A Complete Solution to Problems in "An Introduction to Quantum Field Theory" by Peskin and Schroeder

Zhong-Zhi Xianyu

Harvard University

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Preface

In this note I provide solutions to all problems and final projects in the book *An Introduction to Quantum Field Theory* by M. E. Peskin and D. V. Schroeder [1], which I worked out and typed into TEX during the first two years of my PhD study at Tsinghua University. I once posted a draft version of them on my personal webpage using a server provided by Tsinghua, which was however closed unfortunately after I graduated. Since then I received quite a number of emails asking for the solutions, so I decided to put them on arXiv.*

Nothing much has been updated in this note compared with the previous draft due to the lack of time, except for some editorial work, as well as a few newly added references. In particular, I don't have enough time to proofread and therefore I cannot guarantee the correctness of them, though I expect that most of them are correct. With that said, any feedback via email† about errors, either physical or typographical, is much appreciated.

I would not claim any novelty or originality of this note, since almost all of problems in the book belong to standard material of quantum field theory. Occasionally, I learned the answer to a problem or the strategy for solving it before I started to work it out. But still, I believe that the problem set in the book will always remain a treasure to any beginner of this subject, and I feel it worthy to write up the solutions.

The contraction macro provided by the authors of the book‡ has been used in this note.

I would like to express my gratitude to Prof. Qing Wang and Prof. Hong-Jian He for their wonderful courses of quantum field theory and their great help in my early days of learning this subject. I would also like to thank Prof. Michael Peskin in particular, for his generous permission and kind encouragement to letting me publish this note.

Comments on notations. All notations and conventions are the same with the book. The book will be cited in the main text as "P&S" for short. The $+i\epsilon$ prescription for Feynman propagators is always assumed and is usually hidden.

*The submission, however, was rejected by one of arXiv volunteer moderators based on the reason that "arXiv does not allow submissions containing solutions to problems in physics textbooks", and that "(the) moderators consider that this type of submissions are harmful for students and instructors". Insofar as I can see, however, the solution can only do harm to those who are willing to do harm to themselves.

†xianyuzhongzhi@gmail.com

‡http://physics.weber.edu/schroeder/qftbook.html

Contents

Preface iii

2 The Klein-Gordon Field 1
- 2.1 Classical electromagnetism . . . 1
- 2.2 The complex scalar field . . . 2
- 2.3 The spacelike correlation function . . . 6

3 The Dirac Field 7
- 3.1 Lorentz group . . . 7
- 3.2 The Gordon identity . . . 9
- 3.3 The spinor products . . . 9
- 3.4 Majorana fermions . . . 11
- 3.5 Supersymmetry . . . 15
- 3.6 Fierz transformations . . . 18
- 3.7 The discrete symmetries P, C and T . . . 19
- 3.8 Bound states . . . 21

4 Interacting Fields and Feynman Diagrams 23
- 4.1 Scalar field with a classical source . . . 23
- 4.2 Decay of a scalar particle . . . 25
- 4.3 Linear sigma model . . . 26
- 4.4 Rutherford scattering . . . 28

5 Elementary Processes of Quantum Electrodynamics 33
- 5.1 Coulomb scattering . . . 33
- 5.2 Bhabha scattering . . . 35
- 5.3 The spinor products (2) . . . 36
- 5.4 Positronium lifetimes . . . 39
- 5.5 Physics of a massive vector boson . . . 45
- 5.6 The spinor products (3) . . . 47

6 Radiative Corrections: Introduction 51
- 6.1 Rosenbluth formula . . . 51

6.2 Equivalent photon approximation . . . 53
6.3 Exotic contributions to $g-2$ . . . 55

7 Radiative Corrections: Some Formal Developments **59**
7.1 Optical theorem in ϕ^4 theory . . . 59
7.2 Alternative regulators in QED . . . 60
7.3 Radiative corrections in QED with Yukawa interaction . . . 63

Final Project I. Radiation of Gluon Jets **67**

9 Functional Methods **73**
9.1 Scalar QED . . . 73
9.2 Statistical field theory . . . 74

10 Systematics of Renormalization **79**
10.1 One-Loop structure of QED . . . 79
10.2 Renormalization of Yukawa theory . . . 80
10.3 Field-strength renormalization in ϕ^4 theory . . . 82
10.4 Asymptotic behavior of diagrams in ϕ^4 theory . . . 83

11 Renormalization and Symmetry **87**
11.1 Spin-wave theory . . . 87
11.2 A zeroth-order natural relation . . . 88
11.3 The Gross-Neveu model . . . 91

12 The Renormalization Group **93**
12.1 Beta Function in Yukawa Theory . . . 93
12.2 Beta Function of the Gross-Neveu Model . . . 93
12.3 Asymptotic Symmetry . . . 95

13 Critical Exponents and Scalar Field Theory **99**
13.1 Correlation-to-scaling exponent . . . 99
13.2 The exponent η . . . 100
13.3 The CP^N model . . . 100

Final Project II. The Coleman-Weinberg Potential **103**

15 Non-Abelian Gauge Invariance **113**
15.1 Brute-force computations in $SU(3)$ . . . 113
15.2 Adjoint representation of $SU(2)$ . . . 113
15.3 Coulomb potential . . . 114
15.4 Scalar propagator in a gauge theory . . . 115
15.5 Casimir operator computations . . . 117

16 Quantization of Non-Abelian Gauge Theories **121**
16.1 Arnowitt-Fickler gauge . . . 121
16.2 Scalar field with non-Abelian charge . . . 122
16.3 Counterterm relations . . . 123

17 Quantum Chromodynamics **133**
17.1 Two-Loop renormalization group relations . . . 133
17.2 A Direct test of the spin of the gluon . . . 134
17.3 Quark-gluon and gluon-gluon scattering . . . 135
17.4 The gluon splitting function . . . 139
17.5 Photoproduction of heavy quarks . . . 141
17.6 Behavior of parton distribution functions at small x . . . 142

18 Operator Products and Effective Vertices **145**
18.1 Matrix element for proton decay . . . 145
18.2 Parity-violating deep inelastic form factor . . . 147
18.3 Anomalous dimensions of gluon twist-2 operators . . . 151
18.4 Deep inelastic scattering from a photon . . . 156

19 Perturbation Theory Anomalies **159**
19.1 Fermion number nonconservation in parallel E and B fields . . . 159
19.2 Weak decay of the pion . . . 161
19.3 Computation of anomaly coefficients . . . 162
19.4 Large fermion mass limits . . . 164

20 Gauge Theories with Spontaneous Symmetry Breaking **167**
20.1 Spontaneous breaking of $SU(5)$ . . . 167
20.2 Decay modes of the W and Z bosons . . . 168
20.3 e^+e^- →hadrons with photon-Z^0 interference . . . 170
20.4 Neutral-current deep inelastic scattering . . . 173
20.5 A model with two Higgs fields . . . 174

21 Quantization of Spontaneously Broken Gauge Theories **177**
21.1 Weak-interaction contributions to the muon $g-2$ . . . 177
21.2 Complete analysis of $e^+e^- \to W^+W^-$ . . . 180
21.3 Cross section for $d\bar{u} \to W^-\gamma$ . . . 183
21.4 Dependence of radiative corrections on the Higgs boson mass . . . 185

Final Project III. Decays of the Higgs Boson **189**

Bibliography **201**

Chapter 2

The Klein-Gordon Field

2.1 Classical electromagnetism

In this problem we derive the field equations and energy-momentum tensor from the following action of classical electrodynamics,

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}, \qquad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.1}$$

(a) Maxwell's equations To take variation of the classical action with respect to the field A_μ, we note,

$$\frac{\delta F_{\mu\nu}}{\delta(\partial_\lambda A_\kappa)} = \delta^\lambda_\mu\delta^\kappa_\nu - \delta^\lambda_\nu\delta^\kappa_\mu, \qquad \frac{\delta F_{\mu\nu}}{\delta A_\lambda} = 0. \tag{2.2}$$

Then from the first equality we get:

$$\frac{\delta}{\delta(\partial_\lambda A_\kappa)}\big(F_{\mu\nu}F^{\mu\nu}\big) = 4F^{\lambda\kappa}. \tag{2.3}$$

Now substitute this into Euler-Lagrange equation, we have,

$$0 = \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\Big) - \frac{\delta\mathcal{L}}{\delta A_\nu} = -\partial_\mu F^{\mu\nu}. \tag{2.4}$$

This is sometimes called the "second pair" of Maxwell's equations. The so-called "first pair" follows directly from the definition of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and reads

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\mu\lambda} = 0. \tag{2.5}$$

The familiar electric and magnetic field strengths can be written as $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$, respectively. From this we deduce the Maxwell's equations in terms of E^i and B^i:

$$\partial^i E^i = 0, \qquad \epsilon^{ijk}\partial^j B^k - \partial^0 E^i = 0, \qquad \epsilon^{ijk}\partial^j E^k = 0, \qquad \partial^i B^i = 0. \tag{2.6}$$

(b) The energy-momentum tensor The energy-momentum tensor can be defined to be the Nöther current of the space-time translational symmetry. Under space-time translation the vector A_μ transforms as,

$$\delta^\mu A^\nu = \partial^\mu A^\nu. \tag{2.7}$$

Thus

$$\tilde{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa}. \tag{2.8}$$

Obviously, this tensor is not symmetric. We can add an additional term $\partial_\lambda K^{\lambda\mu\nu}$ to $\tilde{T}^{\mu\nu}$ with $K^{\lambda\mu\nu}$ antisymmetric with its first two indices. It's easy to see that this term does not affect the conservation of $\tilde{T}^{\mu\nu}$. So if we choose $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, then,

$$T^{\mu\nu} = \tilde{T}^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} = F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa}. \tag{2.9}$$

Now this tensor is symmetric and is sometimes called the Belinfante tensor in literature. We can also rewrite it in terms of E^i and B^i,

$$T^{00} = \frac{1}{2}(E^iE^i + B^iB^i), \quad T^{i0} = T^{0i} = \epsilon^{ijk}E^jB^k, \text{ etc.} \tag{2.10}$$

2.2 The complex scalar field

The Lagrangian is given by,

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{2.11}$$

(a) The conjugate momenta of ϕ and ϕ^*:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*, \quad \tilde{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi} = \pi^*. \tag{2.12}$$

The canonical commutation relations:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}), \tag{2.13}$$

The rest of commutators are all zero.

The Hamiltonian:

$$H = \int \mathrm{d}^3x\left(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}\right) = \int \mathrm{d}^3x\left(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi\right). \tag{2.14}$$

(b) Now we Fourier transform the field ϕ as:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip\cdot x} + b^\dagger_\mathbf{p}e^{ip\cdot x}\right), \tag{2.15}$$

thus:

$$\phi^*(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(b_\mathbf{p}e^{-ip\cdot x} + a^\dagger_\mathbf{p}e^{ip\cdot x}\right). \tag{2.16}$$

Substitute the mode expansion into the Hamiltonian:

$$
\begin{aligned}
H &= \int \mathrm{d}^3x \left(\dot{\phi}^*\dot{\phi} + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right) \\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \\
&\quad \times \Big[E_{\mathbf{p}}E_{\mathbf{q}}\Big(a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x} - b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big) \\
&\quad + \mathbf{p}\cdot\mathbf{q}\Big(a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x} - b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big) \\
&\quad + m^2\Big(a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x} + b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x} + b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big)\Big] \\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \\
&\quad \times \Big[(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} + m^2)\Big(a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{\mathrm{i}(p-q)\cdot x} + b_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{-\mathrm{i}(p-q)\cdot x}\Big) \\
&\quad - (E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} - m^2)\Big(b_{\mathbf{q}}a_{\mathbf{q}}e^{-\mathrm{i}(p+q)\cdot x} + a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{\mathrm{i}(p+q)\cdot x}\Big)\Big] \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \\
&\quad \times \Big[(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} + m^2)\Big(a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t} + b_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{-\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&\quad - (E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} - m^2)\Big(b_{\mathbf{q}}a_{\mathbf{q}}e^{-\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}})t} + a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big] \\
&= \int \mathrm{d}^3x \, \frac{E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2}{2E_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right) \\
&= \int \mathrm{d}^3x \, E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + [b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}]\right),
\end{aligned}
\tag{2.17}
$$

where we have used the mass-shell condition $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. Note that the last term contributes an infinite constant, which can be interpreted as the vacuum energy and can be dropped, for instance, by the prescription of normal ordering. Then we get a finite Hamiltonian,

$$
H = \int \mathrm{d}^3x \, E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right),
\tag{2.18}
$$

Hence we get two sets of particles with the same mass m.

(c) The theory is invariant under the global transformation: $\phi \to e^{i\theta}\phi$, $\phi^* \to e^{-i\theta}\phi^*$. The corresponding conserved charge is:

$$
Q = i\int \mathrm{d}^3x \left(\phi^*\dot{\phi} - \dot{\phi}^*\phi\right).
\tag{2.19}
$$

Rewrite this in terms of the creation and annihilation operators:

$$
Q = \mathrm{i}\int \mathrm{d}^3x \left(\phi^*\dot{\phi} - \dot{\phi}^*\phi\right)
$$

$$
\begin{aligned}
&= \mathrm{i}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\Big[\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}+a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x}\Big)\frac{\partial}{\partial t}\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}+b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big)\\
&\quad -\frac{\partial}{\partial t}\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}+a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x}\Big)\cdot\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}+b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big)\Big]\\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\Big[E_{\mathbf{q}}\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}+a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}-b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big)\\
&\quad -E_{\mathbf{p}}\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}-a^\dagger_{\mathbf{p}}e^{\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}+b^\dagger_{\mathbf{q}}e^{\mathrm{i}q\cdot x}\Big)\Big]\\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\Big[(E_{\mathbf{q}}-E_{\mathbf{p}})\Big(b_{\mathbf{p}}a_{\mathbf{q}}e^{-\mathrm{i}(p+q)\cdot x}-a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{\mathrm{i}(p+q)\cdot x}\Big)\\
&\quad +(E_{\mathbf{q}}+E_{\mathbf{p}})\Big(a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{\mathrm{i}(p-q)\cdot x}-b_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{-\mathrm{i}(p-q)\cdot x}\Big)\Big]\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\\
&\quad \times\Big[(E_{\mathbf{q}}-E_{\mathbf{p}})\Big(b_{\mathbf{p}}a_{\mathbf{q}}e^{-\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}})t}-a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}}t)}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})\\
&\quad +(E_{\mathbf{q}}+E_{\mathbf{p}})\Big(a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t}-b_{\mathbf{p}}b^\dagger_{\mathbf{q}}e^{-\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big]\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\cdot 2E_{\mathbf{p}}(a^\dagger_{\mathbf{p}}a_{\mathbf{p}}-b_{\mathbf{p}}b^\dagger_{\mathbf{p}})\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\big(a^\dagger_{\mathbf{p}}a_{\mathbf{p}}-b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\big),
\end{aligned}
\tag{2.20}
$$

where the last equal sign holds up to an infinitely large constant term, as we did when calculating the Hamiltonian in (b). Then the commutators follow straightforwardly:

$$
[Q,a^\dagger]=a^\dagger,\qquad [Q,b^\dagger]=-b^\dagger. \tag{2.21}
$$

We see that the particle a carries one unit of positive charge, and b carries one unit of negative charge.

(d) Now we consider the case with two complex scalars of same mass. In this case the Lagrangian is given by

$$
\mathcal{L}=\partial_\mu\Phi_i^\dagger\partial^\mu\Phi_i-m^2\Phi_i^\dagger\Phi_i, \tag{2.22}
$$

where Φ_i with $i=1,2$ is a two-component complex scalar. Then it is straightforward to see that the Lagrangian is invariant under the $U(2)$ transformation $\Phi_i\to U_{ij}\Phi_j$ with U_{ij} a matrix in fundamental representation of $U(2)$ group. The $U(2)$ group, locally isomorphic to $SU(2)\times U(1)$, is generated by 4 independent generators 1 and $\frac{1}{2}\tau^a$, with τ^a Pauli matrices. Then 4 independent Nöther currents are associated, which are given by,

$$
j_\mu=-\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_i)}\Delta\Phi_i-\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_i^*)}\Delta\Phi_i^*=-(\partial_\mu\Phi_i^*)(\mathrm{i}\Phi_i)-(\partial_\mu\Phi_i)(-\mathrm{i}\Phi_i^*),
$$

$$j^a_\mu = -\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_i)}\Delta^a\Phi_i - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi^*_i)}\Delta^a\Phi^*_i = -\frac{\mathrm{i}}{2}\Big[(\partial_\mu\Phi^*_i)\tau_{ij}\Phi_j - (\partial_\mu\Phi_i)\tau_{ij}\Phi^*_j\Big]. \tag{2.23}$$

The overall sign is chosen such that the particle carry positive charge, as will be seen in the following. Then the corresponding Nöther charges are given by,

$$\begin{aligned} Q =& -\mathrm{i}\int \mathrm{d}^3x\,\big(\dot\Phi^*_i\Phi_i - \Phi^*_i\dot\Phi_i\big), \\ Q^a =& -\frac{\mathrm{i}}{2}\int \mathrm{d}^3x\,\big[\dot\Phi^*_i(\tau^a)_{ij}\Phi_j - \Phi^*_i(\tau^a)_{ij}\dot\Phi_j\big]. \end{aligned} \tag{2.24}$$

Repeating the derivations above, we can also rewrite these charges in terms of creation and annihilation operators, as,

$$\begin{aligned} Q =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(a^\dagger_{i\mathbf{p}}a_{i\mathbf{p}} - b^\dagger_{i\mathbf{p}}b_{i\mathbf{p}}\Big), \\ Q^a =& \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(a^\dagger_{i\mathbf{p}}\tau^a_{ij}a_{i\mathbf{p}} - b^\dagger_{j\mathbf{p}}\tau^a_{ij}b_{j\mathbf{p}}\Big). \end{aligned} \tag{2.25}$$

The generalization to N-component complex scalar is straightforward. In this case we only need to replace the generators $\tau^a/2$ of $SU(2)$ group to the generators t^a in the fundamental representation of $SU(N)$ group with commutation relation $[t^a, t^b] = \mathrm{i}f^{abc}t^c$.

Then we are ready to calculate the commutators among all these Nöther charges and the Hamiltonian. Firstly we show that all charges of the $U(N)$ group commute with the Hamiltonian. For the $U(1)$ generator, we have

$$\begin{aligned} [Q,H] =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,E_{\mathbf{q}}\Big[\big(a^\dagger_{i\mathbf{p}}a_{i\mathbf{p}} - b^\dagger_{i\mathbf{p}}b_{i\mathbf{p}}\big), \big(a^\dagger_{j\mathbf{q}}a_{j\mathbf{q}} + b^\dagger_{j\mathbf{q}}b_{j\mathbf{q}}\big)\Big] \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,E_{\mathbf{q}}\Big(a^\dagger_{i\mathbf{p}}[a_{i\mathbf{p}}, a^\dagger_{j\mathbf{q}}]a_{j\mathbf{q}} + a^\dagger_{j\mathbf{q}}[a^\dagger_{i\mathbf{p}}, a_{j\mathbf{q}}]a_{i\mathbf{p}} \;+\; (a\to b)\Big) \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,E_{\mathbf{q}}\Big(a^\dagger_{i\mathbf{p}}a_{i\mathbf{q}} - a^\dagger_{i\mathbf{q}}a_{i\mathbf{p}} \;+\; (a\to b)\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ =& \,0. \end{aligned} \tag{2.26}$$

Similar calculation gives $[Q^a, H] = 0$. Then we consider the commutation among internal $U(N)$ charges:

$$\begin{aligned} [Q^a,Q^b] =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,\Big[\big(a^\dagger_{i\mathbf{p}}t^a_{ij}a_{j\mathbf{p}} - b^\dagger_{i\mathbf{p}}t^a_{ij}b_{j\mathbf{p}}\big), \big(a^\dagger_{k\mathbf{q}}t^b_{k\ell}a_{\ell\mathbf{q}} - b^\dagger_{k\mathbf{q}}t^b_{k\ell}b_{\ell\mathbf{q}}\big)\Big] \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,\Big(a^\dagger_{i\mathbf{p}}t^a_{ij}t^b_{j\ell}a_{\ell\mathbf{q}} - a^\dagger_{k\mathbf{q}}t^b_{k\ell}t^a_{\ell j}a_{j\mathbf{p}} \;+\; (a\to b)\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ =& \,\mathrm{i}f^{abc}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(a^\dagger_{i\mathbf{p}}t^c_{ij}a_{j\mathbf{p}} - b^\dagger_{i\mathbf{p}}t^c_{ij}b_{j\mathbf{p}}\Big) \\ =& \,\mathrm{i}f^{abc}Q^c, \end{aligned} \tag{2.27}$$

and similarly, $[Q, Q] = [Q^a, Q] = 0$.

2.3 The spacelike correlation function

We evaluate the correlation function of a scalar field at two points,

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle, \tag{2.28}$$

with $x-y$ being spacelike. Since any spacelike interval $x-y$ can be transformed to a form such that $x^0 - y^0 = 0$, thus we will simply take:

$$x^0 - y^0 = 0, \qquad \text{and} \qquad |\mathbf{x}-\mathbf{y}|^2 = r^2 > 0. \tag{2.29}$$

Now:

$$\begin{aligned} D(x-y) =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\sqrt{m^2+p^2}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ =& \frac{1}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\cos\theta\int_0^{\infty}\mathrm{d}p\frac{p^2}{2\sqrt{m^2+p^2}}e^{ipr\cos\theta} \\ =& \frac{-\mathrm{i}}{2(2\pi)^2 r}\int_{-\infty}^{\infty}\mathrm{d}p\frac{pe^{ipr}}{\sqrt{m^2+p^2}} \end{aligned} \tag{2.30}$$

Now we make the path deformation on p-complex plane, as is shown in Figure 2.3 of P&S. Then the integral becomes,

$$D(x-y) = \frac{1}{4\pi^2 r}\int_m^{\infty}\mathrm{d}\rho\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} = \frac{m}{4\pi^2 r}\mathrm{K}_1(mr). \tag{2.31}$$

Chapter 3

The Dirac Field

3.1 Lorentz group

The generators of Lorentz group satisfy the following commutation relation,

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathrm{i}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{3.1}$$

(a) Let us redefine the generators as $L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ (All Latin indices denote spatial components), with L^i generate rotations, and K^i generate boosts. The commutators of them can be derived straightforwardly to be,

$$[L^i, L^j] = \mathrm{i}\epsilon^{ijk}L^k, \qquad [K^i, K^j] = -\mathrm{i}\epsilon^{ijk}L^k. \tag{3.2}$$

If we further define $J^i_\pm = \frac{1}{2}(L^i \pm \mathrm{i}K^i)$, then the commutators become,

$$[J^i_\pm, J^j_\pm] = \mathrm{i}\epsilon^{ijk}J^k_\pm, \qquad [J^i_+, J^j_-] = 0. \tag{3.3}$$

Thus we see that the algebra of the Lorentz group is a direct sum of two identical algebra $\mathfrak{su}(2)$.

(b) It follows that we can classify the finite dimensional representations of the Lorentz group by a pair (j_+, j_-), where $j_\pm = 0, 1/2, 1, 3/2, 2, \cdots$ are labels of irreducible representations of $SU(2)$.

We study two specific cases.

1. $(\frac{1}{2}, 0)$. Following the definition, we have J^i_+ represented by $\frac{1}{2}\sigma^i$ and J^i_- represented by 0. This implies

$$L^i = (J^i_+ + J^i_-) = \tfrac{1}{2}\sigma^i, \qquad K^i = -\mathrm{i}(J^i_+ - J^i_-) = -\tfrac{\mathrm{i}}{2}\sigma^i. \tag{3.4}$$

Hence a field ψ under this representation transforms as:

$$\psi \to e^{-\mathrm{i}\theta^i\sigma^i/2 - \eta^i\sigma^i/2}\psi. \tag{3.5}$$

2. $(\frac{1}{2},0)$. In this case, $J_+^i \to 0$, $J_-^i \to \frac{1}{2}\sigma^i$. Then

$$L^i = (J_+^i + J_-^i) = \tfrac{1}{2}\sigma^i, \qquad K^i = -\mathrm{i}(J_+^i - J_-^i) = \tfrac{\mathrm{i}}{2}\sigma^i. \tag{3.6}$$

Hence a field ψ under this representation transforms as:

$$\psi \to e^{-\mathrm{i}\theta^i\sigma^i/2+\eta^i\sigma^i/2}\psi. \tag{3.7}$$

We see that a field under the representation $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are precisely the left-handed spinor ψ_L and right-handed spinor ψ_R, respectively.

(c) Let us consider the case of $(\frac{1}{2},\frac{1}{2})$. To put the field associated with this representation into a familiar form, we note that a left-handed spinor can also be rewritten as row, which transforms under the Lorentz transformation as:

$$\psi_L^T\sigma^2 \;\to\; \psi_L^T\sigma^2\big(1 + \tfrac{\mathrm{i}}{2}\theta^i\sigma^i + \tfrac{1}{2}\eta^i\sigma^i\big). \tag{3.8}$$

Then the field under the representation $(\frac{1}{2},\frac{1}{2})$ can be written as a tensor with spinor indices:

$$\psi_R\psi_L^T\sigma^2 \equiv V^\mu\bar{\sigma}_\mu = \begin{pmatrix} V^0+V^3 & V^1-\mathrm{i}V^2 \\ V^1+\mathrm{i}V^2 & V^0-V^3 \end{pmatrix}. \tag{3.9}$$

In what follows we will prove that V^μ is in fact a Lorentz vector.

A quantity V^μ is called a Lorentz vector, if it satisfies the following transformation law:

$$V^\mu \to \Lambda^\mu{}_\nu V^\nu, \tag{3.10}$$

where $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu$ in its infinitesimal form. We further note that:

$$(J^{\rho\sigma})_{\mu\nu} = \mathrm{i}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu). \tag{3.11}$$

and also, $\omega_{ij} = \epsilon_{ijk}\theta^k$, $\omega_{0i} = -\omega_{i0} = \eta^i$, then the combination $V^\mu\bar{\sigma}_\mu = V^i\sigma^i + V^0$ transforms according to

$$\begin{aligned} V^i\sigma^i \to& \Big(\delta^i_j - \frac{\mathrm{i}}{2}\omega_{mn}(J^{mn})^i{}_j\Big)V^j\sigma^i + \Big(-\frac{\mathrm{i}}{2}\omega_{0n}(J^{0n})^i{}_0 - \frac{\mathrm{i}}{2}\omega_{n0}(J^{n0})^0{}_i\Big)V^0\sigma^i \\ =& \Big(\delta^i_j - \tfrac{\mathrm{i}}{2}\epsilon_{mnk}\theta^k(-\mathrm{i})(\delta^m_i\delta^n_j - \delta^m_j\delta^n_i)\Big)V^j\sigma^i + \big(-\mathrm{i}\eta^i(-\mathrm{i})(-\delta^n_i)\big)V^0\sigma^i \\ =& V^i\sigma^i - \epsilon^{ijk}V^i\theta^j\sigma^k + V^0\eta^i\sigma^i, \end{aligned}$$

$$\begin{aligned} V^0 \to& V^0 + \Big(-\frac{\mathrm{i}}{2}\omega_{0n}(J^{0n})_{0i} - \frac{\mathrm{i}}{2}\omega_{n0}(J^{n0})_{0i}\Big)V^i \\ =& V^0 + \big(-\mathrm{i}\eta^i(i\delta^n_i)\big)V^i = V^0 + \eta^iV^i. \end{aligned}$$

In total, we have

$$V^\mu\bar{\sigma}_\mu \to \big(\sigma^i - \epsilon^{ijk}\theta^j\sigma^k + \eta^i\big)V^i + (1+\eta^i\sigma^i)V^0. \tag{3.12}$$

If we can reach the same conclusion by treating the combination $V^\mu \bar{\sigma}_\mu$ a matrix transforming under the representation $(\frac{1}{2}, \frac{1}{2})$, then our original statement will be proved. In fact:

$$\begin{aligned} V^\mu \bar{\sigma}_\mu &\to \Big(1 - \frac{\mathrm{i}}{2}\theta^j\sigma^j + \frac{1}{2}\eta^j\sigma^j\Big) V^\mu \sigma_\mu \Big(1 + \frac{\mathrm{i}}{2}\theta^j\sigma^j + \frac{1}{2}\eta^j\sigma^j\Big) \\ &= \Big(\sigma^i + \frac{\mathrm{i}}{2}\theta^j[\sigma^i, \sigma^j] + \frac{1}{2}\eta^j\{\sigma^i, \sigma^j\}\Big) V^i + (1 + \eta^i\sigma^i)V^0 \\ &= \big(\sigma^i - \epsilon^{ijk}\theta^j\sigma^k + \eta^i\big)V^i + (1 + \eta^i\sigma^i)V^0, \end{aligned} \tag{3.13}$$

as expected. Hence we proved that V^μ is a Lorentz vector.

3.2 The Gordon identity

In this problem we derive the Gordon identity,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\Big(\frac{p'^\mu + p^\mu}{2m} + \frac{\mathrm{i}\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m}\Big)u(p). \tag{3.14}$$

Let us start from the right hand side,

$$\begin{aligned} &\bar{u}(p')\Big(\frac{p'^\mu + p^\mu}{2m} + \frac{\mathrm{i}\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m}\Big)u(p) \\ &= \frac{1}{2m}\bar{u}(p')\Big((p'^\mu + p^\mu) + \mathrm{i}\sigma^{\mu\nu}(p'_\nu - p_\nu)\Big)u(p) \\ &= \frac{1}{2m}\bar{u}(p')\Big(\eta^{\mu\nu}(p'_\nu + p_\nu) - \frac{1}{2}[\gamma^\mu, \gamma^\nu](p'_\nu - p_\nu)\Big)u(p) \\ &= \frac{1}{2m}\bar{u}(p')\Big(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}(p'_\nu + p_\nu) - \frac{1}{2}[\gamma^\mu, \gamma^\nu](p'_\nu - p_\nu)\Big)u(p) \\ &= \frac{1}{2m}\bar{u}(p')\Big(\not{p}'\gamma^\mu + \gamma^\mu\not{p}\Big)u(p) = \bar{u}(p')\gamma^\mu u(p), \end{aligned}$$

where we have used the commutator and anti-commutators of gamma matrices, as well as the Dirac equation.

3.3 The spinor products

In this problem, together with the Problems 5.3 and 5.6, we will develop a formalism that can be used to calculating scattering amplitudes involving massless fermions or vector particles. This method can profoundly simplify the calculations, especially in the calculations of QCD. Here we will derive the basic fact that the spinor products can be treated as the square root of the inner product of lightlike Lorentz vectors. Then, in Problem 5.3 and 5.6, this relation will be put in use in calculating the amplitudes with external spinors and external photons, respectively.

To begin with, let k_0^μ and k_1^μ be fixed four-vectors satisfying $k_0^2 = 0$, $k_1^2 = -1$ and $k_0 \cdot k_1 = 0$. With these two reference momenta, we define the following spinors:

1. Let u_{L0} be left-handed spinor with momentum k_0;

2. Let $u_{R0} = \not{k}_1 u_{L0}$;

3. For any lightlike momentum p ($p^2 = 0$), define:

$$u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_{R0}, \qquad u_R(p) = \frac{1}{\sqrt{2p \cdot k}} \not{p} u_{L0}. \tag{3.15}$$

(a) We show that $\not{k}_0 u_{R0} = 0$ and $\not{p} u_L(p) = \not{p} u_R(p) = 0$ for any lightlike p. That is, u_{R0} is a massless spinor with momentum k_0, and $u_L(p)$, $u_R(p)$ are massless spinors with momentum p. This is quite straightforward,

$$\not{k}_0 u_{R0} = \not{k}_0 \not{k}_1 u_{L0} = (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) k_{0\mu} k_{1\nu} u_{L0} = 2k_0 \cdot k_1 u_{L0} - \not{k}_1 \not{k}_0 u_{L0} = 0, \tag{3.16}$$

and, by definition,

$$\not{p} u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} \not{p} u_{R0} = \frac{1}{\sqrt{2p \cdot k_0}} p^2 u_{R0} = 0. \tag{3.17}$$

In the same way, we can show that $\not{p} u_R(p) = 0$.

(b) Now we choose $k_{0\mu} = (E, 0, 0, -E)$ and $k_{1\mu} = (0, 1, 0, 0)$. Then in the Weyl representation, we have:

$$\not{k}_0 u_{L0} = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E \\ 2E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u_{L0} = 0. \tag{3.18}$$

Thus u_{L0} can be chosen to be $(0, \sqrt{2E}, 0, 0)^T$, and:

$$u_{R0} = \not{k}_1 u_{L0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} u_{L0} = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2E} \\ 0 \end{pmatrix}. \tag{3.19}$$

Let $p_\mu = (p_0, p_1, p_2, p_3)$, then:

$$\begin{aligned} u_L(p) &= \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_{R0} \\ &= \frac{1}{\sqrt{2E(p_0 + p_3)}} \begin{pmatrix} 0 & 0 & p_0 + p_3 & p_1 - ip_2 \\ 0 & 0 & p_1 + ip_2 & p_0 - p_3 \\ p_0 - p_3 & -p_1 + ip_2 & 0 & 0 \\ -p_1 - ip_2 & p_0 + p_3 & 0 & 0 \end{pmatrix} u_{R0} \\ &= \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} -(p_0 + p_3) \\ -(p_1 + ip_2) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.20}$$

In the same way, we get:

$$u_R(p) = \frac{1}{\sqrt{p_0+p_3}} \begin{pmatrix} 0 \\ 0 \\ -p_1+ip_2 \\ p_0+p_3 \end{pmatrix}. \tag{3.21}$$

(c) We construct explicitly the spinor product $s(p,q)$ and $t(p,q)$.

$$s(p,q) = \bar{u}_R(p)u_L(q) = \frac{(p_1+ip_2)(q_0+q_3)-(q_1+iq_2)(p_0+p_3)}{\sqrt{(p_0+p_3)(q_0+q_3)}}; \tag{3.22}$$

$$t(p,q) = \bar{u}_L(p)u_R(q) = \frac{(q_1-iq_2)(p_0+p_3)-(p_1-ip_2)(q_0+q_3)}{\sqrt{(p_0+p_3)(q_0+q_3)}}. \tag{3.23}$$

It can be easily seen that $s(p,q) = -s(q,p)$ and $t(p,q) = (s(q,p))^*$.

Now we calculate the quantity $|s(p,q)|^2$:

$$\begin{aligned} |s(p,q)|^2 =& \frac{\big(p_1(q_0+q_3)-q_1(p_0+p_3)\big)^2+\big(p_2(q_0+q_3)-q_2(p_0+p_3)\big)^2}{(p_0+p_3)(q_0+q_3)} \\ =& (p_1^2+p_2^2)\frac{q_0+q_3}{p_0+p_3} + (q_1^2+q_2^2)\frac{p_0+p_3}{q_0+q_3} - 2(p_1q_1+p_2q_2) \\ =& 2(p_0q_0-p_1q_1-p_2q_2-p_3q_3) = 2p\cdot q. \end{aligned} \tag{3.24}$$

Where we have used the lightlike properties $p^2 = q^2 = 0$. Thus we see that the spinor product can be regarded as the square root of the 4-vector dot product for lightlike vectors.

3.4 Majorana fermions

(a) We at first study a two-component massive spinor χ lying in $(\frac{1}{2},0)$ representation, transforming according to $\chi \to U_L(\Lambda)\chi$. It satisfies the following equation of motion:

$$\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi - \mathrm{i}m\sigma^2\chi^* = 0. \tag{3.25}$$

To show this equation is indeed an admissible equation, we need to justify: 1) It is relativistically covariant; 2) It is consistent with the mass-shell condition (namely the Klein-Gordon equation).

To show the condition 1) is satisfied, we note that γ^μ is invariant under the simultaneous transformations of its Lorentz indices and spinor indices. That is $\Lambda^\mu{}_\nu U(\Lambda)\gamma^\nu U(\Lambda^{-1}) = \gamma^\mu$. This implies

$$\Lambda^\mu{}_\nu U_R(\Lambda)\bar{\sigma}^\nu U_L(\Lambda^{-1}) = \bar{\sigma}^\mu,$$

as can be easily seen in chiral basis. Then, the combination $\bar{\sigma}^\mu\partial_\mu$ transforms as $\bar{\sigma}^\mu\partial_\mu \to U_R(\Lambda)\bar{\sigma}^\mu\partial_\mu U_L(\Lambda^{-1})$. As a result, the first term of the equation of motion transforms as

$$\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi \to \mathrm{i}U_R(\Lambda)\bar{\sigma}^\mu\partial_\mu U_L(\Lambda^{-1})U_L(\Lambda)\chi = U_R(\Lambda)\big[\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi\big]. \tag{3.26}$$

To show the full equation of motion is covariant, we also need to show that the second term $\mathrm{i}\sigma^2\chi^*$ transforms in the same way. To see this, we note that in the infinitesimal form,

$$U_L = 1 - i\theta^i\sigma^i/2 - \eta^i\sigma^i/2, \qquad U_R = 1 - i\theta^i\sigma^i/2 + \eta^i\sigma^i/2.$$

Then, under an infinitesimal Lorentz transformation, χ transforms as:

$$\chi \to (1 - i\theta^i\sigma^i/2 - \eta^i\sigma^i/2)\chi, \qquad \Rightarrow \qquad \chi^* \to (1 + i\theta^i\sigma^i/2 - \eta^i\sigma^i/2)\chi^*$$
$$\Rightarrow \qquad \sigma^2\chi^* \to \sigma^2(1 + i\theta^i(\sigma^*)^i/2 - \eta^i(\sigma^*)^i/2)\chi^* = (1 - i\theta^i\sigma^i/2 + \eta^i\sigma^i/2)\sigma^2\chi^*.$$

That is to say, $\sigma^2\chi^*$ is a right-handed spinor that transforms as $\sigma^2\chi^* \to U_R(\Lambda)\sigma^2\chi^*$. Thus we see the the two terms in the equation of motion transform in the same way under the Lorentz transformation. In other words, this equation is Lorentz covariant.

To show the condition 2) also holds, we take the complex conjugation of the equation:

$$-\mathrm{i}(\bar{\sigma}^*)^\mu\partial_\mu\chi^* - \mathrm{i}m\sigma^2\chi = 0.$$

Combining this and the original equation to eliminate χ^*, we get

$$(\partial^2 + m^2)\chi = 0, \tag{3.27}$$

which has the same form with the Klein-Gordon equation.

(b) Now we show that the equation of motion above for the spinor χ can be derived from the following action through the variation principle:

$$S = \int \mathrm{d}^4x \left[\chi^\dagger\mathrm{i}\bar{\sigma}\cdot\partial\chi + \frac{\mathrm{i}m}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\right]. \tag{3.28}$$

Firstly, let us check that this action is real, namely $S^* = S$. In fact,

$$S^* = \int \mathrm{d}^4x \left[(\chi^\dagger\mathrm{i}\bar{\sigma}\cdot\partial\chi)^\dagger - \frac{\mathrm{i}m}{2}(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi)\right],$$

where the first term $(\chi^\dagger\mathrm{i}\bar{\sigma}\cdot\partial\chi)^\dagger = -\mathrm{i}(\partial\chi^\dagger)\mathrm{i}\bar{\sigma}\chi$ is identical to the original kinetic term upon integration by parts. Thus we see that $S^* = S$.

Now we vary the action with respect to $\chi^\dagger$, that gives

$$0 = \frac{\delta S}{\delta\chi^\dagger} = \mathrm{i}\bar{\sigma}\cdot\partial\chi - \frac{\mathrm{i}m}{2}\cdot 2\sigma^2\chi^* = 0, \tag{3.29}$$

which is exactly the Majorana equation.

(c) Let us rewrite the Dirac Lagrangian in terms of two-component spinors:

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi \\
&= \begin{pmatrix}\chi_1^\dagger & -\mathrm{i}\chi_2^T\sigma^2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-m & \mathrm{i}\sigma^\mu\partial_\mu\\ \mathrm{i}\bar{\sigma}^\mu\partial_\mu & -m\end{pmatrix}\begin{pmatrix}\chi_1\\ \mathrm{i}\sigma^2\chi_2^*\end{pmatrix} \\
&= \mathrm{i}\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + \mathrm{i}\chi_2^T\bar{\sigma}^{\mu*}\partial_\mu\chi_2^* - \mathrm{i}m\big(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\big) \\
&= \mathrm{i}\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + \mathrm{i}\chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2 - \mathrm{i}m\big(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\big),
\end{aligned} \tag{3.30}$$

where the equality should be understood to hold up to a total derivative term.

(d) The familiar global $U(1)$ symmetry of the Dirac Lagrangian $\psi \to e^{\mathrm{i}\alpha}\psi$ now becomes $\chi_1 \to e^{\mathrm{i}\alpha}\chi_1$, $\chi_2 \to e^{-\mathrm{i}\alpha}\chi_2$. The associated Nöther current is

$$J^\mu = \bar{\psi}\gamma^\mu\psi = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2. \tag{3.31}$$

To show its divergence $\partial_\mu J^\mu$ vanishes, we make use of the equations of motion:

$$\begin{aligned}
&\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_1 - \mathrm{i}m\sigma^2\chi_2^* = 0,\\
&\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_2 - \mathrm{i}m\sigma^2\chi_1^* = 0,\\
&\mathrm{i}(\partial_\mu\chi_1^\dagger)\bar{\sigma}^\mu - \mathrm{i}m\chi_2^T\sigma^2 = 0,\\
&\mathrm{i}(\partial_\mu\chi_2^\dagger)\bar{\sigma}^\mu - \mathrm{i}m\chi_1^T\sigma^2 = 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
\partial_\mu J^\mu &= (\partial_\mu\chi_1^\dagger)\bar{\sigma}^\mu\chi_1 + \chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 - (\partial_\mu\chi_2^\dagger)\bar{\sigma}^\mu\chi_2 - \chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2\\
&= m\big(\chi_2^T\sigma^2\chi_1 + \chi_1^\dagger\sigma^2\chi_2^* - \chi_1^T\sigma^2\chi_2 - \chi_2^\dagger\sigma^2\chi_1^*\big) = 0.
\end{aligned} \tag{3.32}$$

In a similar way, one can also show that the Nöther currents associated with the global symmetries of Majorana fields have vanishing divergence.

(e) To quantize the Majorana theory, we introduce the canonical anticommutation relation,

$$\big\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\big\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

and also expand the Majorana field χ into modes. To motivate the mode expansion, we note that the Majorana Langrangian can be obtained by replacing the spinor χ_2 in the Dirac Lagrangian (3.30) with χ_1. Then, according to our experience in Dirac theory, it can be found that

$$\chi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_a\Big[\xi_a a_a(\mathbf{p})e^{-\mathrm{i}p\cdot x} + (-\mathrm{i}\sigma^2)\xi_a^* a_a^\dagger(\mathbf{p})e^{\mathrm{i}p\cdot x}\Big]. \tag{3.33}$$

Then with the canonical anticommutation relation above, we can find the anticommutators between annihilation and creation operators:

$$\{a_a(\mathbf{p}), a_b^\dagger(\mathbf{q})\} = \delta_{ab}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \qquad \{a_a(\mathbf{p}), a_b(\mathbf{q})\} = \{a_a^\dagger(\mathbf{p}), a_b^\dagger(\mathbf{q})\} = 0. \tag{3.34}$$

On the other hand, the Hamiltonian of the theory can be obtained by Legendre transforming the Lagrangian:

$$H = \int \mathrm{d}^3x\left(\frac{\delta\mathcal{L}}{\delta\dot{\chi}}\dot{\chi} - \mathcal{L}\right) = \int \mathrm{d}^3x\left[\mathrm{i}\chi^\dagger\boldsymbol{\sigma}\cdot\nabla\chi + \frac{\mathrm{i}m}{2}\big(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\big)\right]. \tag{3.35}$$

Then we can also represent the Hamiltonian H in terms of modes:

$$H = \int \mathrm{d}^3x\int\frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6\sqrt{2E_p 2E_q}}\sum_{a,b}\Big[\Big(\xi_a^\dagger a_a^\dagger(\mathbf{p})e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \xi_a^T(\mathrm{i}\sigma^2)a_a(\mathbf{p})e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)$$

$$
\begin{aligned}
&\times (\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}\Big(\xi_b a_b(\mathbf{q})e^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} - (-\mathrm{i}\sigma^2)\xi_b^* a_b^\dagger(\mathbf{q})e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\Big)\\
&+\frac{\mathrm{i}m}{2}\Big(\xi_a^\dagger a_a^\dagger(\mathbf{p})e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \xi_a^T(\mathrm{i}\sigma^2)a_a(\mathbf{p})e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)\\
&\times (\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*\Big(\xi_b^* a_b^\dagger(\mathbf{q})e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}} + (-\mathrm{i}\sigma^2)\xi_b a_b(\mathbf{q})e^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\Big)\\
&-\frac{\mathrm{i}m}{2}\Big(\xi_a^T a_a(\mathbf{p})e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \xi_a^\dagger(\mathrm{i}\sigma^2)a_a^\dagger(\mathbf{p})e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)\\
&\times (\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}\Big(\xi_b a_b(\mathbf{q})e^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} + (-\mathrm{i}\sigma^2)\xi_b^* a_b^\dagger(\mathbf{q})e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\Big)\Big]\\
=&\int \mathrm{d}^3x\int\frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6\sqrt{2E_p 2E_q}}\sum_{a,b}\Big\{a_a^\dagger(\mathbf{p})a_b(\mathbf{q})\xi_a^\dagger\Big[(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}\\
&+\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*(-\mathrm{i}\sigma^2) - \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}\Big]\xi_b e^{-\mathrm{i}(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\\
&+a_a^\dagger(\mathbf{p})a_b^\dagger(\mathbf{q})\xi_a^\dagger\Big[-(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^* e^{-\mathrm{i}(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\\
&+a_a(\mathbf{p})a_b(q)\xi_a^T\Big[(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma} + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*(-\mathrm{i}\sigma^2)\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}\Big]\xi_b e^{\mathrm{i}(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\\
&+a_a(\mathbf{p})a_b^\dagger(\mathbf{q})\xi_a^T\Big[-(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^* e^{\mathrm{i}(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\Big\}\\
=&\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\sum_{a,b}\Big\{a_a^\dagger(\mathbf{p})a_b(\mathbf{p})\xi_a^\dagger\Big[(\sqrt{p\cdot\sigma})^\dagger(-p\cdot\boldsymbol{\sigma})\sqrt{p\cdot\sigma}\\
&+\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{p\cdot\sigma})^*(-\mathrm{i}\sigma^2) - \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{p\cdot\sigma}\Big]\xi_b\\
&+a_a^\dagger(\mathbf{p})a_b^\dagger(-\mathbf{p})\xi_a^\dagger\Big[-(\sqrt{p\cdot\sigma})^\dagger(\mathbf{p}\cdot\boldsymbol{\sigma})\sqrt{\tilde{p}\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{\tilde{p}\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{\tilde{p}\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^*\\
&+a_a(\mathbf{p})a_b(-\mathbf{p})\xi_a^T\Big[(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(\mathbf{p}\cdot\boldsymbol{\sigma})\sqrt{\tilde{p}\cdot\sigma} + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{\tilde{p}\cdot\sigma})^*(-\mathrm{i}\sigma^2)\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{\tilde{p}\cdot\sigma}\Big]\xi_b\\
&+a_a(\mathbf{p})a_b^\dagger(\mathbf{p})\xi_a^T\Big[-(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{p}\cdot\boldsymbol{\sigma})\sqrt{p\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{p\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{p\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^*\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \sum_{a,b} \frac{1}{2}\left(E_p^2 + |\mathbf{p}|^2 + m^2\right)\left[a_a^\dagger(\mathbf{p}) a_b(\mathbf{p}) \xi_a^\dagger \xi_b - a_a(\mathbf{p}) a_b^\dagger(\mathbf{p}) \xi_a^T \xi_b^*\right] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{E_p}{2} \sum_a \left[a_a^\dagger(\mathbf{p}) a_a(\mathbf{p}) - a_a(\mathbf{p}) a_a^\dagger(\mathbf{p})\right] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \sum_a a_a^\dagger(\mathbf{p}) a_a(\mathbf{p}).
\end{aligned} \tag{3.36}
$$

In the calculation above, each step goes as follows in turn: (1) Substituting the mode expansion for χ into the Hamiltonian. (2) Collecting the terms into four groups, characterized by $a^\dagger a$, $a^\dagger a^\dagger$, aa and $aa^\dagger$. (3) Integrating over $\mathrm{d}^3 x$ to produce a delta function, with which one can further finish the integration over $\mathrm{d}^3 q$. (4) Using the following relations to simplify the spinor matrices:

$$
(p\cdot\sigma)^2 = (p\cdot\bar{\sigma})^2 = E_p^2 + |\mathbf{p}|^2, \qquad (p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2, \qquad \mathbf{p}\cdot\boldsymbol{\sigma} = \tfrac{1}{2}(p\cdot\bar{\sigma} - p\cdot\sigma).
$$

In this step, the $a^\dagger a^\dagger$ and aa terms vanish, while the $aa^\dagger$ and $a^\dagger a$ terms remain. (5) Using the normalization $\xi_a^\dagger \xi_b = \delta_{ab}$ to eliminate spinors. (6) Using the anticommutator $\{a_a(\mathbf{p}), a_a^\dagger(\mathbf{p})\} = \delta^{(3)}(\mathbf{0})$ to further simplify the expression. In this step we have throw away a constant term $-\frac{1}{2}E_p\delta^{(3)}(\mathbf{0})$ in the integrand. The minus sign of this term indicates that the vacuum energy contributed by Majorana field is negative. With these steps done, we find the desired result, as shown above.

3.5 Supersymmetry

(a) In this problem we briefly study the Wess-Zumino model, which may be the simplest supersymmetric field theory in 4 dimensional spacetime. Firstly let us consider the massless case, in which the Lagrangian is given by

$$
\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + F^* F, \tag{3.37}
$$

where ϕ is a complex scalar field, χ is a Weyl fermion, and F is a complex auxiliary scalar field. By auxiliary we mean a field with no kinetic term in the Lagrangian and thus it does not propagate, or equivalently, it has no particle excitation. However, in the following, we will see that it is crucial to maintain the off-shell supersymmtry of the theory.

The supersymmetry transformation in its infinitesimal form is given by:

$$
\delta\phi = -i\epsilon^T \sigma^2 \chi, \tag{3.38a}
$$

$$
\delta\chi = \epsilon F + \sigma^\mu (\partial_\mu \phi)\sigma^2 \epsilon^*, \tag{3.38b}
$$

$$
\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi, \tag{3.38c}
$$

where ϵ is a 2-component Grassmann variable. Now let us show that the Lagrangian is invariant (up to a total divergence) under this supersymmetric transformation. This can be checked term by term, as follows:

$$
\delta(\partial_\mu \phi^* \partial^\mu \phi) = \mathrm{i}\big(\partial_\mu \chi^\dagger \sigma^2 \epsilon^*\big)\partial^\mu \phi + (\partial_\mu \phi^*)\big(-\mathrm{i}\epsilon^T \sigma^2 \partial^\mu \chi\big),
$$

$$\begin{aligned}
\delta(\chi^\dagger \mathrm{i}\bar\sigma^\mu\partial_\mu\chi) &= \big(F^*\epsilon^\dagger + \epsilon^T\sigma^2\sigma^\nu\partial_\nu\phi^*\big)\mathrm{i}\bar\sigma^\mu\partial_\mu\chi + \chi^\dagger\mathrm{i}\bar\sigma^\mu\big(\epsilon\partial_\mu F + \sigma^\nu\sigma^2\epsilon^*\partial_\mu\partial_\nu\phi\big)\\
&= \mathrm{i}F^*\epsilon^\dagger\bar\sigma^2\partial_\mu\chi + \mathrm{i}\partial_\mu\big[\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu(\partial_\nu\phi^*)\chi\big] - \mathrm{i}\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu(\partial_\nu\partial_\mu\phi^*)\chi\\
&\quad + \mathrm{i}\chi^\dagger\bar\sigma^\mu\epsilon\partial_\mu F + \mathrm{i}\chi^\dagger\bar\sigma^\mu\sigma^\nu\sigma^2\epsilon^*\partial_\mu\partial_\nu\phi\\
&= \mathrm{i}F^*\epsilon^\dagger\bar\sigma^2\partial_\mu\chi + \mathrm{i}\partial_\mu\big[\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu(\partial_\nu\phi^*)\chi\big] - \mathrm{i}\epsilon^T\sigma^2(\partial^2\phi^*)\chi\\
&\quad + \mathrm{i}\chi^\dagger\bar\sigma^\mu\epsilon\partial_\mu F + \mathrm{i}\chi^\dagger\sigma^2\epsilon^*\partial^2\phi,\\
\delta(F^*F) &= \mathrm{i}(\partial_\mu\chi^\dagger)\bar\sigma^\mu\epsilon F - \mathrm{i}F^*\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi,
\end{aligned}$$

where we have used $\bar\sigma^\mu\sigma^\nu\partial_\mu\partial_\nu = \partial^2$. Now summing the three terms above, we get:

$$\delta\mathcal{L} = \mathrm{i}\partial_\mu\Big[\chi^\dagger\sigma^2\epsilon^*\partial^\mu\phi + \chi^\dagger\bar\sigma^\mu\epsilon F + \phi^*\epsilon^T\sigma^2\big(\sigma^\mu\sigma^\nu\sigma_\nu - \partial^\mu\big)\chi\Big], \tag{3.39}$$

which is indeed a total derivative.

(b) Now let us add the mass term in to the original massless Lagrangian:

$$\Delta\mathcal{L} = \big(m\phi F + \tfrac{1}{2}\mathrm{i}m\chi^T\sigma^2\chi\big) + \text{c.c.} \tag{3.40}$$

Let us show that this mass term is also invariant under the supersymmetry transformation, up to a total derivative:

$$\begin{aligned}
\delta(\Delta\mathcal{L}) =& -\mathrm{i}m\epsilon^T\sigma^2\chi F - \mathrm{i}m\phi\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi + \tfrac{1}{2}\mathrm{i}m[\epsilon^T F + \epsilon^\dagger(\sigma^2)^T(\sigma^\mu)^T\partial_\mu\phi]\sigma^2\chi\\
&+ \tfrac{1}{2}\mathrm{i}m\chi^T\sigma^2[\epsilon F + \sigma^\mu(\partial_\mu\phi)\sigma^2\epsilon^*] + \text{c.c.}\\
=& -\tfrac{1}{2}imF(\epsilon^T\sigma^2\chi - \chi^T\sigma^2\epsilon) - \mathrm{i}m\phi\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi\\
&- \tfrac{1}{2}\mathrm{i}m(\partial_\mu\phi)\epsilon^\dagger\bar\sigma^\mu\chi + \tfrac{1}{2}\mathrm{i}m(\partial_\mu\phi)\chi^T(\bar\sigma^\mu)^T\epsilon^* + \text{c.c.}\\
=& -\tfrac{1}{2}imF(\epsilon^T\sigma^2\chi - \chi^T\sigma^2\epsilon) - \mathrm{i}m\partial_\mu(\phi\epsilon^\dagger\bar\sigma^\mu\chi)\\
&+ \tfrac{1}{2}\mathrm{i}m(\partial_\mu\phi)[\epsilon^\dagger\bar\sigma^\mu\chi + \chi^T(\bar\sigma^\mu)^T\epsilon^*] + \text{c.c}\\
=& -\mathrm{i}m\partial_\mu(\phi\epsilon^\dagger\bar\sigma^\mu\chi) + \text{c.c}
\end{aligned} \tag{3.41}$$

where we have used the following relations:

$$(\sigma^2)T = -\sigma^2, \qquad \sigma^2(\sigma^\mu)^T\sigma^2 = \bar\sigma^\mu, \qquad \epsilon^T\sigma^2\chi = \chi^T\sigma^2\epsilon, \qquad \epsilon^\dagger\bar\sigma^\mu\chi = -\chi^T(\bar\sigma^\mu)^T\epsilon^*.$$

Now let us write down the Lagrangian with the mass term:

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger\mathrm{i}\bar\sigma^\mu\partial_\mu\chi + F^*F + \big(m\phi F + \tfrac{1}{2}\mathrm{i}m\chi^T\sigma^2\chi + \text{c.c.}\big). \tag{3.42}$$

Varying the Lagrangian with respect to F^*, we get the corresponding equation of motion:

$$F = -m\phi^*. \tag{3.43}$$

Substitute this algebraic equation back into the Lagrangian to eliminate the field F, we get

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \chi^\dagger\mathrm{i}\bar\sigma^\mu\partial_\mu\chi + \tfrac{1}{2}\big(\mathrm{i}m\chi^T\sigma^2\chi + \text{c.c.}\big). \tag{3.44}$$

Thus we see that the scalar field ϕ and the spinor field χ have the same mass.

(c) We can also include interactions into this model. Generally, we can write a Lagrangian with nontrivial interactions containing fields ϕ_i, χ_i and F_i $(i = 1, \cdots, n)$, as

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger \mathrm{i}\bar{\sigma}^\mu \partial_\mu \chi_i + F_i^* F_i + \left[F_i \frac{\partial W[\phi]}{\partial \phi_i} + \frac{\mathrm{i}}{2} \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right], \tag{3.45}$$

where $W[\phi]$ is an arbitrary function of ϕ_i.

To see this Lagrangian is supersymmetry invariant, we only need to check the interactions terms in the square bracket:

$$\begin{aligned}
&\delta \left[F_i \frac{\partial W[\phi]}{\partial \phi_i} + \frac{\mathrm{i}}{2} \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right] \\
=& -\mathrm{i}\epsilon^\dagger \bar{\sigma}^\mu (\partial_\mu \chi_i) \frac{\partial W}{\partial \phi_i} + F_i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} (-\mathrm{i}\epsilon^T \sigma^2 \chi_j) + \frac{\mathrm{i}}{2} \frac{\partial^3 W}{\partial \phi_i \partial \phi_j \partial \phi_k} (-\mathrm{i}\epsilon^T \sigma^2 \chi_k) \chi_i^T \sigma^2 \chi_j \\
&+ \frac{\mathrm{i}}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \Big[\big(\epsilon^T F_i + \epsilon^\dagger (\sigma^2)^T (\sigma^\mu)^T \partial_\mu \phi_i \big) \sigma^2 \chi_j + \chi_i^T \sigma^2 \big(\epsilon F_j + \sigma^\mu \partial_\mu \phi_j \sigma^2 \epsilon^* \big) \Big] + \text{c.c.}.
\end{aligned}$$

The term proportional to $\partial^3 W / \partial \phi^3$ vanishes. To see this, we note that the partial derivatives with respect to ϕ_i are commutable, hence $\partial^3 W / \partial \phi_i \partial \phi_j \partial \phi_k$ is totally symmetric on i, j, k. However, we also have the following identity:

$$(\epsilon^T \sigma^2 \chi_k)(\chi_i^T \sigma^2 \chi_j) + (\epsilon^T \sigma^2 \chi_i)(\chi_j^T \sigma^2 \chi_k) + (\epsilon^T \sigma^2 \chi_j)(\chi_k^T \sigma^2 \chi_i) = 0, \tag{3.46}$$

which can be directly checked by brute force. Then it can be easily seen that the $\partial^3 W / \partial \phi^3$ term vanishes indeed. On the other hand, the terms containing F also sum to zero, which is also straightforward to justify. Hence the terms left now are

$$\begin{aligned}
&-\mathrm{i}\epsilon^\dagger \bar{\sigma}^\mu (\partial_\mu \chi_i) \frac{\partial W}{\partial \phi_i} + \mathrm{i} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \epsilon^\dagger (\sigma^2)^T (\sigma^\mu)^T (\partial_\mu \phi_i) \sigma^2 \chi_j \\
=& -\mathrm{i}\partial_\mu \Big(\epsilon^\dagger \bar{\sigma}^\mu \chi_i \frac{\partial W}{\partial \phi_i} \Big) + \mathrm{i}\epsilon^\dagger \bar{\sigma}^\mu \chi_i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j - \mathrm{i} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \epsilon^\dagger \bar{\sigma}^\mu (\partial_\mu \phi_i) \chi_j \\
=& -\mathrm{i}\partial_\mu \Big(\epsilon^\dagger \bar{\sigma}^\mu \chi_i \frac{\partial W}{\partial \phi_i} \Big),
\end{aligned} \tag{3.47}$$

which is a total derivative. Thus we conclude that the Lagrangian (3.45) is supersymmetrically invariant up to a total derivative.

Let us end up with a explicit example, in which we choose $n = 1$ and $W[\phi] = g\phi^3/3$. Then the Lagrangian (3.45) becomes

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^* F + \Big(gF\phi^2 + i\phi \chi^T \sigma^2 \chi + \text{c.c.} \Big). \tag{3.48}$$

We can eliminate F by solving it from its field equation,

$$F + g(\phi^*)^2 = 0. \tag{3.49}$$

Substituting this back into the Lagrangian, we get

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi - g^2 (\phi^* \phi)^2 + ig(\phi \chi^T \sigma^2 \chi - \phi^* \chi^\dagger \sigma^2 \chi^*). \tag{3.50}$$

This is a Lagrangian of massless complex scalar and a Weyl spinor, with ϕ^4 and Yukawa interactions. The field equations can be easily got by the variation.

3.6 Fierz transformations

In this problem, we derive the generalized Fierz transformation, with which one can express $(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4)$ as a linear combination of $(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2)$, where Γ^A is any normalized Dirac matrices from the following set:

$$\{\ 1, \gamma^\mu, \sigma^{\mu\nu} = \tfrac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu], \gamma^5\gamma^\mu, \gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\ \}.$$

(a) The Dirac matrices Γ^A are normalized according to

$$\mathrm{tr}\,(\Gamma^A\Gamma^B) = 4\delta^{AB}. \tag{3.51}$$

For instance, the unit element 1 is already normalized, since $\mathrm{tr}\,(1\cdot 1) = 4$. For Dirac matrices containing one γ^μ, we calculate the trace in Weyl representation without loss of generality. Then the representation of

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

gives $\mathrm{tr}\,(\gamma^\mu\gamma^\mu) = 2\,\mathrm{tr}\,(\sigma^\mu\bar{\sigma}^\mu)$ (no sum on μ). For $\mu = 0$, we have $\mathrm{tr}\,(\gamma^0\gamma^0) = 2\,\mathrm{tr}\,(1_{2\times 2}) = 4$, and for $\mu = i = 1, 2, 3$, we have $\mathrm{tr}\,(\gamma^i\gamma^i) = -2\,\mathrm{tr}\,(\sigma^i\sigma^i) = -2\,\mathrm{tr}\,(1_{2\times 2}) = -4$ (no sum on i). Thus the normalized gamma matrices are γ^0 and $\mathrm{i}\gamma^i$.

In the same way, we can work out the rest of the normalized Dirac matrices, as:

$$\begin{aligned}
&\mathrm{tr}\,(\sigma^{0i}\sigma^{0i}) = -2\,\mathrm{tr}\,(\sigma^i\sigma^i) = -4, && (\text{no sum on } i)\\
&\mathrm{tr}\,(\sigma^{ij}\sigma^{ij}) = 2\,\mathrm{tr}\,(\sigma^k\sigma^k) = 4, && (\text{no sum on } i, j, k)\\
&\mathrm{tr}\,(\gamma^5\gamma^5) = 4, &&\\
&\mathrm{tr}\,(\gamma^5\gamma^0\gamma^5\gamma^0) = -4, && \mathrm{tr}\,(\gamma^5\gamma^i\gamma^5\gamma^i) = 4.
\end{aligned}$$

Thus the 16 normalized elements are:

$$\{\ 1,\ \gamma^0,\ i\gamma^i,\ i\sigma^{0i},\ \sigma^{ij},\ \gamma^5,\ i\gamma^5\gamma^0,\ \gamma^5\gamma^i\ \}. \tag{3.52}$$

(b) Now we derive the desired Fierz identity, which can be written as:

$$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{C,D} C^{AB}{}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2). \tag{3.53}$$

Left-multiplying the equality by $(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)$, we get:

$$(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{CD} C^{AB}{}_{CD}\,\mathrm{tr}\,(\Gamma^E\Gamma^C)\,\mathrm{tr}\,(\Gamma^F\Gamma^D). \tag{3.54}$$

The left hand side:

$$(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \bar{u}_4\Gamma^E\Gamma^A\Gamma^F\Gamma^B u_4 = \mathrm{tr}\,(\Gamma^E\Gamma^A\Gamma^F\Gamma^B);$$

the right hand side:

$$\sum_{C,D} C^{AB}{}_{CD}\,\mathrm{tr}\,(\Gamma^E\Gamma^C)\,\mathrm{tr}\,(\Gamma^F\Gamma^D) = \sum_{C,D} C^{AB}{}_{CD} 4\delta^{EC} 4\delta^{FD} = 16 C^{AB}{}_{EF},$$

thus we conclude:

$$C^{AB}{}_{CD} = \tfrac{1}{16}\,\mathrm{tr}\,(\Gamma^C\Gamma^A\Gamma^D\Gamma^B). \tag{3.55}$$

(c) Now we derive two Fierz identities as particular cases of the results above. The first one is:

$$(\bar{u}_1 u_2)(\bar{u}_3 u_4) = \sum_{C,D} \frac{\mathrm{tr}\,(\Gamma^C \Gamma^D)}{16} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^D u_2). \tag{3.56}$$

The traces on the right hand side do not vanish only when $C = D$, thus we get:

$$\begin{aligned}(\bar{u}_1 u_2)(\bar{u}_3 u_4) =& \sum_C \tfrac{1}{4} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^C u_2) \\ =& \tfrac{1}{4} \Big[(\bar{u}_1 u_4)(\bar{u}_3 u_2) + (\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2) + \tfrac{1}{2} (\bar{u}_1 \sigma^{\mu\nu} u_4)(\bar{u}_3 \sigma_{\mu\nu} u_2) \\ & - (\bar{u}_1 \gamma^5 \gamma^\mu u_4)(\bar{u}_3 \gamma^5 \gamma_\mu u_2) + (\bar{u}_1 \gamma^5 u_4)(\bar{u}_3 \gamma^5 u_2) \Big]. \end{aligned} \tag{3.57}$$

The second example is:

$$(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4) = \sum_{C,D} \frac{\mathrm{tr}\,(\Gamma^C \gamma^\mu \Gamma^D \gamma_\mu)}{16} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^D u_2). \tag{3.58}$$

Again, the traces vanish if $\Gamma^C \gamma^\mu \neq C \propto \Gamma^D \gamma^\mu$ with C a commuting number, which implies that $\Gamma^C = \Gamma^D$. That is,

$$\begin{aligned}(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4) =& \sum_C \frac{\mathrm{tr}\,(\Gamma^C \gamma^\mu \Gamma^C \gamma_\mu)}{16} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^C u_2) \\ =& \tfrac{1}{4} \Big[4(\bar{u}_1 u_4)(\bar{u}_3 u_2) - 2(\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2) \\ & - 2(\bar{u}_1 \gamma^5 \gamma^\mu u_4)(\bar{u}_3 \gamma^5 \gamma_\mu u_2) - 4(\bar{u}_1 \gamma^5 u_4)(\bar{u}_3 \gamma^5 u_2) \Big]. \end{aligned} \tag{3.59}$$

We note that the normalization of Dirac matrices has been properly taken into account by raising or lowering of Lorentz indices.

3.7 The discrete symmetries P, C and T

(a) In this problem, we will work out the C, P and T transformations of the bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Firstly,

$$P\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})P = \tfrac{\mathrm{i}}{2}\bar{\psi}(t,-\mathbf{x})\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\psi(t,-\mathbf{x}).$$

With the relations $\gamma^0[\gamma^0,\gamma^i]\gamma^0 = -[\gamma^0,\gamma^i]$ and $\gamma^0[\gamma^i,\gamma^j]\gamma^0 = [\gamma^i,\gamma^j]$, we get:

$$P\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})P = \begin{cases} -\bar{\psi}(t,-\mathbf{x})\sigma^{0i}\psi(t,-\mathbf{x}); \\ \bar{\psi}(t,-\mathbf{x})\sigma^{ij}\psi(t,-\mathbf{x}). \end{cases} \tag{3.60}$$

Secondly,

$$T\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})T = -\tfrac{\mathrm{i}}{2}\bar{\psi}(-t,\mathbf{x})(-\gamma^1\gamma^3)[\gamma^\mu,\gamma^\nu]^*(\gamma^1\gamma^3)\psi(-t,\mathbf{x}).$$

Note that gamma matrices keep invariant under transposition, except γ^2, which changes the sign. Thus we have:

$$T\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})T = \begin{cases} \bar{\psi}(-t,\mathbf{x})\sigma^{0i}\psi(-t,\mathbf{x}); \\ -\bar{\psi}(-t,\mathbf{x})\sigma^{ij}\psi(-t,\mathbf{x}). \end{cases} \tag{3.61}$$

Thirdly,

$$C\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})C = -\tfrac{\mathrm{i}}{2}(-\mathrm{i}\gamma^0\gamma^2\psi)^T\sigma^{\mu\nu}(-\mathrm{i}\bar{\psi}\gamma^0\gamma^2)^T = \bar{\psi}\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\psi.$$

Note that γ^0 and γ^2 are symmetric while γ^1 and γ^3 are antisymmetric, we have

$$C\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})C = -\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x}). \tag{3.62}$$

(b) Now we work out the C, P and T transformation properties of a scalar field ϕ. Our starting point is

$$Pa_{\mathbf{p}}P = a_{-\mathbf{p}}, \qquad Ta_{\mathbf{p}}T = a_{-\mathbf{p}}, \qquad Ca_{\mathbf{p}}C = b_{\mathbf{p}}.$$

Then, for a complex scalar field

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[a_{\mathbf{k}}e^{-\mathrm{i}k\cdot x} + b^\dagger_{\mathbf{k}}e^{\mathrm{i}k\cdot x}\Big], \tag{3.63}$$

we have

$$P\phi(t,\mathbf{x})P = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[a_{-\mathbf{k}}e^{-\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})} + b^\dagger_{-\mathbf{k}}e^{\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})}\Big] = \phi(t,-\mathbf{x}). \tag{3.64a}$$

$$T\phi(t,\mathbf{x})T = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[a_{-\mathbf{k}}e^{\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})} + b^\dagger_{-\mathbf{k}}e^{-\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})}\Big] = \phi(-t,\mathbf{x}). \tag{3.64b}$$

$$C\phi(t,\mathbf{x})C = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[b_{\mathbf{k}}e^{-\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})} + a^\dagger_{\mathbf{k}}e^{\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})}\Big] = \phi^*(t,\mathbf{x}). \tag{3.64c}$$

As a consequence, we can deduce the C, P, and T transformation properties of the current $J^\mu = \mathrm{i}\big(\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi\big)$, as follows:

$$\begin{aligned} PJ^\mu(t,\mathbf{x})P &= (-1)^{s(\mu)}\mathrm{i}\big[\phi^*(t,-\mathbf{x})\partial^\mu\phi(t,-\mathbf{x}) - \big(\partial^\mu\phi^*(t,-\mathbf{x})\big)\phi(t,-\mathbf{x})\big] \\ &= (-1)^{s(\mu)}J^\mu(t,-\mathbf{x}), \end{aligned} \tag{3.65a}$$

where $s(\mu)$ is the label for space-time indices that equals to 0 when $\mu = 0$ and 1 when $\mu = 1,2,3$. In the similar way, we have

$$TJ^\mu(t,\mathbf{x})T = (-1)^{s(\mu)}J^\mu(-t,\mathbf{x}); \tag{3.65b}$$

$$CJ^\mu(t,\mathbf{x})C = -J^\mu(t,\mathbf{x}). \tag{3.65c}$$

One should be careful when playing with T — it is antihermitian rather than hermitian, and anticommutes, rather than commutes, with $\sqrt{-1}$.

(c) Any Lorentz-scalar hermitian local operator $\mathcal{O}(x)$ constructed from $\psi(x)$ and $\phi(x)$ can be decomposed into groups, each of which is a Lorentz-tensor hermitian operator and contains either $\psi(x)$ or $\phi(x)$ only. Thus to prove that $\mathcal{O}(x)$ is an operator of $CPT = +1$, it is enough to show that all Lorentz-tensor hermitian operators constructed from either $\psi(x)$ or $\phi(x)$ have correct CPT value. For operators constructed from $\psi(x)$, this has been done as listed in Table on Page 71 of Peskin & Schroeder; and for operators constructed from $\phi(x)$, we note that all such operators can be decomposed further into a product (including Lorentz inner product) of operators of the form

$$(\partial_{\mu_1} \cdots \partial_{\mu_m} \phi^\dagger)(\partial_{\mu_1} \cdots \partial_{\mu_n} \phi) + \text{c.c}$$

together with the metric tensor $\eta^{\mu\nu}$. But it is easy to show that any operator of this form has the correct CPT value, namely, has the same CPT value as a Lorentz tensor of rank $(m+n)$. Therefore we conclude that any Lorentz-scalar hermitian local operator constructed from ψ and ϕ has $CPT = +1$.

3.8 Bound states

(a) A positronium bound state with orbital angular momentum L and total spin S can be build by linear superposition of an electron state and a positron state, with the spatial wave function $\Psi_L(\mathbf{k})$ as the amplitude. Symbolically we have

$$|L, S\rangle \sim \sum_k \Psi_L(\mathbf{k}) a^\dagger(\mathbf{k}, s) b^\dagger(-\mathbf{k}, s')|0\rangle.$$

Then, apply the space-inversion operator P, we get

$$P|L, S\rangle = \sum_k \Psi_L(-\mathbf{k}) \eta_a \eta_b a^\dagger(-\mathbf{k}, s) b^\dagger(\mathbf{k}, s')|0\rangle = (-1)^L \eta_a \eta_b \sum_k \Psi_L(\mathbf{k}) a^\dagger(\mathbf{k}, s) b^\dagger(\mathbf{k}, s')|0\rangle. \tag{3.66}$$

Note that $\eta_b = -\eta_a^*$, we conclude that $P|L, S\rangle = (-)^{L+1}|L, S\rangle$. Similarly,

$$C|L, S\rangle = \sum_k \Psi_L(\mathbf{k}) b^\dagger(\mathbf{k}, s) a^\dagger(-\mathbf{k}, s')|0\rangle = (-1)^{L+S} \sum_k \Psi_L(\mathbf{k}) b^\dagger(-\mathbf{k}, s') a^\dagger(\mathbf{k}, s)|0\rangle. \tag{3.67}$$

That is, $C|L, S\rangle = (-1)^{L+S}|L, S\rangle$. Then its easy to find the P and C eigenvalues of various states, listed as follows:

${}^S L$	${}^1S$	${}^3S$	${}^1P$	${}^3P$	${}^1D$	${}^3D$
P	$-$	$-$	$+$	$+$	$-$	$-$
C	$+$	$-$	$-$	$+$	$+$	$-$

(b) We know that a photon has parity eigenvalue -1 and C-eigenvalue -1. Thus we see that the decay into 2 photons are allowed for ${}^1S$ state but forbidden for ${}^3S$ state due to C-violation. That is, ${}^3S$ has to decay into at least 3 photons.

Chapter 4

Interacting Fields and Feynman Diagrams

4.1 Scalar field with a classical source

In this problem we consider the theory with the following Hamiltonian:

$$H = H_0 - \int \mathrm{d}^3\, j(t,\mathbf{x})\phi(x), \tag{4.1}$$

where H_0 is the Hamiltonian for free Klein-Gordon field ϕ, and j is a classical source.

(a) We calculate the probability that the source creates no particles. The corresponding amplitude is given by the inner product between the in-state and the out-state, both of which are vacuum in our case. Therefore,

$$\begin{aligned} P(0) =& \left|{}_{\text{out}}\langle 0|0\rangle_{\text{in}}\right|^2 = \lim_{t\to(1-i\epsilon)\infty} \left|\langle 0|e^{-i2Ht}|0\rangle\right|^2 \\ =& \left|\langle 0|T\exp\Big\{-i\int \mathrm{d}^4x\, \mathcal{H}_{\text{int}}\Big\}|0\rangle\right|^2 = \left|\langle 0|T\exp\Big\{i\int \mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle\right|^2. \end{aligned} \tag{4.2}$$

(b) Now we expand this probability $P(0)$ to j^2. The amplitude reads,

$$\begin{aligned} \langle 0|T\exp\Big\{i\int \mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle =& 1 - \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, j(x)\langle 0|T\phi_I(x)\phi_I(y)|0\rangle j(y) + O(j^4) \\ =& 1 - \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, j(x)j(y)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} + O(j^4) \\ =& 1 - \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2 + O(j^4). \end{aligned} \tag{4.3}$$

Thus the probability is given by,

$$P(0) = |1 - \frac{1}{2}\lambda + O(j^4)|^2 = 1 - \lambda + O(j^4), \tag{4.4}$$

where

$$\lambda \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2. \tag{4.5}$$

(c) We can calculate the probability $P(0)$ exactly, by working out the j^{2n} term of the expansion as,

$$\begin{aligned}
&\frac{i^{2n}}{(2n)!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_{2n}\, j(x_1)\cdots j(x_{2n})\langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle \\
=&\frac{i^{2n}(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_{2n}\, j(x_1)\cdots j(x_{2n}) \\
&\int \frac{\mathrm{d}^3p_1\cdots\mathrm{d}^3p_n}{(2\pi)^{3n}}\frac{1}{2^nE_{\mathbf{p}_1}\cdots E_{\mathbf{p}_n}}e^{i\mathbf{p}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)}\cdots e^{i\mathbf{p}_n\cdot(\mathbf{x}_{2n-1}-\mathbf{x}_{2n})} \\
=&\frac{(-1)^n}{2^nn!}\Big(\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{|\tilde{j}(p)|^2}{2E_{\mathbf{p}}}\Big)^n=\frac{(-\lambda/2)^2}{n!}.
\end{aligned}\tag{4.6}$$

Then,

$$P(0)=\Big(\sum_{n=0}^{\infty}\frac{(-\lambda/2)^n}{n!}\Big)^2=e^{-\lambda}.\tag{4.7}$$

(d) Now we calculate the probability that the source creates one particle with momentum $\mathbf{k}$, which is given by,

$$P(k)=\Big|\langle\mathbf{k}|T\exp\Big\{i\int\mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle\Big|^2\tag{4.8}$$

Expanding the amplitude to the first order in j, we get:

$$\begin{aligned}
P(k)=&\Big|\langle\mathbf{k}|0\rangle+i\int\mathrm{d}^4x\, j(x)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{ip\cdot x}}{\sqrt{2E_{\mathbf{p}}}}\langle\mathbf{k}|a^\dagger_{\mathbf{p}}|0\rangle+O(j^2)\Big|^2 \\
=&\Big|i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\tilde{j}(p)}{\sqrt{2E_{\mathbf{p}}}}\sqrt{2E_{\mathbf{p}}}(2\pi)^3\delta(\mathbf{p}-\mathbf{k})\Big|^2=|\tilde{j}(k)|^2+O(j^3).
\end{aligned}\tag{4.9}$$

If we go on to work out all the terms, we get,

$$P(k)=\Big|\sum_n\frac{i(2n+1)(2n+1)(2n-1)\cdots 3\cdot 1}{(2n+1)!}\tilde{j}^{n+1}(k)\Big|^2=|\tilde{j}(k)|^2e^{-|\tilde{j}(k)|}.\tag{4.10}$$

(e) To calculate the probability that the source creates n particles, we write down the relevant amplitude,

$$\int\frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n}{(2\pi)^{3n}\sqrt{2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}}\langle\mathbf{k}_1\cdots\mathbf{k}_n|T\exp\Big\{i\int\mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle.\tag{4.11}$$

Expanding this amplitude in terms of j, we find that the first nonvanishing term is the one of n'th order in j. Repeat the similar calculations above, we can find that the amplitude is:

$$\begin{aligned}
&\frac{i^n}{n!}\int\frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n}{(2\pi)^{3n}\sqrt{2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}}\int\mathrm{d}^4x_1\cdots\mathrm{d}^4x_n\, j_1\cdots j_n\langle\mathbf{k}_1\cdots\mathbf{k}_n|\phi_1\cdots\phi_n|0\rangle+O(j^{n+2}) \\
=&\frac{i^n}{n!}\int\frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n\,\tilde{j}^n(k)}{(2\pi)^{3n}\sqrt{2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^nn!}\Big(\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{|\tilde{j}(p)|^2}{2E_{\mathbf{p}}}\Big)^n.
\end{aligned}\tag{4.12}$$

Then we see the probability is given by,

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \tag{4.13}$$

which is a Poisson distribution.

(f) It's easy to check that the Poisson distribution $P(n)$ satisfies the following identities:

$$\sum_n P(n) = 1. \tag{4.14}$$

$$\langle N \rangle = \sum_n n P(n) = \lambda. \tag{4.15}$$

The first one is almost trivial, and the second one can be obtained by acting $\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}$ to both sides of the first identity. If we apply $\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}$ again to the second identity, we get:

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \lambda. \tag{4.16}$$

4.2 Decay of a scalar particle

This problem is based on the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2} M^2 \Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \mu \Phi \phi \phi. \tag{4.17}$$

When $M > 2m$, a Φ particle can decay into two ϕ particles. We want to calculate the lifetime of the Φ particle to lowest order in μ.

The two-body decay rate is given in (4.86) of P&S,

$$\int \mathrm{d}\Gamma = \frac{1}{2M} \int \frac{\mathrm{d}^3 p_1 \mathrm{d}^3 p_2}{(2\pi)^6} \frac{1}{4 E_{\mathbf{p}_1} E_{\mathbf{p}_2}} \left| \mathcal{M}(\Phi(0) \to \phi(p_1)\phi(p_2)) \right|^2 (2\pi)^4 \delta^{(4)}(p_\Phi - p_1 - p_2). \tag{4.18}$$

To lowest order in μ, the amplitude $\mathcal{M}$ is given by,

$$i\mathcal{M} = -2i\mu. \tag{4.19}$$

The delta function in our case reads,

$$\delta^{(4)}(p_\Phi - p_1 - p_2) = \delta(M - E_{\mathbf{p}_1} - E_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2), \tag{4.20}$$

thus,

$$\Gamma = \frac{1}{2} \cdot \frac{2\mu^2}{M} \int \frac{\mathrm{d}^3 p_1 \mathrm{d}^3 p_2}{(2\pi)^6} \frac{1}{4 E_{\mathbf{p}_1} E_{\mathbf{p}_2}} (2\pi)^4 \delta(M - E_{\mathbf{p}_1} - E_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2), \tag{4.21}$$

where an additional factor of $1/2$ takes account of two identical ϕ's in final state. Furthermore, there are two mass-shell constraints,

$$m^2 + \mathbf{p}_i^2 = E_{\mathbf{p}_i}^2. \qquad (i = 1, 2) \tag{4.22}$$

Hence,

$$\Gamma = \frac{\mu^2}{M}\int \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}_1}^2}(2\pi)\delta(M-2E_{\mathbf{p}_1}) = \frac{\mu^2}{8\pi M}\Big(1-\frac{4m^2}{M^2}\Big)^{1/2}. \tag{4.23}$$

Then the lifetime τ of Φ is,

$$\tau = \Gamma^{-1} = \frac{8\pi M}{\mu^2}\Big(1-\frac{4m^2}{M^2}\Big)^{-1/2}. \tag{4.24}$$

4.3 Linear sigma model

In this problem, we study the linear sigma model described by the following Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\Phi^i\partial^\mu\Phi^i - \tfrac{1}{2}m^2\Phi^i\Phi^i - \tfrac{1}{4}\lambda(\Phi^i\Phi^i)^2. \tag{4.25}$$

Where Φ is an N-component scalar.

(a) We firstly compute the differential cross sections to the leading order in λ for the following three processes,

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1. \tag{4.26}$$

Since the masses of all incoming and outgoing particles are identical, the cross section is simply given by

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\mathrm{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 s}, \tag{4.27}$$

where s is the square of center-of-mass energy, and $\mathcal{M}$ is the scattering amplitude. From the Feynman rules it's easy to get,

$$\begin{aligned}&\mathcal{M}(\Phi^1\Phi^2 \to \Phi^1\Phi^2) = \mathcal{M}(\Phi^1\Phi^1 \to \Phi^2\Phi^2) = -2\mathrm{i}\lambda,\\ &\mathcal{M}(\Phi^1\Phi^1 \to \Phi^1\Phi^1) = -6\mathrm{i}\lambda.\end{aligned} \tag{4.28}$$

It follows immediately that

$$\begin{aligned}&\sigma(\Phi^1\Phi^2 \to \Phi^1\Phi^2) = \sigma M(\Phi^1\Phi^1 \to \Phi^2\Phi^2) = \frac{\lambda^2}{16\pi^2 s},\\ &\sigma(\Phi^1\Phi^1 \to \Phi^1\Phi^1) = \frac{9\lambda^2}{16\pi^2 s}.\end{aligned} \tag{4.29}$$

(b) Now we study the symmetry broken case, that is, $m^2 = -\mu^2 < 0$. Then, the scalar multiplet Φ can be parameterized as

$$\Phi = (\pi^1, \cdots, \pi^{N-1}, \sigma + v)^T, \tag{4.30}$$

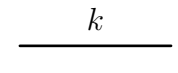
where v is the VEV of $|\Phi|$, and equals to $\sqrt{\mu^2/\lambda}$ at tree level.

Substitute this into the Lagrangian, we get

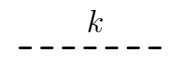
$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\pi^k)^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\sigma\pi^k\pi^k$$

$$-\tfrac{\lambda}{4}\sigma^4 - \tfrac{\lambda}{2}\sigma^2(\pi^k\pi^k) - \tfrac{\lambda}{4}(\pi^k\pi^k)^2. \tag{4.31}$$

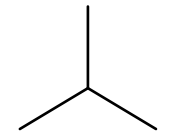
Then it's easy to read the Feynman rules from this expression:



$$= \frac{\mathrm{i}}{k^2 - 2\mu^2}; \tag{4.32a}$$



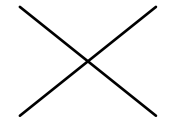
$$= \frac{\mathrm{i}\delta^{ij}}{k^2}; \tag{4.32b}$$



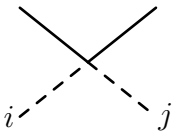
$$= 6\mathrm{i}\lambda v; \tag{4.32c}$$



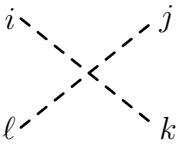
$$= -2\mathrm{i}\lambda v\delta^{ij}; \tag{4.32d}$$



$$= -6\mathrm{i}\lambda; \tag{4.32e}$$



$$= -2\mathrm{i}\lambda\delta^{ij}; \tag{4.32f}$$



$$= -2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}). \tag{4.32g}$$

(c) With the Feynman rules derived in (b), we can compute the amplitude

$$\mathcal{M}\big[\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^\ell(p_4)\big],$$

as:

$$\begin{aligned}\mathcal{M} = (-2\mathrm{i}\lambda v)^2\Big[&\frac{\mathrm{i}}{s-2\mu^2}\delta^{ij}\delta^{k\ell} + \frac{\mathrm{i}}{t-2\mu^2}\delta^{ik}\delta^{j\ell} + \frac{\mathrm{i}}{u-2\mu^2}\delta^{i\ell}\delta^{jk}\Big] \\ &- 2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}),\end{aligned} \tag{4.33}$$

where s, t, u are Mandelstam variables (See Section 5.4 of P&S). Then, at the threshold $p_i = 0$, we have $s = t = u = 0$, and $\mathcal{M}$ vanishes.

On the other hand, if $N = 2$, then there is only one component in π, thus the amplitude reduces to

$$\mathcal{M} = -2\mathrm{i}\lambda\Big[\frac{2\mu^2}{s-2\mu^2} + \frac{2\mu^2}{t-2\mu^2} + \frac{2\mu^2}{u-2\mu^2} + 3\Big]$$

$$= 2\mathrm{i}\lambda\Big[\frac{s+t+u}{2\mu^2} + \mathcal{O}(p^4)\Big]. \tag{4.34}$$

In the second line we perform the Taylor expansion on s, t and u, which are of order $\mathcal{O}(p^2)$. Note that $s+t+u = 4m_\pi^2 = 0$, thus we see that $\mathcal{O}(p^2)$ terms are also canceled out.

(d) We minimize the potential with a small symmetry breaking term:

$$V = -\mu^2\Phi^i\Phi^i + \tfrac{\lambda}{4}(\Phi^i\Phi^i)^2 - a\Phi^N, \tag{4.35}$$

which yields the following equation that determines the VEV:

$$\big(-\mu^2 + \lambda\Phi^i\Phi^i\big)\Phi^i = a\delta^{iN}. \tag{4.36}$$

Thus, up to linear order in a, the VEV $\langle\Phi^i\rangle = (0, \cdots, 0, v)$ is

$$v = \sqrt{\frac{\mu^2}{\lambda}} + \frac{a}{2\mu^2}. \tag{4.37}$$

Now we repeat the derivation in (b) with this new VEV, and write the Lagrangian in terms of new field variable π^i and σ, as

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}(\partial_\mu\pi^k)^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}\tfrac{\sqrt{\lambda}a}{\mu}\pi^k\pi^k - \tfrac{1}{2}(2\mu^2)\sigma^2 \\ & - \lambda v\sigma^3 - \lambda v\sigma\pi^k\pi^k - \tfrac{1}{4}\lambda\sigma^4 - \tfrac{\lambda}{2}\sigma^2(\pi^k\pi^k) - \tfrac{\lambda}{4}(\pi^k\pi^k)^2.\end{aligned} \tag{4.38}$$

The $\pi^i\pi^j \to \pi^k\pi^\ell$ amplitude is still given by

$$\begin{aligned}\mathcal{M} = {} & (-2\mathrm{i}\lambda v)^2\Big[\frac{\mathrm{i}}{s-2\mu^2}\delta^{ij}\delta^{k\ell} + \frac{\mathrm{i}}{t-2\mu^2}\delta^{ik}\delta^{j\ell} + \frac{\mathrm{i}}{u-2\mu^2}\delta^{i\ell}\delta^{jk}\Big] \\ & - 2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}).\end{aligned} \tag{4.39}$$

However this amplitude does not vanish at the threshold. Since the vertices $\lambda\nu \neq \sqrt{\lambda}\mu$ exactly even at tree level, and also s, t and u are not exactly zero in this case due to nonzero mass of π^i. Both deviations are proportional to a, thus we conclude that the amplitude $\mathcal{M}$ is also proportional to a.

4.4 Rutherford scattering

The Rutherford scattering is the scattering of an election by the coulomb field of a nucleus. In this problem, we calculate the cross section by treating the electromagnetic field as fixed classical background given by potential $A_\mu(x)$. Then the interaction Hamiltonian is,

$$H_I = \int \mathrm{d}^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{4.40}$$

(a) We first calculate the T-matrix to lowest order,

$$\begin{aligned}{}_{\text{out}}\langle p'|p\rangle_{\text{in}} =&\langle p'|T\exp(-i\int \mathrm{d}^4x\, H_I)|p\rangle = \langle p'|p\rangle - ie\int \mathrm{d}^4x\, A_\mu(x)\langle p'|\bar\psi\gamma^\mu\psi|p\rangle + O(e^2)\\ =&\langle p'|p\rangle - ie\int \mathrm{d}^4x\, A_\mu(x)\bar u(p')\gamma^\mu u(p)e^{i(p'-p)\cdot x} + O(e^2)\\ =&(2\pi)^4\delta^{(4)}(p-p') - ie\bar u(p')\gamma^\mu u(p)\tilde A_\mu(p'-p) + O(e^2). \end{aligned} \tag{4.41}$$

On the other hand,

$${}_{\text{out}}\langle p'|p\rangle_{\text{in}} = \langle p'|S|p\rangle = \langle p'|p\rangle + \langle p'|iT|p\rangle. \tag{4.42}$$

Thus to the first order of e, we get,

$$\langle p'|iT|p\rangle = -ie\bar u(p')\gamma^\mu u(p)\tilde A_\mu(p'-p). \tag{4.43}$$

(b) Now we calculate the cross section $\mathrm{d}\sigma$ in terms of the matrix elements $i\mathcal{M}$.

The incident wave packet $|\psi\rangle$ is defined to be:

$$|\psi\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{-i\mathbf{b}\cdot\mathbf{k}}}{\sqrt{2E_{\mathbf{k}}}}\psi(\mathbf{k})|\mathbf{k}\rangle, \tag{4.44}$$

where $\mathbf{b}$ is the impact parameter.

