Physics 330 – Problem Set # 1

(due Wednesday, October 5)

The purpose of this problem set is to help you review some items of mathematics and physics that will be used in the course. If it turns out that this is not review, and that in fact you have never done these sorts of calculations before, please take special care that you are familiar with these methods.

1. Relativistic kinematics:

   (a) A particle of mass M decays to a particle of mass m and a massless particle. Find the energies and momenta of the two final particles.

   (b) A particle of mass M decays to two particles, one with mass m_1 and the other with mass m_2. Find the energies and momenta of the two final particles.

   (c) A particle of mass M decays to three massless particles. Let variables $x_i = 2E_i/M$ parametrize the energies of the final particles. Note that $\sum_i x_i = 2$. Find the kinematically allowed region for (x_1, x_2).

   (d) A particle of mass M decays to two massless particles (particles 1 and 2) and one particle of mass m (particle 3). Let variables $x_i = 2E_i/M$ parametrize the energies of the final particles. Again, $\sum_i x_i = 2$. Find the kinematically allowed region for (x_1, x_2). Show that the boundary in this case is formed from a straight line and a hyperbola.

2. Rotations and spin:

   (a) Write the 3×3 rotation matrix for a rotation by θ about the $\hat{2}$ (y) axis that carries the vector $(0, 0, 1)$ into $(\sin\theta, 0, \cos\theta)$.

   (b) Show that this matrix is represented by $R = \exp[-i\theta J^2]$, where $J^2_{13} = -J^2_{31} = i$ and all other elements are 0.

   (c) The corresponding rotation matrix in the spin-$\frac{1}{2}$ representation is generated by $J^2 = \sigma^2/2$. Write this matrix explicitly as a 2×2 matrix.

   (d) Compute the rotation matrix for the following sequence of operations: (1) rotate about $\hat{3}$ by ϕ; (2) rotate about $\hat{2}$ by θ; (3) rotate about $\hat{3}$ by $-\phi$. Perform this computation both in the vector (spin-1) and the spin-$\frac{1}{2}$ representation. Show that the same rotation is produced in the two cases, that is, that the two matrices represent a rotation about the same axis by the same angle.

3. Complex variables and contour integration:

(a) Evaluate

$$I_1 = \int_0^\infty dx \, \frac{x^2}{(a^2 + x^2)(b^2 + x^2)} \tag{1}$$

(b) Evaluate

$$I_2 = \int_0^1 dx \, \frac{1}{\sqrt{x(1-x)}\,(x-2)} \tag{2}$$

(c) Evaluate

$$I_3 = \int_0^\infty dx \, \frac{x \sin x}{(a^2 + x^2)} \tag{3}$$

(d) Evaluate

$$I_4 = \int_{-\infty}^\infty \frac{d\omega}{2\pi} \, \frac{e^{-i\omega t}}{\omega - (\Omega - i\Gamma)} \ , \tag{4}$$

where Ω and Γ are real and positive. Consider the cases $t > 0$ and $t < 0$ separately.

Physics 330 - Problem Set 1
Solutions

1.) I set $c = 1$, so

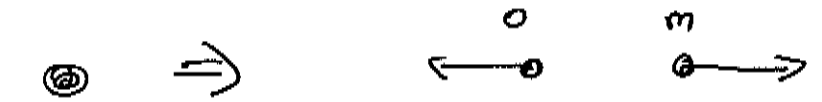
for a massless particle moving $\parallel \hat{3}$:

$$p^\mu = (p, 0, 0, p)$$

for a massive particle moving $\parallel \hat{3}$

$$p^\mu = (E_p, 0\ 0\ p) \qquad E_p = [|\vec{p}|^2 + m^2]^{1/2}$$

a).



$p^\mu = (p, 0\ 0\ -p)$ $\qquad p^\mu = (E_p, 0, 0, p)$

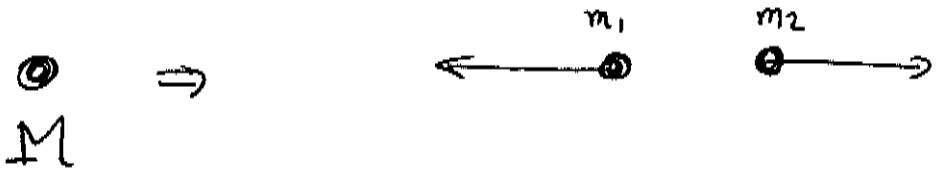
$p + E_p = M$ conserv. of energy.

then $[p^2 + m^2]^{1/2} = M - p$

$p^2 + m^2 = M^2 - 2Mp + p^2$

$$p = \frac{M^2 - m^2}{2M} \qquad E_p = \frac{M^2 + m^2}{2M}$$

↑ energy + mom of massless particle $\qquad$ energy of massive particle

b.) 

$$p_1 = (E_1, 0, 0, p) \qquad p_2 = (E_2, 0, 0, p)$$

$$E_1 = (p^2 + m_1^2)^{1/2} \qquad E_2 = (p^2 + m_2^2)^{1/2}$$

$$E_1 + E_2 = M$$

$$E_1^2 + E_2^2 + 2E_1E_2 = M^2$$

$$4E_1^2E_2^2 = (M^2 - E_1^2 - E_2^2)^2$$

$$4E_1^2E_2^2 = M^4 - 2M^2(E_1^2 + E_2^2) + (E_1^2 + E_2^2)^2$$

$$0 = M^4 - 2M^2(E_1^2 + E_2^2) + (E_1^2 - E_2^2)^2$$

$$0 = M^4 - 2M^2(2p^2 + m_1^2 + m_2^2) + (p^2 + m_1^2 - p^2 + m_2^2)^2$$

so

$$4p^2M^2 = M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)$$

$$p = \frac{[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)]^{1/2}}{2M}$$

$$E_1^2 = p^2 + m_1^2 = \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 + 4M^2m_1^2}{4M^2}$$

$$= \frac{M^4 + 2M^2(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2}{4M^2} = \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)^2$$

So
$$p = \frac{\left[M^4 - 2M^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2\right]^{\frac{1}{2}}}{2M}$$

$$E_1 = \frac{M^2+m_1^2-m_2^2}{2M} \qquad E_2 = \frac{M^2+m_2^2-m_1^2}{2M}$$

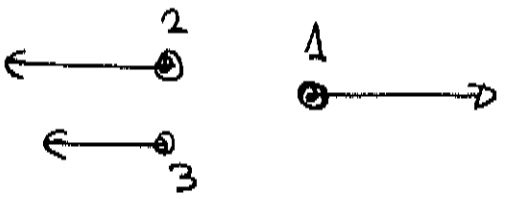
One sometimes sees the notation:

$$\lambda(a,b,c) = a^2+b^2+c^2 - 2ab - 2ac - 2bc$$

$$p = \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{2M}$$

c) To answer this question, we should ask: What configurations allow the energies E_1 E_2 E_3 to take their maximum values?

For E_1, this configuration is



1 recoils against 2 + 3: $p_1 = p_2 + p_3$

For all massless particles $p_1 = E_1$ $p_2 = E_2$ $p_3 = E_3$

$$E_1 + E_2 + E_3 = M \quad \text{so} \quad p_1 = M/2$$

In terms of the x_i $\qquad x_i = 2E_i/M$

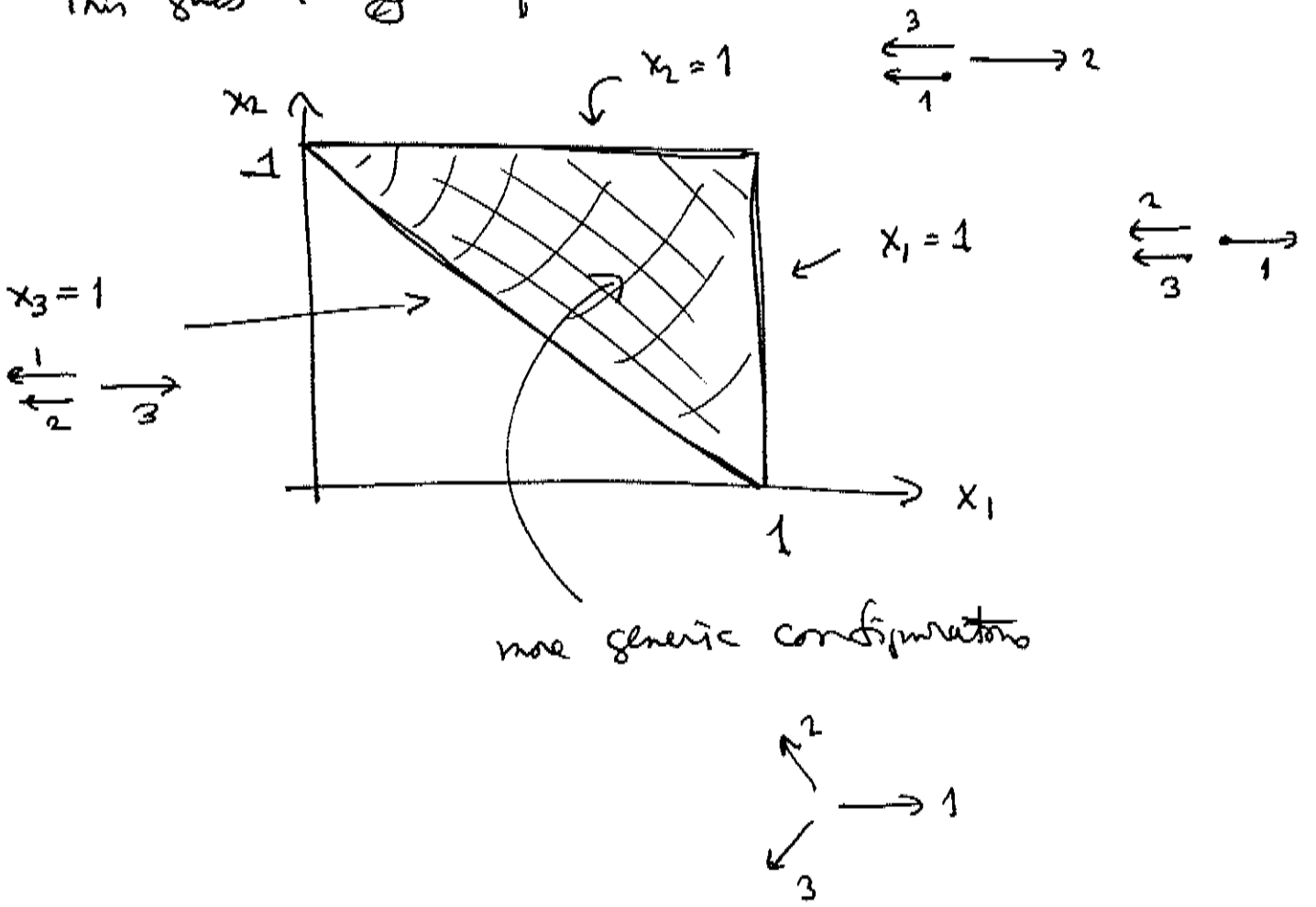
$x_1 = 1 \qquad x_2 + x_3 = 1$

The other boundaries correspond to

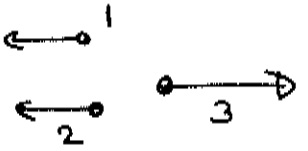
$x_2 = 1 \qquad x_1 + x_3 = 1$

$x_3 = 1 \qquad x_1 + x_2 = 1$

This gives a region of the $x_1\ x_2$ plane:



d.) If particle 3 has mass m:



Now has $p_3 = p_1 + p_2$ $\quad E_3 = (p_3^2 + m^2)^{1/2}$

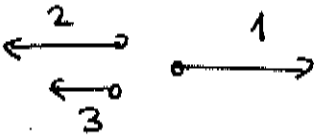
1+2 is a massless 4-vector $(p_1+p_2)^2 = 0$ or

$(p_1+p_2)^\mu = (p_3, 0, 0, -p_3)$. Then this is the problem we solved in part (a).

