supersymmetric. If at all realized, supersymmetry should be a (spontaneously) broken symmetry in the vacuum state chosen by Nature. However, in order to solve the hierarchy problem without fine-tuning this scale should be lower than (or about) 1 TeV. Including lower bounds from present day experiments,

it turns out that the SUSY breaking scale should be in the following energy range

$$100 \text{ GeV} \leq \text{SUSY breaking scale} \leq 1000 \text{ GeV} \ .$$

This is the basic reason why it is believed SUSY to show-up at the LHC.

Let us notice that these bounds are just a crude and rough estimate, as they depend very much on the specific SSM one is actually considering. In particular, the upper bound can be made higher by enriching the structure of the SSM in various ways, as well as defining the concept of naturalness in a less restricted way. There are ongoing discussions on these aspects nowadays, including the idea that naturalness should not be taken as a such important guiding principle, at least in this context.

- *Gauge coupling unification.* There is another reason to believe in supersymmetry; possibly stronger, from a phenomenological point of view, then that provided by the hierarchy problem. Forget about supersymmetry for a while, and consider the SM as it stands. Interesting enough, besides the EW scale, the SM contains in itself a new scale of order 10^{15} GeV. The three SM gauge couplings run according to RG equations like

$$\frac{4\pi}{g_i^2(\mu)} = \frac{b_i}{2\pi} \ln \frac{\mu}{\Lambda_i} \qquad i = 1, 2, 3 \ . \tag{1.9}$$

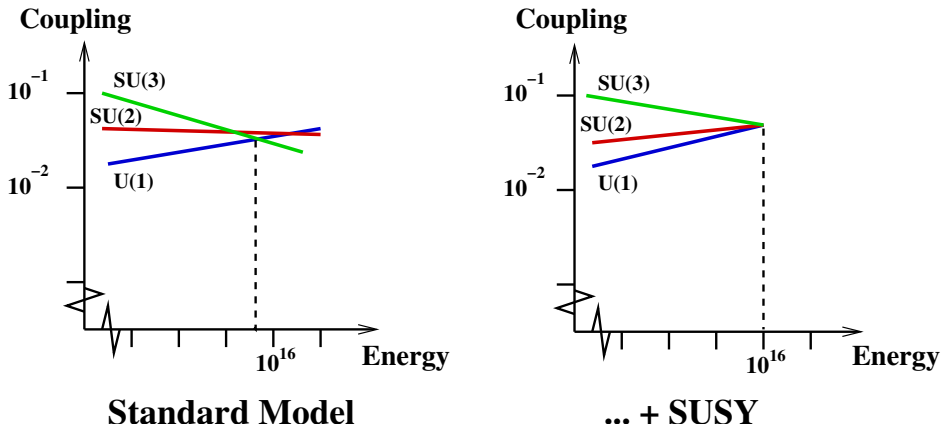
At the EW scale, $\mu = M_Z$, there is a hierarchy between them, $g_1(M_Z) < g_2(M_Z) < g_3(M_Z)$. But RG equations make this hierarchy changing with the energy scale. In fact, supposing there are no particles other than the SM ones, at a much higher scale, $M_{GUT} \sim 10^{15}$GeV, the three couplings tend to meet! This naturally calls for a Grand Unified Theory (GUT), where the three interactions are unified in a single one, two possible GUT gauge groups being $SU(5)$ and $SO(10)$. The symmetry breaking pattern one should have in mind would then be as follows

$$\begin{array}{ccccc} SU(5) & \to & SU(3) \times SU(2)_L \times U(1)_Y & \to & SU(3) \times U(1)_{\text{em}} \\ & \phi & & H & \end{array}$$

where ϕ is an *heavy* Higgs inducing spontaneous symmetry breaking at energies $M_{GUT} \sim 10^{15}$GeV, and H the SM *light* Higgs, inducing EW spontaneous symmetry breaking around the TeV scale. This idea poses several problems.

First, there is a new hierarchy problem (generically, the SM Higgs mass is expected to get corrections from the heavy Higgs ϕ). Second, there is a proton decay problem: some of the additional gauge bosons mediate baryon number violating transitions, allowing processes as $p \to e^+ + \pi_0$. This makes the proton not fully stable and it turns out that its expected lifetime in such GUT framework is violated by present experimental bounds. On a more theoretical side, if we do not allow for new particles besides the SM ones to be there at some intermediate scale, the three gauge couplings only *approximately* meet. The latter is an unpleasant feature: small numbers are unnatural from a theoretical view point, unless there are specific reasons (as symmetries) justifying their otherwise unnatural smallness.

Remarkably, making the GUT supersymmetric (SGUT) solves all of these problems in a glance! If one just allows for the minimal supersymmetric extension of the SM spectrum, known as MSSM, the three gauge couplings exactly meet, and the GUT scale is raised enough to let proton decay rate being compatible with present experimental bounds.



<u>*Disclaimer*</u>: the MSSM is not the only possible option for supersymmetry beyond the SM, just the most economic one. In the MSSM one just adds a superpartner to each SM particle, therefore introducing the higgsino, the wino, the zino, together with all squarks and sleptons, and no more. [There is in fact an exception. To have a meaningful model one has to double the Higgs sector, and have two Higgs doublets. One reason for that is gauge anomaly cancellation: the higgsinos are fermions in the fundamental rep of $SU(2)_L$ hence two of them are needed, with opposite hypercharge, not to spoil

the anomaly-free properties of the SM. A second reason is that in the SM the field H gives mass to down quarks and charged leptons while its charge conjugate, $H^c(\sim \bar{H})$ gives mass to up quarks. As we will see, in a SUSY model $\bar{H}$ cannot enter in the potential, which is a function of H, only. Therefore, in a supersymmetric scenario, to give mass to up quarks one needs a second, independent Higgs doublet.] There exist many non-minimal supersymmetric extensions of the Standard Model (which, in fact, are in better shape against experimental constraints with respect to the MSSM). One can in principle construct any SSM one likes. In doing so, however, several constraints are to be taken into account. For example, it is not so easy to make such non-minimal extensions keeping the nice exact gauge coupling unification enjoyed by the MSSM.

The important lesson we get out of all this discussion can be summarized as follows: in a SUSY quantum field theory radiative corrections are suppressed. Quantities that are small (or vanishing) at tree level tend to remain so at quantum level. This is at the basis of the solutions of all problems we mentioned: the hierarchy problem, the proton life-time, and gauge couplings unification.

iii. Supersymmetry and Cosmology.

- Let me briefly mention yet another context where supersymmetry might play an important role. There are various evidences which indicate that around 26% of the energy density in the Universe should be made of *dark matter*, i.e. non-luminous and non-baryonic matter. The only SM candidates for dark matter are neutrinos, but they are disfavored by available experimental data. Supersymmetry provides a valuable and very natural dark matter candidate: the neutralino. Neutralinos are mass eigenstates of a linear superposition of the SUSY partners of the neutral Higgs and of the SU(2) and U(1) neutral gauge bosons

$$\chi_i = \alpha_{i1}\tilde{B}^0 + \alpha_{i2}\tilde{W}^0 + \alpha_{i3}\tilde{H}_u^0 + \alpha_{i4}\tilde{H}_d^0 \ . \tag{1.10}$$

  Interestingly, in most SUSY frameworks the neutralino is the lightest supersymmetric particle (LSP), and fully stable, as a dark matter candidate should be.

iv. Supersymmetry as a *theoretical laboratory* for strongly coupled gauge dynamics.

- What if supersymmetry will turn out not to be the correct theory to describe (low energy) beyond the SM physics? Or, worse, what if supersymmetry will turn out not to be realized at all, in Nature (something we could hardly ever being able to prove, in fact)? Interestingly, there is yet another reason which makes it worth studying supersymmetric theories, independently from the role supersymmetry might or might not play as a theory describing high energy physics.

  Let us consider non-abelian gauge theories, which strong interactions are an example of. Every time a non-abelian gauge group remains unbroken at low energy, we have to deal with strong coupling. The typical questions one should try and answer (in QCD or similar theories) are:

  - The bare Lagrangian is described in terms of quark and gluons, which are UV degrees of freedom. Which are the IR (light) degrees of freedom of QCD? What is the effective Lagrangian in terms of such degrees of freedom?
  - Strong coupling physics is very rich. Typically, one has to deal with phenomena like confinement, charge screening, the generation of a mass gap, etc.... Is there any theoretical understanding of such phenomena?
  - The QCD vacuum is populated by *vacuum condensates* of fermion bilinears, $\langle\Omega|\bar{\psi}\psi|\Omega\rangle \neq 0$, which induce chiral symmetry breaking. What is the microscopic mechanism behind this phenomenon?

  Most of the IR properties of QCD have eluded so far a clear understanding, since we lack analytical tools to deal with strong coupling dynamics. Most results come from lattice computations, but these do not furnish a theoretical, first principle understanding of the above phenomena. Moreover, they are formulated in Euclidean space and are not suited to discuss, e.g. transport properties.

  Because of their nice renormalization properties, supersymmetric theories are more constrained than ordinary field theories and let one have a better control on strong coupling regimes, sometime. Therefore, one might hope to use them as toy models where to study properties of more realistic theories, such as QCD, in a more controlled way. Indeed, as we shall see, supersymmetric theories do provide examples where some of the above strong coupling effects can be studied exactly! This is possible due to powerful non-renormalizations

theorems supersymmetric theories enjoy, and because of a very special property of supersymmetry, known as *holomorphy*, which in certain circumstances lets one compute several non-perturbative contributions to the Lagrangian. We will spend a sizeable amount of time discussing these issues in the second part of this course.

This is all we wanted to say in this short introduction, that should be regarded just as an invitation to supersymmetry and its fascinating world. Let us end by just adding a curious historical remark. Supersymmetry did not first appear in ordinary four-dimensional quantum field theories but in string theory, at the very beginning of the seventies. Only later it was shown to be possible to have supersymmetry in ordinary quantum field theories.

1.3 Some useful references

The list of references in the literature is endless. I list below few of them, including books as well as some archive-available reviews. Some of these references may be better than others, depending on the specific topic one is interested in (and on personal taste). In preparing these lectures I have used most of them, some more, some less. At the end of each lecture I list those references (mentioning corresponding chapters and/or pages) which have been used to prepare it. This will let students having access to the original font, and me give proper credit to authors.

1. **Historical references**

   - J. Wess and J. Bagger
     Supersymmetry and supergravity
     Princeton, USA: Univ. Pr. (1992) 259 p.
   - P. C. West
     Introduction to supersymmetry and supergravity
     Singapore: World Scientific (1990) 425 p.
   - M. F. Sohnius
     Introducing Supersymmetry
     Phys. Rep. 128 (1985)

2. **More recent books**

- S. Weinberg
  The quantum theory of fields. Vol. 3: Supersymmetry
  Cambridge, UK: Univ. Pr. (2000) 419 p.
- J. Terning
  Modern supersymmetry: Dynamics and duality
  Oxford, UK: Clarendon (2006) 324 p.
- M. Dine
  Supersymmetry and string theory: Beyond the standard model
  Cambridge, UK: Cambridge Univ. Pr. (2007) 515 p.
- H.J. Müller-Kirsten and A. Wiedemann
  Introduction to Supersymmetry
  Singapore: World Scientific (2010) 439 p.

3. **On-line reviews: bases**

- J. D. Lykken
  Introduction to Supersymmetry
  arXiv:hep-th/9612114
- S. P. Martin
  A Supersymmetry Primer
  arXiv:hep-ph/9709356
- A. Bilal
  Introduction to supersymmetry
  arXiv:hep-th/0101055
- J. Figueroa-O'Farrill
  BUSSTEPP Lectures on Supersymmetry
  arXiv:hep-th/0109172
- M. J. Strassler
  An Unorthodox Introduction to Supersymmetric Gauge Theory
  arXiv:hep-th/0309149
- R. Argurio, G. Ferretti and R. Heise
  An introduction to supersymmetric gauge theories and matrix models
  Int. J. Mod. Phys. A **19** (2004) 2015 [arXiv:hep-th/0311066]

4. **On-line reviews: advanced topics**

- K. A. Intriligator and N. Seiberg
  Lectures on supersymmetric gauge theories and electric-magnetic duality
  Nucl. Phys. Proc. Suppl. 45BC (1996) 28 [arXiv:hep-th/9509066]
- A. Bilal
  Duality in N=2 SUSY SU(2) Yang-Mills Theory: A pedagogical introduction to the work of Seiberg and Witten
  arXiv:hep-th/9601007
- M. E. Peskin
  Duality in Supersymmetric Yang-Mills Theory
  arXiv:hep-th/9702094
- M. Shifman
  Non-Perturbative Dynamics in Supersymmetric Gauge Theories
  Prog. Part. Nucl. Phys. 39 (1997) 1 [arXiv:hep-th/9704114]
- P. Di Vecchia
  Duality in supersymmetric $N = 2, 4$ gauge theories
  arXiv:hep-th/9803026
- M. J. Strassler
  The Duality Cascade
  arXiv:hep-th/0505153
- P. Argyres
  Lectures on Supersymmetry
  available at `http://www.physics.uc.edu/~argyres/661/index.html`

5. **On-line reviews: supersymmetry breaking**

- G. F. Giudice and R. Rattazzi
  Theories with Gauge-Mediated Supersymmetry Breaking
  Phys. Rept. 322 (1999) 419 [arXiv:hep-ph/9801271]
- E. Poppitz and S. P. Trivedi
  Dynamical Supersymmetry Breaking
  Ann. Rev. Nucl. Part. Sci. 48 (1998) 307[arXiv:hep-th/9803107]
- Y. Shadmi and Y. Shirman
  Dynamical Supersymmetry Breaking
  Rev. Mod. Phys. 72 (2000) 25 [arXiv:hep-th/9907225]

- M. A. Luty
  2004 TASI Lectures on Supersymmetry Breaking
  arXiv:hep-th/0509029
- Y. Shadmi
  Supersymmetry breaking
  arXiv:hep-th/0601076
- K. A. Intriligator and N. Seiberg
  Lectures on Supersymmetry Breaking
  Class. Quant. Grav. **24** (2007) S741 [arXiv:hep-ph/0702069]

2 The supersymmetry algebra

In this lecture we introduce the supersymmetry algebra, which is the algebra encoding the set of symmetries a supersymmetric theory should enjoy.

2.1 Lorentz and Poincaré groups

The Lorentz group $SO(1,3)$ is the subgroup of matrices Λ of $GL(4,R)$ with unit determinant, $\det\Lambda = 1$, and which satisfy the following relation

$$\Lambda^T \eta \, \Lambda = \eta \tag{2.1}$$

where η is the (mostly minus in our conventions) flat Minkowski metric

$$\eta_{\mu\nu} = \text{diag}(+,-,-,-) \ . \tag{2.2}$$

The Lorentz group has six generators (associated to space rotations and boosts) enjoying the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad , \quad [J_i, K_j] = i\epsilon_{ijk} K_k \quad , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \ . \tag{2.3}$$

Notice that while the J_i are hermitian, the boosts K_i are anti-hermitian, this being related to the fact that the Lorentz group is non-compact (topologically, the Lorentz group is $\mathbb{R}_3 \times S_3/\mathbb{Z}_2$, the non-compact factor corresponding to boosts and the doubly connected $S_3/\mathbb{Z}_2$ corresponding to rotations). In order to construct representations of this algebra it is useful to introduce the following complex linear combinations of the generators J_i and K_i

$$J_i^{\pm} = \frac{1}{2} \left(J_i \pm i K_i \right) \ , \tag{2.4}$$

where now the $J_i^{\pm}$ are hermitian. In terms of $J_i^{\pm}$ the algebra (2.3) becomes

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk} J_k^{\pm} \quad , \quad [J_i^{\pm}, J_j^{\mp}] = 0 \ . \tag{2.5}$$

This shows that the Lorentz algebra is equivalent to two $SU(2)$ algebras. As we will see later, this simplifies a lot the study of the representations of the Lorentz group, which can be organized into (couples of) $SU(2)$ representations. This isomorphism comes from the theory of Lie Algebra which says that at the level of complex algebras

$$SO(4) \simeq SU(2) \times SU(2) \ . \tag{2.6}$$

In fact, the Lorentz algebra is a specific real form of that of $SO(4)$. This difference can be seen from the defining commutation relations (2.3): for $SO(4)$ one would have had a plus sign on the right hand side of the third such commutation relations. This difference has some consequence when it comes to study representations. In particular, while in Euclidean space all representations are real or pseudoreal, in Minkowski space complex conjugation interchanges the two $SU(2)$'s. This can also be seen at the level of the generators $J_i^{\pm}$. In order for all rotation and boost parameters to be real, one must take all the J_i and K_i to be imaginary and hence from eq. (2.4) one sees that

$$(J_i^{\pm})^* = -J_i^{\mp} \ . \tag{2.7}$$

In terms of algebras, all this discussion can be summarized noticing that for the Lorentz algebra the isomorphism (2.6) changes into

$$SO(1,3) \simeq SU(2) \times SU(2)^* \ . \tag{2.8}$$

For later purpose let us introduce a four-vector notation for the Lorentz generators, in terms of an anti-symmetric tensor $M_{\mu\nu}$ defined as

$$M_{\mu\nu} = -M_{\nu\mu} \quad \text{with} \quad M_{0i} = K_i \quad \text{and} \quad M_{ij} = \epsilon_{ijk} J_k \ , \tag{2.9}$$

where $\mu = 0, 1, 2, 3$. In terms of such matrices, the Lorentz algebra reads

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho} + i\eta_{\nu\rho} M_{\mu\sigma} \ . \tag{2.10}$$

Another useful relation one should bear in mind is the relation between the Lorentz group and $SL(2,\mathbb{C})$, the group of 2×2 complex matrices with unit determinant. More precisely, there exists a homomorphism between $SL(2,\mathbb{C})$ and $SO(1,3)$, which means that for any matrix $A \in SL(2,\mathbb{C})$ there exists an associated Lorentz matrix Λ, and that

$$\Lambda(A)\,\Lambda(B) = \Lambda(AB) \ , \tag{2.11}$$

where A and B are $SL(2,\mathbb{C})$ matrices. This can be proved as follows. Lorentz transformations act on four-vectors as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \ , \tag{2.12}$$

where the matrices Λ's are a representation of the generators $M_{\mu\nu}$ defined above. Let us introduce 2×2 matrices σ_μ where σ_0 is the identity matrix and σ_i are the Pauli matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{2.13}$$

Let us also define the matrices with upper indices, σ^μ, as

$$\sigma^\mu = (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i) \ . \tag{2.14}$$

The matrices σ_μ are a complete set, in the sense that any 2×2 complex matrix can be written as a linear combination of them. For every four-dimensional vector x^μ let us construct the 2×2 complex matrix

$$\rho : x^\mu \to x^\mu \sigma_\mu = X \ . \tag{2.15}$$

The matrix X is hermitian, since the Pauli matrices are hermitian, and has determinant equal to $x^\mu x_\mu$, which is a Lorentz invariant quantity. Therefore, ρ is a map from Minkowski space to H, the space of 2×2 hermitian complex matrices

$$M_4 \underset{\rho}{\longrightarrow} H \ . \tag{2.16}$$

Let us now act on X with a $SL(2, \mathbb{C})$ transformation A

$$A : X \to AXA^\dagger = X' \ . \tag{2.17}$$

This transformation preserves the determinant since $\det A = 1$ and also preserves the hermicity of X since

$$X'^\dagger = (AXA^\dagger)^\dagger = AX^\dagger A^\dagger = AXA^\dagger = X' \ . \tag{2.18}$$

Therefore A is a map between H and itself

$$H \underset{A}{\longrightarrow} H \ . \tag{2.19}$$

We finally apply the inverse map ρ^{-1} to X' and get a four-vector x'^μ. The inverse map is defined as

$$\rho^{-1} = \frac{1}{2} \operatorname{Tr}[\bullet \ \bar{\sigma}^\mu] \tag{2.20}$$

(where, as we will later see more rigorously, as a complex 2×2 matrix $\bar{\sigma}^\mu$ is the same as σ_μ). Indeed

$$\rho^{-1} X = \frac{1}{2} \operatorname{Tr}[X \bar{\sigma}^\mu] = \frac{1}{2} \operatorname{Tr}[x_\nu \sigma^\nu \bar{\sigma}^\mu] = \frac{1}{2} \operatorname{Tr}[\sigma^\nu \bar{\sigma}^\mu] x_\nu = \frac{1}{2} \, 2 \, \eta^{\mu\nu} x_\nu = x^\mu \ . \tag{2.21}$$

Assembling everything together we then get a map from Minkowski space into itself via the following chain

$$\begin{array}{ccccccc} M_4 & \underset{\rho}{\longrightarrow} & H & \underset{A}{\longrightarrow} & H & \underset{\rho^{-1}}{\longrightarrow} & M_4 \\ x_\nu & \underset{\rho}{\longrightarrow} & x_\nu \sigma^\nu & \underset{A}{\longrightarrow} & A x_\nu \sigma^\nu A^\dagger & \underset{\rho^{-1}}{\longrightarrow} & \frac{1}{2} \operatorname{Tr}[A x_\nu \sigma^\nu A^\dagger \bar{\sigma}^\mu] = x'^\mu \end{array} \tag{2.22}$$

This is nothing but a Lorentz transformation obtained by the $SL(2,\mathbb{C})$ transformation A as

$$\Lambda^{\mu}{}_{\nu}(A) = \frac{1}{2}\mathrm{Tr}[\bar{\sigma}^{\mu} A\, \sigma_{\nu} A^{\dagger}] \ . \tag{2.23}$$

It is now a trivial exercise, provided eq. (2.23), to prove the homomorphism (2.11).

Notice that the relation (2.23) can in principle be inverted, in the sense that for a given Λ one can find a corresponding $A \in SL(2,\mathbb{C})$. However, the relation is not an isomorphism, since it is double valued. The isomorphism holds between the Lorentz group and $SL(2,\mathbb{C})/\mathbb{Z}_2$ (in other words $SL(2,\mathbb{C})$ is a double cover of the Lorentz group). This can be seen as follows. Consider the 2×2 matrix

$$M(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{2.24}$$

which corresponds to a Lorentz transformation producing a rotation by an angle θ about the z-axis. Taking $\theta = 2\pi$ which corresponds to the identity in the Lorentz group, one gets $M = -\mathbf{1}$ which is a non-trivial element of $SL(2,\mathbb{C})$. It then follows that the elements of $SL(2,\mathbb{C})$ are identified two-by-two under a $\mathbb{Z}_2$ transformation in the Lorentz group. Note that this $\mathbb{Z}_2$ identification holds also in Euclidean space: at the level of groups $SU(2) \times SU(2) = Spin(4)$, where $Spin(4)$ is a double cover of $SO(4)$ as a group (it has an extra $\mathbb{Z}_2$).

The Poincaré group is the Lorentz group augmented by the space-time translation generators P_μ. In terms of the generators $P_\mu, M_{\mu\nu}$ the Poincaré algebra reads

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \\ [M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu \ . \end{aligned} \tag{2.25}$$

2.2 Spinors and representations of the Lorentz group

We are now ready to discuss representations of the Lorentz group. Thanks to the isomorphism (2.8) they can be easily organized in terms of those of $SU(2)$ which can be labeled by the spins. In this respect, let us introduce two-component spinors as the objects carrying the basic representations of $SL(2,\mathbb{C})$. There exist two such representations. A spinor transforming in the self-representation $\mathcal{M}$ is a two complex component object

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.26}$$

where ψ_1 and ψ_2 are complex Grassmann numbers, which transform under a matrix $\mathcal{M} \in SL(2,\mathbb{C})$ as

$$\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha^{\ \beta}\, \psi_\beta \qquad \alpha, \beta = 1, 2 \ . \tag{2.27}$$

The complex conjugate representation is defined from $\mathcal{M}^*$, where $\mathcal{M}^*$ means complex conjugation, as

$$\bar{\psi}_{\dot{\alpha}} \to \bar{\psi}'_{\dot{\alpha}} = \mathcal{M}^*{}_{\dot{\alpha}}^{\ \dot{\beta}}\, \bar{\psi}_{\dot{\beta}} \qquad \dot{\alpha}, \dot{\beta} = 1, 2 \ . \tag{2.28}$$

These two representations are *not* equivalent, that is it does not exist a matrix C such that $\mathcal{M} = C\mathcal{M}^* C^{-1}$.

There are, however, other representations which are equivalent to the former. Let us first introduce the invariant tensor of $SU(2)$, $\epsilon_{\alpha\beta}$, and similarly for the other $SU(2)$, $\epsilon_{\dot{\alpha}\dot{\beta}}$, which one uses to raise and lower spinorial indices as well as to construct scalars and higher spin representations by spinor contractions

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{2.29}$$

We can then define

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad , \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \quad , \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \quad , \quad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \ . \tag{2.30}$$

The convention here is that adjacent indices are always contracted putting the epsilon tensor on the left.

Using above conventions one can easily prove that $\psi'^\alpha = (\mathcal{M}^{-1T})^\alpha{}_\beta \psi^\beta$. Since $\mathcal{M}^{-1T} \simeq \mathcal{M}$ (the matrix C being in fact the epsilon tensor $\epsilon_{\alpha\beta}$), it follows that the fundamental (ψ_α) and anti-fundamental (ψ^α) representations of $SU(2)$ are equivalent (note that this does not hold for $SU(N)$ with $N > 2$, for which the fundamental and anti-fundamental representations are not equivalent). A similar story holds for $\bar{\psi}^{\dot{\alpha}}$ which transforms in the representation $\mathcal{M}^{*-1T}$, that is $\bar{\psi}'^{\dot{\alpha}} = (\mathcal{M}^{*-1T})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}$, which is equivalent to the complex conjugate representation $\bar{\psi}_{\dot{\alpha}}$ (the matrix C connecting $\mathcal{M}^{*-1T}$ and $\mathcal{M}^*$ is now the epsilon tensor $\epsilon_{\dot{\alpha}\dot{\beta}}$). From our conventions one can easily see that the complex conjugate matrix $(\mathcal{M}_\alpha^{\ \beta})^*$ (that is, the matrix obtained from $\mathcal{M}_\alpha^{\ \beta}$ by taking the complex conjugate of each entry), once expressed in terms of dotted indices, is not $\mathcal{M}^*{}_{\dot{\alpha}}^{\ \dot{\beta}}$, but rather $(\mathcal{M}_\alpha^{\ \beta})^* = (\mathcal{M}^{*-1T})^{\dot{\alpha}}{}_{\dot{\beta}}$. Finally, lower undotted indices are row indices, while upper ones are column indices. Dotted indices follow instead the opposite convention. This implies that $(\psi_\alpha)^* = \bar{\psi}^{\dot{\alpha}}$, while

under hermitian conjugation (which also includes transposition), we have, e.g. $\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^\dagger$, as operator identity.

Due to the homomorphism between $SL(2,\mathbb{C})$ and $SO(1,3)$, it turns out that the two spinor representations ψ_α and $\bar{\psi}^{\dot{\alpha}}$ are representations of the Lorentz group, and, because of the isomorphism (2.8), they can be labeled in terms of $SU(2)$ representations as

$$\psi_\alpha \equiv \left(\frac{1}{2}, 0\right) \tag{2.31}$$

$$\bar{\psi}^{\dot{\alpha}} \equiv \left(0, \frac{1}{2}\right) . \tag{2.32}$$

To understand the identifications above just note that $\sum_i (J_i^+)^2$ and $\sum_i (J_i^-)^2$ are Casimir of the two $SU(2)$ algebras (2.5) with eigenvalues $n(n+1)$ and $m(m+1)$ with $n, m = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ being the eigenvalues of J_3^+ and J_3^-, respectively. Hence we can indeed label the representations of the Lorentz group by pairs (n,m) and since $J_3 = J_3^+ + J_3^-$ we can identify the spin of the representation as $n+m$, its dimension being $(2n+1)(2m+1)$. The two spinor representations (2.31) and (2.32) are just the basic such representations.

Recalling that Grassmann variables anticommute (that is $\psi_1\chi_2 = -\chi_2\psi_1$, $\psi_1\bar{\chi}_{\dot{2}} = -\bar{\chi}_{\dot{2}}\psi_1$, etc...) we can now define a scalar product for spinors as

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \tag{2.33}$$

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\bar{\chi}^{\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} . \tag{2.34}$$

Under hermitian conjugation we have

$$(\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = \chi_\alpha{}^\dagger\psi^{\alpha\dagger} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} . \tag{2.35}$$

In our conventions, undotted indices are contracted from upper left to lower right while dotted indices from lower left to upper right (this rule does not apply when raising or lowering indices with the epsilon tensor). Recalling eq. (2.17), namely that under $SL(2,\mathbb{C})$ the matrix $X = x^\mu\sigma_\mu$ transforms as $AXA^\dagger$ and that the index structure of A and $A^\dagger$ is $A_\alpha{}^\beta$ and $A^{*\dot{\beta}}{}_{\dot{\alpha}}$, respectively, we see that σ_μ naturally has a dotted and an undotted index and can be contracted with an undotted and a dotted spinor as

$$\psi\sigma^\mu\bar{\chi} \equiv \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} . \tag{2.36}$$

Similarly one can define $\bar{\sigma}^\mu$ as

$$\bar{\sigma}^{\mu\ \dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} = (\sigma_0, \sigma_i)\ , \tag{2.37}$$

and define the product of $\bar{\sigma}^\mu$ with a dotted and an undotted spinor as

$$\bar{\psi}\bar{\sigma}^\mu\chi \equiv \bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\ \dot{\alpha}\beta}\chi_\beta\ . \tag{2.38}$$

A number of useful identities one can prove are

$$\begin{aligned}
\psi^\alpha\psi^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi\ \ ,\ \ (\theta\phi)\,(\theta\psi) = -\frac{1}{2}\,(\phi\psi)\,(\theta\theta) \\
\chi\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\chi\ \ ,\ \ \chi\sigma^\mu\bar{\sigma}^\nu\psi = \psi\sigma^\nu\bar{\sigma}^\mu\chi \\
(\chi\sigma^\mu\bar{\psi})^\dagger &= \psi\sigma^\mu\bar{\chi}\ \ ,\ \ (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger = \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi} \\
(\theta\psi)\left(\theta\sigma^\mu\bar{\phi}\right) &= -\frac{1}{2}\,(\theta\theta)\left(\psi\sigma^\mu\bar{\phi}\right)\ \ ,\ \ \left(\bar{\theta}\bar{\psi}\right)\left(\bar{\theta}\bar{\sigma}^\mu\phi\right) = -\frac{1}{2}\left(\bar{\theta}\bar{\theta}\right)\left(\bar{\psi}\bar{\sigma}^\mu\phi\right) \\
(\phi\psi)\cdot\bar{\chi}_{\dot{\alpha}} &= \frac{1}{2}\,(\phi\sigma^\mu\bar{\chi})\,(\psi\sigma_\mu)_{\dot{\alpha}}\ .
\end{aligned} \tag{2.39}$$

As some people might be more familiar with four component spinor notation, let us close this section by briefly mentioning the connection with Dirac spinors. In the Weyl representation Dirac matrices read

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\ \ ,\ \ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{2.40}$$

and a Dirac spinor is

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}\ \ \text{implying}\ \ r(\psi) = \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\ . \tag{2.41}$$

This shows that a Dirac spinor carries a reducible representation of the Lorentz algebra. Using this four component spinor notation one sees that

$$\begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}\quad \text{and}\quad \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.42}$$

are Weyl (chiral) spinors, with chirality $+1$ and -1, respectively. One can easily show that a Majorana spinor ($\psi^{\mathcal{C}} = \psi$) is a Dirac spinor such that $\chi_\alpha = \psi_\alpha$. To prove this, just recall that in four component notation the conjugate Dirac spinor

is defined as $\bar{\psi} = \psi^\dagger \gamma_0$ and the charge conjugate is $\psi^{\mathcal{C}} = \mathcal{C}\bar{\psi}^T$ with the charge conjugate matrix in the Weyl representation being

$$\mathcal{C} = \begin{pmatrix} -\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} . \tag{2.43}$$

Finally, Lorentz generators are

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu} \quad , \quad \gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \tag{2.44}$$

while the 2-index Pauli matrices are defined as

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}(\bar{\sigma}^\nu)^{\dot{\gamma}\beta} - (\mu \leftrightarrow \nu)\right) \quad , \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left((\bar{\sigma}^\mu)^{\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - (\mu \leftrightarrow \nu)\right) \tag{2.45}$$

From the last equations one then sees that $i\sigma^{\mu\nu}$ acts as a Lorentz generator on ψ_α while $i\bar{\sigma}^{\mu\nu}$ acts as a Lorentz generator on $\bar{\psi}^{\dot{\alpha}}$.

2.3 The supersymmetry algebra

As we have already mentioned, a no-go theorem provided by Coleman and Mandula implies that, under certain assumptions (locality, causality, positivity of energy, finiteness of number of particles, etc...), the only possible symmetries of the S-matrix are, besides C, P, T,

- Poincaré symmetries, with generators $P_\mu, M_{\mu\nu}$
- Some *internal* symmetry group with generators B_l which are Lorentz scalars, and which are typically related to some conserved quantum number like electric charge, isopin, etc...

The full symmetry algebra hence reads

$$[P_\mu, P_\nu] = 0 \tag{2.46}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \tag{2.47}$$

$$[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu \tag{2.48}$$

$$[B_l, B_m] = if_{lm}{}^n B_n \tag{2.49}$$

$$[P_\mu, B_l] = 0 \tag{2.50}$$

$$[M_{\mu\nu}, B_l] = 0 \ , \tag{2.51}$$

where ${f_{lm}}^n$ are structure constants and the last two commutation relations simply say that the full algebra is the direct sum of the Poincaré algebra and the algebra G spanned by the scalar bosonic generators B_l, that is

$$ISO(1,3) \times G \ . \tag{2.52}$$

at the level of groups (a nice proof of the Coleman-Mandula theorem can be found in Weinberg's book, Vol. III, chapter 24.B).

The Coleman-Mandula theorem can be evaded by weakening one (or more) of its assumptions. For instance, the theorem assumes that the symmetry algebra involves only commutators. Haag, Lopuszanski and Sohnius generalized the notion of Lie algebra to include algebraic systems involving, in addition to commutators, also *anticommutators*. This extended Lie algebra goes under the name of Graded Lie algebra. Allowing for a graded Lie algebra weakens the Coleman-Mandula theorem enough to allow for supersymmetry, which is nothing but a specific graded Lie algebra.

Let us first define what a graded Lie algebra is. Recall that a Lie algebra is a vector space (over some field, say $\mathbb{R}$ or $\mathbb{C}$) which enjoys a composition rule called product

$$[\quad , \quad] : L \times L \to L \tag{2.53}$$

with the following properties

$$\begin{aligned}
&[v_1, v_2] \in L \\
&[v_1, (v_2 + v_3)] = [v_1, v_2] + [v_1, v_3] \\
&[v_1, v_2] = -[v_2, v_1] \\
&[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0 \ ,
\end{aligned}$$

where v_i are elements of the algebra. A *graded* Lie algebra of grade n is a vector space

$$L = \oplus_{i=0}^{i=n} \ L_i \tag{2.54}$$

where L_i are all vector spaces, and the product

$$[\quad , \quad \} : L \times L \to L \tag{2.55}$$

has the following properties

$$
\begin{aligned}
&[L_i, L_j\} \in L_{i+j} \mod n+1 \\
&[L_i, L_j\} = -(-1)^{ij}[L_j, L_i\} \\
&[L_i, [L_j, L_k\}\}(-1)^{ik} + [L_j, [L_k, L_i\}\}(-1)^{ij} + [L_k, [L_i, L_j\}\}(-1)^{jk} = 0 \ .
\end{aligned}
$$

First notice that from the first such properties it follows that L_0 is a Lie algebra while all other L_i's with $i \neq 0$ are not. The second property is called supersymmetrization while the third one is nothing but the generalization to a graded algebra of the well known Jacobi identity any algebra satisfies.

The supersymmetry algebra is a graded Lie algebra of grade one, namely

$$
L = L_0 \oplus L_1 \ , \tag{2.56}
$$

where L_0 is the Poincaré algebra and $L_1 = (Q^I_\alpha \ , \ \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, N$ where $Q^I_\alpha \ , \ \bar{Q}^I_{\dot{\alpha}}$ is a set of $N + N = 2N$ anticommuting fermionic generators transforming in the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group, respectively. Haag, Lopuszanski and Sohnius proved that this is the only possible consistent extension of the Poincaré algebra, given the other (very physical) assumptions one would not like to relax of the Coleman-Mandula theorem. For instance, generators with spin higher than one, like those transforming in the $(\frac{1}{2}, 1)$ representation of the Lorentz group, cannot be there.

The generators of L_1 are spinors and hence they transform non-trivially under the Lorentz group. Therefore, supersymmetry is not an internal symmetry. Rather it is an extension of Poincaré space-time symmetries. Moreover, acting on bosons, the supersymmetry generators transform them into fermions (and viceversa). Hence, this symmetry naturally mixes radiation with matter.

The supersymmetry algebra, besides commutators (2.46)-(2.51), contains the

following (anti)commutators

$$\begin{aligned}
\left[P_\mu, Q^I_\alpha\right] &= 0 && (2.57)\\
\left[P_\mu, \bar{Q}^I_{\dot\alpha}\right] &= 0 && (2.58)\\
\left[M_{\mu\nu}, Q^I_\alpha\right] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta && (2.59)\\
\left[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}\right] &= i(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta}\bar{Q}^{I\dot\beta} && (2.60)\\
\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ} && (2.61)\\
\left\{Q^I_\alpha, Q^J_\beta\right\} &= \epsilon_{\alpha\beta}Z^{IJ} \quad , \quad Z^{IJ} = -Z^{JI} && (2.62)\\
\left\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\right\} &= \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^* && (2.63)
\end{aligned}$$

Several comments are in order at this point.

- Eqs. (2.59) and (2.60) follow from the fact that Q^I_α and $\bar{Q}^I_{\dot\alpha}$ are spinors of the Lorentz group, recall eq.(2.45). From these same equations one also sees that, since $M_{12} = J_3$, we have

$$\left[J_3, Q^I_1\right] = \frac{1}{2}Q^I_1 \quad , \quad \left[J_3, Q^I_2\right] = -\frac{1}{2}Q^I_2 \, . \tag{2.64}$$

  Taking the hermitian conjugate of the above relations we get

$$\left[J_3, \bar{Q}^I_{\dot1}\right] = -\frac{1}{2}\bar{Q}^I_{\dot1} \quad , \quad \left[J_3, \bar{Q}^I_{\dot2}\right] = \frac{1}{2}\bar{Q}^I_{\dot2} \tag{2.65}$$

  and so we see that Q^I_1 and $\bar{Q}^I_{\dot2}$ rise the z-component of the spin by half unit while Q^I_2 and $\bar{Q}^I_{\dot1}$ lower it by half unit.

- Eq.(2.61) has a very important implication. First notice that given the transformation properties of Q^I_α and $\bar{Q}^J_{\dot\beta}$ under Lorentz transformations, their anticommutator should be symmetric under $I \leftrightarrow J$ and should transform as

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) \, . \tag{2.66}$$

  The obvious such candidate is P_μ which is the only generator in the algebra with such transformation properties (the δ^{IJ} in eq. (2.61) is achieved by diagonalizing an arbitrary symmetric matrix and rescaling the Q's and the $\bar{Q}$'s). Hence, the commutator of two supersymmetry transformations is a translation. In theories with *local* supersymmetry (*i.e.* where the spinorial

infinitesimal parameter of the supersymmetry transformation depends on x^μ), the commutator is an infinitesimal translation whose parameter depends on x^μ. This is nothing but a theory invariant under general coordinate transformation, namely a theory of gravity! The upshot is that theories with local supersymmetry automatically incorporate gravity. Such theories are called *supergravity* theories, SUGRA for short.

- Eqs.(2.57) and (2.58) are not at all obvious. Compatibility with Lorentz symmetry would imply the right hand side of eq. (2.57) to transform as

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right)=\left(0,\frac{1}{2}\right)\oplus\left(1,\frac{1}{2}\right)\ , \tag{2.67}$$

and similarly for eq. (2.58). The second term on the right hand side cannot be there, due to the theorem of Haag, Lopuszanski and Sohnius which says that the only allowed fermionic generators in the algebra are supersymmetry generators, which are spin $\frac{1}{2}$. In other words, there cannot be a consistent extension of the Poincaré algebra including generators transforming in the $(1,\frac{1}{2})$ under the Lorentz group. Still, group theory arguments by themselves do not justify eqs.(2.57) and (2.58) but rather something like

$$\left[P_\mu,Q^I_\alpha\right] = C^I{}_J\sigma_{\mu\,\alpha\dot\beta}\,\bar Q^{J\dot\beta} \tag{2.68}$$

$$\left[P_\mu,\bar Q^I_{\dot\alpha}\right] = (C^I{}_J)^*\bar\sigma_{\mu\,\dot\alpha\beta}\,Q^{J\beta}\ . \tag{2.69}$$

where $C^I{}_J$ is an undetermined matrix. We want to prove that this matrix vanishes. Let us first consider the generalized Jacobi identity which the supersymmetry algebra should satisfy and let us apply it to the (Q,P,P) system. We get

$$\begin{aligned}\left[\left[Q^I_\alpha,P_\mu\right],P_\nu\right]+\left[\left[P_\mu,P_\nu\right],Q^I_\alpha\right]+\left[\left[P_\nu,Q^I_\alpha\right],P_\mu\right] &= \\ -C^I{}_J\sigma_{\mu\,\alpha\dot\beta}\left[\bar Q^{J\dot\beta},P_\nu\right]+C^I{}_J\sigma_{\nu\,\alpha\dot\beta}\left[\bar Q^{J\dot\beta},P_\mu\right] &= \\ C^I{}_JC^J{}_K{}^*\sigma_{\mu\,\alpha\dot\beta}\,\bar\sigma_\nu{}^{\dot\beta}{}_\gamma Q^{K\gamma}-C^I{}_JC^J{}_K{}^*\sigma_{\nu\,\alpha\dot\beta}\,\bar\sigma_\mu{}^{\dot\beta}{}_\gamma Q^{K\gamma} &= \\ 4\,(C\;C^*)^I{}_K\;(\sigma_{\mu\nu})_{\alpha\gamma}Q^{K\gamma} &= 0\ .\end{aligned}$$

This implies that

$$C\;C^*=0\ , \tag{2.70}$$

as a matrix equation. Note that this is not enough to conclude, as we would, that $C=0$. For that, we also need to show, in addition, that C is symmetric. To this aim we have to consider other equations, as detailed below.

- Let us now consider eqs. (2.62) and (2.63). As for the first, from Lorentz representation theory we would expect

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right)=(0,0)\oplus(1,0)\ , \tag{2.71}$$

which explicitly means

$$\left\{Q^I_\alpha,Q^J_\beta\right\}=\epsilon_{\alpha\beta}Z^{IJ}+\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma M_{\mu\nu}Y^{IJ}\ . \tag{2.72}$$

The Z^{IJ}, being Lorentz scalars, should be some linear combination of the internal symmetry generators B_l and, given the antisymmetric properties of the epsilon tensor under $\alpha\leftrightarrow\beta$, should be anti-symmetric under $I\leftrightarrow J$. On the contrary, given that $\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma$ is symmetric in $\alpha\leftrightarrow\beta$, the matrix Y^{IJ} should be symmetric under $I\leftrightarrow J$. Let us now consider the generalized Jacobi identity between (Q,Q,P), which can be written as

$$[\{Q_{I\alpha},Q_{J\beta}\},P_\mu]=\{Q_{I\alpha},[Q_{J\beta},P_\mu]\}+\{Q_{J\beta},[Q_{I\alpha},P_\mu]\}\ . \tag{2.73}$$

If one multiplies it by $\epsilon^{\alpha\beta}$, only the anti-symmetric part under $\alpha\leftrightarrow\beta$ of the left hand side survives, which is 0, since the matrix Z_{IJ}, see eq. (2.72), commutes with P_μ. So we get

$$\begin{aligned}
0&=\epsilon^{\alpha\beta}\{Q_{I\alpha},[Q_{J\beta},P_\mu]\}+\epsilon^{\alpha\beta}\{Q_{J\beta},[Q_{I\alpha},P_\mu]\}\\
&=\epsilon^{\alpha\beta}C_I{}^K\sigma_{\mu\,\beta\dot\beta}\{Q_{J\alpha},\bar Q^{\dot\beta}_K\}-\epsilon^{\alpha\beta}C_J{}^K\sigma_{\mu\,\beta\dot\beta}\{Q_{I\alpha},\bar Q^{\dot\beta}_K\}\sim(C_{IJ}-C_{JI})\,\bar\sigma_\mu^{\dot\gamma\alpha}\sigma^\nu_{\alpha\dot\gamma}P_\nu\\
&=2\,(C_{IJ}-C_{JI})P_\mu\ ,
\end{aligned}$$

which implies that the matrix C is symmetric. So the previously found equation $C\,C^*=0$ can be promoted to $CC^\dagger=0$, which implies $C=0$ and hence eq. (2.57). A similar rationale leads to eq. (2.58).

Let us now come back to eq. (2.62), which we have not yet proven. To do so, we should plug eq. (2.57) into the Jacobi identity (2.73), getting

$$[\{Q_{I\alpha},Q_{J\beta}\},P_\mu]=0\ , \tag{2.74}$$

which implies, by (2.72), that the matrix Y^{IJ} vanishes because P_μ does not commute with $M_{\mu\nu}$. This finally proves eq. (2.62). Seemingly, one can prove eq. (2.63), which is just the hermitian conjugate of (2.62).

What about the commutation relations between supersymmetry generators and internal symmetry generators, if any? In general, the Q's will carry a representation of the internal symmetry group G. So one expects something like

$$\begin{aligned} \left[Q^I_\alpha, B_l\right] &= (b_l)^I{}_J Q^J_\alpha & (2.75) \\ \left[\bar{Q}_{I\dot{\alpha}}, B_l\right] &= -\bar{Q}_{J\dot{\alpha}}(b_l)^J{}_I \ . & (2.76) \end{aligned}$$

The second commutation relation comes from the first under hermitian conjugation, recalling that the b_l are hermitian, because so are the generators B_l. The largest possible internal symmetry group which can act non trivially on the Q's is thus $U(N)$, and this is called the R-symmetry group (recall that the relation between a Lie algebra with generators S and the corresponding Lie group with elements U is $U = e^{iS}$; hence hermitian generators, $S^\dagger = S$, correspond to unitary groups, $U^\dagger = U^{-1}$). In fact, in presence of non-vanishing central charges one can prove that the R-symmetry group reduces to $USp(N)$, the compact version of the symplectic group $Sp(N)$, $USp(N) \cong U(N) \cap Sp(N)$.

As already noticed, the operators Z_{IJ}, being Lorentz scalars, should be some linear combination of the internal symmetry group generators B_l of the compact Lie algebra G, say

$$Z^{IJ} = a^{l|IJ} B_l \ . \qquad (2.77)$$

Using the above equation together with eqs.(2.62), (2.75) and (2.76) we get

$$\begin{aligned} \left[Z^{IJ}, B_l\right] &= (b_l)^I_K Z^{KJ} + (b_l)^J_K Z^{IK} \\ \left[Z^{IJ}, Z^{KL}\right] &= a^{l|KL}(b_l)^I_M Z^{MJ} + a^{l|KL}(b_l)^J_M Z^{IM} \end{aligned}$$

This implies that the Z's span an invariant subalgebra of G. Playing with Jacobi identities one can see that the Z are in fact *central charges*, that is they commute with the whole supersymmetry algebra, and within themselves. In other words, they span an invariant *abelian* subalgebra of G and commute with all other generators

$$\begin{aligned} &\left[Z^{IJ}, B_l\right] = 0 \quad \left[Z^{IJ}, Z^{KL}\right] = 0 \\ &\left[Z^{IJ}, P_\mu\right] = 0 \quad \left[Z^{IJ}, M_{\mu\nu}\right] = 0 \\ &\left[Z^{IJ}, Q^K_\alpha\right] = 0 \quad \left[Z^{IJ}, \bar{Q}^K_{\dot{\alpha}}\right] = 0 \ . \end{aligned}$$

Notice that this does not at all imply they are uneffective. Indeed, central charges are not numbers but quantum operators and their value may vary from state to

state. For a supersymmetric vacuum state, which is annihilated by all supersymmetry generators, they are of course trivially realized, recall eqs. (2.62) and (2.63). However, they do not need to vanish in general. For instance, as we will see in a subsequent lecture, massive representations are very different if Z^{IJ} vanishes or if it is non-trivially realized on the representation.

Let us end this section with a few more comments. First, if $N = 1$ we have two supersymmetry generators, which correspond to one Majorana spinor, in four component notation. In this case we speak of unextended supersymmetry (and we do not have central charges whatsoever). For $N > 1$ we have extended supersymmetry (and we can have a central extension of the supersymmetry algebra, too). From an algebraic point of view there is no limit to N; but, as we will later see, increasing N the theory must contain particles of increasing spin. In particular we have

- $N \leq 4$ for theories without gravity (spin ≤ 1)
- $N \leq 8$ for theories with gravity (spin ≤ 2)

For $N = 1$ the R-symmetry group is just $U(1)$ (one can see it from the Jacobi identity between (Q, B, B) which implies that the $f_{lm}{}^{n}$ are trivially realized on the supersymmetry generators). In this case the hermitian matrices b_l are just real numbers and by rescaling the generators B_l one gets

$$[R, Q_\alpha] = -Q_\alpha \quad , \quad [R, \bar{Q}_{\dot{\alpha}}] = +\bar{Q}_{\dot{\alpha}} \ . \tag{2.78}$$

This implies that supersymmetric partners (which are related by the action of the Q's) have different R-charge. In particular, given eqs. (2.78), if a particle has $R = 0$ then its superpartner has $R = \pm 1$. An important physical consequence of this property is that in a theory where R-symmetry is preserved, the lightest supersymmetric particle (LSP) is stable.

Let us finally comment on the relation between two and four component spinor notation, when it comes to supersymmetry. In four component notation the $2N$ supersymmetry generators Q^I_α , $\bar{Q}^I_{\dot{\alpha}}$ constitute a set of N Majorana spinors

$$Q^I = \begin{pmatrix} Q^I_\alpha \\ \bar{Q}^{I\dot{\alpha}} \end{pmatrix} \qquad \bar{Q}^I = \begin{pmatrix} Q^{I\alpha} & \bar{Q}^I_{\dot{\alpha}} \end{pmatrix} \tag{2.79}$$

and the supersymmetry algebra reads

$$\begin{aligned} &\{Q^I, \bar{Q}^J\} = 2\delta^{IJ}\gamma^\mu P_\mu - i\,\mathbb{I}\,\mathrm{Im}Z^{IJ} - \gamma_5\,\mathrm{Re}Z^{IJ} \\ &[Q^I, P_\mu] = 0 \quad [Q^I, M_{\mu\nu}] = \frac{i}{2}\gamma_{\mu\nu}Q^I \quad [Q^I, R] = i\gamma_5 Q^I \ . \end{aligned} \tag{2.80}$$

Depending on what one needs to do, one notation can be more useful than the other. In the following we will mainly stick to the two component notation, though.

2.4 Exercises

1. Prove the following spinor identities

$$\begin{aligned}
\psi^\alpha\psi^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi \quad , \quad (\theta\phi)\,(\theta\psi) = -\frac{1}{2}\,(\phi\psi)\,(\theta\theta) \\
\chi\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\chi \quad , \quad \chi\sigma^\mu\bar{\sigma}^\nu\psi = \psi\sigma^\nu\bar{\sigma}^\mu\chi \\
(\chi\sigma^\mu\bar{\psi})^\dagger &= \psi\sigma^\mu\bar{\chi} \quad , \quad (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger = \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi} \\
(\theta\psi)\left(\theta\sigma^\mu\bar{\phi}\right) &= -\frac{1}{2}\,(\theta\theta)\left(\psi\sigma^\mu\bar{\phi}\right) \quad , \quad \left(\bar{\theta}\bar{\psi}\right)\left(\bar{\theta}\bar{\sigma}^\mu\phi\right) = -\frac{1}{2}\left(\bar{\theta}\bar{\theta}\right)\left(\bar{\psi}\bar{\sigma}^\mu\phi\right) \\
(\phi\psi)\cdot\bar{\chi}_{\dot{\alpha}} &= \frac{1}{2}\,(\phi\sigma^\mu\bar{\chi})\,(\psi\sigma_\mu)_{\dot{\alpha}} \ .
\end{aligned}$$

2. The operators Z^{IJ} are linear combinations of the internal symmetries generators B_l. Hence, they commute with P_μ and $M_{\mu\nu}$. Prove that Z_{IJ} are in fact central charges of the supersymmetry algebra, namely that it also follows that

$$\left[Z^{IJ}, B_l\right] = 0 \ , \ \left[Z^{IJ}, Z^{KL}\right] = 0 \ , \ \left[Z^{IJ}, Q^K_\alpha\right] = 0 \ , \ \left[Z^{IJ}, \bar{Q}^K_{\dot{\alpha}}\right] = 0 \ .$$

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3 Representations of the supersymmetry algebra

In this lecture we will discuss representations of the supersymmetry algebra. Let us first briefly recall how things go for the Poincaré algebra. The Poincaré algebra (2.25) has two Casimir (*i.e.* two operators which commute with all generators)

$$P^2 = P_\mu P^\mu \quad \text{and} \quad W^2 = W_\mu W^\mu \ , \tag{3.1}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the so-called Pauli-Lubanski vector. Casimir operators are useful to classify irreducible representations of a group. In the case of the Poincaré group such representations are nothing but what we usually call *particles*. Let us see how this is realized for massive and massless particles, respectively.

Let us first consider a massive particle with mass m and go to the rest frame, $P_\mu = (m, 0, 0, 0)$. In this frame it is easy to see that the two Casimir reduce to $P^2 = m^2$ and $W^2 = -m^2 j(j+1)$ where j is the spin. The second equality can be proven by noticing that $W_\mu P^\mu = 0$ which implies that in the rest frame $W_0 = 0$. Therefore in the rest frame $W_\mu = (0, \frac{1}{2}\,\epsilon_{i0jk} m\, M^{jk})$ from which one immediately gets $W^2 = -m^2 \vec{J}^2$. So we see that massive particles are distinguished by their mass and their spin.

Let us now consider massless particles. In the rest frame $P_\mu = (E, 0, 0, E)$. In this case we have that $P^2 = 0$ and $W^2 = 0$, and $W^\mu = M_{12} P^\mu$. In other words, the two operators are proportional for a massless particle, the constant of proportionality being $M_{12} = \pm j$, the helicity. For these representations the spin is then fixed and the different states are distinguished by their energy and by the sign of the helicity (e.g. the photon is a massless particle with two helicity states, ± 1).

Now, as a particle is an irreducible representation of the Poincaré algebra, we call superparticle an irreducible representation of the supersymmetry algebra. Since the Poincaré algebra is a subalgebra of the supersymmetry algebra, it follows that any irreducible representation of the supersymmetry algebra is a representation of the Poincaré algebra, which in general will be reducible. This means that a superparticle corresponds to a collection of particles, the latter being related by the action of the supersymmetry generators Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ and having spins differing by units of half. Being a multiplet of different particles, a superparticle is often called *supermultiplet*.

Before discussing in detail specific representations of the supersymmetry algebra, let us list three generic properties any such representation enjoys, all of them having very important physical implications.

1. As compared to the Poincaré algebra, in the supersymmetry algebra P^2 is still a Casimir, but W^2 is not anymore (this follows from the fact that $M_{\mu\nu}$ does not commute with the supersymmetry generators). Therefore, particles belonging to the same supermultiplet have the same mass and different spin, since the latter is not a conserved quantum number of the representation. The mass degeneracy between bosons and fermions is something we do not observe in known particle spectra; this implies that supersymmetry, if at all realized, must be broken in Nature.

2. In a supersymmetric theory the energy of *any* state is always ≥ 0. Consider an arbitrary state $|\phi\rangle$. Using the supersymmetry algebra, we easily get

$$\begin{aligned} \langle\phi| \left\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right\} |\phi\rangle &= 2\sigma^\mu_{\alpha\dot{\alpha}} \langle\phi|P_\mu|\phi\rangle \ \delta^{II} \\ \left(\bar{Q}^I_{\dot{\alpha}} = (Q^I_\alpha)^\dagger\right) &= \langle\phi| \left(Q^I_\alpha (Q^I_\alpha)^\dagger + (Q^I_\alpha)^\dagger Q^I_\alpha\right) |\phi\rangle \\ &= ||(Q^I_\alpha)^\dagger|\phi\rangle||^2 + ||Q^I_\alpha|\phi\rangle||^2 \geq 0 \ . \end{aligned}$$

   The last inequality follows from positivity of the Hilbert space. Summing over $\alpha = \dot{\alpha} = 1, 2$ and recalling that Tr $\sigma^\mu = 2\delta^{\mu 0}$ we get

$$4 \ \langle\phi|P_0|\phi\rangle \geq 0 \ , \tag{3.2}$$

   as anticipated.

3. A supermultiplet contains an equal number of bosonic and fermionic d.o.f., $n_B = n_F$. Define a fermion number operator

$$(-1)^{N_F} = \begin{cases} -1 & \text{fermionic state} \\ +1 & \text{bosonic state} \end{cases} \tag{3.3}$$

   N_F can be taken to be twice the spin, $N_F = 2s$. Such an operator, when acting on a bosonic, respectively a fermionic state, gives indeed

$$(-1)^{N_F}|B\rangle = |B\rangle \quad , \quad (-1)^{N_F}|F\rangle = -|F\rangle \ . \tag{3.4}$$

   We want to show that Tr $(-1)^{N_F} = 0$ if the trace is taken over a finite dimensional representation of the supersymmetry algebra. First notice that

$$\left\{Q^I_\alpha, (-1)^{N_F}\right\} = 0 \rightarrow Q^I_\alpha (-1)^{N_F} = -(-1)^{N_F} Q^I_\alpha \ . \tag{3.5}$$

Using this property and the cyclicity of the trace one easily sees that

$$\begin{aligned} 0 &= \mathrm{Tr}\left(-Q^I_\alpha(-1)^{N_F}\bar{Q}^J_{\dot\beta} + (-1)^{N_F}\bar{Q}^J_{\dot\beta}Q^I_\alpha\right) \\ &= \mathrm{Tr}\left((-1)^{N_F}\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\}\right) = 2\sigma^\mu_{\alpha\dot\beta}\mathrm{Tr}\left[(-1)^{N_F}\right]P_\mu\delta^{IJ} \ . \end{aligned}$$

Summing on I, J and choosing any $P_\mu \neq 0$ it follows that $\mathrm{Tr}\ (-1)^{N_F} = 0$, which implies that $n_B = n_F$.

In the following, we discuss (some) representations in detail. Since the mass is a conserved quantity in a supermultiplet, it is meaningful distinguishing between massless and massive representations. Let us start from the former.

3.1 Massless supermultiplets

Let us first assume that all central charges vanish, i.e. $Z^{IJ} = 0$ (we will see later that this is the only relevant case, for massless representations). Notice that in this case it follows from eqs. (2.62) and (2.63) that all Q's and all $\bar{Q}$'s anticommute among themselves. The steps to construct the irreps are as follows:

1. Go to the rest frame where $P_\mu = (E, 0, 0, E)$. In such frame we get

$$\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix} \tag{3.6}$$

Pluggin this into eq. (2.61) we get

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta}\delta^{IJ} \quad \longrightarrow \quad \left\{Q^I_1, \bar{Q}^J_{\dot 1}\right\} = 0 \ . \tag{3.7}$$

Due to the positiveness of the Hilbert space, this implies that both Q^I_1 and $\bar{Q}^I_{\dot 1}$ are trivially realized. Indeed, from the equation above we get

$$0 = \langle\phi|\left\{Q^I_1, \bar{Q}^I_{\dot 1}\right\}|\phi\rangle = ||Q^I_1|\phi\rangle||^2 + ||\bar{Q}^I_{\dot 1}|\phi\rangle||^2 \ , \tag{3.8}$$

whose only solution is $Q^I_1 = \bar{Q}^I_{\dot 1} = 0$. We are then left with just Q^I_2 and $\bar{Q}^I_{\dot 2}$, hence only half of the generators.

2. From the non-trivial generators we can define

$$a_I \equiv \frac{1}{\sqrt{4E}}Q^I_2 \quad , \quad a^\dagger_I \equiv \frac{1}{\sqrt{4E}}\bar{Q}^I_{\dot 2} \ . \tag{3.9}$$

These operators satisfy the anticommutation relations of a set of N creation and N annihilation operators

$$\left\{a_I, a_J^\dagger\right\} = \delta^{IJ} \quad , \quad \{a_I, a_J\} = 0 \quad , \quad \left\{a_I^\dagger, a_J^\dagger\right\} = 0 \ . \tag{3.10}$$

These are the basic tools we need in order to construct irreps of the supersymmetry algebra. Notice that when acting on some state, the operators Q_2^I and $\bar{Q}_{\dot{2}}^I$ (and hence a_I and $a_I^\dagger$) lower respectively raise the helicity of half unit, since

$$\left[M_{12}, Q_2^I\right] = i(\sigma_{12})_2{}^2 Q_2^I = -\frac{1}{2} Q_2^I \quad , \quad \left[M_{12}, \bar{Q}_{\dot{2}}^I\right] = \frac{1}{2}\bar{Q}_{\dot{2}}^I \ , \tag{3.11}$$

and $J_3 = M_{12}$.

3. To construct a representation, one can start by choosing a state annihilated by all a_I's (known as the Clifford vacuum): such state will carry some irrep of the Poincaré algebra. Besides having $m = 0$, it will carry some helicity λ_0, and we call it $|E, \lambda_0\rangle$ ($|\lambda_0\rangle$ for short). For this state

$$a_I|\lambda_0\rangle = 0 \ . \tag{3.12}$$

Note that this state can be either bosonic or fermionic, and should not be confused with the actual vacuum of the theory, which is the state of minimal energy: the Clifford vacuum is a state with quantum numbers (E, λ_0) and which satisfies eq. (3.12).

4. The full representation (aka supermultiplet) is obtained acting on $|\lambda_0\rangle$ with the creation operators $a_I^\dagger$ as follows

$$\begin{aligned} &|\lambda_0\rangle \quad , \quad a_I^\dagger|\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I \quad , \quad a_I^\dagger a_J^\dagger|\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ} \quad , \\ &\ldots \quad , \quad a_1^\dagger a_2^\dagger \ldots a_N^\dagger|\lambda_0\rangle \equiv |\lambda_0 + \frac{N}{2}\rangle \ . \end{aligned}$$

Hence, starting from a Clifford vacuum with helicity λ_0, the state with highest helicity in the representation has helicity $\lambda = \lambda_0 + \frac{N}{2}$. Due to the antisymmetry in $I \leftrightarrow J$, at helicity level $\lambda = \lambda_0 + \frac{k}{2}$ we have

$$\#\text{ of states with helicity } \lambda_0 + \frac{k}{2} = \binom{N}{k} \ , \tag{3.13}$$

where $k = 0, 1, \ldots, N$. The total number of states in the irrep will then be

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N = \left(2^{N-1}\right)_B + \left(2^{N-1}\right)_F \ , \tag{3.14}$$

half of them having integer helicity (bosons), half of them half-integer helicity (fermions).

5. CPT flips the sign of the helicity. Therefore, unless the helicity is distributed symmetrically around 0, which is not the case in general, a supermultiplet is not CPT-invariant. This means that in order to have a CPT-invariant theory one should in general double the supermultiplet we have just constructed adding its CPT conjugate. This is not needed if the supermultiplet is self-CPT conjugate, which can happen only if $\lambda_0 = -\frac{N}{4}$ (in this case the helicity is indeed distributed symmetrically around 0).

Let us now apply the above procedure and construct several (physically interesting) irreps of the supersymmetry algebra.

N = 1 supersymmetry

- Matter multiplet (aka chiral multiplet):

$$\lambda_0 = 0 \ \rightarrow \ \left(0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0\right) \ . \tag{3.15}$$

  The degrees of freedom of this representation are those of one Weyl fermion and one complex scalar (on shell; recall we are constructing supersymmetry representations on states!). In a $N = 1$ supersymmetric theory this is the representation where matter sits; this is why such multiplets are called matter multiplets. For historical reasons, these are also known as Wess-Zumino multiplets.

- Gauge (or vector) multiplet:

$$\lambda_0 = \frac{1}{2} \ \rightarrow \ \left(+\frac{1}{2}, +1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}\right) \ . \tag{3.16}$$

  The degrees of freedom are those of one vector and one Weyl fermion. This is the representation one needs to describe gauge fields in a supersymmetric theory. Notice that since internal symmetries (but the R-symmetry) commute

with the supersymmetry algebra, the representation the Weyl fermion should transform under gauge transformations should be the same as the vector field, i.e. the adjoint. Hence, usual SM matter (quarks and leptons) cannot be accommodated in these multiplets.

Although in this course we will focus on rigid supersymmetry and hence not consider supersymmetric theories with gravity, let us list for completeness (and future reference) also representations containing states with higher helicity.

- Spin 3/2 multiplet:

$$\lambda_0 = 1 \;\rightarrow\; \left(1, +\frac{3}{2}\right) \underset{CPT}{\oplus} \left(-\frac{3}{2}, -1\right) \,. \tag{3.17}$$

  The degrees of freedom are those of a spin 3/2 particle and one vector.

- Graviton multiplet:

$$\lambda_0 = \frac{3}{2} \;\rightarrow\; \left(+\frac{3}{2}, +2\right) \underset{CPT}{\oplus} \left(-2, -\frac{3}{2}\right) \,. \tag{3.18}$$

  The degrees of freedom are those of a graviton, which has helicity 2, and a particle of helicity 3/2, known as the gravitino (which is indeed the supersymmetric partner of the graviton).

Representations constructed from a Clifford vacuum with higher helicity, will inevitably include states with helicity higher than 2. Hence, if one is interested in *interacting local field theories*, the story stops here. Recall that massless particles with helicity higher than $\frac{1}{2}$ should couple to conserved quantities at low momentum. The latter are: conserved internal symmetry generators for (soft) massless vectors, supersymmetry generators for (soft) gravitinos and four-vector P_μ for (soft) gravitons. The supersymmetry algebra does not allow for generators other than these ones; hence, supermultiplets with helicity $\lambda \geq \frac{5}{2}$ are ruled out: they may exist, but they cannot have couplings that survive in the low energy limit.

The above discussion also implies that in a local interacting field theory a spin 3/2 particle is inevitably associated to local supersymmetry and hence, in turn, with gravity. Hence there is no much meaning for a theory without the graviton multiplet an a spin 3/2 multiplet, which would be a non-intercating one in fact. In other words, the physical gravitino is the one in the graviton multiplet.

N = 2 supersymmetry

- Matter multiplet (aka hypermultiplet):

$$\lambda_0 = -\frac{1}{2} \;\rightarrow\; \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) . \tag{3.19}$$

  The degrees of freedom are those of two Weyl fermions and two complex scalars. This is where matter sits in a $N = 2$ supersymmetric theory. In $N = 1$ language this representation corresponds to two Wess-Zumino multiplets with opposite chirality (CPT flips the chirality). Notice that in principle this representation enjoys the CPT self-conjugate condition $\lambda_0 = -\frac{N}{4}$. However, a closer look shows that an hypermultiplet cannot be self-conjugate (that's why we added the CPT conjugate representation). The way the various states are constructed out of the Clifford vacuum, shows that under $SU(2)$ R-symmetry the helicity 0 states behave as a doublet while the fermionic states are singlets. If the representation were CPT self-conjugate the two scalar degrees of freedom would have been both real. Such states cannot form a $SU(2)$ doublet since a two-dimensional representation of $SU(2)$ is pseudoreal, and hence the doublet should be complex.

- Gauge (or vector) multiplet:

$$\lambda_0 = 0 \;\rightarrow\; \left(0, +\frac{1}{2}, +\frac{1}{2}, +1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) . \tag{3.20}$$

  The degrees of freedom are those of one vector, two Weyl fermions and one complex scalar. In $N = 1$ language this is just a vector and a matter multiplet (beware: both transforming in the same, i.e. adjoint representation of the gauge group).

- Spin 3/2 multiplet:

$$\lambda_0 = -\frac{3}{2} \;\rightarrow\; \left(-\frac{3}{2}, -1, -1, -\frac{1}{2}\right) \underset{CPT}{\oplus} \left(+\frac{1}{2}, +1, +1, +\frac{3}{2}\right) . \tag{3.21}$$

  The degrees of freedom are those of a spin 3/2 particle, two vectors and one Weyl fermion.

- Graviton multiplet:

$$\lambda_0 = -2 \;\rightarrow\; \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \underset{CPT}{\oplus} \left(+1, +\frac{3}{2}, +\frac{3}{2}, +2\right) . \tag{3.22}$$

  The degrees of freedom are those of a graviton, two gravitini and a vector, which is usually called graviphoton in the supergravity literature.

N = 4 supersymmetry

- Gauge (or vector) multiplet:

$$\lambda_0 = -1 \;\rightarrow\; \left(-1, \mathbf{4} \times -\frac{1}{2}, \mathbf{6} \times 0, \mathbf{4} \times +\frac{1}{2}, +1\right) \; . \tag{3.23}$$

  The degrees of freedom are those of one vector, four Weyl fermions and three complex scalars. In $N = 1$ language this corresponds to one vector multiplet and three matter multiplets (all transforming in the adjoint). Notice that this multiplet is CPT self-conjugate. This time there are no problems with R-symmetry transformations. The vector is a singlet under $SU(4)$, fermions transform under the fundamental representation, and scalars under the two times anti-symmetric representation, which is the fundamental of $SO(6)$, and is real. The fact that the representation under which the scalars transform is real also explains why for $\mathcal{N} = 4$ supersymmetry, the R-symmetry group is not $U(4)$ but actually $SU(4)$.

For $N = 4$ it is not possible to have matter in the usual sense, since the number of supersymmetry generators is too high to avoid helicity one states. Therefore, $N = 4$ supersymmetry cannot accommodate fermions transforming in fundamental representations. Besides the vector multiplet there are of course also representations with higher helicity, but we refrain to report them here.

One might wonder why we did not discuss $N = 3$ representations. This is just because as far as non-gravitational theories are concerned, $N = 3$ and $N = 4$ are equivalent: when constructing $N = 3$ representations, once the CPT conjugate representation is added (in this case we cannot satisfy the condition $\lambda_0 = -\frac{N}{4}$) one ends up with a multiplet which is the same as the $N = 4$ vector multiplet.

N > 4 supersymmetry

In this case one can easily get convinced that it is not possible to avoid gravity since there do not exist representations with helicity smaller than $\frac{3}{2}$ when $N > 4$. Hence, theories with $N > 4$ are all supergravity theories. It is interesting to note that $N = 8$ supergravity allows only one possible representation with highest helicity smaller than $\frac{5}{2}$ and that for higher N one cannot avoid states with helicity $\frac{5}{2}$ or higher. Therefore, $N = 8$ is an upper bound on the number of supersymmetry generators, as far as interacting local field theories are concerned. Beware: as stated, all

these statements are valid in four space-time dimensions. The way to count supersymmetries depends on the dimension of space-time, since spinorial representations get bigger, the more the dimensions. Obviously, completely analogous statements can be made in higher dimensions. For instance, in ten space-time dimensions the maximum allowed supersymmetry to avoid states with helicity $\frac{5}{2}$ or higher is $N = 2$. A dimension-independent statement can be made counting the number of single component supersymmetry generators. Using this language, the maximum allowed number of supersymmetry generators for non-gravitational theories is 16 (which is indeed $N = 4$ in four dimensions) and 32 for theories with gravity (which is $N = 8$ in four dimensions).

The table below summarizes all results we have discussed.

N	$\lambda_{max} = 1$	$\lambda_{max} = \frac{1}{2}$	$\lambda_{max} = 2$	$\lambda_{max} = \frac{3}{2}$
8	none	none	$\left[(2), 8(\frac{3}{2}), 28(1), 56(\frac{1}{2}), 70(0)\right]$	none
6	none	none	$\left[(2), 6(\frac{3}{2}), 16(1), 26(\frac{1}{2}), 30(0)\right]$	$\left[(\frac{3}{2}), 6(1), 15(\frac{1}{2}), 20(0)\right]$
5	none	none	$\left[(2), 5(\frac{3}{2}), 10(1), 11(\frac{1}{2}), 10(0)\right]$	$\left[(\frac{3}{2}), 6(1), 15(\frac{1}{2}), 20(0)\right]$
4	$\left[(1), 4(\frac{1}{2}), 6(0)\right]$	none	$\left[(2), 4(\frac{3}{2}), 6(1), 4(\frac{1}{2}), 2(0)\right]$	$\left[(\frac{3}{2}), 4(1), 7(\frac{1}{2}), 8(0)\right]$
3	$\left[(1), 4(\frac{1}{2}), 6(0)\right]$	none	$\left[(2), 3(\frac{3}{2}), 3(1), (\frac{1}{2})\right]$	$\left[(\frac{3}{2}), 3(1), 3(\frac{1}{2}), 2(0)\right]$
2	$\left[(1), 2(\frac{1}{2}), 2(0)\right]$	$\left[2(\frac{1}{2})4(0)\right]$	$\left[(2), 2(\frac{3}{2}), (1)\right]$	$\left[(\frac{3}{2}), 2(1), (\frac{1}{2})\right]$
1	$\left[(1), (\frac{1}{2})\right]$	$\left[(\frac{1}{2})2(0)\right]$	$\left[(2), (\frac{3}{2})\right]$	$\left[(\frac{3}{2}), (1)\right]$

The numbers in parenthesis represent the helicity, while other numbers represent the multiplicity of states with given helicity. Notice that, as anticipated, any supermultiplet contains particles with spin at least as large as $\frac{1}{4}N$.

A final, very important comment regards chirality. The SM is a chiral theory, in the sense that there exist particles in the spectrum whose chiral and anti-chiral components transform differently under the gauge group (weak interactions are chiral). When it comes to supersymmetric extensions, it is easy to see that only $N = 1$ theories allow for chiral matter. That $N = 1$ irreps can be chiral is obvious: Wess-Zumino multiplets contain one single Weyl fermion. Therefore, in $N = 1$ supersymmetric extensions of the Standard Model (SM) one can accommodate left and right components of leptons and quarks in different multiplets, which can then transform differently under the $SU(2)$ gauge group. Let us now consider extended supersymmetry. First notice that all helicity $\frac{1}{2}$ states belonging to multi-

plets containing vector fields should transform in the adjoint representation of the gauge group, which is real. Therefore, the only other representation which might allow for helicity $\frac{1}{2}$ states transforming in fundamental representations is the $N = 2$ hypermultiplet. However, as already noticed, a hypermultiplet contains two Wess-Zumino multiplets with opposite chirality. Since for any internal symmetry group G, we have that $[G, \text{SuperPoincaré}] = 0$, these two Wess-Zumino multiplets transform in the same representation under G. Therefore, $N = 2$ is non-chiral: left and right components of leptons and quarks would belong to the same matter multiplet and could not transform differently under the $SU(2)$ SM gauge group. Summarizing, if extended supersymmetry is realized in Nature, it should be broken at some high enough energy scale to an effective $N = 1$ model. This is why at low energy people typically focus just on $N = 1$ extensions of the SM.

3.2 Massive supermultiplets

The logical steps one should follow for massive representations are similar to previous ones. There is however one important difference. Let us consider a state with mass m in its rest frame $P_\mu = (m, 0, 0, 0)$. One can easily see that, differently from the massless case, the number of non-trivial generators gets not diminished: there remain the full set of $2N$ creation and $2N$ annihilation operators. Indeed, eq. (2.61) is now

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\} = 2m\,\delta_{\alpha\dot\beta}\,\delta^{IJ} \tag{3.24}$$

and no supersymmetric generators are trivially realized. This means that, generically, massive representations are *longer* than massless ones. Another important difference is that we better speak of spin rather than helicity, now. A given Clifford vacuum will be defined by mass m and spin j, with $j(j+1)$ being the eigenvalue of J^2. Hence, the Clifford vacuum will have itself degeneracy $2j + 1$ since j_3 takes values from $-j$ to $+j$.

N = 1 supersymmetry

The annihilation and creation operators, satisfying the usual oscillator algebra, now read

$$a_{1,2} \equiv \frac{1}{\sqrt{2m}} Q_{1,2} \quad , \quad a^\dagger_{1,2} \equiv \frac{1}{\sqrt{2m}} \bar{Q}_{\dot 1,\dot 2}\ . \tag{3.25}$$

As anticipated these are twice those for the massless case. Notice that $a^\dagger_1$ lowers the spin by half unit while $a^\dagger_2$ raises it. We can now define a Clifford vacuum as a state

with mass m and spin j_0 which is annihilated by both a_1 and a_2 and act with the creation operators to construct the corresponding massive representations.

- Matter multiplet:

$$j = 0 \;\rightarrow\; \left(-\frac{1}{2}, 0, 0', +\frac{1}{2}\right) \;. \tag{3.26}$$

  The number of degrees of freedom is the same as the massless case (but with no need to add any CPT conjugate, of course). It is worth noticing that the second scalar state, dubbed $0'$, has opposite parity with respect to the state 0, that is, it is a pseudoscalar (the proof is left to the reader as a trivial exercise; hint: use anticommutation properties of the creation operators). So the scalar is indeed a complex scalar. Summarizing, the multiplet is made of a massive complex scalar and a massive Majorana fermion.

- Gauge (or vector) multiplet:

$$j = \frac{1}{2} \;\rightarrow\; \left(-1, \mathbf{2} \times -\frac{1}{2}, \mathbf{2} \times 0, \mathbf{2} \times +\frac{1}{2}, 1\right) \;. \tag{3.27}$$

  The degrees of freedom one ends-up with are those of one massive vector, one massive Dirac fermion and one massive real scalar (recall the comment after eq. (3.24)!). The representation is longer than that of a massless vector supermultiplet, as expected. Notice that these degrees of freedom are the same as those of a massless vector multiplet plus one massless matter multiplet. This is self-consistent, since we do not like massive vectors to start with, and only allow Higgs-like mechanisms to generate masses for vector fields. In a renormalizable supersymmetric theory, one can generate massive vector multiplets by the supersymmetric generalization of the Higgs mechanism, in which a massless vector multiplet eats-up a chiral multiplet.

Since we cannot really make sense of massive particles with spin higher than one (and we are not much interested in supergravity theories in this course, anyway), we stop here and move to extended supersymmetry representations.

Extended supersymmetry

Let us then consider $N > 1$ and allow also for non-trivial central charges. A change of basis in the space of supersymmetry generators turns out to be useful for the following analysis. Since the central charge matrix Z^{IJ} is antisymmetric, with a $U(N)$ rotation one can put it in the standard block-diagonal form

$$Z^{IJ} = \begin{pmatrix} 0 & Z_1 & & & & & & \\ -Z_1 & 0 & & & & & & \\ & & 0 & Z_2 & & & & \\ & & -Z_2 & 0 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & \cdots & \cdots & & \\ & & & & & & 0 & Z_{N/2} \\ & & & & & & -Z_{N/2} & 0 \end{pmatrix} \tag{3.28}$$

(we have assumed for simplicity that N is even). One can now define

$$\begin{aligned} a^1_\alpha &= \frac{1}{\sqrt{2}} \left(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \\ b^1_\alpha &= \frac{1}{\sqrt{2}} \left(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \\ a^2_\alpha &= \frac{1}{\sqrt{2}} \left(Q^3_\alpha + \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right) \\ b^2_\alpha &= \frac{1}{\sqrt{2}} \left(Q^3_\alpha - \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right) \\ \ldots &= \ldots \\ \ldots &= \ldots \\ a^{N/2}_\alpha &= \frac{1}{\sqrt{2}} \left(Q^{N-1}_\alpha + \epsilon_{\alpha\beta}(Q^N_\beta)^\dagger\right) \\ b^{N/2}_\alpha &= \frac{1}{\sqrt{2}} \left(Q^{N-1}_\alpha - \epsilon_{\alpha\beta}(Q^N_\beta)^\dagger\right) \end{aligned}$$

which satisfy the oscillator algebra

$$\begin{aligned} \left\{a^r_\alpha, (a^s_\beta)^\dagger\right\} &= (2m + Z_r)\ \delta_{rs}\, \delta_{\alpha\beta} \\ \left\{b^r_\alpha, (b^s_\beta)^\dagger\right\} &= (2m - Z_r)\ \delta_{rs}\, \delta_{\alpha\beta} \\ \left\{a^r_\alpha, (b^s_\beta)^\dagger\right\} &= \left\{a^r_\alpha, a^s_\beta\right\} = \cdots = 0\ . \end{aligned}$$

where $r, s = 1, \ldots, N/2$. As anticipated, we have now $2N$ creation operators

$$(a^r_\alpha)^\dagger \quad , \quad (b^r_\alpha)^\dagger \qquad r = 1, \ldots, N/2 \quad , \quad \alpha = 1, 2 \tag{3.29}$$

which we can use to construct massive representations starting from some given Clifford vacuum. Notice that, from their very definition, it follows that creation

operators with spinorial index $\alpha = 1$ lower the spin by half unit, while those with spinorial index $\alpha = 2$ raise it.

Several important comments are in order, at this point. Due to the positiveness of the Hilbert space, from the oscillator algebra above we get

$$2m \geq |Z_r| \quad , \quad r = 1, \ldots, \frac{N}{2} \ . \tag{3.30}$$

This means that the mass of a given irrep is always larger (or equal) than (half) the modulus of any central charge eigenvalue. The first important consequence of the bound (3.30) is that for massless representations (for which the left hand side of the above equation is identically 0) the central charges are always trivially realized, i.e. $Z^{IJ} = 0$. That's why we did not discuss massless multiplets with non vanishing central charges in the previous section.

Suppose none of the central charge eigenvalues saturates the bound (3.30), namely $2m > |Z_r| \ , \ \forall r$. Proceeding as before, starting from a Clifford vacuum λ_0 annihilated by all (undaggered version of) operators (3.29), one creates 2^{2N} states, 2^{2N-1} bosonic and 2^{2N-1} fermionic, with spin going from $\lambda_0 - N/2$ to $\lambda_0 + N/2$. Therefore, the representation has states with spins spanning $2N + 1$ half-integer values.

Suppose instead that some Z_r saturate the bound (3.30), say $k \leq N/2$ of them do so. Looking at the oscillator algebra one immediately sees that k b-type operators become trivial (we are supposing, without loss of generality, that all Z_r are positive), and the dimension of the representation diminishes accordingly. The multiplet contains only $2^{2(N-k)}$ states now. These are called *short multiplets*. The extreme case is when all Z_r saturate the bound ($k = N/2$). In this case half the creation operators trivialize and we get a multiplet, known as *ultrashort*, whose dimension is identical to that of a massless one: the number of states is indeed 2^N, 2^{N-1} bosonic and 2^{N-1} fermionic.

The upshot of the discussion above is that in theories with extended supersymmetry one can have massive multiplets with different lengths: degrees of freedom):

$$\begin{aligned}
&\text{long multiplets} && 2^{2N} = \left(2^{2N-1}\right)_B + \left(2^{2N-1}\right)_F \\
&\text{short multiplets} && 2^{2N-2k} = \left(2^{2(N-k)-1}\right)_B + \left(2^{2(N-k)-1}\right)_F \\
&\text{ultra-short multiplets} && 2^{2N-2\frac{N}{2}} = 2^N = \left(2^{N-1}\right)_B + \left(2^{N-1}\right)_F \ .
\end{aligned}$$

One final comment is that all states belonging to some representation of supersymmetry also transform into given representations of the R-symmetry group,

since the supercharges do so. This can be checked case by case, knowing how states are constructed out of the Clifford vacuum. One should just remember that the R-symmetry group is $U(N)$ in absence of central charges and $USp(N)$ if central charges are present.

N = 2 supersymmetry

Let us first consider the case of long multiplets, namely a situation in which the (only one in this case) central charge eigenvalue does not saturate the bound. In this case we cannot have massive matter since we have too many creation operators to avoid spins higher than $\frac{1}{2}$. So the only possibility are (massive) vector multiplets.

- Gauge (or vector) multiplet:

$$j = 0 \;\rightarrow\; \left(-1, \mathbf{4} \times -\frac{1}{2}, \mathbf{6} \times 0, \mathbf{4} \times +\frac{1}{2}, 1\right) \;. \tag{3.31}$$

  The degrees of freedom correspond to a massive vector, two Dirac fermions, and five real scalars. Their number equals that of a massless $N = 2$ vector multiplet and a massless $N = 2$ hypermultiplet. As before, such massive vector multiplet should be thought of as obtained via some supersymmetric Higgs-like mechanism.

Let us now consider shorter representations. Since the central charge matrix is 2×2, we have only one eigenvalue, Z, and the only possible short representation is in fact the ultrashort, whose length should equal that of the corresponding massless representations.

- Matter multiplet (short rep.):

$$j = 0 \;\rightarrow\; \left(\mathbf{2} \times -\frac{1}{2}, \mathbf{4} \times 0, \mathbf{2} \times +\frac{1}{2}\right) \;, \tag{3.32}$$

  (where the doubling of states arises for similar reasons as for the massless hypermultiplet). The degrees of freedom are those of one massive Dirac fermion and two massive complex scalars. As expected the number of degrees of freedom equals those of a massless hypermultiplet.

- Vector multiplet (short rep.):

$$j = \frac{1}{2} \;\rightarrow\; \left(-1, \mathbf{2} \times -\frac{1}{2}, \mathbf{2} \times 0, \mathbf{2} \times +\frac{1}{2}, +1\right) \;. \tag{3.33}$$

The degrees of freedom are those of one massive vector, one Dirac fermion and one real scalar. While rearranged differently in terms of fields, the number of bosonic and fermionic degrees of freedom equals that of a massless vector multiplet. What's interesting here is that a massive ultrashort vector multiplet can arise dynamically, via some Higgs-like mechanism involving *only* a massless vector multiplet, something peculiar to $N = 2$ supersymmetry and related to the fact that massless vector multiplets contain scalars, and can then self-Higgs.

N = 4 supersymmetry

For $N = 4$ supersymmetry long multiplets are not allowed since the number of states (actually 256!) would include at least spin 2 states; such a theory would then include a massive spin 2 particle which is not believed to be possible in a local quantum field theory. However, in $N = 4$ superymmetry one can allow for ultrashort vector multiplets, whose construction is left to the reader and whose field content simply amounts to rearrange the fields characterizing a massless vector multiplet into massive states: one would get a massive vector, two Dirac fermions and five real scalars.

Let us end this section with an important remark. All short multiplets are *supersymmetry preserving*. Indeed, they are annihilated by some supersymmetry generators (those whose corresponding central charge eigenvalue saturates the bound). In general one can then have $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N/2}{N}$ supersymmetry preserving multiplets (the numerator is nothing but the integer k previously defined). For instance, ultrashort multiplets, for which $k = N/2$, are $\frac{1}{2}$ supersymmetry preserving states. Such multiplets have very important properties at the quantum level; most notably, it turns out that they are more protected against quantum corrections with respect to long multiplets. Short multiplets are also called BPS, since the bound (3.30) is very much reminiscent of the famous Bogomonlyi-Prasad-Sommerfeld bound which is saturated by solitons, tipically. This is not just a mere analogy, since the bound (3.30) is in fact not just an algebraic relation but it has a very concrete physical meaning: it is nothing but a specific BPS-like bound. Indeed, short multiplets often arise as solitons in supersymmetric field theories, and central charges correspond to physical (topological) charges. We will see examples of BPS states later in this course.

3.3 Representation on fields: a first try

So far we have discussed supersymmetry representations on states. However, we would like to discuss supersymmetric *field* theories, eventually. Therefore, we need to construct supersymmetric representations in terms of multiplets of fields rather than multiplets of states. In principle, following our previous strategy this can be done quite easily.

Let us focus on $N = 1$ supersymmetry, for simplicity. To build a representation of the supersymmetry algebra on fields, we start from some field $\phi(x)$ for which

$$\left[\bar{Q}_{\dot{\alpha}}, \phi(x)\right] = 0 \ . \tag{3.34}$$

The field ϕ is the analogous of the Clifford vacuum $|\lambda_0\rangle$ we used previously, the ground state of the representation. Similarly as before, acting on this ground state $\phi(x)$ with the supersymmetry generator Q_α, we can generate new fields out of it, all belonging to the same representation.

For definiteness, we choose $\phi(x)$ to be a scalar field, but one can also have ground states which are fields with some non-trivial tensor structure, as we will later see. Not much of what we want to say here depends on this choice. The first thing to notice is that the scalar field $\phi(x)$ is actually complex. Suppose it were real. Then, taking the hermitian conjugate of eq. (3.34) one would have obtained

$$[Q_\alpha, \phi(x)] = 0 \ . \tag{3.35}$$

One can now use the generalized Jacobi identity for $(\phi, Q, \bar{Q})$ and get

$$\begin{aligned} &\left[\phi(x), \left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\}\right] + \left\{Q_\alpha, \left[\bar{Q}_{\dot{\alpha}}, \phi(x)\right]\right\} - \left\{\bar{Q}_{\dot{\alpha}}, [\phi(x), Q_\alpha]\right\} = 0 \\ \longrightarrow \quad &2\sigma^\mu \left[\phi(x), P_\mu\right] = 0 \quad \longrightarrow \quad [P_\mu, \phi(x)] \sim \partial_\mu \phi(x) = 0 \ , \end{aligned} \tag{3.36}$$

which should then imply that the field is actually a constant (not a field, really!). So better $\phi(x)$ to be complex. In this case eq. (3.35) does not hold, but rather

$$[Q_\alpha, \phi(x)] \equiv \psi_\alpha(x) \ . \tag{3.37}$$

This automatically defines a new field ψ_α belonging to the same representation (since ϕ is a scalar, ψ is a Weyl spinor). The next step is to see whether acting with the supersymmetry generators on ψ_α gives new fields or just derivatives (or combinations) of fields already present in the representation. In principle we have

$$\{Q_\alpha, \psi_\beta(x)\} = F_{\alpha\beta}(x) \tag{3.38}$$

$$\left\{\bar{Q}_{\dot{\alpha}}, \psi_\beta(x)\right\} = X_{\dot{\alpha}\beta}(x) \ . \tag{3.39}$$

Enforcing the generalized Jacobi identity on $(\phi, Q, \bar{Q})$, and using eq. (3.37), after some trivial algebra one gets

$$X_{\dot{\alpha}\beta} = \left\{\psi_\beta(x), \bar{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\beta\dot{\alpha}} \left[P_\mu, \phi\right] \sim \partial_\mu \phi \ , \tag{3.40}$$

which implies that $X_{\dot{\alpha}\beta}$ is not a new field but just the space-time derivative of the scalar field ϕ. Let us now enforce the generalized Jacobi identity on (ϕ, Q, Q). Since the Q's anticommute (recall we are considering $N = 1$ supersymmetry and hence there are no central charges) one simply gets

$$\{Q_\alpha, [Q_\beta, \phi]\} - \{Q_\beta, [\phi, Q_\alpha]\} = 0 \quad \longrightarrow \quad F_{\alpha\beta} + F_{\beta\alpha} = 0 \ . \tag{3.41}$$

This says that the field $F_{\alpha\beta}$ is antisymmetric under $\alpha \leftrightarrow \beta$, which implies that

$$F_{\alpha\beta}(x) = \epsilon_{\alpha\beta} F(x) \ . \tag{3.42}$$

So we find here a new scalar field F. Acting on it with the supersymmetry generators produces new (?) fields as

$$\left[Q_\alpha, F\right] \ = \ \lambda_\alpha \tag{3.43}$$

$$\left[\bar{Q}_{\dot{\alpha}}, F\right] \ = \ \bar{\chi}_{\dot{\alpha}} \ . \tag{3.44}$$

Using the generalized Jacobi identities for (ψ, Q, Q) and $(\psi, Q, \bar{Q})$, respectively, one can easily prove that λ_α is actually vanishing and that $\bar{\chi}_{\dot{\alpha}}$ is proportional to the space-time derivative of the field ψ. So no new fields here: after a certain number of steps the representation closes. The multiplet of fields we have found is then

$$(\phi, \psi, F) \ . \tag{3.45}$$

If ϕ is a scalar field, as we have supposed here, this multiplet is a matter multiplet since it contains particles with spin 0 and 1/2 only. It is called chiral or Wess-Zumino multiplet and it is indeed the field theory counterpart of the chiral multiplet of states we have constructed before. Notice that the equality of the number of fermionic and bosonic states for a given representation still holds: we are now off-shell, and the spinor ψ_α has four degrees of freedom; this is the same number of bosonic degrees of freedom, two coming from the scalar field ϕ and two from the scalar field F

$$\left(\mathrm{Re}\phi, \mathrm{Im}\phi, \mathrm{Re}F, \mathrm{Im}F\right)_B \quad , \quad \left(\mathrm{Re}\psi_1, \mathrm{Im}\psi_1, \mathrm{Re}\psi_2, \mathrm{Im}\psi_2\right)_F \ . \tag{3.46}$$

While we see the expected degeneracy between bosonic and fermonic degrees of freedom, they do not match those of the chiral multiplet of states we have constructed

before, which are just $2_B + 2_F$. This is because we are off-shell, now. In fact, going on-shell, the 4 fermionic degrees of freedom reduce to just 2 propagating degrees of freedom, due to Dirac equation. The same sort of reduction should occur for the bosonic degrees of freedom, so to match the $2_B + 2_F$ on-shell condition. But Klein-Gordon equation does not diminish the number of independent degrees of freedom! What happens is that F turns out to be a non-dynamical *auxiliary* field: as we will see when constructing Lagrangians, the equation of motions for F simply tells that this scalar field is not an independent field but rather a (specific) function of other fields, $F = F(\phi, \psi)$. This is not specific to the chiral multiplet we have constructed, but it is in fact a completely general phenomenon. We will come back to this point in next lectures.