The probability that a scattered electron will be found within an infinitesimal element d^3p centered at $\mathbf{p}$ is,

$$\begin{aligned}P =&\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big|{}_{\text{out}}\langle\mathbf{p}|\psi\rangle_{\text{in}}\Big|^2\\ =&\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6\sqrt{2E_{\mathbf{k}}2E_{\mathbf{k}'}}}\psi(\mathbf{k})\psi^*(\mathbf{k}')\Big({}_{\text{out}}\langle\mathbf{p}|\mathbf{k}\rangle_{\text{in}}\Big)\Big({}_{\text{out}}\langle\mathbf{p}|\mathbf{k}'\rangle_{\text{in}}\Big)^* e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}\\ =&\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6\sqrt{2E_{\mathbf{k}}2E_{\mathbf{k}'}}}\psi(\mathbf{k})\psi^*(\mathbf{k}')\Big(\langle\mathbf{p}|iT|\mathbf{k}\rangle\Big)\Big(\langle\mathbf{p}|iT|\mathbf{k}'\rangle\Big)^* e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}. \end{aligned} \tag{4.45}$$

In the last equality we throw away the trivial scattering part from the S-matrix. Note that,

$$\langle p'|iT|p\rangle = i\mathcal{M}(2\pi)\delta(E_{\mathbf{p}'} - E_{\mathbf{p}}), \tag{4.46}$$

so we have,

$$P = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6\sqrt{2E_{\mathbf{k}}2E_{\mathbf{k}'}}}\psi(\mathbf{k})\psi^*(\mathbf{k}')|i\mathcal{M}|^2(2\pi)^2\delta(E_{\mathbf{p}}-E_{\mathbf{k}})\delta(E_{\mathbf{p}}-E_{\mathbf{k}'})e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}. \tag{4.47}$$

The cross section $\mathrm{d}\sigma$ is given by:

$$\mathrm{d}\sigma = \int \mathrm{d}^2b\, P(\mathbf{b}), \tag{4.48}$$

thus the integration over $\mathbf{b}$ gives a delta function:

$$\int \mathrm{d}^2b\, e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')} = (2\pi)^2\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp). \tag{4.49}$$

The other two delta functions in the integrand can be modified as follows,

$$\delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) = \frac{E_{\mathbf{k}}}{k_\parallel}\delta(k_\parallel - k'_\parallel) = \frac{1}{v}\delta(k_\parallel - k'_\parallel), \tag{4.50}$$

where we have used $|\mathbf{v}| = v = v_\parallel$. Taking all these delta functions into account, we get,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\frac{1}{v}\psi(\mathbf{k})\psi^*(\mathbf{k})|i\mathcal{M}|^2(2\pi)\delta(E_{\mathbf{p}} - E_{\mathbf{k}}). \tag{4.51}$$

Since the momentum of the wave packet should be localized around its central value, we can pull out the quantities involving energy $E_{\mathbf{k}}$ outside the integral,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}(2\pi)|\mathcal{M}|^2\delta(E_{\mathbf{p}} - E_{\mathbf{k}})\int \frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})\psi^*(\mathbf{k}). \tag{4.52}$$

Recall the normalization of the wave packet,

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})^*\psi(\mathbf{k}) = 1, \tag{4.53}$$

then,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\delta(E_{\mathbf{p}} - E_{\mathbf{k}}). \tag{4.54}$$

We can further integrate over $|\mathbf{p}|$ to get the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$,

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} =& \int \frac{\mathrm{d}p\,p^2}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\delta(E_{\mathbf{p}} - E_{\mathbf{k}})\\ =& \int \frac{\mathrm{d}p\,p^2}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\frac{E_{\mathbf{k}}}{k}\delta(p-k)\\ =& \frac{1}{(4\pi)^2}|\mathcal{M}(k,\theta)|^2.\end{aligned} \tag{4.55}$$

In the last line we work out the integral by virtue of delta function, which constrains the outgoing momentum $|\mathbf{p}| = |\mathbf{k}|$ but leave the angle θ between $\mathbf{p}$ and $\mathbf{k}$ arbitrary. Thus the amplitude $\mathcal{M}(k,\theta)$ is a function of momentum $|\mathbf{k}|$ and angle θ.

(c) We work directly for the relativistic case. Firstly the Coulomb potential $A^0 = Ze/4\pi r$ in momentum space is

$$A^0(\mathbf{q}) = \frac{Ze}{|\mathbf{q}|^2}. \tag{4.56}$$

This can be easily worked out by Fourier transformation, with a "regulator" e^{-mr} inserted:

$$A^0(\mathbf{q}, m) \equiv \int \mathrm{d}^3x\, e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}e^{-mr}\frac{Ze}{4\pi r} = \frac{Ze}{|\mathbf{q}|^2 + m^2}. \tag{4.57}$$

This is simply Yukawa potential, and Coulomb potential is a limiting case when $m \to 0$.

The amplitude is given by

$$\mathrm{i}\mathcal{M}(k,\theta) = \mathrm{i}e\bar{u}(p)\gamma^\mu\tilde{A}_\mu(\mathbf{q})u(p) \qquad \text{with } \mathbf{q} = \mathbf{p} - \mathbf{k}. \tag{4.58}$$

Then we have the squared amplitude with initial spin averaged and final spin summed (See §5.1 of P&S for details), as,

$$\begin{aligned}\tfrac{1}{2}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}(k,\theta)|^2 =& \tfrac{1}{2}e^2\tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\sum_{\text{spin}}\bar{u}(p)\gamma^\mu u(k)\bar{u}(k)\gamma^\nu u(p)\\ =& \tfrac{1}{2}e^2\tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\,\mathrm{tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right]\\ =& 2e^2\big[2(p\cdot\tilde{A})(k\cdot\tilde{A})+\big(m^2-(k\cdot p)\big)\tilde{A}^2\big].\end{aligned} \tag{4.59}$$

Note that

$$\tilde{A}^0(\mathbf{q}) = \frac{Ze}{|\mathbf{p}-\mathbf{k}|^2} = \frac{Ze}{4|\mathbf{k}|^2\sin^2(\theta/2)}, \tag{4.60}$$

thus

$$\tfrac{1}{2}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}(k,\theta)|^2 = \frac{Z^2e^4\big(1-v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^4v^2\sin^4(\theta/2)}, \tag{4.61}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2\big(1-v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^2v^2\sin^4(\theta/2)} \tag{4.62}$$

. In non-relativistic case, this formula reduces to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2}{4m^2v^4\sin^4(\theta/2)} \tag{4.63}$$

Chapter 5

Elementary Processes of Quantum Electrodynamics

5.1 Coulomb scattering

In this problem we continue our study of the Coulomb scattering in Problem 4.4. Here we consider the relativistic case. Let's first recall some main points considered before. The Coulomb potential $A^0 = Ze/4\pi r$ in momentum space is

$$A^0(\mathbf{q}) = \frac{Ze}{|\mathbf{q}|^2}. \tag{5.1}$$

Then the scattering amplitude is given by

$$\mathrm{i}\mathcal{M}(k,\theta) = \mathrm{i}e\bar{u}(p)\gamma^\mu \tilde{A}_\mu(\mathbf{q})u(p) \qquad \text{with } \mathbf{q} = \mathbf{p} - \mathbf{k}. \tag{5.2}$$

Then we can derive the squared amplitude with initial spin averaged and final spin summed, as:

$$\begin{aligned}\frac{1}{2}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}(k,\theta)|^2 &= \frac{1}{2}e^2 \tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q}) \sum_{\text{spin}} \bar{u}(p)\gamma^\mu u(k)\bar{u}(k)\gamma^\nu u(p) \\ &= \frac{1}{2}e^2 \tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\,\mathrm{tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right] \\ &= 2e^2\Big[2(p\cdot\tilde{A})(k\cdot\tilde{A}) + \big(m^2 - (k\cdot p)\big)\tilde{A}^2\Big]. \end{aligned} \tag{5.3}$$

Note that

$$\tilde{A}^0(\mathbf{q}) = \frac{Ze}{|\mathbf{p}-\mathbf{k}|^2} = \frac{Ze}{4|\mathbf{k}|^2\sin^2(\theta/2)}, \tag{5.4}$$

thus

$$\frac{1}{2}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}(k,\theta)|^2 = \frac{Z^2e^4\big(1 - v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^4 v^2 \sin^4(\theta/2)}, \tag{5.5}$$

Now, from the result of Problem 4.4(b), we know that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2}\Big(\tfrac{1}{2}\sum_{\text{spin}} |\mathcal{M}(k,\theta)|^2\Big) = \frac{Z^2\alpha^2\big(1 - v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^2 v^2\sin^4(\theta/2)}. \tag{5.6}$$

This is the formula for relativistic electron scatted by Coulomb potential, and is called Mott formula.

Now we give an alternative derivation of the Mott formula, by considering the cross section of $e^-\mu^{-Z} \to e^-\mu^{-Z}$. When the mass of μ goes to infinity and the charge of μ is taken to be Ze, this cross section will reduces to Mott formula.

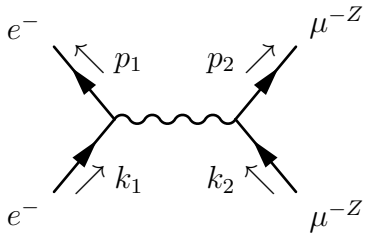


Figure 5.1: The scattering of an electron by a charged heavy particle μ^{-Z}. All initial momenta go inward and all final momenta go outward.

The corresponding Feynman diagram is shown in Figure 5.1, which reads,

$$\mathrm{i}\mathcal{M} = Z(-\mathrm{i}e)^2\bar{u}(p_1)\gamma^\mu u(k_1)\frac{-\mathrm{i}}{t}\bar{U}(p_2)\gamma_\mu U(k_2), \tag{5.7}$$

where u is the spinor for electron and U is the spinor for muon, $t = (k_1 - p_1)^2$ is one of three Mandelstam variables. Then the squared amplitude with initial spin averaged and final spin summed is

$$\begin{aligned}\tfrac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2 &= \frac{Z^2e^4}{t^2}\operatorname{tr}\Big[\gamma^\mu(\not{k}_1 + m)\gamma^\nu(\not{p}_1 + m)\Big]\operatorname{tr}\Big[\gamma_\mu(\not{k}_2 + M)\gamma_\nu(\not{p}_2 + M)\Big] \\ &= \frac{Z^2e^4}{t^2}\Big[16m^2M^2 - 8M^2(k_1\cdot p_1) + 8(k_1\cdot p_2)(k_2\cdot p_1) \\ &\qquad - 8m^2(k_2\cdot p_2) + 8(k_1\cdot k_2)(p_1\cdot p_2)\Big].\end{aligned} \tag{5.8}$$

Note that the cross section is given by

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}} = \frac{1}{2E_e 2E_\mu|v_{\mathbf{k}_1} - v_{\mathbf{k}_2}|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\text{CM}}}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big). \tag{5.9}$$

When the mass of μ goes to infinity, we have $E_\mu \simeq E_{\text{CM}} \simeq M$, $v_{\mathbf{k}_2} \simeq 0$, and $|\mathbf{p}_1| \simeq |\mathbf{k}_1|$. Then the expression above can be simplified to

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}} = \frac{1}{16(2\pi)^2M^2}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big). \tag{5.10}$$

When $M \to \infty$, only terms proportional to M^2 are relevant in $|\mathcal{M}|^2$. To evaluate this squared amplitude further, we assign each momentum a specific value in CM frame,

$$\begin{aligned} k_1 &= (E, 0, 0, k), & p_1 &\simeq (E, \sin\theta, 0, k\cos\theta), \\ k_2 &\simeq (M, 0, 0, -k), & p_2 &\simeq (M, -k\sin\theta, 0, -k\cos\theta), \end{aligned} \tag{5.11}$$

then $t = (k_1 - p_1)^2 = 4k^2 \sin^2 \frac{\theta}{2}$, and,

$$\tfrac{1}{4} \sum |\mathrm{i}\mathcal{M}|^2 = \frac{Z^2 e^4 (1 - v^2 \sin^2 \frac{\theta}{2})}{k^2 v^2 \sin^2 \frac{\theta}{2}} M^2 + \mathcal{O}(M). \tag{5.12}$$

Substituting this into the cross section, and sending $M \to \infty$, we reach the Mott formula again,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2 \alpha^2 \big(1 - v^2 \sin^2 \frac{\theta}{2}\big)}{4|\mathbf{k}|^2 v^2 \sin^4(\theta/2)}. \tag{5.13}$$

5.2 Bhabha scattering

The Bhabha scattering is the process $e^+e^- \to e^+e^-$. At the tree level, it consists of two diagrams, as shown in Figure 5.2.

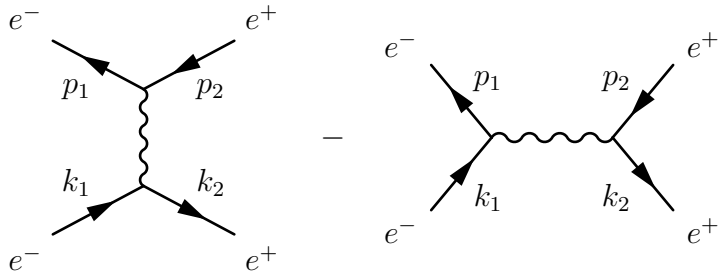


Figure 5.2: Bhabha scattering at tree level. All initial momenta go inward and all final momenta go outward.

The minus sign before the t-channel diagram comes from the exchange of two fermion field operators when contracting with in and out states. In fact, the s- and t-channel diagrams correspond to the following two ways of contraction, respectively,

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \bar{\psi} \not{A} \psi \bar{\psi} \not{A} \psi | \mathbf{k}_1 \mathbf{k}_2 \rangle, \qquad \langle \mathbf{p}_1 \mathbf{p}_2 | \bar{\psi} \not{A} \psi \bar{\psi} \not{A} \psi | \mathbf{k}_1 \mathbf{k}_2 \rangle. \tag{5.14}$$

In the high energy limit, we can omit the mass of electrons, then the amplitude for the whole scattering process is,

$$i\mathcal{M} = (-\mathrm{i}e)^2 \left[\bar{v}(k_2) \gamma^\mu u(k_1) \frac{-\mathrm{i}}{s} \bar{u}(p_1) \gamma_\mu v(p_2) - \bar{u}(p_1) \gamma^\mu u(k_1) \frac{-\mathrm{i}}{t} \bar{v}(k_2) \gamma_\mu v(p_2) \right], \tag{5.15}$$

where we have used the Mandelstam variables s, t and u. They are defined as,

$$s = (k_1 + k_2)^2, \quad t = (p_1 - k_1)^2, \quad u = (p_2 - k_1)^2. \tag{5.16}$$

In the massless case, $k_1^2 = k_2^2 = p_1^2 = p_2^2 = 0$, thus we have,

$$s = 2k_1 \cdot k_2 = 2p_1 \cdot p_2, \quad t = -2p_1 \cdot k_1 = -2p_2 \cdot k_2, \quad u = -2p_2 \cdot k_1 = -2p_1 \cdot k_2. \tag{5.17}$$

We want to get the unpolarized cross section, thus we must average the ingoing spins and sum over outgoing spins. That is,

$$\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{4s^2} \sum \left| \bar{v}(k_2) \gamma^\mu u(k_1) \bar{u}(p_1) \gamma_\mu v(p_2) \right|^2$$

$$
\begin{aligned}
&+\frac{e^4}{4t^2}\sum\left|\bar{u}(p_1)\gamma^\mu u(k_1)\bar{v}(k_2)\gamma_\mu v(p_2)\right|^2\\
&-\frac{e^4}{4st}\sum\Big[\bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(k_1)\gamma^\mu v(k_2)\bar{u}(p_1)\gamma^\nu u(k_1)\bar{v}(k_2)\gamma_\nu v(p_2)\ +\ \text{c.c.}\Big]\\
=&\ \frac{e^4}{4s^2}\,\mathrm{tr}\,(\not{k}_1\gamma^\mu\not{k}_2\gamma^\nu)\,\mathrm{tr}\,(\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu)+\frac{e^4}{4t^2}\,\mathrm{tr}\,(\not{k}_1\gamma^\mu\not{p}_1\gamma^\nu)\,\mathrm{tr}\,(\not{p}_2\gamma_\mu\not{k}_2\gamma_\nu)\\
&-\frac{e^4}{4st}\Big[\,\mathrm{tr}\,(\not{k}_1\gamma^\nu\not{k}_2\gamma_\mu\not{p}_2\gamma_\nu\not{p}_1\gamma^\mu)+\ \text{c.c.}\Big]\\
=&\ \frac{2e^4(u^2+t^2)}{s^2}+\frac{2e^4(u^2+s^2)}{t^2}+\frac{4e^4u^2}{st}\\
=&\ 2e^4\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\right].
\end{aligned}
\tag{5.18}
$$

In the center-of-mass frame, we have $k_1^0 = k_2^0 \equiv k^0$, and $\mathbf{k}_1 = -\mathbf{k}_2$, thus the total energy $E_{\rm CM}^2 = (k_1^0 + k_2^0)^2 = 4k^2 = s$. According to the formula for the cross section in the four identical particles' case (Eq.4.85):

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\rm CM} = \frac{1}{64\pi^2 E_{\rm CM}}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big),
\tag{5.19}
$$

thus

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\rm CM} = \frac{\alpha^2}{2s}\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\right],
\tag{5.20}
$$

where $\alpha = e^2/4\pi$ is the fine structure constant. We integrate this over the angle φ to get:

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\rm CM} = \frac{\pi\alpha^2}{s}\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\right].
\tag{5.21}
$$

5.3 The spinor products (2)

In this problem we continue our study of spinor product method in last chapter. The formulae needed in the following are:

$$
u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0}, \qquad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0}.
\tag{5.22}
$$

$$
s(p_1,p_2) = \bar{u}_R(p_1)u_L(p_2), \qquad t(p_1,p_2) = \bar{u}_L(p_1)u_R(p_2).
\tag{5.23}
$$

For detailed explanation for these relations, see Problem 3.3.

(a) Firstly, we prove the following relation,

$$
|s(p_1,p_2)|^2 = 2p_1\cdot p_2.
\tag{5.24}
$$

We make use of the another two relations,

$$
u_{L0}\bar{u}_{L0} = \frac{1-\gamma^5}{2}\not{k}_0, \qquad u_{R0}\bar{u}_{R0}\frac{1+\gamma^5}{2}\not{k}_0,
\tag{5.25}
$$

which are direct consequences of the familiar spin-sum formula $\sum u_0\bar{u}_0 = \not{k}_0$. We now generalize this to,

$$u_L(p)\bar{u}_L(p) = \frac{1-\gamma^5}{2}\not{p}, \qquad u_R(p)\bar{u}_R(p) = \frac{1+\gamma^5}{2}\not{p}. \tag{5.26}$$

We prove the first one:

$$\begin{aligned} u_L(p)\bar{u}_L(p) =& \frac{1}{2p\cdot k_0}\not{p}u_{R0}\bar{u}_{R0}\not{p} = \frac{1}{2p\cdot k_0}\not{p}\frac{1+\gamma^5}{2}\not{k}_0\not{p} \\ =& \frac{1}{2p\cdot k_0}\frac{1-\gamma^5}{2}\not{p}\not{k}_0\not{p} = \frac{1}{2p\cdot k_0}\frac{1-\gamma^5}{2}(2p\cdot k - \not{k}_0\not{p})\not{p} \\ =& \frac{1-\gamma^5}{2}\not{p} - \frac{1}{2p\cdot k_0}\frac{1-\gamma^5}{2}\not{k}_0 p^2 = \frac{1-\gamma^5}{2}\not{p}. \end{aligned} \tag{5.27}$$

The last equality holds because p is lightlike. Then we get,

$$\begin{aligned} |s(p_1,p_2)|^2 =& |\bar{u}_R(p_1)u_L(p_2)|^2 = \operatorname{tr}\Big(u_L(p_2)\bar{u}_L(p_2)u_R(p_1)\bar{u}_R(p_1)\Big) \\ =& \frac{1}{4}\operatorname{tr}\big((1-\gamma^5)\not{p}_2(1-\gamma^5)\not{p}_1\big) = 2p_1\cdot p_2. \end{aligned} \tag{5.28}$$

(b) Now we prove the relation,

$$\operatorname{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) = \operatorname{tr}(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}), \tag{5.29}$$

where $\mu_i = 0,1,2,3,5$.

To make things easier, let us perform the proof in Weyl representation, without loss of generality. Then it's easy to check that

$$(\gamma^\mu)^T = \begin{cases} \gamma^\mu, & \mu = 0,2,5; \\ -\gamma^\mu, & \mu = 1,3. \end{cases} \tag{5.30}$$

Then, we define $M = \gamma^1\gamma^3$, and it can be easily shown that $M^{-1}\gamma^\mu M = (\gamma^\mu)^T$, and $M^{-1}M = 1$. Then we have,

$$\begin{aligned} \operatorname{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) =& \operatorname{tr}(M^{-1}\gamma^{\mu_1}MM^{-1}\gamma^{\mu_2}M\cdots M^{-1}\gamma^{\mu_n}M) \\ =& \operatorname{tr}\big[(\gamma^{\mu_1})^T(\gamma^{\mu_2})^T\cdots(\gamma^{\mu_n})^T\big] = \operatorname{tr}\big[(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1})^T\big] \\ =& \operatorname{tr}(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}). \end{aligned} \tag{5.31}$$

With this formula in hand, we can derive the equality,

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2) = \bar{u}_R(p_2)\gamma^\mu u_R(p_1), \tag{5.32}$$

as follows,

$$\begin{aligned} \text{LHS} &= C\bar{u}_{R0}\not{p}_1\gamma^\mu\not{p}_2 u_{R0} = C\operatorname{tr}(\not{p}_1\gamma^\mu\not{p}_2) \\ &= C\operatorname{tr}(\not{p}_2\gamma^\mu\not{p}_1) = C\bar{u}_{L0}\not{p}_2\gamma^\mu\not{p}_1 u_{L0} = \text{RHS}, \end{aligned}$$

in which $C \equiv \big(2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}\big)^{-1}$.

(c) The way of proving the Fierz identity

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2)[\gamma_\mu]_{ab} = 2\big[u_L(p_2)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_2)\big]_{ab} \tag{5.33}$$

has been indicated in P&S. The right hand side of this identity, as a Dirac matrix, which we denoted by M, can be written as a linear combination of 16 Γ matrices listed in Problem 3.6. In addition, it is easy to check directly that $\gamma^\mu M = -M\gamma^5$. Thus M must have the form

$$M = \Big(\frac{1-\gamma^5}{2}\Big)\gamma_\mu V^\mu + \Big(\frac{1+\gamma^5}{2}\Big)\gamma_\mu W^\mu.$$

Each of the coefficients V^μ and W^μ can be determined by projecting out the other one with the aid of trace technology, that is,

$$V^\mu = \frac{1}{2}\,\mathrm{tr}\,\Big[\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)M\Big] = \bar{u}_L(p_1)\gamma^\mu u_L(p_2), \tag{5.34}$$

$$W^\mu = \frac{1}{2}\,\mathrm{tr}\,\Big[\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)M\Big] = \bar{u}_R(p_2)\gamma^\mu u_R(p_1) = \bar{u}_L(p_1)\gamma^\mu u_L(p_2). \tag{5.35}$$

The last equality follows from (5.32). Substituting V^μ and W^μ back, we finally get the left hand side of the Fierz identity, which completes the proof.

(d) The amplitude for the process at leading order in α is given by,

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e^2)\bar{u}_R(k_2)\gamma^\mu u_R(k_1)\frac{-\mathrm{i}}{s}\bar{v}_R(p_1)\gamma_\mu v_R(p_2). \tag{5.36}$$

To make use of the Fierz identity, we multiply (5.33), with the momenta variables changed to $p_1 \to k_1$ and $p_2 \to k_2$, by $\big[\bar{v}_R(p_1)\big]_a$ and $\big[v_R(p_2)\big]_b$, and also take account of (5.32), which leads to,

$$\begin{aligned}
&\bar{u}_R(k_2)\gamma^\mu u_R(k_1)\bar{v}_R(p_1)\gamma_\mu v_R(p_2)\\
&= 2\big[\bar{v}_R(p_1)u_L(k_2)\bar{u}_L(k_1)v_R(p_2) + \bar{v}_R(p_1)u_R(k_1)\bar{u}_R(k_2)v_R(p_2)\big]\\
&= 2s(p_1,k_2)t(k_1,p_2).
\end{aligned} \tag{5.37}$$

Then,

$$|\mathrm{i}\mathcal{M}|^2 = \frac{4e^4}{s^2}|s(p_1,k_2)|^2|t(k_1,p_2)|^2 = \frac{16e^4}{s^2}(p_1\cdot k_2)(k_1\cdot p_2) = e^4(1+\cos\theta)^2, \tag{5.38}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to \mu_L^+\mu_R^-) = \frac{|\mathrm{i}\mathcal{M}|^2}{64\pi^2E_{\mathrm{cm}}} = \frac{\alpha^2}{4E_{\mathrm{cm}}}(1+\cos\theta)^2. \tag{5.39}$$

It is straightforward to work out the differential cross section for other polarized processes in similar ways. For instance,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to \mu_R^+\mu_L^-) = \frac{e^4|t(p_1,k_1)|^2|s(k_2,p_2)|^2}{64\pi^2E_{\mathrm{cm}}} = \frac{\alpha^2}{4E_{\mathrm{cm}}}(1-\cos\theta)^2. \tag{5.40}$$

(e) Now we recalculate the Bhabha scattering studied in Problem 5.2, by evaluating all the polarized amplitudes. For instance,

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^+e_R^- \to e_L^+e_R^-) \\
&= (-\mathrm{i}e)^2\Big[\bar{u}_R(k_2)\gamma^\mu u_R(k_1)\frac{-\mathrm{i}}{s}\bar{v}_R(p_1)\gamma_\mu v_R(p_2) \\
&\qquad - \bar{u}_R(p_1)\gamma^\mu u_R(k_1)\frac{-\mathrm{i}}{t}\bar{v}_R(k_2)\gamma_\mu v_R(p_2)\Big] \\
&= 2\mathrm{i}e^2\left[\frac{s(p_1,k_2)t(k_1,p_2)}{s} - \frac{s(k_2,p_1)t(k_1,p_2)}{t}\right].
\end{aligned} \tag{5.41}$$

Similarly,

$$\mathrm{i}\mathcal{M}(e_L^+e_R^- \to e_R^+e_L^-) = 2\mathrm{i}e^2\frac{t(p_1,k_1)s(k_2,p_2)}{s}, \tag{5.42}$$

$$\mathrm{i}\mathcal{M}(e_R^+e_L^- \to e_L^+e_R^-) = 2\mathrm{i}e^2\frac{s(p_1,k_1)t(k_2,p_2)}{s}, \tag{5.43}$$

$$\mathrm{i}\mathcal{M}(e_R^+e_L^- \to e_R^+e_L^-) = 2\mathrm{i}e^2\left[\frac{t(p_1,k_2)s(k_1,p_2)}{s} - \frac{t(k_2,p_1)s(k_1,p_2)}{t}\right], \tag{5.44}$$

$$\mathrm{i}\mathcal{M}(e_R^+e_R^- \to e_R^+e_R^-) = 2\mathrm{i}e^2\frac{t(k_2,k_1)s(p_1,p_2)}{t}, \tag{5.45}$$

$$\mathrm{i}\mathcal{M}(e_L^+e_L^- \to e_L^+e_L^-) = 2\mathrm{i}e^2\frac{s(k_2,k_1)t(p_1,p_2)}{t}. \tag{5.46}$$

Squaring the amplitudes and including the kinematic factors, we find the polarized differential cross sections as,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to e_L^+e_R^-) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_R^+e_L^- \to e_R^+e_L^-) = \frac{\alpha^2u^2}{2s}\Big(\frac{1}{s}+\frac{1}{t}\Big)^2, \tag{5.47}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to e_R^+e_L^-) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_R^+e_L^- \to e_L^+e_R^-) = \frac{\alpha^2}{2s}\frac{t^2}{s^2}, \tag{5.48}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_R^+e_R^- \to e_R^+e_R^-) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_L^- \to e_L^+e_L^-) = \frac{\alpha^2}{2s}\frac{s^2}{t^2}. \tag{5.49}$$

Therefore we recover the result obtained in Problem 5.2,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e^+e^- \to e^+e^-) = \frac{\alpha^2}{2s}\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\right]. \tag{5.50}$$

5.4 Positronium lifetimes

In this problem we study the decay of positronium (Ps) in its S and P states. To begin with, we recall the formalism developed for bound states with nonrelativistic quantum mechanics in P&S. The positronium state $|\mathrm{Ps}\rangle$, as a bound state of an electron-positron pair, can be represented in terms of electron and positron's state vectors, as,

$$|\mathrm{Ps}\rangle = \sqrt{2M_\mathrm{P}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})C_{ab}\frac{1}{\sqrt{2m}}|e_a^-(\mathbf{k})\rangle\frac{1}{\sqrt{2m}}|e_b^+(-\mathbf{k})\rangle, \tag{5.51}$$

where m is the electron's mass, $M_{\rm P}$ is the mass of the positronium, which can be taken to be $2m$ as a good approximation, a and b are spin labels, the coefficient C_{ab} depends on the spin configuration of $|{\rm Ps}\rangle$, and $\psi(\mathbf{k})$ is the momentum space wave function for the positronium in nonrelativistic quantum mechanics. In real space, we have,

$$\psi_{100}(r) = \sqrt{\frac{(\alpha m_{\rm r})^3}{\pi}}\exp(-\alpha m_{\rm r} r), \tag{5.52}$$

$$\psi_{21i}(r) = \sqrt{\frac{(\alpha m_{\rm r}/2)^5}{\pi}} x^i \exp(-\alpha m_{\rm r} r/2). \tag{5.53}$$

where $m_{\rm r} = m/2$ is the reduced mass. Then the amplitude of the decay process ${\rm Ps} \to 2\gamma$ can be represented in terms of the amplitude for the process $e^+e^- \to 2\gamma$ as,

$$\mathcal{M}({\rm Ps} \to 2\gamma) = \frac{1}{\sqrt{m}}\int \frac{{\rm d}^3k}{(2\pi)^3}\psi(\mathbf{k}) C_{ab}\widehat{\mathcal{M}}\big(e_a^-(\mathbf{k})e_b^+(-\mathbf{k}) \to 2\gamma\big). \tag{5.54}$$

We put a hat on the amplitude of $e^+e^- \to 2\gamma$. In the following we always use a hat to denote the amplitude of this process.

(a) In this part we study the decay of the S-state positronium. As stated above, we have to know the amplitude of the process $e^+e^- \to 2\gamma$, which is illustrated in Figure 5.3 with the B replaced with γ, and is given by,

$$\begin{aligned}{\rm i}\widehat{\mathcal{M}} = {} & (-{\rm i}e)^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2) \\ & \times \bar{v}(k_2)\left[\gamma^\nu \frac{{\rm i}(\not{k}_1 - \not{p}_1 + m)}{(k_1-p_1)^2 - m^2}\gamma^\mu + \gamma^\mu \frac{{\rm i}(\not{k}_1 - \not{p}_2 + m)}{(k_1-p_2)^2 - m^2}\gamma^\nu\right]u(k_1),\end{aligned} \tag{5.55}$$

where the spinors can be written in terms of two-component spinors ξ and ξ' in the chiral representation as,

$$u(k_1) = \begin{pmatrix}\sqrt{k_1\cdot\sigma}\xi \\ \sqrt{k_1\cdot\bar{\sigma}}\xi\end{pmatrix}, \qquad v(k_2) = \begin{pmatrix}\sqrt{k_2\cdot\sigma}\xi' \\ -\sqrt{k_2\cdot\bar{\sigma}}\xi'\end{pmatrix}. \tag{5.56}$$

We also write γ^μ as,

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix},$$

where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$ with σ^i the three Pauli matrices. Then the amplitude can be brought into the following form,

$${\rm i}\widehat{\mathcal{M}} = -{\rm i}e^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\xi'^\dagger\left[\frac{\Gamma_t^{\mu\nu}}{(k_1-p_1)^2-m^2} + \frac{\Gamma_u^{\mu\nu}}{(k_1-p_2)^2-m^2}\right]\xi, \tag{5.57}$$

with

$$\begin{aligned}\Gamma_t^{\mu\nu} = {} & \Big(\sqrt{k_2\cdot\sigma}\bar{\sigma}^\nu\sigma^\mu\sqrt{k_1\cdot\bar{\sigma}} - \sqrt{k_2\cdot\bar{\sigma}}\sigma^\nu\bar{\sigma}^\mu\sqrt{k_1\cdot\sigma}\Big)m \\ & + \Big(\sqrt{k_2\cdot\sigma}\bar{\sigma}^\nu\sigma^\lambda\bar{\sigma}^\mu\sqrt{k_1\cdot\sigma} - \sqrt{k_2\cdot\bar{\sigma}}\sigma^\nu\bar{\sigma}^\lambda\sigma^\mu\sqrt{k_1\cdot\bar{\sigma}}\Big)(k_1-p_1)_\lambda,\end{aligned}$$

$$\Gamma_u^{\mu\nu} = \Big(\sqrt{k_2\cdot\sigma}\bar\sigma^\mu\sigma^\nu\sqrt{k_1\cdot\bar\sigma} - \sqrt{k_2\cdot\bar\sigma}\sigma^\mu\bar\sigma^\nu\sqrt{k_1\cdot\sigma}\Big)m \\ + \Big(\sqrt{k_2\cdot\sigma}\bar\sigma^\mu\sigma^\lambda\bar\sigma^\nu\sqrt{k_1\cdot\sigma} - \sqrt{k_2\cdot\bar\sigma}\sigma^\mu\bar\sigma^\lambda\sigma^\nu\sqrt{k_1\cdot\bar\sigma}\Big)(k_1-p_2)_\lambda.$$

In the rest of the part (a), we take the nonrelativistic limit, with the momenta chosen to be

$$k_1^\mu = k_2^\mu = (m,0,0,0), \qquad p_1^\mu = (m,0,0,m), \qquad p_2^\mu = (m,0,0,-m). \tag{5.58}$$

Accordingly, we can assign the polarization vectors for final photons to be,

$$\epsilon_\pm^\mu(p_1) = \tfrac{1}{\sqrt2}(0,1,\pm\mathrm{i},0), \qquad \epsilon_\pm^\mu(p_2) = \tfrac{1}{\sqrt2}(0,-1,\pm\mathrm{i},0). \tag{5.59}$$

Now substituting the momenta (5.58) into (5.57), noticing that $\sqrt{k_i\cdot\sigma} = \sqrt{k_i\cdot\bar\sigma} = \sqrt{m}$ ($i = 1,2$), and $(k_1-p_1)^2 = (k_1-p_2)^2 = -m^2$, and also using the trick that one can freely make the substitution $\bar\sigma^\mu \to -\sigma^\mu$ since the temporal component of the polarization vectors ϵ_μ always vanishes, we get a much more simplified expression,

$$\mathrm{i}\widehat{\mathcal{M}} = \mathrm{i}e^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\xi'^\dagger\big(\sigma^\nu\sigma^3\sigma^\mu - \sigma^\mu\sigma^3\sigma^\nu\big)\xi. \tag{5.60}$$

The positronium can lie in spin-0 (singlet) state or spin-1 (triplit) state. In the former case, we specify the polarizations of final photons in all possible ways, and also make the substitution $\xi\xi'^\dagger \to \frac{1}{\sqrt2}$ (See (5.49) of P&S), which leads to,

$$\mathrm{i}\widehat{\mathcal{M}}^s_{++} = -\mathrm{i}\widehat{\mathcal{M}}^s_{--} = \mathrm{i}2\sqrt2e^2, \qquad \mathrm{i}\widehat{\mathcal{M}}^s_{+-} = \mathrm{i}\widehat{\mathcal{M}}^s_{-+} = 0, \tag{5.61}$$

where the subscripts denote final photons' polarizations and s means singlet. We show the mid-step for calculating $\mathrm{i}\mathcal{M}^s_{++}$ as an example,

$$\mathrm{i}\widehat{\mathcal{M}}^s_{++} = \frac{\mathrm{i}e^2}{2}\,\mathrm{tr}\Big[\Big((\sigma^1+\mathrm{i}\sigma^2)\sigma^3(-\sigma^1+\mathrm{i}\sigma^2) - (-\sigma^1+\mathrm{i}\sigma^2)\sigma^3(\sigma^1+\mathrm{i}\sigma^2)\Big)\xi\xi'^\dagger\Big] = \mathrm{i}2\sqrt2e^2.$$

In the same way, we can calculate the case of triplet initial state. This time, we make the substitution $\xi\xi'^\dagger \to \mathbf{n}\cdot\boldsymbol{\sigma}/\sqrt2$, with $\mathbf{n} = (\hat x \pm \mathrm{i}\hat y)/\sqrt2$ or $\mathbf{n} = \hat z$, corresponding to three independent polarizations. But it is straightforward to show that the amplitudes with these initial polarizations all vanish, which is consistent with our earlier results by using symmetry arguments in Problem 3.8.

Therefore it is enough to consider the singlet state only. The amplitude for the decay of a positronium in its ${}^1S_0$ state into 2γ then follows directly from (5.54), as

$$\mathcal{M}_{\pm\pm}({}^1S_0 \to 2\gamma) = \frac{\psi(\mathbf{x}=0)}{\sqrt m}\widehat{\mathcal{M}}^s_{\pm\pm}, \tag{5.62}$$

where $\psi(\mathbf{x}=0) = \sqrt{(m\alpha/2)^3/\pi}$ according to (5.52). Then the squared amplitude with final photons' polarizations summed is

$$\sum_{\text{spin}}\big|\mathcal{M}({}^1S_0\to2\gamma)\big|^2 = \frac{|\psi(0)|^2}{2m}\Big(|\mathcal{M}^s_{++}|^2 + |\mathcal{M}^s_{--}|^2\Big) = 16\pi\alpha^5m^2. \tag{5.63}$$

Finally we find the decay width of the process $\mathrm{Ps}(^1S_0) \to 2\gamma$, to be

$$\begin{aligned}\Gamma(^1S_0 \to 2\gamma) &= \frac{1}{2}\frac{1}{4m}\int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 2E_1 2E_2}\sum\left|\mathcal{M}(^1S_0 \to 2\gamma)\right|^2(2\pi)^4\delta^{(4)}(p_{\mathrm{Ps}}-p_1-p_2)\\ &= \frac{1}{2}\frac{1}{4m}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3 4m^2}\sum\left|\mathcal{M}(^1S_0 \to 2\gamma)\right|^2(2\pi)\delta(m-E_1)\\ &= \frac{1}{2}\alpha^5 m, \end{aligned} \tag{5.64}$$

where an additional factor of $1/2$ follows from the fact that the two photons in the final state are identical particles.

(b) To study the decay of P state ($l = 1$) positronium, we should keep one power of 3-momenta of initial electron and positron. Thus we set the momenta of initial and final particles, and also the polarization vectors of the latter, in $e^-e^+ \to 2\gamma$, to be

$$\begin{aligned} &k_1^\mu = (E,0,0,k), && k_2^\mu = (E,0,0,-k),\\ &p_1^\mu = (E,E\sin\theta,0,E\cos\theta), && p_2^\mu = (E,-E\sin\theta,0,-E\cos\theta),\\ &\epsilon_\pm^\mu(p_1) = \tfrac{1}{\sqrt{2}}(0,\cos\theta,\pm\mathrm{i},-\sin\theta), && \epsilon_\pm^\mu(p_2) = \tfrac{1}{\sqrt{2}}(0,-\cos\theta,\pm\mathrm{i},\sin\theta). \end{aligned} \tag{5.65}$$

Here we have the approximate expression up to linear order in k:

$$\begin{aligned}\sqrt{k_1\cdot\sigma} &= \sqrt{k_2\cdot\bar\sigma} = \sqrt{m} - \frac{k}{2\sqrt{m}}\sigma^3 + \mathcal{O}(k^2),\\ \sqrt{k_2\cdot\sigma} &= \sqrt{k_1\cdot\bar\sigma} = \sqrt{m} + \frac{k}{2\sqrt{m}}\sigma^3 + \mathcal{O}(k^2),\\ \frac{1}{(k_1-p_1)^2-m^2} &= -\frac{1}{2m^2} - \frac{k\cos\theta}{2m^3} + \mathcal{O}(k^2),\\ \frac{1}{(k_1-p_2)^2-m^2} &= -\frac{1}{2m^2} + \frac{k\cos\theta}{2m^3} + \mathcal{O}(k^2).\end{aligned}$$

Consequently,

$$\begin{aligned}\Gamma_t^{\mu\nu} &= 2m^2\sigma^\nu(\sigma^1 s_\theta + \sigma^3 c_\theta)\sigma^\mu - mk\big(\sigma^3\sigma^\nu\sigma^\mu + \sigma^\nu\sigma^\mu\sigma^3 + 2\sigma^\nu\sigma^3\sigma^\mu\big) + \mathcal{O}(k^2),\\ \Gamma_u^{\mu\nu} &= -2m^2\sigma^\mu(\sigma^1 s_\theta + \sigma^3 c_\theta)\sigma^\nu - mk\big(\sigma^3\sigma^\mu\sigma^\nu + \sigma^\mu\sigma^\nu\sigma^3 + 2\sigma^\mu\sigma^3\sigma^\nu\big) + \mathcal{O}(k^2),\end{aligned}$$

where we use the shorthand notation $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. We can use these expansion to find the terms in the amplitude $\mathrm{i}\widehat{\mathcal{M}}$ of linear order in k, to be

$$\begin{aligned}\mathrm{i}\widehat{\mathcal{M}}\big|_{\mathcal{O}(k)} = &-\mathrm{i}e^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\frac{k}{2m}\xi'^\dagger\Big[-2c_\theta\sigma^\mu(\sigma^1 s_\theta+\sigma^3 c_\theta)\sigma^\nu - 2c_\theta\sigma^\nu(\sigma^1 s_\theta+\sigma^3 c_\theta)\sigma^\mu\\ &+\big(\sigma^3\sigma^\mu\sigma^\nu+\sigma^\mu\sigma^\nu\sigma^3+2\sigma^\mu\sigma^3\sigma^\nu\big)+\big(\sigma^3\sigma^\nu\sigma^\mu+\sigma^\nu\sigma^\mu\sigma^3+2\sigma^\nu\sigma^3\sigma^\mu\big)\Big]\xi,\end{aligned} \tag{5.66}$$

Feeding in the polarization vectors of photons, and also make the substitution $\xi\xi'^\dagger \to \mathbf{n}\cdot\boldsymbol{\sigma}/\sqrt{2}$ or $1/\sqrt{2}$ for triplet and singlet positronium, respectively, as done in last part, we get

$$\mathrm{i}\widehat{\mathcal{M}}^{\downarrow\downarrow}_{\pm\pm}|_{\mathcal{O}(k)} = 0, \qquad \mathrm{i}\widehat{\mathcal{M}}^{\downarrow\downarrow}_{\pm\mp}|_{\mathcal{O}(k)} = -\mathrm{i}2s_\theta(\mp1+c_\theta)e^2k/m,$$

$$
\begin{aligned}
&\mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow+\downarrow\uparrow}_{\pm\pm}|_{\mathcal{O}(k)} = \mathrm{i}2\sqrt{2}e^2k/m, && \mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow+\downarrow\uparrow}_{\pm\mp}|_{\mathcal{O}(k)} = \mathrm{i}2\sqrt{2}s_\theta^2e^2k/m,\\
&\mathrm{i}\widehat{\mathcal{M}}^{\uparrow\uparrow}_{\pm\pm}|_{\mathcal{O}(k)} = 0, && \mathrm{i}\widehat{\mathcal{M}}^{\uparrow\uparrow}_{\pm\mp}|_{\mathcal{O}(k)} = -\mathrm{i}2s_\theta(\pm1+c_\theta)e^2k/m,\\
&\mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow-\downarrow\uparrow}_{\pm\pm}|_{\mathcal{O}(k)} = 0, && \mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow-\downarrow\uparrow}_{\pm\mp}|_{\mathcal{O}(k)} = 0.
\end{aligned}
\tag{5.67}
$$

The vanishing results in the last line indicate that $S=0$ state of P-wave positronium cannot decay to two photons.

(c) Now we prove that the state,

$$
|B(\mathbf{k})\rangle = \sqrt{2M_{\mathrm{P}}}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\psi_i(\mathbf{p})a^\dagger_{\mathbf{p}+\mathbf{k}/2}\Sigma^i b^\dagger_{-\mathbf{p}+\mathbf{k}/2}|0\rangle \tag{5.68}
$$

, is a properly normalized state for the P-wave positronium. In fact,

$$
\begin{aligned}
\langle B(\mathbf{k})|B(\mathbf{k})\rangle &= 2M_{\mathrm{P}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi^*_j(\mathbf{p}')\psi_i(\mathbf{p})\\
&\qquad\times\langle 0|b_{-\mathbf{p}'+\mathbf{k}/2}\Sigma^{j\dagger}a_{\mathbf{p}'+\mathbf{k}/2}a^\dagger_{\mathbf{p}+\mathbf{k}/2}\Sigma^i b^\dagger_{-\mathbf{p}'+\mathbf{k}/2}|0\rangle\\
&= 2M_{\mathrm{P}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi^*_j(\mathbf{p}')\psi_i(\mathbf{p})\\
&\qquad\times\langle 0|b_{-\mathbf{p}'+\mathbf{k}/2}\Sigma^{j\dagger}\Sigma^i b^\dagger_{-\mathbf{p}'+\mathbf{k}/2}|0\rangle(2\pi)^3\delta^{(3)}(\mathbf{p}'-\mathbf{p})\\
&= 2M_{\mathrm{P}}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi^*_j(\mathbf{p})\psi_i(\mathbf{p})\langle 0|b_{-\mathbf{p}+\mathbf{k}/2}\Sigma^{j\dagger}\Sigma^i b^\dagger_{-\mathbf{p}+\mathbf{k}/2}|0\rangle\\
&= 2M_{\mathrm{P}}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi^*_j(\mathbf{p})\psi_i(\mathbf{p})\langle 0|\,\mathrm{tr}\,(\Sigma^{j\dagger}\Sigma^i)|0\rangle(2\pi)^3\delta^{(3)}(\mathbf{0})\\
&= 2M_{\mathrm{P}}\cdot(2\pi)^3\delta^{(3)}(\mathbf{0}),
\end{aligned}
\tag{5.69}
$$

which is precisely the needed normalization of a state. In this calculation we have used the anticommutation relations of creation and annihilation operators, as well as the normalization of the wave function and the Σ matrices.

(d) Now we evaluate the partial decay rate of the $S=1$ P-wave positronium of definite J into two photons. The states for the positronium is presented in (c), with the Σ matrices chosen as

$$
\Sigma = \begin{cases} \dfrac{1}{\sqrt{6}}\sigma^i, & J=0,\\[2ex] \dfrac{1}{2}\epsilon^{ijk}n^j\sigma^k, & J=1,\\[2ex] \dfrac{1}{\sqrt{3}}h^{ij}\sigma^j, & J=2, \end{cases}
\tag{5.70}
$$

and the wave function given by (5.53).

Firstly, consider the $J=0$ state, in which case we have,

$$
\mathrm{i}\mathcal{M}({}^3P_0\to\gamma_\alpha\gamma_\beta) = \frac{1}{\sqrt{m}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi_i(\mathbf{k})\Big(\frac{1}{\sqrt{6}}\sigma^i\Big)_{ab}\mathrm{i}\widehat{\mathcal{M}}\big(e^-_a(\mathbf{k})e^+_b(-\mathbf{k})\to\gamma_\alpha\gamma_\beta\big), \tag{5.71}
$$

where $\alpha, \beta = +$ or $-$ are labels of photons' polarizations and $a, b = \uparrow$ or $\downarrow$ are spinor indices. For amplitude $\mathrm{i}\widehat{\mathcal{M}}$, we only need the terms linear in $\mathbf{k}$, as listed in (5.67). Let us rewrite this as,

$$\mathrm{i}\widehat{\mathcal{M}}\big(e_a^-(\mathbf{k})e_b^+(-\mathbf{k}) \to \gamma_\alpha\gamma_\beta\big) = F^{ab}_{\alpha\beta,i}k^i.$$

In the same way, the wave function can also be put into the form of $\psi_i(\mathbf{x}) = x^i f(r)$, with $r = |\mathbf{x}|$. Then the integration above can be carried out to be,

$$\begin{aligned}\mathrm{i}\mathcal{M}({}^3P_0 \to \gamma_\alpha\gamma_\beta) &= \frac{\mathrm{i}}{\sqrt{6m}}\sigma^i_{ab}F^{ab}_{\alpha\beta,j}\left[\mathrm{i}\frac{\partial}{\partial x^j}\psi^i(\mathbf{x})\right]_{\mathbf{x}=0} \\ &= \frac{\mathrm{i}}{\sqrt{6m}}\sigma^i_{ab}F^{ab}_{\alpha\beta,i}f(0). \end{aligned} \tag{5.72}$$

On the other hand, we have chose the direction of $\mathbf{k}$ to be in the x^3-axis, then $F^{ab}_{\alpha\beta,1} = F^{ab}_{\alpha\beta,2} = 0$ as a consequence. Therefore,

$$\mathrm{i}\mathcal{M}({}^3P_0 \to \gamma_\pm\gamma_\mp) = \frac{\mathrm{i}}{\sqrt{6m}}f(0)\big(F^{\uparrow\uparrow}_{\pm\mp,3} - F^{\downarrow\downarrow}_{\pm\mp,3}\big) = \pm\sqrt{\frac{\pi\alpha^7}{24}}m\sin\theta. \tag{5.73}$$

Square these amplitudes, sum over the photons' polarizations, and finish the phase space integration in the same way as what we did in (a), we finally get the partial decay rate of the $J = 0$ P-wave positronium into two photons to be,

$$\Gamma({}^3P_0) = \frac{1}{576}\alpha^7 m. \tag{5.74}$$

The positronium in ${}^3P_1$ state, namely the case $J = 1$, cannot decay into two photons by the conservation of the angular momentum, since the total angular momentum of two physical photons cannot be 1. Therefore let us turn to the case of $J = 2$. In this case we should average over the initial polarizations of the positronium, which can be represented by the symmetric and traceless polarization tensors h^{ij}_n, with $n = 1, 2, \cdots, 5$ the labeled of 5 independent polarizations. Let us choose these tensors to be,

$$\begin{aligned} h^{ij}_1 &= \tfrac{1}{\sqrt{2}}(\delta^{i2}\delta^{j3} + \delta^{i3}\delta^{j2}), & h^{ij}_2 &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j3} + \delta^{i3}\delta^{j1}), \\ h^{ij}_3 &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j2} + \delta^{i2}\delta^{j1}), & h^{ij}_4 &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j1} - \delta^{i2}\delta^{j2}), \\ h^{ij}_5 &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j1} - \delta^{i3}\delta^{j3}). \end{aligned} \tag{5.75}$$

Then the decay amplitude for a specific polarization of $J = 2$ Ps can be represented as,

$$\begin{aligned}\mathrm{i}\mathcal{M}_n({}^3P_2 \to \gamma_\alpha\gamma_\beta) &= \frac{1}{\sqrt{m}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi_i(\mathbf{k})\Big(\frac{1}{\sqrt{3}}h^{ij}_n\sigma^j\Big)_{ab}\mathrm{i}\widehat{\mathcal{M}}\big(e_a^-(\mathbf{k})e_b^+(-\mathbf{k}) \to \gamma_\alpha\gamma_\beta\big) \\ &= \frac{1}{\sqrt{3m}}h^{ij}_n\sigma^j_{ab}F^{ab}_{\alpha\beta,i}f(0). \end{aligned} \tag{5.76}$$

Now substituting all stuffs in, we find the nonvanishing components of the decay amplitude to be,

$$\mathrm{i}\mathcal{M}_2({}^3P_2 \to \gamma_\pm\gamma_\pm) = \sqrt{\frac{\pi\alpha^7}{48}}\mathrm{i}m,$$

$$\mathrm{i}\mathcal{M}_2(^3P_2 \to \gamma_\pm\gamma_\mp) = \sqrt{\frac{\pi\alpha^7}{48}}\mathrm{i}m\sin^2\theta,$$
$$\mathrm{i}\mathcal{M}_5(^3P_2 \to \gamma_\pm\gamma_\mp) = \mp 2\sqrt{\frac{\pi\alpha^7}{48}}\mathrm{i}m\sin^2\theta. \tag{5.77}$$

Squaring these amplitudes, summing over photon's polarizations and averaging the initial polarization of the positronium (by dividing the squared and summed amplitude by 5), we get,

$$\frac{1}{5}\sum_{\text{spin}}\left|\mathcal{M}_n(^3P_2 \to 2\gamma)\right|^2 = \frac{\pi\alpha^7 m^2}{120}(1+\sin^2\theta+4\sin^4\theta). \tag{5.78}$$

Finally, we finish the phase space integration and get the partial decay rate of $^3P_2$ positronium into 2 photons to be

$$\Gamma(^3P_2) = \frac{19}{19200}\alpha^7 m. \tag{5.79}$$

5.5 Physics of a massive vector boson

In this problem, the mass of electron is always set to zero.

(a) We firstly compute the cross section $\sigma(e^+e^- \to B)$ and the decay rate $\Gamma(B \to e^+e^-)$. For the cross section, the squared amplitude can be easily found to be

$$\frac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spin}}\left|\mathrm{i}g\epsilon_\mu^{*(i)}\bar{v}(p')\gamma^\mu u(p)\right|^2 = 2g^2(p\cdot p'). \tag{5.80}$$

Note that we have set the mass of electrons to be zero. Then the cross section can be deduced from (4.79). Let's take the initial momenta to be,

$$p = \tfrac{1}{2}(E,0,0,E), \qquad p' = \tfrac{1}{2}(E,0,0,-E), \tag{5.81}$$

with E being the center-of-mass energy. Then it's easy to get,

$$\sigma = \frac{g^2}{4E}(2\pi)\delta(M_B - E) = \frac{g^2}{4E}(2\pi)2M_B\delta(M_B^2 - s) = \pi g^2\delta(M_B^2 - s), \tag{5.82}$$

where $s = E^2$.

To deduce the decay rate, we should average polarizations of massive vector B instead of two electrons. Thus the squared amplitude in this case reads,

$$\frac{1}{3}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2 = \frac{8}{3}g^2(p\cdot p'). \tag{5.83}$$

The decay rate can be found from (4.86),

$$\Gamma = \frac{1}{2M_B}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3p'}{2\pi^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{p}'}}\left(\tfrac{1}{3}\sum|\mathcal{M}|^2\right)(2\pi)^4\delta^{(4)}(p_B - p - p')$$

$$
\begin{aligned}
=&\frac{1}{2M_B}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}^2}\Big(\tfrac{16}{3}g^2E_{\mathbf{p}}^2\Big)(2\pi)\delta(M_B-2E_{\mathbf{p}})\\
=&\frac{4\pi}{(2\pi)^2 2M_B}\int\mathrm{d}p\,\tfrac{4}{3}g^2E_{\mathbf{p}}^2\tfrac{1}{2}\delta(\tfrac{1}{2}M_B-E_{\mathbf{p}})=\frac{g^2M_B}{12\pi}.
\end{aligned}
\tag{5.84}
$$

We see the cross section and the decay rate satisfy the following relation, as expected,

$$
\sigma(e^+e^-\to B)=\frac{12\pi^2}{M_B}\Gamma(B\to e^+e^-)\delta(s-M^2).
\tag{5.85}
$$

(b) Now we calculate the cross section $\sigma(e^-e^+\to\gamma+B)$ in COM frame. The related diagrams are shown in Figure 5.3.

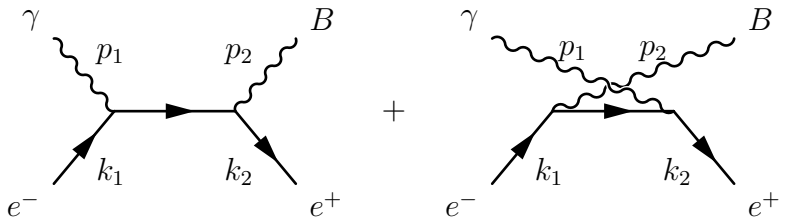


Figure 5.3: The tree diagrams of the process $e^-e^+\to\gamma+B$. All initial momenta go inward and all final momenta go outward.

The amplitude reads,

$$
\mathrm{i}\mathcal{M}=(-\mathrm{i}e)(-\mathrm{i}g)\epsilon_\mu^*(p_1)e_\nu^*(p_2)\bar v(k_2)\Big[\gamma^\nu\frac{\mathrm{i}}{\not k_1-\not p_1}\gamma^\mu+\gamma^\mu\frac{\mathrm{i}}{\not k_1-\not p_2}\gamma^\nu\Big]u(k_1),
\tag{5.86}
$$

where ϵ_μ is the polarization of photon while e_μ is the polarization for B. Now we square this amplitude,

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2=&\frac{1}{4}e^2g^2g_{\mu\rho}g_{\nu\sigma}\,\mathrm{tr}\Big[\Big(\frac{\gamma^\nu(\not k_1-\not p_1)\gamma^\mu}{t}+\frac{\gamma^\mu(\not p_1-\not k_2)\gamma^\nu}{u}\Big)\not k_1\\
&\times\Big(\frac{\gamma^\rho(\not k_1-\not p_1)\gamma^\sigma}{t}+\frac{\gamma^\sigma(\not p_1-\not k_2)\gamma^\rho}{u}\Big)\not k_2\Big]\\
=&\,8e^2g^2\Big[\frac{(k_1\cdot p_1)(k_2\cdot p_1)}{t^2}+\frac{(k_1\cdot p_1)(k_2\cdot p_1)}{u^2}\\
&+\frac{2(k_1\cdot k_2)(k_1\cdot k_2-k_1\cdot p_1-k_2\cdot p_1)}{tu}\Big]\\
=&\,2e^2g^2\Big[\frac{u}{t}+\frac{t}{u}+\frac{2s(s+t+u)}{tu}\Big]\\
=&\,2e^2g^2\Big[\frac{u}{t}+\frac{t}{u}+\frac{2sM_B^2}{tu}\Big].
\end{aligned}
\tag{5.87}
$$

Then the cross section can be evaluated as,

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\mathrm{CM}}=\frac{1}{2E_{\mathbf{k}_1}2E_{\mathbf{k}_2}|v_{\mathbf{k}_1}-v_{\mathbf{k}_2}|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\mathrm{CM}}}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big)
$$

$$=\frac{e^2g^2}{32\pi^2 s}\Big(1-\frac{M_B^2}{s}\Big)\bigg[\frac{u}{t}+\frac{t}{u}+\frac{2sM_B^2}{tu}\bigg]. \tag{5.88}$$

We can also write this differential cross section in terms of squared COM energy s and scattering angle θ. To do this, we note,

$$s=E_{\rm CM}^2, \qquad t=(M_B^2-E_{\rm CM}^2)\sin^2\tfrac{\theta}{2}, \qquad u=(M_B^2-E_{\rm CM}^2)\cos^2\tfrac{\theta}{2}. \tag{5.89}$$

Then we have,

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\rm CM}=\frac{e^2g^2(1-M_B^2/s)}{16\pi^2 s\sin^2\theta}\bigg[1+\cos^2\theta+\frac{4sM_B^2}{(s-M_B^2)^2}\bigg], \tag{5.90}$$

and,

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\rm CM}=\frac{\alpha g^2(1-M_B^2/s)}{2s\sin^2\theta}\bigg[1+\cos^2\theta+\frac{4sM_B^2}{(s-M_B^2)^2}\bigg]. \tag{5.91}$$

(c) The differential cross obtained in (b) diverges when $\theta\to 0$ or $\theta\to\pi$. Now let us study the former case, namely $\theta\to 0$.

If we cut of the integral from $\theta_c^2\simeq m_e^2/s$, then we have,

$$\begin{aligned}\int_{\theta_c}\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\rm CM}\sin\theta\mathrm{d}\theta &\simeq \frac{\alpha g^2(1-M_B^2/s)}{2s}\bigg[2+\frac{4sM_B^2}{(s-M_B^2)^2}\bigg]\int^{1-m_e^2/s}\frac{\mathrm{d}t}{1-t^2}\\ &\simeq \frac{\alpha g^2(1-M_B^2/s)}{4s}\bigg[2+\frac{4sM_B^2}{(s-M_B^2)^2}\bigg]\log\Big(\frac{s}{m_e^2}\Big)\\ &= \frac{\alpha g^2}{2}\frac{1+M_B^4/s^2}{s-M_B^2}\log\Big(\frac{s}{m_e^2}\Big).\end{aligned} \tag{5.92}$$

Now we calculate the following expression,

$$\begin{aligned}&\int_0^1\mathrm{d}x\, f(x)\sigma(e^+e^-\to B)\big|_{E_{\rm CM}=(1-x)s}\\ &=\int_0^1\mathrm{d}x\bigg[\frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\log\Big(\frac{s}{m_e^2}\Big)\bigg]\pi g^2\delta\big(M_B^2-(1-x)s\big)\\ &=\frac{\alpha g^2}{2}\frac{1+M_B^4/s^2}{s-M_B^2}\log\Big(\frac{s}{m_e^2}\Big).\end{aligned} \tag{5.93}$$

5.6 The spinor products (3)

This problem generalize the spinor product formalism to the processes involving external photons.

(a) Firstly we can represent photon's polarization vectors in terms of spinors of definite helicity. Let the momentum of the photon be k, and p be a lightlike momentum such that $p\cdot k\neq 0$. Then, the polarization vector $\epsilon_\pm^\mu(k)$ of the photon can be taken to be,

$$\epsilon_+^\mu(k)=\frac{1}{\sqrt{4p\cdot k}}\bar{u}_R(k)\gamma^\mu u_R(p), \qquad \epsilon_-^\mu(k)=\frac{1}{\sqrt{4p\cdot k}}\bar{u}_L(k)\gamma^\mu u_L(p), \tag{5.94}$$

where the spinors $u_{L,R}(k)$ have been introduced in Problems 3.3 and 5.3. Now we use this choice to calculate the polarization sum,

$$\begin{aligned}
&\epsilon_+^\mu(k)\epsilon_+^{\nu*}(k) + \epsilon_-^\mu(k)\epsilon_-^{\nu*}(k) \\
&= \frac{1}{4p\cdot k}\Big[\bar u_R(k)\gamma^\mu u_R(p)\bar u_R(p)\gamma^\nu u_R(k) + \bar u_L(k)\gamma^\mu u_L(p)\bar u_L(p)\gamma^\nu u_L(k)\Big] \\
&= \frac{1}{4p\cdot k}\,\mathrm{tr}\left[\not p\gamma^\nu \not k\gamma^\mu\right] = -g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{p\cdot k}.
\end{aligned} \tag{5.95}$$

When dotted into an amplitude with external photon, the second term of the result vanishes. This justifies the definitions above for photon's polarization vectors.

(b) Now we apply the formalism to the process $e^+e^- \to 2\gamma$ in the massless limit. The relevant diagrams are similar to those in Figure 5.3, except that one should replace the label 'B' by 'γ'. To simplify expressions, we introduce the standard shorthand notations as follows:

$$p\rangle = u_R(p), \qquad p] = u_L(p), \qquad \langle p = \bar u_L(p), \qquad [p = \bar u_R(p). \tag{5.96}$$

Then the spin products become $s(p_1,p_2) = [p_1p_2]$ and $t(p_1,p_2) = \langle p_1p_2\rangle$. Various expressions get simplified with this notation. For example, the Fierz identity (5.37) now reads $[k_2\gamma^\mu k_1\rangle[p_1\gamma_\mu p_2\rangle = 2[p_1k_2]\langle k_1p_2\rangle$. Similarly, we also have $\langle k_1\gamma^\mu k_2]\langle p_1\gamma_\mu p_2] = 2\langle k_1p_1\rangle[p_2k_2]$.

Now we write down the expression for tree amplitude of $e_R^+e_L^- \to \gamma_R\gamma_L$. For illustration, we still keep the original expression as well as all explicit mid-steps. The auxiliary lightlike momenta used in the polarization vectors are arbitrarily chosen such that the calculation can be mostly simplified.

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_R^+e_L^- \to \gamma_L\gamma_R) \\
&= (-\mathrm{i}e)^2\epsilon_{-\mu}^*(p_1)\epsilon_{+\nu}^*(p_2)\bar u_L(k_2)\left[\gamma^\nu\frac{\mathrm{i}}{\not k_1 - \not p_1}\gamma^\mu + \gamma^\mu\frac{\mathrm{i}}{\not k_1 - \not p_1}\gamma^\nu\right]u_L(k_1) \\
&= -\,\mathrm{i}e^2\frac{\langle k_2\gamma_\mu p_1][k_1\gamma_\nu p_2\rangle}{4\sqrt{(k_2\cdot p_1)(k_1\cdot p_2)}}\left[\frac{\langle k_2\gamma^\nu(\not k_1 - \not p_1)\gamma^\mu k_1]}{t} + \frac{\langle k_2\gamma^\mu(\not k_1 - \not p_2)\gamma^\nu k_1]}{u}\right] \\
&= -\,\mathrm{i}e^2\frac{\langle k_2\gamma_\mu p_1][k_1\gamma_\nu p_2\rangle}{2u}\left[\frac{\langle k_2\gamma^\nu k_1\rangle\langle k_1\gamma^\mu k_1] - \langle k_2\gamma^\nu p_1\rangle\langle p_1\gamma^\mu k_1]}{t}\right. \\
&\qquad\qquad\left. + \frac{\langle k_2\gamma^\mu k_1\rangle\langle k_1\gamma^\nu k_1] - \langle k_2\gamma^\mu p_2\rangle\langle p_2\gamma^\nu k_1]}{u}\right] \\
&= \frac{-2\mathrm{i}e^2}{u}\left[\frac{\langle k_1k_2\rangle[p_1k_1]\langle k_2p_2\rangle[k_1k_1] - \langle k_2p_1\rangle[k_1p_1]\langle k_2p_2\rangle[k_1p_1]}{t}\right. \\
&\qquad\qquad\left. + \frac{\langle k_2k_2\rangle[k_1p_1]\langle k_1p_2\rangle[k_1k_1] - \langle k_2k_2\rangle[p_2p_1]\langle p_2p_2\rangle[k_1k_1]}{u}\right] \\
&= 2\mathrm{i}e^2\frac{\langle k_2p_1\rangle[k_1p_1]\langle k_2p_2\rangle[k_1p_1]}{tu},
\end{aligned} \tag{5.97}$$

where we have used the spin sum identity $\not{p} = p]\langle p + p\rangle[p$ in the third equality, and also the Fierz transformations. Note that all spinor products like $\langle pp\rangle$ and $[pp]$, or $\langle p\gamma^\mu k\rangle$ and $[p\gamma^\mu k]$ vanish. Square this amplitude, we get

$$\left|\mathrm{i}\mathcal{M}(e_R^+ e_L^- \to \gamma_L\gamma_R)\right|^2 = 4e^4\frac{t}{u}. \tag{5.98}$$

In the same way, we calculate other polarized amplitudes,

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_R^+ e_L^- \to \gamma_R\gamma_L) \\
=& -\mathrm{i}e^2\frac{[k_1\gamma_\mu p_1\rangle\langle k_2\gamma_\nu p_2]}{4\sqrt{(k_1\cdot p_1)(k_2\cdot p_2)}}\left[\frac{[k_2\gamma^\nu(\not{k}_1 - \not{p}_1)\gamma^\mu k_1\rangle}{t} + \frac{[k_2\gamma^\mu(\not{k}_1 - \not{p}_2)\gamma^\nu k_1\rangle}{u}\right] \\
=& \ 2\mathrm{i}e^2\frac{\langle k_2 p_1\rangle[k_1 p_2]\langle k_2 p_2\rangle[k_1 p_2]}{tu}.
\end{aligned} \tag{5.99}$$

Note that we have used a different set of auxiliary momenta in photons' polarizations. After evaluating the rest two nonvanishing amplitudes, we get the squared polarized amplitudes, as follows:

$$\left|\mathcal{M}(e_R^+ e_L^- \to \gamma_L\gamma_R)\right|^2 = \left|\mathcal{M}(e_L^+ e_R^- \to \gamma_R\gamma_L)\right|^2 = 4e^4\frac{t}{u}, \tag{5.100}$$

$$\left|\mathcal{M}(e_L^+ e_R^- \to \gamma_L\gamma_R)\right|^2 = \left|\mathcal{M}(e_L^+ e_R^- \to \gamma_L\gamma_R)\right|^2 = 4e^4\frac{u}{t}. \tag{5.101}$$

Then the differential cross section follows straightforwardly,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{1}{16\pi s}\left(\frac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2\right) = \frac{2\pi\alpha^2}{s}\left(\frac{t}{u} + \frac{u}{t}\right), \tag{5.102}$$

which is in accordance with (5.107) of P&S.

Chapter 6

Radiative Corrections: Introduction

6.1 Rosenbluth formula

In this problem we derive the differential cross section for the electron-proton scattering in the lab frame, assuming that the scattering energy is much higher than electron's mass, and taking account of the form factors of the proton. The result is known as Rosenbluth formula. The relevant diagram is shown in Figure 6.1.

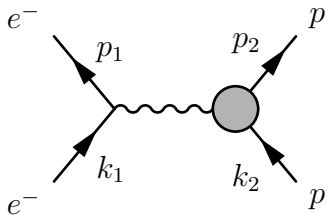


Figure 6.1: The electron-proton scattering. The blob denotes form factors that includes the effect of strong interaction. All initial momenta go inward and all final momenta go outward.

Let us firstly work out the kinematics. In the lab frame, the momenta can be parameterized as

$$k_1 = (E, 0, 0, E), \qquad p_1 = (E', E'\sin\theta, 0, E'\cos\theta), \qquad k_2 = (M, 0, 0, 0), \tag{6.1}$$

and p_2 can be found by momentum conservation, $k_1 + k_2 = p_1 + p_2$. With the on-shell condition $p_2^2 = M^2$, we find that

$$E' = \frac{ME}{M + 2E\sin^2\frac{\theta}{2}}. \tag{6.2}$$

We also use $q = k_1 - p_1$ to denote the momentum transfer and $t = q^2$ its square. Note that we have set the electron mass to zero.

Now we write down the amplitude $\mathcal{M}$.

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2\bar{U}(p_2)\left[\gamma^\mu F_1(q^2) + \frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\right]U(k_2)\frac{-\mathrm{i}}{t}\bar{u}(p_1)\gamma_\mu u(k_1), \tag{6.3}$$

where U is the spinor for the proton and u is for the electron, M is the mass of the proton. At this stage, we convert this expression into a more convenient form by means of the Gordon identity (see Problem 3.2),

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2\bar{U}(p_2)\left[\gamma^\mu(F_1+F_2) - \frac{(p_2+k_2)^\mu}{2M}F_2\right]U(k_2)\frac{-\mathrm{i}}{t}\bar{u}(p_1)\gamma_\mu u(k_1). \tag{6.4}$$

Now, the squared amplitude with initial spins averaged and final spins summed is,

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\,\mathrm{tr}\left[\left(\gamma^\mu(F_1+F_2) - \frac{(p_2+k_2)^\mu}{2M}F_2\right)(\not{k}_2+M)\right.\\ &\qquad\left.\times\left(\gamma^\rho(F_1+F_2) - \frac{(p_2+k_2)^\rho}{2M}F_2\right)(\not{p}_2+M)\right]\mathrm{tr}\left[\gamma_\mu\not{k}_1\gamma_\rho\not{p}_1\right]\\ &= \frac{4e^4M^2}{q^4}\left[\left(2E^2+2E'^2+q^2\right)(F_1+F_2)^2\right.\\ &\qquad\left.-\left(2F_1F_2+F_2^2\left(1+\tfrac{q^2}{4M^2}\right)\right)\left((E+E')^2+q^2\left(1-\tfrac{q^2}{4M^2}\right)\right)\right].\end{aligned} \tag{6.5}$$

There are two terms in the square bracket in the last expression. We rewrite the first factor in the second term as

$$2F_1F_2+F_2^2\left(1+\tfrac{q^2}{4M^2}\right) = (F_1+F_2)^2 - F_1^2 + \tfrac{q^2}{4M^2}F_2^2,$$

and combine the $(F_1+F_2)^2$ part into the first term, which leads to,

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{4e^4M^2}{q^4}\left[\tfrac{q^4}{2M^2}(F_1+F_2)^2 + 4\left(F_1^2 - \tfrac{q^2}{4M^2}F_2^2\right)EE'\cos^2\tfrac{\theta}{2}\right],$$

where we have used the following two relations which can be easily justified,

$$E'-E = \tfrac{q^2}{2m}, \tag{6.6}$$

$$q^2 = -4E'E\sin^2\tfrac{\theta}{2}. \tag{6.7}$$

Now we can put the squared amplitude into its final form,

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{16e^4E^2M^3}{q^4\left(M+2E\sin^2\frac{\theta}{2}\right)}\\ &\qquad\times\left[\left(F_1^2-\tfrac{q^2}{4M^2}F_2^2\right)\cos^2\tfrac{\theta}{2} - \frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\tfrac{\theta}{2}\right].\end{aligned} \tag{6.8}$$

On the other hand, we can derive the $A+B\to 1+2$ differential cross section in the lab frame as

$$\mathrm{d}\sigma_{\mathrm{L}} = \frac{1}{2E_A2E_B|\mathbf{v}_A-\mathbf{v}_B|}\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^62E_12E_2}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_A-p_B). \tag{6.9}$$

In our case, $E_A=E$, $E_B=M$, $E_1=E'$, and $|\mathbf{v}_A-\mathbf{v}_B|\simeq 1$, thus,

$$\mathrm{d}\sigma_{\mathrm{L}} = \frac{1}{4EM}\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^62E_12E_2}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_A-p_B)$$

$$
\begin{aligned}
&= \frac{1}{4EM}\int \frac{E'^2 \mathrm{d}E' \mathrm{d}\cos\theta \mathrm{d}\varphi}{(2\pi)^3 2E' 2E_2} |\mathcal{M}|^2 (2\pi)\delta\big(E' + E_2(E') - E - M\big) \\
&= \frac{1}{4EM}\int \frac{E'^2 \mathrm{d}E' \mathrm{d}\cos\theta \mathrm{d}\varphi}{(2\pi)^2 2E' 2E_2} |\mathcal{M}|^2 \left[1 + \frac{E' - E\cos\theta}{E_2(E')}\right]^{-1} \delta\left(E' - \frac{ME}{M + 2E\sin^2\frac{\theta}{2}}\right) \\
&= \frac{1}{4EM}\int \frac{\mathrm{d}\cos\theta}{8\pi} |\mathcal{M}|^2 \frac{E'}{M + 2E\sin^2\frac{\theta}{2}},
\end{aligned}
$$

where we use the notation $E_2 = E_2(E')$ to emphasize that E_2 is a function of E'. That is,

$$E_2 = \sqrt{M^2 + E^2 + E'^2 - 2E'E\cos\theta}.$$

Then,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)_{\mathrm{L}} = \frac{1}{32\pi\big(M + 2E\sin^2\frac{\theta}{2}\big)^2} |\mathcal{M}|^2. \tag{6.10}$$

So finally we get the differential cross section, the Rosenbluth formula,

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)_{\mathrm{L}} = &\frac{\pi\alpha^2}{2E^2\big(1 + \frac{2E}{M}\sin^2\frac{\theta}{2}\big)\sin^4\frac{\theta}{2}} \\
&\times \left[\big(F_1^2 - \tfrac{q^2}{4M^2}F_2^2\big)\cos^2\tfrac{\theta}{2} - \frac{q^2}{2M^2}(F_1 + F_2)^2 \sin^2\tfrac{\theta}{2}\right].
\end{aligned} \tag{6.11}
$$

6.2 Equivalent photon approximation

In this problem we study the scattering of a very high energy electron from a target in the forward scattering limit. The relevant matrix element is,

$$\mathcal{M} = (-\mathrm{i}e)\bar{u}(p')\gamma^\mu u(p)\frac{-\mathrm{i}g_{\mu\nu}}{q^2}\widehat{\mathcal{M}}^\nu(q). \tag{6.12}$$

(a) First, the spinor product in the expression above can be expanded as,

$$\bar{u}(p')\gamma^\mu u(p) = A\cdot q^\mu + B\cdot \bar{q}^\mu + C\cdot \epsilon_1^\mu + D\cdot \epsilon_2^\mu. \tag{6.13}$$

Now, using the fact that $q_\mu u(p')\gamma^\mu u(p) = 0$, we have,

$$0 = Aq^2 + Bq\cdot\tilde{q} \simeq -4AEE'\sin^2\tfrac{\theta}{2} + Bq\cdot\tilde{q} \qquad \Rightarrow \qquad B \sim \theta^2. \tag{6.14}$$

(b) It is easy to find that

$$\epsilon_1^\mu = N(0, p'\cos\theta - p, 0, -p'\sin\theta), \qquad\qquad \epsilon_2^\mu = (0,0,1,0),$$

where $N = (E^2 + E'^2 - 2EE'\cos\theta)^{-1/2}$ is the normalization constant. Then, for the right-handed electron with spinor $u_+(p) = \sqrt{2E}(0,0,1,0)^T$ and left-handed electron with $u_-(p) = \sqrt{2E}(0,1,0,0)^T$, it is straightforward to show that

$$u_+(p') = \sqrt{2E'}(0,0,\cos\tfrac{\theta}{2},\sin\tfrac{\theta}{2})^T, \qquad u_-(p') = \sqrt{2E'}(-\sin\tfrac{\theta}{2},\cos\tfrac{\theta}{2},0,0), \tag{6.15}$$

and,

$$\bar{u}_{\pm}(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_1 u_{\pm}(p) \simeq -\sqrt{EE'}\frac{E+E'}{|E-E'|}\theta, \tag{6.16}$$

$$\bar{u}_{\pm}(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_2 u_{\pm}(p) \simeq \pm\mathrm{i}\sqrt{EE'}\theta, \tag{6.17}$$

$$\bar{u}_{\pm}(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_1 u_{\mp}(p) = \bar{u}_{\pm}(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_2 u_{\mp}(p) = 0. \tag{6.18}$$

That is to say, we have,

$$C_{\pm} = -\sqrt{EE'}\frac{E+E'}{|E-E'|}\theta, \qquad\qquad D_{\pm} = \pm\mathrm{i}\sqrt{EE'}\theta. \tag{6.19}$$

(c) The squared amplitude is given by,

$$|\mathcal{M}_{\pm\pm}|^2 = \frac{e^2}{(q^2)^2}\widehat{\mathcal{M}}_{\mu}(q)\widehat{\mathcal{M}}_{\nu}(q)\big(C_{\pm}\epsilon_1^{\mu} + D_{\pm}\epsilon_2^{\mu}\big)\big(C_{\pm}^{*}\epsilon_1^{\nu *} + D_{\pm}^{*}\epsilon_2^{\nu *}\big). \tag{6.20}$$

Averaging and summing over the initial and final spins of the electron respectively, we get,

$$\begin{aligned}\frac{1}{2}\sum|\mathcal{M}|^2 &= \frac{e^2}{2(q^2)^2}\widehat{\mathcal{M}}_{\mu}(q)\widehat{\mathcal{M}}_{\nu}(q)\bigg[\big(|C_+|^2+|C_-|^2\big)\epsilon_1^{\mu}\epsilon_1^{\nu *} + \big(|D_+|^2+|D_-|^2\big)\epsilon_2^{\mu}\epsilon_2^{\nu *} \\ &\qquad + \big(C_+D_+^{*}+C_-D_-^{*}\big)\epsilon_1^{\mu}\epsilon_2^{\nu *} + \big(C_+^{*}D_+ + C_-^{*}D_-\big)\epsilon_2^{\mu}\epsilon_1^{\nu *}\bigg] \\ &= \frac{e^2}{(q^2)^2}\widehat{\mathcal{M}}_{\mu}(q)\widehat{\mathcal{M}}_{\nu}(q)EE'\theta^2\bigg[\Big(\frac{E+E'}{E-E'}\Big)^2\epsilon_1^{\mu}\epsilon_1^{\nu *} + \epsilon_2^{\mu}\epsilon_2^{\nu *}\bigg].\end{aligned} \tag{6.21}$$

Then the cross section reads,

$$\begin{aligned}\int \mathrm{d}\sigma &= \frac{1}{2E2M_{\mathrm{t}}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}\frac{\mathrm{d}^3p_{\mathrm{t}}}{(2\pi^3)2E_{\mathrm{t}}}\bigg(\frac{1}{2}\sum|\mathcal{M}|^2\bigg)(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big) \\ &= \frac{e^2}{2E2M_{\mathrm{t}}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}\frac{EE'\theta^2}{3(q^2)^2}\bigg[\Big(\frac{E+E'}{E-E'}\Big)^2+1\bigg] \\ &\qquad\times\int\frac{\mathrm{d}^3p_{\mathrm{t}}}{(2\pi^3)2E_{\mathrm{t}}}|\widehat{\mathcal{M}}_{\mu}(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big) \\ &= -\frac{1}{2E2M_{\mathrm{t}}}\frac{\alpha}{2\pi}\int\mathrm{d}x\bigg[1+\Big(\frac{2-x}{x}\Big)^2\bigg]\int_0^{\pi}\mathrm{d}\theta\frac{\theta^2\sin\theta}{4(1-\cos\theta)^2} \\ &\qquad\times\int\frac{\mathrm{d}^3p_{\mathrm{t}}}{(2\pi^3)2E_{\mathrm{t}}}|\widehat{\mathcal{M}}_{\mu}(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big).\end{aligned} \tag{6.22}$$

where we have used the trick described in the final project of Part I (radiation of gluon jets) to separate the contractions of Lorentz indices, and $x \equiv (E-E')/E$. Now let us focus on the integral over the scattering angle θ in the last expression, which is contributed from the following factor,

$$\int_0^{\pi}\mathrm{d}\theta\,\frac{\theta^2\sin\theta}{4(1-\cos\theta)^4} \sim \int_0\frac{\mathrm{d}\theta}{\theta}. \tag{6.23}$$

which is logarithmically divergent as $\theta\to 0$.

(d) We reintroduce the mass of the electron into the denominator to cut off the divergence, namely, let $q^2 = -2(EE' - pp'\cos\theta) + 2m^2$. Then we can expand q^2, treating m^2 and θ as small quantities, as,

$$q^2 \simeq -(1-x)E^2\theta^2 - \frac{x^2}{1-x}m^2.$$

Then the polar angle integration near $\theta = 0$ becomes,

$$\int_0 \mathrm{d}\theta\,\theta^3\left[\theta^2 + \frac{x^2}{(1-x)^2}\frac{m^2}{E^2}\right]^{-2} \sim -\frac{1}{2}\log\frac{E^2}{m^2}. \tag{6.24}$$

(e) Combining the results above, the cross section can be expressed as,

$$\begin{aligned}\int \mathrm{d}\sigma =& -\frac{1}{2E2M_\mathrm{t}}\frac{\alpha}{2\pi}\int \mathrm{d}x\left[1+\Big(\frac{2-x}{x}\Big)^2\right]\int_0 \mathrm{d}\theta\,\theta^3\left[\theta^2+\frac{x^2}{(1-x)^2}\frac{m^2}{E^2}\right]^{-2}\\ &\times\int\frac{\mathrm{d}^3p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}|\widehat{\mathcal{M}}_\mu(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big)\\ =&\frac{1}{2E2M_\mathrm{t}}\frac{\alpha}{2\pi}\int\mathrm{d}x\,\frac{1+(1-x)^2}{x^2}\log\frac{E^2}{m^2}\\ &\times\int\frac{\mathrm{d}^3p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}|\widehat{\mathcal{M}}_\mu(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big).\end{aligned} \tag{6.25}$$

6.3 Exotic contributions to $g-2$

(a) The 1-loop vertex correction from Higgs boson is,

$$\begin{aligned}\bar u(p')\delta\Gamma^\mu u(p) =& \Big(\frac{\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_h^2}\bar u(p')\frac{\mathrm{i}}{\not k+\not q-m}\gamma^\mu\frac{\mathrm{i}}{\not k-m}u(p)\\ =&\frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{2\bar u(p')N^\mu u(p)}{(k'^2-\Delta)^3},\end{aligned} \tag{6.26}$$

with

$$N^\mu = (\not k+\not q+m)\gamma^\mu(\not k+m), \tag{6.27}$$

$$k' = k - xp + yq, \tag{6.28}$$

$$\Delta = (1-x)m^2 + xm_h^2 - x(1-x)p^2 - y(1-y)q^2 + 2xyp\cdot q. \tag{6.29}$$

To put this correction into the following form,

$$\Gamma^\mu = \gamma^\mu F_1(q) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q), \tag{6.30}$$

we first rewrite N^μ as,

$$N^\mu = A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu, \tag{6.31}$$

where term proportional to $(p'-p)$ can be thrown away by Ward identity $q_\mu\Gamma^\mu(q) = 0$. This can be done by gamma matrix calculations, leading to the following result,

$$N^\mu = \Big[\big(\tfrac{2}{d}-1\big)k'^2 + (3+2x-x^2)m^2 + (y-xy-y^2)q^2\Big]\gamma^\mu + (x^2-1)m(p'+p)^\mu. \tag{6.32}$$

Then, using Gordon identity, we find,

$$N^\mu = \Big[\big(\tfrac{2}{d}-1\big)k'^2 + (x+1)^2m^2 + (y-y^2-xy)q^2\Big]\gamma^\mu + \frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\cdot 2m^2(1-x^2). \tag{6.33}$$

Comparing this with (6.30), we see that

$$\begin{aligned}\delta F_2(q=0) &= 2\mathrm{i}\lambda^2m^2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{1-x^2}{(k'^2-\Delta)^3}\\ &= \frac{\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}x\,\frac{(1-x)^2(1+x)}{(1-x)^2+x(m_h/m)^2}.\end{aligned} \tag{6.34}$$

To carry out the integration over x, we use the approximation that $m_h \gg m$. Then,

$$\begin{aligned}\delta F_2(q=0) &\simeq \frac{\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}x\left[\frac{1}{1+x(m_h/m)^2}-\frac{1+x-x^2}{(m_h/m)^2}\right]\\ &\simeq \frac{\lambda^2}{(4\pi)^2(m_h/m)^2}\Big[\log(m_h^2/m^2)-\tfrac{7}{6}\Big].\end{aligned} \tag{6.35}$$

(b) According to (a), the limits on λ and m_h is given by,

$$\delta F_2(q=0) = \frac{\lambda^2}{(4\pi)^2(m_h/m)^2}\Big[\log(m_h^2/m^2)-\tfrac{7}{6}\Big] < 1\times 10^{-10}. \tag{6.36}$$

For electron, $\lambda \simeq 3\times10^{-6}$, $m\simeq 0.511$MeV, and with $m_h \sim 60\,$GeV, we have $\delta F_2(q=0)\sim 10^{-22} \ll 10^{-10}$. For realistic case $m_h \simeq 125$GeV the effect is even smaller. On the other hand, for muon, we have $\lambda = 6\times10^{-4}$, $m\simeq 106$MeV, and with the input $m_h\simeq 125$GeV, we have $\delta F_2(q=0)\sim 10^{-14}$. At present the experimentally measured muon's anomalous magnetic moment is a bit different from the prediction of Standard Model, and the difference is of order 10^{-9}, a not decisive but still noteworthy "anomaly". More can be found in [3].

(c) The 1-loop correction from the axion is given by,

$$\begin{aligned}\bar u(p')\delta\Gamma^\mu u(p) =& \Big(\frac{-\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_h^2}\bar u(p')\gamma^5\frac{\mathrm{i}}{\not k+\not q-m}\gamma^\mu\frac{\mathrm{i}}{\not k-m}\gamma^5u(p)\\ =& -\frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{2\bar u(p')N^\mu u(p)}{(k'^2-\Delta)^3},\end{aligned} \tag{6.37}$$

in which k' and Δ are still defined as in (a) except the replacement $m_h\to m_a$, while N^μ is now given by,

$$N^\mu = \gamma^5(\not k+\not q+m)\gamma^\mu(\not k+m)\gamma^5 = -(\not k+\not q-m)\gamma^\mu(\not k-m). \tag{6.38}$$

Repeating the same derivation as was done in (a), we get,

$$N^\mu = \big[-\big(\tfrac{2}{d}-1\big)k'^2-(1-x-y)yq^2+(1-x)^2m^2\big]\gamma^\mu-(1-x)^2m(p'+p)^2. \tag{6.39}$$

Again, using Gordon identity, we get,

$$\begin{aligned}
\delta F_2(q=0) =& -2\mathrm{i}\lambda^2 m^2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{(1-x)^2}{(k'^2-\Delta)^3} \\
=& -\frac{\lambda^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \, \frac{(1-x)^3}{(1-x)^2 + x m_a^2/m^2} \\
\simeq& -\frac{\lambda^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \left[\frac{1}{1+x m_a^2/m^2} - \frac{3-3x+x^2}{m_a^2/m^2} \right] \\
=& -\frac{\lambda^2}{(4\pi)^2 (m_a^2/m^2)} \Big[\log(m_a^2/m^2) - \tfrac{11}{6} \Big].
\end{aligned} \tag{6.40}$$

For order-of-magnitude estimation, it's easy to see that $\lambda m/m_a \gtrsim 10^{-5}$ is excluded.

Chapter 7

Radiative Corrections: Some Formal Developments

7.1 Optical theorem in ϕ^4 theory

In this problem we check the optical theorem in phi-4 theory to order λ^2. Firstly, the total cross section $\sigma_{\rm tot}$ at this order receives contributions from tree level only. The squared amplitude is simply λ^2. Then its easy to get the total cross section by complementing kinematic factors. That is,

$$\sigma_{\rm tot} = \frac{\lambda^2}{16\pi s}, \tag{7.1}$$

where $s = E_{\rm CM}^2$ and $E_{\rm CM}$ is the COM energy. Then, consider the imaginary part of the scattering amplitude. The contribution comes from 1-loop diagram in s-channel this time. Let's evaluate this amplitude directly,

$$\begin{aligned} \mathrm{i}\mathcal{M} &= \frac{1}{2}(-\mathrm{i}\lambda)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} \frac{\mathrm{i}}{(k-p)^2 - m^2} = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{1}{(k'^2 - \Delta)^2} \\ &= \frac{\mathrm{i}\lambda^2}{2(4\pi)^2} \Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \int_0^1 \mathrm{d}x \, \log\big(m^2 - x(1-x)s\big) \Big]. \end{aligned} \tag{7.2}$$

Therefore,

$$\mathrm{Im}\mathcal{M} = -\frac{\lambda^2}{2(4\pi)^2} \int_0^1 \mathrm{d}x \, \mathrm{Im}\Big[\log\big(m^2 - x(1-x)s\big) \Big]. \tag{7.3}$$

The argument in the logarithm is real, thus the imaginary part of the logarithm equals to 0 or $-\pi$ depending on the argument is positive or negative. (Strictly speaking the imaginary part is $-\pi$ but not π due to our iϵ prescription.) Then we see this logarithm contributes an constant imaginary part $-\pi$, only when

$$\frac{1 - \sqrt{1 - 4m^2/s}}{2} < x < \frac{1 + \sqrt{1 - 4m^2/s}}{2}.$$

Thus we have

$$\mathrm{Im}\mathcal{M} = \frac{\lambda^2}{32\pi} \sqrt{1 - 4m^2/s} = \frac{\lambda^2 p_{\rm CM}}{16\pi E_{\rm CM}}. \tag{7.4}$$

7.2 Alternative regulators in QED

In this problem we compute the first order corrections to Z_1 and Z_2 in QED, using cut-off regularization and dimensional regularization respectively. By definition, we have,

$$\Gamma^\mu(q=0) = Z_1^{-1}\gamma^\mu, \tag{7.5}$$

$$Z_2^{-1} = 1 - \frac{\mathrm{d}\Sigma}{\mathrm{d}\not{p}}\Big|_{\not{p}=m}. \tag{7.6}$$

Let's begin with dimensional regularization instead of momentum cut-off.