$$p_3 = \frac{M^2-m^2}{2M} \Rightarrow x_1+x_2 = \left(1 - \frac{m^2}{M^2}\right)$$

$$E_3 = \frac{M^2+m^2}{2M}$$

For the configuration



$$p_1 = p_2 + p_3$$

$$M = p_1 + p_2 + (p_3^2 + m^2)^{1/2}$$

$$M - p_1 - p_2 = \left((p_1 - p_2)^2 + m^2\right)^{1/2}$$

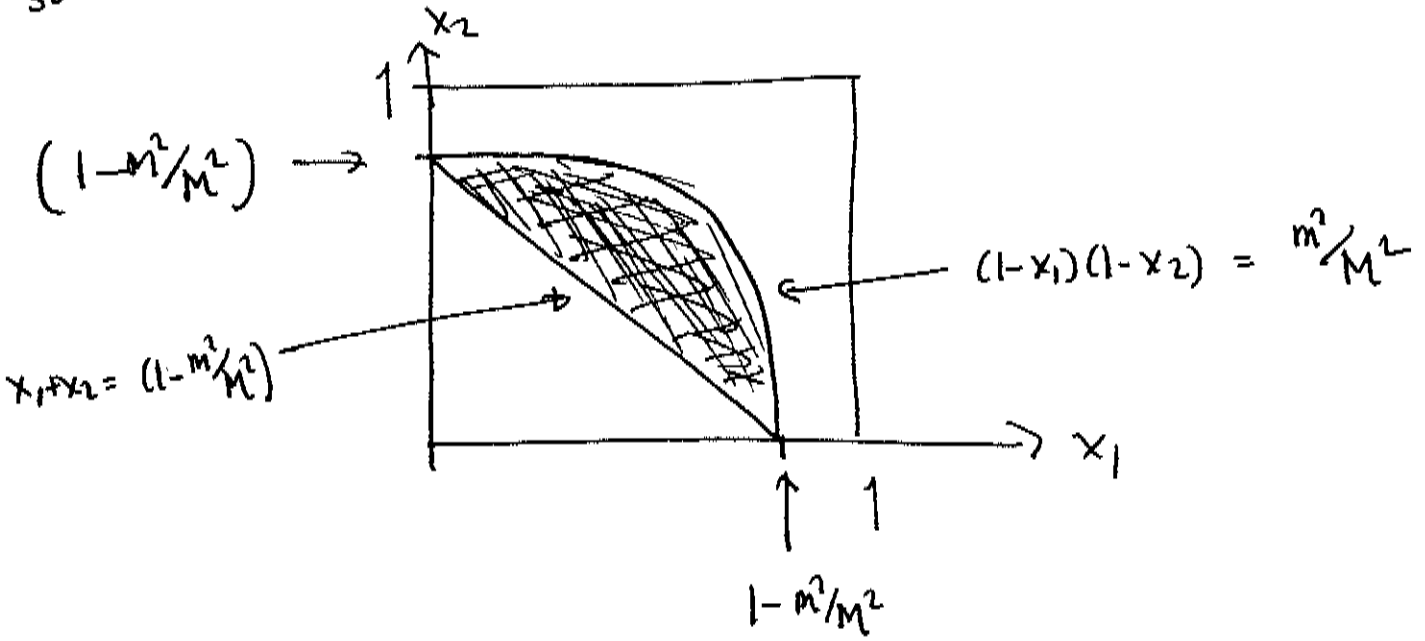
$$M^2 - 2M(p_1+p_2) + (p_1+p_2)^2 = (p_1-p_2)^2 + m^2$$

$$M^2 - 2M(p_1+p_2) + 4p_1p_2 = m^2$$

$$(M-2p_1)(M-2p_2) = m^2$$

$$(1-x_1)(1-x_2) = m^2/M^2$$

So the allowed kinematic region is now:



2.) a.) $$R = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix} \qquad R\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\0\\-\sin\theta\end{pmatrix}$$

$$R\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$ this is the axis of the rotation

b.) $$\frac{d}{d\theta} R = \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

so $R(\theta)$ satisfies

$$\frac{d}{d\theta} R(\theta) = -i \left[\quad J^2 \right] R(\theta)$$

where $$J^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

the solution of this equation with initial condition $R(\theta=0) = 1$

is $$R(\theta) = \exp\left[-i J^2 \theta \right]$$

c.) In the spin-½ representation:

$$e^{-i J^2 \theta} = e^{-i \sigma^2/2 \, \theta} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma^2)^2 = 1$$

$$= \cos\frac{\theta}{2} - i \sigma^2 \sin\frac{\theta}{2} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

d) In the spin-1 representation:

rotate by ϕ about $\hat{3}$ = R_1 = $\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

rotate by θ about $\hat{2}$ = R_2 = $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

rotate by $-\phi$ about $\hat{3}$: R_3 = $\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

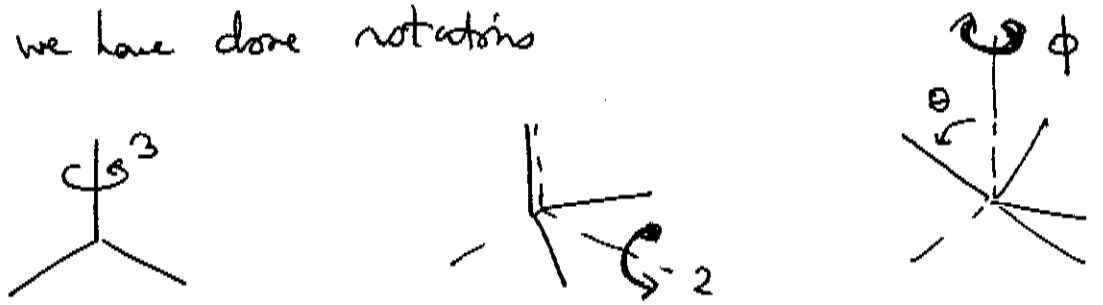
$$R_3 R_2 R_1 = \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & & \sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & \\ \sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & \\ \sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi\cos\theta + \sin^2\phi & -\sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi & \cos\phi\sin\theta \\ \sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi & \sin^2\phi\cos\theta + \cos^2\phi & -\sin\phi\sin\theta \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

What is this?

we have done rotations



It is obvious that the $\hat{3}$ axis should go into

rotate of $(\sin\theta, 0, \cos\theta)$

$= (\cos\phi \sin\theta, -\sin\phi \sin\theta, \cos\theta)$

but the rest is less obvious.

Try the spin-½ case:

$$R_1 = e^{-i\sigma_3^2\phi} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & +\cos\theta/2 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} e^{+i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

so

$$R_3 R_2 R_1 = \begin{pmatrix} \cos\theta/2\, e^{+i\phi/2} & -\sin\theta/2\, e^{+i\phi/2} \\ \sin\theta/2\, e^{-i\phi/2} & \cos\theta/2\, e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{\pm i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2\, e^{+i\phi} \\ \sin\theta/2\, e^{-i\phi} & \cos\theta/2 \end{pmatrix}$$

now

$$e^{-i\vec{\theta}\cdot\vec{\sigma}/2} = \cos\frac{\theta}{2} - i\,\hat{\theta}\cdot\vec{\sigma}\,\sin\theta/2 \qquad \theta = |\vec{\theta}|$$

the above is:

$$R_3R_2R_1 = \cos\theta/2 \cdot 1 - \cos\phi\sin\theta/2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - i\sin\phi\sin\theta/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \cos\theta/2 - i\left[\sin\phi\,\sigma^1 + \cos\phi\,\sigma^2\right]\sin\theta/2$$

so this is a rotation about the axis

$$\hat{n} = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

by an angle θ.

There are some relatively easy ways to check that p. 8 is also a rotation of θ about the axis n

For any rotation by θ: $\mathrm{tr}\,R = \cos\theta + \cos\theta + 1 = 1 + 2\cos\theta$

For any rotation about $\hat{n}$: $R\hat{n} = \hat{n}$

Let's check

$$\mathrm{tr}(R_3R_2R_1) = \cos^2\phi\cos\theta + \sin^2\phi + \sin^2\phi\cos\theta + \cos^2\phi + \cos\theta$$

$$= 2\cos\theta + 1 \quad \checkmark$$

$$(R_3R_2R_1)\begin{pmatrix}\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

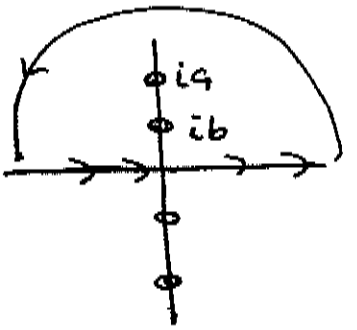
$$= \begin{pmatrix} (\cos^2\phi\cos\theta + \sin^2\phi)\sin\phi + (-\sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi)\cos\phi \\ (-\sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi)\sin\phi + (\sin^2\phi\cos\theta + \cos^2\phi)\cos\phi \\ (-\sin\theta\cos\phi)\sin\phi + (\sin\theta\sin\phi)\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\phi + \sin\phi\cos^2\phi \\ \sin^2\phi\cos\phi + \cos^3\phi \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix}\sin\phi\\ \cos\phi\\ 0\end{pmatrix} = \hat{n} \quad \checkmark$$

∴ $R_3R_2R_1$ in the spin ½ and spin 1 cases are rotations about the same axis by the same angle.

3) a) $I_1 = \int_0^\infty dx \; \frac{x^2}{(a^2+x^2)(b^2+x^2)}$



$= \frac{1}{2} \int_{-\infty}^{\infty} dx \; \frac{x^2}{(a^2+x^2)(b^2+x^2)}$

$= 2\pi i \cdot \frac{1}{2} \{ (\text{Residue at } x=ia) + (\text{Residue at } x=ib) \}$

$= \pi i \left\{ \frac{(ia)^2}{2ia\,(b^2-a^2)} + \frac{(ib)^2}{(a^2-b^2)\,(2ib)} \right\}$

$= \frac{\pi}{2(a^2-b^2)} \{ a-b \}$

so

$I_1 = \frac{\pi}{2} \frac{1}{(a+b)}$

a more direct method: $I_1 = \frac{1}{2} \int_{-\infty}^{\infty} dx \left(\frac{a^2}{a^2+x^2} - \frac{b^2}{b^2+x^2} \right) \frac{1}{(a^2-b^2)}$

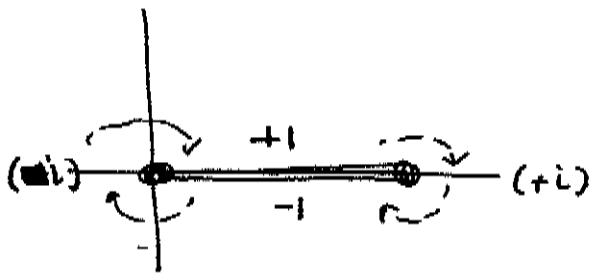
now $\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{a}$

so $I_1 = \frac{\pi}{2} (a-b) \frac{1}{a^2-b^2} = \frac{\pi}{2} \frac{1}{(a+b)}$ ✓

b) To do this exercise, first look at the analytic structure of $\frac{1}{\sqrt{x(1-x)}}$. Define this so that

$$\frac{1}{\sqrt{x(1-x)}} = + \frac{1}{\sqrt{|x(1-x)|}} \quad \text{(real + positive)}$$

just above the real axis in the x plane. Then the phase of $\frac{1}{\sqrt{x(1-x)}}$ at other points is:



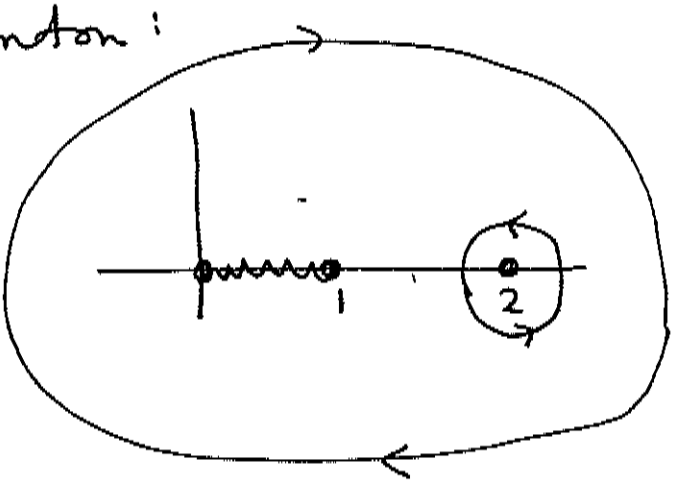
Notice that

$$I_2 = \int_0^1 dx \; \frac{1}{\sqrt{x(1-x)}} \; \frac{1}{x-2}$$

$$= \tfrac{1}{2} \int_C dx \; \frac{1}{\sqrt{x(1-x)}} \; \frac{1}{x-2}$$



Now deform the contour:



The big contour at ∞ gives $\int dx \frac{1}{x^2} \longrightarrow 0$

The pole at $x=2$ gives

$$I_2 = \frac{1}{2} 2\pi i \left[\frac{1}{\sqrt{x(1-x)}}\right]_{x=2}$$

$$= \frac{1}{2} 2\pi i \; (+i) \frac{1}{\sqrt{|2 \cdot 1|}}$$

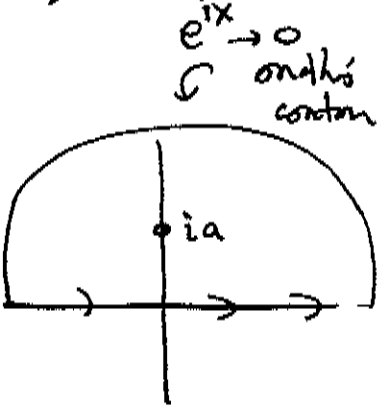
[phase from p.13]

$$I_2 = -\frac{\pi}{\sqrt{2}}$$

c.) $$I_3 = \int_0^\infty dx \; \frac{x \sin x}{a^2+x^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \; \frac{x}{a^2+x^2} \left(\frac{e^{ix} - e^{-ix}}{2i}\right)$$

$$= \int_{-\infty}^{\infty} dx \; \frac{x}{a^2+x^2} \; \frac{e^{ix}}{2i}$$



$$= \frac{2\pi i}{2i} \; \frac{ia}{2ia} \; e^{i(ia)}$$

$$I_3 = \frac{\pi}{2} e^{-a}$$

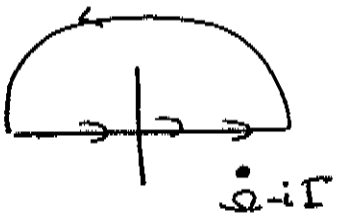
d.) The important part for the integral is that

$e^{-i\omega t} \to 0$ for $\mathrm{Im}\,\omega \to \infty$ when $t < 0$

for $\mathrm{Im}\,\omega \to -\infty$ when $t > 0$

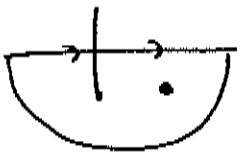
so

$t < 0$



$I_4 = 0$

$t > 0$



$$I_4 = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega - (\Omega - i\Gamma)}$$

$$= -2\pi i \frac{e^{-it(\Omega - i\Gamma)}}{2\pi}$$

so

$$I_4 = \begin{cases} 0 & t < 0 \\ -i\, e^{-i\Omega t} e^{-\Gamma t} & t > 0 \end{cases}$$

Physics 330 – Problem Set # 2

(due Wednesday, October 12)

The first two problems are close to Peskin and Schroeder, problem 2.2. The changes in notation are intentional; hopefully, they will make the exercise more clear.