The procedure we have followed to construct the multiplet (3.45) can be easily generalized. Modifying the condition (3.34) one can construct other kind of multiplets, like linear multiplets, vector multiplets, etc... And/or, construct chiral multiplets with different field content, simply by defining a ground state carrying some space-time index, letting ϕ being a spinor, a vector, etc...

Out of a set of multiplets with the desired field content, one can construct supersymmetric field theories via a suitable Lagrangian made out of these fields. In order for the theory to be supersymmetric, this Lagrangian should (at most) transform as a total space-time derivative under supersymmetry transformations. Indeed, in this case, the action constructed out of it

$$S = \int d^4x \, \mathcal{L} \, , \tag{3.47}$$

will be supersymmetric invariant.

In practice, to see whether a given action is invariant under supersymmetry is rather cumbersome: one should take any single term in the Lagrangian, act on it with supersymmetry transformations and prove that the variations of all (possibly very many) terms sum-up to a total space-time derivative. This turns out to be very involved, in general. This difficulty is due to the fact that the formulation above is a formulation in which supersymmetry is not manifest.

Theoretical physicists came up with a brilliant idea to circumvent this problem. Ordinary field theories are naturally defined in Minkowski space and in such formulation it is easy to construct Lagrangians respecting Poincaré symmetry. It turns out that supersymmetric field theories are naturally defined on an extension of Minkoswki space, known as *superspace*, which, essentially, takes into account the

extra space-time symmetries associated to the supersymmetry generators. In such extended space it is much easier to construct supersymmetric Lagrangians, and indeed the superspace formalism is what is most commonly used nowadays to discuss supersymmetric field theories. This is the formalism we will use along this course, and next lecture will be devoted to a throughout introduction of superspace.

3.4 Exercises

1. Prove that P^2 and W^2 are Casimir of the Poincaré algebra.

2. Prove that CPT flips the sign of the helicity.

3. Construct explicitly the $N = 4$, 1/2 BPS vector multiplet. Discuss its (massive) field content and its relation with the massless vector multiplet. Can one construct a 1/4 BPS $N = 4$ vector multiplet?

4. Enforcing the generalized Jacobi identity on (ψ, Q, Q) and $(\psi, Q, \bar{Q})$, using eqs. (3.38), (3.39), (3.43) and (3.44), prove that

$$\lambda_\alpha = 0 \quad , \quad \bar{\chi}_{\dot{\beta}} \sim \partial_\mu (\sigma^\mu \psi)_{\dot{\beta}} \ . \tag{3.48}$$

References

[1] A. Bilal, *Introduction to supersymmetry*, Chapter 3, arXiv:hep-th/0101055.

[2] J.D. Lykken, *Introduction to Supersymmetry*, Chapter 2, arXiv:hep-th/9612114.

[3] M.F. Sohnius, *Introducing Supersymmetry*, Chapters 3 and 5.4, Phys. Rep. 128 (1985)

4 Superspace and superfields

The usual space-time Lagrangian formulation is not the most convenient one for describing supersymmetric field theories. This is because in ordinary space-time supersymmetry is not manifest. In fact, an extension of ordinary space-time, known as *superspace*, happens to be the best and most natural framework in which to formulate supersymmetric theories. Basically, the idea of $(N = 1)$ superspace is to enlarge the space-time labelled with coordinates x^μ, associated to the generators P_μ, by adding $2 + 2$ anti-commuting Grassman coordinates $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$, associated to the supersymmetry generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$, and obtain a eight coordinate superspace labelled by $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. In such apparently exotic space many mysterious (or hidden) properties of supersymmetric field theories become manifest. As we will see, at the price of learning a few mathematical new ingredients, the goal of constructing supersymmetric field theories will be gained much easily, and within a framework where many classical and quantum properties of supersymmetry will be more transparent.

In this lecture we will introduce superspace and superfields. In subsequent lectures we will use this formalism to construct supersymmetric field theories and study their dynamics.

4.1 Superspace as a coset

Let us start recalling the relation between ordinary Minkowski space and the Poincaré group. Minkowski space is a four-dimensional coset space defined as

$$\mathcal{M}_{1,3} = \frac{ISO(1,3)}{SO(1,3)} \ , \tag{4.1}$$

where $ISO(1,3)$ is the Poincaré group and $SO(1,3)$ the Lorentz group. The Poincaré group $ISO(1,3)$ is nothing but the isometry group of this coset space, which means that any point of $\mathcal{M}_{1,3}$ can be reached from the origin $\mathcal{O}$ with a Poincaré transformation. This transformation, however, is defined up to Lorentz transformations. Therefore, each coset class ($\equiv$ a point in space-time) has a *unique* representative which is a translation and can be parametrized by a coordinate x^μ

$$x^\mu \longleftrightarrow e^{(x^\mu P_\mu)} \ . \tag{4.2}$$

Superspace can be defined along similar lines. The first thing we need to do is to extend the Poincaré group to the so-called superPoincaré group. In order to do this,

given that a group is the exponent of the algebra, we have to re-write the whole supersymmetry algebra in terms of commutators, namely as a Lie algebra. This is easily achieved by introducing a set of *constant* Grassmann numbers $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ which anti-commute with everything fermionic and commute with everything bosonic

$$\{\theta^\alpha, \theta^\beta\} = 0 \quad , \quad \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0 \quad , \quad \{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = 0 \ . \tag{4.3}$$

This allows to transform anti-commutators of the supersymmetry algebra into commutators, and get

$$\left[\theta Q, \bar{\theta}\bar{Q}\right] = 2\ \theta\sigma^\mu\bar{\theta}P_\mu \quad , \quad [\theta Q, \theta Q] = \left[\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}\right] = 0 \ , \tag{4.4}$$

where as usual $\theta Q \equiv \theta^\alpha Q_\alpha$, $\bar{\theta}\bar{Q} \equiv \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$. This way, one can write the supersymmetry algebra solely in terms of commutators. Exponentiating this Lie algebra one gets the superPoincaré group. A generic group element can then be written as

$$G(x, \theta, \bar{\theta}, \omega) = \exp(ixP + i\theta Q + i\bar{\theta}\bar{Q} + \frac{1}{2}i\omega M) \ , \tag{4.5}$$

where xP is a shorthand notation for $x^\mu P_\mu$ and ωM a shorthand notation for $\omega^{\mu\nu}M_{\mu\nu}$.

The superPoincaré group, mathematically, is $\overline{Osp(4|1)}$. Let us open a brief parenthesis and explain such notation. Let us define the graded Lie algebra $Osp(2p|N)$ as the grade one Lie algebra $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$ whose generic element can be written as a matrix of complex dimension $(2p + N) \times (2p + N)$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.6}$$

where A is a $(2p \times 2p)$ matrix, B a $(2p \times N)$ matrix, C a $(N \times 2p)$ matrix and D a $(N \times N)$ matrix. An element of $\mathbb{L}_0$ respectively $\mathbb{L}_1$ has entries

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{respectively} \quad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{4.7}$$

where

$$A^T\Omega_{(2p)} + \Omega_{(2p)}A = 0$$

$$D^T\Omega_{(N)} + \Omega_{(N)}D = 0$$

$$C = \Omega_{(N)}B^T\Omega_{(2p)}$$

and

$$\Omega_{(2p)}^2 = -\mathbb{I} \quad , \quad \Omega_{(N)}^T = \Omega_{(N)} \quad , \quad \Omega_{(2p)}^T = -\Omega_{(2p)} \ . \tag{4.8}$$

This implies that the matrices A span a $Sp(2p, \mathbb{C})$ algebra and the matrices D a $O(N, \mathbb{C})$ algebra. Therefore we have that

$$\mathbb{L}_0 = Sp(2p) \otimes O(N) \ , \tag{4.9}$$

hence the name $Osp(2p|N)$ for the whole superalgebra. A generic element of the superalgebra has the form

$$Q = q^a t_a + q^l t_l \ , \tag{4.10}$$

where $t_a \in \mathbb{L}_0$ and $t_l \in \mathbb{L}_1$ are a basis of the corresponding vector spaces, and we have introduced complex numbers q^a for $\mathbb{L}_0$ and Grassman numbers q^l for $\mathbb{L}_1$ (recall why and how we introduced the fermionic parameters $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ before).

Taking now $p = 2$ we have the algebra $Osp(4|N)$. This is not yet what we are after, though. The last step, which we do not describe in detail here, amounts to take the so-called Inonu-Wigner contraction. Essentially, one has to rescale (almost) all generators by a constant $1/\bar{e}$, rewrite the algebra in terms of the rescaled generators and take the limit $\bar{e} \to 0$. What one ends up with is the N-extended supersymmetry algebra in Minkowski space we all know, dubbed $\overline{Osp(4|N)}$, where in the limit one gets the identification

$$A \to P_\mu \ , M_{\mu\nu} \qquad D \to Z^{IJ} \qquad B, C \to Q_I, \bar{Q}_I \ . \tag{4.11}$$

while all other generators vanish. Taking $N = 1$ one finally gets the unextended supersymmetry algebra $\overline{Osp(4|1)}$.

Given the generic group element of the superPoincaré group (4.5), the $N = 1$ *superspace* is defined as the (4+4 dimensional) group coset

$$\mathcal{M}_{4|1} = \frac{\overline{Osp(4|1)}}{SO(1,3)} \tag{4.12}$$

where, as in eq. (4.1), by some abuse of notation, both factors above refer to the groups and not the algebras.

A point in superspace (point in a loose sense, of course, given the non-commutative nature of the Grassman parameters $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$) gets identified with the coset representative corresponding to a so-called super-translation through the one-to-one map

$$\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) \longleftrightarrow e^{(x^\mu P_\mu)} \ e^{\left(\theta Q + \bar{\theta}\bar{Q}\right)} \ . \tag{4.13}$$

The $2+2$ anti-commuting Grassman numbers $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ can then be thought of as coordinates in superspace (in four-component notation they correspond to a Marojana spinor θ). For these Grassman numbers all usual spinor identities hold.

Thus far we have introduced what is known as $N = 1$ superspace. If discussing extended supersymmetry one should introduce, in principle, a larger superspace. There exist (two, at least) formulations of $N = 2$ superspace. However, these formulations present some subtleties and problems whose discussion is beyond the scope of this course. And no formulation is known of $N = 4$ superspace. In this course we will use $N = 1$ superspace even when discussing extended supersymmetry, as it is typically done in most of the literature.

4.2 Superfields as fields in superspace

Superfields are nothing but fields in superspace: functions of the superspace coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. Since $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ anticommute, any product involving more than two θ's or two $\bar{\theta}$'s vanishes: given that $\theta_\alpha\theta_\beta = -\theta_\beta\theta_\alpha$, we have that $\theta_\alpha\theta_\beta = 0$ for $\alpha = \beta$ and therefore $\theta_\alpha\theta_\beta\theta_\gamma = 0$, since at least two indices in this product are the same. Hence, the most general superfield $Y = Y(x, \theta, \bar{\theta})$ has the following simple Taylor-like expansion

$$\begin{aligned} Y(x,\theta,\bar{\theta}) \;&=\; f(x) \;+\; \theta\psi(x) \;+\; \bar{\theta}\bar{\chi}(x) \;+\; \theta\theta\; m(x) \;+\; \bar{\theta}\bar{\theta}\; n(x) \;+ \\ &+\; \theta\sigma^\mu\bar{\theta}\; v_\mu(x) \;+\; \theta\theta\;\bar{\theta}\bar{\lambda}(x) \;+\; \bar{\theta}\bar{\theta}\;\theta\rho(x) \;+\; \theta\theta\;\bar{\theta}\bar{\theta}\; d(x)\;. \end{aligned} \tag{4.14}$$

Each entry above is a field (possibly with some non-trivial tensor structure). In this sense, a superfield it is nothing but a finite collection (a multiplet) of ordinary fields.

We aim at constructing supersymmetric Lagrangians out of superfields. In such Lagrangians superfields get multiplied by each other, sometime we should act on them with derivatives, etc... Moreover, integration in superspace will be needed, eventually. Therefore, it is necessary to pause a bit and recall how operations of this kind work for Grassman variables.

Derivation in superspace is defined as follows

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} \quad \text{and} \quad \partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta \quad , \quad \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad \text{and} \quad \bar{\partial}^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\beta}}\;, \tag{4.15}$$

where

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \quad , \quad \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \quad , \quad \partial_\alpha\bar{\theta}_{\dot{\beta}} = 0 \quad , \quad \bar{\partial}^{\dot{\alpha}}\theta^\beta = 0\;. \tag{4.16}$$

For a Grassman variable θ (either $\theta_1, \theta_2, \bar{\theta}_{\dot{1}}$ or $\bar{\theta}_{\dot{2}}$ in our case), integration is defined as follows

$$\int d\theta = 0 \qquad \int d\theta\ \theta = 1\ . \tag{4.17}$$

This implies that for a generic function $f(\theta) = f_0 + \theta f_1$, the following results hold

$$\int d\theta\ f(\theta) = f_1 \quad , \quad \int d\theta\ \delta(\theta) f(\theta) = f_0 \quad \longrightarrow \int = \partial \quad , \quad \theta = \delta(\theta)\ . \tag{4.18}$$

These relations can be easily generalized to $N = 1$ superspace, provided

$$d^2\theta \equiv \frac{1}{2} d\theta^1 d\theta^2 \quad , \quad d^2\bar{\theta} \equiv \frac{1}{2} d\bar{\theta}^{\dot{2}} d\bar{\theta}^{\dot{1}}\ . \tag{4.19}$$

With these definitions one can prove the following useful identities

$$\begin{aligned} &\int d^2\theta\ \theta\theta = \int d^2\bar{\theta}\ \bar{\theta}\bar{\theta} = 1\ , \quad \int d^2\theta d^2\bar{\theta}\ \theta\theta\ \bar{\theta}\bar{\theta} = 1 \\ &\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta \quad , \quad \int d^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}}\ . \end{aligned} \tag{4.20}$$

Another crucial question we need to answer is: how does a superfield transform under supersymmetry transformations? As it is the case for all operators of the Poincaré algebra (translations, rotations and boosts), we want to realize the supersymmetry generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ as *differential* operators. In order to make this point clear, we will use momentarily calligraphic letters for the abstract operator and latin ones for the representation of the same operator as a differential operator in field space.

Let us first recall how the story goes in ordinary space-time and consider a translation, generated by $\mathcal{P}_\mu$ with infinitesimal parameter a^μ, on a field $\phi(x)$. This is defined as

$$\phi(x + a) = e^{-ia\mathcal{P}}\phi(x)e^{ia\mathcal{P}} = \phi(x) - ia^\mu\left[\mathcal{P}_\mu, \phi(x)\right] + \dots\ . \tag{4.21}$$

On the other hand, Taylor expanding the left hand side we get

$$\phi(x + a) = \phi(x) + a^\mu\partial_\mu\phi(x) + \dots \tag{4.22}$$

Equating the two right hand sides we then get

$$[\phi(x), \mathcal{P}_\mu] = -i\partial_\mu\phi(x) \equiv P_\mu\phi(x)\ , \tag{4.23}$$

where $\mathcal{P}_\mu$ is the generator of translations and P_μ is its representation as a differential operator in field space (recall that ∂_μ is an operator and from $(\partial_\mu)^* = \partial_\mu$ one gets

that $(\partial_\mu)^\dagger = -\partial_\mu$; hence P_μ is indeed hermitian). Therefore, a translation of a field by parameter a^μ induces a change on the field itself as

$$\delta_a \phi = \phi(x+a) - \phi(x) = i a^\mu P_\mu \ \phi \ . \tag{4.24}$$

Notice that here and below we are using right multiplication, when acting on fields.

We now want to apply the same procedure to a superfield. A translation in superspace (i.e. a supersymmetry transformation) on a superfield $Y(x,\theta,\bar\theta)$ by a quantity $(\epsilon_\alpha, \bar\epsilon_{\dot\alpha})$, where $\epsilon_\alpha, \bar\epsilon_{\dot\alpha}$ are spinorial parameters, is defined as

$$Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) = e^{-i\left(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}}\right)} Y(x,\theta,\bar\theta) e^{i\left(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}}\right)} \ , \tag{4.25}$$

with

$$\delta_{\epsilon,\bar\epsilon} Y(x,\theta,\bar\theta) \equiv Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) - Y(x,\theta,\bar\theta) \tag{4.26}$$

the variation of the superfield under the supersymmetry transformation.

What is the explicit expression for $\delta x, \delta\theta, \delta\bar\theta$? Why are we supposing here $\delta x \neq 0$, given we are not acting with the generator of space-time translations $\mathcal{P}_\mu$, but just with supersymmetry generators? What is the representation of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ as differential operators?

In order to answer these questions we should first recall the Baker-Campbell-Hausdorff formula for non-commuting objects which says that

$$e^A e^B = e^C \quad \text{where} \quad C = \sum_{n=1}^{\infty} \frac{1}{n!} C_n(A,B) \tag{4.27}$$

with

$$C_1 = A + B \quad , \quad C_2 = [A,B] \quad , \quad C_3 = \frac{1}{2}[A,[A,B]] - \frac{1}{2}[B,[B,A]] \quad \dots \ . \tag{4.28}$$

Eq. (4.25) can be written as

$$Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) = e^{-i\left(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}}\right)} e^{-i\left(x\mathcal{P}+\theta\mathcal{Q}+\bar\theta\bar{\mathcal{Q}}\right)} Y(0;0,0) e^{i\left(x\mathcal{P}+\theta\mathcal{Q}+\bar\theta\bar{\mathcal{Q}}\right)} e^{i\left(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}}\right)} \tag{4.29}$$

Let us now evaluate the last two exponentials. We have

$$\begin{aligned}
&\exp\{i\left(x\mathcal{P}+\theta\mathcal{Q}+\bar\theta\bar{\mathcal{Q}}\right)\} \ \exp\{i\left(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}}\right)\} = \\
&= \exp\{i x^\mu \mathcal{P}_\mu + i(\epsilon+\theta)\mathcal{Q} + i(\bar\epsilon+\bar\theta)\bar{\mathcal{Q}} - \frac{1}{2}\left[\bar\theta\bar{\mathcal{Q}}, \epsilon\mathcal{Q}\right] - \frac{1}{2}\left[\theta\mathcal{Q}, \bar\epsilon\bar{\mathcal{Q}}\right]\} \\
&= \exp\{i x^\mu \mathcal{P}_\mu + i(\epsilon+\theta)\mathcal{Q} + i(\bar\epsilon+\bar\theta)\bar{\mathcal{Q}} + \epsilon\sigma^\mu\bar\theta\,\mathcal{P}_\mu - \theta\sigma^\mu\bar\epsilon\,\mathcal{P}_\mu\} \\
&= \exp\{i(x^\mu + i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta)\mathcal{P}_\mu + i(\epsilon+\theta)\mathcal{Q} + i(\bar\epsilon+\bar\theta)\bar{\mathcal{Q}}\}
\end{aligned} \tag{4.30}$$

which means that

$$\begin{cases} \delta x^\mu = i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta \\ \delta\theta^\alpha = \epsilon^\alpha \\ \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha} \end{cases} \tag{4.31}$$

This answers the first question.

Notice the $\epsilon, \bar\epsilon$-depending piece in δx^μ. This is needed to be consistent with the supersymmetry algebra, $\left\{Q_\alpha, \bar Q_{\dot\alpha}\right\} \sim P_\mu$: two subsequent supersymmetry transformations generate a space-time translation. This answers the second question.

We can now address the third question and look for the representation of the supersymmetry generators $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot\alpha}$ as differential operators. To see this, let us consider eq. (4.26) and, recalling eqs. (4.31), let us first Taylor expand the right hand side which becomes

$$\begin{aligned} \delta_{\epsilon,\bar\epsilon} Y(x,\theta,\bar\theta) &= Y(x,\theta,\bar\theta) + i\left(\theta\sigma^\mu\bar\epsilon - \epsilon\sigma^\mu\bar\theta\right)\partial_\mu Y(x,\theta,\bar\theta) + \\ &+ \epsilon^\alpha\partial_\alpha Y(x,\theta,\bar\theta) + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} Y(x,\theta,\bar\theta) + \cdots - Y(x,\theta,\bar\theta) \\ &= \left[\epsilon^\alpha\partial_\alpha + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} + i\left(\theta\sigma^\mu\bar\epsilon - \epsilon\sigma^\mu\bar\theta\right)\partial_\mu + \dots\right] Y(x,\theta,\bar\theta) \end{aligned} \tag{4.32}$$

On the other hand, from eq. (4.25) we get

$$\begin{aligned} \delta_{\epsilon,\bar\epsilon} Y(x,\theta,\bar\theta) &= \left(1 - i\epsilon\mathcal{Q} - i\bar\epsilon\bar{\mathcal{Q}} + \dots\right) Y(x,\theta,\bar\theta)\left(1 + i\epsilon\mathcal{Q} + i\bar\epsilon\bar{\mathcal{Q}} + \dots\right) - Y(x,\theta,\bar\theta) \\ &= -i\epsilon^\alpha\left[\mathcal{Q}_\alpha, Y(x,\theta,\bar\theta)\right] + i\bar\epsilon^{\dot\alpha}\left[\bar{\mathcal{Q}}_{\dot\alpha}, Y(x,\theta,\bar\theta)\right] + \dots \ , \end{aligned} \tag{4.33}$$

(recall that $i\bar\epsilon\bar Q \equiv i\bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha} = -i\bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha}$). Defining

$$[Y, \mathcal{Q}_\alpha] \equiv Q_\alpha Y \quad , \quad \left[Y, \bar{\mathcal{Q}}_{\dot\alpha}\right] \equiv \bar Q_{\dot\alpha} Y \ , \tag{4.34}$$

the previous result implies that the supersymmetry variation of a superfield by parameters $\epsilon, \bar\epsilon$ is represented as

$$\delta_{\epsilon,\bar\epsilon} Y = \left(i\epsilon Q + i\bar\epsilon\bar Q\right)\ Y \ . \tag{4.35}$$

Comparing with eq. (4.32) we get the following expression for the differential operators $Q_\alpha, \bar Q_{\dot\alpha}$

$$\begin{cases} Q_\alpha = -\, i\partial_\alpha - \ \sigma^\mu_{\alpha\dot\beta}\ \bar\theta^{\dot\beta}\partial_\mu \\ \bar Q_{\dot\alpha} = +\, i\bar\partial_{\dot\alpha} + \ \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \end{cases} \tag{4.36}$$

Notice that, consistently, $Q_\alpha^\dagger = \bar Q_{\dot\alpha}$ (recall that $(\sigma^\mu_{\alpha\dot\beta})^\dagger = \sigma^\mu_{\beta\dot\alpha}$).

One can check the validity of the expressions (4.36) by showing that the two differential operators close the supersymmetry algebra, namely that

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \ , \ \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \ . \tag{4.37}$$

To close this section we can now give a more precise definition of a superfield: a superfield is *a field in superspace which transforms under a super-translation according to eq. (4.25)*. This implies, in particular, that a product of superfields is still a superfield.

4.3 Supersymmetric invariant actions - general philosophy

Having seen that a supersymmetry transformation is simply a translation in superspace, it is now easy to construct supersymmetric invariant actions. In order for an action to be invariant under superPoincaré transformations it is enough that the Lagrangian is Poincaré invariant (actually, it should transform as a scalar density) and that its supersymmetry variation is a total space-time derivative.

Here is where the formalism we have introduced starts to manifest its powerfulness. The basic point is that the integral in superspace of *any* arbitrary superfield is a supersymmetric invariant quantity. In other words, the following integral

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \; Y(x, \theta, \bar{\theta}) \tag{4.38}$$

is manifestly supersymmetric invariant, if Y is a superfield. This can be proven as follows. The integration measure is translationally invariant by construction since

$$\int d\theta\theta = \int d(\theta + \xi)(\theta + \xi) = 1 \tag{4.39}$$

This implies that

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x \, d^2\theta \, d^2\bar{\theta} \; Y(x, \theta, \bar{\theta}) = \int d^4x \; d^2\theta \; d^2\bar{\theta} \; \delta_{\epsilon,\bar{\epsilon}} Y(x, \theta, \bar{\theta}) \ . \tag{4.40}$$

Now, using eqs. (4.35) and (4.36) we get

$$\delta_{\epsilon,\bar{\epsilon}} Y = \epsilon^\alpha \partial_\alpha Y + \bar{\epsilon}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} Y + \partial_\mu \left[-i \left(\epsilon\sigma\bar{\theta} - \theta\sigma\bar{\epsilon} \right) Y \right] \ . \tag{4.41}$$

Integration in $d^2\theta d^2\bar{\theta}$ kills the first two terms since they do not have enough θ's or $\bar{\theta}$'s to compensate for the measure, and leaves *only* the last term, which is a total derivative. In other words, under supersymmetry transformations the integrand in

eq. (4.40) transforms as a total space-time derivative plus terms which get killed by integration in superspace. Hence the full integral is supersymmetric invariant

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x\, d^2\theta\, d^2\bar{\theta}\; Y(x,\theta,\bar{\theta}) = 0\ . \tag{4.42}$$

Supersymmetric invariant actions are constructed this way, i.e. by integrating in superspace a suitably defined superfield. Such superfield, call it $\mathcal{A}$, should not be generic, of course. It should have the right structure to give rise, upon integration on Grassman coordinates, to a Lagrangian density, which is a real, dimension-four operator, transforming as a scalar density under Poincaré transformations. The end result will be a supersymmetric invariant action $\mathcal{S}$

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \mathcal{A}(x;\theta,\bar{\theta}) = \int d^4x\; \mathcal{L}\left(\phi(x), \psi(x), A_\mu(x), \dots\right)\ . \tag{4.43}$$

Let us emphasize again: one does not need to prove $\mathcal{S}$ to be invariant under supersymmetry transformations. If it comes from an integral of a superfield in superspace, this is just automatic: by construction, the Lagrangian $\mathcal{L}$ on the r.h.s. of eq. (4.43), an apparently innocent-looking function of ordinary fields, is guaranteed to be Poincaré and supersymmetric invariant, up to total space-time derivatives.

The superfield $\mathcal{A}$ will be in general a product of superfields (recall that a product of superfields is still a superfield). However, the general superfield (4.14) cannot be the basic object of this construction: it contains too many field components to correspond to an irreducible representation of the supersymmetry algebra. We have to put (supersymmetric invariant) constraints on Y and restrict its form to contain only a subset of fields. Being the constraint supersymmetric invariant this reduced set of fields will still be a superfield, and hence will carry a representation of the supersymmetry algebra. In what follows, we will start discussing two such constraints, the so-called chiral and real constraints. These will be the relevant ones for our purposes, as they will lead to chiral and vector superfields, the right superfields to accommodate matter and radiation, respectively.

4.4 Chiral superfields

One can construct *covariant derivatives* $D_\alpha, \bar{D}_{\dot{\alpha}}$ defined as

$$\begin{cases} D_\alpha = \partial_\alpha + i\ \sigma^\mu_{\alpha\dot{\beta}}\ \bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\ \theta^\beta \sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{cases} \tag{4.44}$$

and which anticommute with the supersymmetry generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$. More precisely we have

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\,\sigma^\mu_{\alpha\dot{\beta}}\,\partial_\mu = -2\sigma^\mu_{\alpha\dot{\beta}}\,P_\mu\ , \tag{4.45}$$

$$\{D_\alpha, D_\beta \text{ or } Q_\beta \text{ or } \bar{Q}_{\dot{\beta}}\} = 0 \quad (\text{similarly for } \bar{D}_{\dot{\alpha}})\ . \tag{4.46}$$

This implies that

$$\delta_{\epsilon,\bar{\epsilon}}\,(D_\alpha Y) = D_\alpha\,(\delta_{\epsilon,\bar{\epsilon}} Y)\ . \tag{4.47}$$

Therefore, if Y is a superfield, that is a field in superspace transforming as dictated by eq. (4.25) under a supersymmetry transformation, so is $D_\alpha Y$. This means that $D_\alpha Y = 0$ is a supersymmetric invariant constraint we can impose on a superfield Y to reduce the number of its components, while still having the field carrying a representation of the supersymmetry algebra (the same holds for the constraint $\bar{D}_{\dot{\alpha}} Y = 0$).

Recall the generic expression (4.14) for Y and consider $\bar{\partial}_{\dot{\alpha}} Y$: this has fewer components with respect to Y itself, since, for instance, there is no $\theta\theta\bar{\theta}\bar{\theta}$ term. However

$$\left[\bar{\partial}_{\dot{\alpha}}, \epsilon Q\right] = \epsilon^\beta \sigma^\mu_{\beta\dot{\alpha}}\,\partial_\mu\ . \tag{4.48}$$

This implies that a supersymmetry transformation on $\bar{\partial}_{\dot{\alpha}} Y$ would generate a $\theta\theta\ \bar{\theta}\bar{\theta}$ term. Hence $\bar{\partial}_{\dot{\alpha}} Y$ is not a true superfield in the sense of providing a basis for a representation of supersymmetry. On the other hand, the covariant derivatives defined in (4.44) anticommute with Q and $\bar{Q}$. Hence, if Y is a superfield, $D_\alpha Y, \bar{D}_{\dot{\alpha}} Y$ are also superfields (and so is $\partial_\mu Y$, since P_μ commutes with Q and $\bar{Q}$).

A *chiral* superfield Φ is a superfield such that

$$\bar{D}_{\dot{\alpha}} \Phi = 0\ . \tag{4.49}$$

Seemingly, an *anti-chiral* superfield Ψ is a superfield such that

$$D_\alpha \Psi = 0\ . \tag{4.50}$$

Notice that if Φ is chiral, $\bar{\Phi}$ is anti-chiral. This implies that a chiral superfield cannot be real (i.e. $\bar{\Phi} = \Phi$). Indeed, in this case it is easy to show that it should be a constant. Taking the hermitian conjugate of eq. (4.49) one would conclude that the field would also be anti-chiral. Acting now on it with the anticommutator in eq.(4.45) one would get $\partial_\mu \Phi = 0$. This is the superfield analogue of what we have seen in the previous lecture, when we constructed the chiral multiplet.

We would like to find the most general expression for a chiral superfield in terms of ordinary fields, as we did for the general superfield (4.14). To this aim, it is useful to define new coordinates

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \quad , \quad \bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} \ . \tag{4.51}$$

It easily follows that

$$\bar{D}_{\dot{\alpha}}\theta_\beta = \bar{D}_{\dot{\alpha}}y^\mu = 0 \quad , \quad D_\alpha\bar{\theta}_{\dot{\beta}} = D_\alpha\bar{y}^\mu = 0 \ . \tag{4.52}$$

Recalling the definition (4.49) this implies that Φ depends only on (y^μ, θ_α) explicitly, but not on $\bar{\theta}_{\dot{\alpha}}$ (the $\bar{\theta}$-dependence is hidden inside y^μ). In this (super)coordinate system the chiral constraint is easily solved by

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y) \ . \tag{4.53}$$

Taylor-expanding the above expression around x we get for the actual $\Phi(x,\theta,\bar{\theta})$

$$\Phi(x,\theta,\bar{\theta}) = \phi(x)+\sqrt{2}\theta\psi(x)+i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x)-\theta\theta F(x)-\frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta}-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) \ , \tag{4.54}$$

which can also be conveniently recast as $\Phi(x,\theta,\bar{\theta}) = e^{i\theta\sigma^\mu\partial_\mu\bar{\theta}}\Phi(x,\theta)$. We see that, as expected, this superfield has less components than the general superfield Y, and some of them are related to each other.

The chiral superfield (4.54) is worth its name, since it is a superfield which encodes precisely the degrees of freedom of the chiral multiplet of fields we have previously constructed. On-shell, it corresponds to a $N = 1$ multiplet of states, hence carrying an irreducible representation of the $N = 1$ supersymmetry algebra.

A similar story holds for an anti-chiral superfield $\bar{\Phi}$ for which we would get

$$\begin{aligned}\bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) - \bar{\theta}\bar{\theta}\bar{F}(\bar{y}) \\ &= \bar{\phi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) - \bar{\theta}\bar{\theta}\bar{F}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\bar{\phi}(x) \ .\end{aligned} \tag{4.55}$$

Let us now try and see how does a chiral (or anti-chiral) superfield transform under supersymmetry transformations. This amounts to compute

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(y;\theta) = \left(i\epsilon Q + i\bar{\epsilon}\bar{Q}\right)\Phi(y;\theta) \tag{4.56}$$

(and similarly for $\bar{\Phi}$). To compute eq. (4.56) it is convenient to write the differential operators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ in the $(y^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ coordinate system. This amounts to trade the

partial derivatives taken with respect to $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ for those taken with respect to the new system $(y^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ and plug this into eqs. (4.36). The final result reads

$$\begin{cases} Q^{new}_\alpha = -i\partial_\alpha \\ \bar{Q}^{new}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu} \end{cases} \tag{4.57}$$

Plugging these expressions into eq. (4.56) one gets

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\Phi(y;\theta) &= \left(\epsilon^\alpha \partial_\alpha + 2i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}} \frac{\partial}{\partial y^\mu}\right) \Phi(y;\theta) \\ &= \sqrt{2}\epsilon\psi - 2\epsilon\theta F + 2i\theta\sigma^\mu\bar{\epsilon}\left(\frac{\partial}{\partial y^\mu}\phi + \sqrt{2}\theta\frac{\partial}{\partial y^\mu}\psi\right) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta\left(-\sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\phi\right) - \theta\theta\left(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\frac{\partial}{\partial y^\mu}\psi\right) . \end{aligned} \tag{4.58}$$

Therefore, the final expression for the supersymmetry variation of the different field components of the chiral superfield Φ reads

$$\begin{cases} \delta\phi = \sqrt{2}\epsilon\psi \\ \delta\psi_\alpha = \sqrt{2}i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi - \sqrt{2}\epsilon_\alpha F \\ \delta F = i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \end{cases} \tag{4.59}$$

It is left to the reader to derive the corresponding expressions for an anti-chiral superfield. In this case, one should write the generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ as functions of $(\bar{y}^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$.

4.5 Real (aka vector) superfields

In order to have gauge interactions we clearly need to find some new supersymmetric invariant projection which saves the vector field v^μ in the general expression (4.14) and makes it real (this was not the case for the chiral projection, for which the vector component is $\sim \partial_\mu\phi$). The right thing to do is to impose a *reality* condition on the general superfield Y. Indeed, under hermitian conjugation, $Y \to \bar{Y}$, one has that $v^\mu \to \bar{v}^\mu$; so imposing a reality condition, not only the vector component survives as a degrees of freedom, but becomes real.

A real (aka *vector*) superfield V is a superfield such that

$$V = \bar{V} . \tag{4.60}$$

Looking at the general expression (4.14) this leads to the following expansion for V

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + i\,\theta\chi(x) - i\,\bar\theta\bar\chi(x) + \theta\sigma^\mu\bar\theta v_\mu + \frac{i}{2}\,\theta\theta\,(M(x)+iN(x)) \\ &- \frac{i}{2}\,\bar\theta\bar\theta\,(M(x)-iN(x)) + i\,\theta\theta\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\,\bar\sigma^\mu\partial_\mu\chi(x)\right) \\ &- i\,\bar\theta\bar\theta\theta\left(\lambda(x) + \frac{i}{2}\,\sigma^\mu\partial_\mu\bar\chi(x)\right) + \frac{1}{2}\,\theta\theta\bar\theta\bar\theta\left(D(x) - \frac{1}{2}\,\partial^2 C(x)\right)\,. \end{aligned} \tag{4.61}$$

Notice that, as such, this superfield has $8_B + 8_F$ degrees of freedom. The next step is to introduce the supersymmetric version of gauge transformations. As we shall see, after gauge fixing, this will reduce the number of off-shell degrees of freedom to $4_B + 4_F$, which become $2_B + 2_F$ on-shell (for a massless representation), as it should be the case for a massless vector multiplet of states.

First notice that $\Phi + \bar\Phi$ is a vector superfield, if Φ is a chiral superfield. Second, notice that under

$$V \to V + \Phi + \bar\Phi \tag{4.62}$$

the vector v_μ in V transforms as $v_\mu \to v_\mu - \partial_\mu\,(2\,\mathrm{Im}\phi)$. This is precisely how an ordinary (abelian) gauge transformation acts on a vector field. Therefore, eq. (4.62) is a natural definition for the supersymmetric version of a gauge transformation. Under eq. (4.62) the component fields of V transform as

$$\begin{cases} C & \to & C + 2\,\mathrm{Re}\phi \\ \chi & \to & \chi - i\sqrt{2}\psi \\ M & \to & M - 2\,\mathrm{Im}F \\ N & \to & N + 2\,\mathrm{Re}F \\ D & \to & D \\ \lambda & \to & \lambda \\ v^\mu & \to & v^\mu - 2\,\partial_\mu\mathrm{Im}\phi \end{cases} \tag{4.63}$$

where the components of Φ have been dubbed (ϕ, ψ, F). From the transformations above one sees that properly choosing Φ, namely choosing

$$\mathrm{Re}\phi = -\frac{C}{2}\quad,\quad \psi = -\frac{i}{\sqrt{2}}\chi\quad,\quad \mathrm{Re}F = -\frac{N}{2}\quad,\quad \mathrm{Im}F = \frac{M}{2}\,. \tag{4.64}$$

one can gauge away (namely put to zero) C, M, N, χ. The choice above is called Wess-Zumino gauge. In this gauge a vector superfield can be written as

$$V_{WZ} = \theta\sigma^\mu\bar\theta\,v_\mu(x) + i\theta\theta\,\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta D(x)\,. \tag{4.65}$$

Therefore, taking into account gauge invariance (that is the redundancy of one of the vector degrees of freedom, the one associated to the transformation $v_\mu \to v_\mu - \partial_\mu (2\ \mathrm{Im}\phi)$), we end-up with 4_B+4_F degrees of freedom off-shell. As we shall see later, D will turn out to be an auxiliary field; therefore, by imposing the equations of motion for D, the spinor λ and the vector v^μ, one will end up with 2_B+2_F degrees of freedom on-shell. Since we like to formulate gauge theories keeping gauge invariance manifest off-shell, the WZ gauge is defined as a gauge where $C = M = N = \chi = 0$, but no restrictions on v^μ. This way, while remaining in the WZ gauge, we still have the freedom to do ordinary gauge transformations. In other words, once in the WZ gauge, we can still perform a supersymmetric gauge transformation (4.62) with parameters $\phi = -\bar{\phi}, \psi = 0, F = 0$.

Let us end this section with two important comments. First notice that in the WZ gauge each term in the expansion of V_{WZ} contains at least one θ. Therefore

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}v_\mu v^\mu \quad , \quad V_{WZ}^n = 0 \ \ n \geq 3 \ . \tag{4.66}$$

These identities will simplify things a lot when it comes to construct supersymmetric gauge actions.

Second, notice that the WZ gauge is not supersymmetric. In other words, it does not commute with supersymmetry. Acting with a supersymmetry transformation on a vector superfield in the WZ gauge, one obtains a new superfield which is not in the WZ gauge. Hence, when working in this gauge, after a supersymmetry transformation, one has to do a compensating supersymmetric gauge transformation (4.62), with a properly chosen Φ, to come back to the WZ gauge. We leave to the reader to check this.

4.6 (Super)Current superfields

The two superfields described above are what we need to describe matter and radiation in a supersymmetric theory, if we are not interested in gravitational interactions. However, in a supersymmetric theory, also composite operators should sit in superfields. These can be, e.g. chiral superfields, but there are at least two other classes of superfields which accommodate important composite operators. They are those describing conserved currents and the supersymmetry current (supercurrent for short), respectively, the latter being ubiquitous in a supersymmetric QFT, as this is the current associated to the supersymmetry charge itself. Both these superfields turn out to be real superfields, as the superfield described in the previous

section, but current conservation implies extra supersymmetric invariant conditions they should satisfy which make them a particular class of real superfields. In what follows, we will briefly describe both of them.

4.6.1 Internal symmetry current superfields

Because of Noether theorem, in a local QFT any continuous symmetry is associated to a conserved current j_μ satisfying $\partial^\mu j_\mu = 0$, and to the corresponding conserved charge Q defined as $Q = \int d^3x\, j^0$. Here we are referring to non-R symmetries; R-symmetry will be discussed later.

As any other operator, in a supersymmetric theory a conserved current should be embedded in a superfield. It turns out that this is a real scalar superfield $\mathcal{J}$ satisfying the following extra constraint

$$D^2\mathcal{J} = \bar{D}^2\mathcal{J} = 0\ . \tag{4.67}$$

A real superfield satisfying the constraint above is called *linear* superfield. Working a little bit one can show that a real superfield subject to the conditions (4.67) has the following component expression

$$\mathcal{J} = J(x)+i\theta j(x)-i\bar{\theta}\bar{j}(x)+\theta\sigma^\mu\bar{\theta}j_\mu(x)+\frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu j(x)-\frac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{j}(x)+\frac{1}{4}\theta^2\bar{\theta}^2\Box J(x)\ , \tag{4.68}$$

where J is a real scalar and j_α a spinor. By imposing eq. (4.67) on the above expression one easily sees that the current j_μ satisfies $\partial^\mu j_\mu = 0$, i.e. is a conserved current. So the constraint (4.67) is indeed the correct supersymmetric generalization of current conservation. Note that while the condition (4.67) is compatible with supersymmetry, as it should (both D^2 and $\bar{D}^2$ commute with supersymmetry transformations), it stands on a slightly different footing with respect to the conditions (4.49) and (4.60). The latter constrain the dependence of a superfield as a function of the fermionic coordinates $(\theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, but they do not say anything about space-time dependence. On the contrary, eq. (4.67) constrains the space-time dependence of some of the fields imposing differential equations in x-space, one obvious example being the conservation equation $\partial^\mu j_\mu = 0$. In this sense, (4.67) is an on-shell constraint.

A few comments are in order. First notice that, as compared to a general real superfield (4.61), a linear superfield has less independent components. This is due to the extra condition (4.67) a linear superfield has to satisfy. Another comment

regards the spin content of $\mathcal{J}$. One condition that $\mathcal{J}$ should (and does) satisfy is that it should not contain fields with spin higher than one. If this were the case, one could not gauge the current j^μ without introducing higher-spin gauge fields, something which is expected not to be consistent in a local interacting QFT with rigid supersymmetry (recall our discussion in the previous lecture). This implies that $\mathcal{J}$ should be a real *scalar* superfield, namely its lowest component J should be a scalar. Finally, it may worth notice that the detailed structure of $\mathcal{J}$ is not uniquely fixed, but in fact defined up to Schwinger terms entering the current algebra. This can be understood as follows. Because the conserved charge Q is a non-R symmetry charge, it commutes with supersymmetry generators, $[Q_\alpha, Q] = 0$. This in turn implies that in the current algebra

$$[Q_\alpha, j_\mu] = \mathcal{O}_{\alpha\mu} \ , \tag{4.69}$$

the operator $\mathcal{O}_{\alpha\mu}$ should be an operator which vanishes when acting with ∂_μ, because so is j_μ, and it should also be a total space-time derivative for $\mu = 0$, say $\mathcal{O}_{\alpha 0} = \partial^\nu A_{\alpha\nu}$, so that it integrates to zero, because so happens to the left hand side since

$$\int d^4x \, [Q_\alpha, j_0] = \int dt \, [Q_\alpha, Q] = 0 \ . \tag{4.70}$$

An operator of this kind is known as Schwinger term. Different Schwinger terms provide different completions of the linear superfield $\mathcal{J}$, which is hence not univocally defined. The superfield defined in eq. (4.68) is one possible such completions, for which $\mathcal{O}_{\alpha\mu} = -2i(\sigma_{\mu\nu})_\alpha^{\ \beta} \, \partial^\nu j_\beta$. This can be easily checked using eqs. (4.34)-(4.35).

4.6.2 Supercurrent superfields

While currents associated to internal symmetries might or might not be there, in any supersymmetric theory there always exists, by definition, a conserved current, the supersymmetry current $S_{\alpha\mu}$, associated to the conservation of the fermionic charge Q_α, for which $\partial^\mu S_{\alpha\mu} = 0$. In terms of the supercurrent, the supersymmetry charge is $Q_\alpha = \int d^3x \, S_\alpha^{\ 0}$. Such supercurrent should be embedded in a superfield.

An equation analogous to eq. (4.69) is imposed by the supersymmetry algebra, which reads

$$\left\{\bar{Q}_{\dot\alpha}, S_{\alpha\nu}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} T_{\mu\nu} + \mathcal{O}_{\alpha\dot\alpha\nu} \ , \tag{4.71}$$

where $T_{\mu\nu}$ is the (conserved) energy-momentum tensor and $\mathcal{O}_{\alpha\dot\alpha\nu}$ is again a Schwinger term. Note that now the $\nu = 0$ component of the left hand side does not integrate to

zero but in fact it is proportional to $\int dt\, P_\mu$ by the supersymmetry algebra, namely to $\int d^4x\, T_\mu{}^0$. This is why, on top of a Schwinger term, the energy-momentum tensor appears on the right hand side of eq. (4.71). This also shows that the supercurrent and the energy-momentum tensor sit in the same superfield, $T_{\mu\nu}$ being the highest spin field of the representation (otherwise, it would be problematic coupling supersymmetry with gravity). This is the current operators counterpart of the fact that the graviton and the gravitino sit in the same multiplet.

The arbitrariness of the Schwinger term gives rise, as before, to different possible completions of the superfield. The most known such completions is due to Ferrara and Zumino. The FZ supermultiplet can be described by a pair of superfields $(\mathcal{J}_\mu, X)$ satisfying the relation

$$2\,\bar{D}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\,\mathcal{J}_\mu = D_\alpha X\ , \tag{4.72}$$

with $\mathcal{J}_\mu$ being a real vector superfield, and X a chiral superfield, $\bar{D}_{\dot{\alpha}}X = 0$. The same comment we made on the on-shell nature of the condition (4.67) holds also in this case. From the defining equation above one can work out the component expression of these two superfields. They read

$$\begin{aligned}\mathcal{J}_\mu =& j_\mu + \theta\left(S_\mu - \frac{1}{3}\sigma_\mu\bar{S}\right) + \bar{\theta}\left(\bar{S}_\mu + \frac{1}{3}\bar{\sigma}_\mu S\right) + \frac{i}{2}\theta^2\partial_\mu x^* - \frac{i}{2}\bar{\theta}^2\partial_\mu x \\ &+ \theta\sigma^\nu\bar{\theta}\left(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}T + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma\right) + \dots\end{aligned} \tag{4.73}$$

and

$$X = x + \frac{2}{3}\theta S + \theta^2\left(\frac{2}{3}T + i\,\partial^\mu j_\mu\right) + \dots\ , \tag{4.74}$$

where $\dots$ stand for the supersymmetric completion and we have defined the trace operators $T \equiv T^\mu{}_\mu$ and $S_\alpha \equiv \sigma^\mu_{\alpha\dot{\alpha}}\bar{S}^{\dot{\alpha}}_\mu$. All in all, the FZ superfield contains a (in general non-conserved) R-current j_μ, a symmetric and conserved $T_{\mu\nu}$, a conserved $S_{\alpha\mu}$, and a complex scalar x. From the above expression one can also see that whenever X vanishes the current j_μ becomes conserved and all trace operators vanish. In this case the theory is conformal and j_μ becomes the always present (and conserved) superconformal R-current. We will have more to say on this issue later.

For theories with an R-symmetry (be it preserved or spontaneously broken), there exists an alternative supermultiplet accommodating the energy-momentum tensor and the supercurrent, the so-called $\mathcal{R}$ multiplet. It turns out this is again defined in terms of a pair of superfields $(\mathcal{R}_\mu,\, \chi_\alpha)$ which now satisfy a different on-

shell condition

$$2\,\bar{D}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\,\mathcal{R}_{\mu}=\chi_{\alpha}\ , \tag{4.75}$$

where $\mathcal{R}_{\mu}$ is a real vector superfield and χ_{α} a chiral superfield which, besides $\bar{D}_{\dot{\alpha}}\chi_{\alpha}=0$, also satisfies the identity $\bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}-D^{\alpha}\chi_{\alpha}=0$. This implies, in turn, that $\partial^{\mu}\mathcal{R}_{\mu}=0$, from which it follows that the lowest component of $\mathcal{R}_{\mu}$ is now a conserved current, the R-current j^{R}_{μ}. The component expression of the superfields making-up the $\mathcal{R}$ multiplet reads

$$\mathcal{R}_{\mu}=j^{R}_{\mu}+\theta S_{\mu}+\bar{\theta}\,\bar{S}_{\mu}+\theta\sigma^{\nu}\bar{\theta}\left(2T_{\mu\nu}+\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(\partial^{\rho}j^{\sigma}+C^{\rho\sigma})\right)+\dots \tag{4.76}$$

and

$$\chi_{\alpha}=-2S_{\alpha}-\left(4\delta^{\beta}_{\alpha}T+2i\,(\sigma^{\rho}\bar{\sigma}^{\tau})^{\beta}_{\alpha}C_{\rho\tau}\right)\theta_{\beta}+2i\,\theta^{2}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu}\bar{S}^{\dot{\alpha}}+\dots \tag{4.77}$$

where again $\dots$ stand for the supersymmetric completion, and $C_{\mu\nu}$ is a closed two-form. That j^{R}_{μ} is an R-current can be easily seen noticing that the current algebra now reads $\left[Q_{\alpha},j^{R}_{\mu}\right]=S_{\alpha\mu}$. Taking the time-component and integrating, this implies that $\int dt\ \left[Q_{\alpha},Q^{R}\right]=\int dt\,Q_{\alpha}$, which is what is expected for a R-symmetry, recall eq. (2.78). Notice, finally, that when $X=0$, the FZ multiplet (4.73) becomes a (special instance of an) R-multiplet. Indeed its lowest component j_{μ} becomes now the conserved superconformal R-current.

The FZ and $\mathcal{R}$ multiplets are the more common supercurrent multiplets. However, there are instances in which a theory does not admit a R-symmetry (and hence the $\mathcal{R}$ multiplet cannot be defined) and the FZ multiplet is not a well-defined operator, e.g. it is not gauge invariant. In these cases, one should consider yet another multiplet where the supercurrent can sit, the so-called $\mathcal{S}$ multiplet, which is bigger than the two above. We will not discuss the $\mathcal{S}$ multiplet here, and refer to the references given at the end of this lecture. On the contrary, there exist theories in which both the FZ and the $\mathcal{R}$ multiplets can be defined. In such cases it turns out the two are related by a so-called shift transformation defined as

$$\mathcal{R}_{\mu}=\mathcal{J}_{\mu}+\frac{1}{4}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\left[D_{\alpha},\,\bar{D}_{\dot{\alpha}}\right]U\quad,\quad X=-\frac{1}{2}\bar{D}^{2}U\quad,\quad\chi_{\alpha}=\frac{3}{2}\bar{D}^{2}D_{\alpha}U\,, \tag{4.78}$$

where U is a real superfield associated to a non-conserved (and non-R) current.

We will encounter examples of current and supercurrent multiplets in later lectures.

4.7 Exercises

1. Prove identities (4.20).

2. Check that the differential operators Q_α and $\bar{Q}_{\dot{\alpha}}$ (4.36) close the supersymmetry algebra (4.37).

   Hint: recall that all θ's and $\bar{\theta}$'s anti-commute between themselves, and that

   $$\{a_i, a_j\} = 0 \quad \longrightarrow \quad \frac{\partial}{\partial a_i} a_j = \frac{\partial a_j}{\partial a_i} - a_j \frac{\partial}{\partial a_i} \ , \tag{4.79}$$

   which implies that, e.g.

   $$\{\partial_\alpha, \bar{\theta}^{\dot{\gamma}}\} = 0 \ , \ \{\partial_\alpha, \theta^\beta\} = \delta_\alpha^\beta \ , \ \{\bar{\partial}_{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \delta_{\dot{\alpha}}^{\dot{\beta}} \ . \tag{4.80}$$

3. Check that the covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ (4.44) anticommute between themselves and with the supercharge operators (4.36).

4. Compute how the field components of an anti-chiral superfield Ψ transform under supersymmetry transformations. Show that if $\Psi = \bar{\Phi}$ one gets the hermitian conjugate of the transformations (4.59).

5. Compute the supersymmetric variation of a vector superfield in the WZ gauge, and find the explicit form of the chiral superfield Φ which, via a compensating gauge transformation (4.62), brings the vector superfield back to WZ gauge.

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5 Supersymmetric actions: minimal supersymmetry

In the previous lecture we have introduced the basic superfields one needs to construct $N = 1$ supersymmetric theories, if one is not interested in describing gravitational interactions. We are now ready to look for supersymmetric actions describing the dynamics of these superfields. We will first concentrate on matter actions and construct the most general supersymmetric action describing the interaction of a set of chiral superfields. Then we will introduce SuperYang-Mills theory which is nothing but the supersymmetric version of Yang-Mills. Finally, we will couple the two sectors with the final goal of deriving the most general $N = 1$ supersymmetric action describing the interaction of radiation with matter. In all these cases, we will consider both renormalizable as well as non-renormalizable theories, the latter being relevant to describe effective low energy theories.

Note: in what follows we will deal with gauge theories, and hence gauge groups, like $U(N)$ and alike. In order to avoid confusion, in the rest of these lectures we will use calligraphic N when referring to the number of supersymmetry, $\mathcal{N} = 1, 2$ or 4.

5.1 $\mathcal{N} = 1$ Matter actions

Following the general strategy outlined in §4.3 we want to construct a supersymmetric invariant action describing the interaction of a (set of) chiral superfield(s). Let us first notice that a product of chiral superfields is still a chiral superfield and a product of anti-chiral superfields is an anti-chiral superfield. Conversely, the product of a chiral superfield with its hermitian conjugate (which is anti-chiral) is a (very special, in fact) real superfield.

Let us start analyzing the theory of a single chiral superfield Φ. Consider the following integral

$$\int d^2\theta \, d^2\bar{\theta} \; \bar{\Phi}\Phi \; . \tag{5.1}$$

This integral satisfies all necessary conditions to be a supersymmetric Lagrangian. First, it is supersymmetric invariant (up to total space-time derivative) since it is the integral in superspace of a superfield. Second, it is real and a scalar object. Indeed, the first component of $\bar{\Phi}\Phi$ is $\bar{\phi}\phi$ which is real and a scalar. Now, the $\theta^2\bar{\theta}^2$ component of a superfield, which is the only term contributing to the above integral, has the same tensorial structure as its first component since $\theta^2\bar{\theta}^2$ does not have any free space-time indices and is real, that is $(\theta^2\bar{\theta}^2)^\dagger = \theta^2\bar{\theta}^2$. Finally, the above integral has

also the right physical dimensions for being a Lagrangian, *i.e.* $[M]^4$. Indeed, from the expansion of a chiral superfield, one can see that θ and $\bar{\theta}$ have both dimension $[M]^{-1/2}$ (compare the first two components of a chiral superfield, $\phi(x)$ and $\theta\psi(x)$, and recall that a spinor in four dimensions has physical dimension $[M]^{3/2}$). This means that the $\theta^2\bar{\theta}^2$ component of a superfield Y has dimension $[Y]+2$ if $[Y]$ is the dimension of the superfield (which is that of its first component). Since the dimension of $\bar{\Phi}\Phi$ is 2, it follows that its $\theta^2\bar{\theta}^2$ component has dimension 4 (notice that $\int d^2\theta d^2\bar{\theta}\,\theta^2\bar{\theta}^2$ is dimensionless since $d\theta$ ($d\bar{\theta}$) has opposite dimensions with respect to θ ($\bar{\theta}$), given that the differential is equivalent to a derivative, for Grassman variables). Summarizing, eq. (5.1) is an object of dimension 4, is real, and transforms as a total space-time derivative under SuperPoincaré transformations.

To perform the integration in superspace one can start from the expression of Φ and $\bar{\Phi}$ in the y (resp. $\bar{y}$) coordinate system, take the product of $\bar{\Phi}(\bar{y},\bar{\theta})\Phi(y,\theta)$, expand the result in the $(x,\theta,\bar{\theta})$ space, and finally pick up the $\theta^2\bar{\theta}^2$ component, only. The computation is left to the reader. The end result is

$$\mathcal{L}_{kin}=\int d^2\theta\, d^2\bar{\theta}\;\bar{\Phi}\Phi=\partial_\mu\bar{\phi}\,\partial^\mu\phi+\frac{i}{2}\left(\partial_\mu\psi\;\sigma^\mu\bar{\psi}-\psi\;\sigma^\mu\partial_\mu\bar{\psi}\right)+\bar{F}F+\text{ total der}\,. \quad (5.2)$$

What we get is precisely the kinetic term describing the degrees of freedom of a free chiral superfield! In doing so we also see that, as anticipated, the F field is an auxiliary field, namely a non-propagating degree of freedom. Integrating it out (which is trivial in this case since its equation of motion is simply $F=0$) one gets a (supersymmetric) Lagrangian describing physical degrees of freedom, only. Notice that after integrating F out, supersymmetry is realized on-shell, only, as it can be easily checked.

The equations of motion for ϕ,ψ and F following from the Lagrangian (5.2) can be easily derived using superfield formalism readily from the expression in superspace. This might not look obvious at a first sight since varying the action (5.1) with respect to $\bar{\Phi}$ we would get $\Phi=0$, which does not provide the equations of motion we would expect, as it can be easily inferred expanding it in components. The point is that the integral in eq. (5.1) is a constrained one, since Φ is a chiral superfield and hence subject to the constraint $\bar{D}_{\dot{\alpha}}\Phi=0$. One can rewrite the above integral as an unconstrained one noticing that

$$\int d^2\theta\, d^2\bar{\theta}\;\bar{\Phi}\Phi=\frac{1}{4}\int d^2\bar{\theta}\;\bar{\Phi}D^2\Phi\,. \quad (5.3)$$

In getting the right hand side we have used the fact that $\int d\theta_\alpha = D_\alpha$, up to total space-time derivative, and that Φ is a chiral superfield (hence $D_\alpha\bar{\Phi}=0$). Now,

varying with respect to $\bar{\Phi}$ we get

$$D^2\Phi = 0 \ , \tag{5.4}$$

which, upon expansion in $(x, \theta, \bar{\theta})$, does correspond to the equations of motion for ϕ, ψ and F one would obtain from the Lagrangian (5.2). The check is left to the reader.

A natural question arises. Can we do more? Can we have a more general Lagrangian than just the one above? Let us try to consider a more generic function of Φ and $\bar{\Phi}$, call it $K(\Phi, \bar{\Phi})$, and consider the integral

$$\int d^2\theta \, d^2\bar{\theta} \; K(\Phi, \bar{\Phi}) \ . \tag{5.5}$$

In order for the integral (5.5) to be a promising object to describe a supersymmetric Lagrangian, the function K should satisfy a number of properties. First, it should be a superfield. This ensures supersymmetric invariance. Second, it should be a real and scalar function. As before, this is needed since a Lagrangian should have these properties and the $\theta^2\bar{\theta}^2$ component of K, which is the only one contributing to the above integral, is a real scalar object, if so is the superfield K. Third, $[K] = 2$, since then its $\theta^2\bar{\theta}^2$ component will have dimension 4, as a Lagrangian should have. Finally, K should be a function of Φ and $\bar{\Phi}$ but not of $D_\alpha\Phi$ and $\bar{D}_{\dot{\alpha}}\bar{\Phi}$. The reason is that, as it can be easily checked, covariant derivatives would provide $\theta\theta\bar{\theta}\bar{\theta}$-term contributions giving a higher derivative theory (third order and higher), which cannot be accepted for a local field theory. It is not difficult to get convinced that the most general expression for K which is compatible with all these properties is

$$K(\Phi, \bar{\Phi}) = \sum_{m,n=1}^{\infty} c_{mn} \bar{\Phi}^m \Phi^n \quad \text{where} \quad c_{mn} = c^*_{nm}. \tag{5.6}$$

where the reality condition on K is ensured by the relation between c_{mn} and c_{nm}. All coefficients c_{mn} with either m or n greater than one have negative mass dimension, while c_{11} is dimensionless. This means that, in general, a contribution as that in eq. (5.5) will describe a supersymmetric but non-renormalizable theory, typically defined below some cut-off scale Λ. Indeed, generically, the coefficients c_{mn} will be of the form

$$c_{mn} \sim \Lambda^{2-(m+n)} \tag{5.7}$$

with the constant of proportionality being a pure number. The function K is called *Kähler potential*. The reason for such fancy name will become clear later (see §5.1.1).

If renormalizability is an issue, the lowest component of K should not contain operators of dimensionality bigger than 2, given that the $\theta^2\bar{\theta}^2$ component has dimensionality $[K]+2$. In this case all c_{mn} but c_{11} should vanish and the Kähler potential would just be equal to $\bar{\Phi}\Phi$, the object we already considered before and which leads to the renormalizable (free) Lagrangian (5.2).

In passing, notice that the combination $\Phi+\bar{\Phi}$ respects all the physical requirements discussed above. However, a term like that would not give any contribution since its $\theta^2\bar{\theta}^2$ component is a total derivative. This means that two Kähler potentials K and K' related as

$$K(\Phi,\bar{\Phi})' = K(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \ , \tag{5.8}$$

where Λ is a chiral superfield (obtained out of Φ) and $\bar{\Lambda}$ is the corresponding anti-chiral superfield (obtained out of $\bar{\Phi}$), are different, but their integrals in full superspace, which is all what matters for us, are the same

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\ K(\Phi,\bar{\Phi})' = \int d^4x\, d^2\theta\, d^2\bar{\theta}\ K(\Phi,\bar{\Phi}) \ . \tag{5.9}$$

This is the reason why we did not consider $m,n=0$ in the expansion (5.6).

Thus far, we have not been able to describe any renormalizable interaction, like non-derivative scalar interactions and Yukawa interactions, which should certainly be there in a supersymmetric theory. How to describe them? As we have just seen, the simplest possible integral in superspace full-filling the minimal and necessary physical requirements, $\bar{\Phi}\Phi$, already gives two-derivative contributions, see eq. (5.2). What can we do, then?

When dealing with chiral superfields, there is yet another possibility to construct supersymmetric invariant superspace integrals. Let us consider a generic chiral superfield Σ (which can be obtained from products of Φ's, in our case). Integrating it in full superspace would give

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\ \Sigma = 0 \ , \tag{5.10}$$

since its $\theta^2\bar{\theta}^2$ component is a total derivative. Consider instead integrating Σ in half superspace

$$\int d^4x\, d^2\theta\ \Sigma \ . \tag{5.11}$$

Differently from the previous one, this integral does not vanish, since now it is the θ^2 component which contributes, and this is not a total derivative for a chiral superfield.

Note that while $\Sigma = \Sigma(y, \theta)$, in computing (5.11) one can safely take $y^\mu = x^\mu$ since the terms one is missing would just provide total space-time derivatives, which do not contribute to $\int d^4x$. Another way to see it, is to notice that in $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ coordinate, the chiral superfield Σ reads $\Sigma(x, \theta, \bar{\theta}) = \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\Sigma(x, \theta)$. Besides being non-vanishing, (5.11) is also supersymmetric invariant, since the θ^2 component of a chiral superfield transforms as a total derivative under supersymmetry transformations, as can be seen from eq. (4.59)!

An integral like (5.11) is more general than an integral like (5.5). The reason is the following. Any integral in full superspace can be re-written as an integral in half superspace. Indeed, for any superfield Y

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\, Y = \frac{1}{4}\int d^4x\, d^2\theta\, \bar{D}^2 Y \ , \tag{5.12}$$

(in passing, let us notice that for any arbitrary Y, $\bar{D}^2 Y$ is manifestly chiral, since $\bar{D}^3 = 0$ identically). This is because when going from $d\bar{\theta}$ to $\bar{D}$ the difference is just a total space-time derivative, which does not contribute to the above integral. On the other hand, the converse is not true in general. Consider a term like

$$\int d^4x\, d^2\theta\, \Phi^n \ , \tag{5.13}$$

where Φ is a chiral superfield. This integral cannot be converted into an integral in full superspace, essentially because there are no covariant derivatives to play with. Integrals like (5.13), which cannot be converted into integral in full superspace, are called F-terms. All others, like (5.12), are called D-terms.

Coming back to our problem, it is clear that since the simplest non-vanishing integral in full superspace, eq. (5.1), already contains field derivatives, we must turn to F-terms. First notice that any holomorphic function of Φ, namely a function $W(\Phi)$ such that $\partial W/\partial\bar{\Phi} = 0$, is a chiral superfield, if so is Φ. Indeed

$$\bar{D}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi}\bar{D}_{\dot{\alpha}}\Phi + \frac{\partial W}{\partial \bar{\Phi}}\bar{D}_{\dot{\alpha}}\bar{\Phi} = 0 \ . \tag{5.14}$$

The proposed term for describing interactions in a theory of a chiral superfield is

$$\mathcal{L}_{int} = \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}) \ , \tag{5.15}$$

where W is a holomorphic function of Φ and the hermitian conjugate has been added to make the whole thing real. The function W is called *superpotential*. Which properties should W satisfy? First, as already noticed, W should be a holomorphic

function of Φ. This ensures it to be a chiral superfield and the integral (5.15) to be supersymmetric invariant. Second, W should not contain covariant derivatives since $D_\alpha\Phi$ is not a chiral superfield, given that D_α and $\bar{D}_{\dot{\alpha}}$ do not (anti)commute. Finally, $[W] = 3$, to make the expression (5.15) have dimension 4. The upshot is that the superpotential should have an expression like

$$W(\Phi) = \sum_{n=1}^{\infty} a_n \, \Phi^n \tag{5.16}$$

If renormalizability is an issue, the lowest component of W should not contain operators of dimensionality bigger than 3, given that the θ^2 component has dimensionality $[W]+1$. Since Φ has dimension one, it follows that to avoid non-renormalizable operators the highest power in the expansion (5.16) should be $n = 3$, so that the θ^2 component will have operators of dimension 4, at most. In other words, a renormalizable superpotential should be at most cubic.

The superpotential is also constrained by R-symmetry. Given a chiral superfield Φ, if the R-charge of its lowest component ϕ is r, then that of ψ is $r-1$ and that of F is $r-2$. This follows from the commutation relations (2.78). Therefore, we have

$$R[\theta] = 1 \quad , \quad R[\bar{\theta}] = -1 \quad , \quad R[d\theta] = -1 \quad , \quad R[d\bar{\theta}] = 1 \tag{5.17}$$

(recall that $d\theta = \partial/\partial\theta$, and similarly for $\bar{\theta}$). In theories where R-symmetry is a (classical) symmetry, it follows that the superpotential should have R-charge equal to 2

$$R[W] = 2 \, , \tag{5.18}$$

in order for the Lagrangian (5.15) to have R-charge 0 and hence be R-symmetry invariant. As far as the Kähler potential is concerned, first notice that the integral measure in full superspace has R-charge 0, because of eqs. (5.17). This implies that for theories with a R-symmetry, the Kähler potential should also have R-charge 0. This is trivially the case for a canonical Kähler potential, since $\bar{\Phi}\Phi$ has R-charge 0. If one allows for non-canonical Kähler potential, that is non-renormalizable interactions, then besides the reality condition, one should also impose that $c_{nm} = 0$ whenever $n \neq m$, see eq. (5.6).

The integration in superspace of the Lagrangian (5.15) is easily done recalling the expansion of the superpotential in powers of θ. We have

$$W(\Phi) = W(\phi) + \sqrt{2}\frac{\partial W}{\partial \phi}\theta\psi - \theta\theta\left(\frac{\partial W}{\partial \phi}F + \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi\right) \, , \tag{5.19}$$

where $\frac{\partial W}{\partial \phi}$ and $\frac{\partial^2 W}{\partial \phi \partial \phi}$ are evaluated at $\Phi = \phi$. So we have, modulo total space-time derivatives

$$\mathcal{L}_{int} = \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}) = -\frac{\partial W}{\partial \phi} F - \frac{1}{2}\frac{\partial^2 W}{\partial \phi \partial \phi}\psi\psi + h.c.\ , \tag{5.20}$$

where, again, the r.h.s. is already evaluated at x^μ.

We can now assemble all what we have found. The most generic supersymmetric matter Lagrangian has the following form

$$\mathcal{L}_{matter} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi})\ . \tag{5.21}$$

For renormalizable theories the Kähler potential is just $\bar{\Phi}\Phi$ and the superpotential at most cubic. In this case we get

$$\begin{aligned} \mathcal{L} &= \int d^2\theta\, d^2\bar{\theta}\, \bar{\Phi}\Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}) \qquad (5.22)\\ &= \partial_\mu \bar{\phi}\ \partial^\mu \phi + \frac{i}{2}\left(\partial_\mu \psi\ \sigma^\mu \bar{\psi} - \psi\ \sigma^\mu \partial_\mu \bar{\psi}\right) + \bar{F}F - \frac{\partial W}{\partial \phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial \phi \partial \phi}\psi\psi + h.c.\ , \end{aligned}$$

where

$$W(\Phi) = \sum_1^3 a_n\, \Phi^n\ . \tag{5.23}$$

We can now integrate the auxiliary fields F and $\bar{F}$ out by substituting in the Lagrangian their equations of motion which read

$$\bar{F} = \frac{\partial W}{\partial \phi} \quad , \quad F = \frac{\partial \overline{W}}{\partial \bar{\phi}}\ . \tag{5.24}$$

Doing so, we get the on-shell Lagrangian

$$\mathcal{L}_{on-shell} = \partial_\mu \bar{\phi}\ \partial^\mu \phi + \frac{i}{2}\left(\partial_\mu \psi\ \sigma^\mu \bar{\psi} - \psi\ \sigma^\mu \partial_\mu \bar{\psi}\right) - |\frac{\partial W}{\partial \phi}|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial \phi \partial \phi}\psi\psi - \frac{1}{2}\frac{\partial^2 \overline{W}}{\partial \bar{\phi} \partial \bar{\phi}}\bar{\psi}\bar{\psi} \tag{5.25}$$

From the on-shell Lagrangian we can now read the scalar potential which is

$$V(\phi, \bar{\phi}) = |\frac{\partial W}{\partial \phi}|^2 = \bar{F}F\ , \tag{5.26}$$

where the last equality holds on-shell, namely upon use of eqs. (5.24).

All what we said so far can be easily generalized to a set of chiral superfields Φ^i where $i = 1, 2, \ldots, n$. In this case the most general Lagrangian reads

$$\mathcal{L} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi^i, \bar{\Phi}_i) + \int d^2\theta\, W(\Phi^i) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}_i)\ . \tag{5.27}$$

For renormalizable theories we have

$$K(\Phi^i, \bar{\Phi}_i) = \bar{\Phi}_i \Phi^i \quad \text{and} \quad W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k \ , \tag{5.28}$$

where summation over dummy indices is understood (notice that a quadratic Kähler potential can always be brought to such diagonal form by means of a $GL(n, \mathbb{C})$ transformation on the most general term $K^i_j \bar{\Phi}_i \Phi^j$, where K^i_j is a constant hermitian matrix). In this case the scalar potential reads

$$V(\phi^i, \bar{\phi}_i) = \sum_{i=1}^{n} |\frac{\partial W}{\partial \phi^i}|^2 = \bar{F}_i F^i \ , \tag{5.29}$$

where

$$\bar{F}_i = \frac{\partial W}{\partial \phi^i} \quad , \quad F^i = \frac{\partial \overline{W}}{\partial \bar{\phi}_i} \ . \tag{5.30}$$

5.1.1 Non-linear sigma model I

The possibility to deal with non-renormalizable supersymmetric field theories we alluded to previously, is not just academic. In fact, one often has to deal with effective field theories at low energy. The Standard Model itself, though renormalizable, is best thought of as an effective field theory, valid up to a scale of order the TeV scale. Not to mention other effective field theories which are relevant beyond the realm of particle physics. In this section we would like to say something more about the Lagrangian (5.27), allowing for the most general Kähler potential and superpotential, and showing that what one ends-up with is nothing but a supersymmetric non-linear σ-model. Though a bit heavy notation-wise, the effort we are going to do here will be very instructive as it will show the deep relation between supersymmetry and geometry.

Since we do not care about renormalizability here, the superpotential is no more restricted to be cubic and the Kähler potential is no more restricted to be quadratic (though it must still be real and with no covariant derivatives acting on the chiral superfields Φ^i). For later purposes it is convenient to define the following quantities

$$K_i = \frac{\partial}{\partial \phi^i} K(\phi, \bar{\phi}) \quad , \quad K^i = \frac{\partial}{\partial \bar{\phi}_i} K(\phi, \bar{\phi}) \quad , \quad K^j_i = \frac{\partial^2}{\partial \phi^i \partial \bar{\phi}_j} K(\phi, \bar{\phi})$$

$$W_i = \frac{\partial}{\partial \phi^i} W(\phi) \quad , \quad W^i = \overline{W_i} \quad , \quad W_{ij} = \frac{\partial^2}{\partial \phi^i \partial \phi^j} W(\phi) \quad , \quad W^{ij} = \overline{W_{ij}} \ ,$$

where in the above fromulæ both the Kähler potential and the superpotential are meant as their restriction to the scalar component of the chiral superfields, while ϕ

stands for the full n-dimensional vector made out of the n scalar fields ϕ^i (similarly for $\bar{\phi}$).

Extracting the F-term contribution in terms of the above quantities is pretty simple. The superpotential can be written as

$$W(\Phi) = W(\phi) + W_i \Delta^i + \frac{1}{2} W_{ij} \Delta^i \Delta^j \ , \tag{5.31}$$

where we have defined

$$\Delta^i(y) = \Phi^i - \phi^i(y) = \sqrt{2}\theta\psi^i(y) - \theta\theta F^i(y) \ , \tag{5.32}$$

and we get for the F-term

$$\int d^2\theta \, W(\Phi^i) + \int d^2\bar{\theta} \, \overline{W}(\bar{\Phi}_i) = \left(-W_i F^i - \frac{1}{2} W_{ij} \psi^i \psi^j \right) + h.c. \ , \tag{5.33}$$

where, see the comment after eq. (5.11), all quantities on the r.h.s. are evaluated in x^μ.

Extracting the D-term contribution is more tricky (but much more instructive). Let us first define

$$\Delta^i(x) = \Phi^i - \phi^i(x) \quad , \quad \bar{\Delta}_i(x) = \bar{\Phi}_i - \bar{\phi}_i(x) \tag{5.34}$$

which read

$$\Delta^j(x) = \sqrt{2}\theta\psi^j(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^j(x) - \theta\theta F^j(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi^j(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\,\bar{\theta}\bar{\theta}\Box\phi^j(x)$$

$$\bar{\Delta}_j(x) = \sqrt{2}\bar{\theta}\bar{\psi}_j(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi}_j(x) - \bar{\theta}\bar{\theta}\bar{F}_j(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}_j(x) - \frac{1}{4}\theta\theta\,\bar{\theta}\bar{\theta}\Box\bar{\phi}_j(x) \ .$$

Note that $\Delta^i\Delta^j\Delta^k = \bar{\Delta}_i\bar{\Delta}_j\bar{\Delta}_k = 0$. With these definitions the Kähler potential can be written as follows

$$\begin{aligned} K(\Phi,\bar{\Phi}) &= K(\phi,\bar{\phi}) + K_i\Delta^i + K^i\bar{\Delta}_i + \frac{1}{2}K_{ij}\Delta^i\Delta^j + \frac{1}{2}K^{ij}\bar{\Delta}_i\bar{\Delta}_j + K_i^j\Delta^i\bar{\Delta}_j + \\ &+ \frac{1}{2}K_{ij}^k\Delta^i\Delta^j\bar{\Delta}_k + \frac{1}{2}K_k^{ij}\bar{\Delta}_i\bar{\Delta}_j\Delta^k + \frac{1}{4}K_{ij}^{kl}\Delta^i\Delta^j\bar{\Delta}_k\bar{\Delta}_l \ . \end{aligned} \tag{5.35}$$

We can now compute the D-term contribution to the Lagrangian. We get

$$\begin{aligned} \int d^2\theta\, d^2\bar{\theta}\; K(\Phi,\bar{\Phi}) &= -\frac{1}{4}K_i\Box\phi^i - \frac{1}{4}K^i\Box\bar{\phi}_i - \frac{1}{4}K_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}K^{ij}\partial_\mu\bar{\phi}_i\partial^\mu\bar{\phi}_j + \\ &+ K_i^j\left(F^i\bar{F}_j + \frac{1}{2}\partial_\mu\phi^i\partial^\mu\bar{\phi}_j - \frac{i}{2}\psi^i\sigma^\mu\partial_\mu\bar{\psi}_j + \frac{i}{2}\partial_\mu\psi^i\sigma^\mu\bar{\psi}_j\right) \\ &+ \frac{i}{4}K_{ij}^k\left(\psi^i\sigma^\mu\bar{\psi}_k\partial_\mu\phi^j + \psi^j\sigma^\mu\bar{\psi}_k\partial_\mu\phi^i - 2i\psi^i\psi^j\bar{F}_k\right) - \frac{i}{4}K_k^{ij}\,(h.c.) + \\ &+ \frac{1}{4}K_{ij}^{kl}\psi^i\psi^j\bar{\psi}_k\bar{\psi}_l \ , \end{aligned} \tag{5.36}$$

up to total derivatives. Notice now that

$$\Box K(\phi,\bar\phi) = K_i \Box\phi^i + K^i \Box\bar\phi_i + 2K_i^j \partial_\mu\bar\phi_j \partial^\mu\phi^i + K_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + K^{ij}\partial_\mu\bar\phi_i\partial^\mu\bar\phi_j \ . \quad (5.37)$$

Using this identity we can eliminate K_{ij} and K^{ij}, and rewrite eq. (5.36) as

$$\begin{aligned}\int d^2\theta\, d^2\bar\theta\ K(\Phi,\bar\Phi) &= K_i^j\left(F^i\bar F_j + \partial_\mu\phi^i\partial^\mu\bar\phi_j - \frac{i}{2}\psi^i\sigma^\mu\partial_\mu\bar\psi_j + \frac{i}{2}\partial_\mu\psi^i\sigma^\mu\bar\psi_j\right) + \\ &+ \frac{i}{4}K_{ij}^k\left(\psi^i\sigma^\mu\bar\psi_k\partial_\mu\phi^j + \psi^j\sigma^\mu\bar\psi_k\partial_\mu\phi^i - 2i\psi^i\psi^j\bar F_k\right) - \frac{i}{4}K_k^{ij}\,(h.c.) + \\ &+ \frac{1}{4}K_{ij}^{kl}\psi^i\psi^j\bar\psi_k\bar\psi_l \ , \end{aligned} \quad (5.38)$$

again up to total derivatives.

A few important comments are in order. As just emphasized, independently whether the fully holomorphic and fully anti-holomorphic Kähler potential components, K_{ij} and K^{ij} respectively, are or are not vanishing, they do not enter the final result (5.38). In other words, from a practical view point it is as if they are not there. The only two-derivative contribution entering the effective Lagrangian is hence K_i^j. This means that the transformation

$$K(\phi,\bar\phi) \to K(\phi,\bar\phi) + \Lambda(\phi) + \overline{\Lambda}(\bar\phi) \ , \quad (5.39)$$

known as Kähler transformation, is a symmetry of the theory (in fact, such symmetry applies to the full Kähler potential, as we have already observed). This is important for our second comment.

The function K_i^j which normalizes the kinetic term of all fields in eq. (5.38), is hermitian, i.e. $K_i^j = K_j^{i\,\dagger}$, since $K(\phi,\bar\phi)$ is a real function. Moreover, it is positive definite and non-singular, because of the correct sign for the kinetic terms of all non-auxiliary fields. That is to say K_i^j has all necessary properties to be interpreted as a metric of a manifold $\mathcal{M}$ of complex dimension n whose coordinates are the scalar fields ϕ^i themselves. The metric K_i^j is in fact the second derivative of a (real) scalar function K, since

$$K_i^j = \frac{\partial^2}{\partial\phi^i\partial\bar\phi_j}K(\phi,\bar\phi) \ . \quad (5.40)$$

In this case we speak of a Kähler metric and the manifold $\mathcal{M}$ is what mathematically is known as Kähler manifold. The scalar fields are maps from space-time to this Riemannian manifold, which supersymmetry dictates to be a Kähler manifold. This is the (supersymmetric) σ-model. Actually, in order to prove that the Lagrangian

is a σ-model, with target space the Kähler manifold $\mathcal{M}$, we should prove that not only the kinetic term but any other term in the Lagrangian can be written in terms of geometric quantities defined on $\mathcal{M}$, e.g. the affine connection and the curvature tensor.

With some work one can compute both of them out of the Kähler metric K_i^j and, using the auxiliary field equations of motion

$$F^i = (K^{-1})^i_k W^k - \frac{1}{2}(K^{-1})^i_k K^k_{lm}\psi^l\psi^m \tag{5.41}$$

(remark: the above equation shows that when the Kähler potential is not canonical, the F-fields can depend also on fermion fields!), get for the Lagrangian

$$\begin{aligned}\mathcal{L} &= K_i^j\left(\partial_\mu\phi^i\partial^\mu\bar{\phi}_j + \frac{i}{2}D_\mu\psi^i\sigma^\mu\bar{\psi}_j - \frac{i}{2}\psi^i\sigma^\mu D_\mu\bar{\psi}_j\right) - (K^{-1})^i_j W_i W^j \qquad (5.42)\\ &- \frac{1}{2}\left(W_{ij} - \Gamma^k_{ij}W_k\right)\psi^i\psi^j - \frac{1}{2}\left(W^{ij} - \Gamma^{ij}_k W^k\right)\bar{\psi}_i\bar{\psi}_j + \frac{1}{4}R^{kl}_{ij}\psi^i\psi^j\bar{\psi}_k\bar{\psi}_l\ ,\end{aligned}$$

where

$$V(\phi,\bar{\phi}) = (K^{-1})^i_j W_i W^j \tag{5.43}$$

is the scalar potential, and the covariant derivatives for the fermions are defined as

$$\begin{aligned}D_\mu\psi^i &= \partial_\mu\psi^i + \Gamma^i_{jk}\partial_\mu\phi^i\psi^k\\ D_\mu\bar{\psi}_i &= \partial_\mu\bar{\psi}_i + \Gamma_i^{kj}\partial_\mu\bar{\phi}_k\bar{\psi}_j\ .\end{aligned}$$

With our conventions on indices, $\Gamma^i_{jk} = (K^{-1})^i_m K^m_{jk}$ while $\Gamma_i^{kj} = (K^{-1})^m_i K_m^{kj}$ and $R^{kl}_{ij} = K^{kl}_{ij} - K^m_{ij}(K^{-1})^n_m K^{kl}_n$.

As anticipated, a complicated component field Lagrangian is uniquely characterized by the geometry of the target space. Once a Kähler potential K is specified, anything in the Lagrangian (masses and couplings) depends geometrically on this potential (and on W). This shows the strong connection between supersymmetry and geometry. There are of course infinitely many Kähler metrics and therefore infinitely many $\mathcal{N}=1$ supersymmetric σ-models. The normalizable case, $K_i^j = \delta_i^j$, is just the simplest such instances.

The more supersymmetry the more constraints, hence one could imagine that there should be more restrictions on the geometric structure of the σ-model for theories with extended supersymmetry. This is indeed the case, as we will see explicitly when discussing the $\mathcal{N}=2$ version of the supersymmetric σ-model. In this case, the scalar manifold is further restricted to be a special class of Kähler

manifolds, known as special-Kähler manifolds. For $\mathcal{N} = 4$ constraints are even sharper. In fact, in this case the Lagrangian turns out to be unique, the only possible scalar manifold being the trivial one, $\mathcal{M} = \mathbb{R}^{6n}$, if n is the number of $\mathcal{N} = 4$ vector multiplets (recall that a $\mathcal{N} = 4$ vector multiplet contains six scalars). So, for $\mathcal{N} = 4$ supersymmetry the only allowed Kähler potential is the canonical one! As we will discuss later, this has immediate (and drastic) consequences on the quantum behavior of $\mathcal{N} = 4$ theories.

5.2 $\mathcal{N} = 1$ SuperYang-Mills

We would like now to find a supersymmetric invariant action describing the dynamics of vector superfields. In other words, we want to write down the supersymmetric version of Yang-Mills theory. Let us start considering an abelian theory, with gauge group $G = U(1)$. The basic object we should play with is the vector superfield V, which is the supersymmetric extension of a spin one field. Notice, however, that the vector v^μ appears explicitly in V so the first thing to do is to find a suitable supersymmetric generalization of the field strength, which is the gauge invariant object which should enter the action. Let us define the following superfield

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \quad , \quad \overline{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \ , \tag{5.44}$$

and see if this can do the job. First, W_α is obviously a superfield, since V is a superfield and both $\bar{D}_{\dot{\alpha}}$ and D_α commute with supersymmetry transformations. In fact, W_α is a chiral superfield, since $\bar{D}^3 = 0$ identically. The chiral superfield W_α is invariant under the gauge transformation (4.62). Indeed

$$\begin{aligned} W_\alpha \ &\to \ W_\alpha - \frac{1}{4}\bar{D}\bar{D}D_\alpha\left(\Phi + \bar{\Phi}\right) = W_\alpha + \frac{1}{4}\bar{D}^{\dot{\beta}}\bar{D}_{\dot{\beta}}D_\alpha\Phi \\ &= \ W_\alpha + \frac{1}{4}\bar{D}^{\dot{\beta}}\{\bar{D}_{\dot{\beta}}, D_\alpha\}\Phi = W_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\bar{D}^{\dot{\beta}}\Phi = W_\alpha \ . \end{aligned} \tag{5.45}$$

This also means that, as anticipated, as far as we deal with W_α, we can stick to the WZ-gauge without bothering about compensating gauge transformations or anything.

In order to find the component expression for W_α it is useful to use the $(y, \theta, \bar{\theta})$ coordinate system, momentarily. In the WZ gauge the vector superfield reads

$$V_{WZ} = \theta\sigma^\mu\bar{\theta}\, v_\mu(y) + i\theta\theta\,\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}\bar{\theta}\,\theta\lambda(y) + \frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}\left(D(y) - i\partial_\mu v^\mu(y)\right) \ . \tag{5.46}$$

It is a simple exercise we leave to the reader to prove that expanding in $(x, \theta, \bar{\theta})$ coordinate system, the above expression reduces to eq. (4.65). Acting with D_α we get

$$D_\alpha V_{WZ} = \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} v_\mu + 2i\theta_\alpha \bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\lambda_\alpha + \theta_\alpha \bar{\theta}\bar{\theta} D + 2i(\sigma^{\mu\nu})_\alpha{}^\beta \theta_\beta \bar{\theta}\bar{\theta}\partial_\mu v_\nu + \theta\theta\bar{\theta}\bar{\theta}\sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \bar{\lambda}^{\dot{\beta}} \tag{5.47}$$

(where we used the identity $\sigma^\mu \bar{\sigma}^\nu - \eta^{\mu\nu} = 2\sigma^{\mu\nu}$ and y-dependence is understood), and finally

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i\left(\sigma^{\mu\nu}\theta\right)_\alpha F_{\mu\nu} + \theta\theta\left(\sigma^\mu \partial_\mu \bar{\lambda}\right)_\alpha \ , \tag{5.48}$$

where $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ is the usual gauge field strength. So it seems this is the right superfield we were searching for! W_α is the so-called supersymmetric field strength and it is an instance of a chiral superfield whose lowest component is not a scalar field, as we have been used to, but in fact a Weyl fermion, λ_α, the gaugino. For this reason, W_α is also called the gaugino superfield.

Given that W_α is a chiral superfield, a putative supersymmetric Lagrangian could be constructed out of the following integral in chiral superspace (notice: correctly, it has dimension four)

$$\int d^2\theta \, W^\alpha W_\alpha \ . \tag{5.49}$$

Plugging eq. (5.48) into the expression above and computing the superspace integral one gets after some simple algebra

$$\int d^2\theta \, W^\alpha W_\alpha = -\frac{1}{2} F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^\mu \partial_\mu \bar{\lambda} + D^2 + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} \ . \tag{5.50}$$

One can get a real object by adding the hermitian conjugate to (5.50), having finally

$$\mathcal{L}_{gauge} = \int d^2\theta \, W^\alpha W_\alpha + \int d^2\bar{\theta} \, \overline{W}^{\dot{\alpha}} \overline{W}_{\dot{\alpha}} = -F_{\mu\nu}F^{\mu\nu} - 4i\lambda\sigma^\mu \partial_\mu \bar{\lambda} + 2D^2 \ . \tag{5.51}$$

This is the supersymmetric version of the abelian gauge Lagrangian (up to an overall normalization to be fixed later). As anticipated, D is an (real) auxiliary field.

The Lagrangian (5.51) has been written as an integral over chiral superspace, so one might be tempted to say it is a F-term. This is wrong since (5.51) is *not* a true F-term. Indeed, it can be re-written as an integral in full superspace (while F-terms cannot)

$$\int d^2\theta \, W^\alpha W_\alpha \ = \int d^2\theta \, d^2\bar{\theta} \ D^\alpha V \cdot W_\alpha \ , \tag{5.52}$$

and so it is in fact a D-term. As we will see later, this fact has important consequences at the quantum level, when discussing renormalization properties of supersymmetric Lagrangians.