(b) Dimensional Regularization We firstly calculate $\delta F_1(0)$:

$$\begin{aligned}
&\bar{u}(p')\delta\Gamma^\mu(p,p')u(p)\\
=&(-ie)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-ig_{\rho\sigma}}{(k-p)^2-\mu^2}\bar{u}(p')\gamma^\sigma\frac{i}{\not{k}+\not{q}-m}\gamma^\mu\frac{i}{\not{k}-m}\gamma^\rho u(p)\\
=&-\mathrm{i}e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\bar{u}(p')\gamma_\rho(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma^\rho u(p)}{\big((k-p)^2-\mu^2\big)\big((k+q)^2-m^2\big)\big(k^2-m^2\big)}\\
=&-\mathrm{i}e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{2\bar{u}(p')N^\mu u(p)}{(k'^2-\Delta)^3},
\end{aligned} \tag{7.7}$$

in which we define

$$\begin{aligned}
&k' = k - xp + yq,\\
&\Delta = (1-x)m^2 + x\mu^2 - x(1-x)p^2 - y(1-y)q^2 + 2xyp\cdot q,\\
&N^\mu = \gamma_\rho(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma^\rho.
\end{aligned}$$

The next step is to put N^μ into the needed form. The calculation is basically in parallel with Peskin's Sec.6.3. Let me show some details. The first step is to finish the summation over dummy Lorentz indices. Note that we are using dimensional regularization, thus we should use Peskin's Eq.(A.55). The result is:

$$N^\mu = -2\not{k}\gamma^\mu(\not{k}+\not{q}) + 4m(2k+q)^\mu - (d-2)m^2\gamma^\mu + (4-d)\Big[(\not{k}+\not{q})\gamma^\mu\not{k} - m(\not{k}+\not{q})\gamma^\mu - m\gamma^\mu\not{k}\Big].$$

It's worth noting here that d will be sent to 4 at the end of the calculation. Thus in the square bracket in this expression, only the combination $\not{k}\gamma^\mu\not{k}$ contributes to the final result. Thus we simply have

$$N^\mu = -2\not{k}\gamma^\mu(\not{k}+\not{q}) + 4m(2k+q)^\mu - (d-2)m^2\gamma^\mu + (4-d)\not{k}\gamma^\mu\not{k}. \tag{7.8}$$

Here and following, we are free to drop off terms in N^μ which contribute nothing to final results. The equal sign should be understood in this way. The next step is to rewrite N^μ in terms of k' instead of k:

$$N^\mu = (2-d)\not{k}'\gamma^\mu\not{k}' - 2\big[x\not{p} - y\not{q}\big]\gamma^\mu\big[x\not{p} + (1-y)\not{q}\big] + 4m\big[2xp + (1-2y)q\big]^\mu - 2m^2\gamma^\mu. \tag{7.9}$$

Terms linear in k' has been dropped since they integrate to zero. The third step is to put N^μ into a linear combination of γ^μ, $(p+p')^\mu$. Terms proportional to $(p-p')^\mu$ will be dropped due to Ward identity. The basic strategy is using substitution $q^\mu = (p'-p)^\mu$, on shell condition $\bar{u}(p')\not{p}' = \bar{u}(p')m$ and $\not{p}u(p) = mu(p)$. Here we show the detailed steps for the second term above:

$$\begin{aligned}
&-2\big[x\not{p}-y\not{q}\big]\gamma^\mu\big[x\not{p}+(1-y)\not{q}\big]\\
=&-2\big[x^2\not{p}\gamma^\mu\not{p}-y(1-y)\not{q}\gamma^\mu\not{q}-xy\not{q}\gamma^\mu\not{p}+x(1-y)\not{p}\gamma^\mu\not{q}\big]\\
=&-2\big[x^2(2p^\mu-\gamma^\mu\not{p})\not{p}-y(1-y)(2q^\mu-\gamma^\mu\not{q})\not{q}-xy\not{q}\gamma^\mu\not{p}+x(1-y)(\not{p}'-\not{q})\gamma^\mu\not{q}\big]\\
=&-2\big[2x^2mp^\mu-x^2m^2\gamma^\mu+y(1-y)q^2\gamma^\mu-2xymq^\mu+xm\gamma^\mu\not{q}-x(1-y)\not{q}\gamma^\mu\not{q}\big]\\
=&-2\big[-x(x+2)m^2\gamma^\mu+(x+y)(1-y)q^2\gamma^\mu+2x^2mp^\mu+2xmp'^\mu\big].
\end{aligned}$$

Combining this with other terms, and also make the momentum symmetrization (Peskin's Eq.(7.87)), which amounts to make the substitution:

$$\not{k}'\gamma^\mu\not{k}' \to \big(\tfrac{2}{d}-1\big)k'^2\gamma^\mu,$$

we get

$$N^\mu = \tfrac{(2-d)^2}{d}\not{k}'\gamma^\mu\not{k}' + \big[2(x^2+2x-1)m^2-2(x+y)(1-y)q^2\big]\gamma^\mu + 2x(1-x)m(p'+p)^\mu. \quad (7.10)$$

Now we employ Gordon's identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\Big[\frac{p'^\mu+p^\mu}{2m}+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big]u(p),$$

to put N^μ into a linear combination of γ^μ and $\sigma^{\mu\nu}$:

$$N^\mu = \big[\tfrac{(2-d)^2}{d}k'^2 - 2(x^2-4x+1)m^2 - 2(x+y)(1-y)q^2\big]\gamma^\mu - 2x(1-x)mi\sigma^{\mu\nu}q_\nu. \quad (7.11)$$

Now we have put the vertex Γ^μ into the following form:

$$\Gamma^\mu = \gamma^\mu F_1(q) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q). \quad (7.12)$$

We are interest in $\delta F_1(q)$, which is related to δZ_1 by $\delta Z_1 = -\delta F_1(q=0)$. Finishing momentum integral:

$$\begin{aligned}
\delta F_1(0) =& -2\mathrm{i}e^2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{1}{(k'^2-\Delta)^3}\big[\tfrac{(2-d)^2}{d}k'^2-2(x^2-4x+1)m^2\big]\\
=&\frac{2e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\Big[\frac{(2-d)^2\Gamma(2-\frac{d}{2})}{4\Delta^{2-d/2}}+(x^2-4x+1)m^2\frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}\Big], \quad (7.13)
\end{aligned}$$

and sending $d = 4-\epsilon \to 4$:

$$\delta F_1(0) = \frac{2e^2}{(4\pi)^2}\int_0^1\mathrm{d}x(1-x)\Big[\tfrac{2}{\epsilon}-\gamma+\log 4\pi$$

$$- \log\big((1-x)^2m^2 + x\mu^2\big) - 2 + \frac{(x^2-4x+1)m^2}{(1-x)^2m^2+x\mu^2}\Big], \tag{7.14}$$

we reach the needed result $\delta Z_1 = -\delta F_1(0)$. Now let us turn to Z_2. The correction of first order is given by $\delta Z_2 = (\mathrm{d}\Sigma/\mathrm{d}\not{p})\big|_{\not{p}=m}$. Therefore we'd better evaluate $\Sigma(\not{p})$ using dimensional regularization.

$$\begin{aligned}
-\mathrm{i}\Sigma(\not{p}) =&(-\mathrm{i}e)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^\mu\frac{\mathrm{i}}{\not{k}-m}\gamma_\mu\frac{-\mathrm{i}}{(p-k)^2-\mu^2}\\
=&-e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(2-d)\not{k}+dm}{(k^2-m^2)\big((p-k)^2-\mu^2\big)} = -e^2\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{(2-d)x\not{p}+dm}{(k'^2-\Delta)^2}\\
=&-\frac{\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big[(2-d)x\not{p}+dm\big],
\end{aligned} \tag{7.15}$$

where

$$\begin{aligned}
&k' = k - xp;\\
&\Delta = (1-x)m^2 + x\mu^2 - x(1-x)p^2.
\end{aligned}$$

Then we can compute

$$\begin{aligned}
\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}} =&\frac{e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\Big[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(2-d)x - \frac{(2-\frac{d}{2})\Gamma(2-\frac{d}{2})}{\Delta^{3-d/2}}\frac{\mathrm{d}\Delta}{\mathrm{d}\not{p}}\big((2-d)x\not{p}+dm\big)\Big]\\
=&\frac{e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\Big[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(2-d)x + \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}2x(1-x)\not{p}\big((2-d)x\not{p}+dm\big)\Big].
\end{aligned} \tag{7.16}$$

Then, setting $\not{p} = m$ and $d = 4-\epsilon$ with $\epsilon\to 0$, we get

$$\begin{aligned}
\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\Big|_{\not{p}=m} =& \frac{-2e^2}{(4\pi)^2}\int_0^1\mathrm{d}x\,x\Big[\tfrac{2}{\epsilon}+\gamma+\log 4\pi\\
&-\log\big((1-x)^2m^2+x\mu^2\big) - 1 - \frac{2(1-x)(2-x)m^2}{(1-x)^2m^2+x\mu^2}\Big]
\end{aligned} \tag{7.17}$$

Now it's still not easy to see $\delta Z_1 = \delta Z_2$ immediately. To make this transparent, we need some more steps. Let's focus on logarithm term:

$$\begin{aligned}
&-\int_0^1\mathrm{d}x(1-x)\log\big((1-x)^2m^2+x\mu^2\big) = -\int_0^1\mathrm{d}x(1-2x+x)\log\big((1-x)^2m^2+x\mu^2\big)\\
=&\int_0^1\mathrm{d}x\Big[(1-x) - \frac{(1-x)(1-x^2)m^2}{(1-x^2)m^2+x\mu^2} - x\log\big((1-x)^2m^2+x\mu^2\big)\Big]
\end{aligned} \tag{7.18}$$

Combining this with other terms, and also using the fact $\int x\mathrm{d}x = \int(1-x)\mathrm{d}x$, we get

$$\begin{aligned}
\delta F_1(0) = \frac{2e^2}{(4\pi)^2}\int_0^1\mathrm{d}x x\Big[\tfrac{2}{\epsilon}-\gamma+\log 4\pi\\
-\log\big((1-x)^2m^2+x\mu^2\big) - 1 - \frac{2(1-x)(2-x)m^2}{(1-x)^2m^2+x\mu^2}\Big].
\end{aligned} \tag{7.19}$$

Now it's clear that $\delta Z_1 = \delta Z_2$. Thus $Z_1 = Z_2$ keeps unaffected at this order when dimensional regularization is used.

(a) Momentum Cut-off Now we repeat the calculation above using momentum cut-off. Now we can directly borrow some results above. All we need to do is setting $d \to 4$ and adding a UV momentum cut-off Λ, as well as the following integral formulae:

$$\begin{aligned}
&\int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\Delta)^2} = \frac{\mathrm{i}}{16\pi^2}\Big[\log\big(1+\tfrac{\Lambda^2}{\Delta}\big) - \tfrac{\Lambda^2}{\Lambda^2+\Delta}\Big],\\
&\int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2}{(k^2-\Delta)^3} = \frac{\mathrm{i}}{16\pi^2}\Big[\log\big(1+\tfrac{\Lambda^2}{\Delta}\big) + \tfrac{\Delta(4\Lambda^2+3\Delta)}{2(\Lambda^2+\Delta)^2} - \tfrac{3}{2}\Big],\\
&\int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\Delta)^3} = -\frac{\mathrm{i}}{32\pi^2}\frac{\Lambda^4}{\Delta(\Lambda^2+\Delta)^2}.
\end{aligned}$$

We begin with (7.13):

$$\begin{aligned}
\delta F_1(0) =& -2\mathrm{i}e^2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int^{\Lambda} \frac{\mathrm{d}^4k'}{(2\pi)^4} \frac{1}{(k'^2-\Delta)^3}\big[k'^2 - 2(x^2-4x+1)m^2\big]\\
=& \frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}x\,(1-x)\left[\log\left(1+\frac{\Lambda^2}{\Delta}\right) + \frac{(x^2-4x+1)m^2}{\Delta} - \frac{3}{2}\right],
\end{aligned} \tag{7.20}$$

In the same way, we get

$$-\mathrm{i}\Sigma(\not{p}) = 2e^2 \int^{\Lambda} \frac{\mathrm{d}^4k'}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{x\not{p} - 2m}{(k'^2-\Delta)^2}, \tag{7.21}$$

and

$$\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\Big|_{\not{p}=m} = \frac{-e^2}{8\pi^2}\int_0^1 \mathrm{d}x \left[x\log\left(1+\frac{\Lambda^2}{\Delta}\right) - x + \frac{2x(1-x)(x-2)m^2}{\Delta}\right]. \tag{7.22}$$

This shows that $\delta Z_1 \neq \delta Z_2$ with cut-off regularization.

7.3 Radiative corrections in QED with Yukawa interaction

(a) Let us calculate the first order corrections to Z_1 and Z_2, as was done in Problem 7.2. Firstly, we calculate $\delta\Gamma^\mu$, which is similar to the corresponding QED correction:

$$\begin{aligned}
&\bar{u}(p')\delta\Gamma^\mu(p,p')u(p)\\
=&(-\mathrm{i}\lambda^2/\sqrt{2})^2 \int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{\mathrm{i}}{(k-p)^2-m_\phi^2} \bar{u}(p') \frac{\mathrm{i}}{\not{k}+\not{q}-m}\gamma^\mu \frac{\mathrm{i}}{\not{k}-m} u(p)\\
=&\frac{\mathrm{i}\lambda^2}{2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{2\bar{u}(p')N^\mu u(p)}{(k'^2-\Delta)^3},
\end{aligned} \tag{7.23}$$

where

$$k' = k - xp + yq,$$

$$\Delta = (1-x)m^2 + xm_\phi^2 - x(1-x)p^2 - y(1-y)q^2 + 2xyp\cdot q,$$
$$N^\mu = (\not k + \not q + m)\gamma^\mu(\not k + m).$$

Then we put this correction into the following form, in parallel with steps of Problem 7.2. That is: (1) replace k by k' in N^μ:

$$N^\mu = \not k'\gamma^\mu \not k' + \big(x\not p + (1-y)\not q + m\big)\gamma^\mu(x\not p - y\not q + m);$$

(2) rewrite the numerator N^μ by gamma matrix relations and equations of motion, as

$$N^\mu = \Big[\big(\tfrac{2}{d}-1\big)k'^2 + (3+2x-x^2)m^2 + (y-xy-y^2)q^2\Big]\gamma^\mu + (x^2-1)m(p'+p)^\mu;$$

(3) use Gordon identity to further transform N^μ into:

$$N^\mu = \Big[\big(\tfrac{2}{d}-1\big)k'^2 + (x+1)^2m^2 + (y-y^2-xy)q^2\Big]\gamma^\mu + \frac{\mathrm{i}\sigma^{\mu\nu}}{2m}\cdot 2m^2(1-x^2).$$

Then, we can read off δF_1 from the coefficient of γ^μ, as:

$$\begin{aligned}\delta F_1(0) &= \mathrm{i}\lambda^2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k'^2-\Delta)^3}\Big[\big(\tfrac{2}{d}-1\big)k'^2 + (x+1)^2m^2\Big]\\ &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\,(1-x)\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi\\ &\qquad - \log\big((1-x)^2m^2 + xm_\phi^2\big) - 1 + \frac{(x+1)^2m^2}{(1-x)^2m^2+xm_\phi^2}\Big].\end{aligned} \tag{7.24}$$

Using the trick identity (7.18) again, we finally get

$$\begin{aligned}\delta F_1(0) &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\,x\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi\\ &\qquad - \log\big((1-x)^2m^2 + xm_\phi^2\big) + \frac{2(1-x^2)m^2}{(1-x)^2m^2+xm_\phi^2}\Big].\end{aligned} \tag{7.25}$$

Now we calculate $\Sigma(\not p)$.

$$\begin{aligned}-\mathrm{i}\Sigma(\not p) &= (-\mathrm{i}\lambda/\sqrt{2})^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{\not k - m}\frac{\mathrm{i}}{(p-k)^2-\mu^2} = \frac{\lambda^2}{2}\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{x\not p + m}{(k'^2-\Delta)^2}\\ &= \frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\Big](x\not p + m),\end{aligned} \tag{7.26}$$

where $k' = k - xp$ and $\Delta = (1-x)m^2 + x\mu^2 - x(1-x)p^2$. Then we have

$$\begin{aligned}\frac{\mathrm{d}\Sigma(\not p)}{\mathrm{d}\not p}\Big|_{\not p = m} &= \frac{-\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\,x\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi\\ &\qquad - \log\big((1-x)^2m^2 + xm_\phi^2\big) + \frac{2(1-x^2)m^2}{(1-x)^2m^2+xm_\phi^2}\Big].\end{aligned} \tag{7.27}$$

Thus we have proved that $\delta Z_1 = \delta Z_2$ holds for 1-loop scalar corrections.

(b) Now consider the 1-loop corrections to Yukawa vertex. We focus on the divergent part only. The equalities below should be understood to be hold up to a finite part. Then, for vertex correction from photon, we have

$$\begin{aligned}\delta\Gamma(p,p')\big|_{\text{photon}} =&(-\mathrm{i}e)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}}{(k-p)^2-\mu^2}\gamma^\mu\frac{\mathrm{i}}{\not k+\not q-m}\frac{\mathrm{i}}{\not k-m}\gamma_\mu\\ =&-\mathrm{i}e^2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2dk'^2}{(k'^2-\Delta)^3}\\ =&\frac{d^2e^2}{2(4\pi)^{d/2}}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}=\frac{4e^2}{(4\pi)^2}\frac{2}{\epsilon}\end{aligned}\tag{7.28}$$

In the same way,

$$\begin{aligned}\delta\Gamma(p,p')\big|_{\text{scalar}} =&\Big(\frac{-\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_\phi^2}\frac{\mathrm{i}}{\not k+\not q-m}\frac{\mathrm{i}}{\not k-m}\\ =&\frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2k'^2}{(k'^2-\Delta)^3}=\frac{-\lambda^2}{2(4\pi)^2}\frac{2}{\epsilon}\end{aligned}\tag{7.29}$$

On the other hand, the 1-loop corrections for electron's self-energy also come from two parts: one is the photon correction, which has been evaluated in Problem 7.1,

$$-\mathrm{i}\Sigma(\not p)\big|_{\text{photon}}=-\frac{\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big[(2-d)x\not p+dm\big]=\frac{\mathrm{i}e^2(\not p-4m)}{(4\pi)^2}\frac{2}{\epsilon},\tag{7.30}$$

and the other is the scalar correction:

$$-\mathrm{i}\Sigma(\not p)\big|_{\text{scalar}}=\Big(\frac{-\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{\not k-m}\frac{\mathrm{i}}{(p-k)^2-m_\phi^2}=\frac{\mathrm{i}\lambda^2(\not p+2m)}{4(4\pi)^2}\frac{2}{\epsilon},\tag{7.31}$$

To sum up, we have got the total vertex correction:

$$\delta\Gamma(p,p')=\delta\Gamma(p,p')\big|_{\text{photon}}+\delta\Gamma(p,p')\big|_{\text{scalar}}=\frac{4e^2-\lambda^2/2}{(4\pi)^2}\frac{2}{\epsilon},\tag{7.32}$$

and also:

$$\frac{\mathrm{d}\Sigma(\not p)}{\mathrm{d}\not p}\Big|_{\not p=m}=\frac{\mathrm{d}\big[\Sigma(\not p)_{\text{photon}}+\Sigma(\not p)_{\text{scalar}}\big]}{\mathrm{d}\not p}\Big|_{\not p=m}=-\frac{e^2+\lambda^2/4}{(4\pi)^2}\frac{2}{\epsilon}.\tag{7.33}$$

Final Project I
Radiation of Gluon Jets

In this final project we do a basic exercise about cancellation of infrared divergence. An excellent pedagogical treatment of infrared divergence can be found in [2]

(a) First we calculate the 1-loop vertex correction to $\mathcal{M}(e^+e^- \to q\bar{q})$ from virtual gluon. The amplitude is given by

$$\mathrm{i}\delta_1\mathcal{M} = Q_f(-\mathrm{i}e)^2(-\mathrm{i}g)^2\bar{u}(p_1)\Big[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\gamma^\nu \frac{\mathrm{i}}{\not{k}}\gamma^\mu \frac{\mathrm{i}}{\not{k}-\not{q}}\gamma_\nu \frac{-\mathrm{i}}{(k-p_1)^2-\mu^2}\Big]v(p_2)\frac{-\mathrm{i}}{q^2}\bar{v}(k_2)\gamma_\mu u(k_1). \tag{7.34}$$

Now we simplify the loop integral in the standard way, as was done in Problem 7.2. The result is

$$\begin{aligned}
\mathrm{i}\delta_1\mathcal{M} =& -\mathrm{i}g^2\Big[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^1 \mathrm{d}x \int_0^{1-x}\mathrm{d}y\, \frac{2\big(\frac{(2-d)^2}{d}k'^2 - 2(1-x)(x+y)q^2\big)}{(k'^2-\Delta)^3}\Big]\mathrm{i}\mathcal{M}_0 \\
=& \frac{2g^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\Big[\frac{(2-d)^2}{4\Delta^{2-d/2}}\Gamma(2-\tfrac{d}{2}) + \frac{(1-x)(x+y)q^2}{\Delta^{3-d/2}}\Gamma(3-\tfrac{d}{2})\Big]\mathrm{i}\mathcal{M}_0 \\
=& \frac{2g^2}{(4\pi)^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta - 2 + \frac{(1-x)(x+y)q^2}{\Delta}\Big]\mathrm{i}\mathcal{M}_0,
\end{aligned} \tag{7.35}$$

where

$$\mathrm{i}\mathcal{M}_0 = Q_f(-\mathrm{i}e)^2\bar{u}(p_1)\gamma^\mu\bar{v}(p_2)\frac{1}{q^2}v(k_2)\gamma_\mu u(k_1) \tag{7.36}$$

is the tree amplitude, and

$$k' = k - xq - yp_1, \qquad \Delta = -x(1-x-y)q^2 - y(1-y)p_1^2 + y\mu^2. \tag{7.37}$$

With the external legs amputated, the result is,

$$\mathrm{i}\delta_1\mathcal{M} = \frac{2g^2}{(4\pi)^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\Big[\log\Big(\frac{y\mu^2}{y\mu^2 - x(1-x-y)q^2}\Big) + \frac{(1-x)(x+y)q^2}{y\mu^2 - x(1-x-y)q^2}\Big]\mathrm{i}\mathcal{M}_0, \tag{7.38}$$

Then the cross section is given by

$$\sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} \cdot 3|F_1(q^2 = s)|^2, \tag{7.39}$$

with

$$F_1(q^2 = s) = Q_f^2 + \frac{Q_f^2\alpha_g}{2\pi} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \Big[\log\Big(\frac{y\mu^2}{y\mu^2 - x(1-x-y)s}\Big) + \frac{(1-x)(x+y)s}{y\mu^2 - x(1-x-y)s}\Big]. \tag{7.40}$$

We will carry out the Feynman integration in (e).

(b) Now we simplify the 3-body phase space integral

$$\int \mathrm{d}\Pi_3 = \int \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2\mathrm{d}^3k_3}{(2\pi)^9 2E_1 2E_2 2E_3}\delta^{(4)}(q - k_1 - k_2 - k_3) \tag{7.41}$$

in the center of mass frame. It is convenient to introduce a new set of variables $x_i = 2k_i\cdot q/q^2$, $(i = 1, 2, 3)$. In COM frame, $x_i = 2E_i/E_q$ Then one can show that all Lorentz scalars involving final states only can be represented in terms of x_i and particles masses. In fact, we only need to check $(k_1 + k_2)^2$, $(k_2 + k_3)^2$ and $(k_3 + k_1)^2$. From instance,

$$(k_1 + k_2)^2 = (q - k_3)^2 = q^2 + m_3^2 - 2q \cdot k_3 = s(1 - x_3) + m_3^2. \tag{7.42}$$

Similarly,

$$(k_2 + k_3)^2 = s(1 - x_1) + m_1^2, \qquad (k_3 + k_1)^2 = s(1 - x_2) + m_2^2. \tag{7.43}$$

To simply the phase integral, we first integrate out k_3 with spatial delta function that restricts $k_3 = k_1 + k_2$:

$$\int \mathrm{d}\Pi_3 = \int \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{(2\pi)^6 2E_1 2E_2 2E_3}(2\pi)\delta(E_q - E_1 - E_2 - E_3). \tag{7.44}$$

The integral measure can be rewritten as

$$\mathrm{d}^3k_1\mathrm{d}^3k_2 = k_1^2k_2^2\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}\Omega_1\mathrm{d}\Omega_{12}, \tag{7.45}$$

where $\mathrm{d}\Omega_1$ is the spherical integral measure associated with d^3k_1, and $\mathrm{d}\Omega_{12}$ is the spherical integral of relative angles between $\mathbf{k}_1$ and $\mathbf{k}_2$. The former spherical integral can be directly carried out and results in a factor 4π. To finish the integral with $\mathrm{d}\Omega_{12} = \mathrm{d}\cos\theta_{12}\mathrm{d}\varphi_{12}$, we make use of the remaining delta function, which can be rewritten as

$$\delta(E_q - E_1 - E_2 - E_3) = \frac{E_3}{k_1k_2}\delta\Big(\cos\theta_{12} - \frac{E_3^2 - k_1^2 - k_2^2 - \mu^2}{2k_1k_2}\Big), \tag{7.46}$$

by means of $E_3 = \sqrt{k_1^2 + k_2^2 + 2k_1k_2\cos\theta + \mu^2}$. Thus

$$\int \mathrm{d}\Omega_1 \mathrm{d}\Omega_{12}\, \delta(E_q - E_1 - E_2 - E_3) = \frac{8\pi^2 E_3}{k_1 k_2}.$$

Now using $k_1\mathrm{d}k_1 = E_1\mathrm{d}E_1$ and $k_2\mathrm{d}k_2 = E_2\mathrm{d}E_2$, we have

$$\int \mathrm{d}\Pi_3 = \int \frac{\mathrm{d}k_1\mathrm{d}k_2\, k_1^2 k_2^2}{8(2\pi)^5 E_1 E_2 E_3}\frac{8\pi^2 E_3}{k_1 k_2} = \frac{1}{32\pi^3}\int \mathrm{d}E_1\mathrm{d}E_2 = \frac{s}{128\pi^3}\int \mathrm{d}x_1\mathrm{d}x_2. \tag{7.47}$$

To determine the integral region for $m_1 = m_2 = 0$ and $m_3 = \mu$, we note that there are two extremal cases: $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel or antiparallel. In the former case, we have

$$E_q = E_1 + E_2 + E_3 = E_1 + E_2 + \sqrt{(E_1 + E_2)^2 + \mu^2}, \tag{7.48}$$

which yields

$$2E_q(E_1 + E_2) = E_q^2 - \mu^2, \tag{7.49}$$

while in the latter case,

$$E_q = E_1 + E_2 + \sqrt{(E_1 - E_2)^2 + \mu^2}, \tag{7.50}$$

which gives

$$(E_q - 2E_1)(E_q - 2E_2) = \mu^2. \tag{7.51}$$

These two boundary cases can be represented by x_i variables as

$$x_1 + x_2 = 1 - \frac{\mu^2}{s}; \tag{7.52}$$

$$(1 - x_1)(1 - x_2) = \frac{\mu^2}{s}. \tag{7.53}$$

The integral thus goes over the region bounded by these two curves.

(c) Now we calculate the differential cross section for the process $e^+e^- \to q\bar{q}g$ to lowest order in α and α_g. First, the amplitude is

$$\mathrm{i}\mathcal{M} = Q_f(-\mathrm{i}e)^2(-\mathrm{i}g)\epsilon_\nu^*(k_3)\bar{u}(k_1)\left[\gamma^\nu\frac{\mathrm{i}}{\not{k}_1 + \not{k}_3}\gamma^\mu - \gamma^\mu\frac{\mathrm{i}}{\not{k}_2 + \not{k}_3}\gamma^\nu\right]v(k_2)\frac{-\mathrm{i}}{q^2}\bar{v}(p_2)\gamma_\mu u(p_1). \tag{7.54}$$

Then, the squared amplitude is

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 &= \frac{Q_f^2 g^2 e^4}{4s^2}(-g_{\nu\sigma})\,\mathrm{tr}\,(\gamma_\mu \not{p}_1 \gamma_\rho \not{p}_2) \\ &\quad\times\ \mathrm{tr}\left[\left(\gamma^\nu\frac{1}{\not{k}_1 + \not{k}_3}\gamma^\mu - \gamma^\mu\frac{1}{\not{k}_2 + \not{k}_3}\gamma^\nu\right)\not{k}_2\left(\gamma^\rho\frac{1}{\not{k}_1 + \not{k}_3}\gamma^\sigma - \gamma^\sigma\frac{1}{\not{k}_2 + \not{k}_3}\gamma^\rho\right)\not{k}_1\right] \\ &= \frac{4Q_f^2 g^2 e^4}{3s^2}(8p_1\cdot p_2)\left[\frac{4(k_1\cdot k_2)(k_1\cdot k_2 + q\cdot k_3)}{(k_1 + k_3)^2(k_2 + k_3)^2}\right.\end{aligned}$$

$$+\Big(\frac{1}{(k_1+k_3)^4}+\frac{1}{(k_2+k_3)^4}\Big)\big(2(k_1\cdot k_3)(k_2\cdot k_3)-\mu^2(k_1\cdot k_2)\big)\Big]. \tag{7.55}$$

We have used the trick described in Peskin's book (P261) when getting through the last equal sign. Now rewrite the quantities of final-state kinematics in terms of x_i, and set $\mu \to 0$, we obtain

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 &= \frac{2Q_f^2g^2e^4}{3s^2}\big(8p_1\cdot p_2\big)\left[\frac{2(1-x_3)}{(1-x_1)(1-x_2)}+\frac{1-x_1}{1-x_2}+\frac{1-x_2}{1-x_1}\right]\\ &= \frac{8Q_f^2g^2e^4}{3s}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}.\end{aligned} \tag{7.56}$$

Thus the differential cross section, with 3 colors counted, reads

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2}\Big|_{\mathrm{COM}} &= \frac{1}{2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}|v_{\mathbf{p}_1}-v_{\mathbf{p}_2}|}\frac{s}{128\pi^3}\Big(\tfrac{1}{4}\textstyle\sum|\mathcal{M}|^2\Big)\\ &= \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)},\end{aligned} \tag{7.57}$$

where we have used the fact that the initial electron and positron are massless, which implies that $2E_{\mathbf{p}_1}=2E_{\mathbf{p}_2}=\sqrt{s}$ and $|v_{\mathbf{p}_1}-v_{\mathbf{p}_2}|=2$ in COM frame.

(d) Now we reevaluate the averaged squared amplitude, with μ kept nonzero in (7.55). The result is

$$\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 = \frac{8Q_f^2g^2e^4}{3s}F(x_1,x_2,\mu^2/s), \tag{7.58}$$

where

$$\begin{aligned}F\big(x_1,x_2,\tfrac{\mu^2}{s}\big) &= \frac{2(x_1+x_2-1+\frac{\mu^2}{s})(1+\frac{\mu^2}{s})}{(1-x_1)(1-x_2)}\\ &\quad+\Big[\frac{1}{(1-x_1)^2}+\frac{1}{(1-x_2)^2}\Big]\Big((1-x_1)(1-x_2)-\tfrac{\mu^2}{s}\Big).\end{aligned} \tag{7.59}$$

The cross section, then, can be got by integrating over $\mathrm{d}x_1\mathrm{d}x_2$:

$$\begin{aligned}\sigma(e^+e^-\to q\bar{q}g) &= \frac{1}{2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}|v_{\mathbf{p}_1}-v_{\mathbf{p}_2}|}\frac{s}{128\pi^3}\int\mathrm{d}x_1\mathrm{d}x_2\Big(\tfrac{1}{4}\textstyle\sum|\mathcal{M}|^2\Big)\\ &= \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\int_0^{1-\frac{\mu^2}{s}}\mathrm{d}x_1\int_{1-x_1-\frac{\mu^2}{s}}^{1-\frac{t}{s(1-x_1)}}\mathrm{d}x_2\,F\big(x_1,x_2,\tfrac{\mu^2}{s}\big)\\ &= \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\Big[\log^2\frac{\mu^2}{s}+3\log\frac{\mu^2}{s}+5-\tfrac{1}{3}\pi^2+\mathcal{O}(\mu^2)\Big].\end{aligned} \tag{7.60}$$

(e) It is straightforward to finish the integration over Feynman parameters in (a), yielding

$$F_1(q^2=s)=Q_f^2-\frac{Q_f^2\alpha_g}{4\pi}\left[\log^2\frac{\mu^2}{s}+3\log\frac{\mu^2}{s}+\tfrac{7}{2}-\tfrac{1}{3}\pi^2-\mathrm{i}\pi\Big(2\log\frac{\mu^2}{s}+7\Big)+\mathcal{O}(\mu^2)\right]. \tag{7.61}$$

Then the cross section, to the order of α_g, is given by

$$\sigma(e^+e^-\to q\bar{q})=\frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\left\{1-\frac{\alpha_g}{2\pi}\left[\log^2\frac{\mu^2}{s}+3\log\frac{\mu^2}{s}+\tfrac{7}{2}-\tfrac{1}{3}\pi^2\right]+\mathcal{O}(\mu^2)\right\}. \tag{7.62}$$

(f) Combining the results in (d) and (e), we reach the final result:

$$\sigma(e^+e^- \to q\bar{q} + q\bar{q}g) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2 \left[1 + \frac{3\alpha_g}{4\pi}\right]. \tag{7.63}$$

It is worth noting that all divergent terms as $\mu \to 0$ cancel out in this expression.

Chapter 9

Functional Methods

9.1 Scalar QED

The Lagrangian for scalar QED reads,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) - m^2\phi^\dagger\phi, \tag{9.1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mathrm{D}_\mu\phi = (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{9.2}$$

(a) Expanding the covariant derivative, it's easy to find the corresponding Feynman's rules:

$$\begin{aligned}(A_\mu\text{-}A_\nu\text{-}\phi^\dagger\text{-}\phi \text{ interaction}) &= 2ie^2\eta^{\mu\nu},\\ (A_\mu\text{-}\phi^\dagger(p_1)\text{-}\phi(p_2) \text{ interaction}) &= -ie(p_1-p_2)^\mu,\end{aligned}$$

with all momenta pointing inwards.

The propagators are standard. We will work in the Feynman gauge and set $\xi = 1$, then the propagator for photon is simply

$$\frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon},$$

and the propagator for scalar is

$$\frac{i}{p^2 - m^2 + i\epsilon}.$$

(b) Now we calculate the spin-averaged differential cross section for the process $e^+e^- \to \phi^*\phi$. The scattering amplitude is given by

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2\bar{v}(k_2)\gamma^\mu u(k_1)\frac{-\mathrm{i}}{s}(p_1-p_2)_\mu. \tag{9.3}$$

Then the spin-averaged and squared amplitude is

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\,\mathrm{tr}\left[(\not{p}_1 - \not{p}_2)\not{k}_1(\not{p}_1 - \not{p}_2)\not{k}_2\right]$$

$$= \frac{e^4}{4s^2}\big[8(k_1\cdot p_1 - k_1\cdot p_2)(k_2\cdot p_1 - k_2\cdot p_2) - 4(k_1\cdot k_2)(p_1-p_2)^2\big]. \tag{9.4}$$

We may parameterize the momenta as

$$\begin{aligned} k_1 &= (E,0,0,E), & p_1 &= (E, p\sin\theta, 0, p\cos\theta),\\ k_2 &= (E,0,0,-E), & p_2 &= (E, -p\sin\theta, 0, -p\cos\theta), \end{aligned}$$

with $p=\sqrt{E^2-m^2}$. Then we have

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4p^2}{2E^2}\sin^2\theta. \tag{9.5}$$

Thus the differential cross section is:

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}} = \frac{1}{2(2E)^2}\frac{p}{8(2\pi)^2E}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big) = \frac{\alpha^2}{8s}\Big(1-\frac{m^2}{E^2}\Big)^{3/2}\sin^2\theta. \tag{9.6}$$

(c)

$$\begin{aligned}
\delta\Pi_{\mu\nu} &= 2\mathrm{i}e^2\eta_{\mu\nu}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2-m^2} - (-\mathrm{i}e)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(p-2k)_\mu(p-2k)_\nu}{(k^2-m^2)((p-k)^2-m^2)}\\
&= -e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2\eta_{\mu\nu}\big((p-k)^2-m^2\big)-(p-2k)_\mu(p-2k)_\nu}{(k^2-m^2)\big((p-k)^2-m^2\big)}\\
&= -e^2\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{2\eta_{\mu\nu}\big(k'^2+(1-x)^2p^2-m^2\big)+(1-2x)^2p_\mu p_\nu+4k'^\mu k'^\nu}{(k'^2-\Delta)^2}\\
&\cdot = -e^2\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{2\eta_{\mu\nu}k'^2(1-\frac{2}{d})+2\eta_{\mu\nu}\big((1-x)^2p^2-m^2\big)-(1-2x)^2p_\mu p_\nu}{(k'^2-\Delta)^2}\\
&= \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\Big[\frac{(1-\frac{d}{2})\Gamma(1-\frac{d}{2})2\eta_{\mu\nu}}{\Delta^{2-d/2}}\\
&\qquad\qquad + \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big(2\eta_{\mu\nu}\big((1-x)^2p^2-m^2\big)-(1-2x)^2p_\mu p_\nu\big)\Big]\\
&= \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big[2\big((1-x)^2-x(1-x)\big)p^2\eta_{\mu\nu}-(1-2x)^2p_\mu p_\nu\big].
\end{aligned} \tag{9.7}$$

We can symmetrize the integrand as $(1-x)^2\to\frac{1}{2}\big((1-x)^2+x^2\big)$, then we get

$$\delta\Pi_{\mu\nu} = \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(1-2x)^2\big(p^2\eta_{\mu\nu}-p_\mu p_\nu\big). \tag{9.8}$$

9.2 Statistical field theory

In this problem we study the path integral formulation in statistical mechanics. The theory can be described by the partition function:

$$Z = \operatorname{tr} e^{-\beta H}, \tag{9.9}$$

where H is the Hamiltonian of the system. It is a function of the generalized coordinates q and the corresponding conjugate momentum p. In this problem, we simply assume the Hamiltonian has the following form:

$$H = \frac{p^2}{2m} + V(q). \tag{9.10}$$

We assume the dimension of the configuration space is d, then both q and p have d components. Then we assume the eigenstates of both q and p form a complete orthonormal basis of the Hilbert space:

$$1 = \int \mathrm{d}^d q \, |q\rangle\langle q|; \qquad 1 = \int \frac{\mathrm{d}^d p}{(2\pi)^d} |p\rangle\langle p|. \tag{9.11}$$

Then the partition function can be written as

$$Z = \operatorname{tr} e^{-\beta H} = \int \mathrm{d}^d q \, \langle q|e^{-\beta H}|q\rangle. \tag{9.12}$$

(a) Now we derive a path integral expression for the partition function. Following the same way of deriving path integral in a quantum field theory, we separate the quantity $e^{-\beta H}$ into N factors:

$$e^{-\beta H} = e^{-\epsilon H} \cdots e^{-\epsilon H}, \qquad \text{(N factors)},$$

then inserting a complete basis between each pair of adjacent factors, as

$$e^{-\beta H} = \int \mathrm{d}^d q_1 \cdots \mathrm{d}^d q_{N-1} \langle q|e^{-\epsilon H}|q_{N-1}\rangle\langle q_{N-1}|e^{-\epsilon H}|q_{N-2}\rangle \cdots \langle q_1|e^{-\epsilon H}|q\rangle.$$

Now we focus on one factors:

$$\langle q_{i+1}|e^{-\epsilon H}|q_i\rangle = \langle q_{i+1}|e^{-\epsilon\left(\frac{1}{2m}p^2 + V(q)\right)}|q_i\rangle = e^{-\epsilon V(q_i)}\langle q_{i+1}|e^{-\frac{\epsilon}{2m}p^2}|q_i\rangle,$$

and

$$\begin{aligned}\langle q_{i+1}|e^{-\frac{\epsilon}{2m}p^2}|q_i\rangle &= \int \frac{\mathrm{d}^d p_{i+1}\mathrm{d}^d p_i}{(2\pi)^d(2\pi)^d} \langle q_{i+1}|p_{i+1}\rangle\langle p_{i+1}|e^{-\epsilon p^2/2m}|p_i\rangle\langle p_i|q_i\rangle \\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d} e^{ip(q_{i+1}-q_i)} e^{-\epsilon p^2/2m} = \left[\tfrac{m}{2\pi\epsilon}\right]^{d/2} e^{-m(q_{i+1}-q_i)^2/2\epsilon}.\end{aligned}$$

Inserting all this into the partition function, we get:

$$Z = \Big[\frac{m}{2\pi\epsilon}\Big]^{Nd/2} \prod_{i=0}^{N} \int \mathrm{d}^d q_i \exp\Big[-\frac{m(q_{i+1}-q_i)^2}{2\epsilon} - \epsilon V(q_i)\Big], \tag{9.13}$$

with $q_{N+1} = q_0$.

Now let $N \to \infty$, then we have

$$Z = \int \mathcal{D}q \exp\Big[-\beta \oint \mathrm{d}\tau L_E(\tau)\Big], \tag{9.14}$$

where the integral measure is defined by

$$\mathcal{D}q = \lim_{N\to\infty}\Big[\frac{m}{2\pi\epsilon(N)}\Big]^{Nd/2}\prod_{i=0}^{N}\mathrm{d}^d q_i, \tag{9.15}$$

and $L_E(\tau)$ is a Lagrangian in Euclidean form:

$$L_E(\tau) = \frac{m}{2}\Big(\frac{\mathrm{d}q}{\mathrm{d}\tau}\Big)^2 + V(q(\tau)). \tag{9.16}$$

Note that the periodic integral on τ comes from the trace in the partition function.

(b) Now we study an explicit example, a simple harmonic oscillator, which can by defined by the Lagrangian

$$L_E = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\omega^2 q^2. \tag{9.17}$$

Our task is to complete the path integral to find a expression for the partition function of harmonic oscillator. This can be easily done by a Fourier transformation of the coordinates $q(\tau)$ with respect to τ. Since the "time" direction is periodic, the Fourier spectrum of q is discrete. That is,

$$q(\tau) = \beta^{-1/2}\sum_n e^{2\pi \mathrm{i} n\tau/\beta} q_n, \tag{9.18}$$

Then we have:

$$\begin{aligned}\int \mathrm{d}\tau\, L_E(\tau) &= \int \mathrm{d}\tau \frac{1}{2\beta}\sum_{m,n}\Big[\Big(\frac{2\pi\mathrm{i}}{\beta}\Big)^2 mn + \omega^2\Big] q_m q_n e^{2\pi\mathrm{i}(m+n)\tau/\beta} \\ &= \frac{1}{2}\sum_{m,n}\Big[\Big(\frac{2\pi\mathrm{i}}{\beta}\Big)^2 mn + \omega^2\Big] q_m q_n \delta_{m,-n} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big[\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \omega^2\Big] q_n q_{-n} \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big[\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \omega^2\Big] |q_n|^2. \end{aligned} \tag{9.19}$$

Then the path integral can be written as,

$$\begin{aligned} Z &= C\int \mathrm{d}q_0\, e^{-\beta\omega^2 q_0^2}\int\prod_{n>0}\mathrm{dRe}q_n \mathrm{dIm}q_n\, \exp\Big[-\frac{\beta}{2}\Big(\frac{4\pi^2 n^2}{\beta^2}+\omega^2\Big)|q_n|^2\Big] \\ &= \frac{C}{\omega}\prod_{n>0}\Big[\frac{4\pi^2 n^2}{\beta^2}+\omega^2\Big]^{-1} = \frac{C}{\omega}\prod_{n>0}\Big[1+\frac{1}{(\pi n)^2}\Big(\frac{\beta\omega}{2}\Big)^2\Big]^{-1} \\ &= C\sinh^{-1}(\beta\omega/2) = C\sum_{n\geq 0}\exp\big[-\beta\omega(n+\tfrac{1}{2})\big]. \end{aligned} \tag{9.20}$$

(c) From now on we will consider the statistics of fields. We study the statistical properties of boson system, fermion system, and photon system.

For a scalar field, the Lagrangian is given by,

$$L_E(\tau) = \int \mathrm{d}^3 x\, \frac{1}{2}\Big[\dot{\phi}^2(\tau,\mathbf{x}) + \big(\nabla\phi(\tau,\mathbf{x})\big)^2 + m^2\phi^2(\tau,\mathbf{x})\Big]. \tag{9.21}$$

Following the method we used to deal with the simple harmonic oscillator, here we decompose the scalar field $\phi(\tau, \mathbf{x})$ into eigenmodes in momentum space:

$$\phi(\tau, \mathbf{x}) = \beta^{-1/2} \sum_n e^{2\pi i n\tau/\beta} \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \phi_{n,\mathbf{k}}. \tag{9.22}$$

Then the Lagrangian can also be rewritten in terms of modes, as,

$$\begin{aligned}
\int \mathrm{d}\tau\, L_E(\tau) &= \int \mathrm{d}\tau \mathrm{d}^3x \sum_{n,n'} \int \frac{\mathrm{d}^3k \mathrm{d}^3k'}{(2\pi)^6} \frac{1}{2\beta} \Big[\Big(\frac{2\pi\mathrm{i}}{\beta}\Big)^2 n'n - \mathbf{k}'\cdot\mathbf{k} + m^2 \Big] \\
&\qquad \times \phi_{n,\mathbf{k}} \phi_{n',\mathbf{k}'} e^{\mathrm{i}2\pi(n'+n)\tau/\beta + \mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \\
&= \frac{1}{2} \sum_n \int \frac{\mathrm{d}^3k}{(2\pi)^3} \Big[\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \mathbf{k}^2 + m^2 \Big] |\phi_{n,\mathbf{k}}|^2 \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \Big[\frac{1}{2}\omega_{\mathbf{k}}^2 |\phi_{0,\mathbf{k}}|^2 + \sum_{n>0} \Big(\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \omega_{\mathbf{k}}^2 \Big) |\phi_{n,\mathbf{k}}|^2 \Big],
\end{aligned} \tag{9.23}$$

where $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$. Then the partition function, as a path integral over the field configurations can be represented by

$$Z = C \int \prod_{n>0,\mathbf{k}} \mathrm{Re}\phi_{n,\mathbf{k}} \mathrm{Im}\phi_{n,\mathbf{k}}\, \exp\Big[-\beta \Big(\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \omega_{\mathbf{k}}^2 \Big) |\phi_{n,\mathbf{k}}|^2 \Big]. \tag{9.24}$$

By the calculation similar to that in (b), we get

$$Z = C \prod_{\mathbf{k}} \Big[\omega_{\mathbf{k}} \prod_{n>0} \Big(\frac{4\pi^2 n^2}{\beta^2} + \omega_{\mathbf{k}}^2 \Big) \Big]^{-1} = C \prod_{\mathbf{k}} \exp\Big[-\beta\omega_{\mathbf{k}} \big(n + \tfrac{1}{2}\big) \Big]. \tag{9.25}$$

This product gives the meaning to the formal expression $\big[\det(-\partial^2 + m^2)\big]^{-1/2}$ with proper regularization.

(d) Then consider the fermionic oscillator. The action is given by,

$$S = \int \mathrm{d}\tau\, L_E(\tau) = \int \mathrm{d}\tau \Big(\bar\psi(\tau)\dot\psi(\tau) + \omega\bar\psi(\tau)\psi(\tau) \Big). \tag{9.26}$$

The antiperiodic boundary condition $\psi(\tau + \beta) = -\psi(\tau)$ is crucial to expanding the fermion into modes:

$$\psi(\tau) = \beta^{-1/2} \sum_{n\in\mathbb{Z}+1/2} e^{2\pi\mathrm{i}\tau/\beta} \psi_n. \tag{9.27}$$

Then the partition function can be evaluated to be

$$\begin{aligned}
Z &= \int \prod_n \mathrm{d}\bar\psi_n \mathrm{d}\psi_n \Big[-\beta \sum_{n\in\mathbb{Z}+1/2} \bar\psi_n \Big(\frac{2\pi\mathrm{i}n}{\beta} + \omega \Big) \psi_n \Big] \\
&= C(\beta) \prod_{n\in\mathbb{Z}+1/2} \Big(\frac{2\pi\mathrm{i}n}{\beta} + \omega \Big) = C(\beta) \prod_{n=0}^{\infty} \Big(\frac{4\pi^2 (n+\frac{1}{2})^2}{\beta^2} + \omega^2 \Big) \\
&= C(\beta) \cosh\big(\tfrac{1}{2}\beta\omega\big) = C(\beta) \Big(e^{\beta\omega/2} + e^{-\beta\omega/2} \Big),
\end{aligned} \tag{9.28}$$

with the form of a two-level system, as expected.

(e) Finally we consider the system of photons. The partition function is given by

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \mathcal{D}b\mathcal{D}c \, \exp\left[\int \mathrm{d}\tau \mathrm{d}^3x \left(-\tfrac{1}{2}A_\mu \partial^2 A^\mu - b\partial^2 c\right)\right] \\ &= C(\beta)\big[\det(-\partial)\big]^{4\cdot(-1/2)} \cdot \det(-\partial^2), \end{aligned} \tag{9.29}$$

where the first determinant comes from the integral over the vector field A_μ while the second one comes from the integral over the ghost fields. Therefore,

$$Z = C(\beta)\big[\det(-\partial^2)\big]^{2\cdot(-1/2)}, \tag{9.30}$$

which shows the contributions from the two physical polarizations of a photon. Here we see the effect of the ghost fields of eliminating the additional two unphysical polarizations of a vector field.

Chapter 10

Systematics of Renormalization

10.1 One-Loop structure of QED

(a) In this problem we show that any photon n-point amplitude with n an odd number vanishes.

Now we evaluate explicitly the one-point photon amplitude and three-point photon amplitude at 1-loop level to check Furry's theorem. The one-point amplitude at 1-loop level is simply given by,

$$\mathrm{i}\Gamma^{(1)} = (-\mathrm{i}e)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}\,\mathrm{tr}\left[\gamma^\mu(\not{k}+m)\right]}{k^2-m^2} = 0, \tag{10.1}$$

and the three-point amplitude consists of two diagrams,

$$\begin{aligned}\mathrm{i}\Gamma^{(3)} = (-\mathrm{i}e)^3\int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\Bigg\{ &\mathrm{tr}\left[\gamma^\mu\frac{\mathrm{i}}{\not{k}-m}\gamma^\nu\frac{\mathrm{i}}{\not{k}+\not{p}_1-m}\gamma^\lambda\frac{\mathrm{i}}{\not{k}+\not{p}_1+\not{p}_2-m}\right]\\ &+\mathrm{tr}\left[\gamma^\mu\frac{\mathrm{i}}{\not{k}+\not{p}_1+\not{p}_2-m}\gamma^\lambda\frac{\mathrm{i}}{\not{k}+\not{p}_1-m}\gamma^\nu\frac{\mathrm{i}}{\not{k}-m}\right]\Bigg\}.\end{aligned} \tag{10.2}$$

(b) Next we will show that the potential logarithmic divergences in photon four-point diagrams cancel with each other. Since the divergence in this case does not depend on external momenta, we will set all external momenta to be zero for simplicity. For the same reason we will also set the fermion's mass to be zero. Then the six diagrams contributing the four-point amplitude can be evaluated as,

$$\begin{aligned}&(\text{Divergent part of } \mathrm{i}\Gamma^{\mu\nu\rho\sigma})\\ =&\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-1}{(k^2)^4}\Big[\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}\right]+\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\sigma\not{k}\gamma^\rho\not{k}\right]+\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\rho\not{k}\gamma^\nu\not{k}\gamma^\sigma\not{k}\right]\\ &+\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}\gamma^\nu\not{k}\right]+\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\sigma\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}\right]+\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\sigma\not{k}\gamma^\rho\not{k}\gamma^\nu\not{k}\right]\Big].\end{aligned} \tag{10.3}$$

Now let's focus on the first trace, which can be worked out explicitly, to be

$$\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}\right] = 32k^\mu k^\nu k^\rho k^\sigma - 8k^2\left(k^\mu k^\nu g^{\rho\sigma}+k^\rho k^\sigma g^{\mu\nu}+k^\mu k^\sigma g^{\nu\rho}+k^\nu k^\rho g^{\mu\sigma}\right)$$

$$+ 4(k^2)^2(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{10.4}$$

Then, we symmetrize the momentum factors according to $k^\mu k^\nu \to k^2 g^{\mu\nu}/4$ and $k^\mu k^\nu k^\rho k^\sigma \to (k^2)^2(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})/24$. (Since the divergence can be at most logarithmic, so it is safe to set spacetime dimension $d = 4$ at this stage.) Then the first trace term reduces to,

$$\mathrm{tr}\,[\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}] \Rightarrow \frac{4}{3}(k^2)^2(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{10.5}$$

The other five terms can be easily got by permuting indices. Then it is straightforward to see that the six terms sum to zero.

10.2 Renormalization of Yukawa theory

In this problem we study the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 + \bar{\psi}(i\not{\partial} - M)\psi - ig\bar{\psi}\gamma^5\psi\phi\,, \tag{10.6}$$

where ϕ is a real scalar and ψ is a Dirac Fermion.

(a) Let's figure out how the superficial degree of divergence D depends on the number of external lines. From power counting, it's easy to see that D can be represented by

$$D = 4L - P_f - 2P_s\,, \tag{10.7}$$

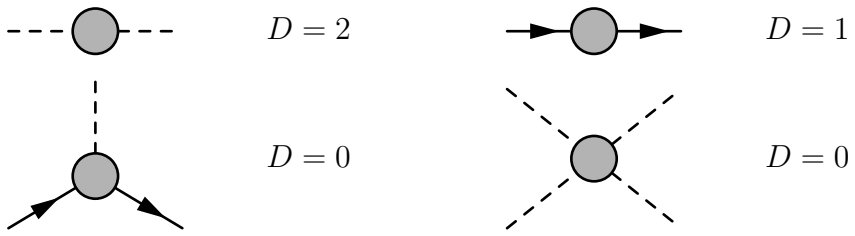
where L is the no. of loops, P_f is the no. of internal fermion lines, and P_s is the no. of internal scalar lines. We also note the following simple relations:

$$\begin{aligned} L &= P_f + P_s - V + 1\,, \\ 2V &= 2P_f + N_f\,, \\ V &= 2P_s + N_s\,. \end{aligned}$$

Then we can deduce

$$D = 4L - P_f - 2P_s = 4(P_f + P_s - V + 1) - P_f - 2P_s = 4 - \tfrac{3}{2}N_f - N_s\,. \tag{10.8}$$

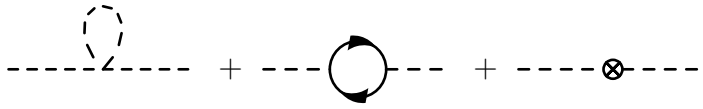
Guided by this result, we can find all divergent amplitudes as follows.



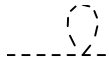
We note that we have ignored the vacuum diagram, which simply contributes an infinitely large constant, the potentially divergent diagrams with odd number of external scalars are also ignored, since they actually vanish. This result shows that the original theory cannot be renormalized unless we including a new ϕ^4 interaction, as

$$\delta\mathcal{L} = -\tfrac{\lambda}{4!}\phi^4\,. \tag{10.9}$$

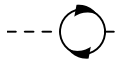
(b) Now let us evaluate the divergent parts of all 1-loop diagrams of Yukawa theory. First we consider the two point function of scalar. The one-loop contribution to this amplitude is shown as follows.



The $d = 4$ pole of first two loop diagrams can be determined as



$$= \frac{-\mathrm{i}\lambda}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} \sim \frac{\mathrm{i}\lambda m^2}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.10}$$

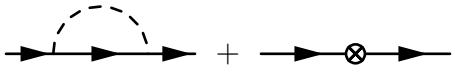


$$= -(-\mathrm{i}g)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \,\mathrm{tr}\left[\frac{\mathrm{i}}{\not{k} - M}\gamma^5 \frac{\mathrm{i}}{(\not{k} - \not{p}) - M}\gamma^5\right] \sim \frac{4\mathrm{i}g^2(p^2 - 2M^2)}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.11}$$

From this we find the divergent part of the counterterm to be

$$\delta_m \sim \frac{(\lambda m^2 - 8g^2 M^2)}{(4\pi)^2} \frac{1}{\epsilon}, \qquad \delta_\phi = \frac{-4g^2}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.12}$$

Then we come to the two point function of fermion, the 1-loop correction of which is given by the following two diagrams.



From the pole of the loop diagram

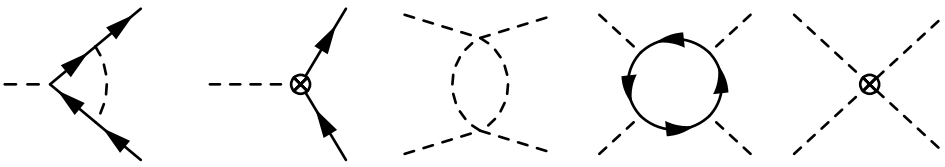


$$= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not{k} - M} \gamma^5 \frac{\mathrm{i}}{(k-p)^2 - m^2} \sim \frac{\mathrm{i}g^2(\not{p} - 2M)}{(4\pi)^2} \frac{1}{\epsilon}, \tag{10.13}$$

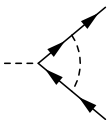
we find the following counterterms:

$$\delta_M \sim \frac{-2g^2 M}{(4\pi)^2} \frac{1}{\epsilon}, \qquad \delta_\psi \sim \frac{-g^2}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.14}$$

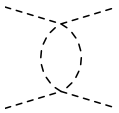
The following two diagrams contribute to 1-loop corrections to Yukawa coupling and ϕ^4 coupling, respectively.



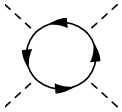
Since the divergent part of diagram is independent of external momenta, we can set all these momenta to be zero. Then the loop diagram is



$$= g^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not{k} - M} \gamma^5 \frac{\mathrm{i}}{\not{k} - M} \gamma^5 \frac{\mathrm{i}}{k^2 - m^2} \sim -\frac{g^3 \gamma^5}{(4\pi)^2} \frac{2}{\epsilon} \tag{10.15}$$



$$= \frac{(-\mathrm{i}\lambda)^2}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \Big(\frac{\mathrm{i}}{k^2 - m^2} \Big)^2 \sim \frac{\mathrm{i}\lambda^2}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.16}$$



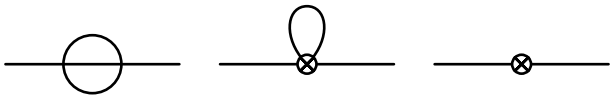
$$= (-1) g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \operatorname{tr} \Big[\Big(\gamma^5 \frac{\mathrm{i}}{\not{k} - M} \Big)^4 \Big] \sim -\frac{8\mathrm{i} g^4}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.17}$$

Note that there are 3 permutations for the first diagram and 6 permutations for the second diagram. Then we can determine the divergent part of counterterm to be

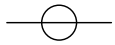
$$\delta_g \sim \frac{2g^3}{(4\pi)^2} \frac{1}{\epsilon}, \qquad \delta_\lambda \sim \frac{3\lambda^2 - 48 g^4}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.18}$$

10.3 Field-strength renormalization in ϕ^4 theory

In this problem we evaluate the two-loop corrections to scalar's two-point function in ϕ^4 theory in the massless limit. There are three diagrams contribute in total.

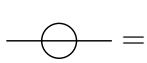


The first diagram reads

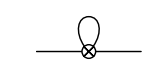


$$\begin{aligned}
&= \frac{(-\mathrm{i}\lambda)^2}{6} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} \frac{\mathrm{i}}{q^2 - m^2} \frac{\mathrm{i}}{(p - k - q)^2 - m^2} \\
&= \frac{\mathrm{i}\lambda^2}{6} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{\mathrm{d}^d q_E}{(2\pi)^d} \frac{\mathrm{i}}{k_E^2 + m^2} \frac{\mathrm{i}}{q_E^2 + m^2} \frac{\mathrm{i}}{(p_E - k_E - q_E)^2 + m^2} \\
&= \frac{\mathrm{i}\lambda^2}{6} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + m^2} \frac{1}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \, \frac{\Gamma(2 - \frac{d}{2})}{[m^2 + x(1-x)(p_E - k_E)^2]^{2 - d/2}} \\
&= \frac{\mathrm{i}\lambda^2 \Gamma(2 - \frac{d}{2})}{6(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \, \mathrm{d}y \\
&\quad \times \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{[x(1-x)]^{d/2-2} (1-y)^{1-d/2} \Gamma(3 - \frac{d}{2}) / \Gamma(2 - \frac{d}{2})}{\big[(k_E - y p_E)^2 + y(1-y) p_E^2 + \big(1 - y + \frac{y}{x(1-x)}\big) m^2 \big]^{3 - d/2}} \\
&= \frac{\mathrm{i}\lambda^2}{6(4\pi)^d} \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \frac{\Gamma(3-d) [x(1-x)]^{d/2-2} (1-y)^{1-d/2}}{\big[y(1-y) p_E^2 + \big(1 - y + \frac{y}{x(1-x)}\big) m^2 \big]^{3-d}}.
\end{aligned} \tag{10.19}$$

Now we take $m^2 = 0$ and $d = 4 - \epsilon \to 4$. Then we have

$$\begin{aligned} \text{} &= \frac{\mathrm{i}\lambda^2}{12(4\pi)^4}\Gamma(-1+\epsilon)(p_E^2)^{1-\epsilon} + \cdots = -\frac{\mathrm{i}\lambda^2}{12(4\pi)^4} p_E^2 \Big[\frac{1}{\epsilon} - \log(p_E^2) + \cdots\Big] \\ &= \frac{\mathrm{i}\lambda^2}{12(4\pi)^4} p^2 \Big[\frac{1}{\epsilon} - \log(-p^2) + \cdots\Big]. \end{aligned} \tag{10.20}$$

The second diagram actually vanishes in $m \to 0$ limit. In fact,

$$\text{} = \frac{-\mathrm{i}\delta_\lambda}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} = \frac{-\mathrm{i}\delta_\lambda}{2(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{m^{1-d/2}} \propto m \to 0. \tag{10.21}$$

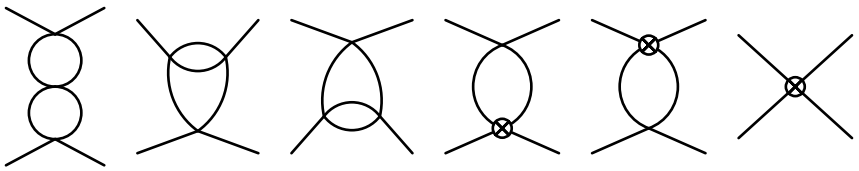
The third diagram reads $\mathrm{i}p^2\delta_Z$. Therefore we can choose the counterterm δ_Z, under the $\overline{MS}$ scheme, to be

$$\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right]. \tag{10.22}$$

Thus the field strength counterterm receives a nonzero contribution at this order. In the massless limit, it is

$$\delta_2\Gamma^{(2)} = \frac{\mathrm{i}\lambda^2}{12(4\pi)^4} p^2 \log\frac{M^2}{-p^2}. \tag{10.23}$$

10.4 Asymptotic behavior of diagrams in ϕ^4 theory



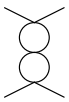
In this problem we calculate the four point amplitude in ϕ^4 theory to 2-loop order in $s \to \infty$, t fixed, limit. The tree level result is simply $-\mathrm{i}\lambda$, and the 1-loop result can be easily evaluated to be

$$\begin{aligned} \mathrm{i}\delta_1\mathcal{M} &= \frac{(-\mathrm{i}\lambda)^2}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} \Big[\frac{\mathrm{i}}{(p_s - k)^2 - m^2} + \frac{\mathrm{i}}{(p_t - k)^2 - m^2} + \frac{\mathrm{i}}{(p_u - k)^2 - m^2}\Big] - \mathrm{i}\delta_\lambda \\ &\simeq \frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\Big[3\big(\tfrac{2}{\epsilon} - \gamma + \log 4\pi\big) - \log s - \log t - \log u\Big] - \mathrm{i}\delta_\lambda \\ &= -\frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\big(\log s + \log t + \log u\big) \sim -\frac{\mathrm{i}\lambda^2}{(4\pi)^2}\log s \end{aligned} \tag{10.24}$$

In the last step we take the limit $s \to \infty$. In this limit t can be ignored and $u \simeq -s$. We see the divergent part of the counterterm coefficient δ_λ at 1-loop order is

$$\delta_\lambda \sim \frac{3\lambda^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{10.25}$$

Now we consider the two-loop correction.



$$= \frac{(-\mathrm{i}\lambda)^3}{4}\Big[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_s-k)^2}\Big]^2 = -\frac{\mathrm{i}\lambda^3}{4(4\pi)^d}\Big[\int_0^1 \mathrm{d}x \frac{\Gamma(2-\frac{d}{2})}{[-x(1-x)s]^{2-d/2}}\Big]^2$$

$$\sim -\frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\log s+\frac{1}{2}\log^2 s\Big). \tag{10.26}$$

In the last line we only keep the divergent terms as $\epsilon \to 0$ and $s \to \infty$.



$$= \frac{(-\mathrm{i}\lambda)^3}{2}\int \frac{\mathrm{d}^d k \mathrm{d}^d q}{(2\pi)^{2d}}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_s-k)^2}\frac{\mathrm{i}}{q^2}\frac{\mathrm{i}}{(k-p_3-q)^2}$$

$$= -\frac{\mathrm{i}\lambda^3}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2(p_s-k)^2}\Big[\frac{\mathrm{i}}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{[x(1-x)(k_E-p_{3E})^2]^{2-d/2}}\Big]$$

$$= \frac{\mathrm{i}\lambda^3}{2(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{[x(1-x)]^{2-d/2}}\int \frac{\mathrm{d}^d k_E}{(2\pi)^d}\frac{1}{k_E^2(p_{sE}-k_E)^2[(k_E-p_{3E})^2]^{2-d/2}}$$

$$= \frac{\mathrm{i}\lambda^3}{2(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\frac{\Gamma(2-\frac{d}{2})}{[x(1-x)]^{2-d/2}}\int \frac{\mathrm{d}^d k_E}{(2\pi)^d}\int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}z\,\frac{z^{1-d/2}}{(k_E^2+\Delta)^{4-d/2}}\frac{\Gamma(4-\frac{d}{2})}{\Gamma(2-\frac{d}{2})}$$

$$= \frac{\mathrm{i}\lambda^3}{2(4\pi)^d}\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\frac{z^{1-d/2}}{[x(1-x)]^{2-d/2}}\frac{\Gamma(4-d)}{\Delta^{4-d}}, \tag{10.27}$$

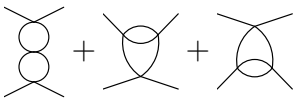
where $\Delta = ys + zp_{3E}^2 - (yp_{sE}+zp_{3E})^2$.

Then we find



$$\sim -\frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\log s+\frac{1}{2}\log^2 s\Big). \tag{10.28}$$

The same result for the third diagram. Then we have



$$\sim -\frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{3}{\epsilon^2}-\frac{3}{\epsilon}\log s+\frac{3}{2}\log^2 s\Big). \tag{10.29}$$

Now we come to the counterterm. The fourth diagram reads

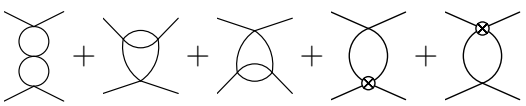


$$= \frac{(-\mathrm{i}\lambda)(-\mathrm{i}\delta_\lambda)}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_s-k)^2}$$

$$\sim \frac{3\lambda^3}{2(4\pi)^2}\frac{1}{\epsilon}\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\Big(1-\frac{\epsilon}{2}\log s+\frac{\epsilon^2}{8}\log^2 s+\cdots\Big)$$

$$\sim \frac{3\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{1}{\epsilon^2}-\frac{1}{2\epsilon}\log s+\frac{1}{8}\log^2 s\Big) \tag{10.30}$$

The same result for the fifth diagram. Then we have



$$\sim \frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{3}{\epsilon^2}-\frac{3}{4}\log^2 s\Big). \tag{10.31}$$

So much for the s-channel. The t and u-channel results can be obtained by replacing s with t and u respectively. In the limit $s \to \infty$ and t-fixed, we can simply ignore t and treating $u \sim -s$, then the total 2-loop correction in this limit is

$$\mathrm{i}\delta_2\mathcal{M} \sim -\frac{3\mathrm{i}\lambda^3}{2(4\pi)^4}\log^2 s. \tag{10.32}$$

The double pole $1/\epsilon^2$ has been absorbed by δ_λ.

In summary, we have the following asymptotic expression for the 4-point amplitude to 2-loop order in the $s \to \infty$ and t-fixed limit:

$$\mathrm{i}\mathcal{M} = -\mathrm{i}\lambda - \frac{\mathrm{i}\lambda^2}{(4\pi)^2}\log s - \frac{3\mathrm{i}\lambda^3}{2(4\pi)^4}\log^2 s + \cdots . \tag{10.33}$$

Chapter 11

Renormalization and Symmetry

11.1 Spin-wave theory

(a) Firstly we prove the following formula:

$$\left\langle T e^{\mathrm{i}\phi(x)} e^{-\mathrm{i}\phi(0)} \right\rangle = e^{[D(x)-D(0)]}. \tag{11.1}$$

Where $D(x) = \langle T\phi(x)\phi(0)\rangle$ is the time-ordered correlation of two scalars. The left hand side of this equation can be represented by path integral, as

$$\frac{1}{Z[0]} \int \mathcal{D}\phi \, e^{\mathrm{i}\phi(x)} e^{-\mathrm{i}\phi(0)} \exp\Big[\mathrm{i} \int \mathrm{d}^d x \mathrm{d}^d y \, \tfrac{1}{2}\phi(x) D^{-1}(x-y)\phi(y)\Big]. \tag{11.2}$$

This expression precisely has the form $Z[J]/Z[0]$, with $J(y) = \delta(y-x) - \delta(0)$. Thus we have

$$Z[J]/Z[0] = -\frac{1}{2} \int \mathrm{d}^d x \mathrm{d}^d y \, J(x) D(x-y) J(y) = \exp\big[D(x) - D(0)\big], \tag{11.3}$$

which is just the right hand side of the formula.

(b) The operator being translational invariant $O[\phi(x)] = O[\phi(x) - \alpha]$ can depend on ϕ only through $\nabla_\mu \phi$. And the only relevant/marginal Lorentz-invariant operator satisfying this condition is $\frac{1}{2}\rho(\nabla\phi)^2$.

(c) From now on we use bold $\mathbf{x}$ to denote coordinate and italic x to denote its length, $x \equiv |\mathbf{x}|$. We can use the result in (a) to evaluate $\langle s(\mathbf{x}) s^*(0)\rangle$, as

$$\langle s(\mathbf{x}) s^*(0)\rangle = A^2 \langle e^{\mathrm{i}\phi(\mathbf{x})} e^{-\mathrm{i}\phi(0)}\rangle = A^2 e^{D(\mathbf{x})-D(0)}. \tag{11.4}$$

Note that the correlation function

$$D(\mathbf{x}) = \frac{1}{\rho} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{k_E^2} e^{\mathrm{i}\mathbf{k}_E \cdot \mathbf{x}} \tag{11.5}$$

is the solution to the following equation:

$$-\rho\nabla^2 D(\mathbf{x}-\mathbf{y}) = \delta^{(d)}(\mathbf{x}-\mathbf{y}). \tag{11.6}$$

Since $D(x)$ is a function of the length only, namely $D(\mathbf{x}) = D(x)$, thus we have

$$-\frac{\rho}{x^{d-1}}\frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x}D(x)\Big) = \frac{\Gamma(1+\frac{d}{2})}{d\pi^{d/2}}\frac{\delta(x)}{x^{d-1}}. \tag{11.7}$$

Then it's easy to find

$$D(x) = \begin{cases} \dfrac{\Gamma(1+\frac{d}{2})}{d(d-2)\pi^{d/2}\rho}\dfrac{1}{x^{d-2}}, & \text{for } \ d\neq 2, \\ -\dfrac{1}{2\pi\rho}\log x, & \text{for } \ d=2. \end{cases} \tag{11.8}$$

Then we have

Dimension d	$d=1$	$d=2$	$d=3$	$d=4$
$D(x)$	$-\dfrac{1}{2\rho}x$	$-\dfrac{1}{2\pi\rho}\log x$	$\dfrac{1}{4\pi\rho x}$	$\dfrac{1}{4\pi^2\rho x^2}$
$\langle ss^*\rangle$	$\sim e^{-x}$	$\sim 1/x^{2\pi\rho}$	$\sim e^{1/x}$	$\sim e^{1/x^2}$

Since $\rho\to 0$ when $d\to 2$, the correlation function $\langle ss^*\rangle$ in this case is independent of length x.

11.2 A zeroth-order natural relation

We study $N=2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \tfrac{1}{2}\mu^2\phi^i\phi^i - \tfrac{1}{4}\lambda(\phi^i\phi^i)^2 + \bar\psi(\mathrm{i}\not\partial)\psi - g\bar\psi(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi, \tag{11.9}$$

with ϕ^i a two-component field, $i=1,2$.

(a) Now, under the following transformation:

$$\phi^1\to\phi^1\cos\alpha - \phi^2\sin\alpha; \qquad \phi^2\to\phi^1\sin\alpha+\phi^2\cos\alpha; \qquad \psi\to e^{-\mathrm{i}\alpha\gamma^2/2}\psi, \tag{11.10}$$

the first three terms involving ϕ^i only keep invariant. The fourth term, as the kinetic term of a chiral fermion, is also unaffected by this transformation. Thus, to show the whole Lagrangian is invariant, we only need to check the last term, and this is really the case:

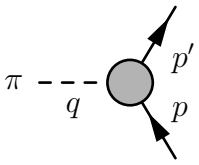
$$\begin{aligned} &-g\bar\psi(\phi^1+\mathrm{i}\gamma^5\phi^2)\psi \\ \to &-g\bar\psi e^{-\mathrm{i}\alpha\gamma^5/2}\big[(\phi^1\cos\alpha-\phi^2\sin\alpha)+\mathrm{i}\gamma^5(\phi^1\sin\alpha+\phi^2\cos\alpha)\big]e^{-\mathrm{i}\alpha\gamma^5/2}\psi \\ = &-g\bar\psi e^{-\mathrm{i}\alpha\gamma^5/2}e^{\mathrm{i}\alpha\gamma^5}(\phi^1+\mathrm{i}\gamma^5\phi^2)e^{-\mathrm{i}\alpha\gamma^5/2}\psi = -g\bar\psi(\phi^1+\mathrm{i}\gamma^5\phi^2)\psi. \end{aligned} \tag{11.11}$$

(b) Now let ϕ acquire a vacuum expectation value v, which equals to $\sqrt{\mu^2/\lambda}$ classically. Then, in terms of new variables $\phi = (v + \sigma(x), \pi(x))$, the Lagrangian reads

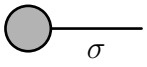
$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 - \mu^2\sigma^2 - \tfrac{1}{4}\lambda(\sigma^4 + \pi^4) \\ & - \tfrac{1}{2}\lambda\sigma^2\pi^2 - \lambda v\sigma^3 - \lambda v\sigma\pi^2 + \bar\psi(\mathrm{i}\partial\!\!\!/ - gv)\psi - g\bar\psi(\sigma + \mathrm{i}\gamma^5\pi)\psi. \end{aligned} \tag{11.12}$$

That is, the fermion acquire a mass $m_f = gv$.

(c) Now we calculate the radiative corrections to the mass relation $m_f = gv$. The renormalization conditions we need are as follows.

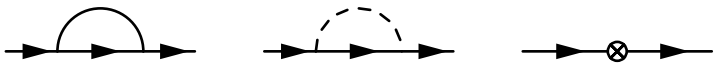


$$= g\gamma^5 \qquad \text{at } q^2 = 0,\ p^2 = p'^2 = m_f^2. \tag{11.13}$$



$$= 0. \tag{11.14}$$

These two conditions fixed g and v so that they receive no radiative corrections. Then we want to show that the mass of the fermion m_f receives *finite* radiative correction at 1-loop. Since the tadpole diagrams of σ sum to zero by the renormalization condition above, the fermion's self-energy receive nonzero contributions from the following three diagrams:



The first two 1-loop diagrams can be evaluated as

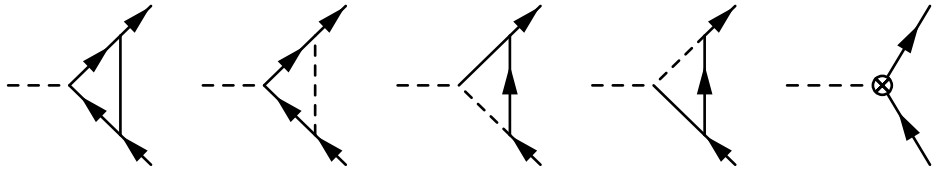
$$\begin{aligned}(\mathrm{e}) &= (-\mathrm{i}g)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{\not k - m_f} \frac{\mathrm{i}}{(k-p)^2 - 2\mu^2} = g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{x\not p + m_f}{(k'^2 - \Delta_1)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\, \frac{\Gamma(2 - \frac{d}{2})}{\Delta_1^{2-d/2}} (x\not p + m_f) \\ &= \frac{\mathrm{i}g^2}{(4\pi)^2} \int_0^1 \mathrm{d}x\, (x\not p + m_f) \Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta_1 \Big] \end{aligned} \tag{11.15}$$

$$\begin{aligned}(\mathrm{f}) &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not k - m_f} \gamma^5 \frac{\mathrm{i}}{(k-p)^2} = g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{x\not p - m_f}{(k'^2 - \Delta_2)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\, \frac{\Gamma(2 - \frac{d}{2})}{\Delta_2^{2-d/2}} (x\not p - m_f) \\ &= \frac{\mathrm{i}g^2}{(4\pi)^2} \int_0^1 \mathrm{d}x\, (x\not p - m_f) \Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta_2 \Big]. \end{aligned} \tag{11.16}$$

This leads to

$$\text{(e)} + \text{(f)} = \frac{\mathrm{i}g^2}{(4\pi)^2}\int_0^1 \mathrm{d}x \left\{ 2x\not{p}\left[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \tfrac{1}{2}\log(\Delta_1\Delta_2)\right] + m_f \log\frac{\Delta_2}{\Delta_1}\right\} \tag{11.17}$$

We see that the correction to the fermions mass m_f from these two diagrams is finite. Besides, the third diagram, namely the counterterm, contributes the mass' correction through $\delta_g v$. The the total correction to m_f is finite only when δ_g is finite. Let us check this by means of the first renormalization condition (11.13) stated above. The 1-loop contributions to (11.13) are as follows.



$$\begin{aligned}
\text{(a)} &= (-\mathrm{i}g)^2 g \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{\not{k} - m_f}\gamma^5\frac{\mathrm{i}}{\not{k} - m_f}\frac{\mathrm{i}}{(k-p)^2 - 2\mu^2} \\
&= \mathrm{i}g^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(\not{k} + m_f)\gamma^5(\not{k} + m_f)}{(k^2 - m_f^2)^2\big((k-p)^2 - 2\mu^2\big)} = -\mathrm{i}g^3\gamma^5 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2 - m_f^2)\big((k-p)^2 - 2\mu^2\big)} \\
&= -\,\mathrm{i}g^3\gamma^5 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\,\frac{1}{(k'^2 - \Delta_1)^2} = \frac{g^3\gamma^5}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\frac{\Gamma(2 - \frac{d}{2})}{\Delta_1^{2-d/2}} \\
&= \frac{g^3\gamma^5}{(4\pi)^2}\int_0^1 \mathrm{d}x\left[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta_1\right]
\end{aligned} \tag{11.18}$$

$$\begin{aligned}
\text{(b)} &= g^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\gamma^5\frac{\mathrm{i}}{\not{k} - m_f}\gamma^5\frac{\mathrm{i}}{\not{k} - m_f}\gamma^5\frac{\mathrm{i}}{(k-p)^2 - 2\mu^2} \\
&= -\,\mathrm{i}g^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\gamma^5(\not{k} + m_f)\gamma^5(\not{k} + m_f)\gamma^5}{(k^2 - m_f^2)^2\big((k-p)^2 - 2\mu^2\big)} = \mathrm{i}g^3\gamma^5 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2 - m_f^2)\big((k-p)^2 - 2\mu^2\big)} \\
&= \mathrm{i}g^3\gamma^5 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\,\frac{1}{(k'^2 - \Delta_2)^2} = \frac{-g^3\gamma^5}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\frac{\Gamma(2 - \frac{d}{2})}{\Delta_2^{2-d/2}} \\
&= \frac{-g^3\gamma^5}{(4\pi)^2}\int_0^1 \mathrm{d}x\left[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta_2\right]
\end{aligned} \tag{11.19}$$

$$\begin{aligned}
\text{(c)} &= (-\mathrm{i}g)g(-2\mathrm{i}\lambda v)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\gamma^5\frac{\mathrm{i}}{\not{k} - m_f}\frac{\mathrm{i}}{(k-p)^2 - 2\mu^2}\frac{\mathrm{i}}{(k-p)^2} \\
&= 4\mathrm{i}g^2\lambda v\gamma^5 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(x+y)\not{p} + m_f}{(k'^2 - \Delta_3)^3} \\
&= \frac{2g^2\lambda v\gamma^5}{(4\pi)^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(x+y)\not{p} + m_f}{\Delta_3}
\end{aligned} \tag{11.20}$$

$$\begin{aligned}(\mathrm{d}) &= (-\mathrm{i}g)g(-2\mathrm{i}\lambda v)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{\not k-m_f}\gamma^5\frac{\mathrm{i}}{(k-p)^2-2\mu^2}\frac{\mathrm{i}}{(k-p)^2}\\ &= 4\mathrm{i}g^2\lambda v\gamma^5\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{-(x+y)\not p+m_f}{(k'^2-\Delta_3)^3}\\ &= \frac{2g^2\lambda v\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{-(x+y)\not p+m_f}{\Delta_3}\end{aligned}\tag{11.21}$$

Thus,

$$(\mathrm{a})+(\mathrm{b})+(\mathrm{c})+(\mathrm{d}) = \frac{g\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\Big[g^2\log\frac{\Delta_2}{\Delta_1}+4\lambda\int_0^{1-x}\mathrm{d}y\,\frac{m_f^2}{\Delta_3}\Big].\tag{11.22}$$

11.3 The Gross-Neveu model

The Gross-Neveu Model is a theory of fermions in $1+1$ dimensional spacetime:

$$\mathcal{L} = \bar\psi_i\mathrm{i}\not\partial\psi_i+\tfrac{1}{2}g^2(\bar\psi_i\psi_i)^2,\tag{11.23}$$

with $i=1,\cdots,N$. The gamma matrices are taken as $\gamma^0=\sigma^2$, $\gamma^1=i\sigma^1$, where σ^i is the familiar Pauli matrices. We also define $\gamma^5=\gamma^0\gamma^1=\sigma^3$.

(a) The theory is invariant under the transformation $\psi_i\to\gamma^5\psi_i$. It is straightforward to check this. We note that:

$$\bar\psi_i=\psi_i^\dagger\gamma^0\to\psi_i^\dagger\gamma^5\gamma^0=-\bar\psi_i\gamma^5,\tag{11.24}$$

thus:

$$\begin{aligned}\mathcal{L}\to&-\bar\psi_i\gamma^5\mathrm{i}\not\partial\gamma^5\psi_i+\tfrac{1}{2}g^2(-\bar\psi_i\gamma^5\gamma^5\psi_i)^2\\ &=\bar\psi_i\mathrm{i}\not\partial\psi_i+\tfrac{1}{2}g^2(\bar\psi_i\psi_i)^2.\end{aligned}\tag{11.25}$$

However, a mass term will transform as $m_i\bar\psi_i\psi_i\to-m_i\bar\psi_i\psi_i$, thus a theory respecting this chiral symmetry does not allow such a mass term.

(b) The superficial renormalizability of the theory (by power counting) is obvious since $[g]=0$.

(c) The model can be phrased in another equivalent way:

$$Z=\int\mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}\sigma\exp\left[\mathrm{i}\int\mathrm{d}^2x\left(\bar\psi_i\mathrm{i}\not\partial\psi_i-\tfrac{1}{2g^2}\sigma^2-\sigma\bar\psi_i\psi_i\right)\right].\tag{11.26}$$

This can be justified by integrating out σ,

$$\int\mathcal{D}\sigma\exp\left[\mathrm{i}\int\mathrm{d}^2x\,(-\tfrac{1}{2g^2}\sigma^2-\sigma\bar\psi_i\psi_i)\right]=N\exp\left[\mathrm{i}\int\mathrm{d}^2x\,\tfrac{g^2}{2}(\bar\psi_i\psi_i)^2\right].\tag{11.27}$$

which recovers the following path integral:

$$Z=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left[\mathrm{i}\int\mathrm{d}^2x\left(\bar\psi_i\mathrm{i}\not\partial\psi_i+\tfrac{1}{2}g^2(\bar\psi_i\psi_i)^2\right)\right].\tag{11.28}$$

(d) We can also integrate out the fermions ψ_i to get the effective potential for the auxiliary field σ:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\mathrm{i}\int \mathrm{d}^2x\big(\bar{\psi}_i \mathrm{i}\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i\big)\right] = \big[\det(\mathrm{i}\partial\!\!\!/ - \sigma)\big]^N = \big[\det(\partial^2+\sigma^2)\big]^N$$
$$= \exp\left[\int \frac{\mathrm{d}^2k}{(2\pi)^2} N\log(-k^2+\sigma^2)\right]. \tag{11.29}$$

The integral is divergent, which should be regularized. We use the dimensional regularization:

$$\int \frac{\mathrm{d}^dk_E}{(2\pi)^d} N\log(k_E^2+\sigma^2) = N\int \frac{\mathrm{d}^dk_E}{(2\pi)^d}\left[\frac{\partial}{\partial\alpha}\frac{1}{k_E^2+\sigma^2}\right]_{\alpha=0}$$
$$= -\,\mathrm{i}N\frac{\Gamma(-d/2)(\sigma^2)^{d/2}}{(4\pi)^{d/2}}. \tag{11.30}$$

Now we set $d = 2-\epsilon$ and send $\epsilon \to 0$,

$$\int \frac{\mathrm{d}^dk_E}{(2\pi)^d} N\log(k_E^2+\sigma^2) = \frac{\mathrm{i}N\sigma^2}{4\pi}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\sigma^2 + 1\right). \tag{11.31}$$

Thus the effective potential is

$$V_{\text{eff}}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right) \tag{11.32}$$

by modified minimal subtraction.

(e) Now we minimize the effective potential:

$$0 = \frac{\partial V_{\text{eff}}}{\partial\sigma} = \frac{1}{g^2}\sigma + \frac{N}{2\pi}\sigma\log\frac{\sigma^2}{\mu^2}, \tag{11.33}$$

and find nonzero vacuum expectation values $\langle\sigma\rangle = \pm\mu e^{-\pi/g^2N}$. The dependence of this result on the renormalization condition is totally in the dependence on the subtraction point μ.

(f) It is well-known that the loop expansion is equivalent to the expansion in powers of $\hbar$ in generic perturbation theory around a classical vacuum. This is true because the integrand of the partition function can be put into the form of $e^{\mathrm{i}S/\hbar}$. That is, $\hbar$ appears as an overall coefficient of the action. In our case, we see that the overall factor N plays the same role. Thus by the same argument, we conclude that the loop expansion is equivalent to the expansion in powers of $1/N$. More details can be found in Section III.3 of [4] and Chapter 8 "$1/N$" of [5].

Chapter 12

The Renormalization Group

12.1 Beta Function in Yukawa Theory

In this problem we calculate the 1-loop beta functions in Yukawa theory. All needed ingredients have been given in Problem 10.2 Here we list the needed counterterms:

$$\delta_\psi = -\frac{g^2}{2(4\pi^2)}\left(\frac{2}{\epsilon} - \log M^2\right); \tag{12.1}$$

$$\delta_\phi = -\frac{2g^2}{(4\pi^2)}\left(\frac{2}{\epsilon} - \log M^2\right); \tag{12.2}$$

$$\delta_g = \frac{g^3}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log M^2\right); \tag{12.3}$$

$$\delta_\lambda = \frac{3\lambda^2 - 48g^4}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \log M^2\right). \tag{12.4}$$

Here Λ is the UV cutoff and M is the renormalization scale. Then, the beta functions to lowest order are given by

$$\beta_g = M\frac{\partial}{\partial M}\Big(- \delta_g + \tfrac{1}{2}g_0\delta_\phi + g_0\delta_\psi\Big) = \frac{5g^3}{(4\pi)^2}; \tag{12.5}$$

$$\beta_\lambda = M\frac{\partial}{\partial M}\Big(- \delta_\lambda + 2\lambda_0\delta_\phi\Big) = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{(4\pi)^2}. \tag{12.6}$$

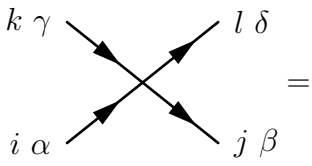
12.2 Beta Function of the Gross-Neveu Model

We evaluate the β function of the 2-dimensional Gross-Neveu model with the Lagrangian

$$\mathcal{L} = \bar\psi_i(\mathrm{i}\partial\!\!\!/)\psi_i + \tfrac{1}{2}g^2(\bar\psi_i\psi_i)^2, \qquad (i = 1, \cdots, N) \tag{12.7}$$

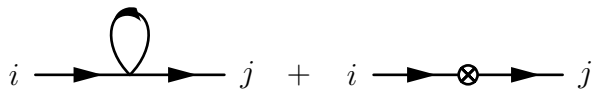
to 1-loop order. The Feynman rules can be easily worked out to be

$$i\ \alpha \xrightarrow[k]{} j\ \beta = \left(\frac{\mathrm{i}}{k\!\!\!/}\right)_{\beta\alpha}\delta_{ij}$$



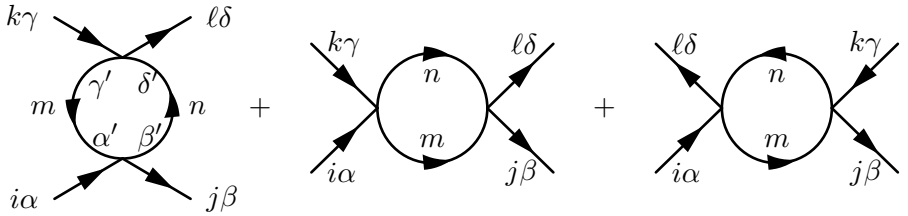
$$= \mathrm{i}g^2(\delta_{ij}\delta_{k\ell}\epsilon_{\beta\alpha}\epsilon_{\delta\gamma} + \delta_{i\ell}\delta_{jk}\epsilon_{\delta\alpha}\epsilon_{\beta\gamma})$$

Now consider the two-point function $\Gamma^{(2)}_{ij}(p)$. The one-loop correction to $\Gamma^{(2)}_{ij}(p)$ comes from the following two diagrams:



It is easy to see the loop diagram contains a factor of $\int \mathrm{d}^2k\ \mathrm{tr}\,[\not{k}^{-1}]$, which is zero under dimensional regularization. Thus the wave function renormalization factor receives no contribution at 1-loop level, namely $\delta_\psi = 0$.

Then we turn to the 4-point function $\Gamma^{(4)}_{ijk\ell}$. There are three diagrams in total, namely,



(a) (b) (c)

We calculate them in turn. The first one:

$$\begin{aligned}
(\mathrm{a}) &= (\mathrm{i}g^2)^2 \int \frac{\mathrm{d}^dk}{(2\pi)^d}(\delta_{mn}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\gamma'\delta'} + \delta_{n\ell}\delta_{mk}\epsilon_{\delta\delta'}\epsilon_{\gamma'\gamma})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\delta'\beta'} \\
&\quad \times (\delta_{ij}\delta_{mn}\epsilon_{\beta'\alpha'}\epsilon_{\beta\alpha} + \delta_{in}\delta_{jm}\epsilon_{\beta'\alpha}\epsilon_{\beta\alpha'})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\alpha'\gamma'} \\
&= g^4\Big((-2N+2)\delta_{ij}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\beta\alpha} + \tfrac{1}{2}\delta_{i\ell}\delta_{jn}(\gamma^\mu)_{\delta\alpha}(\gamma_\mu)_{\beta\gamma}\Big)\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}
\end{aligned} \tag{12.8}$$

The second diagram reads:

$$\begin{aligned}
(\mathrm{b}) &= \frac{1}{2}\cdot(\mathrm{i}g^2)^2 \int \frac{\mathrm{d}^dk}{(2\pi)^d}(\delta_{mj}\delta_{n\ell}\epsilon_{\delta\delta'}\epsilon_{\beta\beta'} + \delta_{m\ell}\delta_{nj}\epsilon_{\delta\beta'}\epsilon_{\beta\delta'})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\beta'\alpha'} \\
&\quad \times (\delta_{im}\delta_{kn}\epsilon_{\alpha'\alpha}\epsilon_{\gamma'\gamma} + \delta_{in}\delta_{km}\epsilon_{\gamma'\alpha}\epsilon_{\alpha'\gamma})\Big(\frac{\mathrm{i}}{-\not{k}}\Big)_{\delta'\gamma'} \\
&= -\frac{g^4}{2}\Big(\delta_{ij}\delta_{k\ell}(\gamma^\mu)_{\delta\gamma}(\gamma_\mu)_{\beta\alpha} + \delta_{i\ell}\delta_{jn}(\gamma^\mu)_{\delta\alpha}(\gamma_\mu)_{\beta\gamma}\Big)\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}
\end{aligned} \tag{12.9}$$

The third diagram:

$$\begin{aligned}
(\mathrm{c}) &= (\mathrm{i}g^2)^2 \int \frac{\mathrm{d}^dk}{(2\pi)^d}(\delta_{mj}\delta_{nk}\epsilon_{\beta\beta'}\epsilon_{\gamma'\gamma} + \delta_{mn}\delta_{jk}\epsilon_{\gamma'\beta'}\epsilon_{\beta\gamma})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\beta'\alpha'} \\
&\quad \times (\delta_{im}\delta_{\ell n}\epsilon_{\delta\delta'}\epsilon_{\alpha'\alpha} + \delta_{i\ell}\delta_{mn}\epsilon_{\alpha'\delta'}\epsilon_{\delta\alpha})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\delta'\gamma'}
\end{aligned}$$

$$=g^4\Big(\tfrac{1}{2}\delta_{ij}\delta_{k\ell}(\gamma^\mu)_{\delta\gamma}(\gamma_\mu)_{\beta\alpha}+(2-2N)\delta_{i\ell}\delta_{jk}\epsilon_{\beta\gamma}\epsilon_{\delta\alpha}\Big)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2} \tag{12.10}$$

Summing up the three diagrams and using dimensional regularization with $d=2-\epsilon$, we get

$$\begin{aligned}&-2g^4(N-1)(\delta_{ij}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\beta\alpha}+\delta_{i\ell}\delta_{jk}\epsilon_{\beta\gamma}\epsilon_{\delta\alpha})\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\\ &\sim\frac{2(N-1)\mathrm{i}g^4}{4\pi}\frac{2}{\epsilon}(\delta_{ij}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\beta\alpha}+\delta_{i\ell}\delta_{jk}\epsilon_{\beta\gamma}\epsilon_{\delta\alpha}).\end{aligned} \tag{12.11}$$

Only the divergent terms are kept in the last expression, from which we can read the counterterm

$$\delta_g=-\frac{(N-1)g^4}{2\pi}\Big(\frac{2}{\epsilon}-\log M^2\Big). \tag{12.12}$$

Thus the β function is

$$\beta(g^2)=M\frac{\partial}{\partial M}(-\delta_g)=-\frac{(N-1)(g^2)^2}{\pi}, \tag{12.13}$$

and

$$\beta(g)=-\frac{(N-1)g^3}{2\pi}. \tag{12.14}$$

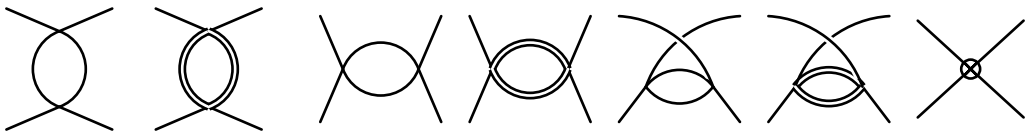
It is interesting to see that the 1-loop β function vanishes for $N=1$. This is because we have the Fierz identity $2(\bar\psi\psi)(\bar\psi\psi)=-(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)$, and the Gross-Neveu model in this case is equivalent to massless Thirring model, which is known to have vanishing β function.

