

粒子是场激发对应的量子

QFT可以描述粒子数改变的过程 < 原子自发辐射
原子核β衰变 $n \to p + e^- + \bar{\nu}_e$ 反中微
(费米引入4粒子相互作
4种场)

一节 历史回顾

固定粒子数的相对论量子力学 困难

1. 负能解问题
→基底→真空不稳定
2. 负几率问题 $\rho(a) = |\psi_a|^2 > 0$
$\rho(a)$不再正定→放弃几率诠释
3. 破坏因果性

4矢量 自由电子 $p^\mu = (\frac{E}{c}, \vec{p})$
物质波-平面波 $e^{i\vec{k}\cdot\vec{x} - i\omega t} = e^{ik^\mu x_\mu} = e^{ikx}$
引入 $k^\mu = (\frac{\omega}{c}, \vec{k})$ $x^\mu = (ct, \vec{x})$

→k^μ也为4矢量 $k^\mu \propto p^\mu$

$p^\mu = \hbar k^\mu$ $\begin{cases} E = \hbar\omega = h\nu \\ p = \hbar k = \frac{h}{\lambda} \end{cases}$

$p^2 = p^\mu p_\mu = m^2 \to E^2 = p^2c^2 + m^2c^4$ (m为静质量) $\to \frac{\omega^2}{c^2} = \vec{k}^2 + (\frac{mc}{\hbar})^2$

$(\frac{E^2}{c^2} - p^2)$

Davison-Germer电子散射晶体→衍射

Klein-Gordan方程→最早薛定谔提出
相对论波动方程

推导：类比
回顾：非相对论薛定谔方程（不考虑V）
$E = \frac{p^2}{2m}$ 作替换

$E=\hbar\omega$ $\vec{p}=\hbar\vec{k}$ → 代回色散关系 → 得到 $i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\nabla^2\psi$

相对论波动方程：

相对论色散关系 $E^2=p^2c^2+m^2c^4$

代换后：$\begin{cases}E\to i\hbar\frac{\partial}{\partial t}\\ \vec{p}=-i\hbar\nabla\end{cases}\Rightarrow\left[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\nabla^2-m^2c^4\right]\psi=0$

利用 $\Box=g^{\mu\nu}\partial_\mu\partial_\nu=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2$ 这叫贝尔，为 Lorantz 不变算符

$\left(\begin{matrix}1&&&\\&-1&&\\&&-1&\end{matrix}\right)$

整理后 $\left[\Box+(mc/\hbar)^2\right]\psi(x,t)=0$ Klein-Gordan 方程

其平面波解：

令 $\psi(x,t)=e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)}$，代回 KG 方程

$\Rightarrow E^2=\vec{p}^2c^2+m^2c^4\Rightarrow E=\pm\sqrt{p^2c^2+m^2c^4}$ 存在负能解

经典中负能解存在不导致严重后果（能量连续变化）

量子世界 → 量子跃迁 → $\frac{E^2}{E}\to\gamma$ 无稳定基态/真空不稳定

定义问题存在性严证：

N-R 情况：$H=\frac{p^2}{2m}+V(x)\Rightarrow\frac{\partial^2}{\partial t^2}|\psi|^2=\nabla\left\{\frac{i\hbar}{2m}(\psi^*\nabla\psi-\psi\nabla\psi^*)\right\}=0$

连续性方程

$|\psi|^2=\rho$（几率密度） j：几率流密度

定义 $Q(t)=\int d^3x\,\rho(x,t)$；$\frac{dQ(t)}{dt}=\int d^3x\,\frac{\partial\rho}{\partial t}$

体积分为面积分高斯公式 $=\int d^3x\,\nabla j$

$=\oint_S j\,dS=0$ 面积足够大

从 K-G 方程出发：

$\psi^* + \left(\frac{mc}{\hbar}\right)^2 \int^2 \psi = 0 \rightarrow \frac{\partial \rho}{\partial t} + \nabla \vec{j} = 0 \rightarrow \begin{cases} \rho = N \mathrm{Im}(\psi^* \frac{\partial}{\partial t} \psi) \\ \vec{j} = N c^2 \mathrm{Re}(\psi^* \nabla \psi) \end{cases}$

问题：ρ是否正定？否

ρ不正定原因：KG方程中包含对t的二阶导

狄拉克方程：给出一个自洽的相对论波动方程，克服以上两个问题

从 $i\hbar \frac{\partial}{\partial t}\psi = H\psi$ 坚持方程只含时间一阶导且不含根号

$\rightarrow$ 波函数ψ扩展成一个列矢量 $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$

H是由$n \times n$矩阵

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi = \left[i\hbar c \vec{\alpha} \cdot \nabla + \beta mc^2 \right] \psi \quad \alpha^2 = (1, 2, 3 \cdots)$$

方程取平方，期待 β为待定矩阵

$\left(i\hbar \frac{\partial}{\partial t}\right)^2 \psi = H^2 \psi$ 回到 K-G方程

做平方

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{1}{2}(\alpha^i \alpha^j + \alpha^j \alpha^i) \partial_i \partial_j \psi - i\hbar mc^3 \sum_{i=1}^{3} (\alpha^i \beta + \beta \alpha^i) \partial_i \psi + m^2 c^4 \beta^2 \psi$$

()为反对易子，[]又对易子

要回到 K-G，则交叉项应为0 $\Rightarrow \{\alpha^i, \beta\} = 0$ 且 β^2为$\hat{I}$

且$\{\alpha^i, \alpha^j\}$ for S_{ij} 3个条件

$i=j$时 $2(\alpha^i)^2 = 2S_{ij}\hat{I} \Rightarrow (\alpha^i)^2 = \hat{I}_{n \times n}$

确定n：

$n=2$时 2×2不满足

$\Rightarrow n=4$

通常定义新矩阵 $\gamma^0 = \beta$；$\vec{\gamma} = \beta\vec{\alpha}$；$\gamma^\mu = (\gamma^0, \vec{\gamma})$ Dirac γ矩阵

狄拉克方程改写为

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right) \psi(\vec{r}, t) = 0$$

利用爱因斯坦求和约定 $\sum_{\mu=0}^{3}$

$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}I_{4\times4}$ $g^{\mu\nu}=(1,-1,-1,-1)=g_{\mu\nu}$

连续性方程: $\partial_t\rho+\nabla\cdot\vec{j}=0$

$\rho=\psi^+\psi$; $\vec{j}=c\psi^+\vec{\alpha}\psi$

$=\psi_1^2+\psi_2^2+\psi_3^2+\psi_4^2\geq0$ 正定

负能解: 依然有负能解 → 发明狄拉克 Sea 负能级填充了整个Sea

电子若被激发 → 产生空穴 → 反电子

另一个巨大成就: 正确预言电子磁矩 : $\mu=\frac{e\hbar}{2mc}$

产生原因: 自旋为$\frac{1}{2}$

氢原子能谱:

$$E_n=\frac{-mc^2\alpha^2}{2n^2}\ ;\ \alpha=\frac{1}{137}$$

方法: 最小耦合 → 引入电子和外电磁场作用

电 $E=i\hbar\frac{\partial}{\partial t}$, $\vec{p}=-i\hbar\nabla$

在外磁场中:

磁矢势: $A^\mu=(\phi,\vec{A})$

$E=i\hbar\frac{\partial}{\partial t}+e\phi$; $\vec{p}=-i\hbar\nabla+\frac{e\vec{A}}{c}$ 最小耦合 $p^\mu=p^\mu+\frac{e}{c}A^\mu$

考察 K-G方程与 Dirac方程在外电磁场下形式

K-G方程: $\{(i\hbar\frac{\partial}{\partial t}+e\phi)^2-c^2(-i\hbar\nabla+e\vec{A})^2+m^2c^4\}\psi=0$

考虑氢原子 Hydrogen atom → 外电磁场为质子产生静电势

由于质子静止, 无矢势

$e\phi=\frac{e^2}{4\pi\varepsilon_0 r}$; $\vec{A}=0$

定态解: $\psi\propto e^{-iEt/\hbar}$ 代入

得到 $\{(E+\frac{e^2}{4\pi\varepsilon_0 r})^2-c^2\hbar^2\nabla^2=m^2c^4\}\psi(\vec{r})=0$ 消去t, $v\sim\frac{c}{137}$ → 非 re

解得, 做个关于α的泰勒展开

$E = mc^2\left[1 + \frac{\alpha}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right) + \cdots\right]$

静止质 玻尔能级 fine structure (L-S耦合相对论修正)

2. 狄拉克方程:

$\left(i\hbar\frac{\partial}{\partial t} + e\phi\right)\psi = \left(-i\hbar\nabla + e\vec{A}\right)\cdot\vec{\alpha}\psi + \beta mc^2\psi$

自由电子 → 最小耦合 $\vec{A}=0$ ↑代入

同上, 求定态解 $\psi \propto e^{-\frac{iEt}{\hbar}}$

得到矩阵波动数平格能:

$E = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{(n-j-\frac{1}{2}+\sqrt{(j+\frac{1}{2})^2-\alpha^2})^2}}}$ $\vec{j} = \vec{l} + \vec{s}$

α 展开 $= mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + \cdots\right]$

与 l 无关了 → j → 与实验完全吻合. (20年代)

存在问题: ① 反常磁矩 ② 兰姆位移 → 量子场论解决.

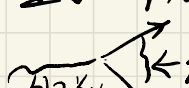
五、相对论性因果破坏

1947年:

1. 实际磁矩: $\mu = \frac{e\hbar}{2mc}$ [1.00118 ± 0.00003] I. Rabi (误差)

2. 预言电子磁矩 $\mu = \frac{e\hbar}{2mc}\left[1 + \frac{\alpha}{2\pi}\right] \approx \frac{e\hbar}{2mc}$ [1.00116] Schwinger

预言依据: 量子电动力学辐射修正

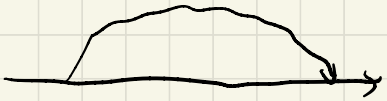


3. $2S_{1/2}$ - $2P_{1/2}$ 狄拉克方程来看为两个态简并 W. Lamb

$n=2$, $l=1$ 但不同 → 发展观测 → 存在1000MHz的能级差

→ 兰姆位移 → 揭示狄拉克方程不完备

H.Bethe
理论计算



外电磁场中电子自能同自由场中差了1000 MHz →
光子与电子相互作用改变自旋 ——→ 需要做更精确计算

→ 计算中出现紫外发散 → 重整化理论
→ 量子场论被接受

单粒子相对论性量子力学的因果性破坏 Causality violation.

量子力学初画：自由粒子从 $\vec{x}_0$ ($t=t_0$) 至 $\vec{x}$ ($t=t$) 的传播振幅

定义：$U(t)=\langle \vec{x}|e^{-\frac{iHt}{\hbar}}|\vec{x}_0\rangle$ ← 时间演化算符

自然单位制 $\begin{cases}\hbar=1\\ c=1\end{cases}\Rightarrow [t]=[L]=E^{-1};\ [E]=[p]=[m]=E^1$

1. 非相对论量子力学

$H=\frac{p^2}{2m}\Rightarrow U(t)=\langle \vec{x}|e^{-i\frac{p^2}{2m}t}|\vec{x}_0\rangle$

$\hat{I}=\int\frac{d^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}|$
$=\int d\vec{x}|\vec{x}\rangle\langle\vec{x}|$
完备性关系

$=\langle\vec{x}|e^{-i\frac{p^2}{2m}t}\int|\vec{p}\rangle\langle\vec{p}|\vec{x}_0\rangle$
$=\int\frac{d^3p}{(2\pi)^3}e^{-i\frac{p^2}{2m}t}\vec{p}_{\vec{x}_0}$
$=\left(\frac{m}{2\pi i t}\right)^{\frac{3}{2}}e^{\frac{im(\vec{x}-\vec{x}_0)^2}{2t}}\neq 0\ \forall\ x,t$

当 $x-x_0\gg t$ 时 $U\neq 0\Rightarrow$ 具有因果性

2. 相对论性单粒子量子力学

$H=\sqrt{p^2+m^2}$，重做上述计算

$U(t)=\langle\vec{x}|e^{-it\sqrt{p^2+m^2}}|\vec{x}_0\rangle$ ← 插入完备集

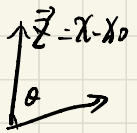
$=\frac{1}{(2\pi)^3}\int d^3p\ e^{-it\sqrt{p^2+m^2}}e^{ip(\vec{x}-\vec{x}_0)}$ 对空间积

写成正负能量之和（对角对称）

$$= \frac{1}{(2\pi)^2} \frac{1}{|x-x_0|} \int_0^\infty dp \cdot p \sin(p|x-x_0|) e^{-it\sqrt{p^2+m^2}}$$

驻波法二 ……

类空间隔 ~ $e^{-m\sqrt{(x-x_0)^2-t^2}}$ $(|x-x_0|^2 \gg t^2)$ 指数压低



很小但并未消失 因果性被破坏

结论：固定粒子数的相对论性量子力学无法实现

解决：Lorentz对称性 → 因果性 $\xrightarrow{QM\ 狭相结合}$ 必须引入反粒子

SR+QM ⟨ 反粒子
spin-statitics theorem 自旋统计定理

六、量子场论的诞生

Born Heisenberg Jorden (1926)

尝试去量子化电磁场 近似：忽略电子极化 假设空间维数为1

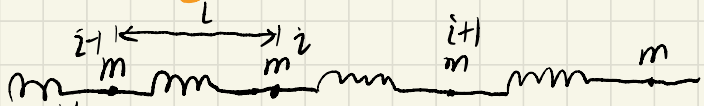
1927年 Dirac 正确量子化电磁场

→ 正确解释原子自发辐射

$$\hat{H} = \frac{(\hat{p}-e\hat{A})^2}{2m} \supset \hat{p}\cdot\hat{A} \rightarrow$$ 电磁场量子化

↳ 描述电偶极矩跃迁

用一维经典弦模拟电磁场



忽略弹簧质量，处于平衡

考虑弹簧发生形变 η

$$T = \sum_{i=1}^{n} \frac{1}{2} m \dot{\eta}^2$$

$$V = \frac{1}{2} \sum_{i=1}^{N} k(\eta_{i+1}-\eta_i)^2$$

考虑连续极限，总长 $L=Nl$，令 $l\to 0$，$N\to\infty$

$$T=\frac{m}{2l}\sum_{i=1}^{N}l\left(\frac{dy_i}{dt}\right)^2=\frac{1}{2}\mu\int_0^L dx\left(\frac{\partial\eta(t,x)}{\partial t}\right)^2$$

$$V=\frac{k}{2}l\sum_{i=1}^{N}l\left(\frac{y_{i+1}-y_i}{l}\right)^2 \qquad T=kl$$

$$=\frac{T}{2}\int_0^L dx\left(\frac{\partial\eta(t,x)}{\partial x}\right)^2$$

Lagrange $L=T-V$

Hamilton $H=T+V$

$$L=\int_0^L dx\underbrace{\left[\frac{\mu}{2}\left(\frac{\partial\eta}{\partial t}\right)^2-\frac{T}{2}\left(\frac{\partial\eta}{\partial x}\right)^2\right]}_{\text{Lagrange density}}$$

$$H=\quad +$$

重新定义 $u(t,x)=\sqrt{\mu}\,\eta(t,x)$

$$L=\frac{1}{2}\int_0^L dx\left[\left(\frac{\partial u}{\partial t}\right)^2-c^2\left(\frac{\partial u}{\partial x}\right)^2\right] \qquad c=\sqrt{\frac{T}{\mu}}\ (\text{波速})$$

下一步：量子化 classical field theory

假设弦两端固定 $\Rightarrow u(t,x=0)=u(t,x=L)=0$

$u(t,x)$ 在位形空间进行傅利叶展开

$$u(t,x)=\sum_{k=1}^{\infty}q_k(t)\sin\left(\frac{\omega_k x}{c}\right) \qquad \omega_k\text{：正则角频率}$$

代回到 Lagarange 方程，交换求和和积分次序

$$L=\frac{L}{4}\sum_{k=1}^{\infty}\left\{\dot{q}_k(t)^2-\omega_k^2 q_k^2(t)\right\}$$

$$H=\frac{L}{4}\sum_{k=1}^{\infty}\left\{\dot{q}_k(t)+\omega_k^2 q_k^2(t)\right\}$$

> 无穷多个谐振子叠加

欧拉-L方程：

$$\ddot{q}_k(t)+\omega_k^2 q_k(t)=0 \qquad \text{解：}\cos\text{及}\sin\omega_k t\text{叠加}$$

$e^{-i\omega_k t}$ $e^{i\omega_k t}$

正则动量：

$P_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}\dot{q}_k(t)$

$H = H(P_k, q_k) = \sum_{k=1}^{\infty}\left[\frac{P_k^2}{1} + \frac{1}{4}\omega_k^2 q_k^2\right]$

量子化条件：$[P_k(t), q_j(t)] = -i\hbar\delta_{kj}$ } 类比单谐振子的 $[p,x] = -i\hbar$

全 $[q_k(t), q_j(t)] = [P_k(t), P_j(t)] = 0$

粒子数表象 $H = \hbar\omega(a^+a + \frac{1}{2})$；$x \propto a + a^+$；$p \propto a - a^+$

$q_k(t) = \sqrt{\frac{\hbar}{2\omega_k}}\left(a_k e^{-i\omega_k t} + a_k^+ e^{i\omega_k t}\right)$

（a 为厄米算符 类似于 $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$）

$[a, a^+] = 1$ $[a_k, a_j^+] = \delta_{kj}$ $[a_k, a_j] = [a_k^+, a_j^+] = 0$

$H = \sum \hbar\omega_k(a_k^+ a_k + \frac{1}{2})$ 自由场论 $\xrightarrow{\text{量点成}}$ 真空能（无穷大）

能谱分析：

单谐振子 $H|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$ $|n\rangle = (a^+)^n|0\rangle/\sqrt{n!}$；$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

量子态本征态：$|n_1, n_2 \cdots n_k \cdots\rangle$（无穷维）

ω_1 模式对应的光子数

引入 Fock 空间（不以 Hilbert 空间）

$F = \bigoplus_n \mathcal{H}_n$（刻画 n 个光子）

直和

以场论中，$|n\rangle$ 为有 n 个全同的角频率为 ω_k 的光子（玻色子）

2) 量子化后，场真空涨发散 → 重整化

后续发展：
- Dirac 量子化之作用，推导自发辐射
- Heisenburg 和 Pauli 引入正则量子化框架（相对：路径积分量子化，费曼博士论文）

1947 QED < 反常 / Lamb shift > 重整化 theory

1967-1973 Standard Model $SU(3)\times SU(2)\times SU(1)$ 自发规范场论

1972 K. Wilson 重整化群

八 洛伦兹变换与场的分类

二章 经典场论

$$L=\frac{1}{2}\int_0^L dx\left[\rho\left(\frac{\partial u}{\partial t}\right)^2-c^2\left(\frac{\partial u}{\partial x}\right)^2\right]=\int dx\,\mathcal{L}$$ ← 拉氏密度

回忆：作用量 $S=\int dt L=\int dt d^3x\,\mathcal{L}$（受尺缩种慢效应影响）

↑ 洛伦兹不变量 $=\int d^4x\,\mathcal{L}$（洛伦兹不变量）

狭义相对论（洛变）：

坐标系变换 $x^\mu \to x'^\mu=\Lambda^\mu_\nu x^\nu$（$\to$ 4×4矩阵）

Λ^μ_ν 3个转动参数

$$\Lambda^\mu_\nu=\begin{pmatrix} 1 & & & \\ & \cos\theta_x & \sin\theta_x & \\ & -\sin\theta_x & \cos\theta_x & \\ & & & 1 \end{pmatrix}\begin{matrix} t \\ x \\ y \\ z \end{matrix}$$ 沿 x 转转 θ_x 角

$$\begin{pmatrix} 1 & & & \\ & \cos\theta_y & & -\sin\theta_y \\ & & 1 & \\ & -\sin\theta_y & & \cos\theta_y \end{pmatrix}$$ y θ_y 角

$$\begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}$$ z θ_z

3个boost参数(有相对速度)

沿x轴boost $\begin{pmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\gamma = \frac{1}{\sqrt{1-v^2}} \geq 1$

若要与转动形式显白的更加对称，定义快度 rapidity β

$\cosh\beta = \gamma = 1/\sqrt{1-v^2}$

$\sinh\beta = v/\sqrt{1-v^2}$ $\Rightarrow$ $\begin{pmatrix} \cosh\beta_x & \sinh\beta & & \\ \sinh\beta & \cosh\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\tan\beta = v$ $\beta = \frac{1}{2}\ln\frac{1+v}{1-v}$ $v \ll 1$时 $\beta \approx v$

沿y轴boost $\begin{pmatrix} \cosh\beta_y & & \sinh\beta_y & \\ & 1 & & \\ \sinh\beta_y & & \cosh\beta_y & \\ & & & 1 \end{pmatrix}$

z $\begin{pmatrix} \cosh\beta_z & & & \sinh\beta_z \\ & 1 & & \\ & & 1 & \\ \sinh\beta_z & & & \cosh\beta_z \end{pmatrix}$

总共6个(3+3)参数刻画lorentz变换

若Λ_1, Λ_2都是lorentz变换，则$\Lambda_1\Lambda_2$，$\Lambda_2\Lambda_1$都是，组成了lorentz群

洛伦兹变换的性质：

四度间隔：$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ = lorentz不变量 $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

在另一个坐标系：$g_{\mu\nu}dx'^\mu dx'^\nu = g_{\rho\sigma}dx^\rho dx^\sigma$ (爱因斯坦求和)

$$\Rightarrow g_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}=g_{\rho\sigma}$$

$$\Rightarrow g_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}=g_{\rho\sigma}$$

逆变换：

$$(\Lambda^{-1})^{\rho}_{\nu}=g_{\nu\mu}g^{\rho\sigma}\Lambda^{\mu}_{\sigma}\ \#$$

验证：$(\Lambda^{-1})^{\rho}_{\nu}\Lambda^{\nu}_{\tau}=\delta^{\rho}_{\tau}$ 其中 $\delta^{\mu}_{\nu}=g^{\mu}_{\nu}=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$

$$g_{\nu\mu}g^{\rho\sigma}\Lambda^{\mu}_{\sigma}\Lambda^{\nu}_{\tau}=g^{\rho\sigma}g_{\sigma\tau}=\delta^{\rho}_{\tau}$$

补充：

$$V=V^{\mu}\partial_{\mu}=V^{\mu}\frac{\partial}{\partial x^{\mu}}=V'^{\mu}\frac{\partial}{\partial x'^{\mu}}=V'^{\mu}\partial_{\mu}$$



范向物体 → 基底

coordinate independent

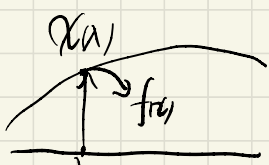
$$\partial'_{\mu}=\frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_{\nu}\qquad V'^{\mu}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}V^{\nu}$$

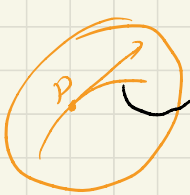
$V: f_{(x)}\to Vf=V^{\mu}\frac{\partial f}{\partial x^{\mu}}$

$f(x): x\to f(x)$ $\quad f:\mathbb{M}\to\mathbb{R}$ scalar function 纯量

↓ scalar field

例：





$\to x^{\mu}(\lambda)$ λ为参数，曲线上每个λ → $x^{\mu}(\lambda)$ → 代表曲线

$f(x)$ 为纯量场 $\quad\frac{d[f(x(\lambda))]}{d\lambda}=\frac{dx^{\mu}}{d\lambda}\frac{df}{dx(x^{\mu})}\Big|_{x=P}$

沿切线方向取微分

定义一曲线 $V_{(x)}=\underbrace{\frac{dx^{\mu}}{d\lambda}}_{V^{\mu}}\frac{\partial}{\partial x^{\mu}}$ 通过P点的切向量

$x^{\mu}(\lambda,\sigma)$ 可以代表一个曲面



tangent vector V_P
在P上的vector为 linear function，从 scalar function（定义在P上，且P可微）
$V: f \to V(f)$

V_P要满足条件：

$$\begin{cases} V_P(af+bg) = aV_P(f) + bV_P(g) \Rightarrow \partial_\mu(f+g) = \partial_\mu f + \partial_\mu g \\ V_P(fg) = f(P)V_P(g) + g(P)V_P(f) \end{cases}$$

Tangent space:

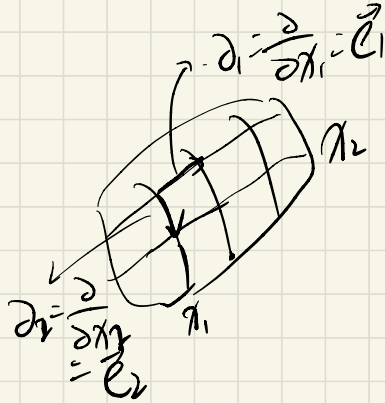
P点上所有的切向量的集合 TM_P

若 $U_P = \alpha V_P + \beta W_P$; $U_P(f) = (\quad\quad)(f) = \alpha V_P(f) + \beta W_P(f)$

$U_P \in TM_P$, $V_P \in TM_P$, $W_P \in TM_P$

先固定基底向量：

$\partial_\mu \Leftrightarrow \vec{e}_\mu$



$\{\partial_\mu\} = \{\vec{e}_\mu\}$

例子：



坐标变换

例子：(r, θ, ϕ)

$$\vec{e}_1 = \frac{\partial}{\partial x} \; ; \; \vec{e}_2 = \frac{\partial}{\partial y} \; ; \; \vec{e}_3 = \frac{\partial}{\partial y}$$

$$x' = x'^{\mu}_{(x)} \to \begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$\partial_r = \vec{e}_r \quad \partial_\theta = \vec{e}_\theta \; ; \; \partial_\phi = \vec{e}_\phi \quad \partial_\nu = \frac{\partial x^\mu}{\partial x'^\nu}\partial_\mu$（链式法则）

$\vec{e}_r = \frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}$

$$\vec{e}_\theta = \frac{\partial}{\partial\theta} = \cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z}$$

$$\vec{e}_\phi = \frac{\partial}{\partial\phi} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y}$$

$\Lambda^\nu_\sigma \Lambda^k_\tau g^{\sigma\tau} = g^{\nu k}$

重写

$\Lambda^\nu_\sigma g^{\sigma\tau}(\Lambda^T)^\tau_k = g^{\nu k} \Rightarrow \Lambda g \Lambda^T = g$

则 $g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} = \Lambda^T g \Lambda = g$

洛变：$\vec{x}' = R\vec{x}$ $R^T R = RR^T = I_{3\times3}$ 为了和洛变对应，插入 I

$= R^T I_{3\times3} R$ $S_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$x^\mu = (x^0, x^1, x^2, x^3)$

$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0\ \partial_1\ \partial_2\ \partial_3) = (\partial_0, \nabla)$

四矢量缩并→标量：

V与W都是4矢量 $V\cdot W = V^\mu W_\mu = g_{\mu\nu}V^\mu W^\nu = V^0W^0 - \vec{V}\cdot\vec{W}$ 是洛…标量

例：$p^2 = p^\mu p_\mu = g_{\mu\nu}p^\mu p^\nu = E^2 - \vec{p}^2 \xrightarrow[\text{满足质能关系}]{\text{在壳}} m^2$

$V^\mu W_\mu Z^\mu$ ✗ 不能出现三次同个指标

$(V\cdot W)^2 = V^\mu W_\mu V^\mu W_\mu = V^\mu W_\mu V^\nu W_\nu$

洛伦兹张量：

$T^{\mu\nu} \xrightarrow{\Lambda} T'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$ =β阶张量

拉氏密度不变→场来构造→场的分类：

按洛变换的形为来分类：

标量场 $\phi(x) \xrightarrow[x'=\Lambda x]{\Lambda} \phi'(x') = \phi(x)$

矢量场 $A^{\mu}(x) \xrightarrow{\Lambda} A'^{\mu}(x') = \Lambda^{\mu}_{\nu} A^{\nu}(x)$

度规场（引力场） $h^{\mu\nu}(x) \longrightarrow h'^{\mu\nu}(x') = \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} T^{\rho\sigma}(x)$

spiner field

旋量场

=Dirac field

第九讲 欧拉-拉氏朗日方程

$\partial_\mu \phi(x) \sim$ 矢 $A^{\mu}(x)$

定义：

Local field theory:

$L = \int d^3x \, \mathcal{L}(x)$ $\quad H = \int d^3x \, \mathcal{H}(x)$

$\mathcal{L}$：洛氏磁量 density

标量

$\mathcal{L}_{K-G} = \partial_\mu \phi(x) \partial^\mu \phi(x)$, $\phi(x)$, $\phi^2(x)$, $\phi^3(x)$, $\phi^4(x)$ ……

但 $\partial_\mu \phi_X \, \partial_\nu \phi \, \partial_\nu \phi_X$ ：$\int d^3y \, \phi(x) \, \phi(y)$ (非定域，放弃)

回顾：

$L = L(q, \dot{q}) \Leftrightarrow \mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

$\mathcal{L}_{KG} = \frac{1}{2} \partial_\nu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \xrightarrow{E\text{-}L\text{方程}} (\Box + m^2)\phi = 0$

不同于相对论波动力学的波动方程，只是形式一样

逻辑：经典场 → 一次量子化

$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi - \frac{1}{2}m^2\phi^2$

类比，引入共轭动量密度：

$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}} \Rightarrow \mathcal{H}(x) = \pi\dot{\phi} - \mathcal{L}$

$L = L(q,\dot{q}) \Rightarrow p = \frac{\partial L}{\partial \dot{q}}$

勒让勒变换 ← 类比

$\pi_x = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(x) \Rightarrow \mathcal{H}(x) = \pi^2 - \mathcal{L} = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$

期待 $\mathcal{H}$：$\mathcal{H}(\pi,\phi)$ $= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \geq 0$ 正定

$\mathcal{H}$为能量密度 $\int d^3x\,\mathcal{H} > 0$ 总能量正定 同于下 → 有稳定基态

补充：$\mathcal{L} = K - V$

(最简场：二次型) 动能 相互作用能(至少三次型)

举例：$\mathcal{L}_K$ $\frac{1}{2}(\partial_\mu\phi)^2$, $\bar{\psi}\gamma^\mu\partial_\mu\psi$, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $\frac{1}{2}m^2\phi^2$, $\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_2$ (不同种类场)

$\phi\partial_\mu A^\mu$ ……

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

若场只有二次型 量子化 → 对应无穷多个独立谐振子 [$\hat{H} = \sum_k \hbar\omega_k(a_k^\dagger a_k + 1/2)$]

→ 自由场

若含有相互作用：$\mathcal{L}_V = \mathcal{L}_{int}$ ⊃ $g\phi^3$, $\lambda\phi^4$, $e\bar{\psi}\gamma^\mu\psi A_\mu$ $Q\neq 0$

耦合常数

$g\partial_\mu\phi A^\mu\phi^*$ (复标量场), $g^2(A_\mu A^\mu)^2$,

$g^{\mu\nu}h_{\mu\nu}\partial^\alpha h^{\alpha\beta}$ (广相中度规相互作用)

回忆：自由场中 运动方程为线性

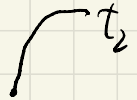
$(\Box + m^2)\phi = 0$

Dirac: $(i\gamma^\mu\partial_\mu - m)\psi = 0$

在相互作用场中：非经典性：E-L方程（运动方程）

点粒子：$L = L(q, \dot{q})$

$\frac{\partial L}{\partial q} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}}$ ← $S = \int dt L$ 最小作用原理



$\delta S = 0$

利用最小作用 E-L方程：

$S = \int dt L = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$

$\delta S = \int d^4x \, \delta \mathcal{L} = 0$

$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right\}$

$[\delta, \partial_\mu] = 0$ $= \int \partial_\mu(\delta\phi) \Big\}$

分部积分 $= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] \right\}$

全导数项 $= 0$

高斯定理 $\delta\phi$ 为0

$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi = 0$

由于 $\delta\phi$ 为任意的 $\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}$ E-L 方程

举例：

$\mathcal{L}_{K\text{-}G} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{1}{2}m^2 \times 2\phi = -m^2\phi$

$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\phi$

$\partial_\mu(\partial_\mu\phi) = \Box\phi$

$\Rightarrow -m^2\phi - \Box\phi = 0$

$(\Box + m^2)\phi_{(x)} = 0$

↓

K-G方程

练习：若 $\mathcal{L}_{KG} = |\partial_\mu\phi|^2 - m^2|\phi|^2 = \partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\phi^*$

推导EL方程

第十讲：对称性及其分类

$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$ → 耦合常数(1) 引入相互作用

其EL：$(\Box + m^2)\phi = -\frac{\lambda}{6}\phi^3$ 不再是线性方程 $[S]=[\hbar]=E^0$

$S=\int d^4x\,\mathcal{L} \Rightarrow [\mathcal{L}]=E^4 \Rightarrow [\phi]=E^1$ $[d^4x]=E^{-4}$ $\to [\lambda]=E^0$

对称性和守恒律：

$\partial_\mu J^{\mu\nu} = eJ^\nu$ 洛伦兹不变

symmetry：对称性变换下动力学不变（洛伦兹，时空平移）

$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$ 6 + $x \to x'^\mu = x^\mu + a^\mu$ 4 = 10 庞加莱变换

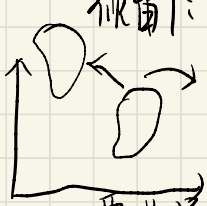
两种视角检验对称性：
- 运动方程(E-g)不变
- $\mathcal{L}(x)$不变或S不变

例：$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m\phi^2 - \frac{1}{4!}\phi^4$

$\phi \to -\phi$ 视角2 ✓

视角1：$(\Box+m^2)\phi = -\frac{\lambda}{6}\phi^3$；$\phi \to -\phi$

例



$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$

$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}(x))$

要满足 $\mathcal{L}'(x) = \mathcal{L}(\Lambda^{-1}x)$

先考察 $\mathcal{L}(x)$ $\partial_\mu\phi(x) \to \partial_\mu\phi'(x) = \partial_\mu\phi(\Lambda^{-1}x)$