All what we said, so far, has to do with abelian interactions. What changes if we consider a non-abelian gauge group? First we have to promote the vector superfield to

$$V = V_a T^a \quad a = 1, \ldots, \dim G \ , \tag{5.53}$$

where T^a are hermitian generators and V_a are $n = \dim G$ vector superfields. Second, we have to define the finite version of the gauge transformation (4.62) which can be written as

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \ . \tag{5.54}$$

One can easily check that at leading order in Λ this indeed reduces to (4.62), upon the identification $\Phi = -i\Lambda$. Again, it is straightforward to set the WZ gauge for which

$$e^V = 1 + V + \frac{1}{2}V^2 \ . \tag{5.55}$$

In what follows this gauge choice is always understood. The gaugino superfield is generalized as follows

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V} D_\alpha e^V\right) \quad , \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD\left(e^V \bar{D}_{\dot{\alpha}} e^{-V}\right) \tag{5.56}$$

which again reduces to the expression (5.44) to first order in V. Let us look at eq. (5.56) more closely. Under the gauge transformation (5.54) W_α transforms as

$$\begin{aligned} W_\alpha &\to -\frac{1}{4}\bar{D}\bar{D}\left[e^{i\Lambda} e^{-V} e^{-i\bar{\Lambda}} D_\alpha \left(e^{i\bar{\Lambda}} e^V e^{-i\Lambda}\right)\right] \\ &= -\frac{1}{4}\bar{D}\bar{D}\left[e^{i\Lambda} e^{-V}\left(\left(D_\alpha e^V\right) e^{-i\Lambda} + e^V D_\alpha e^{-i\Lambda}\right)\right] \\ &= -\frac{1}{4} e^{i\Lambda}\bar{D}\bar{D}\left(e^{-V} D_\alpha e^V\right) e^{-i\Lambda} = e^{i\Lambda} W_\alpha e^{-i\Lambda} \ , \end{aligned} \tag{5.57}$$

where we used the fact that, given that Λ (and products thereof) is a chiral superfield, $\bar{D}_{\dot{\alpha}} e^{-i\Lambda} = 0$, $D_\alpha e^{i\bar{\Lambda}} = 0$ and also $\bar{D}\bar{D} D_\alpha e^{-i\Lambda} = 0$. The end result is that W_α transforms covariantly under a finite gauge transformation, as it should. Similarly, one can prove that

$$\bar{W}_{\dot{\alpha}} \to e^{i\bar{\Lambda}} \bar{W}_{\dot{\alpha}} e^{-i\bar{\Lambda}} \ . \tag{5.58}$$

Let us now expand W_α in component fields. We would expect the non-abelian generalization of eq. (5.48). We have

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\left[\left(1-V+\frac{1}{2}V^2\right)D_\alpha\left(1+V+\frac{1}{2}V^2\right)\right] \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{8}\bar{D}\bar{D}D_\alpha V^2 + \frac{1}{4}\bar{D}\bar{D}VD_\alpha V \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{8}\bar{D}\bar{D}VD_\alpha V - \frac{1}{8}\bar{D}\bar{D}D_\alpha V\cdot V + \frac{1}{4}\bar{D}\bar{D}VD_\alpha V \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \frac{1}{8}\bar{D}\bar{D}\left[V, D_\alpha V\right] \ . \end{aligned}$$

The first term is the same as the one we already computed in the abelian case. As for the second term we get

$$\frac{1}{8}\bar{D}\bar{D}\left[V, D_\alpha V\right] = \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha\left[v_\mu, v_\nu\right] - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot{\beta}}\left[v_\mu, \bar{\lambda}^{\dot{\beta}}\right] \ . \tag{5.59}$$

Adding everything up simply amounts to turn ordinary derivatives into covariant ones, finally obtaining

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i\left(\sigma^{\mu\nu}\theta\right)_\alpha F_{\mu\nu} + \theta\theta\left(\sigma^\mu D_\mu\bar{\lambda}(y)\right)_\alpha \tag{5.60}$$

with

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}\left[v_\mu, v_\nu\right] \quad , \quad D_\mu = \partial_\mu \ - \frac{i}{2}\left[v_\mu, \ \right] \ , \tag{5.61}$$

which provides the correct non-abelian generalization for the field strength and the (covariant) derivatives.

In view of coupling the pure SYM Lagrangian with matter, it is convenient to introduce the coupling constant g explicitly, making the redefinition

$$V \to 2gV \ \Leftrightarrow \ v_\mu \to 2gv_\mu \ \ , \ \ \lambda \to 2g\lambda \ \ , \ \ D \to 2gD \ , \tag{5.62}$$

which implies the following changes in the final Lagrangian. First, we have now

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig\left[v_\mu, v_\nu\right] \ \ , \ \ D_\mu = \partial_\mu \ - ig\left[v_\mu, \ \right] \ . \tag{5.63}$$

Moreover, the gaugino superfield (5.60) should be multiplied by $2g$ and the (non-abelian version of the) Lagrangian (5.51) by $1/4g^2$. The end result for the SuperYang-Mills Lagrangian (SYM) reads

$$\begin{aligned} \mathcal{L}_{SYM} &= \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{Tr}W^\alpha W_\alpha\right) \\ &= \mathrm{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}D^2\Big] + \frac{\theta_{YM}}{32\pi^2}\ g^2\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}, \end{aligned} \tag{5.64}$$

where we have introduced the complexified gauge coupling

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2} \tag{5.65}$$

and the dual field strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \tag{5.66}$$

while gauge group generators are normalized as $\mathrm{Tr}\, T^a T^b = \delta^{ab}$.

5.3 $\mathcal{N} = 1$ Gauge-matter actions

We want to couple radiation with matter in a supersymmetric consistent way. To this end, let us consider a chiral superfield Φ transforming in some representation R of the gauge group G, $T^a \to (T^a_R)^i_j$ where $i, j = 1, 2, \ldots, \dim R$. Under a gauge transformation we expect

$$\Phi \to \Phi' = e^{i\Lambda}\Phi \quad , \quad \Lambda = \Lambda_a T^a_R \quad , \tag{5.67}$$

where Λ must be a chiral superfield, otherwise the transformed Φ would not be a chiral superfield anymore. Notice, however, that in this way the chiral superfield kinetic action we have derived previously would not be gauge invariant since

$$\bar{\Phi}\Phi \to \bar{\Phi}e^{-i\bar{\Lambda}}e^{i\Lambda}\Phi \neq \bar{\Phi}\Phi \ . \tag{5.68}$$

So we have to change the kinetic action. The correct expression is in fact

$$\bar{\Phi}e^{V}\Phi \ , \tag{5.69}$$

which is indeed gauge invariant and also supersymmetric invariant (modulo total space-time derivatives), when integrated in superspace. Therefore, the complete Lagrangian for charged matter reads

$$\mathcal{L}_{matter} = \int d^2\theta\, d^2\bar{\theta}\ \bar{\Phi}e^{V}\Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}) \ . \tag{5.70}$$

Obviously the superpotential should be compatible with the gauge symmetry, *i.e.* it should be gauge invariant itself. This means that a term like

$$a_{i_1 i_2 \ldots i_n}\Phi^{i_1}\Phi^{i_2}\ldots\Phi^{i_n} \tag{5.71}$$

is allowed only if $a_{i_1 i_2 \ldots i_n}$ is an invariant tensor of the gauge group and if $R\times R\times\cdots\times R$ n times contains the singlet representation of the gauge group G.

As an explicit example, take the gauge group of strong interactions, $G = SU(3)$, and consider quarks as matter field. In this case R is the fundamental representation, $R = \underline{3}$. Since $\underline{3} \times \underline{3} \times \underline{3} = \underline{1} + \dots$ and ϵ_{ijk} is an invariant tensor of $SU(3)$, while $\underline{1} \not\subset \underline{3} \times \underline{3}$ it follows that a supersymmetric and gauge invariant cubic term is allowed, but a mass term is not. In order to have mass terms for quarks, one needs $R = \underline{3} + \bar{\underline{3}}$ corresponding to a chiral superfield Φ in the $\underline{3}$ (quark) and a chiral superfield $\tilde{\Phi}$ in the $\bar{\underline{3}}$ (anti-quark). In this case $\tilde{\Phi}\Phi$ is gauge invariant and does correspond to a mass term. This is consistent with the fact that a chiral superfield contains a Weyl fermion only and quarks are described by Dirac fermions. The lesson is that to construct supersymmetric actions with colour charged matter, one needs to introduce two sets of chiral superfields which transform in conjugate representations of the gauge group. This is just the supersymmetric version of what happens in ordinary QCD or in any non-abelian gauge theory with fermions transforming in complex representations ($G = SU(2)$ is an exception because $\underline{2} \simeq \bar{\underline{2}}$) .

Let us now compute the D-term of the Lagrangian (5.70). We have (as usual we work in the WZ gauge)

$$\bar{\Phi} e^V \Phi = \bar{\Phi}\Phi + \bar{\Phi} V \Phi + \frac{1}{2}\bar{\Phi} V^2 \Phi \ . \tag{5.72}$$

The first term is the one we have already calculated, so let us focus on the D-term contribution of the other two. After some algebra we get

$$\begin{aligned} \bar{\Phi} V \Phi|_{\theta\theta\,\bar{\theta}\bar{\theta}} &= \frac{i}{2}\bar{\phi} v^\mu \partial_\mu \phi - \frac{i}{2}\partial_\mu \bar{\phi} v^\mu \phi - \frac{1}{2}\bar{\psi}\bar{\sigma}^\mu v_\mu \psi + \frac{i}{\sqrt{2}}\bar{\phi}\lambda\psi - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}\phi + \frac{1}{2}\bar{\phi} D \phi \\ \bar{\Phi} V^2 \Phi|_{\theta\theta\,\bar{\theta}\bar{\theta}} &= \frac{1}{2}\bar{\phi} v^\mu v_\mu \phi \ . \end{aligned}$$

Putting everything together we finally get (up to total derivatives)

$$\bar{\Phi} e^V \Phi|_{\theta\theta\,\bar{\theta}\bar{\theta}} = \left(\overline{D_\mu \phi}\right) D^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + \bar{F} F + \frac{i}{\sqrt{2}}\bar{\phi}\lambda\psi - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}\phi + \frac{1}{2}\bar{\phi} D \phi \ , \tag{5.73}$$

where $D_\mu = \partial_\mu - \frac{i}{2} v_\mu^a T_R^a$.

Performing the rescaling $V \to 2gV$ and rewriting $\bar{\psi}\bar{\sigma}^\mu D_\mu \psi = \psi \sigma^\mu D_\mu \bar{\psi}$ (recall the spinor identity $\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi$) we get finally

$$\bar{\Phi} e^{2gV} \Phi|_{\theta\theta\,\bar{\theta}\bar{\theta}} = \left(\overline{D_\mu \phi}\right) D^\mu \phi - i\psi\sigma^\mu D_\mu \bar{\psi} + \bar{F} F + i\sqrt{2} g \bar{\phi}\lambda\psi - i\sqrt{2} g \bar{\psi}\bar{\lambda}\phi + g\bar{\phi} D \phi \ , \tag{5.74}$$

where now $D_\mu = \partial_\mu - ig v_\mu^a T_R^a$. So we see that the D-term in the Lagrangian (5.70) not only provides matter kinetic terms but also interaction terms between matter

fields ϕ, ψ and gauginos λ, where it is understood that

$$\bar{\phi}\lambda\psi = \bar{\phi}_i (T_R^a)^i_j \lambda_a \psi^j \ , \tag{5.75}$$

and similarly for the other couplings.

To get the most general action there is one term still missing: the so called Fayet-Iliopulos term. Suppose that the gauge group is not semi-simple, i.e. it contains $U(1)$ factors. Let V^A be the vector superfields corresponding to the abelian factors, $A = 1, 2, \ldots, n$, where n is the number of abelian factors. The D-term of V^A transforms as a total derivative under super-gauge transformations, since

$$V^A \to V^A - i\Lambda + i\bar{\Lambda} \quad : \quad D^A \to D^A + \partial_\mu \partial^\mu (\ldots) \ . \tag{5.76}$$

Therefore a Lagrangian of this type

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta}\, V^A = \frac{1}{2} \sum_A \xi_A D^A \tag{5.77}$$

is supersymmetric invariant (since V^A are superfields) and gauge invariant, modulo total space-time derivatives.

We can now assemble all ingredients and write down the most general $\mathcal{N} = 1$ supersymmetric Lagrangian (with canonical Kähler potential, hence renormalizable, if the superpotential is at most cubic)

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{SYM} + \mathcal{L}_{matter} + \mathcal{L}_{FI} = \\
&= \frac{1}{32\pi} \mathrm{Im} \left(\tau \int d^2\theta \, \mathrm{Tr} W^\alpha W_\alpha \right) + 2g \sum_A \xi_A \int d^2\theta d^2\bar{\theta}\, V^A + \\
&+ \int d^2\theta d^2\bar{\theta}\, \bar{\Phi} e^{2gV} \Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}) \\
&= \mathrm{Tr}\Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \Big] + \frac{\theta_{YM}}{32\pi^2}\, g^2 \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \\
&+ g \sum_A \xi_A D^A + \big(\overline{D_\mu \phi}\big) D^\mu \phi - i\psi\sigma^\mu D_\mu \bar{\psi} + \bar{F} F + i\sqrt{2} g \bar{\phi} \lambda \psi \\
&- i\sqrt{2} g \bar{\psi} \bar{\lambda} \phi + g \bar{\phi} D \phi - \frac{\partial W}{\partial \phi^i} F^i - \frac{\partial \overline{W}}{\partial \bar{\phi}_i} \bar{F}_i - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi^i \psi^j - \frac{1}{2} \frac{\partial^2 \overline{W}}{\partial \bar{\phi}_i \partial \bar{\phi}_j} \bar{\psi}_i \bar{\psi}_j \ .
\end{aligned} \tag{5.78}$$

Notice that both D^a and F^i are auxiliary fields and can be integrated out. The equations of motion of the auxiliary fields read

$$\bar{F}_i = \frac{\partial W}{\partial \phi^i} \quad , \quad D^a = -g \bar{\phi} T^a \phi - g\xi^a \quad (\xi^a = 0 \ \text{ if } \ a \neq A) \ . \tag{5.79}$$

These can plugged back into (5.78) leading to the following on-shell Lagrangian

$$\begin{aligned}\mathcal{L} &= \mathrm{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda}\Big] + \frac{\theta_{YM}}{32\pi^2}\, g^2 \mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} + \overline{D_\mu\phi}D^\mu\phi - i\psi\sigma^\mu D_\mu\bar{\psi} + \\ &+ i\sqrt{2}g\bar{\phi}\lambda\psi - i\sqrt{2}g\bar{\psi}\bar{\lambda}\phi - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\psi^i\psi^j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial\bar{\phi}_i\partial\bar{\phi}_j}\bar{\psi}_i\bar{\psi}_j - V(\phi,\bar{\phi})\ , \qquad (5.80)\end{aligned}$$

where the scalar potential $V(\phi,\bar{\phi})$ is

$$\begin{aligned}V\left(\phi,\bar{\phi}\right) &= \frac{\partial W}{\partial\phi^i}\frac{\partial\overline{W}}{\partial\bar{\phi}_i} + \frac{g^2}{2}\sum_a |\bar{\phi}_i(T^a)^i_j\phi^j + \xi^a|^2 = \qquad (5.81)\\ &= \bar{F}F + \frac{1}{2}D^2\Big|_{\text{on the solution}} \geq 0\ .\end{aligned}$$

We see that the potential is a semi-positive definite quantity in supersymmetric theories. *Supersymmetric vacua*, if any, are described by its zero's which are described by sets of scalar field VEVs which solve simultaneously the so-called D-term and F-term equations

$$\bar{F}_i(\phi) = 0 \quad , \quad D^a(\phi,\bar{\phi}) = 0\ . \qquad (5.82)$$

That supersymmetric vacua correspond to the zeros of the potential can be easily seen as follows. First recall that a vacuum is a Lorentz invariant state configuration. This means that all field derivatives and all fields but scalar ones should vanish in a vacuum state. Hence, the only non trivial thing of the Hamiltonian which can be different from 0 is the non-derivative scalar part, which, by definition, is the scalar potential. Therefore, the vacua of a theory, which are the minimal energy states, are in one-to-one correspondence with the (global or local) minima of the scalar potential. Now, as we have already seen, in a supersymmetric theory the energy of any state is semi-positive definite. This holds also for vacua (which are the minimal energy states). Hence for a vacuum Ω we have

$$\langle\Omega|P^0|\Omega\rangle \sim \sum_\alpha \left(||Q_\alpha|\Omega\rangle||^2 + ||Q^\dagger_\alpha|\Omega\rangle||^2\right) \geq 0\ . \qquad (5.83)$$

This means that the vacuum energy is 0 if and only if it is a supersymmetric state, that is $Q_\alpha|\Omega\rangle = 0, \bar{Q}_{\dot{\alpha}}|\Omega\rangle = 0 \quad \forall\alpha$. Conversely, supersymmetry is broken (in the perturbative theory based on this vacuum) if and only if the vacuum energy is positive. Hence, supersymmetric vacua are in one-to-one correspondence with the zero's of the scalar potential.

To find such zero's, one first looks for the space of scalar field VEVs such that

$$D^a(\phi,\bar{\phi}) = 0\ , \qquad (5.84)$$

which is called the space of D-flat directions. If a superpotential is present, one should then consider the F-term equations, which will put further constraints on the subset of scalar field VEVs already satisfying the D-term equations (5.84). The subspace of the space of D-flat directions which is also F-flat, i.e. which also satisfies the equations

$$\bar{F}^i(\phi) = 0 \ , \tag{5.85}$$

is called (classical) *moduli space* and represents the space of (classical) supersymmetric vacua. Clearly, in solving for the D-term equations, one should mod out by gauge transformations, since solutions which are related by gauge transformations are physically equivalent and describe the same vacuum state.

That the moduli space parametrizes the flat directions is because it represents the space of fields the potential does not depend on, and each such flat direction has a massless particle associated to it, a *modulus*. The moduli represent the lightest degrees of freedom of the low energy effective theory (think about the supersymmetric σ-model we discussed in §5.1.1). As one moves along the moduli space one spans physically inequivalent (supersymmetric) vacua, since the mass spectrum of the theory changes from point to point, as generically particles masses will depend on scalar field VEVs.

In passing, let us notice that while in a non-supersymmetric theory (or in a supersymmetry breaking vacuum of a supersymmetric theory) the space of classical flat directions, if any, is generically lifted by radiative corrections (which can be computed at leading order by e.g. the Coleman-Weinberg potential), in supersymmetric theories this does not happen. If the ground state energy is zero at tree level, it remains so at all orders in perturbation theory. This is because perturbations around a supersymmetric vacuum are themselves described by a supersymmetric Lagrangian and quantum corrections are protected by cancellations between fermionic and bosonic loops. This means that the only way to lift a classical supersymmetric vacuum, namely to break supersymmetry, if not at tree level, are non-perturbative corrections. We will have much more to say about this issue in later lectures.

5.3.1 Classical moduli space: examples

To make concrete the previous discussion on moduli space, in what follows we would like to consider two examples explicitly. Before we do that, however, we would like to rephrase our definition of moduli space, presenting an alternative (but equivalent) way to describe it.

Suppose we are considering a theory without superpotential. For such a theory the space of D-flat directions coincides with the moduli space. The space of D-flat directions is defined as the set of scalar field VEVs satisfying the D-flat conditions

$$\mathcal{M}_{cl} = \{\langle\phi_i\rangle \ / \ D^a = 0 \ \ \forall a\}/\text{gauge transformations} \ . \tag{5.86}$$

Generically it is not at all easy to solve the above constraints and find a simple parametrization of the $\mathcal{M}_{cl}$. An equivalent, though less transparent definition of the space of D-flat directions can help in this respect. It turns out that the same space can be defined as the space spanned by all (single trace) gauge invariant operator VEVs made out of scalar fields, modulo classical relations between them

$$\mathcal{M}_{cl} = \{\langle\text{Gauge invariant operators} \ \equiv X_r(\phi)\rangle\}/\text{classical relations} \ . \tag{5.87}$$

The latter parametrization is very convenient since, up to classical relations, the construction of the moduli space is unconstrained. In other words, the gauge invariant operators provide a direct parametrization of the space of scalar field VEVs satisfying the D-flat conditions (5.84).

Notice that if a superpotential is present, this is not the end of the story: F-equations will put extra constraints on the $X_r(\phi)$'s and may lift part of (or even all) the moduli space of supersymmetric vacua. In later lectures we will discuss some such instances in detail. Below, in order clarify the equivalence between definitions (5.86) and (5.87), we will consider two concrete models with no superpotential term, instead.

Massless SQED. The first example we want to consider is (massless) SQED, the supersymmetric version of quantum electrodynamics. This is a SYM theory with gauge group $U(1)$ and F (couples of) chiral superfields $(Q_i, \tilde{Q}_i)$ having opposite charge with respect to the gauge group (we will set for definiteness the charges to be ± 1) and no superpotential, $W = 0$. The vanishing of the superpotential implies that for this system the space of D-flat directions coincides with the moduli space of supersymmetric vacua. The Lagrangian is an instance of the general one we derived before and reads

$$\mathcal{L}_{SQED} = \frac{1}{32\pi}\text{Im}\left(\tau \int d^2\theta \ W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar{\theta} \ \left(Q_i^\dagger \, e^{2V} Q^i + \tilde{Q}_i{}^\dagger \, e^{-2V} \tilde{Q}^i\right) \tag{5.88}$$

(in order to ease the notation, we have come back to the most common notation for indicating the hermitian conjugate of a field).

The (only one) D-equation reads

$$D = Q_i^\dagger Q^i - \tilde{Q}_i^\dagger \tilde{Q}^i = 0 \tag{5.89}$$

where here and in the following a $\langle\ \rangle$ is understood whenever Q or $\tilde{Q}$ appear.

What is the moduli space? Let us first use the definition (5.86). The number of putative complex scalar fields parametrizing the moduli space is $2F$. We have one D-term equation only, which provides one real condition, plus gauge invariance

$$Q^i \to e^{i\alpha} Q^i \quad , \quad \tilde{Q}^i \to e^{-i\alpha} \tilde{Q}^i \ , \tag{5.90}$$

which provides another real condition. Therefore, the dimension of the moduli space is

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = 2F - \frac{1}{2} - \frac{1}{2} = 2F - 1 \ . \tag{5.91}$$

At a generic point of the moduli space the gauge group $U(1)$ is broken. Indeed, the -1 above corresponds to the complex field which, together with its fermionic superpartner, gets eaten by the vector superfield to give a massive vector superfield (recall the content of a massive vector multiplet). One component of the complex scalar field provides the third polarization to the otherwise massless photon; the other real component provides the real physical scalar field a massive vector superfield has; finally, the Weyl fermion provides the extra degrees of freedom to let the photino become massive. This is nothing but the supersymmetric version of the Higgs mechanism. As anticipated, the vacua are physically inequivalent, generically, since e.g. the mass of the photon depends on the VEV of the scalar fields.

Let us now repeat the above analysis using the definition (5.87). The only gauge invariants we can construct are

$$M^i_{\ j} = Q^i \tilde{Q}_j \ , \tag{5.92}$$

the mesons. As for the classical relation, this can be found as follows. The meson matrix M is a $F \times F$ matrix with rank one since so is the rank of Q and $\tilde{Q}$ (Q and $\tilde{Q}$ are vectors of length F, since the gauge group is abelian). This implies that the meson matrix has only one non-vanishing eigenvalue which means

$$\det\ (M - \lambda \mathbf{1}) = \lambda^{F-1}(\lambda - \lambda_0)(-1)^F \ . \tag{5.93}$$

Recalling that for a matrix A

$$\epsilon_{i_1 i_2 \ldots i_F} A^{i_1}_{j_1} A^{i_2}_{j_2} \ldots A^{i_F}_{j_F} = \det A\ \epsilon_{j_1 j_2 \ldots j_F} \tag{5.94}$$

with $\epsilon_{i_1 i_2 \ldots i_F}$ the fully antisymmetric tensor with F indices, we have

$$\epsilon_{i_1 i_2 \ldots i_F}(M^{i_1}_{j_1} - \lambda\delta^{i_1}_{j_1}) \ldots (M^{i_F}_{j_F} - \lambda\delta^{i_F}_{j_F}) = \lambda^{F-1}(\lambda - \lambda_0)(-1)^F \epsilon_{j_1 j_2 \ldots j_F} \tag{5.95}$$

which means that from the left hand side only the coefficients of the terms λ^F and λ^{F-1} survive. The next contribution, proportional to λ^{F-2}, should vanish, that is

$$\epsilon_{i_1 i_2 \ldots i_F} M^{i_1}_{j_1} M^{i_2}_{j_2} \epsilon^{j_1 j_2 \ldots j_F} = 0 \ . \tag{5.96}$$

These are the classical relations: $(F-1)^2$ complex conditions the meson matrix M should satisfy. Since the meson matrix is a complex $F \times F$ matrix, we finally get that

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = F^2 - (F-1)^2 = 2F - 1 \ . \tag{5.97}$$

which coincides with what we have found before!

The parametrization in terms of (single trace) gauge invariant operators is very useful if one wants to find the low energy effective theory around the supersymmetric vacua. Indeed, up to classical relations, these gauge invariant operators (in fact, their fluctuations) directly parametrize the massless degrees of freedom of the perturbation theory constructed upon these same vacua. Using equation (5.89) and the fact that the meson matrix has rank one, one can easily show that on the moduli space

$$\mathrm{Tr}\, Q^\dagger Q = \mathrm{Tr}\, \tilde{Q}^\dagger \tilde{Q} = \mathrm{Tr}\, \sqrt{M^\dagger M} \ . \tag{5.98}$$

Therefore, the Kähler potential, which is canonical in terms of the microscopic UV degrees of freedom Q and $\tilde{Q}$, once projected on the moduli space reads

$$K = \mathrm{Tr}\, \left[Q^\dagger Q + \tilde{Q}^\dagger \tilde{Q} \right] = 2\mathrm{Tr}\, \sqrt{M^\dagger M} \ . \tag{5.99}$$

The Kähler metric of the non-linear σ-model hence reads

$$ds^2 = K_{MM^\dagger} dM dM^\dagger = \frac{1}{2} \frac{1}{\sqrt{M^\dagger M}} dM dM^\dagger \tag{5.100}$$

which is manifestly non-canonical. Notice that the (scalar) kinetic term

$$\frac{1}{2} \int d^4x \ \frac{1}{\sqrt{M^\dagger M}} \partial_\mu M \partial^\mu M^\dagger \tag{5.101}$$

is singular at the origin, since the Kähler metric diverges there. This has a clear physical interpretation: at the origin the theory is un-higgsed, the photon becomes massless and the correct low energy effective theory should include it in the description. This is a generic feature in all this business: singularities showing-up at

specific points of the moduli space are a signal of extra-massless degrees of freedom that, for a reason or another, show up at those specific points

$$\text{Singularities} \longleftrightarrow \text{New massless d.o.f.} . \tag{5.102}$$

The correct low energy, singularity-free, effective description of the theory should include them. The singular behavior of $K_{MM^\dagger}$ at the origin is simply telling us that.

Massless SQCD. Let us now consider the non-abelian version of the previous theory. We have now a gauge group $SU(N)$ and F flavors. The quarks superfields Q and $\tilde{Q}$ are $F \times N$ matrices, and again there is no superpotential, $W = 0$. Looking at the Lagrangian, which is the obvious generalization of (5.88), we see there are two independent flavor symmetries, one associated to Q and one to $\tilde{Q}$, $SU(F)_L$ and $SU(F)_R$ respectively

$$\begin{array}{ccccc} & SU(N) & SU(F)_L & SU(F)_R \\ Q^i_a & N & F & 1 \\ \tilde{Q}^b_j & \overline{N} & 1 & \overline{F} \end{array} \tag{5.103}$$

where $i, j = 1, 2, \ldots, F$ and $a, b = 1, 2, \ldots, N$. The convention for gauge indices is that lower indices are for an object transforming in the fundamental representation and upper indices for an object transforming in the anti-fundamental. The convention for flavor indices is chosen to be the opposite one. Given these conventions, the D-term equations read

$$\begin{aligned} D^A &= Q^{\dagger}{}_i^b \, (T^A_N)^c_b \, Q^i_c + \tilde{Q}^{\dagger}{}_b^i \, (T^A_{\bar{N}})^b_c \, \tilde{Q}^c_i = Q^{\dagger}{}_i^b \, (T^A)^c_b \, Q^i_c - \tilde{Q}^{\dagger}{}_b^i \, (T^A)^b_c \, \tilde{Q}^c_i \\ &= Q^{\dagger}{}_i^b \, (T^A)^c_b \, Q^i_c - \tilde{Q}_i^b \, (T^A)^c_b \, \tilde{Q}^{\dagger}{}_c^i = 0 \ , \end{aligned} \tag{5.104}$$

where $A = 1, 2, \ldots, N^2 - 1$, and we used the fact that $(T^A_N)^a_b = -(T^A_{\bar{N}})^a_b \equiv (T^A)^a_b$.

Let us first focus on the case $F < N$. Using the two $SU(F)$ flavor symmetries and the (global part of the) gauge symmetry $SU(N)$, one can show that on the moduli space (5.104) the matrices Q and $\tilde{Q}$ can be put, at most, in the following form (recall that the maximal rank of Q and $\tilde{Q}$ is F in this case, since $F < N$)

$$Q = \begin{pmatrix} v_1 & 0 & \ldots & 0 & \ldots \\ 0 & v_2 & \ldots & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & v_F & \ldots \end{pmatrix} = \tilde{Q}^T \tag{5.105}$$

This means that at a generic point of the moduli space the gauge group is broken to $SU(N-F)$. So, the complex dimension of the classical moduli space is

$$\dim_{\mathbb{C}}\mathcal{M}_{cl} = 2FN - \left\{N^2 - 1 - \left[(N-F)^2 - 1\right]\right\} = F^2 \ . \tag{5.106}$$

Let us now use the parametrization in terms of gauge invariant single trace operators (5.87). In this case we have

$$M^i_{\ j} = Q^i_a \tilde{Q}^a_j \tag{5.107}$$

(notice the contraction on the N gauge indices). The meson matrix has now maximal rank, since $F < N$, so there are no classical constraints it has to satisfy: its F^2 entries are all independent. In terms of the meson matrix the classical moduli space dimension is then (trivially) F^2, in agreement with eq. (5.106). Again, playing with global symmetries, M can be diagonalized in terms of F complex eigenvalues V_i, which, obviously, are nothing but the square of the ones in (5.105), $V_i = v_i^2$.

A similar reasoning as the one working for SQED would hold about possible singularities in the moduli space. On the moduli space we have $Q^{\dagger a}_{\ \ i} Q^i_b = \tilde{Q}^a_i \tilde{Q}^{\dagger\, i}_{\ \ b}$. Using this identity we get

$$\left(M^\dagger M\right)^i_{\ j} = \tilde{Q}^{\dagger\, i}_{\ \ a} Q^{\dagger a}_{\ \ k}\ Q^k_b \tilde{Q}^b_j = \tilde{Q}^{\dagger\, i}_{\ \ a} \tilde{Q}^a_k\ \tilde{Q}^{\dagger\, k}_{\ \ b} \tilde{Q}^b_j \tag{5.108}$$

which implies $\tilde{Q}^\dagger \tilde{Q} = \sqrt{M^\dagger M}$ as a matrix equation. So the Kähler potential is

$$K = 2\mathrm{Tr}\,\sqrt{M^\dagger M} \ . \tag{5.109}$$

The Kähler metric is singular whenever the meson matrix M is not invertible. This does not only happen at the origin of field space as for SQED, but actually at subspaces where some of the $N^2 - 1 - \left[(N-F)^2 - 1\right] = (2N-F)F$ massive gauge bosons parametrizing the coset $SU(N)/SU(N-F)$ become massless, and they need to be included in the low energy effective description.

Let us now consider the case $F \geq N$. Following a similar procedure as the one before, the matrices Q and $\tilde{Q}$ can be brought to the following form on the moduli space

$$Q = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_N \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{Q}^T = \begin{pmatrix} \tilde{v}_1 & 0 & \dots & 0 \\ 0 & \tilde{v}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{v}_N \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.110}$$

where $|v_i|^2 - |\tilde{v}_i|^2 = a$, with a a i-independent number. Since $F \geq N$, at a generic point on the moduli space the gauge group is now completely higgsed. Therefore, the dimension of the classical moduli space is now

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = 2NF - \left(N^2 - 1\right) \ . \tag{5.111}$$

The parametrization in terms of gauge invariant operators is slightly more involved, in this case. The mesons are still there, and defined as in eq. (5.107). However, there are non-trivial classical constraints one should take into account, since the rank of the meson matrix, which is at most N, is now smaller than its dimension, F. Moreover, besides the mesons, there are now new gauge invariant single trace operators one can build, the baryons, which are operators made out of N fields Q respectively N fields $\tilde{Q}$, with fully anti-symmetrized indices.

As an explicit example of this richer structure, let us apply the above rationale to the case $N = F$. According to eq. (5.111), in this case $\dim_{\mathbb{C}} \mathcal{M}_{cl} = F^2 + 1$. The gauge invariant operators are the meson matrix plus two baryons, B and $\tilde{B}$, defined as

$$\begin{aligned} B &= \epsilon^{a_1 a_2 \ldots a_N} \, Q^{\ 1}_{a_1} Q^{\ 2}_{a_2} \ldots Q^{\ N}_{a_N} \\ \tilde{B} &= \epsilon_{a_1 a_2 \ldots a_N} \, \tilde{Q}^{a_1}_{\ 1} \tilde{Q}^{a_2}_{\ 2} \ldots \tilde{Q}^{a_N}_{\ N} \ . \end{aligned}$$

Since $F = N$ the anti-symmetrization on the flavor indices is automatically taken care of, once anti-symmetrization on the gauge indices is imposed. All in all, we have apparently $F^2 + 2$ complex moduli space directions. There is however one classical constraints between them which reads

$$\det M - B\tilde{B} = 0 \ , \tag{5.112}$$

as can be easily checked from the definition of the meson matrix (5.107) and that of the baryons above. Hence, the actual dimension of the moduli space is $F^2 + 2 - 1 = F^2 + 1$, as expected. As for the case $F < N$, there is a subspace in the moduli space, which includes the origin, in which some fields become massless and the low energy effective analysis should be modified to include them.

In fact, all we said, so far, is true classically. As we will later see, (non-perturbative) quantum corrections sensibly change this picture and the final structure of SQCD moduli space differs in many respects from the one above. For instance, for $F = N$ SQCD it turns out that the classical constraint (5.112) is modified at the quantum level; this has the effect of excising the origin of field space from the

actual quantum moduli space, removing, in turn, all singularities and corresponding new massless degrees of freedom, which are then just an artifact of the classical analysis, in this case.

We will have much more to say about SQCD and its classical and quantum properties at some later stage.

5.3.2 The SuperHiggs mechanism

We have alluded several times to a supersymmetric version of the Higgs mechanism. In the following, by considering a concrete example, we would like to show how the superfield degrees of freedom rearrangement explicitly works upon higgsing. As we already noticed, it is expected that upon supersymmetric higgsing a full vector superfield becomes massive, eating up a chiral superfield.

Let us consider once again SQCD with gauge group $SU(N)$ and F flavors and focus on a point of the moduli space where all scalar field VEVs v_i are 0 but v_1. At such point of the moduli space the gauge group is broken to $SU(N-1)$ and flavor symmetries are broken to $SU(F-1)_L \times SU(F-1)_R$. The number of broken generators is

$$N^2 - 1 - \left[(N-1)^2 - 1\right] = 2(N-1) + 1 \tag{5.113}$$

which just corresponds to the statement that if we decompose the adjoint representation of $SU(N)$ into $SU(N-1)$ representations we get

$$\text{Adj}_N = 1 \ \oplus \ \square_{N-1} \ \oplus \ \overline{\square}_{N-1} \ \oplus \ \text{Adj}_{N-1} \ \rightarrow \ G^A = X^0 \ , \ X_1^\alpha \ , \ X_2^\alpha \ , \ T^a \tag{5.114}$$

where G^A are the generators of $SU(N)$, T^a those of $SU(N-1)$ and the X's the generators of the coset $SU(N)/SU(N-1)$ ($A = 1, 2, \ldots, N^2-1$, $a = 1, 2, \ldots, (N-1)^2 - 1$ and $\alpha = 1, 2, \ldots, N-1$).

Upon this decomposition, the matter fields matrices can be re-written schematically as

$$Q = \begin{pmatrix} \omega^0 & \psi \\ \omega & Q' \end{pmatrix} \quad , \quad \tilde{Q} = \begin{pmatrix} \tilde{\omega}^0 & \tilde{\psi} \\ \tilde{\omega} & \tilde{Q}' \end{pmatrix} \tag{5.115}$$

where, with respect to the surviving gauge and flavor symmetries, ω^0 and $\tilde{\omega}^0$ are singlets, ω and $\tilde{\omega}$ are flavor singlets but carries the fundamental (resp anti-fundamental) representation of $SU(N-1)$, ψ (resp $\tilde{\psi}$) are gauge singlets and transform in the fundamental representation of $SU(F-1)_L$ (resp $SU(F-1)_R$), and finally Q' (resp

$\tilde{Q}'$) are in the fundamental (resp anti-fundamental) representation of $SU(N-1)$ and in the fundamental representation of $SU(F-1)_L$) (resp $SU(F-1)_R$).

By expanding the scalar fields around their VEVs (which are all vanishing but v_1) and plugging them back into the SQCD Lagrangian, after some tedious algebra one finds all v_1-dependent fermion and scalar masses, together with massive gauge bosons and corresponding massive gauginos. On top of this, there remains a set of massless fields, belonging to the massless vector superfields spanning $SU(N-1)$ and the massless chiral superfields Q' and $\tilde{Q}'$. We refrain to perform this calculation explicitly and just want to show that the number of bosonic and fermionic degrees of freedom, though arranged differently in the supersymmetry algebra representations, are the same before and after Higgsing.

We just focus on bosonic degrees of freedom, since, due to supersymmetry, the same result holds for the fermionic ones. For $v_1 = 0$ we have a fully massless spectrum. As far as bosonic degrees of freedom are concerned we have $2(N^2-1)$ of them coming from the gauge bosons and $4NF$ coming from the complex scalars. All in all there are

$$2(N^2-1) + 4NF \tag{5.116}$$

bosonic degrees of freedom.

For $v_1 \neq 0$ things are more complicated. As for the vector superfield degrees of freedom, we have $(N-1)^2 - 1$ massless ones, which correspond to $2\left[(N-1)^2-1\right]$ bosonic degrees of freedom, and $1 + 2(N-1)$ massive ones which correspond to $4 + 8(N-1)$ bosonic degrees of freedom. As for the matter fields (5.115), the massive ones are not there since they have been eaten by the (by now massive) vectors. These are ω and $\tilde{\omega}$, which are eaten by the vector multiplets associated to the generators X_1^α and X_2^α, and the combination $\omega^0 - \tilde{\omega}^0$ which is eaten by the vector multiplet associated to the generator X^0. We have already taken them into account, then. The massless chiral superfields Q' and $\tilde{Q}'$ provide $2(N-1)(F-1)$ bosonic degrees of freedom each, the symmetric combination $S = \omega^0 + \tilde{\omega}^0$ another 2 bosonic degrees of freedom, and finally the massless chiral superfields ψ and $\tilde{\psi}$ $2(F-1)$ each. All in all we get

$$2\left[(N-1)^2-1\right] + 4 + 8(N-1) + 4(N-1)(F-1) + 2 + 4(F-1) = 2(N^2-1) + 4NF\,, \tag{5.117}$$

which are exactly the same as those of the un-higgsed phase (5.116).

5.3.3 Non-linear sigma model II

In section 5.1.1 we discussed the supersymmetric non-linear σ-model for matter fields, which is relevant to describe supersymmetric low energy effective theories. Though it is not often the case, it may happen to face effective theories with some left over propagating gauge degrees of freedom at low energy. Therefore, in this section we will generalize the σ-model of section 5.1.1 to such a situation: a supersymmetric but non-renormalizable effective theory coupled to gauge fields. Note that the choice of the gauge group cannot be arbitrary here. In order to preserve the structure of the non-linear σ-model one can gauge only a subgroup G of the isometry group of the scalar manifold.

Following the previous strategy, one gets easily convinced that the pure SYM part changes simply by promoting the (complexified) gauge coupling τ to a holomorphic function of the chiral superfields, getting

$$\tau \int d^2\theta \ \mathrm{Tr} W^\alpha W_\alpha \quad \longrightarrow \quad \int d^2\theta \, \mathcal{F}_{ab}(\Phi) \, W^{\alpha a} \, W^b_\alpha \ , \tag{5.118}$$

where the chiral superfield $\mathcal{F}_{ab}(\Phi)$ should transform in the $Adj \otimes Adj$ of the gauge group G in order for the whole action to be G-invariant. Notice that for $\mathcal{F}_{ab} = \tau \, \mathrm{Tr} T_a T_b$ one gets back the usual result (recall that we have normalized the gauge group generators as $\mathrm{Tr}\, T_a T_b = \delta_{ab}$). For this reason the function $\mathcal{F}_{ab}$ (actually its restriction to the scalar fields) is dubbed generalized complex gauge coupling.

As for the matter Lagrangian, given what we have already seen, namely that whenever one has to deal with charged matter fields the gauge invariant combination is $(\bar{\Phi} e^{2gV})_i \Phi^i$, one should simply observe that the same holds for any real G-invariant function of Φ and $\bar{\Phi}$. In other words, the σ-model Lagrangian for charged chiral superfields is obtained from the one we derived in section 5.1.1 upon the substitution

$$K(\Phi^i, \bar{\Phi}_i) \quad \longrightarrow \quad K(\Phi^i, (\bar{\Phi} e^{2gV})_i) \ . \tag{5.119}$$

The end result is then

$$\begin{aligned} \mathcal{L} \ &= \ \frac{1}{32\pi} \, \mathrm{Im} \left[\int d^2\theta \, \mathcal{F}_{ab}(\Phi) \, W^{\alpha a} \, W^b_\alpha \right] + \\ &+ \ \int d^2\theta d^2\bar{\theta} K(\Phi^i, (\bar{\Phi} e^{2gV})_i) + \int d^2\theta \, W(\Phi^i) + \int d^2\bar{\theta} \, \overline{W}(\bar{\Phi}_i) \ . \end{aligned} \tag{5.120}$$

By expanding and integrating in superspace one gets the final result. The derivation is a bit lengthy and we omit it here. Let us just mention some important differences

with respect to our previous results. The gauge part has the imaginary part of $\mathcal{F}_{ab}$ multiplying the kinetic term (the generalized gauge coupling) and the real part multiplying the instanton term (generalized θ-angle). Moreover, there are higher order couplings between fields belonging to vector and scalar multiplets which are proportional to derivatives of $\mathcal{F}_{ab}$ with respect to the scalar fields, and which are obviously absent for the renormalizable Lagrangian (5.78). As for the matter part, one important difference with respect to the σ-model Lagrangian (5.42) is that all derivatives are (also) gauge covariantized. More precisely we have

$$\begin{aligned}\tilde{D}_\mu\psi^i &= \partial_\mu\psi^i - igv^a_\mu T^a_R\psi^i + \Gamma^i_{jk}\partial_\mu\phi^i\psi^k \\ \tilde{D}_\mu\bar{\psi}_j &= \partial_\mu\bar{\psi}_j - igv^a_\mu T^a_R\bar{\psi}_j + \Gamma^{ki}_j\partial_\mu\bar{\phi}_k\bar{\psi}_i \ ,\end{aligned}$$

which are covariant both with respect to the σ-model metric and the gauge connection. As compared to the Lagrangian (5.78) the Yukawa-like couplings have the Kähler metric inserted, that is

$$\bar{\phi}\lambda\psi \quad\longrightarrow\quad K^i_j\bar{\phi}_i\lambda\psi^j = K^i_j\bar{\phi}_i T^a_R\lambda_a\psi^j = K^i_j(\bar{\phi}_i)_M(T^a_R)^M_N\lambda_a(\psi^j)^N \ , \tag{5.121}$$

where M, N are gauge indices. Moreover, the term $g\bar{\phi}D\phi$ is also modified into $g\bar{\phi}_i DK^i$, where as usual $K^i = \frac{\partial}{\partial\phi_i}K$.

All these changes are important to keep in mind. However, it is worth noticing that in $\mathcal{N}=1$ supersymmetry vectors belong to different multiplets with respect to those where scalars sit. Hence, any geometric operation on the scalar manifold $\mathcal{M}$ will not have much effect on the vectors, and viceversa. In other words, the structure of the $\mathcal{N}=1$ non-linear σ-model is essentially unchanged by gauging some of the isometries of the scalar manifold. This is very different from what happens in models with extended supersymmetry, as we will see in the next lecture.

After solving for the auxiliary fields which read (with obvious notation)

$$\begin{aligned}F^i &= (K^{-1})^i_j W^j - \frac{1}{2}\Gamma^i_{jk}\psi^j\psi^k - i\frac{g^2}{16\pi}(K^{-1})^i_j(\mathcal{F}_{ab,j})^\dagger\bar{\lambda}^a\bar{\lambda}^b \\ D^a &= -\frac{4\pi}{g^2}\left(\mathrm{Im}\,\mathcal{F}\right)^{-1}_{ab}\left(g\bar{\phi}_i T^b K^i + g^2\frac{1}{8\pi\sqrt{2}}\left[(\mathcal{F}_{bc,i})\psi^i\lambda^c + h.c.\right]\right)\end{aligned} \tag{5.122}$$

one finds for the potential

$$V(\phi,\bar{\phi}) = (K^{-1})^i_j\, W_i W^j + 2\pi\left(\mathrm{Im}\,\mathcal{F}\right)^{-1}_{ab}\,\left(\bar{\phi}_i T^a K^i\right)\left(\bar{\phi}_j T^b K^j\right) \ , \tag{5.123}$$

which is the σ-model version of the potential (5.81).

As far as the potential, we cannot resist making a comment which will actually be relevant later, when we will discuss supersymmetry breaking. Whenever the effective theory one is dealing with does not have any propagating gauge degrees of freedom (due to Higgs mechanism, confinement or alike) the scalar potential (5.123) gets contributions from the first term, only. In this case the zero's of the potential, which correspond to the supersymmetric vacua of the theory, are described just by

$$W_i = 0 \ , \tag{5.124}$$

as in cases where the Kähler potential is canonical, since K is a positive definite matrix (provided the integrating out procedure has been done correctly along the whole moduli space; cf. our previous discussion). This means that it is possible to see whether supersymmetry is broken/unbroken independently of any knowledge of the Kähler potential! Still, other important features, as field VEVs, the exact value of the vacuum energy (if not zero), the mass and the interactions of the lightest excitations, etc... do depend on K. With an abuse of notation, eqs. (5.124) are usually referred to as F-term equations, even though for a theory with non-canonical Kähler potential the contribution to the scalar potential by the F-terms is really given by the first term of eq. (5.123).

5.4 Exercises

1. Consider a chiral superfield Φ with components ϕ, ψ and F. Compute the D-term of the real superfield $\bar{\Phi}\Phi$ (up to total derivatives). Using eqs. (4.59), show that the resulting expression transform as a total space-time derivative under supersymmetry transformations.

2. Compute $D^2\Phi = 0$ and show that the different components provide the equations of motion for a free massless WZ multiplet.

3. Consider a theory of a chiral superfield Φ with canonical Kähler potential, $K = \bar{\Phi}\Phi$ and superpotential $W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3$. This is the renowned Wess-Zumino (WZ) model. Compute the off-shell and on-shell Lagrangians, which is obtained from the latter integrating out auxiliary fields, and the scalar potential. Finally, show that supersymmetry closes only on-shell, namely that the algebra closes on the on-shell representation only upon use of (some of) the equations of motion.

4. Consider the theory of a single chiral superfield Φ and Kähler potential $K = \ln(1 + \bar{\Phi}\Phi)$. Compute the off-shell and on-shell Lagrangians and study the geometry of the one-dimensional supersymmetric non-linear σ model.

5. Using all possible available symmetries, show that in $SU(N)$ SQCD with $F < N$ flavors, the complex scalar field matrices parametrizing the moduli space can be put in the form (5.105). Using the same procedure, show that the structure (5.110) holds for $F \geq N$.

6. Consider the following matter theories

   1. $K = \bar{Q}Q \ , \ W = \frac{1}{2}mQ^2$
   2. $K = \bar{X}X + \bar{Y}Y \ , \ W = (X - m)\, Y^2$
   3. $K = \bar{X}X + \bar{Y}Y + \bar{Z}Z \ , \ W = gXYZ$
   4. $K = \Lambda\sqrt{\bar{X}X} \ , \ W = \lambda X \ .$

   Determine whether there are supersymmetric vacua and, if they exist, compute the mass spectrum of the theory around them.

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6 Theories with extended supersymmetry

Until now we have discussed theories with $\mathcal{N} = 1$ supersymmetry. In this lecture we will discuss the structure of theories with extended supersymmetry. This will also let us emphasize the basic differences which arise at the quantum level between theories with different number of supersymmetries.

6.1 $\mathcal{N} = 2$ supersymmetric actions

In this section we would like to construct the most general $\mathcal{N} = 2$ supersymmetric action in four dimensions. We will follow the same logic of the previous lecture, but we will not develop the corresponding $\mathcal{N} = 2$ superspace approach, whose formulation is beyond our present scope. Rather, we will use the (by now familiar) $\mathcal{N} = 1$ superspace formalism and see which specific properties does more supersymmetry impose on an otherwise generic $\mathcal{N} = 1$ Lagrangian.

We have two kinds of $\mathcal{N} = 2$ multiplets we have to deal with, vector multiplets and hypermutiplets. What we noticed at the level of representations of the supersymmetry algebra on states, lecture 3, holds also at the field level. In particular, using a $\mathcal{N} = 1$ language, a $\mathcal{N} = 2$ vector superfield can be seen as the direct sum of a vector superfield V and a chiral superfield Φ (with same internal quantum numbers, of course). Similarly, in terms of degrees of freedom a hypermultiplet can be constructed out of two $\mathcal{N} = 1$ chiral superfields, H_1 and H_2. Schematically, we have

$$[\mathcal{N} = 2 \text{ vector multiplet}] \quad : \quad V = (\lambda_\alpha, A_\mu, D) \ \oplus \ \Phi = (\phi, \psi_\alpha, F)$$

$$[\mathcal{N} = 2 \text{ hypermultiplet}] \quad : \quad H_1 = (H_1, \psi_{1\alpha}, F_1) \ \oplus \ \bar{H}_2 = \left(\bar{H}_2, \bar{\psi}_{2\dot{\alpha}}, \bar{F}_2\right)$$

(notice that H_1 and $\bar{H}_2$ transform in the same representations of internal symmetries, while H_2 transforms in the complex conjugate representation).

Let us start considering pure SYM, with gauge group G. There are two minimal requirements we should impose. As already stressed, the chiral multiplet Φ should transform in the adjoint representation of the gauge group. Moreover, we have now a larger R-symmetry group, whose compact component, $SU(2)_R$, should be a symmetry of the Lagrangian. All bosonic fields A_μ, D, F and ϕ are singlets under $SU(2)_R$, but $(\lambda_\alpha, \psi_\alpha)$ transform as a doublet. This is because (Q^1_α, Q^2_α) transform under the fundamental representation of $SU(2)_R$, and the same should hold for λ_α and ψ_α (recall that they are obtained acting with the two supersymmetry generators on the Clifford vacuum).

The Lagrangian reads

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=2}_{SYM} &= \frac{1}{32\pi}\,\mathrm{Im}\,\left(\tau\int d^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha\right)+\int d^2\theta d^2\bar\theta\,\mathrm{Tr}\,\bar\Phi e^{2gV}\Phi = \\ &= \mathrm{Tr}\,\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-i\lambda\sigma^\mu D_\mu\bar\lambda-i\psi\sigma^\mu D_\mu\bar\psi+\overline{D_\mu\phi}D^\mu\phi+\frac{\theta}{32\pi^2}g^2F_{\mu\nu}\tilde F^{\mu\nu}+\right. \\ &+ \left.\frac{1}{2}D^2+\bar F F+i\sqrt{2}g\bar\phi\{\lambda,\psi\}-i\sqrt{2}g\{\bar\psi,\bar\lambda\}\phi+gD\left[\phi,\bar\phi\right]\right\} \qquad (6.1)\end{aligned}$$

where

$$\phi=\phi^a T_a\quad,\quad \psi_\alpha=\psi^a_\alpha T_a\quad,\quad F=F^aT_a$$
$$\lambda_\alpha=\lambda^a_\alpha T_a\quad,\quad D=D^aT_a\quad,\quad v_\mu=v^a_\mu T_a$$

with $a=1,2,\ldots,\dim G$. The reason why commutators and anti-commutators appear in the Lagrangian (6.1) is just that all fields transform in the adjoint representation of G. Indeed, given that $(T^a_{adj})_{bc}=-if_{abc}$, we have that, e.g. $\bar\phi\lambda\psi$ really is

$$\bar\phi^b\lambda^a(T^a_{adj})_{bc}\psi^c=-i\bar\phi^b\lambda^a f_{abc}\psi^c=i\bar\phi^b\lambda^a f_{bac}\psi^c=\bar\phi^b\lambda^a\psi^c\ \mathrm{Tr}\ T_b\left[T_a,T_c\right]=\mathrm{Tr}\ \bar\phi\{\lambda,\psi\}\ , \qquad (6.2)$$

and similarly for all other contributions in eq. (6.1).

As compared to $\mathcal{N}=1$ Lagrangians describing matter coupled SYM theory, the above Lagrangian is special in many respects. A necessary and sufficient condition for $\mathcal{N}=2$ supersymmetry is the existence of a $SU(2)_R$ rotating the two generators Q^1_α, Q^2_α into each other. This has several consequences in the structure of the Lagrangian. The kinetic terms for λ and ψ have the same normalization. Moreover, and more importantly, the Lagrangian has no superpotential, $W=0$. Indeed, a superpotential would give ψ interactions and/or mass terms, that are absent for λ. This is clearly forbidden by the $SU(2)_R$ symmetry. While we do not have a superpotential, we do have a potential, which comes from D-terms. Indeed, the auxiliary fields equations of motion are in this case

$$F^a=0\quad,\quad D^a=-g\left[\phi,\bar\phi\right]^a \qquad (6.3)$$

(the auxiliary fields F^a appear only in the non-dynamical kinetic term $\bar F_a F^a$ and therefore are trivial). The potential hence reads

$$V(\phi,\bar\phi)=\frac{1}{2}D^aD_a=\frac{1}{2}g^2\ \mathrm{Tr}\left[\phi,\bar\phi\right]^2\ . \qquad (6.4)$$

The above expression shows that pure $\mathcal{N}=2$ SYM enjoys a huge moduli space of supersymmetric vacua. Indeed, the potential vanishes whenever the fields ϕ belong to the Cartan subalgebra of the gauge group G. At a generic point of the moduli space the scalar field matrix can be diagonalized and the gauge group is broken to $U(1)^r$, with r the rank of G. The low energy effective dynamics is that of r massless vector multiplets and $dim(G)-r$ massive vector multiplets whose masses depend on the scalar fields VEVs. The theory is said to be in a Coulomb phase, since charged external sources will feel a Coulomb-like potential. The (classical) moduli space is a r-dimensional complex manifolds, parametrized by r massless complex scalars. Singularities arise whenever some VEVs become degenerate and the theory gets partially un-higgsed (in particular, at the origin of the moduli space one recovers the full G gauge symmetry). In some later lectures we will see how this classical picture gets modified once (non-perturbative) quantum corrections are taken into account.

The $SU(2)_R$ symmetry (in fact just the center of the group, $\mathbb{Z}_2$) can be used also to check that the Lagrangian (6.1) is invariant under two independent supersymmetries, as it should. Eq. (6.1) is written in terms of two $\mathcal{N}=1$ superfields and, correspondingly, it is obviously $\mathcal{N}=1$ invariant. Acting now with a $\mathbb{Z}_2$ R-symmetry rotation which acts as $\psi_\alpha \to \lambda_\alpha$ and $\lambda_\alpha \to -\psi_\alpha$, while leaving the bosonic fields invariant, one sees that the same Lagrangian shows an invariance under an *independent* $\mathcal{N}=1$ supersymmetry acting on two different superfields with entries $(A_\mu, -\psi_\alpha, D)$ and $(\phi, \lambda_\alpha, F)$. So we conclude that the Lagrangian is indeed $\mathcal{N}=2$ supersymmetric invariant.

Let us now consider the addition of hypermultiplets. In this case the scalar fields, H_1 and $\bar{H}_2$ form a $SU(2)_R$ doublet (again, recall how they are constructed from the ground state of the corresponding $\mathcal{N}=2$ supersymmetry representation, lecture 3). Hypermultiplets cannot interact between themselves since no cubic $SU(2)$ invariant is possible. Therefore, for renormalizable theories a superpotential is not allowed and interactions turn out to be all gauge interactions.

Let us then couple charged matter to the above Lagrangian. We get for the $\mathcal{N}=2$ hypermultiplet Lagrangian

$$\mathcal{L}^{\mathcal{N}=2}_{Matter} = \int d^2\theta d^2\bar{\theta}\left(\bar{H}_1 e^{2gV_R} H_1 + \bar{H}_2 e^{-2gV_R} H_2\right) + \int d^2\theta \sqrt{2} g H_1 \Phi H_2 + \text{ h.c. }, \quad (6.5)$$

where the suffix R on the vector superfield V refers to the representation of G carried by the hypermultiplets. The F-term coupling the hypermultiplets with the

chiral multiplet Φ belonging to the $\mathcal{N}=2$ vector multiplet is there because of $\mathcal{N}=2$ supersymmetry (it is somehow the supersymmetric partner of the kinetic terms which couple the hypermultiplet to V). So we see that eventually a cubic interaction does arise, but it is a gauge interaction, in the sense that it vanishes once the gauge coupling $g \to 0$.

Eliminating the auxiliary fields F_1 and F_2, the scalar potential for the hypermultiplets can be recast as a D-term contribution only and reads

$$V(H_1, H_2) = \frac{1}{2} D^2 = \frac{1}{2} g^2 |\bar{H}_1 T_R^a H_1 - \bar{H}_2 T_R^a H_2|^2 \ , \ \ D^a = g \ \mathrm{Tr} \left(\bar{H}_1 T_R^a H_1 - \bar{H}_2 T_R^a H_2 \right) \ . \tag{6.6}$$

Notice finally that a mass term can be present and has the form

$$m H_1 H_2 \ . \tag{6.7}$$

However, a term of this sort can be there only for BPS hypermultiplets (which as discussed time ago are short enough to close the algebra within maximal spin 1/2 particle states).

6.1.1 Non-linear sigma model III

It is also possible to get the $\mathcal{N}=2$ version of the σ-model, once renormalizability is relaxed. $\mathcal{N}=2$ supersymmetry will make it special, as compared to the $\mathcal{N}=1$ case. We do not want to enter into much details and will just sketch the end result.

Let us start with pure SYM. Differently from the $\mathcal{N}=1$ case, this is a meaningful thing to do, since scalar fields are present in a $\mathcal{N}=2$ vector multiplet. To write down the $\mathcal{N}=2$ σ-model it is sufficient to take the $\mathcal{N}=1$ σ-model Lagrangian (5.120), set the superpotential to zero, and take into account that the chiral superfield Φ transforms in the adjoint representation. On general grounds, one expects that the Kähler potential K should be related, in a $\mathcal{N}=2$ consistent way, to the generalized complexified gauge coupling $\mathcal{F}_{ab}$, since the scalars spanning the manifold $\mathcal{M}$ sit in the same multiplets where the vectors sit (in particular, one would expect that an isometry transformation on $\mathcal{M}$ should have effects on the vectors, too). Equivalently, one can notice from (5.120) that the imaginary part of the generalized complexified gauge coupling multiplies the gaugino kinetic term while the Kähler metric that of the matter fermion fields. These should transform as a doublet under $SU(2)_R$ and then one would expect $\mathcal{F}_{ab}$ and K_{ab} to be related one another. What one finds is that $\mathcal{F}_{ab}$ and K can be written in terms of one and the same holomorphic function

$\mathcal{F}(\Phi)$, dubbed *prepotential* and read

$$\mathcal{F}_{ab}(\Phi) = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial\Phi^a \partial\Phi^b} \tag{6.8}$$

$$K(\Phi,\bar{\Phi}) = -\frac{i}{32\pi}\bar{\Phi}_a \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^a} + \text{ h.c.} = -\frac{i}{32\pi}\bar{\Phi}_a \mathcal{F}^a(\Phi) + \frac{i}{32\pi}\overline{\mathcal{F}_a(\Phi)}\Phi^a \ , \tag{6.9}$$

which is the very non-trivial statement that the full $\mathcal{N}=2$ σ-model action is uniquely determined by a single holomorphic function, the prepotential $\mathcal{F}(\Phi)$. The end result for the σ-model action reads

$$\begin{aligned}\mathcal{L}_{eff}^{\mathcal{N}=2} &= \frac{1}{64\pi i}\int d^2\theta \mathcal{F}_{ab}(\Phi) W^{\alpha a} W^b_\alpha + \frac{1}{32\pi i}\int d^2\theta d^2\bar{\theta}\left(\bar{\Phi}e^{2gV}\right)^a \mathcal{F}_a(\Phi) + \text{ h.c.} \\ &= \frac{1}{32\pi}\ \text{Im}\ \left[\int d^2\theta \mathcal{F}_{ab}(\Phi) W^{\alpha a} W^b_\alpha + 2\int d^2\theta d^2\bar{\theta}\left(\bar{\Phi}e^{2gV}\right)^a \mathcal{F}_a(\Phi)\right] \ . \end{aligned} \tag{6.10}$$

Using eqs. (6.8)-(6.9) we can compute the Kähler metric and see its relation with the complexified gauge coupling which is

$$K^b_a(\phi,\bar{\phi}) = \frac{\partial^2 K(\phi,\bar{\phi})}{\partial\phi^a \partial\bar{\phi}_b} = -\frac{i}{32\pi}\left(\frac{\partial^2 \mathcal{F}(\phi)}{\partial\phi^a\partial\phi^b} - \frac{\partial^2\overline{\mathcal{F}(\phi)}}{\partial\bar{\phi}_a\partial\bar{\phi}_b}\right) = \frac{1}{16\pi}\ \text{Im}\mathcal{F}_{ab}(\phi) \ . \tag{6.11}$$

Therefore, we finally get for the potential

$$V(\phi,\bar{\phi}) = -\frac{1}{2\pi}\left(\text{Im }\mathcal{F}_{ab}(\phi)\right)^{-1}\left[\bar{\phi},\mathcal{F}_c(\phi)T^c\right]^a\left[\bar{\phi},\mathcal{F}_d(\phi)T^d\right]^b \ . \tag{6.12}$$

A Kähler manifold where the Kähler potential can be written in terms of a holomorphic function as in eq. (6.9) is called *special* Kähler manifold. From a geometric point of view this corresponds to a Kähler manifold endowed with a symplectic structure (a $2n_v$ symplectic bundle, where n_v is the number of vector multiplets). One can recover the renormalizable Lagrangian (6.1) by taking $\mathcal{F}(\Phi) = \frac{1}{2}\tau\,\text{Tr}\Phi^2$. The check is left to the reader.

This is not the end of the story, though. To the σ-model action we have constructed one can add hypermultiplets. We refrain to present its structure here and just make two comments. Hypermultiplets contain two complex scalars. What one finds is that the corresponding σ-model is defined on a quaternionic manifold, known as HyperKähler manifold, which is, essentially, the quaternionic extension of a Kähler manifold (in particular, there are three rather than just one complex structures). So in $\mathcal{N}=2$ supersymmetry, due to the existence of two sets of scalars, those belonging to matter multiplets and those belonging to gauge multiplets, the most general scalar manifold is (classically) of the form

$$\mathcal{M} = \mathcal{M}^V \otimes \mathcal{M}^H \ , \tag{6.13}$$

where $\mathcal{M}^V$ is a special Kähler manifold and $\mathcal{M}^H$ a HyperKähler manifold. Notice that, once renormalizability is relaxed, quartic (and higher, if $SU(2)_R$ singlets) superpotential couplings are possible. We will have much more to say about $\mathcal{N}=2$ σ-models in later lectures.

6.2 $\mathcal{N}=4$ supersymmetric actions

Let us now discuss the structure of the $\mathcal{N}=4$ Lagrangian. We have only one kind of multiplet in this case, the vector multiplet. So, from a $\mathcal{N}=4$ perspective, we can only have pure SYM theories. The decomposition of the $\mathcal{N}=4$ vector superfield in terms of $\mathcal{N}=1$ representations is as follows

$$[\mathcal{N}=4 \text{ vector multiplet}] \;\; : \;\; V=(\lambda_\alpha, A_\mu, D) \;\; \oplus \;\; \Phi_A = \left(\phi^A, \psi^A_\alpha, F^A\right) \quad A=1,2,3 \; .$$

The propagating degrees of freedom are a vector field, six real scalars (two for each complex scalar ϕ_A) and four gauginos. The Lagrangian is very much constrained by $\mathcal{N}=4$ supersymmetry. First, the chiral superfields Φ_A should transform in the adjoint representation of the gauge group G, since internal symmetries commute with supersymmetry. Moreover, we have now a large R-symmetry group, $SU(4)_R$. The four Weyl fermions transform in the fundamental of $SU(4)_R$, while the six real scalars in the two times anti-symmetric representation, which is nothing but the fundamental representation of $SO(6)$. The auxiliary fields are singlets under the R-symmetry group. Using $\mathcal{N}=1$ superfield formalism the Lagrangian reads

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=4}_{SYM} &= \frac{1}{32\pi}\,\mathrm{Im}\,\left(\tau\int d^2\theta\;\mathrm{Tr}\;W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar\theta\;\mathrm{Tr}\;\sum_A \bar\Phi_A\, e^{2gV}\Phi_A \\ &- \int d^2\theta\sqrt{2}g\;\mathrm{Tr}\;\Phi_1\,[\Phi_2,\Phi_3] + \;\mathrm{h.c.} \end{aligned} \tag{6.14}$$

where the commutator in the third term appears for the same reason as for the $\mathcal{N}=2$ Lagrangian (6.1). Notice that the choice of a single $\mathcal{N}=1$ supersymmetry generator breaks the full $SU(4)_R$ R-symmetry to $SU(3)\times U(1)_R$. The three chiral superfields transform in the $\underline{3}$ of $SU(3)$ and have R-charge $R=2/3$ under the $U(1)_R$. It is an easy but tedious exercise to perform the integration in superspace and get an explicit expression in terms of fields. Finally, one can solve for the auxiliary fields and get an expression where only propagating degrees of freedom are present, and where $SU(4)_R$ invariance is manifest (the fact that the scalar fields transform under the fundamental representation of $SO(6)$, which is real, makes the R-symmetry

group of the $\mathcal{N}=4$ theory being at most $SU(4)$ and not $U(4)$, in fact). We refrain to perform the calculation here. We would only like to point out that the scalar potential can be written in a rather compact form in terms of the six real scalars X_i making up the three complex scalars ϕ_A and reads

$$V = \frac{1}{2} g^2 \ \mathrm{Tr} \ \sum_{i,j=1}^{6} [X_i, X_j]^2 \ . \tag{6.15}$$

From the above expression we see that $\mathcal{N}=4$ SYM enjoys a large moduli space of vacua. Except for the origin, where all X_i VEVs vanish, the gauge group is generically broken along the moduli space and the theory is in a Coulomb phase, very much like pure $\mathcal{N}=2$ SYM. At a generic point of the moduli space the gauge group breaking pattern $G \to U(1)^r$, where r is the rank of G, and the dynamics is that of r copies of $\mathcal{N}=4$ $U(1)$ theory.

One might ask whether a σ-model action is possible for $\mathcal{N}=4$ models. After all, we are plenty of scalar fields, actually $3n$ complex scalars, if n is the dimension of G. The answer is that there is only one possible σ-model compatible with $\mathcal{N}=4$ supersymmetry (the stringent constraint comes from the large R-symmetry group), the trivial one: $\mathcal{M} = \mathbb{R}^{6n}$. So, the Lagrangian (6.14) is actually the only possible $\mathcal{N}=4$ Lagrangian one can build. This also implies that, unlike pure $\mathcal{N}=2$ SYM, the moduli space of vacua has a trivial topology.

6.3 On non-renormalization theorems

One of the advantages, in fact *the* advantage of supersymmetry is that it makes quantum corrections much better behaved with respect to ordinary field theories.

Many relevant results about UV properties of supersymmetric field theories were obtained more than thirty years ago and can be summarized in terms of powerful non-renormalization theorems. At that time, a very efficient approach was developed to deal with supersymmetric quantum field theories, a version of Feynman rules, known as *supergraphs* techniques, which let one work directly with superfields in superspace with no need to expand into component fields. Most non-renormalization theorems were proved using such techniques whose description, however, is beyond the scope of these lectures. Here I just want to mention what is possibly the main result so obtained: in a supersymmetric quantum field theory containing chiral and vector superfields, the most general term that can be generated by loop diagrams

has only one Grassman integral over *all* superspace

$$\int d^4x_1 \dots d^4x_n \, d^2\theta d^2\bar{\theta} \; G(x_1, \dots, x_n) F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{6.16}$$

where $G(x_1, \dots, x_n)$ is a translationally invariant function and the F_i's are products of superfields and their covariant derivatives. Such term is a D-term and does not contribute to superpotential terms, which are F-terms, implying that the superpotential is tree-level exact, i.e. it is not renormalized at any order in perturbation theory. The only possible corrections may arise at the non-perturbative level (and in some cases, namely when only chiral superfields are present, the latter also vanish, as we will see later). On the contrary, D-terms, that is the Kähler potential, are renormalized, and generically do receive corrections at any order in perturbation theory (and non-perturbatively).

Let us try and see what non-renormalization theorems imply for theories with different number of supersymmetries.

Let us first focus on a renormalizable $\mathcal{N} = 1$ action describing a chiral superfield Φ (the generalization to many chiral superfields is straightforward and does not present any relevant difference). There are three supersymmetric contributions to the action. One, the kinetic term, is a D-term, and undergoes renormalizations. Two are F-terms (the mass term and the cubic term) and are hence exact, perturbatively. Concretely

$$\int d^4x d^2\theta d^2\bar{\theta} \; \bar{\Phi}\Phi \;\; \to \;\; Z_\Phi \int d^4x d^2\theta d^2\bar{\theta} \; \bar{\Phi}\Phi \tag{6.17}$$

$$m \int d^4x d^2\theta \; \Phi^2 + h.c. \;\; \to \;\; m \int d^4x d^2\theta \; \Phi^2 + h.c. \tag{6.18}$$

$$\lambda \int d^4x d^2\theta \; \Phi^3 + h.c \;\; \to \;\; \lambda \int d^4x d^2\theta \; \Phi^3 + h.c \; . \tag{6.19}$$

This means that m and λ do get renormalized but only logarithmically at one loop, instead of quadratically and linearly, respectively. Hence

$$Z_m Z_\Phi = 1 \quad , \quad m \to Z_\Phi^{-1} m$$
$$Z_\lambda Z_\Phi^{3/2} = 1 \quad , \quad \lambda \to Z_\Phi^{-3/2} \lambda \; .$$

Something similar happens for the renormalization of the factor e^{2gV}. However, things are more subtle, here. Notice that

$$\int d^2\theta d^2\bar{\theta} \bar{\Phi} e^{2gV} \Phi \tag{6.20}$$

is a D-term and hence it renormalizes. However, what one can see using the background field method to renormalize gauge theories is that gV does not renormalize. An independent renormalization for g and V leading to a kinetic term of the form

$$\bar{\Phi} e^{2Z_g g Z_V^{1/2} V} \Phi \tag{6.21}$$

would correspond to counterterms of the form

$$\bar{\Phi} V e^{gV} \Phi \tag{6.22}$$

which are not gauge invariant. This implies that the integral (6.20) should renormalize as the kinetic term (6.17) (not because of supersymmetry, but just due to gauge invariance!), meaning that g and V compensate each other upon renormalization. In other words, gV is not renormalized. Another way to see this is the following. Consider pure SYM. In this theory the only possible counterterm would correspond to something proportional to the action itself

$$\int d^2\theta \ \mathrm{Tr}\ W^\alpha W_\alpha + h.c. \ , \tag{6.23}$$

which would then correspond to a wave-function renormalization of the full Lagrangian (this is certainly there since the integral above is not a F-term, but rather a D-term, as already noticed). This means that one should multiply by the same function both the kinetic terms $dVdV$ as well as the interaction terms $gVVdV$ and g^2V^4 to keep gauge invariance (recall we are considering a non-abelian gauge group). In order for this to be the case one needs that if

$$V \to Z_V^{1/2} V \tag{6.24}$$

then

$$g \to Z_V^{-1/2} g \ , \tag{6.25}$$

which implies that gV is not renormalized, as anticipated. The conclusion is that in renormalizable theories with $\mathcal{N} = 1$ supersymmetry there are only two independent renormalization, Z_Φ and Z_V, which are just logaritmically divergent at one-loop, and correspond to wave-function renormalization of chiral and vector superfields, respectively.

In passing, notice that the fact that $Z_V \neq 1$ means that the integral (6.23) is renormalized. This does not contradict non-renormalization theorems, since, as already observed, (6.23) is not a F-term, really, but actually a D-term, and then

it does renormalize. That $Z_V \neq 1$ also implies that $Z_g \neq 1$, meaning that in $\mathcal{N} = 1$ SYM theories the gauge coupling runs, and can get corrections at all loops, in general.

What about higher supersymmetry? All what we said, so far, still applies, since any extended supersymmetry model is also $\mathcal{N} = 1$. However, extended supersymmetry imposes further constraints. In what follows, we stick to the notation we have used in sections 6.1 and 6.2 when discussing $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Lagrangians, respectively.

Let us start from $\mathcal{N} = 2$ supersymmetry. For one thing, since V and Φ belong now to the same multiplet, we have that

$$Z_V = Z_\Phi \ . \tag{6.26}$$

As for the hypermultiplets, from the cubic interaction $gH_1\Phi H_2$ which appears in the superpotential (and which is then tree level exact, being a F-term) we get the following condition

$$Z_g^2 Z_\Phi Z_{H_1} Z_{H_2} = 1 \ . \tag{6.27}$$

The first two contributions cancel since $Z_V = Z_\Phi$ and, as for $\mathcal{N} = 1$ supersymmetry, gV is not renormalized, meaning $Z_g = Z_V^{-1/2}$. So we get

$$Z_{H_1} Z_{H_2} = 1 \ . \tag{6.28}$$

Hence the wave-functions of the two chiral superfields making up an hypermultiplet are not independent. All in all, we have then only two independent renormalizations in $\mathcal{N} = 2$ supersymmetry, Z_V and, say, Z_{H_1}. In fact, for massless representations there is the $SU(2)_R$ symmetry rotating the scalar components of H_1 and H_2 into each other. Hence, they should have the same renormalization, which means, using eq. (6.28), that $Z_{H_1} = Z_{H_2} = 1$. Actually, the same holds for massive (BPS) representations. In this case the existence of a non-trivial central charge does break the R-symmetry group to $USp(2)$; however, the algebra of such group is the same as that of $SU(2)$ and one can again conclude that $Z_{H_1} = Z_{H_2} = 1$.

It turns out that, because of the relation between Z_Φ and Z_g, not only $\mathcal{N} = 2$ SYM has a unique renormalization but it is one-loop exact in perturbation theory. In other words, the gauge coupling β-function gets only one-loop contributions, perturbatively. We will derive this important result in a later lecture, when discussing the dynamics of supersymmetric gauge theory. There, we will use a very powerful

approach which is based on a crucial property of supersymmetry, known as *holomorphy*. For the time being let us just stress that this one-loop exactness of $\mathcal{N}=2$ SYM gauge coupling does not hold for $\mathcal{N}=1$ SYM, whose physical gauge coupling receives corrections at all orders in perturbation theory.

Let us finally consider $\mathcal{N}=4$ supersymmetry. Here we have a single superfield, the vector superfield which, in $\mathcal{N}=1$ language, can be seen as one vector superfield V and three chiral superfields Φ_A (all transforming in the adjoint representation of the gauge group). The $SU(3)$ symmetry rotating the three chiral superfields implies that the latter should have all and the same wave-function renormalization

$$Z_{\Phi_1} = Z_{\Phi_2} = Z_{\Phi_3} = Z \ . \tag{6.29}$$

On the other hand, due to eq. (6.26), this Z should equal Z_V, the wave-function of the vector superfield. Plugging this into eq. (6.27), which for the $\mathcal{N}=4$ Lagrangian (6.14) is

$$Z_g^2 Z_{\Phi_1} Z_{\Phi_2} Z_{\Phi_3} = 1 \ , \tag{6.30}$$

and recalling that $Z_g = Z_V^{-1/2}$ it follows that

$$Z_V = Z_{\Phi_1} = Z_{\Phi_1} = Z_{\Phi_1} = 1 \ , \tag{6.31}$$

which means that $\mathcal{N}=4$ SYM is perturbatively finite; in other words, the theory does not exhibit ultraviolet divergences in the correlation functions of canonical fields! Though we are not going to prove it here, it turns out that in fact $\mathcal{N}=4$ is finite also once non-perturbative corrections are taken into account. Indeed, the latter give finite contributions, only, and therefore the theory is believed to be UV finite.

There is yet another important property of $\mathcal{N}=4$ SYM we would like to mention. The theory is superconformal invariant, and it is so at the full quantum level. Let us see how this goes. A theory whose Lagrangian contains only dimension four operators, like the $\mathcal{N}=4$ Lagrangian (and many others, in fact) is classically scale invariant. For any relativistic field theory this implies a larger symmetry algebra, the conformal Poincaré algebra which, besides Poincaré generators, includes also dilations and special conformal transformations, the corresponding group being $SO(2,4) \simeq SU(2,2)$. The generators associated to dilations and special conformal transformations, D and K^μ, respectively, act as follows

$$\begin{aligned} D \ &: \ x^\mu \ \longrightarrow \ \lambda \, x^\mu \\ K^\mu \ &: \ x^\mu \ \longrightarrow \ \frac{x^\mu + a^\mu x^2}{1 + 2 x^\nu a_\nu + a^2 x^2} \ , \end{aligned}$$

and have the following commutation relations between themselves and with the generators of the Poincaré algebra

$$[P_\mu, D] = iP_\mu \quad , \quad [D, M_{\mu\nu}] = 0 \quad , \quad [K_\mu, D] = -iK_\mu \quad , \quad [K_\mu, K_\nu] = 0$$
$$[P_\mu, K_\nu] = 2i\left(M_{\mu\nu} - \eta_{\mu\nu} D\right) \quad , \quad [K_\mu, M_{\rho\sigma}] = i\left(\eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho\right) \ . \qquad (6.32)$$

Supersymmetry enlarges further the symmetry group. A conformal invariant supersymmetric theory enjoys an even larger algebra, the *superconformal* algebra, which includes, besides dilations and special conformal transformations, also conformal supersymmetry transformations $S^I_\alpha, \bar{S}^I_{\dot\alpha}$ (which appear in the commutator of the supersymmetry charges Q^I_α with the generators of special conformal transformations K^μ), and the generators associated to R-symmetry transformations, T^I_J (where $I, J = 1, \ldots, N$), which are now part of the algebra and do not act just as external automorphisms (they appear in the anti-commutator of the supersymmetry charges with the S^I_α's). The associated supergroup is $SU(2,2|N)$. The non-vanishing (anti)commutators involving the new generators are

$$\left[K_\mu, Q^I_\alpha\right] = 2i\sigma_{\mu\alpha\dot\beta}\bar{S}^{\dot\beta I} \ , \ \{S^I_\alpha, \bar{S}_{\dot\beta\, J}\} = 2\sigma^\mu_{\alpha\dot\beta} K_\mu \delta^I_J \quad , \quad \left[D, Q^I_\alpha\right] = -\frac{i}{2} Q^I_\alpha \ , \ \left[D, S^I_\alpha\right] = \frac{i}{2} S^I_\alpha$$
$$\{Q^I_\alpha, S_{\beta J}\} = \epsilon_{\alpha\beta}\left(\delta^I_{\ J} D + T^I_{\ J}\right) + \frac{1}{2}\delta^I_{\ J}\sigma^{\mu\nu}_{\alpha\beta} M_{\mu\nu} \quad , \quad \left[P_\mu, S^I_\alpha\right] = \sigma_{\mu\alpha\dot\beta}\bar{Q}^{\dot\beta I} \ . \qquad (6.33)$$

The $\mathcal{N} = 4$ SYM action is invariant under this larger symmetry algebra, $SU(2,2|4)$ in this case, but it is certainly not the only theory having this property, at the classical level. Classical superconformal invariance is shared by any supersymmetric Lagrangian made solely by dimension four operators (in other words, with dimensionless, hence classically marginal couplings), well-known examples being the massless WZ model, and in fact any SYM theory, like $\mathcal{N} = 1$ SQCD discussed in the previous lecture.

What makes $\mathcal{N} = 4$ SYM special is that, as we have observed above, the Lagrangian does not renormalize (recall that essential to this proof was the use of the $SU(3)$ subgroup of the R-symmetry group rotating the three scalar superfields Φ^A). In particular, as we have seen before, $Z_g = 1$. In other words, the $\mathcal{N} = 4$ β-function vanishes identically: the theory remains scale invariant at the quantum level, and the superconformal symmetry $SU(2,2|4)$ is then an exact symmetry of the theory. An equivalent conclusion can be reached by observing that $\mathcal{N} = 4$ SYM is a (very special) $\mathcal{N} = 2$ theory. The $\mathcal{N} = 2$ gauge coupling is one-loop exact and since in $\mathcal{N} = 4$ SYM this is the only coupling appearing in the Lagrangian, it

is enough to compute the one-loop β-function for g. One can easily see that such one-loop coefficient vanishes, concluding that the theory is superconformal also at the quantum level. In fact, the equivalence of these proofs lies in the fact that the gauge coupling β-function and the R-symmetry are in the same supermultiplet, the ($\mathcal{N} = 4$) supercurrent multiplet.

This non-renormalization property is not shared by other theories, in general: typically, the superconformal algebra is broken by quantum corrections and couplings run. For instance, in the massless WZ model, the coupling, which is classically marginal, becomes irrelevant quantum mechanically (i.e., it coupling flows to zero and the theory becomes free in the IR). On the contrary, UV-free supersymmetric gauge theories, like pure $\mathcal{N} = 1, 2$ SYM, enjoy dimensional transmutation and a dynamically scale is generated at the quantum level.

What we said above about the finiteness of $\mathcal{N} = 4$ does not mean that any operator has protected dimension. The scaling dimension of canonical fields (gauge fields, gauginos and adjoint scalars) is unaffected by quantum corrections, but this does not happen, in general, to composite gauge invariant operators. Yet, in a superconformal theory there are special operators whose dimension is protected. To see how this comes, let us start considering the conformal algebra (6.32). In unitary theories there is a lower bound for the scaling dimension Δ of a field (e.g. $\Delta \geq 1$ for a scalar field in four dimensions). Since K_μ lowers the scaling dimension of a field, any representation of the conformal algebra should admit an operator with minimal dimension Δ which is annihilated by K_μ (at $x_\mu = 0$). Such states are called conformal primary operators. Since the conformal algebra is a subalgebra of the superconformal algebra, representations of the latter in general decompose into representations of the former. By definition, a superconformal primary operator is an operator which is annihilated (at $x_\mu = 0$) both by K_μ and $S^I_\alpha, \bar{S}^I_{\dot{\alpha}}$. From the commutator $\left[K_\mu, Q^I_\alpha\right]$ in (6.33) it also follows that any operator which is obtained from a superconformal primary by the action of Q^I_α, and hence sits in the same supermultiplet, is a primary operator of the conformal algebra. Superconformal primary operators which are annihilated by some of the supercharges are called *chiral primaries* and, most importantly, their dimension is fixed by their R-symmetry representation, and as such are protected against quantum corrections (this can be proven by playing a bit with the superconformal algebra, which lets express the scaling dimension Δ of a chiral primary operator in terms of Lorentz and R-symmetry representations). By supersymmetry, this implies that in a superconformal theory operators belong-

ing to supersymmetry representations which include a chiral primary operator do not renormalize. For instance, in $\mathcal{N} = 4$ SYM, a class of superconformal primaries are all operators made of symmetric traceless products of the scalar fields X_i's, e.g. $\mathrm{Tr}\,(X^{\{i}X^{j\}}) = \mathrm{Tr}\,(X^iX^j) - \frac{1}{6}\delta^{ij}\mathrm{Tr}\,(X^kX^k)$.

As a final comment, let us just notice that in $\mathcal{N} = 4$ SYM superconformal invariance is/is not realized depending on the point of the moduli space one is sitting. The phase where all scalar field VEVs $\langle X_i \rangle$ vanish is called superconformal phase since at the origin of the moduli space the gauge group remains unbroken and superconformal invariance is preserved. In other words, physical states are not only gauge invariant, but carry unitary representations of $SU(2,2|4)$. On the contrary, on the Coulomb branch, where gauge symmetry is broken, also superconformal symmetry is broken since scalar VEVs $\langle X_i \rangle$ set a dimensionful scale in the theory.

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7 Supersymmetry breaking

If supersymmetry is at all realized in Nature, it must be broken at low enough energy: we do not see any mass degeneracy in the elementary particle spectrum, at least at energies of order 10^2 GeV or lower. The idea is then that supersymmetry is broken at some scale M_s, such that at energies $E > M_s$ the theory behaves in a supersymmetric way, while at energies $E < M_s$ it does not. On general ground, there are two ways supersymmetry can be broken, either spontaneously or explicitly.

- *Spontaneous supersymmetry breaking*: the theory is supersymmetric but has a scalar potential admitting (stable, or metastable but sufficiently long-lived) supersymmetry breaking vacua. In such vacua one or more scalar fields acquire a VEV of order M_s, which then sets the scale of supersymmetry breaking.

- *Explicit supersymmetry breaking*: the Lagrangian contains terms which do not preserve supersymmetry by themselves. In order for them not to ruin the nice and welcome UV properties of supersymmetric theories, these terms should have positive mass dimension, in other words they should be irrelevant in the far UV. In this case we speak of *soft* supersymmetry breaking. In such scenario, the scale M_s enters explicitly in the Lagrangian.

As we will later show, soft supersymmetry breaking models can (and typically do) actually arise as low energy effective descriptions of models where supersymmetry is broken spontaneously. Therefore, we will start focusing on spontaneous supersymmetry breaking. Only later we will discuss supersymmetry breaking induced by soft terms.

7.1 Vacua in supersymmetric theories

We have already seen that supersymmetric vacua are in one-to-one correspondence with the zero's of the scalar potential. In other words, the vacuum energy is zero if and only if the vacuum preserves supersymmetry. Hence, non-supersymmetric vacua correspond to minima of the potential which are not zero's. In this case supersymmetry is broken in the perturbative theory based on these positive energy vacua (aka spontaneous supersymmetry breaking).

Notice how different is spontaneous supersymmetry breaking with respect to spontaneous breaking of e.g. gauge symmetries. There, what matters is the location

of the minima of the potential in field space, while here it is the absolute *value* of the potential at such minima. This implies that in a supersymmetric gauge theory there can be minima which preserve both gauge symmetry and supersymmetry, others which break both, and others which preserve gauge symmetry and break supersymmetry, or viceversa. A schematic picture of these different situations is reported in figure 7.1.

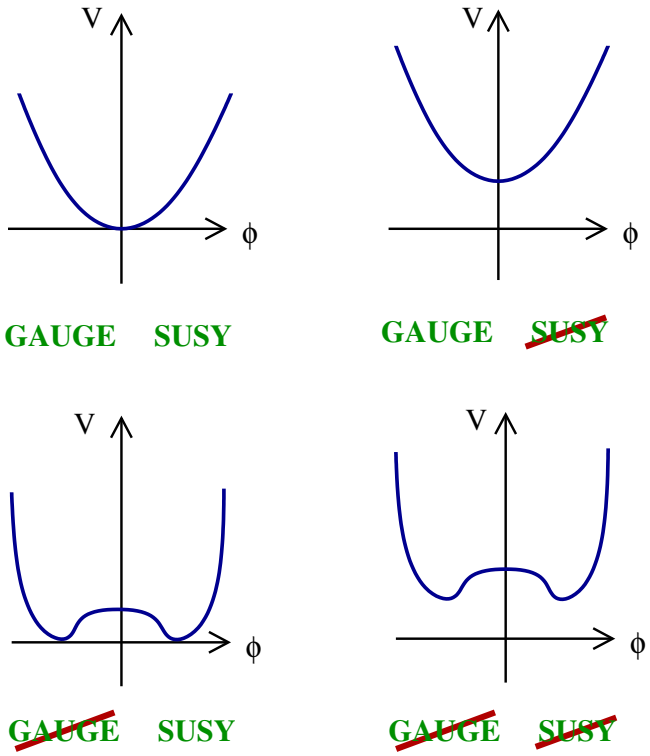

Figure 7.1: A schematic picture of possible patterns of spontaneous gauge symmetry and supersymmetry breakings. The potential on the upper left does not admit any symmetry breaking vacuum. The one on the lower right, instead, admits two vacua breaking *both* gauge symmetry and supersymmetry. The other two represent mixed situations where either supersymmetry or gauge symmetry are broken.

While non-supersymmetric vacua can be either global or local minima of the potential (corresponding to stable or metastable vacua, respectively), supersymmetric vacua, if present, are obviously global minima of the potential.

Recall the expression (5.81), that is

$$V(\phi, \bar{\phi}) = \bar{F}F + \frac{1}{2}D^2 \tag{7.1}$$

(we focus here on models with canonical Kähler potential; later we will also discuss situations where the Kähler potential is not canonical) where

$$\bar{F}_i = \frac{\partial W}{\partial \phi^i} \quad , \quad D^a = -g\left(\bar{\phi}_i (T^a)^i_{\ j}\, \phi^j + \xi^a\right) \ . \tag{7.2}$$

Supersymmetric vacua are described by all possible set of scalar field VEVs satisfying the D and F-term equations

$$\bar{F}^i(\phi) = 0 \quad , \quad D^a(\bar{\phi}, \phi) = 0 \ . \tag{7.3}$$

If there exist more than one possible solution, then we have more supersymmetric vacua, generically a moduli space of vacua, if these are not isolated. If there does not exist a set of scalar field VEVs for which eqs. (7.3) are satisfied, then supersymmetry is broken, and the minima of the potential are all necessarily positive, $V_{min} > 0$.

Notice, on the contrary, that on any vacuum, supersymmetric or not, global or local, the following equations always hold

$$\frac{\partial V(\phi, \bar{\phi})}{\partial \phi^i} = 0 \quad , \quad \frac{\partial V(\phi, \bar{\phi})}{\partial \bar{\phi}_i} = 0 \ . \tag{7.4}$$

An equivalent statement about supersymmetric vacua is that on supersymmetric vacua the supersymmetric variations of the fermion fields vanish. This can be seen as follows. Due to Lorentz invariance, on a vacuum any field's VEV or its derivative should vanish, but scalar fields. Recalling how each field component of a chiral or vector superfield transforms under supersymmetry transformations, it then follows that on a vacuum state we have

$$\delta\langle\phi^i\rangle = 0 \quad , \quad \delta\langle F^i\rangle = 0 \quad , \quad \delta\langle\psi^i_\alpha\rangle \sim \epsilon_\alpha \langle F^i\rangle$$

$$\delta\langle F^a_{\mu\nu}\rangle = 0 \quad , \quad \delta\langle D^a\rangle = 0 \quad , \quad \delta\langle\lambda^a_\alpha\rangle \sim \epsilon_\alpha \langle D^a\rangle \ .$$

Therefore, in a generic vacuum the supersymmetric variations of the fermions is not zero: it is actually proportional to F and D-terms. A supersymmetric vacuum state is by definition supersymmetric invariant (!). Hence, from above equations it follows that on a supersymmetric vacuum also the supersymmetric variations of the fermions should be zero, the latter being equivalent to the D and F-term equations (7.3), as anticipated.

7.2 Goldstone theorem and the goldstino

When a global symmetry is spontaneously broken, Goldstone theorem says that there is a massless mode in the spectrum, the Goldstone field, whose quantum num-

bers should be related to the broken symmetry. We should expect this theorem to work also for spontaneously broken supersymmetry. In fact, given that supersymmetry is a fermionic symmetry, the Goldstone field should be in this case a Majorana spin 1/2 fermion, the so-called *goldstino*.