12.3 Asymptotic Symmetry

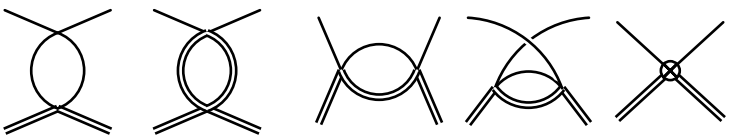
In this problem we study a bi-scalar model, given by the following Lagrangian:

$$\mathcal{L}=\tfrac{1}{2}\big((\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2\big)-\tfrac{\lambda}{4!}(\phi_1^4+\phi_2^4)-\tfrac{\rho}{12}\phi_1^2\phi_2^2. \tag{12.15}$$

(a) First, we calculate the 1-loop beta functions β_λ and β_ρ. The relevant 1-loop diagrams for calculating β_λ are:



The relevant diagrams for calculating β_ρ are:



Here the single line represents ϕ_1 and double line represents ϕ_2. Since the divergent parts of these diagrams are all independent of external momenta, we can therefore simply ignore them. Then it's easy to evaluate them, as follows.



$$= \frac{(-\mathrm{i}\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{k^2} \sim \frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\frac{2}{\epsilon}, \tag{12.16}$$

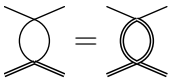


$$= \frac{(-\mathrm{i}\rho/3)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{k^2} \sim \frac{\mathrm{i}\rho^2}{18(4\pi)^2}\frac{2}{\epsilon}. \tag{12.17}$$

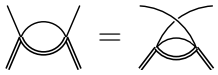
The t-channel and u-channel give the same result. Thus we can determine δ_λ to be

$$\delta_\lambda \sim \frac{9\lambda^2+\rho^2}{6(4\pi)^2}\frac{2}{\epsilon}. \tag{12.18}$$

On the other hand,



$$= \frac{(-\mathrm{i}\lambda)(-\mathrm{i}\rho/3)}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{k^2} \sim \frac{\mathrm{i}\lambda\rho}{6(4\pi)^2}\frac{2}{\epsilon}, \tag{12.19}$$



$$= (-\mathrm{i}\rho/3)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{k^2} \sim \frac{\mathrm{i}\rho^2}{9(4\pi)^2}\frac{2}{\epsilon}. \tag{12.20}$$

Then we have

$$\delta_\rho \sim \frac{3\lambda\rho+2\rho^2}{3(4\pi^2)}\frac{2}{\epsilon}. \tag{12.21}$$

It's easy to see that field strengths for both ϕ_1 and ϕ_2 receives no contributions from 1-loop diagrams. Thus the 1-loop beta functions can be evaluated as

$$\beta_\lambda = -\mu\frac{\mathrm{d}\delta_\lambda}{\mathrm{d}\mu} = \frac{3\lambda^2+\rho^2/3}{(4\pi)^2}; \tag{12.22}$$

$$\beta_\rho = -\mu\frac{\mathrm{d}\delta_\rho}{\mathrm{d}\mu} = \frac{2\lambda\rho+4\rho^2/3}{(4\pi)^2}. \tag{12.23}$$

(b) Now we derive the renormalization equation for ρ/λ:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\Big(\frac{\rho}{\lambda}\Big) = \frac{1}{\lambda}\beta_\rho - \frac{\rho}{\lambda^2}\beta_\lambda = \frac{\rho}{3(4\pi)^2}\Big[-(\rho/\lambda)^2+4(\rho/\lambda)-3\Big]. \tag{12.24}$$

Then it is easy to see that $\rho/\lambda = 1$ is an IR fixed point.

(c) In $4-\epsilon$ dimensions, the β functions for ρ and λ are shifted as

$$\beta_\lambda = -\epsilon\lambda + \frac{3\lambda^2+\rho^2/3}{(4\pi)^2}; \tag{12.25}$$

$$\beta_\rho = -\epsilon\rho + \frac{2\lambda\rho+4\rho^2/3}{(4\pi)^2}. \tag{12.26}$$

But it is easy to show that the terms containing ϵ cancel out in the β function for ρ/λ, and the result is the same as (12.24). This is true because ρ/λ still remains dimensionless in $4-\epsilon$ dimensions. Therefore we conclude that there are three fixed points of the RG flow for ρ/λ at 0, 1, and 3. We illustrate this in the diagram of RG flow in the ρ-λ plane, with the deviation of dimension $\epsilon = 0.01$, in Figure 12.1.

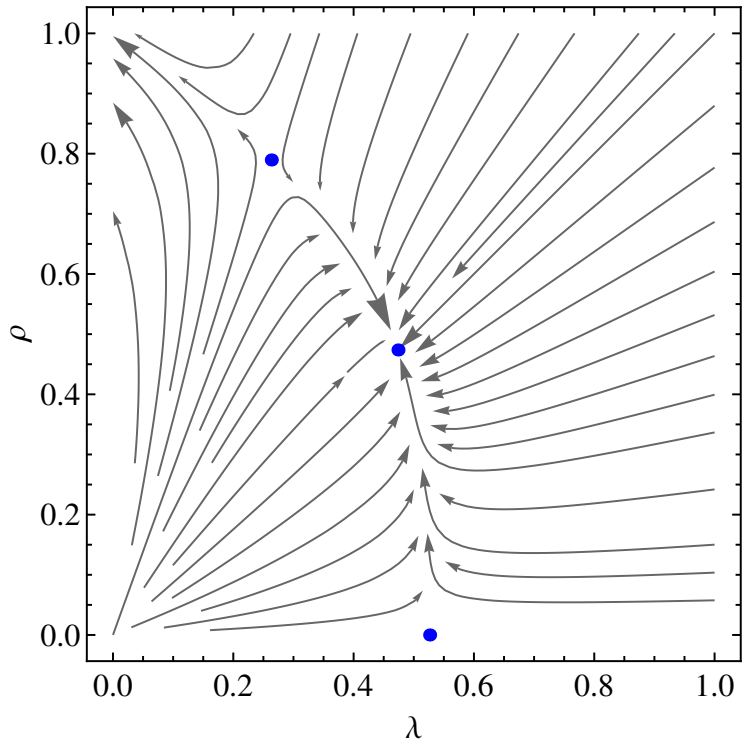


Figure 12.1: The RG flow of the theory (12.15) in $4-\epsilon$ dimensions with $\epsilon = 0.01$. Three nontrivial fixed points are shown by blue dots.

Chapter 13

Critical Exponents and Scalar Field Theory

13.1 Correlation-to-scaling exponent

In this problem we consider the effect of the deviation of the coupling λ from its fixed point to the two-point correlation function $G(M,t)$ in $d = 4 - \epsilon$ dimensions. Symbolically, we can always write

$$G(M,t) = G_*(M,t) + \left.\frac{\delta G(M,t)}{\delta\bar{\lambda}}\right|_{\bar{\lambda}=\lambda_*} \delta\bar{\lambda}, \tag{13.1}$$

where $\bar{\lambda}$ is the running coupling, defined to be the solution of the following renormalization group equation:

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\bar{\lambda} = \frac{2\beta_\lambda(\bar{\lambda})}{d-2+2\gamma(\bar{\lambda})}. \tag{13.2}$$

As the first step, let us expand the β function of $\bar{\lambda}$ around the fixed point, as

$$\begin{aligned}\beta(\bar{\lambda}) &= \beta(\lambda_*) + \left.\frac{\mathrm{d}\beta(\bar{\lambda})}{\mathrm{d}\bar{\lambda}}\right|_{\bar{\lambda}=\lambda_*} (\bar{\lambda}-\lambda_*) + \mathcal{O}((\bar{\lambda}-\lambda_*)^2) \\ &= \omega(\bar{\lambda}-\lambda_*) + \mathcal{O}((\bar{\lambda}-\lambda_*)^2).\end{aligned} \tag{13.3}$$

Then the renormalization group equation reads

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\bar{\lambda} \simeq \frac{2\omega(\bar{\lambda}-\lambda_*)}{d-2+2\gamma(\lambda_*)} = \frac{\omega\nu}{\beta}(\bar{\lambda}-\lambda_*), \tag{13.4}$$

where β and ν on the right hand side are critical exponents, which in our case are defined to be

$$\beta = \frac{d-2+2\gamma(\lambda_*)}{d(2-\gamma_{\phi^*}(\lambda_*))}, \qquad\qquad \nu = \frac{1}{2-\gamma_{\phi^2}(\lambda_*)}.$$

Don't confuse the critical exponent β with the β function. Now, from this equation we can solve the running coupling $\bar{\lambda}$ to be

$$\bar{\lambda} = \lambda_* + \big(\bar{\lambda}(\mu_0) - \lambda_*\big)\Big(\frac{\mu}{\mu_0}\Big)^{\omega\nu/\beta}. \tag{13.5}$$

Now let μ_0 be the scale at which the bare coupling is defined. Then we get

$$\delta\bar{\lambda} \propto (\lambda - \lambda_*)\mu^{\omega\nu/\beta}. \tag{13.6}$$

13.2 The exponent η

We have found the counterterm δ_Z to $\mathcal{O}(\lambda^2)$ with $\overline{MS}$ scheme in Problem 10.3, to be

$$\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right]. \tag{13.7}$$

Then the anomalous dimension γ to $\mathcal{O}(\lambda^2)$ is given by

$$\gamma = \frac{1}{2}M\frac{\partial}{\partial M}\delta_Z = \frac{\lambda^2}{12(4\pi)^4}. \tag{13.8}$$

This result can be easily generalized to the $O(N)$-symmetric ϕ^4 theory, by replacing the Feynman rule of the ϕ^4 coupling $-\mathrm{i}\lambda$ with

$$-2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}),$$

which is equivalent to multiplying the diagram (10.19) by the following factor:

$$4\cdot\left(\delta^{ik}\delta^{\ell m} + \delta^{i\ell}\delta^{km} + \delta^{im}\delta^{k\ell}\right)\left(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\right) = 12(N+2)\delta^{ij}, \tag{13.9}$$

and the anomalous dimension (13.8) obtained above should be multiplied by $12(N+2)$, which leads to

$$\gamma = (N+2)\frac{\lambda^2}{(4\pi)^4}, \tag{13.10}$$

which is the same as (13.47) of Peskin&Schroeder.

13.3 The CP^N model

(a) The Lagrangian of the CP^N model can be written as

$$\mathcal{L} = \frac{1}{g^2}\left(\sum_j |\partial_\mu z_j|^2 - \left|\sum_j z_j^*\partial_\mu z_j\right|^2\right), \tag{13.11}$$

with z_j $(j = 1, \cdots N+1)$ the components of a vector in $(N+1)$ dimensional complex space, subject to the constraint

$$\sum_j |z_j|^2 = 1 \tag{13.12}$$

and the identification

$$(e^{\mathrm{i}\alpha}z_1, \cdots, e^{\mathrm{i}\alpha}z_{N+1}) \sim (z_1, \cdots, z_{N+1}). \tag{13.13}$$

Now we prove that the Lagrangian given above is invariant under the following local transformation:

$$z_j(x) \to e^{\mathrm{i}\alpha(x)} z_j(x), \tag{13.14}$$

as,

$$\begin{aligned} g^2\mathcal{L} &\to \left|\partial_\mu(e^{\mathrm{i}\alpha} z_j)\right|^2 + \left|e^{-\mathrm{i}\alpha} z_j^* \partial_\mu(e^{\mathrm{i}\alpha} z_j)\right|^2 \\ &= \Big(|\partial_\mu z_j|^2 + |\partial_\mu\alpha|^2 + 2\mathrm{Re}\big(-\mathrm{i}(\partial_\mu\alpha) z_j^* \partial_\mu z_j\big)\Big) \\ &\quad - \Big(|z_j^*\partial_\mu z_j|^2 + |\partial_\mu\alpha|^2 + 2\mathrm{Re}\big(-\mathrm{i}(\partial_\mu\alpha) z_i z_i^* z_j^* \partial_\mu z_j\big)\Big) \\ &= g^2\mathcal{L}. \end{aligned} \tag{13.15}$$

Then we show that the nonlinear σ model with $n = 3$ is equivalent to the CP^N model with $N = 1$. To see this, we substitute $n^i = z^*\sigma^i z$ into the Lagrangian of the nonlinear sigma model, $\mathcal{L} = \frac{1}{2g^2}|\partial_\mu n^i|^2$, to get

$$\begin{aligned} \mathcal{L} &= \frac{1}{2g^2}\left|(\partial_\mu z^*)\sigma^i z + z^*\sigma^i\partial_\mu z\right|^2 \\ &= \frac{1}{2g^2}\sigma^i\sigma^i\Big[2(\partial_\mu z^*)(\partial^\mu z) z^* z + (\partial_\mu z)^2 z^{*2} + (\partial_\mu z^*)^2 z^2\Big] \\ &= \frac{1}{2g^2}\sigma^i\sigma^i\Big[2(\partial_\mu z^*)(\partial^\mu z) + \big(z^*\partial_\mu z + z\partial_\mu z^*\big)^2 - 2(z^*\partial_\mu z)(z\partial^\mu z^*)\Big] \\ &= \frac{1}{2g^2}\sigma^i\sigma^i\Big[2(\partial_\mu z^*)(\partial^\mu z) + \big[\partial_\mu(z^* z)\big]^2 - 2(z^*\partial_\mu z)(z\partial^\mu z^*)\Big]. \end{aligned} \tag{13.16}$$

Then after a proper normalization of the field z, it is straightforward to see that the Lagrangian above reduces to

$$\mathcal{L} = \frac{1}{g^2}\Big(|\partial_\mu z|^2 - 2|z^*\partial_\mu z|^2\Big), \tag{13.17}$$

which is indeed the CP^1 model.

(b) The Lagrangian (13.11) can be obtained by the following Lagrangian with a gauge field A_μ and a Lagrange multiplier which expresses the local gauge symmetry and the constraint explicitly:

$$\mathcal{L} = \frac{1}{g^2}\Big(|\mathrm{D}_\mu z_j|^2 - \lambda\big(|z_j|^2 - 1\big)\Big), \tag{13.18}$$

with $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}A_\mu$. Now let us verify this by functionally integrating out the gauge field A_μ as well as the Lagrange multiplier λ to get

$$\begin{aligned} Z &= \int \mathcal{D}^2 z_i \mathcal{D}A_\mu \mathcal{D}\lambda \, \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x \, \Big(|\mathrm{D}_\mu z_j|^2 - \lambda\big(|z_j|^2 - 1\big)\Big)\right] \\ &= \int \mathcal{D}^2 z_i \mathcal{D}A_\mu \, \delta\big(|z_j|^2 - 1\big) \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x \, |\mathrm{D}_\mu z_j|^2\right] \\ &= \int \mathcal{D}^2 z_i \mathcal{D}A_\mu \, \delta\big(|z_j|^2 - 1\big) \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x \, \Big(A_\mu A^\mu + 2\mathrm{i}A^\mu(\partial_\mu z_j^*) z_j + |\partial_\mu z_j|^2\Big)\right] \\ &= N\int \mathcal{D}^2 z_i \, \delta\big(|z_j|^2 - 1\big) \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x \, \Big(|\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|\Big)^2\right]. \end{aligned} \tag{13.19}$$

(c) On the other hand one can also integrate out z_i field in the Lagrangian (13.18), as

$$\begin{aligned} Z &= \int \mathcal{D}z_i \mathcal{D}A_\mu \mathcal{D}\lambda \, \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x \left(|\mathrm{D}_\mu z_j|^2 - \lambda\big(|z_j|^2-1\big)\right)\right] \\ &= \int \mathcal{D}A_\mu \mathcal{D}\lambda \, \exp\left[- N \,\mathrm{tr}\, \log(-D^2-\lambda) + \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\, \lambda\right] \end{aligned} \tag{13.20}$$

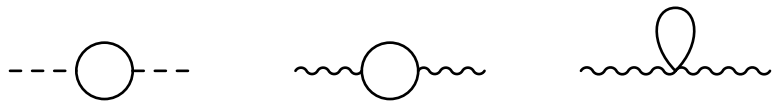
We assume that the expectation values for A_μ and λ are constants. Then the exponent can be evaluated by means of dimensional regularization, as

$$\begin{aligned} \mathrm{i}S = & - N \,\mathrm{tr}\, \log(-D^2-\lambda) + \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\, \lambda \\ =& \left[- N \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log\left(k^2 + A_\mu A^\mu - \lambda\right) + \frac{\mathrm{i}}{g^2}\lambda\right]\cdot V^{(2)} \\ \Rightarrow \mathrm{i}&\left[- \frac{N}{4\pi}\Big(\log\frac{M^2}{\lambda - A^2} + 1\Big)(\lambda - A^2) + \frac{1}{g^2}\lambda\right]\cdot V^{(2)}, \end{aligned} \tag{13.21}$$

where $V^{(2)} = \int \mathrm{d}^2x$, and we have used the $\overline{MS}$ scheme to subtract the divergence. Now we can minimize the quantity in the square bracket in the last line to get

$$A_\mu = 0, \qquad\qquad \lambda = M^2 \exp\Big(-\frac{4\pi}{gN^2}\Big). \tag{13.22}$$

(d) The meaning of the effective action S is most easily seen from its diagrammatic representations. For instance, at the 1-loop level, we know that the logarithmic terms in the effective action is simply the sum of a series of 1-loop diagrams with $n \geq 0$ external legs, where the number of external legs n is simply the power of corresponding fields in the expansion of S. Therefore, to the second order in A and in λ, the effective action is represented precisely by the following set of diagrams,



where the dashed lines represent λ, curved lines represent A_μ, and the internal loop are z field. Then it is straightforward to see that the correct kinetic terms for λ and A_μ are generated from these diagrams. That is, the gauge field A_μ becomes dynamical due to quantum corrections. The gauge invariance of the resulted kinetic term $F_{\mu\nu}F^{\mu\nu}$ can also be justified by explicit calculation as was done in Problem 9.1.

Final Project II
The Coleman-Weinberg Potential

In this final project, we work out some properties of Coleman-Weinberg model, illustrating basic techniques of the renormalization group. The original paper [6] by S. Coleman and E. Weinberg is always a good read, while a recent and very insightful treatment of the model can be found in [7].

Simply put, the Coleman-Weinberg model is a theory of scalar electrodynamics, described by the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) - m^2\phi^\dagger\phi - \tfrac{\lambda}{6}(\phi^\dagger\phi)^2, \tag{13.23}$$

with ϕ a complex scalar and $D_\mu\phi = (\partial_\mu + \mathrm{i}eA_\mu)\phi$.

(a) Consider the case of spontaneous breaking of the $U(1)$ gauge symmetry $\phi(x) \to e^{\mathrm{i}\alpha(x)}\phi(x)$, caused by a negative squared mass, namely $m^2 = -\mu^2 < 0$. The scalar then acquires a nonzero vacuum expectation value (VEV) $\phi_0 = \sqrt{\langle|\phi|^2\rangle}$. We split this VEV out of the scalar field, namely,

$$\phi = \phi_0 + \frac{1}{\sqrt{2}}\big[\sigma(x) + \mathrm{i}\pi(x)\big], \tag{13.24}$$

with the new field $\sigma(x)$ and $\pi(x)$ being real. At the tree level, it is easy to find $\phi_0 = \sqrt{3\mu^2/\lambda}$ by minimize the scalar potential $V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda}{6}(\phi^\dagger\phi)^2$. We also introduce $v = \sqrt{2}\phi_0$ for convenience. Then, rewrite the Lagrangian in terms of these new field variables, we get,

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 + \tfrac{1}{2}e^2v^2A_\mu A^\mu - \tfrac{1}{2}(2\mu^2)\sigma^2 \\ &- \tfrac{\lambda}{24}(\pi^4 + \sigma^4 + 2\pi^2\sigma^2 + 4v\pi^2\sigma + 4v\sigma^3) + evA_\mu\partial^\mu\pi \\ &+ eA_\mu(\sigma\partial^\mu\pi - \pi\partial^\mu\sigma) + \tfrac{1}{2}e^2A_\mu A^\mu(\pi^2 + \sigma^2 + 2v\sigma).\end{aligned} \tag{13.25}$$

Then we see that the vector field A_μ acquires a mass, equal to $m_A = ev$ at the classical level.

(b) Now we calculate the 1-loop effective potential of the model. We know that 1-loop correction of the effective Lagrangian is given by,

$$\Delta\mathcal{L} = \frac{\mathrm{i}}{2}\log\det\left[-\frac{\delta^2\mathcal{L}}{\delta\varphi\delta\varphi}\right]_{\varphi=0} + \delta\mathcal{L}, \tag{13.26}$$

where φ is the fluctuating fields and $\delta\mathcal{L}$ denotes counterterms.

Let the background value of the complex scalar be $\phi_{\rm cl}$. By the assumption of Poincaré symmetry, $\phi_{\rm cl}$ must be a constant. For the same reason, the background value of the vector field A_μ must vanish. In addition, we can set $\phi_{\rm cl}$ to be real without loss of generality. Then we have,

$$\phi(x) = \phi_{\rm cl} + \varphi_1(x) + \mathrm{i}\varphi_2(x),$$

where $\varphi_1(x)$, $\varphi_2(x)$, together with $A_\mu(x)$, now serve as fluctuating fields. Expanding the Lagrangian around the background fields and keeping terms quadratic in fluctuating fields only, we get,

$$\begin{aligned}
\mathcal{L} =& -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \big|(\partial_\mu + \mathrm{i}eA_\mu)(\phi_{\rm cl} + \varphi_1 + \mathrm{i}\varphi_2)\big|^2 \\
& - m^2\big|\phi_{\rm cl} + \varphi_1 + \mathrm{i}\varphi_2\big|^2 - \tfrac{\lambda}{6}\big|\phi_{\rm cl} + \varphi_1 + \mathrm{i}\varphi_2\big|^4 \\
=& \tfrac{1}{2}A_\mu\big[g^{\mu\nu}(\partial^2 + 2e^2\phi_{\rm cl}^2) - \partial^\mu\partial^\nu\big]A_\nu + \tfrac{1}{2}\varphi_1\big(-\partial^2 - m^2 - \lambda\phi_{\rm cl}^2\big)\varphi_1 \\
& + \tfrac{1}{2}\varphi_2\big(-\partial^2 - m^2 - \tfrac{\lambda}{3}\phi_{\rm cl}^2\big)\varphi_2 - 2e\phi_{\rm cl}A_\mu\partial^\mu\varphi_2 + \cdots,
\end{aligned} \tag{13.27}$$

where "$\cdots$" denotes terms other than being quadratic in fluctuating fields. Now we impose the Landau gauge condition $\partial_\mu A^\mu = 0$ to the Lagrangian, which removes the off-diagonal term $-2e\phi_{\rm cl}A_\mu\partial^\mu\varphi_2$. Then, according to (13.26), the 1-loop effective Lagrangian can be evaluated as,

$$\begin{aligned}
\frac{\mathrm{i}}{2}\log\det\bigg[-\frac{\delta^2\mathcal{L}}{\delta\varphi\delta\varphi}\bigg]_{\varphi=0} =& \frac{\mathrm{i}}{2}\bigg[\log\det\big(-\eta^{\mu\nu}(\partial^2 + 2e^2\phi_{\rm cl}^2) + \partial^\mu\partial^\nu\big) \\
& + \log\det\big(\partial^2 + m^2 + \lambda\phi_{\rm cl}^2\big) + \log\det\big(\partial^2 + m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2\big)\bigg] \\
=& \frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\bigg[\mathrm{tr}\,\log(-k^2 + 2e^2\phi_{\rm cl}^2)^3 \\
& + \mathrm{tr}\,\log(-k^2 + m^2 + \lambda\phi_{\rm cl}^2) + \mathrm{tr}\,\log(-k^2 + m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2)\bigg] \\
=& \frac{\Gamma(-\frac{d}{2})}{2(4\pi)^{d/2}}\bigg[3(2e^2\phi_{\rm cl}^2)^{d/2} + (m^2 + \lambda\phi_{\rm cl}^2)^{d/2} + (m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2)^{d/2}\bigg].
\end{aligned} \tag{13.28}$$

In the second equality we use the following identity,

$$\det(\lambda I + AB) = \lambda^{n-1}(\lambda + BA), \tag{13.29}$$

where A and B are matrices of $n\times 1$ and $1\times n$, respectively, λ is an arbitrary complex number and I is the $n\times n$ identity matrix. In our case, this gives,

$$\det\big(-\eta^{\mu\nu}(\partial^2 + 2e^2\phi_{\rm cl}^2) + \partial^\mu\partial^\nu\big) = -2e^2\phi_{\rm cl}^2(\partial^2 + 2e^2\phi_{\rm cl}^2)^3. \tag{13.30}$$

Then the second equality follows up to an irrelevant constant term. The third equality makes use of the trick in (11.72) of P&S. Then, for $d = 4-\epsilon$ and $\epsilon\to 0$, we have,

$$\frac{\mathrm{i}}{2}\log\det\bigg[-\frac{\delta^2\mathcal{L}}{\delta\varphi\delta\varphi}\bigg]_{\varphi=0} = \frac{1}{4(4\pi)^2}\Big[3(2e^2\phi_{\rm cl}^2)^2\big(\Delta - \log(2e^2\phi_{\rm cl}^2)\big)$$

$$+ (m^2 + \lambda\phi_{\rm cl}^2)^2\big(\Delta - \log(m^2 + \lambda\phi_{\rm cl}^2)\big)$$
$$+ (m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2)^2\big(\Delta - \log(m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2)\big)\Big], \tag{13.31}$$

where we define $\Delta \equiv \frac{2}{\epsilon} - \gamma + \log 4\pi + \frac{3}{2}$ for brevity.

Now, with $\overline{MS}$ scheme, we can determine the counterterms in (13.26) to be

$$\delta\mathcal{L} = \frac{-1}{4(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \log 4\pi - \log M^2\right]\Big(3(2e^2\phi_{\rm cl}^2)^2 + (m^2 + \lambda\phi_{\rm cl}^2)^2 + (m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2)^2\Big). \tag{13.32}$$

where M is the renormalization scale. Now the effective potential follows directly from (13.26), (13.31) and (13.32),

$$V_{\rm eff}[\phi_{\rm cl}] = m^2\phi_{\rm cl}^2 + \frac{\lambda}{6}\phi_{\rm cl}^4 - \frac{1}{4(4\pi)^2}\left[3(2e^2\phi_{\rm cl}^2)^2\Big(\log\frac{M^2}{2e^2\phi_{\rm cl}^2} + \frac{3}{2}\Big)\right.$$
$$\left. + (m^2 + \lambda\phi_{\rm cl}^2)^2\Big(\log\frac{M^2}{m^2 + \lambda\phi_{\rm cl}^2} + \frac{3}{2}\Big) + (m^2 + \tfrac{\lambda}{3}\phi_{\rm cl}^2)^2\Big(\log\frac{M^2}{m^2 + \frac{\lambda}{3}\phi_{\rm cl}^2} + \frac{3}{2}\Big)\right]. \tag{13.33}$$

(c) Now taking the mass parameter $\mu^2 = -m^2 = 0$, then the effective potential (13.33) becomes

$$V_{\rm eff}[\phi_{\rm cl}] = \frac{\lambda}{6}\phi_{\rm cl}^4 + \frac{1}{4(4\pi)^2}\left[12e^4\phi_{\rm cl}^4\Big(\log\frac{2e^2\phi_{\rm cl}^2}{M^2} - \frac{3}{2}\Big) + \frac{10}{9}\lambda^2\phi_{\rm cl}^4\Big(\log\frac{\lambda\phi_{\rm cl}^2}{M^2} - \frac{3}{2}\Big)\right]$$
$$\simeq \frac{\lambda}{6}\phi_{\rm cl}^4 + \frac{3e^4\phi_{\rm cl}^4}{(4\pi)^2}\Big(\log\frac{2e^2\phi_{\rm cl}^2}{M^2} - \frac{3}{2}\Big). \tag{13.34}$$

In the second line we use the fact that λ is of the order e^4 to drop the λ^2 term. Then the minimal point of this effective potential can be easily worked out to be,

$$\phi_{\rm cl}^2 = \frac{M^2}{2e^2}\exp\Big(1 - \frac{8\pi^2\lambda}{9e^4}\Big). \tag{13.35}$$

As $\lambda \sim e^4$, we see that $\phi_{\rm cl}$ is of the same order with $e^{-1}M$. Thus the effective potential remains valid at this level of perturbation theory.

(d) We plot the effective potential as a function of $\phi_{\rm cl}$ in Figure (13.1). The purple curve with $m^2 = 5 \times 10^{-7}M^2$ corresponds the case with no spontaneous symmetry breaking. The blue curve shows that as m^2 goes to 0 from above, new local minima is formed. Finally, the orange and red curves correspond to broken symmetry, and in the case of the orange curve with $m^2 = 0$, the symmetry is dynamically broken.

(e) Now we calculate β functions of the Coleman-Weinberg model to 1-loop level at high energies, where we can send the mass parameter m^2 to zero. It is convenient to work in the Feynman gauge $\xi = 1$. Then the relevant Feynman rules can be read from the Lagrangian (13.25) to be,

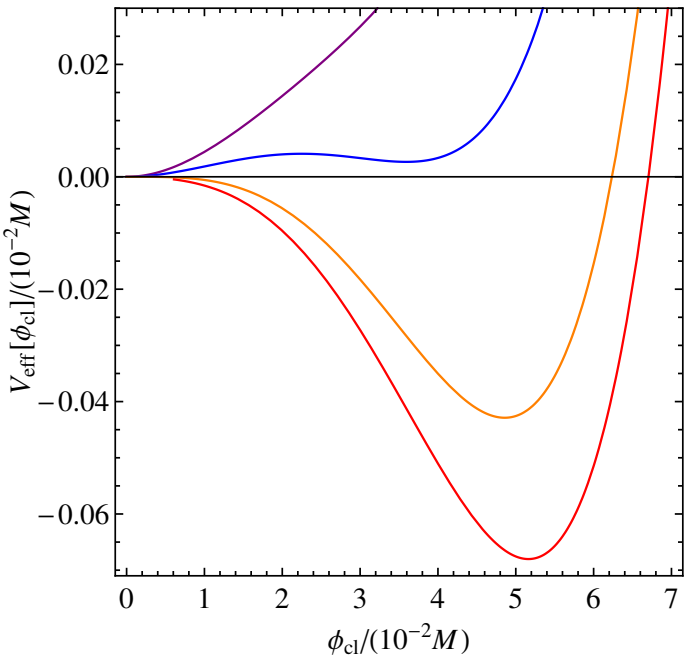
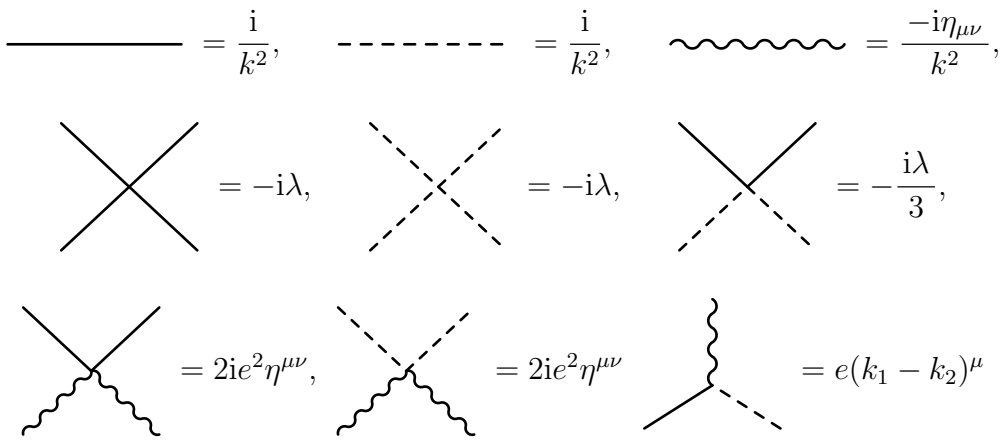
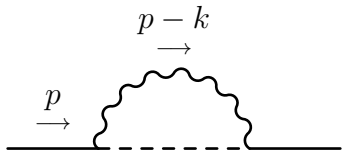


Figure 13.1: The effective potential V_{eff} as a function of ϕ_{cl}, with different values of $m^2/M^2 = 5 \times 10^{-7}$, 2.4×10^{-7}, 0 and -1×10^{-7} from top to bottom, respectively.



We first find the 1-loop wave function renormalization. For σ field, there is only one diagram with nonzero contribution,



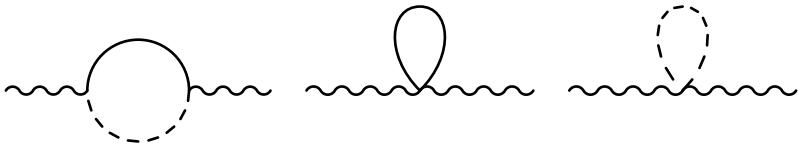
which reads,

$$e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p-k)^2} (p+k)_\mu (-p-k)^\mu \sim -\frac{2\mathrm{i}e^2 p^2}{(4\pi)^2} \frac{2}{\epsilon}. \tag{13.36}$$

Then we have,

$$\delta_\sigma = \frac{2e^2}{(4\pi)^2} \left(\frac{2}{\epsilon} - M^2 \right), \tag{13.37}$$

and it is straightforward to see that $\delta_\pi = \delta_\sigma$. For photon's wave function renormalization (vacuum polarization), we need to evaluate the following three diagrams,



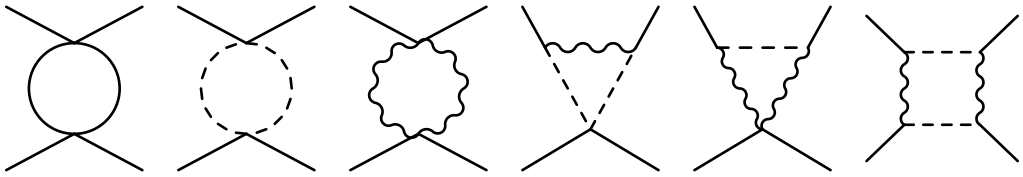
The sum of the three diagrams is,

$$
\begin{aligned}
&-e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p-k)^2}(2k-p)_\mu(2k-p)_\nu+2\cdot\frac{1}{2}\cdot 2\mathrm{i}e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\cdot\frac{(p-k)^2}{(p-k)^2}\\
&\sim-\frac{\mathrm{i}e^2}{3(4\pi)^2}\frac{2}{\epsilon}(p^2\eta_{\mu\nu}-p_\mu p_\nu),
\end{aligned}
\tag{13.38}
$$

which gives,

$$
\delta_A=-\frac{e^2}{3(4\pi)^2}\Big(\frac{2}{\epsilon}-\log M^2\Big). \tag{13.39}
$$

Then we turn to the 1-loop corrections to couplings. For scalar self-coupling λ, we consider the 1-loop corrections to σ^4 term in the Lagrangian. There are six types of diagrams contributing, listed as follows, and we label them by (a) to (f) from left to right,



For each type there are several different permutations of internal lines giving identical result, or more concretely, 3 permutations for each of the first three types, and 6 permutations for each of the last three types. Now we evaluate them in turn. We set all external momenta to zero to simplify the calculation. Then,

$$
\text{(a)}=\frac{(-\mathrm{i}\lambda)^2}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2\sim\frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\frac{2}{\epsilon}, \tag{13.40}
$$

$$
\text{(b)}=\frac{(-\mathrm{i}\lambda/3)^2}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2\sim\frac{\mathrm{i}\lambda^2}{18(4\pi)^2}\frac{2}{\epsilon}, \tag{13.41}
$$

$$
\text{(c)}=\frac{(2\mathrm{i}e^2)^2}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2\eta_{\mu\nu}\eta^{\mu\nu}\sim\frac{8\mathrm{i}e^4}{(4\pi)^2}\frac{2}{\epsilon}, \tag{13.42}
$$

$$
\text{(d)}=\frac{-\mathrm{i}\lambda e^2}{3}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\Big(\frac{\mathrm{i}}{k^2}\Big)^2(-k_\mu k^\mu)\sim-\frac{\mathrm{i}\lambda e^2}{3(4\pi)^2}\frac{2}{\epsilon}, \tag{13.43}
$$

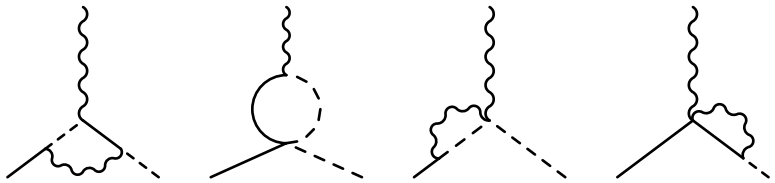
$$
\text{(e)}=(2\mathrm{i}e^2)e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2(-k_\mu k^\mu)\sim-\frac{2\mathrm{i}e^4}{(4\pi)^2}\frac{2}{\epsilon}, \tag{13.44}
$$

$$
\text{(f)}=e^4\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2\Big(\frac{-\mathrm{i}}{k^2}\Big)^2(-k_\mu k^\mu)^2\sim\frac{\mathrm{i}e^4}{(4\pi)^2}\frac{2}{\epsilon}, \tag{13.45}
$$

Then multiplying (a)∼(c) by 3 and (d)∼(f) by 6, we find

$$\delta_\lambda = \frac{5\lambda^2/3 - 2\lambda e^2 + 18e^4}{(4\pi)^2}\frac{2}{\epsilon}. \tag{13.46}$$

Finally we consider the 1-loop corrections to e. For this purpose we calculate 1-loop diagrams with three external lines with 1 A_μ, 1 σ and 1 π respectively, shown as follows, labeled again by (a) to (d) from left to right,



Now we calculate them in turn.

$$\begin{aligned}
(\mathrm{a}) &= e^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\Big(\frac{\mathrm{i}}{(p-k)^2}\Big)^2 (2p-k)^2(2k-2p)^\mu \\
&= \mathrm{i}e^3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{4x[(2-x)p-k']^2[k'-(1-x)p]^\mu}{[k'^2+x(1-x)p^2]^3} \\
&\sim \mathrm{i}e^3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{-k'^2(1-x)p^\mu - 2(2-x)(p\cdot k')k'^\mu}{[k'^2+x(1-x)p^2]^3} \\
&= \mathrm{i}e^3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{[-(1-x)-\frac{2}{d}(2-x)]k'^2 p^\mu}{[k'^2+x(1-x)p^2]^3} \\
&\sim \frac{2e^3}{(4\pi)^2}\frac{2}{\epsilon}p^\mu,
\end{aligned} \tag{13.47}$$

$$(\mathrm{b}) = e\Big(\frac{-\mathrm{i}\lambda}{3}\Big)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2 \cdot 2k^\mu = 0, \tag{13.48}$$

$$(\mathrm{c}) = (\mathrm{d}) = e(2\mathrm{i}e^2)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(p-k)^2}(k+p)^\mu \sim -\frac{3e^3}{(4\pi)^2}\frac{2}{\epsilon}p^\mu. \tag{13.49}$$

Summing the four diagrams, we find that

$$\delta_e = \frac{2e^3}{(4\pi)^2}\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{13.50}$$

Now we are ready to calculate β functions,

$$\beta_e = M\frac{\partial}{\partial M}\Big(-\delta_e + \frac{1}{2}(\delta_A + \delta_\sigma + \delta_\pi)\Big) = \frac{e^3}{48\pi^2}, \tag{13.51}$$

$$\beta_\lambda = M\frac{\partial}{\partial M}\big(-\delta_\lambda + 2\delta_\sigma\big) = \frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{24\pi^2}. \tag{13.52}$$

The trajectory of renormalization group flows generated from these β functions are shown in Figure 13.2.

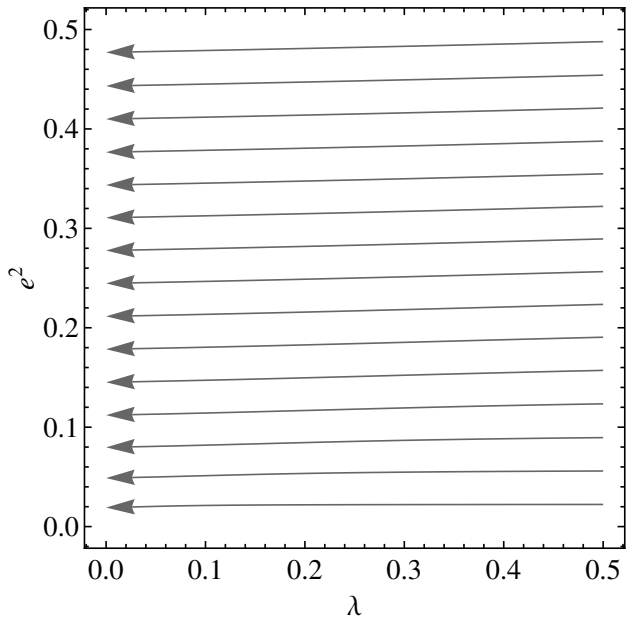


Figure 13.2: The renormalization group flow of Coleman-Weinberg model.

(f) The effective potential obtained in (c) is not a solution to the renormalization group equation, since it is only a first order result in perturbation expansion. However, it is possible to find an effective potential as a solution to the RG equation, with the result in (c) serving as a sort of "initial condition". The effective potential obtained in this way is said to be RG improved.

The Callan-Symansik equation for the effective potential reads

$$\left(M\frac{\partial}{\partial M}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_e\frac{\partial}{\partial e}-\gamma_\phi\phi_{\rm cl}\frac{\partial}{\partial\phi_{\rm cl}}\right)V_{\rm eff}(\phi_{\rm cl},\lambda,e;M)=0. \tag{13.53}$$

The solution to this equation is well known, that is, the dependence of the sliding energy scale M is described totally by running parameters,

$$V_{\rm eff}(\phi_{\rm cl},\lambda,e;M)=V_{\rm eff}\big(\bar\phi_{\rm cl}(M'),\bar\lambda(M'),\bar e(M');M'\big), \tag{13.54}$$

where barred quantities satisfy

$$M\frac{\partial\bar\lambda}{\partial M}=\beta_\lambda(\bar\lambda,\bar e),\qquad M\frac{\partial\bar e}{\partial M}=\beta_e(\bar\lambda,\bar e),\qquad M\frac{\partial\bar\phi_{\rm cl}}{\partial M}=-\gamma_\phi(\bar\lambda,\bar e)\bar\phi_{\rm cl}. \tag{13.55}$$

The RG-improved effective potential should be such that when expanded in terms of coupling constants λ and e, it will recover the result in (c) at the given order. For simplicity here we work under the assumption that $\lambda\sim e^4$, so that all terms of higher orders of coupling constants than λ and e^4 can be ignored. In this case, the perturbative calculation in (c) gives

$$V_{\rm eff}=\frac{\lambda}{6}\phi_{\rm cl}^4+\frac{3e^4\phi_{\rm cl}^4}{(4\pi)^2}\Big(\log\frac{2e^2\phi_{\rm cl}^2}{M^2}-\frac{3}{2}\Big). \tag{13.56}$$

Now we claim that the RG-improved edition of this result reads

$$V_{\text{eff}} = \frac{\bar{\lambda}}{6}\bar{\phi}_{\text{cl}}^4 + \frac{3\bar{e}^4\bar{\phi}_{\text{cl}}^4}{(4\pi)^2}\left(\log 2\bar{e}^2 - \frac{3}{2}\right). \tag{13.57}$$

To see this, we firstly solve the renormalization group equations (13.55),

$$\bar{\lambda}(M') = \bar{e}^4\left(\frac{\lambda}{e^4} + \frac{9}{4\pi^2}\log\frac{M'}{M}\right), \tag{13.58}$$

$$\bar{e}^2(M') = \frac{e^2}{1-(e^2/24\pi^2)\log(M'/M)}, \tag{13.59}$$

$$\bar{\phi}_{\text{cl}}(M') = \phi_{\text{cl}}\left(\frac{M'}{M}\right)^{2e^2/(4\pi)^2}, \tag{13.60}$$

where the unbarred quantities λ, e and ϕ_{cl} are evaluated at scale M. Now we substitute these results back into the RG-improved effective potential (13.57) and expand in terms of coupling constants. Then it is straightforward to see that the result recovers (13.56). To see the spontaneous symmetry breaking still occurs, we note that the running coupling $\bar{\lambda}(M')$ flows to negative value rapidly for small $M' = \phi_{\text{cl}}$, while $\bar{e}(M')$ changes mildly along the ϕ_{cl} scale, as can be seen directly from Figure 13.2. Therefore the the coefficient before ϕ_{cl}^4 is negative for small ϕ_{cl} and positive for large ϕ_{cl}. As a consequence, the minimum of this effective potential should be away from $\phi_{\text{cl}} = 0$, namely the $U(1)$ symmetry is spontaneously broken.

To find the scalar mass m_σ in this case (with $\mu = 0$), we calculate the second derivative of the effective potential V_{eff} with respect to ϕ_{cl}. Since the renormalization scale M can be arbitrarily chosen, we set it to be $M^2 = 2e^2\langle\phi_{\text{cl}}^2\rangle$ to simplify the calculation. Then the vanishing of the first derivative of V_{eff} at $\phi_{\text{cl}} = \langle\phi_{\text{cl}}\rangle$ implies that $\lambda = 9e^4/8\pi^2$. Insert this back to V_{eff} in (13.56), we find that

$$V_{\text{eff}} = \frac{3e^4\phi_{\text{cl}}^4}{16\pi^2}\left(\log\frac{\phi_{\text{cl}}^2}{\langle\phi_{\text{cl}}^2\rangle} - \frac{1}{2}\right). \tag{13.61}$$

Then, taking the second derivative of this expression with respect to ϕ_{cl}, we get the scalar mass $m_\sigma^2 = 3e^4\langle\phi_{\text{cl}}^2\rangle/4\pi^2 = 3e^4v^2/8\pi^2$. Recall that the gauge boson's mass m_A is given by $m_A = e^2v^2$ at the leading order, thus we conclude that $m_\sigma^2/m_A^2 = 3e^2/8\pi^2$ at the leading order in e^2.

(g) Now we consider the effect of finite mass, by adding a positive quadratic term into the effective potential (13.56). For simplicity we still work with $\lambda = 9e^4/8\pi^2$, which is always attainable without tuning. Then the effective potential reads,

$$V_{\text{eff}} = m_r^2\phi_{\text{cl}}^2 + \frac{3e^4\phi_{\text{cl}}^4}{(4\pi)^2}\left(\log\frac{2e^2\phi_{\text{cl}}^2}{M^2} - \frac{1}{2}\right), \tag{13.62}$$

in which the mass m_r is not identical to the bare mass parameter appeared in the classical Lagrangian, but has include 1-loop correction. As m_r increases from zero, the energy of

symmetry breaking vacuum at $\langle\phi_{\rm cl}\rangle \neq 0$ also increases, until it reaches zero when $m = m_c > 0$, and has the same vacuum energy with the symmetric vacuum $\langle\phi_{\rm cl}\rangle = 0$. Then for $m > m_c$, there will be no stable symmetry breaking vacuum.

Using the effective potential above, we can find the position of the symmetry breaking vacuum by solving the equation $\partial V_{\rm eff}/\partial\phi_{\rm cl}\big|_{\phi_{\rm cl}=\phi_v} = 0$, with the following solution,

$$\phi_v^2 = \frac{M^2}{2e^2}\exp\left[\mathrm{W}\Big(-\frac{(4\pi)^2 m_r^2}{3e^2M^2}\Big)\right], \tag{13.63}$$

in which $\mathrm{W}(z)$ is the Lambert W function, defined as the solution of $z = \mathrm{W}(z)e^{\mathrm{W}(z)}$. Then we can use the condition $V_{\rm eff}(\phi_v) = 0$ to determine m_c to be,

$$m_c^2 = \frac{3e^2M^2}{32\pi^2\mathrm{e}^{1/2}}, \tag{13.64}$$

where the e in denominator is 2.718... and should not be confused with electric charge e.

Now we can evaluate the mass ratio $m_\sigma^2/m_A^2 = \frac{1}{2}V''_{\rm eff}(\phi_v)/(e\phi_v)^2$, as a function of mass parameter m_r, to be,

$$m_\sigma^2/m_A^2 = \frac{3e^2}{8\pi^2}\left[1 + \mathrm{W}\Big(-\frac{(4\pi)^2 m_r^2}{3e^2M^2}\Big)\right]. \tag{13.65}$$

This is a monotonically decreasing function of m_r, and when $m_r = 0$, it recovers the previous result $3e^2/8\pi^2$. On the other hand, when $m_r = m_c$, the mass ratio m_σ^2/m_A^2 reaches its minimum value, given by $\frac{3e^2}{8\pi^2}[1 + \mathrm{W}(-\frac{1}{2\mathrm{e}})] = 3e^2/16\pi^2$, which is one half of the massless case.

(h) When the spacetime dimension is shifted from 4 as $d = 4 - \epsilon$, the β functions β_e and β_λ are also shifted to be

$$\beta_e = -\epsilon e + \frac{e^3}{48\pi^2}, \qquad \beta_\lambda = -\epsilon\lambda + \frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{24\pi^2}. \tag{13.66}$$

We plot the corresponding RG flow diagrams for several choice of ϵ in Figure 13.3, where we also extrapolate the result to $\epsilon = 1$.

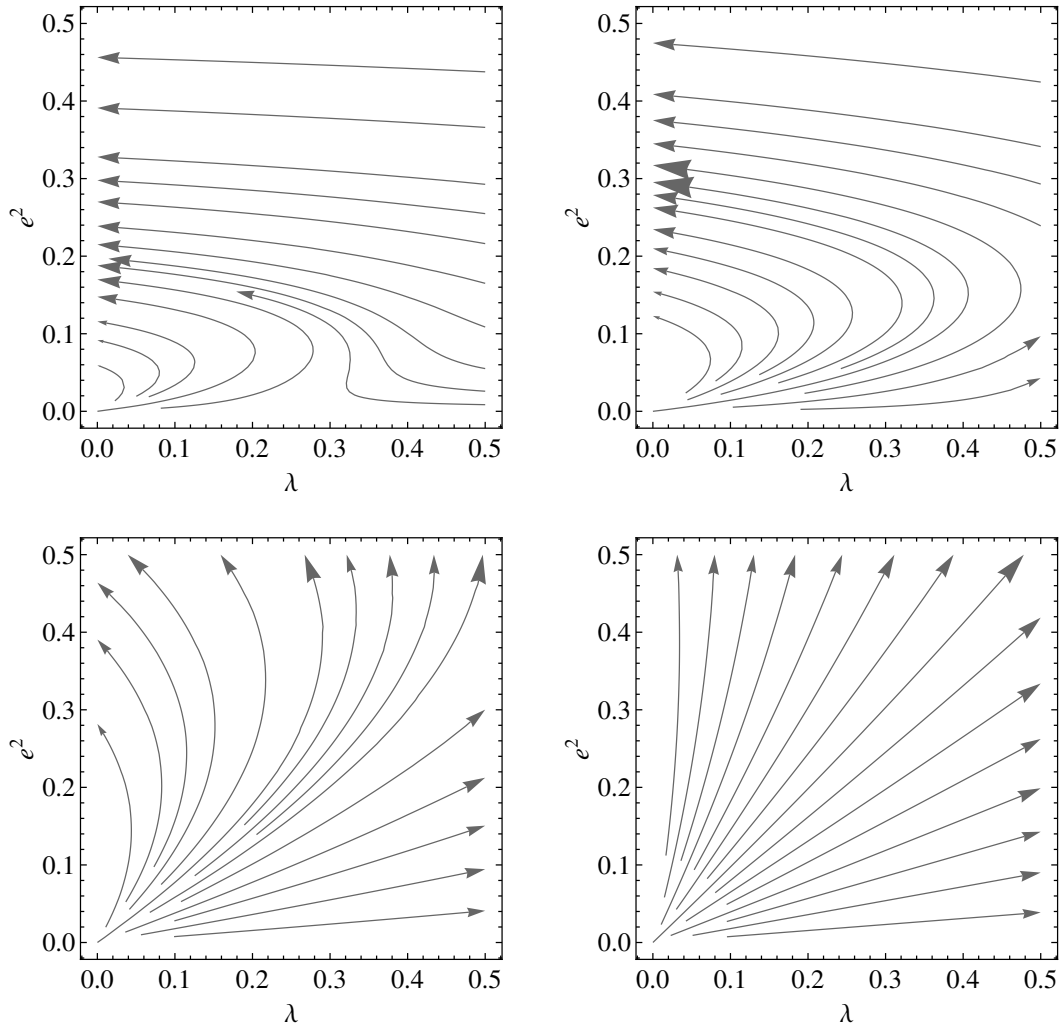


Figure 13.3: The renormalization group flows of Coleman-Weinberg model in $d = 4 - \epsilon$ spacetime dimensions, with $\epsilon = 0.005$, 0.01, 0.1 and 1 in the upper-left, upper-right, lower-left and lower-right diagram, respectively.

Chapter 15

Non-Abelian Gauge Invariance

15.1 Brute-force computations in $SU(3)$

(a) The dimension of $SU(N)$ group is $d = N^2 - 1$, when $N = 3$ we get $d = 8$.

(b) It's easy to see that t^1, t^2, t^3 generate a $SU(2)$ subgroup of $SU(3)$. Thus we have $f^{ijk} = \epsilon^{ijk}$ for $i, j, k = 1, 2, 3$. Just take another example, let's check $[t^6, t^7]$:

$$[t^6, t^7] = \mathrm{i}(-\tfrac{1}{2}t^3 + \tfrac{\sqrt{3}}{2}t^8),$$

thus we get

$$f^{678} = \tfrac{\sqrt{3}}{2}, \quad f^{673} = -\tfrac{1}{2}.$$

Then what about f^{376}?

$$[t^3, t^7] = \tfrac{\mathrm{i}}{2}t^6 \quad \Rightarrow \quad f^{376} = \tfrac{1}{2} = -f^{673}.$$

(c) $C(F) = \frac{1}{2}$. Here F represents fundamental representation.

(d) $C_2(F) = \frac{4}{3}$, $d(F) = 3$, $d(G) = 8$, thus we see that $d(F)C_2(F) = d(G)C(F)$.

15.2 Adjoint representation of $SU(2)$

The structure constants for $SU(2)$ is $f^{abc} = \epsilon^{abc}$, thus we can write down the representation matrices for its generators directly from

$$(t_G^b)_{ac} = \mathrm{i}f^{abc} = \mathrm{i}\epsilon^{abc}.$$

More explicitly,

$$t_G^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad t_G^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad t_G^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{15.1}$$

Then,

$$C(G) = \mathrm{tr}\,(t_G^1 t_G^1) = \mathrm{tr}\,(t_G^2 t_G^2) = \mathrm{tr}\,(t_G^3 t_G^3) = 2,$$

$$C_2(G) I_3 = t_G^1 t_G^1 + t_G^2 t_G^2 + t_G^3 t_G^3 = 2 I_3 \quad \Rightarrow \quad C_2(G) = 2.$$

Here I_3 is the 3×3 unit matrix.

15.3 Coulomb potential

(a) We calculate vacuum expectation value for Wilson loop $U_P(z,z)$, defined by

$$U_P(z,z) = \exp\Big[-\mathrm{i}e \oint_P \mathrm{d}x^\mu A_\mu(x) \Big]. \tag{15.2}$$

By definition, we have

$$\langle U_P(z,z) \rangle = \int \mathcal{D}A_\mu \, \exp\Big[\mathrm{i}S[A_\mu] - \mathrm{i}e \oint_P \mathrm{d}x^\mu A_\mu(x) \Big], \tag{15.3}$$

where

$$S[A_\mu] = \int \mathrm{d}^4x \Big[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2\xi} (\partial^\mu A_\mu)^2 \Big]. \tag{15.4}$$

Thus $\langle U_P(z,z) \rangle$ is simply a Gaussian integral, and can be worked out directly, as

$$\langle U_P(z,z) \rangle = \exp\Big[-\frac{1}{2} \big(-\mathrm{i}e \oint_P \mathrm{d}x^\mu \big) \big(-\mathrm{i}e \oint_P \mathrm{d}y^\nu \big) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{-\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}k\cdot(x-y)} \Big] \tag{15.5}$$

Here we have set $\xi \to 0$ to simplify the calculation. Working out the momentum integral, we get

$$\langle U_P(z,z) \rangle = \exp\Big[-\frac{e^2}{8\pi^2} \oint_P \mathrm{d}x^\mu \oint_P \mathrm{d}y^\nu \, \frac{g_{\mu\nu}}{(x-y)^2} \Big]. \tag{15.6}$$

The momentum integration goes as follows

$$\begin{aligned}
&\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-\mathrm{i}k\cdot(x-y)}}{k^2 + \mathrm{i}\epsilon} = \mathrm{i} \int \frac{\mathrm{d}^4k_E}{(2\pi)^4} \frac{e^{\mathrm{i}k_E\cdot(x-y)}}{-k_E^2} \\
&= \frac{\mathrm{i}}{(2\pi)^4} \int_0^{2\pi} \mathrm{d}\psi \int_0^{\pi} \mathrm{d}\phi \sin\phi \int_0^{\pi} \mathrm{d}\theta \sin^2\theta \int_0^{\infty} \mathrm{d}k_E \, k_E^3 \frac{e^{\mathrm{i}k_E|x-y|\cos\theta}}{-k_E^2} \\
&= -\frac{\mathrm{i}}{4\pi^3} \int_0^{\infty} \mathrm{d}k_E \, k_E \int_0^{\pi} \mathrm{d}\theta \sin^2\theta e^{\mathrm{i}k_E|x-y|\cos\theta} \\
&= -\frac{\mathrm{i}}{4\pi^2} \int_0^{\infty} \mathrm{d}k_E \, k_E \frac{J_1\big(k_E|x-y|\big)}{k_E|x-y|} = -\frac{\mathrm{i}}{4\pi^2(x-y)^2}.
\end{aligned} \tag{15.7}$$

Where $J_1(x)$ is Bessel function and we use the fact that $\int_0^\infty \mathrm{d}x \, J_1(x) = 1$.

(b) Now taking a narrow rectangular Wilson loop P with width R in x^1 direction ($0 < x^1 < 1$) and length T in x^0 direction ($0 < x^0 < T$) and evaluate $\langle U_P \rangle$. When the integral over $\mathrm{d}x$ and $\mathrm{d}y$ go independently over the loop, divergence will occur as $|x-y|^2 \to 0$. But what we want to show is the dependence of $\langle U_P \rangle$ on the geometry of the loop, namely the width R and length T, which should be divergence free. Therefore, when $T \gg R$, the integral in Wilson loop is mainly contributed by time direction and can be expressed as

$$\langle U_P(z,z) \rangle \simeq \exp\bigg[-\frac{2e^2}{8\pi^2} \int_0^T \mathrm{d}x^0 \int_T^0 \mathrm{d}y^0 \frac{1}{(x^0-y^0)^2 - R^2 - \mathrm{i}\epsilon} \bigg], \tag{15.8}$$

and we have add a small imaginary part to the denominator for the reason that will be clear. Carry out the integration, we find

$$\int_0^T \mathrm{d}x^0 \int_T^0 \mathrm{d}y^0 \frac{1}{(x^0-y^0)^2 - R^2 - \mathrm{i}\epsilon} \xrightarrow{T \gg R} \frac{2T}{R} \mathrm{arctanh}\Big(\frac{T}{R+\mathrm{i}\epsilon}\Big) = -\frac{\mathrm{i}}{\pi}\frac{T}{R}.$$

Therefore,

$$\langle U_P \rangle = \exp\Big(\frac{\mathrm{i}e^2}{4\pi R} \cdot T\Big) = e^{-\mathrm{i}V(R)T}, \tag{15.9}$$

which gives the familiar result $V(R) = -e^2/4\pi R$.

(c) For the Wilson loop of a non-Abelian gauge group, we have

$$U_P(z,z) = \mathrm{tr}\bigg\{ \mathrm{P} \exp\Big[-\mathrm{i}g \oint_P \mathrm{d}x^\mu A^a_\mu(x) t^a_r \Big] \bigg\}, \tag{15.10}$$

where t^a_r is the matrices of the group generators in representation r. We expand this expression to the order of g^2,

$$\begin{aligned} U_P(z,z) &= \mathrm{tr}\,(1) - g^2 \oint_P \mathrm{d}x^\mu \oint_P \mathrm{d}y^\nu A^a_\mu(x) A^b_\nu(y)\, \mathrm{tr}\,(t^a_r t^b_r) + \mathcal{O}(g^3) \\ &= \mathrm{tr}\,(1) \bigg[1 - g^2 C_2(r) \oint_P \mathrm{d}x^\mu \oint_P \mathrm{d}y^\nu A^a_\mu(x) A^b_\nu(y) \bigg] + \mathcal{O}(g^3). \end{aligned} \tag{15.11}$$

Compared with the Abelian case, we see that to order g^2, the non-Abelian result is given by making the replacement $e^2 \to g^2 C_2(r)$. Therefore we conclude that $V(R) = -g^2 C_2(r)/4\pi R$ in non-Abelian case.

15.4 Scalar propagator in a gauge theory

In this problem we study very briefly the heat kernel representation of Green functions/propagator of a scalar field living within a gauge field background.

(a) To begin with, we consider the simplest case, in which the background gauge field vanishes. Then we can represent the Green function $D_F(x,y)$ of the Klein-Gordon equation, defined to be

$$(\partial^2 + m^2)D_F(x,y) = -\mathrm{i}\delta^{(4)}(x-y) \tag{15.12}$$

with proper boundary conditions, by the following integral over the heat kernel function $D(x,y,T)$:

$$D_F(x,y) = \int_0^\infty \mathrm{d}T\, D(x,y,T). \tag{15.13}$$

The heat kernel satisfies the following "Schrödinger equation":

$$\Big[\mathrm{i}\frac{\partial}{\partial T} - (\partial^2 + m^2)\Big]D(x,y,T) = \mathrm{i}\delta(T)\delta^{(4)}(x-y). \tag{15.14}$$

The solution to this equation can be represented by

$$\begin{aligned} D(x,y,T) = \langle x|e^{-\mathrm{i}HT}|y\rangle &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\langle x|k\rangle\langle k|e^{-\mathrm{i}HT}|k'\rangle\langle k'|y\rangle \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\, e^{-\mathrm{i}(-k^2+m^2)T}e^{-\mathrm{i}k\cdot x+\mathrm{i}k'\cdot y}(2\pi)^4\delta^{(4)}(k-k') \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{\mathrm{i}(k^2-m^2)T}e^{-\mathrm{i}k\cdot(x-y)}, \end{aligned} \tag{15.15}$$

with $H = \partial^2 + m^2$. Integrating this result over T, with the $+\mathrm{i}\epsilon$ prescription, we recover the Feynman propagator for a scalar field:

$$\begin{aligned} \int_0^\infty \mathrm{d}T\, D(x,y,T) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-\mathrm{i}k\cdot(x-y)}\int_0^\infty \mathrm{d}T\, e^{\mathrm{i}(k^2-m^2+\mathrm{i}\epsilon)T} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}e^{-\mathrm{i}k\cdot(x-y)}}{k^2-m^2+\mathrm{i}\epsilon}. \end{aligned} \tag{15.16}$$

(b) Now let us turn on a background Abelian gauge field $A_\mu(x)$. The corresponding "Schrödinger equation" then becomes

$$\left[\mathrm{i}\frac{\partial}{\partial T} - \Big(\big(\partial_\mu - \mathrm{i}eA_\mu(x)\big)^2 + m^2\Big)\right]D(x,y,T) = \mathrm{i}\delta(T)\delta^{(4)}(x-y), \tag{15.17}$$

the solution of which, $\langle x|e^{-\mathrm{i}HT}|y\rangle$, can also be expressed as a path integral,

$$\langle x|e^{-\mathrm{i}HT}|y\rangle = \lim_{N\to\infty}\int\prod_{i=1}^{N}\Big(\mathrm{d}x_i\big\langle x_i\big|\exp\Big\{-\mathrm{i}\Delta t\big[\big(\partial_\mu - \mathrm{i}eA_\mu(x)\big)^2 + m^2\big]\Big\}\big|x_{i-1}\big\rangle\Big), \tag{15.18}$$

where we have identify $x = x_N$, $y = x_0$, and $\Delta t = T/N$. Then,

$$\begin{aligned} &\langle x_i|e^{-\mathrm{i}\Delta t[(\partial_\mu - \mathrm{i}eA_\mu(x))^2+m^2]}|x_{i-1}\rangle \\ &= \int \frac{\mathrm{d}^4k_i}{(2\pi)^4}\langle x_i|e^{-\mathrm{i}\Delta t[\partial^2 - \mathrm{i}eA_\mu(x)\partial^\mu + m^2]}|k_i\rangle\langle k_i|e^{-\mathrm{i}\Delta t[-\mathrm{i}e\partial^\mu A_\mu(x) - e^2A^2(x)]}|x_{i-1}\rangle \end{aligned}$$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \langle x_i | e^{-\mathrm{i}\Delta t[-k_i^2 + eA_\mu(x_i)k_i^\mu + m^2]} | k_i \rangle \langle k_i | e^{-\mathrm{i}\Delta t[ek_i^\mu A_\mu(x_{i-1}) - e^2 A^2(x_{i-1})]} | x_{i-1} \rangle \\
&= \int \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \, e^{-\mathrm{i}\Delta t[-k_i^2 + ek_i \cdot (A(x_i) + A(x_{i-1})) - e^2 A^2(x_{i-1}) + m^2 - \mathrm{i}\epsilon]} e^{-\mathrm{i}k_i \cdot (x_i - x_{i-1})} \\
&= C \exp\left[-\frac{\mathrm{i}\Delta t}{4} \Big(\frac{x_i - x_{i-1}}{\Delta t} + eA(x_i) + eA(x_{i-1}) \Big)^2 - \mathrm{i}\Delta t (m^2 - e^2 A^2(x_{i-1})) \right] \\
&\Rightarrow C \exp\left[-\frac{\mathrm{i}\Delta t}{4} \Big(\frac{\mathrm{d}x}{\mathrm{d}t} \Big)^2 - \mathrm{i}\Delta t e A(x) \cdot \frac{\mathrm{d}x}{\mathrm{d}t} - \mathrm{i}\Delta t m^2 \right]. \qquad (15.19)
\end{aligned}
$$

In the last line we take the continuum limit, and C is an irrelevant normalization constant. Then we get

$$
D(x, y, T) = \int \mathcal{D}x \, \exp\left[-\mathrm{i} \int_0^T \mathrm{d}t \left(\Big(\frac{\mathrm{d}x}{\mathrm{d}t} \Big)^2 + m^2 \right) - \mathrm{i}e \int_0^T \mathrm{d}x(t) \cdot A(x(t)) \right]. \qquad (15.20)
$$

15.5 Casimir operator computations

(a) In the language of angular momentum theory, we can take common eigenfunctions of $J^2 = \sum_a T^a T^a$ and $J_z = T^3$ to be the representation basis. Then the representation matrix for T^3 is diagonal:

$$
t_j^3 = \mathrm{diag}\,(-j, -j+1, \cdots, j-1, j).
$$

Thus

$$
\mathrm{tr}\,(t_j^3 t_j^3) = \sum_{m=-j}^{j} m^2 = \tfrac{1}{3} j(j+1)(2j+1).
$$

Then we have

$$
\mathrm{tr}\,(t_r^3 t_r^3) = \sum_i \mathrm{tr}\,(t_{j_i}^3 t_{j_i}^3) = \tfrac{1}{3} \sum_i j_i(j_i+1)(2j_i+1) = C(r),
$$

which implies that

$$
3C(r) = \sum_i j_i(j_i+1)(2j_i+1). \qquad (15.21)
$$

(b) Let the $SU(2)$ subgroup be spanned by T^1, T^1 and T^3. Then in fundamental representation, the representation matrices for $SU(2)$ subgroup of $SU(N)$ can be taken as

$$
t_N^i = \begin{pmatrix} \tau_i/2 & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}. \qquad (15.22)
$$

Where τ_i $(i = 1, 2, 3)$ are Pauli matrices. We see that the representation matrices for $SU(2)$ decomposes into a doublet and $(N-2)$ singlet. Then it's easy to find that

$$
C(N) = \tfrac{1}{3}\big(\tfrac{1}{2}(\tfrac{1}{2} + 1)(2 \cdot \tfrac{1}{2} + 1) \big) = \tfrac{1}{2}, \qquad (15.23)
$$

by formula in (a).

In adjoint representation, the representation matrices $(t^i)_{ab} = \mathrm{i}f^{aib}$ $(a, b = 1, \cdots, N^2 - 1,\ i = 1, 2, 3)$. Thus we need to know some information about structure constants. Here we give a handwaving illustration by analyzing the structure of fundamental representation matrices a little bit more. Note that there're three types of representation matrices, listed as follows. For convenience, let's call them t_A, t_B and t_C:

$$t_A = \begin{pmatrix} A_{2\times 2} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.24}$$

$$t_B = \begin{pmatrix} A_{2\times 2} & B_{2\times(N-2)} \\ B^\dagger_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.25}$$

$$t_C = \begin{pmatrix} -\frac{1}{2}\,\mathrm{tr}\,(C) I_{2\times 2} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & C_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.26}$$

In which, t_A is just the representation matrices for $SU(2)$ subgroup. Thus we see that there are $3\ t_A$, $2(N-2)\ t_B$ and $(N-2)^2\ t_C$ in total. It's also obvious that there is no way to generate a t_A from commutators between two t_C or between a t_B and t_C, the only way to generate t_A are commutators between two t_A or between to t_B. Then, t_A commutators correspond to the triplet representation os $SU(2)$ subgroup, and $2(N-2)$-t_B commutators correspond to the doublet representation of $SU(2)$. In this way we see that adjoint representation matrices for $SU(2)$ subgroup decompose into 1 triplet, $2(N-1)$ doublets and $(N-2)^2$ singlets.

Then we can calculate $C(G)$, again, by using formula in (a), as:

$$C(G) = \tfrac{1}{3}\big[1(1+1)(2\cdot 1+1) + 2(N-2)\cdot\tfrac{1}{2}(\tfrac{1}{2}+1)(2\cdot\tfrac{1}{2}+1)\big] = N. \tag{15.27}$$

(c) Let $U \in SU(N)$ be $N \times N$ unitary matrix, S be a symmetric $N \times N$ matrix, and A be an antisymmetric $N \times N$ matrix. Then we can use S and A to build two representations for $SU(N)$ respectively, as

$$S \to USU^T, \qquad A \to UAU^T.$$

It's easy to verify that they are indeed representations. Let's denote these two representation by s and a. It's also obvious to see that the dimensions of s and a are $d(s) = N(N+1)/2$ and $d(a) = N(N-1)/2$ respectively.

Accordingly, the generator T^a acts on S and A as:

$$S \to T^a S + S(T^a)^T, \qquad A \to T^a A + A(T^a)^T. \tag{15.28}$$

To get $C_2(s)$ and $C_2(a)$, we can make use of the formula

$$d(r)C_2(r) = d(G)C(r). \tag{15.29}$$

Thus we need to calculate $C(r)$ and $C(a)$. By formula in (a), we can take an generator in $SU(2)$ subgroup to simplify the calculation. Let's take

$$t^3_N = \tfrac{1}{2}\mathrm{diag}(1, -1, 0, \cdots, 0),$$

Then we have:

$$S=\begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{pmatrix} \quad \rightarrow \quad t_N^3 S + S(t_N^3)^T = \frac{1}{2}\begin{pmatrix} 2S_{11} & 0 & S_{13} & \cdots & S_{1n} \\ 0 & 2S_{22} & S_{23} & \cdots & S_{2n} \\ S_{31} & S_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & 0 & \cdots & 0 \end{pmatrix}$$

$$A=\begin{pmatrix} 0 & A_{12} & \cdots & A_{1n} \\ A_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n-1,n} \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix} \quad \rightarrow \quad t_N^3 A + A(t_N^3)^T = \frac{1}{2}\begin{pmatrix} 0 & 0 & A_{13} & \cdots & A_{1n} \\ 0 & 0 & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & 0 & \cdots & 0 \end{pmatrix}$$

Thus we see that the representation matrices for T^3, in both s representation and a representation, are diagonal. They are:

$$t_s^3 = \mathrm{diag}(1, 0, \underbrace{\tfrac{1}{2}\cdots,\tfrac{1}{2}}_{N-2}, 1, \underbrace{\tfrac{1}{2},\cdots,\tfrac{1}{2}}_{N-2}, \underbrace{0,\cdots,0}_{(N-2)(N-1)/2}); \tag{15.30}$$

$$t_a^3 = \mathrm{diag}(0, \underbrace{\tfrac{1}{2},\cdots,\tfrac{1}{2}}_{2(N-2)}, \underbrace{0,\cdots,0}_{(N-2)(N-3)/2}). \tag{15.31}$$

Here we have rearrange the upper triangular elements of S and A by line.

Then we get

$$C(s) = \mathrm{tr}\,(t_s^3)^2 = \tfrac{1}{2}(N+2); \tag{15.32}$$

$$C(a) = \mathrm{tr}\,(t_a^3)^2 = \tfrac{1}{2}(N-2). \tag{15.33}$$

Then,

$$C_2(s) = \frac{d(G)C(s)}{d(s)} = \frac{(N^2-1)(N+2)/2}{N(N+1)/2} = \frac{(N-1)(N+2)}{N}; \tag{15.34}$$

$$C_2(a) = \frac{d(G)C(a)}{d(a)} = \frac{(N^2-1)(N-2)/2}{N(N-1)/2} = \frac{(N+1)(N-2)}{N}. \tag{15.35}$$

At last let's check the formula implied by (15.100) and (15.101):

$$\big(C_2(r_1)+C_2(r_2)\big)d(r_1)d(r_2) = \sum C_2(r_i)d(r_i), \tag{15.36}$$

in which the tensor product representation $r_1 \times r_2$ decomposes into a direct sum of irreducible representations r_i. In our case, the direct sum of representation s and a is equivalent to the tensor product representation of two copies of N. That is,

$$N \times N \cong s + a.$$

Thus, we have,

$$\big(C_2(N)+C_2(N)\big)d(N)d(N)=\Big[\frac{N^2-1}{2N}+\frac{N^2-1}{2N}\Big]N^2=N(N^2-1);$$

and

$$C_2(s)d(s)+C_2(a)d(a)=\big[C(s)+C(a)\big]d(G)=N(N^2-1).$$

Thus formula (15.36) indeed holds in our case.

Chapter 16

Quantization of Non-Abelian Gauge Theories

16.1 Arnowitt-Fickler gauge

In this problem we perform the Faddeev-Popov quantization of Yang-Mills theory in Arnowitt-Fickler gauge (also called axial gauge), namely $A^{3a} = 0$. More generally, we may write the gauge condition as $n_\mu A^{\mu a} = 0$ with n^μ an arbitrary space-like vector of unit norm ($n^2 = -1$). The condition $A^{3a} = 0$ corresponds simply to the choice $n^\mu = g^{\mu 3}$. This gauge has the advantage that the Faddeev-Popov ghosts do not propagate and do not couple to gauge fields, as we will show below.

Our starting point, the partition function, reads

$$Z = \int \mathcal{D}A_\mu \, \delta(n \cdot A^a) e^{\mathrm{i}S[A_\mu]} \det\Big(\frac{\delta_\alpha (n \cdot A^a)}{\partial \alpha^b}\Big), \tag{16.1}$$

with $S = -\frac{1}{4}\int \mathrm{d}^4x \, (F^a_{\mu\nu})^2$ the classical action for the gauge field, and the Faddeev-Popov determinant is given by

$$\begin{aligned} \det\Big(\frac{\delta_\alpha (n \cdot A^a)}{\partial \alpha^b}\Big) &= \det\Big(\frac{1}{g} n_\mu \partial^\mu \delta^{ab} - f^{abc} n_\mu A^{\mu c}\Big) \\ &= \int \mathcal{D}b \mathcal{D}c \exp\Big[\mathrm{i}\int \mathrm{d}^4x \, b^a \big(n_\mu \partial^\mu \delta^{ab} - f^{abc} n_\mu A^{\mu c}\big) c^b\Big]. \end{aligned} \tag{16.2}$$

When multiplied by the delta function $\delta(n \cdot A^a)$, the second term in the exponent above vanishes, which implies that the ghost and antighost do not interact with gauge field. Meanwhile, they do not propagator either, since there does not exist a canonical kinetic term for them. Therefore we can safely treat the Faddeev-Popov determinant as an overall normalization of the partition function and ignore it. Then, the partition function reduces to

$$\begin{aligned} Z &= \lim_{\xi \to 0} \int \mathcal{D}A_\mu \, \exp\Big[\mathrm{i}\int \mathrm{d}^4x \Big(-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(n \cdot A^a)^2\Big)\Big] \\ &= \lim_{\xi \to 0} \int \mathcal{D}A_\mu \, \exp\Big[\mathrm{i}\int \mathrm{d}^4x \Big(\tfrac{1}{2} A^a_\mu \big(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu - \tfrac{1}{\xi} n^\mu n^\nu\big) A^b_\nu \end{aligned}$$

$$- gf^{abc}(\partial_\kappa A^a_\lambda)A^{\kappa b}A^{\lambda c} - \tfrac{1}{4}g^2 f^{eab}f^{ecd}A^a_\kappa A^b_\lambda A^{\kappa c}A^{\lambda d}\Big)\Big]. \tag{16.3}$$

where we have convert the delta function $\delta(n\cdot A^a)$ into a limit of Gaussian function. Then we see that the three-point or four-point gauge boson vertices share the same Feynman rules with the ones in covariant gauge. The only difference arises from the propagator. Let us parameterize the propagator in momentum space as

$$D^{\mu\nu}(k) = Ak^2 g^{\mu\nu} + Bk^\mu k^\nu + C(k^\mu n^\nu + n^\nu k^\mu) + Dn^\mu n^\nu. \tag{16.4}$$

Then, the equation of motion satisfied by the propagator,

$$\big(g^{\mu\nu}k^2 - k^\mu k^\nu - \tfrac{1}{\xi}n^\mu n^\nu\big)D_{\nu\lambda}(k) = g^\mu_\lambda \tag{16.5}$$

gives

$$A = -\frac{\mathrm{i}}{k^2}, \qquad B = -\frac{1+\xi k^2}{k\cdot n}C, \qquad C = -\frac{1}{k\cdot n}A, \qquad D = 0. \tag{16.6}$$

Note that the gauge fixing parameter ξ should be sent to 0. Therefore the propagator reads

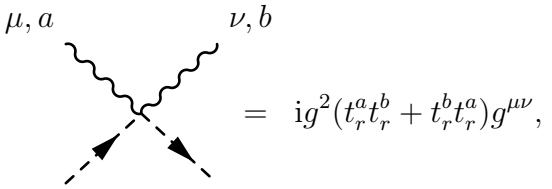
$$D(k)^{\mu\nu} = -\frac{\mathrm{i}}{k^2}\left(g^{\mu\nu} + \frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right). \tag{16.7}$$

16.2 Scalar field with non-Abelian charge

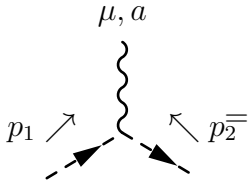
(a) Firstly we write down the Lagrangian for the Yang-Mills theory with charged scalar field, as

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi), \tag{16.8}$$

where the covariant derivative $\mathrm{D}_\mu\phi = \big(\partial_\mu + \mathrm{i}gA^a_\mu t^a_r\big)\phi$ with t^a_r the matrices of gauge group generators in representation r. For simplicity we ignore the possible mass term for the scalar. Then, it is straightforward to derive the Feynman rules for this theory by expanding this Lagrangian. The rules for the propagator and self-interactions of gauge boson are independent of matter content and are the same with the ones given in Figure 16.1 in Peskin&Schroeder. The only new ingredients here are the gauge boson-scalar field interactions, which generate the following Feynman rules:

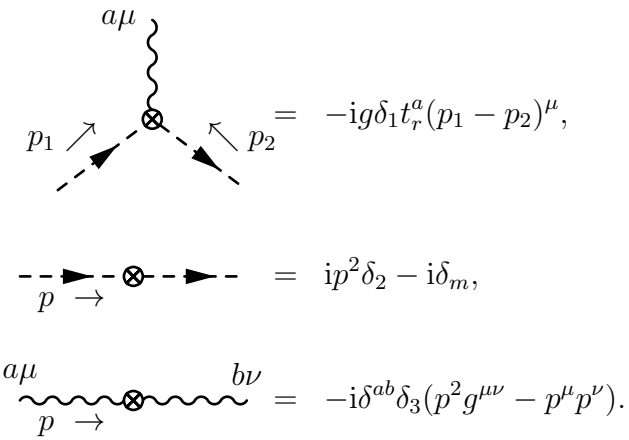


$$= \mathrm{i}g^2(t^a_r t^b_r + t^b_r t^a_r)g^{\mu\nu},$$



$$= -\mathrm{i}gt^a_r(p_1 - p_2)^\mu,$$

(b) To compute the β function of coupling g, we introduce some additional Feynman rules involving counterterms:



Then the β function is given by

$$\beta = gM\frac{\partial}{\partial M}\big(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\big). \tag{16.9}$$

To determine the counterterms, we evaluate the following relevant 1-loop diagrams. But the calculations can be simplified a lot if we observe that the combination $\delta_1 - \delta_2$ is determined by pure gauge sector, and is independent of matter content. This may be most easily seen from the counterterm relation $\delta_1 - \delta_2 = \delta_1^c - \delta_2^c$, where the right hand side comes from ghost contribution which is a pure gauge quantity. We will demonstrate this counterterm relation explicitly in the next problem for fermionic matter. Therefore, we can borrow directly the result of Peskin & Schroeder, or from the result of Problem 16.3(a),

$$\delta_1 - \delta_2 = -\frac{g^2}{(4\pi)^2}C_2(G)\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.10}$$

On the other hand, δ_3 can be found by evaluating the loop corrections to the gauge boson's self-energy. The contributions from the gauge boson loop and ghost loop have already been given in eq.(16.71) in Peskin&Schroeder, while the rest of the contributions is from the scalar-loop, and is simply the result we have found in Problem 9.1(c), multiplied by the gauge factor $\mathrm{tr}\,(t_r^a t_r^b) = C(r)$ and the number of scalar n_s. Combining these two parts gives the divergent part of δ_3:

$$\delta_3 = \frac{g^2}{(4\pi)^2}\big[\frac{5}{3}C_2(G) - \frac{1}{3}n_s C(r)\big]\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.11}$$

Then it is straightforward to see that

$$\beta = -\frac{g^3}{(4\pi)^2}\Big(\frac{11}{3}C_2(G) - \frac{1}{3}n_s C(r)\Big). \tag{16.12}$$

16.3 Counterterm relations

In this problem we calculate the divergent parts of counterterms in Yang-Mills theory with Dirac spinors at 1-loop level, to verify the counterterm relations, which is a set of

constraints set by gauge invariance. To begin with, let us rewrite the Lagrangian in its renormalized form, with counterterms separated, as

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \bar\psi(\mathrm{i}\not\partial - m)\psi - \bar c^a \partial^2 c^a \\ &+ g A^a_\mu \bar\psi \gamma^\mu t^a \psi - g f^{abc}(\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} \\ &- \tfrac{1}{4} g^2 (f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{c\mu} A^{d\nu}) - g\bar c^a f^{abc} \partial^\mu (A^b_\mu c^c) \\ &- \tfrac{1}{4}\delta_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \bar\psi(\mathrm{i}\delta_2 \not\partial - \delta_m)\psi - \delta_2^c \bar c^a \partial^2 c^a \\ &+ g\delta_1 A^a_\mu \bar\psi \gamma^\mu \psi - g\delta_1^{3g} f^{abc}(\partial_\mu A^a_\nu) A_{b\mu} A^{c\nu} \\ &- \tfrac{1}{4}\delta_1^{4g}(f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{c\mu} A^{d\nu}) - g\delta_1^c \bar c^a f^{abc} \partial^\mu (A^b_\mu c^c). \end{aligned} \tag{16.13}$$

Then the counterterm relations we will verify are

$$\delta_1 - \delta_2 = \delta_1^{3g} - \delta_3 = \tfrac{1}{2}(\delta_1^{4g} - \delta_3) = \delta_1^c - \delta_2^c. \tag{16.14}$$

Note that δ_1 and δ_2 have been given in (16.84) and (16.77) in Peskin&Schroeder. Here we simply quote the results:

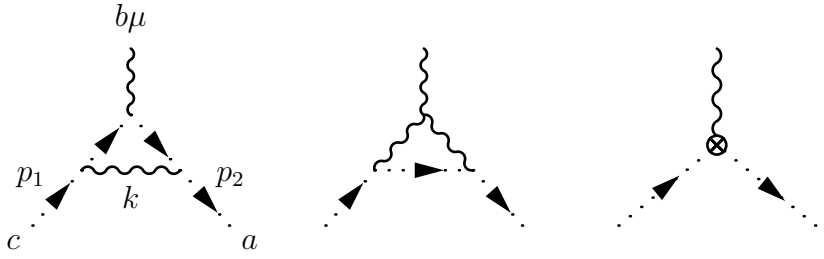
$$\delta_1 = -\frac{g^2}{(4\pi)^2}\big[C_2(r) + C_2(G)\big]\Big(\frac{2}{\epsilon} - \log M^2\Big), \tag{16.15}$$

$$\delta_2 = -\frac{g^2}{(4\pi)^2} C_2(r)\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.16}$$

Therefore,

$$\delta_1 - \delta_2 = -\frac{g^2}{(4\pi)^2} C_2(G)\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.17}$$

(a) Firstly let us check the equality $\delta_1 - \delta_2 = \delta_1^c - \delta_2^c$. The 1-loop contributions to δ_1^c come from the following three diagrams:



The first diagram reads

$$\begin{aligned} &(-g)^3 f^{ade} f^{ebf} f^{fdc} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-\mathrm{i}}{k^2} \frac{\mathrm{i}}{(p_1-k)^2} \frac{\mathrm{i}}{(p_2-k)^2} \cdot p_2^\nu (p_2-k)^\mu (p_1-k)_\nu \\ \Rightarrow& -g^3 f^{ade} f^{ebf} f^{fdc} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu (p_2\cdot k)}{k^6} = -\frac{1}{4} g^3 f^{ade} f^{ebf} f^{fdc} p_2^\mu \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^4} \\ \Rightarrow& -\frac{\mathrm{i}}{4} g^3 f^{ade} f^{ebf} f^{fdc} p_2^\mu \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}. \end{aligned} \tag{16.18}$$

The second diagram reads

$$\begin{aligned}
&(-g)^2 g f^{ade} f^{ebf} f^{fdc} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p_1-k)^2} \frac{-\mathrm{i}}{(p_2-k)^2} \\
&\quad \times p_{2\rho} k_\sigma \left[g^{\mu\rho}(k-p_2-q)^\sigma + g^{\mu\sigma}(q-p_1+k)^\rho + g^{\sigma\rho}(p_1+p_2-2k)^\mu \right] \\
\Rightarrow &-\mathrm{i} g^3 f^{ade} f^{ebf} f^{fdc} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^6} \left[p_2^\mu k^2 + k^\mu (k\cdot p_2) - 2k^\mu (k\cdot p_2) \right] \\
\Rightarrow &-\frac{3}{4} \mathrm{i} g^3 f^{ade} f^{ebf} f^{fdc} p_2^\mu \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^4} \Rightarrow -\frac{3}{4} \mathrm{i} g^3 f^{ade} f^{ebf} f^{fdc} p_2^\mu \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
\end{aligned} \tag{16.19}$$

To simplify the structure constant product, we make use of the Jacobi identity,

$$0 = f^{ebf}(f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe}) = 2 f^{abd} f^{dce} f^{ebf} - f^{caf} C_2(G),$$

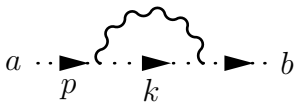
then we have

$$f^{ade} f^{ebf} f^{fdc} = -\frac{1}{2} f^{abc} C_2(G). \tag{16.20}$$

Note that the third diagram reads $-g\delta_1^c f^{abc} p_2^\mu$, thus we see that to make the sum of these three diagrams finite, the counterterm coefficient δ_1^c should be

$$\delta_1^c \sim -\frac{g^2 C_2(G)}{2(4\pi)^2} \Big(\frac{2}{\epsilon} - \log M^2 \Big). \tag{16.21}$$

Then consider δ_2^c. This coefficient should absorb the divergence from the following diagram:



This diagram reads

$$\begin{aligned}
&(-g)^2 f^{bcd} f^{dca} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p-k)^2} (p\cdot k) \\
=&-g^2 C_2(G) \delta^{ab} \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{p\cdot(k'+xp)}{(k'^2-\Delta)^2} \\
\Rightarrow &-\frac{g^2}{2} C_2(G) \delta^{ab} p^2 \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon} + \text{terms indep. of } p^2.
\end{aligned} \tag{16.22}$$

The corresponding counterterm contributes $\mathrm{i}\delta_2^c p^2$, therefore we have

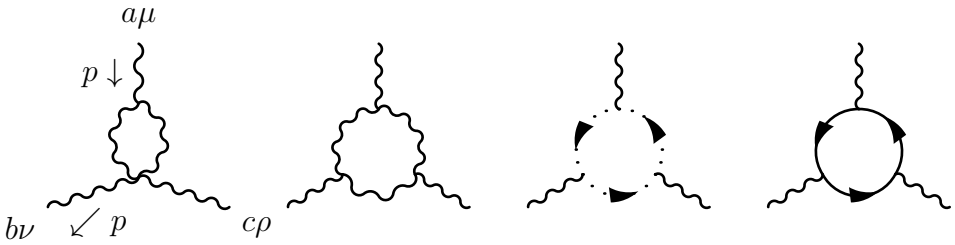
$$\delta_2^c \sim \frac{g^2 C_2(G)}{2(4\pi)^2} \Big(\frac{2}{\epsilon} - \log M^2 \Big). \tag{16.23}$$

Combining (16.21), (16.23) and (16.17), we see that the equality $\delta_1 - \delta_2 = \delta_1^c - \delta_2^c$ is satisfied.

(b) Now let's verify the equality $\delta_1 - \delta_2 = \delta_1^{3g} - \delta_3$. In this case the calculation turns out to be more cumbersome, though. The coefficient δ_3 has been given by (16.74) in Peskin&Schroeder. The result is

$$\delta_3 = \frac{g^2}{(4\pi)^2}\Big[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\Big]\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.24}$$

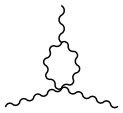
Thus we only need to calculate δ_1^{3g}. The relevant loop diagrams are listed as follows.