1. Consider the field theory of a free complex scalar field. The action of the theory is

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \tag{1}$$

where $\phi(x)$ is a complex-valued field and m is real-valued. It is easiest to deal with this field by treating $\phi(x)$ and $\phi^*(x)$ as the independent variables rather than by using the real and imaginary parts of $\phi(x)$.

(a) By varying S, show that $\phi(x)$ and $\phi^*(x)$ satisfy the Klein-Gordon equation.

(b) Show that the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ are

$$\pi^*(x) = \dot{\phi}^*(x) \qquad \pi(x) = \dot{\phi}(x) \ , \tag{2}$$

respectively. Show that the Hamiltonian of the theory is

$$H = \int d^3x \left(\pi^* \pi + \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi + m^2 \phi^* \phi \right) \tag{3}$$

(c) Using the canonical commutation relations

$$[\phi(\vec{x}), \pi^*(\vec{y})] = i\delta(\vec{x} - \vec{y}) \qquad [\phi^*(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}) \tag{4}$$

(with all other pairs commuting), show that the Heisenberg equation of motion for $\phi(x)$ is the Klein-Gordon equation.

(d) Introducing appropriate representations for the operators $\phi(x)$, $\phi^*(x)$, $\pi(x)$, $\pi^*(x)$ in terms of creation and annihilation operators, diagonalize H. Show that two sets of creation and annihilation operators are needed to represent the canonical commutation relations. Show that these create particles of equal mass. Argue that the corresponding excitations can be considered particle and antiparticle.

(e) Consider the quantity

$$Q = -i \int d^3x \, (\pi^* \phi - \phi^* \pi) \tag{5}$$

Express Q in terms of creation and annihilation operators. Show that Q commutes with H. Show explicitly that single-particle eigenstates of H are eigenstates of Q. How do you interpret the eigenvalues?

2. Generalize the preceding problem to the case of two complex Klein-Gordon fields ϕ_a, $a = 1, 2$ with equal mass m.

   (a) Write down an appropriate Hamiltonian and canonical commutation relations, and show that $\phi_a(x)$ obeys the Klein-Gordon equation as its equation of motion.

   (b) Show that the charge

   $$Q = -i \int d^3x \, (\pi_a^* \phi_a - \phi_a^* \pi_a) \tag{6}$$

   commutes with H. (Sums over repeated indices should be understood.)

   (c) Show that, if σ^j_{ab} are the Pauli sigma matrices, the charges

   $$Q^j = -\frac{i}{2} \int d^3x \left(\pi_a^* \sigma^j_{ab} \phi_a - \phi_a^* \sigma^j_{ab} \pi_b \right) \tag{7}$$

   commute with H. Compute

   $$[Q^j, Q^k] \tag{8}$$

3. Peskin and Schroeder, problem 3.1.

Physics 330 – Problem Set #2

Solutions

1.) a.) $$\delta S = \int d^4x \{ \partial_\mu \delta\phi^* \partial^\mu \phi + \partial_\mu \phi^* \partial^\mu \delta\phi - m^2 \delta\phi^* \phi - m^2 \phi^* \delta\phi \}$$

$$= \int d^4x [\delta\phi^* \{ -\partial_\mu \partial^\mu \phi - m^2 \phi \} + \{ \partial^\mu \partial_\mu \phi^* - m^2 \phi^* \} \delta\phi]$$

so

$$\delta S = 0 \Rightarrow \text{coeff's of } \delta\phi^*, \delta\phi = 0$$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0 = (\partial^\mu \partial_\mu + m^2)\phi^*$$

$\therefore$

b.) c.) Consistently with the discussion in class, it suffices to show that, using the given commutation relations of Hamiltonian, ϕ and ϕ^* satisfy the Klein-Gordon eq. as their Heisenberg equations of motion.

So, compute

$$i\frac{\partial}{\partial t}\phi = [\phi(x), H]$$

$$= \int d^3y \; [\phi(x), \pi^*(y)\pi(y)]$$

$$= \int d^3y \; i\delta(\vec{x}-\vec{y})\,\pi(y) = i\pi(x)$$

$$i\frac{\partial}{\partial t}\phi^* = [\phi^*(x), H]$$

$$= \int d^3y \; [\phi^*(x), \pi^*(y)\pi(y)]$$

$$= \int d^3y \; \pi^*(y) \; [\phi^*(x), \pi(y)]$$

$$= \int d^3y \; \pi^*(y)\, i\delta(\vec{x}-\vec{y}) = i\pi^*(x)$$

$$i\frac{\partial}{\partial t}\pi(x) = [\pi(x), H]$$

$$= [\pi(x), \int d^3y \; (\vec{\nabla}\phi^* \vec{\nabla}\phi + m^2\phi^*\phi)]$$

integrate by parts $= [\pi(x), \int d^3y \; \phi^*(y)(-\nabla^2\phi + m^2\phi)(y)]$

$$= (-i)(-\nabla^2\phi + m^2\phi)(x)$$

$$i\frac{\partial}{\partial t}\pi^*(x) = [\pi^*(x), H]$$

$$= [\pi^*(x), \int d^3y \; \{-\nabla^2\phi^*\,\phi + m^2\phi^*\phi\}]$$

$$= (-i)(-\nabla^2\phi^* + m^2\phi^*)(x)$$

$$\frac{\partial}{\partial t}\phi = \pi \qquad \frac{\partial}{\partial t}\pi = \nabla^2\phi - m^2\phi$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

$$\text{or} \quad \partial^\mu\partial_\mu\phi + m^2\phi = 0$$

similarly for ϕ^*

$$\frac{\partial}{\partial t}\phi^* = \pi^* \qquad \frac{\partial}{\partial t}\pi^* = \nabla^2\phi^* - m^2\phi^*$$

$$\Rightarrow \quad (\partial_\mu\partial^\mu + m^2)\phi^* = 0$$

d.) Making a slight generalization of what we did in class, write

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}} + b^\dagger_{-\vec{p}}\right) e^{+i\vec{p}\cdot\vec{x}}$$

$$\phi^*(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(b_{\vec{p}} + a^\dagger_{-\vec{p}}\right) e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(-iE_p)\left(a_{\vec{p}} - b^\dagger_{-\vec{p}}\right) e^{-i\vec{p}\cdot\vec{x}}$$

$$\pi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(-iE_p)\left(b_{\vec{p}} - a^\dagger_{-\vec{p}}\right) e^{-i\vec{p}\cdot\vec{x}}$$

with $[a_p, a^\dagger_q] = (2\pi)^3\delta(p-q) = [b_p, b^\dagger_q]$

and $a, a^\dagger$ commuting with $b, b^\dagger$

then manifestly $[\phi(x), \pi(y)] = [\phi^\dagger(x), \pi^\dagger(y)] = 0$

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} e^{i\vec{p}\cdot x} e^{i\vec{k}\cdot y} \frac{1}{\sqrt{2\varepsilon_p 2E_k}}$$

$$\cdot \quad [(a_p + b^\dagger_{-p}), (b_{\vec{k}} + a^\dagger_{-\vec{k}})]$$

the second like is

$$(2\pi)^3 \delta(\vec{p}+\vec{k}) - (2\pi)^3 \delta(p+\vec{k}) = 0$$

similarly: $[\pi(x), \pi^\dagger(y)] \propto [(a_p - b^\dagger_{-\vec{p}}), (b_{\vec{k}} - a^\dagger_{-\vec{k}})] = 0$

$$[\phi(x), \pi^\dagger(y)] = \int \frac{d^3p\, d^3k}{(2\pi)^3 (2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p 2\varepsilon_k}} e^{i\vec{p}\cdot x} e^{i\vec{k}\cdot y} (-iE_k)$$

$$[\, a_{\vec{p}} + b^\dagger_{-\vec{p}} \,,\, b_{\vec{k}} - a^\dagger_{-\vec{k}} \,]$$

$$= \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2\varepsilon_p 2\varepsilon_k}} e^{ipx} e^{iky} (-iE_k) (2\pi)^3 \delta(p+\vec{k}) \cdot (-2)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i\, 2E_p}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} = i\, \delta(\vec{x}-\vec{y})$$

and

$$[\phi^\dagger(x), \pi(y)] = i\, \delta(x-y) \quad \text{similarly}$$

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_k}} e^{i(\vec{p}+\vec{k})\cdot\vec{x}}$$

$$\Big\{ (-iE_p)(iE_k)\,(b_{\vec{p}} - a^\dagger_{-\vec{p}})(a_k - b^\dagger_{-\vec{k}})$$

$$+ (i\vec{p}\,)(i\vec{k})\;(b_{\vec{p}} + a^\dagger_{-\vec{p}})(a_k + b^\dagger_{-\vec{k}})$$

$$+ \quad m^2 \qquad (b_{\vec{p}} + a^\dagger_{-\vec{p}})(a_k + b^\dagger_{-\vec{k}}) \Big\}$$

now, the integral over d^3x gives $(2\pi)^3\delta(p+k)$, so set

$$p = -k$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \Big\{ -E_k^2 \{ -b_{-\vec{k}} b^\dagger_{-\vec{k}} - a^\dagger_k a_k + b_{-\vec{k}} a_k + a^\dagger_{\vec{k}} b^\dagger_{-\vec{k}} \}$$

$$+ (k^2+m^2)\{ b_{-k} b^\dagger_{-k} + a^\dagger_k a_k + b_{-\vec{k}} a_k + a^\dagger_{\vec{k}} b^\dagger_{-\vec{k}} \} \Big\}$$

the ba $ab^\dagger$ terms cancel, since $E_k^2 = (k^2+m^2)$. Then

$$= \int \frac{d^3k}{(2\pi)^3} E_k \,(b_k b^\dagger_k + a^\dagger_k a_k)$$

$$H = \int \frac{d^3k}{(2\pi)^3} E_k \,(b^\dagger_k b_k + a^\dagger_k a_k) \quad + \text{(const.)}$$

Let $|0\rangle$ be s.t. $a_k|0\rangle = b_k|0\rangle = 0$; then there are two types of single-particle states:

$|b(\vec{k})\rangle = \sqrt{2E_k}\, a_k^\dagger |0\rangle \qquad |\bar{b}(\vec{k})\rangle = \sqrt{2E_k}\, b_k^\dagger |0\rangle$

then state both have $E = E_k = [k^2 + m^2]^{\frac{1}{2}}$. Since $\phi(x)$ destroys $|b(\vec{k})\rangle$ and creates $|\bar{b}(k)\rangle$, these are particle and antiparticle.

e.) $$Q = -i \int d^3x \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_k}} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}}$$

$$\{ (-iE_p)(b_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{\vec{k}} + b^\dagger_{-\vec{k}})$$
$$- (-iE_k)(b_p + a^\dagger_{-p})(a_k - b^\dagger_{-k}) \}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \{ -E_p (b_{\vec{p}} b^\dagger_{\vec{p}} - a^\dagger_{-\vec{p}} a_{-\vec{p}} + b_{\vec{p}} a_{-\vec{p}} - a^\dagger_{-\vec{p}} b^\dagger_p)$$
$$+ E_p (-b_p b^\dagger_p + a^\dagger_{-p} a_p + b_p a_{-p} - a^\dagger_{-p} b^\dagger_p) \}$$

again ba $a^\dagger b^\dagger$ terms cancel.

$$= \int \frac{d^3p}{(2\pi)^3} \frac{2E_p}{2E_p} (a^\dagger_p a_p - b_p b^\dagger_p)$$

up to a constant

$$Q = \int \frac{d^3p}{(2\pi)^3} (a^\dagger_p a_p - b^\dagger_p b_p)$$

and obviously $[Q, H] = 0$

so

$$Q\, a_p^\dagger |0\rangle = +1 \cdot a_p^\dagger |0\rangle$$

$$Q\, b_p^\dagger |0\rangle = -1 \cdot b_p^\dagger |0\rangle$$

Q is a charge : particles have charge $+1$, antiparticles have charge -1.