$= (\Lambda^{-1})^\nu_\mu (\partial_\nu\phi)(\Lambda^{-1}x)$

$[\partial_\mu \phi(x)]^2 \xrightarrow{\Lambda} g^{\mu\nu}\partial_\mu\phi'(x)\partial_\nu\phi'(x)$

$= g^{\mu\nu}[(\Lambda^{-1})^\rho_\mu \partial_\rho\phi][(\Lambda^{-1})^\sigma_\nu \partial_\sigma\phi](\Lambda^{-1}x)$

由 $\Lambda g \Lambda^T = \Lambda^T g \Lambda = g$

则 $(\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma}$

$\hookrightarrow = g^{\rho\sigma}(\partial_\rho\phi)(\partial_\sigma\phi)(\Lambda^{-1}x) = (\partial\phi)^2(\Lambda^{-1}x)$ 第一项满足

剩下不带导数的项十分平庸，易证

即：$\mathcal{L}(x) \to \mathcal{L}(x') = \mathcal{L}(\Lambda^{-1}x)$

从运动方程不变视角：

是否满足 $(\Box + m^2)\phi'(x) = 0$

验证：$(\Box^2 + m^2)\phi'(x) = (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi(\Lambda^{-1}x)$

$= [(\Lambda^{-1})^\nu_\mu \partial_\nu (\Lambda^{-1})^\sigma_\mu \partial_\sigma + m^2]\phi[\Lambda^{-1}x]$

$= [g^{\nu\sigma}\partial_\nu\partial_\sigma + m^2]\phi[\Lambda^{-1}x] = (\Box + m^2)\phi(\Lambda^{-1}x) = 0$

对称性分类：

- 时空对称性 $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$ 如：洛伦兹对称性 时空平移~ 旋转伸缩共形
  $\phi(x) \to \phi'(x') = \phi_{(x)} + \delta\phi$
- 内禀对称性 internal $x'^\mu = x^\mu$ $\phi'(x) = \phi(x) + \delta\phi$
  例：① $\phi \to -\phi$
  ② $\mathcal{L}_{KG} = (\partial_\mu\phi)\partial^\mu\phi^* - m^2\phi\phi^*$
  变换：$\phi \to \phi' = e^{i\theta}\phi$
  $\phi^* \to \phi'^* = e^{-i\theta}\phi$
  U(1)对称性 直接作用在场的傅立叶 与时空坐标无关

③ 同位旋
④ 味对称性 与异数对称性

另一种分类视角
- 连续对称性 continious：$x'^\mu = \Lambda^\mu_\nu x^\nu = (\delta^\mu_\nu + \omega^\mu_\nu)x^\nu$ 时空平移都是连续对称性
- 离散 ~ $K(x,\phi) \to -\phi$ discreate

1-讲 诺特定理 (Noether theorem)

含义：Symmetry → Conservation law

若经典场论中的系统具有某种连续对称性，且当运动方程满足时，则该系统存在某个相应的守恒流/律. $\partial_\mu j^\mu = 0$

证明：

概念回顾：

对称性 ／ 主动视角 → 坐标系变换
对称性 ＼ 被动 $P(x^\mu)$ 用被动视角更简单

时空点 P，坐标 x^μ $f(x^\mu)$(场)

变化后：x'^μ ; $f'(x'^\mu)$ $x \to x + \delta x$ (无穷小) $\delta f = f'(x) - f(x)$

$$\delta f = f'(x+\delta x) - f(x) = f(x) - f(x) + \delta x^\mu \partial_\mu f + O(\delta x)^2$$

精确到 $O(\delta x)^1$ 阶

$$\delta f = \underbrace{f(x) - f(x)}_{\delta_0 f} + \delta x^\mu \partial_\mu f = \delta_0 f + \delta x^\mu \partial_\mu f$$

$$\delta = \delta_0 + \delta x^\mu \partial_\mu$$

例：时空平移 $x^\mu \to x'^\mu = x^\mu + a^\mu$



显然 $\delta\phi = 0$

被动变换

但是 $\delta_0\phi = \delta\phi - \delta x^\mu \partial_\mu \phi = -\alpha^\mu \partial_\mu \phi$ (同个坐标点的改变)

验证改变量:

$$\phi(x) = \phi(x-a)$$
$$\simeq \phi(x) - a_\mu \partial_\mu \phi$$

术语(主动变换视角)

$$S = \int d^4x \mathcal{L}$$

要求 变分 $\delta S = 0 = \int [(\delta d^4x)\mathcal{L} + d^4x \delta \mathcal{L}]$

考虑其既适用于时空对称性,也适用于内禀对称性

1)时空变换:

平移: $\delta(d^4x) = d^4x (\partial_\mu \delta x^\mu)$

L: $x'^\mu = \Lambda^\mu_\nu x^\nu$ (6个独立参数)

单位变换附近的微小变换

$= (\delta^\mu_\nu + \omega^\mu_\nu) x^\nu$ (利用 $\Lambda g \Lambda^T = g$ 得 $\omega^\mu_\nu = -\omega_{\nu\mu}$ 二阶反对称)

代入得到: $\delta x^\mu = \omega^{\mu\nu} x_\nu$ $\partial_\mu(\delta x^\mu) = g_{\mu\nu}\omega^{\mu\nu} = 0$

$$\delta S = \int d^4x [\partial_\mu \delta x^\mu \mathcal{L} + \delta \mathcal{L}]$$
$$= \int d^4x (\partial_\mu \delta x^\mu \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0 \phi])$$

(满足运动方程第二项去掉)

$$= \int d^4x \partial_\mu [\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0 \phi]$$

→ 被积函数处处为0

→ 对应守恒流 $\partial_\mu j^\mu = 0$

→ 喜欢用同一个物理点的场的变动

→ $\delta_0\phi \to \delta\phi$

利用 $\delta = \delta_0 + \delta x^\mu \partial_\mu$ 可写成

$\delta \mathcal{L} = \delta x^\mu \partial_\mu \mathcal{L} + \delta_0 \mathcal{L}$

δ_0 为坐标点的变分

$= \delta x^\mu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_0 \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0(\partial_\mu \phi)$

注意:以能对 δ_0 操作,对 δ 亦同

由 $[\partial_\mu, \delta_0] = 0$ → 交换次序 → E-L方程

$$= \delta x^\mu \partial_\mu \mathcal{L} + [\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}] \delta_0 \phi + \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0 \phi)$$

(分部积全导数项)

改写：$0=\delta S=\int d^4x\,\partial_\mu\left[\left(\mathcal{L}\delta^\mu_\nu-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi\right)\delta x^\nu+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right]=\int d^4x\,\partial_\mu j^\mu=0$

利用 $\delta_0=\delta-\delta x^\mu\partial_\mu$

若等：$0=\int d^4x\,\partial_\mu j^\mu=\int_{t_1}^{t_2}dx_0\int_\infty d^3\vec{x}\,(\partial_0 j^0+\nabla\cdot\vec{j})$ ← 高斯定理

$=\int_{t_1}^{t_2}dx_0\frac{\partial}{\partial x_0}\int_\infty d^3\vec{x}\,j^0(x,t)=\int_{t_1}^{t_2}dx_0\frac{\partial}{\partial x_0}Q(t)=Q(t_2)-Q(t_1)=0$

或写成 $\frac{dQ(t)}{dt}=0$ → Q为守恒荷 至此，已证明对应守恒量

应用：

时空平移变换 $\delta x^\mu=a^\mu$；$\delta\phi=0$

$j^\mu=\left(\mathcal{L}\delta^\mu_\nu-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi\right)a^\nu$ 去掉a^ν依然不变

极重要：定义一个 $T^{\mu\nu}=-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi+\mathcal{L}g^{\mu\nu}$ 能量-动量-张量

讨论其对应守恒率：

$\partial_\mu T^{\mu\nu}=0$（守恒流）

守恒荷 $P^\nu=\int d^3x\,T^{0\nu}$ 当$\nu=0$时 $P^0=\int d^3x\,T^{00}=\int d^3x\,\mathcal{H}=H$（总能量）

当 $\nu=i$ 时 $P^i=\int d^3x\,T^{0i}=$ 总动量

$x^0\to x'^0=x^0+a_0$ 时间平移 → 能量守恒 → H为守恒荷

$x_i\to x^{i\prime}=x^i+a_0$ 空间～ → 动——→ 产——

$P^\mu=(H,\vec{P})$ 为守恒量（4个守恒荷）

举例：Lorentz变换 → 空间转动不变性 → 角动量守恒（太难，不推了）

$\partial_\mu j^\mu=0$ $j^\mu=(\quad)\delta x^\mu+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi$

对于内禀变换：$j^\mu=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi$

例1：$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2$ 变换 $\phi=\phi'=\phi+\alpha$ 对场平移

则 $\delta\phi=\alpha$；$j^\mu=\partial^\mu\phi\alpha$（$\alpha$可看常数）

验证其是不是守恒流：

$\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = \Box \phi = 0$（对于on shell的场，运用E-L方程）

例2：考虑U(1)的相位对称性

$\mathcal{L}_{K-G} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$

$\phi \to \phi' = e^{i\alpha}\phi$；$\phi^* \to \phi'^* = e^{-i\alpha}\phi^*$

$\alpha \to$ 无穷小 $\delta\phi = i\alpha\phi$；$\delta\phi^* = -i\alpha\phi^*$ → 代入守恒流

$j^\mu = i[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)]\alpha$

易证：$0 = \partial_\mu j^\mu$ [运用 $(\Box + m^2)\phi = (\Box + m^2)\phi^* = 0$]

$Q = \int d^3x\, j^0 = i\int d^3x [\dot{\phi}^*\phi - \phi^*\dot{\phi}]$ → 电荷

电荷守恒来源于U(1)的相位对称性

第三章 量子化 Klein-Gordan场论

$\mathcal{L}_{K-G} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$；$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$；

$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = T^{00} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$ 如何量子化？

NRQM：习惯从H出发 $H = H(p,q) = \int d^3x \mathcal{H}$

量子化：$[q_i, p_j] = i\delta_{ij}$ $[q_i, q_j] = [p_i, p_j] = 0$ ？

guess：量子化过程：广义坐标 → 场变量；动量 → 动量密度

$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$

另外两个量子化条件：$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$

场方程：$(\Box + m^2)\phi(\vec{x}, t) = 0$

傅变：$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p}, t)$ 在谐振子方向靠

代入协方程：

$$\left[\frac{\partial^2}{\partial t^2}+\omega_{\vec{p}}^2\right]\hat{\phi}(\vec{p},t)=0 \quad \text{其中 } \omega_p=\sqrt{\vec{p}^2+m^2}$$

得到两线性独解：

$e^{+i\omega t}$ $\quad e^{-i\omega t}$ 类似于谐时振子 → 利用解决K-G

对比：回忆谐时振子

$$H_{H.O}=\frac{p^2}{2}+\frac{1}{2}\omega^2x^2 \quad (m=1)$$

$$\hat{x}=\frac{1}{\sqrt{2\omega}}(a+a^\dagger) \qquad \hat{p}=-i\sqrt{\frac{\omega}{2}}(a-a^\dagger)$$

$$[\hat{x},\hat{p}]=i \Leftrightarrow [a,a^\dagger]=1$$

则用 $a, a^\dagger$ 写出 H：

$$H=\omega\left(a^\dagger a+\frac{1}{2}\right)$$

则 $|n\rangle=(a^\dagger)^n|0\rangle$

$$H|n\rangle=\left(n+\frac{1}{2}\right)\omega|n\rangle$$

类比 ↓

对比K-G

$$H=\int d^3x\,\mathcal{H}=\int d^3x\left(\frac{1}{2}\pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2\right)$$

$\pi\sim p$ $\quad \phi\sim x$ 则对 ϕ 进行Fourier变换

$$\phi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right)$$

保证共轭

$$\pi(x)=\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}-a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right)$$

至此，类比过程结束，下面讨论为了满足量子化条件 a_p 与 $a_p^\dagger$ 需要？

结论：$[a_{\vec{p}},a_{\vec{p}'}^\dagger]=(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$ ⇔ 量子化条件

推导过程：重写：

$$\phi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_{\vec{p}}+a_{-\vec{p}}^\dagger)e^{i\vec{p}\cdot\vec{x}}$$

使傅利叶回归一体

类似 $\pi(x')=\int\frac{d^3p'}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{p'}}{2}}(a_{\vec{p}'}-a_{-\vec{p}'}^\dagger)e^{i\vec{p}'\cdot\vec{x}'}$

考察：$[\phi(x),\pi(x')]=?$

$$=\iint\frac{d^3pd^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{p'}}{\omega_p}}\left\{[a_{-\vec{p}}^\dagger,a_{\vec{p}'}]-[a_{\vec{p}},a_{-\vec{p}'}^\dagger]\right\}$$

已知 $[a_p,a_{p'}]=0=[a_p^\dagger,a_{p'}^\dagger]$ 四项只剩两项

$$=-(2\pi)^3\delta^3(\vec{p}+\vec{p}')-(2\pi)^3\delta^3(\vec{p}+\vec{p}')$$

$\times e^{i(p\cdot x+p'\cdot x')}$ → $(2\pi)^3\delta^3(p+p')$

$=\int \frac{d^3p}{(2\pi)^3}\left(-\frac{i}{2}\right)(-2)\,e^{i\vec{p}(x-x')}$

$= i\iint \frac{d^3p}{(2\pi)^3}e^{i\vec{p}(x-x')} = i\delta^{(3)}(x-x')$

正好对应于 $[\phi(x),\pi(y)]=i\delta^{(3)}(x-y)$ 至此，已证毕，第二个可以作练习

下一个思考，是否可以像谐振子一样，用产生湮灭算符写出 H

过程：

$$H=\int d^3x\,\mathcal{H}=\int d^3x\left\{\frac{1}{2}\pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2\right\}$$

三重～ + 六重积分 → 九重积分

$$=\int d^3x\int\frac{d^3p\,d^3p'}{(2\pi)^6}e^{i(p+p')x}\left\{-\frac{\sqrt{\omega_p\omega_{p'}}}{4}(a_p-a^\dagger_{-p})(a_{p'}-a^\dagger_{-p'})\right.$$

$$\left.+\frac{-p\cdot p'}{4\sqrt{\omega_p\omega_{p'}}}(a_p+a^\dagger_{-p})(a_{p'}+a^\dagger_{-p'})\right\}$$

$=(2\pi)^3\delta^3(p+p')\cdots$

① 改变积分次序 d^3x

② 算符只有两项 aa，$a^\dagger a^\dagger$

经过推算 $=\int\frac{d^3p}{(2\pi)^3}\omega_p\left(a_p^\dagger a_p+\frac{1}{2}[a_p,a_p^\dagger]\right)$ → 高度发散（零点能、真空能）

由于动量发散，考虑用此记号表示，但至此我们已经把 H 写成对角形式

K-G场论的谱

回顾：

$\mathcal{L}\xrightarrow[\text{变换}]{\text{勒让德}}\mathcal{H}$（正定）；荷

$$H=\int d^3x=\int d^3x\left[\frac{1}{2}\pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2\right]$$

$$=\int\frac{d^3p}{(2\pi)^3}\omega_p\underbrace{\left(a_p^\dagger a_p+\frac{1}{2}(2\pi)^3\delta^3(0)\right)}_{\text{谐振子}}$$ → 来自于对易子

（基态）

$|0\rangle$ 满足 $a_p|0\rangle=0$ $(\forall p)$

$\hat{H}|0\rangle=0\,|0\rangle$

讨论基态：

$H|0\rangle = \omega_0|0\rangle$ ⇒ 找性基态，使 $H|0\rangle = 0$

构造激发态：

回忆：$[H, a^\dagger] = \omega a^\dagger$，$[H, a] = -\omega a$

定义：$|n\rangle = (a^\dagger)^n |0\rangle$

K-G 理论中：

$[H, a_p^\dagger] = \omega_p a_p^\dagger$，$\omega_p = \sqrt{p^2 + m^2}$

$[H^\dagger, a_p^\dagger] = -\omega_p a_p$

$H = \int \frac{d^3q}{(2\pi)} \omega_q a_q^\dagger a_q$

根据 $[A, B, C] = A[B, C] + [A, C]B$

$[a_p^\dagger, a_q] = (2\pi)^3 \delta(p-q)$

$[a, a] = [a^\dagger, a^\dagger] = 0$

构造单粒子态（H的第一激发态）

构造态：$a_p^\dagger|0\rangle$，要证明他是H的本征态

代入

$$H a_p^\dagger |0\rangle = a_p^\dagger H|0\rangle + [H, a_p^\dagger]|0\rangle$$
$$= \omega_p a_p^\dagger |0\rangle = \omega_p (a_p^\dagger |0\rangle)$$

即得证：其本征值为 ω_p

问题若 $a_p^\dagger a_q^\dagger |0\rangle$ 是否也为本征态，本征值？

$$H(a_p^\dagger a_q^\dagger |0\rangle) = a_p^\dagger H a_q^\dagger |0\rangle + [H, a_p^\dagger] a_q^\dagger |0\rangle$$
$$= \omega_q a_p^\dagger a_q^\dagger |0\rangle + \omega_p a_p^\dagger a_q^\dagger |0\rangle$$
$$= (\omega_p + \omega_q)(a_p^\dagger a_q^\dagger |0\rangle)$$ 得证，本征值 $\omega_p + \omega_q$

称之为双粒子态，三粒子态、n粒子态……

下面考察动量算符（空间平移的诺特荷）

$$\vec{P} = \int d^3x \, \pi(\vec{x}) \nabla \phi(x)$$

将其视为算符，用 $a, a^\dagger$ 算符描述（过程略）

总动量算符 $\hat{P}=\int\frac{d^3p}{(2\pi)^3}\vec{p}\,a_p^\dagger a_p$ ← 积分变量

类似于 H $[\hat{\vec{P}},a_p^\dagger]=\vec{p}a_p^\dagger$; $[\hat{\vec{P}},a_p]=-\vec{p}a_p$

可以证: $\hat{\vec{P}}a_p^\dagger|0\rangle=\vec{p}a_p^\dagger|0\rangle$

$\hat{\vec{P}}\,a_p^\dagger a_q^\dagger\cdots|0\rangle=(\vec{p}+\vec{q}+\cdots)(a_p^\dagger a_q^\dagger\cdots|0\rangle)$ → 为 $\hat{\vec{P}}$ 与 $\vec{p}$ 的共同本征态

总结上面: $\hat{\vec{P}}\,a_p^\dagger a_q^\dagger\cdots|0\rangle=(\vec{p}+\vec{q}+\cdots)(a_p^\dagger a_q^\dagger\cdots|0\rangle)$

⇒ $\hat{\vec{P}}$ 与 $\vec{p}$ 为广延量

Bose-Einstein 统计: (协论解释)

$[a_p^\dagger,a_q^\dagger]=0$ → 考察双粒子态: $a_p^\dagger a_q^\dagger|0\rangle=|p,q\rangle$ 全同玻色子

$=a_q^\dagger a_p^\dagger|0\rangle=|q,p\rangle=|p,q\rangle$

~凝聚: (态对称数目)

$(a_p^\dagger)^n|0\rangle$

费米狄拉克统计:

$\{a_p^\dagger,a_q^\dagger\}=0$

单粒子态的归一化条件:

通常把 $\langle p|q\rangle\doteq(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$, 但 $|p\rangle$ 是 Lorentz 不变的

如何巧妙定义单粒子态, 使其自动满足 Lorentz 不变

$\vec{p}\xrightarrow{\Lambda}$ 正转由 boost 与 p $\quad p_3'=\gamma(p_3+\beta E)$

$E'=\gamma(E+\beta p_3)$

还需要 δ^3: $\delta^3(\vec{p}'-\vec{q}')\doteq\delta^3(\vec{p}-\vec{q})$ 比例系数?

回忆：$\delta[f(x)-f(x_0)]=\frac{1}{|f'(x_0)|}\delta(x-x_0)$

则 $\delta^3(\vec{p}'-\vec{q}')=\frac{\delta^3(\vec{p}-\vec{q})}{dp_1'/dp_1\,dp_2'/dp_2\,dp_3'/dp_3}$

$\Rightarrow \frac{dp_3'}{dp_3}\delta(\vec{p}'-\vec{q}')=\delta^3(\vec{p}-\vec{q})$

$\Rightarrow \gamma(1+\beta\frac{dz}{dp_3})\delta(\vec{p}'-\vec{q}')=\delta^3(\vec{p}-\vec{q})$

$\Rightarrow \frac{E'}{E}\delta^3(\vec{p}'-\vec{q}')=\delta^3(\vec{p}-\vec{q})$

$\Rightarrow E'\delta^3(\vec{p}'-\vec{q}')=E\delta^3(\vec{p}-\vec{q})$

由 $p_3'=\gamma(p_3+\beta E)$，$E=\sqrt{p_1^2+p_2^2+p_3^2}$，$\frac{dE}{dp_3}=\frac{p_3}{E}$

$\gamma(1+\beta\frac{p_3}{E})=\frac{\gamma}{E}(E+\beta p_3)$

为洛伦兹不变的

规定：单粒子态归一化等价于 $\langle\vec{p}|\vec{q}\rangle=2E_p(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$

构造单粒子态 $|\vec{p}\rangle=\sqrt{2E_p}\,a^\dagger_{\vec{p}}|0\rangle$ $(E_{\vec{p}}=\omega_{\vec{p}})$

代入 $\langle\vec{p}|\vec{q}\rangle=\sqrt{2E_p2E_q}\langle0|a_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle$

真空矩阵元

$(2\pi)^3\delta^3(\vec{p}-\vec{q})\Rightarrow p=q$

调换次序 $=\sqrt{4E_pE_q}\{\langle0|a^\dagger_{\vec{q}}a_{\vec{p}}|0\rangle+\langle0|[a_{\vec{p}},a^\dagger_{\vec{q}}]|0\rangle\}$

$=(2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q})$

$\phi(x)$ 的物理意义：

首先作用在真空态上：

反求 $\vec{x}$：$\phi(x)|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}})|0\rangle$

$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-i\vec{p}\cdot\vec{x}}|\vec{p}\rangle=|\vec{x}\rangle$？ → 得到单粒子态 $E_p=\sqrt{p^2+m^2}\overset{NR}{=}m$

回忆：$|\vec{x}\rangle=\int\frac{d^3p}{(2\pi)^3}e^{-i\vec{p}\cdot\vec{x}}|\vec{p}\rangle$ (NRQM) 差了一个 $\frac{1}{\sqrt{2E_p}}$

↪ 视为在 x 处产生了一个粒子

☆强调：在量子场论中，矢永远是算符，是一个标记空间点

定义：$\langle 0|\phi(x)|p\rangle = \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p e^{ipx} + a_p^\dagger e^{-ipx})\sqrt{2E_p}\,a_p^\dagger|0\rangle$

$= e^{ipx}$ （$px = \vec{p}\vec{x}$）

NRQM：$\langle x|p\rangle = e^{ipx}$ ⟩ 类比（自由场 理论）

下面量子化：$t=0$时，薛丁格绘景，算符不含时，态可以含时

但可以重新约定算符随着t的演化（海森堡绘景）

KG场在海森堡绘景：

$\phi(\vec{x},t=0) \Rightarrow \phi(\vec{x},t)$

$Q_H(t,\vec{x}) = e^{iHt}\,Q(\vec{x},t=0)\,e^{-iHt}$

$\phi(x) = \phi(\underbrace{\vec{x},t}_{x^\mu}) = e^{iHt}\,\phi(\vec{x})\,e^{-iHt}$ x为4度坐标

由此作泊松括号/海森堡运动方程

$i\frac{\partial}{\partial t}\Theta_H = [\Theta, H]$

$\phi(x)$满足什么样的时间演化规律？

$i\frac{\partial}{\partial t}\phi(x) = [\phi(\vec{x},t), \int d^3x\,[\frac{1}{2}\pi^2(\vec{x},t) + \frac{1}{2}[\nabla\phi(\vec{x},t)]^2 + \frac{1}{2}m^2\phi^2]]$ H的诺特荷

量子算符的对易

$= [\phi(\vec{x},t), \int d^3x\,\frac{1}{2}\pi^2(\vec{x},t)]$

$= \int d^3\vec{x}\; i\delta^{(3)}(\vec{x}-\vec{x})\,\pi(\vec{x},t)$

$= i\,\pi(\vec{x},t)$ ①

在经典场论中 $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \frac{\partial \phi}{\partial t}$

ϕ与ϕ本身对易

$[A, B^2] = [A,B]B + B[A,B]$

接下来看动量算符元随时间的演化

$i\frac{\partial}{\partial t}\pi(\vec{x},t) = [\pi(\vec{x},t), H]$，过程略

$= i(\nabla^2 - m^2)\phi(\vec{x},t)$ ②

整合①②考虑 $\frac{\partial^2}{\partial t^2}\phi(\vec{x},t) = (\nabla^2 - m^2)\phi(\vec{x},t)$

$\Leftrightarrow (\Box + m^2)\phi = 0$

在 QM level ϕ 依然满足 K-G 方程

之前考虑 $t=0$ 时对 ϕ 傅展，是否可以在 $t\neq 0$ 时展开且用 a 与 $a^\dagger$ 表示

引入 $\begin{cases} a_{\vec{p}}(t) = e^{iHt} a_{\vec{p}} e^{-iHt} \\ a^\dagger_{\vec{p}}(t) = e^{iHt} a^\dagger_{\vec{p}} e^{-iHt} \end{cases}$

$\phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt}$

$= e^{iHt}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})e^{-iHt}$

$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}(t)e^{ipx} + a^\dagger_{\vec{p}}(t)e^{-ipx})$

$a_{\vec{p}}(\vec{x},t) = e^{iHt} a_{\vec{p}}(\vec{x}) e^{-iHt}$

$a^\dagger_{\vec{p}}(\vec{x},t) = e^{iHt} a^\dagger_{\vec{p}}(\vec{x}) e^{iHt}$

考察 t 时刻的 a $a^\dagger$ 与 $t=0$ 时刻的关系：

$\frac{d}{dt}a_{\vec{p}}(t) = e^{iHt} i[H, a_{\vec{p}}] e^{-iHt}$

$= -iE_p a_{\vec{p}}$ 一阶常微分方程

H 不含时 $[H, a_{\vec{p}}] = -E_p a_{\vec{p}}$

其解 $\begin{cases} a_{\vec{p}}(t) = a_{\vec{p}} e^{-iE_p t} \\ a^\dagger_{\vec{p}}(t) = a^\dagger_{\vec{p}} e^{iE_p t} \end{cases}$ 代回 $\phi(\vec{x},t)$

类似的

$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}} e^{-iE_p t + i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{+iE_p t - i\vec{p}\cdot\vec{x}})$ $(E_p = \sqrt{\vec{p}^2 + m^2})$

平面波相位

$e^{-ip\cdot x}$ $\quad e^{ip\cdot x}$

$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x})\Big|_{p_0=E_p}$

正频 $E_p>0$　　负频 $E_p<0$

产生或 $\pi(x) = \frac{\partial}{\partial t}\phi(x)$

若 K-G场不是实的，则可以写成

$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipx} + b_p^\dagger e^{ip\cdot x})$

$b_p^\dagger$ 产生一个反粒子

$b_p^\dagger|0\rangle$ 产生一个反粒子

$\phi(x) = e^{i\hat{p}\cdot x}\phi(0)e^{-i\hat{p}\cdot x}$ 空间平移 → 三矢量

利用 $e^{-i\hat{p}\cdot x} a_{\vec{p}} \cdot e^{i\hat{p}\cdot x} = a_{\vec{p}} e^{i\vec{p}\cdot x}$

$e^{-i\hat{p}\cdot x} a_{\vec{p}}^\dagger \cdot e^{i\hat{p}\cdot x} = a_{\vec{p}}^\dagger e^{i\vec{p}\cdot x}$

则 $\phi(x,t) = e^{i(Ht-\vec{p}\cdot\vec{x})}\phi(0)e^{-i(Ht-\vec{p}\cdot\vec{x})} = e^{i\hat{p}\cdot x}\phi(0)e^{-i\hat{p}\cdot x}$

H与p为诺特荷

一个重要含义：波粒二象性（定量化）

$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x})$

$a_p e^{-ip\cdot x}$：粒子　　$a_p^\dagger e^{ip\cdot x}$：波

因果性：

定义两点关联函数

洛伦兹标量 ← $D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$

y点产生一个粒子，传播到x点的几率

→ 用Q进行傅利叶分解

$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x})$

海森堡绘景

标：

$= \langle 0|(a+a^\dagger)(a+a^\dagger)|0\rangle$

$= \langle 0|aa^\dagger|0\rangle$

引入4动量

$p^\mu = (p^0, \vec{p}) = (E_p, \vec{p})$

正负频：$E_p = H_4 = E_4$

4为一本征态 $\propto e^{iEt}$

$E>0$，随时间演化

相对论QM有负能解

在场论中 $a^\dagger|0\rangle$

负频解对应正p的态

原式 $= D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$ 洛伦兹不变量

在壳条件

写成让 $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4} \delta(p^2-m^2)\theta(p^0)$ 阶跃函数

↓壳 ↓壳

计算 $\int dp^0 \delta(p^2-m^2)\theta(p^0)$

$= \int dp^0 \delta(p_0^2 - E_p^2)\theta(p^0) = \int dp^0 \frac{1}{2E_p}\delta(p_0-E_p)\theta(p^0)$

$= \frac{1}{2E_p}$

为什么自变量写成 $x-y$ 形式：

$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|e^{iP\cdot y}\phi(x)e^{-iP\cdot y}\, e^{iP\cdot y}\phi(0)e^{-iP\cdot y}|0\rangle$

泰勒展开 $=1$ $(P|0\rangle = 0)$

$= \langle 0|\phi(x-y)\phi_0|0\rangle$ 只依赖于 $(x-y)$

$x-y$ 为类时间隔：$(x-y)^2 > 0$

总有一个参照系使 $x^0 - y^0 = t$，$\vec{x}-\vec{y} = 0$

积分：$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-iEpt}$ 球坐标 $\frac{4\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$

对 E 积分 $= \frac{4\pi}{(2\pi)^3}\int_0^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$

$t\to\infty$ $= e^{-imt}$

$x-y$ 为类空间隔（无因果联系）：$(x-y)^2 < 0$ 等时

总可找到一个参照系使 $x^0 - y^0 = 0$；$\vec{x}-\vec{y} = r$

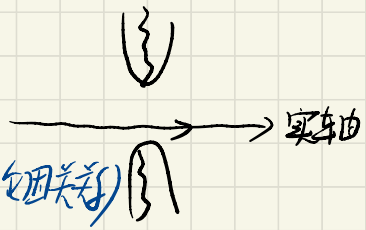
$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}}$ z 轴为 $\vec{r}$ 方向

$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2E_p}\frac{e^{ipr}-e^{-ipr}}{ipr}$

$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{+\infty} \frac{p}{\sqrt{p^2+m^2}} \exp(ip\cdot r)$

令 $\rho = -i\rho$

$D = \frac{1}{4\pi^2 r}\int_0^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2+m^2}}$



$r\to\infty = e^{-mr}$ 是否不为0(与相对论因果)

无因果性的要求：$[\phi(x),\phi(y)]=0$

$\langle 0|\phi(x)|0\rangle = 0$ 场算符的真空平均值(一点关联函数)

$D(x-y) \neq \langle 0|\phi(x)|0\rangle\langle 0|\phi(y)|0\rangle = 0$

因果性的体现：

两个测量分别在x点与y点，若x,y为类空间隔，则在x处的测量必须和y处测量彼此分独立

公式表达：$[O_1(x), O_2(y)]=0$ (玻色型算符) KG场不满足玻色型

举例：$[\phi(x),\phi(y)]$ 在x,y处对易子平移

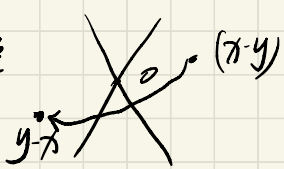
$$[\phi(x),\phi(y)] = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\int\frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}}\left[a_p e^{-ipx}+a_p^\dagger e^{ip\cdot x}, a_q e^{-iqx}+a_q^\dagger e^{iqx}\right]$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_q}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right]$$ 算符 → C数

$= D(x-y) - D(y-x)$

当 $(x-y)^2<0$ (类空)时，总可以做一系列连续洛伦兹变换

使 $x-y$ 变为 $(y-x)$，$D(x-y) = D(y-x)$



$\Rightarrow [\phi(x),\phi(y)]=0$ → 保留因果性

上节课：$x_0=y_0$，$x-y=r$

$$D(x-y) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\vec{p}\cdot\vec{r}}$$

要证 $D(x-y) = D(y-x)$：

令 $\vec{p}=-\vec{p} \Rightarrow \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\vec{p}(-\vec{r})} = D(y-x)$

$\exists D(r) = D(-r)$

但类时间隔：$[\phi(x), \phi(y)] \xrightarrow[t\to\infty]{x=y} e^{-imt} - e^{+imt}$ 扣

两点测量的值会相互影响

以上只是选一个例子（场算符作取揭子）进行因果性验证，其实任意算符都满足一关系：（任意两个玻色子算符）

类空：$[\phi(x), \pi(y)] = 0 \qquad [\phi(x), \pi(y)] = 0$

因果性带：Complex k-G theory（在实 K-G 场中把实不清楚）

$\mathcal{L} = |\partial\phi|^2 - m^2|\phi|^2 \quad \phi = \phi_1 + i\phi_2$；（自由度加倍）

具有 U(1) 相位转动对称性

考虑 $[\phi(x), \phi(y)]$（类空间隔）对其展开

(复) $\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip\cdot x} - b_p^+ e^{ip\cdot x})$ 复数量子化后不是厄米；

a_p 产生一个反粒子

有对易关系 $[a_p, a_q^+] = \delta^3(p-q)$

$[b_q, b_q^+] = \delta^3(p-q)$

$[a, a] = [a^+, a^+] = [b, b] = [b^+, b^+] = [a^+, b] = [a, b^+] = 0$

$a_p|0\rangle = b_p|0\rangle = 0$

$\langle 0|[\phi(x), \phi(y)]|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle - \langle 0|\phi(y)\phi(x)|0\rangle$