For simplicity, we have set the external momenta to be p, $-p$ and 0 for the three external gauge boson lines labeled with $(a\mu)$, $(b\nu)$ and $(c\rho)$. Then the contribution of the counterterm to this vertex is given by

$$g\delta_1^{3g} f^{abc}(2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\rho\mu}p^\nu). \tag{16.25}$$

To extract the divergent part from δ_1^{3g}, we have to evaluate the loop diagrams shown above. Let us calculate them now in turn. The first diagram reads



$$\begin{aligned}
&= \tfrac{1}{2}g(-\mathrm{i}g^2)f^{ade}\big[f^{def}f^{bcf}(g^{\lambda\nu}g^{\kappa\rho} - g^{\lambda\rho}g^{\kappa\nu}) \\
&\quad + f^{dbf}f^{ecf}(g^{\lambda\kappa}g^{\nu\rho} - g^{\lambda\rho}g^{\kappa\nu}) + f^{dcf}f^{ebf}(g^{\lambda\kappa}g^{\nu\rho} - g^{\lambda\nu}g^{\kappa\rho})\big] \\
&\quad \times \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(k-p)^2}\big[g^\mu_\lambda(p+k)_\kappa + g_{\lambda\kappa}(-2k+p)^\mu + g^\mu_\kappa(k-2p)_\lambda\big] \\
&= \tfrac{1}{2}\mathrm{i}g^3 f^{abc}C_2(G)\cdot\tfrac{3}{2}(g^{\lambda\nu}g^{\kappa\rho} - g^{\lambda\rho}g^{\kappa\nu}) \\
&\quad \times \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k-p)^2}\big[g^\mu_\lambda(p+k)_\kappa + g_{\lambda\kappa}(-2k+p)^\mu + g^\mu_\kappa(k-2p)_\lambda\big] \\
&\Rightarrow \tfrac{1}{2}\mathrm{i}g^3 f^{abc}C_2(G)\cdot\tfrac{9}{2}(g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^4} \\
&\Rightarrow \frac{9}{4}\mathrm{i}g^3 f^{abc}C_2(G)(g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}.
\end{aligned} \tag{16.26}$$

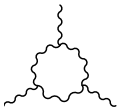
There are two additional diagrams associated with this diagram by the two cyclic permutations of the three external momenta. One gives

$$\frac{9}{4}\mathrm{i}g^3 f^{abc}C_2(G)(g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon},$$

while the other yields zero. Therefore the sum of these three diagrams gives:

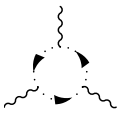
$$\frac{9}{4}\mathrm{i}g^3 f^{abc}C_2(G)(2g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu - g^{\nu\rho}p^\mu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}, \tag{16.27}$$

Then we come to the second diagram, which reads



$$
\begin{aligned}
&= g^3 f^{adf} f^{bed} f^{cfe} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-\mathrm{i}}{k^2} \frac{-\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p+k)^2} \\
&\quad \times \left[g^{\mu\sigma}(2p+k)^\kappa + g^{\sigma\kappa}(-p-2k)^\mu + g^{\kappa\mu}(k-p)^\sigma \right] \\
&\quad \times \left[g^{\nu\lambda}(-p+k)_\sigma + g^\lambda_\sigma (-2k-p)^\nu + g^\nu_\sigma (2p+k)^\lambda \right] \\
&\quad \times \left[g^\rho_\kappa k_\lambda - 2 g_{\kappa\lambda} k^\rho + g^\rho_\lambda k_\kappa \right] \\
&\Rightarrow \mathrm{i}g^3 \left[-\tfrac{1}{2} f^{abc} C_2(G) \right] \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{2(1-x)k'^2}{(k'^2-\Delta)^3} \\
&\quad \times \frac{1}{4} \Big[2(8+15x) g^{\mu\nu} p^\rho + (30x-23) \big(g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu \big) \Big] \\
&\Rightarrow \mathrm{i}g^3 \left[-\tfrac{1}{2} f^{abc} C_2(G) \right] \cdot \frac{13}{4} \big(2 g^{\mu\nu} p^\rho - g^{\mu\rho} p^\nu - g^{\nu\rho} p^\mu \big) \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \frac{1}{k'^4} \\
&\Rightarrow -\frac{13}{8} \mathrm{i} g^3 f^{abc} C_2(G) \big(2 g^{\mu\nu} p^\rho - g^{\mu\rho} p^\nu - g^{\nu\rho} p^\mu \big) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
\end{aligned}
\tag{16.28}
$$

The third diagram reads



$$
\begin{aligned}
&= (-g)^3 f^{daf} f^{ebd} f^{fce} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (-1) \Big(\frac{\mathrm{i}}{k^2} \Big)^2 \frac{\mathrm{i}}{(k+p)^2} \cdot (p+k)^\mu k^\nu k^\rho \\
&= -\mathrm{i}g^3 \cdot \tfrac{1}{2} f^{abc} C_2(G) \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{2(1-x)k'^2}{(k'^2-\Delta)^3} \\
&\quad \times \tfrac{1}{d} \left[-x g^{\mu\nu} p^\rho - x g^{\mu\rho} p^\nu + (1-x) g^{\nu\rho} p^\mu \right] \\
&\Rightarrow \frac{1}{24} \mathrm{i} g^3 f^{abc} C_2(G) (g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu - 2 g^{\nu\rho} p^\mu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
\end{aligned}
\tag{16.29}
$$

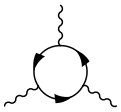
There is again a similar diagram with ghost loop running reversely, which gives

$$
\frac{1}{24} \mathrm{i} g^3 f^{abc} C_2(G) (g^{\mu\nu} p^\rho - 2 g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
$$

Then these two diagrams with ghost loops sum to

$$
\frac{1}{24} \mathrm{i} g^3 f^{abc} C_2(G) (2 g^{\mu\nu} p^\rho - g^{\mu\rho} p^\nu - g^{\nu\rho} p^\mu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}. \tag{16.30}
$$

Finally we consider the fourth diagram with fermion loop. There are also two copies with fermions running in opposite directions. One (shown in the figure) gives



$$
\begin{aligned}
&= n_f (\mathrm{i}g)^3 \operatorname{tr}(t^a t^c t^b) \int \frac{\mathrm{d}^d k}{(2\pi)^d} (-1) \operatorname{tr} \left[\gamma^\mu \frac{\mathrm{i}}{\not{k}} \gamma^\rho \frac{\mathrm{i}}{\not{k}} \gamma^\nu \frac{\mathrm{i}}{\not{k} + \not{p}} \right] \\
&\Rightarrow \frac{4}{3} n_f g^3 \operatorname{tr}(t^a t^c t^b) (2 g^{\mu\nu} p^\rho - g^{\nu\rho} p^\mu - g^{\mu\rho} p^\nu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon},
\end{aligned}
\tag{16.31}
$$

while the other gives

$$
-\frac{4}{3} n_f g^3 \operatorname{tr}(t^a t^b t^c) (2 g^{\mu\nu} p^\rho - g^{\nu\rho} p^\mu - g^{\mu\rho} p^\nu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
$$

Thus they sum to

$$\begin{aligned}&\frac{4}{3}n_f g^3\,\mathrm{tr}\left(t^a[t^c,t^b]\right)(2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\\ =&-\frac{4}{3}\mathrm{i}n_f g^3 C(r) f^{abc}(2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}.\end{aligned}\tag{16.32}$$

Now, sum up the four groups of diagrams, we get

$$\frac{g^3}{(4\pi)^2}\frac{2}{\epsilon}f^{abc}\left[\Big(-\frac{9}{4}+\frac{13}{8}-\frac{1}{24}\Big)C_2(G)+\frac{4}{3}n_f C(r)\right](2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu),\tag{16.33}$$

and consequently,

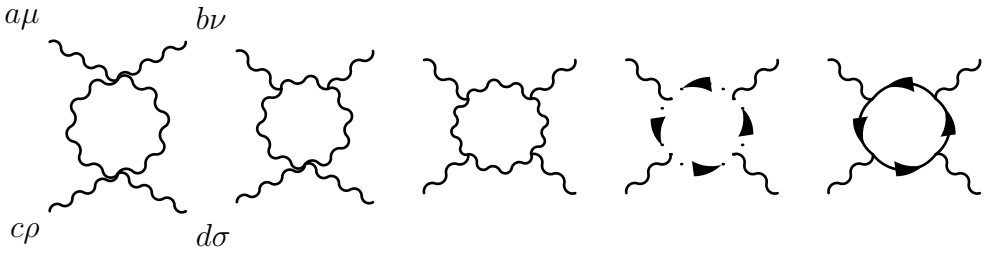
$$\delta_1^{3g}=\frac{g^2}{(4\pi)^2}\Big[\frac{2}{3}C_2(G)-\frac{4}{3}n_f C(r)\Big]\cdot\Big(\frac{2}{\epsilon}-\log M^2\Big).\tag{16.34}$$

Thus,

$$\delta_1^{3g}-\delta_3=-\frac{g^2}{(4\pi)^2}C_2(G)\Big(\frac{2}{\epsilon}-\log M^2\Big),\tag{16.35}$$

which equals to $\delta_1-\delta_2$ (16.17), as expected.

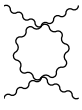
(c) Now let's move to the relation $\delta_1-\delta_2=\frac{1}{2}(\delta_1^{4g}-\delta_3)$. This time we have to evaluate δ_1^{4g}, which is determined by the divergent part of the following five types of diagrams:



Firstly the counterterm itself contributes to the 1-loop corrections with

$$\begin{aligned}&-\delta_1^{4g}\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})\\ &\quad+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big].\end{aligned}\tag{16.36}$$

To evaluate the loop diagrams, we set all external momenta to zero for simplicity. The first diagram then reads



$$\begin{aligned}&=\frac{1}{2}(-\mathrm{i}g^2)^2\Big[f^{abg}f^{efg}(g^{\mu\lambda}g^{\nu\kappa}-g^{\mu\kappa}g^{\nu\lambda})+f^{aeg}f^{bfg}(g^{\mu\nu}g^{\lambda\kappa}-g^{\mu\kappa}g^{\nu\lambda})\\ &\quad+f^{afg}f^{beg}(g^{\mu\nu}g^{\lambda\kappa}-g^{\mu\lambda}g^{\nu\kappa})\Big]\Big[f^{efh}f^{cdh}(g^\rho_\lambda g^\sigma_\kappa-g^\lambda_\sigma g^\kappa_\rho)\\ &\quad+f^{ech}f^{fdh}(g_{\lambda\kappa}g^{\rho\sigma}g^\lambda_\sigma g^\rho_\kappa)+f^{edh}f^{fch}(g_{\lambda\kappa}g^{\rho\sigma}-g^\rho_\lambda g^\sigma_\kappa)\Big]\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2\\ \Rightarrow&\frac{\mathrm{i}g^4}{2(4\pi)^2}\frac{2}{\epsilon}\Big[f^{abg}f^{efg}f^{efh}f^{cdh}(2g^{\mu\rho}g^{\nu\sigma}-2g^{\mu\sigma}g^{\nu\rho})\end{aligned}$$

$$
\begin{aligned}
&+ f^{abg}f^{efg}f^{ech}f^{fdh}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{abg}f^{efg}f^{edh}f^{fch}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})\\
&+ f^{aeg}f^{bfg}f^{efh}f^{cdh}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{afg}f^{beg}f^{efh}f^{cdh}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})\\
&+ \operatorname{tr}(t^at^bt^dt^c)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) + \operatorname{tr}(t^at^bt^ct^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho})\\
&+ \operatorname{tr}(t^at^bt^ct^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) + \operatorname{tr}(t^at^bt^dt^c)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma})\Big]\\
= \frac{\mathrm{i}g^4}{2(4\pi)^2}\frac{2}{\epsilon}\Big[&\operatorname{tr}(t^at^bt^ct^d)(4g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 10g^{\mu\sigma}g^{\nu\rho})\\
&+ \operatorname{tr}(t^at^bt^dt^c)(4g^{\mu\nu}g^{\rho\sigma} + 10g^{\mu\rho}g^{\nu\sigma} - 8g^{\mu\sigma}g^{\nu\rho})\Big],
\end{aligned}
\tag{16.37}
$$

where we have used (16.20) and $f^{abc} = \mathrm{i}(t^a)_{bc}$ with t^a the generators in adjoint representation. There are two additional diagrams similar to this one, which can be obtained by exchange of labels as $(b\nu \leftrightarrow c\rho)$ and $(b\nu \leftrightarrow d\sigma)$. Therefore the total contribution from these three diagrams is

$$
\begin{aligned}
&\frac{\mathrm{i}g^4}{(4\pi)^2}\frac{2}{\epsilon}\Big[\operatorname{tr}(t^at^bt^ct^d)(7g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 7g^{\mu\sigma}g^{\nu\rho})\\
&\quad+ \operatorname{tr}(t^at^bt^dt^c)(7g^{\mu\nu}g^{\rho\sigma} + 7g^{\mu\rho}g^{\nu\sigma} - 8g^{\mu\sigma}g^{\nu\rho})\\
&\quad+ \operatorname{tr}(t^at^ct^bt^d)(-8g^{\mu\nu}g^{\rho\sigma} + 7g^{\mu\rho}g^{\nu\sigma} + 7g^{\mu\sigma}g^{\nu\rho})\Big].
\end{aligned}
\tag{16.38}
$$

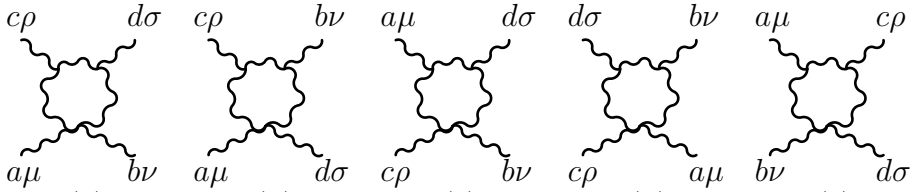
The second diagram has five additional counterparts. The one displayed in the figure reads



$$
\begin{aligned}
&= (-\mathrm{i}g^2)g^2 f^{aeg}f^{bgf}\big[f^{efh}f^{cdh}(g^\rho_\lambda g^\sigma_\kappa - g^\sigma_\lambda g^\rho_\kappa)\\
&\qquad + f^{ech}f^{fdh}(g_{\lambda\kappa}g^{\rho\sigma} - g^\sigma_\lambda g^\kappa_\rho) + f^{edh}f^{fch}(g_{\lambda\kappa}g^{\rho\sigma} - g^\lambda_\rho g^\sigma_\kappa)\big]\\
&\qquad \times \int \frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^3(g^{\mu\lambda}k^\tau - 2k^\mu + g^{\tau\mu}k^\lambda)(g^\nu_\tau k^\kappa - 2g^\kappa_\tau k^\nu + g^{\kappa\nu}k_\tau)\\
&= g^4 f^{aeg}f^{bgf}\big[f^{efh}f^{cdh}(g^\rho_\lambda g^\sigma_\kappa - g^\sigma_\lambda g^\rho_\kappa)\\
&\qquad + f^{ech}f^{fdh}(g_{\lambda\kappa}g^{\rho\sigma} - g^\sigma_\lambda g^\kappa_\rho) + f^{edh}f^{fch}(g_{\lambda\kappa}g^{\rho\sigma} - g^\lambda_\rho g^\sigma_\kappa)\big]\\
&\qquad \times \int \frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{1}{k^2}\Big)^2\big(2g^\mu_\lambda g^\nu_\kappa + 5g^{\mu\nu}g_{\lambda\kappa} - 4g^\mu_\kappa g^\nu_\lambda\big)\\
\Rightarrow\ &\frac{g^4}{4}\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\Big[f^{aeg}f^{bgf}f^{efh}f^{cdh}(6g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})\\
&\qquad + f^{aeg}f^{bgf}f^{ech}f^{fdh}(13g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho})\\
&\qquad + f^{aeg}f^{bgf}f^{edh}f^{fch}(13g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho})\Big]\\
&= \frac{g^4}{4}\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\Big[\mathrm{i}f^{cdh}\operatorname{tr}(t^at^bt^h)(6g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})\\
&\qquad - \operatorname{tr}(t^at^bt^dt^c)(13g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho})\\
&\qquad - \operatorname{tr}(t^at^bt^ct^d)(13g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho})\Big]\\
&= -\frac{g^4}{4}\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\Big[\operatorname{tr}(t^at^bt^ct^d)(13g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 10g^{\mu\sigma}g^{\nu\rho})
\end{aligned}
$$

$$+\ \mathrm{tr}\,(t^a t^b t^d t^c)(13g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\sigma}g^{\nu\rho})\Big], \tag{16.39}$$

where we have used the fact that $f^{aeg}f^{bgf}f^{efh}f^{cdh}=\mathrm{i}f^{cdh}\,\mathrm{tr}\,(t^at^bt^h)=\mathrm{tr}\,\big(t^at^b[t^c,t^d]\big)$, and $f^{aeg}f^{bgf}f^{ech}f^{fdh}=-\mathrm{tr}\,(t^at^bt^dt^c)$, etc. There are additional five diagrams associated with this one, namely,



In the diagrams above, (a) gives the identical result to the one have just evaluated, while (b) and (c) give identical expression, so do (d) and (e). We can find (b) from the result above by the exchange $(a\mu\leftrightarrow c\rho)$, and (d) by the exchange $(a\mu\leftrightarrow d\sigma)$. Then we sum up all six diagrams, which is equivalent to summing the original one with (b) and (d) and multiplying the result by 2:

$$\begin{aligned}
&-2\cdot\frac{\mathrm{i}g^4}{4(4\pi)^2}\frac{2}{\epsilon}\Big[\,\mathrm{tr}\,(t^at^bt^ct^d)(13g^{\mu\nu}g^{\rho\sigma}-8g^{\mu\rho}g^{\nu\sigma}+10g^{\mu\sigma}g^{\nu\rho})\\
&\qquad+\mathrm{tr}\,(t^at^bt^dt^c)(13g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\sigma}g^{\nu\rho})\\
&\qquad+\mathrm{tr}\,(t^ct^bt^at^d)(13g^{\mu\sigma}g^{\nu\rho}-8g^{\mu\rho}g^{\nu\sigma}+10g^{\mu\nu}g^{\rho\sigma})\\
&\qquad+\mathrm{tr}\,(t^ct^bt^dt^a)(13g^{\mu\sigma}g^{\nu\rho}-8g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\rho}g^{\nu\sigma})\\
&\qquad+\mathrm{tr}\,(t^dt^bt^ct^a)(13g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\sigma}g^{\nu\rho})\\
&\qquad+\mathrm{tr}\,(t^dt^bt^at^c)(13g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\sigma}g^{\nu\rho}+10g^{\mu\nu}g^{\rho\sigma})\Big]\\
=&-\frac{\mathrm{i}g^4}{2(4\pi)^2}\frac{2}{\epsilon}\Big[\,\mathrm{tr}\,(t^at^bt^ct^d)(23g^{\mu\nu}g^{\rho\sigma}-16g^{\mu\rho}g^{\nu\sigma}+23g^{\mu\sigma}g^{\nu\rho})\\
&+\mathrm{tr}\,(t^at^bt^dt^c)(23g^{\mu\nu}g^{\rho\sigma}+23g^{\mu\rho}g^{\nu\sigma}-16g^{\mu\sigma}g^{\nu\rho})\\
&+\mathrm{tr}\,(t^at^ct^bt^d)(-16g^{\mu\nu}g^{\rho\sigma}+23g^{\mu\rho}g^{\nu\sigma}+23g^{\mu\sigma}g^{\nu\rho})\Big],
\end{aligned}\tag{16.40}$$

where we use the cyclic symmetry of trace and also the relation $\mathrm{tr}\,(t^at^bt^ct^d)=\mathrm{tr}\,(t^dt^bt^ct^a)$.

The third diagram reads



$$\begin{aligned}
=\,&g^4f^{aeh}f^{bhg}f^{cfe}f^{dgf}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^4(g^{\mu\lambda}k^\eta-2g^{\lambda\eta}k^\mu+g^{\eta\mu}k^\lambda)\\
&\times(g^\nu_\eta k^\xi-2g^\xi_\eta k^\nu+g^{\xi\nu}k_\eta)(g^{\rho\kappa}k_\lambda-2g^\kappa_\lambda k^\rho+g^\rho_\lambda k^\kappa)\\
&\times(g^\sigma_\xi k_\kappa-2g_{\xi\kappa}k^\sigma+g^\sigma_\kappa k_\xi)\\
=\,&g^4\,\mathrm{tr}\,(t^at^bt^dt^c)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2)^4}\big[34k^\mu k^\nu k^\rho k^\sigma+k^4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma})\\
&+3k^2(g^{\mu\nu}k^\rho k^\sigma+g^{\mu\rho}k^\nu k^\sigma+g^{\nu\sigma}k^\mu k^\rho+g^{\rho\sigma}k^\mu k^\nu)\big]\\
=\,&g^4\,\mathrm{tr}\,(t^at^bt^dt^c)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2)^2}\Big[\frac{34}{24}(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})
\end{aligned}$$

$$
\begin{aligned}
&+ (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) + \frac{3}{2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma})\Big] \\
&= \frac{\mathrm{i}g^4}{12(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}\,(t^a t^b t^d t^c)(47g^{\mu\nu}g^{\rho\sigma} + 47g^{\mu\rho}g^{\nu\sigma} + 17g^{\mu\sigma}g^{\nu\rho}). \qquad (16.41)
\end{aligned}
$$

Combined with the other two similar diagrams, we get

$$
\begin{aligned}
&\frac{\mathrm{i}g^4}{12(4\pi)^2}\frac{2}{\epsilon}\big[\,\mathrm{tr}\,(t^a t^b t^c t^d)(47g^{\mu\nu}g^{\rho\sigma} + 17g^{\mu\rho}g^{\nu\sigma} + 47g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + \,\mathrm{tr}\,(t^a t^b t^d t^c)(47g^{\mu\nu}g^{\rho\sigma} + 47g^{\mu\rho}g^{\nu\sigma} + 17g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + \,\mathrm{tr}\,(t^a t^c t^b t^d)(17g^{\mu\nu}g^{\rho\sigma} + 47g^{\mu\rho}g^{\nu\sigma} + 47g^{\mu\sigma}g^{\nu\rho})\big] \qquad (16.42)
\end{aligned}
$$

The fourth diagram with ghost loop is given by



$$
\begin{aligned}
&= (-g)^4 f^{eah} f^{hbg} f^{gdf} f^{fce} \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\Big(\frac{\mathrm{i}}{k^2}\Big)^4 k^\mu k^\nu k^\rho k^\sigma \\
&\Rightarrow -\frac{\mathrm{i}g^4}{24(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}\,(t^a t^b t^d t^c)(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \qquad (16.43)
\end{aligned}
$$

There are six distinct diagrams with ghost loops, with different permutations of external labels (Lorentz and gauge). They sum to

$$
\begin{aligned}
&-\frac{\mathrm{i}g^4}{12(4\pi)^2}\frac{2}{\epsilon}\big[\,\mathrm{tr}\,(t^a t^b t^c t^d) + \,\mathrm{tr}\,(t^a t^b t^d t^c) + \,\mathrm{tr}\,(t^a t^c t^b t^d)\big] \\
&\quad \times (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \qquad (16.44)
\end{aligned}
$$

Finally the diagram with fermion loop reads



$$
\begin{aligned}
&= (\mathrm{i}g)^4 n_f\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)\int\frac{\mathrm{d}^d k}{(2\pi)^d}(-)\,\mathrm{tr}\left[\gamma^\mu\frac{\mathrm{i}}{\not k}\gamma^\mu\frac{\mathrm{i}}{\not k}\gamma^\sigma\frac{\mathrm{i}}{\not k}\gamma^\rho\frac{\mathrm{i}}{\not k}\right] \\
&= -\,g^4 n_f\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2)^4}\big[4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})(k^2)^2 \\
&\qquad - 8(g^{\mu\nu}k^\rho k^\sigma + g^{\mu\rho}k^\nu k^\sigma + g^{\nu\sigma}k^\mu k^\rho + g^{\rho\sigma}k^\mu k^\nu)k^2 + 32k^\mu k^\nu k^\rho k^\sigma\big] \\
&= -\,g^4 n_f\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2)^2}\big[4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\qquad - 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) + \tfrac{4}{3}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\big] \\
&= -\,\frac{4\mathrm{i}g^4 n_f}{3(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}). \qquad (16.45)
\end{aligned}
$$

Combined with the similar diagrams with different permutations, we get

$$
\begin{aligned}
&-\frac{8\mathrm{i}g^4 n_f}{3(4\pi)^2}\frac{2}{\epsilon}\big[\,\mathrm{tr}\,(t_r^a t_r^b t_r^c t_r^d)(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + \,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + \,\mathrm{tr}\,(t_r^a t_r^c t_r^b t_r^d)(-2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \qquad (16.46)
\end{aligned}
$$

Now, we sum up the first four types of diagrams, namely, (16.38), (16.40), (16.42), and (16.44), and find the result to be

$$\begin{aligned}
&\frac{2\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}\Big[\,\mathrm{tr}\,(t^at^bt^ct^d)(-g^{\mu\nu}g^{\rho\sigma}+2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+\,\mathrm{tr}\,(t^at^bt^dt^c)(-g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+2g^{\mu\sigma}g^{\nu\rho})\\
&\quad+\,\mathrm{tr}\,(t^at^ct^bt^d)(2g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\Big]\\
=\;&\frac{2\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}\Big[g^{\mu\nu}g^{\rho\sigma}\big(2\,\mathrm{tr}\,(t^at^ct^bt^d)-\,\mathrm{tr}\,(t^at^bt^ct^d)-\,\mathrm{tr}\,(t^at^bt^dt^c)\big)\\
&\quad+g^{\mu\rho}g^{\nu\sigma}\big(2\,\mathrm{tr}\,(t^at^bt^ct^d)-\,\mathrm{tr}\,(t^at^bt^dt^c)-\,\mathrm{tr}\,(t^at^ct^bt^d)\big)\\
&\quad+g^{\mu\sigma}g^{\nu\rho}\big(2\,\mathrm{tr}\,(t^at^bt^dt^c)-\,\mathrm{tr}\,(t^at^bt^ct^d)-\,\mathrm{tr}\,(t^at^ct^bt^d)\big)\Big]\\
=\;&\frac{\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}C_2(G)\big[g^{\mu\nu}g^{\rho\sigma}(-f^{ade}f^{bce}-f^{ace}f^{bde})\\
&\quad+g^{\mu\rho}g^{\nu\sigma}(f^{ade}f^{bce}-f^{abe}f^{cde})+g^{\mu\sigma}g^{\nu\rho}(f^{ace}f^{bde}+f^{abe}f^{cde})\big]\\
=\;&-\frac{\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}C_2(G)\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big].
\end{aligned}\tag{16.47}$$

Similar manipulations on (16.46) gives

$$\begin{aligned}
&-\frac{4\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}n_fC_2(r)\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big].
\end{aligned}\tag{16.48}$$

Therefore, we finally find δ_1^{4g} to be

$$-\frac{g^2}{3(4\pi)^2}\big[C_2(G)+4n_fC_2(r)\big]\Big(\frac{2}{\epsilon}-\log M^2\Big),\tag{16.49}$$

and it is straightforward to see that $\delta_1^{4g}-\delta_3=2(\delta_1-\delta_2)$.

Chapter 17

Quantum Chromodynamics

17.1 Two-Loop renormalization group relations

(a) In this problem we study the higher orders of QCD β function. Formally, we have

$$\beta(g) = -\frac{b_0}{(4\pi)^2}g^3 - \frac{b_1}{(4\pi)^4}g^5 - \frac{b_2}{(4\pi)^6}g^7 + \cdots . \tag{17.1}$$

The we can deduce the corresponding function for $\alpha_s \equiv g^2/(4\pi)$, namely,

$$\mu\frac{\partial\alpha_s}{\partial\mu} = -\frac{2b_0}{4\pi}\alpha_s^2 - \frac{2b_1}{(4\pi)^2}\alpha_s^3 - \frac{2b_2}{(4\pi)^3}\alpha_s^4 + \cdots . \tag{17.2}$$

Integrate this equation, we get

$$\int_\Lambda^Q \frac{\mathrm{d}\mu}{2\mu} = -\int_\infty^{\alpha_s(Q^2)} \mathrm{d}\alpha_s \left[\frac{b_0}{4\pi}\alpha_s^2 + \frac{b_1}{(4\pi)^2}\alpha_s^3 + \frac{b_2}{(4\pi)^3}\alpha_s^4 + \cdots\right]^{-1}. \tag{17.3}$$

The integral can be carried out approximately, as

$$\log(Q/\Lambda)^2 = \frac{4\pi}{b_0}\left[\frac{1}{\alpha_s(Q^2)} + \frac{b_1}{4\pi b_0}\log\frac{\alpha_s(Q^2)}{1+\frac{b_1}{4\pi b_0}\alpha_s(Q^2)} + \cdots\right]. \tag{17.4}$$

Then the running coupling $\alpha_s(Q^2)$ can be solved iteratively, to be,

$$\alpha_s(Q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\log(Q/\Lambda)^2} - \frac{b_1}{b_0^2}\frac{\log\log(Q/\Lambda)^2}{[\log(Q/\Lambda)^2]^2} + \cdots\right]. \tag{17.5}$$

(b) Now we substitute (17.5) into the e^+e^- annihilation cross section, we get

$$\begin{aligned}
&\sigma(e^+e^- \to \text{hadrons}) \\
&= \sigma_0 \cdot \left(3\sum_f Q_f^2\right) \cdot \left[1 + \frac{\alpha_s}{\pi} + a_2\Big(\frac{\alpha_s}{\pi}\Big)^2 + \mathcal{O}(\alpha_s^3)\right] \\
&= \sigma_0 \cdot \left(3\sum_f Q_f^2\right) \cdot \left[1 + \frac{4}{b_0}\frac{1}{\log(Q/\Lambda)^2} - \frac{4b_1}{b_0^3}\frac{\log\log(Q/\Lambda)^2}{[\log(Q/\Lambda)^2]^2} + \cdots\right].
\end{aligned} \tag{17.6}$$

Since the expression for the cross section is independent of renormalization scheme to the order showed above, we conclude that the β function coefficients b_0 and b_1 are also independent of the renormalization scheme.

17.2 A Direct test of the spin of the gluon

(a) We repeat the calculations in Part (c) of the Final Project I, with the gluon-quark vertex replaced by a Yukawa vertex.

$$\mathrm{i}\mathcal{M} = Q_q(-\mathrm{i}e)^2(-\mathrm{i}g)\bar{u}(k_1)\left[\frac{\mathrm{i}}{\not k_1+\not k_3}\gamma^\mu - \gamma^\mu\frac{\mathrm{i}}{\not k_2+\not k_3}\right]v(k_2)\frac{-\mathrm{i}}{q^2}\bar{v}(p_2)\gamma_\mu u(p_1) \tag{17.7}$$

Then, use the trick described in Final Project I, we have

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 &= \frac{Q_q^2g^2e^4}{4s^2}\,\mathrm{tr}\,(\gamma_\mu\not p_1\gamma_\rho\not p_2)\\ &\quad\times\,\mathrm{tr}\left[\Big(\frac{1}{\not k_1+\not k_3}\gamma^\mu-\gamma^\mu\frac{1}{\not k_2+\not k_3}\Big)\not k_2\Big(\gamma^\rho\frac{1}{\not k_1+\not k_3}-\frac{1}{\not k_2+\not k_3}\gamma^\rho\Big)\not k_1\right]\\ &= \frac{32Q_q^2g^2e^4}{3s^2}(p_1\cdot p_2)(k_1\cdot k_3)(k_2\cdot k_3)\left[\frac{1}{(k_1+k_3)^2}+\frac{1}{(k_2+k_3)^2}\right]^2.\end{aligned} \tag{17.8}$$

Rewrite this in terms of x_q, $x_{\bar q}$ and x_3, we get

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 &= \frac{4Q_q^2g^2e^4}{3s^2}(1-x_q)(1-x_{\bar q})\left[\frac{1}{1-x_q}+\frac{1}{1-x_{\bar q}}\right]^2\\ &= \frac{4Q_q^2g^2e^4}{3s^2}\frac{x_3^2}{(1-x_q)(1-x_{\bar q})}.\end{aligned} \tag{17.9}$$

Note the phase space integral for 3-body final state is deduced in Final Project 1 to be

$$\int \mathrm{d}\Pi_3 = \frac{s}{128\pi^3}\int \mathrm{d}x_q\mathrm{d}x_{\bar q},$$

thus the differential cross section is given by

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_1\mathrm{d}x_2}(e^+e^-\to q\bar qS) = \frac{s}{128\pi^3}\cdot\tfrac{1}{4}|\mathcal{M}|^2 = \frac{4\pi\alpha^2Q_q^2}{3s}\cdot\frac{\alpha_g}{4\pi}\frac{x_3^2}{(1-x_q)(1-x_{\bar q})}. \tag{17.10}$$

(b) Now let $x_a > x_b > x_c$. Then there are six ways to associated the original three variables x_q, $x_{\bar q}$ and x_3 to these three ordered ones. Note that the integral measure $\mathrm{d}x_a\mathrm{d}x_b$ does not change for different possibilities since the change of integral variables $(x_q, x_{\bar q}) \to (x_q, x_3)$ or $\to (x_{\bar q}, x_3)$ generate an Jacobian whose absolute value is 1, due to the constraint $x_q + x_{\bar q} + x_3 = 2$. Therefore, summing up all 6 possibilities, we get

$$\begin{aligned}&\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_a\mathrm{d}x_b}(e^+e^-\to q\bar qS)\\ &\propto \frac{x_c^2}{(1-x_a)(1-x_b)}+\frac{x_b^2}{(1-x_c)(1-x_a)}+\frac{x_a^2}{(1-x_b)(1-x_c)},\end{aligned} \tag{17.11}$$

for $q\bar qS$ final state, and

$$\begin{aligned}&\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_a\mathrm{d}x_b}(e^+e^-\to q\bar qS)\\ &\propto \frac{x_a^2+x_b^2}{(1-x_a)(1-x_b)}+\frac{x_b^2+x_c^2}{(1-x_b)(1-x_c)}+\frac{x_c^2+x_a^2}{(1-x_c)(1-x_a)},\end{aligned} \tag{17.12}$$

We plot these two distributions on the $x_a - x_b$ plain with the range $x_a > x_b > x_c$, as shown in Figure

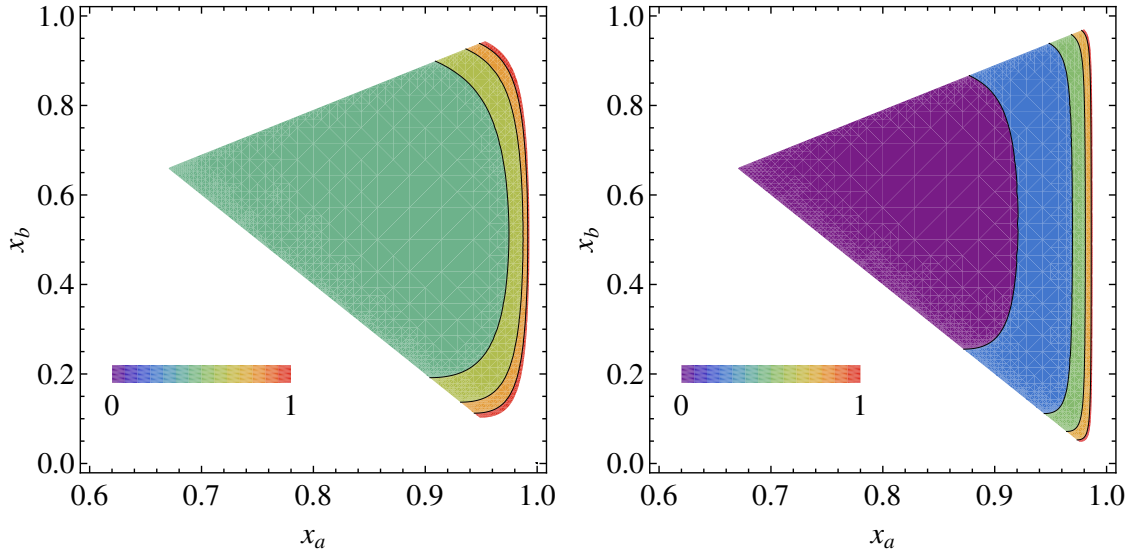


Figure 17.1: The differential cross sections of $e^+e^- \to q\bar{q}g$ as a function of x_1 and x_2, assuming gluon is a vector/scalar particle in left/right diagram.

17.3 Quark-gluon and gluon-gluon scattering

In this problem we evaluate the cross sections for two processes: (a) $q\bar{q} \to gg$, (b) $gg \to gg$.

(a) There are three diagrams contributing the process $q(k_1)\bar{q}(k_2) \to g(p_1)g(p_2)$ at the tree level, as shown in Fig. 17.11 of Peskin&Schroeder. The amplitudes associated with these diagrams are listed as follows:

$$\mathrm{i}\mathcal{M}_1 = (\mathrm{i}g)^2 \bar{v}(k_2) \not{\epsilon}^*(p_2) \frac{\mathrm{i}(\not{k}_1 - \not{p}_1)}{(k_1 - p_1)^2} \not{\epsilon}^*(p_1) u(k_1) t^b t^a, \tag{17.13a}$$

$$\mathrm{i}\mathcal{M}_2 = (\mathrm{i}g)^2 \bar{v}(k_2) \not{\epsilon}^*(p_1) \frac{\mathrm{i}(\not{k}_1 - \not{p}_2)}{(k_1 - p_2)^2} \not{\epsilon}^*(p_2) u(k_1) t^a t^b, \tag{17.13b}$$

$$\begin{aligned}\mathrm{i}\mathcal{M}_3 = {} & (\mathrm{i}g) f^{abc} \left[g^{\mu\nu}(p_2 - p_1)^\rho - g^{\nu\rho}(2p_2 + p_1)^\mu + g^{\rho\mu}(p_2 + 2p_1)^\nu \right] \\ & \times \frac{-\mathrm{i}}{(k_1 + k_2)^2} \bar{v}(k_2) \gamma_\rho u(k_1) \epsilon^*_\mu(p_1) \epsilon^*_\nu(p_2) t^c.\end{aligned} \tag{17.13c}$$

It is convenient to evaluate these diagrams with initial and final states of definite helicities. By P and CP symmetry of QCD, there are only two independent processes, namely $q_L\bar{q}_R \to g_R g_R$ and $q_L\bar{q}_R \to g_R g_L$, that could be nonzero. Let's evaluate them in turn for the three diagrams. To begin with, we set up the kinematics:

$$\begin{aligned} k_1^\mu &= (E, 0, 0, E), & p_1^\mu &= (E, E\sin\theta, 0, E\cos\theta), \\ k_2^\mu &= (E, 0, 0, -E), & p_2^\mu &= (E, -E\sin\theta, 0, -E\cos\theta). \end{aligned} \tag{17.14}$$

Then,

$$u_L(k_1) = \sqrt{2E}(0, 1, 0, 0), \qquad v_L(k_2) = \sqrt{2E}(1, 0, 0, 0).$$

$$\epsilon^*_{L\mu}(p_1) = \tfrac{1}{\sqrt{2}}(0, -\cos\theta, -\mathrm{i}, \sin\theta), \qquad \epsilon^*_{R\mu}(p_1) = \tfrac{1}{\sqrt{2}}(0, -\cos\theta, \mathrm{i}, \sin\theta),$$
$$\epsilon^*_{L\mu}(p_2) = \tfrac{1}{\sqrt{2}}(0, \cos\theta, -\mathrm{i}, -\sin\theta), \qquad \epsilon^*_{R\mu}(p_2) = \tfrac{1}{\sqrt{2}}(0, \cos\theta, \mathrm{i}, -\sin\theta). \tag{17.15}$$

Now we begin the calculation. (In the following we use $s_\theta \equiv \sin\theta$ and $c_\theta = \cos\theta$.)

$$\begin{aligned}
\mathrm{i}\mathcal{M}_1(q_L\bar{q}_R \to g_R g_R) &= \frac{-\mathrm{i}g^2E^2t^bt^a}{t}(0,0,1,0)\begin{pmatrix} & & -s_\theta & 1+c_\theta \\ & & -1+c_\theta & s_\theta \\ s_\theta & -1-c_\theta & & \\ 1-c_\theta & -s_\theta & & \end{pmatrix} \\
&\times \begin{pmatrix} & & 1-c_\theta & -s_\theta \\ & & -s_\theta & -1+c_\theta \\ -1+c_\theta & s_\theta & & \\ s_\theta & 1-c_\theta & & \end{pmatrix}\begin{pmatrix} & & s_\theta & 1-c_\theta \\ & & -1-c_\theta & -s_\theta \\ -s_\theta & -1+c_\theta & & \\ 1+c_\theta & s_\theta & & \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
&= \mathrm{i}g^2t^bt^a\frac{2E^2}{t}(1-\cos\theta)\sin\theta = -\mathrm{i}g^2t^bt^a\sin\theta.
\end{aligned} \tag{17.16}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_2(q_L\bar{q}_R \to g_R g_R) &= \frac{-\mathrm{i}g^2E^2t^at^b}{u}(0,0,1,0)\begin{pmatrix} & & s_\theta & 1-c_\theta \\ & & -1-c_\theta & -s_\theta \\ -s_\theta & -1+c_\theta & & \\ 1+c_\theta & s_\theta & & \end{pmatrix} \\
&\times \begin{pmatrix} & & 1+c_\theta & s_\theta \\ & & s_\theta & -1-c_\theta \\ -1-c_\theta & -s_\theta & & \\ -s_\theta & 1+c_\theta & & \end{pmatrix}\begin{pmatrix} & & -s_\theta & 1+c_\theta \\ & & -1+c_\theta & s_\theta \\ s_\theta & -1-c_\theta & & \\ 1-c_\theta & -s_\theta & & \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
&= -\mathrm{i}g^2t^bt^a\frac{2E^2}{u}(1+\cos\theta)\sin\theta = \mathrm{i}g^2t^at^b\sin\theta.
\end{aligned} \tag{17.17}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_3(q_L\bar{q}_R \to g_R g_R) &= \frac{g^2f^{abc}t^cE^2}{s}(0,0,1,0)\left[-4\begin{pmatrix} & & c_\theta & s_\theta \\ & & s_\theta & -c_\theta \\ -c_\theta & -s_\theta & & \\ -s_\theta & c_\theta & & \end{pmatrix}\right]\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
&= g^2f^{abc}t^c\sin\theta = -\mathrm{i}g^2[t^a,t^b]\sin\theta.
\end{aligned} \tag{17.18}$$

Thus we find that

$$\mathrm{i}\mathcal{M}(q_L\bar{q}_R \to g_R g_R) = \big(\mathrm{i}\mathcal{M}_1 + \mathrm{i}\mathcal{M}_2 + \mathrm{i}\mathcal{M}_3\big)(q_L\bar{q}_R \to g_R g_R) = 0. \tag{17.19}$$

In the same manner, we calculate the amplitude for $q_L\bar{q}_R \to g_R g_L$. This time, we find:

$$\mathrm{i}\mathcal{M}_1(q_L\bar{q}_R \to g_R g_L) = -\,\mathrm{i}g^2t^bt^a\sin\theta, \tag{17.20a}$$
$$\mathrm{i}\mathcal{M}_2(q_L\bar{q}_R \to g_R g_L) = -\,\mathrm{i}g^2t^at^b\frac{t}{u}\sin\theta, \tag{17.20b}$$

$$\mathrm{i}\mathcal{M}_3(q_L\bar{q}_R \to g_R g_L) = 0, \tag{17.20c}$$

Therefore,

$$\mathrm{i}\mathcal{M}(q_L\bar{q}_R \to g_R g_L) = -\mathrm{i}g^2\Big(t^b t^a + t^a t^b \frac{t}{u}\Big)\sin\theta, \tag{17.21}$$

and by crossing symmetry,

$$\mathrm{i}\mathcal{M}(q_L\bar{q}_R \to g_L g_R) = -\mathrm{i}g^2\Big(t^b t^a + t^a t^b \frac{u}{t}\Big)\sin\theta. \tag{17.22}$$

There are two more nonzero amplitudes with $q_R\bar{q}_L$ initial states, which are identical to the amplitudes above. Then we find the spin- and color-summed/averaged squared amplitude to be

$$\begin{aligned}
&\frac{1}{3^2}\cdot\frac{1}{2^2}\sum_{\text{spin,color}}|\mathcal{M}|^2 \\
&= \frac{1}{36}\cdot 2\cdot g^4\sin^2\theta\bigg[\Big(\operatorname{tr}(t^b t^a t^a t^b) + 2\operatorname{tr}(t^b t^a t^b t^a)\frac{t}{u} + \operatorname{tr}(t^a t^b t^b t^a)\frac{t^2}{u^2}\Big) + (t\leftrightarrow u)\bigg] \\
&= \frac{8\pi^2\alpha_s^2}{9}(1-\cos^2\theta)\bigg[\Big(\frac{16}{3}\Big(1+\frac{t^2}{u^2}\Big) - \frac{4t}{3u}\Big) + (t\leftrightarrow u)\bigg] \\
&= \frac{512\pi^2\alpha_s^2}{27}\bigg[\frac{t}{u}+\frac{u}{t}-\frac{9(t^2+u^2)}{4s^2}\bigg].
\end{aligned} \tag{17.23}$$

Therefore the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{32\pi\alpha_s^2}{27s^2}\bigg[\frac{t}{u}+\frac{u}{t}-\frac{9(t^2+u^2)}{4s^2}\bigg]. \tag{17.24}$$

(b) Now consider the process $g(k_1)g(k_2)\to g(p_1)g(p_2)$. The four tree level diagrams are shown in Fig. 17.12 of Peskin&Schroeder. Their amplitudes are given by:

$$\begin{aligned}
\mathrm{i}\mathcal{M}_1 = {}& g^2 f^{abc}f^{cde}\frac{-\mathrm{i}}{s}\big[g^{\mu\nu}(k_1-k_2)^\lambda + g^{\nu\lambda}(k_1+2k_2)^\mu - g^{\lambda\mu}(2k_1+k_2)^\nu\big] \\
&\times\big[g^{\rho\sigma}(p_2-p_1)_\lambda - g^\sigma_\lambda(p_1+2p_2)^\rho + g^\rho_\lambda(p_1+2p_2)^\rho\big]\epsilon_\mu(k_1)\epsilon_\nu(k_2)\epsilon^*_\rho(p_1)\epsilon^*_\sigma(p_2),
\end{aligned} \tag{17.25a}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_2 = {}& g^2 f^{ace}f^{bde}\frac{-\mathrm{i}}{t}\big[g^{\mu\rho}(k_1+p_1)^\lambda - g^{\rho\lambda}(2p_1-k_1)^\mu - g^{\lambda\nu}(2k_1-p_1)^\rho\big] \\
&\times\big[g^{\nu\sigma}(k_2+p_2)_\lambda - g^\sigma_\lambda(2p_2-k_2)^\nu + g^\nu_\lambda(p_2-2k_2)^\sigma\big]\epsilon_\mu(k_1)\epsilon_\nu(k_2)\epsilon^*_\rho(p_1)\epsilon^*_\sigma(p_2),
\end{aligned} \tag{17.25b}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_3 = {}& g^2 f^{ade}f^{bce}\frac{-\mathrm{i}}{u}\big[g^{\mu\sigma}(k_1+p_2)^\lambda - g^{\sigma\lambda}(2p_2-k_1)^\mu - g^{\lambda\nu}(2k_1-p_2)^\sigma\big] \\
&\times\big[g^{\nu\rho}(k_2+p_1)_\lambda - g^\rho_\lambda(2p_1-k_2)^\nu + g^\nu_\lambda(p_1-2k_2)^\rho\big]\epsilon_\mu(k_1)\epsilon_\nu(k_2)\epsilon^*_\rho(p_1)\epsilon^*_\sigma(p_2),
\end{aligned} \tag{17.25c}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_4 = {}& -\mathrm{i}g^2\big[f^{abe}f^{cde}\big(\epsilon(k_1)\cdot\epsilon^*(p_1)\epsilon(k_2)\cdot\epsilon^*(p_2) - \epsilon(k_1)\cdot\epsilon^*(p_2)\epsilon(k_2)\cdot\epsilon^*(p_1)\big) \\
&+ f^{ace}f^{bde}\big(\epsilon(k_1)\cdot\epsilon(k_2)\epsilon^*(k_1)\cdot\epsilon^*(p_2) - \epsilon(k_1)\cdot\epsilon^*(p_2)\epsilon(k_2)\cdot\epsilon^*(p_1)\big) \\
&+ f^{ade}f^{bce}\big(\epsilon(k_1)\cdot\epsilon(k_2)\epsilon^*(p_1)\cdot\epsilon^*(p_2) - \epsilon(k_1)\cdot\epsilon^*(p_1)\epsilon(k_2)\cdot\epsilon^*(p_2)\big)\big].
\end{aligned} \tag{17.25d}$$

The choice for all external momenta and final states polarizations are the same with that in (a). Now to evaluate the amplitude $g_R g_R \to g_R g_R$, we also need the initial states polarization vectors for right-handed gluons with momenta k_1 and k_2, which are given by

$$\epsilon_R^\mu(k_1) = \tfrac{1}{\sqrt{2}}(0,1,\mathrm{i},0), \qquad \epsilon_R^\mu(k_2) = \tfrac{1}{\sqrt{2}}(0,-1,\imath,0). \tag{17.26}$$

Then after some calculations, we find,

$$\mathrm{i}\mathcal{M}_1 = -\,\mathrm{i}g^2 f^{abe} f^{cde} \cos\theta, \tag{17.27a}$$

$$\mathrm{i}\mathcal{M}_2 = \mathrm{i}g^2 f^{ace} f^{bde} \frac{19 + 7\cos\theta - 11\cos^2\theta + \cos^3\theta}{4(1-\cos\theta)}; \tag{17.27b}$$

$$\mathrm{i}\mathcal{M}_3 = \mathrm{i}g^2 f^{ade} f^{bce} \frac{19 - 7\cos\theta - 11\cos^2\theta - \cos^3\theta}{4(1+\cos\theta)}; \tag{17.27c}$$

$$\begin{aligned}\mathrm{i}\mathcal{M}_4 = -\,\mathrm{i}g^2\Big[& f^{abe} f^{cde}\cos\theta + \tfrac{1}{4} f^{ace} f^{bde}(3 + 2\cos\theta - \cos^2\theta) \\ & + \tfrac{1}{4} f^{ade} f^{bce}(3 - 2\cos\theta - \cos^2)\Big].\end{aligned} \tag{17.27d}$$

The sum of these four amplitudes is

$$\begin{aligned}\mathrm{i}\mathcal{M}(g_R g_R \to g_R g_R) &= -\,2\mathrm{i}g^2\Big[f^{abe} f^{cde}\cos\theta - f^{ace} f^{bde}\Big(\frac{2}{1-\cos\theta} + \cos\theta\Big) \\ &\qquad - f^{ade} f^{bce}\Big(\frac{2}{1+\cos\theta} - \cos\theta\Big)\Big] \\ &= 4\mathrm{i}g^2\Big[f^{ace} f^{bde}\frac{1}{1-\cos\theta} + f^{ade} f^{bce}\frac{1}{1+\cos\theta}\Big] \\ &= -\,2\mathrm{i}g^2\Big[f^{ace} f^{bde}\frac{s}{t} + f^{ade} f^{bce}\frac{s}{u}\Big].\end{aligned} \tag{17.28}$$

We can also obtain the amplitudes for $g_L g_R \to g_L g_R$ and $g_L g_R \to g_R g_L$ from the result above by crossing symmetry, namely the change of variables $(s,b) \leftrightarrow (u,d)$ and $(s,b) \leftrightarrow (t,c)$, which gives

$$\mathrm{i}\mathcal{M}(g_L g_R \to g_L g_R) = 2\mathrm{i}g^2\Big[f^{ace} f^{bde}\frac{u}{t} + f^{abe} f^{cde}\frac{u}{s}\Big], \tag{17.29}$$

$$\mathrm{i}\mathcal{M}(g_L g_R \to g_R g_L) = -\,2\mathrm{i}g^2\Big[f^{abe} f^{cde}\frac{t}{s} - f^{ade} f^{bce}\frac{t}{u}\Big]. \tag{17.30}$$

The amplitudes for $g_L g_L \to g_L g_L$, $g_R g_L \to g_R g_L$ and $g_R g_L \to g_L g_R$ are identical to the amplitudes for $g_R g_R \to g_R g_R$, $g_L g_R \to g_L g_R$ and $g_L g_R \to g_R g_L$, respectively, due to parity conservation of QCD. It can be shown by the conservation of angular momentum that other helicity amplitudes all vanish. Therefore we have found all required amplitude. To get the cross section, we take the square of these results.

$$\begin{aligned}&\sum |\mathcal{M}(g_R g_R \to g_R g_R)|^2 \\ &= 4g^4\Big[f^{ace} f^{bde} f^{acf} f^{bdf}\frac{s^2}{t^2} + f^{ade} f^{bce} f^{adf} f^{bcf}\frac{s^2}{u^2} + 2 f^{ace} f^{bde} f^{adf} f^{bcf}\frac{s^2}{tu}\Big] \\ &= 4g^4\Big[\operatorname{tr}(t^a t^a t^b t^b)\Big(\frac{s^2}{t^2} + \frac{s^2}{u^2}\Big) + 2\operatorname{tr}(t^a t^b t^a t^b)\frac{s^2}{tu}\Big]\end{aligned}$$

$$= 288g^4\Big(\frac{s^2}{t^2}+\frac{s^2}{u^2}+\frac{s^2}{tu}\Big), \tag{17.31}$$

where t^a is the generator of $SU(3)$ group in adjoint representation which is related to the structure constants by $f^{abc} = \mathrm{i}(t^a)_{bc}$. Thus $\mathrm{tr}\,(t^a t^a t^b t^b) = (C_2(G))^2 d(G) = 72$, and

$$\begin{aligned}\mathrm{tr}\,(t^a t^b t^a t^b) =&\ \mathrm{tr}\,(t^a t^b [t^a, t^b]) + \mathrm{tr}\,(t^a t^a t^b t^b) = \tfrac{1}{2}\mathrm{i} f^{abc}\,\mathrm{tr}\,([t^a, t^b]t^c) + (C_2(G))^2 d(G)\\ =&-\tfrac{1}{2} f^{abc} f^{abd}\,\mathrm{tr}\,(t^c t^d) + (C_2(G))^2 d(G) = \tfrac{1}{2}(C_2(G))^2 d(G),\end{aligned}$$

which is 36 for $SU(3)$. Similarly, we can work out the square of other amplitudes, to be

$$\sum |\mathcal{M}(g_L g_R \to g_L g_R)|^2 = 288g^4\Big(\frac{u^2}{t^2}+\frac{u^2}{s^2}+\frac{u^2}{st}\Big), \tag{17.32}$$

$$\sum |\mathcal{M}(g_L g_R \to g_R g_L)|^2 = 288g^4\Big(\frac{t^2}{s^2}+\frac{t^2}{u^2}+\frac{t^2}{su}\Big). \tag{17.33}$$

Therefore, the spin-averaged and squared amplitudes is

$$\begin{aligned}&\frac{1}{8^2}\cdot\frac{1}{2^2}\sum|\mathcal{M}^2| = \frac{1}{8^2\cdot 2^2}\cdot 2\cdot 288g^4\Big(6-\frac{2tu}{s^2}-\frac{2us}{t^2}-\frac{2st}{u^2}\Big)\\ &= 72\pi^2\alpha_s^2\Big(3-\frac{tu}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\Big).\end{aligned} \tag{17.34}$$

Thus the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(gg\to gg) = \frac{9\pi\alpha_s^2}{2s^2}\Big(3-\frac{tu}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\Big). \tag{17.35}$$

17.4 The gluon splitting function

In this problem we calculate the gluon splitting function $P_{g\leftarrow g}(z)$ by evaluating the amplitude of the virtual process $g \to gg$, as shown in Fig. 17.2.

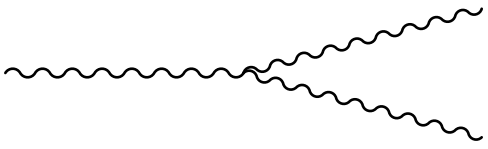


Figure 17.2: The Gluon splitting process.

The momenta of initial and final states are taken to be the same with that of Fig. 17.16 of Peskin&Schroeder. That is, we have

$$p = (p, 0, 0, p), \qquad q = (zp, p_\perp, 0, zp), \qquad k = ((1-z)p, -p_\perp, 0, (1-z)p), \tag{17.36}$$

and the polarization vectors associated with gluons are,

$$\begin{aligned}\epsilon_L^i(p) &= \tfrac{1}{\sqrt{2}}(1, -\mathrm{i}, 0), & \epsilon_R^i(p) &= \tfrac{1}{\sqrt{2}}(1, \mathrm{i}, 0),\\ \epsilon_L^i(q) &= \tfrac{1}{\sqrt{2}}(1, -\mathrm{i}, -\frac{p_\perp}{zp}), & \epsilon_R^i(q) &= \tfrac{1}{\sqrt{2}}(1, \mathrm{i}, -\frac{p_\perp}{zp}),\end{aligned}$$

$$\epsilon_L^i(k) = \tfrac{1}{\sqrt{2}}(1, -\mathrm{i}, \frac{p_\perp}{(1-z)p}), \qquad \epsilon_R^i(k) = \tfrac{1}{\sqrt{2}}(1, \mathrm{i}, \frac{p_\perp}{(1-z)p}). \tag{17.37}$$

Then we can evaluate the amplitude for the process $g \to gg$ directly, which is given by

$$\begin{aligned}\mathrm{i}\mathcal{M}^{abc} = gf^{abc}\big[&\big(\epsilon^*(q)\cdot\epsilon(p)\big)\big((p+q)\cdot\epsilon^*(k)\big) + \big(\epsilon^*(q)\cdot\epsilon^*(k)\big)\big((k-q)\cdot\epsilon(p)\big)\\ &- \big(\epsilon^*(k)\cdot\epsilon(p)\big)\big((p+k)\cdot\epsilon^*(q)\big)\big].\end{aligned} \tag{17.38}$$

We evaluate the amplitudes with definite initial and final polarizations in turn:

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p) \to g_L(q)g_L(k)) = \sqrt{2}\Big(\frac{1}{1-z} + \frac{1}{z}\Big)gf^{abc}p_\perp, \tag{17.39a}$$

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p) \to g_L(q)g_R(k)) = \frac{\sqrt{2}z}{1-z}gf^{abc}p_\perp, \tag{17.39b}$$

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p) \to g_R(q)g_L(k)) = \frac{\sqrt{2}(1-z)}{z}gf^{abc}p_\perp, \tag{17.39c}$$

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p) \to g_R(q)g_R(k)) = 0. \tag{17.39d}$$

By parity invariance, the amplitudes with right-handed initial gluon are dictated by the results above. Note further that $f^{abc}f^{abc} = 24$, thus we have

$$\begin{aligned}\tfrac{1}{2}\cdot\tfrac{1}{8}\sum_{\text{spin,color}}|\mathcal{M}|^2 &= \tfrac{1}{2}\cdot\tfrac{1}{8}\cdot 2\cdot 24\cdot 2g^2p_\perp^2\\ &\quad\times\Big[\Big(\frac{1}{1-z}+\frac{1}{z}\Big)^2 + \frac{z^2}{(1-z)^2} + \frac{(1-z)^2}{z^2}\Big]\\ &= \frac{12g^2p_\perp^2}{z(1-z)}\Big[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\Big]\\ &= \frac{2e^2p_\perp^2}{z(1-z)}\cdot P_{g\leftarrow g}^{(0)}(z),\end{aligned} \tag{17.40}$$

where the superscript (1) represents the part of the splitting function contributed from the diagram calculated above, in parallel with the notation of Peskin&Schroeder. (See 17.100. for instance.) Therefore we get

$$P_{g\leftarrow g}^{(1)} = 6\Big[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\Big]. \tag{17.41}$$

Besides, there should be a term proportional to $\delta(1-z)$ in $P_{g\leftarrow g}$, which comes from the zeroth order, as well as the corrections from $P_{q\leftarrow g}$ and $P_{g\leftarrow g}$, where $P_{q\leftarrow g}(z) = \frac{1}{2}(z^2 + (1-z)^2)$. Now let's take it to be $A\delta(1-z)$, then the coefficient A can be determined by the following normalization condition (namely the momentum conservation):

$$1 = \int_0^1 \mathrm{d}z\, z\big[2n_f P_{q\leftarrow g}(z) + P_{g\leftarrow g}^{(1)}(z) + A\delta(1-z)\big], \tag{17.42}$$

where n_f is the number of fermion types, and the coefficient 2 is from contributions of both quarks and anti-quarks. To carry out the integral, we use the prescription $\frac{1}{1-z} \to \frac{1}{(1-z)_+}$, then it is straightforward to find that $A = \frac{11}{2} - \frac{1}{3}n_f$. Therefore,

$$P_{g\leftarrow g} = 6\Big[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\Big] + \Big(\frac{11}{2} - \frac{n_f}{3}\Big)\delta(1-z). \tag{17.43}$$

17.5 Photoproduction of heavy quarks

In this problem we study the production of a pair of heavy quark-antiquark by the scattering of a photon off a proton. At the leading order at the parton level, the process is contributed from the photon-gluon scattering, as shown in Figure 17.3.

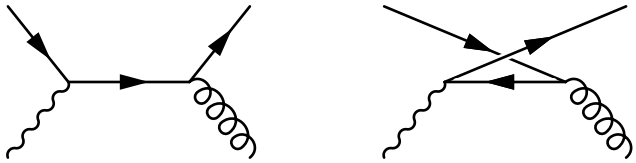


Figure 17.3: Tree diagrams for the photoproduction of heavy quarks at the parton level.

The corresponding amplitude can be read from a similar process $\gamma\gamma \to e^+e^-$ in QED. From (5.105) of Peskin & Schroeder, we have the amplitude for $e^+e^- \to 2\gamma$, which reads (adapted to our notation for external momenta)

$$\frac{1}{4}\sum |\mathcal{M}(e^+e^- \to 2\gamma)|^2 = 2e^4\bigg[\frac{k_1\cdot p_2}{k_1\cdot p_1} + \frac{k_1\cdot p_1}{k_1\cdot p_2} + 2m^2\Big(\frac{1}{k_1\cdot p_1} + \frac{1}{k_1\cdot p_2}\Big) - m^4\Big(\frac{1}{k_1\cdot p_1} + \frac{1}{k_1\cdot p_2}\Big)^2\bigg]. \tag{17.44}$$

Then the amplitude $\mathcal{M}(\gamma g \to Q\bar{Q})$ can be obtained by making the exchange $(k_1, k_2) \leftrightarrow (p_1, p2)$, replacing e^4 by e^2g^2, and also including the factor $\frac{1}{8}Q_q^2\,\mathrm{tr}\,(t^at^a) = \frac{1}{2}Q_q^2$ taking account of the color average, the electric charge of quarks, and the summation of color indices, respectively. Then the amplitude in the present case is

$$\frac{1}{4\cdot 8}\sum |\mathcal{M}(\gamma g \to Q\bar{Q})|^2 = e^2g^2Q_q^2\bigg[\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} + 2m^2\Big(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\Big) - m^4\Big(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\Big)^2\bigg]. \tag{17.45}$$

In parton's center-of-mass frame, we have $k_1 = (E, 0, 0, E)$, $k_2 = (E, 0, 0, -E)$, $p_1 = (E, p\sin\theta, 0, p\cos\theta)$ and $p_2 = (E, -p\sin\theta, 0, -p\cos\theta)$, with $p^2 = E^2 - m^2$. Then $p_1\cdot k_1 = E(E - p\cos\theta)$ and $p_1\cdot k_2 = E(E + p\cos\theta)$. Then the differential cross section is

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta} = \frac{\pi\alpha\alpha_s Q_q^2}{16}\frac{p}{E^3}\bigg[\frac{E^2 + p^2\cos^2\theta - 2m^2}{E^2 - p^2\cos^2\theta} - \frac{2m^4}{(E^2 - p^2\cos^2\theta)^2}\bigg]. \tag{17.46}$$

Then the cross section for photon and proton initial state is given by

$$\sigma(\gamma(k_1) + p(k_2) \to Q\bar{Q}) = \int \mathrm{d}x\, f_g(x)\hat{\sigma}(\gamma(k_1) + g(xk_2) \to Q\bar{Q}). \tag{17.47}$$

17.6 Behavior of parton distribution functions at small x

(a) In this problem we study the solution of A-P equations at small x with certain approximations. Firstly, we show that the A-P equations,

$$\frac{\mathrm{d}}{\mathrm{d}\log Q} f_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{g\leftarrow q}(z)\sum_f\left(f_f\Big(\frac{x}{z},Q\Big)+f_{\bar{f}}\Big(\frac{x}{z},Q\Big)\right) + P_{g\leftarrow g}(z) f_g\Big(\frac{x}{z},Q\Big)\right], \tag{17.48}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q} f_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z) f_f\Big(\frac{x}{z},Q\Big)+P_{q\leftarrow g}(z) f_g\Big(\frac{x}{z},Q\Big)\right], \tag{17.49}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q} f_{\bar{f}}(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z) f_{\bar{f}}\Big(\frac{x}{z},Q\Big)+P_{q\leftarrow g}(z) f_g\Big(\frac{x}{z},Q\Big)\right], \tag{17.50}$$

can be rewritten as a differential equation with variable $\xi = \log\log(Q^2/\Lambda^2)$. To see this, we note that $\mathrm{d}/\mathrm{d}\log Q = 2e^{-\xi}\mathrm{d}/\mathrm{d}\xi$, and to 1-loop order, $\alpha_s(Q) = 2\pi/\big(b_0\log(Q/\Lambda)\big) = (4\pi/b_0)e^{-\xi}$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi} f_g(x,\xi) = \frac{2}{b_0}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{g\leftarrow q}(z)\sum_f\left(f_f\Big(\frac{x}{z},\xi\Big)+f_{\bar{f}}\Big(\frac{x}{z},\xi\Big)\right) + P_{g\leftarrow g}(z) f_g\Big(\frac{x}{z},\xi\Big)\right], \tag{17.51}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} f_f(x,\xi) = \frac{2}{b_0}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z) f_f\Big(\frac{x}{z},\xi\Big)+P_{q\leftarrow g}(z) f_g\Big(\frac{x}{z},\xi\Big)\right], \tag{17.52}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} f_{\bar{f}}(x,\xi) = \frac{2}{b_0}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z) f_{\bar{f}}\Big(\frac{x}{z},\xi\Big)+P_{q\leftarrow g}(z) f_g\Big(\frac{x}{z},\xi\Big)\right]. \tag{17.53}$$

(b) Now we apply the approximation that 1) gluon PDF dominates the integrand in the A-P equations and 2) the function $\tilde{g}(x,Q) = xf_g(x,Q)$ is slowly varying with x when x is small. Then, define $w = \log(1/x)$, which gives $\mathrm{d}/\mathrm{d}w = -x\mathrm{d}/\mathrm{d}x$, we can calculate

$$\begin{aligned}\frac{\partial^2}{\partial w\partial\xi}\tilde{g}(x,\xi) &= -x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\partial}{\partial\xi} f_g(x,Q)\right)\\ &\simeq -x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{b_0}\int_x^1\frac{\mathrm{d}z}{z}P_{g\leftarrow g}(z) f_g\Big(\frac{x}{z},Q\Big)\right)\\ &= \frac{2}{b_0}\cdot xP_{g\leftarrow g}(x) f_g(x,Q) - \frac{2x}{b_0}\int_x^1 \mathrm{d}z P_{g\leftarrow g}(z)\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{x}{z} f_g\Big(\frac{x}{z},Q\Big)\Big]\\ &\simeq \frac{2}{b_0}\cdot xP_{g\leftarrow g}(x) f_g(x,Q).\end{aligned} \tag{17.54}$$

From the result of Problem 17.4 we know that $xP_{g\leftarrow g}(x) = 6$ as $x\to 0$. Therefore the A-P equation for f_g becomes

$$\frac{\partial^2}{\partial w\partial\xi}\tilde{g}(x,\xi) = \frac{12}{b_0}\tilde{g}(x,\xi). \tag{17.55}$$

Then we verify that

$$\tilde{g} = K(Q^2) \cdot \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) \tag{17.56}$$

is an approximation solution to the differential equation above when $w\xi \gg 1$, where $K(Q^2)$ is an initial condition. We apply $\partial^2/\partial w \partial \xi$ on this expression, to get

$$\begin{aligned}\frac{\partial^2}{\partial w \partial \xi}\tilde{g}(w,\xi) = {} & \frac{1}{4}\sqrt{\frac{48}{b_0 w(\xi-\xi_0)}} \cdot \exp\left(\left[\frac{48}{b_0} w(\xi-\xi_0)\right]^{1/2}\right) \\ & \times \left[2(\xi-\xi_0)\frac{\partial K(Q^2)}{\partial \xi} + \left(1 + \sqrt{\frac{48}{b_0} w(\xi-\xi_0)}\right) K(Q^2)\right].\end{aligned} \tag{17.57}$$

In the limit $w\xi \gg 1$, the square root term in the last line dominates, thus

$$\frac{\partial^2}{\partial w \partial \xi}\tilde{g}(w,\xi) \simeq \frac{b_0}{12} K(Q^2) \exp\left(\left[\frac{48}{b_0} w(\xi-\xi_0)\right]^{1/2}\right) = \frac{12}{b_0}\tilde{g}(w,\xi). \tag{17.58}$$

(c) Then we consider the A-P equation for quarks. If we adopt the approximation in (b) again, namely, the gluon PDF dominates and the function $\tilde{q}(x,\xi) = x f_f(x,Q)$ is slowly varying, then we have

$$\begin{aligned}\frac{\partial}{\partial \xi}\tilde{q}(x,\xi) = x\frac{\partial}{\partial \xi} f_f(x,\xi) &= \frac{2x}{b_0}\int_x^1 \frac{\mathrm{d}z}{z} P_{q\leftarrow g}(z) f_g\Big(\frac{x}{z},\xi\Big) \\ &= \frac{2}{b_0}\int_x^1 \mathrm{d}z P_{q\leftarrow g}(z)\tilde{g}\Big(\frac{x}{z},\xi\Big) = \frac{1}{b_0}\int_x^1 \mathrm{d}z\big(z^2+(1-z)^2\big)\tilde{g}\Big(\frac{x}{z},\xi\Big) \\ &\simeq \frac{2}{b_0}\left[\frac{1}{6}(2z^3-2z^2+3z)\tilde{g}\Big(\frac{x}{z},\xi\Big)\right]_x^1 \simeq \frac{2}{3b_0}\tilde{g}(x,\xi),\end{aligned} \tag{17.59}$$

where we have used $x \ll 1$ and $\partial\tilde{q}(x,\xi)/\partial x \simeq 0$. Then, we verify that

$$\tilde{q} = \sqrt{\frac{\xi-\xi_0}{27 b_o w}} K(Q^2) \cdot \exp\left(\left[\frac{48}{b_0} w(\xi-\xi_0)\right]^{1/2}\right) \tag{17.60}$$

is again an approximate solution to the equation derived above, in the limit $w\xi \gg 1$. In fact,

$$\begin{aligned}\frac{\partial}{\partial \xi}\tilde{q}(x,\xi) = {} & \exp\left(\left[\frac{48}{b_0} w(\xi-\xi_0)\right]^{1/2}\right)\left[\frac{2}{3b_0}K(Q^2)\right. \\ & \left. + \frac{1}{18}\sqrt{\frac{3}{b_0 w(\xi-\xi_0)}}\Big(K(Q^2) + 2(\xi-\xi_0)\frac{\partial K(Q^2)}{\partial \xi}\Big)\right] \\ \simeq {} & \frac{2}{3b_0}K(Q^2)\exp\left(\left[\frac{48}{b_0} w(\xi-\xi_0)\right]^{1/2}\right) = \frac{2}{3b_0}\tilde{g}(x,\xi).\end{aligned} \tag{17.61}$$

(d) We use the fitted formula of $K(Q^2)$ to plot the PDFs of gluon and quarks in Figure.

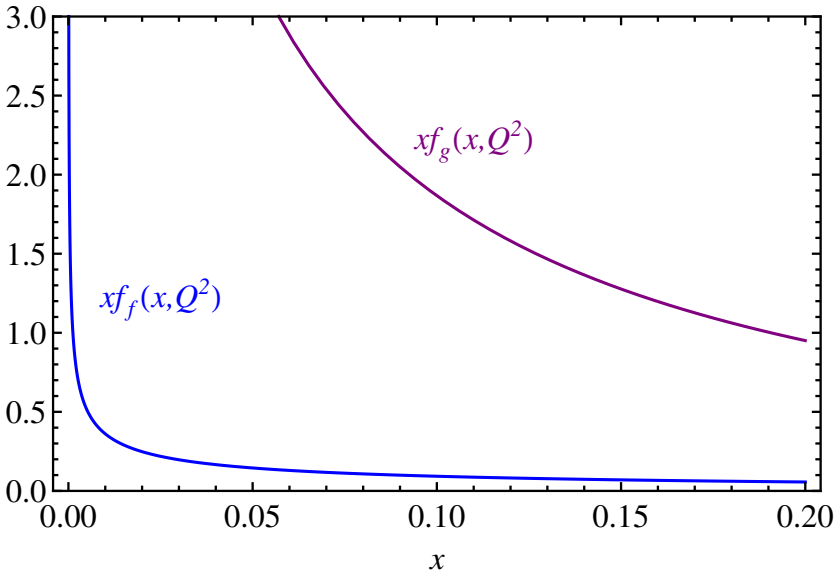


Figure 17.4: Approximate parton distribution functions at small x with $Q = 500\text{GeV}$.

Chapter 18

Operator Products and Effective Vertices

18.1 Matrix element for proton decay

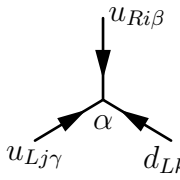
(a) We estimate the order of magnitude of the proton lifetime, through the decay $p \to e^+\pi^0$, based on the following operator,

$$\mathcal{O}_X = \frac{2}{m_X^2}\epsilon_{ijk}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}e_{R\alpha}u_{Ri\beta}u_{Lj\gamma}d_{Lk\delta}, \tag{18.1}$$

where m_X is the scale of this higher dimensional operator, whose typical value is around 10^{16}GeV, and i, j, k are color indices for quarks, $\alpha, \beta, \cdots$ are spinor indices. Then, the amplitude $\mathcal{M}$ of this decay process should be proportional to m_X^{-2}. Note that the amplitude $\mathcal{M}$ has mass dimension 1, thus we should have $\mathcal{M} \sim m_p^3 m_X^{-2}$ with m_p the proton mass. Now take $m_X \sim 10^{16}$GeV and $m_p \sim 1$GeV, we have the decay width

$$\Gamma \sim \frac{1}{8\pi}\frac{1}{2m_p}|\mathcal{M}|^2 \sim \frac{1}{16\pi}\frac{m_p^5}{m_X^4} \sim 10^{-65}\text{GeV} \sim 10^{33}\text{yr}^{-1}. \tag{18.2}$$

(b) Now we consider the first order QCD correction to the estimation above. The correction comes from virtual gluon exchange among three quarks in the operator. To evaluate these 1-loop diagrams, we firstly fixed the renormalization condition of $\mathcal{O}_X$ to be



$$= \mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta}. \tag{18.3}$$

The 1-loop diagrams are shown in Figure 18.1. The Feynman rules can be written in two-component spinor notations. The left-handed spinor's propagator reads $\mathrm{i}(p\cdot\sigma)/p^2$, the right-handed spionr's propagator is $\mathrm{i}(p\cdot\bar{\sigma})/p^2$, the QCD interaction between quark and gluon is $\mathrm{i}[\psi_{Li}^\dagger\bar{\sigma}^\mu(t^a)_{ij}\psi_{Lj} + \psi_{Ri}^\dagger\sigma^\mu(t^a)_{ij}\psi_{Rj}]$, and the vertex corresponding $\mathcal{O}_X$ reads $\mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta}$. Then

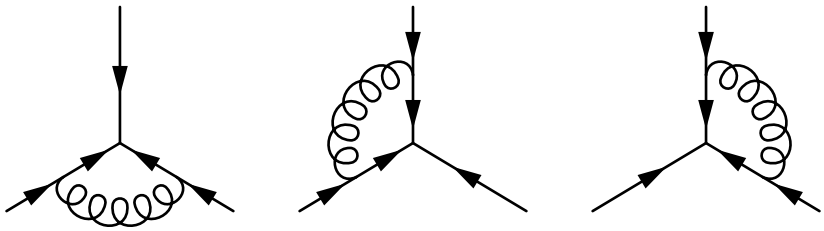


Figure 18.1: 1-loop QCD correction to the effective operator of proton decay.

the first diagram reads

$$\begin{aligned}
\text{(a)} &= \mathrm{i}(\mathrm{i}g)^2\epsilon_{imn}(t^a)_{mj}(t^a)_{nk}\delta^{\alpha\beta}\epsilon^{\gamma'\delta'}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{-\mathrm{i}}{q^2}\left(\frac{-\mathrm{i}q\cdot\sigma}{q^2}\bar{\sigma}_\mu\right)_{\gamma'\gamma}\left(\frac{\mathrm{i}q\cdot\sigma}{q^2}\bar{\sigma}^\mu\right)_{\delta'\delta}\\
&= -\,g^2\cdot\Big(-\frac{2}{3}\Big)\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma'\delta'}(\sigma_\rho\bar{\sigma}_\mu)_{\gamma'}{}^{\gamma}(\sigma_\sigma\bar{\sigma}^\mu)_{\delta'}{}^{\delta}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{q^\rho q^\sigma}{q^6}\\
&= -\,g^2\cdot\Big(-\frac{2}{3}\Big)\epsilon_{ijk}\delta^{\alpha\beta}\cdot 16\epsilon^{\gamma\delta}\cdot\frac{\mathrm{i}}{4(4\pi)^2}\frac{2}{\epsilon}\\
&= \frac{8g^2}{3(4\pi)^2}\frac{2}{\epsilon}\cdot\mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta},
\end{aligned}\tag{18.4}$$

where the Pauli matrices is simplified as follows,

$$\begin{aligned}
&\epsilon^{\gamma'\delta'}(\sigma_\rho\bar{\sigma}_\mu)_{\gamma'}{}^{\gamma}(\sigma_\sigma\bar{\sigma}^\mu)_{\delta'}{}^{\delta} = (\bar{\sigma}_\mu^T\sigma_\rho^T\epsilon\sigma^\rho\bar{\sigma}^\mu)^{\gamma\delta} = -(\bar{\sigma}_\mu^T\epsilon\sigma_\rho\sigma^\rho\bar{\sigma}^\mu)^{\gamma\delta}\\
&= -4(\bar{\sigma}_\mu^T\epsilon\bar{\sigma}^\mu)^{\gamma\delta} = -4(\epsilon\bar{\sigma}_\mu\bar{\sigma}^\mu)^{\gamma\delta} = 16\epsilon^{\gamma\delta},
\end{aligned}\tag{18.5}$$

in which we used the fact that $\epsilon^{\alpha\beta} = \mathrm{i}\sigma^2$ and $\sigma_\mu^T\sigma^2 = -\sigma^2\sigma_\mu$. In the computation of this diagram, we also used $\epsilon_{imn}(t^a)_{mj}(t^a)_{nk} = -(2/3)\epsilon_{ijk}$. The coefficient of this equality can be easily justified by contracting both sides with ϵ_{ijk}. Similarly, we compute the second diagram, as follows,

$$\begin{aligned}
\text{(b)} &= \mathrm{i}(\mathrm{i}g)^2\epsilon_{imn}(t^a)_{mj}(t^a)_{nk}\delta^{\alpha\beta'}\epsilon^{\gamma'\delta}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{-\mathrm{i}}{q^2}\left(\frac{-\mathrm{i}q\cdot\bar{\sigma}}{q^2}\sigma_\mu\right)_{\beta'\beta}\left(\frac{\mathrm{i}q\cdot\sigma}{q^2}\bar{\sigma}^\mu\right)_{\gamma'\gamma}\\
&= -\,g^2\cdot\Big(-\frac{2}{3}\Big)\epsilon_{ijk}\cdot 4\delta^{\alpha\beta}\epsilon^{\gamma\delta}\cdot\frac{\mathrm{i}}{4(4\pi)^2}\frac{2}{\epsilon}\\
&= \frac{2g^2}{3(4\pi)^2}\frac{2}{\epsilon}\cdot\mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta},
\end{aligned}\tag{18.6}$$

where we used the identity $(\bar{\sigma}_\mu)_{\alpha\beta}(\sigma^\mu)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma}$. The third diagram gives the same result as the second one. Therefore, we get the counterterm for the operator $\mathcal{O}_X$ in $\overline{MS}$ scheme to be

$$\delta_{\mathcal{O}_X} = -\frac{4g^2}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log M^2\right),\tag{18.7}$$

where M^2 is the renormalization scale. We further recall that the field strength renormalization counterterm for quarks in QCD is given by

$$\delta_2 = -\frac{4g^2}{3(4\pi)^2}\left(\frac{2}{\epsilon} - \log M^2\right),\tag{18.8}$$

then the anomalous dimension of operator $\mathcal{O}_X$ is given by

$$\gamma = M\frac{\partial}{\partial M}\big(-\delta_{\mathcal{O}_X}+\frac{3}{2}\delta_2\big) = -\frac{4g^2}{(4\pi)^2}. \tag{18.9}$$

Therefore this QCD correction will enhance the operator strength by a factor of

$$\left(\frac{\log(m_X^2/\Lambda^2)}{\log(m_p^2/\Lambda^2)}\right)^{a_0/2b_0}, \tag{18.10}$$

where $\Lambda \simeq 200\text{GeV}$, $a_0 = 4$ is the coefficient from anomalous dimension, and $b_0 = 11 - (2/3)n_f = 7$ is the 1-loop coefficient of QCD β function. Taking $m_X = 10^{16}\text{GeV}$ and $m_p = 1\text{GeV}$, this factor is about 2.5. Then the decay rate of proton is enhanced by a factor of $2.5^2 \simeq 6.3$.

18.2 Parity-violating deep inelastic form factor

(a) We firstly compute the amplitude of the neutrino deep inelastic scattering through charged current interaction, which reads

$$\begin{aligned}\mathrm{i}\mathcal{M}(\nu p\to\mu^- X) = {}& \frac{\mathrm{i}g^2}{2m_W^2}\bar{u}(k')\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)u(k)\\ &\times\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\langle X|\big(J_+^\mu(x)+J_-^\mu(x)\big)|P\rangle.\end{aligned} \tag{18.11}$$

Then the squared amplitude with initial proton's spins averaged and final state X summed is

$$\begin{aligned}\frac{1}{2}\sum|\mathcal{M}|^2 = {}& \frac{1}{2}\frac{g^4}{4m_W^4}\sum_{\text{spin}}\left(\bar{u}(k')\gamma_\nu\Big(\frac{1-\gamma_5}{2}\Big)u(k)\bar{u}(k)\Big(\frac{1+\gamma_5}{2}\Big)\gamma_\mu u(k')\right)\\ &\times\sum_X\int \mathrm{d}\Pi_X\langle P|\big(J_+^\mu(x)+J_-^\mu(x)\big)|X\rangle\langle X|\big(J_+^\nu(0)+J_-^\nu(0)\big)|P\rangle.\end{aligned} \tag{18.12}$$

The trace factor can be straightforwardly worked out to be

$$\begin{aligned}L_{\mu\nu} \equiv {}& \sum_{\text{spin}}\left(\bar{u}(k')\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)u(k)\bar{u}(k)\Big(\frac{1+\gamma_5}{2}\Big)\gamma_\nu u(k')\right)\\ ={}& \mathrm{tr}\left[\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)\not{k}\Big(\frac{1+\gamma_5}{2}\Big)\gamma_\nu\not{k}'\right]\\ ={}& 2\big(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' + \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}k_\rho k'_\sigma\big).\end{aligned} \tag{18.13}$$

Then, use the optical theorem, we have

$$\begin{aligned}&L_{\mu\nu}\sum_X\int \mathrm{d}\Pi_X\langle P|\big(J_+^\mu(x)+J_-^\mu(x)\big)|X\rangle\langle X|\big(J_+^\nu(0)+J_-^\nu(0)\big)|P\rangle\\ &= 2\,\mathrm{Im}\,\big(L_{\mu\nu}W^{\mu\nu(\nu)}\big),\end{aligned} \tag{18.14}$$

with

$$W^{\mu\nu(\nu)} = 2\mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\langle P|T\big\{J_-^\mu(x)J_+^\nu(0)\big\}|P\rangle, \tag{18.15}$$

Therefore, the cross section is

$$\begin{aligned}\sigma(\nu p\to\mu^- X) &= \frac{1}{2s}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{2k'}\cdot\frac{1}{2}\sum|\mathcal{M}|^2\\ &= \frac{1}{2s}\int \mathrm{d}x\mathrm{d}y\frac{ys}{(4\pi)^2}\cdot\frac{g^4}{4m_W^4}\,\mathrm{Im}\left(L_{\mu\nu}W^{\mu\nu(\nu)}\right),\end{aligned} \tag{18.16}$$

and the differential cross section is thus given by

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu p\to\mu^- X) = \frac{yG_F^2}{2\pi^2}\,\mathrm{Im}\left[\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' + \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}k_\rho k'_\sigma\right)W^{\mu\nu(\nu)}\right]. \tag{18.17}$$

(b) The lepton momentum tensor obtained in (a) is

$$L_{\mu\nu} = 2(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' + \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}k_\rho k'_\sigma). \tag{18.18}$$

Then it is straightforward to see that $q^\mu L_{\mu\nu} = (k-k')^\mu L_{\mu\nu} = 0$ and $q^\nu L_{\mu\nu} = 0$. As a consequence, any term in $W^{\mu\nu(\nu)}$ proportional to q^μ or q^ν is irrelevant. Therefore we can rewrite the tensor $W^{\mu\nu(\nu)}$ in terms of three form factors $W_i^{(\nu)}$ $(i=1,2,3)$. That is,

$$W^{\mu\nu(\nu)} = -g^{\mu\nu}W_1^{(\nu)} + P^\mu P^\nu W_2^{(\nu)} + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}P_\rho q_\sigma W_3^{(\nu)} + \cdots. \tag{18.19}$$

Then the deep inelastic scattering cross section becomes

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu p\to\mu^- X) = \frac{yG_F^2}{2\pi^2}\Big[&2(k\cdot k')\,\mathrm{Im}\,W_1^{(\nu)} + 2(P\cdot k)(P\cdot k')\,\mathrm{Im}\,W_2^{(\nu)}\\ &- 4\Big((P\cdot k)(q\cdot k') - (q\cdot k)(P\cdot k')\Big)\,\mathrm{Im}\,W_3^{(\nu)}\Big].\end{aligned} \tag{18.20}$$

(c) Now we evaluate $\mathrm{Im}\,W_{1,2,3}^{(\nu)}$ in the parton model. Firstly, $W^{\mu\nu(\nu)}$ can be written as

$$W^{\mu\nu(\nu)} = 2\mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\int_0^1\mathrm{d}\xi\sum_f f_f(\xi)\frac{1}{\xi}\big\langle q_f(\xi P)\big|T\{J_-^\mu(x)J_+^\nu(0)\}\big|q_f(\xi P)\big\rangle, \tag{18.21}$$

and be evaluated in terms of Feynman diagrams displayed in Fig. 18.10 of Peskin & Schroeder. For the first diagram, we have

$$\begin{aligned}2\mathrm{i}\int_0^1\mathrm{d}\xi\Bigg[&f_d(\xi)\frac{1}{\xi}\bar u(p)\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not p+\not q+\mathrm{i}\epsilon}\gamma^\nu\Big(\frac{1-\gamma^5}{2}\Big)u(p)\\ &+f_{\bar u}(\xi)\frac{1}{\xi}\bar u(p)\gamma^\nu\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not p+\not q+\mathrm{i}\epsilon}\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)u(p)\Bigg],\end{aligned} \tag{18.22}$$

where $p=\xi P$. Then, averaging/summing over initial/final spin states gives

$$2\int_0^1\mathrm{d}\xi\left[f_d(\xi)\frac{1}{\xi}\cdot\frac{1}{2}\,\mathrm{tr}\left(\not p\gamma^\mu\frac{1-\gamma^5}{2}(\not p+\not q)\gamma^\nu\frac{1-\gamma^5}{2}\right)\right.$$

$$+ f_{\bar{u}}(\xi)\frac{1}{\xi}\cdot\frac{1}{2}\,\mathrm{tr}\left(\not{p}\gamma^\mu\frac{1-\gamma^5}{2}(\not{p}+\not{q})\gamma^\nu\frac{1-\gamma^5}{2}\right)\Bigg]\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}$$

$$\Rightarrow\int_0^1\frac{\mathrm{d}\xi}{\xi}\Bigg[\big(f_d(\xi)+f_{\bar{u}}(\xi)\big)\big(4\xi^2P^\mu P^\nu-2\xi P\cdot q g^{\mu\nu}\big)$$

$$+\big(f_d(\xi)-f_{\bar{u}}(\xi)\big)2\mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}P^\rho q^\sigma\Bigg]\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon},\tag{18.23}$$

where we have dropped terms containing q^μ or q^ν in the last line. Then it is easy to read from this expression that

$$\mathrm{Im}\,W_1^{(\nu)}=2P\cdot q\int_0^1\mathrm{d}\xi\,\big[f_d(\xi)+f_{\bar{u}}(\xi)\big]\,\mathrm{Im}\left(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\right),\tag{18.24}$$

$$\mathrm{Im}\,W_2^{(\nu)}=\int_0^1\mathrm{d}\xi\,4\xi\big[f_d(\xi)+f_{\bar{u}}(\xi)\big]\,\mathrm{Im}\left(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\right),\tag{18.25}$$

$$\mathrm{Im}\,W_3^{(\nu)}=2\int_0^1\mathrm{d}\xi\,\big[f_d(\xi)-f_{\bar{u}}(\xi)\big]\,\mathrm{Im}\left(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\right),\tag{18.26}$$

where $2P\cdot q=ys$, and

$$\mathrm{Im}\left(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\right)=\frac{\pi}{ys}\delta(\xi-x).\tag{18.27}$$

Note that the second diagram in Fig. 18.10 of Peskin & Schroeder does not contributes, as explained in the book. Therefore we conclude that

$$\mathrm{Im}\,W_1^{(\nu)}=\pi\big[f_d(x)+f_{\bar{u}}(x)\big],\tag{18.28}$$

$$\mathrm{Im}\,W_2^{(\nu)}=\frac{4\pi x}{ys}\big[f_d(x)+f_{\bar{u}}(x)\big],\tag{18.29}$$

$$\mathrm{Im}\,W_3^{(\nu)}=\frac{2\pi}{ys}\big[f_d(x)-f_{\bar{u}}(x)\big].\tag{18.30}$$

(d) The analysis above can be easily repeated for the left-handed current J^μ_{fL} of single flavor f, defined by $J^\mu_{fL}=\bar{f}\gamma^\mu P_L f$ where $P_L\equiv(1-\gamma^5)/2$. Then, define

$$W^{\mu\nu}_{fL}=2\mathrm{i}\int\mathrm{d}^4x\,e^{\mathrm{i}q\cdot x}\big\langle P\big|T\{J^\mu_{fL}(x)J^\nu_{fL}(0)\}\big|P\big\rangle,\tag{18.31}$$

and its decomposition,

$$W^{\mu\nu}_{fL}=-g^{\mu\nu}W_{1fL}+P^\mu P^\nu W_{2fL}+\mathrm{i}\epsilon^{\mu\nu\rho\sigma}P_\rho q_\sigma W_{3fL}+\cdots.\tag{18.32}$$

We see that it amounts to the replacement in the final result that $d\to f$ and $\bar{u}\to\bar{f}$. Therefore,

$$\mathrm{Im}\,W_{1fL}^{(\nu)}=\pi\big[f_f(x)+f_{\bar{f}}(x)\big],\tag{18.33}$$

$$\mathrm{Im}\,W_{2fL}^{(\nu)}=\frac{4\pi x}{ys}\big[f_f(x)+f_{\bar{f}}(x)\big],\tag{18.34}$$

$$\mathrm{Im}\,W_{3fL}^{(\nu)}=\frac{2\pi}{ys}\big[f_f(x)-f_{\bar{f}}(x)\big].\tag{18.35}$$

(e) Now we perform OPE on $W_{fL}^{\mu\nu}$. Firstly,

$$\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x} J_{fL}^{\mu}(x) J_{fL}^{\nu}(0) \simeq \int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x} \Big(\bar{q}\gamma^{\mu} P_L \overbrace{q(x)\bar{q}}\gamma^{\nu} P_L q(0) + \overbrace{\bar{q}\gamma^{\mu} P_L q(x)\bar{q}\gamma^{\nu} P_L q}(0) \Big)$$
$$= \bar{q}\gamma^{\mu} P_L \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^{\nu} P_L q + \Big(\mu \leftrightarrow \nu, q \to -q\Big). \tag{18.36}$$

Then, the first term in the last line can be written as

$$\bar{q}\gamma^{\mu} P_L \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^{\nu} P_L q = \frac{1}{2}\left(\bar{q}\gamma^{\mu} \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^{\nu} q - \bar{q}\gamma^{\mu} \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^{\nu}\gamma^5 q \right)$$
$$= -\frac{\mathrm{i}}{2}\bar{q}\big[2\gamma^{(\mu}(\mathrm{i}\partial^{\nu)}) - g^{\mu\nu} q\!\!\!/ - \mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}(\mathrm{i}\partial + q)^{\rho}\gamma^{\sigma}\big]$$
$$\times \frac{1}{Q^2}\sum_{n=0}^{\infty}\left(\frac{2\mathrm{i}q\cdot\partial}{Q^2}\right)^n q, \tag{18.37}$$

where we have symmetrize the $\mu\nu$ indices for the first two terms in the square bracket and antisymmetrize the indices for the third term, by using the equalities $\frac{1}{2}(\gamma^{\mu}\gamma^{\lambda}\gamma^{\nu} + \gamma^{\nu}\gamma^{\lambda}\gamma^{\mu}) = g^{\mu\lambda}\gamma^{\nu} + g^{\nu\lambda}\gamma^{\mu} - g^{\mu\nu}\gamma^{\lambda}$ and $\frac{1}{2}(\gamma^{\mu}\gamma^{\lambda}\gamma^{\nu}\gamma^5 - \gamma^{\nu}\gamma^{\lambda}\gamma^{\mu}\gamma^5) = -\mathrm{i}\epsilon^{\mu\nu\lambda}{}_{\rho}\gamma^{\rho}$, and terms proportional to q^{μ} or q^{ν} have also been dropped. The (anti)symmetrization can be understood by looking at (18.32), where the terms with no γ^5 are symmetric on $\mu\nu$ while the term involving γ^5 is antisymmetric on $\mu\nu$. Therefore, when including the second term in (18.36), we should keep terms of even powers in q for symmetric $\mu\nu$ indices and of odd powers in q for antisymmetric $\mu\nu$.