Consider the most general supersymmetric Lagrangian with gauge and matter fields, eq. (5.80), and suppose it admits some vacuum where supersymmetry is broken. In this vacuum eqs. (7.4) hold, while (some of) eqs. (7.3) do not. Recalling eqs. (5.79) and (5.81) we have in this vacuum that

$$\frac{\partial V(\phi,\bar{\phi})}{\partial \phi^i} = F^j(\bar{\phi})\frac{\partial^2 W}{\partial\phi^i\partial\phi^j} - gD^a\bar{\phi}_j(T^a)^j_{\ i} = 0 \ . \tag{7.5}$$

On the other hand, since the superpotential is gauge invariant, we have that

$$\delta^a W = \frac{\partial W}{\partial\phi^i}\delta^a\phi^i = \bar{F}_i(T^a)^i_{\ j}\phi^j = 0 \ . \tag{7.6}$$

Combining the former equation with the complex conjugate of the latter evaluated in the vacuum, we easily get a matrix equation

$$M\begin{pmatrix}\langle F^j\rangle\\ \langle D^a\rangle\end{pmatrix} = 0 \quad \text{where} \quad M = \begin{pmatrix}\langle\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\rangle & -g\langle\bar{\phi}_l\rangle(T^a)^l_{\ i}\\ -g\langle\bar{\phi}_l\rangle(T^b)^l_{\ j} & 0\end{pmatrix} \ . \tag{7.7}$$

This implies that the matrix M has an eigenvector with zero eigenvalue. Now, this matrix is nothing but the fermion mass matrix of the Lagrangian (5.80)! This can be seen looking at the non-derivative fermion bilinears of (5.80), which on the vacuum get contributions also from the cubic coupling between scalar fields, their superpartners and gauginos, and which can be written as

$$\cdots - \frac{1}{2}\left(\psi^i \ , \ \sqrt{2}i\lambda^b\right) M \begin{pmatrix}\psi^j\\ \sqrt{2}i\lambda^a\end{pmatrix} + h.c. + \ldots \tag{7.8}$$

Hence, on the supersymmetry breaking vacuum the spectrum necessarily admits a massless fermion, the goldstino. In terms of spin 1/2 particles belonging to the different multiplets, the goldstino ψ^G_α corresponds to the following linear combination

$$\psi^G_\alpha \sim \langle F^i\rangle\psi^i_\alpha + \langle D^a\rangle\lambda^a_\alpha \ . \tag{7.9}$$

Our proof of the goldstino theorem has been based on the Lagrangian (5.80). In fact, one can provide a similar proof using properties of the supercurrent and Ward

identities, which dooes not rely on the existence of an explicit classical Lagrangian. The supersymmetry Ward identity reads

$$\langle \partial^\mu S_{\mu\alpha}(x)\bar{S}_{\nu\dot{\beta}}(0)\rangle = -\delta^4(x)\langle \delta_\alpha S_{\nu\dot{\beta}}\rangle = -2\,\sigma^\mu_{\alpha\dot{\beta}}\langle T_{\mu\nu}\rangle\,\delta^4(x)\ , \tag{7.10}$$

where the last equality follows from the current algebra, see eq. (4.71). This implies, for the two-point function of the supercurrent, that

$$\langle S_{\mu\alpha}(x)\bar{S}_{\nu\dot{\beta}}(0)\rangle = \cdots + (\sigma_\mu\bar{\sigma}^\rho\sigma_\nu)_{\alpha\dot{\beta}}\frac{x_\rho}{x^2}\langle T\rangle\ , \tag{7.11}$$

where $\langle T\rangle = \eta^{\mu\nu}\langle T_{\mu\nu}\rangle$ and the $\ldots$ are terms which are not relevant to the present discussion. Upon Fourier transforming eq. (7.11) we get

$$\langle S_{\mu\alpha}(k)\bar{S}_{\nu\dot{\beta}}(-k)\rangle = \cdots + (\sigma_\mu\bar{\sigma}^\rho\sigma_\nu)_{\alpha\dot{\beta}}\,k_\rho\frac{\langle T\rangle}{k^2}\ , \tag{7.12}$$

which shows the presence of a massless pole (the goldstino), proportional to the vacuum energy density, in the supercurrent two-point function. The above equation shows that the goldstino is the lowest energy excitation of the supercurrent, and it is so if and only if the vacuum energy is non-vanishing. This shows, as anticipated, that the goldstino theorem holds universally, *i.e.*, also for vacua where supersymmetry is broken in a strongly coupled phase of the theory, where classical arguments may not apply.

From our discussion it should be clear that given a generic Lagrangian, there are two a priori independent ways we can break supersymmetry: either giving a non vanishing expectation value to (some) F-terms or to (some) D-terms. In what follows, we will consider the two possibilities in turn.

7.3 F-term breaking

Since we are focusing here on F-term breaking, let us assume for the time being to deal with a theory with chiral superfields, only. The most general renormalizable Lagrangian of this sort reads

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\,\bar{\Phi}_i\Phi^i + \int d^2\theta\,W(\Phi^i) + \int d^2\bar{\theta}\,\overline{W}(\bar{\Phi}_i)\ , \tag{7.13}$$

where

$$W(\Phi^i) = a_i\Phi^i + \frac{1}{2}m_{ij}\Phi^i\Phi^j + \frac{1}{3}g_{ijk}\Phi^i\Phi^j\Phi^k\ . \tag{7.14}$$

The equations of motions for the auxiliary fields read

$$\bar{F}_i(\phi) = \frac{\partial W}{\partial \phi^i} = a_i + m_{ij}\phi^j + g_{ijk}\phi^j\phi^k \ , \tag{7.15}$$

and the potential is

$$V(\phi, \bar{\phi}) = \sum_i |a_i + m_{ij}\phi^j + g_{ijk}\phi^j\phi^k|^2 \ . \tag{7.16}$$

Supersymmetry is broken if and only if there does not exist a set of scalar field VEVs such that all F-terms vanish, $\langle F^i \rangle = 0$. This implies that in order for supersymmetry to be broken, it is necessary some a_i to be different from zero. If not, the trivial solution $\langle \phi^i \rangle = 0$ solves all F-equations. So, any model of F-term supersymmetry breaking needs a superpotential admitting linear terms (notice that this conclusion applies also for a superpotential with higher non-renormalizable couplings).

This result holds also in presence of a non-canonical Kähler potential. This can be easily seen recalling the expression of the scalar potential when a non-canonical Kähler metric is present

$$V(\phi, \bar{\phi}) = \left(K^{-1}\right)^i_{\ j} \frac{\partial W}{\partial \phi^i} \frac{\partial \overline{W}}{\partial \bar{\phi}_j} \ . \tag{7.17}$$

From this expression it is clear that unless it were singular (something signalling, as already discussed, an inconsistency of the effective theory analysis), a non-trivial Kähler metric could not influence the existence/non existence of supersymmetric vacua, which is still dictated by the possibility/impossibility to make the first derivatives of the superpotential vanish. The only (and important) properties which get modified by a non-trivial Kähler potential would be the value of the vacuum energy (for non-supersymmetric vacua, only!) and the particle spectrum around a given vacuum.

Given this minimal necessary requirement, in what follows we will consider several examples of F-term breaking.

Example 1 : The Polonyi model.

Let us consider the theory of a single chiral superfield with canonical Kähler potential and a linear superpotential

$$K = \bar{X}X \quad , \quad W = \lambda X \ . \tag{7.18}$$

This is the most minimal set-up one can imagine for a F-term supersymmetry breaking model. The potential reads

$$V = |\frac{\partial W}{\partial X}|^2 = |\lambda|^2 \ , \tag{7.19}$$

Supersymmetry is clearly broken for any $|X|$, and the latter is in fact a flat direction. The supersymmetry breaking scale is set by the modulus of λ itself, $|\lambda| = M_s^2$. In Figure 7.2 we report the (trivial) shape of the scalar potential.

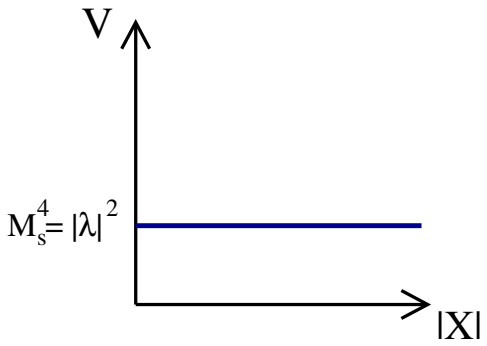

Figure 7.2: The potential of the Polonyi model.

A few comments are in order. First notice that the theory possesses an R-symmetry, the R-charge of X being $R(X) = 2$. At a generic point of the moduli space, then, both supersymmetry and R-symmetry are broken. Second, notice that although supersymmetry is broken, the spectrum is degenerate in mass: $|X|$, its phase α, and ψ_X are all massless. The fermion field has a good reason to be massless: it is the goldstino predicted by Goldstone theorem. Seemingly, the phase of X is expected to be massless: it is nothing but the goldstone boson associated to the broken R-symmetry. Finally, the modulus of the scalar field $|X|$ is massless since it parametrizes the (non-supersymmetric) moduli space. As already noticed, however, such moduli space is not protected, in principle, against quantum corrections, since supersymmetry is broken. Hence, generically, one would expect it to be lifted at the quantum level and $|X|$ to get a mass. This is not the case in this simple theory, since it is a non-interacting theory, and there are no quantum corrections whatsoever. In general, however, things are different: a non-supersymmetric moduli space gets typically lifted at one or higher loops, and the putative moduli get a mass. For this reason, non-supersymmetric moduli spaces are dubbed *pseudo-moduli* spaces, and the moduli parametrizing them, pseudo-moduli. We will see examples of this sort soon.

Let us now consider the following innocent-looking modification of the model above. Let's add a mass term to X,

$$\Delta W = \frac{1}{2} m X^2 \ . \tag{7.20}$$

Things drastically change, since supersymmetry is now restored. Indeed, the F-term equation now reads

$$\bar{F}(X) = mX + \lambda \ = 0 \ , \tag{7.21}$$

which admits the solution $\langle X \rangle_{SUSY} = -\frac{\lambda}{m}$. Hence, there is a choice of scalar field VEV which makes the potential $V = |\lambda + mX|^2$, depicted in figure 7.3, vanish.

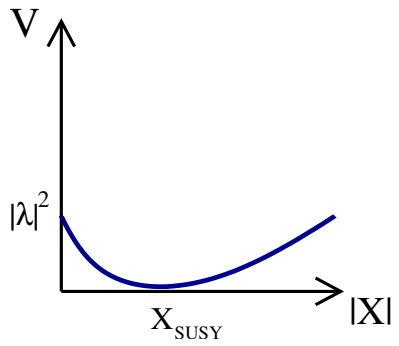

Figure 7.3: The potential of the massive Polonyi model.

The spectrum is supersymmetric *and* massive: all fields have mass m. This agrees with physical expectations: ψ_X is no more the goldstino, since this is not expected to be there now; $|X|$ is no more a (pseudo)modulus since the supersymmetric vacuum is isolated (the VEV of $|X|$ is not a flat direction); finally, α is not anymore the goldstone boson associated to the broken R-symmetry since the superpotential term ΔW breaks the R-symmetry explicitly (more precisely, $W = \lambda X + \frac{1}{2} m X^2$ does not admit any R-charge assignment for X such that $R(W) = 2$).

Things might also change (both qualitatively and quantitatively) if one allows the Kähler potential not being canonical. Suppose we keep $W = \lambda X$ but we let the Kähler metric be non-trivial, that is

$$V = \left(K_{X\bar{X}}\right)^{-1} |\lambda|^2 \quad \text{with} \quad K_{X\bar{X}} = \frac{\partial^2 K}{\partial X \partial \bar{X}} \neq 1 \ . \tag{7.22}$$

A non-trivial Kähler metric can deform sensibly the pseudo-moduli space of figure 7.2. A sample of possible different behaviors, which depend on the asymptotic properties (or singularities) of the Kähler metric, is reported in Figure 7.4.

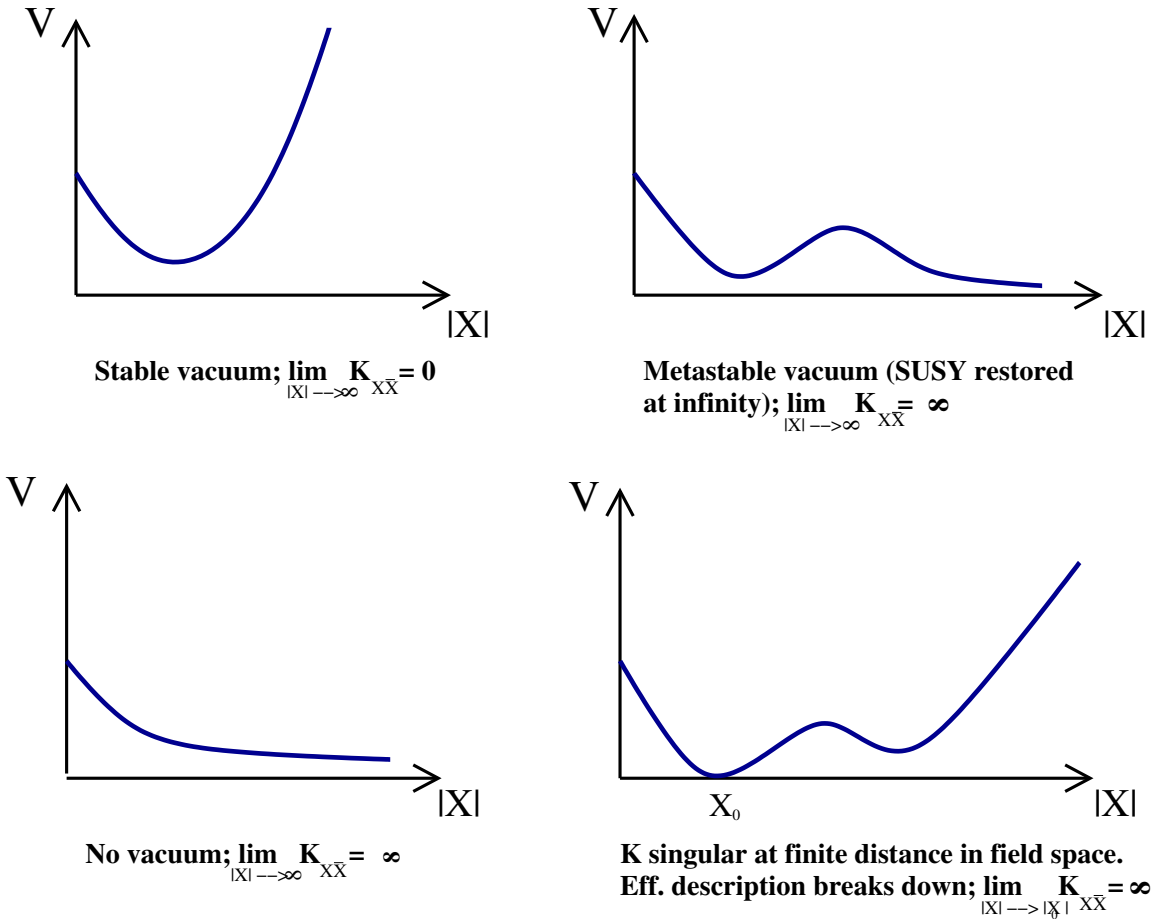

Figure 7.4: Different potentials of non-canonical Polonyi-like models.

Physically, the different behaviors reported in figure 7.4 should be understood as follows. In presence of a classical pseudomoduli space like the one in figure 7.2, the lifting of the pseudomoduli at quantum level occurs because the massless particles which bring from a vacuum to another get a mass at one loop. Sometime, such effect can be mimicked by a non-canonical Kähler potential, like the one above. In fact, these seemingly ad-hoc theories can and sometime do arise at low energies as *effective theories* of more complicated UV-renormalizable ones: the mass scale entering the Kähler potential is nothing but the UV cut-off of these low energy effective theories. Let us try to make the above discussion more concrete by considering an explicit example.

Example 2 : A Polonyi model with quartic Kähler potential.

Let us consider the following model

$$K = \bar{X}X - \frac{c}{\Lambda^2}\left(\bar{X}X\right)^2 \quad , \quad W = \lambda X \ , \tag{7.23}$$

where $c > 0$. Notice that the R-symmetry is not broken by the non-canonical Kähler potential (7.23), which is R-symmetry invariant. So the $U(1)_R$ symmetry is a symmetry of the theory. The Kähler metric and the scalar potential now read

$$K_{X\bar{X}} = 1 - \frac{4c}{\Lambda^2}\bar{X}X \quad , \quad V = K^{X\bar{X}}|\lambda|^2 = \frac{|\lambda|^2}{1 - \frac{4c}{\Lambda^2}X\bar{X}} \ . \tag{7.24}$$

The Kähler potential is an instance of models like those in the upper-left diagram of Figure 7.4: indeed the Kähler metric vanishes for large enough $|X|$, $|X| \to |\Lambda|/2\sqrt{c}$, which is order the natural cut-off of the theory. The potential, which is depicted in figure 7.5, admits a (unique) mimimum at $\langle X \rangle = 0$. There is an *isolated* vacuum

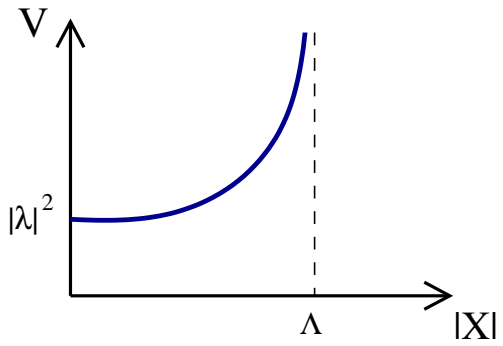

Figure 7.5: The potential of a Polonyi model with quartic Kähler potential (with $c = 1/4$).

now, and it is a supersymmetry breaking one. One can compute the spectrum around such vacuum and find that ψ_X is consistently massless (it is the goldstino), while now the scalar field is massive, $m_X^2 \sim c|\lambda|^2/\Lambda^2$.

Example 3 : Supersymmetry restoration by new degrees of freedom.

Let us now deform the basic Polonyi model by adding a new superfield, Y, while keeping the Kähler potential canonical. The superpotential reads

$$W = \lambda X + \frac{1}{2}hXY^2 \ . \tag{7.25}$$

Notice that this model has an R-symmetry, with R-charge assignment $R(X) = 2$, $R(Y) = 0$. From the F-equations one can compute the potential which reads

$$V = |hXY|^2 + |\frac{1}{2}hY^2 + \lambda|^2 \ , \tag{7.26}$$

implying that there are two supersymmetric vacua at

$$\langle X\rangle_{SUSY} = 0 \quad , \quad \langle Y\rangle_{SUSY} = \pm\sqrt{-2\frac{\lambda}{h}} \ . \tag{7.27}$$

So we see that the additional degrees of freedom have restored supersymmetry. Interestingly, there are other *local* minima of the potential, a pseudo-moduli space in fact, where supersymmetry is broken

$$\langle X\rangle_{SB} = \text{any} \quad , \quad \langle Y\rangle_{SB} = 0 \quad \text{where} \quad V = |\lambda|^2 \ . \tag{7.28}$$

The physical interpretation is as follows. For large $\langle X\rangle$, the superfield Y gets a large mass and affects the low energy theory lesser and lesser. The theory reduces effectively to the original Polonyi model, which breaks supersymmetry and whose vacuum energy is indeed $V = |\lambda|^2$. It is a simple but instructive exercise to compute the mass spectrum around the non-supersymmetric minima. The chiral superfield X is obviously massless while Y gets a mass. There is a first (obvious) contribution to both the scalar and the fermion components of Y from $h\langle X\rangle$, and a second contribution which affects only the scalar component of Y coming from F_X, which is non-vanishing. The end result is

$$m_Y^2 = |h\langle X\rangle|^2 \pm |h\lambda| \quad , \quad m_{\psi_Y} = h\langle X\rangle \ . \tag{7.29}$$

From the above expressions, we see that the supersymmetry breaking pseudomoduli space has a tachyonic mode which develops (and destabilizes the vacuum) for

$$|X|^2 < |\lambda/h| \equiv |X_c|^2 \ . \tag{7.30}$$

In such region the potential decreases along the $\langle Y\rangle$ direction towards the supersymmetry vacua. A qualitative picture of the potential is reported in Figure 7.6.

Example 4 : Runaway behavior.

A minimal modification of the above theory gives a completely different dynamics. Let us suppose the cubic term of the superpotential (7.25) has the squared shifted from Y to X. We get

$$W = \lambda X + \frac{1}{2}hX^2Y \ . \tag{7.31}$$

The potential reads

$$V = |\frac{1}{2}hX^2|^2 + |hXY + \lambda|^2 \ , \tag{7.32}$$

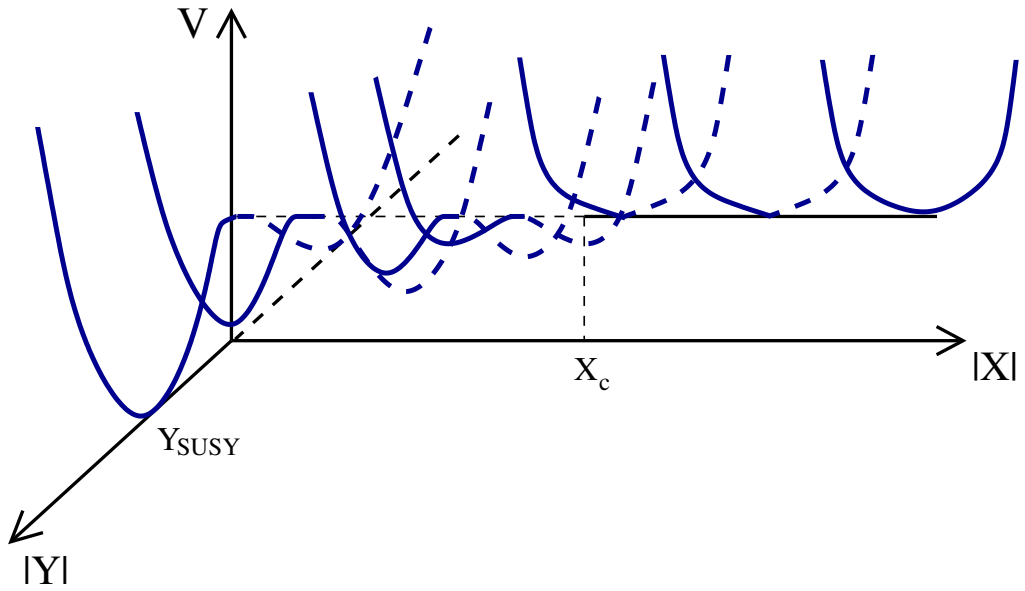

Figure 7.6: The potential of the model of Example 3: supersymmetry restoration.

and there are no supersymmetric ground states! Notice again that R-symmetry is preserved by the superpotential, with charge assignment $R(X) = 2$ and $R(Y) = -2$. Now the question is: where is the minimum of the potential? An analysis of V shows that the minimum is reached for $Y \to \infty$. This can be seen by setting $X = -\frac{\lambda}{hY}$, which kills the second contribution to the potential. By plugging this back into V one gets

$$V = |\frac{\lambda^2}{2hY^2}|^2 \underset{Y\to\infty}{\longrightarrow} 0 \ . \tag{7.33}$$

In other words, there is no stable vacuum but actually a *runaway* behavior and supersymmetry is restored at infinity in field space.

Notice that for large $|Y|$ the amount of supersymmetry breaking gets smaller and smaller and X mass larger and larger. Hence the theory can be described by a theory where X is integrated out. Doing so, the superpotential becomes

$$W_{\text{eff}} = -\frac{\lambda^2}{2hY} \ , \tag{7.34}$$

which gives the runaway behavior described by the potential (7.33).

Of all models we have been considering so far, the only renormalizable one which breaks supersymmetry in a stable vacuum is the original Polonyi model (in fact there is a all pseudo-moduli space). This model is however rather uninteresting per sé, since it describes a non-interacting theory. One might wonder whether there exist

reasonably simple models which are renormalizable, interacting and break supersymmetry in stable vacua. The simplest such model is the re-known O'Raifeartaigh model, which we now describe.

Example 5 : The O'Raifeartaigh model.

Let us consider the theory of three chiral superfields with canonical Kähler potential and a superpotential given by

$$W = \frac{1}{2} h X \Phi_1^2 + m \Phi_1 \Phi_2 - \mu^2 X \ . \tag{7.35}$$

The superpotential respects the R-symmetry, the R-charge assignment being $R(X) = 2, R(\Phi_1) = 0$ and $R(\Phi_2) = 2$. The F-term equations read

$$\begin{cases} \bar{F}_X = \frac{1}{2} h \phi_1^2 - \mu^2 \\ \bar{F}_1 = h X \phi_1 + m \phi_2 \\ \bar{F}_2 = m \phi_1 \end{cases} \tag{7.36}$$

Clearly the first and the third equations cannot be solved simultaneously. Hence supersymmetry is broken. Let us try to analyze the theory a bit further. There are two dimensionful scales, μ and m. Let us choose in what follows $|\mu| < |m|$ (nothing crucial of the following analysis would change choosing a different regime). In this regime one can show that the minimum of the potential is at

$$\phi_1 = \phi_2 = 0 \quad , \quad X = \text{any} \tag{7.37}$$

and the vacuum energy is $V = |\mu^2|^2$. Again, we find a pseudo-moduli space of vacua since X is not fixed by the minimal energy condition. In Figure 7.7 we depict the potential as a function of the scalar fields.

Let us compute the (classical) spectrum around the supersymmetry breaking vacua. The full chiral superfield X is massless, right in the same way as for the Polonyi model (notice that for larger and larger $|X|$ the model gets closer and closer to the Polonyi model since all other fields get heavier and heavier). The massless fermion mode ψ_X is nothing but the goldstino (notice that the only non vanishing F-term in the vacuum is indeed F_X). The phase α of the scalar field $X = |X| e^{i\alpha}$ is the Goldstone boson associated to R-symmetry, which is spontaneously broken in the vacuum. Finally, $|X|$ is massless since it is a modulus (at least at classical level). One can easily compute the ($|X|$-dependent) mass spectrum of all other fields and

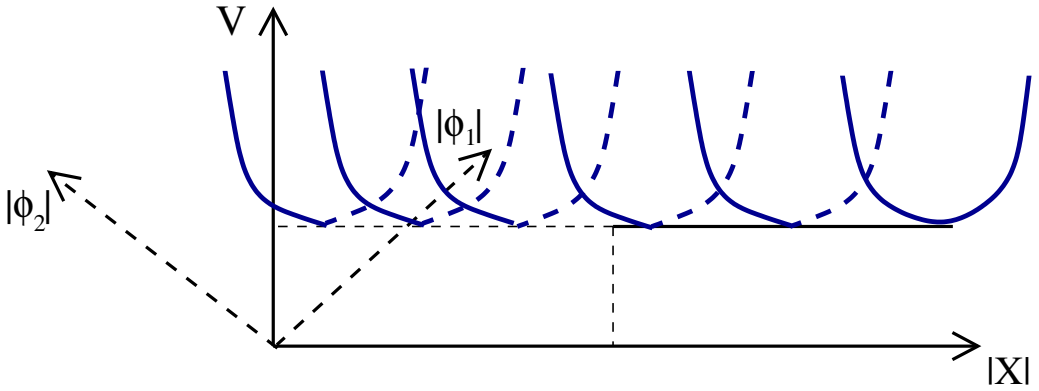

Figure 7.7: The (classical) potential of the O'Raifeartaigh model.

get

$$\begin{aligned} m_0^2(|X|) &= |m|^2 + \frac{1}{2}\eta|h\mu^2| + \frac{1}{2}|hX|^2 \\ &\quad \pm\frac{1}{2}\sqrt{|h\mu^2|^2 + 2\eta|h\mu^2||hX|^2 + 4|m|^2|hX|^2 + |hX|^4} \\ m_{1/2}^2(|X|) &= \frac{1}{4}\left(|hX| \pm \sqrt{|hX|^2 + 4|m|^2}\right)^2 \ . \end{aligned} \tag{7.38}$$

where $\eta = \pm 1$, giving different masses to the four real scalar modes belonging to Φ_1 and Φ_2. As expected, the spectrum is manifestly non-supersymmetric. Notice that $m_{1/2}^2(|X|) = m_0^2(|X|)|_{\mu^2=0}$. Recall finally, that these infinitely many vacua are in fact physically inequivalent, since the mass spectrum depends on $|X|$.

Example 6 : A modified O'Raifeartaigh model.

Let us end this overview of supersymmetry breaking models by considering a modification of the previous model. Let us add a (small) mass perturbation for Φ_2

$$\Delta W = \frac{1}{2}\epsilon\, m\Phi_2^2 \quad \text{with} \quad \epsilon << 1 \ . \tag{7.39}$$

Notice that this term breaks the R-symmetry enjoyed by the original O'Raifeartaigh model. The only F-equation which gets modified is the one for Φ_2 which now reads

$$\bar{F}_2 = m\phi_1 + \epsilon m\phi_2 \ . \tag{7.40}$$

The presence of the second term removes the conflict we had before between this equation and the F-equation for X. Hence we can solve all F-term equations simultaneously and supersymmetry is not broken anymore. The (two) supersymmetric

vacua are at

$$X = \frac{m}{h\epsilon} \quad , \quad \phi_1 = \pm\sqrt{\frac{2\mu^2}{h}} \quad , \quad \phi_2 = \mp\frac{1}{\epsilon}\sqrt{\frac{2\mu^2}{h}} \ . \tag{7.41}$$

Notice that for $\epsilon << 1$ these vacua are far away, in field space, from where the supersymmetry breaking vacua of the O'Raifeartaigh model sit (the VEV of ϕ_2 becomes larger and larger and hence very far from $\phi_2 = 0$, the value of ϕ_2 in O'Raifeartaigh model supersymmetry breaking vacua). In fact, near the origin of field space the potential of the present model is practically identical to the one of the original O'Raifeartaigh model and one can show by direct computation that a classically marginal pseudo-moduli space of supersymmetry breaking minima is present, there.

Computing the mass spectrum near the origin one gets now

$$\begin{aligned} m_0^2(|X|) &= \frac{1}{2}\Big\{|hX|^2 + |m|^2\left(2+|\epsilon|^2\right) + \eta|h\mu^2| \\ &\quad \pm\sqrt{\left[|hX|^2 + |m|^2\left(2+|\epsilon|^2\right) + \eta|h\mu^2|\right]^2 - 4|m|^2\left[|hX\epsilon - m|^2 + \eta|h\mu^2|\left(1+|\epsilon|^2\right)\right]}\Big\} \\ m_{1/2}^2(|X|) &= m_0^2(|X|)|_{\mu^2=0} \ . \end{aligned} \tag{7.42}$$

A close look to the above spectrum shows that in order for mass eigenvalues being all positive, the following inequality should be satisfied

$$\left|1 - \frac{\epsilon h X}{m}\right|^2 > (1+|\epsilon|^2)\left|\frac{h\mu^2}{m^2}\right| \ . \tag{7.43}$$

For small ϵ and μ/m, the marginally stable region described by the above inequality includes a large neighborhood around the origin, and the tachyonic mode develops only for $|X|$ (parametrically) larger than a critical value $|X_c|$ (notice that this is quite the opposite of what we got in Example 4, where the marginally stable region was *above* a critical value; as we will see, this difference has crucial consequences at the quantum level)

$$|X| < |X_c| \ . \tag{7.44}$$

For $\epsilon \to 0$ one gets that $X_c, X_{SUSY} \to \infty$ and the supersymmetric vacua are pushed all the way to infinity. This is consistent with the fact that for $\epsilon = 0$ one recovers the original O'Raifeartaigh model potential where supersymmetric vacua are not there. A rough picture of the potential is given in Figure 7.8.

For future reference, let me notice the following interesting fact. In all models we have been considering so far, the existence of an R-symmetry was a necessary condition for the existence of (stable) supersymmetry breaking vacua (think of the original

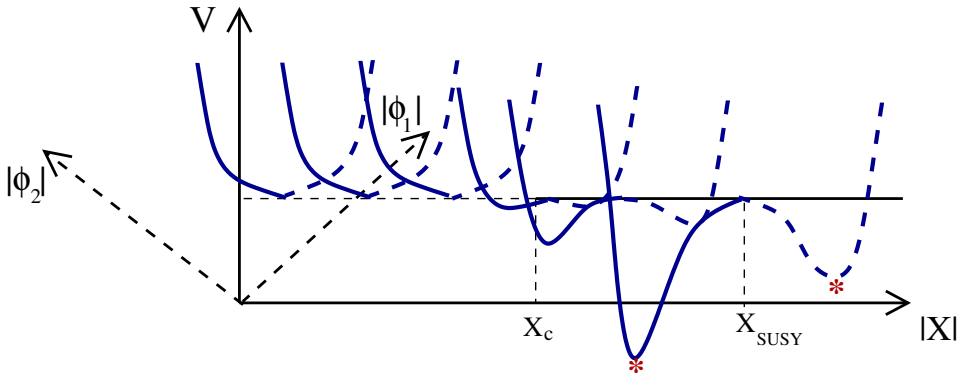

Figure 7.8: The (classical) potential of the *modified* O'Raifeartaigh model.

Polonyi model in Example 1, the model in Example 2, the O'Raifeartaigh model of Example 5, and to some extent the model of Example 4). It is not a sufficient condition, however: think of the counterexample of the model of Example 3. On the contrary, whenever superpotential terms explicitly breaking the R-symmetry were introduced (the massive Polonyi model of Example 1, and Example 6), supersymmetric vacua were found. Finally, every time we found locally stable supersymmetry breaking vacua (Example 6 again), in the vicinity of such vacua an approximate R-symmetry, which the theory does not possess as an exact symmetry, was recovered (essentially, in Examples 6 the superpotential perturbation responsible for the explicit breaking of the R-symmetry becomes negligible near the metastable vacua). All this suggests some sort of relation between R-symmetry and supersymmetry breaking. We will discuss this issue later in this lecture, and put such apparent connection on a firm ground.

7.4 Pseudomoduli space: quantum corrections

In most supersymmetry breaking models we discussed, we found a pseudo-moduli space of non supersymmetric vacua. Associated to this, we found a massless scalar mode $|X|$. While the masslessness of the goldstino and of the Goldstone boson associated to R-symmetry breaking are protected by symmetries, there are no symmetries protecting the pseudo-modulus from getting a mass There isn't any symmetry relating the (degenerate in energy) non supersymmetric vacua. Therefore, by computing quantum corrections, one might expect this field to get a mass, somehow. Let us stress the difference with respect to a moduli space of supersymmetric vacua. Think about the harmonic oscillator. When we quantize the bosonic har-

monic oscillator, the energy of the ground state gets a $\frac{1}{2}\hbar\omega$ contribution. On the contrary, if the ground state is fermionic, the contribution is the same in modulus but with opposite sign (fermions tend to push the energy down). In a supersymmetric situation, the mass degeneracy between bosonic and fermionic degrees of freedom provides equal but opposite contribution to the vacuum energy and the total energy hence remains zero. In a non supersymmetric vacuum the degeneracy is not there anymore (think about the spectrum we computed in the O'Raifeartaigh model) so one expects things to change. In what follows we will try to make this intuition concrete by computing the one-loop Coleman-Weinberg effective potential for both the O'Raifeartaigh and the modified O'Raifertaigh models. In practice, what we have to do is to compute corrections in the coupling h at one loop in the background where the pseudo-modulus $|X|$ has a non-vanishing VEV.

For a supersymmetric theory the Coleman-Weinberg potential reads

$$V_{\text{eff}} = \frac{1}{64\pi^2}\ \text{STr}\ \mathcal{M}^4 \log\frac{\mathcal{M}^2}{\Lambda^2} = \frac{1}{64\pi^2}\left(\ \text{Tr}\ m_B^4 \log\frac{m_B^2}{\Lambda^2} -\ \text{Tr}\ m_F^4 \log\frac{m_F^2}{\Lambda^2}\right)\ , \quad (7.45)$$

where $\mathcal{M} = \mathcal{M}(|X|)$ is the full tree level mass matrix, m_B and m_F correspond to boson and fermion masses respectively, and Λ is a UV cut-off.

There are a few terms missing in the expression (7.45) of the effective potential, if compared to a generic non-supersymmetric theory. Let us consider them in turn. First, we miss the cosmological constant term

$$\sim \Lambda^4\ . \quad (7.46)$$

This term is missing since it only depends on the spectrum, and not on the masses of the different modes. In a supersymmetric theory the spectrum admits an equal number of bosonic and fermionic degrees of freedom at any non-vanishing energy level, no matter whether one is in a supersymmetric or non supersymmetric vacuum. Since bosons and fermions contribute opposite to this term, this degeneracy ensures this term to be vanishing. A second term which is missing is the one proportional to

$$\sim \Lambda^2\ \text{STr}\ \mathcal{M}^2\ . \quad (7.47)$$

This is not expected to vanish in our supersymmetry breaking vacuum since particles have different masses. In other words, the mass spectrum is not supersymmetric along the pseudo-moduli space, recall for instance eqs. (7.38). In fact, an explicit computation shows that also this term is vanishing. This is not specific to this

model. As we will show later, every time supersymmetry is broken spontaneously at tree level, provided the Kähler potential is canonical and in the absence of FI terms, cancellations occur so to give $\mathrm{STr}\ \mathcal{M}^2 = 0$. Finally, there is another divergent term

$$\sim \log \Lambda^2\ \mathrm{STr}\ \mathcal{M}^4\ , \tag{7.48}$$

which however does not depend on $|X|$ and, as we will see momentarily, can be reabsorbed in the renormalization of the tree level vacuum energy $|\mu^2|^2$. So the only non-trivial $|X|$-dependent term is the finite term

$$\sim\ \mathrm{STr}\ \mathcal{M}^4 \log \mathcal{M}^2\ . \tag{7.49}$$

Let us first focus on the O'Raifertaigh model. We should simply plug the tree level masses (7.38) into formula (7.45). A lengthy but straightforward computation shows that V_{eff} is a monotonic increasing function of $|X|$ and can hence be expanded in a power series in $|X|^2$. For small $|X|$ we get

$$V_{\text{eff}}(|X|) = V_0 + m_X^2 |X|^2 + \mathcal{O}(|X|^4) \tag{7.50}$$

where

$$V_0 = |\mu^2|^2 \left[1 + \frac{|h|^2}{32\pi^2}\left(\log\frac{|m|^2}{\Lambda^2} + v(y) + \frac{3}{2}\right) + \mathcal{O}(h^4)\right]\ , \tag{7.51}$$

with

$$y = \left|\frac{h\mu^2}{m^2}\right| < 1 \quad \text{and} \quad v(y) = -\frac{y^2}{12} + \mathcal{O}(y^4)\ , \tag{7.52}$$

and

$$m_X^2 = \frac{1}{32\pi^2}\left|\frac{h^4\mu^4}{m^2}\right| z(y) \quad \text{where} \quad z(y) = \frac{2}{3} + \mathcal{O}(y^2)\ . \tag{7.53}$$

The minimum of the potential is at $|X| = 0$, and besides the tree level contribution $|\mu^2|^2$ it gets a contribution proportional $\sim |h^2|$ which is just a constant, $|X|$-independent shift. As anticipated, the UV cut-off dependence can be reabsorbed in a renormalization of the vacuum energy. Indeed, we can define a running coupling

$$\mu^2(E) \equiv \mu_{bare}^2 \left[1 + \frac{|h|^2}{64\pi^2}\left(\log\frac{E^2}{\Lambda^2} + \frac{3}{2}\right) + \mathcal{O}(h^4)\right] \tag{7.54}$$

getting

$$V_0 = |\mu^2(E=m)|^2 \left(1 + \frac{|h|^2}{32\pi^2} v(y) + \mathcal{O}(h^4)\right) \tag{7.55}$$

and the Λ-dependence has disappeared from the potential.

The upshot of this analysis is that loop corrections have lifted the classical pseudo-moduli space, leaving just one isolated non supersymmetric vacuum. In

this vacuum the scalar field X gets a (one-loop) mass while ψ_X, which is in fact the goldstino, remains massless (notice that in the unique supersymmetry breaking vacuum R-symmetry is preserved, as in Example 2). The shape of the potential in the X-direction becomes at all similar to that of Figure 7.5.

Let us now see what quantum corrections say about the marginally stable supersymmetry breaking vacua of the modified O'Raifeartaigh model of Example 6, the one including the superpotential perturbation (7.39). We will just briefly sketch the main results. The interested reader could try to work out all data in detail. One should again evaluate the Coleman-Weinberg potential, using the tree level spectrum computed near the origin of field space, where the putative marginally stable vacua live, eqs. (7.42). Plugging the latter into formula (7.45), what one finds is that, again, the vacuum degeneracy is lifted and a (locally) stable non-supersymmetric vacuum survives at $|X| = |X_{min}|$ where in our regime, $\epsilon << 1$, X_{min} is near the origin and very far, in field space, from the two supersymmetric vacua sitting at $|X| = |X_{SUSY}|$. More precisely we get

$$V_{\text{eff}}(|X|) = V_0 + m_X^2 |X - X_{min}|^2 + \mathcal{O}(\epsilon^2, |X - X_{min}|^4) \ , \tag{7.56}$$

where $X_{min} \sim \frac{\epsilon m}{h} f(y) + \mathcal{O}(\epsilon^3)$. The spectrum in the supersymmetry breaking vacuum enjoys a massless fermion, ψ_X, the goldstino, while the X-field gets a (ϵ-independent) mass, as in the original O'Raifeartaigh model. The effective potential, once projected into X-direction, looks roughly like that in Figure 7.9 below.

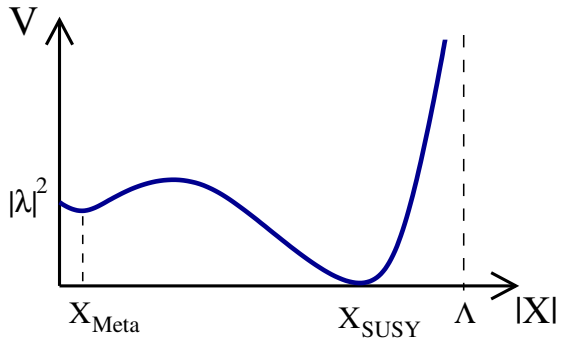

Figure 7.9: The effective potential of the modified O' Raifeartaigh model project onto the $|X|$ direction.

One might ask whether such locally stable supersymmetry breaking minimum is of any physical relevance. An estimate of its lifetime τ can be given looking at the decay rate

$$\Gamma \sim e^{-S_B} \tag{7.57}$$

(recall that $\tau \sim 1/\Gamma$) where S_B is the so-called bunch action, the difference between the Euclidean action of the tunneling configuration and that of remaining in the metastable vacuum. Its exact form depends on the details of the potential, but a simple estimate can be given in the so-called thin wall approximation, which is justified when $|X_{SUSY} - X_{Max}|^4 >> V_{Meta}$, and which is the case here. In this approximation the bunch action reads

$$S_B \sim \frac{\langle \Delta X \rangle^4}{\Delta V} \quad \text{where} \quad \langle \Delta X \rangle = \langle X \rangle_{SUSY} - \langle X \rangle_{Meta} \ , \ \Delta V = V_{Meta} - V_{SUSY} = V_{Meta} \ . \tag{7.58}$$

An explicit computation shows that $S_B \sim \epsilon^{-\alpha}$ where $\alpha > 0$. Hence, the locally stable vacuum is parametrically long-lived, so a metastable one. The upshot of this analysis is that at the quantum level the classically marginal pseudo-moduli space of the modified O'Raifertaigh model is lifted but a local, parametrically long-lived supersymmetry breaking vacuum survives. It is an instructive exercise, which is left to the reader, to repeat this quantum analysis for the classically marginal pseudo-moduli space of Example 3. In this case, the pseudo-moduli space gets completely lifted, and no locally stable supersymmetric minimum survives quantum corrections.

Let us close this section stressing again that nothing we said (and computed) about quantum corrections, pseudomoduli lifting, etc... affects the supersymmetry breaking mechanism itself. All models we have been discussing so far, if breaking supersymmetry, were doing it at tree level. We have not encountered examples where supersymmetry was unbroken at tree level and one-loop quantum corrections induced supersymmetry breaking. Everything coming from the one-loop potential can and does modify the classical supersymmetry breaking vacua (which are not protected by supersymmetry), while it leaves completely unaffected the supersymmetric ones, if any. This agrees with non-renormalization theorems and our claims about the robustness of supersymmetric moduli spaces against (perturbative) quantum corrections.

We will have more to say about F-term breaking in due time. Let us pause a bit now, and consider the other possibility we have alluded to, *i.e.*, spontaneous supersymmetry breaking induced at tree level by D-terms.

7.5 D-term breaking

In a generic theory, where chiral and vector superfields are present, in absence of FI terms it is F-term dynamics which governs supersymmetry breaking. This is

because whenever one can set all F-terms to zero, using (global) gauge invariance acting on the scalar fields one can set to zero all D-terms, too. So, if one wants to consider genuine D-term breaking, one should consider FI-terms, hence abelian gauge factors. In what follows, we will review the most simple such scenario, where two massive chiral superfields with opposite charge are coupled to a single $U(1)$ factor, and a FI-term is present in the Lagrangian. The Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{32\pi}\,\mathrm{Im}\,\left(\tau\int d^2\theta\, W^\alpha W_\alpha\right) + \int d^2\theta\, d^2\bar\theta\,\left(\xi V + \bar\Phi_+ e^{2eV}\Phi_+ + \bar\Phi_- e^{-2eV}\Phi_-\right)\\ &+ m\int d^2\theta\,\Phi_+\Phi_- + h.c.\ ,\end{aligned} \tag{7.59}$$

where under a gauge transformation the two chiral superfields transform as $\Phi_\pm \to e^{\pm ie\Lambda}\Phi_\pm$. The equations of motion for the auxiliary fields read

$$\begin{cases}\bar F_\pm = m\phi_\mp\\ D = -\frac{1}{2}\left[\xi + 2e\left(|\phi_+|^2 - |\phi_-|^2\right)\right]\end{cases} \tag{7.60}$$

It is clearly impossible to satisfy all auxiliary fields equations, due to the presence of the FI-parameter ξ. Hence supersymmetry is broken, as anticipated. The scalar potential reads

$$\begin{aligned}V &= \frac{1}{8}\left[\xi + 2e\left(|\phi_+|^2 - |\phi_-|^2\right)\right]^2 + m^2\left(|\phi_+|^2 + |\phi_-|^2\right)\\ &= \frac{1}{8}\xi^2 + \left(m^2 - \frac{1}{2}e\,\xi\right)|\phi_-|^2 + \left(m^2 + \frac{1}{2}e\,\xi\right)|\phi_+|^2 +\\ &+ \frac{1}{2}e^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2\ .\end{aligned} \tag{7.61}$$

The vacuum structure and the low energy dynamics depends on the sign of $m^2 - \frac{1}{2}e\,\xi$. We will consider the two cases in turn.

- $m^2 > \frac{1}{2}e\,\xi$. All terms in the potential are positive and the minimum of V is at $\langle\phi_\pm\rangle = 0$, where $V = \frac{1}{8}\xi^2$. Supersymmetry is broken but gauge symmetry is preserved. The *only* auxiliary field which gets a VEV is D, so one speaks of *pure* D-term breaking. We are in a situation like the one depicted in the upper right diagram of Figure 7.1.

  One can compute the spectrum and find agreement with expectations. The two fermions belonging to the two chiral multiplets have (supersymmetric) mass m and hence form a massive Dirac fermion. The two scalar fields ϕ_+

and ϕ_- have masses $\sqrt{m^2 + 1/2e\xi}$ and $\sqrt{m^2 - 1/2e\xi}$, respectively. Finally, both the photon A_μ and the photino λ remain massless. The former, because gauge symmetry is preserved, the latter because supersymmetry is broken and a massless fermionic mode, the goldstino (which in this case gets contribution from the photino, only, $\psi^G_\alpha \sim \langle D \rangle \lambda_\alpha$), is expected.

- $m^2 < \frac{1}{2} e\, \xi$. Now the sign of the mass term for ϕ_- is negative. The minimum of the potential is at $\langle \phi_+ \rangle = 0$, $\langle \phi_- \rangle = \sqrt{\frac{\xi}{2e} - \frac{m^2}{e^2}} \equiv h$. Hence both supersymmetry and gauge symmetry are broken. Both the D-field and F_+ get a VEV: in this case we have *mixed* D-term and F-term breaking. The value of the potential at its minumum is $V = \frac{1}{8}\xi^2 - \frac{1}{2}e^2 h^4$. We are in a situation of the type depicted in the lower right diagram of Figure 7.1.

  In order to compute the mass spectrum one should expand the potential around $\langle \phi_+ \rangle = 0$ and $\langle \phi_- \rangle = h$. A lengthy but simple computation gives the following answer. The complex scalar field ϕ_+ has mass $m_{\phi_+} = \sqrt{2}m$. The real part of ϕ_-, ϕ^R_-, gets a mass $m_{\phi^R_-} = \sqrt{2}eh$, while the imaginary part ϕ^I_- disappears from the spectrum (in fact, it is eaten by the photon, which becomes massive, $m_{A_\mu} = \sqrt{2}eh$). The three fermions mix between themselves (there is a mixing induced from Yukawa couplings). One eigenfunction is massless, and is nothing but the goldstino $\psi^G_\alpha \sim \langle D \rangle \lambda_\alpha + \langle \bar{F}_+ \rangle \psi_{+\alpha}$. The other two get equal mass $m_{\tilde{\psi}_\pm} = \sqrt{2e^2h^2 + m^2} = \sqrt{e\xi - m^2}$.

Figure 7.10 gives a summary of the mass spectrum of the FI model as a function of $\frac{1}{2}e\xi$ which, for fixed m, is the order parameter of the supersymmetry breaking transition.

One of the most attractive features of supersymmetric fields theories is the stability of masses under quantum corrections. In models where the FI mechanism plays a role, the physical mass spectrum depends on ξ which is not protected, a priori, since it appears in a D-term. Therefore, it is important to investigate the circumstances under which the FI-term does not get renormalized. This can be done, and the upshot is that the contribution renormalizing the FI-term is proportional to the trace of the $U(1)$ generator taken over all chiral superfields present in the model. However, in order not to have gravitational anomalies, this trace must vanish. Therefore the FI-term does not renormalize for theories free of gravitational anomalies.

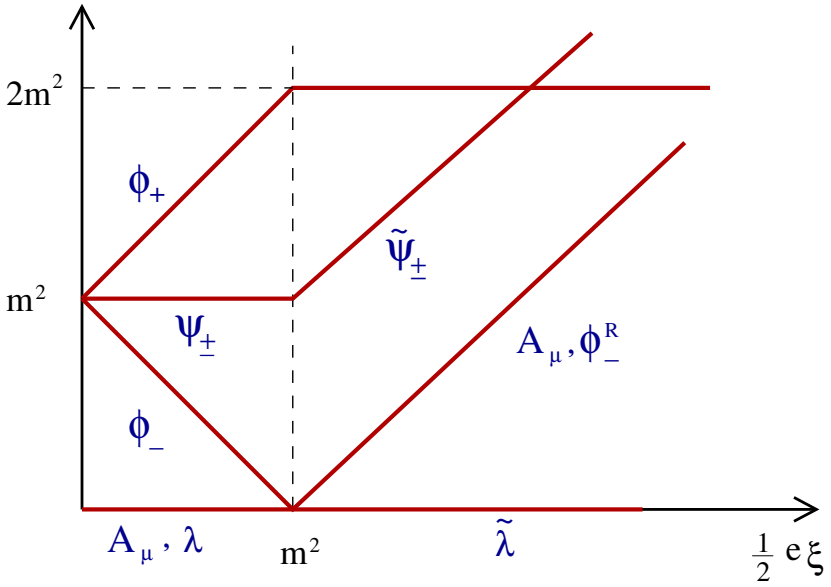

Figure 7.10: The mass spectrum of the Fayet-Iliopoulos model as a function of the FI-parameter ξ.

7.6 Indirect criteria for supersymmetry breaking

We have already alluded to some possible relation between supersymmetry breaking and R-symmetry. In what follows, we will try to make this intuition precise and, more generally, present a few general criteria one can use to understand whether a theory might or might not break supersymmetry, without having a precise knowledge of the details of the theory itself. These criteria might be useful as guiding principles when trying to construct models of supersymmetry breaking in a bottom-up approach and, at the same time, they allow to have a handle on theories which are more involved than the simple ones we analyzed in previous sections. Finally, having some general criteria, possibly being valid also beyond the realm of perturbative physics might also be useful when one has to deal with theories in strongly coupled phases, where a perturbative, semi-classical approach is not possible, and where the direct study of the zero's of the potential is not easy or even not possible.

7.6.1 Supersymmetry breaking and global symmetries

Let us consider a supersymmetric theory which has a spontaneously broken global symmetry and which does not admit (non compact) classical flat directions. This theory, generically, breaks supersymmetry. This can be easily proven as follows.

Since there is a broken global symmetry, the theory admits a goldstone boson (a massless particle with no potential). If supersymmetry were unbroken then one should expect a scalar companion of this goldstone boson, which, being in the same multiplet of the latter, would not admit a potential either. But then, the theory would admit a flat direction, contrary to one of the hypotheses. This is a *sufficient* condition for supersymmetry breaking.

In the above reasoning we have assumed that the second massless scalar corresponds to a non-compact flat direction. This is typically the case since the Goldstone boson is the phase of the order parameter, and its scalar companion corresponds to a dilation of the order parameter, and therefore represents indeed a non-compact flat direction.

Consider now a theory of F chiral superfields Φ^i with superpotential W. Supersymmetry is unbroken if

$$\bar{F}_i = \frac{\partial W}{\partial \phi^i} = 0 \quad , \quad \forall i = 1, 2, \ldots, F \ . \tag{7.62}$$

These are F holomorphic conditions on F complex variables. Therefore, if the superpotenital W is *generic*, one expects (typically distinct) solutions to exist. Hence, supersymmetry is unbroken. By the superpotential being generic we mean the following. The superpotential is a function of the Φ^i's of degree, say, n. It is generic if all possible polynomials of degree n or lower compatible with the symmetries of the theory are present.

Suppose now that W preserves some global non-R symmetry. Hence, W is a function of singlet combinations of the Φ^i's. It is easy to see that in terms of these reduced number of variables, eqs. (7.62) impose an equal number of independent conditions. Let us see how this goes. Suppose, for definiteness, that the global symmetry is a $U(1)$ symmetry and call q_i the corresponding charge of the i-th chiral superfield Φ^i. Hence, we can rewrite the superpotential as e.g.

$$W = W(X_i) \quad \text{where} \quad X_i = \Phi_i \Phi_1^{-q_i/q_1} \quad , \quad i = 2, 3, \ldots, F \ . \tag{7.63}$$

If we now consider eqs. (7.62) we have

$$\begin{cases} j \neq 1 & \frac{\partial W}{\partial \phi^j} = \phi_1^{\frac{-q_j}{q_1}} \frac{\partial W(X_i)}{\partial X_j} = 0 \\ j = 1 & \frac{\partial W}{\partial \phi^1} = \frac{\partial W(X_i)}{\partial X_k} \frac{\partial X_k}{\partial \phi_1} = 0 \ . \end{cases} \tag{7.64}$$

We see that the equation for $\bar{F}_1$ is automatically satisfied if the others $F-1$ are satisfied. Hence, having a system of $F-1$ holomorphic equations in $F-1$ variables,

generically the system allows for solutions. The same reasoning holds for a generic global symmetry. The global symmetry, under which the superpotential is a singlet, diminishes the number of independent variables, but it diminishes also the number of independent F-equations by the same amount. Hence, again, if W is generic, eqs. (7.62) can be solved and supersymmetry is unbroken.

Suppose now that the global symmetry under consideration is a R-symmetry. The crucial difference here is that the superpotential is charged under this symmetry, $R(W) = 2$. Let us call r_i the R-charge of the i-th superfield Φ^i. We can now rewrite the superpotential as

$$W = \Phi_1^{2/r_1} f(X_i) \quad \text{where} \quad X_i = \Phi_i \Phi_1^{-r_i/r_1} \quad , \quad i = 2, 3, \ldots, F \ . \tag{7.65}$$

If we now compute eqs. (7.62) we get

$$\begin{cases} j \neq 1 & \frac{\partial W}{\partial \phi^j} = \phi_1^{\frac{2-r_j}{r_1}} \frac{\partial f(X_i)}{\partial X_j} = 0 \\ j = 1 & \frac{\partial W}{\partial \phi^1} = \frac{2}{r_1} \phi_1^{\frac{2}{r_1}-1} f(X_i) + \phi_1^{\frac{2}{r_1}} \frac{\partial f(X_i)}{\partial X_k} \frac{\partial X_k}{\partial \phi_1} = 0 \ . \end{cases} \tag{7.66}$$

Once the first $F - 1$ equations are satisfied, the latter reduces to $f(X_i) = 0$, which is not at all trivial. So now we have F independent equations in $F - 1$ variables and hence, generically, solutions do not exist. Therefore, supersymmetry is broken, generically. This implies that the existence of an R-symmetry is a *necessary* condition for supersymmetry breaking, if the potential is generic. And, if it is then spontaneously broken, it is a *sufficient* condition (if there are no classical flat directions).

This is known as the Nelson-Seiberg criterium. The O'Raifeartaigh model meets this criterium. It possesses an R-symmetry (which is then spontaneously broken along the pseudo-moduli space), the superpotential is generic, and it breaks supersymmetry. The modified O'Raifeartaigh model instead enjoys supersymmetry restoration. Indeed, R-symmetry is absent since the mass perturbation ΔW breaks it explicitly. So, one would expect the model not to break supersymmetry. And in fact it doesn't. Notice however that somewhere else in the space of scalar field VEVs this model admits non supersymmetric vacua which, if the mass perturbation is small enough, we have proven to be long-lived. In this region the perturbation is negligible and an *approximate* R-symmetry (O'Raifeartaigh model's original one) is recovered. This property is not specific to the modified O'Raifeartaigh model, but is a generic feature of supersymmetry breaking metastable vacua.

Summarizing, a rough guideline in the quest for supersymmetry breaking theories can be as follows:

No R-symmetry	$\longrightarrow$	SUSY unbroken
R-symmetry	$\longrightarrow$	SUSY (maybe) broken
Approximate R-symmetry	$\longrightarrow$	SUSY (maybe) broken locally, restored elsewhere

Since necessary conditions are quite powerful tools, let me stress again one important point. The existence of an R-symmetry is a necessary condition for supersymmetry breaking under the assumption that the superpotential is generic. If this is not the case, supersymmetry can be broken even if the R-symmetry is absent. Another possibility, which typically occurs when gauge degrees of freedom are present in the Lagrangian, is that R-symmetry is absent, but then it arises as an accidental symmetry in the low energy effective theory. Also in this case supersymmetry can be broken even if R-symmetry was absent in the UV Lagrangian. We will see examples of this sort later in this course.

7.6.2 Topological constraints: the Witten Index

Another powerful criterium exists which helps when dealing with theories with complicated vacuum structure and for which it is then difficult to determine directly whether supersymmetry is broken, i.e. to find the zero's of the potential. This criterium, which again provides a necessary condition for supersymmetry breaking, has to do with the so-called Witten index, which, for a supersymmetric theory, is a topological invariant quantity.

The Witten index, let us dub it I_W, is an *integer number* which measures the difference between the number of bosonic and fermionic states, for any given energy level. In a supersymmetric theory, for any *positive* energy level there is an equal number of bosonic and fermionic states. This is obvious if supersymmetry is unbroken, but it also holds if supersymmetry is broken: every state is degenerate with the state obtained from it by adding a zero-momentum goldstino (which is certainly there, if supersymmetry is broken), i.e. a state $|\Omega\rangle$ is paired with $|\Omega + \{p_\mu = 0 \text{ goldstino}\}\rangle$. On the contrary, zero energy states can be unpaired since, due to the supersymmetry algebra, such states are annihilated by the supercharges. Therefore, in a supersymmetric theory the Witten index can get contribution from the zero energy states only, regardless the vacuum one is considering preserves supersymmetry or it does not.

Disclaimer. Strictly speaking what's above holds only if we put the theory in a finite volume. In an infinite volume, when supersymmetry is broken one has to deal with IR singularity issues. In particular, the (broken) supercharge diverges and acting with it on a physical state gives a non-normalizable state. This can be seen as follows. From the current algebra

$$E\,\eta^{\mu\nu} = \langle T^{\mu\nu}\rangle = \frac{1}{4}\,\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\langle\{Q_\alpha, \bar{S}^\nu_{\dot{\alpha}}\}\rangle \tag{7.67}$$

one sees that if the vacuum energy is non vanishing, the vacuum is transformed into one goldstino states (recall that the supercurrent creates a goldstino when acting on the vacuum). In an infinite volume a non-vanishing vacuum energy density corresponds to an infinite total energy, implying that the supercharge indeed diverges and that the zero-momentum goldstino state is not defined (the corresponding state does not exist in the Hilbert space). Putting the theory in a finite volume is a way to regularize (translational invariance can be maintained imposing periodic boundary conditions on all fields). Therefore, in what follows we will start considering the theory in a finite volume V and only later take the infinite volume limit. As we will see, what is relevant for the argument we want to convey is not affected by these issues and holds true also when $V \to \infty$.

A theory in a finite volume has a discrete energy spectrum, all states in the Hilbert space are discrete and normalizable and can be counted unambiguously. First notice that in a supersymmetric theory the energy is $\geq$ than the momentum $|\vec{P}|$, so zero energy states have $\vec{P} = 0$. In what follows we can then restrict to the zero-momentum subspace of the Hilbert space, since all what we are concerned with is to look for the existence of zero energy states. In such subspace the supersymmetry algebra simplifies. In particular, using four-component spinor notation, we have $Q_1^2 = Q_2^2 = Q_3^2 = Q_4^2 = H$, where H is the Hamiltonian of the system and Q_i are the four components of the supercharge.

Suppose to have a bosonic state $|b\rangle$ for which $Q^2|b\rangle = E|b\rangle$, where Q is one of the Q_i's. Then the fermionic state obtained from $|b\rangle$ as

$$|f\rangle \equiv \frac{1}{\sqrt{E}} Q|b\rangle \ , \tag{7.68}$$

has also energy E. This does not apply to zero-energy states since they are annihilated by Q, and hence are not paired. So, as anticipated, the Witten index receives contributions only from the zero energy states. In Figure 7.11 we report the general form of the spectrum of a supersymmetric theory.

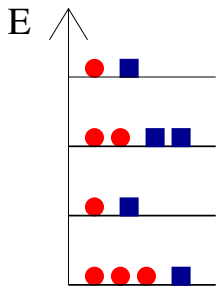

Figure 7.11: The spectrum of a supersymmetric theory in a finite volume. Circles indicate bosons, squares indicate fermions. The zero energy level is the only one where there can exist a different number of circles and squares.

In order to appreciate its topological nature, let us define the index a bit more rigorously. A supersymmetric theory is a unitary representation of the Poincaré superalgebra on some Hilbert space $\mathcal{H}$. Let us assume that

$$\mathcal{H} = \bigoplus_{E \geq 0} \mathcal{H}_E \ . \tag{7.69}$$

The Witten index is defined as

$$I_W(\beta) = \ \mathrm{STr}_{\mathcal{H}} e^{\beta H} \equiv \ \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{\beta H} \quad , \quad \beta \in \mathbb{R}^+ \ . \tag{7.70}$$

It follows that

$$\begin{aligned} I_W(\beta) \ &= \ \sum_{E \geq 0} e^{\beta E} \ \mathrm{Tr}_{\mathcal{H}_E} (-1)^F = \sum_{E \geq 0} e^{\beta E} \left[n_B(E) - n_F(E) \right] = \\ &= n_B(0) - n_F(0) = \ \mathrm{Tr}_{\mathcal{H}_0} (-1)^F = I_W(0) \ . \end{aligned} \tag{7.71}$$

We have been rewriting what we have already shown to hold. The point is that this way it is clear that the index does not depend on β: its value does not vary if we vary β. More generally, one can prove that the Witten index does not depend on *any* parameter (i.e. couplings) and can then be computed in appropriate corner of the parameter space (e.g. at weak coupling) and the result one gets is exact. In other words, the Witten index is a topological invariant.

Suppose one starts from a situation like the one depicted in Figure 7.11. Varying the parameters of the theory, like masses, couplings, etc..., it may very well be that some states move around in energy. The point is that they must do it in pairs, in a supersymmetric theory. Hence, it can happen that a pair of non-zero energy states moves down to zero energy; or, viceversa, that some zero energy states may acquire

non-zero energy. But again, this can only happen if an equal number of bosonic and fermionic zero energy states moves towards a non-zero energy level. The upshot is that the Witten index does not change. All what we have just said is summarized in Figure 7.12.

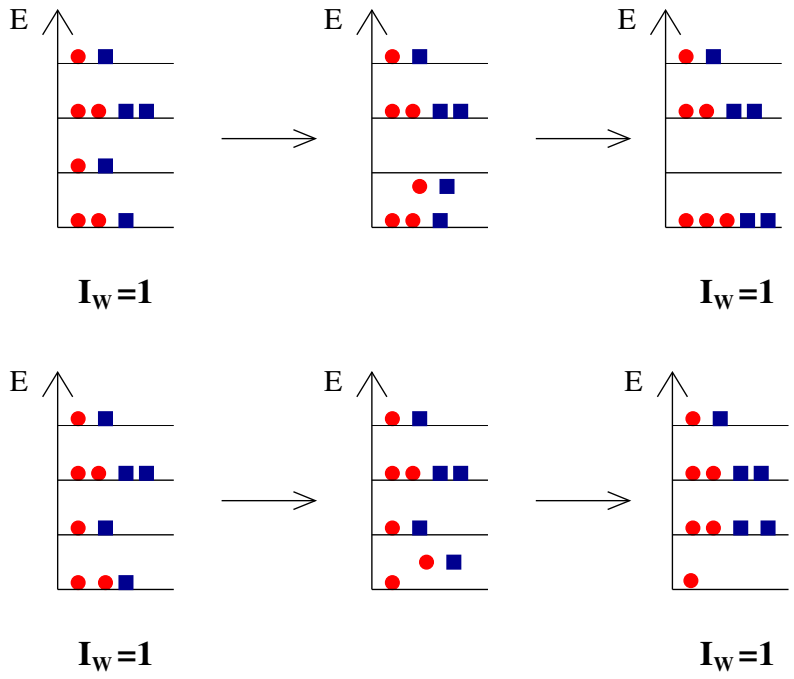

Figure 7.12: Supersymmetric theory dynamics. Upon modifications of parameters of the theory, the number of zero energy states can change; but the Witten index remains the same.

What is this useful for? The crucial point is that the Witten index measures the difference between zero-energy states only. Suppose it is different from zero, $I_W \neq 0$. This means that there exists some zero-energy state, hence supersymmetry is unbroken. But, because of the topological nature of I_W, this conclusion holds at any order in perturbation theory and even non-perturbatively! A theory with non vanishing Witten index cannot break supersymmetry. Suppose instead that $I_W = 0$. Now one cannot conclude anything, just that the number of bosonic and fermionic zero-energy state is the same; but one cannot tell whether this number is zero (broken supersymmetry) or different from zero (unbroken supersymmetry).

So we conclude that having a non-vanishing Witten index is a sufficient condition for the existence of supersymmetric vacua, which also means that having it vanishing is a *necessary* condition for supersymmetry breaking. And a robust one, since I_W is

an exact quantity.

Few comments are in order at this point.

First, the fact that we have been working in a finite volume does not question our main conclusions. If supersymmetry is unbroken in an arbitrary finite volume it means the ground state energy $E(V)$ is zero for any V. Since the large-V limit of zero is still zero, supersymmetry is unbroken also in the infinite volume limit. If one can explicitly compute the Witten index at finite volume and find that it is not vanishing, one can safely conclude that supersymmetry is not broken even in the actual theory, i.e. at infinite volume. On the contrary, the converse is not necessarily true. It might be that supersymmetry is broken at finite volume and restored in the infinite volume limit. Suppose that $I_W = 0$ and that one knows that supersymmetry is broken, that is the minimal energy states have positive energy. The energy density goes as $E(V)/V$ and it may very well be that for $V \to \infty$ the increase of E is not enough to compensate for the larger and larger volume. So the energy density can very well become zero in the infinite volume limit and supersymmetry restored. But this does not hurt much, since all what the vanishing of the Witten index provides is a necessary condition for supersymmetry breaking, not a sufficient one.

A second comment regards the relation between classical and quantum results. Suppose one can explicitly check at tree level that a given theory has non-vanishing Witten index, $I_W \neq 0$. For what we said above, this implies that supersymmetry is unbroken classically *and* that it cannot be broken whatsoever, neither perturbatively nor non-perturbatively. On the contrary, if $I_W = 0$ at tree level and we know that supersymmetry is unbroken classically, it can very well be that (non-perturbative) quantum effects may break it.

The theorem we have discussed may find very useful applications. For one thing, it turns out that pure SYM theories have non-vanishing Witten index (for SYM with gauge group G, the index equals the dual Coxeter number of G, which for SU and Sp groups is just $r + 1$, where r is the rank of G). So pure SYM theories cannot break supersymmetry. As a corollary, SYM theories with massive matter (like massive SQCD) cannot break supersymmetry either. This is because for low enough energy all massive fields can be integrated out and the theory flows to pure SYM, which has non vanishing index. More generally, non-chiral theories, for which a mass term can be given to all matter fields, are not expected to break supersymmetry.

What about chiral theories, instead? Chiral theories behave differently. In this case some chiral superfield cannot get a mass anyway, and so these theories cannot

be obtained from deformation of vector-like theories, as massive SQCD. Hence, one cannot conclude that these theories cannot break supersymmetry. As we will see, most known examples of theories breaking supersymmetry are, in fact, chiral theories.

There is a subtlety in all what we said, so far, which is sort of hidden in some of our claims. We said that the Witten index is robust against any continuous change of parameters. But it turns out that a perturbation that changes the asymptotic behavior of the potential may induce a change in I_W. This is related to the topological nature of the index, which makes it depending on boundary effects. Consider the following simple potential for a scalar field ϕ,

$$V(\phi) = \left(m\phi - g\phi^2\right)^2 \ . \tag{7.72}$$

For $g = 0$ low energy states correspond to $\phi \sim 0$. For $g \neq 0$ low energy states may correspond to $\phi = 0$ but also to $\phi \sim m/g$ (no matter how small g is). So we see here that $I_W(g = 0) \neq I_W(g \neq 0)$. What is going on? The point is that switching on and off g changes the asymptotic behavior of V for large ϕ (that is, at the boundary of field space): in the large ϕ region, for $g = 0$ $V \sim \phi^2$ while for $g \neq 0$ $V \sim \phi^4$. The punchline is that the Witten index is invariant under any change in the parameters of a theory in which, in the large field regime, the potential changes by terms no bigger than the terms already present. If this is not the case, the Witten index can indeed change discontinuously. In other words, I_W is independent of numerical values of parameters as long as these are non-zero. When sending a set of parameters to zero, or switching on some couplings which were absent, one should check that the asymptotic behavior of the potential is unchanged, in order to avoid new states coming in from (or going out to) infinity. Coming back to our SQCD example, we see that for massive SQCD the potential in the large field regime is quadratic, while for massless SQCD is flat: the two theories do not have a priori the same Witten index. Therefore, while massive SQCD is expected to be in the same equivalence class of pure SYM (as far as supersymmetry breaking is concerned), this is not guaranteed for massless SQCD. In other words, no conclusions can be drawn for the massless regime by the analysis in the massive regime. We will see explicit examples of this phenomenon in later lectures.

Let us finally notice that Witten index argument limits a lot the landscape of possible supersymmetry breaking theories; for instance, non-chiral gauge theories most likely (cfr. the subtlety above) cannot break supersymmetry.

7.6.3 Genericity and metastability

Before concluding this section, there is yet another important conclusion we can draw from all what we have learned. Both the Nelson-Seiberg criterium and Witten index argument seem to favor, at least statistically, supersymmetry breaking into metastable vacua.

For one thing, thinking about R-symmetry one might have the impression to fall into a vicious circle. Having an R-symmetry (which is a necessary condition for supersymmetry breaking) forbids a mass term for the gaugino which, being a fermion in a real representation and having R-charge $R(\lambda) = 1$, would have a R-symmetry breaking mass term. But we do not see any massless gauginos around, so gauginos should be massive. If we have an R-symmetry which is spontaneously broken, we could have gaugino masses but we should also have an R-axion, which is not observed. This suggests that R-symmetry should be broken explicitly. But then, generically, we cannot break supersymmetry! The conclusion is that asking for stable vacua compatible with phenomenological observations implies one should look for non-generic theories, which are obviously much less than generic ones. If one admits that supersymmetry might be broken in metastable vacua, then R-symmetry would not be an exact symmetry but only an approximate one. In this case gaugino mass and supersymmetry breaking would be compatible, generically, at least in the metastable vacuum (it is worth noticing that in concrete models there is, not unexpectedly, some tension between the magnitude of gaugino mass and the lifetime of such supersymmetry breaking vacua).

Regardless the R-axion problem mentioned above (which can also get a mass by gravitational effects), there exists another argument favoring metastability. This is related with the computations we performed in section 7.4 when we studied one loop corrections of the O'Raifeartaigh model. We have seen that quantum corrections lift the classical pseudomoduli space, rendering one unique supersymmetry breaking vacuum. However, such vacuum is (the only) one where R-symmetry is in fact not broken! This is not a specific feature of the O'Raifeartaigh model but applies to any model where the R-charges of superfields are either 0 or 2. It has been proven that a necessary condition for having the true vacuum to break the R-symmetry is to have fields in the Lagrangian having R-charge different from 0 and 2. Models of this kind exist and have been constructed. However, in all such models supersymmetry preserving vacua also exist. Hence, supersymmetry breaking vacua where also R-symmetry is in the end broken, are actually metastable.

Also Witten index argument favors metastability, statistically. If accepting we leave in a metastable vacuum, we would allow supersymmetry preserving vacua elsewhere in field space. Hence, all theories with non-vanishing Witten index would not be anymore excluded from the landscape of possible supersymmetry breaking and phenomenologically sensible theories. For example, non-chiral theories would be back in business.

The punchline is that, generically, it may be more likely we leave in a metastable vacuum rather than in a fully stable one. Just... we need to ensure that its lifetime is long enough to be safe!

7.7 Exercises

1. Consider a theory of n chiral superfields Φ^i with superpotential as in eq. (7.14). Prove that, for an interacting theory (that is, some g_{ijk} should be non-vanishing), in order to have spontaneous supersymmetry breaking one needs at least three chiral superfields. Derive the generic form a three-superfield superpotential should have in order to break supersymmetry.

2. Compute the one-loop effective potential on the classically marginal, non supersymmetric vacua of the model in Example 3. What is the fate of these vacua after quantum corrections are taken into account? How is Figure 7.6 modified?

3. Compute the mass spectrum of the FI model both in the pure D-term as well as in the mixed D and F-terms breaking phases, and check explicitly that the spectrum satisfies the so-called supertrace mass formula, that is $\mathrm{STr}\ \mathcal{M}^2 = 0$. (Notice that this identity is trivially satisfied in supersymmetric vacua).

4. Consider all models of F-term breaking of section 7.3 and discuss whether and how the Nelson-Seiberg criterium applies or not.

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8 Mediation of supersymmetry breaking

In this lecture we would like to elaborate a little bit on how the machinery we have been constructing can be used to describe physics beyond the Standard Model.

The basic idea is that the SM should be viewed as an effective theory, valid only up to, say, the TeV scale or slightly higher, and that Nature, at energy above such scale, is described by some suitable $\mathcal{N} = 1$ supersymmetric extension of the SM itself. The most economic option we can think of would be a $\mathcal{N} = 1$ Lagrangian which just includes known particles (gauge bosons, Higgs fields, leptons and quarks) and their superpartners (up to the Higgs sector, which should be doubled, for reasons I mentioned in my first lecture). Then, we might ask whether is it possible to break supersymmetry spontaneously in this theory and be consistent with phenomenological constraints and expectations. Addressing this question will be our main concern, in this lecture.

So far we have discussed possible supersymmetry breaking scenarios at *tree level*: the Lagrangian is supersymmetric but the classical potential is such that the vacuum state breaks supersymmetry (or at least there exist metastable but sufficiently long-lived supersymmetry breaking vacua, besides supersymmetric ones). Can these scenarios, i.e. either F or D-term supersymmetry breaking at tree level, occur in such a minimal supersymmetric extension of the Standard Model (MSSM)? In what follows, we will claim the answer is no: things cannot be as simple as that.

8.1 Towards dynamical supersymmetry breaking

As far as tree level supersymmetry breaking is concerned, from a purely theoretical view point there is at least one point of concern. For several reasons, most notably the hierarchy problem, we would like to have the sparticle masses around the TeV scale (which is not much different from the EW scale, in fact roughly of the same order). This scale is much smaller than any natural UV cut-off one can think of, like the Planck mass. If supersymmetry is broken at tree level, the mass setting the scale of supersymmetry breaking, M_s, would be some mass parameter entering the bare Lagrangian, which would in turn set the scale of all other masses of the SM. For instance, in the O'Raifeartaigh model this scale is μ, the coefficient of the linear term in the superpotential. This way, we would have a scenario where an unnaturally small mass scale has been introduced in a theory in order to solve the unnatural hierarchy between the EW scale and, say, the Planck scale, and avoid a

fine-tuning problem for the Higgs mass. What we gain introducing supersymmetry, would then just be that this small parameter, put by hand into the Lagrangian, would be protected against quantum corrections. Is this a satisfactory solution of the hierarchy problem?

It would be much more natural for this small mass parameter to be explained in some dynamical way. This is possible, in fact. In order to understand how it comes, we should first recall two pieces of knowledge.

First, recall that due to non-renormalization theorems, if supersymmetry is unbroken at tree level, then it cannot be broken at any order in perturbation theory, but only non-perturbatively. In other words, the exact superpotential of a generic supersymmetric theory describing interactions of chiral and vector superfield looks like

$$W_{\text{eff}} = W_{\text{tree}} + W_{\text{non-pert}} \ . \tag{8.1}$$

The second piece of knowledge we need comes from a well-known property that many gauge theories share, i.e. dimensional transmutation. Due to the running of the gauge coupling, which becomes bigger and bigger towards the IR, any UV-free gauge theory possesses an intrinsic (dynamical) scale, Λ, which governs the strong-coupling IR dynamics of the theory. This scale is naturally very small with respect to the scale M_X at which the theory is weakly coupled, according to

$$\Lambda \sim M_X e^{-\frac{\#}{g^2(M_X)}} << M_X \ , \tag{8.2}$$

where # is a number which depends on the details of the specific theory, but is roughly of order 1.

Suppose now we have some complicated supersymmetric gauge theory which does not break supersymmetry at tree level (so all F-terms coming from W_{tree} are zero), but whose strong coupling dynamics generates a contribution to the superpotential $W_{\text{non-pert}}$ which does provide a non-vanishing F-term. This F-term will be order the dynamical scale Λ, and so will be the scale of supersymmetry breaking. This would imply

$$M_s \sim \Lambda << M_X \ , \tag{8.3}$$

hence giving a natural hierarchy between M_s and the UV scale M_X (which can be the GUT scale, $M_{\text{GUT}} \sim 10^{15}$ GeV, or any other scale of the UV-free theory under consideration). This idea is known as *Dynamical Supersymmetry Breaking* (DSB). For the reasons I outlined above, it can be considered as the most natural way we can think of supersymmetry breaking in a fully satisfactory way. We will discuss

several DSB models in later lectures. For the time being, let me simply notice that even if supersymmetry were broken dynamically in Nature, what we have learned in the previous lecture has not been by any means a waste of time. As we will see, in DSB models the effective superpotential W_{eff} has typically a O'Raifeartaigh-like structure: at low enough energy gauge degrees of freedom typically disappear from the low energy spectrum (because of confinement, higgsing and alike) and the effective theory ends-up being a theory of chiral superfields, only. The analysis will then follow the one of the previous lecture, but with the great advantage that the mass parameter setting the scale of supersymmetry breaking and sparticle masses has been dynamically generated (and with the complication that in general the Kähler potential will be non-canonical, of course).

8.2 The Supertrace mass formula

There is yet another reason against tree-level supersymmetry breaking in the MSSM, which is more phenomenological in nature, and related to the so-called supertrace mass formula.

Let us consider the most general $\mathcal{N} = 1$ renormalizable Lagrangian (5.78) and suppose that supersymmetry is spontaneously broken at tree level. We want to compute the trace over all bosonic and fermionic fields of the mass matrix squared in an arbitrary supersymmetry breaking vacuum. To this aim, let us suppose that, generically, all F and D auxiliary fields have some non-vanishing VEV.

- *Vectors*

  If F and D-fields are non vanishing, this means that some scalar fields ϕ^i have acquired a non-vanishing VEV. If such fields are charged under the gauge group, due to gauge covariant derivatives in the scalar kinetic terms, a mass for some vector fields will be induced: $\overline{D_\mu \phi^i} D_\mu \phi^i \to g^2 \langle \bar{\phi} T^a T^b \phi \rangle A_{a,\mu} A_b^\mu$. Hence, we have for the mass matrix squared of vector bosons

  $$\left[(\mathcal{M}_1)^2\right]^{ab} = 2g^2 \langle \bar{\phi} T^a T^b \phi \rangle \ , \tag{8.4}$$

  where the lower index refers to the spin, which is one for vectors. The above formula can be efficiently rewritten as

  $$\left[(\mathcal{M}_1)^2\right]^{ab} = 2\langle D_i^a D^{bi} \rangle = 2\langle D_i^a \rangle \langle D^{bi} \rangle \ , \tag{8.5}$$

  where $D_i^a = \partial D^a / \partial \phi^i$, $D^{ai} = \partial D^a / \partial \bar{\phi}_i$.

- *Fermions*

  The fermion mass matrix can be easily read from the Lagrangian (5.78) to be

$$\mathcal{M}_{1/2} = \begin{pmatrix} \langle F_{ij} \rangle & \sqrt{2} i \langle D_i^b \rangle \\ \sqrt{2} i \langle D_j^a \rangle & 0 \end{pmatrix} , \tag{8.6}$$

  where $F_{ij} = \partial^2 W / \partial \phi^j \partial \phi^i$. The matrix squared reads

$$\mathcal{M}_{1/2} \mathcal{M}_{1/2}^\dagger = \begin{pmatrix} \langle F_{il} \rangle \langle F^{lj} \rangle + 2 \langle D_i^b \rangle \langle D_b^j \rangle & -\sqrt{2} i \langle F_{il} \rangle \langle D_b^l \rangle \\ \sqrt{2} i \langle D_l^a \rangle \langle F^{lj} \rangle & 2 \langle D_l^a \rangle \langle D_b^l \rangle \end{pmatrix} \tag{8.7}$$

  where, with obvious notation, $F^{ij} = \partial^2 \overline{W} / \partial \bar{\phi}_j \partial \bar{\phi}_i$.

- *Scalars*

  The scalar mass matrix squared is instead

$$(\mathcal{M}_0)^2 = \begin{pmatrix} \langle \frac{\partial^2 V}{\partial \phi^i \partial \bar{\phi}_k} \rangle & \langle \frac{\partial^2 V}{\partial \phi^i \partial \phi^l} \rangle \\ \langle \frac{\partial^2 V}{\partial \bar{\phi}_j \partial \bar{\phi}_k} \rangle & \langle \frac{\partial^2 V}{\partial \bar{\phi}_j \partial \phi^l} \rangle \end{pmatrix} . \tag{8.8}$$

  Recalling that $V = F^i F_i + \frac{1}{2} D^a D_a$, one can write it as

$$(\mathcal{M}_0)^2 = \begin{pmatrix} \langle F_{ip} \rangle \langle F^{kp} \rangle + \langle D^{ak} \rangle \langle D_i^a \rangle + \langle D^a \rangle D_i^{ak} & \langle F^p \rangle \langle F_{ilp} \rangle + \langle D_i^a \rangle \langle D_l^a \rangle \\ \langle F_p \rangle \langle F^{jkp} \rangle + \langle D^{aj} \rangle \langle D^{ak} \rangle & \langle F_{lp} \rangle \langle F^{jp} \rangle + \langle D^{aj} \rangle \langle D_l^a \rangle + \langle D^a \rangle D_l^{aj} \end{pmatrix} , \tag{8.9}$$

  where $D_j^{ai} = -g T_j^{ai}$, $F_{ijk} = \partial^3 W / \partial \phi^i \partial \phi^j \partial \phi^k$, $F^{ijk} = \partial^3 \overline{W} / \partial \bar{\phi}_i \partial \bar{\phi}_j \partial \bar{\phi}_k$.

Taking the trace over gauge and flavor indexes of the three matrices (8.5), (8.7) and (8.9) we finally get for the supertrace

$$\mathrm{STr}\ \mathcal{M}^2 = -2g \langle D^a \rangle\, \mathrm{Tr}\ T^a , \tag{8.10}$$

which is the re-known supertrace mass formula.

This formula puts severe phenomenological constraints. First notice that, because of the trace on gauge generators, the r.h.s. is non vanishing only in presence of $U(1)$ factors. If this is the case, then one needs non trivial FI terms to let the r.h.s. being non-vanishing, since we know that if $\xi = 0$ then also $\langle D^a \rangle = 0$. Now, suppose supersymmetry is broken spontaneously, at tree level, in the MSSM. We have only two $U(1)$ factors we can play with, the hypercharge generator $U(1)_Y$ and,

eventually, $U(1)_{\rm em}$. The latter cannot be of any use since if the corresponding FI parameter ξ were non-vanishing, some squarks or sleptons would get a VEV and hence would break EM interactions (comparing with the FI model, being all MSSM scalars massless at tree level, we will be in the mixed F and D-term phase, and hence the potential would have a minimum at non-vanishing value of some scalar field VEV). As for the hypercharge, this again cannot work, since the trace of $U(1)_Y$ taken over all chiral superfields vanishes in the SM (this is just telling us that the SM is free of gravitational anomalies). The upshot is that within the MSSM, formula (8.10) reduces to

$$\mathrm{STr}\ \mathcal{M}^2 = 0\ . \tag{8.11}$$

It is easy to see that this formula is hardly compatible with observations. Since supersymmetry commutes with internal quantum numbers the vanishing of the supertrace would imply that for any given SM set of fields with equal charge, we should observe at least a real component of a sparticle with a mass smaller than all particles with the same charge. Take a charged $SU(3)$ sector. Gluons are massless, since $SU(3)$ is unbroken. From (8.5) it then follows that $\langle D_i^a \rangle = \langle D^{bi} \rangle = 0$, which, by (8.6), implies that the corresponding gluinos are also massless. Then, in such charged sector, only quarks and squarks can contribute non-trivially to (8.11). Since they contribute with opposite sign, the squarks cannot all be heavier than the heaviest quark, and some must be substantially lighter. For instance, in the color-triplet sector with electric charge $e = -1/3$, to which down, strange and bottom quarks belong, a charged scalar with mass smaller than 7 GeV should exist! This is clearly excluded experimentally, not to mention the existence of massless gluinos.

The upshot of this discussion is that we should give up with the idea that the whole story is as simple as (just) tree level supersymmetry breaking in the MSSM.

8.3 Beyond Minimal Supersymmetric Standard Model

The supertrace condition derived above holds a tree-level, and masses get modified by loop effects. However, within the MSSM such modifications are small since the Standard Model is a weakly interacting theory at the electro-weak scale. So this does not help much. A way to avoid the supertrace mass formula severe constraints, while still keeping the MSSM, would be to allow for supersymmetry breaking beyond tree-level, that is dynamical supersymmetry breaking. If supersymmetry breaking were transmitted to the MSSM by quantum corrections, there would be effective

corrections to kinetic terms from wave-function renormalization which could violate the supertrace mass formula by in principle large amounts, hence allowing for phenomenologically meaningful sparticle spectra. We do have a dynamical scale we can play with in the Standard Model, the $SU(3)$ strong coupling scale $\Lambda_{\rm QCD}$. However, DSB driven by QCD strong coupling dynamics could not work either. Looking at eq. (8.3) we would expect in this case a supersymmetry breaking scale of order 300 MeV, which is by far too low for accommodating any sensible phenomenology.

The punchline is that we need something more than just the MSSM, to describe beyond the Standard Model physics. We might need new particles and fields and/or new strong interactions. The options we can play with are many, and understanding the correct path of supersymmetry breaking beyond the SM has been, and still is, a matter of concern and great challenge for theoretical physicists. There are, however, at least two basic properties a competitive model should have. Supersymmetry should be broken dynamically, so to generate the low scale we need (much lower than, say, the Planck scale) in a natural way. Second, in order to avoid the unpleasant constraints coming from the supertrace mass formula, we should better rely on non-renormalizable couplings, or loop effects, to transmit this breaking to the MSSM. As we will see shortly, besides invalidating formula (8.11), such an option would also have the free-bonus of providing an extra suppression between the natural scale of the underlying UV theory and the scale of MSSM sparticle masses. Essentially, supersymmetry breaking from either these sources would be suppressed by either loop factors and/or high masses setting the scale of non-renormalizable terms in the Kähler potential. Hence, the primordial supersymmetry breaking scale would not need to be comparable with electro-weak scale; it could be sensibly higher.

8.4 Spurions, soft terms and the messenger paradigm

Let us deviate, for a while, from what we have been saying so far, and come back to what we said at the very beginning of the previous lecture about possible mechanisms for supersymmetry breaking. We have a second option we have not yet considered: explicit supersymmetry breaking by soft terms. Let us suppose we add explicit supersymmetry breaking terms to the MSSM Lagrangian. In order to save the nice UV properties of supersymmetry, these terms should be UV irrelevant. For instance, if we were to add non-supersymmetric dimensionless couplings, like Yukawa couplings and scalar quartic couplings, we would certainly destroy the pattern of UV cancellations which makes supersymmetry solving, e.g. the hierarchy

problem. We can instead add mass terms, and more generally, positive dimension couplings, like cubic scalar couplings. These would simply tell us below which scale UV cancellations will stop working. The most general such soft supersymmetry breaking Lagrangian will schematically be of the form

$$\mathcal{L}_{\rm soft} = m_\lambda \lambda\lambda - m^2 \bar{\phi}\phi + b\,\phi\phi + a\,\phi^3 \ , \tag{8.12}$$

where λ represents gauginos and ϕ any possible scalar of the MSSM. The first two terms provide masses for gauginos (wino, zino, photino, gluino) and scalar particles (squarks and sleptons, Higgs particles), respectively. The third term, known as B-term, may arise in the Higgs sector and couples the up and down scalar Higgs H_u and H_d. Finally the fourth, known as A-term, corresponds to cubic gauge and flavor singlet combinations of MSSM scalars, e.g. Higgs and left and right squark components. A-terms are in one-to-one correspondence with Yukawa couplings (which belong to the supersymmetric part of the MSSM Lagrangian): each quark and lepton is just substituted by its scalar superpartner.

All terms appearing in eq. (8.12) are UV irrelevant and renormalizable, and it was indeed shown time ago that the full Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm MSSM} + \mathcal{L}_{\rm soft} \tag{8.13}$$

is free of quadratic divergences to all orders in perturbation theory. Notice, in passing, that such a Lagrangian would automatically solve the supertrace mass formula problem. Indeed, a Lagrangian like the one above would violate eq.(8.11) precisely by terms of order the sparticle masses, see the expression (8.12), which is, by construction, compatible with observations.

There is a number of very important issues one should discuss regarding the Lagrangian (8.13), including a number of potential problems some of the soft terms could pose, like the so-called supersymmetry flavor, CP and fine-tuning problems, to name a few. This is however beyond the scope of this course, for which we refer instead to the complementary course on Beyond the Standard Model (BSM) physics.

Here we would like instead to emphasize a crucial point. We want to reconnect to our previous discussion and show how such a rather ad hoc soft Lagrangian, where supersymmetry is broken explicitly, can actually be generated by *spontaneous* supersymmetry breaking in a bigger theory, including fields and interactions beyond the MSSM ones.