$= \langle 0|a\ b^+|0\rangle = 0$

正粒子 反粒子

正反粒子内积为 0

$\langle 0|[\phi(x), \phi^+(y)]|0\rangle = \langle 0|\phi(x)\phi^+(y)|0\rangle - \langle 0|\phi^+(y)\phi(x)|0\rangle$

$= \langle 0|(a+b^+)(a^+ + b)|0\rangle - \langle 0|(a^+ + b)(a + b^+)|0\rangle$

$= \langle 0|aa^+|0\rangle - \langle 0|bb^+|0\rangle$

单粒子态内积 — 因果性要求必须存在反粒子

由 $D(x-y)=\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$ $E_p=\sqrt{p^2+m^2}$

要两项相等 → 两类子体系有相同 m

总结：因果性要求 {① 体系存在粒子 ② 反~ m与正~严格相等

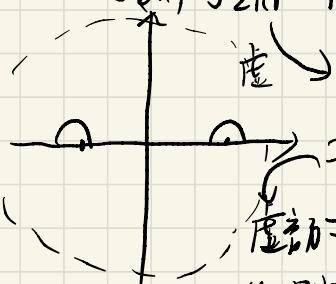
K-G场论 propagator（传播子）

两类 { Retarded ~ / Feynmann ~ 费

$\langle 0|[\phi(x),\phi(y)]|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(e^{-ip\cdot(x-y)}-e^{ip(x-y)})\Big|_{p^0=E_p=\sqrt{m^2+p^2}}$

假设 $x^0>y^0=\int\frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2E_p}e^{-ip(x-y)}\Big|_{p^0=E_p}+\frac{1}{-2E_p}e^{ip(x-y)}\Big|_{p^0=E_p}\right\}$

$=\int\frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip(x-y)}$ $(x_0>y_0)$

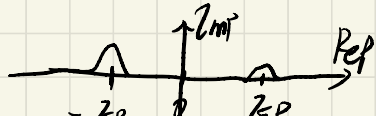


$x_0>y_0$ 则应取下半平面积分

则 $\int\frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2E_p}e^{-ip(x-y)}\Big|_{p_0=E_p}+\frac{1}{2E_p}e^{-ip(x-y)}\Big|_{p_0=-E_p}\right\}$

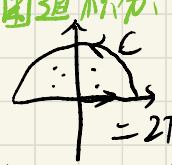
$\frac{1}{p^2-m^2}=\frac{1}{p_0^2-E_p^2}$

$=\frac{1}{(p^0+E_p)(p_0-E_p)}$



未见定在奇点附近从上积

围道积分：



$\oint_C dz\, f(z)$

$=2\pi i\sum \mathrm{Res}[f(z)]$

柯西定理：留数定理

定义推迟传播子/格林函数

$D_R(x-y)=\langle 0|[\phi(x),\phi(y)]|0\rangle$

要求 $x_0>y_0$，$D_R(x-y)\neq 0$ $x_0<y_0=0$

用阶跃函数：

$=\theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)}$



同上：格林函数

无源 $\Box\phi=0$

有源 $\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2+J(x)\phi(x)$ $(\Box\phi+m^2)\phi_x=J(x)$

在时间 t 时 $\phi(x)=\phi_0(x)+i\int d^4y\,D_R(x-y)J(y)$ (通解)

$(\Box^2+m^2)D_R(x-y)=-i\delta^{(4)}(x-y)$ 格林~

验证：若存在这样一个 $D_R(x-y)$，其中 ϕ 是否满足运动方程

$(\Box_x+m^2)\phi(x)=(\Box_x+m^2)\phi_0+i\int d^4y\,(\Box_x+m^2)D_R(x-y)J(y)$

满足自由场 $=0$ $=-i\delta^4(x-y)$

$=J(y)$

$\phi(x)$ 可换成经典电动力学中的麦克斯韦场 → 推迟势

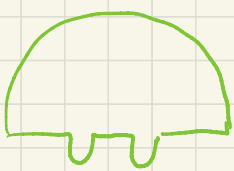
量子场论中

$D_F(x-y)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\widetilde{D_R}(p)$ 动量空间的推迟传播子

联立 D_R 与 D_F，读出 $\widetilde{D_R}(p)\Rightarrow$

代入运动方程，

$(-p^2+m^2)\widetilde{D_R}(p)=-i\Rightarrow\widetilde{D_R}(p)=\frac{i}{p^2-m^2}$



回忆：如何得到一个一个洛伦兹不变的传播子

奇点存在 → 留数 → 闭上积分

若 $x^0<y^0$，→ 闭下积 → 超前传播

若一个上一个下 但超前传播与滞后传播都不重要

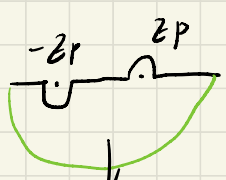
费曼传播子

积分形式相同

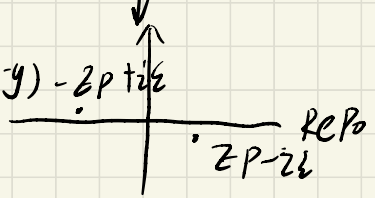
$D_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)}$

优势：不论 x^0 与 y^0 关系，$D_F\neq0$

作用：费曼图



$=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-ip(x-y)}$



$$= \int \frac{d^4p}{(2\pi)^4} \tilde{D}_F(p)\, e^{-ip(x-y)}$$

则 $\tilde{D}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$ 动量空间中的费曼传播子，满足 $(-p^2+m^2)\tilde{D}(p) = -i$

在坐标空间中要求 $(\Box+m^2)D_F(x-y) = -i\delta^4(x-y)$

算符定义：

$$D_F(x-y) = \begin{cases} D(x-y) & \text{当 } x^0 > y^0 \\ D(y-x) & \text{当 } x^0 < y^0 \end{cases}$$

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

观测：$x^0 > y^0$，x^0 事件晚于 y^0

$$\Rightarrow = \langle 0|T\phi(x)\phi(y)|0\rangle$$ T为编时算符 较早放右侧

则 $T[\phi(x)\phi(y)\phi(z)] = \phi(x)\phi(y)\phi(z)$ $(x^0>y^0>z^0)$

$= \phi(z)\phi(y)\phi(x)$ $(z^0>y^0>x^0)$

若两事件等时，两事件类空，→ 两算符对易 → 次序不重要

费曼传播子会如何出现在计算中：

$e^+e^- \longrightarrow \mu^+\mu^-$（此电子重200倍）



外线
内线 ~ 费曼传播子

光子的费曼传播子

$$D_F^{\mu\nu}(x-y) = \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \tilde{D}_F^{\mu\nu}(p)\, e^{-ip(x-y)}$$

$$D_F^{\mu\nu}(p) = \frac{i}{p^2-m^2+i\varepsilon}(-g^{\mu\nu})$$

第四章 Dirac field theory

复习：Lorentz symmetry / Lorentz group 群的表示 representation.
～ 生成元 generator → 李代数

1. 物理按 Lorentz 变换性质来分类：

K-G：标量场 $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$

矢量场 $A^\mu(x) \to A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$

张量场 $h^{\mu\nu}(x) = h'^{\mu\nu}(x) = \Lambda^\mu_\rho \Lambda^\nu_\sigma h^{\rho\sigma}(\Lambda^{-1}x)$

问：有无其他类型的场，满足洛变换性质

考虑转动变换：(矢量场)

$A^i \to A'^i(x) = R^{ij} A^j(R^{-1}x)$ R为3×3转动矩阵 $R^TR = RR^T = I_{3\times3}$

转动怎么写

$R = e^{-i\theta\cdot J}$ 指数映射

等价于 $X \neq 1$

3×3转动性质：

满足 $[J^i, J^j] = i\epsilon^{ijk}$ $(i,j,k = 1,2,3)$

$\epsilon^{123}=1$, $\epsilon^{213}=-1$, $\epsilon^{112}=0$, ϵ 为3阶全反对称张量

ϵ 为123分别转动时才不为0

→ 群的结构常数 决定 李代数

z轴转θ角：$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ $R(2\pi) = R(0)$

举例：以自旋为例

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 描述 $U = e^{-i\theta\cdot\frac{\vec{\sigma}}{2}}$ $\vec{\sigma}^1\sigma^2\sigma^3$ 为泡利矩阵

且 $[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\epsilon^{ijk}\frac{\sigma^k}{2}$ → 相同结构常数

↳生成元

发现 $SO(2N)$ 与 SU 相似，其实 U 为转动群的自己的李代群

于为生成元，生成元的对易子满足李代数，转动群必须满足一样的李代数

举例：更高维代数群：

$R = e^{-i\vec{\theta}\cdot\vec{J}}$（任意确定实常数）$\xrightarrow{\text{在单位元附近}} 1 - i\vec{\theta}\cdot\vec{J} + O(\theta^2)$

生成元决定一切

以自旋为例讲述其转动：$\vec{\theta} = (0, 0, \theta)$

$$U(\theta) = 1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \frac{1}{2!}\left(\frac{-1}{2}\right)^2(\theta\cdot\vec{\sigma})^2 + \cdots$$

$$= 1 - \frac{1}{2}\theta^2 - i\theta\frac{\sigma^3}{2} + \cdots$$

$$= \underbrace{\cos\left(\frac{\theta}{2}\right)}_{\theta\text{偶次幂}} - \underbrace{i\sigma^3\sin\left(\frac{\theta}{2}\right)}_{\text{奇}}\ (\text{乎格表达式})$$

$$U(\theta + 2\pi) = \cos\left(\frac{\theta+2\pi}{2}\right) - i\sigma^3\sin\left(\frac{\theta+2\pi}{2}\right)$$

$$= -\cos\left(\frac{\theta}{2}\right) + i\sigma^3\sin\left(\frac{\theta}{2}\right)$$

$$= -U(\theta)$$

含义：向量空间转2π为负号，旋量表示

$U(\theta + 4\pi) = U(\theta)$

下面考虑洛伦兹变换有无类似表示

转动 → 角动量 $\vec{J} = \vec{x}\times\vec{p} = \vec{x}\times(-i\nabla)$

其中 $[J^i, J^j] = i\varepsilon^{ijk}J^k$

推广：$J^{ij} = -i(x^i\nabla^j - x^j\nabla^i)$ 反对称

推广到4维 $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$

↑二阶反对称张量，6个独立的 $J^{\mu\nu}$

Lorentz群的李代数：

系数（为度规张量）

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

理解：Lorentz群满足相同李代数

以矢量场为例（标量场过于平庸）

$A^\alpha \xrightarrow{\Lambda} A'^\alpha = \Lambda^\alpha_\beta A^\beta$

$\theta^i\theta^j\sigma^i\sigma^j$ 球和

$= \frac{1}{2}\theta^i\theta^j\{\sigma^i, \sigma^j\}$

$= 2\delta^{ij}$ = 反对易子

$= \theta^2$

对从θ都有确定表达

无迷转动变换只是洛伦兹变换的一部分

思路：指数映射 → 利用L变换的参数

$\Lambda^{\mu}{}_{\alpha} = [e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}]^{\alpha}_{\beta} A^{\beta}$

↳ 矢量4维表示对应的生成元

$(J^{\mu\nu})^{\alpha}_{\beta} = i(g^{\mu\nu}g^{\nu}_{\beta} - g^{\mu}_{\beta}g^{\nu\alpha})$

经过验证 $J^{\mu\nu}$ 也满足L群的李代数

未完成总结：

V场 $\phi_a(x) \xrightarrow{\Lambda}$ 时空坐标 $M^{(V)}_{ab}\phi_b(\Lambda^{-1}x)$ or $\phi \to M(\Lambda)\phi$

连续 $\phi \to M(\Lambda_1)M(\Lambda_2)\phi$；$\Lambda_1\Lambda_2$ → 群的封闭

$= M(\Lambda')\phi$；$M(\Lambda)$ 为 L 群的一个有限维表示

例：对标量场 $M(\Lambda) = 1$

一矢 $M(\Lambda) = \Lambda$ } 有限维表示

对旋量场 $M(\Lambda)$

$\forall M(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$

↳ 生成元

由于旋转变换 360° → 负号，720° → 不变，无法用指数表示寻找矩阵表示

思路：先找一种满足李的生成元 → 2×2

称为 spinor representation $SL(2,\mathbb{C})_{complex}$

从L群角度构造 Dirac field 及 Dirac function

L群生成元：$J^{\mu\nu} = J^{\nu\mu}$ （矢量表示下，生成元 $(J^{\mu\nu})^{\alpha}_{\beta}$）

旋量表示：

假设存在 4 个 Dirac matrix γ^{μ}，$\mu = 0123$，（全维数待定 $n\times n$）

满足 $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}I_{n\times n}$（狄拉克 agebra）

定义 $S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$

求 $[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\mu\rho}S^{\nu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$

与洛伦兹生成元满足相同李代数 ⇒ S^{ij} 为2阶张量的一组数目的生成元

之后寻找n的值

n=1，S为一个number不满足

n=2，尝试构造 $\gamma^i = i\sigma^i$，是否满足以下克代 algebra？

$\{\gamma^i, \gamma^j\} = -2\delta^{ij} = 2g^{ij}I_{2\times2}$（反映张量的空间对称）

$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$

$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$

检验其转动变换生成元：

$S^{ij} = \frac{i}{2}\epsilon^{ijk}\sigma^k$ 满足

出现问题：Dirac algebra为4个矩阵的反对称关系，现在只有3个空间措施矩阵，差一个 γ^0？

试一下 $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 不可行 ⇒ $\{\gamma^0, \gamma^i\} = 2g^{0i} = 0$ 不可行

且 $[X, \gamma^i\gamma^i + \gamma^i X] = 2\gamma^i \neq 0$

∴不加以n≥2矩阵阶，接着试n=3

试过不行

试：4×4矩阵：

$\gamma^0 = \begin{pmatrix} 1_{2\times2} & \\ & -1_{2\times2} \end{pmatrix}$，$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ 手征-外尔矩阵 适用于高速或高承受之子

之所以不直接规定Dirac矩阵，是因为γ矩阵不唯一：

若找到一组γ矩阵满足Dirac代数 $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$，总是可以令 $\gamma' = U\gamma U^{-1}$

则 $\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}$ ⇒ 不唯一，无穷多种一样好的矩阵

Dirac矩阵厄米性

$(\gamma^0)^\dagger = \gamma^0$ $(\gamma^i)^\dagger = -\gamma^i$；$(\gamma^0)^2 = 1$；$(\gamma^i)^2 = -1$

$\gamma^0(\gamma^i)^\dagger\gamma^0 = \gamma^0$

生成元 $S^{\mu\nu}$：

转动：$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\sum^k - \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

其他常用矩阵：

② $\gamma^0 = \begin{pmatrix} 1_{2\times2} & \\ & -1_{2\times2} \end{pmatrix}$

$\gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$

Dirac-Pauli表象 适用于低承受之子

$S^{12}=\frac{1}{2}\epsilon^{12k}\begin{pmatrix}\sigma_k & \\ & \sigma_k\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\sigma_3 & \\ & \sigma_3\end{pmatrix}$ $U=e^{-i\vec{\theta}\left(\frac{\vec{\sigma}}{2}\right)}$ 分区对角

boost: $S^{0i}=\frac{i}{4}[\gamma^0,\gamma^i]=\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma_i\end{pmatrix}$

→ Weyl 旋量

有3生成元，具体构造 Dirac 场在L变换下的有限维表示

4×4 $\Lambda_{\frac{1}{2}}=\left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)$

$\Lambda_{\frac{1}{2}}$ 旋量表示

Dirac 旋量场 $\psi(x)$ —— Λ(时空) → $\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)$

区分：A^μ_ν 4×4，洛伦兹指标，作用于时空坐标，矢量表示

$\Lambda_{\frac{1}{2}}$ 4×4，作用于 $\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ 的4分量（四个矢量混合在一起）旋量表示

举例：沿 z 轴转 θ 角：

$\omega^{12}=\theta$ $\omega_{21}=\omega_{12}$ $S^{21}=-S^{12}$

$\Lambda_{\frac{1}{2}}=\exp\left[-\frac{i}{2}\omega^{12}S^{12}-\frac{i}{2}\omega^{21}S^{21}\right]$

$=\exp\left[-i\omega^{12}S_{12}\right]$

$=\exp\left[-i\theta_z/2\begin{pmatrix}\sigma_3 & \\ & \sigma_3\end{pmatrix}\right]$

$=\begin{pmatrix}e^{-\frac{i}{2}\theta_z\sigma_3} & \\ & e^{-\frac{i}{2}\theta_z\sigma_3}\end{pmatrix}$ 依然分区对角

$=\begin{pmatrix}e^{-\frac{i}{2}\theta_z} & & & \\ & e^{\frac{i}{2}\theta_z} & & \\ & & e^{-\frac{i}{2}\theta_z} & \\ & & & e^{\frac{i}{2}\theta_z}\end{pmatrix}=\Lambda_{\frac{1}{2}}(\theta_z)$

z 对称的定，反对称也要求和0

由 $U_\theta=e^{-i\theta_z\frac{\sigma_3}{2}}$ 正角平

③ Majorana 表象

$\gamma^0=\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix}$ $\gamma^1=\begin{pmatrix}i\sigma_3 & \\ & i\sigma_3\end{pmatrix}$

$\gamma^2=\begin{pmatrix}-\sigma_2 & \\ & \sigma_2\end{pmatrix}$ $\gamma^3=\begin{pmatrix}-i\sigma_1 & \\ & -i\sigma_1\end{pmatrix}$

纯虚矩阵

适用于 Majorana 费米子（电中性）中微子（maybe）

验证：$\theta_z=0$ 平庸：$\Lambda_{\frac12}(\theta_z)=1$

$\theta_z=2\pi$ $\quad \Lambda_{\frac12}=\begin{pmatrix}-1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}=-1_{4\times4}=-\Lambda_{\frac12}(0)$

转过2π角，多了个负号 ☆

Dirac表示的其他性质：

$$[\gamma^\mu, S^{\rho\sigma}]=(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta}=i(g^\mu_\alpha g^\nu_\beta-g^\mu_\beta g^\nu_\alpha)\quad ①$$

证明①：Dirac代数+$[A,[B,C]]=\{A,B\}C+C\{A,B\}-B\{A,C\}-\{A,C\}B$

①的应用：

$$\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12}=\Lambda^\mu_\nu\gamma^\nu$$

证明：在单位圆附近无穷小

$$\Lambda_{\frac12}=\exp[-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}]\approx1-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}+O(\omega^2)$$

$$\Lambda_{\frac12}^{-1}=\exp[\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}]$$

$$\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12}=(1+\tfrac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta})\gamma^\mu(1-\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})$$

$$=(1-\tfrac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu$$

$$=\Lambda^\mu_\nu$$

$[\gamma^\mu,S^{\rho\sigma}]=$

$\Lambda^\mu_\nu=\exp(-\tfrac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}]^\mu_\nu$

$\approx\delta^\mu_\nu-\tfrac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu_\nu+\cdots$

不仅对单位圆邻域，对ω有限Λ^μ_ν都成立

应用：为什么克莱因戈登方程$(i\gamma^\mu\partial^2+m^2)\psi=0$考察其在L变换下是否保证不变

在另参照系中：$\psi(x)\to\psi'(x)=\Lambda_{\frac12}\psi(\Lambda^{-1}x)$，$\psi'(x)$是否仍然满足Dirac方程

$$(i\gamma^\mu\partial_\mu-m)\Lambda_{\frac12}\psi(\Lambda^{-1}x)=[i\gamma^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu-m]\Lambda_{\frac12}\psi(\Lambda^{-1}x)$$

$$=\Lambda_{\frac12}\Lambda_{\frac12}^{-1}[i\gamma^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu-m]\Lambda_{\frac12}\psi(\Lambda^{-1}x)$$

乘一个单位矩阵

$= \Lambda_{\frac{1}{2}}\left[i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}(\Lambda^{-1})^\nu_\mu\partial_\nu - m\right]\psi(\Lambda^{-1}x)$

$= \Lambda_{\frac{1}{2}}\left[i\gamma^\sigma\Lambda^\mu_\sigma(\Lambda^{-1})^\nu_\mu\partial_\nu - m\right]\psi(\Lambda^{-1}x)$ $\quad \Lambda^{-1}\Lambda = \delta$

$= \Lambda_{\frac{1}{2}}\left[i\gamma^\sigma\delta^\nu_\sigma\partial_\nu - m\right]\psi(\Lambda^{-1}x)$

$= \Lambda_{\frac{1}{2}}\left[i\gamma^\nu\partial_\nu - m\right]\psi(\Lambda^{-1}x)$ 得证

缩并指标

尝试构造 Dirac 场自旋拉氏量

要求 $\psi(x) \xrightarrow{\Lambda} \psi'(x') = \psi(\Lambda^{-1}x)$ 标量变换 $\quad \mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

$\psi(x)$ 必为一实数

尝试：$\psi(x) = \psi^\dagger\psi = (\cdots)\begin{pmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}$ 不行，因为 ψ 分量为 complex；

改进 $\psi^+ = (\psi^*)^T$ 厄米共轭

$\psi(x) = \psi^+\psi$ 验证其在 L 变换下为标量变换

证明：

有 $\psi \xrightarrow{\Lambda} \Lambda_{\frac{1}{2}}\psi$，$\psi^+ \xrightarrow{\Lambda} \psi^+\Lambda^+_{\frac{1}{2}}$

则 $\psi^+\psi \xrightarrow{\Lambda} \psi^+\Lambda^+_{\frac{1}{2}}\Lambda_{\frac{1}{2}}\psi$ 但 $\Lambda^+_{\frac{1}{2}}\Lambda_{\frac{1}{2}} \neq I$

因为 $\Lambda^+_{\frac{1}{2}} \neq \Lambda^{-1}_{\frac{1}{2}}$（即不是幺正矩阵），所以不能这样构造

已知转动群任何维矩阵表示是幺正的，为什么 Dirac 表示不幺正：

数学上证明：

$\Lambda^+_{\frac{1}{2}} = \left(\exp\left[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right]\right)^+ \xrightarrow{\text{指数展开}} \exp\left[+\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^+\right]$

但 $S^{\mu\nu}$（生成元）不厄米，不能等：

$(S^{\mu\nu})^+ = \left(\frac{i}{4}[\gamma^\mu,\gamma^\nu]\right)^+ = -\frac{i}{4}[(\gamma^\nu)^+,(\gamma^\mu)^+] = +\frac{i}{4}[\gamma^{\mu+},\gamma^{\nu+}]$

已知：γ^0 厄米 $\quad \vec{\gamma}^+ = -\vec{\gamma}$（反厄米）

所以 $(S^{ij})^+ = S^{ij}$ 转动生成元厄米

数学上，若生成元全厄米，则这层李群（如转动群）幺正的

$(S^{0i})^\dagger = S^{0i}$ boost —— 反厄米 ⇒ L群的Dirac表示矩阵不幺正

L群不存在幺正的有限维度表示 → 会不会带来一些后果

boost生成元反厄米
→ L群非紧致群 幺正
$(\Lambda^\dagger)^{-1} = \Lambda^{-1} + Z_{4\times4}$
$\Lambda g \Lambda^\dagger = g$；但 $\Lambda\Lambda^\dagger \neq 1$
$\gamma^{0\dagger} = \gamma^0$，$(\gamma^i)^\dagger = -\gamma^i$
$\Rightarrow \gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$

作用在场本身，场并不对应一个可观测量

虽然 $S^{\mu\nu}$ 不厄米，但考虑 $\gamma^0(S^{\mu\nu})^\dagger\gamma^0$ $(\gamma^0)^2 = 1_{4\times4}$

$= \gamma^0\frac{i}{4}[(\gamma^\mu)^\dagger,(\gamma^\nu)^\dagger]\gamma^0 = \frac{i}{4}[(\gamma^0)(\gamma^\mu)^\dagger\gamma^0, \gamma^0(\gamma^\nu)^\dagger\gamma^0]$

$= \frac{i}{4}[\gamma^\mu,\gamma^\nu] = S^{\mu\nu}$ 回到本身

所以，$\mathcal{L}$ 不包含 $\psi^\dagger\psi$，但包含 $\bar\psi = \psi^\dagger\gamma^0$

$\bar\psi\psi$ 为L洛兹标量 $\bar\psi\psi \subset \mathcal{L}$

↳ 证：$\bar\psi \xrightarrow{\Lambda} \psi^\dagger\gamma^0 = \psi^\dagger\Lambda_{\frac12}^\dagger\gamma^0$ （$1 = \gamma_0^2$）

$= \bar\psi\gamma^0\Lambda_{\frac12}^\dagger\gamma^0 = \bar\psi\gamma^0\exp[+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger}]\gamma^0$ $\quad \gamma S^{\mu\nu\dagger}\gamma^0 = S^{\mu\nu}$

$= \bar\psi\Lambda_{\frac12}^{-1}$

所以 $\bar\psi\psi \xrightarrow{\Lambda} \bar\psi\Lambda_{\frac12}^{-1}\Lambda_{\frac12}\psi = \bar\psi\psi$ L标量 → 构造合理

$\mathcal{L}$ 除了非导数项 $\bar\psi\psi$ 还有一个导数项：

先考察：$\bar\psi\gamma^\mu\psi \xrightarrow{\Lambda} \bar\psi\Lambda_{\frac12}\gamma^\mu\Lambda_{\frac12}\psi$ $\quad \Lambda_{\frac12}\gamma^\mu\Lambda_{\frac12} = \Lambda^\mu{}_\nu\gamma^\nu$

$= \Lambda^\mu{}_\nu(\bar\psi\gamma^\nu\psi)$ 矢量协变换

尝试 $\mathcal{L} \supset \bar\psi\gamma^\mu\psi$，但 $\mathcal{L}$ 为标量，所以需要携带 μ 的与之缩并

考虑 $\mathcal{L} \supset \bar\psi\gamma^\mu\partial_\mu\psi$，以上两项拼凑出

$\mathcal{L}_{\text{Dirac}} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$，描述自由度：$\psi$，$\bar\psi$

E-L方程 对 $\bar\psi$：$\frac{\partial\mathcal{L}}{\partial\bar\psi} = 0$ （没有因为含导数作用在 $\bar\psi$ 上）

$(i\gamma^\mu\partial_\mu - m)\psi = 0$ （狄拉克方程）

对 $\bar{\psi}$: E-L方程:

$$-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0$$ 重新诠释七方程

有的教科书

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \overleftrightarrow{\partial_\mu} - m)\psi \qquad \mathcal{L} = \mathcal{L}_{\text{动}} + \partial_\mu J^\mu$$ 全导数

$$\mathcal{L}_{K\text{-}G} = \frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi \;;\; (\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2)\text{等价}$$

外尔旋量

$$\psi_{Dirac} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}_{4分量} \text{写成2个Weyl} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

这样写原因: $\psi_D' \xrightarrow{\Lambda} \Lambda_{\frac{1}{2}}\psi_D \;;\; \Lambda_{\frac{1}{2}} = \exp\left[-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\right]$

$$S^{0i} = \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} \;;\; S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$ block-diagnal

说明L变换下, ψ_L, ψ_R 独立变换, 因为 $S^{\mu\nu}$(生成元)分块对角

$\Leftrightarrow$ Dirac表示为一个可约表示 (reducible repre)

→ 考虑找到基本单元, 将Dirac更细分→找不到分块对角

→ Weyl填充一个不可约表示

→ 刻画旋量场基本表示是Dirac表示, 而是Weyl (2×2)表示

$\psi_L \to (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2})\psi_L$

↑转动3个参数 → boost 左手外尔场

$\psi_R \xrightarrow{\Lambda} (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2})\psi_R$ 右 ——

考虑: 由于Weyl为基本场, 用 ψ_L 能否构造一个Dirac场

原则上可以 ψ_L 做 $\xrightarrow{\Lambda}$ 按右手 ψ_R 变换 $\to \psi_D = \begin{pmatrix} \psi_L \\ \sigma_2\psi_L \end{pmatrix}$

用Weyl重新理解Dirac方程:

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

m为对角 ↗ ↘ ∂_μ为非对角

$\longrightarrow$ $\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0 \end{cases}$ Weyl equation

m项所有在混合 $\psi_L\psi_R$ 简化理论 $m=0$

手征极限

手征的费米子(中微子)经用

定义4维把到矢的矩阵

$\sigma^\mu = (1, \vec{\sigma})$ $\bar{\sigma}^\mu = (1, -\vec{\sigma})$

则 $\gamma^\mu \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ $\mu = 1,2,3$

Dirac equation 泡利矩阵 $= i\sigma^\mu\partial^\mu$

$$\begin{pmatrix} -m & i\bar{\sigma}\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

分解为组:

$\begin{cases} i\bar{\sigma}\partial\psi_L = 0 \\ i\sigma\partial\psi_R = 0 \end{cases}$

为什么克方程的自由粒子解:

Dirac方程:

$(-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi = 0$

$= (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = 0$

缩并对称 $(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi = 0$

$= (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = 0$

$= (\Box^2 + m^2)\psi = 0$ (K-G方程)

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

平面波解: $(\Box^2 + m^2)e^{-ip\cdot x} = 0 \Rightarrow p^2 = m^2$, 在壳条件 $p^0 = E = \pm\sqrt{p^2 + m^2}$

假设 Dirac $\psi(x) \propto e^{-ip\cdot x}$ $(p^2 = m^2)$ ($\psi(x)$为D平面波解,且每分量都满足K-G equation)

但Dirac的4个分量的列矢量 $\rightarrow$ 引入4分量旋量

令 $\psi(x) = u(p)e^{-ip\cdot x}$ $u(p)$为 Dirac Spinor, 仅是 p^μ 的函数

代入 $(i\gamma^\mu\partial_\mu - m)u(p)e^{-ip\cdot x} = 0$ 有4个本征值 $\rightarrow$ 4个$u(p)$ $\begin{cases} 2个\ p^0 = \sqrt{p^2+m^2} & 正能解 \\ 2个\ p^0 = -\sqrt{\ } & 负一 \end{cases}$

先考虑正能解 $(p^0 > 0)$

方程:

$(p_\mu\gamma^\mu - m)u(p)=0$

静止粒子 $p_0^\mu=(m,\vec{0})$，方程变为

$(p_0\gamma^0-m)u(p)=0$

$\begin{pmatrix}-m & m\\ m & -m\end{pmatrix}u(p)=0 = m\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$

其解为：$u(p_0)=\sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}$

粒子沿 z 轴运动为

$p^\mu=(E,0,0,p_z)\quad E=\sqrt{p^2+m^2}$

Schwartz 习题

$(\not{p}-m)u(p)=0\quad \not{p}=p_\mu\gamma^\mu=\begin{pmatrix}0 & p\cdot\sigma\\ p\cdot\bar{\sigma} & 0\end{pmatrix}=\begin{pmatrix}p_0\sigma_0-p_z\sigma_3 & \\ & p^0\sigma^0+p_z\sigma^3\end{pmatrix}$

$=\left(\begin{array}{cc|cc} 0 & 0 & E-p_0 & 0\\ 0 & 0 & 0 & E+p_0\\ \hline E+p_z & 0 & 0 & 0\\ 0 & E-p_z & 0 & 0\end{array}\right)$

令 $a=\sqrt{E-p_z}\quad b=\sqrt{E+p_z}$

易证：$a^2b^2=E^2-p_z^2=m^2\Rightarrow m=ab$

$\left(\begin{array}{cc|cc} -ab & 0 & a^2 & 0\\ 0 & -ab & 0 & b^2\\ \hline b^2 & 0 & -ab & 0\\ 0 & a^2 & 0 & -ab\end{array}\right)u(p)=0$ 有解 $\begin{bmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\xi^s\\ \begin{pmatrix}b & 0\\ 0 & a\end{pmatrix}\xi^s\end{bmatrix}_p$

$u^{s=1}(p)=\begin{pmatrix}\sqrt{E-p_z}\\ 0\\ \sqrt{E+p_z}\\ 0\end{pmatrix}\quad u^{s=2}(p)=\begin{pmatrix}0\\ \sqrt{E+p_z}\\ 0\\ \sqrt{E-p_z}\end{pmatrix}$

自洽性检验：

静止：$p=0$，$E=m$

$u^{s=1}(p)=\begin{pmatrix}\sqrt{m}\\ 0\\ \sqrt{m}\\ 0\end{pmatrix}\quad u^{s=2}(p)=\begin{pmatrix}0\\ \sqrt{m}\\ 0\\ \sqrt{m}\end{pmatrix}$

记号：$\not{p}=p_\mu\gamma^\mu$

$\not{A}=A_\mu\gamma^\mu=A^\mu\gamma_\mu$

式 $p\cdot\gamma$

$p_0=\begin{pmatrix}-m & \\ & -m\end{pmatrix}$

类似于自旋

$\xi=\begin{pmatrix}1\\ 0\end{pmatrix},\begin{pmatrix}0\\ 1\end{pmatrix}$

高动极限：

$$u^{s=1}=\begin{pmatrix}0\\0\\\sqrt{2E}\\0\end{pmatrix} \qquad u^{s=2}(p)=\begin{pmatrix}0\\\sqrt{2E}\\0\\0\end{pmatrix}$$

螺旋态：$h=\vec{p}\cdot\vec{S}$（带质量粒子，指明参照系）

对0质量粒子，h为Lorentz不变量（好量子数）

例如对光子 $|\vec{p},\lambda=1\rangle$ $\lambda=1$右旋光；$\lambda=-1$左旋光

无内禀旋度为0的自旋粒子；对0质量粒子，自旋为J，只有两个物理态 $h=\pm J$

对有质量粒子（ J）2J+1个简并物理态

0质量条件下，手征性与螺旋度重合

扩展：（高动极限）

有 $\sigma=(1,\vec{\sigma})$ $\bar{\sigma}^\mu=(1,-\vec{\sigma})$

$$p\cdot\sigma=p^0\sigma^0-p^3\sigma^3=\begin{pmatrix}E-p_z & 0\\0 & E+p_z\end{pmatrix}$$