2.) a.) Generalize the above!

$$[\phi_a(\vec{x}), \pi_b^*(\vec{y})] = i\,\delta(\vec{x}-\vec{y})\,\delta_{ab} = [\phi_a^*(\vec{x}), \pi_b(\vec{y})]$$

$$[\phi_a(\vec{x}), \phi_b^{(*)}(\vec{y})] = 0 = [\pi_a(\vec{x}), \pi_b^{(*)}(\vec{y})]$$

$$H = \int d^3y \left\{ \pi_a^* \pi_a + \vec{\nabla}\phi_a^* \vec{\nabla}\phi_a + m^2 \phi_a^* \phi_a \right\}$$

sum over a understood.

since ϕ_a π_a^* etc. commute for different values of a, the above construction goes through separately for $a=1,2$.

thus with

$$[a_{pa}, a^{\dagger}_{qb}] = (2\pi)^3 \delta(\vec{p}-\vec{q})\,\delta_{ab} = [b_{pa}, b^{\dagger}_{qb}]$$

$$\phi_a(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{pa} + b^{\dagger}_{-\vec{p}a})\, e^{i\vec{p}\cdot\vec{x}}$$

etc

diagonalizes H to the form

$$H = \int \frac{d^3p}{(2\pi)^3} E_p (a^{\dagger}_{pa} a_{pa} + b^{\dagger}_{pa} b_{pa}) + \text{const.}$$

the time-dependent field $\phi(x)$ is

$$\phi_a(x) = e^{iHx^0} \phi_a(\vec{x})\, e^{-iHx^0}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{pa} e^{-ip\cdot x} + b^{\dagger}_{pa} e^{ip\cdot x})$$

so $\phi_a(x)$ satisfies the Klein Gordon eq.

b.) Follow the argument on p. 6

$$Q = -i \int d^3x \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_k}} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}}$$

$$\cdot \{ -i E_p (b_{pa} - a^\dagger_{-pa})(a_{ka} + b^\dagger_{-ka}) + i E_k (b_{pa} + a^\dagger_{-pa})(a_{ka} - b^\dagger_{-ka}) \}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \{ -E_p (b_{pa} b^\dagger_{pa} - a^\dagger_{-pa} a_{-pa} + b_{pa} a_{-pa} - a^\dagger_{-pa} b^\dagger_{-pa}) + E_p (-b_{pa} b^\dagger_{pa} + a^\dagger_{-pa} a_{-pa} + b_{pa} a_{-pa} - a^\dagger_{-pa} b^\dagger_{pa}) \}$$

$$= \int \frac{d^3p}{(2\pi)^3} (b^\dagger_{pa} b_{pa} - a^\dagger_{pa} a_{pa}) + (\text{const.})$$

this manifestly commutes with H on p. 8.

c.) we can compute Q^j by the same method.

$$Q^j = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \{ -E_p (b_{pa} \blacksquare a^\dagger_{-pa}) \frac{\sigma^j_{ab}}{2} (a_{-pb} + b^\dagger_{pb}) + E_p (b_{pa} + a^\dagger_{-pa}) \frac{\sigma^j_{ab}}{2} (a_{-pb} - b^\dagger_{pb}) \}$$

again the $a^\dagger b^\dagger$, ba terms cancel.

$$Q^j = \int \frac{d^3p}{(2\pi)^3} \left(a^+_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} - b_{pa} \frac{\sigma^j_{ab}}{2} b^+_{pb} \right)$$

Let me show carefully that this commutes with H

$$[H, Q^j] = \left[\int \frac{d^3k}{(2\pi)^3} E_k \left(a^+_{kc} a_{kc} + b^+_{kc} b_{kc} \right), Q^j \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} E_k \int \frac{d^3p}{(2\pi)^3} \left\{ \left[a^+_{kc} a_{kc}, a^+_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} \right] - \left[b^+_{kc} b_{kc}, b_{pa} \frac{\sigma^j_{ab}}{2} b^+_{pb} \right] \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} E_k \left\{ a^+_{kc} \left[(2\pi)^3 \delta(\vec{k}-\vec{p}) \delta_{ca} \right] \frac{\sigma^j_{ab}}{2} a_{pb} + a^+_{pa} \frac{\sigma^j_{ab}}{2} \left[-(2\pi)^3 \delta(\vec{k}-\vec{p}) \delta_{cb} \right] a_{kc} - b^+_{kc} b_{pa} \frac{\sigma^j_{ab}}{2} \left[(2\pi)^3 \delta(\vec{k}-\vec{p}) \delta_{bc} \right] - \left(-(2\pi)^3 \delta(k-p) \delta_{ca} \right) \frac{\sigma^j_{ab}}{2} b^+_{pb} b_{kc} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left\{ a^+_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} - a^+_{pa} \frac{\sigma^j_{ab}}{2} a_{pb} - b^+_{pb} b_{pa} \frac{\sigma^j_{ab}}{2} + b^+_{pb} b_{pa} \frac{\sigma^j_{ab}}{2} \right\}$$

$$= 0$$

$$[Q^j, Q^k] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\left\{\left[a^+_{pa}\,\frac{\sigma^j_{ab}}{2}\,a_{pb},\ a^+_{kc}\,\frac{\sigma^k_{cd}}{2}\,a_{kd}\right]\right.$$
$$\left. + \left[b_{pa}\,\frac{\sigma^j_{ab}}{2}\,b^+_{pb},\ b_{kc}\,\frac{\sigma^k_{cd}}{2}\,b^+_{kd}\right]\right\}$$

$$= \int \frac{d^3p\,d^3k}{(2\pi)^6}\left\{ a^+_{pa}\,\frac{\sigma^j}{2}_{ab}\,\left((2\pi)^3\delta(\vec{p}-\vec{k})\delta_{bc}\right)\frac{\sigma^k}{2}_{cd}\,a_{kd}\right.$$
$$+ a^+_{kc}\,\frac{\sigma^k_{cd}}{2}\left(-(2\pi)^3\delta(\vec{p}-\vec{k})\delta_{ad}\right)\frac{\sigma^j}{2}_{ab}\,a_{pb}$$
$$+ b_{pa}\,\frac{\sigma^j_{ab}}{2}\left(-(2\pi)^3\delta(\vec{p}+\vec{k})\delta_{bc}\right)\frac{\sigma^k_{cd}}{2}\,b^+_{kd}$$
$$\left. + b_{kc}\,\frac{\sigma^k_{cd}}{2}\left((2\pi)^3\delta(\vec{p}+\vec{k})\,\delta_{ad}\right)\frac{\sigma^j_{ab}}{2}\,b^+_{pb}\right\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{ a^+_{pa}\left[\frac{\sigma^j}{2}\frac{\sigma^k}{2} - \frac{\sigma^k}{2}\frac{\sigma^j}{2}\right]_{ab} a_{pb}\right.$$
$$\left. + b_{pa}\left[-\frac{\sigma^j}{2}\frac{\sigma^k}{2} + \frac{\sigma^k}{2}\frac{\sigma^j}{2}\right]_{ab} b^+_{pb}\right\}$$

now $\quad [\sigma^j, \sigma^k] = 2i\,\epsilon^{jkl}\,\sigma^l$

or $\quad \left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i\,\epsilon^{jkl}\,\frac{\sigma^l}{2}$ $\quad$ so

$$= \int \frac{d^3p}{(2\pi)^3}\; i\,\epsilon^{jkl}\left\{ a^+_{pa}\,\frac{\sigma^l}{2}_{ab}\,a_{pb} - b_{pa}\,\frac{\sigma^l_{ab}}{2}\,b^+_{pb}\right\}$$

so

$$[Q^j, Q^k] = i\epsilon^{jkl} Q^l$$

this is the algebra of SU(2) or the rotation group.

3.) a.) $L^1 = \frac{1}{2}(J^{23} - J^{32}) = J^{23}$

$L^2 = J^{31} = -J^{13}$

$L^3 = J^{12} = -J^{21}$

so compute:

$$[L^1, L^2] = [J^{23}, J^{31}] = i(\overbrace{g^{33}}^{-1} J^{21} - \overbrace{g^{23}}^{0} J^{31} - \overbrace{g^{31}}^{0} J^{23} + \underbrace{g^{21}}_{0} J^{33})$$

$$= -i J^{21} = +i L^3$$

$$[L^2, L^3] = [J^{31}, J^{12}] = i(g^{11} J^{32} + \text{zero})$$

$$= i L^1$$

similarly $[L^i, L^j] = i\epsilon^{ijk} L^k$

$$[K^1, K^2] = [J^{01}, J^{02}] = i[g^{10} J^{02} - g^{00} J^{12} - g^{12} J^{00} + g^{02} J^{10}]$$

$$= -i g^{00} J^{12} = -i L^3$$

similarly: $[K^i, K^j] = -i\epsilon^{ijk} L^k$

$[K^1, L^2] = [J^{01}, J^{31}]$

$= i(g^{13} J^{01} - g^{03} J^{11} - g^{11} J^{03} + g^{01} J^{13})$

$= +i J^{03} = i K^3$

similarly $[K^i, L^j] = [L^i, K^j] = i\epsilon^{ijk} K^k$

in all

$[L^i, L^j] = i\epsilon^{ijk} L^k$

$[L^i, K^j] = [K^i, L^j] = i\epsilon^{ijk} K^k$

$[K^i, K^j] = -i\epsilon^{ijk} L^k$

then if $J_+^i = \frac{1}{2}(L^i + iK^i)$

$[J_+^i, J_+^j] = \frac{1}{4}[(L^i + iK^i), (L^j + iK^j)]$

$= \frac{1}{4} i\epsilon^{ijk} \{ L^k + 2iK^k + (i)^2(-L^k) \}$

$= i\epsilon^{ijk} \frac{1}{2}(L^k + iK^k) = i\epsilon^{ijk} J_+^k$

similarly if $J_-^i = \frac{1}{2}(L^i - iK^i)$

$$[J_-^i, J_-^j] = i\epsilon^{ijk} J_-^k$$

finally

$$[J_+^i, J_-^j] = \frac{1}{4}[L^i + iK^i, L^j - iK^j]$$

$$= 2\epsilon^{ijk}\frac{1}{4}[L^k + iK^k - iK^k + (-L^k)]$$

$$= 0$$

so J_+^i J_-^i commute ad separately satisfy the CR's of angular momentum.

b.) In the $(\frac{1}{2}, 0)$ representation of (J_+, J_-)

$$J_+^i = \frac{\sigma^i}{2} \qquad J_-^i = 0$$

then $$L^i = J_+^i + J_-^i = \frac{\sigma^i}{2}$$

$$K^i = -i(J_+^i - J_-) = -i\frac{\sigma^i}{2}$$

the Lorentz transformation in this representation is

$$(1 - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{K}) = \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

i.e. $\psi \longrightarrow (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2})\psi$

(2-component)

this is the ψ_L transformation.

In the $(0, \frac{1}{2})$ representation

$$J_+^i = 0 \qquad J_-^i = \frac{\sigma^i}{2}$$

$$L^i = \frac{\sigma^i}{2} \qquad K^i = +i\frac{\sigma^i}{2}$$

$$\psi \longrightarrow (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2})\psi$$

this is the ψ_R transformation.

c.) In the $(\frac{1}{2}, \frac{1}{2})$ representation, we have a 2×2 matrix V that transforms under Lorentz transformations as

$$V \longrightarrow (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2})\, V\, (1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2})$$

Let's write this more explicitly:

$$V = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^0 + \vec{V}\cdot\vec{\sigma}$$

then under rotations:

$$V \to \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2\right)\left(V^0 + \vec{V}\cdot\vec{\sigma}\right)\left(1 + i\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$$= V^0 + \vec{V}\cdot\vec{\sigma} - i\theta^i V^j \left[\frac{\sigma^i}{2}, \sigma^j\right]$$

$$= V^0 + \vec{V}\cdot\vec{\sigma} - i\theta^i V^j \, i\epsilon^{ijk}\sigma^k$$

$$= V^0 + \left(\vec{V} + \vec{\theta}\times\vec{V}\right)\cdot\vec{\sigma}$$

under boosts.

$$V \to \left(1 + \vec{\beta}\cdot\vec{\sigma}/2\right)\left(V^0 + \vec{V}\cdot\vec{\sigma}\right)\left(1 + \vec{\beta}\cdot\vec{\sigma}/2\right)$$

$$= V^0 + \vec{V}\cdot\vec{\sigma} + V^0\vec{\beta}\cdot\vec{\sigma} + \frac{\beta^i}{2} V^j \underbrace{\{\sigma^i, \sigma^j\}}_{2\delta^{ij}}$$

$$= \left(V^0 + \vec{\beta}\cdot\vec{V}\right) + \left(\vec{V} + \beta V^0\right)\cdot\vec{\sigma}$$

under rotations

$$V^0 \to V^0 \qquad \vec{V} \to \vec{V} + \vec{\theta}\times\vec{V}$$

under boosts

$$\begin{pmatrix} V^0 \\ \hat{\beta}\cdot\vec{V} \end{pmatrix} \to \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} V^0 \\ \hat{\beta}\cdot\vec{V} \end{pmatrix}$$

this is the transformation law of a 4-vector!

Physics 330 – Problem Set # 3

(due Wednesday, October 19)

1. Peskin and Schroeder, problem 3.2. Use the fact that the $u(p)$ satisfy $(\gamma^\mu p_\mu)u(p) = mu(p)$.

2. Peskin and Schroeder, problem 3.4.

3. Peskin and Schroeder, problem 3.5.

Problems 2 and 3 illustrate that it is advantageous to consider even the *classical* spin-$\frac{1}{2}$ field as an anticommuting object. Classical anticommuting numbers ('Grassmann numbers') a, b satisfy

$$ab = -ba \ , \qquad a^2 = b^2 = 0 \ , \qquad (ab)^\dagger = b^\dagger a^\dagger = -a^\dagger b^\dagger \ . \tag{1}$$

The last part of Problem 2, concerning the quantization of the Majorana fermion theory, is tricky. Here are some hints:

(1) The canonical anticommutation relations should be

$$\{\chi_a(\vec{x}), \chi_b^\dagger(\vec{y})\} = \delta_{ab}\delta(\vec{x} - \vec{y}) \tag{2}$$

The Hamiltonian is

$$H = \int d^3x \Big\{ \chi^\dagger i\vec{\sigma}\cdot\vec{\nabla}\chi - \frac{i}{2}m(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \Big\} \tag{3}$$

Show that this Hamiltonian leads to the correct Heisenberg equation of motion for $\chi(x)$.