First, let us introduce the idea of *spurion* fields. The basic observation is that in a supersymmetric theory any constant, non-zero value for the lowest component of

a superfield (a VEV) does not break supersymmetry. Hence, in a supersymmetric Lagrangian each coupling constant can be promoted to a background superfield, a spurion in fact, with a non-vanishing such VEV. Let us consider, for concreteness, the WZ model

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\, Z\bar{\Phi}\Phi + \int d^2\theta \left(\frac{1}{2}M\Phi^2 + \frac{1}{6}\lambda\Phi^3\right) + h.c.\ , \tag{8.14}$$

and think of Z, M and λ as real and chiral background superfields, respectively. If only their lowest components have a non-vanishing VEV, this is just the WZ model itself. We can include supersymmetry breaking terms in the above Lagrangian by allowing these superfields having higher (scalar!) component VEVs. We can have in general

$$\begin{aligned} \langle Z\rangle &= 1 + \theta^2 B + h.c. + c\,\theta^2\bar{\theta}^2 \\ \langle M\rangle &= \mu - \theta^2 F_M \\ \langle \lambda\rangle &= \lambda - \theta^2 F_\lambda\ . \end{aligned}$$

Plugging these expressions into the Lagrangian (8.14), after integrating out the auxiliary fields of Φ we find for the potential

$$V = V_{\text{SUSY}} - \left(c - |B|^2\right)\bar{\phi}\phi + \left[\left(F_M + B\mu\right)\phi^2 + \left(\frac{1}{3}F_\lambda + \frac{1}{2}B\mu\right)\phi^3 + h.c.\right]\ , \tag{8.15}$$

where $V_{\text{SUSY}} = \left|\mu\phi + \frac{1}{2}\lambda\phi^2\right|^2$. We see that the non supersymmetric contribution to the potential exactly reproduces the second, third and fourth soft terms of the Lagrangian (8.12), upon the trivial identifications

$$\begin{aligned} m^2 &= c - |B|^2 \\ b &= F_M + B\mu \\ a &= \frac{1}{3}F_\lambda + \frac{1}{6}B\mu\ . \end{aligned}$$

Following the same logic for the SYM action

$$\mathcal{L} = \int d^2\theta\ \tau\, W_a^\alpha W_\alpha^a\ , \tag{8.16}$$

one can seemingly reproduce gaugino masses by promoting the complexified gauge coupling τ to a chiral superfield and provide a non-vanishing constant VEV for its F-term

$$\langle\tau\rangle = \tau + \theta^2 m_\lambda\ . \tag{8.17}$$

Applying this logic to the Lagrangian (8.13), one can actually write all soft terms by means of spurion couplings, and re-write (8.13) using a pure supersymmetric formalism. This turns out to be a very convenient thing to do when it comes to compute the divergence structure of the theory and prove, e.g., the absence of quadratic divergences.

The whole picture we get, although phenomenologically viable and logically consistent, is still not completely satisfactory. The softly broken MSSM Lagrangian (8.13) has more than 100 free parameters (masses, phases, mixing angles, etc...), meaning there are few unambiguous predictions one can really make. One might want to find some organizing principle, where these many parameters may be naturally explained in terms of some simpler underlying theory.

Here is where we can finally close the gap between soft term breaking and spontaneous supersymmetry breaking. It is enough to promote spurions to fully fledged superfields with their own Lagrangian and kinetic terms. By some suitable and for the time being unspecified mechanism, they acquire non-vanishing F and D-terms spontaneously, and then generate soft terms by their interactions with the MSSM fields via couplings of the kind (8.14). This is the basic idea of the so-called *messenger paradigm*: one imagines a fully renormalizable theory where supersymmetry is broken spontaneously in some hidden sector at some high scale and then communicated to the MSSM fields by non-renormalizable interactions and/or loop effects. After integrating out heavy fields, this will generate effective couplings precisely as those in the Lagrangian (8.14), with non-vanishing F and D-components for some fields. These F and D-terms will then give rise to soft terms through a procedure like the one above. This way, all specific properties that MSSM supersymmetric breaking soft terms should have, will be ultimately generated (and explained) by a larger theory in which supersymmetry breaking occurs spontaneously.

8.5 Mediating the breaking

What are the possible ways in which a scenario as the one outlined above can actually be realized?

An obvious candidate as messenger of supersymmetry breaking is gravity, since any sort of particle couples universally to it. Gravity is inherently non-renormalizable, at least as it manifests itself at energies lower than the Planck scale. Hence, couplings like those appearing in eqs. (8.14) and (8.16) are precisely what one expects,

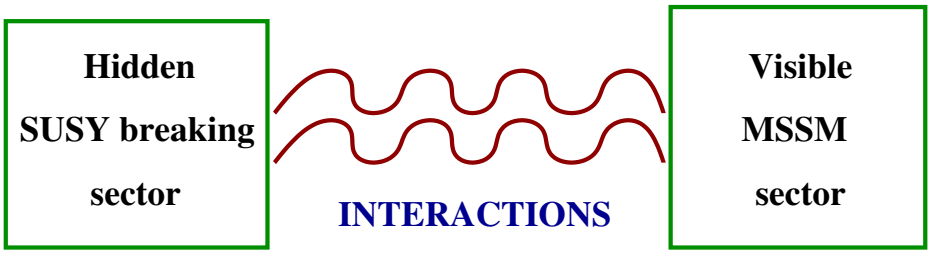

Figure 8.1: The messenger paradigm: supersymmetry is broken in a hidden sector and then communicated to the visible MSSM sector (or any viable supersymmetric extension of the Standard Model) via interactions felt by the MSSM particles.

in this scenario.

Another possibility is that supersymmetry breaking is mediated by gauge interactions. We can imagine that supersymmetry is broken in the hidden sector, and that some fields, known as *messenger fields*, feeling (or directly participating in, this is a model-dependent property) supersymmetry breaking are also charged under SM gauge interactions. Gauginos will get a mass at one-loop, by direct coupling with messenger fields. Scalar MSSM sparticles, instead, would get mass at two loops, interacting with messenger fields via intermediate MSSM vector superfields, to which gauginos belong to. In this scenario, soft terms will be generated after integrating out heavy fields, ending-up again with effective couplings of the kind (8.14) and (8.16). Obviously, the main source of mediation can be gauge interactions only in a regime where the always present gravity mediation is suppressed. Below we will give an estimate of the regime where such a situation can occur.

In what follows, we are not going to discuss these two mediation mechanisms in detail, nor any of their diverse phenomenological benchmarks, neither the many variants of the basic models which have appeared in the literature, with their pros and cons. For this, we refer again to the BSM course. Here, we just want to show how such scenarios may naturally generate, at low energy, spurion-like couplings with MSSM fields and, eventually, give rise to soft terms.

8.5.1 Gravity mediation

From a low energy point of view, one can parameterize the effect of unknown physics at the Planck scale M_P by higher order operators, suppressed by M_P. Suppose that

some hidden sector field X gets a non-vanishing F-term, that is

$$\langle X \rangle = 0 \quad , \quad \langle F_X \rangle \neq 0 \ . \tag{8.18}$$

The most general form of the Lagrangian describing the gravitational interaction between X and the visible sector fields will be something like

$$\begin{aligned} \mathcal{L}_{\rm int} &= \int d^2\theta d^2\bar{\theta} \left(\frac{c}{M_P^2} X^\dagger X Q_i^\dagger Q^i + \frac{b'}{M_P^2} X^\dagger X H_u H_d + \frac{b}{M_P} X^\dagger H_u H_d + h.c. \right) \\ &+ \int d^2\theta \left(\frac{s}{M_P} X W_a^\alpha W_\alpha^a + \frac{a}{M_P} X Q^i H_u \tilde{Q}_i + h.c. \right) \end{aligned} \tag{8.19}$$

plus, possibly, higher order operators. The Q_i's represent the up and down Higgs chiral superfields plus all matter superfields, while H_u and H_d obviously refer to the up and down Higgs only. For the sake of simplicity, we have taken all order one dimensionless coefficients in each term to be the same, that is i-independent.

Plugging the values (8.18) into the above Lagrangian one gets all possible MSSM soft terms! The first term on the r.h.s. of eq. (8.19) gives rise to non-supersymmetric masses for all sfermions (squarks, sleptons and scalar Higgs particles), while the second and third terms provide mass terms for the scalar Higgs only (more below). The first term of the second line provides gaugino masses; finally, the last term generates all A-terms. We see that we get a rather simple pattern of soft terms. Up to order one coefficients, they share one and the same mass scale, which we dub $m_{\rm SOFT}$

$$m_{\rm SOFT} \sim \frac{\langle F_X \rangle}{M_P} \ . \tag{8.20}$$

Imposing $m_{\rm SOFT}$ to be order the TeV scale we see that in a gravity mediated scenario the primordial supersymmetry breaking scale, the so-called intermediate scale, is order

$$M_s = \sqrt{\langle F_X \rangle} \sim \sqrt{m_{\rm SOFT}\, M_P} \sim 10^{11} \text{ GeV} \ , \tag{8.21}$$

somewhat in between the EW scale and the Planck scale.

Let us spend a few more words on Higgs mass terms. From the Lagrangian (8.19) we see three contributions to scalar Higgs mass . The first gives rise to mass terms for the up and down Higgs, respectively (they are proportional to $H_u^\dagger H_u$ and $H_d^\dagger H_d$). The second term is the B-term, which gives rise to a quadratic term mixing H_u and H_d. Finally, as for the third term, notice that it can be re-written as

$$\int d^2\theta d^2\bar{\theta}\, \frac{b}{M_P} X^\dagger H_u H_d = b \frac{\langle F_X^\dagger \rangle}{M_P} \int d^2\theta H_u H_d \ . \tag{8.22}$$

This contribution is a so-called μ-term contribution and upon integration in chiral superspace it gives a quadratic contribution similar in structure to the first term.

Notice that *all* these three couplings are needed in order to trigger EW breaking. The first such terms gives masses to scalar Higgs particle, and it can actually give a negative mass square to some of them, something we certainly need to trigger spontaneous symmetry breaking. The second one is also necessary. One can show the B-term to be proportional to $\sin 2\beta$ where $\tan\beta$ is the ratio between the VEVs of the up and down Higgses, $\tan\beta = v_u/v_d$. Clearly, if $B = 0$, either the up or the down Higgs do not get a VEV, and therefore one cannot provide masses to all SM particles. Finally, the μ-term is the only possible contribution which can provide higgsino a mass, and therefore should certainly be there. The way we have re-written the μ-term makes it clear that it can also be (and generically is) generated from a perfectly supersymmetric superpotential coupling in the MSSM Lagrangian

$$W = \mu_{\rm SUSY} H_u H_d \ . \tag{8.23}$$

There is no a priori reason why the above term, which comes from a supersymmetric contribution and is then not related to the dynamics driving the breaking of supersymmetry, should come to be the same scale of the soft terms, as it should. In principle, it could be any scale between $m_{\rm SOFT}$ and M_P. This is the famous μ (or $\mu/B\mu$) problem: how to avoid large μ-terms *and* have them the same order of magnitude of B-terms. Gravity mediation provides an elegant and simple way to solve this problem. First, one can impose some (discrete) symmetry on the MSSM Lagrangian which forbids a tree level μ-term in the superpotential, $\mu_{\rm SUSY} = 0$. This way, both the μ and the B-terms are generated radiatively. The non trivial thing is to make them be the same order of magnitude. However, as we have seen above, this is exactly what happens in a gravity mediation scenario: up to coefficients of order unity, all soft terms, including the B and μ-terms, are the same order, eq. (8.20)!

A typical problem of gravity mediation scenarios, instead, is the so-called supersymmetry flavor problem. In order not to spoil the excellent agreement between FCNC effects predicted by the SM and known experimental bounds, any sort of new physics should not induce any sensible extra FCNC. In order for this to be the case the interactions mediating supersymmetry breaking better be flavor-blind. This is not the case for gravity whose high energy UV completion is not actually guaranteed to couple universally to flavor. Therefore, in general, in gravity mediation scenarios one has to confront with the flavor problem. We will not discuss this further here. Let us just remark that there exist different proposals on how to overcome this

problem, the most compelling and natural one being possibly the so-called anomaly mediation scenario.

8.5.2 Gauge mediation

Any gauge mediation model is characterized by the assumption that there exist messenger fields. The latter, by definition, are those hidden sector fields which are charged under the SM gauge group. The basic idea of gauge mediation is as follows.

Messengers couple (in a model-dependent way) to hidden sector supersymmetry breaking dynamics and this affects their mass matrix which, besides a supersymmetric contribution (which is supposed to be large enough not to make messengers appear at energies of order the EW scale), receives a non-supersymmetric contribution. By coupling radiatively with MSSM fields, supersymmetry breaking is communicated to MSSM fields and provides soft terms for MSSM sparticles. As we are going to show in the following, gaugini get a mass at one-loop while squarks, sleptons and Higgs fields feel supersymmetry breaking at two loops through ordinary $SU(3) \times SU(2) \times U(1)_Y$ gauge boson and gauginos interactions. One of the beauties of gauge mediation as opposed to gravity mediation, is that gauge mediation supersymmetry breaking can be understood entirely in terms of loop effects in a renormalizable framework. Hence, it has a high level of reliability and calculability.

There are different schemes for gauge mediation, e.g. minimal, direct, semi-direct gauge mediation, which differ, ultimately, by the way the messenger mass matrix is affected by the hidden sector supersymmetry breaking dynamics. This provides different patterns for the MSSM soft terms texture. As an exemplification, in what follows we will briefly discuss minimal gauge mediation (MGM) which is a simple, still rich enough scenario to let us get a feeling on how things work. In MGM all complicated hidden sector dynamics is parameterized in terms of a single chiral superfield X which couples to the messenger sector. The latter is made of two set of chiral superfields Φ and $\tilde{\Phi}$ transforming in complex conjugate representation of the SM gauge group, so not to generate gauge anomalies. The interaction term is as simple as

$$W = X\tilde{\Phi}\Phi \ . \tag{8.24}$$

A rough scheme of MGM is depicted in Figure 8.2.

The spurion-like field X inherits non-vanishing F and lower component term

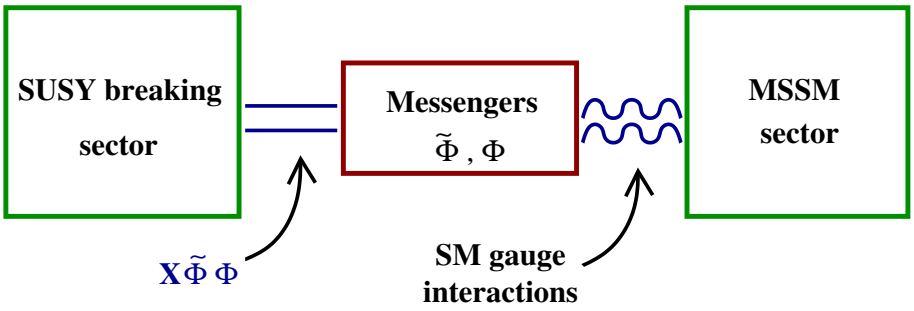

Figure 8.2: Minimal gauge mediation. Messengers feel supersymmetry breaking via a cubic coupling with a spurion-like chiral superfield X which has a non-vanishing F-term VEV inherited from the hidden sector non supersymmetric dynamics.

VEVs from the hidden sector,

$$\langle X \rangle = M + \theta^2 \langle F_X \rangle \ . \tag{8.25}$$

Once plugged into the messenger Lagrangian, this gives a splitted messenger mass spectrum

$$m^2_{\phi,\tilde{\phi}} = M^2 \pm \langle F_X \rangle \quad , \quad m_{\psi,\tilde{\psi}} = M \ . \tag{8.26}$$

While fermions receive only the supersymmetric contribution, scalars receive both supersymmetric and non supersymmetric contributions. Recalling that messenger fields are charged under the SM gauge group we see there is a stability bound which forces us to take $M^2 > \langle F_X \rangle$ (if not, some messenger scalars would get a non-vanishing VEV and would break part of the SM gauge group). If M is large enough we can then integrate the messengers out and the effective low energy theory at scale lower than M breaks supersymmetry. The net low energy effect boils down to radiative corrections to gaugino propagators, which get a mass at one loop, while gauge bosons remain massless since they are protected by gauge invariance. Via intermediate SM gauge coupling interactions, also MSSM scalar fields will eventually get a non supersymmetric mass contribution at two loop order (such contributions come from inserting messenger loop corrections in the one-loop sfermions mass diagrams, which in the MSSM, that is without contribution from the messenger sector, consistently sum-up to zero). Feynman diagrams contributing to gaugino and scalar masses are reported in Figures 8.3 and 8.4, respectively.

The gaugino mass computation is rather easy, since only one type of diagram contributes. Summing-up all two-loop contributions renormalizing scalar masses is instead not an easy task. However, the end result is surprisingly simple and reads

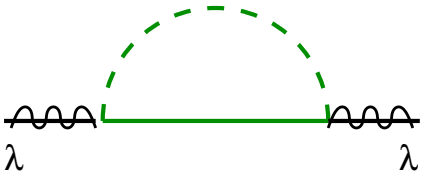

Figure 8.3: The one-loop diagram providing gaugino mass. Black lines are MSSM fields, green lines are messenger fields. Dashed lines correspond to scalar fields and continuous lines to fermion fields.

(quite interestingly, and in agreement with the general philosophy advocated in section 8.4, one can equivalently get these results upon integrating out messenger fields and using the RG)

$$m_\lambda \sim \frac{g^2}{16\pi^2}\frac{\langle F_X\rangle}{M}\left[1+\mathcal{O}(\left|\frac{\langle F_X\rangle}{M^2}\right|^2)\right] \tag{8.27}$$

$$m_{sf}^2 \sim \left(\frac{g^2}{16\pi^2}\right)^2\left|\frac{\langle F_X\rangle}{M}\right|^2\left[1+\mathcal{O}(\left|\frac{\langle F_X\rangle}{M^2}\right|^2)\right] . \tag{8.28}$$

We see that in MGM all soft terms come naturally of the same order of magnitude

$$m_{\rm SOFT} \sim \frac{g^2}{16\pi^2}\frac{\langle F_X\rangle}{M} . \tag{8.29}$$

Imposing again that soft masses are order the TeV scale one then gets

$$\frac{\langle F_X\rangle}{M} \sim 10^5 \text{GeV} , \tag{8.30}$$

which implies that in MGM the primordial supersymmetry breaking scale M_s can be as low as

$$M_s = \sqrt{\langle F_X\rangle} \sim 10\sqrt{m_{\rm SOFT}M} \geq 10^5 \text{ GeV} , \tag{8.31}$$

where the lower bound is reached for $M^2 \sim \langle F_X\rangle$.

As we have already observed, gravity mediation is an always present contribution to supersymmetry breaking mediation mechanisms (the field X would also interact gravitationally with the visible sector via a Lagrangian like (8.19), in general). Hence, it is only when its contribution is suppressed with respect to that of gauge mediation that the latter can play a role. In order for gravity effects to be negligible, say to contribute no more than 1/1000 to soft mass squared, one gets an upper bound for the scale M (and hence for $M_s \sim \sqrt{\langle F_X\rangle}$)

$$\frac{g^2}{16\pi^2}\frac{\langle F_X\rangle}{M} \geq 10^{3/2}\frac{\langle F_X\rangle}{M_P} \longrightarrow M \leq \frac{g^2}{16\pi^2}10^{-3/2}M_P \sim 10^{15} \text{ GeV} . \tag{8.32}$$

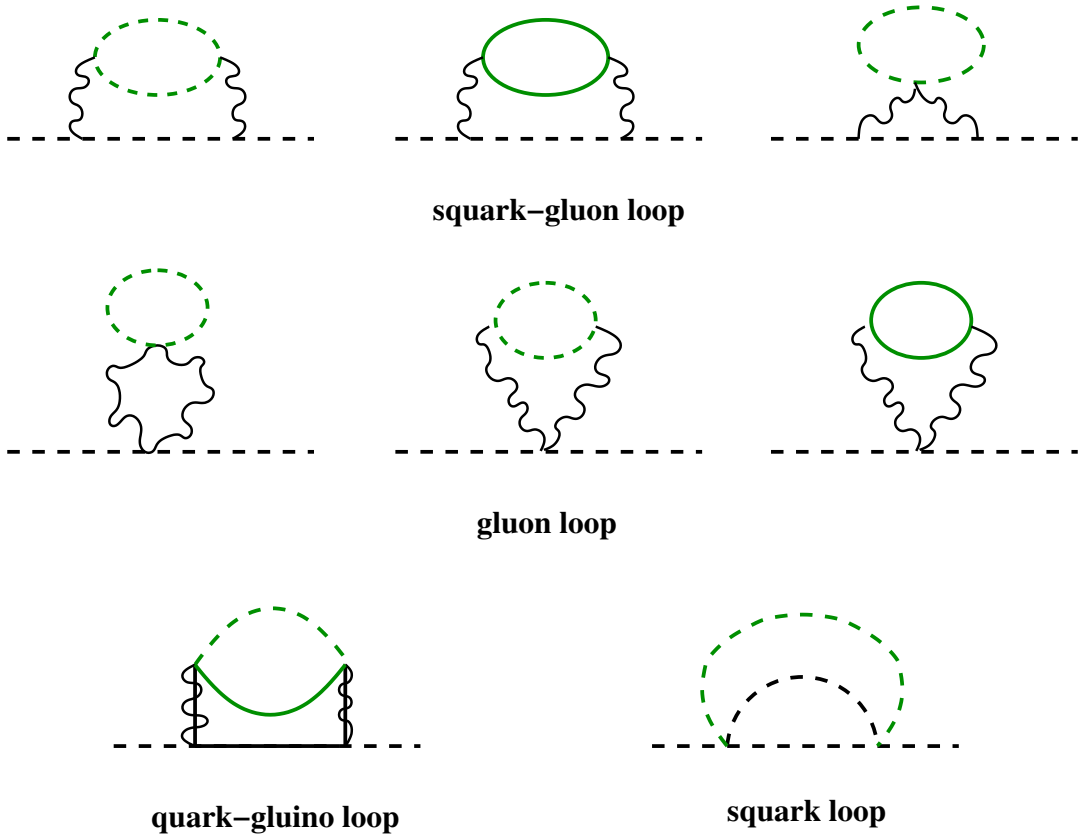

Figure 8.4: Two-loops diagrams providing scalar sparticles mass. There are four different class of diagrams. Conventions are as in Figure 8.3.

Using (8.29) this gives an upper bound for M_s of order 10^{10} GeV. Together with the lower bound (8.31) this implies that the supersymmetry breaking scale M_s can range from 10^5 to up to 10^{10} GeV, in gauge mediation scenarios.

In passing, let us notice that the simple mass pattern (8.27)-(8.28) is not a generic feature of gauge mediation. Indeed, in another popular scheme, direct gauge mediation, as well as in semi-direct gauge mediation, the soft spectrum tends to be split, that is gauginos are typically suppressed with respect to scalar particles.

Let us close this brief overview on gauge mediation saying a few words about flavor and μ problems. We are in a sort of reversed situation with respect to gravity mediation. Gauge interactions are intrinsically flavor-blind. Hence, gauge mediation does not provide any further FCNC contribution to the SM, and the flavor problem is automatically solved in this framework. On the contrary, the μ problem is much harder. One can again avoid a supersymmetric μ-term by means of some discrete symmetry to be imposed on the Higgs sector supersymmetric Lagrangian. What

is problematic, though, is to generate radiatively μ and B-terms of the same order of magnitude (note that the two-loops diagrams we have discussed above do not provide B and μ-terms). The simplest possible way one can think of, does not work. Indeed, allowing a cubic coupling between H_u, H_d and the field X

$$W_{\rm H} = \lambda_{\rm H} X H_u H_d \ , \tag{8.33}$$

one could in principle generate both a μ and a B-term from supersymmetry breaking dynamics. In order for the μ-term being of the order of other soft masses, as it should be, we need

$$\mu = \lambda_H M \sim 1 \text{ Tev} \ . \tag{8.34}$$

This implies that λ_H is order 10^{-2} or smaller. This enhances the B-term. Indeed, recalling that $\langle F_X \rangle \leq M^2$, the non-supersymmetric to supersymmetric mass ratio contribution coming from the superpotential coupling (8.33) is

$$\frac{B}{\mu^2} \sim \frac{\lambda_{\rm H} \langle F_X \rangle}{\lambda_{\rm H}^2 M^2} \sim \frac{\langle F_X \rangle}{\mu M} \sim 10^2 \ , \tag{8.35}$$

giving an unacceptably large B-term. This problem is not specific to MGM nor to the actual way we have generated μ and B-terms here. It is a problem which generically plagues any gauge mediation scenario. Even though several proposals has been put forward to solve the μ-problem in gauge mediation, it is fair to say that a fully satisfactory and natural framework to solve this problem is not yet available.

Let me conclude stressing again what is the main point of this all business. What all these mediation models are about, is to provide a theory of the soft terms, a predictive pattern for these extra terms that one can (and has to) add to the MSSM Lagrangian or any desired supersymmetric extension of the Standard Model. We have been trying to give an idea on how things might work, and reviewed few aspects of the most basic mediation mechanisms. A throughout analysis of the phenomenology of these schemes and their variants is not our goal here. In the reminder of these lectures, we will instead focus on the hidden sector dynamics, trying to deepen our understanding of supersymmetric dynamics at strong coupling. Equipped with new tools to attack non-perturbative regimes of supersymmetric theories, we will eventually be able to study concrete models of dynamical supersymmetry breaking.

8.6 Exercises

1. Compute formula (8.27) from the Feynman diagram of Figure 8.3.

2. Compute the contribution of two diagrams arbitrarily chosen out of those depicted in Figure 8.4 to the sfermion mass formula (8.28).

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9 Non-perturbative effects and holomorphy

In this lecture we will start our program regarding the study of the non-perturbative regime of supersymmetric theories. The main point of this first lecture will be to introduce *holomorphy*, or better put holomorphy, which is an intrinsic property of supersymmetric theories, at work. Before doing that, however, there are a few standard non-perturbative field theory results we need to review.

9.1 Instantons and anomalies in a nutshell

Gauge theories might contain a θ-term, which is

$$\mathcal{S}_\theta = \frac{\theta_{\rm YM}}{32\pi^2}\int d^4x\ {\rm Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} \quad \text{where} \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\ . \tag{9.1}$$

This term is a total derivative. Indeed

$$\frac{1}{2}\int d^4x\ {\rm Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \int d^4x\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu {\rm Tr}\left[A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right]\ , \tag{9.2}$$

which implies that the θ-term does not have any effect on the classical equations of motion. However, when one quantizes a theory one has to average over all fluctuations, not just those satisfying the classical equations of motions and therefore the θ-term can in fact be relevant in some cases.

One such configurations are *instantons*, classical solutions of the Euclidean action that approach pure gauge for $|x| \to \infty$ (so to make the action finite). In this case the relevant integral is in fact an integer number, the so-called instanton number of the configuration (aka winding number)

$$\mathcal{S}_\theta = \frac{\theta_{\rm YM}}{32\pi^2}\int d^4x\ {\rm Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = n\,\theta_{\rm YM} \quad \text{where} \quad n\in\mathbb{Z}\ . \tag{9.3}$$

The instanton number is a topological quantity, in the sense that it does not change upon continuous deformations of the gauge field configuration. Moreover, since the action enters the path integral as $\int \mathcal{D}\phi e^{i\mathcal{S}_\theta}$, the θ-angle indeed behaves as an angle, in the sense that the shift

$$\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi\ , \tag{9.4}$$

is a symmetry of the theory.

An instanton field configuration interpolates between different vacua of the gauge theory. Both vacua are gauge equivalent to the usual vacuum with zero gauge potential but the corresponding gauge transformation cannot be deformed to the identity

(these are known as *large gauge transformations*). If this were the case it would have been possible to let the field strength being vanishing in all space-time, contradicting eq. (9.3). The fact that $F_{\mu\nu}$ cannot vanish identically for configurations with $n \neq 0$ implies that there is a field energy associated with the gauge field configuration interpolating between the different vacua. An energy barrier and an associated quantum mechanical tunneling amplitude proportional to $e^{-\mathcal{S}_E}$ where $\mathcal{S}_E$ is the Euclidean action of a field configuration which interpolates between the different vacua. Instantons are nothing but just such interpolating field configurations. Notice that configurations related by large gauge transformations are weighted differently in the action, because of eq. (9.3), implying that they should not be identified as physically equivalent configurations.

Instantons have an intrinsic non-perturbative nature. Recall that the RG-equation for the gauge coupling g reads

$$\mu \frac{\partial g}{\partial \mu} = -\frac{b_1}{16\pi^2} g^3 + \mathcal{O}(g^5) \ , \tag{9.5}$$

where b_1 is a numerical coefficient which depends on the theory. The solution of this equation at one loop is

$$\frac{1}{g^2(\mu)} = -\frac{b_1}{8\pi^2} \log \frac{\Lambda}{\mu} \ . \tag{9.6}$$

where the scale Λ is defined as the scale where the one-loop coupling diverges. It sets the scale where higher-loop and non-perturbative effects should be taken into account. For any scale μ_0 we have that

$$\Lambda \equiv \mu_0 e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} \ . \tag{9.7}$$

It is important to stress that Λ does *not* depend on the energy scale: it is a RG-invariant quantity. Indeed

$$\begin{aligned} \frac{\partial \Lambda}{\partial \mu_0} &= e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} + \mu_0 \left[-\frac{8\pi^2}{b_1 g^3(\mu_0)} \frac{2}{\mu_0} \left(\frac{b_1}{16\pi^2} g^3(\mu_0) + \mathcal{O}(g^5) \right) \right] e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} \\ &= e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} + \mu_0 (-\frac{1}{\mu_0}) e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} = 0 \ . \end{aligned} \tag{9.8}$$

up to higher-order corrections. This can be reiterated order by order in perturbation theory, getting the same result, namely that $\partial \Lambda / \partial \mu_0 = 0$.

Let us now consider an instanton field configuration. There exists a lower bound on the Euclidean action of an instanton. Indeed,

$$0 \leq \int d^4x \, \mathrm{Tr} \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right)^2 = \int d^4x \left[2 \, \mathrm{Tr} \, F_{\mu\nu} F^{\mu\nu} \pm 2 \, \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right] \tag{9.9}$$

which implies

$$\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \geq \left| \int d^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} \right| = 32\pi^2 n \ , \tag{9.10}$$

where the last equality holds for an instanton configuration with instanton number n. This implies that there is a lower bound to an instanton action: instanton contributions to amplitudes are suppressed at least by (multiply above equation by $1/4g^2$)

$$e^{-\mathcal{S}_{\mathrm{inst}}} = \left(e^{-\frac{8\pi^2}{g^2(\mu)}} \right)^n = \left(\frac{\Lambda}{\mu} \right)^{n\, b_1} \tag{9.11}$$

where in the last step we have used eq. (9.6). This shows that instantons are inherently non-perturbative effects, since they vanish for $\Lambda \to 0$, and are very weak, if not negligible, in the perturbative regime.

Anomalies are classical symmetries of the action which are broken by quantum effects. In other words, we have

$$\partial_\mu j^\mu_A = 0 \ \xrightarrow[\text{corrections}]{\text{quantum}} \ \partial_\mu j^\mu_A \neq 0 \ , \tag{9.12}$$

where j_A is the current associated to the anomalous symmetry.

In what follows we will focus on chiral anomalies, that is anomalies associated to chiral currents. These arise in field theories in which fermions with chiral symmetries are coupled to gauge fields. Recall that local currents cannot be anomalous, since they would imply violation of unitarity of the theory (we only know how to couple spin-1 fields in a way respecting unitarity to conserved currents). Hence a quantum field theory, in order to make sense, should not have any gauge anomaly. On the contrary, global chiral currents can be anomalous. This is what is usually meant as chiral anomaly.

Let us review a few basic facts about anomalies. Anomalies get contribution at one loop, only, by evaluation of triangle diagrams like those reported in Figure 9.1. Let us first consider a set of global currents j_A. The one-loop three-point function corresponding to diagram a of Figure 9.1 will be

$$\langle j^\mu_A(x_1) j^\nu_B(x_2) j^\rho_C(x_3) \rangle = \ \mathrm{Tr}\,(t_A t_B t_C)\, f_{\mu\nu\rho}(x_i) \ , \tag{9.13}$$

where the trace comes from contraction of the group generators around the loop. This correlator has important properties, as we will see momentarily, but it does not provide by itself any anomaly: the corresponding classical conservation law is not violated quantum mechanically.

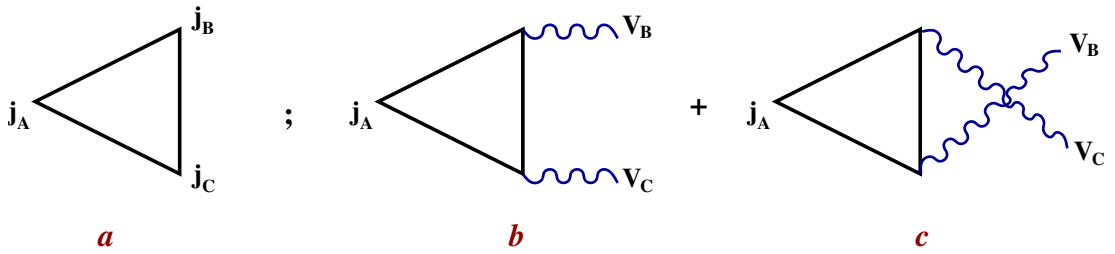

Figure 9.1: One-loop diagrams contributing to correlators of one global current with two global or local currents. Diagram a does not provide any anomaly. Diagrams b and c, instead, contribute to the anomaly of the global current j_A.

Suppose now to gauge some (or all) global currents, by coupling the original Lagrangian with gauge fields as

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \sum_B V_\mu^B \, j_B^\mu \ , \tag{9.14}$$

and let us compute the correlator $\langle j_A V_B V_C \rangle$. The one-loop diagrams contributing to such correlator are diagrams b and c of Figure 9.1. By differentiating the result one gets

$$\partial_\mu \, j_A^\mu \sim \ \text{Tr}\left(t_A \left\{t_B, t_C\right\}\right) F_B^{\mu\nu} \tilde{F}_{C,\mu\nu} \ , \tag{9.15}$$

and we do have an anomaly now. We clearly see here that j_A should be a global current, since if this were not the case we would have had a violation of unitarity in the quantum theory.

From the above formula we see that the anomaly coefficient is proportional to $\text{Tr}\left(t_A \left\{t_B, t_C\right\}\right)$. This vanishes for real and pseudoreal representations. Indeed, for real or pseudoreal representations we have that $t_A = -(t_A)^T$ and it then easily follows that

$$\text{Tr}_r \left(t_A \left\{t_B, t_C\right\}\right) = - \, \text{Tr}_r \left(t_A \left\{t_B, t_C\right\}\right) \ . \tag{9.16}$$

Therefore, only massless chiral fermions can contribute to the anomaly coefficient. This result, once applied to local currents, which cannot be anomalous, provides severe restrictions on the massless fermion content of a quantum field theory.

Suppose to have a theory with gauge group G with generators t_A, a global symmetry group $\tilde{G}$ with generators $\tilde{t}_A$, and a set of Weyl fermions ψ_i, transforming in the representations $(r_i, \tilde{r}_i)$ of the gauge and global symmetry groups, respectively. In this case, the ABJ anomaly computation gives

$$\partial_\mu \, j_A^\mu \sim \sum_i \, \text{Tr}_{\tilde{r}_i} \tilde{t}_A \, \text{Tr}_{r_i} \left(t_B t_C\right) F_B^{\mu\nu} \tilde{F}_{C\mu\nu} \ . \tag{9.17}$$

Since $\mathrm{Tr}_{\tilde{r}_i} \tilde{t}_A = 0$ for any simple algebra, only abelian factors $U(1) \subset \tilde{G}$ can be anomalous. On the other hand, $\mathrm{Tr}_{r_i}(t_B t_C) = C(r_i)\delta_{BC}$, where $C(r_i)$ is the quadratic invariant (Casimir) of the representation r_i. Working everything out, paying attention to numerical coefficients, one finally gets for an abelian group

$$\partial_\mu j^\mu = -\frac{A}{16\pi^2} F_B^{\mu\nu} \tilde{F}^B_{\mu\nu} \ , \tag{9.18}$$

where $A = \sum_i q_i C(r_i)$ is the anomaly coefficient, q_i being the $U(1)$ global charges of fermion fields ψ_i.

This result shows the connection between anomalies, instantons and the θ-angle. First, we see that the anomaly is proportional to the instanton number we have previously defined. Indeed, integrating eq. (9.18) in space-time, and comparing with eq. (9.3), we get

$$\Delta Q = 2A\, n \ , \tag{9.19}$$

where n is the instanton number and ΔQ the amount of charge violation due to the anomaly. So we see that anomalous symmetries are violated by a specific amount, given by eq. (9.19), in an instanton background. This also shows that anomalies are IR effects, since the violation is very mild at weak coupling.

As far as equation (9.18) is concerned, the effect of the anomalous $U(1)$ symmetry corresponds to a shift in the θ-angle as

$$\psi_i \to e^{i q_i \alpha} \psi_i \quad \Longrightarrow \quad \theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} - 2\alpha A \ . \tag{9.20}$$

That is, the anomalous breaking can be seen as an explicit breaking: a term in the action, the θ-term, in fact, is not invariant under the anomalous symmetry.

Notice that if we perform a $U(1)$ transformation but assign transformation properties to θ_{YM} as to compensate for the shift, then the anomalous $U(1)$ is promoted to an actual symmetry of a larger theory (where the complexified gauge coupling is promoted to a dynamical field). This symmetry, however, is spontaneously broken by the coupling constant VEV (θ_{YM} in this case). As we will later see, this way of looking at anomalous symmetries can be efficiently used to put constraints on the construction of low-energy effective Lagrangians.

9.2 't Hooft anomaly matching condition

The usefulness of correlators between global currents has been pointed out by 't Hooft. As we will review below, they compute scale independent information about

a quantum theory (hence providing a powerful tool to understand some of its non-perturbative properties).

Let us consider a Lagrangian $\mathcal{L}$ defined at some scale μ, with some non-anomalous global symmetry group G generated by currents j_A^μ. Compute the triangle diagram for three global currents (which is not an anomaly!) and call $A_{\rm UV}$ the result. Weakly gauge the global symmetry group G by adding new gauge fields V_μ^A and define a new Lagrangian

$$\mathcal{L}' = \mathcal{L} - \frac{1}{4g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + j_A^\mu V_\mu^A \ . \tag{9.21}$$

This theory is inconsistent since it has a gauge anomaly, $A_{\rm UV}$, because we have gauged G. Let us then add some spectator free massless fermion fields ψ_s (spectator in the sense that they couple only through the G gauge coupling) transforming in representations of G so to exactly cancel the anomaly, i.e. $A_s = -A_{\rm UV}$. The resulting theory

$$\mathcal{L}'' = \mathcal{L} - \frac{1}{4g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_s \not{\partial}\, \psi_s + \left(j_{s,A}^\mu + j_A^\mu\right) V_\mu^A \ , \tag{9.22}$$

where $j_{s,A}$ are the currents associated to the spectator fermions ψ_s, is non-anomalous. Consider this anomaly-free theory at some scale $\mu' < \mu$. Since the spectator fermion fields and gauge fields can be made arbitrarily weakly coupled by taking $g \to 0$, the IR dynamics of the enlarged theory (9.22) is just the IR dynamics of the original theory plus the arbitrarily weakly coupled spectator theory. Therefore, A_s should be the same and since the theory is anomaly free, we should have that $A_{\rm IR} + A_s = 0$, which implies

$$A_{\rm IR} = A_{\rm UV} \ . \tag{9.23}$$

Taking $g \to 0$ spectators fields completely decouple and (9.23) should still hold. The punchline is that in a quantum field theory, anomaly coefficients associated to global currents are scale independent quantities, and their UV and IR values should match. This is known as 't Hooft anomaly matching condition.

A simple equation such as (9.23) puts severe constraints on the IR dynamics of a quantum field theory, in particular as far as its massless spectrum. In particular, it implies that a theory with global conserved currents but with 't Hooft anomaly (that is a non vanishing triangular anomaly associated to these global currents), does not have a mass gap. This remarkable result can be claimed without doing any sort of non-perturbative computation; just a one-loop one! If the global symmetry is preserved, a non-vanishing 't Hooft anomaly implies the existence of massless (maybe composite) fermions in the effective IR theory. It might happen that there

does not exist any choice of quantum number for composite states to match this anomaly. But eq. (9.23) still holds. This indicates that the global symmetry should be spontaneously broken - the corresponding goldstone bosons, which transform non-linearly under the broken symmetry, being the massless particle needed to match the anomaly. Actually, it is precisely this argument which originally suggested that the $SU(3)_L \times SU(3)_R$ global symmetry of QCD should be spontaneously broken, the pion being the corresponding (pseudo) goldstone bosons.

9.3 Holomorphy

In what follows, we want to discuss a bit further a property of supersymmetric theories, known as holomorphy, which plays a crucial role when it comes to understand the quantum properties of supersymmetric theories and to what extent they differ from non-supersymmetric ones. Let us first briefly recall the concept of Wilsonian effective action.

When dealing with effective theories we deal with effective actions. The transition from a fundamental (bare) Lagrangian down to an effective one, involves integrating out high-momentum degrees of freedom. The effective action (aka Wilsonian action) is defined from the bare action $\mathcal{S}_{\mu_0}$ defined at some UV scale μ_0, as

$$e^{i\mathcal{S}_\mu} = \int_{\phi(p)\,,\,p>\mu} \mathcal{D}\phi\, e^{i\mathcal{S}_{\mu_0}} \tag{9.24}$$

where $\mathcal{S}_\mu$ is the effective action, of which we review below few basic properties.

The Wilsonian action correctly describes a theory's degrees of freedom at energies below a given scale μ (the cut-off). It is local on length scales larger than $1/\mu$, and describes in a unitary way physical processes involving energy-momentum transfers less than μ. As far as processes are concerned:

- at energies $E \sim \mu$, the effective action couplings and masses are given by the tree-level couplings in the effective action (effects of all higher energy degrees of freedom have already been integrated out),
- at energies $E << \mu$ there will be quantum corrections due to fluctuations of modes of the fields in $\mathcal{S}_\mu$ with energies between E and μ.

The upshot is that the Wilsonian action $\mathcal{S}_\mu$ is the action which describes the physics at the scale μ by its *classical* couplings.

As we have already noticed, any parameter in a supersymmetric Lagrangian can be thought of as a VEV of a superfield. In particular, if we focus on the F-term part of the Lagrangian (i.e. the superpotential) it follows that each coupling, which is complex in general, can be thought of as the lowest component VEV of a (very heavy) chiral superfield. The theory can then be viewed as an effective theory of a bigger theory where these heavy fields have been integrated out and they act as spurions at low energy.

This makes it manifest that the (F-term) bare Lagrangian is not only holomorphic in the fields but also in the couplings. This property is important since, as we will see, promoting coupling constants to chiral superfields one can often extend symmetries of the superpotential and put severe constraints on the form (and sometime the very existence) of quantum corrections. This important result can also be proven by means of a (supersymmetric) Ward identity which implies that all coupling constants appearing in the tree level superpotential must only appear holomorphically in quantum corrections to the superpotential.

The crucial observation, which is a consequence of such Ward identities and, more in general, of holomorphy, is that the Wilsonian action is also holomorphic in the coupling constants. In other words, the couplings appearing in $\mathcal{S}_\mu$ should be holomorphic functions of the UV couplings. Basically, holomorphy makes the restrictions on possible quantum corrections allowed by supersymmetry apparent. It provides a supersymmetric version of selection rules.

Let us recall that the Wilsonian effective action is not the effective action Γ which is obtained by integrating out *all* degrees of freedom down to $\mu = 0$. The latter is the generating functional of 1PI graphs and calculates the Green functions of the original, UV theory. It is not holomorphic in the coupling constants and suffers from holomorphic anomalies. It is not the correct thing to look at in asymptotically free gauge theories since it is not well defined. The two effective actions are one and the same only if there are no interacting massless particles, which make the 1PI effective action Γ suffer from IR divergences.

In order to make this rather abstract discussion more explicit, let us consider a concrete example. Suppose we have a given supersymmetric theory and, following the logic outlined previously, let us consider an enlarged symmetry group which includes a spurious $U(1)$ symmetry, associated to a coupling constant λ, which breaks this symmetry spontaneously (a spurion) and has unit charge with respect to it, $Q(\lambda) = 1$. This simply means that we have in the tree-level superpotential a

term like

$$W_{\rm tree} \supset \lambda\, \mathcal{O}_{-1} \ , \tag{9.25}$$

where the operator $\mathcal{O}$ has charge -1 with respect to the spurious $U(1)$ symmetry.

Suppose we are interested in the appearance of a given operator $\mathcal{O}_{-10}$ among quantum corrections. In general, we expect it to appear at tenth or higher order as

$$\Delta W \sim \lambda^{10}\, \mathcal{O}_{-10} + \lambda^{11} \bar{\lambda}\, \mathcal{O}_{-10} + \cdots + \lambda^{10} e^{-1/|\lambda|^2} \mathcal{O}_{-10} \ , \tag{9.26}$$

where we have assumed that the classical limit, $\lambda \to 0$ is well defined and so we do not expect any negative powers of λ to appear. Holomorphy implies that *only* the first term appears. All other terms cannot be there since are non-holomorphic in nature (both λ and $\bar{\lambda}$ appear). A corollary of the above discussion is that any operator with positive charge with respect to the $U(1)$ is also disallowed. For one thing, we cannot have negative powers of λ because we are supposing the theory is well defined in the classical limit (in other words, we are assuming that the physics is smooth for $\lambda \to 0$), while any power of $\bar{\lambda}$ is forbidden by holomorphy. Notice that the latter property is due to supersymmetry, and it is not shared by an ordinary field theory.

The basic message we want to convey here is that holomorphicity in the coupling constants (and usual selection rules for symmetries under which coupling constants may transform) and the requirement of smoothness of physics in various weak-coupling limits, provide severe constraints on the structure of the effective superpotential of a supersymmetric quantum field theory.

9.4 Holomorphy and non-renormalization theorems

Using holomorphy one can prove many non-renormalization theorems (and go beyond them, as we will see).

Example 1: the WZ model (and its many siblings). The tree level superpotential of the WZ model has the following structure

$$W_{\rm tree} = \frac{1}{2} m \Phi^2 + \frac{1}{3} \lambda \Phi^3 \ . \tag{9.27}$$

The question one might ask is: what is the form of the effective superpotential $W_{\rm eff}$, once quantum corrections (both perturbative and non-perturbative) are taken into account? Let us try to answer this question using holomorphy. First, promote m

and λ to spurion superfields. This makes the theory enlarging its symmetries by a flavor $U(1)$ symmetry and a R-symmetry, according to the table below

$$\begin{array}{ccc} & U(1)_R & U(1) \\ \Phi & 1 & 1 \\ m & 0 & -2 \\ \lambda & -1 & -3 \end{array} \tag{9.28}$$

The superpotential has (correctly) R-charge 2 and flavor $U(1)$ charge 0. Notice that both symmetries are spontaneously broken whenever the spurion superfields, m and λ have a non-vanishing lower component VEV.

Because of what we discussed in the previous section, the effective (that is, exact) superpotential should be some holomorphic function of Φ, m and λ, with R-charge equal to 2 and flavor charge equal to 0. Its most general form can be written as a function of $\lambda\Phi/m$ as follows

$$W_{\text{eff}} = m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right) = \sum_{n=-\infty}^{\infty} a_n \lambda^n \, m^{1-n} \, \Phi^{n+2} \ , \tag{9.29}$$

where $f_{\text{free}} = \frac{1}{2} + \frac{1}{3}\lambda\Phi/m$, and a_n are arbitrary coefficients.

The form of f can be fixed as follows. First, in the classical limit, $\lambda \to 0$, we should recover the tree level result. This implies that there cannot appear negative powers of λ; hence $n \geq 0$ and, in order to agree with (9.27) at tree-level, $a_0 = \frac{1}{2}$ and $a_1 = \frac{1}{3}$. Taking also the massless limit at the same time, $m \to 0$, restricts n further, i.e. $n \leq 1$. The upshot is that the effective superpotential should be nothing but the tree level one: holomorphy (plus some obvious physical requirements, more below) tells us that the superpotential of the WZ model is not renormalized at any order in perturbation theory *and* non-perturbatively!

The requirement about finiteness in the massless limit requires a few more comments. Taking the massless limit at finite λ does not lead to a weakly-coupled theory, so one could not use smoothness arguments so naively. However, taking both $m, \lambda \to 0$ such that $m/\lambda \to 0$ we do achieve the result above, since the theory is free in this case. One may still wonder whether this conclusion is correct since in this limit there is a massless particle and so the effective theory should have some IR divergences. This is not the case since we do not run the RG-flow down to $\mu = 0$: there are no IR divergences in the Wilsonian effective action, as opposed to the 1PI effective action.

Another, equivalent way to see the absence of negative powers of m in the effective superpotential is to observe that all terms with $n \geq 0$ are generated by *tree-level* diagrams only, in the UV theory (it is a matter of number of vertices and propagators), see Figure 9.2 below. All diagrams of the kind of the one depicted in Figure



Figure 9.2: Tree level (super)graph producing terms of the series (9.29) for $n > 1$.

9.2 are not 1PI for $n > 1$; they cannot be produced from loops, and they should not be included in the effective action for finite m. So the integer n in eq. (9.29) is indeed restricted to be either 0 or 1.

What we have just proven, namely that the superpotential of the WZ model is not renormalized at any order in perturbation theory and non-perturbatively, is not specific to the WZ model. It actually applies to all models where only chiral superfields are present: in these cases, that is in the absence of gauge interactions, the tree-level superpotential is an exact quantity.

Example 2: As a second example, we want to illustrate what holomorphy can tell us about the running of gauge coupling in supersymmetric gauge theories. Let us focus, for definiteness, on SQCD. Recall that this is a supersymmetric gauge theory with gauge group $SU(N)$ and F flavors, described by F pairs of chiral superfields $(Q, \tilde{Q})$ transforming in the fundamental respectively anti-fundamental representation of the gauge group and vanishing superpotential, $W(Q, \tilde{Q}) = 0$. At the classical level, the global symmetries are as detailed below

	$SU(F)_L$	$SU(F)_R$	$U(1)_B$	$U(1)_A$	$U(1)_{R_0}$
Q_a^i	F	$\bullet$	1	1	a
$\tilde{Q}_j^b$	$\bullet$	$\overline{F}$	-1	1	a
λ	$\bullet$	$\bullet$	0	0	1

where the convention on indices is the same as in lecture 5, cf the discussion below eq.(5.103), and the R-charges of Q and $\tilde{Q}$ are the same since under charge conjugation (which commutes with supersymmetry) $Q \leftrightarrow \tilde{Q}$. For later convenience, we

have also written down the charges of the gaugino field. The axial current and the R_0 current are anomalous, the anomaly coefficients being

$$\begin{aligned} A_A &= \frac{1}{2}(+1)F + \frac{1}{2}(+1)F = F \\ A_{R_0} &= \frac{1}{2}\left[(a-1)F + (a-1)F\right] + N = N + (a-1)F \end{aligned}$$

These two anomalous symmetries admit an anomaly-free combination (which is obviously an R-symmetry)

$$j^R_\mu = j^{R_0}_\mu + \frac{(1-a)F - N}{F} j^A_\mu \ , \tag{9.30}$$

under which the matter fields have the following charges $R(Q^i_a) = R(\tilde{Q}^b_j) = \frac{F-N}{F}$ (note that the anomaly-free R-charge of matter fields does not depend on a, while the gaugino has always R-charge equal to 1). Therefore, the group of continuos global symmetries at the quantum level is $G_F = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$. Notice that for $F = 0$, namely for pure SYM, there do not exist an axial current and in turn the R-symmetry is anomalous. This difference will play a crucial role later on.

What we are interested in is the gauge coupling running, namely the β function which, to leading order in the gauge coupling, is

$$\beta = \mu \frac{\partial g}{\partial \mu} = -\frac{b_1}{16\pi^2} g^3 + \mathcal{O}(g^5) \ . \tag{9.31}$$

The one-loop coefficient b_1 can be easily computed from the field content of the classical Lagrangian and reads $b_1 = 3N - F$. The question we would like to answer is whether holomorphy can tell us something about higher-loop (and non-perturbative) corrections to the gauge coupling running.

Let us consider pure SYM, first, whose action is

$$\mathcal{L} = \frac{1}{16\pi i} \int d^2\theta \ \tau \, \mathrm{Tr}\, W^\alpha W_\alpha + h.c. \ , \tag{9.32}$$

where τ is the complexified gauge coupling, $\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}$. Notice that τ appears holomorphically in the action above, but the gauge fields are not canonically normalized (to go to a basis where gauge fields are canonically normalized, instead, one should transform $V \to gV$, as we did already when constructing matter-coupled SYM actions).

We want to discuss quantum corrections due to gauge coupling running, using holomorphy. To this purpose, since we want to treat all parameters as complex ones, we can also trade the dynamical generated scale Λ for a complex parameter. For $G = SU(N)$ the one-loop running of the gauge coupling is

$$\frac{1}{g^2(\mu)} = -\frac{3N}{8\pi^2}\log\left(\frac{|\Lambda|}{\mu}\right) \quad , \quad |\Lambda| = \mu_0\, e^{-\frac{8\pi^2}{3Ng^2(\mu_0)}} \ , \tag{9.33}$$

where $3N$ is the one-loop coefficient of pure SYM β-function and $|\Lambda|$ what we previously called Λ. We can then define a holomorphic intrinsic scale Λ as

$$\Lambda = |\Lambda| e^{i\frac{\theta_{\rm YM}}{3N}} = \mu\, e^{\frac{2\pi i\tau}{3N}} \ , \tag{9.34}$$

in terms of which the one-loop complexified gauge coupling reads

$$\tau_{\text{1-loop}} = \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} \ . \tag{9.35}$$

What about higher order corrections? Suppose we integrate down to a scale μ, then

$$W_{\rm eff} = \frac{\tau(\Lambda;\mu)}{16\pi i}\,\mathrm{Tr}\, W^\alpha W_\alpha \ . \tag{9.36}$$

Since physics is periodic under $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi$, the following rescaling

$$\Lambda \to e^{\frac{2\pi i}{3N}}\Lambda \ , \tag{9.37}$$

is a symmetry of the theory, under which $\tau \to \tau + 1$. The most general form for τ transforming this way under (9.37) is

$$\tau(\Lambda;\mu) = \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} + f(\Lambda;\mu) \tag{9.38}$$

with f a holomorphic function of Λ having the following properties:

- f should have a positive Taylor expansion in Λ in such a way that in the limit $\Lambda \to 0$, which is a classical limit, we get back the one-loop result.
- Plugging the transformation (9.37) into the expression (9.35) shows that $\tau_{\text{1-loop}}$ already accounts for the shift of θ-angle by 2π. Hence, the function f should be invariant under (9.37).

These two properties imply that the effective coupling (9.38) should have the following form

$$\tau(\Lambda;\mu) = \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{3Nn} \ . \tag{9.39}$$

Recalling that the instanton action is

$$e^{-\mathcal{S}_{\rm inst}} = \left(\frac{\Lambda}{\mu}\right)^{3N} \quad , \tag{9.40}$$

we conclude that the function f receives only non-perturbative corrections and these corrections come from n-instantons contributions. The upshot is that τ is one-loop exact, in perturbation theory.

The one-loop exactness of the SYM gauge coupling can be equivalently proven as follows. The θ-term is a topological term so it does not get renormalized perturbatively. Therefore the β-function, $\beta = \beta(\tau)$ can only involve $\mathrm{Im}\,\tau$. If β should be a holomorphic function of τ this implies that it can only be a imaginary constant (a holomorphic function $f(z)$, which is independent of $\mathrm{Re}\,z$, is an imaginary constant). Therefore

$$\beta(\tau) \equiv \mu \frac{d}{d\mu}\tau = ia \ , \tag{9.41}$$

which implies

$$\mu \frac{d}{d\mu}\theta_{\rm YM} = 0 \quad , \quad \mu \frac{d}{d\mu} g = -\frac{a}{8\pi} g^3 + 0 \ . \tag{9.42}$$

So we see that, indeed, the gauge coupling does not receive corrections beyond one-loop, in perturbation theory (for the theory at hand $a = 3N/2\pi$).

All what we said above applies identically to SQCD (again, working in the basis where gauge fields are not canonically normalized and the complexified gauge coupling enters holomorphically in the action), the only difference being that the one-loop coefficient of the β-function is now proportional to $3N - F$, with F being the number of flavors.

Remarkably, in some specific cases one can show that also non-perturbative corrections are absent. One such instances is pure SYM, and the argument goes as follows. As already noticed, the R-symmetry of pure SYM is anomalous

$$\partial_\mu j^\mu_R = 0 \ \xrightarrow[\text{corrections}]{\text{quantum}} \ \partial_\mu j^\mu_R = -\frac{2N}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a \ . \tag{9.43}$$

The $U(1)_R$, however, is not fully broken. This can be seen as follows. A R-symmetry transformation with parameter α, under which the gaugino transforms as

$$\lambda \to e^{i\alpha}\lambda \tag{9.44}$$

is equivalent to a shift of the θ-angle

$$\theta_{\rm YM} \to \theta_{\rm YM} - 2N\alpha \ . \tag{9.45}$$

The point is that the transformation $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi k$ where $k \in \mathbb{Z}$, is a symmetry of the theory. So, whenever the $U(1)_R$ parameter α equals $\pi k/N$, the theory is unchanged also at the quantum level. This implies that a discrete subgroup of the original continuos abelian symmetry is preserved,

$$U(1)_R \longrightarrow \mathbb{Z}_{2N} \ . \tag{9.46}$$

Treating the complexified gauge coupling τ as a spurion field, we can define a spurious symmetry given by

$$\lambda \to e^{i\alpha}\lambda \quad , \quad \tau \to \tau + \frac{N\alpha}{\pi} \ . \tag{9.47}$$

This constrains the coefficients a_n in the expansion (9.39). Indeed, under the spurious symmetry the holomorphic scale $\Lambda = \mu\, e^{\frac{2\pi i \tau}{3N}}$ transforms as

$$\Lambda \to e^{i\frac{2N\alpha}{3N}}\Lambda \ . \tag{9.48}$$

Hence we have

$$\tau(\Lambda;\mu) \to \frac{N\alpha}{\pi} + \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{3Nn} e^{2iN\alpha n} \ . \tag{9.49}$$

Since $\forall n \neq 0$ certainly $e^{-2iN\alpha n} \neq 0$, it follows that to match the spurious symmetry we need to have

$$a_n = 0 \quad \forall n > 0 \ . \tag{9.50}$$

Hence in pure SYM also non-perturbative corrections to the gauge coupling are absent! This does not hold in presence of matter, namely for SQCD, since there the R-symmetry is not anomalous and running the above argument one would not get any constraint on the coefficients a_n (the r.h.s. of eq. (9.43) is zero in this case and θ_{YM} would be insensitive to R-symmetry transformations).

If we collect all what we have learned about the SQCD gauge coupling we might have the feeling that something wrong is going on. There are three apparently incompatible results regarding the running of the gauge coupling.

- Due to holomorphy, the supersymmetric gauge coupling runs only at one-loop in perturbation theory, and the full perturbative β-function hence reads

$$\beta = -\frac{g^3}{16\pi^2}(3N - F) \ , \tag{9.51}$$

- There exists a well known result in the literature which claims that the exact, all-loops β-function of SQCD is

$$\beta = -\frac{g^3}{16\pi^2}\frac{\left[3N - \sum_{i=1}^{F}(1-\gamma_i)\right]}{1-\frac{Ng^2}{8\pi^2}} \tag{9.52}$$

  where $\gamma_i = d\log Z_i(\mu)/d\log\mu$ are matter fields anomalous dimensions. This result gets contribution at all loops and is in clear contradiction with the previous result. Eq. (9.52) is sometime called the NSVZ β-function.

- Another piece of knowledge we have about the β-function of SQCD (and, in general, of any gauge theory) is that its one and two-loop coefficients are universal, in the sense that are renormalization scheme independent. This can be easily proven as follows. Changing renormalization scheme amounts to define a new coupling g' which is related to g as

$$g' = g + ag^3 + \mathcal{O}(g^5) \ . \tag{9.53}$$

  Suppose that the β-function for g is

$$\beta_g = b_1 g^3 + b_2 g^5 + \mathcal{O}(g^7) \ . \tag{9.54}$$

  We get for the β-function for g'

$$\beta_{g'} = \beta_g \frac{\partial g'}{\partial g} = \beta_g \left(1 + 3ag^2 + \mathcal{O}(g^4)\right) = b_1 g^3 + (b_2 + 3ab_1)g^5 + \mathcal{O}(g^7) \ . \tag{9.55}$$

  We can invert the relation between g and g' and get

$$g = g' - ag'^3 + \mathcal{O}(g'^5) \ , \tag{9.56}$$

  and finally

$$\beta_{g'} = b_1 g'^3 - 3ab_1 g'^5 + (b_2 + 3ab_1)\, g'^5 + \mathcal{O}(g'^7) = b_1 g'^3 + b_2 g'^5 + \mathcal{O}(g'^7) \ , \tag{9.57}$$

  which shows that the first two coefficients of the β-function are indeed universal. Given the universality of the β-function up to two loops, the discrepancy between the two expressions for the SQCD β-function, (9.51) and (9.52), cannot just be a matter of renormalization scheme.

How can we reconcile this apparent contradiction? The answer turns out to be surprisingly simple. Let us first consider pure SYM whose action is

$$\mathcal{L} = \frac{1}{16\pi i}\int d^2\theta\, \tau\, \mathrm{Tr}\, W^\alpha W_\alpha + h.c.\ . \tag{9.58}$$

As we already noticed, if one integrates in superspace one gets a space-time action where gauge fields are not canonically normalized

$$\mathcal{L} = -\frac{1}{4g^2}\ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \dots\ . \tag{9.59}$$

Let us call the gauge coupling defined in this frame *holomorphic* gauge coupling g_h, defined via the complexified gauge coupling as $\tau = 4\pi i/g_h^2$. In order to get a Lagrangian in terms of canonically normalized fields one should rescale $V \to gV$. In other words, we should perform the change of variables $V_h = g_p V_p$. In terms of this physical gauge coupling g_p the Lagrangian reads

$$\mathcal{L} = \frac{1}{4}\int d^2\theta \left(\frac{1}{g_p^2} - i\frac{\theta_{\rm YM}}{8\pi^2}\right)\ \mathrm{Tr}\, W^\alpha(g_pV_p)W_\alpha(g_pV_p) + h.c.\ , \tag{9.60}$$

Notice that the Lagrangian above is not holomorphic in the physical coupling since g_p is real as g_pV_p should also be real. The crucial point now is that the two Lagrangians (9.58) and (9.60) are *not* equivalent under the change of variables $V_h = g_pV_p$ in the path integral, since there is a rescaling anomaly (there is an anomalous Jacobian in passing from V_h to V_p), that is $D(g_pV_p) \neq DV_p$. In particular, one can show that

$$D(g_pV_p) = DV_p\ \exp\left[-\frac{i}{4}\int d^2\theta \left(\frac{2T(Adj)}{8\pi^2}\log g_p\right)\ \mathrm{Tr}\, W^\alpha(g_pV_p)W_\alpha(g_pV_p) + h.c.\right]\ . \tag{9.61}$$

Hence we get for the partition function

$$\begin{aligned} Z &= \int DV_h\ \exp\left[\frac{i}{4}\int d^2\theta \frac{1}{g_h^2}\ \mathrm{Tr}\, W^\alpha(V_h)W_\alpha(V_h) + h.c.\right] = [V_h = g_pV_p] \\ &= \int DV_p\ \exp\left[\frac{i}{4}\int d^2\theta \left(\frac{1}{g_h^2} - \frac{2T(Adj)}{8\pi^2}\log g_p\right)\ \mathrm{Tr}\, W^\alpha(g_pV_p)W_\alpha(g_pV_p) + h.c.\right]\ , \end{aligned} \tag{9.62}$$

which implies

$$\frac{1}{g_p^2} = \ \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Adj)}{8\pi^2}\log g_p = \ \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2N}{8\pi^2}\log g_p\ . \tag{9.63}$$

where in the last equality we used the fact that for $SU(N)$ the Dynkin index for the adjoint $T(Adj) = N$. Differentiating with respect to $\log \mu$, and using the expression (9.51) for the holomorphic gauge coupling β-function (setting $F = 0$), one gets for the physical gauge coupling g_p precisely the (pure SYM) NSVZ β-function (9.52).

One can repeat an identical reasoning for SQCD where the relation between the physical and the holomorphic gauge couplings reads

$$\frac{1}{g_p^2} = \text{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Adj)}{8\pi^2}\log g_p - \sum_i \frac{T(r_i)}{8\pi^2}\log Z_i \ . \tag{9.64}$$

Differentiating with respect to $\log \mu$ (using again $T(Adj) = N$ and taking matter to be in the fundamental, for which $T(r) = 1/2$), one gets for the physical gauge coupling exactly the expression (9.52).

We now see why there is no contradiction with two-loops universality of the β-function. The point is simply that the relation between the holomorphic and the physical gauge coupling is not analytic. In other words, one cannot be Taylor-expanded in the other, because of the log-term (it is a *singular* change of renormalization scheme: the so-called holomorphic scheme is not related continuously to any other physical renormalization scheme). Furthermore, we also now understand why the physical β-function gets contribution at all loops. This is just because of wave function renormalization (both of the vector superfield as well as of matter superfields): the physical gauge coupling differs from the holomorphic gauge coupling by effects coming from wave-function renormalization, which get contribution at all loops. Consistently, the physical β-function can be expressed exactly in terms of anomalous dimension of fields, once the one-loop coefficient (which agrees with that of the holomorphic β-function) has been calculated.

One can repeat the same kind of reasoning for gauge theories with extended supersymmetry which, after all, are just (very) special cases of $\mathcal{N} = 1$ theories. In doing so one immediately gets the result we anticipated when discussing non-renormalization theorems. Let us start from $\mathcal{N} = 2$ pure SYM. Using $\mathcal{N} = 1$ language we have a vector and a chiral multiplet, the latter transforming in the adjoint of the gauge group. As we have already seen, due to $\mathcal{N} = 2$ supersymmetry, the kinetic terms of V and Φ are both changed according to the holomorphic gauge coupling. Hence, going to canonical normalization for all fields we must rescale them the same way, $V_h = g_p V_p$ and $\Phi_h = g_p \Phi_p$. The crucial point is that the Jacobian for V cancels exactly that from Φ! In other words, $D(g_p V_p) D(g_p \Phi_p) = DV_p \, D\Phi_p$, implying that the holomorphic and physical gauge couplings coincide. Adding matter, nothing

changes since, as we have already noticed, kinetic terms for hypermultiplets do not renormalize. Hence, $\mathcal{N}=2$ is (perturbatively) one-loop finite. Applying this result to $\mathcal{N}=4$ we conclude that the latter is tree-level exact, in fact, since the $\mathcal{N}=4$ one-loop β-function vanishes.

9.5 Holomorphic decoupling

Holomorphy helps also in getting effective superpotentials when one has to integrate out some massive modes and study the theory at scales lower than the corresponding mass scale.

Let us consider a model of two chiral superfields, L and Φ, interacting via the following superpotential

$$W = \frac{1}{2}M\Phi^2 + \frac{\lambda}{2}L^2\Phi \ , \tag{9.65}$$

The above superpotential does not suffer from quantum corrections, because of holomorphy (neither perturbatively nor non-perturbatively). The spectrum is that of a massless chiral superfield and a massive one. If we want to study the system at energies $\mu < M$, we have to integrate Φ out. In order to do so we can use holomorphicity arguments, and proceed as we did when proving the exactness of the WZ superpotential. Let us first promote the couplings to spurion fields and, consequently, enlarge the global symmetries as follows

$$\begin{array}{cccc} & U(1)_a & U(1)_b & U(1)_R \\ L & 0 & 1 & 1 \\ \Phi & 1 & 0 & 0 \\ M & -2 & 0 & 2 \\ \lambda & -1 & -2 & 0 \end{array} \tag{9.66}$$

The low energy effective superpotential should be a dimension-three function of λ, M and L respecting the above symmetries. This implies that

$$W_{\text{eff}} = a\frac{\lambda^2 L^4}{M} \ , \tag{9.67}$$

where a is an undetermined constant of order one.

The same result can be obtained by using the ordinary integrating out procedure. At scales well below M, the chiral superfield Φ is frozen at its VEV (we do not have enough energy to make it fluctuate). Therefore, we can integrate the field out by

solving its equation of motion, which is just an algebraic one, involving only the F-term, since the kinetic term (the D-term) is trivially zero. The equation of motion is

$$\frac{\partial W}{\partial \Phi} = M\Phi + \frac{\lambda}{2}L^2 = 0 \quad \rightarrow \quad \Phi = -\frac{\lambda}{2M}L^2 \ . \tag{9.68}$$

Substituting back into the superpotential we get

$$W_{\text{eff}} = -\frac{1}{8}\frac{\lambda^2 L^4}{M} \ , \tag{9.69}$$

which is the same as (9.67) (with the undetermined coefficient being fixed). Notice that the superpotential we have just obtained is the effective superpotential one generates in perturbation theory, in the limit of small λ, see Figure 9.3.

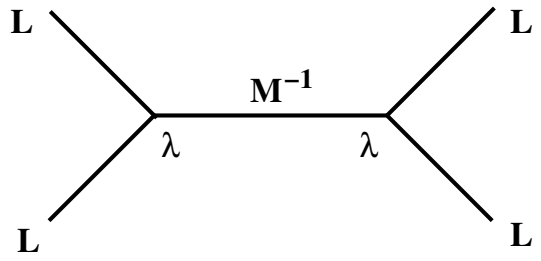

Figure 9.3: The tree level (super)graph which produces the effective superpotential (9.67) in the weak coupling limit.

Let us emphasize that in this case, differently from the case discussed in the previous section, we have allowed for negative powers of M. In other words, we have not required any smoothness in the $M \to 0$ limit. The reason is that the effective theory we are considering is valid only at energies lower than M, which is indeed a UV cut-off for the theory. Hence, we can accept (and actually do expect) singularities as we send M to zero: new massless degrees of freedom are expected to arise; those associated to Φ, the superfield we have integrated out.

As a final instructive example, let us consider a perturbation of the previous model. The superpotential we would like to analyze is

$$W = \frac{1}{2}M\Phi^2 + \frac{\lambda}{2}L^2\Phi + \frac{\epsilon}{6}\Phi^3 \ . \tag{9.70}$$

Again, if we want to study the system at energies $\mu < M$, we have to integrate the massive field out. The equation of motion for Φ gives

$$\Phi = -\frac{M}{\epsilon}\left(1 \mp \sqrt{1 - \frac{\epsilon\lambda L^2}{M^2}}\right) \ . \tag{9.71}$$

We have now two possible solutions, hence two different vacua. Consistently, as we send ϵ to zero one of the two vacua approaches the one of the unperturbed model while the second is pushed all the way to infinity.

The effective superpotential now reads

$$W_{\text{eff}} = \frac{M^3}{3\epsilon^2}\left[1 - \frac{3}{2}\frac{\epsilon\lambda L^2}{M^2} \mp \left(1 - \frac{\epsilon\lambda L^2}{M^2}\right)\sqrt{1 - \frac{\epsilon\lambda L^2}{M^2}}\right] . \tag{9.72}$$

There are again singularities in the effective superpotential, both in parameter space as well as in field L space, now. Comparing to the unperturbed case one can suspect that at these points extra massless degrees of freedom show up. Indeed, computing the (effective) mass for the field we have integrated out, we get

$$\frac{\partial^2 W}{\partial \Phi^2} = M + \epsilon\Phi = \text{ (on the solution) } = \pm M\sqrt{1 - \frac{\epsilon\lambda L^2}{M^2}} . \tag{9.73}$$

The field Φ becomes massless at $\langle L\rangle = \pm M/\sqrt{\epsilon\lambda}$, precisely the two singularities of the effective superpotential (9.72). In the limit $\epsilon \to 0$, keeping M fixed, one recovers, again, the result of the unperturbed theory.

Again, the same result could have been obtained just using holomorphicity arguments. Promoting also the coupling ϵ to a spurion field with charges

$$\begin{array}{cccc} & U(1)_a & U(1)_b & U(1)_R \\ \epsilon & -3 & 0 & 2 \end{array} \tag{9.74}$$

and repeating the same argument as in the previous example, one could conclude that the effective superpotential should have the following structure

$$W_{\text{eff}} = \frac{M^3}{\epsilon^2} f\left(\frac{\epsilon\lambda L^2}{M^2}\right) , \tag{9.75}$$

which has precisely the structure of the exact expression (9.72). Taking various limits one can actually fix also the form of the function f (modulo an overall numerical coefficient, as before).

This way of integrating out in supersymmetric theories, which preserves holomorphy, is called *holomorphic decoupling*. We will heavily use holomorphic decoupling when studying the quantum properties of SQCD in our next lecture. For instance, using this technique it is possible to get the effective superpotential for SQCD with an arbitrary number of flavors once the exact expression (including numbers like a in eq. (9.67)) for a given number of flavors is known. Everything amounts to integrate flavors in and out (more later).

9.6 Exercises

1. Using holomorphy and (spurious) symmetries, show that the superpotential

$$W = \mu_1 \Phi + \mu_2 \Phi^2 + \cdots + \mu_n \Phi^n \ , \tag{9.76}$$

is not renormalized at any order in perturbation theory and non-perturbatively.

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10 Supersymmetric gauge dynamics: $\mathcal{N} = 1$

The very basic questions one should ask about a quantum field theory regard the way its symmetries are realized in its vacua, and what the dynamics around such vacua is.

- Given a QFT with gauge group G and global symmetry group $G_{\rm F}$, how are these realized in the vacuum?
- Which phases may enjoy such a theory?
- Are there tools to give not only qualitative but also quantitative answers to these questions?

It is very difficult to fully or even partially answer these questions, in general. However, as we will discuss in this lecture, for supersymmetric theories this *is* possible, sometime.

10.1 Confinement and mass gap in QCD, YM and SYM

Asymptotically free gauge theories are expected to enjoy many interesting and fascinating phenomena at low energy, like confinement, the generation of mass gap, chiral symmetry breaking etc... However, such phenomena may be realized very differently, for different theories. Below we want to consider three specific theories, namely QCD, YM and SYM, which are all UV-free and all said to be confining, and show how different the IR dynamics of these theories actually is.

• QCD, the theory of strong interactions.

At high energy QCD is a weakly coupled theory, a $SU(3)$ gauge theory of weakly interacting quarks and gluons. It grows, through renormalization effects, to become strong in low energy processes. So strong so to bind quarks into nucleons. The strong coupling scale of QCD is

$$\Lambda_{\rm QCD} \sim 300 \text{ MeV} . \tag{10.1}$$

Note that as compared to protons and neutrons, constituent quarks are relatively light (the u and d quarks are order of a few MeV; the s quark is order 100 MeV). Most of the mass of nucleons comes from quark kinetic energy and the interactions binding quarks together.

The reason why we cannot see free quarks, we usually say, is confinement: quarks are bound into nucleons and cannot escape. In fact, this statement is not completely correct: if we send an electron deep into a proton, we can make the quark escape! The processes is summarized in Figure 10.1.

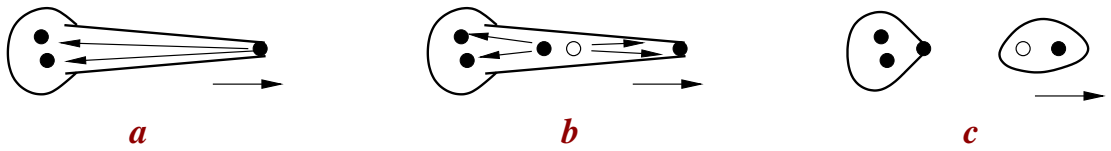


Figure 10.1: Charge screening: confinement in QCD.

If the electron is energetic enough, a large amount of energy, in the form of chromoelectric field, appears in the region between the escaping quark and the rest of the proton. When the field becomes strong enough, of order $\Lambda^4_{\rm QCD} \sim (300\ {\rm MeV})^4$, flux lines can break and produce $q - \bar{q}$ pairs (this is a familiar phenomenon also in electromagnetism: electric fields beyond a certain magnitude cannot survive; strong fields with energy density bigger that $m_e^4 \sim (1\ {\rm Mev})^4$ decay by producing $e^+ - e^-$ pairs). The $\bar{q}$ quark binds to the escaping quark while the q quark binds to the other two quarks in the proton. Therefore, the original quark does escape, the force between it and the remaining proton constituent drops to zero. Just, the escaping quark is not alone, it is bound into a meson. This phenomenon should be better called charge screening, rather than confinement.

Can we have confinement in a more strict sense? Suppose that quarks were much more massive, say $m_q \sim 1$ TeV. Now proton mass would be order the TeV. The dynamics drastically changes, now. Repeating the previous experiment, when the chromoelectric field becomes order $\Lambda^4_{\rm QCD}$, there is not enough energy now to produce $q-\bar{q}$ pairs. The force between the escaping quark and the proton goes to a constant: a string, or a tube of chromoelectric flux of thickness $\sim \Lambda^{-1}_{\rm QCD}$ and tension (energy per unit length) of order $\Lambda^2_{\rm QCD}$ connects the two, see Figure 10.2. Not only the quark is confined, it is the flux itself which is confined. This is certainly a more precise definition of confinement: it holds regardless of quarks, in the sense that it holds also in the limit $m_q \to \infty$, namely when the quarks disappear from the spectrum. It is a property of the pure glue. In summary, strict confinement would be a property of QCD only if the quarks were very massive, more precisely in the limit $F/N << 1$, where F is the number of light quarks and N the number of colors. Real-life quarks are light enough to let the chromoelectric flux tube break. Hence, actual QCD does

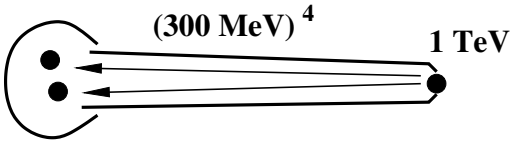


Figure 10.2: Strict confinement in QCD, in the limit of infinite (very large) quark masses.

not confine in the strict sense.

Let us discuss the structure of QCD vacua in more detail, looking at how the global symmetry group of QCD is realized in the vacuum. In what follows, we consider only the three light quarks, u, d and s and forget the other ones (which are dynamically less important). So we have $F = 3$ flavors. Moreover, we will first put ourselves in the limit where the light quarks are massless, which is approximately true for u, d and s quarks constituting ordinary matter (protons and neutrons). Only later we will consider the effect of the small quark masses. In this massless limit the QCD Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \sum_i \bar{q}_L^i\, i\not{D} q_L^i + \sum_i \bar{q}_R^i\, i\not{D} q_R^i \quad , \quad i = 1,2,3\ . \tag{10.2}$$

Quark quantum numbers under the global symmetry group are

$$\begin{array}{cccccc} & SU(3)_L & SU(3)_R & U(1)_A & U(1)_B \\ q_L & 3 & 1 & 1 & 1 \\ q_R & 1 & \bar{3} & 1 & -1 \end{array} \tag{10.3}$$

As well known there is an axial anomaly, in the sense that the $U(1)_A$ symmetry is broken to $\mathbb{Z}_{2F}$ at the quantum level. Therefore, the continuos global symmetries at the quantum level are just

$$G_{\rm F} = SU(3)_L \times SU(3)_R \times U(1)_B\ . \tag{10.4}$$

Experimental and theoretical considerations plus several numerical simulations on the lattice lead to a definite picture of the realization of the global symmetry, as well as the $SU(3)$ gauge symmetry, at low energy. As we have already discussed, the theory undergoes confinement (or better charge screening) and quarks and gluons are bounded into color singlet states. What about the global symmetry group? It is believed that at low energy only a subgroup survives

$$SU(3)_D \times U(1)_B\ , \tag{10.5}$$

under which hadrons are classified: $SU(3)_D$ gives the flavor quantum numbers and $U(1)_B$ is the baryon charge. The remaining generators must be broken, somehow. The intuitive picture is as follows. Due to confinement, at strong coupling quarks and anti-quarks are bound into pairs, and the vacuum is filled by a condensate of these color singlet quark bilinears

$$\langle q_L^i q_R^j \rangle = \Delta\, \delta^{ij} \ , \tag{10.6}$$

where $\Delta \sim \Lambda_{\rm QCD}^3$. This condensate is invariant under a diagonal $SU(3)$ subgroup of the original $SU(3)_R \times SU(3)_L$ group and is then responsible for the spontaneous breaking of the chiral symmetry of the original symmetry group $G_{\rm F}$

$$SU(3)_L \times SU(3)_R \times U(1)_B \to SU(3)_D \times U(1)_B \ . \tag{10.7}$$

Eight global symmetries are broken by the quark condensates and hence we would expect eight Goldstone bosons. The latter are indeed observed experimentally, and correspond to the eight pseudoscalar mesons, the pions, $\pi^{0,\pm}, K^{0,1}, \bar{K}^{0,+}, \eta$

$$\begin{array}{llllll} \pi^+ = u\bar{d} & , & \pi^- = d\bar{u} & , & \pi^0 = d\bar{d} - u\bar{u} & , \quad \eta = u\bar{u} + d\bar{d} - 2s\bar{s} \\ K^0 = s\bar{d} & , & K^- = \bar{u}s & , & \bar{K}^0 = \bar{s}d & , \quad \bar{K}^+ = \bar{s}u \end{array} \tag{10.8}$$

Let us first notice that if $U(1)_A$ were not anomalous, we would have had a ninth meson, the so-called η' meson, which would have corresponded to a shift in the phase of the condensate (10.6). The condensate breaks spontaneously the $\mathbb{Z}_{2F}$ symmetry down to $\mathbb{Z}_2$, but this does not give rise to any massless particle. The $\eta\,'$ has a periodic potential with F minima, each of them being $\mathbb{Z}_2$ invariant, and related one another by $\mathbb{Z}_F$ rotations. These minima are not isolated, though, since they are connected via $SU(F)_L \times SU(F)_R$ rotations. This means that there is a moduli space of vacua. Via a $SU(3)$ rotation acting separately on q_L and q_R, the condensate (10.6) can be put in the form

$$\langle q_L^i q_R^j \rangle = \Delta\, U^{ij} \tag{10.9}$$

where U^{ij} is a $SU(3)$ matrix on which a $SU(3)_L \times SU(3)_R$ rotation acts as

$$U \to A_L^\dagger U A_R \ , \tag{10.10}$$

which shows there exists a $SU(3)_D$ rotation ($A_L = A_R$) under which the matrix U is invariant. So the moduli space of vacua is a $SU(3)$ manifold.

The quantum fluctuations of the entries of this matrix hence represent the massless excitations around the vacua of massless QCD. An effective Lagrangian for such

excitations (the pions) can be written in terms of $U(x)$ and its derivatives. This Lagrangian should be invariant under the full global symmetry group $G_{\rm F}$, hence non-derivative terms are not allowed (the only $G_{\rm F}$-invariant function would be $U^\dagger U = 1$), which is simply saying that the pions are massless in the massless QCD limit we are considering. The structure of the effective Lagrangian hence reads

$$\mathcal{L}_{\rm eff} = f_\pi^2 \left(\partial_\mu U^\dagger \partial^\mu U\right) + \kappa\, \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U + \dots \ , \tag{10.11}$$

where $\kappa \sim 1/M^2$, with M being some intrinsic mass scale of the theory, and traces on flavor indices are understood. At low momenta only the first term contributes and we then get a definite prediction for mesons scattering amplitudes, in terms of a single parameter f_π.

Now, quarks are not exactly massless and therefore the above picture is only approximate. In reality, the $SU(3)_L \times SU(3)_R$ symmetry is only approximate since quark masses correspond to (weak) $G_{\rm F}$-breaking terms. This has the effect to make the pions be only pseudo-Goldstone bosons. Hence, one would expect them to be massive, though pretty light, and this is indeed what we observe in Nature.

In principle, one should have gotten the chiral Lagrangian (10.11) from the UV Lagrangian (10.2) by integrating out high momentum modes. This is difficult (next-to impossible, in fact). However, we know in advance the expression (10.11) to be right, since that is the most general Lagrangian one can write describing pion dynamics and respecting the original symmetries of the problem. Combining the expression (10.11) with weak $G_{\rm F}$-breaking terms induced by actual quark masses, one gets a Lagrangian which, experimentally, does a good job.

Summarizing, we see that combining symmetry arguments, lattice simulations, experimental observations and some physical reasoning, we can reach a rather reasonable understanding of the low energy dynamics of QCD. This is all very nice but one would like to gain, possibly, a theoretical (i.e. more microscopic) understanding of QCD phenomena. As of today, this is still an open question for QCD. And, more generally, it is so for any generic gauge theory. As we will later see, supersymmetry lets one have more analytical tools to answer this kind of questions, having sometime the possibility to derive strong coupling phenomena like confinement, chiral symmetry breaking and the generation of a mass gap, from first principles.

• Yang-Mills (YM) theory, gauge interactions without matter fields.

Let us focus, for definiteness, on YM theory with gauge group $SU(N)$. This is

again a UV-free theory, the one-loop β-function being

$$\beta_g = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}N\right) \ . \tag{10.12}$$

There are two claims about this theory, coming from a combination of experimental and theoretical reasoning, analytic and lattice calculations.

1. The theory has a *mass gap*, i.e. there are no massless fields in the spectrum. Rather, there is a discrete set of states with masses of order Λ, the scale where the one-loop gauge coupling diverges (higher loop and non-perturbative effects do not change the actual value of Λ in any sensible way)

$$\Lambda = \mu\, e^{-\frac{8\pi^2}{g^2(\mu)b_1}} \quad \text{where} \quad b_1 = \frac{11}{3}N \ . \tag{10.13}$$

   The low energy spectrum consists of glueballs. These are sort of gluons bound states which however do not consist of a fixed number of gluons (gluon number is not a conserved quantum number in strong interactions), but rather of a shifting mass of chromoelectric flux lines. Unlike gluons, for which a mass term is forbidden (because they have only two polarizations), glueballs include scalars and vectors with three polarizations (as well as higher spin particle states), for which a mass term is allowed. Such mass should clearly be of order of the dynamical scale, $m \sim \Lambda$, so not to contradict perturbation theory.

   The low energy spectrum is very different from QCD. In QCD there is a mass gap just because quarks are massive. If u, d and s quarks were massless, we would not have had a mass gap in QCD since pions would have been exact Goldstone bosons and hence massless. Here instead there is a genuine mass gap, see Figure 10.3.

2. The theory undergoes *confinement* (now in the strict sense). The chromoelectric flux is confined, it cannot spread out in space over regions larger than about Λ^{-1} in radius. How can we see confinement, namely the presence of strings which contain the chromoelectric flux? Let us add some heavy quarks to the theory and let us see whether these quarks are confined, as it was the case for very massive QCD. The Lagrangian would read

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi - M\bar{\psi}\psi \quad , \quad M >> \Lambda \ . \tag{10.14}$$

   In the limit $M \to \infty$ the test particles become chromoelectrostatic sources, and play no role in the dynamics.

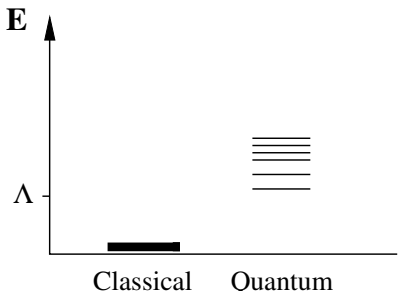


Figure 10.3: The spectrum of pure YM theory. Classically the theory is a theory of weakly coupled massless gluons. At the quantum level, a mass gap is generated, and the physical spectrum consists of a discrete set of glueball states, whose masses are order Λ.

If confinement occurs, we would expect a linear potential between the two quarks. Indeed, in an unconfined theory, the electric flux is uniformly distributed over a sphere surrounding a charge, and falls-off as $1/r^2$. In a confining theory with flux tubes, the flux tube has a fixed cross-sectional area $\sim \Lambda^{-2}$. Thus, for any sphere of radius $r >> R \equiv \Lambda^{-1}$, the flux is zero except in a region of area Λ^{-2}, see Figure 10.4.

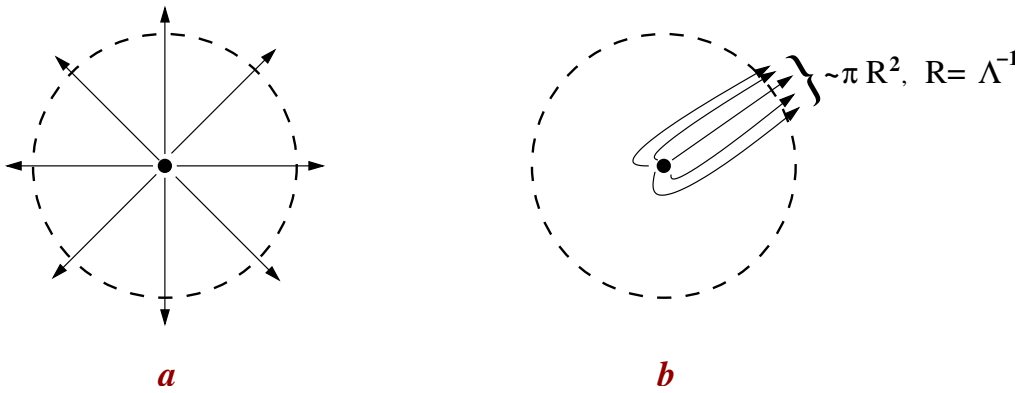


Figure 10.4: Gauss law for unconfined and confined flux.

Hence, the electric field in that region has a magnitude which is r-independent, which implies that the force it generates on a test charge is also r-independent, and so the potential V between charges would grow linearly in r. The force goes to a constant, it never drops to zero, see Figure 10.5 (that this is the behavior of YM theory it is something which has not been proven analytically, but via numerical lattice simulations).

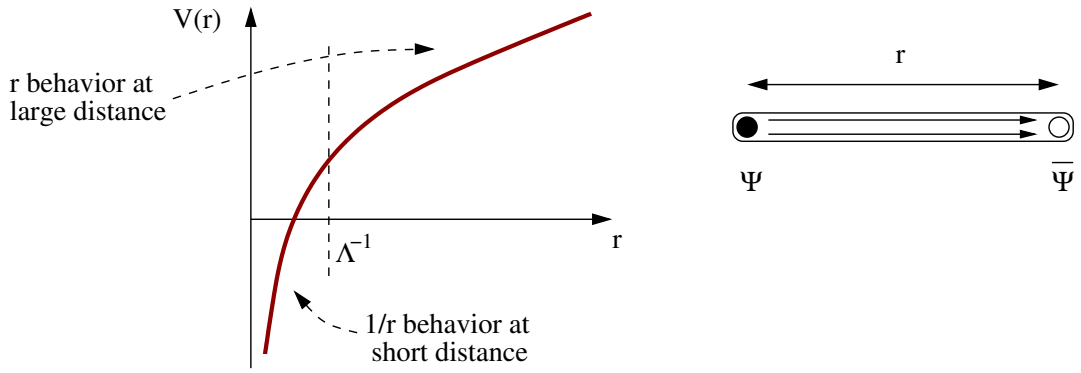


Figure 10.5: The potential between two test charges in YM theory.

Let us take a closer look to the potential, which generically reads

$$V(r) = T_{\rm R}\, r \ . \tag{10.15}$$

The proportionality coefficient has dimension of an energy per unit length, and it is the so-called string tension. On general ground, one would expect the string tension to depend in some way on the gauge group representation the test charges transform. This is pretty obvious since, e.g., for the singlet representation $T_{\rm R}$ is clearly zero, while for actual quarks, which transform in the fundamental representation, it is not. In fact, as we are going to show below, the string tension does not depend on the representation itself, but actually on what is called the *N-ality* of the representation.

Let us consider a (gauge) group G. Its center, $C_{\rm G}$ is defined as the part of G which commutes with all generators. For $G = SU(N)$, we have that

$$C_{\rm G} = \left\{ U^{\alpha}_{\bar\beta} = e^{2\pi i k/N} \delta^{\alpha}_{\bar\beta} \ \ ; \ \ k = 0, 1, \ldots, N-1 \ \mathrm{mod}\, N \right\} \ , \tag{10.16}$$

where α an index in the fundamental and $\bar\beta$ in the anti-fundamental of the gauge group. Hence, in this case, $C_{\rm G} = \mathbb{Z}_N$. Let us now consider some representation R. An element ρ of this representation is labeled by n upper indices α_i and $\bar n$ lower indices $\bar\beta_i$, each upper index transforming in the fundamental and each lower index transforming in the anti-fundamental representation. If one acts with the center of the group on ρ one gets

$$C_{\rm G} \ : \ \rho \to e^{2\pi i k (n-\bar n)/N} \rho \ . \tag{10.17}$$

The coefficient $n - \bar n$ is called the N-ality of the representation ρ. If ρ has N-ality p, then the complex conjugate representation ρ^C has N-ality $N - p$ (from eq. (10.17) it follows that the N-ality is defined modulo N). For instance, while the

adjoint representation and the trivial representation have $p = 0$, the fundamental representation has $p = 1$ and the anti-fundamental has $p = N - 1$.