引入$\sqrt{\ }$（对角阵操作）

$$\sqrt{p\cdot\sigma}=\begin{pmatrix}\sqrt{E-p_z} & \\ & \sqrt{E+p_z}\end{pmatrix} \qquad \sqrt{p\cdot\bar{\sigma}}=\begin{pmatrix}\sqrt{E+p_z} & \\ & \sqrt{E-p_z}\end{pmatrix}$$

则 $u^s(p)=\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$ 是正确的解（对于ψ）

可验证其满足Dirac方程

$$(\not{p}-m)u(p)=\begin{pmatrix}-m & p\cdot\sigma\\ p\cdot\bar{\sigma} & -m\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$$

$$=\begin{pmatrix}(-m\sqrt{p\cdot\sigma}+p\cdot\sigma\sqrt{p\cdot\bar{\sigma}})\xi^s\\ (p\cdot\bar{\sigma}\sqrt{p\cdot\sigma}-m\sqrt{p\cdot\bar{\sigma}})\xi^s\end{pmatrix}=\begin{pmatrix}\sqrt{p\cdot\sigma}\,(-m+\sqrt{p\cdot\sigma}\cdot(p\cdot\bar{\sigma}))\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,(\sqrt{(p\cdot\bar{\sigma})}\cdot(p\cdot\sigma)-m)\xi^s\end{pmatrix}$$

$$\gamma^\mu=\begin{pmatrix}&\sigma^\mu\\ \bar{\sigma}^\mu&\end{pmatrix}$$

手征-外尔表象

$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = p_\mu p_\nu \frac{1}{2}\{\sigma^\mu, \bar{\sigma}^\nu\}$

$= g_{\mu\nu} p^\mu p^\nu I_{2\times 2} = p^2 I_{2\times 2} = m^2 I_{2\times 2}$

开根号 $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ 满足 Dirac 方程

$\sigma^\mu = (1, \vec{\sigma})$

$\{\sigma^i, \sigma^j\} = 2\delta^{ij} I_{2\times 2}$

$\{\sigma^\mu, \bar{\sigma}^\nu\} = 2g^{\mu\nu} I_{2\times 2}$

Dirac 场量的归一化和完备关系 (Dirac 方程的负频解)

$(i\gamma\partial - m)\psi(x) = 0$

$\psi = u^s(p) e^{-ip\cdot x}$

$(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla - m) u^s(p) e^{-ip\cdot x} = 0$

$\Rightarrow (\underbrace{\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p}}_{\not{p}} - m) u^s(p) e^{-ip\cdot x} = 0$

若 $p_0 = \pm\sqrt{\vec{p}^2 + m^2} = \pm E_p$，$p_0 = -E_p$ 时，代入

$(-\gamma_0 E_p + \vec{\gamma}\cdot(-\vec{p}) - m) u^s(p) e^{iE_p x^0 - i(-\vec{p})\vec{x}}$

$= -(\gamma_0 E_p - \vec{\gamma}\cdot(-\vec{p}) + m) u^s(p) \exp^{+i\tilde{p}x}$

$= -(\not{\tilde{p}} + m) u^s(p) e^{i\tilde{p}x} = 0$

定义 $\tilde{p}^\mu = (E_p, -\vec{p})$

定义 v 旋量 $v^s(\tilde{p}) = u^s(p)(p^0 = E_p, \vec{p})$

满足 $(\not{\tilde{p}} + m) v^s(\tilde{p}) = 0$ 去掉 ~，$(\not{p} + m) v^s(p) = 0$ $(p^0 = E_p, -\vec{p})$

$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$ $s = 1, 2$，$\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$(p\cdot\sigma)(p\cdot\bar{\sigma}) = m^2$ 引入记号，$v^s(p)$ 也满足 Dirac 方程

考虑两个特例

$v^s(p) = \sqrt{m}\begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix}$

沿 z 轴的 $\uparrow$ 个 p $\bar{v}^s(p) = \begin{pmatrix} (\sqrt{E-p_z}\,\sqrt{E+p_z})\eta^s \\ -(\sqrt{E+p_z}\,\sqrt{E-p_z})\eta^s \end{pmatrix}$ (只有 η^3)

$$v^{s=1}_{(p)} = \begin{pmatrix} \sqrt{E+p_z} \\ 0 \\ -\sqrt{E+p_z} \\ 0 \end{pmatrix} \qquad v^{s=2}_{(p)} = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ \sqrt{E+p_z} \end{pmatrix}$$

也可验证在高动极限下，手征性体现

u与v正交归一性质：

$$u^\dagger u = (\xi^\dagger\sqrt{p\cdot\sigma}, \xi^\dagger\sqrt{p\cdot\bar\sigma}) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger p\cdot\sigma\,\xi + \xi^\dagger p\cdot\bar\sigma\,\xi = 2\xi^\dagger p^0 I_{2\times 2}\,\xi$$

$$= 2E_p \xi^\dagger\xi \quad (E_p\text{是}1\text{不变量}, \xi^\dagger\xi\text{不是}1\text{，但是}\delta)$$

ξ为厄米的

$$\bar u = u^\dagger\gamma^0$$

$$\bar u u = (\xi^\dagger\sqrt{p\sigma}, \xi^\dagger\sqrt{p\cdot\bar\sigma}) \begin{pmatrix} & I \\ I & \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar\sigma}\,\xi \\ \sqrt{p\cdot\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger \left(\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} + \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)}\right)\xi$$

$$= 2\xi^\dagger m\xi = 2m\xi^\dagger\xi$$

$\xi^{r\dagger}\xi^s = \delta^{rs}$
$r,s = 1,2$

总结：
$$\begin{cases} u^{r\dagger}u^s = 2E_p\delta^{rs} \\ \bar u^r_{(p)} u^s_{(p)} = 2m\delta^{rs} \end{cases}$$
→ 用于0质量粒子归一化

推导$v^\dagger v$：

$$v^\dagger v = \eta^\dagger(p\cdot\sigma + p\cdot\bar\sigma)\eta = 2E_p\eta^\dagger\eta$$

$$\bar v = v^\dagger\gamma^0 v = (\eta^\dagger\sqrt{p\cdot\sigma}, -\eta^\dagger\sqrt{p\cdot\bar\sigma}) \begin{pmatrix} & I \\ I & \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ \sqrt{p\cdot\bar\sigma}\,\eta \end{pmatrix}$$

$$= -\eta^\dagger\left(\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} + \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)}\right)\eta^\dagger$$

$$= -2m\eta^\dagger\eta$$

$\eta^{s=1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\eta^{s=2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

u与v的正交归一关系：

$$u^{s\dagger}(p)\,v^{s}(p)=v^{s\dagger}(p)\,u^{s}(p)=0$$

完备性关系：自旋求和公式

$$\sum_{s=1,2}u^{s}(p)\,\bar{u}^{s}(p)=\sum_{s}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^{s}\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^{s}\end{pmatrix}\left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

$$=\begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\end{pmatrix}=\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar{\sigma} & m\end{pmatrix}=\not{p}+m$$

$\xi=\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$\sum\xi\xi^{\dagger}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\not{p}=p_\mu\gamma^\mu$

对负能解，类似推导

$$\sum_{s}v^{s}(p)\,\bar{v}^{s}(p)=\not{p}-m$$

至此，Dirac方程平面波解已清晰

Dirac矩阵与Dirac场的双线性

4阶矩阵类型（找4×4矩阵的完备基）

$\Gamma_{4\times4}$		个
$I_{4\times4}$		1
γ^μ		4
$S^{\mu\nu}=\frac{i}{4}[\gamma^\mu,\gamma^\nu]$		6
$\gamma^{\mu\nu\rho}=\gamma[\mu\gamma^\nu\gamma^{\rho]}$		4
$\gamma^{\mu\nu\rho\sigma}=\gamma[\mu\gamma^\nu\gamma^\rho\gamma^{\sigma]}$	1	赝量

验证 $\bar{\psi}S^{\mu\nu}\psi\xrightarrow{\Lambda}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar{\psi}S^{\alpha\beta}\psi$

$\Lambda_{\frac12}\gamma^\mu\Lambda_{\frac12}=\Lambda^\nu{}_\mu\gamma^\mu$

验证其他矩阵的变换性质，首先定义γ^5

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

$\epsilon^{0123}=1$

交换4个指标，有$4!$种全反对称张量

为什么叫γ^5：$x^\mu=(x^0\ x^1\ x^2\ x^3)$

$=(x\ x^2\ x^3\ x^4)$ 欧氏矢量

$x^\mu\gamma_\mu$

$(\gamma^1\gamma^2\gamma^3\gamma^4)\gamma^4$被用掉

$$\gamma^5=\begin{pmatrix}-1_{2\times2} & \\ & 1_{2\times2}\end{pmatrix}$$

其性质：① 厄米矩阵

② $(\gamma^5)^2=1_{4\times4}$

不依赖于表象

③ $\{\gamma^5, \gamma^\mu\}=0$

$\psi_D\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$ $\gamma^5\begin{pmatrix}\psi_L\\ 0\end{pmatrix}=\begin{pmatrix}-\psi_L\\ 0\end{pmatrix}$ 本征值为-1

$\gamma^5\begin{pmatrix}0\\ \psi_R\end{pmatrix}=\begin{pmatrix}0\\ \psi_R\end{pmatrix}$ ——1

左手投影算符：

$P_L=\frac{1-\gamma^5}{2}=\begin{pmatrix}1_{2\times2}&0\\0&0\end{pmatrix}$; $P_R=\frac{1+\gamma^5}{2}=\begin{pmatrix}0&0\\0&1_{2\times2}\end{pmatrix}$

性质：$P_L+P_R=\begin{pmatrix}1_{2\times2}\\ &1_{2\times2}\end{pmatrix}$; $P_L\cdot P_R=P_R\cdot P_L=0$; $P_L^2=P_L$; $P_R^2=P_R$

考虑 $P_L\psi_D=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}=\begin{pmatrix}\psi_L\\0\end{pmatrix}$; $P_R\psi_D=\begin{pmatrix}0\\ \psi_R\end{pmatrix}$

又有$\{\gamma^5,\gamma^\mu\}=0\Rightarrow[\gamma^5, S^{\mu\nu}]=0$

最后 $\bar\psi\gamma^5\psi \xrightarrow{\Lambda} \bar\psi\Lambda_{\frac12}^{-1}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\Lambda_{\frac12}\psi$

$\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12}=\Lambda^\mu_{\ \nu}\gamma^\nu$

矢量表示 $\Lambda_{\frac12}\Lambda_{\frac12}^{-1}$

$=\bar\psi\,\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu_{\ \mu'}\Lambda^\nu_{\ \nu'}\Lambda^\rho_{\ \rho'}\Lambda^\sigma_{\ \sigma'}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\rho'}\gamma^{\sigma'}\psi$

$=\det\Lambda=+1$ 洛伦兹不变张量

$=\bar\psi\gamma^5\psi$ 赝标量流　　也记L相量类似于$\bar\psi\psi$标量流

电中性粒子为厚宇称为自

回到那张表

$\gamma^{\mu\nu\rho}=\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$ 构造方法 $\gamma^5\cdot\gamma^\mu$ $\mu=\mu\nu\rho\sigma$

$\Gamma^{4\times4}$			
	1	1	标量
	γ	4	矢量流
	$S^{\mu\nu}$	6	张量流
	$\gamma^\mu\gamma^5$	4	赝矢流（在L变换下为4矢量）
	γ^5	1	赝标量

对称性分析：

$j^\mu(x)=\bar\psi(x)\gamma^\mu\psi(x)$　　$j^{\mu5}=\bar\psi(x)\gamma^\mu\gamma^5\psi(x)$

$\mathcal{L}=\bar\psi(i\partial\!\!\!/-m)\psi$

$\mathcal{L}$ 之为零 $\bar\psi(i\partial\!\!\!/-m)\psi=0$

检验其是否守恒及对称性

$\partial_\mu j^\mu=(\partial_\mu\bar\psi)\gamma^\mu\psi+\bar\psi\gamma^\mu(\partial_\mu\psi)$

$=im\bar\psi\psi+\bar\psi(-im)\psi=0$ 守恒流 诺特流

$(\partial_\mu\bar\psi)\gamma^\mu-im\bar\psi=0$

什么对称性？

$U(1)$ 的内禀对称性 $\psi\to\psi'=e^{i\alpha}\psi$

$\bar\psi=\bar\psi'=e^{-i\alpha}\bar\psi$ } 对应 $U(1)$ 的相位转动对称性

考虑：

$\partial_\mu j^{\mu5}=\partial_\mu\bar\psi\gamma^\mu\gamma^5\psi+\bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$

$=(im\bar\psi)\gamma^5\psi+\bar\psi(-\gamma^5\gamma^\mu)\partial_\mu\psi$

$=(im\bar\psi)\gamma^5\psi-\bar\psi\gamma^5(-im\psi)=2im\bar\psi\gamma^5\psi$ 赝标流

$\{\gamma^5,\gamma^\mu\}=0$

$m\to0$ 时，近似守恒 PCAC（部分守恒流）→ QCD

$m=0$ } $\mathcal{L}_{Dirac}=\bar\psi i\partial\!\!\!/\psi$ 不变

$U(1)_V:\psi\to e^{i\alpha}\psi\to j^\mu$

$U(1)_A:\psi\to e^{i\alpha\gamma^5}\psi\to j^{\mu5}$ 手征荷守恒，也是 $U(1)$ 内禀对称性

对称性

各种宏线组合（也是守恒流）

定义：$j_L^\mu=\bar\psi\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi$　　$j_R^\mu=\bar\psi\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi$

手征极限下 $\partial_\mu j_L^\mu=\partial_\mu j_R^\mu=0$

$m\to0$，Dirac理论增多对称性

$\mathcal{L}_{Dirac}=\bar\psi i\partial\!\!\!/\psi-m\bar\psi\psi$　$(1=P_L+P_R)$

$=\bar\psi i\partial\!\!\!/(P_L^2\psi)+\bar\psi i\partial\!\!\!/P_R^2\psi-m\bar\psi P_L^2\psi-m\bar\psi P_R^2\psi$

$=\bar\psi i\gamma^\mu P_L\partial_\mu\psi_L+\cdots$

$=\bar\psi_R i\gamma^\mu\partial_\mu\psi_L+\cdots$

$\gamma^\mu P_L=P_R\gamma^\mu$

$= \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R$ 分成left-handed 分离 $u_{1L}\otimes u_{1R}$ → j^μ_L 诺特流

$\bar{\psi}_L = \psi_L^\dagger \gamma^0 = (P_L\psi)^\dagger \gamma^0$
$= \psi^\dagger P_L^\dagger \gamma^0$
$= \psi^\dagger P_L \gamma^0$
$= \psi^\dagger \gamma^0 P_R = \bar{\psi} P_R$

即外场变换: $\psi_L \to \psi_L' = e^{i\alpha_L}\psi_L$

$\psi_R \to \psi_R' = e^{i\alpha_R}\psi_R$ $u_{1L}\otimes u_{1R} \to j^\mu_R$

右质质量项:

$\cdots - m\bar{\psi}P_L\psi_L - m\bar{\psi}P_R\psi_R$

$= -m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R = -m(\bar{\psi}_R\psi_L - \bar{\psi}_L\psi_R)$ 混合左右手场

Fierz 重排:

$\bar{\psi}_1\Gamma\psi_2\ \bar{\psi}_3\Gamma'\psi_4$ 想把4与4结合并

$= \sum_{\gamma=1}^{16} \bar{\psi}_1\Gamma'\psi_4\ \bar{\psi}_3\Gamma'\psi_2$

表示: $(\bar{u}_{1R}\sigma^\mu u_{2R})(\bar{u}_{3R}\sigma^\mu u_{4R})$ $\alpha\beta\gamma\delta = 12$

$= \bar{u}_{1R\alpha}(\sigma^\mu)_{\alpha\beta}u_{2R\beta}\ \bar{u}_{3R\delta}(\sigma^\mu)_{\gamma\delta}u_{4R\delta}$

$= 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\bar{u}_{1R\alpha}u_{2R\beta}\bar{u}_{3R\gamma}u_{4R\delta}$

重组 $= -2\epsilon_{\alpha\gamma}\epsilon_{\delta\beta}$

$= -(\bar{u}_{1R}\sigma^\mu u_{4R})(\bar{u}_{3R}\sigma^\mu u_{2R})$

$(\sigma^\mu)_{\alpha\beta}(\sigma^\mu)_{\gamma\delta}$
$= 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$
$\epsilon_{12} = 1;\ \epsilon_{21} = -1$
$\epsilon = i\sigma^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$
$\sigma^\mu\bar{\sigma}^\nu = 4X_{2\times2}$

Dirac 场的量子化

复习:

$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$; $[\phi] = \frac{3}{2}$; $\mathcal{L}_{K\text{-}G} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$ $[\mathcal{L}] \sim 1$

坐标变换 $x^\mu \to x'^\mu = \lambda x^\mu$ (标度变换):

K-G $\phi(x) \Rightarrow \phi'(x') = \lambda^{\frac{d-2}{2}}\phi(\lambda x)$ engenering demension

Dirac $\psi(x) \Rightarrow \psi'(x') = \lambda^{\frac{3}{2}}\psi(\lambda x)$ (scaling 对称性)

反常量纲
来源于辐射修正

共轭动量密度：

$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i\gamma^0 = \psi^\dagger \gamma^0 i\gamma^0 = i\psi^\dagger$

$\gamma^{0\dagger} = 1$

$\pi = \psi^\dagger \gamma^0$

$H = \int d^3x \mathcal{H} = \int d^3x T^{00} \xrightarrow{\text{勒让德}} \int d^3x (\pi\dot{\psi} - \mathcal{L}_{Dirac})$

$= \int d^3x (\pi\bar{\psi}\gamma^0\partial_0\psi - \bar{\psi}i\gamma^0\dot{\psi} - \bar{\psi}i\vec{\gamma}\cdot\vec{\partial}\psi + m\bar{\psi}\psi)$

$= \int d^3x\ \psi^\dagger[-i\gamma^0\vec{\gamma}\cdot\vec{\partial} + m]\psi$ 第一节相对论波动方程构造 Dirac 方程

$i\frac{\partial}{\partial t}\psi = H\psi = (-i\vec{\alpha}\cdot\vec{\partial} + m\beta)\psi$ ；$\vec{\alpha} = \gamma^0\vec{\gamma}$；$\beta = \gamma^0$

薛定谔绘景：等时

$[\psi(\vec{x}), \pi(\vec{y})] = i\hbar\delta^{(3)}(\vec{x}-\vec{y})$（不同时刻协变其与时间无关）

$\pi(y) = i\psi^\dagger(\vec{y}) \Rightarrow [\psi(\vec{x}), \psi^\dagger(\vec{y})] = \hbar\delta^3(\vec{x}-\vec{y})$ 对偶子量子化

思路：用满足 K-G 场量子化条件 $\longrightarrow$ 先取

回忆：量子化 K-G 场

$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x})$ 厄米算符

$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} e^{ip\cdot x}(a_p + a_{-p}^\dagger)$

对 Dirac 场

$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_{s=1,2}(a_p^s u^s(p) e^{ip\cdot x} + b_p^s v^s(p) e^{-ip\cdot x})$

$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_{s=1,2} e^{ip\cdot x}(a_p^s u_p + b_{(-p)}^s v_{(-p)}^s)$

a 为湮灭动量为 p，能量为 E_p 的电子

$b^\dagger$ 为 p，$-E_p$ 的电子

对 $\bar{\psi}(x)$ ……

思路，a b 满足什么条件，才能满足正则量子化条件。

类似 K-G，假设

$[a_p^r, a_q^{s\dagger}] = [b_p^r\ b_p^{s\dagger}]\{(2\pi)^3\delta^3(\vec{p}-\vec{q})\delta^{rs}$ $r,s=1,2$

$= [b_p, b_q^\dagger]$

其余对易子为0

$[a,b]=[a^\dagger,b]=[a^\dagger,a^\dagger]=[b^\dagger,b^\dagger]=0$

整合：

$[\psi(x),\psi^\dagger(y)]=\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_p 2E_q}}e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\times\sum_{rs}\left([a_p^r,a_q^{s\dagger}]u^r(p)\bar{u}^s(q)\right.$

$\left.+[b_{-p}^r,b_{-q}^{s\dagger}]v^r(-p)\bar{v}^s(-q)\right)\gamma^0$

$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\sum_{r=1,2}\left(u^r(p)\bar{u}^r(p)+v^r(-p)\bar{v}^r(-p)\right)\gamma^0$

$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip(\vec{x}-\vec{y})}\times(\not{p}+m+\not{p}-m)\gamma^0$ // $\not{p}\gamma^0=2E_p$

$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip(\vec{x}-\vec{y})}\left[2E_p\gamma_0-\vec{p}\vec{\gamma}+\gamma_i E_p+\vec{p}\cdot\vec{p}\right]\gamma^0$

$=\delta^3(x-y)\times 1_{4\times4}$（由4+4个量决定）

或写成 $[\psi_a,\psi_b^\dagger]=\delta^3(x-y)1_{4\times4ab}$

完备关系：

$\sum_r u^r(p)\bar{u}^r(p)=\not{p}+m$

$\sum_r v^r(p)\bar{v}^r(p)=\not{p}-m$

令 $\sum_r p^r=E_p-\vec{p}$

则：

$\sum_r v^r(p)\bar{v}^r(p)=\not{p}-m$

目前记问题

则

$H=\int d^3x\,\psi^\dagger(-i\vec{\alpha}\cdot\nabla+m)\psi$

$=\int\frac{d^3p}{(2\pi)^3}\sum_s\left(E_p a_p^{s\dagger}a_p^s-E_p b_p^{s\dagger}b_p^s\right)$ ⇒负能产生一个负粒子→质量↓→不固有

→真空不稳定　$b_p^\dagger|>$ 质量降 E_p

再看因果性：

类比 $[\psi(x),\psi^\dagger(y)]=0$（若 $(x-y)^2<0$）

Haisenburg pic：

$\psi(x)=e^{iHt}\psi(\vec{x},t=0)e^{-iHt}$

$\vec{x}$ 为空间坐标

HPic $\begin{cases}a_p^\dagger(t)=e^{iHt}a_p^{s\dagger}e^{-iHt}=a_p^\dagger e^{-iE_pt}\\ b_p^s(t)=e^{iHt}b_p^s e^{-iHt}=b_p^s e^{iE_pt}\end{cases}$ 负能粒子湮灭算符；

Hpic

$$\psi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(a_p^s u^s(p)e^{-ip\cdot x}+b_p^{s\dagger}v^s(p)e^{ip\cdot x}\right)$$

正频 负频

$$\bar\psi(x)=\int\cdots\left(a_p^{s\dagger}\bar u^s(p)e^{ip\cdot x}+b_p^s\bar v^s(p)e^{-ip\cdot x}\right)$$

因果性

$[\psi_a(x),\bar\psi_b(y)]$ 当 $(x-y)^2<0$ 是否为 0

代入 $=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s\left(u_a^s(p)\bar u_b^s(p)e^{-ip(x-y)}+v_a^s(p)\bar v_b^s(p)e^{ip(x-y)}\right)$

$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left[(\not p+m)_{ab}e^{-ip(x-y)}+(\not p-m)_{ab}e^{ip(x-y)}\right]$

$=(i\partial_x+m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip(x-y)}-e^{ip(x-y)}\right)$ (可以合证)

$=(i\partial_x+m)_{ab}[\hat\varphi(x),\hat\varphi(y)]$ 实 K-G 场

则 $[\psi(x),\bar\psi(y)]=0$ 当 $(x-y)^2<0$，因果性成立

之后再深入考查因果性

$[\psi_a(x),\bar\psi_b(y)]=c\text{number}$

$=\langle0|[\psi_a(x),\bar\psi_b(y)]|0\rangle=\langle0|\psi_a(x)\bar\psi_b(y)|0\rangle+\langle0|\bar\psi_b(y)\psi_a(x)|0\rangle=0$

$\psi_a(x)$ 含 $(a+b)$，$\bar\psi_b(y)$ 含 $(a^\dagger+b^\dagger)$，$\bar\psi_b(y)$ 含 $(a^\dagger+b^\dagger)$，$\psi_a(x)$ 含 $(a+b)$

物理含义：

定义真空：$a_p^s|0\rangle=b_p^s|0\rangle=0\quad\forall p,s=1,2$

第一项 $\langle0|a^\dagger a|0\rangle$ $\langle0|bb^\dagger|0\rangle$ 均 $\neq0$（因果性来源）→ 只有一个矩阵元

第二项 a,b 作用在 $|0\rangle$ 上都已湮灭，$a^\dagger,b^\dagger$ 均从左侧湮灭真空，平庸 $=0$ → 不符合

但因果性（复 K-G 场）

$[\phi(x),\phi^\dagger(y)]=0\quad(x-y)^2<0$

$=\langle0|\phi(x)\phi^\dagger(y)|0\rangle-\langle0|\phi^\dagger(y)\phi(x)|0\rangle$

$a+b^\dagger$ $a^\dagger+b$ $a^\dagger+b$ $a+b^\dagger$

b 不湮灭 $\propto\langle0|aa^\dagger|0\rangle$ $\quad\langle0|bb^\dagger|0\rangle$

→ 两矢量矩阵元都已贡献

完备性公式

对 K-G

$[\phi(x),\phi(y)]=D(x-y)-D(y-x)$

$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(e^{-ip\cdot(x-y)}-e^{+ip(x-y)})$

$=0$

反思之前假设：

1. ①对场关系进行量子化
   ②对真空假设 $a_p^s|0\rangle = b_p^s|0\rangle = 0$ 全能持两个矩阵元者贡献

重新审视量子化假设：

Dirac早期理论：

Dirac海 $E<0$ 自洽地把负能态全部填满

→ $a_p^s|0\rangle = 0$ 无正能电子，但要修改 $b_p^s|0\rangle = 0 \Rightarrow b_p^{s\dagger}|0\rangle = 0$

但可以少一个电子

不能再填充任何一个新电子

可以 $b_p^s|0\rangle \neq 0$

之后再看两项

第一项：$\langle 0|aa^\dagger|0\rangle$，第二项 $\langle 0|b^\dagger b|0\rangle$ 负能电子从无→有

二者孰否精细假设？

不考虑第一个假设，直接计算（放弃(1)）

$$\langle 0|\psi(x)|0\rangle = \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s a_p^s u(p) e^{-ip\cdot x} \times \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\sum_s a_q^{s\dagger}\bar{u}^s(q) e^{i(q\cdot y)}|0\rangle$$

计算 $\langle 0|a_p^s a_q^{s\dagger}|0\rangle$ 不写成 $[a, a^\dagger] + a^\dagger a$ 如何计算？

空间平移算符 $e^{-i\hat{p}\cdot\vec{x}} a_p e^{i\hat{p}\cdot\vec{x}} = a_p e^{ip\cdot x}$ 空间平移算符

$e^{-i\hat{p}\cdot\vec{x}} a_p^\dagger e^{i\hat{p}\cdot\vec{x}} = a_p^\dagger e^{-ip\cdot x}$ $e^{i\hat{p}\cdot\vec{x}}|0\rangle = |0\rangle$

$\hat{p}|0\rangle = 0$；$\hat{J}|0\rangle = 0$ 真空是空间平移，转动不变 $\Rightarrow$ 三动量为0

$$\langle 0|a_p^r a_p^{s\dagger}|0\rangle = \langle 0|a_p^r a_p^{s\dagger} e^{i\hat{p}\cdot\vec{x}}|0\rangle$$

$$1 = e^{i\hat{p}\cdot\vec{x}}\cdot e^{-i\hat{p}\cdot\vec{x}}$$

$$= e^{-iq\cdot x}\langle 0|e^{-i\hat{p}\cdot\vec{x}} a_p^r e^{i\hat{p}\cdot x} a_q^{s\dagger}|0\rangle$$

$$= e^{i(p-q)\cdot x}\langle 0|a_p^r a_q^{s\dagger}|0\rangle \quad \text{当且仅当 } p=q$$

则 $\langle 0|a_p^r a_q^{s\dagger}|0\rangle \propto \delta(p-q)$

再考虑插入空间转动算符

得 $\langle 0|a_p^r a_q^{s\dagger}|0\rangle \propto \delta^{rs}$

含 $\langle 0|\vec{a}_{\vec{p}}^{r} a_{\vec{q}}^{s\dagger}|0\rangle$

$= (2\pi)^3 \delta(\vec{p}-\vec{q})\ \delta^{rs}$ $A(\vec{p}) = \not{p} + m$

则 $\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{s=1,2} u^s(\vec{p})\bar{u}^s(\vec{p}) A(\vec{p}) e^{-ip(x-y)}$

$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\not{p}+m)A(\vec{p})e^{-ip(x-y)}$ $(p^2=m^2)$

插入 $u(\lambda)$: $u(\lambda)|\vec{p}\rangle = |\lambda\vec{p}\rangle$

要求 $A(p)=A(p^2)=A(m^2)=$ constant $\propto \langle 0|a_{\vec{p}}^{s} a_{\vec{p}}^{s\dagger}|0\rangle \propto \langle \vec{p},s|\vec{p},s\rangle$ 模态正定

$= (i\gamma_\alpha - m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip(x-y)} A$ $= i(\partial_x - m) D(x-y) A$

之后计算 $\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$ a与$a^\dagger$不湮灭 $\Rightarrow$ VEV

$\Rightarrow \langle 0|b^\dagger b|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\not{p}-m) e^{ip(x-y)} B(\vec{p})$ $B(m^2) = \overset{>0}{\text{constant}}$

$= -(i\gamma_\alpha + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{ip(x-y)} B$ $\overset{\text{令}}{=} D(y-x)$

若要因果性

$(x-y)^2 < 0$ $D(x-y) = D(y-x)$ 只剩常数 A B

令 A 与 B 一

当 $(x-y)^2 < 0$ (类空), $\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$

$\Rightarrow \{\psi_a(x), \bar{\psi}_b(y)\} = 0$ $[\psi(x), \bar{\psi}(y)] \neq 0$

是否和类空平面: $\psi(x)$, $\bar{\psi}$ 并不是独立构成 可观测量

可观测量: $\vec{P} = \int d^3x\, \psi^\dagger(-i\partial)\psi$ 至少含2个ψ (玻色型算符) ← 偏导数

例如要 $[\bar{\psi}\Gamma_1\psi(x), \bar{\psi}\Gamma_2\psi(y)] = 0$ 对 $(x-y)^2<0$ 因果性依然成立

利用[AB]C = A[BC] + [AC]B − C[A,B]

则等时量子化条件: $\{\psi_a(\vec{x}), \bar{\psi}_b(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$

$\{\psi_a(x), \psi_b(y)\} = \{\bar{\psi}_a(x), \bar{\psi}_b(y)\} = 0$

等价 $\Leftrightarrow \{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\delta^{rs}$

其他反对易子为0 $\{a, a\} \{b, b\} = \{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = \{a, b^\dagger\} = \{a^\dagger, b\} = 0$

代入 H

$$H = \int \frac{d^3p}{(2\pi)^3} \hat{\bar{\psi}}^\dagger [-i\gamma^0 \vec{\gamma}\cdot\vec{\partial} + m\gamma^0] \hat{\psi}$$

展开 $= \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - \tilde{b}^s_{\vec{p}} \tilde{b}^{s\dagger}_{\vec{p}})$

$= \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + \underbrace{\tilde{b}^{s\dagger}_{\vec{p}} \tilde{b}^s_{\vec{p}}}_{\text{正能级发}} - \underbrace{E_{\vec{p}} (2\pi)^3 \delta^3_{(0)} \delta^{ss}}_{\text{负无穷大}})$

重新校准真空 → 解决负能级问题

令 $\tilde{b}^s_{\vec{p}} = b^s_{\vec{p}}$ 反空穴

令 $\tilde{b}^{s\dagger}_{\vec{p}} = b^s_{\vec{p}}$ 产正粒子

$\Rightarrow \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \delta^{rs}$

真空: $\tilde{b}^s_{\vec{p}} |0\rangle = 0$

Dirac theory 真空能 $-\infty$

K-G —— $+\infty$

宇宙学常数 → super symmetry

→ 真空能为0

费米-Dirac统计

$\{a^{r\dagger}_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{\tilde{b}^{r\dagger}_{\vec{p}}, \tilde{b}^{s\dagger}_{\vec{q}}\} = 0$ (Dirac场量子化条件)

$a^{r\dagger}_{\vec{q}} a^{s\dagger}_{\vec{p}} |0\rangle \propto |\vec{p}, s\rangle |\vec{q}, r\rangle$ 直积

$= -a^{s\dagger}_{\vec{p}} a^{r\dagger}_{\vec{q}} |0\rangle = -|\vec{p}, s; \vec{q}, r\rangle$

若指标一样

$(a^{s\dagger}_{\vec{p}})^2 = 0$ $\{a^{s\dagger}_{\vec{p}}\}^2 |0\rangle = |\vec{p}, s\rangle = 0$

Pauli不相容原理

自旋统计定理

$$\vec{P} = \int d^3x T^{0i} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \sum_s (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}})$$

单粒子态：$\sqrt{2E_p}\, a^{s\dagger}_{\vec p}|0\rangle = |\vec p, s\rangle$ ← 自旋指标

满足 $\langle p,r|q,s\rangle = 2E_p(2\pi)^3\delta^3(\vec p-\vec q)\delta^{rs}$ 归一化条件

$|\vec p, s\rangle = \sqrt{2E_p}\, b^{s\dagger}_{\vec p}|0\rangle$ 反粒子

电子的自旋 1/2 验证：

考虑沿 z 轴转 θ $\Lambda^\mu{}_\nu$ θ 很小

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$; $\Lambda g \Lambda^T = g \Rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu}$