Now, with these understood, and using the definition of twist-2, spin-n operator,

$$\mathcal{O}_f^{(n)\mu_1\cdots\mu_n} = \bar{q}_f\gamma^{(\mu_1}(\mathrm{i}\mathrm{D}^{\mu_2})\cdots(\mathrm{i}\mathrm{D}^{\mu_n)})q_f - \text{traces}, \tag{18.38}$$

we have,

$$\mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x} J_{fL}^{\mu}(x) J_{fL}^{\nu}(0) = \sum_{n>0,\ \text{even}} 2\frac{(2q_{\mu_1})\cdots(2q_{\mu_{n-2}})}{(Q^2)^{n-1}}\mathcal{O}_f^{(n)\mu\nu\mu_1\cdots\mu_{n-2}}$$
$$- \frac{1}{2}g^{\mu\nu}\sum_{n>0,\ \text{even}}\frac{(2q_{\mu_1})\cdots(2q_{\mu_n})}{(Q^2)^n}\mathcal{O}_f^{(n)\mu_1\cdots\mu_n}$$
$$- \mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}q^{\rho}\sum_{n>0\ \text{odd}}\frac{(2q_{\mu_1})\cdots(2q_{\mu_{n-1}})}{(Q^2)^n}\mathcal{O}_f^{(n)\sigma\mu_1\cdots\mu_{n-1}}. \tag{18.39}$$

Then, using $\langle P|\mathcal{O}_f^{(n)\mu_1\cdots\mu_n}|P\rangle = 2A_f^n P^{\mu_1}\cdots P^{\mu_n}$, we can get $W_{fL}^{\mu\nu}$ to be

$$W_{fL}^{\mu\nu} = 8P^{\mu}P^{\nu}\sum_{n>0,\ \text{even}}\frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-1}}A_f^n - 2g^{\mu\nu}\sum_{n>0,\ \text{even}}\frac{(2q\cdot P)^n}{(Q^2)^n}A_f^n$$
$$+ 4\mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}P^{\rho}q^{\sigma}\sum_{n>0,\ \text{odd}}\frac{(2q\cdot P)^{n-1}}{(Q^2)^n}A_f^n. \tag{18.40}$$

So we can read out

$$W_{1fL} = 2 \sum_{n>0,\ \text{even}} \frac{(2q\cdot P)^n}{(Q^2)^n} A_f^n, \tag{18.41}$$

$$W_{2fL} = 8 \sum_{n>0,\ \text{even}} \frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-1}} A_f^n, \tag{18.42}$$

$$W_{3fL} = 4 \sum_{n>0,\ \text{odd}} \frac{(2q\cdot P)^{n-1}}{(Q^2)^n} A_f^n. \tag{18.43}$$

(f) Now we use W_{3fL} obtained above to derive a sum rule for parton distribution f_f^-, defined by

$$f_f^-(x,Q^2) = \frac{ys}{2\pi} \operatorname{Im} W_{3fL}(x,Q^2), \tag{18.44}$$

where $x = Q^2/\nu$ and $\nu = 2P\cdot q = ys$. The the analytic behavior of W_{3fL} on the v-complex plane is shown in Fig. 18.11 of Peskin & Schroeder. Thus we can define the contour integral

$$I_{3n} = \int \frac{\mathrm{d}\nu}{2\pi\mathrm{i}} \frac{1}{\nu^n} W_{3fL}(\nu, Q^2), \tag{18.45}$$

where the contour is a small circle around the origin $\nu = 0$. This integral picks up the coefficient of $\nu^{(n-1)}$ term, namely, $I_{3n} = 4A_f^n/(Q^2)^n$. On the other hand, the contour can be deformed as shown in Fig. 18.12 of Peskin & Schroeder. Then the integral can be evaluated as

$$I_{3n} = 2\int_{Q_f^2}^{\infty} \frac{\mathrm{d}\nu}{2\pi\mathrm{i}} \frac{1}{\nu^n} (2\mathrm{i}) \operatorname{Im} W_{3fL}(\nu, Q^2) = \frac{4}{(Q^2)^n} \int_0^1 \mathrm{d}x\, x^{n-1} f_f^-(x, Q^2). \tag{18.46}$$

Therefore we get the sum rule,

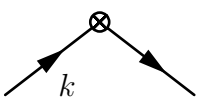
$$\int_0^1 \mathrm{d}x\, x^{n-1} f_f^-(x,Q^2) = A_f^n. \tag{18.47}$$

18.3 Anomalous dimensions of gluon twist-2 operators

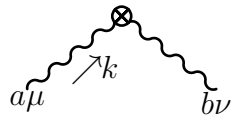
In this problem we finish evaluating anomalous dimension matrix γ^n in (18.180) of Peskin & Schroeder, given by

$$\gamma^n = -\frac{g^2}{(4\pi)^2} \begin{pmatrix} a_{ff}^n & a_{fg}^n \\ a_{gf}^n & a_{gg}^n \end{pmatrix} \tag{18.48}$$

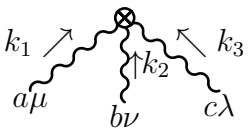
where a_{ff}^n has already been evaluated explicitly in the book. Here we evaluate the remaining three elements. The needed Feynman rules involving operators $\mathcal{O}_f^{(n)}$ and $\mathcal{O}_g^{(n)}$ are listed as follows:



$$= \not{\Lambda}(\Lambda\cdot k)^{n-1},$$



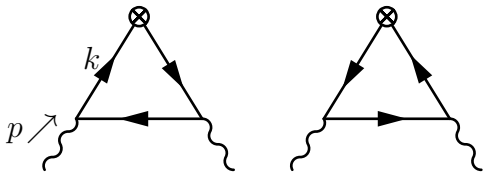
$$= -2\Big[g^{\mu\nu}(\Lambda\cdot k)^n + k^2\Lambda^\mu\Lambda^\nu(\Lambda\cdot k)^{n-2} - 2k^{(\mu}\Lambda^{\nu)}(\Lambda\cdot k)^{n-1}\Big]\delta^{ab},$$



$$= -2\mathrm{i}gf^{abc}g^{\mu\nu}\Lambda^\lambda\sum_{j=1}^{n}(\Lambda\cdot k_1)^{j-1}(-\Lambda\cdot k_2)^{n-j}$$
$$+ (\text{cyclic permutations on } \mu a k_1, \nu b k_2, \lambda c k_3) + \cdots .$$

In the last expression, we list only terms containing a metric tensor $g^{\mu\nu}$, and the ignored terms (marked by $\cdots$) are irrelevant in the following calculations. To be clear, we have introduced a source $J^{(n)}$ to these operators, namely, we write $\Delta\mathcal{L} = J^{(n)}_{\mu_1\cdots\mu_n}\mathcal{O}^{(n)\mu_1\cdots\mu_n}_{f,g}$, with $J^{(n)}_{\mu_1\cdots\mu_n} = \Lambda_{\mu_1}\cdots\Lambda_{\mu_n}$, and $\Lambda^2 = 0$. As can be easily seen, this source automatically projects the operator $\mathcal{O}^{(n)}_{f,g}$ to its symmetric and traceless part.

(a) Firstly, we consider a^n_{fg}, which can be got by evaluating the following two diagrams.



With the Feynman rules listed above, the first diagram reads,

$$\begin{aligned}
&(\mathrm{i}g)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}(-1)\,\mathrm{tr}\left[t^b\gamma^\nu\frac{\mathrm{i}}{\not k}\not\Lambda\frac{\mathrm{i}}{\not k}t^a\gamma^\mu\frac{\mathrm{i}}{\not k-\not p}\right](\Lambda\cdot k)^{n-1}\\
=&-\mathrm{i}g^2\,\mathrm{tr}\,[t^at^b]\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{2(1-x)}{(k'^2-\Delta)^3}(\Lambda\cdot k)^{n-1}\,\mathrm{tr}\,[\gamma^\nu\not k\not\Lambda\not k\gamma^\mu(\not k-\not p)].
\end{aligned}\tag{18.49}$$

We need to extract terms of proportional to $g^{\mu\nu}(\Lambda\cdot p)^n$ and of logarithmical divergence. This needs some manipulations on the numerator of the integrand. We firstly evaluate the gamma trace, keep terms containing at least two powers of k, and shift the variable $k^\mu \to k'^\mu = k^\mu - xp^\mu$. Then we pick up terms containing two k', which contributes to logarithmical divergence. At last we symmetrize the indices according to $k'^\mu k'^\nu \to k'^2 g^{\mu\nu}/4$. The detailed steps are given as follows.

$$\begin{aligned}
&(\Lambda\cdot k)^{n-1}\,\mathrm{tr}\,[\gamma^\nu\not k\not\Lambda\not k\gamma^\mu(\not k-\not p)]\\
\Rightarrow&\Big[16(\Lambda\cdot k)^nk^\mu k^\nu\Big]-\Big[4(\Lambda\cdot k)^n(k-2p)\cdot kg^{\mu\nu}\Big]-\Big[4(\Lambda\cdot k')^{n-1}(\Lambda\cdot p)k^2g^{\mu\nu}\Big]\\
\Rightarrow&\Big[16x^n(\Lambda\cdot p)^nk'^\mu k'^\nu\Big]-\Big[4nx^n(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}g^{\mu\nu}\\
&\quad+4n(x-2)x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}g^{\mu\nu}+4x^n(\Lambda\cdot p)^nk'^2g^{\mu\nu}\Big]\\
&-\Big[8(n-1)x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}g^{\mu\nu}+4x^{n-1}(\Lambda\cdot p)^nk'^2g^{\mu\nu}\Big]\\
\Rightarrow&\Big[4x^n\Big](\Lambda\cdot p)^nk'^2g^{\mu\nu}-\Big[nx^n+n(x-2)x^{n-1}+4x^n\Big](\Lambda\cdot p)^nk'^2g^{\mu\nu}
\end{aligned}$$

$$
\begin{aligned}
&- \Big[2(n-1)x^{n-1} + 4x^{n-1}\Big](\Lambda\cdot p)^n k'^2 g^{\mu\nu} \\
=& - (2nx^n + 2x^{n-1})(\Lambda\cdot p)^n k'^2 g^{\mu\nu}.
\end{aligned} \tag{18.50}
$$

Then it is straightforward to finish the loop integral,

$$
\begin{aligned}
&\mathrm{i}g^2\,\mathrm{tr}\,[t^a t^b](\Lambda\cdot p)^n g^{\mu\nu}\int_0^1 \mathrm{d}x\, 2(1-x)(2nx^n+2x^{n-1})\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{k'^2}{(k'^2-\Delta)^3} \\
=& -\frac{g^2}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n+1)(n+2)}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(\Lambda\cdot p)^n\delta^{ab}g^{\mu\nu}.
\end{aligned} \tag{18.51}
$$

The second diagram contributes an identical term for n even. The two diagrams sum to

$$
\frac{g^2}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n+1)(n+2)}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\cdot\Big(-2(\Lambda\cdot p)^n\delta^{ab}g^{\mu\nu}\Big). \tag{18.52}
$$

Therefore the corresponding counterterm reads

$$
\delta_{fg} = -\frac{g^2}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n+1)(n+2)}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}}, \tag{18.53}
$$

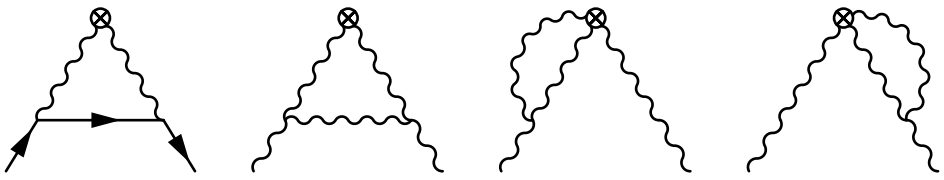
and the anomalous dimension element reads

$$
\gamma_{fg}^n = -M\frac{\partial}{\partial M}\delta_{fg} = -\frac{g^2}{(4\pi)^2}\frac{4(n^2+n+2)}{n(n+1)(n+2)}, \tag{18.54}
$$

and thus,

$$
a_{fg}^n = \frac{4(n^2+n+2)}{n(n+1)(n+2)}. \tag{18.55}
$$

(b) Then we consider a_{gf}^n and a_{gg}^n. This time we need to evaluate the following four diagrams.



The first diagram contributes to a_{gf}^n, which reads

$$
\begin{aligned}
&-2(\mathrm{i}g)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}t^a\gamma^\nu\frac{\mathrm{i}}{\not p-\not k}t^a\gamma^\mu\Big(\frac{\mathrm{i}}{k^2}\Big)^2 \\
&\quad\times\Big[g_{\mu\nu}(\Lambda\cdot k)^n + k^2\Lambda_\mu\Lambda_\nu(\Lambda\cdot k)^{n-2} - 2k_{(\mu}\Lambda_{\nu)}(\Lambda\cdot k)^{n-1}\Big] \\
=& -2\mathrm{i}g^2C_2(N)\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{2(1-x)}{(k'^2-\Delta)^3}\Big[\gamma^\mu(\not p-\not k)\gamma_\mu(\Lambda\cdot k)^n \\
&\quad+\not\Lambda(\not p-\not k)\not\Lambda(\Lambda\cdot k)^{n-2}k^2 - \Big(\not\Lambda(\not p-\not k)\not k+\not k(\not p-\not k)\not\Lambda\Big)(\Lambda\cdot k)^{n-1}\Big]
\end{aligned} \tag{18.56}
$$

To find the pieces proportional to $\not{\Lambda}(\Lambda\cdot p)^{n-1}$ and of logarithmical divergence, we manipulate on the expression in the square bracket, shifting the variable $k^\mu \to k'^\mu = k^\mu - xp^\mu$, extracting terms with two factors of k'^μ, symmetrizing the integrand with $k'^\mu k'^\nu \to k'^2 g^{\mu\nu}/4$, and throwing away terms proportional to $\Lambda^2(=0)$. This gives

$$
\begin{aligned}
&\Big[\gamma^\mu(\not{p}-\not{k})\gamma_\mu(\Lambda\cdot k)^n\Big] + \Big[\not{\Lambda}(\not{p}-\not{k})\not{\Lambda}(\Lambda\cdot k)^{n-2}k^2\Big] \\
&\quad - \Big[\Big(\not{\Lambda}(\not{p}-\not{k})\not{k} + \not{k}(\not{p}-\not{k})\not{\Lambda}\Big)(\Lambda\cdot k)^{n-1}\Big] \\
&= \Big[-2(\not{p}-\not{k})(\Lambda\cdot k)^n\Big] + \Big[2\not{\Lambda}\big(\Lambda\cdot(p-k)\big)(\Lambda\cdot k)^{n-2}k^2\Big] \\
&\quad - \Big[2\Big(\big(\Lambda\cdot(p-k)\big)\not{k} + \big((p-k)\cdot k\big)\not{\Lambda} - (\not{p}-\not{k})(\Lambda\cdot k)\Big)(\Lambda\cdot k)^{n-1}\Big] \\
&\Rightarrow \Big[2nx^{n-1}\not{k}'(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}\Big] + \Big[-2x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2} \\
&\qquad + 2(n-2)(1-x)x^{n-2}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2} \\
&\qquad + (1-x)x^{n-2}(\Lambda\cdot p)^{n-1}k'^2\Big] + \Big[2x^{n-1}\not{k}'(\Lambda\cdot k')(\Lambda\cdot p)^{n-1} \\
&\qquad - 2(n-1)(1-x)x^{n-2}(\Lambda\cdot k')(\Lambda\cdot p)^{n-1} - 2\not{\Lambda}\Big(-x^{n-1}k'^2(\Lambda\cdot p)^{n-1} \\
&\qquad - (n-1)x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2} \\
&\qquad + (n-1)(1-x)x^{n-2}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2}\Big) - 2nx^{n-1}\not{k}'(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}\Big] \\
&\Rightarrow \Big[\frac{n}{2}x^{n-1}\Big]\not{\Lambda}k'^2(\Lambda\cdot p)^{n-1} + \Big[-x^{n-1} + n(1-x)x^{n-2}\Big]\not{\Lambda}k'^2(\Lambda\cdot p)^{n-1} \\
&\qquad + \Big[-(n-1)(1-x)x^{n-2} + 2x^{n-1}\Big]\not{\Lambda}k'^2(\Lambda\cdot p)^{n-1} \\
&= \Big[\frac{n}{2}x^{n-1} + x^{n-2}\Big]\not{\Lambda}k'^2(\Lambda\cdot p)^{n-1}. \qquad (18.57)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&-2\mathrm{i}g^2C_2(N)\not{\Lambda}(\Lambda\cdot p)^{n-1}\int_0^1 \mathrm{d}x\, 2(1-x)\Big(\frac{n}{2}x^{n-1} + x^{n-2}\Big)\int \frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{k'^2}{(k'^2-\Delta)^3} \\
&= \frac{g^2C_2(N)}{(4\pi)^2}\,\frac{2(n^2+n+2)}{n(n^2-1)}\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\not{\Lambda}(\Lambda\cdot p)^{n-1}, \qquad (18.58)
\end{aligned}
$$

which gives the counterterm coefficient,

$$
\delta^n_{gf} = -\frac{g^2C_2(N)}{(4\pi)^2}\,\frac{2(n^2+n+2)}{n(n^2-1)}\,\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \qquad (18.59)
$$

Then, in a similar way as in (a), we get

$$
\gamma^n_{gf} = -\frac{g^2C_2(N)}{(4\pi)^2}\,\frac{4(n^2+n+2)}{n(n^2-1)}, \qquad (18.60)
$$

and for $N=3$, $C_2(N)=4/3$, we get

$$
a^n_{gf} = \frac{16}{3}\,\frac{(n^2+n+2)}{n(n^2-1)}. \qquad (18.61)
$$

The second to fourth diagrams contribute to a_{gg}^n. Now we evaluate them in turn. The second one reads,

$$
\begin{aligned}
&-2g^2 f^{ace} f^{bde} \delta^{cd} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \Big(\frac{-\mathrm{i}}{k^2}\Big)^2 \frac{-\mathrm{i}}{(p-k)^2} \\
&\times \Big[g^{\mu\rho}(p+k)^\lambda + g^{\rho\lambda}(p-2k)^\mu + g^{\lambda\mu}(k-2p)^\rho \Big] \\
&\times \Big[-g^{\nu\sigma}(p+k)_\lambda + g^\sigma_\lambda (2k-p)^\nu + g^\nu_\lambda (2p-k)^\sigma \Big] \\
&\times \Big[g_{\rho\sigma}(\Lambda\cdot k)^n + k^2 \Lambda_\rho \Lambda_\sigma (\Lambda\cdot k)^{n-2} - 2k_{(\rho}\Lambda_{\sigma)} (\Lambda\cdot k)^{n-1} \Big] \\
\Rightarrow &-2\mathrm{i} g^2 C_2(G) \delta^{ab} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \int_0^1 \mathrm{d}x \, \frac{2(1-x)}{(k'^2-\Delta)^3} \Big[-8(\Lambda\cdot k)^n k^\mu k^\nu - \Big((\Lambda\cdot k)^n k^2 \\
&+ 2(\Lambda\cdot k)^n (k\cdot p) - 8(\Lambda\cdot k)^{n-1}(\Lambda\cdot p)(k\cdot p) + 4(\Lambda\cdot k)^{n-2}(\Lambda\cdot p)^2 k^2 \Big) g^{\mu\nu} \Big] \\
\Rightarrow &-2\mathrm{i} g^2 C_2(G) g^{\mu\nu} \delta^{ab} (\Lambda\cdot p)^n \int_0^1 \mathrm{d}x \, 2(1-x) \Big[-\Big(3+\frac{n}{2}\Big) x^n - \frac{1}{2} n x^{n-1} - 2x^{n-2} \Big] \\
&\times \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{k'^2}{(k'^2-\Delta)^3} \\
=&-\frac{g^2 C_2(G)}{(4\pi)^2} \left(\frac{4}{n+2} - \frac{6}{n+1} + \frac{4}{n} - \frac{4}{n-1} \right) \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} (-2) g^{\mu\nu} \delta^{ab} (\Lambda\cdot p)^n .
\end{aligned}
\tag{18.62}
$$

The third diagram reads (where an additional $1/2$ is the symmetry factor),

$$
\begin{aligned}
&-\frac{1}{2}\cdot 2\mathrm{i} g^2 f^{acd} f^{bcd} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p-k)^2} \\
&\times \Big[g^{\mu\rho}(p+k)^\sigma + g^{\rho\sigma}(p-2k)^\mu + g^{\sigma\mu}(k-2p)^\rho \Big] \\
&\times \sum_{j=1}^{n} \Big[g^\nu_\sigma \Lambda_\rho (\Lambda\cdot(p-k))^{j-1} (\Lambda\cdot p)^{n-j} - g^\nu_\rho \Lambda_\sigma (\Lambda\cdot p)^{j-1} (\Lambda\cdot k)^{n-j} \Big] \\
\Rightarrow &-\mathrm{i} g^2 C_2(G) g^{\mu\nu} \delta^{ab} (\Lambda\cdot p)^n \sum_{j=1}^{n} \int_0^1 \mathrm{d}x \, \Big[(1+x) x^{n-j} - (x-2)(1-x)^{j-1} \Big] \\
&\times \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{(k'^2-\Delta)^2} \\
=&\frac{g^2 C_2(G)}{(4\pi)^2} g^{\mu\nu} \delta^{ab} \frac{\Gamma(2-\frac{2}{d})}{\Delta^{2-d/2}} (\Lambda\cdot p)^n \\
&\times \sum_{j=1}^{n} \Big(\frac{1}{j} + \frac{1}{j+1} + \frac{1}{n-j+1} + \frac{1}{n-j+2} \Big) \\
\Rightarrow &-\frac{g^2 C_2(G)}{(4\pi)^2} \left[2\sum_{j=2}^{n} \frac{1}{j} + \frac{1}{n+1} + 1 \right] \frac{\Gamma(2-\frac{2}{d})}{\Delta^{2-d/2}} (-2) g^{\mu\nu} (\Lambda\cdot p)^n .
\end{aligned}
\tag{18.63}
$$

The contribution from fourth diagram is identical to the one from the third diagram. Summing the last three diagram together, we get

$$
\frac{g^2 C_2(G)}{(4\pi)^2} \left(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n} \frac{1}{j} - 2 \right) \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} (-2) g^{\mu\nu} \delta^{ab} (\Lambda\cdot p)^n . \tag{18.64}
$$

Thus the corresponding counterterm is

$$\delta_g = -\frac{g^2 C_2(G)}{(4\pi)^2}\left(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n}\frac{1}{j} - 2\right)\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}}. \tag{18.65}$$

As a result,

$$\begin{aligned}\gamma_{gg}^n &= M\frac{\partial}{\partial M}(-\delta_g + \delta_3)\\ &= -\frac{2g^2}{(4\pi)^2}\left[\left(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n}\frac{1}{j} - \frac{1}{3}\right)C_2(G) - \frac{4}{3}n_f C(N)\right],\end{aligned} \tag{18.66}$$

therefore, for $N = 3$, $C_2(N) = 4/3$ and $C(N) = 1/2$, we have,

$$a_{gg}^n = 6\left(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n}\frac{1}{j} - \frac{1}{3} - \frac{2}{9}n_f\right). \tag{18.67}$$

18.4 Deep inelastic scattering from a photon

(a) The A-P equation for parton distributions in the photon can be easily written down by using the QED splitting functions listed in (17.121) of Peskin & Schroeder. Taking account of quarks' electric charge properly, we have,

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_q(x,Q) = \frac{3Q_q^2\alpha}{\pi}\int_x^1\frac{\mathrm{d}z}{z}\left\{P_{e\leftarrow e}(z)f_q\Big(\frac{x}{z},Q\Big) + P_{e\leftarrow\gamma}(z)f_\gamma\Big(\frac{x}{z},Q\Big)\right\}, \tag{18.68}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_{\bar{q}}(x,Q) = \frac{3Q_q^2\alpha}{\pi}\int_x^1\frac{\mathrm{d}z}{z}\left\{P_{e\leftarrow e}(z)f_{\bar{q}}\Big(\frac{x}{z},Q\Big) + P_{e\leftarrow\gamma}(z)f_\gamma\Big(\frac{x}{z},Q\Big)\right\}, \tag{18.69}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\log Q}f_\gamma(x,Q) = \sum_q\frac{3Q_q^2\alpha}{\pi}\int_x^1\frac{\mathrm{d}z}{z}\Bigg\{&P_{\gamma\leftarrow e}(z)\left[f_q\Big(\frac{x}{z},Q\Big) + f_{\bar{q}}\Big(\frac{x}{z},Q\Big)\right]\\ &+ P_{\gamma\leftarrow\gamma}(z)f_\gamma\Big(\frac{x}{z},Q\Big)\Bigg\},\end{aligned} \tag{18.70}$$

where the splitting functions are

$$P_{e\leftarrow e}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z), \tag{18.71}$$

$$P_{\gamma\leftarrow e}(z) = \frac{1+(1-z)^2}{z}, \tag{18.72}$$

$$P_{e\leftarrow\gamma}(z) = z^2 + (1-z)^2, \tag{18.73}$$

$$P_{\gamma\leftarrow\gamma}(z) = -\frac{2}{3}\delta(1-z). \tag{18.74}$$

We take $q = u, d, c, s$, and $Q_{u,c} = +2/3$, $Q_{d,s} = -1/3$. The factor 3 in the A-P equations above takes account of 3 colors. Since no more leptons appear in final states other than original e^+e^-, they are not included in the photon structure. With the initial condition

$f_\gamma(x,Q_0)=\delta(1-x)$ and $f_{q,\bar{q}}(x,Q_0)=0$ where $Q_0=0.5\text{GeV}$, these distribution functions can be solved from the equations above to the first order in α, to be

$$f_q(x,Q)=f_{\bar{q}}(x,Q)=\frac{3Q_q^2\alpha}{2\pi}\log\frac{Q^2}{Q_0^2}\big[x^2+(1-x)^2\big], \tag{18.75}$$

$$f_\gamma(x,Q)=\bigg(1-\sum_q\frac{Q_q^2\alpha}{\pi}\log\frac{Q^2}{Q_0^2}\bigg)\delta(1-x). \tag{18.76}$$

(b) The formulation of deep inelastic scattering from a photon is similar to the one for the proton, as described in Peskin & Schroeder. The process can be formulated as a two-photon scattering, with one photon being hard and the other one play the role of proton, which has the internal structure as shown in (a). Therefore, we can write down the corresponding current product as

$$W^{\mu\nu}=\mathrm{i}\int\mathrm{d}^4x\,e^{\mathrm{i}q\cdot x}\langle\gamma|\mathrm{T}\{J^\mu(x)J^\nu(0)\}|\gamma\rangle, \tag{18.77}$$

which can be again expanded in terms of scalar form factors,

$$W^{\mu\nu}=\Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big)W_1+\Big(P^\mu-q^\mu\frac{P\cdot q}{q^2}\Big)\Big(P^\nu-q^\nu\frac{P\cdot q}{q^2}\Big)W_2. \tag{18.78}$$

After operator product expansion, the form factor W_2 can be expressed as

$$W_2=3\sum_q Q_q^2\sum_n\frac{8}{Q^2}\frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-2}}A_q^n(Q^2), \tag{18.79}$$

and $A_q^n(Q^2)$ is a scale-dependent quantity, whose scaling behavior is dictated by the anomalous dimension matrix γ. This matrix can be evaluated again from the diagrams in Fig. 18.13 in Peskin & Schroeder and in figures of last problem. The only difference is that we should replace the gluon field with photon field. Therefore it is straightforward to see that $a_{\gamma\gamma}^n=0$. For a_{qq}^n and $a_{\gamma q}^n$, we should take away the group factor $C_2(N)=4/3$, while for $a_{q\gamma}^n$, we should take away the factor $\mathrm{tr}\,(t^at^b)=\delta^{ab}/2$. In addition, we should also include the factor Q_q^2 corresponding to electric charge of each quark. Then we have,

$$a_{qq}^n=-2Q_f^2\Big[1+4\sum_{j=2}^n\frac{1}{j}-\frac{2}{n(n+1)}\Big], \tag{18.80}$$

$$a_{q\gamma}^n=\frac{8Q_f^2(n^2+n+2)}{n(n+1)(n+2)}, \tag{18.81}$$

$$a_{\gamma q}^n=\frac{4Q_f^2(n^2+n+2)}{n(n^2-1)}, \tag{18.82}$$

$$a_{\gamma\gamma}^n=0. \tag{18.83}$$

(c) The $n=2$ moment photon structure function can be worked out through the moment sum rules (18.154) in Peskin & Schroeder, where the matrix elements A_q^n in our case is a scale-dependent quantity. This dependence can be found by evaluating the anomalous

dimension matrix of operator $\mathcal{O}_q^{(2)}$ as is done below (18.185) of Peskin & Schroeder, but with different entries, given by

$$\gamma = -\frac{\alpha}{4\pi}\begin{pmatrix} a_{uu}^2 & 0 & 3\times 2a_{u\gamma}^2 \\ 0 & a_{dd}^2 & 3\times 2a_{u\gamma}^2 \\ a_{\gamma u}^2 & a_{\gamma d}^2 & a_{\gamma\gamma}^2 \end{pmatrix} = -\frac{\alpha}{4\pi}\begin{pmatrix} -\frac{64}{27} & 0 & \frac{32}{27} \\ 0 & -\frac{16}{27} & \frac{8}{27} \\ \frac{64}{27} & \frac{16}{27} & 0 \end{pmatrix}. \tag{18.84}$$

(d) As can be inferred from (a), the photon structure function $f_\gamma(x, Q)$ is originally peaked at $x = 1$ for $Q = Q_0$, and the peak shifts toward smaller x and the peak goes lower and broader, as Q goes large from Q_0.

Chapter 19

Perturbation Theory Anomalies

19.1 Fermion number nonconservation in parallel E and B fields

(a) In this problem we investigate the effect of chiral anomaly on the (non)conservation of fermion number with definite chirality. Let us begin with the Adler-Bell-Jackiw anomaly equation,

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{19.1}$$

Integrating the left hand side over the whole spacetime, we get the difference between the numbers of right-handed fermions N_R and of left-handed fermions N_L, namely,

$$\int \mathrm{d}^4x\,\partial_\mu j^{\mu 5} = \int \mathrm{d}^4x\,\partial_\mu(j_R^\mu - j_L^\mu) = \int \mathrm{d}^3x\,(j_R^0 - j_L^0)\Big|_{t_1}^{t_2} = \Delta N_R - \Delta N_L, \tag{19.2}$$

where we assume that the integral region for time is $[t_1, t_2]$ and that $\partial_i j^i$ integrates to zero with suitable boundary conditions (i.e. vanishing at spatial infinity or periodic boundary condition). On the other hand,

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4\epsilon^{0ijk}F_{0i}F_{jk} = -8F_{0i}\big(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\big) = -8\mathbf{E}\cdot\mathbf{B}. \tag{19.3}$$

Therefore, the ABJ anomaly equation gives,

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2}\int \mathrm{d}^4x\,\mathbf{E}\cdot\mathbf{B}. \tag{19.4}$$

(b) The Hamiltonian for massless charged fermions with background electromagnetic field is given by

$$H = \int \mathrm{d}^3x\,\big(\pi \mathrm{D}_0\psi - \mathcal{L}\big) = -\int \mathrm{d}^3x\,\mathrm{i}\bar\psi\gamma^i\mathrm{D}_i\psi, \tag{19.5}$$

where $\pi = \mathrm{i}\psi^\dagger$ is the canonical conjugate momentum of ψ, $\mathcal{L} = \mathrm{i}\bar\psi\not{\mathrm{D}}\psi$ is the Lagrangian for the fermion, and $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}eA_\mu$ is the covariant derivative. Now we expand the Hamiltonian

in the chiral basis,

$$H = -\int \mathrm{d}^3x \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} -\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D} & 0 \\ 0 & \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$
$$= \int \mathrm{d}^3x \Big[\psi_L^\dagger(\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D})\psi_L - \psi_R^\dagger(\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D})\psi_R\Big]. \tag{19.6}$$

(c) Now we focus on the eigenvalue problem of the right-handed fermion ψ_R, namely the equation $-\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D}\psi_R = E\psi_R$. To be definite, we set the background electromagnetic potential to be $A^\mu = (0, 0, Bx^1, A)$ with B and A two constants. To seek the eigenfunction of the form $\psi_R = \big(\phi_1(x^1), \phi_2(x^1)\big)^T e^{\mathrm{i}(k_2x^2+k_3x^3)}$, we substitute it into the equation above and get

$$\phi_1' = (k_2 - eBx^1)\phi_1 + \mathrm{i}(E + k_3 - eA)\phi_2, \tag{19.7a}$$
$$\phi_2' = \mathrm{i}(E - k_3 + eA)\phi_1 - (k_2 - eBx^1)\phi_2. \tag{19.7b}$$

Eliminating ϕ_2 from these two equations, we get a single differential equation in the form of the harmonic oscillator,

$$\phi_1'' - \Big[e^2B^2\Big(x^1 - \frac{k_2}{eB}\Big)^2 - E^2 + (k_3 - eA)^2 - eB\Big]\phi_1 = 0. \tag{19.8}$$

(d) Now we specify the spatial boundary condition to be the box of length L in each side and periodic boundary condition. Then the condition $\psi_R(x^1, x^2, x^3) = \psi_R(x^1, x^2 + L, x^3) = \psi_R(x_1, x_2, x_3 + L)$ implies that k_2 and k_3 are quantized according to $k_i = 2\pi n_i/L$ $(i = 2, 3)$. On the other hand, k_2 also has an upper bound since (19.8) shows that the center of the oscillator would be out of the box if k_2 is too large. This condition implies that $k_2/eB < L$, which further gives the maximum value of n_2 to be $(n_2)_{\max} = eBL^2/2\pi$. Note also that the energy eigenvalue does not depend on k_2, thus each energy level consists of $eBL^2/2\pi$ degenerate states. Furthermore, we can also write down explicitly the energy eigenvalue associated with the state labeled by (n_1, n_3):

$$E = \pm\Big[\Big(\frac{2\pi n_3}{L} - eA\Big)^2 - (n_1 + \tfrac{3}{2})eB\Big]^{1/2}. \tag{19.9}$$

(e) Now we consider the case with $n_1 = 0$ for simplicity. Then the spectrum reads $E = 2\pi n_3/L - eA$. Suppose that the background potential changes by $\Delta A = 2\pi/eL$. Then it is easy to see that all state with energy marked by n_3 will turn to states with energy marked by $n_3 - 1$. Note that each energy eigenvalue is $eBL^2/2\pi$-degenerate, thus the net change of right-handed fermion number is $-eBL^2/2\pi$. Similar analysis shows that the left-handed fermion number get changed by $eBL^2/2\pi$. Therefore the total change is $\Delta N_R - \Delta N_L = -eBL^2/\pi$.

19.2 Weak decay of the pion

(a) In this problem we study the decay of charged pion. So let us work out the amplitude for $\pi^+ \to \ell\nu$, with the effective four-fermion interaction

$$\Delta\mathcal{L} = \frac{4G_F}{\sqrt{2}}(\bar{\ell}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L) + \text{h.c.} \tag{19.10}$$

and the relation

$$\langle 0|j^{\mu 5a}(x)|\pi^b(p)\rangle = -\mathrm{i}p^\mu f_\pi \delta^{ab} e^{-\mathrm{i}p\cdot x} \tag{19.11}$$

as inputs. Firstly we recall that

$$j^{\mu a} = \bar{Q}_L\gamma^\mu\tau^a Q_L + \bar{Q}_R\gamma^\mu\tau^a Q_R, \tag{19.12a}$$

$$j^{\mu 5a} = -\bar{Q}^\gamma_L\mu\tau^a Q_L + \bar{Q}_R\gamma^\mu\tau^a Q_R, \tag{19.12b}$$

where $Q_L = (u_L, d_L)^T$ and $Q_R = (u_R, d_R)^T$. Thus,

$$\frac{1}{2}(j^{\mu 1} + \mathrm{i}j^{\mu 2} - j^{\mu 51} - \mathrm{i}j^{\mu 52}) = \bar{Q}_L\gamma^\mu(\tau^1 + \mathrm{i}\tau^2)Q_L = \bar{u}_L\gamma^\mu d_L. \tag{19.13}$$

Then we find the decay amplitude $\mathcal{M}\big(\pi^+(p) \to \ell^+(k)\nu(q)\big)$ to be

$$\mathrm{i}\mathcal{M} = \frac{4\mathrm{i}G_F}{\sqrt{2}}\bar{u}(q)\gamma^\mu\Big(\frac{1-\gamma_5}{2}\Big)v(k)\cdot\frac{1}{\sqrt{2}}f_\pi \mathrm{i}p_\mu = -G_F f_\pi \bar{u}(q)\not{p}(1-\gamma_5)v(k). \tag{19.14}$$

(b) Now let us calculate the decay rate of the charged pion. We note that the amplitude above can be further simplified to

$$\mathrm{i}\mathcal{M} = -G_F f_\pi \bar{u}(q)(\not{q} + \not{k})(1-\gamma_5)v(k) = -G_F f_\pi m_\ell \bar{u}(q)(1+\gamma_5)v(k). \tag{19.15}$$

Therefore we have

$$\sum|\mathcal{M}|^2 = G_F^2 f_\pi^2 m_\ell^2 \,\mathrm{tr}\,\big(\not{q}(1+\gamma_5)(\not{k} - m_\ell)(1-\gamma_5)\big) = 8G_F^2 f_\pi^2 m_\ell^2 q\cdot k, \tag{19.16}$$

where the summation goes over all final spins. We choose the momenta to be

$$p = (m_\pi, 0, 0, 0), \qquad k = (E_k, 0, 0, k), \qquad q = (E_q, 0, 0, -k). \tag{19.17}$$

Then the kinematics can be easily worked out to be

$$E_k = \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}, \qquad E_q = k = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} \tag{19.18}$$

The decay rate then follows straightforwardly,

$$\begin{aligned}\Gamma &= \frac{1}{2m_\pi}\int\frac{\mathrm{d}\Omega}{16\pi^2}\frac{k^2}{E_k E_q}\Big(\frac{k}{E_k} + \frac{k}{E_q}\Big)^{-1}\cdot 8G_F^2 f_\pi^2 m_\ell^2 (q\cdot k)\\ &= \frac{G_F^2 f_\pi^2}{4\pi m_\pi}\Big(\frac{m_\ell}{m_\pi}\Big)^2(m_\pi^2 - m_\ell^2)^2,\end{aligned} \tag{19.19}$$

and we have the ratio between two decay channels,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \simeq 10^{-4}. \tag{19.20}$$

Thus to determine the pion decay constant f_π, we can consider the channel $\mu^+\nu_\mu$ only as a good approximation. With the lifetime of charged pion $\tau_\pi = 2.6 \times 10^{-8}$sec as well as m_π and m_μ, we find that

$$f_\pi = \sqrt{\frac{4\pi m_\pi}{G_F^2 \tau_\pi}}\Big(\frac{m_\pi}{m_\mu}\Big)(m_\pi^2 - m_\mu^2)^{-1} \simeq 90.6\text{MeV}. \tag{19.21}$$

19.3 Computation of anomaly coefficients

(a) By definition, $\mathcal{A}^{abc} = \operatorname{tr}[t^a, \{t^b, t^c\}]$. Then for the product representation $r_1 \times r_2$, we have

$$\begin{aligned}
\mathcal{A}^{abc}(r_1 \times r_2) &= \operatorname{tr}_{r_1\times r_2}\Big[t^a \otimes 1 + 1 \otimes t^a, \big\{t^b \otimes 1 + 1 \otimes t^b, t^c \otimes 1 + 1 \otimes t^c\big\}\Big] \\
&= \operatorname{tr}_{r_1\times r_2}\Big[t^a \otimes 1 + 1 \otimes t^a, \{t^b, t^c\} \otimes 1 + t^b \otimes t^c + t^c \otimes t^b + 1 \otimes \{t^b, t^c\}\Big] \\
&= \operatorname{tr}_{r_1\times r_2}\Big([t^a\{t^b, t^c\}] \otimes 1 + [t^a, t^b] \otimes t^c + [t^a, t^c] \otimes t^b \\
&\quad + t^b \otimes [t^a, t^c] + t^b \otimes [t^a, t^c] + 1 \otimes [t^a, \{t^b, t^c\}]\Big) \\
&= \operatorname{tr}_{r_1}[t^a, \{t^b, t^c\}] \operatorname{tr}_{r_2}(1) + \operatorname{tr}_{r_2}[t^a, \{t^b, t^c\}] \operatorname{tr}_{r_1}(1) \\
&= \mathcal{A}^{abc}(r_1)d(r_2) + \mathcal{A}^{abc}(r_2)d(r_1).
\end{aligned} \tag{19.22}$$

On the other hand, as we decompose the representation $r_1 \times r_2$ into a direct product of irreducible representations $\sum_i r_i$, we have

$$\begin{aligned}
\mathcal{A}^{abc}\Big(\sum_i r_i\Big) &= \operatorname{tr}_{\Sigma r}\Big[\sum_i t_i^a, \Big\{\sum_j t_j^b, \sum_k t_k^c\Big\}\Big] = \operatorname{tr}_{\Sigma r}\Big(\sum_i \sum_j \sum_k [t_i^a, \{t_j^b, t_k^c\}]\Big) \\
&= \sum_i \operatorname{tr}_{r_i}[t_i^a, \{t_i^b, t_i^c\}] = \sum_i \mathcal{A}^{abc}(r_i)
\end{aligned} \tag{19.23}$$

Note that $\mathcal{A}^{abc}(r) = \frac{1}{2}A(r)d^{abc}$ where d^{abc} is the unique symmetric gauge invariant. Then equating the two expressions above, we get

$$d(r_2)A(r_1) + d(r_1)A(r_2) = \sum_i A(r_i). \tag{19.24}$$

(b) In this part we show that the representation $(\mathbf{3} \times \mathbf{3})_a$ of $SU(3)$ is equivalent to $\bar{\mathbf{3}}$. Let ψ_i be the base vectors of $\mathbf{3}$ representation. Then, a set of base vectors of $(\mathbf{3} \times \mathbf{3})_a$ can be chosen to be $\epsilon_{ijk}\psi_j\psi_k$. From the transformation rule $\psi_i \to U_{ij}\psi_j$, we know that the $(\mathbf{3} \times \mathbf{3})_a$ base vectors transform according to $\epsilon_{ijk}\psi_j\psi_k \to \epsilon_{imn}U_{mj}U_{nk}\psi_j\psi_k$. Now, it

is easy to show that $\epsilon_{\ell mn}U_{\ell i}U_{mj}U_{nk}$ is totally antisymmetric, and thus is proportional to ϵ_{ijk}. Let us write $\epsilon_{\ell mn}U_{\ell i}U_{mj}U_{nk} = C\epsilon_{ijk}$, then taking $U = I$ shows that $C = 1$. Now we multiply both sides of this equality by $(U^\dagger)_{ip}$. Since U is unitary, $(U^\dagger)_{ip} = (U^{-1})_{ip}$, so we get $\epsilon_{pmn}U_{mj}U_{nk} = \epsilon_{ijk}(U^\dagger)_{ip}$. That is, the base vector $\epsilon_{ijk}U_jU_k$ transforms according to $\epsilon_{ijk}U_jU_k \to (U^\dagger)_{\ell i}\epsilon_{\ell jk}\psi_j\psi_k = (U^*)_{i\ell}\epsilon_{\ell jk}\psi_j\psi_k$, which is exactly the transformation rule of $\bar{\mathbf{3}}$.

Now from $A(\mathbf{3}) = 1$, it follows that $A(\bar{\mathbf{3}}) = -1$. Therefore $A((\mathbf{3}\times\mathbf{3})_a) = -1$, and by using the equation derived in (a), we have $A((\mathbf{3}\times\mathbf{3})_s) = 6 - (-1) = 7$.

(c) Now we compute the anomaly coefficients for a and s representations of the $SU(N)$ group. As indicated in Peskin & Schroeder, it is enough to consider an $SU(3)$ subgroup of $SU(N)$. Then the fundamental representation $\mathbf{N}$ is decomposed into a direct sum of irreducible representations when restricted to $SU(3)$, that is, $\mathbf{N} = \mathbf{3} + (N-3)\mathbf{1}$. This decomposition is easily justified by considering the upper-left 3×3 block of a matrix in fundamental representation of $SU(N)$. When this block is treated as a transformation of $SU(3)$, the first three components of the vector on which the matrix acts form a fundamental representation vector of $SU(3)$, while the other $(N-3)$ components of the column vector are obviously invariant. With this known, we have,

$$\mathbf{N}\times\mathbf{N} = \big(\mathbf{3} + (N-3)\mathbf{1}\big)\times\big(\mathbf{3} + (N-3)\mathbf{1}\big) = \mathbf{3}\times\mathbf{3} + 2(N-3)\mathbf{3} + (N-3)^2\mathbf{1}. \quad (19.25)$$

On the other hand, we know that $\mathbf{N} = s + a$ while s and a are irreducible. Then we have, by (a), $2N\cdot A(\mathbf{N}) = A(s) + A(a)$. But we already know that $A(\mathbf{N}) = 1$. Thus $A(s)+A(a) = 2N$. Now, to compute $A(a)$, we make use of the $SU(3)$ restriction,

$$(\mathbf{N}\times\mathbf{N})_a = (\mathbf{3}\times\mathbf{3})_a + (N-3)\mathbf{3} + \tfrac{1}{2}(N-3)(N-4)\mathbf{1}. \quad (19.26)$$

Then,

$$A(a) = A\big((\mathbf{3}\times\mathbf{3})_a\big) + (N-3)A(\mathbf{3}) = A(\bar{\mathbf{3}}) + (N-3)A(\mathbf{3}) = N-4, \quad (19.27)$$

and $A(s) = 2N - A(a) = N + 4$.

Now consider totally antisymmetric rank-j tensor representation. Again we decompose the fundamental representation as $\mathbf{N} = \mathbf{3} + (n-3)\mathbf{1}$. Then the rank-$j$ totally antisymmetric tensor can be decomposed as

$$\begin{aligned}(\mathbf{N}^j)_a = {} & \frac{(N-j)\cdots(N-j+1)}{(j-3)!}(\mathbf{3}\times\mathbf{3}\times\mathbf{3})_a + \frac{(N-3)\cdots(N-j)}{(j-2)!}(\mathbf{3}\times\mathbf{3})_a \\ & + \frac{(N-3)\cdots(N-j-1)}{(j-1)!}\mathbf{3} + \mathbf{1}\text{'s}. \end{aligned} \quad (19.28)$$

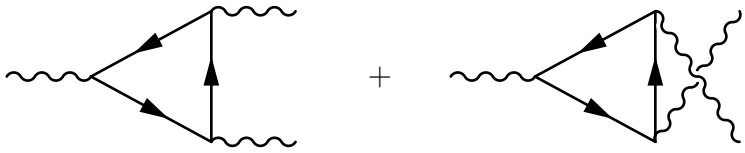
Therefore,

$$\begin{aligned} A(a) &= -\frac{(N-3)\cdots(N-j)}{(j-2)!} + \frac{(N-3)\cdots(N-j-1)}{(j-1)!} \\ &= \frac{(N-3)\cdots(N-j)(N-2j)}{(j-1)!}. \end{aligned} \quad (19.29)$$

19.4 Large fermion mass limits

In this problem we study the chiral anomaly and the trace anomaly in triangle diagrams with Pauli-Villars regularization.

(a) and (c) Firstly we evaluate the expectation value of the divergence of the chiral current $j^{\mu 5}$ between the vacuum and the two-photon state, namely the matrix element $\langle p,k|j^{\mu 5}|0\rangle$. This matrix element receives contributions at 1-loop level from the following two diagrams:



In momentum space, the divergence of the first diagram reads

$$\begin{aligned} \mathrm{i}q_\mu \mathcal{M}_1^{\mu\nu\lambda} = (-1)(-\mathrm{i}e)^2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \Bigg\{ & \operatorname{tr}\left[\not{q}\gamma^5 \frac{\mathrm{i}}{\not{\ell}-\not{k}}\gamma^\lambda \frac{\mathrm{i}}{\not{\ell}}\gamma^\nu \frac{\mathrm{i}}{\not{\ell}+\not{p}}\right] \\ & - \operatorname{tr}\left[\not{q}\gamma^5 \frac{\mathrm{i}}{\not{\ell}-\not{k}-M}\gamma^\lambda \frac{\mathrm{i}}{\not{\ell}-M}\gamma^\nu \frac{\mathrm{i}}{\not{\ell}+\not{p}-M}\right]\Bigg\} \end{aligned} \tag{19.30}$$

The integral is finite, thus we are allowed to shift the integral variable. For the first trace and the second trace above, we rewrite the $\not{q}\gamma^5$ factors, respectively, as follows,

$$\begin{aligned} \not{q}\gamma^\mu &= (\not{\ell}+\not{p}-\not{\ell}+\not{k})\gamma^\mu = (\not{\ell}+\not{p})\gamma^5 + \gamma^5(\not{\ell}-\not{k}), \\ \not{q}\gamma^\mu &= (\not{\ell}+\not{p}-M-\not{\ell}+\not{k}+M)\gamma^\mu = (\not{\ell}+\not{p}-M)\gamma^5 + \gamma^5(\not{\ell}-\not{k}-M) + 2M\gamma^5. \end{aligned}$$

Then, the loop integral becomes

$$\begin{aligned} \mathrm{i}q_\mu \mathcal{M}_1^{\mu\nu\lambda} = e^2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \Bigg\{ & \operatorname{tr}\left[\gamma^5 \frac{1}{\not{\ell}-\not{k}}\gamma^\lambda \frac{1}{\not{\ell}}\gamma^\nu + \gamma^5\gamma^\lambda \frac{1}{\not{\ell}}\gamma^\nu \frac{1}{\not{\ell}+\not{p}}\right] \\ & - \operatorname{tr}\left[\gamma^5 \frac{1}{\not{\ell}-\not{k}-M}\gamma^\lambda \frac{1}{\not{\ell}-M}\gamma^\nu + \gamma^5\gamma^\lambda \frac{1}{\not{\ell}-M}\gamma^\nu \frac{1}{\not{\ell}+\not{p}-M}\right] \\ & + 2M \operatorname{tr}\left[\gamma^5 \frac{1}{\not{\ell}-\not{k}-M}\gamma^\lambda \frac{1}{\not{\ell}-M}\gamma^\nu \frac{1}{\not{\ell}+\not{p}-M}\right]\Bigg\} \end{aligned} \tag{19.31}$$

In the expression above, the first and the second lines are canceled by the corresponding terms from the second diagram with $(k,\lambda \leftrightarrow p,\nu)$, while the third line is doubled. Therefore the sum of two diagrams gives

$$\begin{aligned} \mathrm{i}q_\mu \mathcal{M}^{\mu\nu\lambda} &= 4e^2 M \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \operatorname{tr}\left[\gamma^5 \frac{1}{\not{\ell}-\not{k}-M}\gamma^\lambda \frac{1}{\not{\ell}+M}\gamma^\nu \frac{1}{\not{\ell}+\not{p}-M}\right] \\ &= 4e^2 M \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{2N_1}{\left[(\ell - xk + yp)^2 - \Delta\right]^3} \\ &= -\frac{4\mathrm{i}e^2 M N_1}{(4\pi)^2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{1}{\Delta} \end{aligned} \tag{19.32}$$

with

$$N_1 = \mathrm{tr}\left[\gamma^5(\not{\ell} - \not{k} + M)\gamma^\lambda(\not{\ell} - M)\gamma^\nu(\not{\ell} + \not{p} + M)\right] = -4\mathrm{i}M\epsilon^{\alpha\beta\lambda\nu}k_\alpha p_\beta,$$
$$\Delta = M^2 - x(1-x)k^2 - y(1-y)p^2 - 2xyk\cdot p.$$

Then the integral can be carried out directly in the $M^2 \to \infty$ limit, to be

$$\mathrm{i}q_\mu \mathcal{M}^{\mu\nu\lambda} = -\frac{e^2}{2\pi^2}\epsilon^{\alpha\beta\lambda\nu}k_\alpha p_\beta, \tag{19.33}$$

as expected.

(b) and (d) For scale anomaly, the diagrams are the same. Now the relevant matrix element is given by $\langle p,k|M\bar{\psi}\psi|0\rangle$. Then the first diagram reads

$$\begin{aligned}\mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \mathrm{i}e^2M\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\Bigg\{&\left[\frac{1}{\not{\ell}-\not{k}}\not{\epsilon}^*(k)\frac{1}{\not{k}}\not{\epsilon}^*(p)\frac{1}{\not{\ell}+\not{p}}\right]\\ &-\mathrm{tr}\left[\frac{1}{\not{\ell}-\not{k}-M}\not{\epsilon}^*(k)\frac{1}{\not{\ell}-M}\not{\epsilon}^*(p)\frac{1}{\not{\ell}+\not{p}-M}\right]\Bigg\}\end{aligned} \tag{19.34}$$

The first trace vanishes upon regularization, then,

$$\mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = -\mathrm{i}e^2M\int\frac{\mathrm{d}^4\ell'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2N_2}{(\ell'^2-\Delta)^3}, \tag{19.35}$$

where $\ell' = \ell - xk + yp$, $\Delta = M^2 - 2xyk\cdot p$, and the trace in the numerator is

$$\begin{aligned}N_2 &= \mathrm{tr}\left[(\not{\ell} - \not{k} + M)\not{\epsilon}^*(k)(\not{\ell}+M)\not{\epsilon}^*(p)(\not{\ell}+\not{p}+M)\right]\\ &= 4M\big[M^2\epsilon^*(k)\cdot\epsilon^*(p) + \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big) - \big(\epsilon^*(k)\cdot\epsilon^*(p)\big)\big(k\cdot p\big)\\ &\qquad + 4\big(\epsilon^*(k)\cdot\ell\big)\big(\epsilon^*(p)\cdot\ell\big) - \big(\epsilon^*(k)\cdot\epsilon^*(p)\big)\ell^2\big]\\ &= 4M\big[M^2\epsilon^*(k)\cdot\epsilon^*(p) + (1-4xy)\big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\\ &\qquad - (1-2xy)\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) + \big(\tfrac{4}{d}-1\big)\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)\ell'^2\big],\end{aligned}$$

where we used the transverse condition $k\cdot\epsilon^*(k) = p\cdot\epsilon^*(p) = 0$, and in the last equality, the substitution $\ell'^\mu\ell'^\nu \to \frac{1}{d}g^{\mu\nu}\ell'^2$. We also dropped all terms linear in ℓ' in the last equality. The integral is then divergent, and we regularize it by dimensional regularization. Then after carrying out the loop integral, we get

$$\begin{aligned}\mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \frac{e^2}{4\pi^2}&\big[\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) - \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\big]\\ &\times\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{(1-4xy)M^2}{M^2-2xyk\cdot p}\end{aligned} \tag{19.36}$$

Then, taking $M^2\to\infty$ limit, we find

$$\mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \frac{e^2}{12\pi^2}\big[\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) - \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\big] \tag{19.37}$$

The second diagram is obtained, again, by the exchange $(k,\lambda \leftrightarrow p,\nu)$, which gives the identical result. Therefore we finally get

$$\mathrm{i}\mathcal{M}^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \frac{e^2}{6\pi^2}\big[\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) - \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\big]. \tag{19.38}$$

Chapter 20

Gauge Theories with Spontaneous Symmetry Breaking

20.1 Spontaneous breaking of $SU(5)$

We consider two patterns of spontaneous breaking of $SU(5)$ gauge symmetry, with an adjoint-representation scalar field Φ picking up vacuum expectation values

$$\langle\Phi\rangle = A\mathrm{diag}(1,1,1,1,-4), \qquad \langle\Phi\rangle = B\mathrm{diag}(2,2,2,-3,-3), \tag{20.1}$$

respectively. The kinetic term of the scalar field in the Lagrangian is

$$\mathcal{L}_{\text{kin.}} = \mathrm{tr}\left((\mathrm{D}_\mu\Phi)^\dagger(\mathrm{D}^\mu\Phi)\right) = \mathrm{tr}\left(\left(\partial_\mu\Phi + g[A_\mu,\Phi]\right)^\dagger\left(\partial^\mu\Phi + g[A^\mu,\Phi]\right)\right). \tag{20.2}$$

Then the mass term of gauge bosons after symmetry breaking is given by

$$\Delta\mathcal{L} = g^2\,\mathrm{tr}\left([A_\mu,\Phi]^\dagger[A^\mu,\Phi]\right) = -g^2A^a_\mu A^{\mu b}\,\mathrm{tr}\left([T^a,\langle\Phi\rangle][T^b,\langle\Phi\rangle]\right). \tag{20.3}$$

To analyze the gauge bosons' spectrum, we note that there are 24 independent generators for $SU(5)$ group, each of which can be represented as a 5×5 traceless hermitian matrix. Then, for the first choice of $\langle\Phi\rangle = \mathrm{diag}(1,1,1,1,-4)$, we see that for the generators of the form

$$T = \begin{pmatrix} T^{(4)} & \\ & 0 \end{pmatrix} \quad \text{and} \quad T = \tfrac{1}{2\sqrt{10}}\mathrm{diag}(1,1,1,1,-4),$$

where $T^{(4)}$ is a 4×4 matrix being any generator of $SU(4)$ group, the commutators vanish. That is, a subgroup $SU(4)\times U(1)$ remains unbroken in this case. Then, for the rest of the generators, namely

$$\frac{1}{2}\begin{pmatrix} 0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0&0&0&0&\mathrm{i}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\-\mathrm{i}&0&0&0&0\end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},$$

etc, it is easy to calculate the commutators to get the trace equal to $-25A^2/2$. Thus the corresponding components of gauge bosons acquire mass $M_A = 5gA$. In the same way, we can also analyze the case of $\langle\Phi\rangle = \mathrm{diag}(2, 2, 2, -3, -3)$. This time the unbroken subgroup is $SU(3) \times SU(2) \times U(1)$, and the remaining 12 components of gauge bosons acquire a mass equal to $M_A = 5gB$, as can be found by evaluating the corresponding commutators.

20.2 Decay modes of the W and Z bosons

(a) The relevant interaction term in the Lagrangian reads

$$\Delta\mathcal{L} = \tfrac{1}{\sqrt{2}} g W_\mu^+ \Big(\sum_i \bar{\nu}_{iL}\gamma^\mu e_{iL} + \sum_{j,c} \bar{u}^c_{jL}\gamma^\mu d^c_{jL} \Big), \tag{20.4}$$

where the sum on i goes over all three generations of leptons, the sum on j goes over the first two generations of quarks, since $m_t > m_W$, and the sum on c is due to 3 colors.

Now consider the decay of W^+ boson. The amplitude of the decay into a pair of fermions is

$$\mathrm{i}\mathcal{M} = \frac{\mathrm{i}g}{\sqrt{2}} \epsilon_\mu(k)\bar{u}(p_1)\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)v(p_2), \tag{20.5}$$

where ϵ_μ is the polarization vector for W_μ^+, and the labels for momenta are shown in Fig. 20.1. Thus the squared amplitude with initial polarizations averaged is

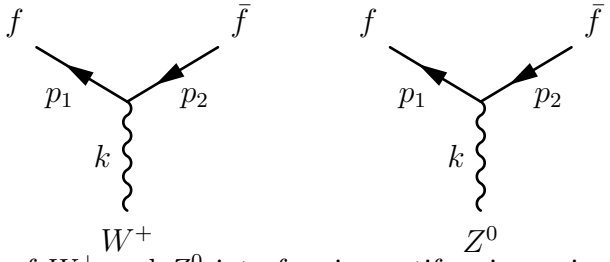


Figure 20.1: The decay of W^+ and Z^0 into fermion-antifermion pairs. All initial momenta go inward and all final momenta go outward.

$$\begin{aligned}\frac{1}{3}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}|^2 &= \frac{g^2}{6}\Big(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\Big)\,\mathrm{tr}\left[\not{p}_2\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)\not{p}_1\gamma^\nu\Big(\frac{1-\gamma^5}{2}\Big)\right] \\ &= \frac{g^2}{3}\left(p_1\cdot p_2 + 2\frac{k\cdot p_1 k\cdot p_2}{m_W^2}\right).\end{aligned} \tag{20.6}$$

The momenta in the center-of-mass frame can be taken to be

$$k = (m_W, 0, 0, 0), \qquad p_1 = (p, 0, 0, p), \qquad p_2 = (p, 0, 0, -p), \tag{20.7}$$

and energy conservation requires that $p = m_W/2$. Thus we get

$$\frac{1}{3}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}|^2 = \frac{1}{3}g^2 m_W^2, \tag{20.8}$$

and the decay rate

$$\int \mathrm{d}\Gamma = \frac{1}{2m_W}\int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 2E_1 2E_2}\left(\frac{1}{3}g^2m_W^2\right)(2\pi)^4\delta^{(4)}(k-p_1-p_2) = \frac{\alpha m_W}{12\sin^2\theta_w}, \tag{20.9}$$

where we have used $g = e/\sin\theta_w$ and $\alpha = e^2/4\pi$. For each quark final state we multiply the result by a QCD correction factor $\left(1+\frac{\alpha_s}{\pi}\right)$. Then, taking account of 3 generations of leptons and 2 generations of quarks with 3 colors, we get the partial decay rate of W^+ into fermions,

$$\Gamma(W^+ \to e_i^+\nu_i) = \frac{\alpha m_W}{12\sin^2\theta_w} \simeq 0.23\mathrm{GeV}; \tag{20.10}$$

$$\Gamma(W^+ \to u_j\bar{d}_j) = \frac{\alpha m_W}{4\sin^2\theta_w}\left(1+\frac{\alpha_s}{\pi}\right) \simeq 0.70\mathrm{GeV}; \tag{20.11}$$

$$\Gamma(W^+ \to \text{fermions}) = \frac{\alpha m_W}{12\sin^2\theta_w}\left(9+6\frac{\alpha_s}{\pi}\right) \simeq 2.08\mathrm{GeV}. \tag{20.12}$$

and also the branching ratios $\mathrm{BR}(W^+ \to e_i^+\nu_i) = 0.11\%$, and $\mathrm{BR}(W^+ \to u_j\bar{d}_j) = 0.34\%$. Note that the fine structure constant at m_W is $\alpha(m_W) \simeq 1/129$.

(b) In the same way, we can also calculate the decay rate of $Z \to$fermions. The relevant term in the Lagrangian is

$$\Delta\mathcal{L} = \frac{g}{\cos\theta_w}Z_\mu\sum_i \bar{f}_i\gamma^\mu\left(I_i^3 - \sin^2\theta_w Q_i\right)f_i, \tag{20.13}$$

where the sum goes over all left- and right-handed fermions, including 3 generations of leptons, and the first two generations of quarks with 3 colors, while I_i^3 and Q_i are associated 3-component of the weak isospin and the electric charge, respectively.

Then we can write down the amplitudes of the decay of Z^0 into a pair of fermions $f\bar{f}$ with specific I^3 and Q, as illustrated in Fig. 20.3,

$$\begin{aligned}
\mathrm{i}\mathcal{M} &= \frac{\mathrm{i}g}{\cos\theta_w}\epsilon_\mu(k)\bar{u}(p_1)\gamma^\mu\left[\left(I^3-\sin^2\theta_w Q\right)\left(\frac{1-\gamma^5}{2}\right) - \sin^2\theta_w Q\left(\frac{1+\gamma^5}{2}\right)\right]v(p_2) \\
&= \frac{\mathrm{i}g}{\cos\theta_w}\epsilon_\mu(k)\bar{u}(p_1)\gamma^\mu\left[I^3\left(\frac{1-\gamma^5}{2}\right) - \sin^2\theta_w Q\right]v(p_2),
\end{aligned} \tag{20.14}$$

the squared matrix elements,

$$\begin{aligned}
\frac{1}{3}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2 &= \frac{g^2}{3\cos^2\theta_w}\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_Z^2}\right) \\
&\quad\times \mathrm{tr}\left[\not{p}_2\gamma^\mu\left(\tfrac{1}{2}I^3(1-\gamma^5)-\sin^2\theta_w Q\right)\not{p}_1\gamma^\nu\left(\tfrac{1}{2}I^3(1-\gamma^5)-\sin^2\theta_w Q\right)\right] \\
&= \frac{4g^2}{3\cos\theta_w^2}\left[\left(\tfrac{1}{2}I^3-\sin^2\theta_w Q\right)^2+\left(\tfrac{1}{2}I^3\right)^2\right]\left(p_1\cdot p_2+\frac{2k\cdot p_1 k\cdot p_2}{m_Z^2}\right) \\
&= \frac{4g^2m_Z^2}{3\cos\theta_w^2}\left[\left(\tfrac{1}{2}I^3-\sin^2\theta_w Q\right)^2+\left(\tfrac{1}{2}I^3\right)^2\right],
\end{aligned} \tag{20.15}$$

and the partial decay rate,

$$\Gamma(Z^0 \to f\bar{f}) = \frac{\alpha m_Z}{3\sin^2\theta_w \cos^2\theta_w}\left[\left(\tfrac{1}{2}I^3 - \sin^2\theta_w Q\right)^2 + \left(\tfrac{1}{2}I^3\right)^2\right]. \tag{20.16}$$

We should also multiply the result for quarks by the QCD factor $\left(1 + \frac{\alpha_s}{\pi}\right)$. Now we list the numerical results of partial width and the branching ratios for various decay products as follows.

$f\bar{f}$	$\Gamma(f\bar{f})$/GeV	BR($f\bar{f}$)
$\nu_e\bar{\nu}_e$, $\nu_\mu\bar{\nu}_\mu$, $\nu_\tau\bar{\nu}_\tau$	0.17	6.7%
e^-e^+, $\mu^-\mu^+$, $\tau^-\tau^+$	0.08	3.4%
$u\bar{u}$, $c\bar{c}$	0.30	11.9%
$d\bar{d}$, $s\bar{s}$, $b\bar{b}$	0.39	15.4%
All fermions	2.51	100%

20.3 e^+e^- →hadrons with photon-Z^0 interference

(a) It is easier to work with amplitudes between initial and final fermions with definite chirality. In this case the relevant amplitude is given by*

$$\mathrm{i}\mathcal{M} = (\mathrm{i}e)^2\bar{v}(k_2)\gamma^\mu u(k_1)\frac{-\mathrm{i}}{q^2}\bar{u}(p_1)\gamma_\mu Q_f v(p_2)\left[-Q_f + \frac{(I_e^3 + s_w^2)(I_f^3 - s_w^2 Q_f)}{s_w^2 c_w^2}\frac{q^2}{q^2 - m_Z^2}\right], \tag{20.17}$$

where $I_e^3 = -1/2$ or 0 when the initial electron is left-handed or right-handed, so as I_f^3 to the final fermion. The momenta is labeled as shown in Fig. 20.2. Then we can find associated

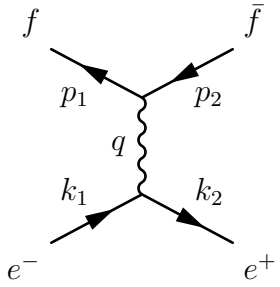


Figure 20.2: The process of $e^+e^- \to f\bar{f}$ via the exchange of a photon/Z^0 in s-channel. The directions of k_i's and p_i's are inward and outward, respectively.

differential cross section to be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_R^+e_L^- \to \bar{f}_R f_L) = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2 F_{LL}(f), \tag{20.18a}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_R^+e_L^- \to \bar{f}_L f_R) = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2 F_{LR}(f), \tag{20.18b}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_L^+e_R^- \to \bar{f}_R f_L) = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2 F_{RL}(f), \tag{20.18c}$$

*In this problem we simplify the notation by $s_w \equiv \sin\theta_w$ and $c_w \equiv \cos\theta_w$.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_L^+e_R^- \to \bar{f}_L f_R) = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2 F_{RR}(f), \tag{20.18d}$$

in which α is the fine structure constant, $s = q^2$ is the center-of-mass energy, and the F factors are defined as follows:

$$F_{LL}(f) = \left| Q_f + \frac{(\frac{1}{2} - s_w^2)(I_f^3 - s_w^2 Q_f)}{s_w^2 c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.19}$$

$$F_{LR}(f) = \left| Q_f - \frac{(\frac{1}{2} - s_w^2) Q_f}{c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.20}$$

$$F_{RL}(f) = \left| Q_f - \frac{(I_f^3 - s_w^2 Q_f)}{c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.21}$$

$$F_{RR}(f) = \left| Q_f + \frac{s_w^2 Q_f}{c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.22}$$

where we have added the correction from resonance by using the Breit-Wigner formula. Summing up the four expressions in (20.18), averaging the initial spins, and integrating over the angle θ, we get finally the unpolarized cross section

$$\sigma(f\bar{f}) = \frac{\pi\alpha^2}{3s}\Big[F_{LL}(f) + F_{LR}(f) + F_{RL}(f) + F_{RR}(f)\Big]. \tag{20.23}$$

When the final state particle f is a quark, one should multiply the result by $3\big(1+\frac{\alpha_s}{\pi}\big)$ where 3 is the color factor, and $\big(1+\frac{\alpha_s}{\pi}\big)$ is the 1-loop QCD correction.

For the final fermion being muon ($I_f^3 = -1/2$, $Q_f = -1$), up quark ($I_f^3 = 1/2$, $Q = 2/3$), and down quark ($I_f^3 = -1/2$, $Q_f = -1/3$), we plot the corresponding cross section as a function of center-of-mass energy $E_{\mathrm{CM}} = \sqrt{s}$ in Fig. 20.3.

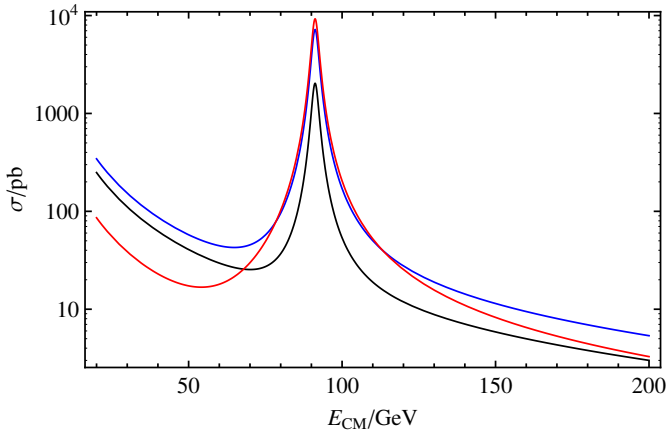


Figure 20.3: The cross section $\sigma(e^+e^- \to f\bar{f})$ as a function of center-of-mass energy E_{CM}. The black, blue, and red curves correspond to $f\bar{f} = \mu^-\mu^+$, $u\bar{u}$ and $d\bar{d}$, respectively.

(b) Now we calculate the forward-backward asymmetry A_{FB}^f, defined to be

$$A_{FB}^f = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{\big(\int_0^1 - \int_{-1}^0\big)\mathrm{d}\cos\theta(\mathrm{d}\sigma/\mathrm{d}\cos\theta)}{\big(\int_0^1 + \int_{-1}^0\big)\mathrm{d}\cos\theta(\mathrm{d}\sigma/\mathrm{d}\cos\theta)}. \tag{20.24}$$

Then from (20.18), we find

$$\sigma_F = \frac{\pi\alpha^2}{24s}\Big[7F_{LL}(f) + F_{LR}(f) + F_{RL}(f) + 7F_{RR}(f)\Big], \tag{20.25}$$

$$\sigma_B = \frac{\pi\alpha^2}{24s}\Big[F_{LL}(f) + 7F_{LR}(f) + 7F_{RL}(f) + F_{RR}(f)\Big]. \tag{20.26}$$

Thus

$$A_{FB}^f = \frac{3}{4}\cdot\frac{F_{LL}(f) - F_{LR}(f) - F_{RL}(f) + F_{RR}(f)}{F_{LL}(f) + F_{LR}(f) + F_{RL}(f) + F_{RR}(f)}. \tag{20.27}$$

Again, we plot A_{FB}^f, as a function of $E_{\rm CM}$, for $f = \mu^-, u, d$, in Fig. 20.4.

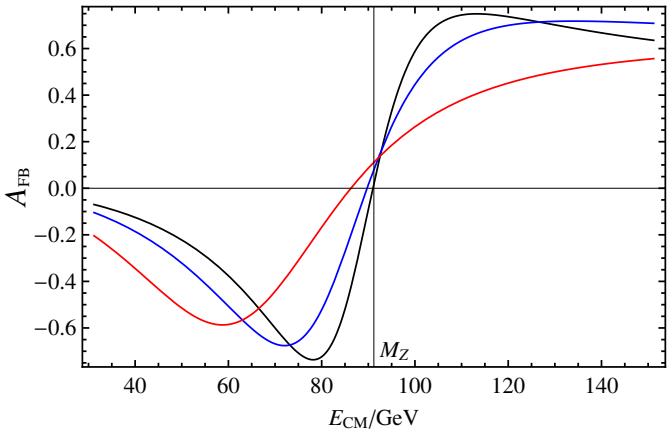


Figure 20.4: The forward-backward asymmetry A_{FB}^f as a function of center-of-mass energy $E_{\rm CM}$. The black, blue, and red curves correspond to $f\bar{f} = \mu^-\mu^+$, $u\bar{u}$ and $d\bar{d}$, respectively.

(c) Recall the definition of F's, we find, on the Z^0 resonance ($s = m_Z$),

$$F_{LL}(f) \simeq \left[\frac{\left(\frac{1}{2} - s_w^2\right)\left(I_f^3 - s_w^2 Q_f\right)}{s_w^2 c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2, \qquad F_{LR}(f) \simeq \left[\frac{\left(\frac{1}{2} - s_w^2\right)Q_f}{c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2,$$

$$F_{RL}(f) \simeq \left[\frac{\left(I_f^3 - s_w^2 Q_f\right)}{c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2, \qquad F_{RR}(f) \simeq \left[\frac{s_w^2 Q_f}{c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2,$$

therefore,

$$A_{FB}^f = \frac{3}{4}\cdot\frac{\left[\left(\frac{1}{2} - s_w^2\right)^2 - s_w^4\right]\left[\left(I_f^3 - s_w^2 Q_f\right)^2 - (s_w^2 Q_f)^2\right]}{\left[\left(\frac{1}{2} - s_w^2\right)^2 + s_w^4\right]\left[\left(I_f^3 - s_w^2 Q_f\right)^2 + (s_w^2 Q_f)^2\right]} = \frac{3}{4}A_{LR}^e A_{LR}^f. \tag{20.28}$$

(d)

$$\begin{aligned}\sigma_{\rm peak} &= \frac{\pi\alpha^2}{3m_Z^2}\cdot\frac{1}{s_w^4 c_w^4}\cdot\frac{m_Z^2}{\Gamma_Z^2}\left[\left(\tfrac{1}{2} - s_w^2\right)^2 + s_w^4\right]\left[\left(I_f^3 - s_w^2 Q_f\right)^2 + (s_w^2 Q_f)^2\right] \\ &= \frac{12\pi}{m_Z^2\Gamma_Z^2}\left(\frac{\alpha m_Z}{6s_w^2 c_w^2}\left[\left(\tfrac{1}{2} - s_w^2\right)^2 + s_w^4\right]\right)\left(\frac{\alpha m_Z}{6s_w^2 c_w^2}\left[\left(I_f^3 - s_w^2 Q_f\right)^2 + (s_w^2 Q_f)^2\right]\right) \\ &= \frac{12\pi}{m_Z^2}\cdot\frac{\Gamma(Z^0 \to e^+e^-)\Gamma(Z^0 \to f\bar{f})}{\Gamma_Z^2}.\end{aligned} \tag{20.29}$$

20.4 Neutral-current deep inelastic scattering

(a) In this problem we study the neutral-current deep inelastic scattering. The process is mediated by Z^0 boson. Assuming m_Z is much larger than the energy scale of the scattering process, we can write down the corresponding effective operators, from the neutral-current Feynman rules in electroweak theory,

$$\begin{aligned}\Delta\mathcal{L} = \frac{g^2}{4m_W^2}\big(\bar\nu\gamma^\mu\big)P_L\nu\Big[\bar u\gamma_\mu\Big(\big(1-\tfrac{4}{3}s_w^2\big)P_L - \tfrac{4}{3}s_w^2P_R\Big)u \\ + \bar d\gamma_\mu\Big((1-\tfrac{2}{3}s_w^2)P_L - \tfrac{2}{3}s_w^2P_R\Big)d\Big] + \text{h.c.},\end{aligned} \tag{20.30}$$

where $P_L = (1-\gamma^5)/2$ and $P_R = (1+\gamma^5)/2$ are left- and right-handed projectors, respectively. Compare the effective operator with the charged-operator in (17.31) of Peskin & Schroeder, we can write down directly the differential cross section by modifying (17.35) in Peskin & Schroeder properly, as

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu p\to\nu X) = \frac{G_F^2sx}{4\pi}\Big\{&\Big[\big(1-\tfrac{4}{3}s_w^2\big)^2 + \tfrac{16}{9}s_w^4(1-y^2)\Big]f_u(x) \\ &+\Big[\big(1-\tfrac{2}{3}s_w^2\big)^2 + \tfrac{4}{9}s_w^4(1-y^2)\Big]f_d(x) \\ &+\Big[\tfrac{16}{9}s_w^4 + \big(1-\tfrac{4}{3}s_w^2\big)^2(1-y^2)\Big]f_{\bar u}(x) \\ &+\Big[\tfrac{4}{9}s_w^4 + \big(1-\tfrac{2}{3}s_w^2\big)^2(1-y^2)\Big]f_{\bar d}(x)\Big\},\end{aligned} \tag{20.31}$$

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\bar\nu p\to\bar\nu X) = \frac{G_F^2sx}{4\pi}\Big\{&\Big[\tfrac{16}{9}s_w^4 + \big(1-\tfrac{4}{3}s_w^2\big)^2(1-y^2)\Big]f_u(x) \\ &+\Big[\tfrac{4}{9}s_w^4 + \big(1-\tfrac{2}{3}s_w^2\big)^2(1-y^2)\Big]f_d(x) \\ &+\Big[\big(1-\tfrac{4}{3}s_w^2\big)^2 + \tfrac{16}{9}s_w^4(1-y^2)\Big]f_{\bar u}(x) \\ &+\Big[\big(1-\tfrac{2}{3}s_w^2\big)^2 + \tfrac{4}{9}s_w^4(1-y^2)\Big]f_{\bar d}(x)\Big\}.\end{aligned} \tag{20.32}$$

(b) For the neutrino scattering from a nucleus A with equal numbers of protons and neutrons, we have $f_u = f_d$ and $f_{\bar u} = f_{\bar d}$. Then the differential cross sections reads

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu A\to\nu X) = \frac{G_F^2sx}{\pi}\Big\{&\big[\tfrac{1}{2} - s_w^2 + \tfrac{5}{9}s_4^4 + \tfrac{5}{9}s_4^4(1-y^2)\big]f_u(x) \\ &+\big[\tfrac{5}{9}s_w^4 + \big(\tfrac{1}{2} - s_w^2 + \tfrac{5}{9}s_w^4\big)(1-y^2)\big]f_{\bar u}(x)\Big\},\end{aligned} \tag{20.33}$$

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\bar\nu p\to\bar\nu X) = \frac{G_F^2sx}{\pi}\Big\{&\big[\tfrac{5}{9}s_w^4 + \big(\tfrac{1}{2} - s_w^2 + \tfrac{5}{9}s_w^4\big)(1-y^2)\big]f_u(x) \\ &+\big[\tfrac{1}{2} - s_w^2 + \tfrac{5}{9}s_4^4 + \tfrac{5}{9}s_4^4(1-y^2)\big]f_{\bar u}(x)\Big\}.\end{aligned} \tag{20.34}$$

Recall that for charged-current neutrino deep inelastic scattering, the differential cross sections are given by (17.35) in Peskin & Schroeder. Thus it is easy to find that

$$R^\nu = \frac{\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y(\nu A\to\nu X)}{\mathrm{d}^\sigma/\mathrm{d}x\mathrm{d}y(\nu A\to\mu^- X)} = \frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4\left(1 + \frac{f_u(x)(1-y^2) + f_{\bar u}(x)}{f_u(x) + f_{\bar u}(x)(1-y)^2}\right), \tag{20.35}$$

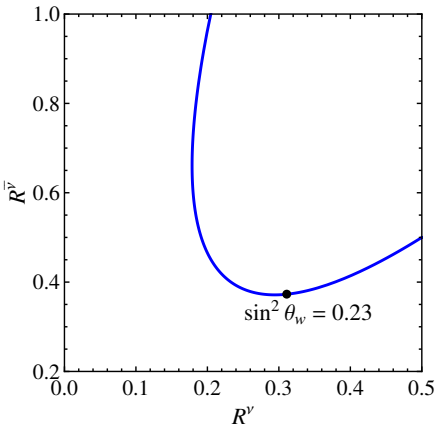


Figure 20.5: Weinberg's nose with $r = 0.4$. See problem 21.4.

$$R^{\bar{u}} = \frac{\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y(\bar{\nu}A \to \bar{\nu}X)}{\mathrm{d}^\sigma/\mathrm{d}x\mathrm{d}y(\bar{\nu}A \to \mu^+X)} = \frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4\left(1 + \frac{f_u(x) + f_{\bar{u}}(x)(1-y)^2}{f_u(x)(1-y^2) + f_{\bar{u}}(x)}\right), \tag{20.36}$$

where

$$\frac{f_u(x)(1-y^2) + f_{\bar{u}}(x)}{f_u(x) + f_{\bar{u}}(x)(1-y)^2} = r. \tag{20.37}$$

(c) The plot "Weinberg's Nose" with $r = 0.4$ is shown in Figure 20.5.

20.5 A model with two Higgs fields

(a) The gauge boson mass matrix comes from the kinetic term of scalar fields,

$$(\mathrm{D}_\mu\phi_1)^\dagger(\mathrm{D}^\mu\phi_1) + (\mathrm{D}_\mu\phi_2)^\dagger(\mathrm{D}^\mu\phi_2),$$

with $\mathrm{D}_\mu\phi_{1,2} = \big(\partial_\mu - \frac{\mathrm{i}}{2}gA_\mu^a\sigma^a - \frac{\mathrm{i}}{2}g'B_\mu\big)\phi_{1,2}$. After $\phi_{1,2}$ acquire the vacuum expectation value $\frac{1}{\sqrt{2}}\binom{0}{v_{1,2}}$, we observe that each of the kinetic terms gives rise to mass terms for gauge bosons similar to the ones in the standard electroweak theory. Thus it is straightforward that the masses of gauge bosons in this model is given by the replacement $v^2 \to v_1^2 + v_2^2$.

(b) The statement that the configuration $\frac{1}{\sqrt{2}}\binom{0}{v_{1,2}}$ is a locally stable minimum, is equivalent to that all particle excitations generated above this solution have positive squared mass m^2. Thus we investigate the mass spectrum of the theory with the vacuum chosen to be $\frac{1}{\sqrt{2}}\binom{0}{v_{1,2}}$. Firstly, we parameterize two scalar doublets as

$$\phi_i = \begin{pmatrix} \pi_i^+, \\ \frac{1}{\sqrt{2}}(v_i + h_i + \mathrm{i}\pi_i^0) \end{pmatrix}, \qquad (i = 1, 2) \tag{20.38}$$

and substitute this parameterization into the potential,

$$\begin{aligned} V = &-\mu_1^2\phi_1^\dagger\phi_1 - \mu_2^2\phi_2^\dagger\phi_2 + \lambda_1(\phi_1^\dagger\phi)^2 + \lambda_2(\phi_2^\dagger\phi_2)^2 \\ &+ \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \lambda_5\big((\phi_1^\dagger\phi_2)^2 + \mathrm{h.c.}\big). \end{aligned} \tag{20.39}$$

Then the mass term of various scalar components can be extracted, as follows.

$$\begin{aligned}\mathcal{L}_{\text{mass}} = {} & (\lambda_4+2\lambda_5)v_1v_2\begin{pmatrix}\pi_1^- & \pi_2^-\end{pmatrix}\begin{pmatrix}v_2/v_1 & -1\\ -1 & v_1/v_2\end{pmatrix}\begin{pmatrix}\phi_1^+\\ \phi_2^+\end{pmatrix}\\ & + 2\lambda_5 v_1v_2\begin{pmatrix}\pi_1^0 & \pi_2^0\end{pmatrix}\begin{pmatrix}v_2/v_1 & -1\\ -1 & v_1/v_2\end{pmatrix}\begin{pmatrix}\phi_1^0\\ \phi_2^0\end{pmatrix}\\ & - v_1v_2\begin{pmatrix}h_1 & h_2\end{pmatrix}\begin{pmatrix}\lambda_1(v_1/v_2) & \lambda_3+\lambda_4+2\lambda_5\\ \lambda_3+\lambda_4+2\lambda_5 & \lambda_2(v_2/v_1)\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix}. \end{aligned} \tag{20.40}$$

The eigenvalues of these matrices are easy to be found. For charged components, there is a zero mode corresponding two broken directions in $SU(2)$, and the mass of the other charged scalar is given by $m_c^2 = -(\lambda_4+2\lambda_5)(v_1^2+v_2^2)$. For pseudoscalar components, there is also a zero mode corresponding to the rest one direction of broken $SU(2)$, and the mass of the other pseudoscalar is $m_p^2 = -4\lambda_5(v_1^2+v_2^2)$. Finally, for neutral scalars, the two mass eigenvalues are given by the roots of following equation,

$$m_n^2 - (\lambda_1 v_1^2 + \lambda_2 v_2^2)m_n^2 + \left[\lambda_1\lambda_2 - (\lambda_3+\lambda_4+2\lambda_5)^2\right] = 0. \tag{20.41}$$

Therefore, to make $m_c^2 > 0$, $m_p^2 > 0$ and $m_n^2 > 0$, it is sufficient that

$$\lambda_4+2\lambda_5 < 0, \qquad \lambda_5 < 0, \qquad \lambda_1,\lambda_2 > 0, \qquad \lambda_1\lambda_2 > (\lambda_3+\lambda_4+2\lambda_5)^2. \tag{20.42}$$

(c) From the mass terms in (b) we can diagonalize the charged scalar mass matrix with the rotation matrix

$$\begin{pmatrix}\pi^+\\ \phi^+\end{pmatrix} = \begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\pi_1^+\\ \pi_2^+\end{pmatrix}, \tag{20.43}$$

where π^+ is the Goldstone mode and ϕ^+ is a physical charged scalar. Given that ϕ^+ to get the physical mass, it is easy to see that the rotation angle can be chosen to be $\tan\beta = v_2/v_1$.

(d) Assuming that the Yukawa interactions between quarks and scalars take the following form,

$$\mathcal{L}_m = -\begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}\left[\lambda_d\begin{pmatrix}\pi_1^+\\ \frac{1}{\sqrt{2}}v_1\end{pmatrix}d_R + \lambda_u\begin{pmatrix}\frac{1}{\sqrt{2}}v_2\\ \pi^-\end{pmatrix}u_R\right] + \text{h.c.}, \tag{20.44}$$

where we have suppressed flavor indices and neglected neutral scalar components. We focus on charged component only. Then, with Peskin & Schroeder's notation, we make the replacement $u_L \to U_u u_L$, $d_L \to U_d d_L$, $u_R \to W_u u_R$, and $d_R \to W_d d_R$. Then, together with $\lambda_d = U_d D_d W_d^\dagger$ and $\lambda_u = U_u D_u W_u^\dagger$ where D_d and D_u are diagonal matrix, we have

$$\begin{aligned}\mathcal{L}_m = {} & -\frac{1}{\sqrt{2}}\big(v_1\bar{d}_L D_d d_R + v_2\bar{u}_L D_u u_R\big)\\ & - \bar{u}V_{\text{CKM}}D_d d_R\pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger D_u u_R\pi_2^- + \text{h.c.}\end{aligned} \tag{20.45}$$

From the first line we see that the diagonal mass matrix for quarks are given by $m_u = (v_1/\sqrt{2})D_u$ and $m_d = (v_2/\sqrt{2})D_d$. We further define $v = \sqrt{v_1^2 + v_2^2}$ and note that $\pi_1^+ = -\phi^+ \sin\beta \cdots$, $\pi_2^+ = \phi^+ \cos\beta + \cdots$, then the Yukawa interactions between charged boson and quarks can be written as

$$\begin{aligned}\mathcal{L}_m \Rightarrow & -\frac{\sqrt{2}}{v_1}\Big(\bar{u}_L V_{\rm CKM} m_d d_R \pi_1^+ + \bar{d}_L V_{\rm CKM}^\dagger m_u u_R \pi_2^-\Big) + \text{h.c.} \\ \Rightarrow & \frac{\sqrt{2}}{v}\Big(\bar{u}_L V_{\rm CKM} m_d d_R \phi^+ \tan\beta + \bar{d}_L V_{\rm CKM}^\dagger m_u u_R \phi^- \cot\beta\Big) + \text{h.c.}. \end{aligned} \tag{20.46}$$

Chapter 21

Quantization of Spontaneously Broken Gauge Theories

21.1 Weak-interaction contributions to the muon $g-2$

In this problem we study the weak-interaction corrections to the muon's anomalous magnetic moment (AMM). The relevant contributions come from the W-neutrino loop and Z-muon loop, together with the diagrams with the gauge bosons replaced with the corresponding Goldstone bosons. Here we will evaluate the W-neutrino loop diagram with Feynman-'t Hooft gauge and general R_ξ gauge in part (a) and part (b) respectively, and Z-muon diagram in part (c).

(a) Now we come to the W-neutrino loop diagram and the corresponding Goldstone boson diagrams, shown in Fig. 21.1.

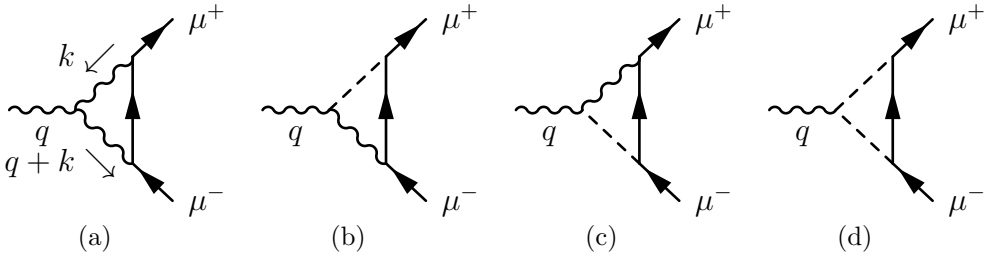


Figure 21.1: The weak-interaction contributions to muon's EM vertex. These four diagrams contain neutrino internal lines in the loops.

The Fig. 21.1(a) with W-neutrino loop reads

$$\begin{aligned}\delta_\nu^{(a)}\Gamma^\mu(q) &= \frac{(\mathrm{i}g)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\big[g^{\rho\lambda}(2k+q)^\mu + g^{\lambda\mu}(-2q-k)^\rho + g^{\rho\mu}(q-k)^\lambda\big] \\ &\quad\times\frac{-\mathrm{i}g_{\rho\sigma}}{k^2-m_W^2}\frac{-\mathrm{i}g_{\lambda\kappa}}{(q+k)^2-m_W^2}\bar{u}(p')\gamma^\sigma\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\gamma^\kappa\Big(\frac{1-\gamma^5}{2}\Big)u(p) \\ &= \frac{\mathrm{i}g^2}{2}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2}{(k'^2-\Delta)^3}\bar{u}(p')\Big[(2k+q)^\mu\gamma^\sigma(\not{p}'+\not{k})\gamma_\sigma\end{aligned}$$

$$+(-\not{k}-2\not{q})(\not{p}'+\not{k})\gamma^\mu+\gamma^\mu(\not{p}'+\not{k})(\not{q}-\not{k})\Big]\Big(\frac{1-\gamma^5}{2}\Big)u(p), \tag{21.1}$$

where

$$k'=k+xq+yp',$$
$$\Delta=(1-y)m_W^2-x(1-x)q^2-y(1-y)p'^2+2xyq\cdot p'.$$

To extract the form factor $F_2(q^2)$, recall that the total diagram can be written as a linear combination of $(p'+p)^\mu$, q^μ, γ^μ, and parity-violating terms containing γ^5. Only the $(p'+p)^\mu$ terms contribute to $F_2(q^2)$ through the Gordon identity. With this in mind, now we try to simplify the expression in the square bracket in (21.1), during which we will drops terms proportional to q^μ or γ^μ freely, and totally ignore the γ^5 terms.

$$\begin{aligned}
&\Big[(2k+q)^\mu\gamma^\sigma(\not{p}'+\not{k})\gamma_\sigma\Big]+\Big[(-\not{k}-2\not{q})(\not{p}'+\not{k})\gamma^\mu\Big]+\Big[\gamma^\mu(\not{p}'+\not{k})(\not{q}-\not{k})\Big]\\
=&\Big[-2\big(2k'+(1-2x)q-2yp'\big)^\mu\big(\not{k}'-x\not{q}+(1-y)\not{p}'\big)\Big]\\
&+\Big[\big(-\not{k}'-(2-x)\not{q}+y\not{p}'\big)\big(\not{k}'-x\not{q}+(1-y)\not{p}'\big)\gamma^\mu\Big]\\
&+\Big[\gamma^\mu\big(\not{k}'-x\not{q}+(1-y)\not{p}'\big)\big(-\not{k}'+(1+x)\not{q}+y\not{p}'\big)\Big]\\
\Rightarrow&\Big[4y(1-y)mp'^\mu\Big]+\Big[2(x+2y-2)mp^\mu\Big]+\Big[2(-1-x+y)mp'^\mu\Big]\\
\Rightarrow&-(1-y)(3-2y)m(p'+p)^\mu\\
\Rightarrow&\,2(1-y)(3-2y)m^2\cdot\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}.
\end{aligned}$$

The steps of this calculation is basically in parallel with the one of Problem 7.2. Here we have written the mass of muon as m instead of m_μ to avoid confusions. Thus the contribution to the muon's AMM from Fig. 21.1(a) is

$$\begin{aligned}
&\frac{\mathrm{i}g^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2}{(k'^2-\Delta)^3}\cdot\frac{1}{2}\cdot2(1-y)(3-2y)m^2\\
\simeq&\,\frac{7}{3}\cdot\frac{g^2m^2}{64\pi^2m_W^2}=\frac{7}{3}\cdot\frac{G_Fm^2}{8\pi^2\sqrt{2}},
\end{aligned}\tag{21.2}$$

where we have used the approximation $m_W\gg m$, and set $q^2=0$ in the second line. The Fermi constant $G_F/\sqrt{2}=g^2/8m_W^2$.