Clearly, representations break into equivalence classes under the center of the gauge group. It turns out that the string tension $T_{\rm R}$ is not a function of the representation but actually of the N-ality (the basic reason for this is that gluon number is not a conserved quantity in YM theories, while N-ality is). Let us consider heavy test particles transforming either in the anti-symmetric representation or in the symmetric representation of the gauge group, $\psi_{\rm S}$ and $\psi_{\rm A}$, respectively. Each of them will have its own string tension, $T_{\rm S}$ and $T_{\rm A}$ (but same N-ality, $p = 2$).

Suppose that $T_{\rm S} > T_{\rm A}$. Since gluon number is not a conserved quantity, we can add a gluon A_μ coming from the chromoelectric flux tube next to $\psi_{\rm S}$. The charge of the bound state $\psi_{\rm S} A_\mu$ is Symmetric $\otimes$ Adj $=\oplus$ Representations, where all representations entering the sum have the same N-ality (the same as the symmetric representation, in fact, since the N-ality is an additive quantity, and that of the adjoint representation is zero). For $G = SU(3)$ we have

$$6 \otimes 8 = \overline{3} + 6 + \overline{15} + 24 \ , \tag{10.18}$$

where the first representation on the r.h.s. is the anti-symmetric representation. Since we have assumed that $T_{\rm S} > T_{\rm A}$ it is energetically favored to pop a gluon out of the vacuum and put it near to $\psi_{\rm S}$ (and another one near to $\bar{\psi}_{\rm S}$) since this has an energy cost (of order Λ) which is lower than the energy gain, proportional to $(T_{\rm S} - T_{\rm A})r$, which for sufficiently large r always wins. In other words, in YM theory the representation of a chromoelectric source is not a conserved quantum number; only the N-ality is. Therefore, for all representations with the same N-ality, there is only one stable configuration of strings, the one with lowest tension, as shown in Figure 10.6. In summary, the tension of *stable* flux tubes are labeled by p, the N-ality, not by R, the representation, as anticipated.

Notice that charge conjugation symmetry ensures that $T_p = T_{N-p}$. Therefore, there are order $N/2$ stable flux tube configurations for $SU(N)$. For $SU(3)$ there is only one single confining string, that with N-ality $p = 1$, since $T_0 = 0$ and $T_2 = T_1$. Multiple flux tubes only arise for larger gauge groups (for instance, for $G = SU(4)$ we have two string tensions, with N-ality $p = 1$ and $p = 2$, respectively).

All what we said let us also understand how to classify gauge singlets bound states. While gluons are not confined by flux tubes, since $T_{\rm Adj} = T_0 = 0$, any heavy quark ψ with N-ality $\neq 0$ will experience a linear potential and a constant force

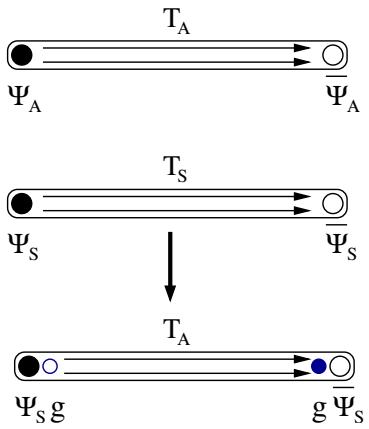


Figure 10.6: The string tensions corresponding to the antisymmetric and symmetric representations. All strings have the same N-ality, $p = 2$. The flux tube in the symmetric representation decays into that of the anti-symmetric one (which has lower energy) by popping-up a gluon out of the vacuum.

which will confine it to an antiquark $\bar{\psi}$ (these are the *mesons*) or, more generally, to some combination of quarks and anti-quarks with opposite N-ality. For instance, such combination can be made of $N-1$ quarks and the bound state is called a *baryon*.

In passing, let us finally notice that fully analogous statements can be made without introducing heavy quarks, but rather computing Wilson loops (in different representations). In a confining theory, the Wilson loop should follow the area law

$$W_C = Tr_R\, e^{-A_C} \ , \tag{10.19}$$

while for unconfined theories (including those enjoying charge screening!) it follows the perimeter law.

• *SYM*, supersymmetric gauge interactions without matter fields.

We will study this theory in detail later. Here, we would just like to emphasize similarities and differences with respect to pure YM theories and QCD.

Similarly to YM, SYM enjoys strict confinement, a mass gap and no pions. Similarly to QCD, it has a sort of chiral symmetry breaking and an anomaly, which makes the corresponding η'-like particle being massive. Finally, it differs from both since it has multiple *isolated* vacua.

Let us recall the structure of the (on-shell) SYM Lagrangian

$$\mathcal{L}_{\rm SYM} = -\,\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\not{D}\lambda\right] \ . \tag{10.20}$$

First notice that gauginos do not break flux tubes since they transform in the Adjoint representation, which is in the same N-ality class of the singlet representation. So gauginos behave very differently from QCD quarks, in this respect. Basically, the presence of these fermion fields does not change the confining behavior of pure YM glue, since gauginos cannot break flux tubes. That is why SYM enjoys strict confinement, differently from QCD. On the other hand, the $U(1)_R$ symmetry resembles the axial symmetry of QCD, since it is anomalous and it is broken to $\mathbb{Z}_{2N}$ at the quantum level (recall gauginos have R-charge equal to one). Finally, also SYM enjoys chiral symmetry breaking, in the sense that gaugino bilinears acquire a non-vanishing VEV in the vacuum. More precisely, we have

$$\langle\lambda\lambda\rangle \sim \Lambda^3 e^{2\pi i k/N} \quad , \quad k = 0, 1, \ldots, N-1 \ , \tag{10.21}$$

which breaks $\mathbb{Z}_{2N} \to \mathbb{Z}_2$. Hence there are N isolated vacua, each of them $\mathbb{Z}_2$ symmetric, related by $\mathbb{Z}_N$ rotations, as shown in Figure 10.7.

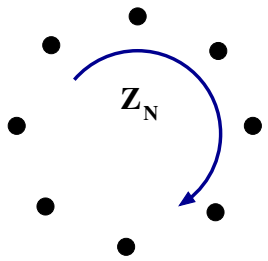


Figure 10.7: The vacuum structure of pure $\mathcal{N} = 1$ SYM. The N vacua are isolated, and related by $\mathbb{Z}_N$ rotations.

The η' is the phase of the condensate (10.21) (similarly to QCD), but the vacua are isolated, so there is a dynamical mass gap (unlike QCD and like YM).

10.2 Phases of gauge theories: examples

We would like now to consider, in more general terms, which kind of phases a generic gauge theory can enjoy. Roughly, there are basically three different such phases

- *Higgs* phase: the gauge group G is spontaneously broken, all vector bosons obtain a mass.

- *Coulomb* phase: vector bosons remain massless and mediate $1/r$ long range interactions (here we are not making any distinction between proper Coulomb phase and free phase, see below).

- *Wilson* or confining phase: color sources, like quarks, gluons, etc..., are bound into color singlets (here we are using the word confinement in its weakest sense, i.e., we are not distinguishing between charge screening and strict confinement).

Notice that the Coulomb phase is not specific to abelian gauge theories, as QED. For example, a non-abelian gauge theory with enough matter fermion content may become IR-free, giving a long-range potential between color charges $V(r) \sim a(r) \times 1/r$, with $a(r)$ a coefficient decreasing logarithmically with r.

There can of course be intermediate situations, where for instance the original gauge group is Higgsed down to a subgroup H, which then confines, or is in a Coulomb phase (this is what happens in the SM of electroweak interactions), etc... In these cases the phase of the gauge theory is defined by what happens to H in the vacua, regardless of the fate of the original gauge group G.

Below we consider two examples which will clarify the meaning of some of above statements, but also point out some subtleties one could encounter when dealing with concrete models.

10.2.1 Coulomb phase and free phase

Let us consider SQED. The scalar potential reads

$$V = m^2|\phi_-|^2 + m^2|\phi_+|^2 + \frac{1}{2}e^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2 \ , \tag{10.22}$$

where ϕ_- and ϕ_+ are the scalar fields belonging to the two chiral superfields Φ_- and Φ_+ with electric charge ± 1, and a superpotential mass term $W = m\Phi_-\Phi_+$ has been allowed. Let us consider massive and massless cases separately.

- $m \neq 0$. In this case the vacuum is at $\langle\phi_-\rangle = \langle\phi_+\rangle = 0$. Heavy static probe charges would experience a potential

$$V \sim \frac{\alpha(r)}{r} \quad , \quad \alpha(r) \sim \frac{1}{\log r} \ . \tag{10.23}$$

However, the logarithmic fall-off is frozen at distance $r = m^{-1}$: for larger distances α stops running. Hence, the asymptotic potential reads

$$V(r) \sim \frac{\alpha_*}{r} \quad , \quad \alpha_* = \alpha(r = m^{-1}) \ , \tag{10.24}$$

which simply says that massive SQED is in a Coulomb phase.

- $m = 0$. In this case the potential gets contributions from D-terms only. Now there are more vacua, actually a moduli space of vacua. Besides the origin of field space, also any $\langle \phi_- \rangle = \langle \phi_+ \rangle \neq 0$ satisfies the D-equations. One can parameterize the supersymmetric vacua in terms of the gauge invariant combination $u = \langle \phi_- \phi_+ \rangle$. We have then two options. When $u \neq 0$ we are in a Higgs phase, the gauge group $U(1)$ is broken and the photon becomes massive (the theory is described by a massive vector multiplet and a massless chiral multiplet). When instead $u = 0$ the gauge group remains unbroken. Still, we are in a different phase with respect to the massive case. The basic difference is that the coupling $\alpha(r)$ does not stop running, now, since $m = 0$, and hence it ends-up vanishing at large enough distances. In other words, the potential again reads

$$V(r) \sim \frac{\alpha(r)}{r} \quad , \quad \alpha(r) \sim \frac{1}{\log r} \tag{10.25}$$

but now $\alpha = 0$ for $r \to \infty$. This is not really a Coulomb phase but actually what is called a *free* phase. At low energy (large enough distances) the theory is a theory of free massless particles.

Let us emphasize again that both the Coulomb phase and the free phase are not specific to abelian gauge theories, and can be enjoyed even by non-abelian theories. We will see examples of this phenomenon later in this lecture.

10.2.2 Continuously connected phases

Sometime there is no gauge-invariant distinction between phases. Let us show this non-trivial fact with a simple, though instructive, example.

Let us consider a $SU(2)$ gauge theory with a $SU(2)$ scalar doublet ϕ (a Higgs field), a $SU(2)$ singlet e_R and a $SU(2)$ doublet $L = (\nu_L, e_L)$, with interaction Lagrangian

$$\mathcal{L}_{\text{int}} = a \bar{L} \phi e_R + h.c. \ . \tag{10.26}$$

This is nothing but a one-family EW theory model. As it happens in standard EW theory, this theory can be realized in the Higgs phase, where the field ϕ gets a non-vanishing VEV

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \ . \tag{10.27}$$

In this phase all three gauge bosons get a mass, the neutrino ν_L remains massless while the electron gets a mass $m_e = av/\sqrt{2}$.

Suppose instead that the theory were realized in a different phase, a confinement phase. In such phase one would not observe massless gauge bosons (as above) while fermions and Higgs bosons would bind into $SU(2)$ singlet combinations

$$E_L = \phi^\dagger L \quad , \quad N_L = \epsilon_{ab}\phi_a L_b \quad , \quad e_R \tag{10.28}$$

in terms of which the interaction Lagrangian becomes

$$\mathcal{L}_{\rm int} = m\bar{E}_L e_R + h.c. \ , \tag{10.29}$$

where $m \sim a\Lambda$. So we see that E_L and e_R pair-up and become massive while N_L remains massless. The spectrum in this phase is the same as that of the Higgs phase: there is no gauge-invariant distinction between the two phases! Obviously, they differ quantitatively: for instance, E_L is a composite field and its pair production would be suppressed by a form factor which is not observed. Still, we need experiments to discern between the two phases and understand which one is actually realized in Nature.

The general lesson we want to convey here is that by adjusting some parameter of a gauge theory, sometime one can move continuously from one type of phase to another. In this non-abelian example, there is no invariant distinction between Higgs and Wilson phase. We will encounter many such situations when studying supersymmetric gauge theory dynamics.

10.3 N=1 SQCD: perturbative analysis

In what follows we will consider SQCD and its quantum behavior (including non-perturbative effects) and try to answer the basic questions about its dynamical properties in the most analytical possible way. Let us first summarize what we already know about SQCD, which is its classical and quantum, though perturbative only, behavior. SQCD is a renormalizable supersymmetric gauge theory with gauge

group $SU(N)$, F flavors $(Q, \tilde{Q})$ and no superpotential. Interaction terms are present and come from D-terms. The group of continuous global symmetries at the quantum level is $G_{\rm F} = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$ with charge assignment

$$\begin{array}{ccccc} & SU(F)_L & SU(F)_R & U(1)_B & U(1)_R \\ Q^i_a & F & \bullet & 1 & \frac{F-N}{F} \\ \tilde{Q}^b_j & \bullet & \overline{F} & -1 & \frac{F-N}{F} \\ \lambda & \bullet & \bullet & 0 & 1 \end{array}$$

As already emphasized, for pure SYM the R-symmetry is anomalous, and only a $\mathbb{Z}_{2N}$ subgroup of $U(1)_R$ survives at the quantum level.

What do we know, already, about the quantum properties of SQCD? We know there is a huge moduli space of supersymmetric vacua, described by the D-term equations

$$D^A = Q^{\dagger}{}_i^b \, (T^A)^c_b \, Q^i_c - \tilde{Q}_i^b \, (T^A)^c_b \, \tilde{Q}^{\dagger}{}_c^i = 0 \ , \tag{10.30}$$

where $A = 1, 2, \ldots, N^2 - 1$ is an index in the adjoint representation of $SU(N)$.

Up to flavor and global gauge rotations, a solution of the above equations can be found for both $F < N$ and $F \geq N$. For $F < N$ one can show that on the moduli space (10.30) the matrices Q and $\tilde{Q}$ can be put, at most, in the following form

$$Q = \begin{pmatrix} v_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & v_2 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & v_F & 0 & \ldots & 0 \end{pmatrix} = \tilde{Q}^T \tag{10.31}$$

Hence, at a generic point of the moduli space the gauge group is broken to $SU(N - F)$. The (classical) moduli space can be parameterized in terms of mesons fields

$$M^i_{\ j} = Q^i_a \tilde{Q}^a_j \tag{10.32}$$

without the need of considering any classical constraint since the meson matrix has maximal rank for $F < N$.

For $F \geq N$ the matrices Q and $\tilde{Q}$ can also be brought to a diagonal form on the

moduli space

$$Q = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_N \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{Q}^T = \begin{pmatrix} \tilde{v}_1 & 0 & \dots & 0 \\ 0 & \tilde{v}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{v}_N \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.33}$$

where $|v_i|^2 - |\tilde{v}_i|^2 = a$, with a a i-independent number. At a generic point of the moduli space the gauge group is now completely broken. The moduli space is efficiently described in terms of mesons and baryons, now, modulo classical constraints between them, which are now non-trivial and increase with F, at fixed N. The mesons are again defined as in eq. (10.32) but the meson matrix does not have maximal rank anymore. Baryons are gauge invariant single trace operators made out of N fields Q respectively N fields $\tilde{Q}$, with fully anti-symmetrized indices and read

$$\begin{aligned} B_{i_1 \dots i_{F-N}} &= \epsilon_{i_1 i_2 \dots i_{F-N} j_1 \dots j_N} \epsilon^{a_1 a_2 \dots a_N} \, Q^{j_1}_{a_1} Q^{j_2}_{a_2} \dots Q^{j_N}_{a_N} \\ \tilde{B}^{i_1 \dots i_{F-N}} &= \epsilon^{i_1 i_2 \dots i_{F-N} j_1 \dots j_N} \epsilon_{a_1 a_2 \dots a_N} \, \tilde{Q}^{a_1}_{j_1} \tilde{Q}^{a_2}_{j_2} \dots \tilde{Q}^{a_N}_{j_N} \, , \end{aligned} \tag{10.34}$$

where a_i are gauge indices and i_l, j_l are flavor indices.

As far as quantum correction are concerned, we know the exact (perturbative) expression for the gauge coupling which, in the holomorphic scheme, reads

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + i \frac{4\pi}{g^2(\mu)} = \frac{b_1}{2\pi i} \log \frac{\Lambda}{\mu} \quad , \quad b_1 = 3N - F \ \text{ and } \ \Lambda = \mu \, e^{2\pi i \tau / b_1} \, . \tag{10.35}$$

10.4 N=1 SQCD: non-perturbative dynamics

Our goal is to understand how this whole picture is modified once non-perturbative corrections are taken into account. Notice that for a UV-free theory, as SQCD is for $F < 3N$, quantum corrections are expected to modify the perturbative analysis only near the origin of field space. Indeed, for large value of scalar field VEVs, the gauge group gets broken (and the gauge coupling hence stops running) for small values of the gauge coupling constant

$$e^{-\frac{8\pi^2}{g^2(\langle Q \rangle)} + i\theta_{\rm YM}} = \left(\frac{\Lambda}{\langle Q \rangle} \right)^{3N-F} \longrightarrow 0 \ \text{ for } \ \langle Q \rangle \to \infty \, . \tag{10.36}$$

This implies that for large field VEVs the gauge coupling freezes at a value g_* where (semi)-classical analysis works properly. The smaller the field VEV the more important are quantum corrections. Hence, generically, we expect non-perturbative dynamics to modify the perturbative answer mostly near the origin of field space.

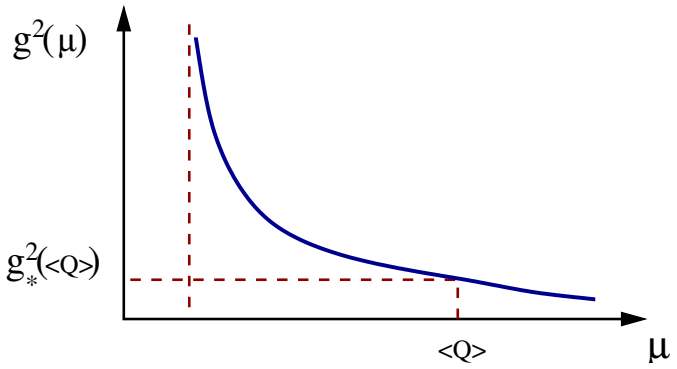


Figure 10.8: The gauge coupling running of a UV-free theory. The large $\langle Q \rangle$ region is a weakly coupled region where classical analysis is correct, since the value at which the gauge coupling stops running, $g = g_*$, is small.

Not surprisingly, for any fixed value of N, several non-perturbative dynamical properties change with the number of flavors, F. Hence, in what follows, we will consider different cases separately.

10.4.1 Pure SYM: gaugino condensation

This case, $F = 0$, was already analyzed, at a qualitative level. Let us first recall that this is the only case in which there does not exist an anomaly-free R-symmetry. At the quantum level, only a discrete $\mathbb{Z}_{2N}$ R-symmetry survives. Promoting τ to a spurion field and using holomorphy arguments, it is easy to see what the structure of the non-perturbative generated superpotential should be. Let us first notice that the operator $e^{2\pi i\tau/N}$ has R-charge $R = 2$. Indeed, due to the transformation properties of $\theta_{\rm YM}$ under R-symmetry transformations, $\theta_{\rm YM} \to \theta_{\rm YM} + 2N\alpha$, we have

$$e^{2\pi i\tau/N} \to e^{2i\alpha} e^{2\pi i\tau/N} \ . \tag{10.37}$$

Because of confinement, *assuming* a mass gap, the effective Lagrangian should depend only on τ, and hence $W_{\rm eff}$, if any, should also depend only on τ. Imposing R-symmetry, by dimensional analysis the only possible term reads

$$W_{\rm eff} = c\,\mu^3 e^{2\pi i\tau/N} = c\,\Lambda^3 \ . \tag{10.38}$$

where c is an undetermined coefficient (which in principle could also be zero, of course). This innocent-looking constant superpotential contribution contains one crucial physical information. Given the presence of massless strong interacting fermion fields (the gaugino) one could wonder whether in SYM theory gauginos undergo pair condensation, as it happens in QCD, where quark bilinears condense. Looking at the SYM Lagrangian

$$\mathcal{L} = \frac{1}{32\pi}\mathrm{Im}\left[\int d^2\theta\, \tau\, \mathrm{Tr}\, W^\alpha W_\alpha\right] \ , \tag{10.39}$$

we see that $\lambda^\alpha\lambda_\alpha$ is the scalar component of $W^\alpha W_\alpha$ and (minus) F_τ acts as a source for it (recall we are thinking of τ as a spurion superfield, $\tau = \tau + \sqrt{2}\theta\psi_\tau - \theta\theta F_\tau$). Therefore, in order to compute the gaugino condensate one should just differentiate the logarithm of the partition function $Z = \int \mathcal{D}V e^{i\int \mathcal{L}}$ with respect to F_τ. In fact, under the assumption of a mass gap, the low energy effective action depends only on τ, since gauge fields have been integrated out, and it coincides with the effective superpotential (10.38), giving for the gaugino condensate

$$\langle\lambda\lambda\rangle = -16\pi\frac{\partial}{\partial F_\tau}\log Z = -16\pi i\frac{\partial}{\partial F_\tau}\int d^2\theta W_{\text{eff}}(\tau) = 16\pi i\frac{\partial}{\partial\tau}W_{\text{eff}}(\tau) \tag{10.40}$$

where in doing the second step we have used the fact that

$$W_{\text{eff}} = w_{\text{eff}}(\tau) + \sqrt{2}\frac{\partial W_{\text{eff}}}{\partial\tau}\theta\psi_\tau - \theta\theta\left(\frac{\partial W_{\text{eff}}}{\partial\tau}F_\tau + \frac{1}{2}\frac{\partial^2 W_{\text{eff}}}{\partial\tau^2}\psi_\tau^2\right) \ .$$

The upshot is that we can compute eq. (10.40) for the effective superpotential (10.38). We get

$$\langle\lambda\lambda\rangle = -\frac{32\pi^2}{N}c\,\mu^3 e^{2\pi i\tau/N} \equiv a\,\Lambda^3 \ . \tag{10.41}$$

which means that if $c \neq 0$ indeed gauginos condense in SYM. Since gauginos have $R = 1$, this implies that in the vacua the $\mathbb{Z}_{2N}$ symmetry is broken to $\mathbb{Z}_2$ and that there are in fact N distinct (and isolated, in this case) vacua. All these vacua appear explicitly in the above formula since the transformation

$$\theta_{\text{YM}} \to \theta_{\text{YM}} + 2\pi k \ , \tag{10.42}$$

which is a symmetry of the theory, sweeps out N distinct values of the gaugino condensate

$$\langle\lambda\lambda\rangle \to e^{2i\alpha}\langle\lambda\lambda\rangle \quad , \quad \theta_{\text{YM}} \to \theta_{\text{YM}} + 2N\alpha \simeq \theta_{\text{YM}} + 2\pi k \tag{10.43}$$

where $k = 0, 1, \ldots, 2N-1$, and $k = i$ and $k = i + N$ give the same value of the gaugino condensate. In other words, we can label the N vacua with N distinct phases of the gaugino condensate $(0, 2\pi\frac{1}{N}, 2\pi\frac{2}{N}, \ldots, 2\pi\frac{N-1}{N})$, recall Figure 10.7.

This ends our discussion of pure SYM. It should be stressed that to have a definitive picture we should find independent ways to compute the constant c in eq. (10.38), since if it were zero, then all our conclusions would have been wrong (in particular, there would not be any gaugino condensate, and hence we would have had a unique vacuum preserving the full $\mathbb{Z}_{2N}$ symmetry). We will come back to this important point later.

10.4.2 SQCD for $F < N$: the ADS superpotential

For $F < N$ classical analysis tells that there is a moduli space of complex dimension F^2, parameterized by meson field VEVs. The question, again, is whether an effective superpotential is generated due to strong coupling dynamics. Let use again holomorphy, and the trick of promoting coupling constants to spurion superfields. The effective superpotential could depend on meson fields and on the complexified gauge coupling, through Λ. The quantum numbers of (well educated functions of) these two basic objects are

	$U(1)_B$	$U(1)_A$	$U(1)_R$
$\det M$	0	$2F$	$2(F-N)$
Λ^{3N-F}	0	$2F$	0

Both above objects are invariant under the non-abelian part of the global symmetry group (notice that $\det M$ is the only $SU(F)_L \times SU(F)_R$ invariant one can make out of M). From the table above it follows that the only superpotential term which can be generated should have the following form

$$W_{\text{eff}} = c_{N,F} \left(\frac{\Lambda^{3N-F}}{\det M} \right)^{\frac{1}{N-F}} , \tag{10.44}$$

where, again, the overall constant, which generically will be some function of N and F, is undetermined. That (10.44) is the only possible term can be understood as follows. The effective superpotential should have R-charge two, $U(1)_A$ and $U(1)_B$ charges equal to zero, and should have dimension three. The chiral superfields that might contribute to it are Λ, W^α and the meson matrix M. Provided what we just stated about non-abelian global symmetries the generic expression for W_{eff} should

be made of terms like

$$W_{\text{eff}} \sim \Lambda^{(3N-F)n} \left(W^\alpha W_\alpha\right)^m (\det M)^p \ , \tag{10.45}$$

where n, m and p are integer numbers. The invariance under the baryonic symmetry is guaranteed by any such term. As for the other two global symmetries we get

$$\begin{cases} 0 = 2nF + 2pF \\ 2 = 2m + 2p(F-N) \end{cases} \longrightarrow \quad \begin{cases} n = -p \\ p = (m-1)/(N-F) \end{cases} \tag{10.46}$$

Since $3N - F > 0$, in order to have a meaningful weak couping limit, we should have $n \geq 0$, which implies that $p \leq 0$ and $m \leq 1$. On the other hand, we should have $m \geq 0$ in order for the Wilsonian action to be local (it needs to have a sensible derivative expansion), which finally implies that $m = 0, 1$. The contribution $m = 1$ and hence $p = n = 0$ is the tree level result (the gauge kinetic term, in fact). The contribution $m = 0$ which implies $p = -1/(N-F)$ and $n = 1/(N-F)$ is precisely the so-called Affleck-Dine-Seiberg (ADS) superpotential contribution (10.44).

In what follows we would like to analyze several properties of the ADS superpotential, possibly understanding where it comes from, physically, and eventually determine the coefficient $c_{N,F}$.

Let us consider again the classical moduli space. At a generic point of the moduli space the $SU(N)$ gauge group is broken to $SU(N-F)$. Suppose for simplicity that all scalar field VEVs are equal, $v_i = v$, recall expression (10.31). Clearly the theory behaves differently at energies higher or lower than v. At energies higher than v the gauge coupling running is that of SQCD with gauge group $SU(N)$ and F massless flavors. At energies lower than v all matter fields become massive (and should be integrated out) while the gauge group is broken down to $SU(N-F)$. Hence the theory runs differently and, accordingly, the dynamical generated scale, Λ_L is also different. More precisely we have

$$\begin{aligned} E > v \qquad & \frac{4\pi}{g^2(\mu)} = \frac{3N-F}{2\pi} \log \frac{\mu}{\Lambda} \\ E < v \qquad & \frac{4\pi}{g_L^2(\mu)} = \frac{3(N-F)}{2\pi} \log \frac{\mu}{\Lambda_L} \ . \end{aligned} \tag{10.47}$$

If supersymmetry is preserved the two above equations should match at $E = v$. This is known as *scale matching* (that there are no threshold factors reflects a choice of subtraction scheme, on which threshold factors depend; this is the correct matching

in, e.g., the $\overline{DR}$ scheme). Hence we get

$$\Lambda_L^{3(N-F)} = \Lambda^{3N-F}\frac{1}{v^{2F}} = \frac{\Lambda^{3N-F}}{\det M} \quad \longrightarrow \quad \Lambda_L^3 = \left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} \ . \tag{10.48}$$

This implies that

$$W_{\text{eff}} = c_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} = c_{N,F}\Lambda_L^3 \ , \tag{10.49}$$

which means that

$$c_{N,F} = c_{N-F,0} \ . \tag{10.50}$$

Besides getting a relation between c's for different theories (recall these are (N,F)-dependent constants, in general), we also get from the above analysis some physical intuition for how the ADS superpotential is generated. One can think of W_{eff} being generated by gaugino condensation of the left over $SU(N-F)$ gauge group (recall that gaugino condensation is in *one-to-one correspondence* with the very existence of an effective superpotential for pure SYM theory: the two are fully equivalent statements).

Let us now start from SQCD with a given number of flavors and suppose to give a mass m to the F-th flavor. At high enough energy this does not matter much. But below the scale m the theory behaves as SQCD with $F-1$ flavors. More precisely, we have

$$\begin{aligned} E > m \qquad & \frac{4\pi}{g^2(\mu)} = \frac{3N-F}{2\pi}\log\frac{\mu}{\Lambda_F} \\ E < m \qquad & \frac{4\pi}{g_L^2(\mu)} = \frac{3N-(F-1)}{2\pi}\log\frac{\mu}{\Lambda_{L,F-1}} \ . \end{aligned} \tag{10.51}$$

Matching the scale at $E = m$ we obtain the following relation between non-perturbative scales

$$\Lambda_{L,F-1}^{3N-F+1} = m\,\Lambda_F^{3N-F} \ , \tag{10.52}$$

which tells that the effective superpotential for SQCD with one massive flavor and $F-1$ massless ones can be written in the following equivalent ways

$$W_{\text{eff}} = c_{N,F-1}\left(\frac{\Lambda_{L,F-1}^{3N-F+1}}{\det\tilde{M}}\right)^{\frac{1}{N-F+1}} = c_{N,F-1}\left(\frac{m\,\Lambda_F^{3N-F}}{\det\tilde{M}}\right)^{\frac{1}{N-F+1}} \ . \tag{10.53}$$

where $\tilde{M}$ is the meson matrix made out of $F-1$ flavors. Let us check this prediction using holomorphic decoupling. The superpotential of SQCD with $F-1$ massless

flavors and one massive one reads

$$W_{\text{eff}} = c_{N,F} \left(\frac{\Lambda_F^{3N-F}}{\det M} \right)^{\frac{1}{N-F}} + m Q^{\text{F}} \tilde{Q}_{\text{F}} \ . \tag{10.54}$$

The F-term equation for $M^{\text{F}}_{\ i}$ for $i \neq F$ implies $M^{\text{F}}_{\ i} = 0$, and similarly for $M^{i}_{\ \text{F}}$. So the meson matrix can be put into the form

$$M = \begin{pmatrix} \tilde{M} & 0 \\ 0 & t \end{pmatrix} \quad , \quad t \equiv M^{\text{F}}_{\ \text{F}} \ . \tag{10.55}$$

The F-term equation for t gives

$$0 = -\frac{c_{N,F}}{N-F} \left(\frac{\Lambda_F^{3N-F}}{\det \tilde{M}} \right)^{\frac{1}{N-F}} \left(\frac{1}{t} \right)^{1+\frac{1}{N-F}} + m \tag{10.56}$$

which implies

$$t = \left[\frac{N-F}{c_{N,F}} \, m \left(\frac{\Lambda_F^{3N-F}}{\det \tilde{M}} \right)^{\frac{1}{F-N}} \right]^{\frac{F-N}{N-F+1}} \ . \tag{10.57}$$

Plugging this back into eq. (10.54) one gets

$$W_{\text{eff}} = (N-F+1) \left(\frac{c_{N,F}}{N-F} \right)^{\frac{N-F}{N-F+1}} \left(\frac{m \Lambda_F^{3N-F}}{\det \tilde{M}} \right)^{\frac{1}{N-F+1}} \ . \tag{10.58}$$

Comparing with the expression (10.53) one finds complete agreement and, as a bonus, the following relation between coefficients

$$c_{N,F-1} = (N-F+1) \left(\frac{c_{N,F}}{N-F} \right)^{\frac{N-F}{N-F+1}} \ . \tag{10.59}$$

Combining this result with the relation we found before, eq. (10.50), one concludes that all coefficients are related one another as

$$c_{N,F} = (N-F) \, c^{\frac{1}{N-F}} \ , \tag{10.60}$$

with a unique common coefficient c to be determined. This result tells that if the ADS superpotential can be computed exactly for a given value of F (hence fixing c), then we know its expression for any other value!

Let us consider the case $F = N-1$, which is the extreme case in the window $F < N$. In this case

$$c_{N,N-1} = c \ . \tag{10.61}$$

Interestingly, for $F = N - 1$ the gauge group is fully broken, so there is no left-over strong IR dynamics. In other words, any term appearing in the effective action should be visible in a weak-coupling analysis. Even more interesting, the ADS superpotential for $F = N-1$ is proportional to Λ^{2N+1} which is nothing but how one-instanton effects contribute to gauge theory amplitudes (recall that for $F = N - 1$ $b_1 = 2N+1$, and $e^{-S_{inst}} \sim \Lambda^{b_1}$), suggesting that in this case the ADS superpotential is generated by instantons. At weak coupling, a reliable one-instanton calculation can indeed be done and gives $c = 1$. Via eq. (10.60) this result hence fixes uniquely $c_{N,F}$ for arbitrary values of N and F as

$$c_{N,F} = N - F \ , \tag{10.62}$$

giving finally for the ADS superpotential the following exact expression

$$W_{\text{ADS}} = (N - F) \left(\frac{\Lambda^{3N-F}}{\det M} \right)^{\frac{1}{N-F}} \ . \tag{10.63}$$

Notice that this also fixes the coefficient of the effective superpotential of pure SYM theory which is

$$W_{\text{SYM}} = N \, \Lambda^3 \ , \tag{10.64}$$

implying, via eq. (10.40), that gauginos do condense!

Let us finally see how does the ADS superpotential affect the moduli space of vacua. From the expression (10.63) we can compute the potential, which is expected not to be flat anymore, since the effective superpotential W_{ADS} depends on scalar fields (through the meson matrix). The potential

$$V_{\text{ADS}} = \sum_i \left| \frac{\partial W_{\text{ADS}}}{\partial Q_i} \right|^2 + \left| \frac{\partial W_{\text{ADS}}}{\partial \tilde{Q}^i} \right|^2 \tag{10.65}$$

is minimized at infinity in field space, namely for $Q = \tilde{Q} \to \infty$, where it reaches zero, see Figure 10.9. This can be easily seen noticing that, qualitatively, $\det M \sim M^F$, which implies that $V_{\text{ADS}} \sim |M|^{-\frac{2N}{N-F}}$, which is indeed minimized at infinity. This means that the theory does not admit any stable vacuum at finite distance in field space: the (huge) classical moduli space is completely lifted at the quantum level! This apparently strange behavior makes sense, in fact, if one thinks about it for a while. For large field VEVs, eventually for $v \to \infty$, we recover pure SYM which has indeed supersymmetric vacua (that is, zero energy states). This is part of the space of D-term solutions of SQCD; any other configuration would have higher energy

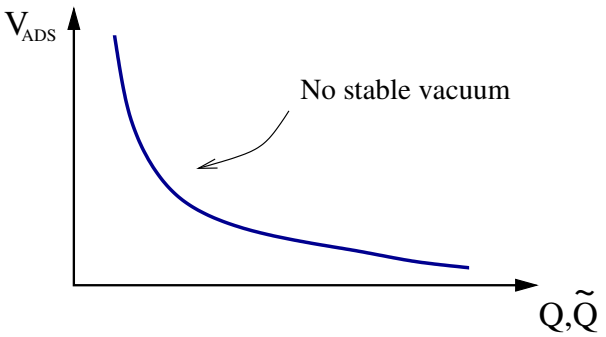


Figure 10.9: The runaway behavior of the quantum corrected potential of $SU(N)$ SQCD with $F < N$.

and would hence be driven to the supersymmetric one. Let us suppose this picture were wrong and that SQCD had a similar behavior as QCD: confinement and chiral symmetry breaking. Then we would have expected a quark condensate to develop $\langle \psi_{Q_i} \psi_{\tilde{Q}^j} \rangle \neq 0$. Such condensate, differently from a gaugino condensate (which we certainly have), would break supersymmetry, since it is nothing but an F-term for the meson matrix M_i^j. Hence this configuration would have $E > 0$ and thus any configuration with $E = 0$ would be preferred. The latter are all configurations like (10.31) which, by sending v_i all the way to infinity, reduce to SYM, which admits supersymmetry preserving vacua. The ADS superpotential simply shows this.

There is a caveat in all this discussion. In our analysis we have not included wave-function renormalization effects. The latter could give rise, in general, to non-canonical Kähler potential terms, which could produce wiggles or even local minima in the potential. However, at most this could give rise to metastable vacua (which our

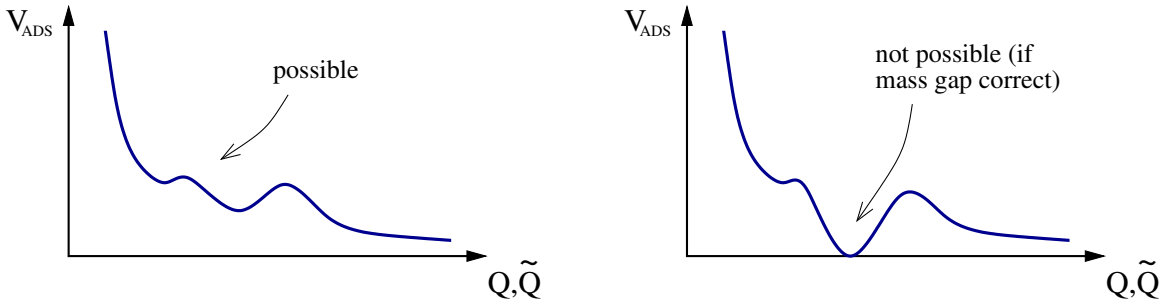


Figure 10.10: The effect of a non-canonical Kähler potential on the ADS potential. The picture on the right cannot hold if the assumption of mass gap for pure SYM is correct.

holomorphic analysis cannot see), but it would not lift the absolute supersymmetric minima at infinity, a region where the Kähler potential is nearly canonical in the UV-variables Q and $\tilde{Q}$. On the other hand, no supersymmetric minima can arise at finite distance in field space. These would correspond to singularities of the Kähler metric, implying that at those specific points in field space extra massless degrees of freedom show-up. This cannot be, if the assumption of mass gap for pure SYM (to which the theory reduces at low enough energy, at generic points in the classical moduli space) is correct.

10.4.3 Integrating in and out: the linearity principle

The superpotential of pure SYM is sometime written as

$$W_{\text{VY}} = NS\left(1 - \log\frac{S}{\Lambda^3}\right) \tag{10.66}$$

where $S = -\frac{1}{32\pi^2}\,\text{Tr}\,W^\alpha W_\alpha$ is the so-called glueball superfield and the subscript VY stands for Veneziano-Yankielowicz. Let us first notice that integrating S out (recall we are supposing pure SYM has a mass gap) we get

$$\frac{\partial W_{\text{VY}}}{\partial S} = N\left(1 - \log\frac{S}{\Lambda^3}\right) + NS\left(-\frac{1}{S}\right) = 0\ , \tag{10.67}$$

which implies

$$\langle S\rangle = \Lambda^3\ . \tag{10.68}$$

Plugging this back into the VY superpotential gives

$$W_{\text{VY}} = N\Lambda^3\ , \tag{10.69}$$

which is nothing but the effective superpotential of pure SYM we have previously derived. From this view point the two descriptions seem to be equivalent: while the above superpotential can be obtained from the VY one by integrating S out, one can say that the VY superpotential is obtained from (10.69) integrating the glueball superfield S *in*.

Analogously, the ADS superpotential is sometime written as

$$W_{\text{TVY}} = (N-F)\,S\left[1 - \frac{1}{N-F}\log\left(\frac{S^{N-F}\,\det M}{\Lambda^{3N-F}}\right)\right]\ . \tag{10.70}$$

where TVY stands for Taylor-Veneziano-Yankielowicz. Integrating S out one recovers the ADS superpotential. And, consistently, adding a mass term for all matter

fields, $\sim \mathrm{Tr}\, mM$, and integrating M out, one gets from the TVY superpotential the VY superpotential. Finally, the TVY superpotential can be obtained from the ADS superpotential integrating the glueball superfield S in.

Let us try to understand better the meaning of all that (and actually the *complete equivalence* between these apparently different descriptions). Naively, one could imagine the VY or TVY superpotentials containing more information than the ADS superpotential, since they include one more dynamical field, the glueball superfield S. As we are going to discuss below, this intuition is not correct.

Let us try to be as general as possible and consider a supersymmetric gauge theory admitting also a tree-level superpotential W_{tree}. Given a set of chiral superfields Φ_i, the generic form of such superpotential is

$$W_{\text{tree}} = \sum_r \lambda_r X_r(\Phi_i) \ , \tag{10.71}$$

where λ_r are coupling constants and X_r gauge invariant combinations of the chiral superfields Φ_i. In general, one would expect the non-perturbative generated superpotential $W_{\text{non-pert}}$ to be a (holomorphic) function of the couplings λ_r, the gauge invariant operators X_r, and of the dynamical generated scales Λ_s (we are supposing, to be as most general as possible, the gauge group not to be simple, hence we allow for several dynamical scales). In fact, as shown by Intriligator, Leigh and Seiberg, $W_{\text{non-pert}}$ does not depend on the couplings λ_r. This fact implies that the full effective superpotential (which includes both the tree level and the non-perturbative contributions) is linear in the couplings, and hence this is sometime referred to as *linearity principle*. The upshot is that, in general, we have

$$W_{\text{eff}} = \sum_r \lambda_r X_r + W_{\text{non-pert}}(X_r, \Lambda_s) \ . \tag{10.72}$$

Let us focus on the dependence on, say, λ_1. At low enough energy (where the superpotential piece dominates - let us suppose for now that X_1 is massive) we can integrate out the field X_1 by solving its F-term equation only, which, because of eq. (10.72), reads

$$\lambda_1 = -\frac{\partial}{\partial X_1} W_{\text{non-pert}} \ . \tag{10.73}$$

The above equation is the same as a Legendre transform. In other words, the coupling λ_r and the gauge invariant operator X_r behave as Legendre dual variables. Solving for X_1 in terms of λ_1 and all other variables, and substituting in eq. (10.72), one obtains an effective superpotential with a complicated dependence on λ_1 but

where X_1 has been integrated out. Repeating the same reasoning for all X_r one can integrate out all fields and end-up with an effective superpotential written in terms of couplings only

$$W_{\text{eff}}(\lambda_r, \Lambda_s) = \left[\sum_r \lambda_r X_r + W_{\text{non-pert}}(X_r, \Lambda_s) \right]_{X_r(\lambda,\Lambda)} . \tag{10.74}$$

The point is that the Legendre transform is invertible. Therefore, as we can integrate out a field, we can also integrate it back in, by reversing the procedure

$$\langle X_r \rangle = \frac{\partial}{\partial \lambda_r} W_{\text{eff}}(\lambda_r, \Lambda_s) \ . \tag{10.75}$$

The reason why the two descriptions, one in terms of the fields, one in terms of the dual couplings, are equivalent is because we have not considered D-terms. D-terms contain the dynamics (e.g. the kinetic term). Hence, if we ignore D-terms, namely if we only focus on holomorphic terms as we are doing here (which is a more and more correct thing to do the lower the energy), integrating out or in a field is an operation which does not make us lose or gain information. As far as the holomorphic part of the effective action is concerned, a field and its dual coupling are fully equivalent.

What about the dynamical scales Λ_s? Can one introduce canonical pairs for them, too? The answer is yes, and this is where the physical equivalence between ADS and TVY superpotentials we claimed about becomes explicit. Let us consider pure SYM, for definiteness. One can write the gauge kinetic term as a contribution to the tree level superpotential in the sense of eq. (10.71)

$$W_{\text{tree}} = \frac{\tau(\mu)}{16\pi i} \operatorname{Tr} W^\alpha W_\alpha = 3N \log\left(\frac{\Lambda}{\mu}\right) S \ , \tag{10.76}$$

where S is a X-like field and $3N \log(\Lambda/\mu)$ the dual coupling. In other words, one can think of S and $\log \Lambda$ as Legendre dual variables. From this view point, the SYM superpotential (10.64) is an expression of the type (10.74), where the field S has been integrated out and the dependence on the dual coupling is hence non-linear. Indeed (10.64) can be re-written as

$$W_{\text{SYM}} = N\Lambda^3 = N\mu^3 e^{\frac{1}{N} 3N \log \frac{\Lambda}{\mu}} \ , \tag{10.77}$$

where the coupling appears non-linearly. Using now eq. (10.75) applied to this dual pair, one gets

$$\langle S \rangle = \frac{1}{3N} \Lambda \frac{\partial}{\partial \Lambda} W_{\text{eff}} = \Lambda^3 \ . \tag{10.78}$$

Therefore

$$W_{\text{non-pert}}(S) = W_{\text{eff}} - W_{\text{tree}} = NS - 3N \log\left(\frac{\Lambda}{\mu}\right) S = NS - NS \log \frac{S}{\mu^3} \ , \qquad (10.79)$$

which is correctly expressed, according to the linearity principle, in terms of S only, and not the coupling, $\log \Lambda$. We can now add the two contributions, the one above and (10.76) and get for the effective superpotential an expression in the form (10.72)

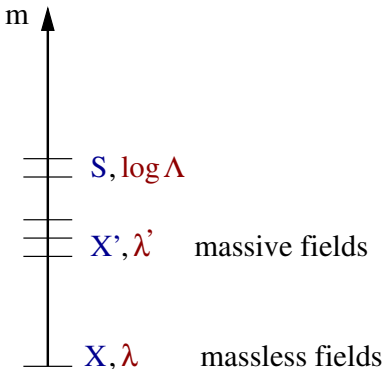
$$W_{\text{eff}} = W_{\text{non-pert}} + W_{\text{tree}} = NS \left(1 - \log \frac{S}{\Lambda^3}\right) \qquad (10.80)$$

which is nothing but the VY superpotential! The same reasoning can be applied to a theory with flavor and/or with multiple dynamical scales. The upshot is one and the same: integrating in (TVY) or out (ADS) fields holomorphically, are operations which one can do at no cost. The two descriptions are physically equivalent.

In the table below we summarize the relation between couplings and dual field variables for the most generic situation

Couplings	$b_1^1 \log \frac{\Lambda_1}{\mu}$	$b_1^2 \log \frac{\Lambda_1}{\mu}$	$\dots$	λ_1	λ_2	$\dots$
Fields	S_1	S_2	$\dots$	X_1	X_2	$\dots$

Suppose that the mass spectrum of above (composite) fields is as follows



where the X's are a set of massless (possibly composite) fields, the X''s are massive ones, and the Λ's are dynamical scales (glueball superfields are all massive because of mass gap of the pure glue theory, i.e. all Λ's $\neq 0$). The most Wilsonian thing to do would be to describe the effective superpotential in terms of fields X, and couplings λ' and Λ

$$W_{\text{eff}} = W_{\text{eff}}(X, \lambda', \Lambda) \ . \qquad (10.81)$$

In this sense, the ADS superpotential is more Wilsonian than the TVY. Seemingly, for pure SYM the most Wilsonian thing to do is to express the effective superpotential as a function of the coupling only (since the glueball superfield is massive), namely as $W_{\text{eff}} = N\Lambda^3$. However, since integrating in and out fields are equivalent operations, one can very well choose to write down the effective superpotential by integrating X fields out and X' and S fields in (or anything in between these two extreme cases)

$$W_{\text{eff}} = W_{\text{eff}}(\lambda, X', S) \ , \tag{10.82}$$

getting an equivalent way of describing the low energy effective theory superpotential. In fact, one should bare in mind that as far as the massless fields X are concerned, there is no actual energy range for which integrating them out makes real physical sense, and this would be indicated by the Kähler potential of the effective theory being ill-defined (in other words, there is no energy range in which the kinetic term of such massless fields is negligible, since the energy is always bigger or equal than the field mass, which is vanishing). On the contrary, in presence of a mass gap, that is in the absence of X-like fields, the two descriptions, one in terms of couplings the other in terms of fields, are equivalent (as far as the F-term!), since now no singularities are expected in the Kähler potential. And this is a more and more exact equivalence the lower the energy.

10.4.4 SQCD for $F = N$ and $F = N + 1$

Let us now go back to our analysis of the IR dynamics of SQCD with gauge group $SU(N)$ and F flavors. What about the case $F \geq N$? As we are going to see, things change drastically. For one thing, a properly defined effective superpotential cannot be generated. There is no way of constructing an object respecting all symmetries, with the correct dimension, and being vanishing in the classical limit, using couplings and fields we have (mesons, baryons and the dynamical scale Λ). This has the effect that for $F \geq N$ the classical moduli space is not lifted. This does not mean nothing interesting happens. For instance, the moduli space can be *deformed* by strong dynamics effects. Moreover, the perturbative analysis does not tell us what the low energy effective theory looks like; as we will see instead (mainly using holomorphy arguments), in some cases we will be able to make very non-trivial statements about the way light degrees of freedom interact, and in turn about the phase the theory enjoys.

In what follows, we will consider qualitatively different cases separately. Let us

start analyzing the case $F = N$. It is easy to see that in this case all gauge invariant operators have R-charge $R = 0$, so one cannot construct an effective superpotential with $R = 2$. However, as we are going to show, something does happen due to strong dynamics.

Besides the mesons, there are now two baryons

$$\begin{aligned} B &= \epsilon^{a_1 a_2 \ldots a_N} Q^1_{a_1} Q^2_{a_2} \ldots Q^N_{a_N} \\ \tilde{B} &= \epsilon_{a_1 a_2 \ldots a_N} \tilde{Q}^{a_1}_1 \tilde{Q}^{a_2}_2 \ldots \tilde{Q}^{a_N}_N \ . \end{aligned}$$

The classical moduli space is parameterized by VEVs of mesons and baryons. There is, however, a classical constraint between them

$$\det M - B\tilde{B} = 0 \tag{10.83}$$

(this comes because for $N = F$ we have that $\det Q = B$ and $\det \tilde{Q} = \tilde{B}$ and the determinant of the product is the product of the determinants). One can ask whether this classical constraint is modified at the quantum level. In general, one could expect the quantum version of the above classical constraint to be

$$\det M - B\tilde{B} = a\,\Lambda^{2N} \ , \tag{10.84}$$

where a is a (undetermined for now) dimensionless and charge-less constant. Let us try to understand why this is the (only) possible modification one can have. First notice that this modification correctly goes to zero in the classical limit, $\Lambda \to 0$. Second, the power with which Λ enters, $2N$, is the one-loop coefficient of the β-function and is exactly that associated with a one instanton correction, since for $N = F$ we have for the instanton action

$$e^{-S_{\rm inst}} \sim e^{-\frac{8\pi^2}{g^2} + i\theta_{\rm YM}} \sim \Lambda^{2N} \ . \tag{10.85}$$

Finally, there are no symmetry reasons not to allow for it (modulo the constant a which can very well be vanishing, after all). So, given that in principle a modification like (10.84) is allowed, everything boils down to determine whether the constant a is vanishing or has a finite value.

The constraint (10.84) can be implemented, formally, by means of a Lagrange multiplier, allowing a superpotential

$$W = A\left(\det M - B\tilde{B} - a\Lambda^{2N}\right) \tag{10.86}$$

where A is the Lagrange multiplier, whose equation of motion is by construction the constraint (10.84). The interesting thing is that one can use holomorphic decoupling to fix the constant a. Adding a mass term for the N-th flavor, $W = mM^{\rm N}_{\rm N}$, the low energy theory reduces to SQCD with $F = N - 1$. Imposing that after having integrated out the N-th flavor one obtains an effective superpotential which matches the ADS superpotential for $F = N - 1$, fixes $a = 1$, that is

$$\det M - B\tilde{B} = \Lambda^{2N} \ . \tag{10.87}$$

So the quantum constraint is there, after all. Actually, it is *necessary* for it to be there in order to be consistent with what we already know about the quantum properties of SQCD with $F < N$!

Several comments are in order at this point.

This is the first case where a moduli space of supersymmetric vacua persists at the quantum level. Still, the quantum moduli space is different from the classical one. The moduli space (10.83) is singular. It has a singular submanifold reflecting the fact that on this submanifold additional massless degrees of freedom arise. This is the submanifold where not only (10.83) is satisfied, but also $d(\det M - B\tilde{B}) = 0$, which makes the tangent space singular and therefore good local coordinates not being well-defined. This happens whenever baryon VEVs vanish, $B = \tilde{B} = 0$, and the meson matrix has rank $k \leq N - 2$. On this subspace a $SU(N - k)$ gauge group remains unbroken, and corresponding gluons (as well as some otherwise massive matter fields) remain massless. The quantum moduli space (10.87) is instead smooth. Basically, when $B = \tilde{B} = 0$ the rank of the meson matrix is not diminished since its determinant does not vanish, now: everywhere on the quantum moduli space the gauge group is fully broken.

Classically, the origin is part of the space of vacua. Hence, chiral symmetry can be unbroken. At the quantum level, instead, the origin is excised so in any allowed vacuum chiral symmetry is broken (like in QCD). Moreover, being the moduli space non-singular, means there are no massless degrees of freedom other than mesons and baryons. But the latter are indeed massless, since are moduli. Hence in SQCD with $N = F$ there is no mass gap (as for massless QCD). By supersymmetry, there are also massless composite fermions.

Obviously, the chiral symmetry breaking pattern is not unique. Different points on the moduli space display different patterns. At a generic point, where all gauge invariant operators get a VEV, all global symmetries are broken. But there are

submanifolds of enhanced global symmetry. For instance, along the *mesonic branch*, defined as

$$M_i^j = \Lambda^2 \delta_i^j \quad , \quad B = \tilde{B} = 0 \ , \tag{10.88}$$

we have that

$$SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R \longrightarrow SU(F)_D \times U(1)_B \times U(1)_R \ , \tag{10.89}$$

a chiral symmetry breaking pattern very much similar to QCD. Along the *baryonic branch*, which is defined as

$$M = 0 \quad , \quad B = -\tilde{B} = \Lambda^N \ , \tag{10.90}$$

we have instead

$$SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R \longrightarrow SU(F)_L \times SU(F)_R \times U(1)_R \ , \tag{10.91}$$

which is very different from QCD (the full non-abelian chiral symmetry is preserved).

Which phases does the theory enjoy? The point where all field VEVs vanish, i.e. the origin, is excised. Therefore, the gauge group is always broken and the theory is hence in a Higgs phase. Still, near the origin the theory can be better thought to be in a confined phase, since the effective theory is smooth in terms of mesons and baryons, and, moreover, we are in the strongly coupled region of field space, where an inherently perturbative Higgs description is not fully appropriate. In fact, there is no order parameter which can distinguish between the two phases; there is no phase transition between them (this is similar to the prototype example of one-family EW theory we discussed already). In this respect, notice that the Wilson loop is not a useful order parameter here since it follows the perimeter law, no matter where one sits on the moduli space: we do not have strict confinement but just charge screening, as in QCD, since we have (light) matter transforming in the fundamental representation of the gauge group, and therefore flux lines can (and do) break. The qualitative difference between classical and quantum moduli spaces, and their interpretation is depicted in Figure 10.11.

A non-trivial consistency check of this whole picture comes from computing 't Hooft anomalies in the UV and in the IR. Let us consider, for instance, the mesonic branch. The charges under the unbroken global symmetries, $SU(F)_D \times U(1)_B \times$

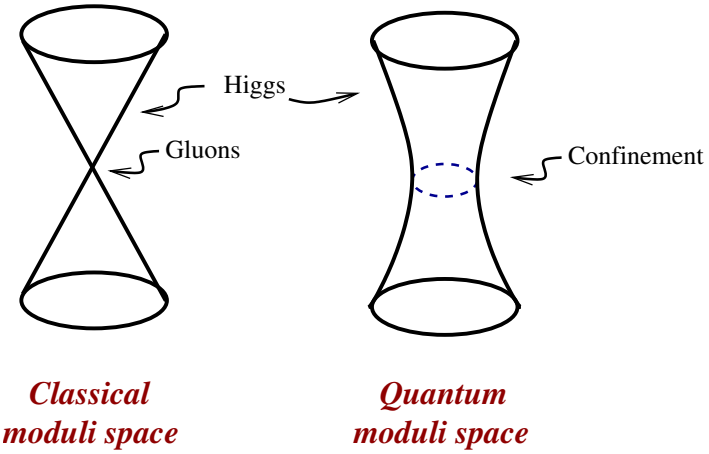


Figure 10.11: Classical picture (left): at the origin gauge symmetry is recovered, and chiral symmetry is not broken. Quantum picture (right): the (singular) origin has been replaced by a circle of theories where chiral symmetry is broken (rather than Higgs phase, this resembles more closely the physics of a confining vacuum).

$U(1)_R$ of the UV (fundamental) and IR (composite) degrees of freedom are as follows

$$\begin{array}{ccccc} & & SU(F)_D & U(1)_B & U(1)_R \\ & \psi_Q & F & 1 & -1 \\ & \psi_{\tilde{Q}} & \overline{F} & -1 & -1 \\ & \lambda & \bullet & 0 & 1 \\ \\ & \psi_M & \text{Adj} & 0 & -1 \\ & \psi_B & \bullet & F & -1 \\ & \psi_{\tilde{B}} & \bullet & -F & -1 \end{array}$$

where we have being using the constraint (10.87) to eliminate the fermionic partner of $\mathrm{Tr}\, M$, so that ψ_M transforms in the Adjoint of $SU(F)_D$. We can now compute diverse triangular anomalies and see whether computations done in terms of UV and IR degrees of freedom agree. We get

$$\begin{array}{ccc} & UV & IR \\ SU(F)_D^2\, U(1)_R & 2N\frac{1}{2}(-1) = -N & F(-1) = -F \\ U(1)_B^2\, U(1)_R & -2NF & -2F^2 \\ U(1)_R^3 & -2NF + N^2 - 1 & -(F^2-1) - 1 - 1 = -F^2 - 1 \end{array} \tag{10.92}$$

Since (crucially!) $F = N$ we see that 't Hooft anomaly matching holds. A similar computation can be done for the baryonic branch finding again perfect agreement

between the UV and IR 't Hooft anomalies. This rather non-trivial agreement ensures that our low energy effective description in terms of mesons and baryons, subject to the constraint (10.87), is most likely correct.

Let us move on and consider the next case, $F = N + 1$. The moduli space is again described by mesons and baryons. We have $N + 1$ baryons of type B and $N + 1$ baryons of type $\tilde{B}$ now

$$\begin{aligned} B_i &= \epsilon_{ij_1\ldots j_N}\, \epsilon^{a_1 a_2 \ldots a_N}\, Q^{j_1}_{a_1} Q^{j_2}_{a_2} \ldots Q^{j_N}_{a_N} \\ \tilde{B}^i &= \epsilon^{ij_1\ldots j_N}\, \epsilon_{a_1 a_2 \ldots a_N}\, \tilde{Q}^{a_1}_{j_1} \tilde{Q}^{a_2}_{j_2} \ldots \tilde{Q}^{a_N}_{j_N} \ . \end{aligned}$$

As we are going to show, differently from the previous case, the classical moduli space not only is unlifted, but is quantum exact, also. In other words, there are no quantum modifications to it.

This apparently surprising result can be proved using holomorphic decoupling. The rationale goes as follows. As proposed by Seiberg, this system can be described, formally, by the following superpotential

$$W_{\text{eff}} = \frac{a}{\Lambda^{2N-1}} \left(\det M - B_i\, M^i_{\ j}\, \tilde{B}^j \right) \ , \tag{10.93}$$

where $i = 1, 2, \ldots, N+1$ is a flavor index, $2N-1$ is the one-loop β-function coefficient and a, as usual, is for now an undetermined coefficient. The above superpotential has all correct symmetry properties, including the R-charge, which is indeed equal to 2. Notice, though, that since the rank of the meson matrix $k \leq N$, then $\det M = 0$, classically. So the above equation should be really thought of as a quantum equation, valid off-shell, so to say.

Let us now add a mass m to the F-th flavor. This gives

$$W_{\text{eff}} = \frac{a}{\Lambda^{2N-1}} \left(\det M - B_i\, M^i_{\ j}\, \tilde{B}^j \right) - m M^{\text{N+1}}_{\ \ \text{N+1}} \ . \tag{10.94}$$

The F-flatness conditions for $M^{\text{N+1}}_{\ \ i}, M^i_{\ \ \text{N+1}}, B_i$ and $\tilde{B}^i$ for $i < N + 1$ reduce the meson matrix and the baryons to

$$M = \begin{pmatrix} \hat{M}_{ij} & 0 \\ 0 & t \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0_i \\ \hat{B} \end{pmatrix} \quad , \quad \tilde{B} = \begin{pmatrix} 0^i \\ \hat{\tilde{B}} \end{pmatrix} \tag{10.95}$$

where now $i, j = 1, \ldots, N$, and $t = M_{\text{N+1,N+1}}$. The F-flatness condition for t reads

$$\frac{a}{\Lambda^{2N-1}} \left(\det \hat{M} - \hat{B}\hat{\tilde{B}} \right) - m = 0 \tag{10.96}$$

which implies

$$\det \hat{M} - \hat{B}\tilde{\hat{B}} = \frac{1}{a}\, m\, \Lambda^{2N-1} = \frac{1}{a}\, \Lambda_L^{2N} \ , \tag{10.97}$$

where in the last step we have used the relation (10.52). This shows that the ansatz (10.93) is correct, since upon holomorphic decoupling we get exactly the quantum constraint of $F = N$ SQCD (and a gets fixed to one). From eq. (10.93), by differentiating with respect to M, B_i and $\tilde{B}^i$ we get the moduli space equations (i.e. the classical, still quantum exact, constraints between baryons and mesons)

$$\begin{cases} M \cdot \tilde{B} = B \cdot M = 0 \\ \det M \cdot (M^{-1})_i^j - B_i \tilde{B}^j = 0 \end{cases} \tag{10.98}$$

where $\det M \cdot (M^{-1})_i^j \equiv \text{minor}\,\{M\}_i^j = (-1)^{i+j} \times$ det of the matrix obtained from M by omitting the i-th row and the j-th column (recall that above equations are on-shell, and on-shell $\det M$ itself vanishes).

As a non-trivial check of this whole picture one can verify, choosing any preferred point in the space of vacua, that 't Hooft anomalies match (and hence that our effective description holds).

Now that we know eq. (10.93) is correct, let us try to understand what does it tell us about the vacuum structure of SQCD with $F = N + 1$. First, unlike $F = N$ SQCD, the origin of field space, $M = B = \tilde{B} = 0$, is now part of the moduli space. In such vacuum chiral symmetry is unbroken. This is an instance of a theory displaying confinement (actually charge screening) without chiral symmetry breaking. Theories with such a property, like $F = N+1$ SQCD at the origin of field space, are known as *s-confining*.

Classically, the singularities at the origin are interpreted as extra massless gluons (and matter fields), since the theory gets unhiggsed for vanishing values of matter field VEVs. At the quantum level, the physical interpretation is different, since because the theory is UV-free, the region around the origin is the more quantum one. Singularities are more naturally associated with additional massless mesons and baryons which pop-up since eqs. (10.98) are trivially realized at the origin, and do not provide any actual constraint between meson and baryon components. In other words, at the origin the number of mesonic and baryonic massless degrees of freedom is larger than the dimension of the moduli space.

Note, finally, that as $F = N$ SQCD, also this theory exhibits *complementarity*, in the sense that one can move smoothly from a confining phase (near the origin)

to a Higgs phase (at large field VEVs) without any order parameter being able to distinguish between them.

One could in principle try to go further, and apply the same logic to $F = N + 2$ (and on). On general ground one would expect M, B^{ij}, $\tilde{B}_{ij}$ (baryons have now two free flavor indices) to be the dynamical degrees of freedom in the IR, and could then try to construct an effective (off-shell) superpotential of the kind of (10.93). This, however, does not work. Looking at the charges of the various gauge invariant operators and dynamical scale Λ one can easily see that an effective superpotential with R-charge equal to 2, correct physical dimensions and symmetries, cannot be constructed. Indeed, the only $SU(F)_L \times SU(F)_R$ invariant superpotential one could construct should be the obvious generalization of (10.93), that is

$$W_{\text{eff}} \sim \det M - B_{il}\, M^i_j M^l_m\, \tilde{B}^{jm} \ , \tag{10.99}$$

which does not have $R = 2$ (things get worse the larger the number of flavors). Even 't Hooft anomaly matching condition can be proven not to work. For instance, choosing (for simplicity) the origin of field space where meson and baryons are unconstrained, one can see that 't Hooft anomalies do not match.

In fact, things turn out to be rather different. As we will show, the correct degrees of freedom to describe the dynamics around SQCD vacua for $F = N + 2$ are those of an IR-free theory (!) described by $SU(2)$ SYM coupled to F chiral superfields q transforming in the fundamental of $SU(2)$, F chiral superfields $\tilde{q}$ transforming in the anti-fundamental and F^2 singlet chiral superfields Φ, plus a cubic tree level superpotential coupling $q, \tilde{q}$ and Φ. What's that?

Two pieces of information are needed in order to understand this apparently weird result and, more generally, to understand what is going on for $F \geq N + 2$. Both are due to Seiberg. In the following we will review them in turn.

10.4.5 Conformal window

A first proposal is that SQCD in the range $\frac{3}{2}N < F < 3N$ flows to an interacting IR fixed point (meaning it does not confine). In other words, even if the theory is UV-free and hence the gauge coupling g increases through the IR, at low energy g reaches a constant RG-fixed value. Let us try to see how such claim comes about. The SQCD β-function for the physical gauge coupling (which hence takes into account

wave-function renormalization effects) is

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N - F\left[1-\gamma(g^2)\right]}{1 - Ng^2/8\pi^2} \ , \tag{10.100}$$

where γ is the anomalous dimension of matter fields and can be computed in perturbation theory to be

$$\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N^2-1}{N} + \mathcal{O}(g^4) \ . \tag{10.101}$$

Expanding formula (10.100) in powers of g^2 we get

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[3N - F + \left(3N^2 - 2FN + \frac{F}{N}\right)\frac{g^2}{8\pi^2} + \mathcal{O}(g^4)\right] \ . \tag{10.102}$$

From the above expression it is clear that there can exist values of F and N such that the one-loop contribution is negative but the two-loops contribution is positive. This suggests that in principle there could be a non-trivial fixed point, a value of the gauge coupling $g = g_*$, for which $\beta(g_*) = 0$.

Let us consider F slightly smaller than $3N$. Defining

$$\epsilon = 3 - \frac{F}{N} << 1 \tag{10.103}$$

we can re-write the β-function as

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\epsilon N - \left[3(N^2-1) + \mathcal{O}(\epsilon)\right]\frac{g^2}{8\pi^2} + \mathcal{O}(g^4)\right] \ . \tag{10.104}$$

The first term inside the parenthesis is positive while the second is negative and hence we see we have a solution $\beta(g) = 0$ at

$$g_*^2 = \frac{8\pi^2}{3}\frac{N}{N^2-1}\,\epsilon \ , \tag{10.105}$$

up to $\mathcal{O}(\epsilon^2)$ corrections. This is called Banks-Zaks (BZ) fixed point. Seiberg argued that an IR fixed point like the one above exists not only for F so near to $3N$ but actually for any F in the range $\frac{3}{2}N < F < 3N$, the so-called *conformal window*. According to this proposal, the IR dynamics of SQCD in the conformal window is described by an interacting superconformal theory: quarks and gluons are *not* confined but appear as interacting massless particles, the Coulomb-like potential being

$$V(r) \sim \frac{g_*^2}{r} \ . \tag{10.106}$$

Hence, according to this proposal, SQCD in the conformal window enjoys a non-abelian Coulomb phase.

Let us try to understand why the conformal window is bounded from below and from above. The possibility of making exact computations in a SCFT shows that for $F < \frac{3}{2}N$ the theory should be in a different phase. In a SCFT the dimension of a field satisfies the following relation

$$\Delta \geq \frac{3}{2}|R| \ , \tag{10.107}$$

where R is the field R-charge (recall that in a SCFT the generator of the R-symmetry enters the algebra, and hence an R-symmetry is always present). The equality holds for chiral (or anti-chiral) operators. This implies that

$$\Delta(M) = \frac{3}{2}R(M) = \frac{3}{2}R(Q\tilde{Q}) = 3\frac{F-N}{F} \equiv 2 + \gamma_* \ , \tag{10.108}$$

given that M is a chiral operator. This means that the anomalous dimension of the meson matrix at the IR fixed point is $\gamma_* = 1 - 3N/F$.

The dimension of a scalar field must satisfy

$$\Delta \geq 1 \ . \tag{10.109}$$

Indeed, when $\Delta < 1$ the operator, which is in a unitary representation of the superconformal algebra, would include a negative norm state which cannot exist in a unitary theory. This implies that $F = \frac{3}{2}N$ is a lower bound since there $\Delta(M) = 1$ and lower values of F make no sense (recall that the lowest component of the superfield M is a scalar field): for $F < \frac{3}{2}N$ the theory should be in a different phase. A clue to what such phase could be is that at $F = \frac{3}{2}N$ the field M becomes free. Indeed, for $F = \frac{3}{2}N$ we get that $\Delta(M) = 1$ which is possible only for free, non-interacting scalar fields. Perhaps it is the whole theory of mesons and baryons which becomes free, somehow. We will make this intuition more precise later.

As for the upper bound, let us notice that for $F \geq 3N$ SQCD is not asymptotically free anymore (the β-function changes sign). The spectrum at large distance consists of elementary quarks and gluons interacting through a potential

$$V \sim \frac{g^2}{r} \quad \text{with} \quad g^2 \sim \frac{1}{\log(r\Lambda)} \ , \tag{10.110}$$

which implies that SQCD is in a non-abelian free phase. It is interesting to notice that for $F = 3N$ the anomalous dimension of M is actually zero, consistent with

the fact that from that value on, the IR dimension of gauge invariant operators is not renormalized since the theory becomes IR-free. A summary of the IR behavior of SQCD for $F > \frac{3}{2}N$ is reported in Figure 10.12.

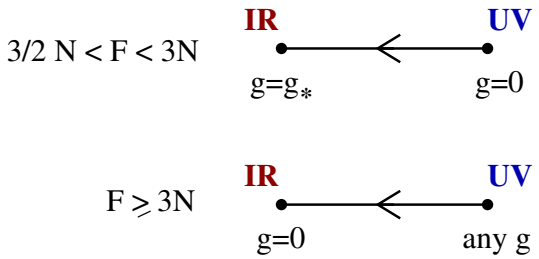


Figure 10.12: The IR behavior of SQCD in the window $\frac{3}{2}N < F < 3N$, where the theory flows to an IR fixed point with $g = g_*$, and for $F \geq 3N$, where $g_* = 0$ and the theory is in a non-abelian IR-free phase.

10.4.6 Electric-magnetic duality (aka Seiberg duality)

The second proposal put forward by Seiberg regards the existence of a electro-magnetic-like duality. The IR physics of SQCD for $F > N + 1$ has an equivalent description in terms of another supersymmetric gauge theory, known as the *magnetic dual* theory. Such dual theory is IR-free for $N + 1 < F \leq \frac{3}{2}N$, and UV-free for $F > \frac{3}{2}N$. In the conformal window defined before, that is for $\frac{3}{2}N < F < 3N$, it has a IR fixed point (the same as the original SQCD theory!), while for $F \geq 3N$, where SQCD becomes IR-free, it enters into a confining phase. So these two theories are very different: as such, the equivalence is an IR equivalence. SQCD, sometime called electric theory in this context, and its magnetic dual are not equivalent in the UV neither along the RG-flow. They just provide two equivalent ways to describe the dynamics around the space of vacua (in fact, perturbing SQCD by suitable operators, *e.g.*, by quartic operators, one can sometime promote this IR duality to a full duality, valid along the whole RG; however, discussing such instances is beyond our present scope).

In order to understand this claim (and its implications), and define such dual theory more precisely, we first need to do a step back. In trying to extend to higher values of F the reasoning about SQCD with $F = N + 1$, one would like to consider the following gauge invariant operators

$$M^i_{\ j} \quad , \quad B_{i_1 i_2 \ldots i_{F-N}} \quad , \quad \tilde{B}^{i_1 i_2 \ldots i_{F-N}} \ . \tag{10.111}$$

The baryons have $\tilde{N} = F - N$ free indices so one might like to view them as bound states of $\tilde{N}$ components, some new quark fields q and $\tilde{q}$ of some SYM theory with gauge group $SU(\tilde{N}) = SU(F-N)$ for which q and $\tilde{q}$ transform in the $\tilde{N}$ and $\overline{\tilde{N}}$ representations, respectively. Then the SQCD baryons would have a dual description as

$$B_{i_1 i_2 \ldots i_{\tilde{N}}} \sim \epsilon_{a_1 a_2 \ldots a_{\tilde{N}}} q_{i_1}^{a_1} q_{i_2}^{a_2} \ldots q_{i_{\tilde{N}}}^{a_{\tilde{N}}} \tag{10.112}$$

and similarly for $\tilde{B}$. Recall that in terms of the original matter fields Q and $\tilde{Q}$, the baryons are composite fields made out of N components.

Seiberg made this naive idea concrete (and physical), putting forward the following proposal: SQCD with gauge group $SU(N)$ and $F > N+1$ flavors, can be equivalently described, in the IR, by a SQCD theory with gauge group $SU(F-N)$ and F flavors plus an additional chiral superfield Φ which is a gauge singlet and which transforms in the fundamental representation of $SU(F)_L$ and in the anti-fundamental representation of $SU(F)_R$, and which interacts with q and $\tilde{q}$ via a cubic superpotential

$$W = q_i \, \Phi^i_j \, \tilde{q}^j \ . \tag{10.113}$$

As bizarre this proposal may look like, let us try to understand it better. Let us first consider the Seiberg dual theory (which from now on we dub mSQCD, where 'm' stands for magnetic) without superpotential term, and let us focus on the SQCD conformal window, $\frac{3}{2}N < F < 3N$, first. For $W = 0$ the field Φ is completely decoupled and mSQCD is just SQCD with gauge group $SU(F-N)$ and F flavors. Interestingly, the SQCD conformal window is a conformal window also for mSQCD! Hence mSQCD (without the singlet Φ) flows to an IR fixed point for $\frac{3}{2}N < F < 3N$. At such fixed point the superpotential coupling, that we now switch-on, is relevant, since $\Delta(W) = \Delta(\Phi) + \Delta(q) + \Delta(\tilde{q}) = 1 + \frac{3}{2}N/F + \frac{3}{2}N/F < 3$. The claim is that the perturbation (10.113) drives the theory to some new fixed point which is actually the same fixed point of SQCD.

How does mSQCD look like for $F \leq \frac{3}{2}N$? The one-loop β-function coefficient of mSQCD is $b_1 = 2F - 3N$. Hence, for $F = \frac{3}{2}N$ the β-function vanishes and for lower values of F it changes its sign and mSQCD becomes IR-free. Hence, the bound $F = \frac{3}{2}N$ has the same role that the bound $F = 3N$ has for SQCD (not surprisingly, one can apply the BZ-fixed point argument to mSQCD for F slightly larger than $\frac{3}{2}N$ and find the existence of a perturbative fixed point). This explains why, if Seiberg duality is correct, the IR dynamics of SQCD in the range $N+1 < F \leq \frac{3}{2}N$ differs from the behavior in the conformal window - something we had some indications

of, when studying the lower bound in F of the SQCD conformal window. Indeed, we now can make our former intuition precise: using a clever set of variables (i.e. the magnetic dual variables), one concludes that for $N+1 < F \leq \frac{3}{2}N$ SQCD IR dynamics is described by a theory of freely interacting (combinations of) meson and baryon fields. These can be described in terms of free dual quarks interacting with a Coulomb-like potential

$$V_{\rm m} \sim \frac{g_{\rm m}^2}{r} \quad \text{with} \quad g_{\rm m}^2 \sim \frac{1}{\log(r\Lambda)} \tag{10.114}$$

This phase of SQCD is dubbed *free magnetic phase*, a theory of freely interacting (dual) quarks. The fact that the IR dynamics of SQCD for $N+1 < F \leq \frac{3}{2}N$, where the theory is confining, can be described this way is a rather powerful statement: since mSQCD is IR-free, in terms of magnetic dual variables the Kähler potential is canonical (up to subleading $1/\Lambda^2$ corrections), meaning that we know the full effective IR Lagrangian of SQCD for $N+1 < F \leq \frac{3}{2}N$, at low enough energies!

As for the conformal window, which variables to use depends on F. The larger F, the nearer to IR-freedom SQCD is, and the more UV-free mSQCD is. In other words, the conformal window IR-fixed point is at smaller and smaller value of the electric gauge coupling the nearer F is to $3N$, and eventually becomes 0 for $F = 3N$. For mSQCD things are reversed. The IR-fixed point arises at weaker coupling the nearer F is to $\frac{3}{2}N$, and for $F = \frac{3}{2}N$ we have that $g_*^m = 0$. Therefore, the magnetic description is the simplest to describe SQCD non-abelian Coulomb phase for F near to $\frac{3}{2}N$; the electric description is instead the most appropriate one when F is near to $3N$.

For $F \geq 3N$ the magnetic theory does not reach anymore an IR interacting fixed point. The value $F = 3N$ plays for mSQCD the same role the value $F = \frac{3}{2}N$ plays for SQCD. Indeed, the mSQCD meson matrix, $U = q\tilde{q}$ has $\Delta = 1$ for $F = 3N$, and becomes a free field, while for larger values of F it would get dimension lower than one, which is not acceptable. For $F \geq 3N$ the theory should enter in a new phase. This is something we know already: in this region we are in the SQCD IR-free phase.

In the remainder of this section, we will provide several consistency checks for the validity of this proposed duality.

First, we note that two basic necessary requirements for its validity are met: the two theories have the same global symmetry group as well as the same number of IR degrees of freedom. In order to see this, let us first make the duality map precise. The mapping between chiral operators of SQCD and mSQCD (at the IR fixed point)

is

$$M \longleftrightarrow \Phi \quad : \quad \Phi^i{}_j = \frac{1}{\mu} M^i{}_j \tag{10.115}$$

$$B \longleftrightarrow b \quad : \quad b^{j_1 j_2 \ldots j_N} = \epsilon^{i_1 i_2 \ldots i_{F-N} j_1 j_2 \ldots j_N} \, B_{i_1 i_2 \ldots i_{F-N}}$$

and similarly for $\tilde{b}$ and $\tilde{B}$. The scale μ relating SQCD mesons with the mSQCD gauge singlet Φ appears for the following reason. In SQCD mesons are composite fields and their dimension in the UV, where SQCD is free, is $\Delta = 2$. On the other hand, Φ is an elementary field in mSQCD and its dimension in the UV is $\Delta = 1$. Hence the scale μ needs to be introduced to match Φ to M in the UV. Clearly, upon RG-flow both fields acquire an anomalous dimension and should flow to one and the same operator in the IR, if the duality is correct. Applying formula (10.107), which for chiral operators is an equality, one easily sees that this is indeed what should happen, since $R(M) = R(\Phi)$.

From the map (10.115) it easily follows that the magnetic theory has a global symmetry group which is nothing but the one of SQCD, $G_{\rm F} = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$, with the following charges for the elementary fields

	$SU(F)_L$	$SU(F)_R$	$U(1)_B$	$U(1)_R$
q_i^a	$\overline{F}$	$\bullet$	$\frac{N}{F-N}$	$\frac{N}{F}$
$\tilde{q}_b^j$	$\bullet$	F	$-\frac{N}{F-N}$	$\frac{N}{F}$
Φ	F	$\overline{F}$	0	$2\frac{F-N}{F}$
$\tilde{\lambda}$	$\bullet$	$\bullet$	0	1

while the superpotential (10.113) has $R = 2$.

We can now use global symmetries to see that SQCD and mSQCD have the same number of IR degrees of freedom. Basically, there is a one-to-one map between gauge invariant operators, and these operators have the same global symmetries (which counts physically distinct degrees of freedom). Indeed, the meson matrix M enjoys the same symmetries as the mSQCD singlet Φ, and the SQCD baryons $B, \tilde{B}$ the same as the mSQCD baryons $b, \tilde{b}$ (the latter being gauge invariant operators constructed in terms of $F - N$ dual quarks $q, \tilde{q}$). One might feel uncomfortable since the mesons of the magnetic dual theory, $U_i^j = q_i \tilde{q}^j$ seem not to match with anything in the electric theory. This is where the superpotential (10.113) comes into play. Recall the supposed equivalence between SQCD and mSQCD is just a IR equivalence. The F-equations for Φ fix the dual meson to vanish on the moduli space: $F_\Phi = q\tilde{q} = U = 0$. Hence, in the IR the two theories do have the same number of degrees of freedom!

In what follows, we are going to present further non-trivial checks for the validity of Seiberg's proposal.

- The first very non-trivial check comes from 't Hooft anomaly matching. The computation of 't Hooft anomalies gives

$$\begin{array}{cc} & SQCD \\ SU(F)_L^2\ U(1)_B & \frac{1}{2}N(+1) = \frac{1}{2}N \\ U(1)_B^2\ U(1)_R & 2NF(+1)(-\frac{N}{F}) = -2N^2 \\ U(1)_R^3 & \left(-\frac{N}{F}\right)^3 2NF + N^2 - 1 = -2\frac{N^4}{F^2} + N^2 - 1 \end{array}$$

$$\begin{array}{cc} & mSQCD \\ SU(F)_L^2\ U(1)_B & \frac{1}{2}(F-N)\frac{N}{F-N} = \frac{1}{2}N \\ U(1)_B^2\ U(1)_R & 2(F-N)F\left(\frac{N}{F-N}\right)^2\left(\frac{N-F}{F}\right) = -2N^2 \\ U(1)_R^3 & \left(\frac{N-F}{F}\right)^3 2(F-N)F + F^2\left(\frac{F-2N}{F}\right)^3 + (F-N)^2 - 1 = -2\frac{N^4}{F^2} + N^2 - 1 \end{array}$$

  which shows there is indeed matching between SQCD and its IR-equivalent mSQCD description.

  Note that for the matching to work it turns out that the presence of dual gauginos is crucial (as well as that of the magnetic superpotential term). This explicitly shows that the description of SQCD baryons in terms of some sort of dual quarks is not just a mere group representation theory accident. There is a truly dynamical dual gauge group, under which dual quarks are charged, and dual vector superfields (which include dual gauginos) which interact with them.

- The duality relation *is* a duality, which means that acting twice with the duality map one recovers the original theory (as far as IR physics!). Let us start from SQCD with N colors and F flavors and act with the duality map twice

$$\begin{array}{rcl} SQCD & : & SU(N)\ ,\ F\ ,\ W = 0 \\ & & \downarrow \text{duality} \\ mSQCD & : & SU(F-N)\ ,\ F\ ,\ W = \frac{1}{\mu}q_i M^i{}_j \tilde{q}^j = q_i \Phi^i{}_j \tilde{q}^j \\ & & \downarrow \text{duality} \\ mmSQCD & : & SU(N)\ ,\ F\ ,\ W = \frac{1}{\mu}q_i M^i{}_j \tilde{q}^j + \frac{1}{\tilde{\mu}} d^i U_i{}^j \tilde{d}_j = q_i \Phi^i{}_j \tilde{q}^j + d^i \Psi_i{}^j \tilde{d}_j \end{array}$$

where ${U_i}^j = q_i \tilde{q}^j$ is the meson matrix of mSQCD, while ${\Psi_i}^j$ is the gauge singlet chiral superfield dual to U and belonging to the magnetic dual of mSQCD. Choosing $\tilde{\mu} = -\mu$, we can rewrite the superpotential of mmSQCD as

$$W = \frac{1}{\mu} \mathrm{Tr} \left[UM - dU\tilde{d} \right] \ . \tag{10.116}$$

The fields U and M are hence massive and can be integrated out (recall we claim the IR equivalence of Seiberg-dual theories, not the equivalence at all scales). This implies

$$\frac{\partial W}{\partial U} = 0 \to {M^i}_j = d^i \tilde{d}_j \quad , \quad \frac{\partial W}{\partial M} = 0 \to U = 0 \tag{10.117}$$

showing that the dual of the dual quarks are nothing but the original quark superfields Q and $\tilde{Q}$, and that $U = 0$ (hence $W = 0$) in the IR . Summarizing, after integrating out heavy fields, we are left with SQCD with gauge group $SU(N)$, F flavors and no superpotential, exactly the theory we have started with! In passing, let us note that in order to make the duality working we have to set $\tilde{\mu} = -\mu$, a mass scale which is not fixed by the duality itself.

- The duality is preserved under mass perturbations, namely upon holomorphic decoupling. Let us again consider SQCD with gauge group $SU(N)$ and F flavors and let us add a mass term to the F-th flavor, $W = m M^{\mathrm{F}}_{\ \mathrm{F}}$. This corresponds to $SU(N)$ SQCD with $F-1$ massless flavors and one massive one. In the dual magnetic theory this gives a superpotential

$$W = \frac{1}{\mu} q_i {M^i}_j \tilde{q}^j + m\, M^{\mathrm{F}}_{\ \mathrm{F}} \ . \tag{10.118}$$

The F-flatness conditions for $M^{\mathrm{F}}_{\ \mathrm{F}}$ and q_{F} and $\tilde{q}^{\mathrm{F}}$ are

$$q^a_{\mathrm{F}}\, \tilde{q}^{\mathrm{F}}_a + \mu\, m = 0 \quad , \quad (M \cdot \tilde{q}_a)^{\mathrm{F}} = (q^a \cdot M)_{\mathrm{F}} = 0 \ , \tag{10.119}$$

where a is a $SU(F-N)$ gauge index. The first equation induces a VEV for the dual quarks with flavor index F, which breaks the gauge group down to $SU(F-N-1)$. The other two equations imply that the F-th row and column of the SQCD meson matrix M vanish. We hence end-up with $SU(F-N-1)$ SQCD with $F-1$ flavors, a gauge singlet M which is a $(F-1) \times (F-1)$ matrix, while the superpotential (10.118) reduces to eq. (10.113) where now i, j run from 1 to $F-1$ only. This is the correct Seiberg dual mSQCD theory at low energy.

This analysis shows that a mass term in the electric theory corresponds to higgsing in the magnetic dual theory, according to the table below.

SQCD		**mSQCD**
$SU(N), F$	$\underset{DUAL}{\longleftrightarrow}$	$SU(F-N), F$
$\downarrow$ mass		$\downarrow$ higgsing
$SU(N), F-1$	$\underset{DUAL}{\longleftrightarrow}$	$SU(F-N-1), F-1$

The converse is also true (though slightly harder to prove): a mass term in mSQCD corresponds to higgsing in SQCD.

- Let us use holomorphic decoupling to go from the last value of F where we have the duality, $F = N+2$, to $F = N+1$. If Seiberg duality is correct, we should recover the description of SQCD with $F = N+1$ flavors we discussed previously. Let us consider mSQCD when $F = N+2$. The magnetic gauge group is $SU(2)$. Upon holomorphic decoupling, an analysis identical to the one we did above produces a cubic superpotential at low energy as

$$W \sim q_i M^i_j \tilde{q}^j \qquad i,j = 1,\ldots,N+1 \tag{10.120}$$

where q_i are the baryons B^i of SQCD with $F = N+1$ and $\tilde{q}_j$ the baryons $\tilde{B}_j$. At the same time, the VEVs for $q_{\rm N+2}$ and $\tilde{q}^{\rm N+2}$ break the $SU(2)$ gauge symmetry completely. From mSQCD view point this is a situation similar to SQCD with $F = N-1$ where the full breaking of gauge symmetry group allowed an exact instanton calculation providing the $\sim \det M$ contribution to the effective superpotential. The same happens here and the final answer one gets for the low energy effective superpotential is

$$W_{\rm eff} \sim \left(q_i M^i_j \tilde{q}^j - \det M\right) \ , \tag{10.121}$$

which is precisely the effective superpotential of SQCD with $F = N+1$! This also shows that by holomorphic decoupling we can actually connect the description of the IR dynamics of SQCD for *any* number of flavors, from $F = 0$ to any larger values of F, at fixed N.

Let us finally notice, in passing, that even for $F = N+1$ we can sort of speak of a magnetic dual theory. Just it is trivial, since there is no magnetic dual gauge group.

- There is yet an important relation between the three a priori different mass scales entering the duality: the electric dynamical scale $\Lambda_{\rm el}$, the magnetic scale $\Lambda_{\rm m}$, and the matching scale μ. This reads

$$\Lambda_{\rm el}^{3N-F}\Lambda_{\rm m}^{3(F-N)-F} = (-1)^{F-N}\mu^F \ . \tag{10.122}$$

That this relation is there, can be seen in different ways. First, one can check that the relation is duality invariant, as it should. Indeed, applying the duality map (recall that $\tilde{\mu} = -\mu$, while $\Lambda_{\rm el}$ and $\Lambda_{\rm m}$ get interchanged by the duality) one gets

$$\Lambda_{\rm m}^{3(F-N)-F}\Lambda_{\rm el}^{3N-F} = (-1)^N\tilde{\mu}^F = (-1)^{N-F}\mu^F \ , \tag{10.123}$$

which is identical to (10.122). One can also verify the consistency of the relation (10.122) upon higgsing and/or holomorphic decoupling. The check is left to the reader.

Eq. (10.122) shows that as the electric theory becomes stronger (i.e. $\Lambda_{\rm el}$ increases), the magnetic theory becomes weaker (i.e. $\Lambda_{\rm m}$ decreases). By using the relation between dynamical scales and gauge couplings, this can be translated into a relation between gauge coupling constants, and gives an inverse relation between them

$$g_{\rm el}^2 \sim g_{\rm m}^{-2} \ , \tag{10.124}$$

showing that large values of the electric gauge coupling $g_{\rm el}$ correspond to small values of the magnetic one, and viceversa. This is why Seiberg duality is an electric-magnetic duality.

Depending on where in the (F, N) space one sits, the meaning of the dynamical scales changes. In the conformal window both SQCD and mSQCD are UV-free. Both theories have a non-trivial RG-flow and, upon non-perturbative effects, driven by $\Lambda_{\rm el}$ and $\Lambda_{\rm m}$, reach an IR fixed point (which is one and the same, in fact). In the free-magnetic phase, mSQCD is IR-free and SQCD is UV-free. Therefore, in this regime $\Lambda_{\rm m}$ should be better thought of as a UV-scale for the magnetic theory, which is an effective theory. In this regime SQCD can be thought of as the (or better, a possible) UV-completion of mSQCD (the electric free phase can be thought of in a similar way, with the role of SQCD and mSQCD reversed). Within this interpretation it is natural to tune the free parameter μ to make the two theories have one single non-perturbative scale, the scale at which non-perturbative SQCD effects come into play and

the scale below which the magnetic effective description takes over. From the relation (10.122) one sees that this is obtained by equating, up to an overall phase, the matching scale μ with $\Lambda_{\rm el}$ and $\Lambda_{\rm m}$

$$\mu = \Lambda_{\rm el} = \Lambda_{\rm m} \ . \tag{10.125}$$

It is left as an exercise to check that using the above relation for $F = N + 2$ and adding a mass term for the F-th flavor, upon holomorphic decoupling one gets the expression (10.121) including the correct power of Λ, that is

$$W_{\rm eff} = \frac{1}{\Lambda^{2N-1}} \left(q_i M^i_j \tilde{q}^j - {\rm det} M \right) \ . \tag{10.126}$$

Figure 10.13 contains a qualitative description of the three different regimes we have just discussed.