$\omega_{12} = -\omega_{21} = \theta$; $\delta x^\mu = \omega^{\mu\nu}x_\nu$

$\delta x^0 = \omega^{0\nu}x_\nu = 0$

$\delta x^3 = \omega^{3\nu}x_\nu = 0$

诺特

$0 = \delta S = \int d^4x\, \partial_\mu [\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial_\mu \phi}\delta_0\phi]$

$\delta_0\phi = \phi'(x') - \phi(x)$ 同坐标

$Q = \int d^3x j^0$

$j^0 = \frac{\partial \mathcal{L}}{\partial_0 \psi}\delta_0\psi$; $\delta_0\psi = \psi'(x') - \psi(x') = \Lambda_{\frac12}\psi(\Lambda^{-1}x) - \psi(x)$

$\Lambda_{\frac12} = \exp\{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\} \approx 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$

$\delta_0\psi = (1 - \frac{i}{2}\theta\Sigma^3)\psi(t, x+\theta y, y-\theta x) - \psi(x)$

$= -\theta(\underbrace{x\partial_y - y\partial_x}_{L_z} + \frac{i}{2}\Sigma^3)\psi(x) + O(\theta^2)$

$L_z = (\vec x \times \vec p)_z$

$j^0 = -i\psi^\dagger\gamma^0(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3)\psi$ 沿 z 轴转 θ 角诺特流 0 分量

$\hat J_z = \int d^3x j^0 \to \hat J = \int d^3x\, \psi^\dagger(\underbrace{\vec x \times (-i\nabla)}_{\text{轨角}} + \underbrace{\frac12\vec\Sigma}_{\text{自角}})\psi$ 总角动量算符

不用算符表示角动量

$J_z|\vec p = 0, s\rangle = ?$ 静止粒子，轨道角不贡献

$\hat J_z a^{s\dagger}_{\vec p = 0}|0\rangle = [\hat J_z, a^{s\dagger}_{\vec p=0}]|0\rangle$

$\hat J_z \supset \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_p 2E_q}} e^{-i\vec q\cdot\vec x} e^{i\vec p\cdot\vec x} \sum_{l,r}(a^{l\dagger}_{\vec q} u_{\vec q}$

$\hat J|0\rangle = 0$ 真空是 z 轴不变

交换 $+b^{L}_{-q}v^{L+}_{(-q)})\frac{\Sigma^3}{2}(a^{r}_{p}u_{rp}+b^{r+}_{-p}v_{(-p)}^{r})$

$\int$次序 $=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{L,r}(a^{L+}_{p}u^{L}_{(p)}+b^{L}_{(-p)}v^{L+}_{(-p)})\frac{\Sigma^3}{2}(a^{r}_{(p)}u_{r(p)}+b^{r+}_{-p}v^{r}_{(-p)})$

与 a^{st}_{p} 对易，只有 a^{r}_{p} 与 a^{L+} 有贡献：

$[a^{L+}_{p}a^{r}_{p},a^{st}_{0}]=a^{L+}_{p}(a^{r}_{p}a^{st}_{0}+\delta^{rs}\delta^{3}(p-0)-\int d^3p\, a^{st}_{0}a^{r}_{p}$ （$=0$）

$=(2\pi)^3\delta^3(p-0)\delta^{rs}a^{L+}_{p}$

$[A,B,C]=A[B,C]+[A,C]B$

则 $[\hat{J}_z,a^{st}_{0}]|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{L,r}u^{L+}_{(p)}\frac{\Sigma^3}{2}u^{r}_{(p)}\delta^3(p)\delta^{rs}a^{L+}_{p}|0\rangle$

$=\frac{1}{2m}\sum_{L=1,2}u^{L+}_{(0)}\frac{\Sigma^3}{2}u_{(0)}\times a^{L+}_{0}|0\rangle$ 求和要求$L=s$，Σ^3为对角，$a^{L+}_{0}\to a^{s+}_{0}$ 为本征

$u^{s}_{(p=0)}=\sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}$，$s=1:\begin{pmatrix}1\\0\end{pmatrix}$ $s=2:\begin{pmatrix}0\\1\end{pmatrix}$

让标示本征值：

$\hat{J}_z|\vec{p}=0,s\rangle=\frac{1}{2m}\sum_{L}\sqrt{m}(\xi^{L+},\xi^{L+})\frac{1}{2}\begin{pmatrix}\sigma^3&\\&\sigma^3\end{pmatrix}\sqrt{m}\begin{pmatrix}\xi^s\\\xi^s\end{pmatrix}a^{L+}_{0}|0\rangle$

$=\sum_{L}(\xi^{L+}\frac{\vec{\sigma}^3}{2}\xi^{s})a^{L+}_{0}|0\rangle$

$s=1$，$\xi^s=\begin{pmatrix}1\\0\end{pmatrix}$ $=(1,0)\frac{1}{2}\begin{pmatrix}1&\\&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=+\frac{1}{2}$ J_z本征值为 $\frac{1}{2}$

$s=2$，$\xi^2=\begin{pmatrix}0\\1\end{pmatrix}$ $=(0,1)\frac{1}{2}\begin{pmatrix}1&\\&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{2}(0,1)\begin{pmatrix}0\\-1\end{pmatrix}=-\frac{1}{2}$ ……为$-\frac{1}{2}$

$|\vec{p}=0,s=1\rangle=|\uparrow\rangle$；$|\vec{p}=0,s=2\rangle=|\downarrow\rangle$

反电子自旋：

$J_z\supset\int(bv\frac{\Sigma^3}{2}b^{+}v)$

$J_z b^{+}_{s}|0\rangle=J_z|\vec{p}=0,s\rangle=[J_z,b^{st}_{0}]|0\rangle$

$=\mp\frac{1}{2}b^{st}_{0}|0\rangle$

$[b,b^{+},b^{+}]=-[b^{+}b,b^{+}]$

$=-b^{+}_{0}$

$v^s=\sqrt{m}\begin{pmatrix}\eta^s\\\chi^s\end{pmatrix}$

$s=1$，$\eta^s=\begin{pmatrix}1\\0\end{pmatrix}$ 本征值 $=-\frac{1}{2}$

$s=2$，$\eta^s=\begin{pmatrix}0\\1\end{pmatrix}$ 〃 $=\frac{1}{2}$

} 与正粒子相反

理解：

举例：U(1)相位对称性

$\mathcal{L}=\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi$ $\psi\to\psi'=e^{i\alpha}\psi$ $\mathcal{L}$不变

$j^\mu=\bar{\psi}\gamma^\mu\psi$

$\hat{Q}=\int d^3x j^0=\int d^3x\,\psi^\dagger\psi$

$=\int\frac{d^3p}{(2\pi)^3}\sum_s(a_p^{s\dagger}a_p^s+b_{-p}^s b_{-p}^{s\dagger})$

$=\int\frac{d^3p}{(2\pi)^3}\sum_s(a^\dagger a-b^\dagger b+\infty\ \text{constant})$

$\hat{Q}|0\rangle=0$ 真空无荷

$\hat{Q}|\text{单电子态}\rangle=+1|\text{单电子态}\rangle$

$\hat{Q}|\text{单反}\ \text{态}\rangle=-1|\text{单反态}\rangle$

$\hat{Q}|5e^-,11e^+\rangle=(5-11)|5e^-,11e^+\rangle=6|\rangle$

回忆：

Noether 守荷 $\hat{Q}$

$\hat{Q}=\int\frac{d^3p}{(2\pi)^3}\sum_s(a_p^{s\dagger}a_p^s-b_p^{s\dagger}b_p^s)$ 电荷算符

$Q|e^+\rangle=e^+|e^+\rangle$

$Q|e^-\rangle=-e|e^-\rangle$

$J_z b_0^{s\dagger}|0\rangle=\mp\frac{1}{2}b_0^{s\dagger}|0\rangle$ s

如何把H写成诉说多个谐振子求和形式：

考虑 $\vec{P}=\int d^3x T^{0i}=\int d^3x\,\psi^\dagger(-i\nabla)\psi$ 玻色型算符 $[P_{(x)},P_{(y)}]=0$ 对(x-y)≥0

$\psi_{(x)}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{i\vec{p}\cdot\vec{x}}\sum_s(a_p^s u_{(p)}+b_p^{s\dagger}v_{(p)})$ $\psi^\dagger_{(x)}=\cdots$

$-i\nabla\psi_{(x)}=\vec{p}\cdot\psi_{(x)}$

$\vec{P}=\int d^3x\int\frac{d^3p\,d^3a}{(2\pi)^6}\frac{e^{i(p-a)x}}{\sqrt{2\omega_p 2\omega_a}}\sum_{r,s}[a_q^{r\dagger}u_q^{r\dagger}+b_{-a}^r v_{(-a)}^{r\dagger}][a_p^s u_p+b_p^{s\dagger}v_{(-p)}^s]\vec{p}$

$$\text{其积分} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{rs} \left(a_p^{rt} u^{rt}(p) + b_{-p}^{r} v^{r}(-p)\right)\left(a_p^{s} u^{s} + b_{-p}^{st} v_{-p}\right)$$

u, v 旋量正交性：

$u^\dagger(p)u(p) = v^\dagger(p)v(p) = 2E_p \delta^{rs}$ 且：

$u^{rt}(p) v^{s}(-p) = v^{rt}(-p) u^{s}(p) = 0$

$u^{rt}(p) v^{s}_{p} \neq 0$

$v^{rt}(p) u^{s}_{p} \neq 0$

$\bar{u}^{r}(p) v^{s}(p) = \bar{v}^{r}(p) u^{s}(p) = 0$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \vec{p} \cdot \left(2E_p \delta^{rs} a_p^{rt} a_p^{s} + 2E_p \delta^{rs} b_p^{r} b_{-p}^{st}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{p} \sum_{s} \left(a_p^{st} a_p^{s} - b_{-p}^{s} b_{-p}^{st}\right) \quad \text{利用} \langle 0|b b^\dagger|0\rangle (2\pi)^3 \delta^3(\vec{p}-\vec{q})$$

↓ 去掉负号 → ∞ constant

$$= \int \frac{d^3p}{(2\pi)^3} \vec{p} \sum_{s} \left[a_p^{st} a_p^{s} + b_p^{st} b_p^{s} - (2\pi)^3 \delta^3(\vec{p}-\vec{q})\right] \quad \text{奇迹发生}$$

Dirac 场论的 propagator

回忆：

两点关联函数 (K-G)

$$D_R(x-y) = \theta(x^0-y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$$

对于 Dirac 场论如何定义费曼传播子：

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{s} u_a^{s}(p) \bar{u}_b^{s}(p) e^{-ip(x-y)}$$

$(\not{p}+m)_{ab}$

$$= (i\partial\!\!\!/_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = (i\partial\!\!\!/_x + m)\langle 0|\phi(x)\phi(y)|0\rangle$$

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = -(i\partial\!\!\!/_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \rightarrow \langle 0|\phi(y)\phi(x)|0\rangle = D(y-x)$$

Dirac 传播子 $S_R^{ab} = \theta(x^0-y^0)\langle 0|[\psi_a(x), \bar{\psi}_b(y)]|0\rangle$

$$= (i\partial\!\!\!/_x + m)_{ab} D_R(x-y)$$

对 K-G $(\Box + m^2) D_R(x-y) = -i\delta^4(x-y)$ 格林 func

是否也满足格林函数：

$$(i\partial\!\!\!/_x - m) S_R(x-y) = (i\partial\!\!\!/_x - m)(i\partial\!\!\!/_x + m) D_R(x-y)$$

$$= (\Box_x + m^2) D_R(x-y) = i\delta^4(x-y) I_{4\times 4}$$ 满足

$\partial\!\!\!/\partial\!\!\!/ = \Box = \partial_\mu \partial^\mu$ 类似于 $\not{p}\not{p} = p^2$

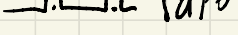
在动量空间 如何得到传播子

$$S_R(x-y)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\tilde{S}_R(p)$$

$$=\int\frac{d^4p}{(2\pi)^4}(\not p+m)e^{-ip\cdot(x-y)}\quad \tilde{S}_R(p)=i\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\ I_{4\times4}\text{左右相等}$$

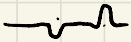
则 $\tilde{S}_F(p)=\frac{i}{\not p+m}=\frac{i(\not p+m)}{p^2-m^2}$

总结：$S_R(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)}{p^2-m^2}e^{-ip\cdot(x-y)}$



积分会避到奇点，

类比 $S_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)}{p^2-m^2}e^{-ip(x-y)}$



$$=\int\frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)}{p^2-m^2+i\varepsilon}e^{-ip(x-y)}=\tilde{S}_F(p)\ (+i\varepsilon\text{很重要})$$

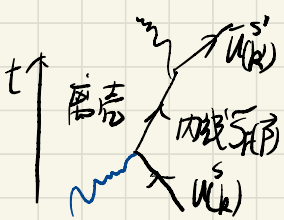
对应于稳定的子 真空变

$$S_F(x-y)=\begin{cases}\langle0|\psi(x)\bar{\psi}(y)|0\rangle & x^0>y^0\\ \langle0|\bar{\psi}(y)\psi(x)|0\rangle & x^0<y^0\end{cases}$$

$$=\langle0|T\psi(x)\bar{\psi}(y)|0\rangle$$

负号为Dirac旋量场特有形式，来自于反对易

举例：康普顿散射：



外线 $u\bar{u}$

内线 $\tilde{S}_F(p)$

重点审视L对称性：时空分离对称性 {空间反射 / 时间反演

$x\to x'^\mu=\Lambda^\mu_\nu x^\nu$ $x^2=x'^\mu$不变（广度时间间隔 $ds^2=g_{\mu\nu}dx^\mu dx^\nu$）

即 $\Lambda g\Lambda^T=\Lambda^T g\Lambda=g$ 求行列式 $|\det\Lambda|^2\det g=\det g\Rightarrow(\det\Lambda)^2=1\ (=\pm1)$

举例 狭义相对论：

$$\Lambda=\begin{pmatrix}\gamma & & & -\gamma\beta\\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma\end{pmatrix}\text{沿x boost}=\begin{pmatrix}\cosh\eta & & & \sinh\eta\\ & 1 & & \\ & & 1 & \\ \sinh\eta & & & \cosh\eta\end{pmatrix}$$

$g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = g_{\rho\sigma}$ 逆变 $\Rightarrow g^{\rho\sigma}\Lambda^\alpha_\rho\Lambda^\beta_\sigma = g^{\alpha\beta}$

$\rho,\sigma = 0,0$ $\quad \alpha,\beta = 0$ $\quad$ 对α,β求和

$(\Lambda^0_0)^2 = 1 + (\Lambda^1_0)(\Lambda^1_0) = 1 + \Lambda^0_i\Lambda^0_i \geq 1$

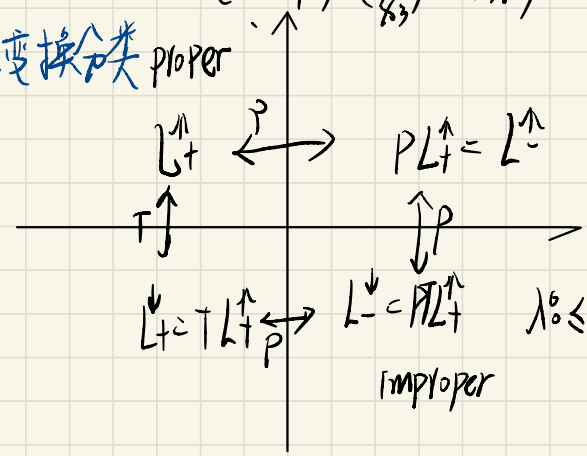
$\det\Lambda = \pm 1$ 且 $\Lambda^0_0 \geq 1$

3从特殊洛伦兹变换 $P^\mu_\nu = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$ $T^\mu_\nu = \begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$ $\det P = -1$; $P^0_0 \geq 1$

则 x': $P^\mu_\nu x^\nu = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}\begin{pmatrix}t\\x\\y\\z\end{pmatrix} = \begin{pmatrix}t\\-x\\-y\\-z\end{pmatrix}$ 空间反演对称 $\to ds^2$不变

x': $T^\mu_\nu x^\nu = \begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}t\\x\\y\\z\end{pmatrix} = \begin{pmatrix}-t\\x\\y\\z\end{pmatrix}$ 时间反演对称 $\det T = -1$; $T^0_0 \leq -1$

对L变换分类 proper



正则时序 $\Lambda^0_0 \geq 1$ $\quad \Lambda^0_0 \leq -1$

$L^\uparrow_+$ + 时空平移 $\to$ 庞加莱对称性 (QRQT的严格对称)

T不守恒 $\to$ CP也不守恒

经典场论中的P和T对称性

两种作用量不变 $\to S$

等价 运动方程不变

考虑L变换

$x \to x' = \Lambda x$ 任意张量场 Φ $\Phi(x) = M(\Lambda)\Phi(\Lambda^{-1}x)$

$x^\mu \to x'^\mu = P^\mu_\nu x^\nu$ 精调 $\Phi(x)$ $\xrightarrow{P}$ $\Phi(x) = P\ \Phi(P^{-1}x)$

考察K-G方程

$(\Box+m^2)\phi(x)=0 \quad x^\mu=(t,\vec{x})$

$\phi'(x')=$ 上 $\phi(x_p)$ 其中 $x_p^\mu=P^\mu_\nu x^\nu=(t,-\vec{x})$

检查 $(\Box+m^2)\phi(x')=0$?

$=\left(\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}+m^2\right)\phi(t,-\vec{x})$

$(\Box'+m^2)\phi(x_p)=0$ 有P对称性 $\quad \Box=\partial_\mu\partial^\mu;\ \partial_\mu=\frac{\partial}{\partial x^\mu}$

考虑 Dirac 方程

$(i\not{\partial}-m)\psi(x)=0$

$\psi(x)\xrightarrow{P}\psi'(x')=P_{4\times4}\psi(x_p)$

$(i\gamma^\mu\partial_\mu-m)\psi'(x')=\left(i\gamma^0\frac{\partial}{\partial t}+i\gamma^i\frac{\partial}{\partial x^i}-m\right)P\psi(t,-\vec{x})$

$=\left(i\gamma^0\frac{\partial}{\partial t}-i\gamma^i\frac{\partial}{\partial(x^i)}-m\right)P\psi(t,-\vec{x})$

若 γ^0 与 P 对易，且 γ^i 与 γ^i 反对易

则 $P\left(i\gamma^0\frac{\partial}{\partial t}+i\gamma^i\frac{\partial}{\partial(x^i)}-m\right)\psi(t,-\vec{x})$

$=P(i\not{\partial}'-m)\psi(t,\vec{x})=0$

求证找到 P：$P=\gamma^0$

时间反演

$i\frac{\partial\psi}{\partial t}=H\psi$ $\qquad \psi(t,\vec{x})=\psi(t,\vec{x})\times \quad \psi'(t,\vec{x})=\psi^*(-t,\vec{x})\surd$

$-i\frac{\partial}{\partial(t)}\psi^*=H\psi^*$ $\qquad$ K-G方程比较平庸

非平庸：Dirac方程

$\psi'(x')=\gamma_{4\times4}\psi^*(-t,\vec{x})$

考虑相互：$(i\not{\partial}-m)\psi(x)=0$

$(i\not{\partial}-m)\psi(x)=\int(i\gamma^0\frac{\partial}{\partial t}+i\vec{\gamma}\cdot\frac{\partial}{\partial\vec{x}}-m)\int\psi^*(t,\vec{x})$

$=\int[i(\gamma^0)^*\frac{\partial}{\partial t}+i(\gamma^i)^*\partial_i-m]\psi^*(x,T)$

$=(i\not{\partial}-m)^*\psi^*(x,T)$

是否找到一个J矩阵

结果：$J=\gamma^0\gamma^3=J^+=J^{-1}$

其中$\not{\partial}=\frac{\partial}{\partial x^\mu}$

$=(i\gamma^\mu\partial_\mu-m)^*$

$=-i(\gamma^\mu)^*\partial_\mu-m$

若$J\gamma^\mu J^{-1}=(\gamma^\mu)^*$

$\gamma^\mu=g^{\mu\nu}\gamma_\nu$

$\gamma_\nu=g_{\mu\nu}\gamma^\nu$

$\gamma^0=\gamma_0$但$\gamma^i=-\gamma_i$

$\Gamma^\mu_{\nu\lambda}x^\nu=(-t,\vec{x})$

↳$(-1,1,1)$

量子的Poincare群及其李代数

L对称性 $x^\mu\to x'^\mu=\Lambda^\mu_\nu x^\nu+a^\mu=(\Lambda,a)$ → 也是L群

再作一次 $x'\to x''^\mu=\bar{\Lambda}^\mu_\rho x'^\rho+\bar{a}^\mu=(\bar{\Lambda}^\mu_\rho\Lambda^\rho_\nu)x^\nu+(\bar{\Lambda}^\mu_\rho a^\rho+\bar{a}^\mu)$ 群 成加群

定义：量子庞加莱群

$\hat{U}(\Lambda,a)$

算符 Λ 平移 a

$U(0,a)|\vec{p}\rangle=e^{i\hat{p}_\mu a^\mu}|\vec{p}\rangle=e^{ip\cdot a}|\vec{p}\rangle$ $\quad\hat{p}^\mu=(H,\vec{p})$

$U(\Lambda,0)|\vec{p}\rangle=|\Lambda\vec{p}\rangle$

$\hat{U}(\Lambda,a)$

记：庞加莱变换

$U(\bar{\Lambda}\bar{a})\cdot U(\Lambda a)=U(\bar{\Lambda}\Lambda,\bar{\Lambda}a+\bar{a})$ 与时空坐标变换同构，满足乘法公式 群的

则$\hat{U}(\Lambda,a)$为庞加莱群的无穷维表示 → 算符，生成元

$U(\Lambda,a)$ 李定理的推广 $=\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-i\epsilon_\rho\hat{P}^\mu+\cdots$

$U(\Lambda,a)\ U(1+\omega,\epsilon)\ U(\Lambda,a)=U(\Lambda(1+\omega)\Lambda^{-1},\Lambda\epsilon-\Lambda\omega\Lambda^{-1}a)$

$\Rightarrow[J^{\mu\nu},J^{\rho\sigma}]=i(g^{\mu\rho}J^{\nu\sigma}-g^{\mu\rho}J^{\nu\sigma}-g^{\nu\sigma}J^{\mu\rho}+g^{\mu\sigma}J^{\nu\rho})$ L群的李代数

$\Lambda^\mu_\nu=\delta^\mu_\nu+\omega^\mu_\nu$

$a^\mu\epsilon$ 小

$[\hat{P}^\mu, \hat{P}^{\rho\sigma}] = i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho)$

$[P^\mu, P^\nu] = 0$

用更熟悉的语言

$\hat{P}^0 = H$; $\vec{P} = (P^1, P^2, P^3)$

转动 $J = \{J^{23}, J^{31}, J^{12}\}$ 角动量算符

$K = \{J^{01}, J^{02}, J^{03}\}$ boost算符

重写为

$[J_i, J_j] = i\epsilon_{ijk}J_k$

$[J_i, K_j] = i\epsilon_{ijk}K_k$

$[K_i, K_j] = -i\epsilon_{ijk}K_k$

$[J_i, P_j] = i\epsilon_{ijk}P_k$

$[K_i, P_j] = -iH\delta_{ij}$

$[K_i, H] = -iP_i$

$[J_i, H] = [P_i, H] = 0$

庞加莱群的李代数

L群有限维表示生成元不厄米：

$(S^{0i})^\dagger = -S^{0i}$

$(\Lambda, a) \xrightarrow{\text{整体}} U(\Lambda, a)$ $P \xrightarrow{\text{空间反演}} U(P)$ $T \xrightarrow{\text{时间}} U(T)$

若一个量子理论为P与T不变，其精确的数学含义是？

$\hat{U}(P,0)\hat{U}(\Lambda,a)\hat{U}(P,0) = U(P\Lambda P^{-1}, Pa)$

$P^\mu_\nu = (-1, 1, 1, 1)$

$\hat{U}(T,0)\hat{U}(\Lambda,a)\hat{U}(T,0) = U(T\Lambda T^{-1}, Ta)$

广义庞加莱变换

满足这两项，该理论才有时间反演和空间反演对称性

令 $\Lambda = I + W$; $a = \epsilon$ $W^{\mu\nu}$, ϵ 为无穷小

$U(\Lambda, a) = I + \frac{i}{2}W_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu + \cdots$

引入简写符号：

$P=U(P_{10})$; $T=\hat{U}(T_{10})$

则 $P\hat{J}^{\mu\nu}P^{-1}=\frac{1}{2}P^{\mu}_{\alpha}P^{\nu}_{\beta}\hat{J}^{\alpha\beta}$ 之求和 J 生成元，P 为四阶矩阵

$P\hat{P}^{\mu}P^{-1}=\frac{1}{2}P^{\mu}_{\nu}\hat{P}^{\nu}$

空间反演算符 四动量算符 4×4矩阵

$T\hat{J}^{\mu\nu}T^{-1}=\frac{1}{2}T^{\mu}_{\alpha}T^{\nu}_{\beta}\hat{J}^{\alpha\beta}$ $T^{\mu}_{\nu}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$

$T\hat{P}^{\mu}T^{-1}=$ 之为 $\hat{P}^{\mu}$ $P^{\mu}_{\nu}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$

如何求量子算符 P,T 的性质：

Wigner's Symmetry 的 fundamental theory：

任何一个对称性对应的量子算符只可属于2种类型，

① 线性并且幺正（连续对称性）

② 反线性且反幺正（分立对称性）

$\begin{cases}\langle\psi_1|\psi_2\rangle=\langle\hat{U}\psi_1|\hat{U}\psi_2\rangle \text{ 幺正线} \\ \hat{U}(\alpha|\psi_1\rangle+\beta|\psi_2\rangle)=\hat{U}\alpha|\psi_1\rangle+\hat{U}\beta|\psi_2\rangle\end{cases}$

$\begin{cases}\langle\hat{U}\psi_1|\hat{U}\psi_2\rangle=\langle\psi_1|\psi_2\rangle^*=\langle\psi_2|\psi_1\rangle \text{ 反幺正} \\ \hat{U}(\alpha|\psi_1\rangle+\beta|\psi_2\rangle)=\alpha^*\hat{U}\psi_1+\gamma^*\hat{U}\psi_2 \text{ 反线性}\end{cases}$

对幺正：$\hat{U}^{+}=\hat{U}^{-1}$

对反~也成立

$\langle\psi_1|\hat{U}^{+}|\psi_2\rangle=\langle U\psi_1|\psi_2\rangle$

对反：

$\langle\psi_1|\hat{U}^{+}|\psi_2\rangle=\langle U\psi_1|\psi_2\rangle^*=\langle\psi_2|\hat{U}\psi_1\rangle$

举例 ①单位算符第①类

②质算 ——①——

问：T,P 属于哪一类？

取 $u=0$，$P^0=H$

$PH\hat{P}^{-1}=\frac{1}{2}H$（以P为反线性反幺正，证不动提到外面

仅证：设P为反线性反幺正

则 $\hat{P}H\hat{P}^{-1}=-H$ 假设存在定态 $H|\psi\rangle=E|\psi\rangle$ 本征态 $E>0$

考察 $\hat{P}^{-1}|\psi\rangle$

$H|P^{-1}|\psi\rangle = -P^{-1}E\psi = -(E)(P^{-1}|\psi\rangle)$

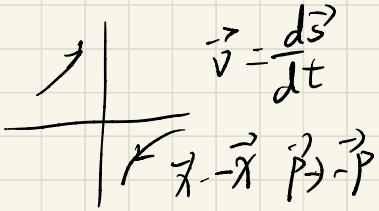
则 $P^{-1}\psi$ 为 H 本征态，但本征值 $E<0$ 矛盾 的结果

物理上不接受，排除 P 为反线性反幺正

$H|P\psi\rangle = -|P^{-1}H\rangle$

P 为线性幺正算符 → 可以放心提之

得到
$$\begin{cases} P\vec{J}P^{-1} = \vec{J} \\ P\vec{P}P^{-1} = -\vec{P} \\ P\vec{K}P^{-1} = -\vec{K} \end{cases}$$



对轨角 $\vec{J} = \vec{x} \times \vec{p}$

$\to (-\vec{x}) \times (-\vec{p})$

自旋推测也变

回头讨论时间反演 取代 $\Rightarrow$

$T: iH T^{-1} = -iH$,

反证：T 为线性幺正：

$H|\psi\rangle = E\psi$

$H(T^{-1}|\psi\rangle) = E T^{-1}\psi$ → T 为一反幺反线性

得到 $T\vec{J}T^{-1} = -\vec{J}$

$T\vec{P}T^{-1} = -\vec{P}$

$T\vec{K}T^{-1} = \vec{K}$



光外 $\to T\vec{x}$ $\vec{J} = \vec{x} \times \vec{p}$

$\vec{v} = \frac{d\vec{s}}{dt} \to -\vec{v}$ $\to -\vec{J}$

$\vec{p} \to -\vec{p}$

从其他角度考虑 T

$[J_i, J_j] = i\varepsilon_{ijk}J_k$ 两边 $T\,T^{-1}$

T $\quad$ $T^{-1}T$ $\quad$ T^{-1}

$\to$ 要求相应为 $= i\varepsilon_{ijk}$

接下来考虑单粒子态在 P 和 J 的性质：$|P\rangle = N|p,\sigma\rangle$

静止粒子：k 为4动量 $k^\mu = (m,0)$

$T|k,\sigma\rangle$ $\xrightarrow{\text{自旋投影不变}}$ $|k,\sigma\rangle$ 本征态，本征态再作用1次 T 仍 $=$

梯减：$J_\pm|k,\sigma\rangle=(J_1\pm iJ_2)|k,\sigma\rangle=\sqrt{(J\mp\sigma)(J\pm\sigma+1)}|k,\sigma\pm1\rangle$　J为自旋

作用P　P平=平P　对易

左侧：$JP|k,\sigma\rangle=\eta_\sigma k_\sigma J_\pm|k,\sigma\rangle$ 本征态
右侧 $\eta_{\sigma\pm1}|k,\sigma\pm1\rangle$
$\Rightarrow \eta_{\sigma\pm1}=\eta_\sigma$

简写：$P|k,\sigma\rangle=\eta|k,\sigma\rangle$　$|\eta|=1$ 复相位　intrinsic Parity 内禀宇称

运动粒子

$P|p,\sigma\rangle=\eta|p,\sigma\rangle$　同一内禀宇称

对于T变换 略

对于L变换：

静止粒子 $k^\mu=(m,\vec0)$

矩阵 $W^\mu{}_\nu k^\nu=k^\mu$　W为转动操作 构成群的子群　(有质：SO(3)转群
　　无：ISO(2)
$SO(3)\in SO(1,3)$ → $(2J+1)\times(2J+1)$　　σ：$-J\cdots J$ (2J+1)个

$U(W)|k,\sigma\rangle=\sum_{\sigma'}D^{(J)}_{\sigma\sigma'}|k\sigma'\rangle$　　$\vec J$ 动量子数不变

可含boost → 静 ↔ 转动群的不可约表示

非静止：$U(\Lambda)|p,\sigma\rangle=\sum_{\sigma'}D^{\vec J}_{\sigma\sigma'}W(\Lambda,p)|\Lambda p,\sigma'\rangle$

W：little group 小群 → 诱导出L变换的矩阵　小群表诱导表示 (induce)

对massless，$U(\Lambda)|\vec p,\lambda\rangle=e^{i\lambda\theta(\Lambda,p)}|\Lambda p,\lambda\rangle$ (ISO(2))　入射粒子的角动量子数(不再是J，因为有车动)

小群在洛伦兹计算 $S=\{R(z)\}_{1N}$

回忆：$|p,\sigma\rangle=\sqrt{2E}\,a^\dagger_{\vec p\sigma}|0\rangle$

$U(\Lambda)a^\dagger_{\vec p\sigma}U^\dagger(\Lambda)=\sum_{\sigma'}D^{\vec J}_{\sigma\sigma'}(W(\Lambda,p))a^{\sigma'\dagger}_{\Lambda\vec p}$　massive

2后　$U(\Lambda)\phi(x)U^\dagger(\Lambda)=?$

经典场　$\phi(x)\xrightarrow{\Lambda}\phi(x')=M(\Lambda)\phi(\Lambda^{-1}x)$

在量子场中：

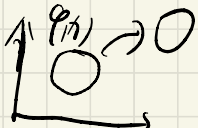
$$U(\Lambda)\hat{\varphi}(x)U^{-1}(\Lambda)=M^{-1}(\Lambda)\hat{\varphi}(\Lambda x)$$

自变量互逆

推导：仿照NRQM

$\langle y|\hat{x}|y\rangle=y$

考察：$\langle\varphi|\hat{\varphi}(x)|\varphi\rangle$ 其期望 $\varphi(x)$



$=\langle\varphi|U(\Lambda)\hat{\varphi}(x)U^{-1}(\Lambda)|\varphi\rangle=\langle\varphi|U^{-1}U(\Lambda)\hat{\varphi}(x)U^{-1}(\Lambda)U(\Lambda)|\varphi\rangle$

$=\langle U(\Lambda)\varphi|M(\Lambda^{-1})\hat{\varphi}(\Lambda x)|U(\Lambda)\varphi\rangle=M^{-1}(\Lambda)\langle\varphi'|\hat{\varphi}(\Lambda x)|\varphi'\rangle$

代入结论，φ'

$=M^{-1}(\Lambda)\varphi'(\Lambda x)$

根据 $\varphi_{\Lambda}=M(\Lambda)\varphi(\Lambda^{-1}x)\Rightarrow\varphi(\Lambda x)=M(\Lambda)\varphi(x)$ 自洽性检验

以上为普适讨论，下面回到 Dirac场

classical：$\psi(x)\xrightarrow{\Lambda}\psi'(x)=\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)$

QM：$U(\Lambda)\hat{\psi}(x)U^{-1}(\Lambda)=\Lambda_{\frac{1}{2}}\hat{\psi}(\Lambda x)$

从算符表示与矩阵表示带来分析后果

$$\hat{\psi}(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(a_p^{\sigma}u_p^{\sigma}e^{-ip\cdot x}+b_p^{\sigma\dagger}v_p^{\sigma}e^{ip\cdot x}\right)$$