Fig. 21.1(b) and 21.1(c) read

$$\begin{aligned}
\delta_\delta^{(b)}\Gamma^\mu(q)=&\frac{\mathrm{i}g}{\sqrt{2}}\cdot\frac{-m_W}{2}\cdot\frac{-\mathrm{i}\sqrt{2}gm}{m_W}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,g^{\mu\rho}\frac{\mathrm{i}}{k^2-m_W^2}\frac{-\mathrm{i}g_{\rho\sigma}}{(q+k)^2-m_W^2}\\
&\times\bar{u}(p')\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\gamma^\sigma\Big(\frac{1-\gamma^5}{2}\Big)u(p).
\end{aligned}\tag{21.3}$$

$$\delta_\delta^{(c)}\Gamma^\mu(q)=\frac{\mathrm{i}g}{\sqrt{2}}\cdot\frac{-m_W}{2}\cdot\frac{-\mathrm{i}\sqrt{2}gm}{m_W}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,g^{\mu\rho}\frac{-\mathrm{i}g_{\rho\sigma}}{k^2-m_W^2}\frac{\mathrm{i}}{(q+k)^2-m_W^2}$$

$$\times \bar{u}(p')\gamma^\sigma\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\Big(\frac{1+\gamma^5}{2}\Big)u(p). \tag{21.4}$$

Through the calculation similar to that of Fig. 21.1(a), it is easy to show that these two diagrams contribute the same to the AMM, which reads

$$\frac{1}{2}\cdot\frac{G_F m^2}{8\pi^2\sqrt{2}}. \tag{21.5}$$

Finally, Fig. 21.1(d) reads

$$\begin{aligned}\delta_\delta^{(d)}\Gamma^\mu(q) = &\Big(\frac{-\mathrm{i}\sqrt{2}gm}{m_W}\Big)^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}(2k-q)^\mu\frac{\mathrm{i}}{k^2-m_W^2}\frac{\mathrm{i}}{(q+k)^2-m_W^2}\\ &\times\bar{u}(p')\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\Big(\frac{1+\gamma^5}{2}\Big)u(p).\end{aligned} \tag{21.6}$$

But it is not difficult to see that the contribution to the muon's AMM from this diagram is proportional to $(m/m_W)^4$, which can be omitted in the limit $m_W \gg m$, compared with the other three diagrams. Therefore we conclude that the AMM of the muon contributed by W-neutrino and corresponding Goldstone boson's 1-loop diagrams is

$$a_\mu(\nu) = \Big[\frac{7}{3}+\frac{1}{2}+\frac{1}{2}+\mathcal{O}\Big(\frac{m^2}{m_W^2}\Big)\Big]\cdot\frac{G_F m^2}{8\pi^2\sqrt{2}} \simeq \frac{10}{3}\cdot\frac{G_F m^2}{8\pi^2\sqrt{2}}. \tag{21.7}$$

(c) Now we come to the second set of diagrams as shown in Fig. 21.2.

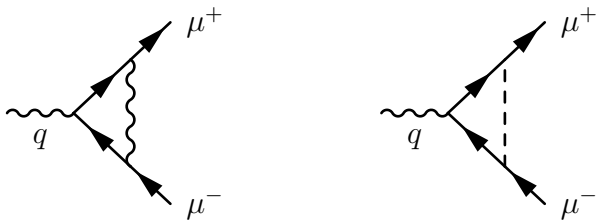


Figure 21.2: The weak-interaction contributions to muon's EM vertex. These two diagrams contains no neutrino internal lines.

Firstly the Fig. 21.2(a) reads

$$\begin{aligned}\delta_Z^{(a)}\Gamma^\mu(q) = &\Big(\frac{\mathrm{i}g}{4c_w}\Big)^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}g_{\rho\sigma}}{(p'+k)^2-m_Z^2}\\ &\times\bar{u}(p')\gamma^\rho\big(4s_w^2-1-\gamma^5\big)\frac{\mathrm{i}}{-\not{k}-m}\gamma^\mu\frac{\mathrm{i}}{-\not{q}-\not{k}-m}\gamma^\sigma\big(4s_w^2-1-\gamma^5\big)u(p)\\ \Rightarrow\ &\frac{-\mathrm{i}g^2}{16c_w^2}\int\frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2}{(k'^2-\Delta)^3}\bar{u}(p')\gamma^\rho\Big[(\not{k}+m)\gamma^\mu(\not{k}+\not{q}+m)\\ &+(4s_w^2-1)^2(\not{k}-m)\gamma^\mu(\not{k}+\not{q}-m)\Big]\gamma_\rho u(p),\end{aligned} \tag{21.8}$$

where we have omitted terms proportional to γ^5, as indicated by "$\Rightarrow$" sign, and

$$k' = k + xq + yp',$$
$$\Delta = (1-y)m^2 + ym_Z^2 - x(1-x)q^2 - y(1-y)p'^2 + 2xyq\cdot p'.$$

We will again focus only on the terms proportional to $(p'+p)^\mu$. Then the spinor part can be reduced to

$$\bar{u}(p')\gamma^\rho\Big[(\not{k}+m)\gamma^\mu(\not{k}+\not{q}+m) + (3s_w^2 - c_w^2)^2(\not{k}-m)\gamma^\mu(\not{k}+\not{q}-m)\Big]\gamma_\rho u(p)$$
$$\Rightarrow \Big[2y(3+y) - (4s_w^2-1)^2\cdot 2y(1-y)\Big]2m^2\cdot\bar{u}(p')\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}u(p).$$

Thus the AMM contribute by this diagram is

$$-\frac{\mathrm{i}g^2}{16c_w^2}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2\cdot 2m^2[2y(3+y)-(4s_w^2-1)^2\cdot 2y(1-y)]}{(k'^2-\Delta)^3}$$
$$= \frac{G_F m^2}{8\pi^2\sqrt{2}}\cdot\frac{1}{3}\Big[(4s_w^2-1)^2 - 5\Big]. \tag{21.9}$$

On the other hand, the Fig. 21.2(b) only contributes terms of order m^4/m_W^4 that can be omitted, as can be seen from the coupling between the Goldstone boson and the muon. Thus we conclude that the total contribution to $a_\mu(Z)$ from the two diagrams in Fig. 21.2 at the leading order is given by (21.9).

21.2 Complete analysis of $e^+e^- \to W^+W^-$

In this problem we calculate the amplitude for the process $e^+e^- \to W^+W^-$ at tree level in standard electroweak theory. There are 3 diagrams contributing in total, as shown in Figure 21.3.

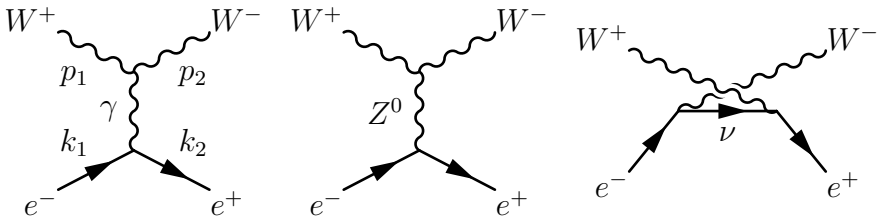


Figure 21.3: The process $e^-e^+ \to W^+W^-$ at tree level. All initial momenta go inward and all final momenta go outward.

We will evaluate these diagrams for definite helicities for initial electrons as well as definite polarizations for final W bosons. The initial and final momenta can be parameterized as

$$k_1^\mu = (E,0,0,E), \qquad p_1^\mu = (E, p\sin\theta, 0, p\cos\theta),$$
$$k_2^\mu = (E,0,0,-E), \qquad p_2^\mu = (E, -p\sin\theta, 0, -p\cos\theta), \tag{21.10}$$

with $E^2 = p^2 + m_W^2$, and electron mass ignored. For initial electron and positron, the spinors with definite helicities can be chosen to be

$$\begin{aligned} u_L(k_1) &= \sqrt{2E}(0,1,0,0)^T, & v_L(k_2) &= \sqrt{2E}(1,0,0,0)^T, \\ u_R(k_1) &= \sqrt{2E}(0,0,1,0)^T, & v_R(k_2) &= \sqrt{2E}(0,0,0,1)^T. \end{aligned} \tag{21.11}$$

For final W bosons, the polarization vectors are

$$\begin{aligned} \epsilon^*_{-\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0,-\cos\theta,-\mathrm{i},\sin\theta), & \epsilon^*_{-\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0,\cos\theta,-\mathrm{i},-\sin\theta), \\ \epsilon^*_{+\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0,-\cos\theta,\mathrm{i},\sin\theta), & \epsilon^*_{+\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0,\cos\theta,\mathrm{i},-\sin\theta), \\ \epsilon^*_{L\mu}(p_1) &= \tfrac{1}{m_W}(p,-E\sin\theta,0,-E\cos\theta), & \epsilon^*_{L\mu}(p_2) &= \tfrac{1}{m_W}(p,E\sin\theta,0,E\cos\theta). \end{aligned} \tag{21.12}$$

It is easy to see that for initial electron-positron pair, only two helicity states $e_L^- e_R^+$ and $e_R^- e_L^+$ contribute nonzero amplitudes. This is because the first two diagrams with s-channel gauge bosons vanish for the other two possibilities $e_L^- e_L^+$ and $e_R^- e_R^+$ due to angular momentum conservation, while the third diagram vanishes since the weak coupling vanishes for right-handed electron and left-handed positron. With this known, we can write down the amplitudes for $e_L^- e_R^+$ and $e_R^- e_L^+$ initial states, as follows. Generally the amplitude reads

$$\begin{aligned} &\mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W^+W^-) \\ &= \Bigg\{ \left[(-\mathrm{i}e)\frac{-\mathrm{i}}{(k_1+k_2)^2}(\mathrm{i}e) + \frac{\mathrm{i}e(-\frac{1}{2}+s_w^2)}{c_w s_w}\frac{-\mathrm{i}}{(k_1+k_2)^2 - m_Z^2}(\mathrm{i}g c_w) \right] \\ &\quad \times \bar v_L(k_2)\gamma_\lambda u_L(k_1)\left[\eta^{\mu\nu}(p_2-p_1)^\lambda + \eta^{\nu\lambda}(-p_1-2p_2)^\mu + \eta^{\lambda\mu}(2p_1+p_2)^\nu\right] \\ &\quad + \left(\frac{\mathrm{i}g}{\sqrt{2}}\right)^2 \bar v_L(k_2)\gamma^\mu \frac{\mathrm{i}}{\not k_1 - \not p_2}\gamma^\nu u_L(k_1) \Bigg\} \epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2) \\ &= \mathrm{i}e^2 \left[\frac{m_Z^2}{s(s-m_Z^2)} - \frac{1}{2s_w^2}\frac{1}{s-M_Z^2} \right] \bar v_L(k_2) \Big(\epsilon^*(p_1)\cdot\epsilon^*(p_2)(\not p_2 - \not p_1) \\ &\quad - (p_1+2p_2)\cdot\epsilon^*(p_1)\not\epsilon^*(p_2) + (2p_1+p_2)\cdot\epsilon^*(p_2)\not\epsilon^*(p_1) \Big) u_L(k_1) \\ &\quad - \frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot \bar v_L(k_2)\not\epsilon^*(p_1)(\not k_1 - \not p_2)\not\epsilon^*(p_2)u_L(k_1), \end{aligned} \tag{21.13}$$

and,

$$\begin{aligned} &\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W^+W^-) \\ &= \left[(-\mathrm{i}e)\frac{-\mathrm{i}}{(k_1+k_2)^2}(\mathrm{i}e) + \frac{\mathrm{i}e s_w^2}{c_w s_w}\frac{-\mathrm{i}}{(k_1+k_2)^2 - m_Z^2}(\mathrm{i}g c_w) \right] \bar v_R(k_2)\gamma_\lambda u_R(k_1) \\ &\quad \times \left[\eta^{\mu\nu}(p_2-p_1)^\lambda + \eta^{\nu\lambda}(-p_1-2p_2)^\mu + \eta^{\lambda\mu}(2p_1+p_2)^\nu\right] \epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2) \\ &= \mathrm{i}e^2 \frac{m_Z^2}{s(s-m_Z^2)} \bar v_R(k_2) \Big(\epsilon^*(p_1)\cdot\epsilon^*(p_2)(\not p_2 - \not p_1) - (p_1+2p_2)\cdot\epsilon^*(p_1)\not\epsilon^*(p_2) \\ &\quad + (2p_1+p_2)\cdot\epsilon^*(p_2)\not\epsilon^*(p_1) \Big) u_R(k_1), \end{aligned} \tag{21.14}$$

In what follows we need the inner products among some of these vectors, as listed below.

$$p_1\cdot p_2 = E^2+p^2 \qquad p_1\cdot\epsilon_0^*(p_2) = p_2\cdot\epsilon_0^*(p_1) = \frac{2Ep}{m_W},$$
$$\epsilon_+^*(p_1)\cdot\epsilon_+^*(p_2) = \epsilon_-^*(p_1)\cdot\epsilon_-^*(p_2) = 1, \qquad \epsilon_0^*(p_1)\cdot\epsilon_0^*(p_2) = \frac{E^2+p^2}{m_W^2}. \tag{21.15}$$

We also need

$$\bar v_L(k_2)\not p_1 u_L(k_1) = -\bar v_L(k_2)\not p_2 u_L(k_1) = 2Ep\sin\theta, \tag{21.16}$$
$$\bar v_L(k_2)\not\epsilon_\pm^*(p_1)u_L(k_1) = 2E\Big(\frac{\mp1+\cos\theta}{\sqrt2}\Big), \tag{21.17}$$
$$\bar v_L(k_2)\not\epsilon_\pm^*(p_2)u_L(k_1) = 2E\Big(\frac{\mp1-\cos\theta}{\sqrt2}\Big), \tag{21.18}$$
$$\bar v_L(k_2)\not\epsilon_0^*(p_1)u_L(k_1) = -\bar v_L(k_2)\not\epsilon_0^*(p_2)u(k_1) = \frac{2E^2\sin\theta}{m_W}, \tag{21.19}$$
$$\bar v_R(k_2)\not p_1 u_R(k_1) = -\bar v_R(k_2)\not p_2 u_R(k_1) = -2Ep\sin\theta, \tag{21.20}$$
$$\bar v_L(k_2)\not\epsilon_\pm^*(p_1)u_L(k_1) = -2E\Big(\frac{\mp1+\cos\theta}{\sqrt2}\Big), \tag{21.21}$$
$$\bar v_L(k_2)\not\epsilon_\pm^*(p_2)u_L(k_1) = -2E\Big(\frac{\mp1-\cos\theta}{\sqrt2}\Big), \tag{21.22}$$
$$\bar v_L(k_2)\not\epsilon_0^*(p_1)u_L(k_1) = -\bar v_L(k_2)\not\epsilon_0^*(p_2)u(k_1) = -\frac{2E^2\sin\theta}{m_W}. \tag{21.23}$$

We first consider $e_L^-e_R^+\to W^+W^-$. In this case we take $u(k_1)=u_L(k_1)=\sqrt{2E}(0,1,0,0)^T$ and $\bar v(k_2)=\bar v_L(k_2)=\sqrt{2E}(0,0,1,0)$. Then each of the final W particle can have polarization $(+,-,0)$, which gives 9 possible combinations for (W^+,W^-). Now we evaluate the corresponding amplitudes in turn.

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^-e_R^+\to W_{(0)}^+W_{(0)}^-)\\
&=\mathrm{i}e^2\bigg[\frac{m_Z^2}{s(s-m_Z^2)}-\frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\bigg]\Big(-\frac{4Ep(E^2+p^2)}{m_W^2}+\frac{16E^3p}{m_W^2}\Big)\sin\theta\\
&\quad+\frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot\frac{2E(-3E^2p+p^3-2E^3\cos\theta)\sin\theta}{m_W^2}\\
&=-\mathrm{i}e^2\cdot\frac{s}{4m_W^2}\bigg\{\frac{m_Z^2}{s-m_Z^2}\cdot\beta(3-\beta^2)\\
&\quad-\frac{1}{2s_w^2}\bigg[\Big(\frac{2}{1+\beta^2+2\beta\cos\theta}-\frac{s}{s-m_Z^2}\Big)\beta(3-\beta^2)+\frac{4\cos\theta}{1+\beta^2+2\beta\cos\theta}\bigg]\bigg\}\sin\theta
\end{aligned}\tag{21.24}$$

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^-e_R^+\to W_{(0)}^+W_{(\pm)}^-)=\mathrm{i}\mathcal{M}(e_L^-e_R^+\to W_{(\mp)}^+W_{(0)}^-)\\
&=\mathrm{i}e^2\bigg[\frac{m_Z^2}{s(s-m_Z^2)}-\frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\bigg]\Big(\frac{8E^2p}{m_W}\frac{\mp1+\cos\theta}{\sqrt2}\Big)\\
&\quad-\frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot\frac{2E}{m_W}(E^2(2\cos\theta\mp1)+2Ep\pm p^2)\frac{\pm1+\cos\theta}{\sqrt2}
\end{aligned}$$

$$= \mathrm{i}e^2\left[\frac{m_Z^2}{s-m_Z^2}\beta - \frac{1}{2s_W^2}\left(\frac{s}{s-m_Z^2}\beta + \frac{\pm 1 - 2\cos\theta - 2\beta \mp \beta^2}{1+\beta^2+2\beta\cos\theta}\right)\right]\frac{\sqrt{s}}{m_W}\frac{\pm 1+\cos\theta}{\sqrt{2}} \tag{21.25}$$

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W_{(\pm)}^+ W_{(\pm)}^-) \\
&= \mathrm{i}e^2\left[\frac{m_Z^2}{s(s-m_Z^2)} - \frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\right]\Big(-4Ep\sin\theta\Big) + \frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot 2E(p+E\cos\theta)\sin\theta \\
&= \mathrm{i}e^2\left[-\frac{m_Z^2}{(s-m_Z^2)}\beta + \frac{1}{2s_w^2}\left(\frac{s}{s-M_Z^2}\beta - \frac{2(\beta+\cos\theta)}{(1+\beta^2+2\beta\cos\theta)}\right)\right]\sin\theta
\end{aligned} \tag{21.26}$$

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W_{(\pm)}^+ W_{(\mp)}^-) \\
&= -\frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot 2E^2(\mp 1+\cos\theta)\sin\theta = \frac{\mathrm{i}e^2}{2s_w^2}\frac{2(\pm 1-\cos\theta)\sin\theta}{(1+\beta^2+2\beta\cos\theta)}.
\end{aligned} \tag{21.27}$$

Though not manifest, these expressions have correct high energy behavior. To see this, we note that $\beta \simeq 1 - 2m_W^2/s$ when $s \gg m_W^2$. Then, for instance, the amplitude for two longitudinal W final state becomes

$$\begin{aligned}
\mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W_{(0)}^+ W_{(0)}^-) &= -\mathrm{i}e^2\cdot\frac{s}{4m_W^2}\left\{\frac{m_Z^2}{s-m_Z^2}\cdot\beta(3-\beta^2)\right. \\
&\left. - \frac{1}{2s_w^2}\left[\Big(\frac{2}{1+\beta^2+2\beta\cos\theta} - \frac{s}{s-m_Z^2}\Big)\beta(3-\beta^2) + \frac{4\cos\theta}{1+\beta^2+2\beta\cos\theta}\right]\right\}\sin\theta \\
&= -\frac{\mathrm{i}e^2}{2s_W^2}\frac{(1+2\cos\theta)\sin\theta}{1+\cos\theta} + \mathcal{O}(1/s).
\end{aligned} \tag{21.28}$$

Then we can plot the azimuthal distribution of the corresponding differential cross section at $s = (1000\mathrm{GeV})^2$, as shown in Figure 21.4.

Next we consider the other case with $e_R^- e_L^+$ initial state. Now there is no contribution from u-channel neutrino exchange. The amplitudes for various polarizations of final W pairs can be worked out to be

$$\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(0)}^+ W_{(0)}^-) = \mathrm{i}e^2\frac{s}{s-m_Z^2}\frac{m_Z^2}{4m_W^2}\beta(\beta^2-3)\sin\theta, \tag{21.29}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(0)}^+ W_{(\pm)}^-) &= \mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(\mp)}^+ W_{(0)}^-) \\
&= \mathrm{i}e^2\frac{m_Z^2}{s-m_Z^2}\frac{\sqrt{s}}{m_W}\beta\frac{\pm 1-\cos\theta}{\sqrt{2}}
\end{aligned} \tag{21.30}$$

$$\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(\pm)}^+ W_{(\pm)}^-) = \mathrm{i}e^2\frac{m_Z^2}{s-m_Z^2}\beta\sin\theta \tag{21.31}$$

$$\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(\pm)}^+ W_{(\mp)}^-) = 0. \tag{21.32}$$

21.3 Cross section for $d\bar{u} \to W^-\gamma$

In this problem we compute the tree amplitude of $d\bar{u} \to W^-\gamma$ at high energies so that the quark masses can be ignored. In this case the left-handed and right-handed spinors

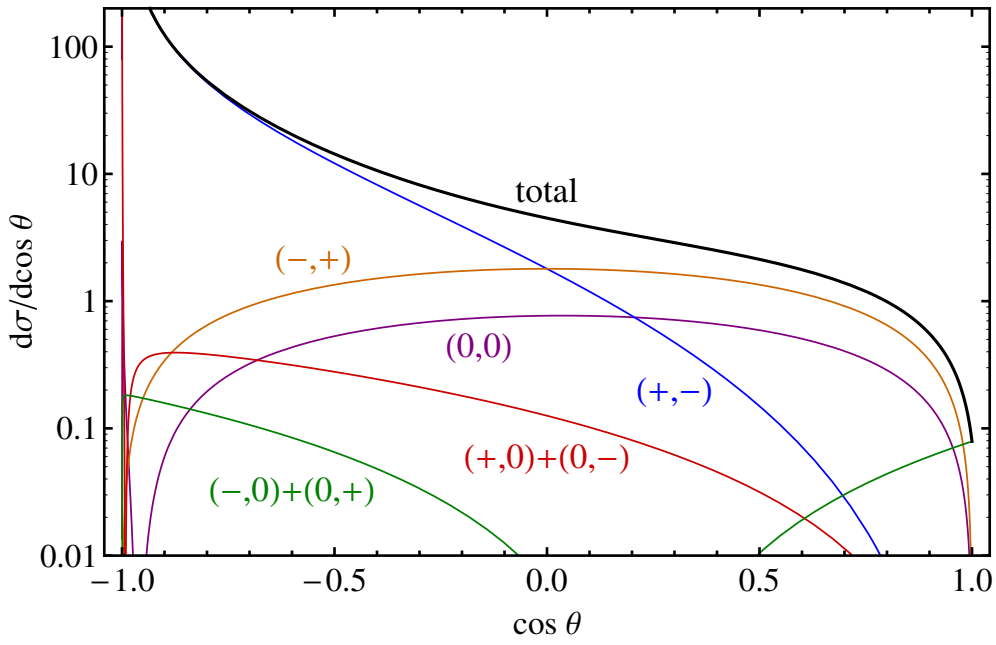


Figure 21.4: The differential cross section of $e^-e^+ \to W^+W^-$ with definite helicity as a function of azimuthal angle at $s = (1000\text{GeV})^2$.

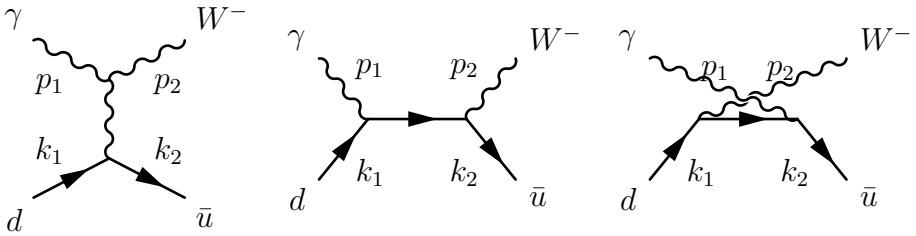


Figure 21.5: The process $d\bar{u} \to W^-\gamma$ at tree level. All initial momenta go inward and all final momenta go outward.

decouple and only the amplitudes with $d_L\bar{u}_R$ initial state do not vanish. To calculate it, we firstly work out the kinematics as follows.

$$\begin{aligned} k_1 &= (E, 0, 0, E), & p_1 &= (p, p\sin\theta, 0, p\cos\theta), \\ k_2 &= (E, 0, 0, -E), & p_2 &= (E_W, -p\sin\theta, 0, -p\cos\theta), \end{aligned} \tag{21.33}$$

where $p = E - m_W^2/4E$ and $E_W = E + m_W^2/4E$. The initial spinors of definite helicities are given by

$$u_L(k_1) = \sqrt{2E}(0,1,0,0)^T, \qquad v_L(k_2) = \sqrt{2E}(1,0,0,0)^T, \tag{21.34}$$

while the polarization vectors for final photon and W^- read

$$\begin{aligned} \epsilon^*_{\pm\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0, -\cos\theta, \pm\mathrm{i}, \sin\theta), & \epsilon^*_{\pm\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0, \cos\theta, \pm\mathrm{i}, -\sin\theta), \\ \epsilon^*_{L\mu}(p_2) &= \tfrac{1}{m_W}(p, E_W\sin\theta, 0, E_W\cos\theta). \end{aligned} \tag{21.35}$$

Then the amplitude is given by

$$\mathrm{i}\mathcal{M}(d_L\bar{u}_R \to \gamma W^-) = \frac{-\mathrm{i}e^2}{\sqrt{2}s_w}\frac{N_s}{s - m_W^2} - \frac{\mathrm{i}e^2}{3\sqrt{2}s_w}\left(\frac{-N_t}{t} + \frac{2N_u}{u}\right), \tag{21.36}$$

where

$$N_s = \bar{v}_L(k_2)\big[\epsilon^*(p_1)\cdot\epsilon^*(p_2)(\not{p}_1 - \not{p}_2) + (p_1+2p_2)\cdot\epsilon^*(p_1)\not{\epsilon}^*(p_2) - (2p_1+p_2)\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)\big]u_L(k_1), \tag{21.37}$$

$$N_t = \bar{v}_L(k_2)\not{\epsilon}^*(p_2)(\not{k}_1 - \not{p}_1)\not{\epsilon}^*(p_1)u_L(k_1), \tag{21.38}$$

$$N_u = \bar{v}_L(k_2)\not{\epsilon}^*(p_1)(\not{k}_1 - \not{p}_2)\not{\epsilon}^*(p_2)u_L(k_1). \tag{21.39}$$

Now, using the physical conditions $\epsilon^*(p_i)\cdot p_i = 0$, $\not{k}_1 u_L(k_1) = 0$ and $\bar{v}_L(k_2)\not{k}_2 = 0$, we can show that $N_s = N_t - N_u$. In fact,

$$\begin{aligned}
N_s &= \bar{v}_L(k_2)\big[2\epsilon^*(p_1)\cdot\epsilon^*(p_2)\not{p}_1 + 2p_2\cdot\epsilon^*(p_1)\not{\epsilon}^*(p_2) - 2p_1\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)\big]u_L(k_1),\\
N_t &= \bar{v}_L(k_2)\big[2k_1\cdot\epsilon_1\not{\epsilon}_2 + 2\epsilon^*(p_1)\cdot\epsilon^*(p_2)\not{p}_1 - \not{\epsilon}^*(p_1)\not{\epsilon}^*(p_2)\not{p}_1\big]u_L(k_1),\\
N_u &= \bar{v}_L(k_2)\big[-2k_2\cdot\epsilon_1\not{\epsilon}_2 + 2p_1\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1) - \not{\epsilon}^*(p_1)\not{\epsilon}^*(p_2)\not{p}_1\big]u_L(k_1).
\end{aligned}$$

Then $N_s = N_t - N_u$ is manifest. Note further that $s - m_W^2 = -(t+u)$, we have

$$\begin{aligned}
\mathrm{i}\mathcal{M}(d_L\bar{u}_R \to \gamma W^-) &= \frac{\mathrm{i}e^2}{\sqrt{2}s_w}\left(\frac{N_t - N_u}{t+u} - \frac{N_t}{3t} + \frac{2N_u}{3u}\right)\\
&= \frac{\mathrm{i}e^2}{\sqrt{2}s_w}\frac{(2t-u)}{3(t+u)}\Big(\frac{N_t}{t} + \frac{N_u}{u}\Big)\\
&= \frac{\mathrm{i}e^2}{6\sqrt{2}s_w}(1-3\cos\theta)\Big(\frac{N_t}{t} + \frac{N_u}{u}\Big).
\end{aligned} \tag{21.40}$$

One can see clearly from this expression that all helicity amplitudes vanish at $\cos\theta = 1/3$. (Note that the definition of scattering angle θ is different from the one in Peskin & Schroeder, which, in our notation, is $\pi - \theta$.) Then, by including all helicity combinations (6 in total), we find the differential cross section, as a function of s and θ, to be

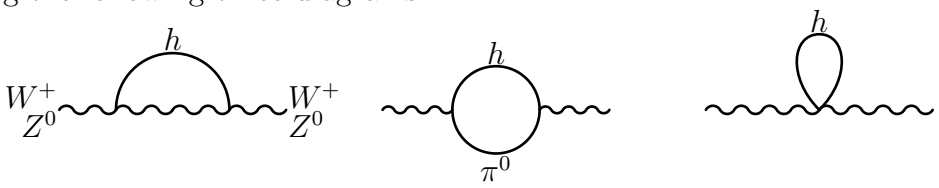
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{32s_w^2}\Big(\frac{1-\cos\theta}{\sin\theta}\Big)^2\frac{x^3+18x^2+9x+24-(x^3-14x^2+9x-8)\cos 2\theta}{36(s-m_W^2)}, \tag{21.41}$$

where $x \equiv m_W^2/s$.

21.4 Dependence of radiative corrections on the Higgs boson mass

(a) We first analyze the radiative corrections to μ decay process at 1-loop level with the Higgs boson in the loop. It is easy to see that if the internal Higgs boson line is attached to one of the external fermions, the resulted vertex will contribute a factor of m_f/v which can be ignored. Therefore only the vacuum polarization diagrams are relevant, and they should sum to a gauge invariant result.

(b) Now we compute the vacuum polarization amplitudes of $W^\pm$, Z^0 and photon with Higgs contribution. We will only consider the pieces proportional to $g^{\mu\nu}$, namely $\Pi_{WW}(q^2)$, $\Pi_{ZZ}(q^2)$, $\Pi_{\gamma\gamma}(q^2)$ and $\Pi_{Z\gamma}(q^2)$. It is easy to show that $\Pi_{\gamma\gamma}(q^2)$ and $\Pi_{ZZ}(q^2)$ receive no contribution from Higgs boson at 1-loop level, while $\Pi_{WW}(q^2)$ and $\Pi_{ZZ}(q^2)$ can be found by computing the following three diagrams:



Now we compute these three diagrams in turn for W^+. The first diagram reads

$$\begin{aligned}
&(\mathrm{i}gm_W)^2 g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m_h^2}\frac{-\mathrm{i}}{(q-k)^2-m_W^2}\\
=&-\frac{\mathrm{i}}{(4\pi)^2}g^2m_W^2g^{\mu\nu}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}(m_W^2,q^2)}\\
\Rightarrow&-\frac{\mathrm{i}}{(4\pi)^2}g^2m_W^2g^{\mu\nu}\left[E+\int_0^1\mathrm{d}x\,\log\frac{M^2}{\Delta(m_W^2,q^2)}\right],
\end{aligned}\tag{21.42}$$

where $\Delta(m_W^2,q^2)=xm_W^2+(1-x)m_h^2-x(1-x)q^2$, $E=2/\epsilon-\gamma+\log 4\pi-\log M^2$, and M^2 is the subtraction scale. The second one reads

$$\begin{aligned}
&(\mathrm{i}g/2)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m_h^2}\frac{-\mathrm{i}}{(q-k)^2-m_W^2}(2k-p)^\mu(2k-p)^\nu\\
\Rightarrow&\frac{g^2}{4}g^{\mu\nu}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{(4/d)k'^2}{(k'^2-\Delta(m_W^2,q^2))^2}\\
\Rightarrow&\frac{\mathrm{i}}{(4\pi)^2}\frac{g^2}{4}g^{\mu\nu}\int_0^1\mathrm{d}x\,2\Delta(m_W^2,q^2)\left[E+1+\log\frac{M^2}{\Delta(m_W^2,q^2)}\right],
\end{aligned}\tag{21.43}$$

in which we have ignored terms proportional to $q^\mu q^\nu$. Then, the last diagram reads

$$\begin{aligned}
&\frac{1}{2}(\mathrm{i}g^2/2)g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m_h^2}=-\frac{g^2}{4}g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m_h^2}\frac{(q-k)^2-m_W^2}{(q-k)^2-m_W^2}\\
=&-\frac{g^2}{4}g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{k'^2+(1-x)^2q^2-m_W^2}{(k'^2-\Delta(m_W^2,q^2))^2}\\
\Rightarrow&-\frac{\mathrm{i}}{(4\pi)^2}\frac{g^2}{4}g^{\mu\nu}\int_0^1\mathrm{d}x\left[\Big(2\Delta(m_W^2,q^2)-m_W^2+(1-x)^2q^2\Big)E\right.\\
&\left.+\Big(2\Delta(m_W^2,q^2)-m_W^2+(1-x)^2q^2\Big)\log\frac{M^2}{\Delta(m_W^2,q^2)}+\Delta(m_W^2,q^2)\right].
\end{aligned}\tag{21.44}$$

Thus we have, when the three diagrams above are taken into account only,

$$\begin{aligned}
\Pi_{WW}(q^2)=\frac{g^2}{4(4\pi)^2}\Bigg[&-\Big(3m_W^2+\frac{1}{3}q^2\Big)E\\
&+\int_0^1\mathrm{d}x\left(\Delta(m_W^2,q^2)-\big[3m_W^2+(1-x)^2q^2\big]\log\frac{M^2}{\Delta(m_W^2,q^2)}\right)\Bigg].
\end{aligned}\tag{21.45}$$

Now we extract Higgs mass contribution from this expression in the large Higgs limit, and also fix the subtraction point at $M^2 = m_W^2$. In this limit we may take $\Delta(m_W^2, q^2) \simeq x m_h^2$, and $\log(M^2/\Delta) \simeq -\log(m_h^2/m_W^2)$. We also throw divergent terms with E, which should be canceled out in the final expression of zeroth order natural relation after including completely loop diagrams with W, Z, and would-be Goldstone boson internal lines. Then we have

$$\Pi_{WW}(q^2) = \frac{g^2}{4(4\pi)^2}\left[\frac{1}{2}m_h^2 + \left(3m_W^2 + \frac{1}{3}q^2\right)\log\frac{m_h^2}{m_W^2}\right]. \tag{21.46}$$

Similarly, we have, for $\Pi_{ZZ}(q^2)$,

$$\Pi_{ZZ}(q^2) = \frac{g^2}{4(4\pi)^2\cos^2\theta_w}\left[\frac{1}{2}m_h^2 + \left(3m_Z^2 + \frac{1}{3}q^2\right)\log\frac{m_h^2}{m_Z^2}\right]. \tag{21.47}$$

(c) Now, we derive the zeroth order natural relation given in (21.134) of Peskin & Schroeder, in the large Higgs mass limit. Note that $\Pi_{\gamma\gamma} = \Pi_{Z\gamma} = 0$. Thus,

$$\begin{aligned} s_*^2 - \sin^2\theta_0^2 &= \frac{\sin^2\theta_w\cos^2\theta_w}{\cos^2\theta_w - \sin^2\theta_w}\left(\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2}\right) \\ &= \frac{\alpha}{48\pi}\frac{1 + 9\sin^2\theta_w}{\cos^2\theta_w - \sin^2\theta_w}\log\frac{m_h^2}{m_W^2}, \end{aligned} \tag{21.48}$$

$$s_W^2 - s_*^2 = -\frac{\Pi_{WW}(m_W^2)}{m_Z^2} + \frac{m_W^2}{m_Z^2}\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} = \frac{5\alpha}{24\pi}\log\frac{m_h^2}{m_W^2}. \tag{21.49}$$

Final Project III
Decays of the Higgs Boson

In this final project, we calculate partial widths of various decay channels of the standard model Higgs boson. Although a standard-model-Higgs-like boson has been found at the LHC with mass around 125GeV, it is still instructive to treat the mass of the Higgs boson as a free parameter in the following calculation.

The main decay modes of Higgs boson include $h^0 \to f\bar{f}$ with f the standard model fermions, $h^0 \to W^+W^-$, $h^0 \to Z^0Z^0$, $h^0 \to gg$ and $h^0 \to \gamma\gamma$. The former three processes appear at the tree level, while the leading order contributions to the latter two processes are at one-loop level. We will work out the decay widths of these processes in the following.

In this problem we only consider the two-body final states. The calculation of decay width needs the integral over the phase space of the two-body final states. By momentum conservation and rotational symmetry, we can always parameterize the momenta of two final particles in CM frame to be $p_1 = (E, 0, 0, p)$ and $p_2 = (E, 0, 0, -p)$, where $E = \frac{1}{2}m_h$ by energy conservation. Then the amplitude $\mathcal{M}$ will have no angular dependence. Then the phase space integral reads

$$\int \mathrm{d}\Pi_2 \, |\mathcal{M}|^2 = \frac{1}{4\pi}\frac{p}{m_h}|\mathcal{M}|^2. \tag{21.50}$$

Then the decay width is given by

$$\Gamma = \frac{1}{2m_h}\int \mathrm{d}\Pi_2 \, |\mathcal{M}|^2 = \frac{1}{8\pi}\frac{p}{m_h^2}|\mathcal{M}|^2. \tag{21.51}$$

In part (d) of this problem, we will also be dealing with the production of the Higgs boson from two-gluon initial state, thus we also write down the formula here for the cross section of the one-body final state from two identical initial particle. This time, the two ingoing particles have momenta $k_1 = (E, 0, 0, k)$ and $k_2 = (E, 0, 0, -k)$, with $E^2 = k^2 + m_i^2$ and $2E = m_f$ where m_i and m_f are masses of initial particles and final particle, respectively. The final particle has momentum $p = (m_f, 0, 0, 0)$. Then, the cross section is given by

$$\begin{aligned}\sigma &= \frac{1}{2\beta s}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p - k_1 - k_2) \\ &= \frac{1}{4m_f\beta s}|\mathcal{M}|^2(2\pi)\delta(2k - m_f) = \frac{\pi}{\beta m_f^2}|\mathcal{M}|^2\delta(s - m_f^2),\end{aligned} \tag{21.52}$$

where $\beta = \sqrt{1-(4m_i/m_f)^2}$ is the magnitude of the velocity of the initial particle in the center-of-mass frame.

(a) The easiest calculation of above processes is $h^0 \to f\bar{f}$, where f represents all quarks and charged leptons. The tree level contribution to this process involves a single Yukawa vertex only. The corresponding amplitude is given by

$$\mathrm{i}\mathcal{M}(h^0 \to f\bar{f}) = -\frac{\mathrm{i}m_f}{v}\bar{u}^*(p_1)v(p_2). \tag{21.53}$$

Then it is straightforward to get the squared amplitude with final spins summed to be

$$\sum |\mathcal{M}(h^0 \to f\bar{f})|^2 = \frac{m_f^2}{v^2}\,\mathrm{tr}\left[(\not{p}_1 + m_f)(\not{p}_2 - m_f)\right] = \frac{2m_f^2}{v^2}(m_h^2 - 4m_f^2). \tag{21.54}$$

In CM frame, the final states momenta can be taken to be $p_1 = (E, 0, 0, p)$ and $p_2 = (E, 0, 0, -p)$, with $E = \frac{1}{2}m_h$ and $p^2 = E^2 - m_f^2$. Then the decay width is given by

$$\Gamma(h^0 \to f\bar{f}) = \frac{1}{8\pi}\frac{p}{m_h^2}|\mathcal{M}|^2 = \frac{m_h m_f^2}{8v^2}\left(1 - \frac{4m_f^2}{m_h^2}\right)^{3/2}. \tag{21.55}$$

This expression can be expressed in terms of the fine structure constant α, the mass of W boson m_w and Weinberg angle $\sin\theta_w$, as

$$\Gamma(h^0 \to f\bar{f}) = \frac{\alpha m_h}{8\sin^2\theta_w}\frac{m_f^2}{m_W^2}\left(1 - \frac{4m_f^2}{m_h^2}\right)^{3/2}. \tag{21.56}$$

(b) Next we consider the decay of h^0 to massive vector bosons W^+W^- and Z^0Z^0. The amplitude for the process $h^0 \to W^+W^-$ is given by

$$\mathrm{i}\mathcal{M}(h^0 \to W^+W^-) = \frac{\mathrm{i}g^{\mu\nu}g^2 v}{2}\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2). \tag{21.57}$$

Then the squared amplitude with final polarizations summed reads

$$\begin{aligned}\sum |\mathcal{M}|^2 &= \frac{g^4v^2}{4}\left(g_{\mu\nu} - \frac{p_{1\mu}p_{1\nu}}{m_W^2}\right)\left(g^{\mu\nu} - \frac{p_2^\mu p_2^\nu}{m_W^2}\right) \\ &= \frac{\pi\alpha}{\sin^2\theta_w}\frac{m_h^4}{m_W^2}\left(1 - \frac{4m_W^2}{m_h^2} + \frac{12m_W^4}{m_h^4}\right).\end{aligned} \tag{21.58}$$

Therefore the decay width is

$$\Gamma(h^0 \to W^+W^-) = \frac{1}{8\pi}\frac{p_1}{m_h^2}|\mathcal{M}|^2 = \frac{\alpha m_h^3}{16\pi m_W^2\sin^2\theta_w}(1 - 4\tau_W^{-1} + 12\tau_W^{-2})(1 - 4\tau_W^{-1})^{1/2}, \tag{21.59}$$

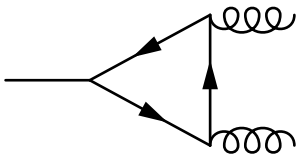
where we have defined $\tau_W \equiv (m_h/m_W)^2$ for brevity. For $h^0 \to Z^0Z^0$ process, it can be easily checked that nothing gets changed in the calculation except that all m_W should be replaced with m_Z, while an additional factor $1/2$ is needed to account for the identical particles in final state. Therefore we have

$$\Gamma(h^0 \to Z^0Z^0) = \frac{\alpha m_h^3}{32\pi m_Z^2\sin^2\theta_w}(1 - 4\tau_Z^{-1} + 12\tau_Z^{-2})(1 - 4\tau_Z^{-1})^{1/2}, \tag{21.60}$$

where $\tau_Z \equiv (m_h/m_Z)^2$.

The calculation above considered "on-shell" decay only, while for realistic 125GeV boson, the off-shell decay turns out to be very important. That is, although $h^0 \to W^+W^-$ and $h^0 \to Z^0Z^0$ are kinetically forbidden when $m_h = 125\text{GeV}$ according to above results, the produced W or Z pair can subsequently decay into lighter fermions, and the process like $h^0 \to W^*(f\bar{f})W$ contributes considerable amount of Higgs decay, where $W^*(f\bar{f})$ means an off-shell W decaying to a pair of fermions. More details can be found in [8]

(c) Now we come to the process $h^0 \to gg$. The leading order contribution comes from diagrams with one quark loop.



The amplitude reads

$$\begin{aligned}
\mathrm{i}\mathcal{M}(h^0 \to gg) = &-\frac{\mathrm{i}m_q}{v}(\mathrm{i}g_s)^2\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\,\mathrm{tr}\,(t^a t^b) \\
&\times \int \frac{\mathrm{d}^d q}{(2\pi)^d}\left\{(-1)\,\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not{q}-m_q}\gamma^\nu \frac{\mathrm{i}}{\not{q}+\not{p}_2-m_q}\frac{\mathrm{i}}{\not{q}-\not{p}_1-m_q}\right]\right. \\
&\left.+(-1)\,\mathrm{tr}\left[\gamma^\nu \frac{\mathrm{i}}{\not{q}-m_q}\gamma^\mu \frac{\mathrm{i}}{\not{q}+\not{p}_1-m_q}\frac{\mathrm{i}}{\not{q}-\not{p}_2-m_q}\right]\right\} \qquad (21.61)
\end{aligned}$$

The first trace in the integrand can be simplified through standard procedure,

$$\begin{aligned}
&\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not{q}-m_q}\gamma^\nu \frac{\mathrm{i}}{\not{q}+\not{p}_2-m_q}\frac{\mathrm{i}}{\not{q}-\not{p}_1-m_q}\right] \\
&= \frac{-\mathrm{i}\,\mathrm{tr}\left[(\not{q}+m_q)(\not{q}+\not{p}_2-m_q)(\not{q}-\not{p}_1-m_q)\right]}{(q^2-m_q^2)\left[(q+p_2)^2-m_q^2\right]\left[(q-p_1)^2-m_q^2\right]} \\
&= -2\mathrm{i}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{N^{\mu\nu}}{(q'^2-\Delta)^3}, \qquad (21.62)
\end{aligned}$$

where

$$q'_\mu = q_\mu - xp_{1\mu} + yp_{2\mu}, \qquad (21.63)$$

$$\begin{aligned}
\Delta &= m_q^2 - x(1-x)p_1^2 - y(1-y)p_2^2 - 2xyp_1\cdot p_2 = m_q^2 - xym_h^2, \\
N^{\mu\nu} &= 4m_q\big(p_1^\nu p_2^\nu - p_1^\mu p_2^\nu + 2p_2^\nu q^\mu - 2p_1^\mu q^\nu + 4q^\mu q^\nu + (m_q^2 - p_1\cdot p_2 - q^2)\eta^{\mu\nu}\big) \qquad (21.64)
\end{aligned}$$

Then we can reexpress $N^{\mu\nu}$ in terms of q', p_1 and p_2 and drop off all terms linear in q which integrates to zero. It is most easy to work with definite helicity states for final gluons. Then the result gets simplified if we dot $N^{\mu\nu}$ with polarization vectors as $N^{\mu\nu}\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)$. Note

that $\epsilon^*(p_i)\cdot p_j = 0$ with $i,j = 1,2$. Note also the on-shell condition $p_1^2 = p_2^2 = 0$, $p_1\cdot p_2 = \frac{1}{2}m_h^2$. then

$$N^{\mu\nu}\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2) = 4m_q\Big[m_q^2 + \big(xy - \tfrac{1}{2}\big)m_h^2 + \big(\tfrac{4}{d} - 1\big)q'^2\Big]\epsilon^*(p_1)\cdot\epsilon^*(p_2). \tag{21.65}$$

The same calculation shows that the second trace in the integrand of (21.61) gives identical result with the first trace. To check the gauge invariance of this result, one can simply replace $\epsilon_\mu^*(p_1)$ with $p_{1\mu}$ in the expression above, then it is straightforward to find that $N^{\mu\nu}p_{1\mu}\epsilon_\nu^*(p_2) = 0$. Similarly, it can also be checked that $N^{\mu\nu}\epsilon_\mu^*(p_1)p_{2\nu} = 0$.

Then the amplitude (21.61) now reads

$$\mathrm{i}\mathcal{M}(h^0 \to gg) = -\frac{2g_s^2 m_q}{v}\delta^{ab}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^d q'}{(2\pi)^d}\frac{N^{\mu\nu}\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)}{(q'^2-\Delta)^3}, \tag{21.66}$$

where the relation $\mathrm{tr}\,(t^a t^b) = \frac{1}{2}\delta^{ab}$ in fundamental representation is also used. The momentum integration is finite as $d \to 4$ under dimensional regularization, and can now be carried out directly to be

$$\begin{aligned}\mathrm{i}\mathcal{M}(h^0 \to gg) &= -\frac{2\mathrm{i}g_s^2 m_q^2}{(4\pi)^2 v}\delta^{ab}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{(1-4xy)m_h^2}{m_q^2 - xym_h^2}\\ &= -\frac{\mathrm{i}\alpha_s m_h^2}{6\pi v}\delta^{ab}\epsilon^*(p_1)\cdot\epsilon^*(p_2)I_f(\tau_q),\end{aligned} \tag{21.67}$$

where $\tau_q \equiv (m_h/m_q)^2$, and

$$I_f(\tau_q) \equiv 3\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1-4xy}{1-xy\tau_q}$$

Note that the inner product between two polarization vectors is nonzero only for $\epsilon_+^*\cdot\epsilon_-^*$ and $\epsilon_-^*\cdot\epsilon_+^*$. Therefore the squared amplitude with final states polarizations, color indices summed ($\delta^{ab}\delta_{ab} = 8$) is,

$$|\mathcal{M}(h^0 \to gg)|^2 = |\mathcal{M}_{+-}(h^0 \to gg)|^2 + |\mathcal{M}_{-+}(h^0 \to gg)|^2 = \frac{4\alpha_s^2 m_h^4}{9\pi^2 v^2}|I_f(\tau_q)|^2, \tag{21.68}$$

and the decay width is

$$\Gamma(h^0 \to gg) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big)\cdot\frac{m_h^2}{m_w^2}\cdot\frac{\alpha_s^2}{9\pi^2}\cdot|I_f(\tau_q)|^2, \tag{21.69}$$

where an additional factor 1/2 should be included in (21.51) when calculating $\Gamma(h^0 \to gg)$ because the two gluons in final states are identical particles. This result is easily generalized for N_q copies of quarks to be

$$\Gamma(h^0 \to gg) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big)\cdot\frac{m_h^2}{m_W^2}\cdot\frac{\alpha_s^2}{9\pi^2}\cdot\Big|\sum_q I_f(\tau_q)\Big|^2, \tag{21.70}$$

(d) Now we calculate the cross section for the Higgs production via gluon fusion at the leading order. The amplitude is simply given by the result in (c), namely (21.67). When we take the square of this amplitude, an additional factor $(\frac{1}{8}\cdot\frac{1}{2})^2$ should be included, to average over helicities and color indices of initial gluons. Then, comparing (21.52) with (21.51), we find that

$$\sigma(gg\to h^0)=\frac{\pi^2}{8m_h}\delta(\hat{s}-m_h^2)\Gamma(h^0\to gg), \tag{21.71}$$

where the hatted variable $\hat{s}$ is the parton level center-of-mass energy. We note again that the correct formula is obtained by including an factor of $(\frac{1}{8}\cdot\frac{1}{2})^2$ in $\sigma(gg\to h^0)$ to average over the initial degrees of freedom of two gluons, and an factor of $1/2$ in $\Gamma(h^0\to gg)$ to count the identical particles in the final state. Then, from (21.70), it is straightforward to find

$$\sigma(gg\to h^0)=\frac{\alpha\alpha_s^2}{576\sin^2\theta_w}\cdot\frac{m_h^2}{m_W^2}\Big|\sum_q I_f(\tau_q)\Big|^2\delta(\hat{s}-m_h^2). \tag{21.72}$$

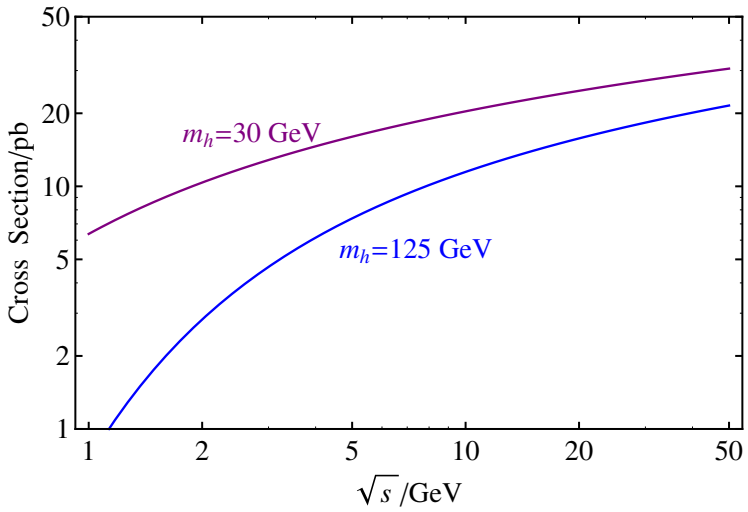
Then the proton-level cross section of Higgs boson production via gluon-gluon fusion is given by

$$\begin{aligned}
&\sigma_{GGF}\big(p(P_1)p(P_2)\to h^0\big)\\
&=\int_0^1 \mathrm{d}x_1\int_0^1 \mathrm{d}x_2\, f_g(x_1)f_g(x_2)\sigma\big(g(x_1P_1)g(x_2P_2)\to h^0\big)\\
&=\int \mathrm{d}M^2\mathrm{d}Y\left|\frac{\partial(x_1,x_2)}{\partial(M^2,Y)}\right| f_g(x_1)f_g(x_2)\sigma\big(g(x_1P_1)g(x_2P_2)\to h^0\big)\\
&=\int \mathrm{d}M^2\mathrm{d}Y\,\frac{1}{M^2}x_1f_g(x_1)x_2f_g(x_2)\sigma\big(g(x_1P_1)g(x_2P_2)\to h^0\big),
\end{aligned} \tag{21.73}$$

where $M^2=x_1x_2s$ is the center-of-mass energy of two initial gluons, while s is the center-of-mass energy of two initial protons, and Y, given by $\exp Y=\sqrt{x_1/x_2}$, is the rapidity of the produced Higgs boson relative to the center-of-mass frame of the proton system. (Note that in our case $M^2=m_h^2$.) The relations between M^2, Y and the momentum fractions x_1, x_2 can be inverted to give $x_1=(M/\sqrt{s})e^Y$ and $x_2=(M/\sqrt{s})e^{-Y}$. Furthermore, f_g is the parton distribution function of the gluon in a proton, which we will take to be $f_g=8(1-x)^7/x$ in the following calculations. Then the cross section can be evaluated to be

$$\begin{aligned}
&\sigma_{GGF}\big(p(P_1)p(P_2)\to h^0\big)\\
&=\frac{\alpha\alpha_s^2}{9\sin^2\theta_w}\cdot\frac{1}{m_W^2}\Big|\sum_q I_f(\tau_q)\Big|^2\int_{-Y_0}^{Y_0}\mathrm{d}Y\left(1-\frac{m_h}{\sqrt{s}}e^Y\right)^7\left(1-\frac{m_h}{\sqrt{s}}e^{-Y}\right)^7,
\end{aligned} \tag{21.74}$$

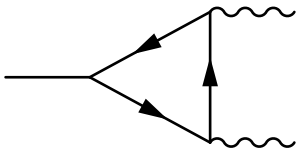
where Y_0, given by $\cosh Y_0=\sqrt{s}/2m_h$ is the largest possible rapidity of a produced Higgs boson. We plot this cross section as a function of the center-of-mass energy $\sqrt{s}$ of the pp pair, with the Higgs boson's mass taken to be $m_h=30\mathrm{GeV}$ and $m_h=125\mathrm{GeV}$, respectively, in Figure



(e) Next we consider the process $h^0 \to 2\gamma$. The contribution to this decay channel at the leading (1-loop) level is from two types of diagrams, one with a fermion loop and the other with a W boson (and related would-be Goldstone boson) loop. The former contribution is easy to find by virtue of the result in (c) for $h^0 \to gg$. The calculation here is in fully parallel, except that we should include the factor for the electric charges of internal fermions Q_f, take away the color factor $\mathrm{tr}\,(t^a t^b)$, change the strong coupling g_s by the electromagnetic coupling e, and sum over all charged fermions. Note that the color factor enters the expression of the decay width as $|\,\mathrm{tr}\,(t^a t^b)|^2 = \frac{1}{2}\delta^{ab}\frac{1}{2}\delta^{ab} = 2$, then it is straightforward to write down the fermion contribution to the $h^0 \to 2\gamma$ to be

$$\mathrm{i}\mathcal{M}(h^0 \to 2\gamma)_f = \left(\frac{\alpha m_h}{8\sin^2\theta_w}\right)\cdot\frac{m_h^2}{m_w^2}\cdot\frac{\alpha_s^2}{18\pi^2}\cdot\left|\sum_f Q_f^2 N_c(f) I_f(\tau_f)\right|^2, \tag{21.75}$$

where $N_c(f)$ is the color factor, equal to 3 for quarks and 1 for charged leptons.



(f) Now we come to the W-loop contributions to $h^0 \to 2\gamma$. In Feynman-'t Hooft gauge, we should also include the corresponding Goldstone loop diagrams. Then there are 13 diagrams in total. We compute them as follows,



(a) (b)

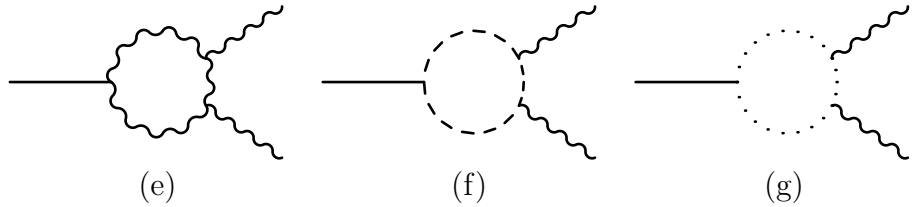
$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{a})} &= \frac{1}{2}\frac{\mathrm{i}g_{\rho\sigma}g^2v}{2}(-\mathrm{i}e^2)(2\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\\
&\quad\times\int\frac{\mathrm{d}^dq}{(2\pi)^d}D_W(q)D_W(k-q)\\
&=-\frac{2\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)(d-1)\Gamma(2-\tfrac{d}{2})\\
&\quad\times\int_0^1\frac{\mathrm{d}x}{[m_W^2-x(1-x)m_h^2]^{2-d/2}},
\end{aligned}
\tag{21.76}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{b})} &= \frac{1}{2}(-2\mathrm{i}\lambda v)(2\mathrm{i}e^2)\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\frac{\mathrm{d}^dq}{(2\pi)^d}D_s(q)D_s(k-q)\\
&=-\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_h^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\\
&\quad\times\int_0^1\frac{\mathrm{d}x}{[m_W^2-x(1-x)m_h^2]^{2-d/2}}.
\end{aligned}
\tag{21.77}
$$

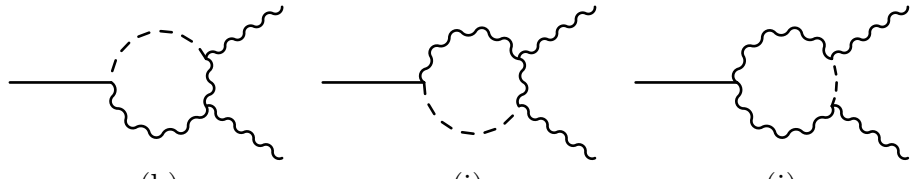


(c) (d)

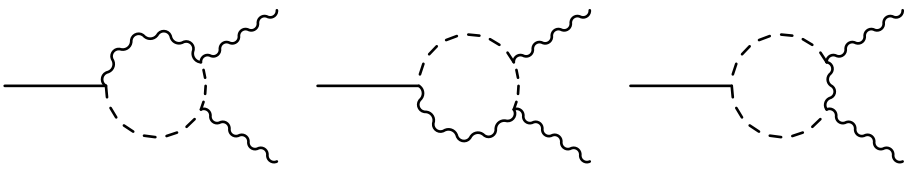
$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{c})}=\mathrm{i}\mathcal{M}^{(\mathrm{d})} &= \frac{\mathrm{i}g^2\sin\theta_w}{2}\cdot\frac{\mathrm{i}g^2v\sin\theta_w}{2}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\frac{\mathrm{d}^dq}{(2\pi)^d}D_s(q)D_W(p_2-q)\\
&=-\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\frac{1}{(m_W^2)^{2-d/2}}.
\end{aligned}
\tag{21.78}
$$



(e) (f) (g)



(h) (i) (j)



(k) (l) (m)

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{e})} =& \frac{\mathrm{i}g^2 v}{2}(-\mathrm{i}e)^2\eta_{\rho\sigma}\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_W(q)D_W(q-p_1)D_W(q+p_2)\\
&\times\left[\eta^{\rho\lambda}(2q-p_1)^\mu+\eta^{\mu\rho}(2p_1-q)^\lambda-\eta^{\lambda\mu}(p_1+q)^\rho\right]\\
&\times\left[\eta_\lambda^\sigma(2q+p_2)^\nu-\eta_\lambda^\nu(q-p_2)^\sigma-\eta^{\sigma\nu}(2p_2+q)_\lambda\right]\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\bigg[\int\mathrm{d}x\mathrm{d}y\frac{(5-x-y+4xy)m_h^2}{m_W^2-xym_h^2}\\
&+6(d-1)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}}\bigg],
\end{aligned}
\tag{21.79}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{f})} =& (-2\mathrm{i}\lambda\nu)(-\mathrm{i}e)^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}(2q-p_1)^\mu(2q+p_2)^\nu\\
&\times D_s(q)D_s(q-p_1)D_s(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_h^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\int\frac{2\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}},
\end{aligned}
\tag{21.80}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{g})} =&\Big(-\frac{\mathrm{i}m_W^2}{v}\Big)(\mathrm{i}e)^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}(-1)(q-p_1)^\mu q^\nu\\
&\times D_s(q)D_s(q-p_1)D_s(q+p_2)\\
=&-\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}},
\end{aligned}
\tag{21.81}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{h})} =\mathrm{i}\mathcal{M}^{(\mathrm{i})} =& \frac{\mathrm{i}g}{2}\frac{\mathrm{i}g^{\mu\lambda}g^2v\sin\theta_w}{2}(-\mathrm{i}e)\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}(q-p_1-k)^\sigma\\
&\times\left[\eta_{\sigma\lambda}(2q+p_2)^\nu-\eta_\lambda^\nu(q-p_2)_\sigma-\eta_\sigma^\nu(2p_2+q)_\lambda\right]\\
&\times D_W(q)D_s(q-p_1)D_W(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\bigg[\int\mathrm{d}x\mathrm{d}y\frac{(1-x)(1+y)m_h^2}{m_W^2-xym_h^2}\\
&-\frac{1}{2}(d-1)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}}\bigg],
\end{aligned}
\tag{21.82}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{j})} =& \frac{\mathrm{i}g^2v}{2}\Big(\frac{\mathrm{i}g^2v\sin\theta_w}{2}\Big)^2\epsilon^*(p_1)\cdot\epsilon^*(p_2)\\
&\times\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_s(q)D_W(q-p_1)D_W(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\mathrm{d}x\mathrm{d}y\frac{2m_W^2}{m_W^2-xym_h^2},
\end{aligned}
\tag{21.83}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{k})} =\mathrm{i}\mathcal{M}^{(\mathrm{l})} =& \frac{\mathrm{i}g}{2}\frac{\mathrm{i}g^2v\sin\theta_w}{2}(-\mathrm{i}e)\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\\
&\times\int\frac{\mathrm{d}^d q}{(2\pi)^d}(p_1+2p_2+q)^\mu(2q+p_2)^\nu D_s(q)D_W(q-p_1)D_s(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}},
\end{aligned}
\tag{21.84}
$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{m})} &= (-2\mathrm{i}\lambda\nu)\Big(\frac{\mathrm{i}g^2 v\sin\theta_w}{2}\Big)^2\epsilon^*(p_1)\cdot\epsilon^*(p_2) \\
&\quad\times\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_W(q)D_s(q-p_1)D_s(q+p_2) \\
&= \frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_h^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\mathrm{d}x\mathrm{d}y\frac{m_W^2}{m_W^2-xym_h^2}.
\end{aligned}
\tag{21.85}
$$

The results can be summarized as,

$$
\mathrm{i}\mathcal{M}^{(X)} = \frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Big[A\cdot\Gamma(2-\tfrac{d}{2})+B\Big],\quad (X=a,b,\cdots,m)
\tag{21.86}
$$

with the coefficients A and B for each diagram listed in Table.

Diagrams	A	B
(a)	$-2(d-1)J_1$	0
(c)+(d)	$-2(m_W^2)^{d/2-2}$	0
(e)	$6(d-1)J_2$	J_3
(g)	$-J_2$	0
(h)+(i)	$-(d-1)J_2$	$2J_4$
(j)	0	$2(m_W/m_h)^2J_5$
(k)+(l)	$2J_2$	0
(m)	0	J_5
(b)	$-(m_h/m_W)^2J_1$	0
(f)	$2(m_h/m_W)^2J_2$	0

where

$$
\begin{aligned}
J_1 &= \int_0^1\mathrm{d}x\frac{1}{[m_W^2-x(1-x)m_h^2]^{2-d/2}} \\
&= 1-\frac{\epsilon}{2}\int_0^1\mathrm{d}x\log\Big(m_W^2-x(1-x)m_h^2\Big)+\mathcal{O}(\epsilon^2),
\end{aligned}
\tag{21.87}
$$

$$
\begin{aligned}
J_2 &= \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{1}{(m_W^2-xym_h^2)^{2-d/2}} \\
&= \frac{1}{2}-\frac{\epsilon}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\log\Big(m_W^2-xym_h^2\Big)+\mathcal{O}(\epsilon^2),
\end{aligned}
\tag{21.88}
$$

$$
J_3 = \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(5-x-y+4xy)m_h^2}{m_W^2-xym_h^2},
\tag{21.89}
$$

$$
J_4 = \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(1-x)(1+y)m_h^2}{m_W^2-xym_h^2},
\tag{21.90}
$$

$$
J_5 = \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{m_h^2}{m_W^2-xym_h^2}.
\tag{21.91}
$$

To see that the divergences of all diagrams cancel among themselves, it just needs to show that sum of all A-coefficients is of order ϵ. This is straightforward by noting that $J_1 = 1+\mathcal{O}(\epsilon)$ and $J_2 = 1/2+\mathcal{O}(\epsilon)$.

Before reaching the complete result, let us first find out the W-loop contribution in the limit $m_h^2 \ll m_W^2$, although it seems unlikely to be true within our current knowledge. To find the amplitude in this limit, we expand the five integrals $J_1, \cdots, J_5$ in terms of m_h/m_W,

$$J_1 \simeq 1 - \frac{\epsilon}{2}\log m_W^2 + \frac{\epsilon}{12}\frac{m_h^2}{m_W^2}, \qquad J_2 \simeq \frac{1}{2} - \frac{\epsilon}{4}\log m_W^2 + \frac{\epsilon}{48}\frac{m_h^2}{m_W^2},$$
$$J_3 \simeq \frac{7}{3}\frac{m_h^2}{m_W^2}, \qquad J_4 \simeq \frac{11}{24}\frac{m_h^2}{m_W^2}, \qquad J_5 \simeq \frac{1}{2}\frac{m_h^2}{m_W^2} + \frac{1}{24}\Big(\frac{m_h^2}{m_W^2}\Big)^2.$$

Then the amplitude can be recast into

$$\mathrm{i}\mathcal{M} = \frac{\mathrm{i}e^2 m_W^2}{(4\pi)^2 v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\left[C\Big(\frac{2}{\epsilon} - \gamma + \log 4\pi\Big) + D\cdot\log m_W^2 + E + F\cdot\frac{m_h^2}{m_W^2}\right] \tag{21.92}$$

Diagrams	C	D	E	F
(a)	-6	3	4	-1
(c)+(d)	-2	1	0	0
(e)	9	$-9/2$	-6	$37/12$
(g)	$-1/2$	$1/4$	0	$-1/24$
(h)+(i)	$-3/2$	$3/4$	1	$19/24$
(j)	0	0	1	$1/12$
(k)+(l)	1	$-1/2$	0	$1/12$
(m)	0	0	0	$1/2$
(b)	$-(m_h/m_W)^2$	$(m_h/m_W)^2/2$	0	0
(f)	$(m_h/m_W)^2$	$-(m_h/m_W)^2/2$	0	0
sum	0	0	0	$7/2$

Therefore, the amplitude in the limit $m_h^2 \ll m_W^2$ is given by

$$\mathrm{i}\mathcal{M}(h^0 \to 2\gamma)_W = 2\cdot\frac{7}{2}\frac{\mathrm{i}\alpha m_h^2}{4\pi v}\epsilon^*(p_1)\cdot\epsilon^*(p_2), \tag{21.93}$$

where the factor 2 counts the identical contributions from the diagrams with two final photons changed. Now we sum up the fermion-loop contribution found in (e) and the result here to get the $h^0 \to 2\gamma$ amplitude in the light Higgs limit,

$$\mathrm{i}\mathcal{M} = -\frac{\mathrm{i}\alpha m_h^2}{3\pi v}\left[\sum_f Q_f^2 N_c(f) - \frac{21}{4}\right]\epsilon^*(p_1)\cdot\epsilon^*(p_2). \tag{21.94}$$

Then the corresponding partial width is given by

$$\Gamma(h^0 \to 2\gamma) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big)\cdot\frac{m_h^2}{m_w^2}\cdot\frac{\alpha^2}{18\pi^2}\cdot\left|\sum_f Q_f^2 N_c(f) - \frac{21}{4}\right|^2, \tag{21.95}$$

Now we retain m_h as a free variable. Then the various diagrams sum into the following full expression for the W-loop contribution to $h^0 \to 2\gamma$,

$$\mathrm{i}\mathcal{M}(h^0 \to 2\gamma)_W = \frac{\mathrm{i}\alpha m_h^2}{2\pi v}\epsilon^*(p_1)\cdot\epsilon^*(p_2) I_W(\tau_W), \tag{21.96}$$

where the factor $I_W(\tau_W)$, as a function of $\tau_W \equiv (m_h/m_W)^2$, is given by

$$I_W(\tau_W) = \frac{1}{\tau_W}\Big[6I_1(\tau_W) - 8I_2(\tau_W) + \tau_W\big(I_1(\tau_W) - I_2(\tau_W)\big) + I_3(\tau_W)\Big], \tag{21.97}$$

where

$$I_1(\tau_W) \equiv \int_0^1 \mathrm{d}x\, \log\big[1 - x(1-x)\tau_W\big], \tag{21.98}$$

$$I_2(\tau_W) \equiv 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \log\big(1 - xy\tau_W\big), \tag{21.99}$$

$$I_3(\tau_W) \equiv \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{(8 - 3x + y + 4xy)\tau_W}{1 - xy\tau_W}. \tag{21.100}$$

Then the full expression for the partial width of $h^0 \to 2\gamma$ at one-loop is

$$\Gamma(h^0 \to 2\gamma) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big) \cdot \frac{m_h^2}{m_w^2} \cdot \frac{\alpha^2}{18\pi^2} \cdot \Big|\sum_f Q_f^2 N_c(f) I_f(\tau_f) - I_W(\tau_W)\Big|^2, \tag{21.101}$$

(h) Collecting all results above (expect the $\gamma\gamma$ channel, which is quite small*), we plot the total width and decay branching fractions of the Higgs boson in Figures 21.6 and 21.7, respectively.

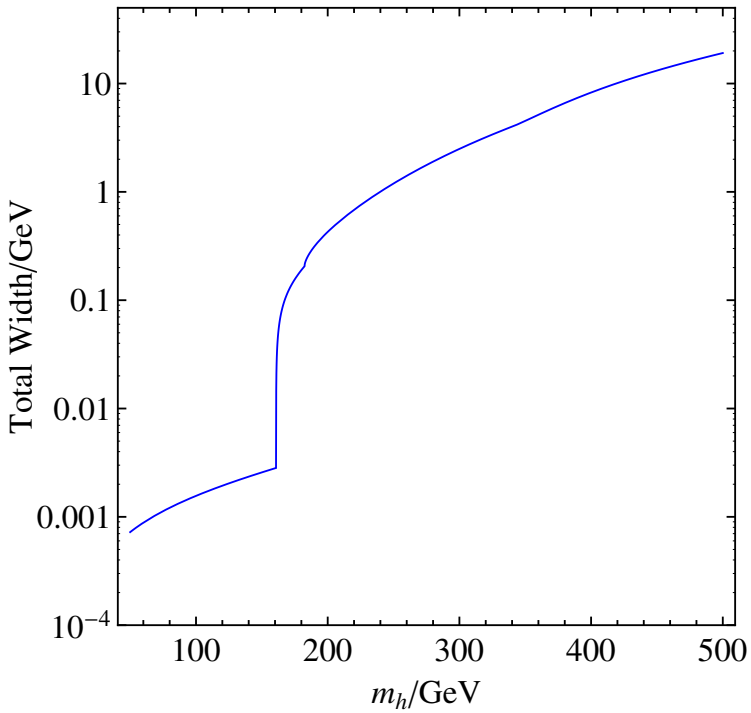


Figure 21.6: The total width of the Higgs boson as a function of its mass.

*– but very important!

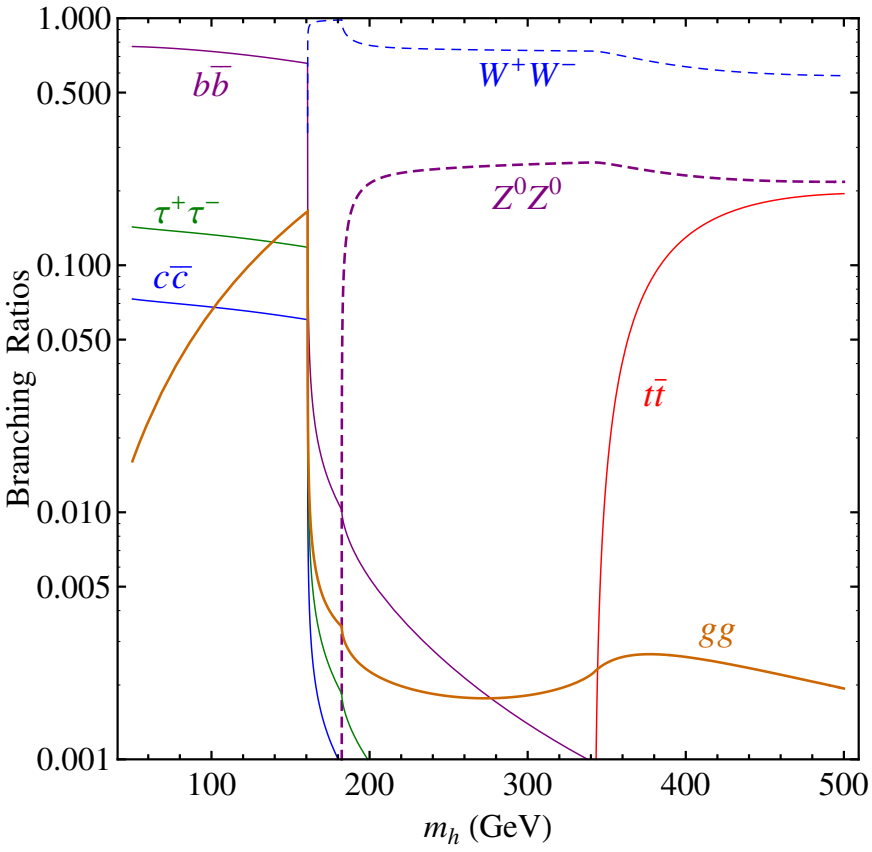


Figure 21.7: The Higgs decay branching fractions of $t\bar{t}$, $b\bar{b}$, $c\bar{c}$, $\tau^+\tau^-$, WW, ZZ and gg channels, as functions of Higgs mass.

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The energy of the system after the source has been turned off is

$$\langle 0|\, H\, |0\rangle = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2} |\tilde{\jmath}(p)|^2, \tag{2.65}$$

where $|0\rangle$ still denotes the ground state of the free theory. We can interpret these results in terms of particles by identifying $|\tilde{\jmath}(p)|^2/2E_{\mathbf{p}}$ as the probability density for creating a particle in the mode p. Then the total number of particles produced is

(2.66)

Only those Fourier components of $j(x)$ that are in resonance with on-mass-shell (i.e., $p^2 = m^2$) Klein-Gordon waves are effective at creating particles.

We will return to this subject in Problem 4.1. In Chapter 6 we will study the analogous problem of photon creation by an accelerated electron (bremsstrahlung).

Problems

2.1 Classical electromagnetism (with no sources) follows from the action

(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu,$$

leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \tfrac{1}{2}(E^2 + B^2); \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

2.2 The complex scalar field. Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \, (\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

(c) Rewrite the conserved charge

$$Q = \int d^3x \, \frac{i}{2}(\phi^*\pi^* - \pi\phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by

$$Q^i = \int d^3x \, \frac{i}{2}(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b),$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.‡

2.3 Evaluate the function

$$\langle 0|\, \phi(x)\phi(y)\, |0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)},$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicitly in terms of Bessel functions.

---

‡With some additional work you can show that there are actually six conserved charges in the case of two complex fields, and $n(2n-1)$ in the case of n fields, corresponding to the generators of the rotation group in four and $2n$ dimensions, respectively. The extra symmetries often do not survive when nonlinear interactions of the fields are included.

we eventually find

$$C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi; \tag{3.149}$$

$$C\bar{\psi}\gamma^\mu\gamma^5\psi C = +\bar{\psi}\gamma^\mu\gamma^5\psi. \tag{3.150}$$

Although the operator C interchanges ψ and $\bar{\psi}$, it does not actually change the order of the creation and annihilation operators. Thus, if $\bar{\psi}\gamma^0\psi$ is defined to subtract the infinite constant noted above Eq. (3.113), this constant does not reappear in the process of conjugation by C.

Summary of C, P, and T

The transformation properties of the various fermion bilinears under C, P, and T are summarized in the table below. Here we use the shorthand $(-1)^\mu \equiv 1$ for $\mu = 0$ and $(-1)^\mu \equiv -1$ for $\mu = 1, 2, 3$.

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

We have included the transformation properties of the tensor bilinear (see Problem 3.7), and also of the derivative operator.

Notice first that the free Dirac Lagrangian $\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under C, P, and T separately. We can build more general quantum systems that violate any of these symmetries by adding to $\mathcal{L}_0$ some perturbation $\delta\mathcal{L}$. But $\delta\mathcal{L}$ must be a Lorentz scalar, and the last line of the table shows that all Lorentz scalar combinations of $\bar{\psi}$ and ψ are invariant under the combined symmetry CPT. Actually, it is quite generally true that one cannot build a Lorentz-invariant quantum field theory with a Hermitian Hamiltonian that violates CPT.[†]

Problems

3.1 Lorentz group. Recall from Eq. (3.17) the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

(a) Define the generators of rotations and boosts as

$$L^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i},$$

[†]This theorem and the spin-statistics theorem are proved with great care in Streater and Wightman, *op. cit.*

where $i, j, k = 1, 2, 3$. An infinitesimal Lorentz transformation can then be written

$$\Phi \rightarrow (1 - i\boldsymbol{\theta} \cdot \mathbf{L} - i\boldsymbol{\beta} \cdot \mathbf{K})\Phi.$$

Write the commutation relations of these vector operators explicitly. (For example, $[L^i, L^j] = i\epsilon^{ijk}L^k$.) Show that the combinations

$$\mathbf{J}_+ = \tfrac{1}{2}(\mathbf{L} + i\mathbf{K}) \quad \text{and} \quad \mathbf{J}_- = \tfrac{1}{2}(\mathbf{L} - i\mathbf{K})$$

commute with one another and separately satisfy the commutation relations of angular momentum.

(b) The finite-dimensional representations of the rotation group correspond precisely to the allowed values for angular momentum: integers or half-integers. The result of part (a) implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers, (j_+, j_-), corresponding to pairs of representations of the rotation group. Using the fact that $\mathbf{J} = \boldsymbol{\sigma}/2$ in the spin-1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. Show that these correspond precisely to the transformations of ψ_L and ψ_R given in (3.37).

(c) The identity $\boldsymbol{\sigma}^T = -\sigma^2\boldsymbol{\sigma}\sigma^2$ allows us to rewrite the ψ_L transformation in the unitarily equivalent form

$$\psi' \rightarrow \psi'(1 + i\boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta} \cdot \frac{\boldsymbol{\sigma}}{2}),$$

where $\psi' = \psi_L^T\sigma^2$. Using this law, we can represent the object that transforms as $(\frac{1}{2}, \frac{1}{2})$ as a 2×2 matrix that has the ψ_R transformation law on the left and, simultaneously, the transposed ψ_L transformation on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}.$$

Show that the object V^μ transforms as a 4-vector.

3.2 Derive the *Gordon identity*,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p),$$

where $q = (p' - p)$. We will put this formula to use in Chapter 6.

3.3 Spinor products. (This problem, together with Problems 5.3 and 5.6, introduces an efficient computational method for processes involving massless particles.) Let k_0^μ, k_1^μ be fixed 4-vectors satisfying $k_0^2 = 0$, $k_1^2 = -1$, $k_0 \cdot k_1 = 0$. Define basic spinors in the following way: Let u_{L0} be the left-handed spinor for a fermion with momentum k_0. Let $u_{R0} = \not{k}_1 u_{L0}$. Then, for any p such that p is lightlike ($p^2 = 0$), define

$$u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}}\not{p}\, u_{R0} \qquad \text{and} \qquad u_R(p) = \frac{1}{\sqrt{2p \cdot k_0}}\not{p}\, u_{L0}.$$

This set of conventions defines the phases of spinors unambiguously (except when p is parallel to k_0).

(a) Show that $\not{k}_0 u_{R0} = 0$. Show that, for any lightlike p, $\not{p}u_L(p) = \not{p}u_R(p) = 0$.

(b) For the choices $k_0 = (E, 0, 0, -E)$, $k_1 = (0, 1, 0, 0)$, construct u_{L0}, u_{R0}, $u_L(p)$, and $u_R(p)$ explicitly.

(c) Define the *spinor products* $s(p_1, p_2)$ and $t(p_1, p_2)$, for p_1, p_2 lightlike, by

$$s(p_1, p_2) = \bar{u}_R(p_1) u_L(p_2), \qquad t(p_1, p_2) = \bar{u}_L(p_1) u_R(p_2).$$

Using the explicit forms for the u_λ given in part (b), compute the spinor products explicitly and show that $t(p_1, p_2) = (s(p_2, p_1))^*$ and $s(p_1, p_2) = -s(p_2, p_1)$. In addition, show that

$$|s(p_1, p_2)|^2 = 2p_1 \cdot p_2.$$

Thus the spinor products are the square roots of 4-vector dot products.

3.4 Majorana fermions. Recall from Eq. (3.40) that one can write a relativistic equation for a massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (ψ_L). Call such a 2-component field $\chi_a(x)$, $a = 1, 2$.

(a) Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma} \cdot \partial \chi - im\sigma^2 \chi^* = 0.$$

That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

(b) Does the Majorana equation follow from a Lagrangian? The mass term would seem to be the variation of $(\sigma^2)_{ab}\chi_a^*\chi_b^*$; however, since σ^2 is antisymmetric, this expression would vanish if $\chi(x)$ were an ordinary c-number field. When we go to quantum field theory, we know that $\chi(x)$ will become an anticommuting quantum field. Therefore, it makes sense to develop its classical theory by considering $\chi(x)$ as a classical anticommuting field, that is, as a field that takes as values *Grassmann numbers* which satisfy

$$\alpha\beta = -\beta\alpha \qquad \text{for any } \alpha, \beta.$$

Note that this relation implies that $\alpha^2 = 0$. A Grassmann field $\xi(x)$ can be expanded in a basis of functions as

$$n$$

where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a product of Grassmann numbers to reverse the order:

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*.$$

This rule imitates the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma} \cdot \partial \chi + \frac{im}{2}(\chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^*) \right],$$

(where $\chi^\dagger = (\chi^*)^T$) is real ($S^* = S$), and that varying this S with respect to χ and χ^* yields the Majorana equation.

(c) Let us write a 4-component Dirac field as

$$\psi(x) = \qquad ,$$

and recall that the lower components of ψ transform in a way equivalent by a unitary transformation to the complex conjugate of the representation ψ_L. In this way, we can rewrite the 4-component Dirac field in terms of two 2-component spinors:

$$\psi_L(x) = \chi_1(x), \qquad \psi_R(x) = i\sigma^2\chi_2^*(x).$$

Rewrite the Dirac Lagrangian in terms of χ_1 and χ_2 and note the form of the mass term.

(d) Show that the action of part (c) has a global symmetry. Compute the divergences of the currents

$$J^\mu = \chi^\dagger\bar{\sigma}^\mu\chi, \qquad J^\mu = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2,$$

for the theories of parts (b) and (c), respectively, and relate your results to the symmetries of these theories. Construct a theory of N free massive 2-component fermion fields with $O(N)$ symmetry (that is, the symmetry of rotations in an N-dimensional space).

(e) Quantize the Majorana theory of parts (a) and (b). That is, promote $\chi(x)$ to a quantum field satisfying the canonical anticommutation relation

$$\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

construct a Hermitian Hamiltonian, and find a representation of the canonical commutation relations that diagonalizes the Hamiltonian in terms of a set of creation and annihilation operators. (Hint: Compare $\chi(x)$ to the top two components of the quantized Dirac field.)

3.5 Supersymmetry. It is possible to write field theories with continuous symmetries linking fermions and bosons; such transformations are called *supersymmetries.*

(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, written in the form

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F.$$

Here F is an auxiliary complex scalar field whose field equation is $F = 0$. Show that this Lagrangian is invariant (up to a total divergence) under the infinitesimal tranformation

$$\begin{aligned}\delta\phi &= -i\epsilon^T\sigma^2\chi,\\ \delta\chi &= \epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^*,\\ \delta F &= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi,\end{aligned}$$

where the parameter ϵ_a is a 2-component spinor of Grassmann numbers.

(b) Show that the term

$$\Delta\mathcal{L} = [m\phi F + \tfrac{1}{2}im\chi^T\sigma^2\chi] + (\text{complex conjugate})$$

is also left invariant by the transformation given in part (a). Eliminate F from the complete Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its field equation, and show that the fermion and boson fields ϕ and χ are given the same mass.

(c) It is possible to write supersymmetric nonlinear field equations by adding cubic and higher-order terms to the Lagrangian. Show that the following rather general field theory, containing the field (ϕ_i, χ_i), $i = 1, \ldots, n$, is supersymmetric:

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma} \cdot \partial \chi_i + F_i^* F_i$$

where $W[\phi]$ is an arbitrary function of the ϕ_i, called the *superpotential.* For the simple case $n = 1$ and $W = g\phi^3/3$, write out the field equations for ϕ and χ (after elimination of F).

3.6 Fierz transformations. Let u_i, $i = 1, \ldots, 4$, be four 4-component Dirac spinors. In the text, we proved the Fierz rearrangement formulae (3.78) and (3.79). The first of these formulae can be written in 4-component notation as

In fact, there are similar rearrangement formulae for any product

$$(\bar{u}_1 \Gamma^A u_2)(\bar{u}_3 \Gamma^B u_4),$$

where Γ^A, Γ^B are any of the 16 combinations of Dirac matrices listed in Section 3.4.

(a) To begin, normalize the 16 matrices Γ^A to the convention

$$\mathrm{tr}[\Gamma^A \Gamma^B] = 4\delta^{AB}.$$

This gives $\Gamma^A = \{1, \gamma^0, i\gamma^j, \ldots\}$; write all 16 elements of this set.

(b) Write the general Fierz identity as an equation

$$(\bar{u}_1 \Gamma^A u_2)(\bar{u}_3 \Gamma^B u_4) = \sum_{C,D} C^{AB}{}_{CD} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^D u_2),$$

with unknown coefficients $C^{AB}{}_{CD}$. Using the completeness of the 16 Γ^A matrices, show that

$$C^{AB}{}_{CD} = \frac{1}{16} \mathrm{tr}[\Gamma^C \Gamma^A \Gamma^D \Gamma^B].$$

(c) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4)$.

3.7 This problem concerns the discrete symmetries P, C, and T.

(a) Compute the transformation properties under P, C, and T of the antisymmetric tensor fermion bilinears, $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. This completes the table of the transformation properties of bilinears at the end of the chapter.

(b) Let $\phi(x)$ be a complex-valued Klein-Gordon field, such as we considered in Problem 2.2. Find unitary operators P, C and an antiunitary operator T (all defined

in terms of their action on the annihilation operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ for the Klein-Gordon particles and antiparticles) that give the following tranformations of the Klein-Gordon field:

$$\begin{aligned} P\,\phi(t,\mathbf{x})\,P &= \phi(t,-\mathbf{x}); \\ T\,\phi(t,\mathbf{x})\,T &= \phi(-t,\mathbf{x}); \\ C\,\phi(t,\mathbf{x})\,C &= \phi^*(t,\mathbf{x}). \end{aligned}$$

Find the transformation properties of the components of the current

$$J^\mu = i(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi)$$

under P, C, and T.

(c) Show that any Hermitian Lorentz-scalar local operator built from $\psi(x)$, $\phi(x)$, and their conjugates has $CPT = +1$.

3.8 Bound states. Two spin-1/2 particles can combine to a state of total spin either 0 or 1. The wavefunctions for these states are odd and even, respectively, under the interchange of the two spins.

(a) Use this information to compute the quantum numbers under P and C of all electron-positron bound states with S, P, or D wavefunctions.

(b) Since the electron-photon coupling is given by the Hamiltonian

where j^μ is the electric current, electrodynamics is invariant to P and C if the components of the vector potential have the same P and C parity as the corresponding components of j^μ. Show that this implies the following surprising fact: The spin-0 ground state of positronium can decay to 2 photons, but the spin-1 ground state must decay to 3 photons. Find the selection rules for the annihilation of higher positronium states, and for 1-photon transitions between positronium levels.

Note that the repulsion in fermion-fermion scattering came entirely from the extra factor $-g_{00} = -1$ in the vector boson propagator. A tensor boson, such as the graviton, would have a propagator

$$\mu\,\nu \sim\sim\sim\sim \rho\,\sigma = \frac{1}{2}\Big((-g_{\mu\rho})(-g_{\nu\sigma}) + (-g_{\mu\sigma})(-g_{\nu\rho})\Big)\left(\frac{i}{q^2 + i\epsilon}\right),$$

which in nonrelativistic collisions gives a factor $(-g_{00})^2 = +1$; this will result in a universally attractive potential. It is reassuring to see that quantum field theory does indeed reproduce the obvious features of the electric and gravitational forces:

Exchanged particle	ff and $\bar{f}\bar{f}$	$f\bar{f}$
scalar (Yukawa)	attractive	attractive
vector (electricity)	repulsive	attractive
tensor (gravity)	attractive	attractive

Problems

4.1 Let us return to the problem of the creation of Klein-Gordon particles by a classical source. Recall from Chapter 2 that this process can be described by the Hamiltonian

$$H = H_0 + \int d^3x\,(-j(t,\mathbf{x})\phi(x)),$$

where H_0 is the free Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon field, and $j(x)$ is a c-number scalar function. We found that, if the system is in the vacuum state before the source is turned on, the source will create a mean number of particles

$$\langle N\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,|\tilde{j}(p)|^2.$$

In this problem we will verify that statement, and extract more detailed information, by using a perturbation expansion in the strength of the source.

(a) Show that the probability that the source creates *no* particles is given by

$$P(0) = \left|\langle 0|\,T\Big\{\exp[\,i\int d^4x\,j(x)\phi_I(x)]\Big\}\,|0\rangle\right|^2.$$

(b) Evaluate the term in P(0) of order j^2, and show that $P(0) = 1 - \lambda + \mathcal{O}(j^4)$, where λ equals the expression given above for $\langle N\rangle$.

(c) Represent the term computed in part (b) as a Feynman diagram. Now represent the whole pertubation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly: $P(0) = \exp(-\lambda)$.

(d) Compute the probability that the source creates one particle of momentum k. Perform this computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of part (c) to sum the series.

(e) Show that the probability of producing n particles is given by

$$P(n) = (1/n!)\lambda^n \exp(-\lambda).$$

This is a *Poisson distribution.*

(f) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1; \qquad \langle N \rangle = \sum_{n=0}^{\infty} n\, P(n) = \lambda.$$

The first identity says that the $P(n)$'s are properly normalized probabilities, while the second confirms our proposal for $\langle N \rangle$. Compute the mean square fluctuation $\left\langle (N - \langle N \rangle)^2 \right\rangle$.

4.2 Decay of a scalar particle. Consider the following Lagrangian, involving two real scalar fields Φ and ϕ:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \Phi)^2 - \tfrac{1}{2}M^2\Phi^2 + \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi.$$

The last term is an interaction that allows a Φ particle to decay into two ϕ's, provided that $M > 2m$. Assuming that this condition is met, calculate the lifetime of the Φ to lowest order in μ.

4.3 Linear sigma model. The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model.* Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x)$, $i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\tfrac{1}{2}(\Pi^i)^2 + \tfrac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

$$V(\Phi^2) = \tfrac{1}{2}m^2(\Phi^i)^2 + \tfrac{\lambda}{4}((\Phi^i)^2)^2$$

is a function symmetric under rotations of $\mathbf{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

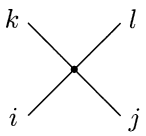
(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, u and d. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d):

This transformation is called an *isospin* rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$\overbrace{\Phi^i(x)\,\Phi^j(y)} = \delta^{ij}\, D_F(x-y),$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

(That is, the vertex between two Φ^1s and two Φ^2s has the value $(-2i\lambda)$; that between four Φ^1s has the value $(-6i\lambda)$.) Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

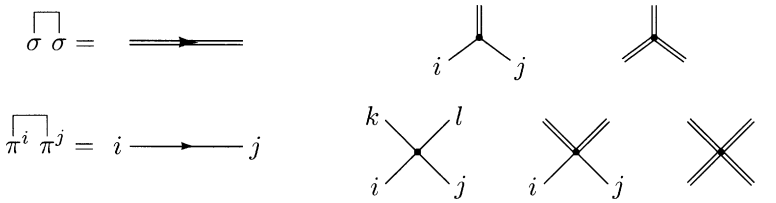
$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \text{and} \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \quad i = 1, \ldots, N-1,$$
$$\Phi^N(x) = v + \sigma(x),$$

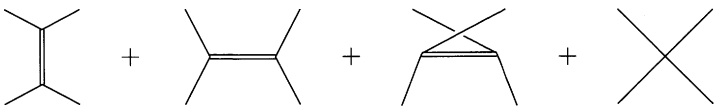
where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:



(c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\,\pi^j(p_2) \to \pi^k(p_3)\,\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

(d) Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same nonvanishing mass.) Find the new value of v that minimizes V, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to a.