(2) Since fields will be expanded in terms of the solutions of the Majorana equation, the first thing to do is to find a representation of these solutions. Show that expressions that look like the upper two components of the usual u and v solutions of the Dirac equation:

$$\chi^{(+)} = \sqrt{p\cdot\sigma}\xi e^{-ip\cdot x} \qquad \chi^{(-)} = \sqrt{p\cdot\sigma}(-i\sigma^2)\xi^* e^{ip\cdot x} \ , \tag{4}$$

where ξ is a 2-component spinor, solve

$$i\overline{\sigma}\cdot\partial\chi^{(+)} = im\sigma^2\chi^{(-)*} \tag{5}$$

and the same equation with $(+)$ and $(-)$ interchanged. This motivates the representation:

$$\chi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big\{ \sqrt{p\cdot\sigma}\xi^s a^s_{\vec{p}} + \sqrt{\hat{p}\cdot\sigma}(-i\sigma^2)\xi^{s*} a^{s\dagger}_{-\vec{p}} \Big\} e^{i\vec{p}\cdot\vec{x}} \ , \tag{6}$$

where $\hat{p} = (E_p, -\vec{p})$. Show that this representation satisfies the canonical anticommutation relations and diagonalizes H.

Physics 330 – Problem Set #3

Solutions

1.) $u(p)$ satisfies $(\not{p}-m)\,u(p)=0$ $\quad \not{p}=\gamma_\mu p^\mu$

$\bar{u}(p')$ satisfies $\bar{u}(p')\,(\not{p}'-m)=0$

so $\bar{u}(p')\,[\not{p}'\gamma^\mu+\gamma^\mu\not{p}\,]\,u(p)$

$= \bar{u}(p')\,(m\gamma^\mu+\gamma^\mu m)\,u(p) = 2m\,\bar{u}\gamma^\mu u$

now

$$\not{p}'\gamma^\mu = \tfrac{1}{2}\left(\{\not{p}',\gamma^\mu\}+[\not{p}',\gamma^\mu]\right)$$

$$= p'^\mu - \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\,p'_\nu$$

$$= p'^\mu + i\sigma^{\mu\nu}p'_\nu$$

$$\gamma^\mu\not{p} = \tfrac{1}{2}\left(\{\gamma^\mu,\not{p}\}+[\gamma^\mu,\not{p}]\right)$$

$$= p^\mu + [\gamma^\mu,\gamma^\nu]\,p_\nu$$

$$= p^\mu - i\sigma^{\mu\nu}p_\nu$$

so

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\,\bar{u}(p')\left\{(p'^\mu+p^\mu)+i\sigma^{\mu\nu}(p'_\nu-p_\nu)\right\}u(p)$$

as required

2.) a.) χ transforms under Lorentz transformations as

$$\chi(x) \to \Lambda_L \chi(\Lambda^{-1}x)$$

where Λ_L is the top 2×2 matrix in $\Lambda_{1/2}$, with infinitesimal form

$$\Lambda_L = (1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2)$$

Assume that $\chi(x)$ satisfies the Majorana equation

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0$$

We need to prove that $\chi'(x) = \Lambda_L\chi(\Lambda^{-1}x)$ satisfies this equation

$$i\bar{\sigma}\cdot\partial\,\Lambda_L\chi(\Lambda^{-1}x) - im\sigma^2\,\Lambda_L^*\chi^*(\Lambda^{-1}x)$$

$$= i\bar{\sigma}^\nu(\Lambda^{-1})^\mu{}_\nu\,\Lambda_L(\partial_\mu\chi)(\Lambda^{-1}x) - im(\sigma^2\Lambda_L^*)\chi^*(\Lambda^{-1}x)$$

in class we proved

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\lambda\gamma^\lambda$$

in 2×2 blocks, this is

$$\begin{pmatrix}\Lambda_L^{-1} & \\ & \Lambda_R^{-1}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\Lambda_L & \\ & \Lambda_R\end{pmatrix} = \Lambda^\mu{}_\lambda\begin{pmatrix}0 & \sigma^\lambda \\ \bar{\sigma}^\lambda & 0\end{pmatrix}$$

or

$$\Lambda_L^{-1}\sigma^\mu\Lambda_R = \Lambda^\mu{}_\lambda\sigma^\lambda$$

$$\Lambda_R^{-1}\bar{\sigma}^\mu\Lambda_L = \Lambda^\mu{}_\lambda\bar{\sigma}^\lambda$$

so $\bar{\sigma}^\nu \Lambda_L = \Lambda_R \Lambda_R^{-1} \bar{\sigma}^\nu \Lambda_L$

$= \Lambda_R \bar{\sigma}^\partial \Lambda^\nu{}_\partial$

Now look at $\sigma^2 \Lambda_L^*$. The infinitesimal form is

$\sigma^2 \Lambda_L^* \cong \sigma^2 (1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\eta}\cdot\frac{\vec{\sigma}^*}{2})$

$= (1 - i\vec{\theta}\frac{\vec{\sigma}}{2} + \vec{\eta}\frac{\vec{\sigma}}{2})\sigma^2 \cong \Lambda_R \sigma^2$

Finite repeated small steps, the same is true for finite transformations:

$$\sigma^2 \Lambda_L^* = \Lambda_R \sigma^2$$

Then the above equation is:

$$[\Lambda_R i\bar{\sigma}^\partial \Lambda^\nu{}_\partial (\Lambda^{-1})^\mu{}_\nu \partial_\mu \chi - im \Lambda_R \sigma^2 \chi^*](\Lambda^{-1}x)$$

$$\Lambda_R (i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^*)(\Lambda^{-1}x) = 0$$

if χ satisfies the original equation.

So the Majorana equation is covariant. Now, if χ satisfies this equation

$$i\bar{\sigma}\cdot\partial\chi = im\sigma^2\chi^*$$

$$* \quad -i\bar{\sigma}^*\partial\chi^* = im\sigma^2\chi$$

$$i\sigma\cdot\partial\, i\bar{\sigma}\cdot\partial\,\chi = -\partial^2\chi$$

$$= i\sigma\cdot\partial\; im\sigma^2\chi^*$$

$$= (i)^2 m\sigma^2(\bar{\sigma}\cdot\partial)^*\chi^*$$

$$= -m\sigma^2(-m\sigma^2\chi) = m^2\chi$$

so $(-\partial^2 - m^2)\chi = 0$ if χ satisfies the Majorana eq.

b) $$S = \int d^4x \left\{\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\right\}$$

$$S^* = \int d^4x \left\{\partial_\mu\chi^\dagger(-i\bar{\sigma}^\mu)\chi - i\frac{m}{2}(\chi^\dagger(\sigma^2)^\dagger\chi^* - \chi^T(\sigma^2)^\dagger\chi)\right\}$$

$$(\sigma^2)^\dagger = \sigma^2$$

$$= \int d^4x \left\{\chi^\dagger i\bar{\sigma}\cdot\partial\chi - i\frac{m}{2}(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi)\right\}$$

$$= S$$

Now vary S with respect to χ, $\chi^\dagger$

$$\delta S = \int d^4x \Big\{ \delta\chi^\dagger i\bar\sigma\cdot\partial\chi + \chi^\dagger i\bar\sigma\cdot\partial\delta\chi + i\frac{m}{2}\left[\delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi - \delta\chi^\dagger\sigma^2\chi^* - \chi^\dagger\sigma^2\delta\chi^*\right]\Big\}$$

Grassmann

now $\delta\chi^T\sigma^2\chi = \chi^T(\sigma^2)^T\delta\chi\,(-1) = +\chi^T\sigma^2\delta\chi$

$\chi^\dagger\sigma^2\delta\chi^* = \delta\chi^\dagger(\sigma^2)^T\chi^*(-1) = +\delta\chi^\dagger\sigma^2\chi^*$

so this organizes itself into:

$$\delta S = \int d^4x \Big\{ \delta\chi^\dagger \Big\{ i\bar\sigma\cdot\partial\chi - im\sigma^2\chi^* \Big\} + \Big\{ -i\partial_\mu\chi^\dagger\bar\sigma^\mu + im\chi^T\sigma^2 \Big\}\delta\chi \Big\}$$

then

$\delta S = 0$ implies $\begin{cases} i\bar\sigma\cdot\partial\chi - im\sigma^2\chi^* = 0 \\ -i\partial_\mu\chi^\dagger\bar\sigma^\mu + im\chi^T\sigma^2 = 0 \end{cases}$

(which are equivalent, conjugate equations)

c.) Write the Dirac field as

$$\psi = \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix}$$

Then $\psi^\dagger = (\chi_1^\dagger, -i\chi_2^T\sigma^2)$ $\qquad \bar\psi = (-i\chi_2^T\sigma^2, \chi_1^\dagger)$

then

$$\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial - m)\psi$$

$$= (-i\chi_2^T\sigma^2,\ \chi_1^\dagger)\left\{\begin{pmatrix} 0 & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right\}\begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix}$$

$$= -i\chi_2^T\sigma^2(i\sigma\cdot\partial)i\sigma^2\chi_2^* + \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1$$

$$+ im\,\chi_2^T\sigma^2\chi_1 - im\chi_1^\dagger\sigma^2\chi_2^*$$

now $\quad \sigma^2\, i\sigma\cdot\partial\, \sigma^2 = +i\,\bar{\sigma}^*\cdot\partial = +i\,(\bar{\sigma}^\dagger)^T\partial_\mu$

integrate by parts

$$-i\chi_2^T\sigma^2\, i\sigma\cdot\partial\, i\sigma^2\chi_2^* = \chi_2^T\, i(\bar{\sigma}^\mu)^T\partial_\mu\,\chi_2^*$$

$$= -i\,\partial_\mu\chi_2^T(\bar{\sigma}^\mu)^T\chi_2^*$$

$$= +i\,\chi_2^\dagger\bar{\sigma}\cdot\partial\chi_2$$

integrating $\chi_2\,\chi_2^*$ w. (-1) from Grassmanian

so!

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^\dagger i\bar{\sigma}\cdot\partial\chi_2$$

$$+ i\frac{m}{2}\left(2\chi_1^T\sigma^2\chi_2 - 2\,\chi_1^\dagger\sigma^2\chi_2^*\right)$$

etc $\rightarrow$

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^\dagger i\bar{\sigma}\cdot\partial\chi_2 + i\tfrac{1}{2}\left(\chi_a^T\sigma^2 m_{ab}\chi_b - \chi_a^\dagger\sigma^2 m_{ab}\chi_b^\dagger\right)$$

where $m_{ab} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$

d.) This Lagrangian has a symmetry

$$\chi_1 \to e^{i\alpha}\chi_1 \qquad \chi_2 \to e^{-i\alpha}\chi_2$$

$$\chi_1^\dagger \to \chi_1^\dagger e^{-i\alpha} \qquad \chi_2^\dagger \to \chi_2^\dagger e^{+i\alpha}$$

the corresponding current is

$$J^\mu = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2$$

and this should be conserved:

$$\partial_\mu J^\mu = \partial_\mu(\chi_1^\dagger\bar{\sigma}^\mu\chi_1) - \partial_\mu\chi_2^\dagger\bar{\sigma}^\mu\chi_2$$

the equations of motion are:

$$i\bar{\sigma}\cdot\partial\chi_1 - i\sigma^2 m\chi_2^* = 0$$

$$-i\partial_\mu\chi_1^\dagger\bar{\sigma}^\mu + im\chi_2^T\sigma^2 = 0$$

$$i\bar{\sigma}\cdot\partial\chi_2 - i\sigma^2 m\chi_1^* = 0$$

$$-i\partial_\mu\chi_2^\dagger\bar{\sigma}^\mu + im\chi_1^T\sigma^2 = 0$$

so $\partial_\mu J^\mu = (\partial_\mu \chi_1^\dagger \bar\sigma^\mu)\,\chi_1 + \chi_1^\dagger(\bar\sigma\cdot\partial\chi_1)$

$\quad - (\partial_\mu \chi_2^\dagger \bar\sigma^\mu)\,\chi_2 - \chi_2^\dagger(\bar\sigma\cdot\partial\chi_2)$

$= m\,\chi_2^T\sigma^2\chi_1 + \chi_1^\dagger\, m\sigma^2\chi_2^*$

$\quad - m\chi_1^T\sigma^2\chi_2 - \chi_2^\dagger\, m\sigma^2\chi_1^*$

$= 0$ since $\chi_2^T\sigma^2\chi_1 = \chi_1^T\sigma^2\chi_2$ etc.

actually $\partial_\mu J^\mu = \partial_\mu(\bar\psi\gamma^\mu\psi)$ in the Dirac thing.

on the other hand

$\partial_\mu(\chi_1^\dagger\bar\sigma^\mu\chi_1 + \chi_2^\dagger\bar\sigma^\mu\chi_2)$

$= 2m(\chi_1^T\sigma^2\chi_2 - \chi_1^\dagger\sigma^2\chi_2^*)$

this current generates $\chi_1 \to e^{i\alpha}\chi_1 \quad \chi_2 \to e^{i\alpha}\chi_2$

which is not a symmetry for $m \neq 0$.