考虑其变 $U(\Lambda)\hat{\psi}(x)U^{-1}(\Lambda)=\Lambda_{\frac{1}{2}}\hat{\psi}(\Lambda x)$

$U(\Lambda)$ 为线，Λ 因为 $U(\Lambda)$ 连续对称 $\rightarrow$ 直接把 $U(\Lambda)$ 与 $U(\Lambda)$ 夹在 a_p^{σ} 与 $a_p^{\sigma\dagger}$ 那俩

对 u,v 有约束：从动量可以唯一确定

考察量子场算符的空间反演下的变换行为

从简单：K-G $P=\hat{U}(P,0)$ $P\hat{\varphi}(x)P^{-1}=\pm\hat{\varphi}(P^{\mu}_{\nu}x^{\nu})$

取决于模型需要 $x^P=(t,-\vec{x})$

考虑Dirac theory:

回忆: $P|\vec{p},\sigma\rangle = |\vec{p},\sigma\rangle$

$\Rightarrow P\, a^{\sigma}_{\vec{p}}\, P^{-1} = \eta_a^{*}\, a^{\sigma}_{-p}$ $\quad |\eta_a|=1$

反粒子湮灭算符 $P\, b^{\sigma}_{\vec{p}}\, P^{-1} = \eta_b^{*}\, a^{\sigma}_{-p}$ $\quad |\eta_b|=1$

(两个 η^{*} 不重要)

对Dirac场

$P\,\psi(x)\,P^{-1} \overset{?}{=} \eta\hat{\psi}(x_p)$ 先猜解 $\hat{\psi}(x_p)$

$$P\psi(x)P^{-1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(u^{\sigma}(\vec{p})\, P a^{\sigma}_{\vec{p}} P\, e^{-ip\cdot x} + v^{\sigma}(\vec{p})\, P b^{\sigma\dagger}_{\vec{p}} P\, e^{ip\cdot x}\right)$$

$$= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(u^{\sigma}(\vec{p})\,\eta_a a^{\sigma}_{\tilde{p}} e^{-i\tilde{p}\cdot x_p} + v^{\sigma}(\vec{p})\,\eta_b^{*} b^{\sigma\dagger}_{\tilde{p}} e^{+i\tilde{p}\cdot x_p}\right)$$

除了 u 和 v, 其他都跟 $\vec{p}$ 有关的函数

$$u_p = \begin{pmatrix}\sqrt{P\cdot\sigma}\,\xi \\ \sqrt{P\cdot\bar{\sigma}}\,\xi\end{pmatrix} \quad \sigma^{\mu}=(1,\vec{\sigma}) \quad \bar{\sigma}^{\mu}=(1,-\vec{\sigma}) \quad \gamma^{\mu}=\begin{pmatrix}0 & \sigma^{\mu}\\ \bar{\sigma}^{\mu} & 0\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{\tilde{p}\cdot\sigma}\,\xi \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi\end{pmatrix} = \gamma^0\begin{pmatrix}\sqrt{\tilde{p}\cdot\sigma}\,\xi \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi\end{pmatrix} = \gamma^0 u(\tilde{p})$$ 代回去

同理 $v(p) = \begin{pmatrix}\sqrt{P\cdot\sigma}\,\eta \\ -\sqrt{P\cdot\bar{\sigma}}\,\eta\end{pmatrix} = \begin{pmatrix}\sqrt{\tilde{p}\cdot\sigma}\,\eta \\ -\sqrt{\tilde{p}\cdot\bar{\sigma}}\,\eta\end{pmatrix} = \gamma^0 v(\tilde{p})$

$$= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(\gamma^0 u(\tilde{p})\,\eta_a a^{\sigma}_{\tilde{p}} e^{-i\tilde{p}\cdot x_p} + \gamma^0 v(\tilde{p})\,\eta_b^{*} b^{\sigma\dagger}_{\tilde{p}} e^{i\tilde{p}\cdot x_p}\right)$$

$= \gamma^0 \cdots$

假设 $\eta_b^{*} = \eta_a$, 回到场算符形式

$= \gamma^0 \eta_a \hat{\psi}(x_p)$ η_a 为内禀宇称

$\eta_a\eta_b = |\eta_a|^2 = 1$, 反粒子和反费米子必须有相反的内禀宇称

经典:

$\psi(x) \xrightarrow{P} \psi'(x) = \gamma^0\psi(P^{-1}x)$

$P^{\mu}_{\ \nu} = \mathrm{diag}(1,-1,-1,-1) = P^{-1}$

相应因子

$P^{\mu} = (E_p, \vec{p})$

$x^{\mu} = (t, \vec{x})$

$\tilde{p} = (E_p, -\vec{p})$

$x^{\mu}_p = (t_1, -\vec{x})$

$p\cdot x = \tilde{p}\cdot\tilde{x}_p$

$\gamma^0 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ Weyl

给出来 $\eta_a\eta_b^{*}$

物理后果：

束缚态：$|e^-,e^+\rangle$ 正负电子偶素

二类

$q\bar{q}$ 介子(介) $|q\bar{q}\rangle$ 基态的正负一对介子的内禀宇称为 -1

奇态仅考虑 $\frac{1}{\sqrt{4\pi}}$ $L=0$ 即 S 波(角分布各向同性) $1=P^\dagger P$

构造 $P|J,M\rangle = \int \frac{d^3p}{(2\pi)^3}\phi(|\vec{p}|)\,Y_{00}(\hat{p})\sum_{\sigma_1\sigma_2}\langle \tfrac{1}{2}\sigma_1\tfrac{1}{2}\sigma_2|J,M\rangle\, a^{\sigma_1\dagger}_{p}\, b^{\sigma_2\dagger}_{-p}|0\rangle$ $P^\dagger$

S波

$P|J,M\rangle$，观察 $|J,M\rangle$ 是否为 P 的本征态，若是则本征值为内禀宇称

$$= \int \eta_a^{*}\, a^{\sigma_1\dagger}_{-p}\,\eta_b^{*}\, b^{\sigma_2\dagger}_{p}|0\rangle$$

$$= -\int \frac{d^3p}{(2\pi)^3}\phi(|\vec{p}|)\sum(\,)\, a^{s\dagger}_{-p}\, b^{s\dagger}_{p}|0\rangle$$

令 $p\to -p$ ϕ 是空间波函数

$$= \int \frac{d^3p}{(2\pi)^3}\phi(|\vec{p}|)\sum(\,)\, a^{\sigma_1\dagger}_{p}\, a^{\sigma_2\dagger}_{p}|0\rangle$$

右侧 $= -|J,M\rangle$ 本征值为负1

$^1S_0 \to \gamma\gamma$

$^3S_1 \to \gamma\gamma\gamma$

$P\psi(x)P^{-1} = \eta_a\gamma^0\psi(\mathcal{P}x)$ 要求 $P\bar{\psi}P\psi P^{-1}=?$ 双线性协算符

$P\bar{\psi}_{(x)}P^{-1} = P\psi^\dagger_{(x)}P^{-1}\gamma^0 = (P^{*}\psi^\dagger_{(x)}P^\dagger)\gamma^0$ 令如何变化

$= (P\psi P^\dagger)^\dagger\gamma^0 = (\eta_a\gamma^0\psi_{(\mathcal{P}x)})^\dagger\gamma^0$

$= \eta_a^{*}\,\psi^\dagger(\gamma^0)^\dagger\gamma^0$ $(\gamma^0)^\dagger=\gamma^0$；$\vec{\gamma}^\dagger=-\vec{\gamma}$

$= \eta_a^{*}\,\bar{\psi}_{(\mathcal{P}x)}\gamma^0$

$\bar{\psi}=\psi^\dagger\gamma^0$

γ^0 为线性算符

$P^{-1}=P^\dagger$

$A^\dagger B^\dagger=(BA)^\dagger$

还有类双线型

$P\bar{\psi}\psi P^{-1}$ $\Sigma=P^{-1}\cdot P$

$= |\det\Lambda|^2 \bar{\psi}\gamma_0\gamma^0\psi(\Lambda p) = \bar{\psi}\psi(\Lambda p)$ → 空间反演下 $\bar{\psi}\psi$ 标量（只有自变量变化）

$\{\gamma^5, \gamma^\mu\} = 0$

对 $\bar{\psi}\gamma^5\psi$

$P\bar{\psi}(x)(P^{-1}P)\gamma^5\psi P^{-1} = |\det\Lambda|^2\bar{\psi}\gamma^0\gamma^5\gamma^0\psi(\Lambda p)$

$= \bar{\psi}\gamma^5(-\gamma^0)\gamma^0\psi(\Lambda p) = -\bar{\psi}\gamma^5\psi(\Lambda p)$ 赝标量

到现在，我们可以解决一个问题

① Dirac 方程 ② $\psi(x) \xrightarrow{P} \psi(\Lambda p)$ P 不变

若加上一项：$\bar{\psi}\gamma^5\psi$ P 变换下有负号，L 变换下为标量 → 自由 Dirac 场的守恒

另一个例子：

矢量流 $P\bar{\psi}\gamma^\mu\psi(x)P^{-1}$

$= \bar{\psi}\gamma^\mu\gamma^0\psi(\Lambda p) = \bar{\psi}(\gamma^\mu)^\dagger\psi(\Lambda p) = \bar{\psi}\gamma_\mu\psi(\Lambda p)$

$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$

$= \begin{cases} +\bar{\psi}\gamma^\mu\psi & \text{当 } \mu=0 \\ -\bar{\psi}\gamma^\mu\psi & \mu=1,2,3 \ (\gamma^i = -\gamma_i) \end{cases}$

类似于动量，称这类矢量为 Polar Vector

赝矢流：

$P\bar{\psi}\gamma^\mu\gamma_5\psi P^{-1} = -\bar{\psi}\gamma_\mu\gamma_5\psi(\Lambda p) = \begin{cases} -\bar{\psi}\gamma^\mu\gamma_5\psi \\ + \text{——} \end{cases}$ $\mu=0$ pseudo-vector；$\mu=1,2,3$

K-G 场内禀宇称可负可正（湮灭）

$L_{\text{Klein-Gordon}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{2}i\bar{\psi}\gamma^5\psi$（空间反演破）

相互作用场合为标量场 ϕ

求 ϕ 与 ψ 对应粒子的内禀宇称：

$P\bar{\psi}\psi P^{-1} = +\bar{\psi}\psi(\Lambda p)$

$= \bar{\psi}\psi(\Lambda p)$ 同 $P\phi(x)P^{-1} = \phi(\Lambda p)$ 宇称为正 耦合标量流

$P\bar{\psi}_{(x)}P^{-1} = -\bar{\psi}_{(x)}$ 宇称为负 —— 质标流

左手Weyl场在空间反演怎么变

$\psi_L = P_L\psi$ $\quad P_L = \frac{1-\gamma_5}{2}$

$\psi_R = P_R\psi$ $\quad P_R = \frac{1+\gamma_5}{2}$

$P\psi_L(x)P^{-1} = \frac{1-\gamma_5}{2}P\psi(x)P^{-1} = \frac{1-\gamma_5}{2}\eta_a\gamma^0\psi(x) = \eta_a\gamma^0 P_R\psi$

$\parallel$

$P_L\psi = \eta_a\gamma^0\psi_R$ 左手 手征变化 右手

0质量粒子

massless $\Leftarrow$ 左手Weyl旋量 空间反演 $\Rightarrow$ 右手Weyl旋量

$\lambda = -\frac{1}{2}$螺旋度

$\Leftarrow$ P $\Rightarrow$ 右手Weyl

$\lambda = \frac{1}{2}$

helicity与自旋重合

$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \psi = \psi_L + \psi_R$

$= \bar{\psi}_L i\partial\!\!\!/\psi_L + \bar{\psi}_R i\partial\!\!\!/\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$

左手Weyl旋量 非手征理论

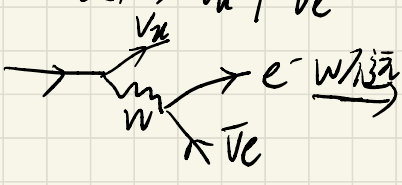
弱相互作用(破坏空间反射对称性)

2-θ puzzle; 李-杨 1956 → 宇称不守恒

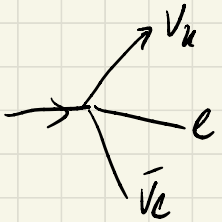
吴健雄住 Co_{60} β衰变,电子角分布(监视)

1957 李-杨第一次Nobel

解释此子(2代轻子) → $\nu_\mu + \bar{\nu}_e$



W远



$\mathcal{L}_{weak} = -\frac{4G_e}{\sqrt{2}}[\bar{\nu}_k\gamma^\mu\nu_L u][\bar{e}\gamma^\mu P_L \nu_e]$ 4费米子相互作用

$= -[\bar{\nu}_{uL}\gamma_\mu u_L][\bar{e}_L\gamma^\mu\nu_{eL}]$ 考虑左手Weyl场

Weak interaction为一个手征理论

改写 $\bar{e}\gamma^\mu P_L\nu_e = \bar{e}\gamma^\mu\frac{1-\gamma^5}{2}\nu_e = \underbrace{\bar{e}\gamma^\mu\nu_e}_{矢量流} - \underbrace{\bar{e}\gamma^\mu\gamma^5\nu_e}_{轴矢流\ pseou\ vector}$

V-A型耦合

第一次提出V-A型：

Gell-Mann-Feynman

Π变换：（反线性，反么正）

单粒子态的时间反演：

回忆：

$$\begin{cases} \Pi\vec{p}\Pi^{-1} = -\vec{p} & 有质 \\ H \quad\quad H \\ \vec{J} \quad = -\vec{J} \end{cases}$$

有质量的静止粒子：

$k^\mu = (m, \vec{0})$，为了H，$\vec{p}$的本征态

$\Pi(k|0\rangle$是否依然为了，$\vec{p}$ H的本征态？

$H\Pi|k,\sigma\rangle = m(\Pi|k,\sigma\rangle$ ；$\vec{p}(\Pi|k,\sigma\rangle) = 0$；

$J_3(\Pi|k,\sigma\rangle) = \sigma\Pi|k,\sigma\rangle$ 答案是yes；

因此 $\Pi|k,\sigma\rangle = e^{i\vec{J}\cdot\vec{R}}|k,-\sigma\rangle$ 转动算子 $\sigma = -J \cdots J$

考虑 $J_\pm|k,\sigma\rangle$ $(J_1 \pm iJ_2)|k,\sigma\rangle = \sqrt{(J\mp\sigma)(J\pm\sigma+1)}|k,\sigma\pm1\rangle$

$\Pi J_\pm|k,\sigma\rangle = \sqrt{\quad}\ \Pi|k,\sigma\pm1\rangle$

$= N^{-} \sum |k_1 - \sigma_1 \rangle$

(一维注记) $\Pi |k_1 \sigma_1\rangle = \xi_\sigma |k_1 - \sigma\rangle$

$\Rightarrow \xi_{\sigma+1} = -\xi_\sigma$ ξ依赖于σ

$\Pi |k_1 \sigma\rangle = \xi (-1)^{j-\sigma} |k_1 - \sigma\rangle$

↳ 内禀宇称 无物理意义

运动学子的Π变换

$\Pi |\vec{p}, \sigma\rangle = \sum (-1)^{j-\sigma} |-\vec{p}, -\sigma\rangle$

$\Pi a^{\sigma\dagger}_{\vec{p}} \Pi^{-1} = \xi (-1)^{j-\sigma} a^{\sigma\dagger}_{(-\vec{p})}$

$\Pi\Pi |\vec{p}, \sigma\rangle = \Pi \xi (-1)^{j-\sigma} |-\vec{p}, -\sigma\rangle = \xi^* (-1)^{j-\sigma} \Pi |-\vec{p}, -\sigma\rangle$

$= |\xi|^2 (-1)^{j+\sigma} |\vec{p}, \sigma\rangle = (-1)^{2j} |\vec{p}, \sigma\rangle$ （$(-1)^{j+\sigma}$）

对0质量子

$\Pi\Pi |\vec{p}, \lambda\rangle = (-1)^{2|\lambda|} |\vec{p}, \lambda\rangle$

j若为半整(Fermi) $\Pi^2 = -1$ (奇由奇数个费米子组成)

考虑 $H|\alpha\rangle = E|\alpha\rangle$ $\Pi|\alpha\rangle$ 和 $|\alpha\rangle$ 是同一态吗?

若是同态: $\Pi|\alpha\rangle = \xi|\alpha\rangle$

$\Pi^2|\alpha\rangle = \xi\Pi|\alpha\rangle = |\xi|^2|\alpha\rangle$ $|\xi| = 1$

和$\Pi^2 = -1$是矛盾的(对费米子)

则对费米子来说 $\Pi|\alpha\rangle$ 与 $|\alpha\rangle$ 为简并但非同一态 → Kramers 简并

至少二重简并

$|k_1, \frac{1}{2}\rangle$

$|k_1, -\frac{1}{2}\rangle$

更有趣的是费米子处于外场

加静电场(右旋转动不变，但宇称不变时间反演不变)

$H = \mu_B \vec{S}\cdot\vec{B} + \mu_D \vec{S}\cdot\vec{E}$

微扰使两能级劈裂 → electric dipole 电偶极矩

$\Delta E = \mu_B B$ $\Delta E = \mu_D E$ → 破坏 Kramer 定理

说明：时间反演不变禁戒了费米子有非0的电偶极矩

目前有人测中子电偶极矩，电子~ → 找到时间反演的证据.

量子场算符的时间反演对称

回忆 $T|p,\sigma\rangle = \xi(-1)^{\frac{1}{2}-\sigma}|-\vec{p},-\sigma\rangle$

$T\hat{\psi}(x)T^{-1} = \psi(x_T)$ or $\psi^*(x_T)$ $x_T = (-t,\vec{x})$

$= T\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma=\pm\frac{1}{2}}\left(a_p^\sigma u_p^\sigma e^{-ip\cdot x} + b_p^\sigma v_p^\sigma e^{+ip\cdot x}\right)T^{-1}$

$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum\left(\xi_a^+(-1)^{\frac{1}{2}-\sigma}a_{-p}^{-\sigma}u_p^{\sigma *}e^{-ip\cdot x_T} + \xi_b(-1)^{\frac{1}{2}-\sigma}(b_{-p}^{-\sigma})^+ v^* e^{+ip\cdot x_T}\right)$ → 应该取 $\psi(x_T)$

$T a_p^{s+}T^{-1} = \xi_a(-1)^{\frac{1}{2}-\sigma}(a_{-p}^{-\sigma})^+$

$T b_p^{s+}T^{-1} = \xi_b(-1)^{\frac{1}{2}-\sigma}(b_{-p}^{-\sigma})^+$

$|\xi_a| = |\xi_b| = 1$

定义 $\tilde{p} = (\vec{p},-p)$

$x_T = (-t,x)$

$p\cdot x = -\tilde{p}\cdot x_T$

下面找 u^* v^* 对应关系：

回忆 $u_S^p = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$ $v_{(p)}^s = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta_s\\ -\sqrt{p\cdot\bar{\sigma}}\,\eta_s\end{pmatrix}$

$\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ $\eta_s = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

重新定义：$\eta^s = i\sigma^2\xi^s = \begin{pmatrix}-i\\ \end{pmatrix}\xi^s \Rightarrow \eta_1 = \begin{pmatrix}0\\-1\end{pmatrix}$ $\eta_2 = \begin{pmatrix}1\\0\end{pmatrix}$

因为 考虑 $\vec{p}\parallel\vec{z}$ $p^\mu = (E,0,0,p_z)$, $E^2 - p_z^2 = m^2$

$u_{(p_z)}^{\sigma=\frac{1}{2}} = \begin{pmatrix}\sqrt{E-p_z}\\0\\ \sqrt{E+p_z}\\0\end{pmatrix}$ $u_{(p_z)}^{\sigma=-\frac{1}{2}} = \begin{pmatrix}0\\ \sqrt{E+p_z}\\0\\ \sqrt{E-p_z}\end{pmatrix}$,

$$V^{\sigma=\frac{1}{2}}_{(p_z)}=\begin{pmatrix}0\\-\sqrt{E+p_z}\\0\\\sqrt{E-p_z}\end{pmatrix}\qquad U^{\sigma=\frac{1}{2}}_{(p_z)}=\begin{pmatrix}\sqrt{E-p_z}\\0\\-\sqrt{E+p_z}\\0\end{pmatrix}$$

回到

$$[u^{\sigma}_{(p)}]^{*}=(-1)^{\frac{1}{2}-\sigma}\,J\,u^{-\sigma}(\tilde{p})$$

$$[v^{\sigma}_{p}]^{*}=(-1)^{\frac{1}{2}-\sigma}\,J\,v^{-\sigma}_{(\tilde{p})}$$

$$J=\gamma^1\gamma^3=\begin{pmatrix}i\sigma_2 & \\ & i\sigma_2\end{pmatrix}_{\text{Weyl}}=\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1\end{pmatrix}$$

验证 $\left(u^{-\frac{1}{2}}_{(p_z)}\right)^{*}=-J\,u^{\frac{1}{2}}(-p_z)=-\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}\sqrt{E+p_z}\\0\\\sqrt{E-p_z}\\0\end{pmatrix}=\begin{pmatrix}0\\\sqrt{E+p_z}\\0\\-\sqrt{E-p_z}\end{pmatrix}$

再验证 $\left(v^{-\frac{1}{2}}_{p_z}\right)^{*}=\begin{pmatrix}\sqrt{E-p_z}\\0\\-\sqrt{E+p_z}\\0\end{pmatrix}=-\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}\sqrt{E-p_z}\\0\\\sqrt{E+p_z}\\0\end{pmatrix}=\begin{pmatrix}\sqrt{E-p_z}\\0\\-\sqrt{E+p_z}\\0\end{pmatrix}$ 也回到

继续量子算符反演的推导：

$$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2Ep}}\sum_{\sigma=\pm\frac{1}{2}}\left[\xi^{*}_{a}a^{\sigma}_{\tilde{p}}(-)^{\frac{1}{2}-\sigma}(-1)^{\frac{1}{2}-\sigma}J\,u^{-\sigma}(\tilde{p})\,e^{-i\tilde{p}\cdot x_T}\right.$$
$$\left.+\xi_b b^{\sigma+}_{\tilde{p}}(-1)^{\frac{1}{2}-\sigma}(-1)^{\frac{1}{2}-\sigma}J\,v^{-\sigma}(\tilde{p})\,e^{i\tilde{p}\cdot x_T}\right]$$

$$=\int\frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2Ep}}\sum_{\sigma=\pm\frac{1}{2}}\left[\xi^{*}_{a}a^{-\sigma}_{\tilde{p}}e^{-i\tilde{p}x_T}+\xi_b(b^{-\sigma}_{\tilde{p}})^{+}v^{\sigma}_{(\tilde{p})}e^{i\tilde{p}\cdot x_T}\right]$$

假设 $\xi^{*}_{a}=\xi_b\Leftrightarrow\xi_a\xi_b=1$

$=\xi^{*}_{a}J\,\bar{\psi}(x_T)$

若干矩阵性质

① 幺正矩阵 $J^{+}=J^{-1}=J$

② $J\gamma^{\mu*}J^{-1}=\gamma_\mu$

证明：$J(\gamma^i)^{*}J^{-1}=\begin{pmatrix}i\sigma_2 & \\ & i\sigma_2\end{pmatrix}\begin{pmatrix}-\sigma_i^{*} & \\ & \sigma_i^{*}\end{pmatrix}\begin{pmatrix}-i\sigma_2 & \\ & -i\sigma_2\end{pmatrix}$

$=\begin{pmatrix}\sigma_2\sigma_i^{*}\sigma_2 & \\ & -\sigma_2\sigma_i^{*}\sigma_2\end{pmatrix}=\begin{pmatrix}-\sigma_i & \\ & \sigma_i\end{pmatrix}=-\gamma^i$

$\sigma_2\vec{\sigma}\sigma_2=-\vec{\sigma}$ 对 σ_2 作共轭变换

$= \gamma_2$

考虑 $\bar{\psi}\psi$ 在 Π 如何变

先求 ψ 在 Π 变换下：

$$\Pi\psi(x)\Pi^{-1} = |\eta_a|^2 \Pi^+(\gamma^0)^* = (\Pi\psi(x)\Pi^+)^+(\gamma^0)^*$$

$$= \int_a^* \int \psi(x_\Pi^+)(\gamma^0)^* = \int_a \psi(x_\Pi) \underbrace{J^+(\gamma^0)^* J}_{=\gamma^0} J^+$$

$$= \int_a \psi(x_\Pi) J^{-1} = \int_a \bar{\psi}(x_\Pi) J^+$$

双线性：

$$\Pi\bar{\psi}\psi(x)\Pi^{-1} \quad I = \Pi^{-1}\Pi \text{ 分别对待 } \psi \text{ 进行变换}$$

$$= |\eta_a|^2 \bar{\psi}(x_\Pi) J^{-1} J \psi(x_\Pi) = \bar{\psi}(x_\Pi)\psi(x_\Pi) \text{ 出现在 } J \text{ 中}$$

$\Pi \hat{J} \Pi^{-1} = \hat{J}(x_\Pi)$ $J(x)$ 为 Π 不变的

其他重要 bilinear：

矢量流：$\Pi\bar{\psi}\gamma_5\psi\Pi^{-1}$ $I = \Pi^{-1}\Pi$

$$= -\eta_a \Pi\bar{\psi}\Pi^{-1} \gamma_5^* \Pi\psi\Pi^{-1} = |\eta_a|^2(-1)\bar{\psi}(x_\Pi) \underbrace{J^+\gamma_5^* J}_{\text{结构无关}} \psi(x_\Pi)$$

$= -\bar{\psi}\gamma_5\psi(x_\Pi)$ 有额外负号，不能出现在 Π 不变 $\mathcal{L}^5$ 的 J 中

$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$

仅需插入，利用 $J^+(\gamma^\mu)^* J = \gamma^\mu$

矢量流 $\Pi\bar{\psi}\gamma^\mu\psi\Pi = \bar{\psi}(x_\Pi) J^+(\gamma^\mu)^* J \psi(x_\Pi)$

$$= +\bar{\psi}\gamma^\mu\psi(x_\Pi) = \begin{cases} \bar{\psi}\gamma^\mu\psi & \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi(x_\Pi) & \mu = 1,2,3 \end{cases}$$

在量子电动力学中

相互作用项 $\mathcal{L}_{int} = e\bar{\psi}\gamma^\mu\psi A_\mu$ ⇒ 对于量子化电磁场

要求 Π 不变 $\quad \Pi A^\mu(x)\Pi^{-1} = A_\mu(\bar{x})$

宇称失流 $\quad \Pi\bar{\psi}\gamma^\mu\gamma_5\psi\Pi^{-1} = +\bar{\psi}\gamma_\mu\gamma_5\psi(\bar{x})$

现在考虑 左手流 (强制 ψ 与 ψ 只有左手，在弱作用中很重要)

$\Pi\ \bar{\psi}\gamma^\mu P_L\psi\Pi^{-1} = \bar{\psi}\gamma^\mu P_L\psi$ 依然是左手流

$= \frac{1-\gamma_5}{2}$

电荷共轭变换 (charge conjugation transformation)

含义：粒子 ⇔ 反粒子 (只有 RQM 才有反粒子，因为高能

Dirac方程：e^- 在经典外电磁场中运动

$$(i\partial\!\!\!/ + eA\!\!\!/^{ext} - m)\psi(x) = 0 \qquad A^{ext} \text{ is fixed.}$$

$$(4\times4)(\text{列}) = 0 \xrightarrow{\text{转置}} (\ )(\ ) = 0$$

等价 ⇒ $\bar{\psi}(i\overleftarrow{\partial\!\!\!/}\gamma^\mu + eA\!\!\!/ + m) = 0$

转置 $(i(\gamma^\mu)^T\partial_\mu - e(\gamma^\mu)^T A_\mu^{ext} + m)\bar{\psi} = 0$

假设存在 C

有 $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$ $\qquad (C = i\gamma^2\gamma^0)$

令 $\psi^C(x) = C\bar{\psi}^T(x)$，方程变为

$(i\partial\!\!\!/ + eA\!\!\!/^{ext} + m)\psi^C(x) = 0 = (i\partial\!\!\!/ - eA\!\!\!/^{ext} - m)\psi^C(x) = 0$

$\psi^C(x)$ 依然满足 Dirac 方程 但符号不同 → 反粒子波函数

以上为 Dirac 理论 (相对论量子力学) 下面从量子场论角度理解

$\hat{C}|\text{粒子}\vec{p},\sigma\rangle = f_a^{*}|\text{反粒子}\vec{p},\sigma\rangle$

$\hat{C}|\text{反粒子}\vec{p},\sigma\rangle = f_b^{*}|\text{粒子},\vec{p},\sigma\rangle$ $\quad |f_a|=|f_b|=1$

则 $\hat{C}a_{\vec{p}}^{\sigma}\hat{C}^{-1} = f_a^{*}b_{\vec{p}}^{\sigma}$

$\hat{C}b_{\vec{p}}^{\sigma}\hat{C}^{-1} = f_b^{*}a_{\vec{p}}^{\sigma}$

考察场算符在C的变换

假设线性幺正

$$\hat{C}\psi(x)\hat{C}^{-1} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(\hat{C}a_p^{\sigma}\hat{C}^{-1}u_p^{\sigma}e^{-ip\cdot x} + \hat{C}b_p^{\sigma\dagger}\hat{C}^{-1}v_p^{\sigma}e^{ip\cdot x}\right)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(f_a^{*}b_p^{\sigma}u\,e^{-ip\cdot x} + f_b^{*}a_p^{\sigma\dagger}v(p)^{\sigma}e^{ip\cdot x}\right)$$

改写 u,v，验证①

$$u_{(p_z)}^{\uparrow} = \begin{pmatrix}\sqrt{E-p_z}\\0\\\sqrt{E+p_z}\\0\end{pmatrix}\qquad \gamma_2 = \begin{pmatrix}&&&-i\sigma_2\\&&i\sigma_2&\\&-i\sigma_2&&\\i\sigma_2&&&\end{pmatrix}_{\text{Weyl}} = \begin{pmatrix}&&&-1\\&&1&\\&-1&&\\1&&&\end{pmatrix}_{\text{Weyl}}$$

正反粒子间的关系

$u^{\sigma}(p) = i\gamma^2 v^{\sigma}(p)$ ①

$v^{\sigma}(p) = i\gamma^2 u^{\sigma}(p)^{*}$ ②

代入得到验证①，同理可验证②，回到④

$$= \int\cdots\, i\gamma^2\left(f_a^{*}b_p^{\sigma}[v_p]^{*}e^{-ip\cdot x} + f_b a_p^{\sigma\dagger}u(p)e^{ip\cdot x}\right)$$

之前Dirac $\psi^c(x) = C\bar{\psi}^T(x)$，在物理水平是否满足（本质上为猜测结果）

$$C\bar{\psi}^T(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\sigma}\left(a_p^{\sigma\dagger}C\bar{u}_p^T e^{ip\cdot x} + b_p^{\sigma\dagger}C\bar{v}_p^{\sigma T}e^{-ip\cdot x}\right)$$

$\bar{u}^T = (u^{\dagger}\gamma^0)^T = \gamma^{0T}u^{*}$；$C\bar{u}^T = C\gamma^{0T}u^{*} = C\gamma^T C^{-1}Cu^{*}$

$= -\gamma^0 Cu^{*}$

$C(\gamma^{\mu})^T C^{-1} = \gamma^{\mu}$

$-\gamma^0 C = -i\gamma^0\gamma^2\gamma^0 = +i\gamma^2$

$$(-\gamma^0)\sum\left(a_p^{\sigma\dagger}(u_p^{\sigma})^{*} + b_p^{\sigma\dagger}(v_p^{\sigma})^{*}e^{-ip\cdot x}\right)$$

将以上2个讨论形式上进行对照

令 $f_a^{*} = f_b$，则

$\hat{C}\psi(x)\hat{C}^{-1} = \int a^3 r^2 \int \frac{}{} - \sum \left(b_p^s v^* e^{-ipx} + a_p^{+} u^* e^{-ipx} \right)$

二)两种形式上阶段的相同

则 $\hat{C}\hat{\psi}(x)\hat{C}^{-1} = \int a^3 \ell \psi^T(x) = -\int a^3 \gamma^0 \ell \psi^*$ $\psi^* = (\psi^+)^T$

Furry定理和CP定理

$\ell = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma_2 & \\ & -i\sigma_2 \end{pmatrix}_{weyl} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}_{weyl}$

有 $\ell^+ = \ell^{-1} = -\ell$ 且 $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$

当 $\mu = 1,2,3$ 时

验证 $C(\gamma^i)^T C^{-1} = \ell(-\gamma^i)^* \ell^{-1} = \begin{pmatrix} & -\sigma_2\sigma_i^*\sigma_2 \\ -\sigma_2\sigma_i^*\sigma_2 & \end{pmatrix}$

$= \begin{pmatrix} & -\sigma_i \\ \sigma_i & \end{pmatrix}_{weyl} = -\gamma^i$ 得证

$(\gamma^i)^+ = -\gamma^i$ 空间部分反厄米

$(\gamma^i)^T = -(\gamma^i)^*$

$\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$

一种特殊费米子：

Majorana 费~，自己为自己反粒子，电中性

Real k-G ⇔ Complex k-G 类似于 Majorana theory ⇔ Dirac theory

$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $i\sigma_2\psi_L^*$ 构造 $\psi_M = \begin{pmatrix} \psi_L \\ i\sigma_2\psi_L^* \end{pmatrix}$ 比 ψ_D 自由度少一半