4.4 Rutherford scattering. The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x\, e\bar{\psi}\gamma^\mu\psi\, A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

(a) Show that the T-matrix element for electron scattering off a localized classical potential is, to lowest order,

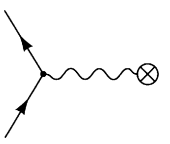
$$\langle p'|iT|p\rangle = -ie\,\bar{u}(p')\gamma^\mu u(p)\cdot\widetilde{A}_\mu(p'-p),$$

where $\widetilde{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p'|iT|p\rangle \equiv i\mathcal{M}\cdot(2\pi)\delta(E_f - E_i),$$

where E_i and E_f are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:



$$= -ie\gamma^\mu\widetilde{A}_\mu(\mathbf{q}),$$

where $\widetilde{A}_\mu(\mathbf{q})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-independent,

localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\,|\mathcal{M}(p_i \to p_f)|^2\,(2\pi)\delta(E_f - E_i),$$

where v_i is the particle's initial velocity. This formula is a natural modification of (4.79). Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.

(c) Specialize to the case of electron scattering from a Coulomb potential ($A^0 = Ze/4\pi r$). Working in the nonrelativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

(With a few calculational tricks from Section 5.1, you will have no difficulty evaluating the general cross section in the relativistic case; see Problem 5.1.)

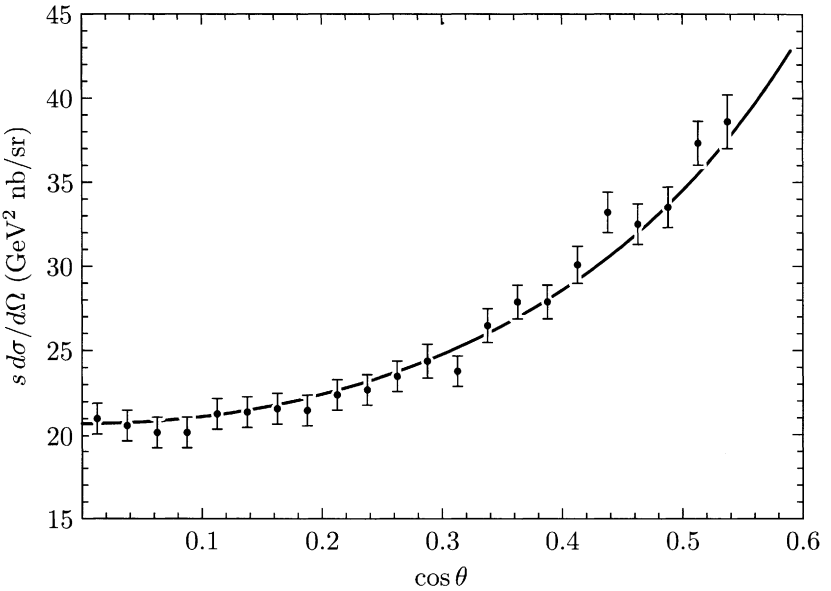


Figure 5.7. Angular dependence of the cross section for $e^+e^- \to 2\gamma$ at $E_{\rm cm} = 29$ GeV, as measured by the HRS collaboration, M. Derrick, et. al., *Phys. Rev.* **D34**, 3286 (1986). The solid line is the lowest-order theoretical prediction, Eq. (5.107).

except when $\sin\theta$ is of order m/p or smaller. Note that since the two photons are identical, we count all possible final states by integrating only over $0 \le \theta \le \pi/2$. Thus the total cross section is computed as

$$\sigma_{\rm total} = \int_0^1 d(\cos\theta)\,\frac{d\sigma}{d\,\cos\theta}. \tag{5.108}$$

Figure 5.7 compares the asymptotic formula (5.107) for the differential cross section to measurements of e^+e^- annihilation into two photons at very high energy.

Problems

5.1 Coulomb scattering. Repeat the computation of Problem 4.4, part (c), this time using the full relativistic expression for the matrix element. You should find, for the spin-averaged cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right),$$

where $\mathbf{p}$ is the electron's 3-momentum and β is its velocity. This is the *Mott formula* for Coulomb scattering of relativistic electrons. Now derive it in a second way, by working

out the cross section for electron-muon scattering, in the muon rest frame, retaining the electron mass but sending $m_\mu \to \infty$.

5.2 Bhabha scattering. Compute the differential cross section $d\sigma/d\cos\theta$ for Bhabha scattering, $e^+e^- \to e^+e^-$. You may work in the limit $E_{\rm cm} \gg m_e$, in which it is permissible to ignore the electron mass. There are two Feynman diagrams; these must be added in the invariant matrix element before squaring. Be sure that you have the correct relative sign between these diagrams. The intermediate steps are complicated, but the final result is quite simple. In particular, you may find it useful to introduce the Mandelstam variables s, t, and u. Note that, if we ignore the electron mass, $s + t + u = 0$. You should be able to cast the differential cross section into the form

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right].$$

Rewrite this formula in terms of $\cos\theta$ and graph it. What feature of the diagrams causes the differential cross section to diverge as $\theta \to 0$?

5.3 The *spinor product* formalism introduced in Problem 3.3 provides an efficient way to compute tree diagrams involving massless particles. Recall that in Problem 3.3 we defined spinor products as follows: Let u_{L0}, u_{R0} be the left- and right-handed spinors at some fixed lightlike momentum k_0. These satisfy

$$u_{L0}\bar{u}_{L0} = \left(\frac{1-\gamma^5}{2}\right)\not{k}_0, \qquad u_{R0}\bar{u}_{R0} = \left(\frac{1+\gamma^5}{2}\right)\not{k}_0. \tag{1}$$

(These relations are just the projections onto definite helicity of the more standard formula $\sum u_0\bar{u}_0 = \not{k}_0$.) Then define spinors for any other lightlike momentum p by

$$u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0}, \qquad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0}. \tag{2}$$

We showed that these spinors satisfy $\not{p}u(p) = 0$; because there is no m around, they can be used as spinors for either fermions or antifermions. We defined

$$s(p_1,p_2) = \bar{u}_R(p_1)u_L(p_2), \qquad t(p_1,p_2) = \bar{u}_L(p_1)u_R(p_2),$$

and, in a special frame, we proved the properties

$$t(p_1,p_2) = (s(p_2,p_1))^*, \quad s(p_1,p_2) = -s(p_2,p_1), \quad |s(p_1,p_2)|^2 = 2p_1\cdot p_2. \tag{3}$$

Now let us apply these results.

(a) To warm up, give another proof of the last relation in Eq. (3) by using (1) to rewrite $|s(p_1,p_2)|^2$ as a trace of Dirac matrices, and then applying the trace calculus.

(b) Show that, for any string of Dirac matrices,

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\cdots] = \mathrm{tr}[\cdots\gamma^\rho\gamma^\nu\gamma^\mu]$$

where $\mu,\nu,\rho,\ldots = 0,1,2,3$, or 5. Use this identity to show that

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2) = \bar{u}_R(p_2)\gamma^\mu u_R(p_1).$$

(c) Prove the Fierz identity

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2)\,[\gamma_\mu]_{ab} = 2\left[u_L(p_2)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_2)\right]_{ab},$$

where $a, b = 1, 2, 3, 4$ are Dirac indices. This can be done by justifying the following statements: The right-hand side of this equation is a Dirac matrix; thus, it can be written as a linear combination of the 16 Γ matrices discussed in Section 3.4. It satisfies

$$\gamma^5[M] = -[M]\gamma^5,$$

thus, it must have the form

$$[M] = \left(\frac{1-\gamma^5}{2}\right)\gamma_\mu V^\mu + \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu W^\mu$$

where V^μ and W^μ are 4-vectors. These 4-vectors can be computed by trace technology; for example,

$$V^\nu = \frac{1}{2}\,\mathrm{tr}[\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)M].$$

(d) Consider the process $e^+e^- \to \mu^+\mu^-$, to the leading order in α, ignoring the masses of both the electron and the muon. Consider first the case in which the electron and the final muon are both right-handed and the positron and the final antimuon are both left-handed. (Use the spinor v_R for the antimuon and $\bar{u}_R$ for the positron.) Apply the Fierz identity to show that the amplitude can be evaluated directly in terms of spinor products. Square the amplitude and reproduce the result for

$$\frac{d\sigma}{d\cos\theta}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)$$

given in Eq. (5.22). Compute the other helicity cross sections for this process and show that they also reproduce the results found in Section 5.2.

(e) Compute the differential cross section for Bhabha scattering of massless electrons, helicity state by helicity state, using the spinor product formalism. The average over initial helicities, summed over final helicities, should reproduce the result of Problem 5.2. In the process, you should see how this result arises as the sum of definite-helicity contributions.

5.4 Positronium lifetimes.

(a) Compute the amplitude $\mathcal{M}$ for e^+e^- annihilation into 2 photons in the extreme nonrelativistic limit (i.e., keep only the term proportional to zero powers of the electron and positron 3-momentum). Use this result, together with our formalism for fermion-antifermion bound states, to compute the rate of annihilation of the $1S$ states of positronium into 2 photons. You should find that the spin-1 states of positronium do not annihilate into 2 photons, confirming the symmetry argument of Problem 3.8. For the spin-0 state of positronium, you should find a result proportional to the square of the $1S$ wavefunction at the origin. Inserting the value of this wavefunction from nonrelativistic quantum mechanics, you should find

$$\frac{1}{\tau} = \Gamma = \frac{\alpha^5 m_e}{2} \approx 8.03 \times 10^9\,\mathrm{sec}^{-1}.$$

A recent measurement[‡] gives $\Gamma = 7.994 \pm .011\,\text{nsec}^{-1}$; the 0.5% discrepancy is accounted for by radiative corrections.

(b) Computing the decay rates of higher-l positronium states is somewhat more difficult; in the rest of this problem, we will consider the case $l = 1$. First, work out the terms in the $e^+e^- \to 2\gamma$ amplitude proportional to one power of the 3-momentum. (For simplicity, work in the center-of-mass frame.) Since

$$\int \frac{d^3p}{(2\pi)^3}\, p^i \psi(\mathbf{p}) = i \frac{\partial}{\partial x^i} \psi(\mathbf{x})\Big|_{\mathbf{x}=0},$$

this piece of the amplitude has overlap with P-wave bound states. Show that the $S = 1$, but not the $S = 0$ states, can decay to 2 photons. Again, this is a consequence of C.

(c) To compute the decay rates of these P-wave states, we need properly normalized state vectors. Denote the three P-state wavefunctions by

$$\psi_i = x^i\, f(|\mathbf{x}|), \qquad \text{normalized to } \int d^3x\, \psi_i^*(\mathbf{x})\psi_j(\mathbf{x}) = \delta_{ij},$$

and their Fourier transforms by $\psi_i(\mathbf{p})$. Show that

$$|B(\mathbf{k})\rangle = \sqrt{2M} \int \frac{d^3p}{(2\pi)^3}\, \psi_i(\mathbf{p})\, a^\dagger_{\mathbf{p}+\mathbf{k}/2}\, \Sigma^i\, b^\dagger_{-\mathbf{p}+\mathbf{k}/2}\, |0\rangle$$

is a properly normalized bound-state vector if Σ^i denotes a set of three 2×2 matrices normalized to

$$\sum_i \text{tr}(\Sigma^{i\dagger}\Sigma^i) = 1.$$

To build $S = 1$ states, we should take each Σ^i to contain a Pauli sigma matrix. In general, spin-orbit coupling will split the multiplet of $S = 1$, $L = 1$ states according to the total angular momentum J. The states of definite J are given by

$$J = 0: \qquad \Sigma^i = \frac{1}{\sqrt{6}}\sigma^i,$$

$$J = 1: \qquad \Sigma^i = \frac{1}{2}\epsilon^{ijk} n^j \sigma^k,$$

$$J = 2: \qquad \Sigma^i = \frac{1}{\sqrt{3}} h^{ij}\sigma^j,$$

where $\mathbf{n}$ is a polarization vector satisfying $|\mathbf{n}|^2 = 1$ and h^{ij} is a traceless tensor, for which a typical value might be $h^{12} = 1$ and all other components zero.

(d) Using the expanded form for the $e^+e^- \to 2\gamma$ amplitude derived in part (b) and the explicit form of the $S = 1$, $L = 1$, definite-J positronium states found in part (c), compute, for each J, the decay rate of the state into two photons.

[‡]D. W. Gidley et. al., *Phys. Rev. Lett.* **49**, 525 (1982).

5.5 Physics of a massive vector boson. Add to QED a massive photon field B_μ of mass M, which couples to electrons via

$$\Delta H = \int d^3x \, (g\bar{\psi}\gamma^\mu \psi B_\mu).$$

A massive photon in the initial or final state has three possible physical polarizations, corresponding to the three spacelike unit vectors in the boson's rest frame. These can be characterized invariantly, in terms of the boson's 4-momentum k^μ, as the three vectors $\epsilon_\mu^{(i)}$ satisfying

$$\epsilon^{(i)} \cdot \epsilon^{(j)} = -\delta^{ij}, \qquad k \cdot \epsilon^{(i)} = 0.$$

The four vectors $(k_\mu/M, \epsilon_\mu^{(i)})$ form a complete orthonormal basis. Because B_μ couples to the conserved current $\bar{\psi}\gamma^\mu\psi$, the Ward identity implies that k_μ dotted into the amplitude for B production gives zero; thus we can replace:

$$\sum_i \epsilon_\mu^{(i)} \epsilon_\nu^{(i)*} \;\to\; -g_{\mu\nu}.$$

This gives a generalization to massive bosons of the Feynman trick for photon polarization vectors and simplifies the calculation of B production cross sections. (Warning: This trick does not work (so simply) for "non-Abelian gauge fields".) Let's do a few of these computations, using always the approximation of ignoring the mass of the electron.

(a) Compute the cross section for the process $e^+e^- \to B$. Compute the lifetime of the B, assuming that it decays only to electrons. Verify the relation

$$\sigma(e^+e^- \to B) = \frac{12\pi^2}{M}\Gamma(B \to e^+e^-)\delta(M^2 - s)$$

discussed in Section 5.3.

(b) Compute the differential cross section, in the center-of-mass system, for the process $e^+e^- \to \gamma + B$. (This calculation goes over almost unchanged to the realistic process $e^+e^- \to \gamma + Z^0$; this allows one to measure the number of decays of the Z^0 into unobserved final states, which is in turn proportional to the number of neutrino species.)

(c) Notice that the cross section of part (b) diverges as $\theta \to 0$ or π. Let us analyze the region near $\theta = 0$. In this region, the dominant contribution comes from the t-channel diagram and corresponds intuitively to the emission of a photon from the electron line before e^+e^- annihilation into a B. Let us rearrange the formula in such a way as to support this interpretation. First, note that the divergence as $\theta \to 0$ is cut off by the electron mass: Let the electron momentum be $p^\mu = (E, 0, 0, k)$, with $k = (E^2 - m_e^2)^{1/2}$, and let the photon momentum be $k^\mu = (xE, xE\sin\theta, 0, xE\cos\theta)$. Show that the denominator of the propagator then never becomes smaller than $\mathcal{O}(m_e^2/s)$. Now integrate the cross section of part (b) over forward angles, cutting off the θ integral at $\theta^2 \sim (m_e^2/s)$ and keeping only the leading logarithmic term, proportional to $\log(s/m_e^2)$. Show that, in this approximation, the cross section for forward photon emission can be written

$$\sigma(e^+e^- \to \gamma + B) \approx \int dx\, f(x) \cdot \sigma(e^+e^- \to B \text{ at } E_{\text{cm}}^2 = (1-x)s),$$

where the annihilation cross section is evaluated for the collision of a positron of energy E and an electron of energy $(1-x)E$, and the function $f(x)$, the *Weizsäcker-Williams distribution function*, is given by

$$f(x) = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\cdot\log\left(\frac{s}{m_e^2}\right).$$

This function arises universally in processes in which a photon is emitted collinearly from an electron line, independent of the subsequent dynamics. We will meet it again, in another context, in Problem 6.2.

5.6 This problem extends the spinor product technology of Problem 5.3 to external photons.

(a) Let k be the momentum of a photon, and let p be another lightlike vector, chosen so that $p\cdot k \neq 0$. Let $u_R(p)$, $u_L(p)$ be spinors of definite helicity for fermions with the lightlike momentum p, defined according to the conventions of Problem 5.3. Define photon polarization vectors as follows:

$$\epsilon_+^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_R(k)\gamma^\mu u_R(p), \qquad \epsilon_-^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_L(k)\gamma^\mu u_L(p).$$

Use the identity

$$u_L(p)\bar{u}_L(p) + u_R(p)\bar{u}_R(p) = \not{p}$$

to compute the polarization sum

$$\epsilon_+^\mu\epsilon_+^{\nu *} + \epsilon_-^\mu\epsilon_-^{\nu *} = -g^{\mu\nu} + \frac{k^\mu p^\nu + k^\nu p^\mu}{p\cdot k}.$$

The second term on the right gives zero when dotted with any photon emission amplitude $\mathcal{M}^\mu$, so we have

$$|\epsilon_+\cdot\mathcal{M}|^2 + |\epsilon_-\cdot\mathcal{M}|^2 = \mathcal{M}^\mu\mathcal{M}^{\nu *}(-g_{\mu\nu});$$

thus, we can use the vectors ϵ_+, ϵ_- to compute photon polarization sums.

(b) Using the polarization vectors just defined, and the spinor products and the Fierz identity from Problem 5.3, compute the differential cross section for a massless electron and positron to annihilate into 2 photons. Show that the result agrees with the massless limit derived in (5.107):

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)$$

in the center-of-mass frame. It follows from the result of part (a) that this answer is independent of the particular vector p used to define the polarization vectors; however, the calculation is greatly simplified by taking this vector to be the initial electron 4-vector.

In this limit, the probability of scattering without emitting a hard photon decreases faster than any power of q^2. The exponential correction factor, containing the Sudakov double logarithm, is known as the *Sudakov form factor.*

To conclude this section, let us calculate the probability, in the same approximation, that some hard scattering process is accompanied by the production of n soft photons, all with energies between E_- and E_+. The phase-space integral for these photons gives $\log(E_+/E_-)$ instead of $\log(E_\ell/\mu)$. If we assign photons with energy greater than E_+ to the "hard" part of the process, we find that the cross section is given by (6.84), times the additional factor

$$\begin{aligned}\text{Prob}(n\gamma \text{ with } E_-<E<E_+) = \frac{1}{n!}\Big[\frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\Big(\frac{E_+^2}{E_-^2}\Big)\Big]^n \\ \times \exp\Big[-\frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\Big(\frac{E_+^2}{E_-^2}\Big)\Big].\end{aligned} \tag{6.86}$$

This expression has the form of a Poisson distribution,

$$P(n) = \frac{1}{n!}\lambda^n e^{-\lambda},$$

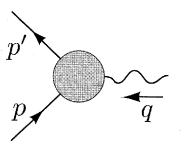
with

$$\lambda = \langle n\rangle = \frac{\alpha}{\pi}\log\Big(\frac{E_+}{E_-}\Big)\mathcal{I}(\mathbf{v},\mathbf{v}').$$

This is precisely the semiclassical estimate of the number of radiated photons that we made in Eq. (6.19).

Problems

6.1 Rosenbluth formula. As discussed Section 6.2, the exact electromagnetic interaction vertex for a Dirac fermion can be written quite generally in terms of two form factors $F_1(q^2)$ and $F_2(q^2)$:



$$= \bar{u}(p')\Big[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\Big]u(p),$$

where $q = p' - p$ and $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu,\gamma^\nu]$. If the fermion is a strongly interacting particle such as the proton, the form factors reflect the structure that results from the strong interactions and so are not easy to compute from first principles. However, these form factors can be determined experimentally. Consider the scattering of an electron with energy $E \gg m_e$ from a proton initially at rest. Show that the above expression for the vertex leads to the following expression (the Rosenbluth formula) for the elastic scattering cross section, computed to leading order in α but to all orders in the strong interactions:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2\Big[(F_1^2 - \frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\Big]}{2E^2[1+\frac{2E}{m}\sin^2\frac{\theta}{2}]\sin^4\frac{\theta}{2}},$$

where θ is the lab-frame scattering angle and F_1 and F_2 are to be evaluated at the q^2 associated with elastic scattering at this angle. By measuring $(d\sigma/d\cos\theta)$ as a function of angle, it is thus possible to extract F_1 and F_2. Note that when $F_1 = 1$ and $F_2 = 0$, the Rosenbluth formula reduces to the Mott formula (in the massless limit) for scattering off a point particle (see Problem 5.1).

6.2 Equivalent photon approximation. Consider the process in which electrons of very high energy scatter from a target. In leading order in α, the electron is connected to the target by one photon propagator. If the initial and final energies of the electron are E and E', the photon will carry momentum q such that $q^2 \approx -2EE'(1-\cos\theta)$. In the limit of forward scattering, whatever the energy loss, the photon momentum approaches $q^2 = 0$; thus the reaction is highly peaked in the forward direction. It is tempting to guess that, in this limit, the virtual photon becomes a real photon. Let us investigate in what sense that is true.

(a) The matrix element for the scattering process can be written as

$$\mathcal{M} = (-ie)\bar{u}(p')\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{q^2}\widehat{\mathcal{M}}^\nu(q),$$

where $\widehat{\mathcal{M}}^\nu$ represents the (in general, complicated) coupling of the virtual photon to the target. Let us analyze the structure of the piece $\bar{u}(p')\gamma^\mu u(p)$. Let $q = (q^0, \mathbf{q})$, and define $\tilde{q} = (q^0, -\mathbf{q})$. We can expand the spinor product as:

$$\bar{u}(p')\gamma^\mu u(p) = A\cdot q^\mu + B\cdot\tilde{q}^\mu + C\cdot\epsilon_1^\mu + D\cdot\epsilon_2^\mu,$$

where A, B, C, D are functions of the scattering angle and energy loss and ϵ_i are two unit vectors transverse to $\mathbf{q}$. By dotting this expression with q_μ, show that the coefficient B is at most of order θ^2. This will mean that we can ignore it in the rest of the analysis. The coefficient A is large, but it is also irrelevant, since, by the Ward identity, $q^\mu\widehat{\mathcal{M}}_\mu = 0$.

(b) Working in the frame where $p = (E, 0, 0, E)$, compute explicitly

$$\bar{u}(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_i u(p)$$

using massless electrons, $u(p)$ and $u(p')$ spinors of definite helicity, and $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ unit vectors parallel and perpendicular to the plane of scattering. We need this quantity only for scattering near the forward direction, and we need only the term of order θ. Note, however, that for $\boldsymbol{\epsilon}$ in the plane of scattering, the small $\hat{3}$ component of $\boldsymbol{\epsilon}$ also gives a term of order θ which must be taken into account.

(c) Now write the expression for the electron scattering cross section, in terms of $|\widehat{\mathcal{M}}^\mu|^2$ and the integral over phase space on the target side. This expression must be integrated over the final electron momentum p'. The integral over p'^3 is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

(d) The divergence as $\theta \to 0$ appears because we have ignored the electron mass in too many places. Show that reintroducing the electron mass in the expression for q^2,

$$q^2 = -2(EE' - pp'\cos\theta) + 2m^2,$$

cuts off the divergence and yields a factor of $\log(s/m^2)$ in its place.

(e) Assembling all the factors, and assuming that the target cross sections are independent of the photon polarization, show that the largest part of the electron-target scattering cross section is given by considering the electron to be the source of a beam of real photons with energy distribution ($x = E_\gamma/E$):

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}\left[1+(1-x)^2\right]\log\left(\frac{s}{m^2}\right).$$

This is the Weizsäcker-Williams *equivalent photon approximation.* This phenomenon allows us, for example, to study photon-photon scattering using e^+e^- collisions. Notice that the distribution we have found here is the same one that appeared in Problem 5.5 when we considered soft photon emission before electron scattering. It should be clear that a parallel general derivation can be constructed for that case.

6.3 Exotic contributions to $g-2$. Any particle that couples to the electron can produce a correction to the electron-photon form factors and, in particular, a correction to $g-2$. Because the electron $g-2$ agrees with QED to high accuracy, these corrections allow us to constrain the properties of hypothetical new particles.

(a) The unified theory of weak and electromagnetic interactions contains a scalar particle h called the *Higgs boson*, which couples to the electron according to

$$H_{\text{int}} = \int d^3x \,\frac{\lambda}{\sqrt{2}}\, h\,\bar{\psi}\psi.$$

Compute the contribution of a virtual Higgs boson to the electron $(g-2)$, in terms of λ and the mass m_h of the Higgs boson.

(b) QED accounts extremely well for the electron's anomalous magnetic moment. If $a = (g-2)/2$,

$$|a_{\text{expt.}} - a_{\text{QED}}| < 1\times 10^{-10}.$$

What limits does this place on λ and m_h? In the simplest version of the electroweak theory, $\lambda = 3\times 10^{-6}$ and $m_h > 60$ GeV. Show that these values are not excluded. The coupling of the Higgs boson to the muon is larger by a factor (m_μ/m_e): $\lambda = 6\times 10^{-4}$. Thus, although our experimental knowledge of the muon anomalous magnetic moment is not as precise,

$$|a_{\text{expt.}} - a_{\text{QED}}| < 3\times 10^{-8},$$

one can still obtain a stronger limit on m_h. Is it strong enough?

(c) Some more complex versions of this theory contain a pseudoscalar particle called the *axion*, which couples to the electron according to

$$H_{\text{int}} = \int d^3x \,\frac{i\lambda}{\sqrt{2}}\, a\,\bar{\psi}\gamma^5\psi.$$

The axion may be as light as the electron, or lighter, and may couple more strongly than the Higgs boson. Compute the contribution of a virtual axion to the $g-2$ of the electron, and work out the excluded values of λ and m_a.

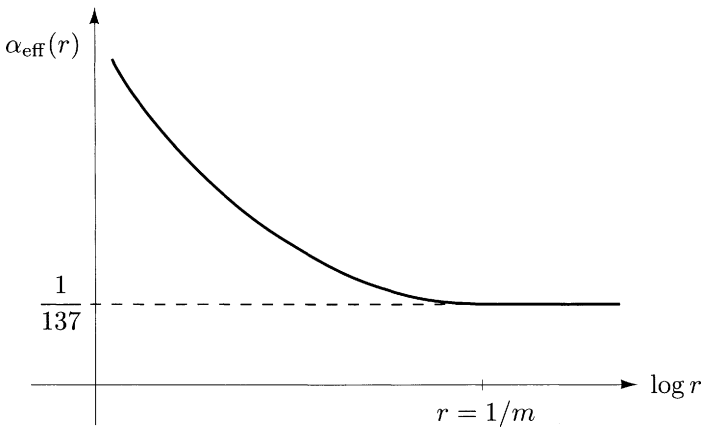


Figure 7.10. A qualitative sketch of the effective electromagnetic coupling constant generated by the one-loop vacuum polarization diagram, as a function of distance. The horizontal scale covers many orders of magnitude.

Problems

7.1 In Section 7.3 we used an indirect method to analyze the one-loop s-channel diagram for boson-boson scattering in ϕ^4 theory. To verify our indirect analysis, evaluate all three one-loop diagrams, using the standard method of Feynman parameters. Check the validity of the optical theorem.

7.2 Alternative regulators in QED. In Section 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Pauli-Villars regularization.

(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

(b) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be 4×4 as usual, but note that, in d dimensions,

$$g_{\mu\nu}\gamma^\mu\gamma^\nu = d.$$

Show that, with this definition, $\delta Z_1 = \delta Z_2$.

7.3 Consider a theory of elementary fermions that couple both to QED and to a Yukawa field ϕ:

$$H_{\text{int}} = \int d^3x \, \frac{\lambda}{\sqrt{2}} \, \phi \bar{\psi}\psi + \int d^3x \, e \, A_\mu \, \bar{\psi}\gamma^\mu\psi.$$

(a) Verify that the contribution to Z_1 from the vertex diagram with a virtual ϕ equals the contribution to Z_2 from the diagram with a virtual ϕ. Use dimensional regularization. Is the Ward identity generally true in this theory?

(b) Now consider the renormalization of the $\phi\bar{\psi}\psi$ vertex. Show that the rescaling of this vertex at $q^2 = 0$ is *not* canceled by the correction to Z_2. (It suffices to compute the ultraviolet-divergent parts of the diagrams.) In this theory, the vertex and field-strength rescaling give additional shifts of the observable coupling constant relative to its bare value.

We have now arrived at a more general understanding of the terms on the right-hand side of the Ward-Takahashi identity. These are the contact terms that we now expect to find when we convert classical equations of motion to Schwinger-Dyson equations for quantum Green's functions. The functional integral formalism allows a simple and elegant derivation of these quantum-mechanical terms.

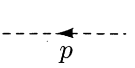
Problems

9.1 Scalar QED. This problem concerns the theory of a complex scalar field ϕ interacting with the electromagnetic field A^μ. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi,$$

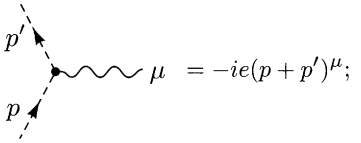
where $D_\mu = \partial_\mu + ieA_\mu$ is the usual gauge-covariant derivative.

(a) Use the functional method of Section 9.2 to show that the propagator of the complex scalar field is the same as that of a real field:

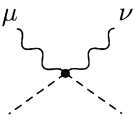


$$= \frac{i}{p^2 - m^2 + i\epsilon}.$$

Also derive the Feynman rules for the interactions between photons and scalar particles; you should find



$$= -ie(p + p')^\mu;$$



$$= 2ie^2 g^{\mu\nu}.$$

(b) Compute, to lowest order, the differential cross section for $e^+e^- \to \phi\phi^*$. Ignore the electron mass (but not the scalar particle's mass), and average over the electron and positron polarizations. Find the asymptotic angular dependence and total cross section. Compare your results to the corresponding formulae for $e^+e^- \to \mu^+\mu^-$.

(c) Compute the contribution of the charged scalar to the photon vacuum polarization, using dimensional regularization. Note that there are two diagrams. To put the answer into the expected form,

$$\Pi^{\mu\nu}(q^2) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2),$$

it is useful to add the two diagrams at the beginning, putting both terms over a common denominator before introducing a Feynman parameter. Show that, for $-q^2 \gg m^2$, the charged boson contribution to $\Pi(q^2)$ is exactly 1/4 that of a virtual electron-positron pair.

9.2 Quantum statistical mechanics.

(a) Evaluate the quantum statistical partition function

$$Z = \mathrm{tr}[e^{-\beta H}]$$

(where $\beta = 1/kT$) using the strategy of Section 9.1 for evaluating the matrix elements of e^{-iHt} in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

(b) Evaluate this integral for a simple harmonic oscillator,

$$L_E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2,$$

by introducing a Fourier decomposition of x(t):

$$x(t) = \sum_n x_n \cdot \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that, up to a (possibly divergent and β-dependent) constant, the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You may find the identity

$$\sinh z = z \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right)$$

useful.]

(c) Generalize this construction to field theory. Show that the quantum statistical partition function for a free scalar field can be written in terms of a functional integral. The value of this integral is given formally by

$$\left[\det(-\partial^2 + m^2)\right]^{-1/2},$$

where the operator acts on functions on Euclidean space that are periodic in the time direction with periodicity β. As before, the β dependence of this expression is difficult to compute directly. However, the dependence on m^2 is unambiguous. (More generally, one can usually evaluate the variation of a functional determinant with respect to any explicit parameter in the Lagrangian.) Show that the determinant indeed reproduces the partition function for relativistic scalar particles.

(d) Now let $\psi(t)$, $\bar{\psi}(t)$ be two Grassmann-valued coordinates, and define a fermionic oscillator by writing the Lagrangian

$$L_E = \bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi.$$

This Lagrangian corresponds to the Hamiltonian

$$H = \omega\bar{\psi}\psi, \quad \text{with } \{\bar{\psi}, \psi\} = 1;$$

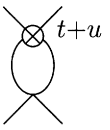
that is, to a simple two-level system. Evaluate the functional integral, assuming that the fermions obey antiperiodic boundary conditions: $\psi(t + \beta) = -\psi(t)$. (Why is this reasonable?) Show that the result reproduces the partition function of a quantum-mechanical two-level system, that is, of a quantum state with Fermi statistics.

(e) Define the partition function for the photon field as the gauge-invariant functional integral

$$Z = \int \mathcal{D}A \, \exp\Big(-\int d^4x_E \, [\tfrac{1}{4}(F_{\mu\nu})^2]\Big)$$

over vector fields A_μ that are periodic in the time direction with period β. Apply the gauge-fixing procedure discussed in Section 9.4 (working, for example, in Feynman gauge). Evaluate the functional determinants using the result of part (c) and show that the functional integral does give the correct quantum statistical result (including the correct counting of polarization states).

process:



$$= (-i\lambda)^3 \cdot -2iV(0) \cdot iV(p^2)$$

$$= \frac{i\lambda^3}{2(4\pi)^d} \int_0^1 dy \, \frac{\Gamma(2-\frac{d}{2})}{\left[m^2\right]^{2-d/2}} \frac{\Gamma(2-\frac{d}{2})}{\left[m^2 - y(1-y)p^2\right]^{2-d/2}}$$

$$\xrightarrow[d\to 4]{} \frac{i\lambda^3}{2(4\pi)^4} \int_0^1 dy \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log m^2\right)$$

$$\times \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log\left[m^2 - y(1-y)p^2\right]\right). \quad (10.63)$$

(Again we have dropped finite terms from the last line.) This expression also contains a nonlocal divergence, given by the first pole times the second logarithm. It exactly cancels the nonlocal divergence in (10.62). The remaining terms are all either finite, or divergent but independent of momentum. This completes the proof that the two-loop contribution to the four-point function is finite.

The two features of the Group I diagrams appear here in Group II as well. The divergent pieces of (10.62) and (10.63) contain double poles that do not cancel, so we again find that the second-order vertex counterterm must contain a double pole. The finite pieces of (10.62) and (10.63) contain double logarithms, so we again find that the two-loop amplitude behaves as $\lambda^3 \log^2 p^2$ as $p \to \infty$.

Problems

10.1 One-loop structure of QED. In Section 10.1 we argued from general principles that the photon one-point and three-point functions vanish, while the four-point function is finite.

(a) Verify directly that the one-loop diagram contributing to the one-point function vanishes. There are two Feynman diagrams contributing to the three-point function at one-loop order. Show that these cancel. Show that the diagrams contributing to any n-point photon amplitude, for n odd, cancel in pairs.

(b) The photon four-point amplitude is a sum of six diagrams. Show explicitly that the potential logarithmic divergences of these diagrams cancel.

10.2 Renormalization of Yukawa theory. Consider the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi,$$

where ϕ is a real scalar field and ψ is a Dirac fermion. Notice that this Lagrangian is invariant under the parity transformation $\psi(t, \mathbf{x}) \to \gamma^0\psi(t, -\mathbf{x})$, $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$,

in which the field ϕ carries odd parity.

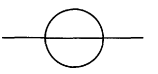
(a) Determine the superficially divergent amplitudes and work out the Feynman rules for renormalized perturbation theory for this Lagrangian. Include all necessary counterterm vertices. Show that the theory contains a superficially divergent 4ϕ amplitude. This means that the theory cannot be renormalized unless one includes a scalar self-interaction,

$$\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4,$$

and a counterterm of the same form. It is of course possible to set the renormalized value of this coupling to zero, but that is not a natural choice, since the counterterm will still be nonzero. Are any further interactions required?

(b) Compute the divergent part (the pole as $d \to 4$) of each counterterm, to the one-loop order of perturbation theory, implementing a sufficient set of renormalization conditions. You need not worry about finite parts of the counterterms. Since the divergent parts must have a fixed dependence on the external momenta, you can simplify this calculation by choosing the momenta in the simplest possible way.

10.3 Field-strength renormalization in ϕ^4 theory. The two-loop contribution to the propagator in ϕ^4 theory involves the three diagrams shown in (10.31). Compute the first of these diagrams in the limit of zero mass for the scalar field, using dimensional regularization. Show that, near $d = 4$, this diagram takes the form:



$$= -ip^2 \cdot \frac{\lambda^2}{12(4\pi)^4}\left[-\frac{1}{\epsilon} + \log p^2 + \cdots\right],$$

with $\epsilon = 4 - d$. The coefficient in this equation involves a Feynman parameter integral that can be evaluated by setting $d = 4$. Verify that the second diagram in (10.31) vanishes near $d = 4$. Thus the first diagram should contain a pole only at $\epsilon = 0$, which can be canceled by a field-strength renormalization counterterm.

10.4 Asymptotic behavior of diagrams in ϕ^4 theory. Compute the leading terms in the S-matrix element for boson-boson scattering in ϕ^4 theory in the limit $s \to \infty$, t fixed. Ignore all masses on internal lines, and keep external masses nonzero only as infrared regulators where these are needed. Show that

$$i\mathcal{M}(s,t) \sim -i\lambda - i\frac{\lambda^2}{(4\pi)^2}\log s - i\frac{5\lambda^3}{2(4\pi)^4}\log^2 s + \cdots.$$

Notice that ignoring the internal masses allows some pleasing simplifications of the Feynman parameter integrals.

Problems

11.1 Spin-wave theory.

(a) Prove the following wonderful formula: Let $\phi(x)$ be a free scalar field with propagator $\langle T\phi(x)\phi(0)\rangle = D(x)$. Then

$$\left\langle Te^{i\phi(x)}e^{-i\phi(0)}\right\rangle = e^{[D(x)-D(0)]}.$$

(The factor $D(0)$ gives a formally divergent adjustment of the overall normalization.)

(b) We can use this formula in Euclidean field theory to discuss correlation functions in a theory with spontaneously broken symmetry for $T < T_C$. Let us consider only the simplest case of a broken $O(2)$ or $U(1)$ symmetry. We can write the local spin density as a complex variable

$$s(x) = s^1(x) + is^2(x).$$

The global symmetry is the transformation

$$s(x) \to e^{-i\alpha}s(x).$$

If we assume that the physics freezes the modulus of $s(x)$, we can parametrize

$$s(x) = Ae^{i\phi(x)}$$

and write an effective Lagrangian for the field $\phi(x)$. The symmetry of the theory becomes the translation symmetry

$$\phi(x) \to \phi(x) - \alpha.$$

Show that (for $d > 0$) the most general renormalizable Lagrangian consistent with this symmetry is the free field theory

$$\mathcal{L} = \tfrac{1}{2}\rho(\vec{\nabla}\phi)^2.$$

In statistical mechanics, the constant ρ is called the *spin wave modulus*. A reasonable hypothesis for ρ is that it is finite for $T < T_C$ and tends to 0 as $T \to T_C$ from below.

(c) Compute the correlation function $\langle s(x)s^*(0)\rangle$. Adjust A to give a physically sensible normalization (assuming that the system has a physical cutoff at the scale of one atomic spacing) and display the dependence of this correlation function on x for $d = 1, 2, 3, 4$. Explain the significance of your results.

11.2 A zeroth-order natural relation. This problem studies an $N = 2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + \bar{\psi}(i\partial\!\!\!/)\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi \qquad (1)$$

where ϕ^i is a two-component field, $i = 1, 2$.

(a) Show that this theory has the following global symmetry:

$$\begin{aligned} \phi^1 &\to \cos\alpha\,\phi^1 - \sin\alpha\,\phi^2, \\ \phi^2 &\to \sin\alpha\,\phi^1 + \cos\alpha\,\phi^2, \\ \psi &\to e^{-i\alpha\gamma^5/2}\,\psi. \end{aligned} \tag{2}$$

Show also that the solution to the classical equations of motion with the minimum energy breaks this symmetry spontaneously.

(b) Denote the vacuum expectation value of the field ϕ^i by v and make the change of variables

$$\phi^i(x) = (v + \sigma(x), \pi(x)). \tag{3}$$

Write out the Lagrangian in these new variables, and show that the fermion acquires a mass given by

$$m_f = g \cdot v. \tag{4}$$

(c) Compute the one-loop radiative correction to m_f, choosing renormalization conditions so that v and g (defined as the $\psi\psi\pi$ vertex at zero momentum transfer) receive no radiative corrections. Show that relation (4) receives nonzero corrections but that these corrections are *finite*. This is in accord with our general discussion in Section 11.6.

11.3 The Gross-Neveu model. The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2$$

with $i = 1, \ldots, N$. The kinetic term of two-dimensional fermions is built from matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma^0 = \sigma^2, \qquad \gamma^1 = i\sigma^1,$$

where σ^i are Pauli sigma matrices. Define

$$\gamma^5 = \gamma^0\gamma^1 = \sigma^3;$$

this matrix anticommutes with the γ^μ.

(a) Show that this theory is invariant with respect to

$$\psi_i \to \gamma^5\psi_i,$$

and that this symmetry forbids the appearance of a fermion mass.

(b) Show that this theory is renormalizable in 2 dimensions (at the level of dimensional analysis).

(c) Show that the functional integral for this theory can be represented in the following form:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\; e^{i\int d^2x\,\mathcal{L}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\; \exp\left[i\int d^2x\left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right],$$

where $\sigma(x)$ (not to be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

(d) Compute the leading correction to the effective potential for σ by integrating over the fermion fields ψ_i. You will encounter the determinant of a Dirac operator; to evaluate this determinant, diagonalize the operator by first going to Fourier components and then diagonalizing the 2×2 Pauli matrix associated with each Fourier mode. (Alternatively, you might just take the determinant of this 2×2 matrix.) This 1-loop contribution requires a renormalization proportional to σ^2 (that is, a renormalization of g^2). Renormalize by minimal subtraction.

(e) Ignoring two-loop and higher-order contributions, minimize this potential. Show that the σ field acquires a vacuum expectation value which breaks the symmetry of part (a). Convince yourself that this result does not depend on the particular renormalization condition chosen.

(f) Note that the effective potential derived in part (e) depends on g and N according to the form

$$V_{\text{eff}}(\sigma_{\text{cl}}) = N \cdot f(g^2 N).$$

(The overall factor of N is expected in a theory with N fields.) Construct a few of the higher-order contributions to the effective potential and show that they contain additional factors of N^{-1} which suppress them if we take the limit $N \to \infty$, $(g^2 N)$ fixed. In this limit, the result of part (e) is unambiguous.

Problems

12.1 Beta functions in Yukawa theory. In the pseudoscalar Yukawa theory studied in Problem 10.2, with masses set to zero,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\partial\!\!\!/)\psi - ig\bar{\psi}\gamma^5\psi\phi,$$

compute the Callan-Symanzik β functions for λ and g:

$$\beta_\lambda(\lambda, g), \qquad \beta_g(\lambda, g),$$

to leading order in coupling constants, assuming that λ and g^2 are of the same order. Sketch the coupling constant flows in the λ-g plane.

12.2 Beta function of the Gross-Neveu model. Compute $\beta(g)$ in the two-dimensional Gross-Neveu model studied in Problem 11.3,

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/\psi_i + \tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2,$$

with $i = 1, \ldots, N$. You should find that this model is asymptotically free. How was that fact reflected in the solution to Problem 11.3?

12.3 Asymptotic symmetry. Consider the following Lagrangian, with two scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L} = \frac{1}{2}((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2) - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\rho}{4!}(\phi_1^2\phi_2^2).$$

Notice that, for the special value $\rho = \lambda$, this Lagrangian has an $O(2)$ invariance rotating the two fields into one another.

(a) Working in four dimensions, find the β functions for the two coupling constants λ and ρ, to leading order in the coupling constants.

(b) Write the renormalization group equation for the ratio of couplings ρ/λ. Show that, if $\rho/\lambda < 3$ at a renormalization point M, this ratio flows toward the condition $\rho = \lambda$ at large distances. Thus the $O(2)$ internal symmetry appears asymptotically.

(c) Write the β functions for λ and ρ in $4 - \epsilon$ dimensions. Show that there are nontrivial fixed points of the renormalization group flow at $\rho/\lambda = 0, 1, 3$. Which is the most stable? Sketch the pattern of coupling constant flows. This flow implies that the critical exponents are those of a symmetric two-component magnet.

four dimensions, any scalar field interaction is irrelevant and the expected behavior is trivial. Just at four dimensions, the coupling constant tends to zero only logarithmically at large scale, giving rise to a renormalizable theory with predictions such as those in Section 13.2. Below four dimensions, the theory is intrinsically a theory of interacting scalar fields, dominated by the Wilson-Fisher fixed point. The coupling at this fixed point is small near four dimensions but grows large as the dimensionality decreases. Finally (for $N > 2$), as $d \to 2$, the fixed-point theory approaches the weak-coupling limit of a completely different Lagrangian with the same symmetries, the nonlinear sigma model.

This evolution of the behavior of the model as a function of d illustrates the main point of the previous two chapters: The qualitative behavior of a quantum field theory is determined not by the fundamental Lagrangian, but rather by the nature of the renormalization group flow and its fixed points. These, in turn, depend only on the basic symmetries that are imposed on the family of Lagrangians that flow into one another. This conclusion signals, at the deepest level, the importance of symmetry principles in determining the fundamental laws of physics.

Problems

13.1 Correction-to-scaling exponent. For critical phenomena in $4-\epsilon$ dimensions, the irrelevant contributions that disappear most slowly are those associated with the deviation of the coupling constant λ from its fixed-point value. This gives the most important nonuniversal correction to the scaling laws derived in Section 13.1. By studying the solution of the Callan-Symanzik equation, show that if the bare value of λ differs slightly from λ_*, the Gibbs free energy receives a correction

$$G(M,t) \to G(M,t) \cdot (1 + (\lambda - \lambda_*) t^{\omega\nu} \hat{k}(tM^{-1/\beta})).$$

This formula defines a new critical exponent ω, called the *correction-to-scaling exponent*. Show that

$$\omega = \frac{d}{d\lambda}\beta\Big|_{\lambda_*} = \epsilon + \mathcal{O}(\epsilon^2).$$

13.2 The exponent η. By combining the result of Problem 10.3 with an appropriate renormalization prescription, show that the leading term in $\gamma(\lambda)$ in ϕ^4 theory is

$$\gamma = \frac{\lambda^2}{12(4\pi)^4}.$$

Generalize this result to the $O(N)$-symmetric ϕ^4 theory to derive Eq. (13.47). Compute the leading-order (ϵ^2) contribution to η.

13.3 The CP^N model. The nonlinear sigma model discussed in the text can be thought of as a quantum theory of fields that are coordinates on the unit sphere. A slightly more complicated space of high symmetry is complex projective space,

CP^N. This space can be defined as the space of $(N+1)$-dimensional complex vectors $(z_1, \ldots, z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1,$$

with points related by an overall phase rotation identified, that is,

$$(e^{i\alpha} z_1, \ldots, e^{i\alpha} z_{N+1}) \quad \text{identified with} \quad (z_1, \ldots, z_{N+1}).$$

In this problem, we study the two-dimensional quantum field theory whose fields are coordinates on this space.

(a) One way to represent a theory of coordinates on CP^N is to write a Lagrangian depending on fields $z_j(x)$, subject to the constraint, which also has the local symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x),$$

independently at each point x. Show that the following Lagrangian has this symmetry:

$$\mathcal{L} = \frac{1}{g^2}[|\partial_\mu z_j|^2 - |z_j^* \partial_\mu z_j|^2].$$

To prove the invariance, you will need to use the constraint on the z_j, and its consequence

$$z_j^* \partial_\mu z_j = -(\partial_\mu z_j^*) z_j.$$

Show that the nonlinear sigma model for the case $N = 3$ can be converted to the CP^N model for the case $N = 1$ by the substitution

$$n^i = z^* \sigma^i z,$$

where σ^i are the Pauli sigma matrices.

(b) To write the Lagrangian in a simpler form, introduce a scalar Lagrange multiplier λ which implements the constraint and also a vector Lagrange multiplier A_μ to express the local symmetry. More specifically, show that the Lagrangian of the CP^N model is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}[|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)],$$

where $D_\mu = (\partial_\mu + iA_\mu)$, by functionally integrating over the fields λ and A_μ.

(c) We can solve the CP^N model in the limit $N \to \infty$ by integrating over the fields z_j. Show that this integral leads to the expression

$$Z = \int \mathcal{D}A\mathcal{D}\lambda \exp\left[-N \operatorname{tr} \log(-D^2 - \lambda) + \frac{i}{g^2}\int d^2x \lambda\right],$$

where we have kept only the leading terms for $N \to \infty$, $g^2 N$ fixed. Using methods similar to those we used for the nonlinear sigma model, examine the conditions for minimizing the exponent with respect to λ and A_μ. Show that these conditions have a solution at $A_\mu = 0$ and $\lambda = m^2 > 0$. Show that, if g^2 is renormalized at the scale M, m can be written as

$$m = M \exp\left[-\frac{2\pi}{g^2 N}\right].$$

(d) Now expand the exponent about $A_\mu = 0$. Show that the first nontrivial term in this expansion is proportional to the vacuum polarization of massive scalar fields. Evaluate this expression using dimensional regularization, and show that it yields a standard kinetic energy term for A_μ. Thus the strange nonlinear field theory that we started with is finally transformed into a theory of $(N+1)$ massive scalar fields interacting with a massless photon.

transform as a general traceless $N \times N$ tensor; the matrices that effect these transformations make up the adjoint representation of $SU(N)$. In this case Eq. (15.98) becomes explicitly

$$N \times \overline{N} = 1 + (N^2 - 1). \tag{15.102}$$

For this decomposition, Eqs. (15.100) and (15.101) imply the identity

$$\left(2 \cdot \frac{N^2 - 1}{2N}\right) N^2 = 0 + C_2(G) \cdot (N^2 - 1). \tag{15.103}$$

Thus, for $SU(N)$,

$$C_2(G) = C(G) = N. \tag{15.104}$$

Some additional examples of the computation of quadratic Casimir operators are given in Problem 15.5. However, the examples we have discussed in this section, combined with the basic group-theoretic concepts that we have reviewed, already provide enough material to carry out the most important computations of physical interest in non-Abelian gauge theories.

Problems

15.1 Brute-force computations in $SU(3)$. The standard basis for the fundamental representation of $SU(3)$ is

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Explain why there are exactly eight matrices in the basis.

(b) Evaluate all the commutators of these matrices, to determine the structure constants of $SU(3)$. Show that, with the normalizations used here, f^{abc} is totally antisymmetric. (This exercise is tedious; you may wish to check only a representative sample of the commutators.)

(c) Check the orthogonality condition (15.78), and evaluate the constant $C(r)$ for this representation.

(d) Compute the quadratic Casimir operator $C_2(r)$ directly from its definition (15.92), and verify the relation (15.94) between $C_2(r)$ and $C(r)$.

15.2 Write down the basis matrices of the adjoint representation of $SU(2)$. Compute $C(G)$ and $C_2(G)$ directly from their definitions (15.78) and (15.92).

15.3 Coulomb potential.

(a) Using functional integration, compute the expectation value of the Wilson loop in pure quantum electrodynamics without fermions. Show that

$$\langle U_P(z,z)\rangle = \exp\left[-e^2 \oint_P dx^\mu \oint_P dy^\nu g_{\mu\nu} \frac{1}{8\pi^2(x-y)^2}\right],$$

with x and y integrated around the closed curve P.

(b) Consider the Wilson loop of a rectangular path of (spacelike) width R and (timelike) length T, $T \gg R$. Compute the expectation value of the Wilson loop in this limit and compare to the general expression for time evolution,

$$\langle U_P\rangle = \exp[-iE(R)T],$$

where $E(R)$ is the energy of the electromagnetic sources corresponding to the Wilson loop. Show that the potential energy of these sources is just the Coulomb potential, $V(R) = -e^2/4\pi R$.

(c) Assuming that the propagator of the non-Abelian gauge field is given by the Feynman gauge expression

$$\left\langle A^a_\mu(x) A^b_\mu(y)\right\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-ig_{\mu\nu}\delta^{ab}}{p^2} e^{-ip\cdot(x-y)},$$

compute the expectation value of a non-Abelian Wilson loop to order g^2. The result will depend on the representation r of the gauge group in which one chooses the matrices that appear in the exponential. Show that, to this order, the Coulomb potential of the non-Abelian gauge theory is $V(R) = -g^2 C_2(r)/4\pi R$.

15.4 Scalar propagator in a gauge theory. Consider the equation for the Green's function of the Klein-Gordon equation:

$$(\partial^2 + m^2) D_F(x,y) = -i\delta^{(4)}(x-y).$$

We can find an interesting representation for this Green's function by writing

$$D_F(x,y) = \int_0^\infty dT\, D(x,y,T),$$

where $D(x,y,T)$ satisfies the Schrödinger equation

$$\left[i\frac{\partial}{\partial T} - (\partial^2 + m^2)\right] D(x,y,T) = i\delta(T)\delta^{(4)}(x-y).$$

Now, represent $D(x,y,T)$ using the functional integral solution of the Schrödinger equation presented in Section 9.1.

(a) Using the explicit formula of the propagator of the Schrödinger equation, show that this integral formula gives the standard expression for the Feynman propagator.

(b) Using the method just described, show that the expression

$$D_F(x,y) = \int_0^\infty dT \int \mathcal{D}x \, \exp\left[i \int dt \, \frac{1}{2}\left(\left(\frac{dx^\mu}{dt} \right)^2 - m^2 \right) - ie \int dt \, \frac{dx^\mu}{dt} A_\mu(x) \right]$$

is a functional integral representation for the scalar field propagator in an arbitrary background electromagnetic field. Show, in particular, that the functional integral satisfies the relevant Schrödinger equation. Notice that this integral depends on A_μ through the Wilson line.

(c) Generalize this expression to a non-Abelian gauge theory. Show that the functional integral solves the relevant Schrödinger equation only if the group matrices in the exponential for the Wilson line are path-ordered.

15.5 Casimir operator computations. An alternative strategy for computing the quadratic Casimir operator is to compute $C(r)$ in the formula

$$\mathrm{tr}[t_r^a t_r^b] = C(r)\delta^{ab}$$

by choosing t^a and t^b to lie in an $SU(2)$ subgroup of the gauge group.

(a) Under an $SU(2)$ subgroup of a general group G, an irreducible representation r of G will decompose into a sum of representations of $SU(2)$:

$$r \to \sum j_i,$$

where the j_i are the spins of $SU(2)$ representations. Show that

$$3C(r) = \sum_i j_i(j_i+1)(2j_i+1).$$

(b) Under an $SU(2)$ subgroup of $SU(N)$, the fundamental representation N transforms as a 2-component spinor ($j = \frac{1}{2}$) and $(N-2)$ singlets. Use this relation to check the formula $C(N) = \frac{1}{2}$. Show that the adjoint representation of $SU(N)$ decomposes into one spin 1, $2(N-2)$ spin-$\frac{1}{2}$'s, plus singlets, and use this decomposition to check that $C(G) = N$.

(c) Symmetric and antisymmetric 2-index tensors form irreducible representations of $SU(N)$. Compute $C_2(r)$ for each of these representations. The direct sum of these representations is the product representation $N \times N$. Verify that your results for $C_2(r)$ satisfy the identity for product representations that follows from Eqs. (15.100) and (15.101).

Problems

16.1 Arnowitt-Fickler gauge. Perform the Faddeev-Popov quantization of Yang-Mills theory in the gauge $A^{3a} = 0$, and write the Feynman rules. Show that there are no propagating ghosts, and that the gauge field is reduced to two positive-metric degrees of freedom. (Although the gauge condition violates Lorentz invariance, this symmetry is restored in the calculation of gauge-invariant S-matrix elements.)

16.2 Scalar field with non-Abelian charge. Consider a non-Abelian gauge theory with gauge group G. Couple to this theory a complex scalar field in the representation r.

(a) Show that the Feynman rules for the scalar field are a simple modification of the Feynman rules displayed for scalar QED in Problem 9.1(a).

(b) Compute the contribution of this scalar field to the β function, and show that the full β function for this theory is

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}C(r)\right).$$

16.3 Counterterm relations. In Section 16.5, we computed the divergent parts of δ_1, δ_2, and δ_3. It is a good exercise to compute the divergent parts of the remaining counterterms in Eq. (16.88) to one-loop order in the Feynman-'t Hooft gauge, and to explicitly verify that the counterterm relations (16.89) are consistent with the removal of ultraviolet divergences.

(a) The ghost counterterms are particularly easy to compute. Work out δ_1^c and δ_2^c, and show that the divergent part of their difference equals the divergent part of $\delta_1 - \delta_2$. This gives a derivation of asymptotic freedom that is slightly easier than the one in Section 16.5.

(b) Compute the counterterm for the 3-gauge-boson vertex and verify the first equality in (16.89).

(c) Compute the counterterm for the 4-gauge-boson vertex and find, when the smoke clears, the second relation in (16.89).

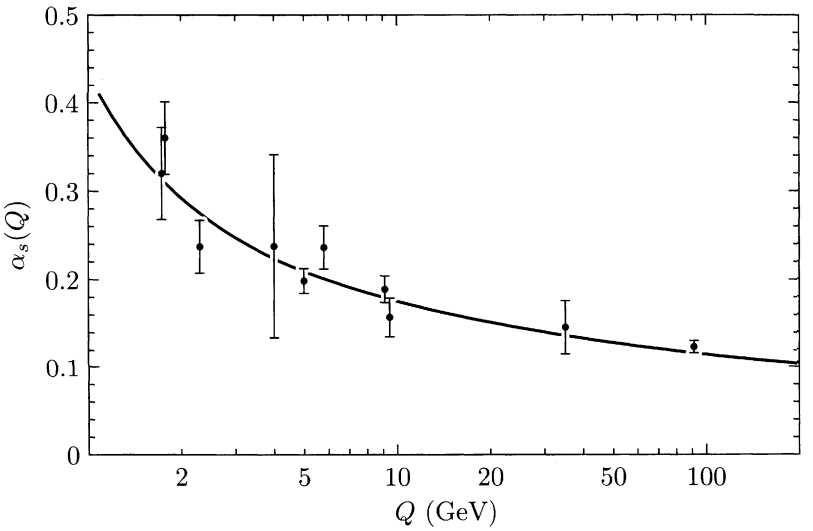


Figure 17.23. Measurements of α_s, plotted against the momentum scale Q at which the measurement was made. This figure was constructed by evolving the values of $\alpha_s(m_Z)$ listed in Table 17.1 back to the values of Q indicated in the table. The value for e^+e^- event shapes has been split into two points corresponding to experiments at the TRISTAN and LEP accelerators. These values are compared to the theoretical expectation from the renormalization group evolution with the initial condition $\alpha_s(m_Z) = 0.117$.

Problems

17.1 Two-loop renormalization group relations.

(a) In higher orders of perturbation theory, the expression for the QCD β function will be a series

$$\beta(g) = -\frac{b_0}{(4\pi)^2}g^3 - \frac{b_1}{(4\pi)^4}g^5 - \frac{b_2}{(4\pi)^6}g^7 + \cdots .$$

Integrate the renormalization group equation and show that the running coupling constant is now given by

$$\alpha_s(Q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\log(Q^2/\Lambda^2)} - \frac{b_1}{b_0^2}\frac{\log\log(Q^2/\Lambda^2)}{(\log(Q^2/\Lambda^2))^2} + \cdots\right],$$

where the omitted terms decrease as $(\log(Q^2/\Lambda^2))^{-2}$.

(b) Combine this formula with the perturbation series for the e^+e^- annihilation cross section:

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \cdot \left(3\sum_f Q_f^2\right) \cdot \left[1 + \frac{\alpha_s}{\pi} + a_2\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right].$$

The coefficient a_2 depends on the details of the renormalization conditions defining α_s. Show that the leading two terms in the asymptotic behavior of $\sigma(s)$ for large s depend only on b_0 and b_1 and are independent of a_2 and b_2. Thus the first *two* coefficients of the QCD β function are independent of the renormalization prescription.

17.2 A direct test of the spin of the gluon. In this problem, we compare the predictions of QCD with those of a model in which the interaction of quarks is mediated by a scalar boson. Let the coupling of the scalar gluon to quarks be given by

$$\delta\mathcal{L} = gS\bar{q}q,$$

and define $\alpha_g = g^2/4\pi$.

(a) Using the technique described in parts (b) and (c) of the Final Project of Part I, compute the cross section for $e^+e^- \to q\bar{q}S$ to the leading order of perturbation theory. This cross section depends on the energies of the q, $\bar{q}$, and S, which we represent as fractions x_1, x_2, x_3 of the electron beam energy, as in Eq. (17.18). Show that

$$\frac{d^2\sigma}{dx_1 dx_2}(e^+e^- \to q\bar{q}S) = \frac{4\pi\alpha^2 Q_q^2}{3s} \cdot \frac{\alpha_g}{4\pi} \frac{x_3^2}{(1-x_q)(1-x_{\bar{q}})}.$$

(b) In practice, it is very difficult to tell quarks from gluons experimentally, since both particles appear as jets of hadrons. Therefore, let x_a be the largest of x_1, x_2, x_3, let x_b be the second largest, and let x_c be the smallest. Sum over the various possibilities to derive an expression for $d^2\sigma/dx_a dx_b$, both in QCD, using Eq. (17.18), and in the scalar gluon model. Show that these models can be distinguished by their distributions in the x_a, x_b plane.

17.3 Quark-gluon and gluon-gluon scattering.

(a) Compute the differential cross section

$$\frac{d\sigma}{d\hat{t}}(q\bar{q} \to gg)$$

for quark-antiquark annihilation in QCD to the leading order in α_s. This is most easily done by computing the amplitudes between states of definite quark and gluon helicity. Ignore all masses. Use explicit polarization vectors and spinors, for example,

$$\epsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$$

for a right-handed gluon moving in the $+\hat{3}$ direction. You need only consider transversely polarized gluons. By helicity conservation, only the initial states $q_L\bar{q}_R$ and $q_R\bar{q}_L$ can contribute; by parity, these two states give identical cross sections. Thus it is necessary only to compute the amplitudes for the three processes

$$q_L\bar{q}_R \to g_R g_R,$$
$$q_L\bar{q}_R \to g_R g_L,$$
$$q_L\bar{q}_R \to g_L g_L.$$

In fact, by CP invariance, the first and third processes have equal cross sections. After computing the amplitudes, square them and combine them properly with

color factors to construct the various helicity cross sections. Finally, combine these to form the total cross section averaged over initial spins and colors.

(b) Compute the differential cross section

$$\frac{d\sigma}{d\hat{t}}(gg \to gg)$$

for gluon-gluon scattering. There are 16 possible combinations of helicities, but many of them are related to each other by parity and crossing symmetry. All 16 can be built up from the three amplitudes for

$$g_R g_R \to g_R g_R,$$
$$g_R g_R \to g_R g_L,$$
$$g_R g_R \to g_L g_L.$$

Show that the last two of these amplitudes vanish. The first can be dramatically simplified using the Jacobi identity. When the smoke clears, only three of the 16 polarized gluon scattering cross sections are nonzero. Combine these to compute the spin- and color-averaged differential cross section.

17.4 The gluon splitting function. Compute the gluon splitting function (17.130) for the Altarelli-Parisi equations. To carry out this computation, first compute the matrix elements of the three-gluon vertex shown in Fig. 17.20 between gluon states of definite helicity. Combine these to derive the splitting function in the region $x < 1$. Then fix the singularity of the splitting function at $x = 1$ to give this function the correct overall normalization.

17.5 Photoproduction of heavy quarks. Consider the process of heavy quark pair photoproduction, $\gamma + p \to Q\bar{Q} + X$, for a heavy quark of mass M and electric charge Q. If M is large enough, any diagram contributing to this process must involve a large momentum transfer; thus a perturbative QCD analysis should apply. This idea applies in practice already for the production of c quark pairs. Work out the cross section to the leading order in QCD. Choose the parton subprocess that gives the leading contribution to this reaction, and write the parton-model expression for the cross section. You will need to compute the relevant subprocess cross section, but this can be taken directly from one of the QED calculations in Chapter 5. Then use this result to write an expression for the cross section for γ-proton scattering.

17.6 Behavior of parton distribution functions at small x. It is possible to solve the Altarelli-Parisi equations analytically for very small x, using some physically motivated approximations. This discussion is based on a paper of Ralston.[†]

(a) Show that the Q^2 dependence of the right-hand side of the A-P equations can be expressed by rewriting the equations as differential equations in

$$\xi = \log\log(\frac{Q^2}{\Lambda^2}),$$

where Λ is the value of Q^2 at which $\alpha_s(Q^2)$, evolved with the leading-order β function, formally goes to infinity.

[†]J. P. Ralston, *Phys. Lett.* **172B**, 430 (1986).

(b) Since the branching functions to gluons are singular as z^{-1} as $z \to 0$, it is reasonable to guess that the gluon distribution function will blow up approximately as x^{-1} as $x \to 0$. The resulting distribution

$$dx\, f_g(x) \sim \frac{dx}{x}$$

is approximately scale invariant, and so its form should be roughly preserved by the A-P equations. Let us, then, make the following two approximations: (1) the terms involving the gluon distribution completely dominate the right-hand sides of the A-P equations; and (2) the function

$$\tilde{g}(x, Q^2) = x f_g(x, Q^2)$$

is a slowly varying function of x. Using these approximations, and the limit $x \to 0$, show that the A-P equation for $f_g(x)$ can be converted to the following differential equation:

$$\frac{\partial^2}{\partial w \partial \xi} \tilde{g}(x, \xi) = \frac{12}{b_0} \tilde{g}(x, \xi),$$

where $w = \log(1/x)$ and $b = (11 - \frac{2}{3} n_f)$. Show that if $w\xi \gg 1$, this equation has the approximate solution

$$\tilde{g} = K(Q^2) \cdot \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0) \right]^{1/2} \right),$$

where $K(Q^2)$ is an initial condition.

(c) The quark distribution at very small x is mainly created by branching of gluons. Using the approximations of part (b), show that, for any flavor of quark, the right-hand side of the A-P equation for $f_q(x)$ can be approximately integrated to yield an equation for $\tilde{q}(x) = x f_q(x)$:

$$\frac{\partial}{\partial \xi} \tilde{q}(x, \xi) = \frac{2}{3 b_0} \tilde{g}(x, \xi).$$

Show, again using $w\xi \gg 1$, that this equation has as its integral

$$\tilde{q} = \left(\frac{\xi - \xi_0}{27 b_0 w} \right)^{1/2} K(Q^2) \cdot \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0) \right]^{1/2} \right).$$

(d) Ralston suggested that the initial condition

$$K(Q^2) = 50.36(\exp(\xi - \xi_0) - 0.957) \cdot \exp\left[-7.597(\xi - \xi_0)^{1/2} \right],$$

with $Q_0^2 = 5\ \text{GeV}^2$, $\Lambda = 0.2\ \text{GeV}$, and $n_f = 5$, gave a reasonable fit to the known properties of parton distributions, extrapolated into the small x region. Use this function and the results above to sketch the behavior of the quark and gluon distributions at small x and large Q^2.

express the manner in which the constituents of hadrons in QCD are resolved, layer by layer, by hard-scattering processes at successively higher values of the momentum transfer. Our understanding that a quark, when studied on a fine scale, is resolved into a set of quarks, antiquarks, and gluons indicates that we have gone far beyond the simple notions of one-particle relativistic mechanics. Our two complementary derivations of this idea reinforce its fundamental character as a prediction of quantum field theory. It is especially pleasing that, as we saw at the end of Section 17.5, Nature apparently accepts this prediction and makes this consequence of quantum field theory an essential part of the structure of hadrons.

Problems

18.1 Matrix element for proton decay. Some advanced theories of particle interactions include heavy particles X whose couplings violate the conservation of baryon number. Integrating out these particles produces an effective interaction that allows the proton to decay to a positron and a photon or a pion. This effective interaction is most easily written using the definite-helicity components of the quark and electron fields: If u_L, d_L, u_R, e_R are two-component spinors, then this effective interaction is

$$\Delta\mathcal{L} = \frac{2}{m_X^2}\epsilon_{abc}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}e_{R\alpha}u_{Ra\beta}u_{Lb\gamma}d_{Lc\delta}.$$

A typical value for the mass of the X boson is $m_X = 10^{16}$ GeV.

(a) Estimate, in order of magnitude, the value of the proton lifetime if the proton is allowed to decay through this interaction.

(b) Show that the three-quark operator in $\Delta\mathcal{L}$ has an anomalous dimension

$$\gamma = -4\frac{g^2}{(4\pi)^2}.$$

Estimate the enhancement of the proton decay rate due to the leading QCD corrections.

18.2 Parity-violating deep inelastic form factor. In this problem, we first motivate the presence of additional deep inelastic form factors that are proportional to *differences* of quark and antiquark distribution functions. Then we define these functions formally and work out their properties.

(a) Analyze neutrino-proton scattering following the method used at the beginning of Section 18.5. Define

$$J_+^\mu = \bar{u}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)d, \qquad J_-^\mu = \bar{d}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u.$$

Let

$$W^{\mu\nu(\nu)} = 2i\int d^4x\, e^{iq\cdot x}\,\langle P|\,T\{J_-^\mu(x)J_+^\nu(0)\}\,|P\rangle\,,$$

averaged over the proton spin. Show that the cross section for deep inelastic neutrino scattering can be computed from $W^{\mu\nu(\nu)}$ according to

$$\frac{d^2\sigma}{dxdy}(\nu p \to \mu^- X) = \frac{G_F^2 y}{2\pi^2} \cdot \mathrm{Im}[(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k\cdot k' - i\epsilon_{\mu\nu}{}^{\alpha\beta} k'_\alpha k_\beta) W^{\mu\nu(\nu)}(P,q)].$$

(b) Show that any term in $W^{\mu\nu(\nu)}$ proportional to q^μ or q^ν gives zero when contracted with the lepton momentum tensor in the formula above. Thus we can expand $W^{\mu\nu(\nu)}$ with three scalar form factors,

$$W^{\mu\nu(\nu)} = -g^{\mu\nu} W_1^{(\nu)} + P^\mu P^\nu W_2^{(\nu)} + i\epsilon^{\mu\nu\lambda\sigma} P_\lambda q_\sigma W_3^{(\nu)} + \cdots,$$

where the additional terms do not contribute to the deep inelastic cross section. Find the formula for the deep inelastic cross section in terms of the imaginary parts of $W_1^{(\nu)}$, $W_2^{(\nu)}$, and $W_3^{(\nu)}$.

(c) Evaluate the form factors $W_i^{(\nu)}$ in the parton model, and show that

$$\begin{aligned}
\mathrm{Im}\, W_1^{(\nu)} &= \pi(f_d(x) + f_{\bar{u}}(x)), \\
\mathrm{Im}\, W_2^{(\nu)} &= \frac{4\pi}{ys} x(f_d(x) + f_{\bar{u}}(x)), \\
\mathrm{Im}\, W_3^{(\nu)} &= \frac{2\pi}{ys}(f_d(x) - f_{\bar{u}}(x)).
\end{aligned}$$

Insert these expressions into the formula derived in part (b) and show that the result reproduces the first line of Eq. (17.35).

(d) This analysis motivates the following definition: For a single quark flavor f, let

$$J^\mu_{fL} = \bar{f}\gamma^\mu \Big(\frac{1-\gamma^5}{2}\Big) f.$$

Define

$$W^{\mu\nu}_{fL} = 2i \int d^4x\, e^{iq\cdot x} \langle P| T\{J^\mu_{fL}(x) J^\nu_{fL}(0)\} |P\rangle .$$

Decompose this tensor according to

$$W^{\mu\nu}_{fL} = -g^{\mu\nu} W_{1fL} + P^\mu P^\nu W_{2fL} + i\epsilon^{\mu\nu\lambda\sigma} P_\lambda q_\sigma W_{3fL} + \cdots,$$

where the remaining terms are proportional to q^μ or q^ν. Evaluate the W_{iL} in the parton model. Show that the quantities W_{1fL} and W_{2fL} reproduce the expressions for W_{1f} and W_{2f} given by Eqs. (18.120) and (18.144), and that W_{3fL} is given by

$$\mathrm{Im}\, W_{3fL} = \frac{2\pi}{ys}(f_f(x) - f_{\bar{f}}(x)).$$

(e) Compute the operator product of the currents in the expression for $W^{\mu\nu}_{fL}$, and write the terms in this product that involve twist-2 operators. Show that the expressions for W_{1fL} and W_{2fL} that follow from this analysis reproduce the expressions for W_{1f} and W_{2f} given by Eqs. (18.144) and (18.145). Find the corresponding expression for W_{3fL}.

(f) Define the parton distribution f_f^- by the relation

$$f_f^-(x, Q^2) = \frac{ys}{2\pi} \operatorname{Im} W_{3fL}(x, Q^2).$$

Show that, by virtue of this definition, the distribution function f_f^- satisfies the sum rule (18.155) for odd n.

18.3 Anomalous dimensions of gluon twist-2 operators.

(a) Compute the divergent parts of the diagrams in Fig. 18.14, and use these to derive the second line of Eq. (18.181). Notice that this result holds only for n even. Show that the two diagrams cancel for n odd.

(b) Compute the divergent parts of the diagrams in Fig. 18.5, and use these to derive the third and fourth lines of Eq. (18.181).

18.4 Deep inelastic scattering from a photon. Consider the problem of deep-inelastic scattering of an electron from a photon. This process can actually be measured by analyzing the reaction $e^+e^- \to e^+e^- + X$ in the regime where the positron goes forward, with emission of a collinear photon, which then has a hard reaction with the electron. Let us analyze this process to leading order in QED and to leading-log order in QCD. To predict the photon structure functions, it is reasonable to integrate the renormalization group equations with the initial condition that the parton distribution for photons in the photon is $\delta(x-1)$ at $Q^2 = (\frac{1}{2}\ \text{GeV})^2$. Take $\Lambda = 150$ MeV. Assume for simplicity that there are four flavors of quarks, u, d, c, and s, with charges $2/3$, $-1/3$, $2/3$, $-1/3$, respectively, and that it is always possible to ignore the masses of these quarks.

(a) Use the Altarelli-Parisi equations to compute the parton distributions for quarks and antiquarks in the photon, to leading order in QED and to zeroth order in QCD. Compute also the probability that the photon remains a photon as a function of Q^2.

(b) Formulate the problem of computing the moments of W_2 for the photon as a problem in operator mixing. Compute the relevant anomalous dimension matrix γ. You should be able to assemble this matrix from familiar ingredients without doing further Feynman diagram computations.

(c) Compute the $n = 2$ moments of the photon structure functions as a function of Q^2.

(d) Describe qualitatively the evolution of the photon structure function as a function of x and Q^2.

As with the axial vector anomaly, the trace anomaly can be found in many different ways. For each possible method of regulating a quantum field theory, there is a derivation of the trace anomaly that exploits the possible pathology of that particular regulator. For example, if one uses a Pauli-Villars regulator with heavy fermions to cancel the divergence of the QED fermion loop diagram, the heavy fermions Ψ contribute a term $M\bar{\Psi}\Psi$ to the trace of $\Theta^{\mu\nu}$. The loop diagram with this term inserted turns out to have a finite limit as $M \to \infty$, which precisely reproduces the trace anomaly. This computation is worked out in Problem 19.4.

As with the axial vector anomaly, each derivation of the anomaly with a different regulator, taken individually, seems artificial, as if there were a problem with the field theory that we are not quite clever enough to fix. Eventually, though, we are forced to conclude that the quantum field theory is trying to tell us something. The anomalous symmetries of the classical theory cannot be promoted to symmetries of the quantum theory. Instead, the anomalous conservation laws require profound and qualitative changes in the theory from the classical to the quantum level.

Problems

19.1 Fermion number nonconservation in parallel E and B fields.

(a) Show that the Adler-Bell-Jackiw anomaly equation leads to the following law for global fermion number conservation: If N_R and N_L are, respectively, the numbers of right- and left-handed massless fermions, then

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2} \int d^4x \, \mathbf{E} \cdot \mathbf{B}.$$

To set up a solvable problem, take the background field to be $A^\mu = (0, 0, Bx^1, A)$, with B constant and A constant in space and varying only adiabatically in time.

(b) Show that the Hamiltonian for massless fermions represented in the components (3.36) is

$$H = \int d^3x \left[\psi_R^\dagger (-i\boldsymbol{\sigma} \cdot \mathbf{D}) \psi_R - \psi_L^\dagger (-i\boldsymbol{\sigma} \cdot \mathbf{D}) \psi_L \right],$$

with $D^i = \nabla^i - ieA^i$. Concentrate on the term in the Hamiltonian that involves right-handed fermions. To diagonalize this term, one must solve the eigenvalue equation $-i\boldsymbol{\sigma} \cdot \mathbf{D}\psi_R = E\psi_R$.

(c) The ψ_R eigenvectors can be written in the form

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i(k_2 x^2 + k_3 x^3)}.$$

The functions ϕ_1 and ϕ_2, which depend only on x^1, obey coupled first-order differential equations. Show that, when one of these functions is eliminated, the other obeys the equation of a simple harmonic oscillator. Use this observation to find the single-particle spectrum of the Hamiltonian. Notice that the eigenvalues do not depend on k_2.

(d) If the system of fermions is set up in a box with sides of length L and periodic boundary conditions, the momenta k_2 and k_3 will be quantized:

$$k_i = \frac{2\pi n_i}{L}.$$

By looking back to the harmonic oscillator equation in part (c), show that the condition that the center of the oscillation is inside the box leads to the condition

$$k_2 < eBL.$$

Combining these two conditions, we see that each level found in part (c) has a degeneracy of

$$\frac{eL^2B}{2\pi}.$$

(e) Now consider the effect of changing the background A adiabatically by an amount (19.37). Show that the vacuum loses right-handed fermions. Repeating this analysis for the left-handed spectrum, one sees that the vacuum gains the same number of left-handed fermions. Show that these numbers are in accord with the global nonconservation law given in part (a).

19.2 Weak decay of the pion.

(a) In the effective Lagrangian for semileptonic weak interactions (18.28), the hadronic part of the operator is a left-handed current involving the u and d quarks. Show that this current is related to the quark currents of Section 19.3 as follows:

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}(j^{\mu 1} + ij^{\mu 2} - j^{\mu 51} - ij^{\mu 52}),$$

where 1, 2 are isospin indices. Using this identification and (19.88), show that the amplitude for the decay $\pi^+ \to \ell^+ \nu$ is given by

$$i\mathcal{M} = G_F f_\pi \bar{u}(q) \not{p}(1 - \gamma^5) v(k),$$

where p, k, q are the momenta of the π^+, ℓ^+, ν.

(b) Compute the decay rate of the pion. Show that this rate vanishes in the limit of zero lepton mass, and that the relative rate of pion decay to muons and electrons is given by

$$\frac{\Gamma(\pi^+ \to e^+ \nu)}{\Gamma(\pi^+ \to \mu^+ \nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \frac{(1 - m_e^2/m_\pi^2)^2}{(1 - m_\mu^2/m_\pi^2)^2} = 10^{-4}.$$

From the measured pion lifetime, $\tau_\pi = 2.6 \times 10^{-8}$ sec, and the pion and muon masses, $m_\pi = 140$ MeV, $m_\mu = 106$ MeV, determine the value of f_π.

19.3 Computation of anomaly coefficients.

(a) Consider a product $r_1 \times r_2$ of $SU(n)$ representations, which is decomposed into irreducible representations as in (15.98). Using the explicit form of the generators given in (15.99), show that the anomaly coefficients satisfy

$$d_1 A(r_2) + d_2 A(r_1) = \sum_i A(r_i).$$

(b) As we saw in Problem 15.5, the two-index symmetric and antisymmetric tensors form irreducible representations of $SU(n)$, which we will call s and a, respectively. In $SU(3)$, the representation a is three-dimensional. Show that it is equivalent to the $\bar{3}$. Compute the anomaly coefficients for a and s, making use of the identity in part (a).

(c) Since $SU(n)$ has a unique three-index symmetric tensor d^{abc} which is already nonvanishing in an $SU(3)$ subgroup, we can compute the anomaly coefficient in $SU(n)$ by restricting our attention to three generators in this subgroup. By decomposing $SU(n)$ representations into $SU(3)$ representations, compute the anomaly coefficients for a and s in $SU(n)$ and derive Eq. (19.141). Find the anomaly coefficient of the j-index totally antisymmetric tensor representation of $SU(n)$. Why does the result always vanish when $2j = n$?

19.4 Large fermion mass limits. In the text, we derived the Adler-Bell-Jackiw and trace anomalies by the use of dimensional regularization. As an alternative, one could imagine deriving these results using Pauli-Villars regularization. In that technique, one regularizes the value of a fermion loop integral by subtracting the value of the same loop diagram computed with fermions Ψ of large mass M. The parameter M plays the role of the cutoff and should be taken to infinity at the end of the calculation. The anomalies arise because some pieces of the diagrams computed for very heavy fermions do not disappear in the limit $M \to \infty$. These nontrivial $M \to \infty$ limits are interesting in their own right and can have physical applications (for example, in part (c) of the Final Project for Part III).

(a) Show that the Adler-Bell-Jackiw anomaly equation is equivalent to the following large-mass limit of a fermion matrix element between the vacuum and a two-photon state:

$$\lim_{M\to\infty} \left\{ \langle p,k|\, 2iM\bar{\Psi}\gamma^5\Psi\,|0\rangle \right\} = -\frac{e^2}{2\pi^2}\epsilon^{\alpha\nu\beta\lambda}p_\alpha\epsilon^*_\nu(p)k_\beta\epsilon^*_\lambda(k).$$

(b) Show that the trace anomaly, at one-loop order, is equivalent to the following large-mass limit:

$$\lim_{M\to\infty} \left\{ \langle p,k|\, M\bar{\Psi}\Psi\,|0\rangle \right\} = +\frac{e^2}{6\pi^2}[p\cdot k\,\epsilon^*(p)\cdot\epsilon^*(k) - p\cdot\epsilon^*(k)\,k\cdot\epsilon^*(p)].$$

(c) Show that the matrix element in part (a) is ultraviolet-finite before the $M \to \infty$ limit is taken. Evaluate the matrix element explicitly at one-loop order and verify the limit claimed in part (a).

(d) The matrix element in part (b) has a potential ultraviolet divergence. However, show that the coefficient of $(p\cdot\epsilon^*(k)k\cdot\epsilon^*(p))$ is ultraviolet-finite, and that the rest of the expression is determined by gauge invariance. Compute the full matrix element using dimensional regularization as a gauge-invariant regulator and verify the result claimed in part (b).

Problems

20.1 Spontaneous breaking of $SU(5)$. Consider a gauge theory with the gauge group $SU(5)$, coupled to a scalar field Φ in the adjoint representation. Assume that the potential for this scalar field forces it to acquire a nonzero vacuum expectation value. Two possible choices for this expectation value are

$$\langle\Phi\rangle = A\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \quad \text{and} \quad \langle\Phi\rangle = B\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}.$$

For each case, work out the spectrum of gauge bosons and the unbroken symmetry group.

20.2 Decay modes of the W and Z bosons.

(a) Compute the partial decay widths of the W boson into pairs of quarks and leptons. Assume that the top quark mass m_t is larger than m_W, and ignore the other quark masses. The decay widths to quarks are enhanced by QCD corrections. Show that the correction is given, to order α_s, by Eq. (17.9). Using $\sin^2\theta_w = 0.23$, find a numerical value for the total width of the W^+.

(b) Compute the partial decay widths of the Z boson into pairs of quarks and leptons, treating the quarks in the same way as in part (a). Determine the total width of the Z boson and the fractions of the decays that give hadrons, charged leptons, and invisible modes $\nu\bar{\nu}$.

20.3 $e^+e^- \to$ hadrons with photon-Z^0 interference

(a) Consider a fermion species f with electric charge Q_f and weak isospin I_L^3 for its left-handed component. Ignore the mass of the f. Compute the differential cross section for the process $e^+e^- \to f\bar{f}$ in the standard electroweak model. Include the effect of the Z^0 width using the Breit-Wigner formula, Eq. (7.60). Plot the behavior of the total cross section as a function of CM energy through the Z^0 resonance, for u, d, and μ.

(b) Compute the forward-backward asymmetry for $e^+e^- \to f\bar{f}$, defined as

$$A_{FB}^f = \frac{(\int_0^1 - \int_{-1}^0) d\cos\theta (d\sigma/d\cos\theta)}{(\int_0^1 + \int_{-1}^0) d\cos\theta (d\sigma/d\cos\theta)},$$

as a function of center of mass energy.

(c) Show that, just on the Z^0 resonance, the forward-backward asymmetry is given by

$$A_{FB}^f = \frac{3}{4} A_{LR}^e A_{LR}^f.$$

(d) Show that the cross section at the peak of the Z^0 resonance is given by

$$\sigma_{\text{peak}} = \frac{12\pi}{m_Z^2} \frac{\Gamma(Z^0 \to e^+e^-)\Gamma(Z^0 \to f\bar{f})}{\Gamma_Z^2},$$

where Γ_Z is the total width of the Z^0. Notice that both the total width of the Z^0 and the peak height are affected by the presence of extra invisible decay modes. Compute the shifts in Γ_Z and σ_{peak} that would be produced by a hypothetical fourth neutrino species, and compare these shifts to the cross section measurements shown in Fig. 20.5.

20.4 Neutral-current deep inelastic scattering.

(a) In Eq. (17.35), we wrote formulae for neutrino and antineutrino deep inelastic scattering with $W^{\pm}$ exchange. Neutrinos and antineutrinos can also scatter by exchanging a Z^0. This process, which leads to a hadronic jet but no observable outgoing lepton, is called the *neutral current* reaction. Compute $d\sigma/dxdy$ for neutral current deep inelastic scattering of neutrinos and antineutrinos from protons, accounting for scattering from u and d quarks and antiquarks.

(b) Next, consider deep inelastic scattering from a nucleus A with equal numbers of protons and neutrons. For such a target, $f_u(x) = f_d(x)$, and similarly for antiquarks. Show that the formulae in part (a) simplify in such a situation. In particular, let R^{ν}, $R^{\bar{\nu}}$ be defined as

$$R^{\nu} = \frac{d\sigma/dxdy(\nu A \to \nu X)}{d\sigma/dxdy(\nu A \to \mu^- X)}, \qquad R^{\bar{\nu}} = \frac{d\sigma/dxdy(\bar{\nu} A \to \bar{\nu} X)}{d\sigma/dxdy(\bar{\nu} A \to \mu^+ X)}.$$

Show that R^{ν} and $R^{\bar{\nu}}$ are given by the following simple formulae:

$$R^{\nu} = \frac{1}{2} - \sin^2\theta_w + \frac{5}{9}\sin^4\theta_w(1+r),$$
$$R^{\bar{\nu}} = \frac{1}{2} - \sin^2\theta_w + \frac{5}{9}\sin^4\theta_w(1+\frac{1}{r}),$$

where

$$r = \frac{d\sigma/dxdy(\bar{\nu} A \to \mu^+ X)}{d\sigma/dxdy(\nu A \to \mu^- X)}.$$

These formulae remain true when R^{ν} and $R^{\bar{\nu}}$ are redefined to be the ratios of neutral- to charged-current cross sections integrated over the region of x and y that is observed in a given experiment.

(c) By setting r equal to the observed value—say, $r = 0.4$—and varying $\sin^2\theta_w$, the relations of part (b) generate a curve in the plane of R^{ν} versus $R^{\bar{\nu}}$ that is known as *Weinberg's nose*. Sketch this curve. The observed values of R^{ν}, $R^{\bar{\nu}}$ lie close to this curve, near the point corresponding to $\sin^2\theta_w = 0.23$.

20.5 A model with two Higgs fields.

(a) Consider a model with two scalar fields ϕ_1 and ϕ_2, which transform as $SU(2)$ doublets with $Y = 1/2$. Assume that the two fields acquire parallel vacuum expectation values of the form (20.23) with vacuum expectation values v_1, v_2. Show that these vacuum expectation values produce the same gauge boson mass matrix that we found in Section 20.2, with the replacement

$$v^2 \to (v_1^2 + v_2^2).$$

(b) The most general potential function for a model with two Higgs doublets is quite complex. However, if we impose the discrete symmetry $\phi_1 \to -\phi_1$, $\phi_2 \to \phi_2$,

the most general potential is

$$V(\phi_1, \phi_2) = -\mu_1^2 \phi_1^\dagger \phi_1 - \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_1 (\phi_1^\dagger \phi_1)^2 + \lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \lambda_4 (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1) + \lambda_5 ((\phi_1^\dagger \phi_2)^2 + \text{h.c.}).$$

Find conditions on the parameters μ_i and λ_i so that the configuration of vacuum expectation values required in part (a) is a locally stable minimum of this potential.

(c) In the unitarity gauge, one linear combination of the upper components of ϕ_1 and ϕ_2 is eliminated, while the other remains as a physical field. Show that the physical charged Higgs field has the form

$$\phi^+ = \sin\beta\, \phi_1^+ - \cos\beta\, \phi_2^+,$$

where β is defined by the relation

$$\tan\beta = \frac{v_2}{v_1}.$$

(d) Assume that the two Higgs fields couple to quarks by the set of fundamental couplings

$$\mathcal{L}_m = -\lambda_d^{ij} \overline{Q}_L^i \cdot \phi_1 d_R^j - \lambda_u^{ij} \epsilon^{ab} \overline{Q}_{La}^i \phi_{2b}^\dagger u_R^j + \text{h.c.}$$

Find the couplings of the physical charged Higgs boson of part (c) to the mass eigenstates of quarks. These couplings depend only on the values of the quark masses and $\tan\beta$ and on the elements of the CKM matrix.

Problems

21.1 Weak-interaction contributions to the muon $g-2$. The GWS model of the weak interactions leads to two new contributions to the anomalous magnetic moments of the leptons. Because these contributions are proportional to $G_F m_\ell^2$, they are extremely small for the electron, but for the muon they might possibly be observable. Both contributions are larger than the contribution of the Higgs boson discussed in Problem 6.3.

(a) Consider first the contribution to the muon electromagnetic vertex function that involves a W-neutrino loop diagram. In the R_ξ gauges, this diagram is accompanied by diagrams in which W propagators are replaced by propagators for Goldstone bosons. Compute the sum of these diagrams in the Feynman-'t Hooft gauge and show that, in the limit $m_W \gg m_\mu$, they contribute the following term to the anomalous magnetic moment of the muon:

$$a_\mu(\nu) = \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{10}{3}.$$

(b) Repeat the calculation of part (a) in a general R_ξ gauge. Show explicitly that the result of part (a) is independent of ξ.

(c) A second new contribution is that from a Z-muon loop diagram and the corresponding diagram with the Z replaced by a Goldstone boson. Show that these diagrams contribute

$$a_\mu(Z) = -\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \left(\frac{4}{3} + \frac{8}{3}\sin^2\theta_w - \frac{16}{3}\sin^4\theta_w\right).$$

21.2 Complete analysis of $e^+e^- \to W^+W^-$.

(a) Using explicit polarization vectors, work out the amplitudes for $e^+e^- \to W^+W^-$ from left- and right-handed electrons to states in which the W^+ and W^- have definite helicity. For the cases in which both W bosons have longitudinal polarization, verify that Eq. (21.99) gives the correct high-energy limit for right-handed electrons, and verify the complete expression (21.108) for left-handed electrons. For the cases in which one W is longitudinally polarized and the second is transversely polarized, show that the individual diagrams give contributions to the amplitudes that grow like $\sqrt{s}$, but that the complete amplitudes fall as $1/\sqrt{s}$.

(b) Show that the contributions to $e_L^- e_R^+ \to W^-W^+$ found in part (a) reproduce Fig. 21.10, and that the differential cross section for $e_R^- e_L^+ \to W^-W^+$ is about 30 times smaller. How many of the qualitative features of the figure can you understand physically?

21.3 Cross section for $d\bar{u} \to W^-\gamma$. Compute the amplitudes for $d\bar{u} \to W^-\gamma$ for the various possible initial and final helicities. Ignore the quark masses. In this approximation, only the annihilation amplitude from $d_L\bar{u}_R$ is nonzero. Show that the scattering amplitudes for all final helicity combinations vanish at $\cos\theta = -1/3$, where θ is the scattering angle in the center-of-mass system. Compute the differential cross section as a function of $\cos\theta$.

21.4 Dependence of radiative corrections on the Higgs boson mass.

(a) Consider the contributions to weak-interaction radiative corrections involving the physical Higgs boson h^0 of the GWS model. The couplings of the h^0 were discussed near the end of Section 20.2. Show that, if we ignore terms proportional to the masses of light fermions, the Higgs boson contributes one-loop corrections to the processes considered in Section 21.3 only through vacuum polarization diagrams. It follows that the contributions to vacuum polarization amplitudes that depend on the Higgs boson mass are gauge invariant.

(b) Draw the vacuum polarization diagrams in Feynman-'t Hooft gauge that involve the Higgs boson, and compute the dependence of the various vacuum polarization amplitudes on the Higgs boson mass m_h.

(c) Show that, for $m_h \gg m_W$, the natural relations discussed in Section 21.3 receive corrections

$$s_*^2 - s_0^2 = \frac{\alpha}{\cos^2\theta_w - \sin^2\theta_w} \frac{(1 + 9\sin^2\theta_w)}{48\pi} \log\frac{m_h^2}{m_W^2},$$

$$s_W^2 - s_*^2 = \alpha \frac{5}{24\pi} \log\frac{m_h^2}{m_W^2}.$$

The effect of varying m_h is displayed in Fig. 21.14 and is included as a theoretical uncertainty in the prediction (21.158). More accurate experiments might allow one to predict m_h from its effect on electroweak radiative corrections.