The comparison to the Dirac thing is

$$\chi_1^\dagger\bar\sigma^\mu\chi_1 - \chi_2^\dagger\bar\sigma^\mu\chi_2 = \bar\psi\gamma^\mu\psi$$

$$\chi_1^\dagger\bar\sigma^\mu\chi_1 + \chi_2^\dagger\bar\sigma^\mu\chi_2 = -\bar\psi\gamma^\mu\gamma^5\psi$$

Now consider a thy of N 2-compont fermions:

$$\mathcal{L} = \chi_a^\dagger i\bar{\sigma}\cdot\partial \chi_a + i\frac{m}{2}(\chi_a^T \sigma^2 \chi_a - \chi_a^\dagger \sigma^2 \chi_a)$$

summed over $a = 1 \text{---} N$

If R_{ab} is a rotatn in N-dim space $x \to Rx$

(elemts of R real) $x\cdot y \to x R^T R y = x\cdot y$

so $R^T R = 1$.

$$\mathcal{L} \to \chi^\dagger R^T i\bar{\sigma}\cdot\partial R\chi + i\frac{m}{2}(\chi^T R^T \sigma^2 R\chi - \chi^\dagger R^T \sigma^2 R\chi)$$

$R^T R = 1$ so

$$= \mathcal{L}$$

the kinetic term $\chi_a^\dagger i\bar{\sigma}\cdot\partial\chi_a$ is invariant to

unitry transformatns $\chi \to U\chi$ $U \in U(N)$

N×N unitry

the full $\mathcal{L}$ is invariant to rotatns

$\chi \to R\chi$ $R \in O(N)$

e.) Follow the method in the problem set. First, check that the Hamiltonian given leads to the Majorana equation as its equation of motion:

$$i\frac{\partial}{\partial t}\chi(x) = [\chi(\vec{x}), H]$$

$$- \left[\chi(\vec{x}), \int d^3y \left(\chi^+ (i\vec{\sigma}\cdot\vec{\nabla})\chi(y) + i\frac{m}{2}\chi^+\sigma^2\chi^*_{(y)} - \cdots\right)\right]$$

$$= i\vec{\sigma}\cdot\vec{\nabla}\chi(x) + i\frac{m}{2}\sigma^2\chi^*(y)\cdot 2$$

$$i\left(\underbrace{1\frac{\partial}{\partial t}}_{\vec{\sigma}\cdot\partial} - \vec{\sigma}\cdot\vec{\nabla}\right)\chi - im\sigma^2\chi^* = 0$$

$$i\vec{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0$$

this equation is solved by elements:

$$\chi^{(+)} = \sqrt{p\cdot\sigma}\,\xi\, e^{-ip\cdot x} \qquad \chi^{(-)} = \sqrt{p\cdot\sigma}\,(-i\sigma^2)\xi^*\, e^{ip\cdot x}$$

$$i\vec{\sigma}\cdot\partial\chi^{(+)} = p\cdot\vec{\sigma}\sqrt{p\sigma}\,\xi\, e^{-ip\cdot x} = m\sqrt{p\bar{\sigma}}\,\xi\, e^{-ip\cdot x}$$

$$im\sigma^2\chi^{(-)*} = im\sigma^2(\sqrt{p\sigma})^*(-i\sigma^2)\xi\, e^{-ip\cdot x} = m\sqrt{p\cdot\bar{\sigma}}\,\xi\, e^{-ip\cdot x}$$

so $$i\vec{\sigma}\cdot\partial\chi^{(+)} = im\sigma^2\chi^{(-)*}$$

so write

$$\chi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\sqrt{p\cdot\sigma}\, \xi^s a_p^s + \sqrt{\hat{p}\cdot\sigma}(-i\sigma^2)\xi^{s*} a_{\hat{p}}^{s\dagger} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$\chi^\dagger(\vec{y}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\xi^{s\dagger}\sqrt{p\cdot\sigma}\, a_p^{s\dagger} + \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^{s} \right) e^{-i\vec{p}\cdot\vec{y}}$$

with $\{ a_p^s, a_q^{s'\dagger} \} = (2\pi)^3 \delta(p-q)\delta^{ss'}$

$$\{ a_p^s, a_q^{s'} \} = 0 = \{ a_p^{\dagger s}, a_q^{\dagger s'} \}$$

$$\{ \chi(\vec{x}), \chi^\dagger(\vec{y}) \} = \int \frac{d^3p\, d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_k}} \sum_s \Big\{ \sqrt{p\cdot\sigma}\, \xi^s \xi^{\dagger}_{s} \sqrt{k\cdot\sigma}\; (2\pi)^3 \delta(\vec{p}-\vec{k}) e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$+ \sqrt{\hat{p}\cdot\sigma}\, -i\sigma^2 \xi^{s*} \xi^{sT} i\sigma^2 \sqrt{\hat{k}\cdot\sigma}\, (2\pi)^3 \delta(\vec{p}-\vec{k}) e^{i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{(p\cdot\sigma + \hat{p}\cdot\sigma)}_{2E_p} e^{i\vec{p}\cdot\vec{x}-\vec{y}}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta(\vec{x}-\vec{y})$$

$$\{ \chi(\vec{x}), \chi(\vec{y}) \} = \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_k}} \sum_s \Big\{ \sqrt{p\cdot\sigma}\, \xi^s \xi^{s\dagger} (i\sigma^2)(\sqrt{\hat{k}\cdot\sigma})^T (2\pi)^3 \delta(\vec{p}+\vec{k}) e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$+ \sqrt{\hat{p}\cdot\sigma}(-i\sigma^2)\xi^{s*}\xi^{sT}(\sqrt{\hat{k}\cdot\sigma})^T 2\pi\delta(\vec{p}+\vec{k}) e^{i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(\underbrace{\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}}_{m} (i\sigma^2) + \underbrace{\sqrt{\hat{p}\cdot\sigma}\sqrt{\hat{p}\cdot\bar{\sigma}}}_{m} (-i\sigma^2) \right) e^{i\vec{p}\cdot\vec{x}-\vec{y}}$$

$$= 0$$

so this formula does represent the anti-commutation relations

Now plug these formulae into H:

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_k}} e^{-ip\cdot x} e^{ik\cdot x} \qquad \hat{p} = (p^0, -\vec{p}),\ \hat{k} = (k^0, -\vec{k})$$

$$\Big\{ (\xi^{\dagger s} \sqrt{p\cdot\sigma}\, a_p^{s\dagger} + \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^{s}) (-\vec{\sigma}\cdot\vec{k})$$
$$(\sqrt{k\cdot\sigma}\, \xi^t a_k^t + \sqrt{\hat{k}\cdot\sigma}\,(-i\sigma^2)\,\xi^{t*} a_{\hat{k}}^{t\dagger})$$

$$-i\frac{m}{2} (\xi^{sT}(\sqrt{\hat{p}\cdot\sigma})^T a_{\hat{p}}^s + \xi^{s\dagger} i\sigma^2 (\sqrt{p\cdot\sigma})^T a_p^{\dagger s}) \sigma^2$$
$$(\sqrt{k\cdot\sigma}\, \xi^t a_k^t + \sqrt{\hat{k}\cdot\sigma}\,(-i\sigma^2)\,\xi^{t*} a_{\hat{k}}^{t\dagger})$$

$$+i\frac{m}{2} (\xi^{s\dagger} \sqrt{p\cdot\sigma}\, a_p^{s\dagger} + \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^{s}) \sigma^2$$
$$((\sqrt{k\cdot\sigma})^T \xi^{t*} a_{\hat{k}}^{t\dagger} + (\sqrt{\hat{k}\cdot\sigma})^T(-i\sigma^2)\,\xi^t a_k^t) \Big\}$$

first collect the aa terms: $\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} e^{+i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta(\vec{p}-\vec{k})$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\, a_{\hat{p}}^s\, (-\vec{\sigma}\cdot\vec{k}) \sqrt{p\cdot\sigma}\, \xi^t a_p^t$$

$$+ \frac{m}{2} \xi^{sT} (\sqrt{\hat{p}\cdot\sigma})^T a_{\hat{p}}^s (-i\sigma^2) \sqrt{p\cdot\sigma}\, \xi^t a_p^t$$

$$- \frac{m}{2} \xi^{sT} (i\sigma^2) \sqrt{\hat{p}\cdot\sigma} (-i\sigma^2) (\sqrt{p\cdot\sigma})^T (-i\sigma^2) \xi^t a_p^t \Big\}$$

$$(\hat{p}\cdot\sigma)^T (-i\sigma^2) = -i\sigma^2 (p\cdot\sigma), \text{ so}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\,T} (i\sigma^2) \sqrt{p\cdot\bar\sigma}\, \Big(p\cdot\sigma - \frac{p\cdot\bar\sigma}{2}\Big) \sqrt{p\cdot\sigma}\, \xi^t$$

$$+ \frac{m}{2}\, \xi^{s\,T} (-i\sigma^2) \sqrt{p\cdot\sigma}\, \sqrt{p\cdot\sigma}\, \xi^t$$

$$- \frac{m}{2}\, \xi^{s\,T} i\sigma^2 \sqrt{p\cdot\bar\sigma}\, \sqrt{p\cdot\bar\sigma}\, (-1)\, \xi^t \Big\}\, a^s_{\hat p}\, a^t_p$$

$p\cdot\bar\sigma \sqrt{p\cdot\sigma} = m\sqrt{p\cdot\bar\sigma}$

$\sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\sigma} = m$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\,T} (i\sigma^2) \Big\{ \frac{m}{2}(p\cdot\sigma - p\cdot\bar\sigma) - \frac{m}{2} p\cdot\sigma + \frac{m}{2} p\cdot\bar\sigma \Big\} \xi^t$$

$$\cdot\, a^s_{\hat p}\, a^t_p$$

$$= 0$$

Similarly, the $a^\dagger a^\dagger$ terms are:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \sqrt{p\cdot\sigma}\, (-\vec\sigma\cdot\vec p)\, \sqrt{\hat p\cdot\sigma}\, (-i\sigma^2)\, \xi^{t\,*}\; a^{s\dagger}_p\, a^{t\dagger}_{\hat p}$$

$$+ \frac{m}{2}\, \xi^{s\dagger} i\sigma^2 (\sqrt{p\cdot\sigma})^T (-i\sigma^2) \sqrt{\hat p\cdot\sigma}\, (-i\sigma^2)\, \xi^{t\,*}\; a^{s\dagger}_p\, a^{t\dagger}_{\hat p}$$

$$- \frac{m}{2}\, \xi^{s\dagger} \sqrt{p\cdot\sigma}\, (-i\sigma^2)\, (\sqrt{\hat p\cdot\sigma})^T\, \xi^{t\,*}\, a^{s\dagger}_p\, a^{t\dagger}_{\hat p} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \sqrt{p\cdot\sigma}\, \Big(p\cdot\sigma - \frac{p\cdot\bar\sigma}{2}\Big) \sqrt{p\cdot\bar\sigma}\, (-i\sigma^2)\, \xi^{t\,*}$$

$$+ \frac{m}{2}\, \xi^{s\dagger} \sqrt{p\cdot\bar\sigma}\, \sqrt{p\cdot\bar\sigma}\, (-i\sigma^2)\, \xi^{t\,*}$$

$$- \frac{m}{2}\, \xi^{s\dagger} \sqrt{p\cdot\sigma}\, \sqrt{p\cdot\sigma}\, (-i\sigma^2)\, \xi^{t\,*} \Big\}\, a^{s\dagger}_p\, a^{t\dagger}_{\hat p}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \Big(\frac{m}{2}(p\cdot\sigma - p\cdot\bar\sigma) + \frac{m}{2} p\cdot\bar\sigma - \frac{m}{2} p\cdot\sigma \Big)^{(-i\sigma^2)} \xi^{t\,*} \Big\}$$

$$a^{s\dagger}_p\, a^{t\dagger}_{\hat p}$$

$$= 0$$

now all that remains is:

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \sqrt{p\cdot\sigma}\ (-\vec{\sigma}\cdot\vec{p})\ \sqrt{p\cdot\sigma}\ \xi^t\ a_p^{s\dagger} a_p^t$$

$$+ \xi^{sT} i\sigma^2 \sqrt{\hat{p}\cdot\sigma}\ (-\vec{\sigma}\cdot\vec{p})\ \sqrt{\hat{p}\cdot\sigma}\ (-i\sigma^2)\ \xi^{t*}\ a_{\hat{p}}^{s}\ a_{\hat{p}}^{t\dagger}$$

$$+ \frac{m}{2} \Big(\xi^{s\dagger} (i\sigma^2) (\sqrt{p\cdot\sigma})^T (-i\sigma^2) \sqrt{p\cdot\sigma}\ \xi^t\ a_p^{s\dagger} a_p^t$$

$$+ \xi^{sT} (\sqrt{\hat{p}\cdot\sigma})^T (-i\sigma^2) \sqrt{\hat{p}\cdot\sigma}\ (-i\sigma^2)\ \xi^{t*}\ a_{\hat{p}}^{s}\ a_{\hat{p}}^{t\dagger} \Big)$$

$$- \frac{m}{2} \Big(\xi^{s\dagger} \sqrt{p\cdot\sigma}\ (-i\sigma^2) (\sqrt{p\cdot\sigma})^T (-i\sigma^2)\ \xi^t\ a_p^{s\dagger} a_p^t$$

$$+ \xi^{sT} (i\sigma^2) \sqrt{\hat{p}\cdot\sigma}\ (-i\sigma^2) (\sqrt{\hat{p}\cdot\sigma})^T\ \xi^{t*}\ a_{\hat{p}}^{s}\ a_{\hat{p}}^{t\dagger} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \Big[\xi^{s\dagger} \sqrt{p\cdot\sigma}\ \Big(\frac{p\cdot\sigma - p\cdot\bar{\sigma}}{2} \Big) \sqrt{p\cdot\sigma}\ \xi^t$$

$$+ \frac{m}{2}\ \xi^{s\dagger} \sqrt{p\cdot\bar{\sigma}}\ \sqrt{p\cdot\sigma}\ \xi^t + \frac{m}{2} \xi^{s\dagger} \sqrt{p\cdot\sigma}\ \sqrt{p\cdot\bar{\sigma}}\ \xi^t \Big]\ a_p^{s\dagger} a_p^t$$

$$+ \Big[\xi^{sT} (\sqrt{p\cdot\sigma})^T (\vec{\sigma}\cdot\vec{p})^T (\sqrt{p\cdot\sigma})^T \xi^{t*}\ a_{-\vec{p}}^{s}\ a_{-\vec{p}}^{t\dagger}$$

$$+ \frac{m}{2} \big(-\xi^{sT} (\sqrt{p\cdot\bar{\sigma}})^T (\sqrt{p\cdot\sigma})^T \xi^{t*} - \xi^{sT} (\sqrt{p\cdot\sigma})^T (\sqrt{p\cdot\bar{\sigma}})^T \xi^{t*} \big) a_{-\vec{p}}^{s}\ a_{-\vec{p}}^{t\dagger} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \Big[\xi^{s\dagger} \Big(p\cdot\sigma \cdot \tfrac{1}{2}(p\cdot\bar{\sigma}) - \frac{m^2}{2} \Big) \xi^t + m\cdot m\cdot\ \xi^{s\dagger} \xi^t \Big]\ a_p^{s\dagger} a_p^t$$

$$+ \Big[\xi^{t\dagger} \sqrt{p\cdot\sigma}\ \vec{\sigma}\cdot\vec{p}\ \sqrt{p\cdot\sigma}\ \xi^s - m\cdot m\ \xi^{t\dagger} \xi^s \Big]\ a_{-\vec{p}}^{s}\ a_{-\vec{p}}^{t\dagger}$$

now, in the second term, let $a_{-\vec{p}}^{s} a_{-\vec{p}}^{t\dagger} = -a_{-\vec{p}}^{t\dagger} a_{-\vec{p}}^{s} + \text{const.}$

drop the const, and let $s \leftrightarrow t$ $\quad \vec{p} \leftrightarrow -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \xi^{s\dagger} \left[\tfrac{1}{2}(p\cdot\sigma)^2 - \frac{m^2}{2} + m^2 \right] \xi^t$$

$$+ \xi^{s\dagger} \left[\sqrt{p\bar{\sigma}}\sqrt{p\bar{\sigma}} \underbrace{\vec{\sigma}\cdot\vec{p}}_{p\cdot\bar{\sigma} - p\cdot\sigma} \sqrt{p\cdot\bar{\sigma}} + m^2 \right] \xi^t \Big\} a_p^{s\dagger} a_p^t$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \xi^{s\dagger} \left[\tfrac{1}{2}(p\cdot\sigma)^2 + \frac{m^2}{2} + \tfrac{1}{2}(p\cdot\bar{\sigma})^2 + m^2/2 \right] \xi^t \; a_p^{s\dagger} a_p^t$$

$$\tfrac{1}{2}\left((p\cdot\sigma)^2 + (p\cdot\bar{\sigma})^2\right) = E_p^2 + |\vec{p}|^2$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \xi^{s\dagger} \underbrace{\left[E_p^2 + |\vec{p}|^2 + m^2\right]}_{2E_p \cdot E_p \cdot \underset{\sim}{(1)}} \xi^t \; a_p^{s\dagger} a_p^t$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \; a_p^{s\dagger} a_p^s + (\text{const})$$

as we might have hoped for !

3.) a.) $\delta\mathcal{L} = \partial_\nu \delta\phi^* \partial^\nu\phi + \partial_\nu\phi^* \partial^\nu \delta\phi$

$+ \delta\chi^\dagger i\bar\sigma\cdot\partial\chi + \chi^\dagger i\bar\sigma\cdot\partial\delta\chi$

$+ \delta F^* F + F^* \delta F$

insert into this:

$\delta\phi = -i\epsilon^T\sigma^2\chi \qquad \delta\phi^* = i\chi^\dagger\sigma^2\epsilon^*$

$\delta\chi = \epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^* \qquad \delta\chi^\dagger = F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\nu\phi^\dagger\sigma^\nu$

$\delta F = -i\epsilon^\dagger\bar\sigma\cdot\partial\chi \qquad \delta F^* = i\partial_\nu\chi^\dagger\bar\sigma^\nu\epsilon$

$\delta\mathcal{L} = \partial_\nu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\nu\phi + \partial_\nu\phi^*\partial^\nu(-i\epsilon^T\sigma^2\chi)$

$+ [(F^*\epsilon^\dagger) + \epsilon^T\sigma^2\partial_\nu\phi^\dagger\sigma^\nu]\, i\bar\sigma\cdot\partial\chi$

$+ \chi^\dagger i\bar\sigma\cdot\partial(\epsilon F + \sigma\cdot\partial\phi\,\sigma^2\epsilon^*)$

$+ (-i\epsilon^\dagger\bar\sigma\cdot\partial\chi)F^* + F^*(i\partial_\nu\chi^\dagger\bar\sigma^\nu\epsilon)$

terms w. ϕ: (w. integr. by parts)

$\partial_\nu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\nu\phi + \chi^\dagger i\bar\sigma\cdot\partial\,\sigma\cdot\partial\phi\,\sigma^2\epsilon^*$

$= -i\chi^\dagger\sigma^2\epsilon^*\partial^2\phi + i\chi^\dagger i\partial^2\phi\,\sigma^2\epsilon^* = 0$

terms w. ϕ^*:

$\partial_\nu\phi^*\partial^\nu(-i\epsilon^T\sigma^2\chi) + \epsilon^T\sigma^2\partial_\nu\phi^\dagger\sigma^\nu\, i\bar\sigma\cdot\partial\chi$

$= +i\partial^2\phi^*\epsilon^T\sigma^2\chi - i\epsilon^T\sigma^2\phi^*\partial^2\chi = 0$ after int. by parts.

terms w. F:

$$\chi^\dagger i\bar\sigma\cdot\partial(\varepsilon F) + i\partial_\mu\chi^\dagger\bar\sigma^\mu\varepsilon F$$

$$= \chi^\dagger i\bar\sigma\partial F\varepsilon - i\chi^\dagger\bar\sigma\cdot\partial F\varepsilon = 0$$

terms w. F^*:

$$F^*\varepsilon^\dagger i\sigma\cdot\partial\chi + (-i)F^*\varepsilon^\dagger i\bar\sigma\cdot\partial\chi = 0$$

so $\quad \delta\mathcal{L} = 0$

b.) $\quad \delta(\Delta\mathcal{L}) = m\delta\phi F + \frac{i}{2}m\delta\chi^T\sigma^2\chi + m\phi\delta F + \frac{i}{2}m\chi^T\sigma^2\delta\chi$

$$= m\delta\phi F + im\chi^T\sigma^2\delta\chi + m\phi\delta F$$

$$= m(-i\varepsilon^T\sigma^2\chi)F + im\chi^T\sigma^2(\varepsilon F + \sigma\partial\phi\sigma^2\varepsilon^*) + m\phi(-i\varepsilon^\dagger\bar\sigma\cdot\partial\chi)$$

$$= -im\epsilon^T\sigma^2\chi F + im\chi^T\sigma^2\varepsilon F + im\chi^\dagger\sigma^2\sigma\cdot\partial\phi\,\sigma^2\varepsilon^* - im\phi\varepsilon^\dagger\bar\sigma\cdot\partial\chi$$

$$= -im\epsilon^T\sigma^2\chi F + im\varepsilon^T\sigma^2\chi F + im(-\partial_\mu\chi^T(\bar\sigma^\mu)^T\sigma^2\varepsilon^*)\phi - im\phi\varepsilon^\dagger\bar\sigma\cdot\partial\chi$$

$$= 0 + im\varepsilon^\dagger\bar\sigma\cdot\partial\chi\,\phi - im\phi\chi\bar\sigma\cdot\partial\chi$$

$$= 0$$

the Full Lagrangian is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F$$
$$+ m\phi F + \tfrac{i}{2} m \chi^T \sigma^2 \chi + m\phi^* F^* - \tfrac{i}{2} m \chi^\dagger \sigma^2 \chi^*$$

$$\delta\mathcal{L} = \delta F^* (F + m\phi^*) + \delta F (m\phi + F^*)$$

so $F = -m\phi^*$ $\quad F^* = -m\phi$

using these equations for F, F^*

$$\mathcal{L} = \underbrace{\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi}_{\text{complex KG boson w. mass } m} + \underbrace{\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^\dagger)}_{\text{Majorana fermion w. mass } m.}$$

c.) $$\Delta\mathcal{L} = F_i \frac{\partial W}{\partial \phi_i} + \frac{1}{2} \frac{\partial^2 W}{\partial\phi_i \partial\phi_j} \chi_i^T \sigma^2 \chi_j$$

$$\delta(\Delta\mathcal{L}) = F_i \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} (-i\epsilon^T\sigma^2\chi_j) + \frac{\partial W}{\partial\phi_i}(-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi_i)$$
$$+ i \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} \chi_j^T\sigma^2(\epsilon F_i + \sigma\cdot\partial\phi_i\,\sigma^2\epsilon^*)$$
$$+ \frac{i}{2} \frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k} \chi_i^T\sigma^2\chi_j\,(-i\epsilon^T\sigma^2\chi_k)$$

the terms w. F are:

$$(-i\epsilon^T\sigma^2\chi_j)\,F_i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} + i\,\chi_j^T\sigma^2\varepsilon\,F_i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}$$

$$= 0$$

the terms w 1χ are:

$$-i\,\frac{\partial W}{\partial\phi_i}\,\epsilon^\dagger\bar\sigma\cdot\partial\chi_i + i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\chi_j^T\sigma^2\sigma\cdot\partial\phi_i\,\sigma^2\varepsilon^*$$

$$= -i\,\frac{\partial W}{\partial\phi_i}\,\epsilon^\dagger\bar\sigma\cdot\partial\chi_i + i\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\underbrace{\chi_j^T(\bar\sigma)^T\partial\phi_i\varepsilon^*}_{=\,-\varepsilon^\dagger\bar\sigma^\mu\chi_j\cdot\partial_\mu\phi_i}$$

$$= -i\,\epsilon^\dagger\bar\sigma\,\partial^\mu\chi_i\,\frac{\partial W}{\partial\phi_i} - i\,\epsilon^\dagger\bar\sigma^\mu\chi_i\,\partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right)$$

$= 0$ after integrate by parts.

finally, the last term involves:

$$\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\quad \chi_{i\alpha}\chi_{j\beta}\chi_{k\gamma}$$

since the derivative is totally symmetric under any interchange of (ijk), but χ anticommute, this object is totally antisymmetric in $[\alpha\beta\gamma]$. But $\alpha,\beta,\gamma = 1,2$ only. So this object must be 0.

$$\delta(\Delta\mathcal{L}) = 0 \quad \text{and so} \quad \delta\mathcal{L} = 0$$

for the special case in the problem set

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^\dagger F$$
$$+ F g\phi^2 + i g\phi\, \chi^T \sigma^2 \chi + \text{h.c.}$$

then $F^* + g\phi^2 = 0$ $\qquad F^* = -g\phi^2$

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - g^2 |\phi^*\phi|^2 + \chi^\dagger i\bar{\sigma}\cdot\partial\chi$$
$$+ i g\phi\, \chi^T \sigma^2 \chi - i g \phi^* \chi^\dagger \sigma^2 \chi^*$$

This is a nonlinear field thy with $\underline{\phi^4}$ and Yukawa interactions, both given by the dimensionless coupling g.

$$-\partial^2\phi - i g \chi^\dagger \sigma^2 \chi^* - 2g^2 \phi^* \phi^2 = 0$$

$$i\bar{\sigma}\cdot\partial\chi - 2i\sigma^2\chi^* = 0$$

are the eqns. of motion.

Physics 330 – Problem Set # 4

(due Wednesday, October 26)

1. Using the transformation properties of $\psi(x)$ and $\overline{\psi}(x)$ under P, T, and C given in eqs. (3.126), (3.128), (3.139), (3.140), (3.145), (3.146) of Peskin and Schroeder, work out explicitly the transformations of the 16 Dirac fermion bilinears under C, P, and T. In this way, verify all of the entries in the table on p. 71 of Peskin and Schroeder.