类似于实 k-G 比 Com k-G 少一半

由于其反粒子是自身

$C\hat{\psi}_M C^{-1} = \xi\hat{\psi}_M$ $\xi = \pm 1$ C宇称

验证：套用上一节 $C\hat{\psi}C^{-1}$ 公式

$= -\int i\gamma^2\psi_M^* = \int \begin{pmatrix} & -i\sigma_2 \\ i\sigma_2 & \end{pmatrix} \begin{pmatrix} \psi_L^* \\ -i\sigma_2^*\psi_L \end{pmatrix} = \int \begin{pmatrix} -\sigma_2\sigma_2^*\psi_L \\ i\sigma_2\psi_L^* \end{pmatrix} = \int \begin{pmatrix} \psi_L \\ i\sigma_2\psi_L^* \end{pmatrix}$ （$\sigma_2\sigma_2$）

$= \int \psi_M$

得证

考虑bilinear行为，首先要求

$C\bar{\psi}C^{-1} = C\psi^{\dagger}C^{-1}\gamma^0 = (C^{\dagger})^{\dagger}\psi^{\dagger}C^{\dagger}\gamma^0 = (\psi C^{-1})^{\dagger}\gamma^0$ 利用之前公式

$= (-\delta a\bar{\psi}^{\dagger}\gamma^0 e\psi^{*})^{\dagger}\gamma^0 = -\delta a\psi^{T}e^{\dagger}\gamma^{0\dagger}\gamma^0 = -\delta a\psi^{T}e^{\dagger}\gamma^0$

至此可以计标任何 bi 行为

求：$\bar{\psi}\psi$ 标量（C变换为内禀变换→不改变时空位置，省略各自变量）

$\hat{C}\bar{\psi}\hat{C}^{-1}\hat{C}\psi\hat{C}^{-1} = (-\delta a\psi^{T}e^{\dagger})\delta a^{*}e\bar{\psi}^{T} = -|\delta a|^2\psi^{T}\bar{\psi}^{T}$

$= -\psi\bar{\psi}^{T} = -(\bar{\psi}\psi)^{T} = -\bar{\psi}\psi$

→ $\mathcal{L}_D$ 包含 $\bar{\psi}\psi$ → 电荷共轭不变？→ 存在问题：

$\psi\psi$ 记调换次序是否可行？

Dirac场要求 $\{\psi_a(x), \psi_b^{\dagger}(y)\} = i\delta^3(x-y)\delta_{ab}$ 交换次序要改号

重写该式 $-\psi_a\bar{\psi}_a = +\bar{\psi}_a\psi_a +$ 天文大 constant $= \bar{\psi}\psi$ 标量流不变

$\Rightarrow \mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ 为C不变

下一个 bi $\hat{C}\bar{\psi}i\gamma_5\psi\hat{C}^{-1}$

$= -i\psi^{T}\underbrace{C^{\dagger}\gamma_5 e}_{C^{-1}C}\bar{\psi}^{T} = -i\psi^{T}\gamma_5^{T}\bar{\psi}^{T} = +i(\bar{\psi}\gamma_5\psi)^{T} = +\bar{\psi}i\gamma_5\psi$ ← 费米统计交换位置

Yukawa 理论：传中间→不行

$\mathcal{L}_{int} = g\bar{\psi}i\gamma_5\psi\phi$ ψ 粒子场算符，ϕ 为 K-G

$\mathcal{L}_{int}$ 为C不变 ϕ 对应 π^0 介子 π^0 自己字称为正

其 $J^{PC} = 0^{-+}$

矢量流自己C变换性质：

$\hat{C}\bar{\psi}\gamma^{\mu}\psi\hat{C}^{-1} = -\psi^{T}\underbrace{C^{\dagger}\gamma^{\mu}C}_{-\gamma^{\mu T}}\bar{\psi}^{T} = \psi^{T}(\gamma^{\mu})^{T}\bar{\psi}^{T} = (\bar{\psi}\gamma^{\mu}\psi)^{T} + \infty$ ← Fermi 统计

$= -\bar{\psi}\gamma^{\mu}\psi$

由场存在导出一个重要定理：

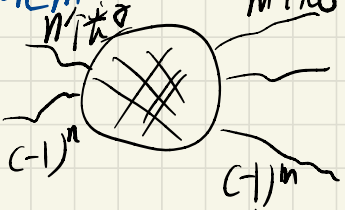
Lagrange 量 $\mathcal{L}$ 要求：$C\mathcal{L}C^{-1}=\mathcal{L}$ 具有 C 对称

则 $C\bar{\psi}\gamma^\mu\psi C^{-1}C A_\mu C^{-1}$

$=-\bar{\psi}\gamma^\mu\psi\hat{C}A_\mu\hat{C}^{-1}=?$ 必须有 $\hat{C}A_\mu\hat{C}^{-1}=-A_\mu$

$\Rightarrow$ 光子的 C 宇称为 -1

Furry theorem：



C 宇称

n+m 必须是偶数

C 不变 $C-1)^n=C-1)^m$

$C|\gamma\rangle=-|\gamma\rangle$ 推论：基态束缚态的内禀宇称为负

$^1S_0\to 2\gamma$ $^3S_1\to 3\gamma$

$C|f\bar{f}\rangle=+|f\bar{f}\rangle$ 允许 $C|f\bar{f}\rangle=-|f\bar{f}\rangle$ 3 个光子

对一个费一个反费构成的中性粒子

$\pi^0\to 2\gamma$； $\phi\to 3\gamma$ $J/\psi\to 3\gamma$

3 光子相互作用及电磁～者尸遵守 C 不变，而 2～ 破坏 C 不变

左曲矢流：

$C\bar{\psi}\gamma^\mu\gamma^5\psi C^{-1}=+\bar{\psi}\gamma^\mu\gamma^5\psi$

若 $C\bar{\psi}\gamma^\mu P_L\psi C^{-1}=-\bar{\psi}\gamma^\mu P_L\psi$ 左手流→右手流（弱只有左）

V-A

CPT 定理（为反线，反粒）

$CPT\ \psi(x)(CPT)^{-1}=CPT\psi(x)T^{-1}P^{-1}\hat{C}^{-1}=\xi_a^*\int(P^*\psi(x_T)P^*C^{-1}$

$=\xi_a^*\int(\eta_a)^*\gamma^0\hat{C}\psi(-x)\hat{C}^{-1}=-\xi_a^*\eta_a^*\xi_a^*\gamma^0\psi^*(-x)$

$=-\xi_a^*\eta_a^*\xi_a^*\gamma^5$

$=\int e$

$=\int^*\int\psi(x_T)$ $=\gamma^*\gamma^3$

$=\eta^*\gamma^0\psi(x_T)$ $\eta^*\psi(P^\mu,x_T)$

$x_T=(t,\vec{x})$

$\to(-t,-\vec{x})=-x_T$

若CPT作用在标量场

$CPT\phi(x)(CPT)^{-1}=\xi^*\eta^*\zeta^*\phi^\dagger(-x)$

矢量场 $CPTV_\mu(x)(CPT)^{-1}=-\xi^*\eta^*\zeta^*V_\mu^\dagger(-x)$

可以全部有 $\xi^*\eta^*\zeta^*=1$

symmetry

陈述：对于一个相对论性的 local QFT（满足 Lorentz + spacetime translation 且是厄米的），这个理论一定是CPT不变的。

$CPT\hat{H}(CPT)^{-1}=\hat{H}(-x)$

产生后果：正反粒子必须有相同质量

不稳定 —— 寿命

若CP破坏 $T\Leftrightarrow$ CP → 常测到 CP；可以借助

宇宙学观测

matter >> 反matter

自然界为什么 — B 重子数破坏；CP；月岩离热平衡

标准模型前2个满足 — 量子反常（小数弱）；KM机制

矢量场的量子化

$$\begin{cases}\nabla\cdot\vec{E}=\rho,\\ \nabla\times\vec{B}=\frac{\partial\vec{E}}{\partial t}+\vec{j},\\ \nabla\cdot\vec{B}=0,\\ \nabla\cdot\vec{E}=-\frac{\partial\vec{B}}{\partial t}\end{cases}\Rightarrow\partial_\mu F^{\mu\nu}=j^\nu$$

$\partial_\mu\tilde{F}^{\mu\nu}=0$

$j^\mu=(\rho,\vec{j})$ 满足守恒

定义场强张量 $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ ← 电磁势

$\tilde{F}^{\mu\nu}=-F^{\nu\mu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ 四阶全反对称张量

$$F^{\mu\nu}=-F^{\nu\mu}=\begin{pmatrix}0 & -E^1 & -E^2 & -E^3\\ & 0 & -B^3 & B^2\\ & & 0 & -B^1\\ & & & 0\end{pmatrix}\begin{matrix}\mu\\0\\1\\2\\3\end{matrix}\quad\begin{matrix}\nu\\0\ 1\ 2\ 3\end{matrix}$$

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ & 0 & E^3 & -E^2 \\ & & 0 & E^1 \\ & & & 0 \end{pmatrix} = F^{\mu\nu}\Big|_{\substack{E\to B \\ B\to -E}}$$

$F_{\mu\nu}F^{\mu\nu} = -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = 2(\vec{E}^2 - \vec{B}^2)$ 不变

$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4(\vec{E}\cdot\vec{B})$ 赝标量

构造

$\mathcal{L}_{Maxw} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$ → 得到

E-L方程: $\partial_\mu F^{\mu\nu} = j^\nu$

耦合

两作用边 $\partial_\nu\partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0$ $F^{\mu\nu}$ 反对 → j^ν 守恒流 → 电磁协变

另两个: $\partial_\mu \tilde{F}^{\mu\nu} = \varepsilon^{\rho\sigma\mu\nu}\partial_\mu\partial_\rho A_\sigma = 0$ (反对 对称 相乘为0) → $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$

有质量的矢量场 (Proca拉氏量)

带质量, S=1 → 3种极化(2J+1), A^μ 有4个 4动

真实存在 spin=1 的粒子: $W^\pm/Z$; ρ介子, J/ψ粒子, γ, ... Spin有助!

如何构造 $\mathcal{L}$?

回忆:

构造 spin=0 $\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

$= -\frac{1}{2}\phi(\Box + m^2)\phi + ?$

能量密度 $\mathcal{E} = T^{00} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} = \frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2]$ 正定

构造矢量场 A^μ

猜测: $\mathcal{L} \overset{?}{=} \frac{1}{2}A^\mu(\Box^2 + m^2)A^\mu$ 负号若则是为了保证空间部分一样

$= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}m^2 A_\mu^2$ (分部积) + 全导

其E-L方程 $(\Box^2+m^2)A_\mu=0$

下面看一下矢量玻色子的Lagrangian长什么样：

$\pi^\mu=\frac{\partial\mathcal{L}}{\partial\dot{A}_\mu}=-\dot{A}^\mu$，$\mathcal{E}=T_{00}=\pi^\mu\dot{A}_\mu-\mathcal{L}=\frac{1}{2}[\vec{A}^2+(\nabla A_i)^2+\frac{1}{2}m\vec{A}^2]$（正定）

缺的是A^μ为4矢量，存在0分量

负定 $-\frac{1}{2}[\vec{\dot{A}}^2+(\nabla A_0)^2+m^2A_0^2]$

→ 能量不bounded from below

→ 本质上copy了4次标量理论 → 4个标量粒子1⊕1⊕1⊕1 → 未体现S=1

→ 被迫放弃

如何体现Lorentz的矢量变换？

$\partial_\mu A_\nu\partial^\nu A^\mu\sim A_\nu\partial^\nu\partial_\mu A^\mu\sim\partial^\nu A_\nu\partial_\mu A^\mu\sim(\partial\cdot A)^2$ ↦ scalar

尝试：$\mathcal{L}=\frac{a}{2}A_\mu\Box A^\mu+\frac{b}{2}A_\mu\partial^\mu\partial^\nu A_\nu+\frac{1}{2}m^2A_\mu^2$（最普适形式）

$=-\frac{a}{2}\partial_\nu A_\mu\partial^\nu A^\mu-\frac{b}{2}\partial_\nu A_\mu\partial^\mu A^\nu+\frac{1}{2}m^2A_\mu^2$

求其E-L方程：

$\frac{\partial\mathcal{L}}{\partial A_\mu}=m^2A^\mu$；$\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}=-a\partial^\nu A^\mu-b\partial^\mu A^\nu$

$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}=\partial_\nu(-a\partial^\nu A^\mu-b\partial^\mu A^\nu)=-a\Box A^\mu-b\partial^\mu(\partial\cdot A)$

E-L方程：$a\Box A^\mu+b\partial^\mu(\partial\cdot A)+m^2A^\mu=0$

连续作用∂_μ：$\left[\Box+\left(\frac{m}{\sqrt{a+b}}\right)^2\right](\partial_\mu A^\mu)=0$

↑spin zero，质量为$m/\sqrt{a+b}$　↖scalar

4个自由度=1⊕3

↑spin 0 不是目标粒子，故去掉一个自由度

$[(a+b)\Box+m^2]\cdot\partial\cdot A=0$

令$a+b=0$；标量粒子质量为无穷大　Lorentz不变形式

$m^2\cdot\partial_\mu A^\mu=0\Rightarrow\partial_\mu A^\mu=0$ → 约束作用：移除A^μ的一个自由度

令 $a=1$ $b=-1$，则

$$\mathcal{L}=-\frac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu+\frac{1}{2}\partial_\nu A_\mu \partial^\mu A^\nu+\frac{1}{2}m^2A_\mu^2$$

$$=-\frac{1}{2}\partial_\nu A_\mu(\partial^\nu A^\mu-\partial^\mu A^\nu)+\frac{1}{2}m^2A_\mu^2$$

$$=-\frac{1}{2}\partial_\nu A_\mu F^{\nu\mu}+\frac{1}{2}m^2A_\mu^2=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu^2=\mathcal{L}_{Proca}$$

$$=-\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}m^2A_\mu^2$$

满足E-L方程 $(\Box+m^2)A^\mu=0$ 满足 $\partial_\mu A^\mu=0$

之后检查其是否满足动量正定

在经典层次：

$$\mathcal{E}=T^{00}=\pi^\mu\dot{A}_\mu-\mathcal{L}_{Procam}$$

$$\pi^\mu=\frac{\partial\mathcal{L}}{\partial\dot{A}_\mu}=-F^{0\mu}=F^{\mu0}\quad T^{00}=0,\ \pi^0=0,\text{式中无}\dot{A}_0\text{项}$$

→ $\mathcal{E}=-\vec{\pi}\dot{\vec{A}}-\mathcal{L}_{Procam}$（只要对3自由度的vector量子化）

$\pi^i=F^{i0}=\partial^iA^0-\partial^0A^i=-\partial_iA^0-\partial_0A^i=E^i$ （$\vec{E}=-\nabla A^0-\dot{\vec{A}}$；$\vec{B}=\nabla\times\vec{A}$）

$\vec{\pi}=\vec{E}$ 代入 $\mathcal{E}$ $=-\frac{1}{4}F^2$

$$\mathcal{E}=-\vec{\pi}\cdot\dot{\vec{A}}-\left[-\frac{1}{2}(\vec{E}^2-\vec{B}^2)+\frac{1}{2}m^2A_0^2-\frac{1}{2}m^2\vec{A}^2\right]$$

$$=+\vec{E}\cdot(\vec{E}+\nabla A_0)-(\qquad)$$

$$=\underbrace{\frac{1}{2}(\vec{E}^2+\vec{B}^2)}_{\text{正定}}+\vec{E}\cdot\nabla A_0+\underbrace{\frac{1}{2}m^2(A_0^2+\vec{A}^2)}_{\text{正定}}-m^2A_0^2$$

Maxwell(0质量的矢量场的Hamilton密度)

考虑 $\vec{E}\cdot\nabla A_0=\nabla\cdot(\vec{E}\cdot A_0)-A_0\nabla\cdot\vec{E}=\nabla\cdot(\vec{E}A_0)+A_0\nabla^2A_0+A_0\nabla\dot{\vec{A}}$

本回 $\mathcal{E}=\frac{1}{2}(\vec{E}^2+\vec{B}^2)+\frac{1}{2}m^2(A_0^2+\vec{A}^2)+\nabla\cdot(\vec{E}\cdot A_0)-A_0(-\nabla^2+m^2)A_0+A_0\partial_0\partial_iA^i$

$+A_0\partial_0\partial_0A_0-A_0\partial_0\partial_0A_0$ → 0及入

$(\Box+m^2)+A_0\partial_0(\partial_\mu A^\mu)$

$=\frac{1}{2}(\vec{E}^2+\vec{B}^2)+\frac{1}{2}m^2(A_0^2+\vec{A}^2)+\nabla\cdot(\vec{E}A_0)-A_0(\Box+m^2)A_0+A_0\partial_0(\partial\cdot A)$

处处正定　　表面项，无穷远边为0　　运动方程=0

则 $H=\int d^3x\,\mathcal{H}(x)\geq 0$ → 经典理论好

接下来量子化理论

寻找平面波解：

令 $A_\mu(x)=\varepsilon_\mu(p)e^{-ip\cdot x}$　代入 $\begin{cases}(\Box+m^2)A_\mu=0\\ \partial_\mu A^\mu=0\end{cases}$ 要求 $\begin{cases}p^2=m^2;\ p_0=\omega_{\vec{p}}=\sqrt{\vec{p}^2+m^2}\\ \hookrightarrow p_\mu\varepsilon^\mu(p)=0\end{cases}$ → 正交 → 3个独立的 $\varepsilon^\mu(p)$ 偏振矢量

静止条件下：$p^\mu=(m,\vec{0})$

$\varepsilon^{(1)\mu}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$　$\varepsilon^{(2)\mu}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$　$\varepsilon^{(3)\mu}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$　正交，类空矢量　$\varepsilon^{(\lambda)}\cdot\varepsilon^{(\lambda')}=-\delta^{\lambda\lambda'}$

若 $\vec{p}\parallel\vec{z}$　$p^\mu=(E,0,0,p_z)$　　z boost

线性组合 $\varepsilon_R^{(+)\mu}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$　$\varepsilon_L^{(-)\mu}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}$　$\varepsilon_L^{(0)\mu}=\begin{pmatrix}|\vec{p}|/m\\0\\0\\E/m\end{pmatrix}$

光：右旋光　左旋光 → 纯偏振

光子：螺旋度 $h=+1$　$h=-1$　横向极化　　$h=0$ → 纵向极化矢量，电磁场无纵向极化（光子为横波）

任何方向的纵向极化矢量：

$\varepsilon^{(0)}(p)=(|\vec{p}|/m,\frac{E}{m}\hat{p})$　$\hat{p}=\frac{\vec{p}}{|\vec{p}|}$

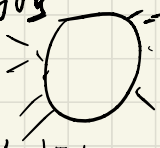
正交性：$\varepsilon^{(\lambda)}(p)\cdot\varepsilon^{(\lambda')}(p)=-\delta^{\lambda\lambda'}=g^{\lambda\lambda'}$　$(\lambda\lambda'=1,2,3)$

完备性：$\sum_{\lambda=1,2,3}\varepsilon_\mu^{(\lambda)}(p)\varepsilon_\nu^{(\lambda)}(p)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}$

若高动量极限 $E\gg m$

$\varepsilon_L^\mu\approx\frac{p^\mu}{m}\begin{pmatrix}1\\0\end{pmatrix}$

高阶对撞 → 产生spin1粒子



crossection $\propto |\mathcal{M}|^2 \sim |g(\text{耦合常数}) \epsilon p|^2$

横向极化 $\sigma_T \sim g^2 \epsilon_T^2 \sim g^2$

纵向 $\sim \lambda=0$ $\sigma_L \sim g^2 \epsilon_L^2 (\frac{E}{m})^2 >> \sigma_T$

产生W玻色子 → 纵向极化主导

→ 阻碍破坏幺正性 → 阻止增长 → ? 玻色子

正式对其量子化：

(厄米) $\hat{A}_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=1}^{3} (a_p \varepsilon_\mu^{(\lambda)} e^{-ip\cdot x} + a_p^\dagger \varepsilon_\mu^{(\lambda)*} e^{ip\cdot x})$

采用对易量子化 $[\hat{a_p}, \hat{a_{p'}^\dagger}] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}') \delta^{\lambda\lambda'}$

$[a\ a] = [a^\dagger, a^\dagger] = 0$

} 类似于之前K-G

对 $\hat{a_p}|0\rangle = 0 \ \forall \vec{p}, \lambda$

定义单粒子态 $|\vec{p}, \lambda\rangle = \sqrt{2E_p} a_p^{\lambda\dagger}|0\rangle$

$\langle 0|\hat{A}_\mu(x)|p,\lambda\rangle = \varepsilon_{\mu(p)}^{(\lambda)} e^{ip\cdot x}$

老套因果性： → C number (数，非算符)

要求：$[\hat{A}_\mu(x), \hat{A}_\nu(y)] = 0$ x y 类空间隔

$= \langle 0|[\quad]|0\rangle = \langle 0|xy|0\rangle - \langle 0|yx|0\rangle$

第一个 $\langle 0|A_\mu(x) A_\nu(y)|0\rangle$ 6重积分，只有 $\langle 0|a\ a^\dagger|0\rangle$ 不为0

$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_\lambda \varepsilon_{\mu(p)}^{(\lambda)} \varepsilon_{\nu(p)}^{(\lambda)*}}_{= -g_{\mu\nu} + p_\mu p_\nu / m^2} e^{-ip(x-y)} \Big|_{p^0 = E_p}$

$= (-g_{\mu\nu} - \partial_\mu \partial_\nu / m^2) D(x-y)$

$\partial_\mu = \frac{\partial}{\partial x^\mu}$

另一个 $\langle 0|A_\mu(y)A_\nu(x)|0\rangle = (-g_{\mu\nu} - \partial_\mu\partial_\nu/m^2) D(y-x)$

$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip(x-y)}$

因果性：$[A_\mu(x), A_\nu(y)]$ $[\phi(x), \phi(y)] \to$ K-G

$= (-g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2})[D(x-y) - D(y-x)]$

$= (-g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2})[\phi(x), \phi(y)]$

当 $(x-y)^2 < 0$ $[\ ] = 0$；满足因果性 ✱

正则量子化：采取等时对易关系（玻色对易关系）

猜想 ① $[A^i(t,\vec{x}), \pi^j(t,\vec{y})] = i\delta^{ij}\delta^3(\vec{x}-\vec{y})$ （$\pi^0 = 0$, A^0 非独立）

假设 ② $[A^\mu(t,\vec{x}), A^\nu(t,\vec{y})] = [\pi^\mu(t,x), \pi^\nu(t,y)] = 0$

先验证：①

回顾：$\vec{\pi} = \vec{E} = -\dot{\vec{A}} - \nabla A_0$

$\vec{\pi}(\vec{x})$ $(t=0$, Schödinger 绘景$)$

$= i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2}\sqrt{2E_p}} e^{i\vec{p}\cdot\vec{x}} \sum_\lambda [(\sqrt{E_p}a^\dagger_{\vec{p}}\vec{\varepsilon}^{(\lambda)}_{(\vec{p})} - \sqrt{E_p}\, a^\dagger_{(-\vec{p})}\vec{\varepsilon}^{(\lambda)*}_{(-\vec{p})}) - \vec{p}(a^\dagger_{\vec{p}}\vec{\varepsilon}^{(\lambda)}_{(\vec{p})} + a^{(\lambda)}_{(-\vec{p})}\vec{\varepsilon}^{(\lambda)*}_{(-\vec{p})})]$

$\vec{A}(y) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} e^{i\vec{q}\cdot\vec{y}} \sum_{r=1}^{3}[a^r_{\vec{q}}\vec{\varepsilon}^{(r)}_{(\vec{q})} + a^{r\dagger}_{(-\vec{q})}\vec{\varepsilon}^{(r)*}_{(-\vec{q})}]$

暴力计算：$\to$ 易证 $=0$

$[\pi^i(\vec{x}), A^j(\vec{y})] = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \sum \{\sqrt{E_p}\varepsilon^{(\lambda)i}_{(\vec{p})}\varepsilon^{(\lambda)*j}_{(\vec{p})} - p^2\varepsilon^{(\lambda)i}_{(\vec{p})}\varepsilon^{(\lambda)*j}_{(\vec{p})}$

$+ \sqrt{E_p}\varepsilon^{(\lambda)*i}_{(-\vec{p})}\varepsilon^{(\lambda)j}_{(-\vec{p})} - p^i\varepsilon^{(\lambda)*0}_{(-\vec{p})}\varepsilon^{(\lambda)j}_{(-\vec{p})}\}$

利用完备性关系 $= i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \{\sqrt{E_p}(-g^{ij} + \frac{p^ip^j}{m^2}) - p^i(-g^{0j} + \frac{p^0p^j}{m^2})$

$+ \sqrt{E_p}(-g^{ij} + \frac{p^ip^j}{m^2}) - p^i(-g^{0j} + \frac{p^0p^j}{m^2})\}$

$\sqrt{E_p}$ ↓ δ^{ij} ；$\frac{p^ip^j}{m^2}$ ↓ 抵消 ；$-p^i(-g^{0j})$ ↓ 0 ；$-p^i\frac{p^0p^j}{m^2}$ ↓ 抵消

$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \cdot 2\omega_p \delta^{ij} = \delta^3(x-y)\delta^{ij}$ ①得证

之后证明②：

$[A_\mu(x), A_\nu(y)] = (-g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{m^2})[\phi(x), \phi(y)]$，令 $x_0 = y_0$

易证：$[A^0(x), A^0(y)] = [A^i(x), A^j(y)] = 0$，但若 A^0 与 A^i 时

$[A^0(x), A^i(y)] = (-g^{0i} + \frac{\partial^0\partial^i}{m^2})[\phi(x), \phi(y)]$

$= \frac{1}{m^2}\partial_i\partial_0[\phi(x), \phi(y)] = \frac{1}{m^2}\partial_i\frac{\partial}{\partial x_0}[\phi(x), \phi(y)]$

$= \frac{1}{m^2}\partial_i[\dot{\phi}(x), \phi(y)] = \frac{1}{m^2}\partial_i[\pi(\vec{x},t), \phi(\vec{y},t)] = -i\frac{1}{m^2}\partial_i\delta^3(\vec{x}-\vec{y})$ 奇怪（不对劲）

考虑为何？A^0 不是协变量，且不含 A^0 的时间导数

在量子化之前，计标传播子（费曼传播子）

$D_{F\mu\nu}(x-y) = \langle 0|T A_\mu(x) A_\nu(y)|0\rangle = \theta(x^0-y^0)\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\langle\rangle$

$= \theta(x^0-y^0)(-g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2})D(x-y) + \theta(y^0-x^0)(-g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2})D(y-x)$

guess $= (-g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2})\underbrace{[\theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)]}_{= \langle 0|T\phi(x)\phi(y)|0\rangle = D_F(x-y)}$

$= (-g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2})\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-ip\cdot(x-y)}$

$= \int\frac{d^4p}{(2\pi)^4}\Big[\underbrace{\frac{i(-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2})}{p^2-m^2+i\varepsilon}}_{\tilde{D}_{F\mu\nu}(p)\text{ 非协变项}} - \frac{i}{m^2}\delta_{\mu0}\delta_{\nu0}\Big]$



$\tilde{D}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$

（因为 guess 不合法）

$= \int[\quad] - \frac{i}{m^2}\delta_{\mu0}\delta_{\nu0}\delta^4(x-y)$

当 μ,ν 中一个分量为 0 时

$\frac{\partial}{\partial x_0}[\theta(x_0-y_0)D(x-y)] = \delta(x^0-y^0)D(x-y) + \theta(x^0-y^0)\partial_0 D(x-y_0)$

$$\partial_0^2[\theta(x^0-y^0)D(x-y)] = \delta(x^0-y^0)D(x-y) + 2\delta(x^0-y^0)\partial_0 D(x-y) + \theta(x^0-y^0)\partial_0^2 D(x-y)$$

第2项 $(y_0 \to x_0)$

$$\partial_0^2[\theta(y^0-x^0)D(y-x)] = -\delta(x^0-y^0)D(y-x) + 2\delta(x^0-y^0)\partial_0 D(y-x) + \theta(y^0-x^0)\partial_0^2 D(y-x)$$

$\therefore \theta(x^0-y^0)\partial_0^2 D(x-y) + \theta(y^0-x^0)\partial_0^2 D(y-x)$

$= \partial_0^2 D_F(x-y) - \delta(x^0-y^0)[\phi(x),\phi(y)] - 2\delta(x^0-y^0)\frac{\partial}{\partial x^0}[\phi(x),\phi(y)]$

$\pi(x)$

对第2项 $D(x-y) - D(y-x) \to 0 \times \infty$ $= -i\delta^3(\vec{x}-\vec{y})$

$(AB)' = A'B + AB' \Rightarrow -\frac{\partial}{\partial x^0}[\delta(x^0-y^0)[\phi(x),\phi(y)]] - \delta(x_0-y_0)\partial_0[\phi(x),\phi(y)]$

等时 + 类空 $\Rightarrow 0$

$= \delta(x^0-y^0)[\dot{\phi}(x),\phi(y)] = -i\delta(x^0-y^0)\delta^3(\vec{x}-\vec{y})$

原式 $= \partial_0^2 D_F(x-y) + (1-2)(-i)\delta^4(x-y)$

$= \partial_0^2 D_F(x-y) + i\delta^4(x-y)$ $\Rightarrow \mu=\nu=0$ 时，将 $\partial_\mu \partial_\nu$ 移到外面有密度外项

思考：当 μ,ν 只有一个为0时（一阶时间导时）

$\theta(x^0-y^0)\partial_0 D(x-y) + \theta(y^0-x^0)\partial_0 D(y-x)$

$= \partial_0 D_F(x-y) - \delta(x^0-y^0)[\phi(x),\phi(y)]$

Dirac（费曼提）否定提：

$S_F(x-y) = \theta(x_0-y_0)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle$

$10 \nearrow \quad \theta(y_0-x_0)\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$

Fermi-Dirac统计

$= \theta(x^0-y^0)(i\not{\partial}+m)D(x-y)$

$+ \theta(y^0-x^0)(i\not{\partial}+m)D(y-x)$

$= (i\not{\partial}+m)D_F(x-y)$

1成实的量子化

$\mathcal{H} = \frac{1}{2}(\vec{E}^2+\vec{B}^2) + \frac{1}{2}m^2(A_0^2+\vec{A}^2)$ 经典层次

$\pi^\mu = F^{\mu 0} \Rightarrow \pi^0 = 0$

E-L方程：$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \xrightarrow{\nu=0} \partial_\mu F^{\mu 0} + m^2 A^0 = 0 \quad (\mu = i = 1,2,3)$

$\pi^i = F^{i0} \to \nabla\cdot\vec{\pi} = -m^2 A^0 \Rightarrow A^0 = -\frac{1}{m^2}\nabla\cdot\vec{\pi}$

A^0 是独立的正则算符

H中的变量应该都为正则变量，考虑把 A^0 换掉

$\mathcal{H}=\frac{1}{2}\vec{\pi}^2+\frac{1}{2m^2}(\nabla\cdot\vec{\pi})^2+\frac{1}{2}(\nabla\times\vec{B})^2+\frac{1}{2}m^2\vec{A}^2$ (正则形式)

正则满足Heisenburg方程：

$i\dot{O}=[O,H]$

$i\dot{\vec{A}}(x)=[\vec{A}(x),\int d^3y\,\mathcal{H}(y)]$

$=[\vec{A}(x),\int d^3y[\frac{1}{2}\vec{\pi}^2(y)+\frac{1}{2m^2}(\nabla\cdot\vec{\pi}(y))^2]]$

$=-i\vec{\pi}(x)+i\nabla(\nabla\cdot\vec{\pi})\Rightarrow\vec{\pi}(x)=-\dot{\vec{A}}-\nabla A^0$ ③ 无其物理含义

$[\vec{A}^i(t,x),\pi^j(t,y)]=-i\delta^{ij}\delta^3(\vec{x}-\vec{y})$

$[A^i,A^j]=[\pi^i,\pi^j]=0$ (不含A^0)

$i\dot{\vec{\pi}}(x)=[\vec{\pi}(x),\int d^3y[\frac{1}{2}(\nabla\times\vec{A}(y))^2+\frac{1}{2}m^2\vec{A}^2(y)]]$

$=im^2\vec{A}-i\nabla^2\vec{A}+i\nabla(\nabla\cdot\vec{A})$

改写 $\dot{\vec{\pi}}=-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})+m^2\vec{A}$ ② ⟷ 耳关立

对②取散

$\nabla\cdot\dot{\vec{\pi}}=-\nabla\cdot\dot{\vec{A}}-\nabla^2\dot{A}^0$ ①

相互等价 $\dot{A}^0=-\frac{1}{m^2}\nabla\cdot\dot{\vec{\pi}}$，则①变为

$-m^2\dot{A}^0=-\nabla\cdot\dot{\vec{A}}-\nabla^2A^0$

③代入②

$-\ddot{\vec{A}}-\nabla\dot{A}^0=-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})+m^2\vec{A}$

$-\nabla\dot{A}^0=\Box\vec{A}+\nabla(\nabla\cdot\vec{A})+m^2\vec{A}$

至此，已完成正则变量满足的Heisenburg与E-L方程自洽

定义 $A^\mu=(A^0,\vec{A})=(-\frac{1}{m^2}\nabla\cdot\vec{\pi},\vec{A})$ 正确的量子化方案；

E-L方程

$\Box A^\mu-\partial^\mu\partial A+m^2A^\mu=0$

$\Box A^\mu-\partial^\mu(\dot{A}^0+\nabla\vec{A})+m^2A^\mu=0$

$\mu=0$ 时

$\Box A^0-\ddot{A}^0-\nabla\dot{\vec{A}}+m^2A^0=0$

$(\partial_t^2-\nabla^2)A^0-\ddot{A}^0-\nabla\dot{\vec{A}}+m^2A^0=0$

$\dot{A}^0=0$

$\Leftrightarrow\nabla^2A^0+\nabla\dot{\vec{A}}-m^2A^0=0$

$\mu=1,2,3$ 时

$\Box\vec{A}+\nabla\dot{A}^0+\nabla(\nabla\cdot\vec{A})+m^2\vec{A}=0$

自洽

自洽

之后，便可以用产生湮灭算符来表示：（对$\vec{\pi}$展开 九重积分→S函数→3重）

$$\mathcal{H}=\int\frac{d^3p}{(2\pi)^3}\sum_{r=1}^{3}\varepsilon_p a_p^{r\dagger}a_p^r+\infty\text{真空能}$$