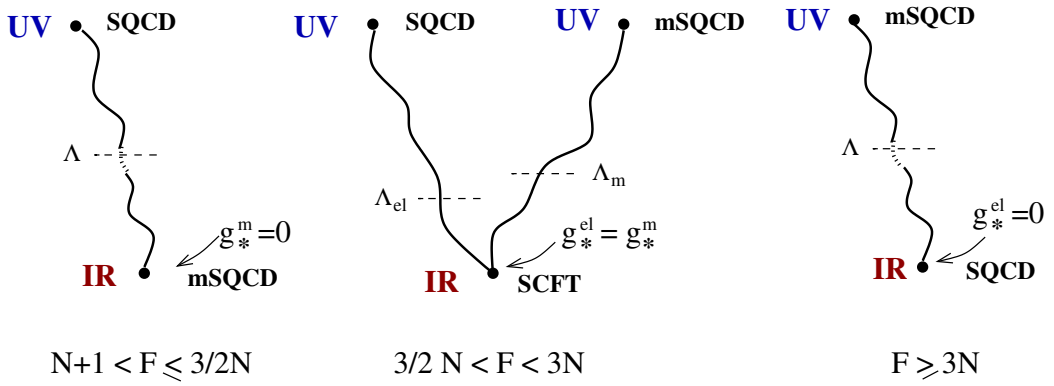


Figure 10.13: The three qualitative different phases of SQCD with $F > N+1$. In the magnetic-free and electric-free phases we have chosen (for convenience) the arbitrary matching scale μ in such a way to make $\Lambda_{\rm el}$ and $\Lambda_{\rm m}$ being identified, eq. (10.125).

10.5 The phase diagram of N=1 SQCD

After this long tour on quantum properties of SQCD, it is time to wrap-up and summarize its phase diagram.

For $F = 0$ SQCD (pure SYM in this case) enjoys strict confinement, displays N isolated supersymmetric vacua and a mass gap. For $0 < F < N$ the theory doesn't exist by its own. The classical moduli space is completely lifted and a runaway potential, with no absolute minima at finite distance in field space, is generated.

For $F = N, N+1$ a moduli space persists at the quantum level and SQCD enjoys confinement with charge screening (the asymptotic states are gauge singlets but flux lines can break) and no mass gap. Asymptotic states are mesons and baryons. The theory displays complementarity, as any theory where there are scalars transforming in the fundamental representation of the gauge group: there is no invariant distinction between Higgs phase, which is the more appropriate description for large field VEVs, and confinement phase, which takes over near the origin of field space. The potential between static test charges goes to a constant asymptotically since in the Higgs phase gauge bosons are massive and there are no long-range forces. As already observed, this holds also in the confining description, since we have charge screening and no area-law for Wilson loops.

For $N+2 \leq F \leq \frac{3}{2}N$ we are still in a confinement phase, but the theory is in the so-called free magnetic phase and can be described at large enough distance in terms of freely interacting dual quarks and gluons. What is amusing here is that while asymptotic massless states are composite of elementary electric degrees of freedom (i.e. mesons and baryons), they are magnetically charged with respect to a gauge group whose dynamics is not visible in the electric description and which is generated, non-perturbatively, by the theory itself.

For $F > \frac{3}{2}N$ SQCD does not confine anymore, not even in the weak sense: asymptotic states are quarks and gluons (and their superpartners). The potential between asymptotic states, though, differs if $F \geq 3N$ or $F < 3N$. In the former case the theory is IR-free and it is described by freely interacting particles. Hence the potential vanishes, at large enough distance. For $\frac{3}{2}N < F < 3N$, instead, the theory (which is still UV-free) is in a non-abelian Coulomb phase. Charged particles are not confined but actually belong to a SCFT, and interact by a $1/r$ potential with coupling $g = g_*$.

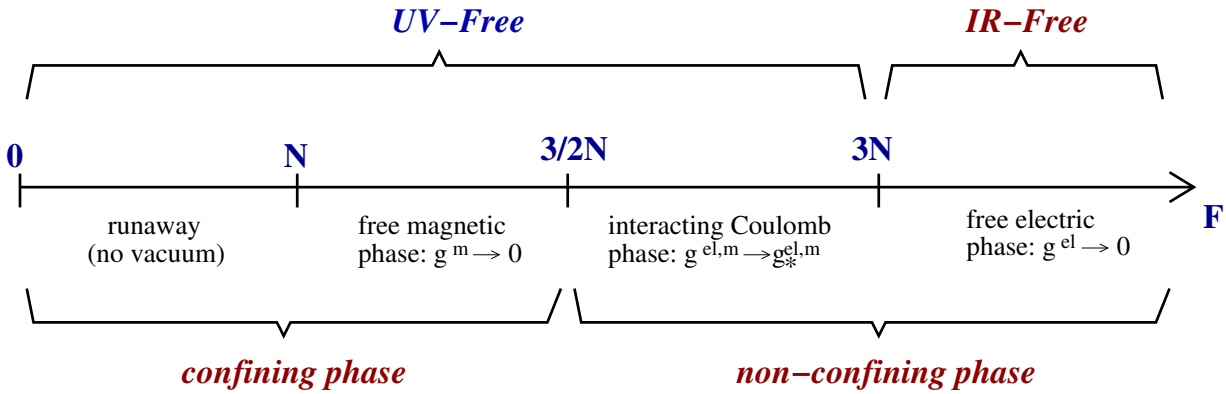
A diagram summarizing the gross features of the quantum dynamics of SQCD is reported below.

10.6 Exercises

1. Consider SQCD with $F = N$ with superpotential

$$W = A\left(\det M - B\tilde{B} - a\Lambda^{2N}\right) + mQ^N\tilde{Q}_N \tag{10.127}$$

By integrating out the massive flavor, show that one recovers the ADS superpotential for $F = N-1$ SQCD if and only if $a = 1$.



2. Check 't Hooft anomaly matching for SQCD with $F = N$ along the baryonic branch, $M = 0, B = -\tilde{B} = \Lambda^N$.

3. Check 't Hooft anomaly matching for SQCD with $F = N + 1$ at the origin of the moduli space.

4. Consider mSQCD for $F = 3(F - N) - \epsilon(F - N)$ with $\epsilon << 1$, and find the BZ perturbative fixed point (i.e. the value of g such that the β-function vanishes).

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11 Dynamical supersymmetry breaking

After this long detour on quantum properties of supersymmetric gauge theories, we can now go back to supersymmetry breaking, and finally discuss models where supersymmetry is broken by strong coupling effects, aka *dynamical supersymmetry breaking* (DSB). As we already emphasized, these models can in principle be used as consistent (and natural) hidden sectors within gravity or gauge mediation scenarios (or any of their possible variants).

We will first focus on models where supersymmetry is broken dynamically in *stable* vacua, that is at absolute minima of the potential. The rough general picture for models of this kind (with exceptions, as we will see) is as follows:

- The supersymmetric theory at hand is a gauge theory. This is because gauge degrees of freedom are the only ones having some chance of generating non-perturbative contributions to the superpotential. As we already discussed, in models of chiral superfields only, the superpotential is tree-level exact.

- The tree level superpotential W_{tree} does not break supersymmetry but lifts all flat directions. Since the superpotential is polynomial in the fields, this typically gives a potential V which vanishes at the origin and grows for large field VEVs. Since the superpotential is classically exact in perturbation theory, supersymmetry is preserved at all orders, perturbatively.

- Strong coupling effects generate a non-perturbative superpotential which provides a contribution to the potential which is strong at the origin of field space but decreases for large field VEVs (recall that the large field VEVs region corresponds to the classical region, where quantum corrections are negligible). An instance of a such a potential is the effective potential of $F < N$ SQCD.

DSB arises because of the interplay between the classical contribution and the non-perturbatively generated one, as shown in Figure 11.1. Generically, supersymmetry will be broken and the exact potential will display a stable non-supersymmetric minimum at finite distance in field space.

11.1 Calculable and non-calculable models: generalities

Once a stable non-supersymmetric minimum is found, one would like to study quantum fluctuations around it. The spectrum around such vacuum is non supersymmet-

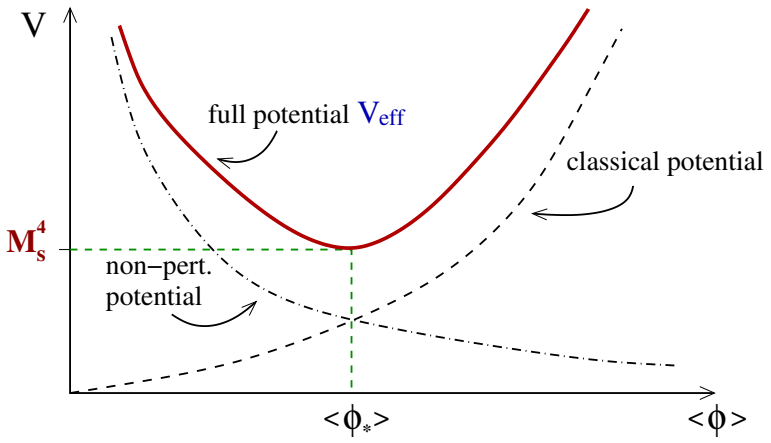

Figure 11.1: A schematic view of dynamical supersymmetry breaking conspiracy. The classical and non-perturbative contributions to the effective potential sum-up and give a stable supersymmetry-breaking minimum at $\langle\phi\rangle = \langle\phi_*\rangle$.

ric and hence quantum corrections will not be protected by supersymmetry. Moreover, besides the superpotential, the knowledge of the Kähler potential will also be important if one wants to make any sort of quantitative statement. A knowledge of the Kähler potential is needed to know the exact point in field space where the supersymmetry breaking vacuum sits, the values of the vacuum energy, i.e. the supersymmetry breaking scale $M_s \sim (V_{\rm min})^{1/4}$, and the masses and interactions of light fields; in other words, one should know the structure of the effective Lagrangian. It is in general a difficult task to control the form of the Kähler potential, since K is corrected at all orders in perturbation theory (and non-perturbatively). There is then a problem of *calculability* around a non-perturbatively generated supersymmetry breaking vacuum, in general.

Looking at Figure 11.1, calling λ the generic tree level coupling(s), it should be clear that if we decrease λ the tree-level potential $V_{\rm tree}$ becomes less and less steep, and the supersymmetry breaking minimum is pushed more and more towards the large VEVs region, where the theory is weakly coupled, see Figure 11.2.

There are three basic reasons why making the tree-level superpotential couplings smaller, calculability is increased.

The smaller λ the smaller M_s, too. Eventually, it might become even smaller than Λ (e.g., curve 4*a* in Figure 11.2). This is useful since at energies lower than Λ gauge degrees of freedom can be safely integrated out giving rise to simpler models (of O'R-like type, so to say). Hence, the analysis of the low energy effective theory

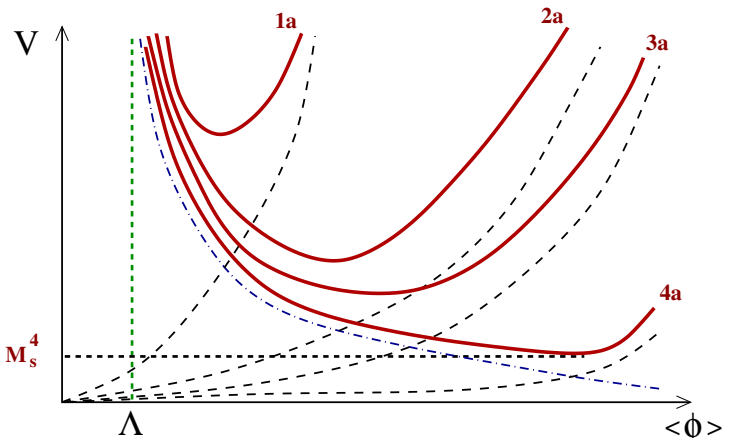

Figure 11.2: Decreasing the perturbative coupling λ the DSB minimum moves towards large field VEVs. For the curve $4a$ the supersymmetry breaking scale M_s is smaller than the dynamical scale Λ.

around supersymmetry breaking vacua might be simpler, the potential having only F-terms contributions

$$V(\phi,\phi^\dagger) = \left[K^{''}(\phi,\phi^\dagger)\right]^{-1} \left|\frac{\partial W}{\partial \phi}\right|^2 . \tag{11.1}$$

Second, as we have discussed at length in the previous lecture, most progresses in understanding supersymmetric theories at the non-perturbative level regard the deep IR, $E < \Lambda$ (structure of vacua, lowest lying state excitations around them, etc...). Hence, having $M_s < \Lambda$ is a welcome feature.

One more reason why having λ small increases calculability has to do with the very possibility of computing the Kähler potential. While the effective superpotential W_{eff} can often be determined exactly, the Kähler potential is in general more difficult to calculate, since it is not protected by holomorphy. Having supersymmetry breaking vacua at large VEVs has the advantage that the theory is more and more classical (i.e. weakly coupled) there. Therefore, one can in principle determine the Kähler potential of the low energy effective fields just by projecting the UV-fields canonical Kähler potential on such operators (which are typically some gauge and flavor invariant combinations of UV fields), getting a correct result up to corrections which, in such semi-classical region, are weak.

According to this general picture, DSB models can be roughly divided into three classes, with increasing level of calculability.

- The worst case scenario is a situation where one cannot get any information

on the full potential due to incapacity of computing both the effective superpotential and the Kähler potential. That supersymmetry is broken can be concluded based on indirect arguments, as those we discussed in Lecture 7 (like R-symmetry and/or global symmetry arguments). In these cases one can reasonably say that supersymmetry is broken and that $M_s \sim \Lambda$, but nothing can be said about the massless excitations around the supersymmetry breaking vacua nor on the effective Lagrangian describing their dynamics.

- A better situation occurs when one can compute the effective superpotential and explicitly see that the latter generates some non-vanishing F-terms which were vanishing at tree-level. In these cases one can safely say that supersymmetry is broken and possibly tell which are the low energy degrees of freedom around the supersymmetry breaking vacua. Still, the Kähler potential cannot be determined. Hence, one cannot calculate any property of the ground states nor determine the dynamics around them. DSB models belonging to this class are known as *non-calculable models*.

- Finally, there can exist models where the scenario summarized in Figure 11.2 can be fully realized. There exists a region in parameter space where the theory is weakly coupled and one can also compute the Kähler potential, then. In these situations one can get also quantitative information about the low energy effective theory, like the precise value of supersymmetry breaking scale, the structure of the light spectrum and interactions. Possibly at an arbitrary high level of accuracy, if supersymmetry breaking vacua can be made parametrically far from the origin of field space. Models of this kind are known as *calculable models*.

In what follows, we will present some concrete examples for each of above three classes.

11.2 The one GUT family SU(5) model

Let us consider a supersymmetric gauge theory with gauge group $SU(5)$, a chiral superfield T transforming in the $\underline{10}$ (i.e. the antisymmetric representation), and another chiral superfield $\tilde{Q}$ transforming in the anti-fundamental representation, $\overline{\underline{5}}$. This theory is UV-free, the one-loop β function coefficient being $b_1 = 13$.

This theory does not have any classical flat direction since it is impossible to construct holomorphic gauge invariant operators out of T and $\tilde{Q}$. For the same reason a superpotential cannot be added. Therefore, at the classical level there exist one supersymmetric vacuum, sitting at the origin of field space, where the gauge group is unbroken, and therefore, the theory most likely confines.

At the origin the theory is strongly coupled and it is difficult to perform any reliable computation. However, one can use indirect arguments to conclude that non-perturbative corrections break supersymmetry. First, one can easily check that there exist two non-anomalous global symmetries, $G_{\rm F} = U(1) \times U(1)_R$, under which the fields have charges $T \simeq (-1, 1)$ and $\tilde{Q} \simeq (3, -9)$, where the charges are fixed by anomaly cancellations. We now use 't Hooft anomaly matching to argue that $G_{\rm F}$ is spontaneously broken. We do not know what the low energy $SU(5)$ invariant degrees of freedom are, but if $G_{\rm F}$ is unbroken, they should reproduce 't Hooft anomalies for $U(1)^3$, $U(1)^2 U(1)_R$, etc... of the original theory. One can be as general as possible and allow for a set of putative low energy fields X_i with charges $\simeq (q_i, r_i)$ under $U(1) \times U(1)_R$. One gets four equations for the q_i's and r_i's. Allowing charges not larger than ~ 50, one needs a least five fields (with rather bizarre charges) to obtain a solution. This sounds quite unnatural. It is therefore quite possible the system not to admit solutions, and the global symmetry group be spontaneously broken. But then, since the theory does not have classical flat directions, according to the indirect criteria we have discussed at the end of Lecture 7, supersymmetry is broken, too.

An independent way to see that supersymmetry most likely is broken is to add one pair of chiral superfields in the 5 and $\bar{5}$ representation. There are now classical flat directions and by adding a mass term for the fundamentals one can show that supersymmetry is broken, in fact. In the limit $m \to \infty$ this theory reduces to the original one, without extra matter. If there are no phase transitions in the limit of large mass, then also the original theory breaks supersymmetry.

This is an instance of the first class of supersymmetry breaking models we discussed before. We do not have direct access to the effective superpotential nor to the Kähler potential, so no quantitative statements can be made. However, symmetry arguments indicate that supersymmetry is most likely broken at the non-perturbative level. In principle, the hidden sector can be a model of this kind.

There exist generalizations of this model which break supersymmetry in a similar manner. They are based on a gauge group $SU(N)$, with N odd, $N - 4$ chiral

superfields transforming in the antifundamental of $SU(N)$, one chiral superfield transforming in the antisymmetric representation of $SU(N)$, and a superpotential which lifts all otherwise present classical flat directions. One can show that at low energy the dynamics of all these models essentially reduces to the one of the $SU(5)$ model described above, and, as the latter, they are therefore expected to break supersymmetry.

11.3 The 3-2 model: instanton driven SUSY breaking

In what follows, we are going to describe an instance of a calculable model.

Let us consider a supersymmetric theory with gauge group $G = SU(3) \times SU(2)$ and the following matter content

$$\begin{array}{cccccc} & SU(3) & SU(2) & U(1)_Y & U(1)_R & \\ Q_{\alpha i} & 3 & 2 & 1/3 & 1 & \\ \bar{U}^i & \bar{3} & \bullet & -4/3 & -8 & \\ \bar{D}^i & \bar{3} & \bullet & 2/3 & 4 & \\ L_\alpha & \bullet & 2 & -1 & -3 & \end{array} \tag{11.2}$$

where i is a $SU(3)$ index and α a $SU(2)$ index, and there are two abelian anomaly-free global symmetries, $U(1)_Y$ and $U(1)_R$.

The above global symmetry charge assignment for matter fields comes from the computation of triangle diagrams with global and gauge currents as detailed in Figure 11.3. Anomaly-free global symmetries require (letters follow diagrams in Figure 11.3)

$$\begin{aligned} a: &\quad 3Y(Q) + Y(L) = 0 \\ b: &\quad 2Y(Q) + Y(\bar{U}) + Y(\bar{D}) = 0 \\ c: &\quad \frac{1}{2}\left[3(R(Q)-1) + R(L) - 1\right] + 2 = 0 \\ d: &\quad \frac{1}{2}\left[2(R(Q)-1) + R(\bar{U}) - 1 + R(\bar{D}) - 1\right] + 3 = 0 \ . \end{aligned}$$

Up to an overall (inessential) normalization, this system of equations admits indeed the solution

$$\begin{aligned} &R(Q) = 1 \quad , \quad R(U) = -8 \quad , \quad R(D) = 4 \quad , \quad R(L) = -3 \\ &Y(Q) = 1/3 \ , \ Y(U) = -4/3 \ , \ Y(D) = 2/3 \ , \ Y(L) = -1 \ . \end{aligned}$$

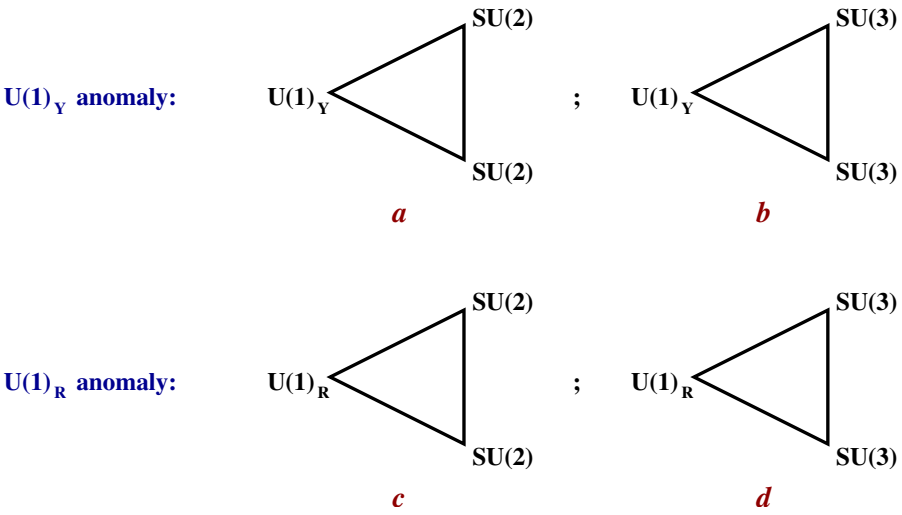

Figure 11.3: One-loop diagrams contributing to anomalies of the abelian global symmetries of the 3-2 model.

Finally, the theory has a tree-level superpotential

$$W_{\text{tree}} = \lambda\, Q\bar{D}L \ , \tag{11.3}$$

which, given the above charge assignment, respects both R and non-R symmetries.

Let us start analyzing this theory at the classical level. The space of D-flat directions has (real) dimension six as it can be seen using the usual parameterization in terms of single trace gauge invariant operators. These read

$$X_A = Q\bar{Q}_A L = Q_{\alpha i}\bar{Q}^i_A L_\beta \epsilon^{\alpha\beta} \ , \ Y = \det\left(Q\bar{Q}\right) = \epsilon^{\alpha\beta}\epsilon^{AB}\left(Q_{\alpha i}\bar{Q}^i_A\right) Q_{\beta j}\bar{Q}^j_B \ , \tag{11.4}$$

where $A = 1, 2$, $\bar{Q}^i_1 \equiv \bar{U}^i$, $\bar{Q}^i_2 \equiv \bar{D}^i$ and $\epsilon^{\alpha\beta}$ is the invariant tensor of $SU(2)$. That these are the correct degrees of freedom to describe the space of D-flat directions can be seen as follows. One can start constructing $SU(3)$ invariants. The only ones are $Q_\alpha \bar{Q}_A$ and L_α, which are both $SU(2)$ doublets. Using them to make (single trace) operators which are also $SU(2)$ invariant, operators X_A and Y follow.

We should now ask whether the superpotential (11.3) affects this space of supersymmetry preserving vacua, looking for the subspace of D-flat directions where all F-terms also vanish. The F-equation for L_α reads

$$\frac{\partial W_{\text{tree}}}{\partial L_\alpha} = \lambda Q_\alpha \bar{Q}_2 = 0 \ . \tag{11.5}$$

Contracting with L_α itself this implies that on the moduli space $X_2 = 0$. Similarly, multiplying by $Q_\alpha \bar{Q}_1$ so to construct the Y invariant, one can show that also $Y = 0$

on the moduli space. Finally, the F-equation for $\bar{Q}_2$ is

$$\frac{\partial W_{\text{tree}}}{\partial \bar{Q}_2^i} = \lambda Q_i L = 0 \ . \tag{11.6}$$

Contracting with $\bar{Q}_1^i$ one can show that also $X_1 = 0$ on supersymmetric vacua. The conclusion is that because of the presence of the superpotential (11.3) there do not exist classical flat directions but rather one single supersymmetric vacuum at the origin of field space.

Notice that there are no classical flat directions. Therefore, if we can prove that some of the global symmetries are spontaneously broken, then we know supersymmetry is broken, too.

Let us start asking whether a dynamical superpotential is generated. In principle, we would expect contributions from $SU(3)$ and/or $SU(2)$ gauge dynamics. Let us choose for now a regime where $\Lambda_3 >> \Lambda_2$ and $\lambda << 1$. In this regime, at scales lower than Λ_3 and bigger than Λ_2, the $SU(2)$ gauge group is weakly coupled while $SU(3)$ confines. Hence, up to subleading corrections, we can consider the $SU(3)$ gauge group as dynamical and the $SU(2)$ gauge group acting as a global symmetry group. Looking at the matter content of the model, we see that from the $SU(3)$ gauge theory view point this is nothing but SQCD with $F = N - 1$, where $N = 3$. Hence a non-perturbative superpotential *is* generated and reads

$$W_{\text{non-pert}} = \frac{\Lambda_3^7}{Y} \ . \tag{11.7}$$

This is enough to conclude that supersymmetry is dynamically broken! Due to (11.7) the minimum of the potential is certainly at some non-zero VEV for Y. Since $R(Y) = -2$ the R-symmetry is then spontaneously broken and since there are no classical flat directions, supersymmetry is broken, too.

Summing up the tree-level and non-perturbative superpotential contributions, we get for the full effective superpotential

$$W_{\text{eff}} = \lambda X_2 + \frac{\Lambda_3^7}{Y} \ . \tag{11.8}$$

From the above expression one can easily see that supersymmetry is broken because, in terms of such low energy fields, we have

$$\frac{\partial W_{\text{eff}}}{\partial X_2} = \lambda \neq 0 \ . \tag{11.9}$$

In this derivation we have implicitly assumed that X_1, X_2 and Y are the correct low energy degrees of freedom, and that no other massless fields show up at any point of field space. If this were the case, one could have met singularities in the Kähler metric at such points, and the vacuum energy would have gone to zero there, hence recovering supersymmetry preserving vacua. For small enough λ, we are safe on this side. First notice that $W_{\text{non-pert}}$ brings the theory away from the origin. For $\lambda << 1$ the minimum of the potential is certainly in the large $Q, \bar{Q}$ region. Since Q and $\bar{Q}$ are charged under both gauge groups, in the supersymmetry breaking vacuum the gauge symmetry is completely broken, and (heavy) gauge bosons can be integrated out. This suggests that X_1, X_2 and Y are indeed the correct low energy degrees of freedom and therefore we do not expect singularities (which correspond to extra massless states) in the Kähler potential.

All what we said so far shows that the 3-2 model belongs, at least, to the second class of supersymmetry breaking models we discussed at the beginning of this section, the so-called non-calculable models. In fact, we can do more.

In the regime we chose, $\lambda << 1$, $\Lambda_3 >> \Lambda_2$, the ground states are in a weakly coupled region, and then we are in a situation similar to curve $4a$ of Figure 11.2. Therefore, the Kähler potential can be safely taken to be canonical in terms of UV-fields

$$K = Q^\dagger Q + \bar{Q}^\dagger \bar{Q} + L^\dagger L \ . \tag{11.10}$$

We can project this potential onto D-flat directions and get

$$K = 24 \, \frac{A + Bx}{x^2} \tag{11.11}$$

where

$$A = \frac{1}{2}\left(X_1^\dagger X_1 + X_2^\dagger X_2\right) \quad , \quad B = \frac{1}{3}\sqrt{Y^\dagger Y} \quad , \quad x = 4\sqrt{B}\cos\left(\frac{1}{3}\arccos\frac{A}{B^{3/2}}\right) \ . \tag{11.12}$$

We can now plug the above expression and that for the effective superpotential, eq. (11.8), into eq. (11.1) and, upon minimization with respect to all scalar fields, find the minima, and hence the vacuum energy $E \sim M_s$.

The computation is doable but rather lengthy, so let us first try to get an estimate of the different scales. The minima will be around a region of field space where the classical and the non-perturbative contributions to the potential are the same order (see Figure 11.1), which is the same to ask the two contributions to the effective superpotential in eq. (11.8) being roughly comparable. In what follows, we think in

terms of fundamental UV fields (an acceptable thing to do, given we are in a weakly coupled region). Calling v the generic VEV of (fundamental) scalar fields at the non-supersymmetric minima, we get that

$$\lambda v^3 \sim \frac{\Lambda_3^7}{v^4} \quad \text{that is} \quad v \sim \frac{\Lambda_3}{\lambda^{1/7}} \ . \tag{11.13}$$

This implies that $W \sim \Lambda_3^3 \lambda^{4/7}$ and hence $\partial W/\partial\phi \sim \Lambda_3^2 \lambda^{5/7}$. Therefore, since the potential is proportional to the derivative of the superpotential squared (recall that the Kähler potential is canonical in the UV fields around the supersymmetry breaking vacua), we finally get

$$M_s^4 \sim \Lambda_3^4 \lambda^{10/7} \quad \text{that is} \quad M_s \sim \Lambda_3 \lambda^{5/14} \ . \tag{11.14}$$

Note that this is a leading order estimate. The Kähler potential receives perturbative and non-perturbative corrections in inverse powers of v. However, in the regime we are considering these are very small, in the sense that v is much larger than any other scale in the theory. Indeed

$$v \sim \lambda^{-1/7}\Lambda_3 \underset{\lambda<<1}{>>} \Lambda_3 >> \Lambda_2 \ . \tag{11.15}$$

From eq.(11.14) we see that $M_s << \Lambda_3$, as well as $M_s << \Lambda_2$, if λ is small enough. This gives an a posteriori justification of our claim that X_1, X_2 and Y were the correct low energy degrees of freedom. Supersymmetry breaking occurs at an energy scale below the confining scale of both non-abelian gauge groups. Therefore, the low energy effective dynamics is certainly not including light gauge degrees of freedom. The effective Lagrangian should be (and actually is) of O'R-like type, with the only complication of a non-canonical Kähler potential (in IR field variables); in fact, a supersymmetric σ-model.

As already stressed, our rough estimates do not prevent to compute everything analytically, by means of eq. (11.1). The answer one gets this way is that the minimum of the potential is at $X_1 = 0$, which means that the $U(1)_Y$ symmetry is unbroken (X_2 and Y are uncharged under this symmetry). On the other hand we know that R-symmetry is broken, since the vacua are at finite value of Y, which is charged under the R-symmetry. This suggests, and confirmed by explicit computations, the massless spectrum to be composed by a goldstino, an R-axion, associated to the breaking of the R-symmetry, and finally a fermionic field with hypercharge $Y = -1$, whose existence can be proved using t'Hooft anomaly matching condition for the unbroken $U(1)_Y$ symmetry. All other fields have masses of order $\sim \lambda v$.

What changes in our analysis if choosing a different regime, namely $\Lambda_2 >> \Lambda_3$? One can derive an effective superpotential also in this case (which is that of SQCD with $F = N$, now) and show that supersymmetry is still broken (though at a different scale with respect to previous regime). However, generically the theory is strongly coupled and hence the Kähler potential is unknown, regardless how small the superpotential coupling λ is. Therefore, in this regime the model is non-calculable.

Finally, one can be as general as possible, and consider the two dynamical scales being the same order, leading to a superpotential of the following form

$$W_{\text{eff}} = \lambda X_2 + \frac{\Lambda_3^7}{Y} + A\left(Z - \Lambda_2^4\right) \tag{11.16}$$

where $Z = \epsilon^{ijk} Q_{i\alpha} Q_{j\beta} \epsilon^{\alpha\beta} Q_{k\gamma} L_\delta \epsilon^{\gamma\delta}$. The latter is nothing but just the gauge invariant expression $\det M - B\tilde{B}$ for the $SU(2)$ theory, which classically is zero, $Z = 0$. This shows why in the regime where the $SU(2)$ gauge group is classical, the superpotential reduces to the expression (11.8) we used before. Notice that since Z is classically zero, the $Z^\dagger Z$ term in the Kahler potential is suppressed by some function of Λ_2/v. Restoring canonical normalization for Z kinetic term implies that the mass of Z is enhanced by the inverse of this function. Therefore at low energy, in the regime where the $SU(2)$ group is nearly classical, one can safely integrate Z out and use just X_A and Y as low energy fields, as we did before. Obviously, the analysis in the regime where both $SU(3)$ and $SU(2)$ have a quantum behavior is more complicated but one can again conclude that supersymmetry is broken.

The 3-2 model is the prototype of calculable DSB models, and many interesting generalizations are available, like the so-called $SU(N) \times SU(2)$ and $SU(N) \times SU(N-1)$ models, plus several others.

A final comment is in order. The 3-2 model is a beautiful instance of a DSB model, and provides a natural way to generate a (small) supersymmetry breaking scale dynamically, without the need of having dimension-full parameters put by hand in the theory, as it was the case for the supersymmetry breaking models we discussed in Lecture 7. This holds at *any* point in the parameter space. Calculability, though, does not. As we have seen, the model is fully calculable in the region of the parameter space where the tree-level dimensionless coupling is parametrically small, something not at all generic, from a naturalness point of view.

11.4 The 4-1 model: gaugino condensation driven SUSY breaking

Let us now consider a model which to some extent is similar to the previous one, but differs in that at low energy the theory is not fully higgsed but reduces to a non-abelian SYM theory. In this case supersymmetry breaking will be driven by gaugino condensation, and not by instanton effects as for the 3-2 model. Let us consider a supersymmetric theory with gauge group $G = SU(4) \times U(1)$ and the following matter content

$$\begin{array}{ccc} & SU(4) & U(1) \\ Q_i & 4 & -3 \\ \tilde{Q}^i & \bar{4} & -1 \\ A_{ij} & 6 & 2 \\ S & \bullet & 4 \end{array} \tag{11.17}$$

plus a tree-level superpotential

$$W_{\text{tree}} = \lambda S \tilde{Q} Q \ . \tag{11.18}$$

As usual, let us start analyzing this theory at the classical level. We first consider the $SU(4)$ dynamics, only, and ignore the $U(1)$ dynamics as well as the superpotential (11.18). The $SU(4)$ D-flat directions can be described by the following gauge invariant operators

$$M = \tilde{Q} Q \quad , \quad \text{Pf}A = \epsilon^{ijkl} A_{ij} A_{kl}/8 \quad , \quad S \ . \tag{11.19}$$

For later purposes let us notice that M has $U(1)$ charge equal to -4, while PfA and S have $U(1)$ charge equal to 4.

Along a generic flat direction a $SU(2) \subset SU(4)$ gauge invariance survives under which no matter is charged (the check is left as an exercise). At scales below the dynamical scale Λ_2 of the effective $SU(2)$ SYM, the theory confines and glueballs and their superpartners can be integrated out: one is only left with M, PfA and S as low energy degrees of freedom. Gaugino condensation of pure SYM leads to the following non-perturbative generated superpotential

$$W_{\text{non-pert}} \sim \Lambda_2^3 = \frac{\Lambda_4^5}{\sqrt{M \text{Pf}A}} \tag{11.20}$$

where the second equality comes from the usual scale-matching condition.

If we now switch-on the $U(1)$ gauge interactions, we have to project the $SU(4)$ D-flat space onto the subspace which is also $U(1)$ D-flat. The latter is parameterized by two moduli, $M\mathrm{Pf}A$ and SM. Note that the tree-level superpotential (11.18), exactly as in the 3-2 model, lifts all classical flat directions and leaves only the origin of field space as a supersymmetric vacuum (this can be seen from the F-term equation for S which sets to zero M and hence both $M\mathrm{Pf}A$ and SM).

Both the $U(1)$ gauge coupling and the superpotential coupling are IR-free, so they would not affect the above IR analysis leading to (11.20). Therefore, we can now consider the full superpotential simply adding up the tree-level and non-perturbative contributions and get

$$W_{\text{eff}} = \frac{\Lambda_4^5}{\sqrt{M\mathrm{Pf}A}} + \lambda SM \ . \tag{11.21}$$

This superpotential is essentially the same as that of the 3-2 model, eq. (11.8). Therefore, from this point on the analysis is the same as the one we performed in the previous section. Supersymmetry is broken because of the interplay between the dynamically generated runaway superpotential term (11.20) and the tree-level contribution (11.18). Differently from the 3-2 model, though, the fact that, for small enough λ, SM and $M\mathrm{Pf}A$ are the correct low energy degrees of freedom, does not follow from complete Higgsing of the gauge group, since on the moduli space there is a surviving $SU(2)$ SYM theory. Still, at energies below Λ_2 the gauge group confines and glueballs and their superpartners can be integrated out. Hence, at low enough energy, the effective theory is indeed given in terms of SM and $M\mathrm{Pf}A$ only.

Similarly to the 3-2 model, one can argue that for small values of the coupling λ the model is calculable. This might look strange, given the left-over non-abelian $SU(2)$ gauge dynamics which is strongly coupled. How can that be? One expects non-perturbative strong coupling dynamics associated to $SU(2)$ to give rise to corrections to the Kähler potential in terms of some function of $\sim \Lambda_2/v$, where v, as before, is taken to be the typical scale of a fundamental field VEV. Balancing the two terms in eq. (11.21), recalling the expression (11.20) one finds

$$v \sim \frac{\Lambda_2}{\lambda^{1/3}} \ \text{ which implies } \ \frac{\Lambda_2}{v} << 1 \ \text{ for } \ \lambda << 1 \ . \tag{11.22}$$

Hence, quantum corrections to the Kähler potential are suppressed in this regime and the model is calculable.

Let us stress again, though, how different the dynamics is with respect to the 3-2 model. There, the smallness of λ ensures that both gauge groups are fully broken

at very high energy, and therefore quantum corrections due to gauge dynamics suppressed. Here, instead, a fully unbroken gauge groups survives at low energy.

The computation of low energy spectrum and interactions goes along similar lines as the 3-2 model, and we do not repeat it here (for instance, also for this model the tree-level superpotential has an R-symmetry which is spontaneously broken in the vacua; hence we expect, as for the 3-2 model, an R-axion in the massless spectrum). Let us summarize, instead, the physical picture one should bare in mind. The theory in the UV is a $SU(4)\times U(1)$ gauge theory. At a scale v this is broken down to $SU(2)$. This left-over non abelian gauge theory confines at a scale $\Lambda_2 << v$, below which we have a low energy effective theory with chiral superfields, only. Gaugino condensation gives rise to a superpotential contribution which induces supersymmetry breaking at a scale M_s. Note that in the limit $\lambda << 1$ the supersymmetry breaking scale is parametrically smaller than Λ_2 (using the same rationale we used for the 3-2 model, one easily sees that $M_s \sim \Lambda_2 \lambda^{1/6}$ which is well below Λ_2, if λ is small). Hence, at the supersymmetry breaking scale all gauge degrees of freedom are heavy and do not contribute to the effective action, which justifies the description in terms of SM and $M\mathrm{Pf}A$, only.

The 4-1 model has several generalizations. The most straightforward ones are theories with gauge group $SU(2l) \times U(1)$ and matter consisting of a chiral superfield transforming in the anti-symmetric representation of $SU(2l)$, $2l-3$ anti-fundamentals $\tilde{Q}$, one fundamental Q, and $2l-3$ singlets S_i. Supersymmetry breaking is again driven by gaugino condensation of a IR left-over $SU(2)$ gauge group, provided a suitable tree-level superpotential is added which lifts all classical flat directions. The 4-1 model corresponds to $l = 2$.

11.5 The ITIY model: SUSY breaking with classical flat directions

Let us now consider an instance of a non-calculable model. Its interest lies in the fact that supersymmetry is broken even though the theory is non-chiral and admits classical flat directions (the latter get lifted by non-perturbative effects *not* leading to runaway behavior).

Let us consider a gauge theory with group $G = SU(2)$, four fundamental fields Q^i (which correspond to two flavors, since for $SU(2)$ the fundamental and anti-

fundamental representations are equivalent) plus six singlets S_{ij} and a superpotential

$$W_{\text{tree}} = \lambda S_{ij} Q^i Q^j \tag{11.23}$$

(notice that the product $Q^i Q^j$ is antisymmetric since what it really means is $Q^i_\alpha Q^j_\beta \epsilon^{\alpha\beta}$ where α, β are $SU(2)$ gauge indices). This theory admits a $SU(4)$ flavor symmetry group (this enhancement of the global non R-symmetry group from $SU(F)_L \times SU(F)_R \times U(1)_B$ to $SU(2F)$ is always there whenever the gauge group is $SU(2)$), under which the Q^i's transform in the fundamental and the singlets in the antisymmetric representations, respectively. Hence, the tree-level superpotential (11.23) respects the flavor symmetry. As usual, let us start studying the classical behavior of the theory. The $SU(2)$ D-flat directions can be parameterized by six meson-like operators $M^{ij} \sim Q^i Q^j$, which transform in the 6 of $SU(4)$ and satisfy the classical constraint of SQCD with $N = F = 2$

$$\text{Pf}M = \epsilon_{ijkl} M^{ij} M^{kl} = 0 \ , \tag{11.24}$$

where indices i, j should be seen as $SO(4)$ indices (recall that $SO(4) \simeq SU(2) \times SU(2)$ and notice that for any nonzero value of M the global symmetry is broken to $SU(2) \times SU(2)$). The F-flatness condition for S_{ij} sets all Q^i's to zero hence all flat directions are lifted *but* the singlets.

At the quantum level the classical constraint (11.24) is modified and the full effective superpotential reads

$$W_{\text{eff}} = \lambda S_{ij} M^{ij} + A \left(\epsilon_{ijkl} M^{ij} M^{kl} - \Lambda^4 \right) \ , \tag{11.25}$$

where A is a Lagrange multiplier. The F-equation for S_{ij} still gives $M^{ij} = 0$ but now this is in conflict with the quantum constraint, i.e. the F-equation for the Lagrange multiplier A. Therefore, supersymmetry is broken. More precisely, working out the potential from the expression (11.25) one can show that, up to symmetry transformations, the minimum is at $M_{ij} = \Lambda_2^2$, $S_{13} = S_{14} = S_{23} = S_{24} = 0$ and $S_{12} = S_{34} \equiv S$. Therefore, there is a pseudoflat direction parametrized by S.

This model is instructive in many respects, which we consider in turn.

Having a flat direction, parametrized by S, one could be worried about where, in field space, the supersymmetry breaking vacua lie, once quantum corrections in the coupling λ are taken into account. In principle, there can also be a runaway. A careful analysis, which we refrain to do here, shows that this is not the case: for small enough λ and large $\lambda\langle S \rangle$ the Kähler potential for S can be shown to grow

logarithmically for large S, hence ensuring that the actual minimum is stabilized at a finite distance in field space.

Notice also that this model is non-chiral. Therefore, one could add a mass term for all fields, lifting all classical flat directions. At low energy one could then integrate all chiral fields out and end-up with pure $SU(2)$ SYM, which does not break supersymmetry (it has two vacua and Witten index equal to 2). How that can be? The answer comes from a careful analysis of the massless limit.

Let us add a mass perturbation to the superpotential (11.25)

$$W_{\text{eff}} = \lambda S_{ij} M^{ij} + m_{ij} M^{ij} + \frac{1}{2}\tilde{m}\,\mathrm{Pf}\, S + A\left(\epsilon_{ijkl} M^{ij} M^{kl} - \Lambda^4\right) \ . \tag{11.26}$$

The F-equations for M^{ij} and S_{ij} set

$$\langle M^{ij} \rangle \sim \epsilon^{ijkl} m_{kl} \left(\frac{\Lambda^4}{\mathrm{Pf}\, m}\right)^{1/2}$$

$$\langle S_{ij} \rangle \sim \frac{m_{ij}}{\tilde{m}} \left(\frac{\Lambda^4}{\mathrm{Pf}\, m}\right)^{1/2} \ .$$

where the square root gets two values, corresponding to the two vacua of pure $SU(2)$ SYM. Take now the limit $\tilde{m}, m_{ij} \to 0$ with the ratio fixed. This way, $\langle M^{ij} \rangle$ has a finite limit, but $\langle S_{ij} \rangle$ is pushed all the way to infinity. This implies that the supersymmetry preserving vacua are also pushed to infinity and disappear from the spectrum, recovering our previous result.

This is an instance of discountinuos change of the Witten index, which moves from 2 to 0 in the limit of vanishing masses. This is because the mass terms change the behavior of the Hamiltonian in the large field region. As the limit $\tilde{m} \to 0$ is taken, the asymptotic behavior of the potential changes since now there are classical flat directions (and the Witten index can, and does, change).

The ITIY model admits many generalizations. An interesting class is based on SQCD with symplectic gauge group $USp(2N)$ and $F = N + 1$ flavors. This theory has a $SU(2F) = SU(2N + 2)$ flavor symmetry, and enjoys a quantum deformed moduli space, very much like $SU(N)$ SQCD with $F = N$ flavors. Coupling the quark superfields to a set of gauge singlets transforming in the antisymmetric representation of the flavor symmetry group via a superpotential like (11.23), one can show supersymmetry is broken in a way identical to that of the original ITIY model (in fact, recalling that $SU(2) \simeq USp(2)$, one sees that the ITIY model corresponds to the case $N = 1$ of the above class).

11.6 DSB into metastable vacua. A case study: massive SQCD

As a final project, we want to discuss the possibility that supersymmetry is broken dynamically into *metastable* vacua.

A model of DSB into metastable vacua share some basic properties with ordinary DSB models. The theory should be a gauge theory and should not break supersymmetry at tree level. Only non-perturbative corrections should. The difference is that the non-perturbative dynamics does not lift classical supersymmetric vacua but just ensure that local minima of the potential whose nature is intrinsically non-perturbative, arise.

On general ground, due to Witten index arguments, R-symmetry arguments, etc... the landscape of theories admitting metastable DSB vacua is much larger than those admitting fully stable DSB vacua. This has been known for a long time, but only more recently it was made concrete. In 2006 Intriligator, Seiberg and Shih (ISS) proved the existence of DSB metastable vacua in the most innocent-looking supersymmetric gauge theory one can imagine: massive SQCD. Note that this is a non-chiral theory, with supersymmetric vacua (a full moduli space, in fact, in the massless limit), non-vanishing Witten index and no R-symmetry (quarks mass terms explicitly break the non-anomalous R-symmetry of massless SQCD). Even more strikingly, the model is calculable, in the sense that around these metastable vacua one can compute both the superpotential and the Kähler potential, and hence the effective Lagrangian describing the dynamics of light fields.

These results have been extended into several directions, and many interesting applications have been found. In what follows, we will just review the basic model, which represents the core of all these developments.

11.6.1 Summary of basic results

Since the derivation is rather lengthy, let us anticipate the upshot of the analysis we are going to perform. This is as follows: $SU(N)$ SQCD with (light) massive flavors in the free magnetic phase (that is for $N+1 \leq F \leq \frac{3}{2}N$) admits metastable supersymmetry breaking vacua which, for $m << \Lambda$, where m is the scale of quark masses and Λ the dynamical scale of the theory, can be made parametrically long lived. More precisely, the theory admits:

- N supersymmetric vacua along the mesonic branch, at

$$\langle M \rangle_{\rm SUSY} = \left(m^{F-N} \Lambda^{3N-F} \right)^{1/N} \ , \ \langle B_{i_1 i_2 \ldots i_{F-N}} \rangle = 0 \ , \ \langle \tilde{B}^{i_1 i_2 \ldots i_{F-N}} \rangle = 0 \ . \quad (11.27)$$

- A compact space of metastable supersymmetry breaking vacua along the baryonic branch

$$\langle B_{i_1 i_2 \ldots i_{F-N}} \rangle, \langle \tilde{B}^{i_1 i_2 \ldots i_{F-N}} \rangle \neq 0 \ , \ \langle M \rangle_{\rm META} = 0 \ , \quad (11.28)$$

  with vacuum energy $V_{\rm META} \sim N |m\Lambda|^2$.

One can also compute the life-time of the metastable vacua and find that

$$\tau \sim e^{S_{\rm B}} \ \text{ where } \ S_{\rm B} \sim \epsilon^{-4(3N-2F)/N} \ \text{ and } \ \epsilon = \sqrt{\frac{m}{\Lambda}} \ , \quad (11.29)$$

with $S_{\rm B}$ the Coleman bounce action. This implies, as anticipated, that for small masses, i.e. $\epsilon << 1$, the metastable vacua can be made arbitrarily long-lived, and hence potentially viable, phenomenologically.

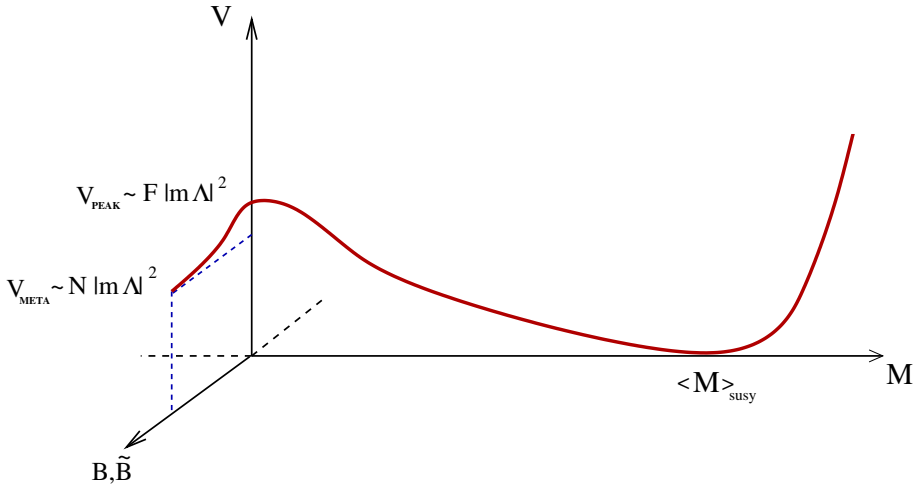

Figure 11.4: The scalar potential of massive SQCD in the free magnetic phase. On the mesonic branch there are supersymmetric vacua. On the baryonic branch there are supersymmetry breaking vacua, which are metastable and parametrically long-lived.

11.6.2 Massive SQCD in the free magnetic phase: electric description

Consider SQCD with gauge group $SU(N)$ in the free magnetic phase, namely for $N+1 \leq F \leq \frac{3}{2}N$. This theory has many supersymmetric vacua, actually a full moduli space. Let us add a mass term for all matter fields

$$W_m = \mathrm{Tr}\, mQ\tilde{Q} \equiv \mathrm{Tr}\, mM \ . \tag{11.30}$$

where the trace is taken on gauge and flavor indices. Notice that (11.30) breaks the SQCD R-symmetry explicitly, while the flavor symmetry group is broken to a subgroup H, whose structure depends on the specific form of the matrix m (more later).

This theory has two mass scales, the quarks mass, which with a slight abuse of language we call again m, and Λ, the dynamical scale of the theory. Let us consider the two obvious possible regimes in turn.

a. $m > \Lambda$

The theory at low energy flows to pure SYM with gauge group $SU(N)$ and has N (isolated) supersymmetric vacua. By scale matching, we obtain

$$\Lambda_{\mathrm{L}}^{3N} = \det m\, \Lambda^{3N-F} \tag{11.31}$$

which implies

$$W_{\mathrm{eff}} = N\Lambda_{\mathrm{L}}^3 = N\left(\det m\, \Lambda^{3N-F}\right)^{\frac{1}{N}} \ , \tag{11.32}$$

an effective superpotential displaying, correctly, the N vacua of pure $SU(N)$ SYM. What's this, really? The mass matrix m and the meson matrix M are Legendre dual variables. The effective superpotential above is nothing but the effective superpotential once the mesons have been integrated out. Hence, using formula (10.75), we get the matrix equation

$$\langle M \rangle_{\mathrm{SUSY}} \ = \ \left(\det m\, \Lambda^{3N-F}\right)^{\frac{1}{N}} \frac{1}{m} \tag{11.33}$$

which tells where in the moduli space the N supersymmetric vacua sit: they correspond to the N roots of the above equation.

b. $m < \Lambda$

In this case, which is actually the one we will be interested in, eventually, it is not completely correct to proceed as before since strong coupling dynamics, driven by Λ, enters before being allowed to integrate the massive quarks out. The more

correct thing to do, in this case, is to notice that m and M are Legendre dual variables, and integrate M in starting from eq. (11.32). In practice, one should take the determinant of eq. (11.33), solve for $\det m$ and follow the procedure outlined in section 10.4.3, getting finally

$$W_{\text{eff}} = (N-F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} + \operatorname{Tr} mM \ . \tag{11.34}$$

Then we can find eq. (11.33) simply solving the F-equations for M. Recall, however, that strictly speaking $\det M = 0$ for $F \geq N+1$, so one has to go a bit off-shell in performing the computation. The final result, eq. (11.33), is of course a perfectly meaningful on-shell result.

The upshot is that, no matter the value of m, there exist N supersymmetric vacua on the mesonic branch. That baryon VEVs are vanishing can be easily argued as follows. As m and M are Legendre dual variables, one can think of b and $\tilde{b}$ as sources for the baryons B and $\tilde{B}$, deforming the theory by $\Delta W = b\,B + \tilde{b}\,\tilde{B}$ (flavor indexes are suppressed, for the ease of notation). At low energy the theory reduces to pure SYM with gauge group $SU(N)$ and effective superpotential (11.32). Using now eq. (10.75) applied to the Legendre dual variables b and B (respectively $\tilde{b}$ and $\tilde{B}$) one concludes that $\langle B \rangle = \langle \tilde{B} \rangle = 0$. Hence, indeed, the supersymmetric vacua (11.33) have zero baryon number.

In general, m is a matrix transforming under the anti-fundamental of $SU(F)_L$ and the fundamental of $SU(F)_R$. This matrix can always be diagonalized via a bi-unitary transformation and, from here on, we choose for simplicity all entries to be equal, $m_i = m$. The superpotential term hence reads

$$W_m = m \operatorname{Tr} M \ , \tag{11.35}$$

where now m is just a number. With this choice, the $SU(F)_L \times SU(F)_R$ flavor symmetry group is broken to $SU(F)_D$. Similarly, eq. (11.33) now reads

$$\langle M \rangle_{\text{SUSY}} = \left(m^F\,\Lambda^{3N-F}\right)^{\frac{1}{N}} \frac{1}{m} = \left(m^{F-N}\Lambda^{3N-F}\right)^{\frac{1}{N}} = \epsilon^{2\frac{F-N}{N}}\Lambda^2 \ , \tag{11.36}$$

where $\epsilon \equiv \sqrt{m/\Lambda}$.

11.6.3 Massive SQCD in the free magnetic phase: magnetic description

So far, we have derived the first part of ISS statement, the easy one. We have obtained, via holomorphic decoupling, the N supersymmetric vacua of massive SQCD,

and found they lie along the mesonic branch. In order to find something more interesting, we have to turn to the Seiberg dual description of the theory, i.e. mSQCD.

Since we are in the magnetic-free phase we choose, in what follows, $\Lambda_{\rm el} = \Lambda_{\rm m} = \mu \equiv \Lambda$. The magnetic dual superpotential, including the mass deformation (11.35) reads

$$W_m = h\mathrm{Tr}\, q\Phi\tilde{q} - m\Lambda\, h\mathrm{Tr}\,\Phi \ , \tag{11.37}$$

where

$$\Phi = \frac{1}{h\Lambda} M \ . \tag{11.38}$$

Let us start by recovering, using magnetic variables, the N supersymmetric vacua we have found before. To this aim, let us suppose we give some non-vanishing VEV to the gauge singlet Φ. This provides a mass to dual quarks, q and $\tilde{q}$, which can then be integrated out. The theory reduces to pure $SU(F-N)$ SYM and the effective superpotential one obtains, upon holomorphic decoupling, reads

$$W_{\rm eff} = -m\Lambda h\mathrm{Tr}\,\Phi + (F-N)\,\Lambda_L^3 \ . \tag{11.39}$$

By scale matching we then find

$$\Lambda_L^{3(F-N)} = h^F \det\Phi\, \Lambda^{3(F-N)-F} \quad \text{that is} \quad \Lambda_L^3 = \left(h^F \det\Phi\, \Lambda^{2F-3N}\right)^{\frac{1}{F-N}} \ . \tag{11.40}$$

We can substitute the above relation into the superpotential (11.39) and get

$$W_{\rm eff} = -m\Lambda h\mathrm{Tr}\,\Phi + (F-N)\left(h^F \det\Phi\Lambda^{2F-3N}\right)^{\frac{1}{F-N}} \ . \tag{11.41}$$

The F-equation for Φ gives

$$\begin{aligned} \langle h\Phi\rangle_{\rm SUSY} &= \sqrt{m\Lambda}\,\epsilon^{-\frac{3N-2F}{N}} >> \sqrt{m\Lambda} \\ &= \Lambda\,\epsilon^{2\frac{F-N}{N}} << \Lambda \ , \end{aligned} \tag{11.42}$$

where the inequalities hold if ϵ is small. The expression in the first line says that the supersymmetric vacua are at a parametrically large distance from the origin of field space in units of $\sqrt{m\Lambda}$, while the second one ensures that the above analysis is meaningful in mSQCD. Indeed, these vacua, though located in a very quantum region from mSQCD point of view, are well below the Landau pole and hence reachable within the magnetic description. Self-consistently, using the map (11.38) one can easily see that the vacua (11.42) are nothing but the vacua (11.36).

Also using magnetic variables one can easily conclude that in the supersymmetric vacua baryon VEVs are vanishing. As already observed, on such vacua the dual

quarks are massive and can be integrated out, hence their VEVs vanish. The VEVs of the magnetic baryons can be easily computed from that of dual quarks. Indeed, the magnetic theory is IR free and in the supersymmetric vacua the VEV of the product of qs is the same as the product of the VEVs of each q. Therefore

$$\langle b^{j_1 j_2 \dots j_N} \rangle = 0 \tag{11.43}$$

and similarly for $\tilde{b}_{j_1 j_2 \dots j_N}$. From the map (10.115) it then follows that $\langle B_{i_1 i_2 \dots i_{F-N}} \rangle = \langle \tilde{B}^{i_1 i_2 \dots i_{F-N}} \rangle = 0$, as anticipated.

Notice that while the ϵ parameter defined here and in the electric description is one and the same, the $\epsilon \to 0$ limit should be understood differently. In the electric description Λ is a dynamical RG-invariant scale and the limit of small ϵ is obtained sending $m \to 0$ keeping Λ fixed. In the magnetic description, Λ is a cut-off scale, above which the theory is not defined. The limit should now be understood as $\Lambda \to \infty$ keeping $\sqrt{m\Lambda}$, the mass scale entering the superpotential (11.37), fixed (notice that $\epsilon = \sqrt{m/\Lambda} = \sqrt{m\Lambda}/\Lambda$). This apparently pedantic observation will be relevant later.

Let us now come back to the expression (11.37) and analyze the properties of deformed mSQCD more closely. We will do this in steps and forget, for a while, that the magnetic group $SU(F-N)$ is gauged. If gauge degrees of freedom are frozen, the vacua of the theory are obtained solving F-equations only. From eq. (11.37) these read

$$\begin{cases} \bar{F}_{\Phi_i^j} = \tilde{q}_a^i q_j^a - m\Lambda \delta_j^i \\ \bar{F}_{q_i} = h \Phi_j^i \tilde{q}^j \\ \bar{F}_{\tilde{q}^j} = h q_i \Phi_j^i \end{cases} \tag{11.44}$$

where a are $SU(F-N)$ indices. We see that the first set of equations cannot be solved. The rank of $\tilde{q}_a^i q_j^a$ is at most $F-N$ while that of δ_j^i is clearly F. Hence we can set to zero at most $(F-N)$ terms of F_Φ-equations: we are left with $F-(F-N) = N$ non-vanishing F-terms. On the other hand, the F-equations for q and $\tilde{q}$'s are easily satisfied. We conclude that supersymmetry is broken, and is so by a *rank condition*. The potential energy gets contribution from the N F-equations that cannot be set to zero and hence reads

$$V_{\text{META}} \sim N |m\Lambda|^2 \ . \tag{11.45}$$

The supersymmetry breaking vacua are at

$$\langle \Phi \rangle = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_0 \end{pmatrix} \quad , \quad \langle q \rangle = \begin{pmatrix} q_0 \\ 0 \end{pmatrix} \quad , \quad \langle \tilde{q}^T \rangle = \begin{pmatrix} \tilde{q}_0 \\ 0 \end{pmatrix} \tag{11.46}$$

where $q_0\tilde{q}_0 = m\Lambda \mathbf{1}_{F-N}$, with q_0 and $\tilde{q}_0$ being $F-N \times F-N$ matrices, and Φ_0 an arbitrary $N \times N$ matrix. Therefore, we find a pseudomoduli space of supersymmetry breaking vacua parameterized by Φ_0, q_0 and $\tilde{q}_0$. If this classical analysis were correct, the picture we would obtained is what is summarized, schematically, in Figure 11.5. So, the question we should now try to answer is to what extent the above results

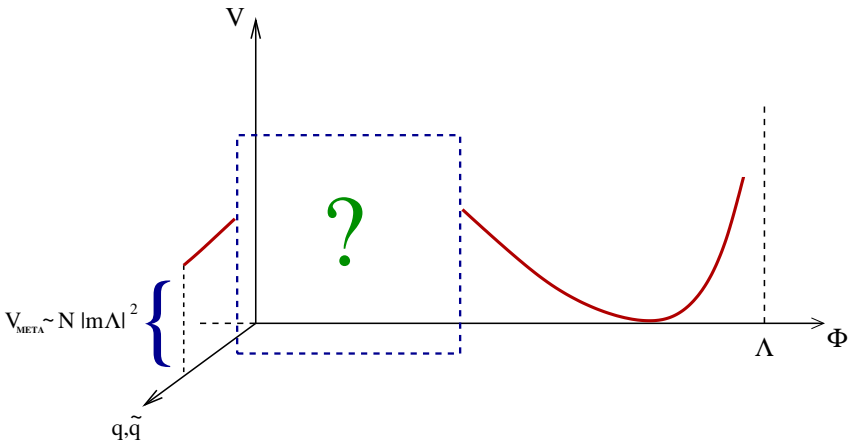

Figure 11.5: Linearly deformed mSQCD *classical* potential.

are solid in the full quantum theory. So far, our analysis was classical, both because we have been ignoring local $SU(F-N)$ gauge dynamics, and because, even within the ungauged model, we have not taken into account quantum corrections coming from the coupling h. Let us start considering quantum effects due to h. Later, we will consider the role of gauge degrees of freedom and interactions.

Let us first notice that the supersymmetry breaking vacua lie relatively near to the origin, which is the more classical region for mSQCD, which is a IR-free theory. Indeed, as already observed, the scale $\sqrt{m\Lambda}$ is set to be the mass scale entering the mSQCD Lagrangian by the superpotential (11.37), the natural mass unit to measure dimensionfull quantities in the magnetic theory. Looking at eqs. (11.45) and (11.46), we see that the energy density of the supersymmetry breaking minima is order one in units of $\sqrt{m\Lambda}$, and so are the values of q_0 and $\tilde{q}_0$ on such minima (the Φ_0 flat direction does not play any role here since, as we will see momentarily, quantum corrections lift this degeneracy and set $\Phi_0 = 0$). On the contrary, looking at eq. (11.42) we see instead that $\langle \Phi \rangle_{\rm SUSY}$ is parametrically large in units of $\sqrt{m\Lambda}$. Since mSQCD is IR-free, we can then safely take the Kähler potential to be canonical in the region where the supersymmetry breaking vacua sit, that is

$$K = \mathrm{Tr}\left(\Phi^\dagger \Phi + q^\dagger q + \tilde{q}^\dagger \tilde{q}\right) \ . \tag{11.47}$$

A second comment regards global symmetries. In the limit where the magnetic gauge group $SU(F-N)$ is taken to be ungauged, mSQCD has a global symmetry group $SU(F-N) \times SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_{R_0}$ which is broken by the second term in (11.37) to $G = SU(F-N) \times SU(F)_D \times U(1)_B \times U(1)_{R_0}$, where under the non-anomalous R-symmetry $U(1)_{R_0}$ the dual quarks are chargeless and Φ has R-charge $R_0 = 2$.

On the supersymmetry breaking vacua (11.46) the group G is spontaneously broken. The vacua with maximal unbroken global symmetry sit at (up to unbroken flavor rotations)

$$\Phi_0 = 0 \quad , \quad q_0 = \tilde{q}_0 = \sqrt{m\Lambda}\ \mathbf{1}_{F-N} \ , \tag{11.48}$$

which preserve $H = SU(F-N)_D \times SU(N) \times U(1)_{B'} \times U(1)_{R_0}$ (notice in particular that the R-symmetry is not broken).

In order to study quantum corrections around the supersymmetry breaking vacua, we can proceed as we did for the O'Raifeartaigh model, and compute the masses of the fluctuations of Φ, q and $\tilde{q}$ as functions of the pseudomoduli Φ_0, q_0 and $\tilde{q}_0$. We can parameterize the various fields as (for ease of comparison we use the same notation of ISS)

$$\Phi = \begin{pmatrix} \delta Y & \delta Z^\dagger \\ \delta \tilde{Z} & \delta \hat{\Phi} \end{pmatrix} \ , \ q = \begin{pmatrix} \sqrt{m\Lambda} + \frac{1}{\sqrt{2}}(\delta\chi_+ + \delta\chi_-) \\ \frac{1}{\sqrt{2}}(\delta\rho_+ + \delta\rho_-) \end{pmatrix} \ , \ \tilde{q}^T = \begin{pmatrix} \sqrt{m\Lambda} + \frac{1}{\sqrt{2}}(\delta\chi_+ - \delta\chi_-) \\ \frac{1}{\sqrt{2}}(\delta\rho_+ - \delta\rho_-) \end{pmatrix} \tag{11.49}$$

where we have expanded around the maximally symmetry preserving vacua (11.48). What one finds is that the model looks as N copies of a O'Raifeartaigh-like model and after computing the one-loop effective potential the spectrum is as follows:

- Some fields have (tree-level) mass $\sim |h\sqrt{m\Lambda}|$ from the classical superpotential (11.37).
- Pseudomoduli are all lifted and get non-tachyonic masses at one-loop $\sim |h^2\sqrt{m\Lambda}|$ from their coupling to massive fields.
- Some fields remain exactly massless. These are: the Goldstone bosons associated to the coset G/H, a goldstino, as well as several fermionic partners of Φ_0 pseudomoduli.

So, after taking into account quantum corrections in the tree-level coupling h we are left with a compact moduli space of stable non-supersymmetric vacua. This moduli space is robust against quantum corrections, because it is protected by symmetries.

What does it change of the above analysis if we now switch-on gauge interactions, namely we let $SU(F-N)$ group being gauged? Interestingly, not much happens around the supersymmetry breaking vacua (11.48).

First, besides F-equations (11.44) we have now to impose D-equations on the supersymmetry breaking vacua (11.48). In fact, these are trivially satisfied, as one can verify plugging VEVs (11.48) into

$$\sum_A \mathrm{Tr}\left(q^\dagger T_A q - \tilde{q} T_A \tilde{q}^\dagger\right) = 0 \ . \tag{11.50}$$

Hence, the compact space parameterized by (11.48) remains a minimum of the potential (D-terms identically vanish and therefore do not contribute to the vacuum energy).

Second, the $SU(F-N)$ gauge group is completely higgsed in the vacua (11.48), since $\langle q_o \rangle = \langle \tilde{q}_0 \rangle \neq 0$. Gauge bosons acquire a mass $\sim g\sqrt{m\Lambda}$, eating some of the previously massless Goldstone bosons of the ungauged model. The only change, then, is that the compact moduli space is smaller since global symmetries in the gauged model are less, to start with. In particular we have now that $G = SU(F)_D \times U(1)_B$ and $H = SU(F-N) \times SU(N) \times U(1)_{B'}$. Notice that the R-symmetry of the ungauged model $U(1)_{R_0}$ is now anomalous, while, as we already observed, the non-anomalous R-symmetry which mSQCD shares with SQCD is explicitly broken by the mass term in the superpotential (i.e. the linear term in Φ, in mSQCD language).

Finally, the gauging does not affect the computation of the one-loop effective potential, either, since the tree level spectrum of massive $SU(F-N)$ fields is supersymmetric and gives no contribution to $\mathrm{Str}M^2$. This happens because, as already observed, D-terms vanish on the vacua (11.48), and the non-zero expectation values of q and $\tilde{q}$ which provide masses to $SU(F-N)$ gauge fields do not couple directly to any non-vanishing F-term.

So we conclude that, up to a restriction of the compact moduli space, the supersymmetry breaking vacua we found classically in mSQCD survive quantum corrections and are hence supersymmetry breaking vacua of our original theory!

Gauging the $SU(F-N)$ group does have (drastic) consequences on other regions of field space, though. We already know, and we have proved it using both electric and magnetic variables, that the theory has supersymmetric vacua on the mesonic branch. Besides other things, this makes the supersymmetry breaking vacua (11.48) being not absolute minima of the potential. Using magnetic language the effect of

gauging is the generation of a non-perturbative superpotential contribution

$$W_{\rm np} \sim (\det\Phi)^{\frac{1}{F-N}} \sim \Phi^{\frac{F}{F-N}} \ . \tag{11.51}$$

This contribution is irrelevant near the origin, where supersymmetry breaking vacua sit, and becomes more and more important the farer we move along the mesonic branch. This operator plays the same role that a mass term for the chiral superfield Φ_2 played in the modified O'Raifeartaigh model: it brings in supersymmetry preserving vacua. The difference is that everything happens dynamically, here. So we conclude that mSQCD has metastable supersymmetry breaking vacua semiclassically, and non-perturbative restoration of supersymmetry by a a dynamical generated superpotential. On the other hand, in terms of the original SQCD theory, the supersymmetry breaking vacua are highly quantum mechanical, since they sit in a region which is, say, highly quantum, from SQCD view point.

The final picture we obtain is represented in Figure 11.6.

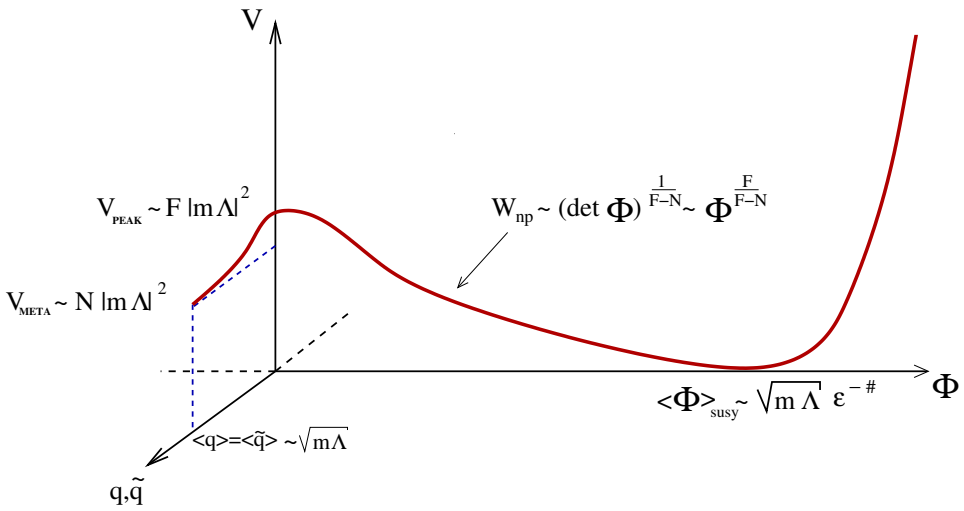

Figure 11.6: The ISS potential in magnetic dual variables.

A final comment regards the R-symmetry breaking pattern. Here again, we find a consistent picture with what we learned in previous lectures. The SQCD original R-symmetry is explicitly broken by the mass term (11.30). Hence, the theory does not satisfy the necessary condition for supersymmetry breaking, and indeed it has N supersymmetric vacua, and non-vanishing Witten index. On the other hand, the anomalous $U(1)_{R_0}$ R-symmetry is restored, approximately, near the origin. This is more transparent using magnetic variables. The superpotential contribution (11.51) breaks R_0 explicitly, but this operator is irrelevant near the origin and this is why

this symmetry arises as an approximate R-symmetry around the vacua (11.48). Therefore, by the (extended) NS criterium, one would expect metastable vacua to arise there, and this is exactly what happens.

It is amusing to notice that in the ISS vacuum is plenty of massless fields, so there is no mass gap: a theory with tree-level masses for all matter fields and strict confinement, admits vacua without a mass gap!

In all our discussion there is one point that we have overlooked. The magnetic theory has a UV cut-off, Λ. Do our results depend on the physics at scale Λ? Luckily, not in the limit we are interested in, namely

$$\epsilon = \sqrt{\frac{m}{\Lambda}} << 1 \quad \longrightarrow \quad \begin{cases} \text{SQCD} & : \ m \to 0 \quad , \quad \Lambda \ \text{ fixed} \\ \text{mSQCD} & : \ \Lambda \to \infty \quad , \quad \sqrt{m\Lambda} \ \text{ fixed} \end{cases} \tag{11.52}$$

First, the analysis within the macroscopic theory (i.e. *ungauged* mSQCD) is valid, since this was done at scales of order $\sqrt{m\Lambda} = \epsilon\Lambda$, which are well below the UV cut-off Λ, if ϵ is small. Second, also the supersymmetry preserving vacua can be seen in the magnetic theory: as we have already observed, second line of eq. (11.42), for small ϵ they are also well below the scale Λ

$$\langle \Phi \rangle_{\text{SUSY}} = \Lambda \, \epsilon^{2\frac{F-N}{N}} << \Lambda \ , \tag{11.53}$$

and hence are very weakly affected by any Λ-physics effects. Finally, the one-loop effective potential gives pseudomoduli mass squares of order $|m\Lambda|$, that is $\sqrt{m\Lambda}\sqrt{\overline{m\Lambda}}$, which is not an holomorphic expression. On the other hand, corrections from Λ-physics are holomorphic in $m\Lambda$ and provide mass contributions of the form

$$\frac{m\Lambda}{\Lambda} \cdot \frac{\overline{m\Lambda}}{\overline{\Lambda}} = |m\Lambda|^2/|\Lambda|^2 = |m\Lambda|\epsilon^2 << |m\Lambda| \ , \tag{11.54}$$

which are again subleading for $\epsilon << 1$. A direct way to see this is to note that corrections in Λ would make the Kähler potential (11.47) not being canonical. In particular, to leading order, we would get a contribution as $\delta K = c/|\Lambda|^2(\Phi\Phi^\dagger)^2$, with c a number of order one. This is reminiscent of the Polonyi model with quartic Kähler potential we discussed previously. A similar computation as the one there gives a contribution to the pseudomoduli mass as in eq. (11.54), $\delta m^2 \sim |m\Lambda|^2/|\Lambda|^2$.

The last important check we have to do regards the life-time of the supersymmetry breaking vacua. The life-time can be computed using the Coleman bounce action. Intuitively, the more the two vacua are far in field space in units of the energy difference between them, the more one might expect the life-time to be long.

This expectation is confirmed by an explicit computation. It turns out that in the present case we are in a situation in which the so-called thin-wall approximation is valid. In such a situation, up to inessential numerical factors, the bounce action is proportional to the ratio between the fourth power of the distance, in field space, between the supersymmetry breaking and the supersymmetry preserving vacua, and the value of the energy difference between them. Using previous formulas S_B hence reads

$$S_B \sim \frac{(\Delta\Phi)^4}{V_{\rm META}} \sim \epsilon^{-4\frac{3N-2F}{N}} \ , \tag{11.55}$$

which is indeed large for $\epsilon << 1$. This ensures that the ISS vacua are parametrically long lived, since $\tau \sim e^{S_B}$. Notice that the largeness of the bounce action is due to different effects depending whether one is working in electric or magnetic variables. From mSQCD view point it is large since $\Delta\Phi$ is parametrically large in units of $\sqrt{m\Lambda}$. From SQCD view point, the bounce action is large because $V_{\rm META}^{1/4}$ is parametrically small in units of Λ.

11.6.4 Summary of the physical picture

Let us summarize the physical picture which emerges from our analysis of massive SQCD in the free magnetic phase.

Since the theory is UV-free, at high energies, larger than the dynamical scale, $E > \Lambda$, the theory is weakly coupled, it can be described in terms of electric variables and the gauge coupling $g_{\rm el}$ increases along the flow. The scale Λ is an IR cut-off for SQCD and a UV cut-off for the IR-free dual magnetic theory. Hence, at scales $E \sim \Lambda$, in order to describe the dynamics of the theory one should better change to magnetic variables. Below Λ but above $\langle\Phi\rangle$ the theory renormalizes as for an IR-free theory, in the sense that the magnetic gauge copling $g_{\rm m}$ decreases. This goes on until one meets the scale $\langle\Phi\rangle$. What happens next depends on the value of such scale. If $\langle\Phi\rangle \neq 0$ at $E \sim \langle\Phi\rangle$ the dual quarks get massive and the theory reduces to pure SYM and leads to N supersymmetric vacua. If instead $\langle\Phi\rangle = 0$ the magnetic theory becomes completely free, gauge degrees of freedom get frozen and one is driven to the supersymmetry breaking vacua at $E \sim \sqrt{m\Lambda}$.

Do exist metastable vacua in massive SQCD for different values of F ? The existence of metastable vacua in massive SQCD has been proven only in the free magnetic phase. For $F \geq 3N$ SQCD the dynamics is trivial and there do not exist ISS-like metastable vacua whatsoever. In the conformal window, $\frac{3}{2}N < F < 3N$,

the analysis is not easy since mSQCD is not IR-free. Moreover, one can show that $\langle\Phi\rangle_{\text{SUSY}}$ turns out to be very near to the origin of field space, hence making metastability difficult to achieve anyway. Finally, the non-perturbative generated superpotential W_{np} is relevant in the IR, indicating the difficulty in treating separately classical and quantum effects. For $F < N$ the runaway is too strong and there are simply no tools to say whether local minima develop along the moduli space. Finally, for $F = N$ the existence of ISS vacua cannot actually be proven using the magnetic dual theory, which does not exists for $F = N$, but can only be inferred using holomorphic decoupling starting from $F = N + 1$. Even though there are convincing arguments in favor of ISS vacua also in this case, given the state we are speaking about is not supersymmetric, the procedure requires some assumptions which are not fully under control; hence, the case $F = N$ is not completely understood, in fact. A natural question is therefore whether is it possible to find ISS-like vacua in theories with a quantum deformed moduli space, as $SU(N)$ SQCD with $F = N$ is. The answer is for the affermative. It has been shown that suitable deformations of the $USp(2N)$ ITIY model we discussed in section 11.5 allow for dynamically generated metastable vacua, in a theory with a quantum deformed moduli space, as the ITIY model and any of its generalizations actually are. Basically, giving supersymmetric masses to some of the singlets S_{ij}, one can show that supersymmetric vacua come in from infinity (because, integrating out massive singlet(s), mesonic flat directions develop) but dynamically generated non-supersymmetric local vacua survive. Moreover, such vacua can be made parametrically long lived in a region of the parameter space which, interestingly enough, coincides with the region where Kähler potential corrections are fully under control.

The ISS model admits many generalizations (including those above). In particular, at the price of some complications and subtleties which we cannot discuss here, one can generalize the model in order to let the emergent IR R-symmetry to be spontaneously broken in the supersymmetry breaking vacua. This is a feature that the original ISS model does not have, since, as we have seen, quantum corrections stabilize $U(1)_{R_0}$ charged moduli at the origin, $\Phi_0 = 0$. And, if one thinks of the ISS model as a hidden sector in gravity or gauge mediation scenarios, having broken R-symmetry is a necessary condition to let gauginos getting (Majorana) mass.

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12 Supersymmetric gauge dynamics: extended supersymmetry

In this lecture we will focus on asymptotically free gauge theories with extended supersymmetry and try to understand their quantum dynamics, in analogy with what we did for $\mathcal{N} = 1$ supersymmetric theories in chapter 10.

Asymptotically free gauge theories can enjoy different phases at low energy. In the case of $\mathcal{N} = 1$ supersymmetry, thanks to powerful non-renormalization theorems and more generally holomorphy, we were able to understand a great deal about the possible phases such field theories can enjoy. This can obviously be done also for theories with extended supersymmetry. In fact, the beauty of theories with extended supersymmetry is that, quite often, one cannot only determine the phase in which the theory is, but also derive the exact expression of the low energy effective action. The main purpose of this lecture is to show when and how this is possible.

12.1 Low energy effective actions: classical and quantum

Let us start making some general comments, independent from supersymmetry. Suppose to start from some matter coupled, asymptotically free gauge theory. At low energy, its dynamics will be described by some (non-renormalizable) effective action whose degrees of freedom will be in general very different from UV ones. What is the structure one would expect for such an action?

Let us assume that in the vacuum we want to expand the theory about, the potential vanishes, $V = 0$. This is not a restriction, since the minimum of the potential can always be chosen to vanish via a constant shift in the Lagrangian. Moreover, in the context of supersymmetric theories, which is what we are eventually interested in, this is not even a choice but a necessary condition, if supersymmetric vacua exist.

The leading dynamics around these vacua is governed by light fields, eventually only massless ones, as we take the cutoff energy characterizing the low energy effective action to be lower than any scale in the theory. In this limit, the physics is, by definition, scale invariant. However, the nature of the corresponding IR fixed point is not unique. If no massless fields are present (like in the case of strict confinement) there are no propagating degrees of freedom in the limit $E \to 0$, so the IR theory is empty and the IR fixed point a trivial one. The theory is gapped. If massless fields

are present, instead, the theory can be in a free or an interacting phase.

A necessary condition for having an interacting conformal field theory at low energy is that massless non-abelian gauge fields are present in the effective action (at least if one assumes that a local Lagrangian description can exist). Indeed, by the Coleman-Gross theorem, in four space-time dimensions any theory of scalars, spinors and abelian gauge fields flows in the IR to a free (or trivial, if everything get a mass) theory. In presence of massless non-abelian gauge fields, one can either end up with, say, confinement, hence a trivial IR fixed point, or, indeed, an interacting conformal field theory (an example being $\mathcal{N} = 1$ SQCD with $3/2\, N < F < 3N$). There are no general tools to describe strongly coupled, interacting conformal field theories, for which, regardless of supersymmetry, typically one cannot easily derive an effective Lagrangian (recall we are assuming that the UV theory is asymptotically free so we are excluding the case in which there are enough massless charged matter fields to make the β function being IR-free to start with, as *e.g.* $\mathcal{N} = 1$ SQCD with $F > 3N$). Sometime duality can help, like for $\mathcal{N} = 1$ SQCD with $N + 1 < F < 3/2\, N$, whose dynamics can be described by a dual, IR-free, magnetic theory. But this is clearly non generic.

Since our aim is to discuss low energy effective actions, in what follows we will focus on effective theories where scalars, spinors and abelian gauge fields enter, only. This is not such a restriction, since, for both $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric theories, a large fraction of the moduli space happens to enjoy precisely such an abelian, IR-free phase.

A few more comments are in order.

In absence of supersymmetry, one expects the minima of the potential to be isolated, and hence the space of vacua to be a set of isolated points in the space of scalar field VEVs (there might exist a classical pseudo-moduli space which, however, is typically lifted once quantum corrections are taken into account). If this is the case, while scalar fields VEVs do parametrize the space of vacua, no scalar fields can be actually massless. Hence, having truly massless scalar fields in the low energy spectrum, implies the existence of a moduli space of vacua on which the potential vanishes identically, $V = 0$ (this includes also the case of spontaneously broken global symmetries, a scenario which can occur also in non-supersymmetric setups, albeit in this case the moduli space, parametrized by goldstone bosons, is compact). Supersymmetric theories typically admit moduli space of supersymmetric vacua. Hence, in the following, we will assume we are in such a situation, and hence we

allow massless scalar fields to be present in the low energy effective action. These scalar fields, or better their VEVs, parametrize the moduli space.

In writing down the most general form of a IR-free effective action, an important simplification occurs. Suppose a charged massless field is present in the theory. Due to one-loop running, the abelian gauge coupling τ under which the massless field is charged vanishes in the far IR, that is $\mathrm{Im}\tau \to \infty$. Hence, the abelian gauge field associated to it is decoupled and does not partecipate to the low energy effective dynamics. Notice, further, that a charged massless scalar field cannot parametrize the moduli space. Indeed, a non vanishing VEV would Higgs the $U(1)$ and thereby give the field a mass, as the gauge field itself. They would both disappear from the low energy effective action. If, on the contrary, all charged fields are massive, they do not appear in the low energy effective action to start with. Therefore, in the limit $E \to 0$ the low energy effective action just contains massless *neutral* fields and abelian gauge fields (plus fermions).

To sum up, the low energy effective action would be something like

$$\mathcal{L} = g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{2}\,\mathrm{Im}[\tau_{IJ}(\phi)\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\,\mu\nu}] + \text{ fermions}\,, \tag{12.1}$$

where i, j run on (neutral and massless) scalar fields, and I, J on (abelian) gauge fields.The complexified gauge coupling matrix τ_{IJ} and field strength are defined, respectively, as

$$\tau_{IJ} = \frac{\theta_{IJ}}{2\pi} + \frac{4\pi i}{g^2_{IJ}} \quad , \quad \mathcal{F}^I_{\mu\nu} = F^I_{\mu\nu} + \frac{i}{2}\,\epsilon_{\mu\nu\rho\sigma}F^{I\rho\sigma}\ . \tag{12.2}$$

The σ-model metric $g_{ij} = g_{ij}(\phi)$ is the metric on the moduli space $\mathcal{M}$, whose coordinates are the massless scalar fields VEVs. Solving the theory boils down to compute the exact expression of the metric g_{ij} and the gauge coupling matrix τ_{IJ} (and of the coefficient functions of fermion kinetic terms which also depend on the scalar fields ϕ^i).

So far, we have been rather qualitative. In what follows we will show, focusing on theories with extended supersymmetry, that one can be quantitative and understand a great deal about actions as (12.1) and their quantum dynamics.

12.1.1 $\mathcal{N} = 2$ effective actions

Let us focus on theories with $\mathcal{N} = 2$ supersymmetry and suppose to start from some asymptotically free $\mathcal{N} = 2$ renormalizable action. Such an action is fully specified by the gauge group G and matter content, cf § 6.1.

If supersymmetry is preserved and a moduli space exists, the lightest excitations are massless. Hence, for low enough energy, lower then any scale in the theory, the dynamics on the moduli space is described by an effective action including these massless fields, only. This action should preserve $\mathcal{N} = 2$ supersymmetry. Hence, it should be nothing but a special instance of the $\mathcal{N} = 2$ non-linear σ-model discussed in § 6.1.1. As such, it would be fully determined by knowing the exact expression of the prepotential $\mathcal{F}(\Phi)$, which gives both the special Kähler metric and the generalized complexified gauge coupling, eqs. (6.8) and (6.9), and by knowing the HyperKähler metric describing the hypermultiplets σ-model.

Scalar fields parametrize a complex manifold which, classically, has the following form

$$\mathcal{M} = \mathcal{M}^V \otimes \mathcal{M}^H \ . \tag{12.3}$$

$\mathcal{M}^V$ is a special Kähler manifold, whose coordinates are the massless scalars ϕ^I belonging to vector multiplets, and $\mathcal{M}^H$ a HyperKähler manifold, whose coordinates are the massless scalars H^i belonging to hypermultiplets (for the ease of notation, we refer collectively to H^i_1 and H^i_2 as H^i here).

Let us look at the structure of the (classical by now) moduli space (12.3) more closely.

The first thing is that, in writing (12.3), we have assumed that the σ-model metric is diagonal, *i.e.* that there are no kinetic terms mixing ϕ^I and H^i. Namely, that the moduli space $\mathcal{M}$ is, locally, a direct product of a special Kähler manifold and a HyperKähler manifold. That this is the case comes from a $\mathcal{N} = 2$ selection rule. If a cross term where there in the Lagrangian, its supersymmetry variation should be canceled (up to total space-time derivatives) by the supersymmetry variation of some other term. Looking at the supersymmetry variations of vector- and hypermultiplet component fields, one can easily see that such a term cannot be constructed. Hence, metric cross terms are zero.

The subspace $\mathcal{M}^V$ where only the complex scalars ϕ^I get a VEV is called Coulomb branch. This is because the scalars belonging to $\mathcal{N} = 2$ vector multiplets transform in the adjoint of the gauge group G and, as such, can at most Higgs G down to $U(1)^n$, where $n = \operatorname{rank} G$. More precisely, the scalar potential is $V \sim \operatorname{Tr}[\phi, \phi^\dagger]$ and its supersymmetric minima are described by the adjoint scalars VEVs being in the Cartan subalgebra of G

$$\langle \phi \rangle = \sum_{I=1}^{n} a^I \ h_I \quad \text{where} \quad h_I \in CSA \ \text{ of } \ G \ . \tag{12.4}$$

For generic a^I the gauge group is broken as $G \to U(1)^n \times W_G$, where W_G is the Weyl subgroup of G, the group of residual gauge transformations which, while acting on ϕ, do not take it out from the Cartan subalgebra. The moduli space is hence, locally, $\mathbf{C}^n/W_G$.

Taking, for example, $G = SU(N)$, the Weyl subgroup is S_{N-1}, the group of permutations of $N-1$ elements. A natural set of $U(1)^{N-1} \times S_{N-1}$ invariant coordinates on the $(N-1)$-dimensional moduli space $\mathcal{M}^V = \mathbf{C}^{N-1}/S_{N-1}$ can be shown to be

$$u_2 = \sum_{i<j} a^i a^j \ , \ u_3 = \sum_{i<j<k} a^i a^j a^k \ , \ \ldots \ , \ u_N = a^1 a^2 \cdots a^N \quad , \quad i,j,k = 1, \ldots, N \tag{12.5}$$

where, in this case, eq. (12.4) is

$$\langle \phi \rangle = \begin{pmatrix} a^1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a^N \end{pmatrix} \quad \text{with} \quad \sum_{i=1}^{N} a^k = 0 \ , \tag{12.6}$$

since $\operatorname{Tr} \phi = 0$. At low energy, the effective Lagrangian describes n massless $\mathcal{N} = 2$ abelian vector superfields V^I. The scalar fields ϕ^I belonging to these massless abelian vector superfields are neutral and the gauge couplings τ_{IJ} are frozen at the value corresponding to the (lightest) massive particles (whose masses are in fact proportional to a^I). Hence the theory is in a IR-free abelian Coulomb phase. The (qualitative) behavior of gauge coupling evolution is shown in figure 12.1, where we refer to a^I collectively as a.

The subspace $\mathcal{M}^H$, where only the scalars H^i get a VEV is called Higgs branch. This is because, for generic values of hyperscalar fields VEVs, the gauge group is fully broken, now. So, on the Higgs branch, one does not expect propagating massless gauge degrees of freedom. Here again, the hyperscalars parametrizing the moduli space $\mathcal{M}^H$ are not only massless but also neutral (if this were not the case, they would acquire a mass by Higgs mechanism and should be integrated out for low enough energy, disappearing from the effective action).

Finally, branches where both ϕ^I and H^i have non-vanishing VEVs are called mixed branches.

All in all, we have to deal with a set of massless, neutral scalar fields and $n = \operatorname{rank} G$ abelian gauge fields (plus fermionic superpartners). So, we are exactly in a situation as the one advocated in the previous general discussion, cf eq. (12.1).

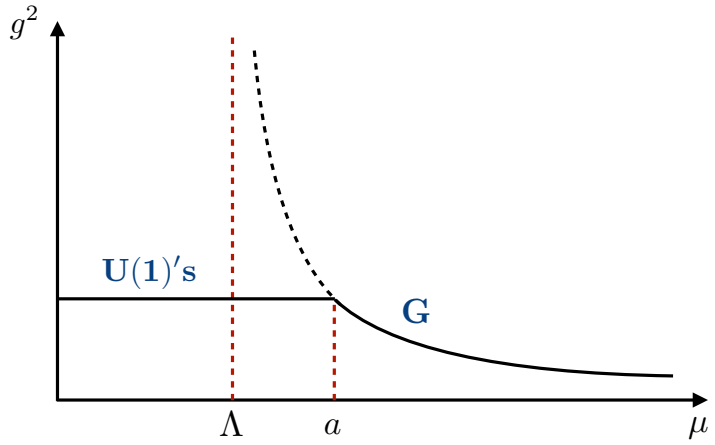

Figure 12.1: Schematic behavior of gauge coupling running of $\mathcal{N} = 2$ SYM with gauge group $SU(N)$, higgsed to $N-1$ $U(1)$'s at scale $a > \Lambda$. The gauge coupling stops running at $\mu \simeq a$. The low energy effective theory is described by abelian vector superfields, containing photons and neutral fields (scalars and fermions) and enjoys an IR-free Coulomb phase.