最后，讨论非协变项：

拓展 $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu^2-j_\mu A^\mu$（引入相互作用项）　加入一个外源

$\xrightarrow{\text{E-L}}\partial_\mu F^{\mu\nu}+m^2A^\nu-j^\nu=0$　　$(\pi^0=0;\ \pi^i=F^{i0}=\dot{E}^i)$

取 $\nu=0$　$\partial_\mu F^{\mu 0}+m^2A^0-j^0=0$　$(\mu=1,2,3=i)$

$\nabla\cdot\vec{\pi}+m^2A^0-j^0=0\Rightarrow A^0=\frac{1}{m^2}(j^0-\nabla\cdot\vec{\pi})$

利用 $\vec{E}=-\dot{\vec{A}}-\nabla A_0=-\dot{\vec{A}}+\frac{1}{m^2}(\nabla\cdot\vec{\pi}-j^0)$

$\dot{\vec{A}}=-\nabla A^0-\vec{\pi}=\frac{1}{m^2}(\nabla\cdot\vec{\pi}-j^0)-\vec{\pi}$

$\mathcal{H}=\pi^\mu\dot{A}_\mu-\mathcal{L}=-\vec{\pi}\cdot\dot{\vec{A}}-\mathcal{L}$

$=\mathcal{H}_0+V$（相互作用），计算得到：

$\mathcal{H}_0=\frac{1}{2}\vec{\pi}^2+\frac{1}{2m^2}(\nabla\cdot\vec{\pi})^2+\frac{1}{2}(\nabla\times\vec{A})^2+\frac{1}{2}m^2\vec{A}^2$

$\hookrightarrow H_0(\vec{A},\vec{\pi})$

$V=-\vec{j}\cdot\vec{A}-\frac{1}{m^2}j^0\nabla\cdot\vec{\pi}+\frac{1}{2m^2}j_0^2$ 又多了一个非协变项

协变 $=j^\mu A_\mu+\frac{1}{2m^2}j_0^2$

3个经子项（相互作用给出）

相互作用，重子的定义为 $\Delta_0=-\frac{1}{m^2}\nabla\cdot\vec{\pi}$

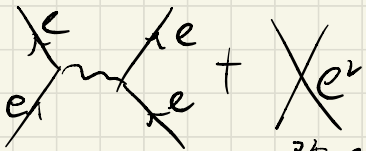
计算子散射的振幅：

引入 $\hat{S}=T\exp\left[i\int_{-\infty}^{+\infty}dt\,V_{相互}\right]$　V中有2次展开2阶

$=-\frac{1}{2}\iint d^4x\,d^4y\,D_F^{\mu\nu}(x-y)\,j_\mu(x)\,j_\nu(y)$（2阶）　$=\frac{1}{m^2}\delta^4(x-y)\,\delta_{\mu 0}\delta_{\nu 0}$ 寄生外项

$-\frac{1}{2m^2}\int d^4x\, j^2$ 得给2阶 $+2\frac{1}{m^2}\int d^4x\, j^{02}$ ← 抵消

非协变项举例：



一阶 $\propto \delta^4(x-y)$

$A^0 = -\frac{1}{m^2}\partial\cdot\vec{\pi}$ 非独立 → 3个自由度

有等时对易关系：$[A^0(t,\vec{x}), \dot{A}(t,\vec{y})] = \frac{1}{m^2}\partial\cdot\delta^3(\vec{x}-\vec{y})$

注意 $A^\mu = (A^0, \vec{A}) = (-\frac{1}{m^2}\partial\cdot\vec{\pi}, \vec{A})$，该等等满足 $\partial^\mu(\partial A) + m^2 A^\mu = 0$

加外源后：$A^0 = \frac{1}{m^2}(-\partial\cdot\vec{\pi} + j^0)$ → $H = H_0 + V$

但 H_0 中自由场仍为 $A^0 = -\frac{1}{m^2}\partial\cdot\vec{\pi}$（自由场中）

Maxwell 场的量子化

如何推导 Maxwell 拉式量

从 $\mathcal{L}_{Proca} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2$ 取 $m\to 0$

但 Proca 有3个自由度，$\mathcal{L}_{max}$ 有2个自由度 → 自由度不连续

去哪了

$\sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)*} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$ → $m\to 0$ 纵从何极化发散 → 应舍去 → 物理上

引入一个 trick，考虑有外源

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu - j^\mu A_\mu$$ 无规范对称性

令 $A_\mu \to A_\mu + \partial_\mu\phi$ ϕ 为实标量场（场的重新定义），代入

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2(A_\mu + \partial_\mu\phi)^2 - A_\mu j^\mu + \phi\partial_\mu j^\mu$

$-A_\mu j^\mu$ 规范不变；$\phi\partial_\mu j^\mu$ 分部（边界）

→ 得出 $U(1)$ 的规范不变性 $\begin{cases} A_\mu \to A_\mu + \partial_\mu\alpha(x) \\ \phi \to \phi - \alpha \end{cases}$ > 抵消

local 对称性 $A_\mu \to A_\mu + \partial_\mu\alpha(x)$ 与 global 不同，不导致任何守恒流

ϕ^2 系数为 m^2，不是正常 K-G，含 ϕ 之质量

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2A_\mu^2 + \frac{1}{2}(\partial_\mu\phi)^2 + mA_\mu\partial^\mu\phi - A_\mu j^\mu + m\phi\partial_\mu j^\mu$

完全消失　　j^μ 若不是守恒流，则 ϕ 与 $\partial_\mu j^\mu$ 是强耦合

对称性变为 $\begin{cases} A_\mu \to A_\mu + \partial_\mu\alpha \\ \phi \to \phi - m\alpha \end{cases}$

$m\to 0$ 极限：若 $\partial_\mu j^\mu = 0$ 为守恒流（无危险性）

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\phi)^2 - A_\mu j^\mu \to$

仍然有 3 个自由度　$3 = 2$（由 1^+ spin 0 的玻色子）

现在，对称性变为 $\begin{cases} A_\mu \to A_\mu + \partial_\mu\alpha \\ \phi \to \phi \end{cases}$ 纵向极化的 spin 1 玻色子

发现 $m\to 0$ 时纵向极化的粒子退耦，与 Maxwell 场不作用

其后果：$\partial_\mu j^\mu = 0$ 守恒流：

$$\sigma \sim |M|^2 = \sum_\lambda |\langle j_\mu\rangle \varepsilon_\mu^{(\lambda)}|^2 = \langle j^\mu\rangle\langle j^\nu\rangle^* \sum_\lambda \varepsilon_\mu^{(\lambda)}(p)\,\varepsilon_\nu^{(\lambda)*}(p)$$

$$= \langle j_\mu\rangle\langle j_\nu\rangle^*\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right)$$ 由于 $\partial_\mu j^\mu = 0 \Leftrightarrow p^\mu\langle j_\mu\rangle = 0$ 故

纵项为 0，不贡献 σ

正式讨论 Maxwell 理论（对应量子光子）

$\mathcal{L}_{Max} = -\frac{1}{4}F_{\mu\nu}^2 - j_\mu A^\mu$ 表明外源，为规范不变性

$\partial_\nu\partial_\mu F^{\mu\nu} = \partial_\nu j^\nu \to$ 给制 j_μ 为守恒流

从另一个角度，规范不变性也要求 ——————

$A^\mu \to A^\mu + \partial^\mu\alpha \to \Delta\mathcal{L} = -j_\mu\partial^\mu\alpha$

$$= \underbrace{-\partial^\mu(j_\mu\alpha)}_{\text{表面项}} + \underbrace{\alpha\partial_\mu j^\mu}_{0}$$

难量子化的原因：

冗余的自由度，又能得到 Feynmann 传播子：

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 = \frac{1}{2}A_\mu(x)\left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu(x)$

对 K-G $\mathcal{L}_{K\text{-}G} = -\frac{1}{2}(\Box + m^2)\phi$

对应的 Feynmann，定义

$$\left(\Box g_{\mu\nu} - \partial_\mu\partial_\nu\right)D_F^{\nu\rho}(x-y) = i\delta_\mu^\rho\,\delta^4(x-y)$$

or $\left(-k^2 g_{\mu\nu} + k_\mu k^\rho\right)\tilde{D}_F^{\nu\rho}(k) = i\delta_\mu^\rho$ 无解（4×4，因为行列式=0，无逆

若要：$\int[DA]\,e^{iS[A]} = \int[DA]e^{i\int d^4x\,A_\mu(x)(\Box g^{\mu\nu}-\partial^\mu\partial^\nu)A_\nu(x)}$

泛函积分 ≠ 路径积分

纯规范：$A_\mu(x) = \partial_\mu\alpha(x) \Rightarrow F_{\mu\nu} = 0 \rightarrow$ 指数=0

又能：massless spin1 粒子 + L不变性 → 必须有规范不变性

回忆：

小群诱导表示：单粒子态 $\hat{P}^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle$

若要 $\Lambda\in\mathcal{L}_+^\uparrow$ $U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'}C_{\sigma\sigma'}|\Lambda p,\sigma'\rangle$

$C = C(\Lambda,p)$

小群：系统的动角定单粒子态在路径变化下的行为，定出 C

小群：定义标准动量 k^μ，对任动量 $p^\mu = L^\mu_\nu(p)\,k^\nu$；

要求 $|p,\sigma\rangle = U(L(p))|k,\sigma\rangle$

$L^\mu_\nu \in \Lambda$

$k \to p \to \Lambda p \to p$

对应的动量还是 k

则 $U(\Lambda)|p,\sigma\rangle = U(\Lambda L(p))|k,\sigma\rangle = U(L(\Lambda p))U(L^{-1}(\Lambda p))U(\Lambda L(p))|k,\sigma\rangle$

$= U(L(\Lambda p))\sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,p))|k,\sigma'\rangle$

$|k,\sigma'\rangle$ 一个小群单位动量单元

小群 $\forall W$ $k^\mu = k^\mu$ k不变 $U(W)|k,\sigma\rangle = \sum_{\sigma'}D_{\sigma\sigma'}(W)|k,\sigma'\rangle$

$= \sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,p))U(L(\Lambda p))|k,\sigma'\rangle$

$= \sum D|\Lambda p,\sigma'\rangle$

对于 massive $k^\mu = (m,\vec{0})$ 则只有空间转动群作用在 k^μ 上不变 $W = SO(3)$

massless $k^\mu = (k, 0, 0, k)$ 沿z轴类光矢量

显然绕z轴转动操作保持k^μ不变 但不完全是

另一种 $t^\mu = (1, \vec{0})$ 类时

$$W(t)^\mu (W(t))_\mu = t^\mu t_\mu = 1$$

$$(W_t)^\mu k_\mu = (W_t)^\mu (W_k)_\mu = t \cdot k = k$$

可以写成 $(W_t)^\mu = (1+\xi, \alpha, \beta, \xi) \Rightarrow \xi = \frac{\alpha^2+\beta^2}{2}$

则 $S(\alpha,\beta)^\mu{}_\nu = \begin{pmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{pmatrix}$ 也是一个合理群元 $\in L^\uparrow_+$

最普适规群 (messless)

$W(\alpha,\beta,\theta) = S(\alpha,\beta)\cdot R(\theta)$ $\quad R(\theta) = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$ 沿z轴θ角

上述小群的性质:

① $R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$ 阿贝尔群

② $S(\bar\alpha,\bar\beta)S(\alpha,\beta) = S(\bar\alpha+\alpha, \bar\beta+\beta)$

③ 被自群: $R(\theta)S(\alpha,\beta)R^{-1}(\theta) = S(\alpha\cos\theta+\beta\sin\theta, -\alpha\sin\theta+\beta\cos\theta)$

相当于矢量自的叠加 (二维平面) $\sim SO(2)$ = 二维欧几里德群

其李代数: $U(W(\theta,\alpha,\beta)) = 1 + i\alpha\hat{A} + i\beta\hat{B} + i\theta\hat{J}_3$ → 角动量第三分量

$\hat{A} = J_2 + K_1$; $\hat{B} = -J_1 + K_2$

↓ 沿y转 ↓ 沿x boost

李: $[A,B] = 0 \longrightarrow$ 引入 A,B 的共同本征态 $|k, a, b\rangle$

$[J_3, A] = +iB$ 使 $A|k,a,b\rangle = a|k,a,b\rangle$; $B|k,a,b\rangle = b|k,a,b\rangle$

$[J_3, B] = -iA$ 定义新态 $|k^0, a, b\rangle = U^{-1}(R(\theta))|\theta, a, b\rangle$ 转动 → 依然为 AB 的共同本征态；

$$\begin{cases} A|k^0, a, b\rangle = (a\cos\theta - b\sin\theta)|k^0, a, b\rangle; \\ B| \quad \rangle = (a\sin\theta + b\cos\theta)| \quad \rangle \end{cases}$$ → 无穷多简并的单态子态 → 但实际上没观测到

→ 要求 $ab = 0$，简并被动破坏 $\Rightarrow A|\rangle = B|\rangle = 0$

除动量外还剩另一个不变量记为 → 生成元剩下 → 为下本征态：

$$J_3|k, \sigma\rangle = \sigma|k, \sigma\rangle$$

→ 粒子螺旋度 → 半整 or 整

指数形式：

$U(S(\alpha,\beta)) = e^{i\alpha A + i\beta B}$ $U(R(\theta)) = e^{iJ_3\theta}$

则小群 $U(W)|k, \sigma\rangle = e^{i\alpha A + i\beta B} \cdot e^{iJ_3\theta}|k, \sigma\rangle$

$= e^{i\alpha A + i\beta B} \cdot e^{i\sigma\theta}|k, \sigma\rangle = e^{i\sigma\theta}|k, \sigma\rangle$ → helicity 为 Lorentz 变

对 massless

$$U(\Lambda)|p, \sigma\rangle = e^{i\sigma\alpha(\Lambda, p)}|\Lambda p, \sigma\rangle$$

或 $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) = S(\alpha(\Lambda, p), \beta(\Lambda, p))R(\theta(\Lambda, p))$

回到 Maxwell 理论，$A^\mu(x)$ 在 Lorentz 变换下如何变

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|p|}}\sum_{\sigma=\pm1}(a_p^\sigma \varepsilon^\mu(p, \sigma)e^{ip\cdot x} + h.c)$$

↳ 极化矢量

$$U(\Lambda)\hat{A}^\mu(x)U^{-1}(\Lambda) = \int \cdots (U(\Lambda)a_p^\sigma U^{-1}(\Lambda) \quad)$$

左侧 $= (\Lambda^{-1})^\mu_\nu \hat{A}^\nu(\Lambda x)$

右侧 $= |p, \sigma\rangle = \sqrt{2|p|}\, a_p^{\sigma\dagger}|0\rangle = \sqrt{2|p|}\, a_p^{\sigma\dagger}U(\Lambda)|0\rangle$ 真空 Lorentz 不变

$\Rightarrow \sqrt{2p^0}\, U(\Lambda)\, a^{\sigma\dagger}_{\vec p}\, U^{-1}(\Lambda) = e^{i\theta(\Lambda,p)\cdot\sigma}\sqrt{2(\Lambda p)^0}\, a^{\sigma\dagger}_{(\Lambda p)}$

利用到主式：

$(\Lambda^{-1})^\nu_\mu A^\mu(\Lambda x) = \int [dp] \frac{1}{2p^0}\sqrt{2p^0}\sum_\sigma \left(e^{-i\theta(\Lambda,p)\cdot\sigma}\frac{}{\sqrt{2(\Lambda p)^0}}\, a^\sigma_{\vec p}\, \varepsilon^{(\sigma)\nu}(p)\, e^{ip\cdot x} + hc\right)$

左侧 $= (\Lambda^{-1})^\nu_\mu \int [dp] \sum_\sigma \sqrt{2p^0}\, a^\sigma_{\vec p}\, \varepsilon^{(\sigma)\mu}(p)\, e^{ip\Lambda x} + hc)$ $\quad \Lambda p = \Lambda p$

$= (\Lambda^{-1})^\nu_\mu \int [d\Lambda p] \sum_\sigma \sqrt{2(\Lambda p)^0}\, a^\sigma_{\Lambda p}\, \varepsilon^{(\sigma)\mu}(p)\, e^{-i\frac{\Lambda p\cdot\Lambda x}{=px}} + hc)$

L变换不改变两矢量内积

比较左右两侧：$e^{-i\theta(\Lambda,p)\cdot\sigma}\, \varepsilon^{(\sigma)\nu}(p) = (\Lambda^{-1})^\nu_\mu\, \varepsilon^{(\sigma)\mu}(\Lambda p)$ (极化矢量的规制)

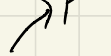
$\Rightarrow$ $e^{i\theta(\Lambda,p)\cdot\sigma}\cdot\varepsilon^{(\sigma)\mu}(\Lambda p) = \Lambda^\mu_\nu\, \varepsilon^{(\sigma)\nu}_p$ 给出极矢约束

小群矢量

回忆 $L(p): k^\mu \to p$；令 $p=k$ $L(p)=1$；令 $\Lambda = L(q)$ 小群 $W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)$

$= L^{-1}(q)L(q) = 1 \to \varepsilon^{(\sigma)\mu}_{(q)} = L(q)^\mu_\nu\, \varepsilon^{(\sigma)\nu}_{(k)}$

$L(\vec p)$ 是什么：



$L(\vec p) = R(\vec p)\, B \Rightarrow \vec k \to \vec p$

令 $p=k$： $\Rightarrow$ ① $\varepsilon^{(\sigma)\mu}_{(k)}\, e^{i\sigma\theta} = R(\theta)^\mu_\nu\, \varepsilon^{(\sigma)\nu}(k)\ (\theta)$ $R\theta \in W$

② $\varepsilon^{(\sigma)\mu}_{(k)} = S(\alpha,\beta)^\mu_\nu\, \varepsilon^{(\sigma)\nu}_{(k)}$ $S(\alpha,\beta)\in W$

} 之所以为大极化矢量性质

因有 ③ $\varepsilon^{(\sigma)\mu}_{(p)} = \left(L(p)\right)^\mu_\nu\, \varepsilon^{(s)\nu}_{(k)}$

发现

$R(\theta) = \frac{1}{\sqrt2}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix} = \begin{pmatrix}1&&&\\&\cos\theta&\sin\theta&\\&-\sin\theta&\cos\theta&\\&&&1\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix} = e^{\pm i\theta}\frac{1}{\sqrt2}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}$

$= \varepsilon^{(\sigma)\mu}_{(k)}$ 满足③

对于④：

$$S(\alpha,\beta)\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}=\begin{pmatrix}\alpha\pm i\beta\\1\\\pm i\\\alpha\pm i\beta\end{pmatrix}=\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}\pm(\alpha\pm i\beta)\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\parallel k^\mu$$

由于$\alpha,\beta\neq0$，但$\alpha,\beta=0$才满足 ⇒ 结论出错，A^μ不为严格的4矢量

→ 放松条件：

$$W(\Omega)(\alpha,\beta)^\mu_\nu\,\varepsilon^{(\pm1)\nu}_{(k)}=S(\alpha,\beta)^\mu_\lambda R(\Omega)^\lambda_\nu\,\varepsilon^{(\pm1)\nu}_{(k)}=e^{\pm i\theta}\Big[\varepsilon^{(\pm1)\mu}_{(k)}+\underbrace{\frac{(\alpha\pm i\beta)}{\sqrt{2}|k|}k^\mu}_{\text{另等项}}\Big]$$

群元

$$A^\mu_{(x)}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=\pm1}\left(\varepsilon^{(\pm)\mu}_{(p)}a^\sigma_p e^{-ip\cdot x}+h.c\right)$$

满足 $D_\mu A^\mu_{(x)}=0$　极化矢量 $\varepsilon^{\sigma\mu}_{(p')}=L(p)^\mu_\nu\,\varepsilon^{(\sigma)\nu}_{(k)}$；记 $R(\theta)$ 为转动，$B(\frac{p}{m})$ 为Boost

$\varepsilon=\frac{1}{\sqrt2}(0,1,\pm i,0)$；沿z轴 boost 有贡献量级→

$\varepsilon^{\sigma\mu}_{(p)}=R(\theta)\frac{1}{\sqrt2}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}$ → ct项恒为0 → $\varepsilon^{(\pm)0}=0$ → 量子化电磁场的0分量=0

→ A^μ不可能是4矢量

换一种方式 $p\cdot\varepsilon^{(\pm)}_{(p)}=0\Rightarrow\partial\cdot A=0$

考虑电磁场的规范变换

$U(\Lambda)A^\mu_{(x)}U(\Lambda)=(\Lambda^{-1})^\nu_\mu A^\mu(\Lambda x)+\partial^\mu\Omega(x,\Lambda)$ → 得数次项

考虑 $U(\Lambda)S(x)U^+(\Lambda)=S(\Lambda x)$（严格要求）$\mathcal{L}$不变

对$\mathcal{L}_{max}=-\frac14F^2_{\mu\nu}-j_\mu A^\mu$；$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\mu A_\nu$

$-j_\mu A^\mu$贡献了零项外 $j^\mu\partial_\mu\Omega(x,\Lambda)$ 等于表面项 $-\partial_\mu(j^\mu\Omega(x,\Lambda))$

若是守恒流，则不造成零外项难

正题：量子化方案；　　4+2

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - J_\mu A^\mu$，固定库伦规范 $\nabla\cdot\vec{A}=0$

经典中，若 A^μ 不满足 $\nabla\cdot\vec{A}=0$，总可以找 $A'^\mu \to A^\mu + \partial^\mu\alpha$ 使 $\nabla^2\alpha = -\nabla\cdot\vec{A}$

Maxwell 方程　$\partial_i F^{i0} = j^0 \Rightarrow \nabla\cdot\vec{E} = j^0$　　$E = -\dot{\vec{A}} - \nabla A_0$

$\Rightarrow \nabla^2 A_0 + \underbrace{\nabla\cdot\dot{\vec{A}}}_{=0} = j^0$

若无源：$j^\mu = 0$ 令 $A^0 = 0$

$\pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F^{\mu 0} \to \pi^0 = 0$ 约束　$\pi^i = E^i = \dot{A}^i - \underbrace{\partial^i A_0}_{\text{难要}}$　库伦势　$A^0(t,\vec{x}) = \int d^3y \frac{j^0(y,t)}{4\pi|\vec{x}-\vec{y}|}$

正则量子化手件（对易）

$[A_\mu(t,\vec{x}), \pi^\nu(t,\vec{y})] = i g_\mu^\nu \delta^3(\vec{x}-\vec{y})$ ①

$[A^i(t,\vec{x}), \pi^j(t,\vec{y})] = -i\delta^{ij}\delta^3(\vec{x}-\vec{y})$ ②

$[A^i(\vec{x}), A^j(\vec{y})] = [\pi^i(\vec{x}), \pi^j(\vec{y})] = 0$ ③

$\nabla^2\frac{1}{|\vec{x}-\vec{y}|} = -4\pi\delta^3(\vec{x}-\vec{y})$

A^0 不是力学正则变量 → 约束

被手物质物分布 → 库伦势

化 $j^\mu = e\bar{\psi}\gamma^\mu\psi$

不对，对①

$\partial_i[\quad] \Rightarrow [\underbrace{\nabla\cdot\vec{A}}_{=0}(\vec{x}), \pi^j(y)] = -i\delta^{ij}\partial_i\delta^3(\vec{x},\vec{y}) \neq 0$

等时量子化必须满足零数为0；

给出正确：

$[\qquad\qquad] = -i\delta_{\perp}^{ij}(\vec{x}-\vec{y})$ 要求 $\partial_i\delta_\perp^{ij}(\vec{x}-\vec{y}) = 0$

其中：$\delta_\perp^{ij}(\vec{x}-\vec{y}) = (\delta^{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{x}-\vec{y})$ 作F变换

$= (\delta^{ij} - \frac{\partial_i\partial_j}{\nabla^2})\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}(\vec{x}-\vec{y})}$

$= \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}(\vec{x}-\vec{y})}(\delta^{ij} - \frac{k^ik^j}{|\vec{k}|^2}) = \delta^{ij}\delta^3(\vec{x}-\vec{y}) + \frac{\partial^2}{\partial x^i\partial x^j}\frac{1}{4\pi|\vec{x}-\vec{y}|}$

平面波

$$\vec{A}(x)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{k}|}}\sum_{\lambda=\pm1}\left(\vec{\varepsilon}^{(\lambda)}(k)\,a_{\hat{k}}\,e^{-ik\cdot x}+h.c\right)$$

$$\vec{\pi}(x)=\dot{\vec{A}}=-\vec{E}(x)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{k}|}}(i|\vec{k}|)\sum_{\lambda=\pm1}\left(\vec{\varepsilon}^{(\lambda)}(k)\,a_{\hat{k}}\,e^{ik\cdot x}-\vec{\varepsilon}^{(\lambda)*}(k)\,a^{\lambda\dagger}_{\hat{k}}\,e^{ikx}\right)$$

验证其满足回

$$[a_{\hat{k}},a^{\lambda'\dagger}_{\hat{k}'}]=(2\pi)^3\delta^{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')$$

$$[a,a]=[a^\dagger a^\dagger]=0$$

$$[A^i(t,\vec{x}),\pi^j(t,\vec{y})]=\int\!\!\int\frac{d^3pd^3p'}{(2\pi)^6}\frac{1}{2\sqrt{|\vec{p}||\vec{p}'|}}i|\vec{p}|\,e^{i\vec{p}\cdot\vec{y}+i\vec{p}'\cdot\vec{x}}\sum_{a\lambda'}\Big\{[a_{\hat{p}},a^{\lambda'\dagger}_{\hat{p}'}]\varepsilon^{\lambda' i}(\vec{p}')$$

$$\cdot\varepsilon^{(\lambda)j*}(\vec{p}')+[a^{\lambda\dagger}_{-\hat{p}},a^{\lambda'}_{\hat{p}'}]\varepsilon^{\lambda' i*}(-\vec{p})\,\varepsilon^{(\lambda)j}(\vec{p}')\Big\}$$

$$=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}(i|\vec{k}|)(-2)\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\underbrace{\sum_{\lambda=\pm1}\varepsilon^{\lambda i}(\vec{k})\,\varepsilon^{(\lambda)j*}(\vec{k})}_{=\delta^{ij}-\hat{k}^i\hat{k}^j/|\vec{k}|^2}$$

$$=-i\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}(\vec{x}-\vec{y})}(\delta^{ij}-\hat{k}^i\hat{k}^j)$$

$$=-i\,\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

$$H=\int d^3x\,\mathcal{E}=\int d^3x\,T^{00}=\int d^3x\,\frac{1}{2}(\vec{E}^2+\vec{B}^2)$$

$$=\int d^3x\,\frac{1}{2}[\vec{\pi}^2+(\nabla\times\vec{A})^2]=\int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=\pm1}|\vec{k}|\,a^{\lambda\dagger}_{\hat{k}}\,a^{\lambda}_{\hat{k}}+\infty\text{个零点能}$$

$$\vec{P}=\int d^3x\,(\vec{E}\times\vec{B})=\int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=\pm1}\vec{k}\,a^{\lambda\dagger}_{\hat{k}}\,a^{\lambda}_{\hat{k}}$$

考虑因果性：

$$[A^i(x),A^j(y)]=\langle0|[\quad]|0\rangle$$

$$=\langle0|A^iA^j|0\rangle-\langle0|A^jA^i|0\rangle$$

第一项：$\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}(\delta^{ij}-\hat{k}^i\hat{k}^j)\,e^{-ik(x-y)}\big|_{k^0=|\vec{k}|}$

$$=\left(\delta^{ij}-\frac{\partial^i\partial^j}{\nabla^2}\right)\underbrace{\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}e^{-ik(x-y)}}_{\text{对massless场D}}$$

类似A⇒

$[A^i(x), A^j(y)] = (\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})(D(x-y) - D(y-x))$

$= (\delta_{ij}, \frac{\partial_i\partial_j}{\nabla^2})[\phi_x, \phi_y] = 0$ 满足

费曼传播子

$D_F^{\mu\nu}(x-y) = \langle 0|T A_x^\mu A_y^\nu|0\rangle$

$= D_F^{ij}(x-y) = \theta(x^0-y^0)(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})D(x-y) + \theta(y^0-x^0)(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})D(y-x)$

$= (\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\langle 0|T\phi(x)\phi(y)|0\rangle \to 0$ 质量K-G场的费曼传播子

$= (\quad) \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2+i\epsilon}$

$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i(\delta^{ij} - \frac{k^ik^j}{k^2})}{k^2+i\epsilon} = \tilde{D}^{ij}(k)$ 动量空间的费曼传播子

则将 $D_F^{\mu\nu}(x-y)$ 写成协变i

$= \int \frac{d^4k}{(2\pi)^4} \frac{iP^{\mu\nu}(\vec{k})}{k^2+i\epsilon}$

对 $P^{\mu\nu}$ $\begin{cases} P_{ij}(\vec{k}) = \delta^{ij} - \frac{k^ik^j}{|\vec{k}|^2} \\ P_{00}(\vec{k}) = P_{0i}(\vec{k}) = P_{i0}(\vec{k}) = 0 \end{cases}$ ⇒涉及3度指标⇒非L协变

库伦规范带来的后果

A^μ 4个变量 → 库伦规范→去掉矢势自由度 ⇒ 4-1-1=2

⇒ A^0 不是一个动力变量

恢复L协变：

物理过程⇒相互作用

加上一个外源 $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - j^\mu A_\mu$ 其中 $j^\mu = (\rho, \vec{j})$

$A^0(t,x) = \frac{1}{4\pi}\int d^3y \frac{j^0(t,\vec{y})}{|\vec{x}-\vec{y}|}$ → 库仑势 物质场

$\vec{E} = -\dot{\vec{A}} - \nabla A^0$

↳ $\vec{E}_\perp$ ↳ $\vec{E}_\parallel$

$\nabla\cdot\vec{E}=0$ 横场部分 $\nabla\times\vec{E}=0$ 纵场部分

其实 $-\pi\dot{\varphi} = \frac{1}{2}(\vec{E}^2-\vec{B}^2) = \frac{1}{2}(\vec{E}_\parallel^2+\vec{E}_\perp^2-\vec{B}^2) + \vec{E}_\parallel\cdot\vec{E}_\perp$

$\vec{E}_\parallel\cdot\vec{E}_\perp = -\nabla A^0\cdot\vec{E}_\perp = -\underbrace{\nabla(A^0\cdot\vec{E}_\perp)}_{\text{surface}} + A^0\cdot\nabla\vec{E}_\perp = 0$

则 $\mathcal{L} = \frac{1}{2}(\vec{E}_\perp^2+\vec{E}_\parallel^2-\vec{B}^2)$

$\vec{\pi}_\perp = -\frac{\partial\mathcal{L}}{\partial\dot{\vec{A}}} = \vec{E}_\perp$ (代回入 A^0)

$\mathcal{H} = \mathcal{E} = \pi_\mu\dot{A}^\mu - \mathcal{L} = -\pi_\perp\dot{\vec{A}} - \mathcal{L}$ (只有空间部分贡献)

$= \vec{E}_\perp^2 - \frac{1}{2}(\vec{E}_\parallel^2+\vec{E}_\perp^2-\vec{B}^2) + j^0A^0 - \vec{j}\cdot\vec{A}$

$= \underbrace{\frac{1}{2}\vec{E}_\perp^2 + \frac{1}{2}(\nabla\times\vec{A})^2}_{\text{自由}\mathcal{H}^0} - \underbrace{\frac{1}{2}\vec{E}_\parallel^2 + j^0A^0 - \vec{j}\cdot\vec{A}}_{\text{相互}}$

应用微扰论

$\mathcal{H} = \mathcal{H}^0 + V$

$\mathcal{H}_0 = \int d^3x \underbrace{\left[\frac{1}{2}\vec{\pi}_\perp^2 + \frac{1}{2}(\nabla\times\vec{A})^2\right]}_{\text{光子场的自由部分 } H_{\text{Dirac}}} + H_{\text{matter}}$ → 物质场~

$V = \int d^3x \left(-\frac{1}{2}\vec{E}_\parallel^2 + j^0A^0 - \vec{j}\cdot\vec{A}\right)$

$= \int d^3x \underbrace{\left(\frac{1}{2}j^0A^0 - \vec{j}\cdot\vec{A}\right)}_{V_{\text{col}}}$ 相互作用

$-\frac{1}{2}\vec{E}_\parallel^2 = -\frac{1}{2}(\nabla A^0)\cdot(\nabla A^0)$

$= \text{表面项} + \frac{1}{2}A^0\nabla^2A^0$

$= -j\cdot A^0$

$\frac{1}{2}j(x)A(x) = \frac{1}{2}\int d^3x\,d^3y$

相互作用绘景:

算符演化 $V_I(t) = e^{iH_0t}Ve^{-iH_0t}$

$A^{\mu}_{2}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|\vec{k}|} \sum_{\lambda=\pm 1} \left(\varepsilon^{(\lambda)\mu}(k) a^{\dagger}_{k} e^{-ikx} + h.c. \right)$ | $J_0(t,\vec{x}) \frac{1}{4\pi|\vec{x}-\vec{y}|} J_0(t,\vec{y})$ 库仑势

$A^{0}_{2} = 0$ (相互作用绘景下 $A^{0}_{2} = 0$)

例



$H = \int d^3x \left[\frac{1}{2}\dot{A}^{0}\dot{A}^{0} - \frac{1}{2}\vec{\nabla}A^{0} \cdot \vec{\nabla}A^{0} \right] = \int d^3x \left(j^{\mu} A_{\mu} + V_{coul} \right)$ $\hat{=} e^2$ $\overbrace{J(y) A^{\nu}(y)}$

$\hat{S} = T\exp\left\{ -i \int_{-\infty}^{+\infty} dt\, V_{I}(t) \right\} = 1 - i \int_{-\infty}^{+\infty} dt\, V_{coul} + \frac{(-i)^2}{2!} \iint dx^0 dy^0\, T\left(j_{\mu}(x) A^{\mu}(y) \right)$

$=$