As we have already seen discussing $\mathcal{N} = 1$ theories, the moduli space needs not to be smooth. There can exist singularities where submanifolds of different dimensions meet. For example, classically, at the origin of field space, where the Coulomb and the Higgs branch meet, the theory is fully un-higgsed and the metric of the moduli space is expected to be singular: extra massless degrees of freedom appear and they should be included in the low energy effective description. Fig. 12.2 provides a qualitative picture of the $\mathcal{N} = 2$ classical moduli space.

All what we said above is the classical part of the story. How do quantum corrections change it? Answering this question will be the basic goal of this lecture. However, already at this stage, a few important properties can be anticipated.

First, the selection rule dictating a direct product for the moduli space $\mathcal{M}$, eq. (12.3), holds also at the quantum level, since it comes from the supersymmetry algebra.

$\mathcal{N} = 2$ supersymmetry implies that the special Kähler metric on $\mathcal{M}^V$ and the (imaginary part of the) generalized complexified gauge coupling, τ_{IJ}, are related. The former is a function of the scalar fields ϕ^I, only, and so is the gauge coupling matrix τ_{IJ}. The latter undergoes renormalization, at one loop and non-perturbatively in $\mathcal{N} = 2$, its quantum corrected expression being some (unknown for the time be-

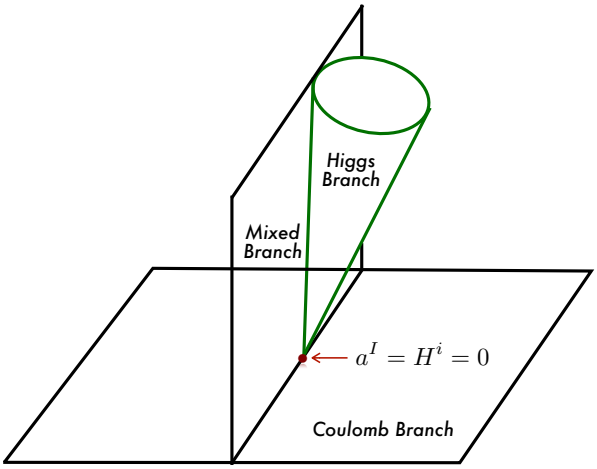

Figure 12.2: $\mathcal{N} = 2$ classical moduli space.

ing) function of the strong coupling scale Λ. Since Λ appears in τ_{IJ}, it appears in the Lagrangian in the same way as a VEV of a scalar belonging to a vector multiplet would (one can think of Λ as a spurion). Since the metric on $\mathcal{M}^H$ does not depend on vector multiplet scalars, it does not depend on Λ, either. Taking the classical limit, $\Lambda \to 0$, one then concludes that the metric on $\mathcal{M}^H$ is classically exact! The upshot is that the metric on the Coulomb branch can receive quantum corrections, while that on the Higgs branch is classically exact. Therefore, the exact low energy effective action will be described by a quantum corrected Coulomb branch and a classically exact Higgs branch. Solving the quantum theory boils down to determine the geometry on the Coulomb branch. Hence, in what follows, we will mostly focus on Coulomb branches.

One more property which makes $\mathcal{N} = 2$ special is that, unlike for $\mathcal{N} = 1$ (an example being SQCD with $F < N$), in $\mathcal{N} = 2$ theories a moduli space always survives at quantum level. In other words, the classical moduli space can be modified, but never completely lifted. As for $\mathcal{M}^H$, this is obvious. The HyperKähler manifold is classically exact so, if it exists classically, it persists quantum mechanically. More interestingly, a Coulomb branch always survives at the quantum level. One way to see this is as follows. For large field VEVs, $a^I >> \Lambda$, we can use classical intuition where, by ordinary Higgs mechanism, the gauge theory is higgsed to $U(1)^n$ at weak coupling, see eq. (12.4). The corresponding n flat directions can be lifted at the quantum level, if given a mass. However, this cannot occur since in such semi-classical region this can happen only by higgsing, and abelian vector multiplets are

neutral and so are the scalar fields ϕ^I, which cannot then Higgs the theory further. Therefore, we conclude that at large fields VEVs the moduli space persists even at quantum level. But then, by analytic continuation, a moduli space persists also in the strongly coupled region, where $a^I \sim \Lambda$ (complex manifolds can become singular only on complex submanifolds, whose dimension is then at least 2 real dimensions smaller, so there is no obstructions against analytic continuation into a region of strong coupling where classical intuition would fail) .

The classical moduli space has singularities of enhanced gauge symmetry and one could wonder if such singularities survive at the quantum level. One of the basic results we will show in the following is that the quantum moduli space does admit singularities where massive particles become massless, but none of them are gauge fields. So, there are no points of enhanced gauge symmetry, and the theory is always in an abelian Coulomb phase. What can exist, instead (for sufficiently large gauge groups and hypermultiplets content), are other type of singularities, known as Argyres-Douglas points, where mutually non-local particles, as monopoles and dyons, become *simultaneously* massless. At these singularities the low energy effective dynamics is described by an interacting conformal field theory which, however, does not admit a Lagrangian description. We will have more to say about this later.

To sum up, apart from special points/curves where a Lagrangian description is not available, the structure of the $\mathcal{N} = 2$ low energy effective action is

$$\mathcal{L} = K_I^J \partial_\mu \bar{\phi}^I \, \partial^\mu \phi_J + \frac{1}{2} \, \mathrm{Im}[\tau_{IJ}(\phi) \mathcal{F}^I_{\mu\nu} \mathcal{F}^{J\mu\nu}] + K_i^j(h, \bar{h}) \, \partial_\mu h^i \, \partial^\mu \bar{h}_j + \text{ fermions} \, , \quad (12.7)$$

where K_J^J is the (special Kähler) metric on the Coulomb branch and K_j^i the (HyperKähler) metric on the Higgs branch. The complexified gauge coupling is related to the prepotential as

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial \phi^I \partial \phi^J} \, . \quad (12.8)$$

We have slightly changed normalizations with respect to previous lectures. With present normalizations, eq. (6.11), which relates the special Kähler metric to the complexified gauge coupling matrix, reads $K_I^J = \mathrm{Im}\, \tau_{IJ}$. Defining

$$\phi_{DJ} \equiv \frac{\partial \mathcal{F}}{\partial \phi^J} \, , \quad (12.9)$$

we can re-write (12.7) as

$$\begin{aligned}\mathcal{L} &= \mathrm{Im}\left[\tau_{IJ}\left(\partial_\mu\bar\phi^I\,\partial^\mu\phi^J+\frac{1}{2}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\mu\nu}\right)\right]+K^j_i(h,\bar h)\,\partial_\mu h^i\,\partial^\mu\bar h_j+\text{ fermions}\\ &= \mathrm{Im}\left(\partial_\mu\bar\phi^I\,\partial^\mu\phi_{DI}\right)+\frac{1}{2}\,\mathrm{Im}[\tau_{IJ}(\phi)\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\mu\nu}]+K^j_i(h,\bar h)\,\partial_\mu h^i\,\partial^\mu\bar h_j+\text{ fermions}\,.\end{aligned} \tag{12.10}$$

Solving the theory boils down to determine the quantum exact expression of the prepotential $\mathcal{F}$ and, via eqs. (12.8), the Lagrangian (12.10).

A cartoon of the quantum corrected moduli space is depicted in fig. 12.3.

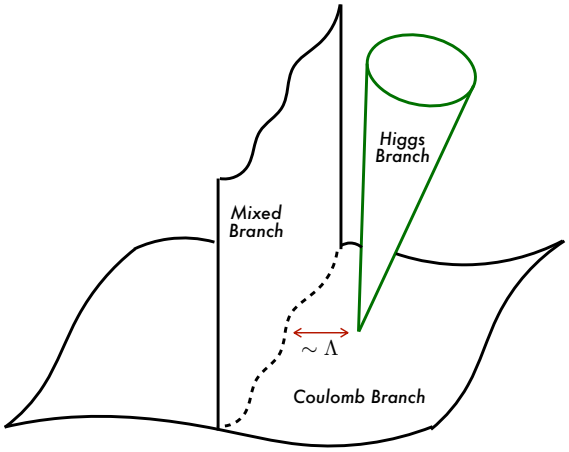

Figure 12.3: $\mathcal{N}=2$ quantum corrected moduli space.

12.1.2 $\mathcal{N}=4$ effective actions

Let us now consider $\mathcal{N}=4$ supersymmetry. The story here is much simpler. First, there exist only one class of scalar fields, all transforming in the adjoint of the gauge group G. So, at a generic point of the moduli space, the low energy dynamics is that of a free $U(1)^n$ $\mathcal{N}=4$ theory, with $n=\mathrm{rank}\,G$, and the moduli space $\mathcal{M}$ is parametrized by $6n$ neutral scalars. Hence, at a generic point on the moduli space, we are in an (IR-free) abelian Coulomb phase. Moreover, as we already discussed in § 6.2, the gauge coupling does not run, neither perturbatively nor non-perturbatively, and then $\mathcal{M}=\mathbb{R}^{6n}$. This is, though, not boring at all. As we will discuss later, $\mathcal{N}=4$ non-renormalization theorems, which are the strongest possible, let one get very interesting exact results.

12.2 Monopoles, dyons and electric-magnetic duality

Before proceeding, we need to recall a few properties that some gauge theories enjoy and discuss how these are realized in supersymmetric contexts.

Let us start from a $U(1)$ gauge theory without matter, namely electro-magnetism. Maxwell equations in the vacuum, which in differential form notation can be written as

$$d * F = 0 \quad , \quad dF = 0 \tag{12.11}$$

are invariant under the transformation $F \to *F$, $* F \to -F$, which corresponds to the exchange of electric and magnetic fields. This transformation is called S-duality transformation. In presence of electric sources Maxwell equations can still be invariant under S duality if one postulates the existence of magnetic sources (aka monopoles) and the associated current j_m, with the following action of S

$$F \to *F \quad , \quad * F \to -F \quad \text{and} \quad j_e \to j_m \quad , \quad j_m \to -j_e. \tag{12.12}$$

where now Maxwell equations read

$$d * F = j_e \quad , \quad dF = -j_m \ . \tag{12.13}$$

The exchange of electric and magnetic currents implies, in particular, that under a S duality transformation electric and magnetic charges are also exchanged.

One crucial consequence of the presence of magnetic monopoles is that the electric charge is quantized. More precisely, as shown by Dirac, it turns out that a theory with both electric and magnetic charges, q and p respectively, can be consistently quantized only if the following condition holds

$$q\, p = 2\pi n \quad \text{with} \quad n \in \mathbf{Z} \ . \tag{12.14}$$

This is the renown Dirac quantization condition, which implies that any electric charge is an integer multiple of an elementary charge $e \equiv (2\pi/p)\, n_0$, for some integer number n_0. Another important consequence of eq. (12.14) is that regimes where the electric charge is small correspond to regimes where the magnetic charge is large and viceversa. Therefore, S duality is a strong-weak coupling duality.

Maxwell equations are not affected if adding to the action a θ-term

$$\frac{\theta\, e^2}{32\pi^2}\, F^a_{\mu\nu}\, \tilde{F}^{\mu\nu}_a \ . \tag{12.15}$$

However, in presence of magnetic monopoles, a θ-term does have an interesting physical effect. As shown by Witten, in this case the magnetic charge of a particle contributes to its electric charge, too. Specifically, a particle with magnetic charge $p = \frac{4\pi}{e}$ and $U(1)$ electric charge $n_e e$, has the following physical charges

$$p = \frac{4\pi}{e} \quad , \quad q = n_e e + \frac{\theta e^2}{8\pi^2} p = n_e e + \frac{\theta e}{2\pi} \ . \tag{12.16}$$

In other words, if a θ-term is present a magnetic monopole always carries an electric charge (even if $n_e = 0$) and such electric charge is not a multiple of some basic unit. This is known as Witten effect. In the following, with some abuse of language, we will refer to the $U(1)$ charge e as the electric charge.

Dirac quantization condition is generalized in presence of dyons, which are states carrying both electric and magnetic charges, as

$$q_1 \, p_2 - q_2 \, p_1 = 2\pi n \ , \tag{12.17}$$

which is known as Dirac-Schwinger-Zwanziger quantization condition. An aspect regarding eq. (12.17) and that will play a relevant rôle later is that only if the right hand side vanishes the corresponding states are local with respect to each other. So, for instance, two electrically charged states are local with respect to each other while an electrically charged state and a magnetic monopole (or a dyon) are not. As such, they cannot be described within one and the same Lagrangian. Therefore, an effective low energy theory where such mutually non-local objects are both present, is believed not to admit a Lagrangian description.

Let us emphasize that the duality transformation (12.12) is not a symmetry of the theory, since it acts on the couplings. Rather, it maps a description of the theory to another description of the same theory. There exists another transformation, known as T-duality transformation, which does not act on the electro-magnetic field but shifts the θ angle by 2π and, as such, is a symmetry of the theory. These two transformations, S and T, generate a full group, $SL(2, \mathbf{Z}) \simeq Sp(2, \mathbf{Z})$, the duality group of electro-magnetism.

As $SL(2, \mathbf{Z})$ 2×2 matrices, S and T are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{12.18}$$

The way S and T act on τ, $\tau \to -1/\tau$ and $\tau \to \tau + 1$, respectively, shows that the group they generate acts on the complexified gauge coupling τ as a fractional linear

transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbf{Z}) \ . \tag{12.19}$$

In Maxwell theory, magnetic monopoles (or dyons) are introduced by hand as extra degrees of freedom, they are pointlike and carry infinite energy. However, monopole (and dyon)-like sources arise as solitons, *i.e.* localized, finite energy, non-singular solutions of equations of motion, in the context of spontaneously broken gauge theories. The first and most famous example is the Giorgi-Glashow model, a $SU(2)$ gauge theory coupled to a scalar ϕ transforming in the adjoint of $SU(2)$ and quartic potential

$$V = \frac{\lambda}{4} \left(\mathrm{Tr}\, \phi^2 - a^2 \right)^2 \ , \tag{12.20}$$

with a some real number. This theory undergoes a Higgs mechanism which breaks $SU(2) \to U(1)$, and admits soliton solutions carrying monopole and/or dyonic charges under the *low energy effective* $U(1)$. For example, there exists a magnetically charged soliton, the 't Hooft-Polyakov soliton with charges $\pm(1, 0)$, as well as a dyon, found by Julia and Zee, with charges $\pm(1, -1)$ (the plus and minus sign solutions should both be there by anomaly cancellation of the low-energy effective $U(1)$ theory). For generic values of the parameters (charge g, scalar field VEV a and quartic coupling λ) these solutions are not known analytically. However, there exists a limit in which the equations of motion can be solved exactly. This is the so-called BPS limit, which corresponds to take $\lambda \to 0$ with g and a fixed while retaining the boundary conditions on the Higgs field, that should tend towards a at spatial infinity. In this limit, the minimal energy configurations satisfy the following relation

$$M = \sqrt{2}\, |\frac{a}{g}(q + ip)| \tag{12.21}$$

where M is the mass of the soliton and $q = n_e g$ and $p = \frac{4\pi}{g} n_m$ its electric and magnetic charges (the reason for the 4π in place of the 2π for the magnetic charge p is just because in this theory we could add fields in the fundamental representation of $SU(2)$, which would carry electric charge $\pm g/2$ and, in terms of such minimal charge, one would get the usual Dirac quantization condition). In fact, in the BPS limit all particles in the spectrum, including fundamental degrees of freedom (gauge bosons and Higgs field), satisfy the mass formula (12.21) and so belong to the BPS spectrum.

In presence of a θ-term, the analysis that lead to eq. (12.21) can be repeated

almost unchanged, the BPS mass formula becoming now

$$M = \sqrt{2}|a\,(n_e + \tau\, n_m)| \quad \text{where} \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\ . \tag{12.22}$$

Note that since $\theta \to \theta + 2\pi n$ is a symmetry of the theory, from the monopole and dyon solutions we found, one can construct a full tower of solutions with charges $\pm(1, -n)$ and $\pm(1, -n-1)$, $n \in \mathbf{Z}$, which are all physically equivalent.

Because of Witten effect, a T-duality transformation acts on the charge vector (n_m, n_e) as

$$T : (n_m, n_e) \to (n_m, -n_m + n_e)\ . \tag{12.23}$$

This implies that under a T-duality transformation, which acts on the complexified gauge coupling as $\tau \to \tau + 1$, the BPS mass formula (12.22) is left invariant. This is consistent with the fact that since masses are physical observables, they should be insensitive to symmetry transformations.

Looking at eq. (12.12), we see that a S transformation, which sends $\tau \to -1/\tau$, should instead act on a charge vector as

$$S : (n_m, n_e) \to (-n_e, n_m)\ . \tag{12.24}$$

If we demand eq. (12.22) to be invariant under S duality, this should then also be accompanied by the shift $a \to a\tau$. More generally, a matrix $A \in Sp(2, \mathbf{Z})$ transforming $(a\,\tau, a)^T$ as $A \cdot (a\,\tau, a)^T$, should correspond to a change of the vector of electric and magnetic charges as $(n_m, n_e) \cdot A^{-1}$. We will re-derive this important result later.

All above analysis has been (semi) classical. In particular, the derivation of the BPS bound and the construction of the monopole and dyon solutions. One might wonder to what extent this still holds at the full quantum level. This is something difficult to check in the Giorgi-Glashow model since an analytical handling of the quantum/strong coupling regime is not possible in such a non-supersymmetric setup. But, as usual, supersymmetry helps.

Let us consider $\mathcal{N} = 2$ pure SYM with gauge group $SU(2)$. Since we are going to use slightly difference normalizations with respect to previous lectures, let us write down the on-shell Lagrangian explicitly

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}g^2 F_{\mu\nu}\tilde{F}^{\mu\nu} + \overline{D_\mu\phi}D^\mu\phi - \frac{1}{2}\left[\phi, \bar{\phi}\right]^2 \\ &-i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\psi\sigma^\mu D_\mu\bar{\psi} + i\sqrt{2}[\bar{\phi}, \psi]\lambda + i\sqrt{2}[\phi, \bar{\lambda}]\bar{\psi}\Big]\ ,\end{aligned} \tag{12.25}$$

where, in present normalizations, $D_\mu = \partial_\mu - iA^a_\mu T_a$. This theory has the same bosonic content of the Giorgi-Glashow model (including a scalar potential which Higgses the theory down to $U(1)$ in the vacuum) and shares with it its basic dynamics. As such, it also admits magnetically charged solitons, as monopoles and dyons.

There are, however, some important differences with respect to the Giorgi-Glashow model. First, in the Giorgi-Glashow model the BPS limit is a rather special limit, since it consists in ignoring the quartic Higgs field potential, just retaining the boundary conditions on the Higgs field at spatial infinity. This is automatically the set-up we have in our $\mathcal{N} = 2$ example, since the potential, $V \sim \left[\phi, \bar{\phi}\right]^2$, is identically 0 on the moduli space (and it is so whenever ϕ is in the Cartan subalgebra, that is $\phi = (0, 0, a)$, for *any* values of a). So here the BPS limit is built in, in a sense. A more important difference regards the BPS mass formula (12.22). This formula is reminiscent of the bound that massive states in the $\mathcal{N} = 2$ spectrum should satisfy and which 1/2 supersymmetry preserving states (short representations) saturate. This suggests that, in presence of charged solitons in the spectrum, the central charge may be related to their electric and magnetic charges. Witten and Olive showed that this is indeed the case. Let us see how it goes.

In present normalizations the $\mathcal{N} = 2$ algebra and the corresponding bound read

$$\{Q^1_\alpha, Q^2_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z \quad , \quad M \geq \sqrt{2}\,|Z| \ . \tag{12.26}$$

Starting from the Lagrangian of pure $\mathcal{N} = 2$ SYM one can compute the corresponding supercurrents $S^1_{\alpha\mu}$ and $S^2_{\alpha\mu}$ by Nöether theorem. Recalling that $Q^I_\alpha = \int d^3x\, S^I_{\alpha\,0}$ one finds (after dropping contributions which trivially vanish at spatial infinity) that

$$\begin{aligned} \{Q^1_\alpha, Q^2_\beta\} &= \frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta} \int d^3x\, \partial_i \left[\left(F^{a0i} - i\tilde{F}^{a0i} \right) \bar{\phi}_a \right] \equiv 2\sqrt{2}\epsilon_{\alpha\beta} Z \\ \{\bar{Q}^1_{\dot\alpha}, \bar{Q}^2_{\dot\beta}\} &= \frac{2\sqrt{2}}{g^2}\epsilon_{\dot\alpha\dot\beta} \int d^3x\, \partial^i \left[\left(-F_{a0i} + i\tilde{F}_{a0i} \right) \phi^a \right] \equiv 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta} Z^* \ . \end{aligned} \tag{12.27}$$

Then, since the electric and magnetic charges of the $U(1)$ low energy effective theory are

$$q = -\frac{1}{ag} \int d^3x\, \partial_i \left(F^{0i}_a \phi^a \right) = g n_e \quad , \quad p = -\frac{1}{ag} \int d^3x\, \partial_i \left(\tilde{F}^{a0i} \bar{\phi}^a \right) = \frac{4\pi}{g} n_m \tag{12.28}$$

one finally finds (after taking into account the effect of a non-trivial θ-term) that

$$\mathrm{Re}\, Z = a n_e \quad , \quad \mathrm{Im}\, Z = a\tau n_m \tag{12.29}$$

and hence eq. (12.22). So we learn that in presence of monopoles (and dyons) the supersymmetry algebra must be modified with the addition of a non-trivial central charge (which measures the electric and magnetic charges of soliton solutions). From a geometric viewpoint this should not come as a surprise. Indeed, supersymmetry charges are space integrals. In calculating their anticommutators one has to deal with surface terms, which one usually neglects. However, as shown in eqs. (12.28), in presence of electric and magnetic charges these surface terms are non-zero and give rise to a non-vanishing Z.

Note also that, unlike the Giorgi-Glashow model, here the relation between masses and charges of BPS states, eq. (12.22), does not come from a (semi) classical analysis but is dictated by the supersymmetry algebra. Hence, it cannot be spoiled quantum mechanically and should remain valid even when perturbative and non-perturbative corrections are taken into account. Indeed, BPS states are in short representations and quantum corrections cannot generate the extra degrees of freedom needed to convert a short multiplet in a long one. So, BPS saturated states remain so also at the quantum level.

That eq. (12.22) persists quantum mechanically, does not mean that the quantities therein do not undergo renormalization. For one thing, in $\mathcal{N} = 2$ we know that the gauge coupling τ runs (at one loop and non perturbatively). Therefore, upon taking into account renormalization effects, while by its very definition the central charge Z is still a linear combination of conserved (electric and magnetic) abelian charges, the coefficients multiplying n_m and n_e will be replaced by some (holomorphic) functions, which we dub a and a_D , of the strong coupling scale Λ and field VEVs

$$Z = a\, n_e + a_D\, n_m \ . \tag{12.30}$$

In the classical limit a is the VEV of the scalar field ϕ and $a_D = a\tau$. But one expects this not to be true at the full quantum level. In particular, the expression of a_D in terms of a could be different. Seiberg and Witten proposed the following exact relation between a and a_D

$$a_D \equiv \frac{\partial \mathcal{F}}{\partial a} \quad \text{that is} \quad \tau = \frac{da_D}{da} \ , \tag{12.31}$$

with $\mathcal{F}$ the prepotential. We will provide evidence for the proposal (12.31) later. Here, just notice that this way eq. (12.30) has the correct semi-classical limit but, unlike (12.22), is, by construction, renormalization group invariant.

One of our main goals in the following will be to check the proposal (12.31) and

to compute the exact expression of a and a_D in terms of the scalar field VEVs and Λ. Given this information, the masses of all BPS states (fundamental fields as well as magnetic monopoles and dyons) will be known exactly in terms of the moduli parameters. More importantly, finding the exact expressions of a and a_D amounts to find the exact expression for τ and hence, by (12.10), the full effective action!

This discussion can be repeated for $\mathcal{N} = 4$ SYM, which is also expected to admit charged solitons in its spectrum. There, however, the relation $a_D = \tau a$ is not renormalized, since in this case τ is classically exact, as so is the moduli space. This has important consequences which we will come back to, when discussing quantum properties of $\mathcal{N} = 4$ SYM.

So far, we have been considering pure $\mathcal{N} = 2$ $SU(2)$ SYM. One may want to add matter fields, *i.e.* hypermultiplets. This amounts to add to the Lagrangian the superpotential term

$$\sum_{i=1}^{F} \left(\sqrt{2} H_1^i \Phi H_2^i + m_i H_1^i H_2^i \right) + h.c. \ . \tag{12.32}$$

For equal masses, the theory has a $SU(F)$ flavor symmetry, which is broken to $U(1)^F$ for generic values of m_i. One can repeat previous computation and calculate the contribution of H_1 and H_2 to the supercurrent and, in turn, to the central charge mass formula (12.30). The end result is

$$Z = a\, n_e + a_D\, n_m + \sum_{i=1}^{F} \frac{1}{\sqrt{2}}\, m_i S_i \ , \tag{12.33}$$

where S_i are global conserved $U(1)$ charges under which H_1^i and H_2^i have charges $+1$ and -1, respectively.

There exist generalizations of this construction. The Giorgi-Glashow model can be generalized to a gauge theory with gauge group G spontaneously broken to a subgroup H by some Higgs-like field transforming in the adjoint representation of G. Thanks to the topological nature of soliton solutions, it turns out that an analysis on their existence can be carried-on in the context of homotopy theory. In particular, inequivalent solutions are classified by the homotopy group $\Pi_2(G/H)$. A necessary condition for this not to be trivial, is that G is simply connected. If this is the case, $\Pi_2(G/H) = \Pi_1(H)$ and so everything boils down to compute the first homotopy group of the unbroken group H. For example, in the Giorgi-Glashow model we have $\Pi_2(G/H) = \Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = \mathbf{Z}$, and one family of magnetic

monopoles with integer charge is indeed present. Conversely, the Standard Model gauge group is not simply connected and so there do not exist monopole-like solutions. In GUT theories, instead, solitons can exist. Taking, *e.g.*, $G_{GUT} = SU(5)$ one has $\Pi_2(G/H) = \Pi_2(SU(5)/SU(3) \times SU(2) \times U(1)) = \Pi_1(SU(3) \times SU(2) \times U(1)) = \Pi_1(U(1)) = \mathbf{Z}$ (in fact, that the electric charge happens to be quantized can be seen as an evidence in favour of the existence of monopoles and, in turn, of GUT theories).

Exactly as for the original Giorgi-Glashow model, this more general story finds a natural embedding in supersymmetric contexts. One such situation is nothing but the low energy effective theories describing the $\mathcal{N} = 2$ (and $\mathcal{N} = 4$) Coulomb branch we are actually interested in. There, we have a gauge theory with gauge group G broken to its Cartan subalgebra $H = U(1)^n$, where $n = \operatorname{rank} G$. From previous general analysis, it follows that magnetically charged solitons are present in the spectrum. Most of what we said above holds unchanged. In particular, the IR-free effective theory is form-invariant under electro-magnetic duality transformations which are the natural generalization to $n > 1$ of eq. (12.19), and act on the couplings as

$$\tau_{IJ} \to \left(A_I^L \tau_{LM} + B_{IM}\right) \left(C^{JN} \tau_{NM} + D_M^J\right)^{-1} \tag{12.34}$$

where now $M \equiv \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2n, \mathbf{Z})$. The vector of electric and magnetic charges is now a $2n$-component row vector (n_m^I, n_{eJ}). The corresponding BPS mass formula which generalizes (12.30) is

$$Z = a^I \, n_{eI} + a_{D,I} \, n_m^I = a \cdot n_e + a_D \cdot n_m \ , \tag{12.35}$$

where, in the second step, matrix multiplication is understood and (12.31) is now $a_{DI} \equiv \partial \mathcal{F} / \partial a^I$. The addition of (massive) flavors changes the central charge formula in a way similar to eq. (12.33).

Let us conclude this section with a comment which will be relevant later. As already noticed, electro-magnetic duality transformations are not symmetries of the theory. They just express the equivalence of abelian theories coupled to massive sources under a $Sp(2n, \mathbf{Z})$ redefinitions of electric and magnetic charges. It is a redundancy of the effective Lagrangian description. The point, though, is that this redundancy can capture important features of the theory, when a moduli space is present (which is the case we will actualy be concerned with). This can be seen as follows.

Suppose there is a moduli space of vacua and that the effective dynamics on

this moduli space is described by n abelian gauge fields and a bunch of massless, neutral scalars, collectively dubbed ϕ, which parametrize $\mathcal{M}$. Upon traversing a closed loop in $\mathcal{M}$ the physics must be the same at the beginning and at the end of the loop. However, the Lagrangian does not need to: it can just be invariant modulo a electro-magnetic duality transformation. Geometrically, this corresponds to say that the matrix of couplings τ_{IJ} is a section of a $Sp(2n, \mathbf{Z})$ bundle. In matrix notation, this means that upon making a circle in the ϕ moduli space, the matrix τ should transform as

$$\tau(e^{2\pi i}\phi) = \frac{A \cdot \tau(\phi) + B}{C \cdot \tau(\phi) + D} \ . \tag{12.36}$$

The element $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2n, \mathbf{Z})$ is called *monodromy* around the loop. If the closed loop does not encircle any singularity, the monodromy is the identity element of $Sp(2n, \mathbf{Z})$. If it does, the monodromy is a non-trivial element of $Sp(2n, \mathbf{Z})$, instead.

Singularities on the moduli space are associated to massive particles becoming massless, there. The monodromy matrix tells about the nature of such particle. Since masses are physical observables, the BPS mass formula (12.35) should be invariant under monodromies. Hence, as already emphasized for the case $n = 1$, a $A \in Sp(2n, \mathbf{Z})$ transforming $(a_D, a)^T$ as $A \cdot (a_D, a)^T$, should correspond to a change of the vector of electric and magnetic charges as $(n_m, n_e) \cdot A^{-1}$. This means that, in general, the action of the monodromy changes the quantum numbers of charged states.

Now, the state of vanishing mass at a given singularity on the moduli space, should be invariant under the action of the monodromy associated to the singularity itself (it is the properties of such massless state which determine the monodromy matrix). That is to say, it should be a left eigenvector of the matrix A with unit eigenvalue. It is easy to see that the vector (n_m, n_e) is an eigenvector of the $Sp(2n, \mathbf{Z})$ matrix

$$A(n_m, n_e) = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} \tag{12.37}$$

with unit eigenvalue. This means that at a singularity with monodromy matrix of the form above, a state with charges $\pm(n_m, n_e)$ becomes massless.

Actually, any state with charges $l(n_m, n_e)$ with $l \in \mathbf{Z}$ is left invariant by the action of (12.37). However, stable dyons require ln_m and ln_e to be relatively prime, which is the case only for $l = \pm 1$. One way to see it is the following. Suppose to start from a BPS saturated state with charges (N_m, N_e) and mass $M = \sqrt{2}|Z|$. Such state can decay into states whose sum of masses should be less or equal M.

For each of these states we have $Z_i = a \cdot n_e^i + a_D \cdot n_m^i$ and $M_i \geq \sqrt{2}|Z_i|$. Since charge conservation implies that $Z = \sum Z_i$, it follows from triangle inequality that $|Z| \leq \sum_i |Z_i|$ which in turn implies that

$$M \leq \sum_i M_i \ . \tag{12.38}$$

In order for the decay to occur the above bound should be saturated, which implies $|Z| = \sum_i |Z_i|$ (so also the states with charges (n_m^i, n_e^i) should be BPS). This can happen if and only if the vectors (N_m, N_e) and (n_m^i, n_e^i) are proportional, that is if N_m and N_e are not relatively prime, $(N_m, N_e) = l(n_m, n_e)$. If they are, instead, the decay cannot occur.

12.3 Seiberg-Witten theory

Following the discussion in § 12.1.1, we would like now to consider asymptotically free $\mathcal{N} = 2$ gauge theories and try to see what can we say about their low energy effective dynamics. As advertised, we will focus on the Coulomb branch, which is the only part of the moduli space which is modified at the quantum level. What this boils down to is to determine the exact expression of the prepotential $\mathcal{F}$, more specifically of the generalized complexified gauge coupling matrix, whose imaginary part is the metric on the Coulomb branch.

Our starting point is some UV-free $\mathcal{N} = 2$ matter-coupled Lagrangian. Recalling that the perturbative gauge coupling β function of $\mathcal{N} = 2$ is one loop exact, this means that if, for example, the gauge group is taken to be $SU(N)$ and matter multiplets transform in the (anti)fundamental representation of $SU(N)$, we must require that $F < 2N$, since the one-loop coefficient is proportional to $2N - F$, in this case.

One thing which will play an important role later is the R-symmetry breaking pattern. Let us first focus on the pure SYM part of the Lagrangian. Besides its compact component, $SU(2)_R$, under which all bosons in the $\mathcal{N} = 2$ vector multiplet are singlets and the two gaugini transform as a doublet, there is also a $U(1)_R$ symmetry, under which (both) gaugini have $R(\lambda, \psi) = 1$. This symmetry is anomalous and, following the same discussion we had for $\mathcal{N} = 1$ SYM, since the anomaly coefficient is now $A = 2\,T(Adj)$ one can see that it gets broken as

$$U(1)_R \ \longrightarrow \mathbf{Z}_{4\,T(Adj)} \ . \tag{12.39}$$

For example, taking $G = SU(N)$ we have that $U(1)_R \to \mathbf{Z}_{4N}$. The fact that $R(\psi) = 1$ implies that the adjoint scalars ϕ have $R(\phi) = 2$, meaning that on the Coulomb branch the residual symmetry gets further broken. For example, we will see later that for $G = SU(2)$ Coulomb branch vacua preserve a $\mathbf{Z}_4$ subgroup of the full $\mathbf{Z}_8$ and, therefore, each point on the Coulomb branch is paired with its mirror under the residual $\mathbf{Z}_2$, which acts non-trivially on $\mathcal{M}$. For $G = SU(3)$ a $\mathbf{Z}_2$ subgroup survives, only, while for higher ranks the $U(1)_R$ is fully broken.

Interestingly, unlike $\mathcal{N} = 1$, the addition of matter does not restore an anomaly-free $U(1)_R$ symmetry, in general. Indeed, given that $R(\phi) = 2$, from the cubic superpotential term $\sim H_1 \Phi H_2$ one sees that the hyperscalars are neutral. Hence, their fermionic partners ψ_1 and ψ_2 have $R(\psi_1, \psi_2) = -1$. So, if the hypermmultiplets transform in the representation $\underline{r}$ of the gauge group G, the $U(1)_R$ is broken at the quantum level as

$$U(1)_R \longrightarrow \mathbf{Z}_{4T(Adj)-4T(r)} \ , \tag{12.40}$$

if adding one hypermultiplet. For example, taking $G = SU(N)$ and F hypers in the fundamental representation, one gets $U(1)_R \to \mathbf{Z}_{4N-2F}$. Note that for $F = 2N$ the R-symmetry is not anomalous, in agreement with the vanishing of the β function and the supposedly conserved R-charge in superconoformal field theories.

Let us start considering pure SYM and let us take, for definiteness, the gauge group to be $G = SU(N)$. Following our general discussion in section § 12.1.1 the low energy Coulomb branch effective (bosonic) Lagrangian looks like

$$\mathcal{L} \ = \ \mathrm{Im}(\partial_\mu \bar{\phi}^I \, \partial^\mu \phi_{DI}) + \frac{1}{2} \, \mathrm{Im}[\tau_{IJ}(\phi) \mathcal{F}^I_{\mu\nu} \mathcal{F}^{J\mu\nu}] \tag{12.41}$$

where $I = 1, 2, \ldots, N-1$. Solving the theory amounts to find the exact expression for the prepotential $\mathcal{F}$ or, which is the same, for the effective abelian gauge coupling matrix τ_{IJ}, as a function of Λ and of (gauge-invariant combination of) scalar field VEVs. Recall that τ_{IJ} gets one-loop and non-perturbative corrections, only, and reads (we refer collectively to a as the common VEV of all scalar fields ϕ^I)

$$\tau_{IJ}(a, \Lambda) = \frac{2N}{2\pi i} C_{IJ} \log \frac{\Lambda}{a} + \sum_{n=1}^{\infty} d_{IJ,n} \left(\frac{\Lambda}{a} \right)^{b_1 n} \ , \tag{12.42}$$

where $2N$ is the one-loop β-function coefficient, C_{IJ} is a matrix which can be computed in perturbation theory and $d_{IJ,n}$'s weight n-instanton corrections. Since the model is Higgsed at a scale a, which can be taken arbitrarily large, these instanton

effects can be made arbitrarily small and are calculable. So, in principle, one could compute τ_{IJ}, and hence solve the low energy effective theory exactly, by evaluating all instanton contributions. In practice, this is hard. Seiberg and Witten came-up with a more physical approach to determine τ_{IJ}, which is the one we will follow.

We start analyzing the simplest case, $\mathcal{N} = 2$ SYM with gauge group $SU(2)$.

12.3.1 $\mathcal{N} = 2$ $SU(2)$ pure SYM

$\mathcal{N} = 2$ SYM with $G = SU(2)$ admits a one-dimensional moduli space in which the gauge group is broken to $U(1)$. The gauge invariant coordinate on the (classical) moduli space can be chosen to be

$$u = \frac{1}{2}\langle \mathrm{tr}\, \phi^2 \rangle = a^2 \ , \tag{12.43}$$

where $\langle \phi \rangle = a\sigma_3$ is the (adjoint) scalar field VEV. The u here corresponds to what we called u_2 in eq. (12.5).

The above formula is valid classically. Quantum corrections may change the relation between u and a. In what follows, we will keep on calling u the coordinate on the quantum moduli space but the above equation will be modified as

$$u = \frac{1}{2}\langle \mathrm{tr}\, \phi^2 \rangle = a^2 + \text{ quantum corrections } . \tag{12.44}$$

While classically $a = \sqrt{u}$, quantum mechanically one could expect a more general relation, $a = a(u)$, which only in the classical limit reduces to $a = \sqrt{u}$.

The abelian low energy effective Lagrangian is

$$\mathcal{L} \ = \ \mathrm{Im}\left[\tau(\phi)\left(\frac{1}{2}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \partial_\mu \bar{\phi}\,\partial^\mu \phi\right)\right] \ , \tag{12.45}$$

and it is univocally determined knowing the exact expression of τ and so of the holomorphic prepotential $\mathcal{F}$, since $\tau = \partial^2\mathcal{F}/\partial\phi\partial\phi$. Our goal, in the following, will then be to find the exact expression for $\mathcal{F}$ and hence of the analogous of (12.42) which in this case becomes

$$\tau(a,\Lambda) = \frac{1}{2\pi i}\log\left(\frac{\Lambda^4}{a^4}\right) + \sum_{n=1}^{\infty} d_n \left(\frac{\Lambda^4}{a^4}\right)^n \ . \tag{12.46}$$

The first thing one should readily notice is that $\mathcal{F}$ cannot be a holomorphic function of a all along $\mathcal{M}$, it should be multivalued. Indeed, if this were not the case,

$\mathrm{Im}\,\tau(a) = \mathrm{Im}\,\partial^2\mathcal{F}(a)/\partial\phi\partial\phi$ would be a harmonic function. As such, it could not be positive definite everywhere (unless it were a constant). Hence, there would necessary be regions in the moduli space where $\mathrm{Im}\,\tau(a)$ would be negative, making the effective gauge coupling squared g^2 being negative, too. This would correspond to propagation of negative norm states, that cannot be. We need $\mathrm{Im}\,\tau > 0$. The way out is to allow for different local descriptions which requires $\mathcal{F}(a)$ to be defined only locally, say in a neighborhood of the classical region $u \to \infty$. In regions where $\tau(a)$ approaches zero, we need a different (but equivalent) description of the theory. Note that this corresponds to regions where the gauge coupling is very large, eventually infinite, so strongly coupled regimes. Geometrically, the moduli space should admit singularities and in the vicinity of such singularities we expect a different coordinate patch with respect to a.

In order to understand how different local descriptions can emerge, we have to understand how electric-magnetic duality is realized in the low energy effective theory. The action (12.45) can be re-written as

$$\mathcal{L} = \frac{1}{2}\,\mathrm{Im}\,[\tau(\phi)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] + \frac{1}{2}\,\partial_\mu \begin{pmatrix} \phi_D \\ \phi \end{pmatrix}^\dagger J\,\partial^\mu \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \tag{12.47}$$

where $\phi_D \equiv \partial\mathcal{F}/\partial\phi$. The scalar kinetic term is invariant under $Sp(2,\mathbf{R})$ transformations acting on ϕ_D and ϕ as

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to M \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \quad \text{where} \quad M^\dagger J M = J \ . \tag{12.48}$$

This is the continuum version of the duality group of electro-magnetism previously defined and it is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad b \in \mathbf{R} \ . \tag{12.49}$$

To see how the duality group acts on the Maxwell term we should first write the Lagrangian introducing a Lagrange multiplier field $A_{D\mu}$, so to have both the equation of motion and the Bianchi identity emerging as field equations (this is needed, since S duality is a non-local transformation in the electro-magnetic fields). Recalling that $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + i\tilde{F}_{\mu\nu}$, we can write the Maxwell term action as

$$\mathcal{S} = \int \mathrm{Im}\left[\frac{1}{2}\,\tau(\phi)\left(F_{\mu\nu} + i\tilde{F}_{\mu\nu}\right)^2\right] + \int A_{D\mu}\partial_\nu \tilde{F}^{\mu\nu} \ . \tag{12.50}$$

where $A_{D\mu}$ should be treated as an independent field, with field strength $F_{D\mu\nu} = \partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}$.

In doing the path integral one should now integrate over F and A_D. Path integrating over A_D we get the equation $dF = 0$, which implies $F = dA$ and we are left with the original path integral. But one can path integrate over F, first. From eq. (12.50), integrating by parts, the second term in eq. (12.50) becomes $F_{D\mu\nu}F^{\mu\nu}$ and completing the square one finally gets

$$\mathcal{S} = \int \mathrm{Im}\left[\frac{1}{2}\tau(\phi)\left\{\left(F_{\mu\nu} + i\tilde{F}_{\mu\nu}\right) + \frac{1}{\tau(\phi)}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)\right\}^2 - \frac{1}{\tau(\phi)}\frac{1}{2}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)^2\right] . \tag{12.51}$$

Performing the Gaussian path integral over F one gets back the original action (up to an overall normalization) but in terms of the dual gauge field $A_{D\mu}$ as

$$\mathcal{S} = \int \mathrm{Im}\left[-\frac{1}{\tau(\phi)}\frac{1}{2}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)^2\right] . \tag{12.52}$$

So we see that, as expected, the effect of a S-duality transformation which transforms the gauge coupling as $\tau \to \tau_D = -1/\tau$, is to replace a gauge field, to which electric sources couple locally, by a dual gauge field, to which magnetic sources couple locally. The other generator of $Sp(2, \mathbf{R})$, T_b, does not act on the gauge field but on the coupling, only, shifting the θ angle. In order for it not to change the physics, one should take $b \in \mathbf{Z}$, hence obtaining the actual E-M duality group, which is $Sp(2, \mathbf{Z})$. We will henceforth call T the generator T_1, to be consistent with conventions in § 12.2.

In our previous discussion, we argued that whenever $\mathrm{Im}\,\tau(a)$ approaches 0, a different description of the (same) physics should hold. The above discussion suggests what that can be: an S-dual description in terms of a magnetic dual gauge field $A_{D\mu}$, with $\tau \to \tau_D = -1/\tau$ and ϕ and ϕ_D exchanged, see eq. (12.48).

Note, in passing, that the way the duality group acts on (ϕ_D, ϕ), and in turn on (a_D, a), eq. (12.48), provides further evidence for the BPS mass formula (12.30) and the proposed relation (12.31). To see this, let us couple the low energy effective theory to a charged hypermultiplet with charge n_e. Its coupling is fixed by $\mathcal{N} = 2$ supersymmetry to be

$$\sqrt{2}\, n_e H_1 \Phi H_2 . \tag{12.53}$$

On the moduli space this induces a mass for the (BPS!) hypermultiplet whose corresponding central charge would then be $Z = n_e a$. By a S-duality transformation,

it is clear that for a magnetic monopole with magnetic charge n_m we would have $Z = n_m a_D$ (with $a_D = \partial\mathcal{F}/\partial a$) and, for dyons, the more general formula (12.30).

Singularities and monodromies

We now have all ingredients to understand the singularity structure of the moduli space and the physical meaning of such singularities.

Let us start looking at the (semi)classical region, namely $u \to \infty$. There one can safely use the classical relation $u = a^2$ and the one-loop expression for the prepotential

$$\mathcal{F}_{1-loop} = \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} \ . \tag{12.54}$$

From this expression we can compute a_D which is

$$a_D = \frac{\partial\mathcal{F}}{\partial a} = \frac{i}{\pi} a \left(\log \frac{a^2}{\Lambda^2} + 1 \right) \ . \tag{12.55}$$

Let us take a counterclockwise contour in the u plane, say $u \to e^{2\pi i} u$, with very large $|u|$. Since in such semiclassical region $u = a^2$ we see that a transforms as $a \to -a$. For a_D, instead, using (12.55), we get

$$a_D \to \frac{i}{\pi}(-a) \left(\log \frac{e^{2\pi i} a^2}{\Lambda^2} + 1 \right) = -a_D + 2a \ . \tag{12.56}$$

So, there is a non-trivial monodromy, which acts on the vector (a_D, a) as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \quad \text{where} \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{12.57}$$

Note that, consistently with previous general discussion, $M_\infty \in Sp(2, \mathbf{Z})$. More specifically, $M_\infty = -T^{-2}$, with T the generator previously defined.

The log term in a_D and the non-trivial monodromy show that a and a_D are multivalued functions: there is a branch cut extending from infinity, due to the log term in the one-loop running. Given the singularity at $u = \infty$, there must be singularities also somewhere else on the u plane, with their associated monodromies M_i. Since a contour circling around infinity can be deformed (it is topologically equivalent) to a contour circling around all other singularities, say we have k of them, the following consistency relation should hold, in general

$$M_\infty = M_1 \, M_2 \, ... \, M_k \ . \tag{12.58}$$

Now, how many singularities, besides that at $u = \infty$, do we have on the u plane? The R-symmetry breaking pattern helps, here. As already discussed, the $U(1)$ R-symmetry of the original theory is anomalous and broken to $\mathbf{Z}_8$ at the quantum level. Since ϕ has R-charge 2, on the moduli space, parametrized by $u \sim \langle \mathrm{Tr}\, \phi^2 \rangle$, this is further broken to $\mathbf{Z}_4$. The residual $\mathbf{Z}_2$ symmetry acting on the moduli space changes u as $u \to -u$. Therefore, singularities should come in pairs on the moduli space, but at the fixed points of the $\mathbf{Z}_2$ action $u = \infty, 0$. We conclude that if we had one only more singularity beside the one at infinity, this should be at $u = 0$.

But this cannot be. If there were just one singularity at $u = 0$, because of (12.58) we would have $M_0 = M_\infty$. But then, since a^2 is left invariant by M_∞, $u = a^2$ would be a good global coordinate on the full moduli space, not just in the classical region. Then, $\mathcal{F}(a)$ would be a holomorphic function of a and so $\mathrm{Im}\, \tau(a)$ a harmonic function. But then, the latter could not be positive definite (unless it were a constant, which we know it is not).

So, we conclude that there must be at least two singularities besides that at infinity, located at, say $u = \pm u_0$. If this is this case, $u = 0$, which is a singular point on the classical moduli space, will not be a singular point anymore in the quantum theory. In order to have a singularity at $u = 0$ one should have at least three singularities on $\mathcal{M}$. As we will see, this cannot be either. Having two (and only two) singularities on $\mathcal{M}$, located at $u = \pm u_0$, seems to be the only consistent possibility.

A natural question to ask is what the nature of the particles becoming massless at $u = \pm u_0$ is. Interestingly, unlike to what happens classically, singularities in the quantum moduli space are not associated to extended gauge symmetry, namely to extra massless gauge bosons. This can be understood as follows. If an interacting non-abelian Coulomb phase were there in the IR, a conserved R-symmetry should be present (the superconformal R-symmetry). As we have already discussed, singularities on $\mathcal{M}$ occur at $u \neq 0$. Hence, if conformal invariance should be preserved, then the dimension of u at the singularity should be zero. In a SCFT the dimension of an operator is proportional to its R-charge, which for the operator u is $R(u) = 4$ since $R(\phi) = 2$. Therefore, at the singularity the operator u would have non-vanishing scaling dimension and a VEV would break conformal invariance. This suggests that a non-abelian Coulomb phase cannot emerge in the IR. The extra massless degrees of freedom cannot be gauge bosons.

The analysis of the previous section, suggests what the other possibility could be.

The only other states in the spectrum (at least as far as we know) are monopoles and dyons. For examples, magnetic monopoles are very heavy at weak coupling, because of the BPS mass formula (12.30) and tend to become light at strong coupling. So it might very well be that these are the states becoming massless at the strong coupling singularities. Note that if this is the case, by the reasoning in the previous paragraph, we conclude that they cannot sit in vector multiplets (which would include spin 1 particles). So they should correspond to hypermultiplets. Indeed, in the $\mathcal{N} = 2$ version of the Giorgi-Glashow model, this was explicitly shown to be the case.

As a corollary, one would expect that the singularity at $a = 0$ of the classical moduli space, where extra massless gauge bosons did become massless, should disappear at quantum level. From the exact expression we will eventually get for $a = a(u)$, we will see that this is indeed the case: the point $a = 0$ does not belong to the moduli space, at quantum level (similarly to what happens for $\mathcal{N} = 1$ SQCD with $F = N$).

Let us then focus on the strong coupling singularities at $u = \pm u_0$. Note that u_0 should be proportional to Λ^2, since in the classical limit, $\Lambda \to 0$, one should recover the (only one) singularity at $u = 0$. Hence, from now on, without loss of generality, we will take $u_0 = \Lambda^2$. The structure of the moduli space, with punctures and associated monodromies, is depicted in figure 12.4.

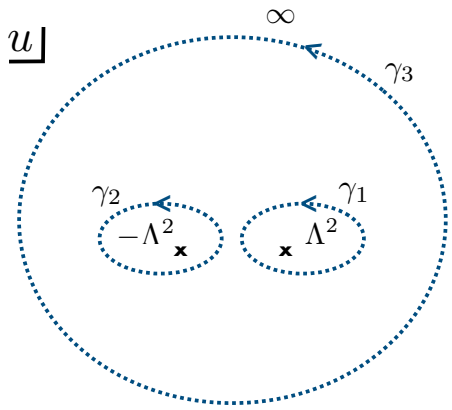

Figure 12.4: The u plane with the three singularities at $\infty, \Lambda^2, -\Lambda^2$. The monodromies associated to the three cycles γ_i must satisfy the consistency relation $M_\infty = M_{\Lambda^2} M_{-\Lambda^2}$.

To find the structure of the monodromy matrices M_{Λ^2} and $M_{-\Lambda^2}$ notice that they

should have a form like (12.37) in terms of the (integer) electric and magnetic charges (n_m, n_e), (n'_m, n'_e) of the corresponding massless states. Imposing the consistency relation $M_\infty = M_{\Lambda^2} M_{-\Lambda^2}$ and using (12.57) one finds that the unique solution (modulo physically equivalent solutions, cf the comment after eq. (12.22)) is

$$(n_m, n_e) = \pm(1, 0) \quad , \quad (n'_m, n'_e) = \pm(1, -1) \tag{12.59}$$

corresponding to monodromy matrices

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad , \quad M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ . \tag{12.60}$$

So, we finally see what the nature of the singularities is: at $u = \Lambda^2$ a monopole with charge $\pm(1, 0)$ becomes massless and at $u = -\Lambda^2$ a dyon with charge $\pm(1, -1)$ does. Note, in passing, that $M_{\Lambda^2} = ST^2S^{-1}$ and $M_{-\Lambda^2} = TST^2S^{-1}T^{-1} = TM_{\Lambda^2}T^{-1}$, which nicely agrees with the fact that the residual $\mathbf{Z}_2$ symmetry connecting $u = \Lambda^2$ and $u = -\Lambda^2$ shift the θ angle by 2π.

One might wonder if there can be more than two singularities on $\mathcal{M}$. For this to be the case, one should be able to solve an equation like (12.58) with $k > 2$, with M_∞ given by (12.57) and the M_k's having a structure as (12.37) with integers (n^i_m, n^i_e). While a general proof is not available, one can explicitly show, for not too large values of k, that there are no solutions for $k > 2$.

Seiberg-Witten curve

Given the knowledge of the singularity structure of the moduli space and its monodromies, we want now to construct holomorphic functions $a = a(u)$ and $a_D = a_D(u)$ satisfying the monodromies (12.57) and (12.60). A holomorphic function is univocally determined by its singularities. Therefore, if we are able to find a function with the correct monodromies around $u = \infty, \Lambda^2, -\Lambda^2$, we can be sure we get the correct answer. In principle, this can be done, but we will follow a different, more geometric pattern. This was the approach originally pursued by Seiberg and Witten and also the one which makes it easier and more natural to understand generalizations to richer theories (more general gauge groups and matter content).

The crucial observation comes from the property τ should have: a complex quantity with positive definite imaginary part, $\mathrm{Im}\,\tau > 0$. Such quantities are fundamental in the theory of Riemann surfaces, where they describe their moduli, the positivity

condition ensuring regularity of the surface. In the case at hand, the relevant Riemann surface is a torus, or equivalently, using the language of algebraic geometry, an elliptic curve. This curve can be written as a complex surface

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) \ , \tag{12.61}$$

where u parametrizes the modulus τ of the torus and x and y are complex coordinates. Varying u we vary τ and hence eq. (12.61) describes a family of tori.

A way to understand that (12.61) describes a torus is as follows. From eq. (12.61) we see that y is the square root of a polynomial in x so we can look at the x-plane consisting into two sheets with branch points at Λ^2, $-\Lambda^2$, u and ∞, gluing along the branch cuts (the two sheets corresponding to the $\pm y$ branches). One can take one branch cut between $-\Lambda^2$ and Λ^2 and the second one between u and ∞. The two sheets can be thought as spheres and the branch cuts as tubes connecting them. Topologically, this is a torus, see figure 12.5.

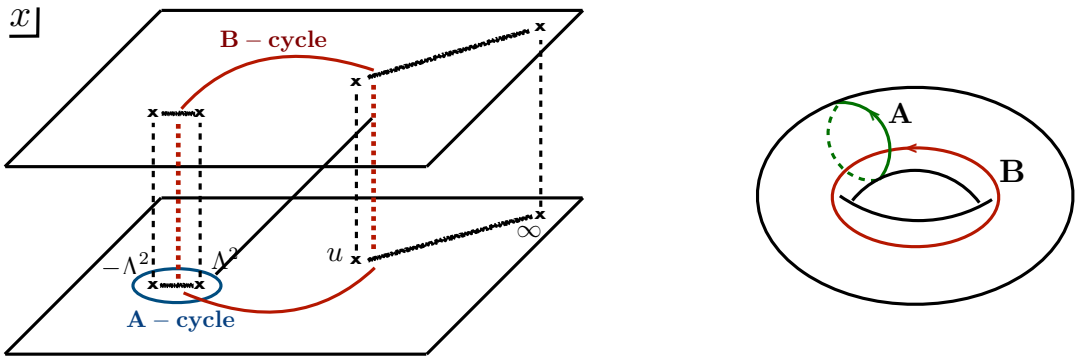

Figure 12.5: On the left the elleptic curve in the (two sheeted) x plane. A and B are the two one-cycles of the torus. On the right the corresponding torus.

On a torus the are two independent, non-trivial homology one-cycles, the A and the B cycles, which we can take as in the figure. Degenerate tori (that is tori where some cycles shrink to zero size) occur when any two zero's of eq. (12.61) coincide. In other words, when one of the branch cuts disappears. In particular, for $u = \Lambda^2$ the B cycle shrinks to zero size, for $u = \infty$ the A cycle shrinks to zero size and for $u = -\Lambda^2$ a linear combination of the two, $A + B$, does.

The basis of one-cycles is not unique, but defined up to $Sp(2, \mathbf{Z})$ transformations

which act as

$$\begin{pmatrix} B \\ A \end{pmatrix} \to M \begin{pmatrix} B \\ A \end{pmatrix} \quad \text{where} \quad M \in Sp(2, \mathbf{Z}) \ . \tag{12.62}$$

The modulus of the torus, $\tau(u)$, corresponds to the ratio of the periods ω and ω_D, the integrals over the A and B cycles of the unique holomorphic (closed) one-form dx/y,

$$\tau(u) = \frac{\omega_D}{\omega} \ , \tag{12.63}$$

where

$$\omega = \oint_A \frac{dx}{y} \quad , \quad \omega_D = \oint_B \frac{dx}{y} \qquad \text{with} \quad \frac{dx}{y} = \frac{dx}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)(x-u)}} \ . \tag{12.64}$$

Note that the periods ω and ω_D inherit from the A and B cycles the transformation properties under $Sp(2, \mathbf{Z})$ and, so, also the same monodromies at the three singular points $u = \infty, \pm\Lambda^2$, where two branch points collide.

The identification between the $SU(2)$ gauge theory and the above family of tori parametrized by u, holds via identifying the modulus of the torus with the complexified gauge coupling, and the periods with the u derivative of a and a_D, that is

$$\tau = \frac{\omega_D}{\omega} \equiv \tau(a) = \frac{\partial a_D}{\partial a} = \frac{\partial a_D / \partial u}{\partial a / \partial u} \ , \tag{12.65}$$

with the identification

$$\frac{\partial a}{\partial u} = \omega = \oint_A \frac{dx}{y} \quad , \quad \frac{\partial a_D}{\partial u} = \omega_D = \oint_B \frac{dx}{y} \ . \tag{12.66}$$

Note, in passing, that since u is globally defined, a and a_D have the same monodromies of the periods ω and ω_D.

Integrating in u on both sides one obtains

$$a = \oint_A d\lambda \quad , \quad a_D = \oint_B d\lambda \ , \tag{12.67}$$

where the one-form differential $d\lambda$ can be easily computed to be

$$d\lambda = \frac{(x-u)dx}{y} \ , \tag{12.68}$$

up to exact forms. Using the above definition of A and B cycles, and deforming them so to lie entirely along the cuts between $-\Lambda^2$ and Λ^2 and between Λ^2 and u,

respectively, one can express the integrals (12.67) as

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2} dx \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} \tag{12.69}$$

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{\Lambda^2}^{u} dx \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} \ , \tag{12.70}$$

where the overall normalization has been fixed by requiring that for $u \to \infty$ one recovers the (semi)classical result (12.55). Using the identity

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dx\, x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha} \ , \tag{12.71}$$

one can finally recast (12.69) and (12.70) in terms of hypergeometric functions

$$a(u) = \sqrt{2}(\Lambda^2+u)^{1/2}\, F(-\frac{1}{2},\frac{1}{2},1;\frac{2}{1+u/\Lambda^2}) \tag{12.72}$$

$$a_D(u) = i\,\frac{\Lambda-u/\Lambda}{2}\, F(\frac{1}{2},\frac{1}{2},2;\frac{1-u/\Lambda^2}{2}) \ . \tag{12.73}$$

One can invert (12.72) to obtain $u(a)$ and insert the result into (12.73) to obtain $a_D(a)$. Integrating with respect to a yields $\mathcal{F}(a)$. Equivalently, deriving with respect to a yields $\tau(a)$ and, hence, the exact expression of the low energy effective action! Let us emphasize again that the expression one gets for $\mathcal{F}(a)$ is not globally defined on the moduli space, and different analytic continuations should be used in different patches. For example, near $u=\Lambda^2$, better to use S-dual coordinates, where the role of what is electric and what is magnetic is inverted. This is represented in figure 12.6.

As a check that the result we got describes the coupling τ entering the effective Lagrangian (12.45), one can expand (12.73) and (12.72) around $u = 0, \Lambda^2$ and $-\Lambda^2$ and show agreement with the expected (singular) behavior for a_D and a, including the monodromies (12.57) and (12.60).

- For $u \to \infty$ we have $a \sim \sqrt{u}$ and $a_D \sim \frac{i}{\pi}\sqrt{u}\log\frac{u}{\Lambda^2} \sim \frac{i}{\pi}a\log\frac{a}{\Lambda}$.

  This reproduces the (semi)classical result (12.55) and so also the correct monodromy (12.57). Note that there is no choice of (n_m, n_e) giving a vanishing mass, in agreement with the fact that at $u \to \infty$ there are no extra massless particles in the spectrum.

- For $u \to \Lambda^2$ we have $a \sim \frac{i}{\pi}a_D\log\frac{a_D}{\Lambda}$ and $a_D \sim (u-\Lambda^2)$.

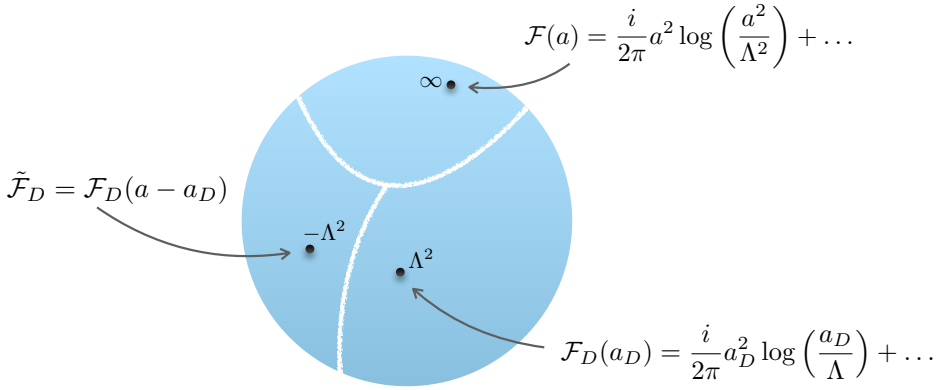

Figure 12.6: The quantum moduli space $\mathcal{M}_q$ of pure SYM with $G = SU(2)$ represented as a sphere, obtained by adding the point at infinity to the complex u plane. The space is covered by three distinct regions where a local, weakly coupled Lagrangian can be written using appropriate coordinates, *i.e.* the appropriate duality frame. No local Lagrangian exists which would be globally defined on $\mathcal{M}_q$. What are classical and strongly coupled regions is not an invariant concept, since it depends on the coordinate frame.

So we see that a is singular at $u \sim \Lambda^2$ while a_D vanishes. This is the correct behavior for a magnetic monopole with charge n_m becoming massless at $u = \Lambda^2$, in agreement with what we previously found. Again, the monodromy M_{Λ^2} in eq. (12.60) is correctly reproduced.

- For $u \to -\Lambda^2$ we have $a - a_D \sim (u + \Lambda^2)$ and $a \sim \frac{i}{\pi}(a_D - a) \log \frac{a_D - a}{\Lambda}$.

  This shows that at $u = -\Lambda^2$ we have a singularity where $a = a_D$, which gives a massless dyon with opposite electric and magnetic charges, $n_e = -n_m$, again in agreement with previous results, including the monodromy $M_{-\Lambda^2}$.

There are other non-trivial checks one can make. For example, one can expand $\tau(u)$, eq. (12.65), in (inverse) powers of u, at large u, and compare with (12.46). This gives perfect agreement with the instanton coefficients d_1 and d_2, which have been independently calculated.

As anticipated, an inspection of the exact solution (12.72) shows that for no values of u the scalar field VEV a becomes 0. So, the point $a = 0$ is not part of the quantum exact moduli space, as anticipated. This is consistent with the claim that nowhere on the moduli space extra massless gauge bosons arise.

12.3.2 Intermezzo: confinement by monopole condensation

Before discussing generalizations of this model, there is one (very nice) consistency check one can do.

Suppose to add a mass m to the superfield Φ belonging to the $\mathcal{N} = 2$ vector multiplet, that is $W = m\,\mathrm{Tr}\Phi^2$. This breaks explicitly $\mathcal{N} = 2$ to $\mathcal{N} = 1$. For $m >> \Lambda$ we can use the UV Lagrangian, integrate Φ out and end-up with pure $\mathcal{N} = 1$ $SU(2)$ SYM at low energy, which admits two isolated supersymmetric vacua with charge confinement and mass gap. By supersymmetry, this same scenario should hold even if $m << \Lambda$. In this regime, the low-energy $\mathcal{N} = 2$ effective description we discussed before should be approximately valid and we should use it, adding to it the small mass perturbation. But how the moduli space can be lifted giving back just two isolate (gapped) vacua? How can the otherwise massless photon get a mass, since, cf our discussion in § 12.1, there are no light charged fields? As we are going to show below, the results we got in the previous section contain the answer.

The addition of a mass term with $m << \Lambda$ makes the effective theory becoming an $\mathcal{N} = 1$ abelian gauge theory with a massive (neutral!) chiral multiplet Φ. Let us dub $U = \mathrm{Tr}\Phi^2$ the chiral superfield whose lowest component VEV u parametrizes the (original) $\mathcal{N} = 2$ moduli space. At a generic value of u, there are no massless chiral superfields other than U so we easily see that the F-term equation we have to impose on the space of D-flat directions

$$\frac{\partial W}{\partial U} = 0 \tag{12.74}$$

cannot be satisfied for $m \neq 0$. Therefore, there are no ($\mathcal{N} = 1$) supersymmetric vacua, in contradiction with what expected. In fact, we have learnt that at special points of the complex u-plane (two points, actually), there are extra massless degrees of freedom. One such points is $u = \Lambda^2$ where a massless magnetically charged hypermultiplet is present and should hence be included in the effective theory. Let us describe our theory near $u = \Lambda^2$. There, better to usual S-dual variables, for which the superpotential reads

$$W = \sqrt{2}\,\Phi_D H_1 H_2 + m\,U\ , \tag{12.75}$$

where Φ_D is the S-dual of Φ and U should be thought of as a function of Φ_D, now. The D-term equations from the coupling to the (magnetic dual) $U(1)$ gauge field imply that $|H_1| = |H_2|$ (recall that H_1 and H_2 have conjugate internal quantum

numbers and hence are oppositely charged under the $U(1)$ gauge symmetry), while the F-term equations read

$$\sqrt{2}H_1H_2 + m\,\frac{du}{da_D} = 0 \quad , \quad a_D H_1 = a_D H_2 = 0 \ . \tag{12.76}$$

Since $du/da_D \neq 0$ (u is a good global coordinate on u!) we get the following answer

$$\begin{aligned} m = 0 \quad &: \quad H_1 = H_2 = 0 \ , \ a_D = \text{ any} \\ m \neq 0 \quad &: \quad a_D = 0 \ , \ H_1 = H_2 = \left(-\frac{m}{\sqrt{2}}\,\frac{du}{da_D}\Big|_{a_D=0} \right)^{1/2} \ . \end{aligned} \tag{12.77}$$

For $m = 0$ we recover (tautologically) the $\mathcal{N} = 2$ moduli space. For $m \neq 0$, since H_1 and H_2 are (magnetically) charged, their VEVs break $U(1)$ and give a mass to the abelian gauge field (to all the $\mathcal{N} = 1$ gauge multiplet, in fact). So, we end up with a supersymmetric vacuum with a mass gap. This same reasoning holds also at $u = -\Lambda^2$, where the role of the massless magnetic monopole is played by a massless dyon. So, the $\mathcal{N} = 2$ moduli space is fully lifted but at two points, where there are supersymmetric vacua with mass gap. Exactly what we expect. The exact answer we got for the original $\mathcal{N} = 2$ model is just right to give what should hold for the $\mathcal{N} = 1$ massive theory under study!

This is nice, but, superficially, there is still a point of concern. The two $\mathcal{N} = 1$ supersymmetric vacua are confining ones. Here, instead, the dynamics is more of a Higgs mechanism, where a scalar monopole or a dyon field condenses. The point is that the Higgs mechanism taking place is not the usual condensation of an electrically charged field, but of a magnetically (or dyonically) charged one.

To understand what that means let us recall some basics of the usual Higgs mechanism, where electrically charged fields condense.

The condensation of the electric charge has the effect that any background electromagnetic field gets screened. This implies that electric sources in the theory are (almost) free, since their electric fields can be absorbed by the vacuum condensate and their interaction energy drops off exponentially. Magnetic charges behave very differently. The magnetic field lines have no condensate source to end on. The result is that magnetic field lines tend to be expelled from the vacuum (this is the well-known Meissner effect taking place in superconductors). The minimum energy configuration is for the magnetic field to be confined to a thin flux tube connecting opposite magnetic charges. Therefore, in the Higgs mechanism, electric charges are screened and magnetic charges are confined (note: this is strict confinement).

In the model above what condenses (let us focus, momentarily, on the vacuum at $u = \Lambda^2$) is not an electric charged state, but a magnetically charged one. By electro-magnetic duality it follows that here magnetic charges are screened, while electric charges are confined. So, eventually, we do have a confining vacuum (due to a magnetic dual of the Meissner effect)! This is a concrete realization of an old idea, due to 't Hooft and others, that confinement in non-Abelian gauge theories maybe due to monopole condensation. The point $u = -\Lambda^2$, where a dyonic field condense, is just related to the latter by a different electro-magnetic duality rotation. There, both electric and magnetic charges are confined but dyonic charges proportional to $(1, -1)$ won't, they will just be screened. This is known as oblique confinement, also proposed by 't Hooft long ago.

The result we got is beautiful from several point of views. First, it shows that the presence of magnetically charged solitons becoming massless somewhere on the moduli space is necessary to match $\mathcal{N} = 2$ dynamics with $\mathcal{N} = 1$ via holomorphic decoupling, one of our guiding principles all along this course. Second, it gives an a posteriori consistency check about the claim that two and only two singularities should be there on $\mathcal{M}$. Finally, it shows that (at least in this softly broken $\mathcal{N} = 2$ model) confinement is due to monopole condensation, providing a concrete realization of the old idea that this could in fact be the way electric charge gets confined.

12.3.3 Seiberg-Witten theory: generalizations

Till now we have been focusing on pure $\mathcal{N} = 2$ SYM with gauge group $SU(2)$. The story can be generalized to gauge groups with higher rank and/or coupled to matter fields.

We are not going to discuss these generalizations in detail and refer the interested reader to the references at the end of this lecture. Still, we want to make a few comments.

For $G = SU(2)$ and no matter, we have learnt that the moduli space, whose complex dimension is one, is the complex plane u with two singularities at $u = \pm u_0$ (beside the singularity at infinity), exchanged by the residual $\mathbf{Z}_2$ R-symmetry. At these two singularities magnetically charged objects, a monopole and a dyon, respectively, become massless. The metric on the moduli space can be described via an auxiliary elleptic curve, a Riemann surface $\Gamma(u)$ of genus $n = 1$ (a torus), whose modulus τ can be explicitly computed and corresponds (in fact, its imaginary part does) to the metric itself, the only unknown in the low energy effective action.

Gauge groups with higher ranks mean moduli spaces $\mathcal{M}$ with complex dimension n, locally $\mathbf{C}^n/W_G$, where W_G is the Weyl group of the gauge group G, n the gauge group rank and u_I, $I = 1, \ldots, n$, gauge invariant coordinates on $\mathcal{M}$, see *e.g.* eqs. (12.5). As already discussed, these theories are generalizations of the ($\mathcal{N} = 2$ supersymmetric version of the) Giorgi-Glashow model. As such, they admit several types of charged soliton-like solutions which, in the BPS limit, satisfy the BPS mass formula (12.35). The low energy effective action is form-invariant under electromagnetic duality rotations, which are generated by $Sp(2n, \mathbf{Z})$. Again, one finds that on the quantum exact moduli space there are singularities where magnetically charged states become massless. The prepotential is not globally defined and different charts (duality frames) should be used. In order to extract the metric on the moduli space, one can again make use of auxiliary elliptic curves $\Gamma(u_I)$, generalizations of the curve (12.61), which are genus n Riemann surfaces, now, and can be described as a double-sheeted x plane with $n + 1$ branch cuts. The period matrix τ of these genus n two-dimensional surfaces, which is defined in analogy to the modulus of a torus and whose imaginary part is in fact positive, gets identified with the gauge coupling matrix τ and hence determines the metric on the moduli space and the exact low energy effective action, eq. (12.41).

A genus n Riemann surface, figure 12.7, can be characterized in terms of n pairs of homology cycles A^I and B_I and, correspondingly, period integrals defined as

$$\omega_{IL} = \oint_{A^I} \lambda_L \quad , \quad (\omega^J_L)_D = \oint_{B_J} \lambda_L \qquad I, J, L = 1, \ldots, n \ , \tag{12.78}$$

where λ_L are n independent holomorphic one-forms, generalizations of the unique holomorphic differential defined on genus 1 surfaces, eqs. (12.64). The period matrix, to be identified with the gauge coupling matrix, is given by

$$\tau = \omega_D \cdot \omega^{-1} \tag{12.79}$$

(to be understood as a matrix equation), which ensures that $\mathrm{Im}\,\tau > 0$.

What about adding matter? The presence of matter fields opens-up the possibility for Higgs branches. However, in what follows we will focus on the Coulomb branch only, since, as already emphasized, that is the only component of the moduli space which gets modified at the quantum level. Note, also, that since we want to keep the theory UV-free, matter is constrained, there cannot be too much. For instance, adding matter to $SU(2)$ SYM one can add up to three hyermutiplets in the fundamental representation or, if allowing a vanishing β function, four hyper-

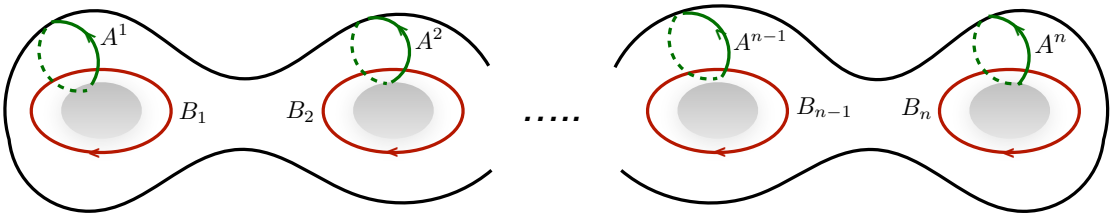

Figure 12.7: A genus n Reimann surface with the basis of homology cycles A^I, B_J.

multiplets in the fundamental representation or one in the adjoint (the latter case corresponds to $\mathcal{N} = 4$ SYM).

While matter fields do not change the dimension of the Coulomb branch, and so the genus of the corresponding Seiberg-Witten curve, they do change its singularity structure. One way to see this is to notice that hypermultiplets enjoy two contributions to their (effective) mass: the bare mass m_i contribution and the one inherited from the $\mathcal{N} = 2$ supersymmetry preserving cubic coupling, eq. (12.32), whenever the chiral superfield Φ gets a VEV. So, one expects new singularities on the moduli space wherever the two mass contributions cancel each other and the (charged) hypermultiplets become effectively massless. Notice, further, that in order to understand the monodromomy associated to these singularities, one should use the generalized BPS mass formula

$$Z = a \cdot n_e + a_D \cdot n_m + \sum_{i=1}^{F} \frac{1}{\sqrt{2}} m_i S_i \ , \tag{12.80}$$

in place of (12.35).

In order to construct the curves $\Gamma(u_I)$ concretely, one should follow the same logical steps we discussed for the $SU(2)$ theory, some of the guiding principles being matching their singularity structure with the appearance of massless particles in the $\mathcal{N} = 2$ theory spectrum, R-symmetry (and in this case also flavor symmetry) considerations as well as agreement, by holomorphic decoupling or scale matching, with curves with less flavors or smaller gauge groups. In any event, one ends up with equations like

$$y^2 = f(x, u_I, m_i, \Lambda) \ , \tag{12.81}$$

where u^I, the moduli space coordinates, parametrize the period matrix τ of the curves, eq. (12.79), m_i, with $i = 1, \ldots, F$, are hypermultiplet bare masses and Λ is

the strong coupling scale (we are assuming, for simplicity, that the gauge group G is simple).

It should be remarked that different parametrizations can be used to represent the curve associated to the moduli space of a given theory. Some may be more useful than others, depending what one wants to look at. For this reason, in the literature (and in the references at the end of this lecture) different parametrizations can be found. Needless to say, they are all physically equivalent, as they should. We will see one such example shortly.

Admittedly, despite the clear logical steps and guiding principles one can follow, some amount of educated guesswork is usually needed to find the correct curves. That is to say, there is not an overall recipe to construct the Seiberg-Witten curve for an arbitrary theory.

This said, a systematic way to construct a large class (but not all!) Seiberg-Witten curves does exist and relies on M-theory, where a physical meaning can be given to the Riemann surfaces (the curves) themselves. Four-dimensional $\mathcal{N} = 2$ theories can be engineered from $M5$ branes wrapped on suitably chosen two-dimensional compact surfaces. At low energy, smaller than the typical size of the surface, the theory becomes effectively four-dimensional and preserves $\mathcal{N} = 2$ supersymmetry. Such Riemann surfaces are nothing but the Seiberg-Witten curves! This makes several properties of the low energy effective theory having a geometrical interpretation which often helps.

As already stressed, there are a number of consistency checks one can make on the curves (12.81). For example, by making one hypermultiplet massive, that is taking its mass m_i large, eventually $m_i \to \infty$, one can integrate the hypermultiplet out and end on the theory with one flavor less, and then show that the limit of the corresponding curve agrees with the curve with one less hypermultiplet. Similarly, by letting one of the vacuum expectation values becoming large, one obtains a limit in which the gauge group is higgsed at high energy, *e.g.* $SU(N) \to SU(N-1)$. Again, the corresponding limit of the $SU(N)$ curve should agree with the curve for $SU(N-1)$.

Let us work out one of the simplest generalizations of the original Seiberg-Witten model, namely $\mathcal{N} = 2$ SQCD with gauge group $SU(2)$ and one massive hypermultiplet in the fundamental representation. The proposed corresponding Seiberg-Witten curve reads

$$y^2 = x^2(x-u) - \Lambda_1^6 - 2m\Lambda_1^3 x \ , \tag{12.82}$$

where, with obvious notation, Λ_1 is the strong coupling scale.

Let us first check that upon holomorphic decoupling one recovers the curve for the unflavored $SU(2)$ gauge theory. If we send $m \to \infty$ keeping $m\Lambda_1^3$ fixed, the flavor decouples and, using the relation $\Lambda_0^4 = m\Lambda_1^3$, one ends up with the curve

$$y^2 = x^2(x-u) - 2\Lambda_0^4\, x \ . \tag{12.83}$$

This is the curve for pure SYM with gauge group $SU(2)$, although in a different parametrization (and normalization) with respect to eq. (12.61). Let us see how this comes about.

First, as previously noticed, in a normalization where charged fields transforming in the adjoint representation of the gauge group have integer charges, those of fields transforming in the fundamental are half integers. Hence, in making a comparison between gauge theory with and without matter, it is convenient to first change the normalization we used to treat $SU(2)$ pure SYM and multiply, in eq. (12.30), n_e by 2, so to ensure n_e to still be an integer, and divide a by 2, so to ensure that (12.30) is unchanged. This change of conventions corresponds to the following transformation on the vector (a_D, a)

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \tag{12.84}$$

which changes monodromy matrices as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix} \ . \tag{12.85}$$

With these normalizations, the monodromy matrices (12.57) and (12.60) read

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \quad , \quad M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad , \quad M_{-\Lambda^2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \tag{12.86}$$

The corresponding elleptic curve reads

$$y^2 = x^2(x-u) + \frac{1}{4}\Lambda_0^4\, x \ , \tag{12.87}$$

which, by a constant rescaling of Λ_0, coincides with (12.83).

As compared to (12.61), the above expression makes it less transparent the points on the complex plane u where singularities arise, namely where two (or more) branch points collide and the curve degenerates. To this aim, regardless

the parametrization one is using, it suffices to compute the discriminant of the x-poynomial, $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)^2$ (where α_i are the roots of the polynomial), and find the values of u such that some roots coincide. For a cubic polynomial of the form

$$x^3 + bx^2 + cx + d \ , \tag{12.88}$$

we have $\Delta = b^2c^2 - 4c^3 - 4b^3d + 18bcd - 27d^2$ which applied to eq. (12.87) gives

$$\Delta = \frac{1}{16}\Lambda_0^8(u^2 - \Lambda_0^4) \ , \tag{12.89}$$

which vanishes at $u = \pm\Lambda_0^2$, in agreement with our previous analysis.

Let us go back to the curve (12.82), and look at the singularity structure of the moduli space more carefully. Let us first suppose the mass m to be very large. Then, at roughly the scale $|m|$ the hypermultiplet decouples and, below that scale the theory behaves, effectively, as pure $SU(2)$ SYM. This suggests that the structure of the moduli space in the region $|u| << |m^2|$ should be the same of the pure $SU(2)$ theory, with two singularities at $u \simeq \pm\Lambda_0^2$. Moreover, following our general discussion, we expect a third singularity where the hypermultiplet becomes effectively massless. Given that in this limit m lies in the region where $u = a^2$ and we can use classical intuition, that should happen at $u \simeq m^2$. Consistently with holomorphic decoupling, sending $m \to \infty$ this third singularity is pushed all the way to infinity and one recovers, correctly, the pure $SU(2)$ SYM moduli space of figure 12.4. Let us now consider the other extreme case, namely $m = 0$. Here, the surviving non-anomalous discrete R-symmetry (which is instead fully broken for $m \neq 0$) helps, as it was for the pure $SU(2)$ model. In the massless case there is now a conserved $\mathbf{Z}_6$ R-symmetry, each point on the u plane being $\mathbf{Z_2}$ invariant. Hence, we expect three strong coupling singularities, related by the broken $\mathbf{Z}_3$.

The above conclusions can be checked analytically from the curve (12.82), by computing the discriminant and the three roots, expanding the result for large, respectively small m. For example, in the massless case one easily sees that the three singularities are located at

$$u = \left(-\frac{27}{4}\Lambda_1^6\right)^{1/3} \ , \tag{12.90}$$

which shows, as anticipated, that the three singularities get transformed one another by $\mathbf{Z}_3$ rotations. Clearly, one can interpolate between these two extreme cases by continuously increasing (decreasing) m. Figure 12.8 shows how the moduli space

changes as we vary the hypermultiplet mass. Note that around any of the singularities, the low energy effective theory is made of one $\mathcal{N} = 2$ massless vector multiplet (where the photon sits) and one charged light (eventually massless) hypermultiplet.

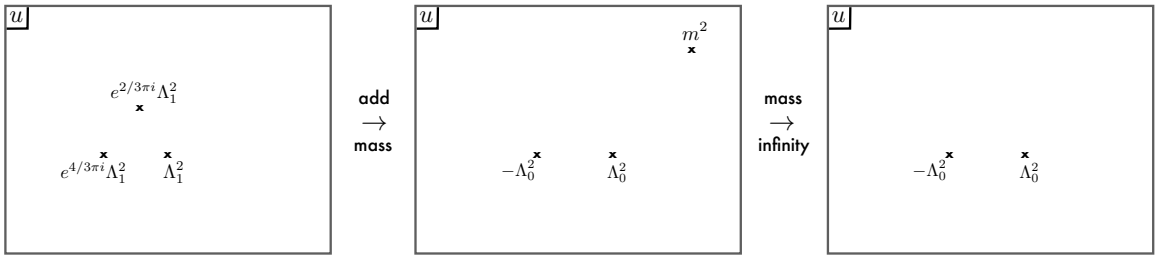

Figure 12.8: The moduli space of $\mathcal{N} = 2$ SQCD with $F = 1$ as we vary the hypermultiplet mass. In the limit of infinite mass one recovers the pure $SU(2)$ moduli space.

The very nature of the hypermultiplet becoming massless at the three singularities depends on m. When m is large, the two strong coupling singularities (and the associated monodromies) are basically the same as the pure theory, a monopole and a dyon. At $u \simeq m^2$ the hypermultiplet becomes massless, so it is a electrically charged object. As we decrease $|m|$ the three singularities become closer and closer and more and more similar to those of the massless theory, related by the $\mathbf{Z}_3$ symmetry. So there is no more a clear distinction between hypermultiplets coming from solitons or from elementary objects. In fact, by rotating m at fixed $|m|$ the three singularities can be continuously exchanged one another.

This model let us also illustrate a novel phenomenon which can occur when the gauge group rank $n > 1$ and/or when matter is added: the existence of special points on the moduli space, known as Argyres-Douglas points, where the theory enjoys an interacting (as opposed to free) conformal phase. This may sound quite surprising, given that the low energy effective theory is abelian and there are no points whatsoever on the *quantum* moduli space where extra gauge bosons (extra with respect to those associated to $U(1)^n$) become massless. Actually, what happens at these points is that cycles having non vanishing intersections (like the A and B cycles of the two-torus) simultaneously shrink. Physically, this corresponds to mutually non-local objects, as *e.g.* a dyon and a monopole, or a dyon and an

electrically charged object, becoming simultaneously massless. When this happens, one can argue that, unlike ordinary abelian gauge theories, the fixed point is an interacting one.

Let us consider again the curve (12.82). One can compute the discriminant of the x-polynomial and get

$$\Delta = 4m^2u^2\Lambda_1^6 + 32m^3\Lambda_1^9 - 4u^3\Lambda_1^6 - 27\Lambda_1^{12} - 36um\Lambda_1^9 \ . \tag{12.91}$$

Choosing the mass as $m = -(3/2)\,\omega\Lambda_1$ with $\omega^3 = 1$ one can check that at $u = 3\,\Lambda_1^2/\omega$ the discriminant Δ vanishes, and all branch points coincide at $x = \omega^{-1}\Lambda_1^2$. This implies that *both* A and B cycles shrink there and all three hypermultiplets become massless. For example, in the massless limit, this corresponds to monopoles with electric charge 0,1 and 2, respectively, becoming simultaneously massless at $u = 0$. So, in this case, the low energy effective theory is an interacting fixed point (despite the gauge group is abelian!).

As already remarked, this phenomenon is not specific to $F = 1$ SQCD. For example, Argyres-Douglas CFTs exist also for the other minimal generalization of the original Seiberg-Witten model, namely pure SYM with gauge group $SU(3)$ (which is actually the first instance where this phenomenon was discovered).

12.4 $\mathcal{N} = 4$: Montonen-Olive duality

In view of discussing $\mathcal{N} = 4$ SYM, it is useful to do a step back and consider again the original Georgi-Glashow model, which, as already stressed, finds a natural embedding both into $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM.

In the BPS limit, all states of the Giorgi-Glashow model satisfy the BPS mass formula (12.22). Combined with Dirac quantization condition, it implies that states carrying magnetic charge are very heavy at weak coupling and states carrying electric charge are heavy at strong coupling, and viceversa. Therefore, one could imagine that at strong coupling the rôles of electric (fundamental) and magnetic (solitonic) sources are interchanged, and that the theory at strong coupling is a theory of light monopoles. This idea was put forward by Montonen and Olive which suggested that the Giorgi-Glashow model could have two completely equivalent descriptions, related by a S-duality transformation, one in terms of electric sources and one in terms of magnetic sources, the two being exchanged under S duality (which indeed interchanges electric and magnetic couplings).

There are two non-trivial evidences in favor of such duality in the Giorgi-Glashow model:

- The BPS mass formula (12.22) is S-duality invariant.
- One can explicitly show that there is no interactions between two monopoles, while there is a non-vanishing interaction between a monopole and an anti-monopole. If duality is correct, since upon S transformations monopoles and gauge bosons are exchanged, the same should hold for the W^+ and W^- bosons in the Giorgi-Glashow model. This has been shown to be the case. Basically, in the BPS limit the (massless) Higgs field contributes exactly the opposite to the photon in the interactions between W's with equal charge, and exactly the same to the photon in the interactions between W's with opposite charge.

These convincing evidences, however, are not enough to conclude that Montonen-Olive duality is realized in the Giorgi-Glashow model. For one thing, as already noticed, there is no guarantee that the (semi)classical BPS mass formula (12.22) holds at quantum level (differently from theories with extended supersymmetry). Second, another necessary condition for the duality to hold is that the $W^{\pm}$ bosons and monopoles carry the same spin. Both these two crucial requirements cannot be verified in the Giorgi-Glashow model (and most likely are not met).

What about the supersymmetric version of this story?

The persistence of (12.22) at the quantum level ensured by the $\mathcal{N} = 2$ supersymmetry algebra, could suggest that exact S duality could be realized in $\mathcal{N} = 2$ theories. However, the other necessary condition, namely that monopoles should have the same quantum numbers of massive gauge bosons, does not hold. Magnetically charged states sit in hypermultiplets, which do not accomodate spin one particles. In fact, we have seen that in $\mathcal{N} = 2$ theories a quite different duality is realized, which is not an exact duality but rather a electro-magnetic duality which holds at the level of the IR effective theory. As we are going to discuss below, $\mathcal{N} = 4$ SYM, instead, is believed to realize exact S duality.

There are several facts which suggest this to be plausible. We enumerate them in turn.

First, we know from the representation of the supersymmetry algebra that in $\mathcal{N} = 4$ massive representations cannot be anything but BPS multiplets containing spin one particles. This implies that monopoles and dyons sit in vector multiplets, as

gauge bosons do, differently from $\mathcal{N}=2$ SYM. In fact, in $\mathcal{N}=4$ SYM *all* physical states (massive or massless) sit in BPS saturated vector multiplets, which is the only possible supersymmetric representation in theories without gravity, see § 3.1 and 3.2.

Second, differently from $\mathcal{N}=2$, in $\mathcal{N}=4$ the BPS bound (12.22) does not only hold true at quantum level, but non-renormalization theorems guarantee that the quantities therein are classically exact, cf lecture 6. Hence, the $U(1)$ couplings entering the effective Lagrangian (12.1) do not renormalize, *i.e.* τ_{IJ} are free parameters and they are all proportional to the UV-coupling τ

$$\tau_{IJ} = C_{IJ}\,\tau \ , \tag{12.92}$$

where C_{IJ} is the Cartan matrix of the gauge group. This suggests that, unlike for $\mathcal{N}=2$ theories, the electro-magnetic duality of the effective theory may propagate all the way to the UV. Indeed, the $Sp(2n,\mathbf{Z})$ transformations (12.34) which leave the low energy effective action form-invariant, contain transformations which would act on the UV coupling τ, via eq. (12.92), as $\tau \to \tau+1$ and $\tau \to -1/\tau$, which generate the group $Sp(2,\mathbf{Z})$. This implies that theories with UV couplings τ related by $Sp(2,\mathbf{Z})$ transformations are physically equivalent. That is to say, not only theories where electrically charged, respectively magnetically charged states are the fundamental degrees of freedom are physically equivalent. All theories whose fundamental degrees of freedom are dyonic states related to purely electric ones by a $Sp(2,\mathbf{Z})$ transformation

$$\tau \to \frac{a\tau+b}{c\tau+d} \ , \ \ (n_m,n_e) \to (n_m,n_e)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ \ \text{with} \ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2,\mathbf{Z}) \ , \tag{12.93}$$

are. Notice how different this duality is with respect to the IR dualities discussed previously, which act at the level of free IR effective $U(1)$'s theories. Here we are claiming that different interacting theories, with different UV couplings and different UV degrees of freedom, are physically equivalent!

Third, if Montonen-Olive duality is an exact symmetry, not only monopoles and dyons should carry the same Lorentz representations as gauge bosons: the whole spectrum of the theory should be duality invariant. In particular, given that massive gauge bosons are BPS states with charges $(n_m,n_e)=\pm(0,1)$, there should also be in the theory all BPS states which can be obtained acting on $\pm(0,1)$ with $Sp(2,\mathbf{Z})$ transformations (one state for all relatively prime choice of electric and magnetic

charges). Several evidence and consistency checks were given showing this to be the case.

Finally, we cannot resist saying that the strongest evidence for Montonen-Olive duality in $\mathcal{N} = 4$ SYM comes, in fact, from string (and M-) theory. There, it exists an intricate set of dualities between different string theories which implies, as a by-product, Montonen-Olive duality of $\mathcal{N} = 4$. The self-consistency of this web of dualities has passed many tests and it is regarded as an independent indication for the S duality of $\mathcal{N} = 4$ SYM.

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