2. (a) Show that the state

$$|\Phi(K)\rangle = \sqrt{2E_K} \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(\vec{k}) a^{\dagger s}_{K/2+k} b^{\dagger t}_{K/2-k} |0\rangle \tag{1}$$

is a correctly normalized bound state of a fermion with $S^3 = s$ and an antifermion with $S^3 = t$, assuming that

$$\int \frac{d^3k}{(2\pi)^3} \left|\tilde{\phi}(\vec{k})\right|^2 = 1 \tag{2}$$

and that $E_K^2 = M^2 + |\vec{K}|^2$, where M is the mass of the state. How is $\tilde{\phi}(\vec{k})$ related to the Schrödinger wavefunction $\phi(\vec{x})$ of the bound state? For the rest of this problem, assume that $\tilde{\phi}$ is S-wave: $\tilde{\phi}(\vec{k}) = \tilde{\phi}(|\vec{k}|)$.

(b) Let s and t both be spin up ($S^3 = +\frac{1}{2}$). For a bound state at rest, $\vec{K} = 0$, compute

$$\langle 0| \overline{\psi} \gamma^\mu \psi(0) |\Phi(K)\rangle \quad \text{and} \quad \langle 0| \overline{\psi} \gamma^\mu \gamma^5 \psi(0) |\Phi(K)\rangle \ , \tag{3}$$

where $\psi(x)$ is the Dirac quantum field. It suffices to work in the leading approximation for non-relativistic fermions:

$$u^s(p) \approx \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \qquad v^s(p) \approx \sqrt{m} \begin{pmatrix} \xi^{-s} \\ -\xi^{-s} \end{pmatrix} \ . \tag{4}$$

Which matrix elements are nonzero? Why?

(c) Let s and t be combined into a spin-0 state. For a bound state at rest, $\vec{K} = 0$, compute

$$\langle 0| \overline{\psi} \psi(0) |\Phi(K)\rangle \quad \text{and} \quad \langle 0| \overline{\psi} \gamma^5 \psi(0) |\Phi(K)\rangle \tag{5}$$

again using the leading nonrelativistic approximation. Which matrix elements are nonzero? Why?

3. The action for the electromagnetic field coupled to a c-number current $j^\mu(x)$ is written in relativistic notation as

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right) , \tag{6}$$

where $A^\mu = (\varphi, \vec{A})$ is the vector potential and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

is the field strength.

(a) Work out the relativistic form of the variational equation of motion for A_μ.

(b) Define $E^i = F^{i0}$, $\epsilon^{ijk} B^k = -F^{ij}$. Show that the equations for E and B are the familiar Maxwell equations (in 'rationalized Heaviside units' with $\epsilon_0 = \mu_0 = c = 1$).

(c) Compute the canonical momentum conjugate to A^μ

$$\frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \tag{8}$$

Show that A^0 has no conjugate momentum. This is related to the fact that the corresponding equation of motion

$$\vec{\nabla} \cdot \vec{E} = j^0 \tag{9}$$

does not contain a time derivative. Coordinates without conjugate momenta occur in equations with a gauge invariance, that is, with the freedom to redefine variables by an arbitrary function of t or x^μ in a way that does not change the physics. For electrodynamics, show that the change of variables

$$A'^\mu = A^\mu - \partial^\mu \alpha(x) \tag{10}$$

does not affect Maxwell's equations.

(d) The Hamiltonian formalism for a system with gauge invariance typically requires specifying a gauge. For electrodynamics, a convenient choice is Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. Then also $(d/dt)(\vec{\nabla} \cdot \vec{A}) = 0$. Show that this allows one to solve for A^0 in terms of the source current j^0.

(e) With A^0 and $\vec{\nabla} \cdot \vec{A}$ eliminated, this system now has only two degrees of freedom per point, the two components of A^i that satisfy $\vec{\nabla} \cdot \vec{A} = 0$. At this point, it is more convenient to go to Fourier space and constrain $\vec{A}(\vec{k})$ by

$$\vec{k} \cdot \vec{A}(\vec{k}) = 0 \tag{11}$$

The conjugate momentum should obey a similar constraint. An appropriate form for the canonical commutation relation is then

$$[A^i(\vec{k}), \Pi^j(-\vec{k}')] = i\left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2}\right)(2\pi)^3\delta(\vec{k} - \vec{k}') \tag{12}$$

Find a representation of this commutation relation in which $\vec{A}$ has the form

$$\vec{A}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p}}e^{i\vec{p}\cdot\vec{x}}\sum_a \left(\vec{\epsilon}_a(\vec{p})a_{\vec{p}a} + \vec{\epsilon}^*_a(-\vec{p})a^\dagger_{-\vec{p}a}\right) \tag{13}$$

where $\vec{\epsilon}_a(\vec{p})$ is a transverse unit vector, $\vec{p}\cdot\vec{\epsilon}_a(\vec{p}) = 0$, and $a = 1, 2$.

(f) What is the relation between $\vec{\Pi}$ and $\vec{E}$?

(g) Show that the representation (12) diagonalizes

$$H = \int d^3x \left[\frac{1}{2}E^2 + \frac{1}{2}B^2\right] \tag{14}$$

The result of these manipulations is a theory of transverse quantum photons and c-number Coulomb potentials. If j^0 is replaced by a quantum operator, we would obtain a fully quantum theory of electromagnetism. The theory is not manifestly relativistically invariant, and it is not manifestly local (since, for example, (11) is nonlocal when transformed back to real space). In class, I will give another description of electrodynamics which is manifestly relativistically invariant and local but which can contain negative probabilities. It is true but nontrivial to show that these descriptions are equivalent in a subspace of their Hilbert spaces; in this subspace, all bad features disappear.

Physics 330 – Problem Set #4

Solutions

1.) Under $\mathcal{P}$: $\quad \mathcal{P}\,\psi(t,\vec{x})\,\mathcal{P} = \gamma^0\psi(t,-\vec{x})$

then

$$\mathcal{P}\,\bar{\psi}\,\mathcal{P} = (\psi^\dagger\gamma^0)\gamma^0 = \bar{\psi}\gamma^0(t,-\vec{x})$$

then

$\mathcal{P}\;\bar{\psi}\psi\;\mathcal{P} = \bar{\psi}\gamma^0\gamma^0\psi = +\bar{\psi}\psi \qquad (\gamma^0)^2 = 1$

$\mathcal{P}\;i\bar{\psi}\gamma^5\psi\;\mathcal{P} = i(\bar{\psi}\gamma^0)\gamma^5(\gamma^0\psi) = -i\bar{\psi}\gamma^5\psi \qquad \gamma^0\gamma^5 = -\gamma^5\gamma^0$

$\mathcal{P}\;\bar{\psi}\gamma^\mu\psi\;\mathcal{P} = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi$

$\mu=0 \quad = \bar{\psi}(\gamma^0)^3\psi = +\bar{\psi}\gamma^0\psi$

$\mu=i \quad = \bar{\psi}\gamma^0\gamma^i\gamma^0\psi = -\bar{\psi}\gamma^i\psi \qquad \gamma^0\gamma^i = -\gamma^i\gamma^0$

$\mathcal{P}\;\bar{\psi}\gamma^\mu\gamma^5\psi\;\mathcal{P} = \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi$

$\mu=0 \quad = \bar{\psi}(\gamma^0)^2\gamma^5\gamma^0\psi = -\bar{\psi}\gamma^0\gamma^5\psi$

$\mu=i \quad = \bar{\psi}\gamma^0\gamma^i\gamma^5\gamma^0\psi = +\bar{\psi}\gamma^i\gamma^5\psi$

γ^0 commutes w. $(\gamma^i\gamma^5)$

$\mathcal{P}\;\bar{\psi}\sigma^{\mu\nu}\psi\;\mathcal{P} = \bar{\psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi$

$\mu\nu = 0i \quad \gamma^0$ anticommutes w. $\gamma^0\gamma^i \Rightarrow \quad -\bar{\psi}\sigma^{\mu\nu}\psi$

$\mu\nu = ij \quad \gamma^0$ commutes w $\gamma^i\gamma^j \Rightarrow \quad +\bar{\psi}\sigma^{\mu\nu}\psi$

so under $\mathbb{P}$:

	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi\sigma^{\mu\nu}\psi$
$\mathbb{P}$	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$

where $(-1)^\mu = \begin{cases} +1 & \mu = 0 \\ -1 & \mu = i \end{cases}$

Under $\mathbb{T}$: $\quad \mathbb{T}\,\psi(t,\vec{x})\,\mathbb{T} = -\gamma^1\gamma^3\,\psi(-t,\vec{x}) \qquad (\gamma^1\gamma^3)(-\gamma^1\gamma^3) = 1$

$$\mathbb{T}\,\bar\psi(t,\vec{x})\,\mathbb{T} = \psi^\dagger(\gamma^3)^\dagger(-\gamma^1)^\dagger(\gamma^0)^* = +\bar\psi\,\gamma^1\gamma^3\,(t,\vec{x})$$

then

$$\mathbb{T}\,\bar\psi\psi\,\mathbb{T} = \bar\psi\,(\gamma^1\gamma^3)\,(-\gamma^1\gamma^3)\,\psi = +\bar\psi\psi$$

$$\mathbb{T}(i\bar\psi\gamma^5\psi)\,\mathbb{T} = -i\,\bar\psi\gamma^1\gamma^3\,(\gamma^5)^*(-\gamma^1\gamma^3)\,\psi = -i\bar\psi\gamma^5\psi$$

(γ^5)

$$\mathbb{T}\,\bar\psi\gamma^\mu\psi\,\mathbb{T} = \bar\psi\,\gamma^1\gamma^3\,(\gamma^\mu)^*(-\gamma^1\gamma^3)\,\psi$$

$\mu = 0$ $\quad (\gamma^0)^* = +\gamma^0$ $\quad \gamma^0$ commutes w. $\gamma^1\gamma^3$ $= +\bar\psi\gamma^0\psi$

$\mu = 1,3$ $\quad (\gamma^i)^* = +\gamma^i$ $\quad \gamma^i$ anticommutes w. $\gamma^1\gamma^3$ $= -\bar\psi\gamma^i\psi$

$\mu = 2$ $\quad (\gamma^2)^* = -\gamma^2$ $\quad \gamma^2$ commutes w. $\gamma^1\gamma^3$ $= -\bar\psi\gamma^2\psi$

$$\mathbb{T}\,\bar\psi\gamma^\mu\gamma^5\psi\,\mathbb{T} = \bar\psi\,\gamma^1\gamma^3\,(\gamma^\mu)^*(\gamma^5)^*(-\gamma^1\gamma^3)\,\psi$$

$\mu = 0$ $\quad (\gamma^0\gamma^5)^* = +\gamma^0\gamma^5$ $\quad \gamma^5$ commutes w. $\gamma^1\gamma^3$ $= +\bar\psi\gamma^0\gamma^5\psi$

$\mu = 1,3$ $\quad (\gamma^i\gamma^5)^* = +\gamma^i\gamma^5$ $\quad \rightarrow \quad -\bar\psi\gamma^i\gamma^5\psi$

$\mu = 2$ $\quad (\gamma^2\gamma^5)^* = -\gamma^2\gamma^5$ $\quad \rightarrow \quad -\bar\psi\gamma^2\gamma^5\psi$

$$T\,\bar\psi\sigma^{\mu\nu}\psi\,T = \bar\psi\,\gamma^1\gamma^3(\sigma^{\mu\nu})^*(-\gamma^1\gamma^3)\psi$$

$(\sigma^{01})^* = \frac{-i}{2}[\gamma^0,\gamma^1] = -\sigma^{01}$ anticommutes w. $\gamma^1\gamma^3$ $\rightarrow$ $+\bar\psi\sigma^{01}\psi$

σ^{02} $\rightarrow$ same story $\dashrightarrow$ $\rightarrow$ $+\bar\psi\sigma^{03}\psi$

$(\sigma^{02})^* = \frac{-i}{2}[\gamma^0,-\gamma^2] = +\sigma^{02}$ commutes w. $\gamma^1\gamma^3$ $\rightarrow$ $+\bar\psi\sigma^{02}\psi$

$(\sigma^{12})^* = \frac{-i}{2}[\gamma^1,-\gamma^2] = +\sigma^{12}$ anticommutes w. $\gamma^1\gamma^3$ $\rightarrow$ $-\bar\psi\sigma^{12}\psi$

(σ^{32}) $\rightarrow$ same story $\rightarrow$ $\rightarrow$ $-\bar\psi\sigma^{32}\psi$

$(\sigma^{13})^* = \frac{-i}{2}[\gamma^1,\gamma^3] = -\sigma^{13}$ commutes w. $\gamma^1\gamma^3$ $\rightarrow$ $-\bar\psi\sigma^{13}\psi$

in all

	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi\sigma^{\mu\nu}\psi$
T:	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$

Under C: $\quad C\,\psi(x)\,C = -i\,(\bar\psi\gamma^0\gamma^2)^T$

$C\,\bar\psi\,C = (i\,\gamma^2\psi)^T\gamma^0 = -i(\gamma^0\gamma^2\psi)^T$

then

$$C\,\bar\psi\psi\,C = (-i)(\gamma^0\gamma^2\psi)^T(-i)(\bar\psi\gamma^0\gamma^2)^T$$

$$= -(-i)^2\,\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi \qquad \gamma^2\gamma^0\gamma^2 = -(\gamma^2)^2\gamma^0 = +\gamma^0$$

$$= +\bar\psi\psi$$

$$C\,i\bar\psi\gamma^5\psi\,C = i\,(-i)^2(\gamma^0\gamma^2\psi)^T\gamma^5(\bar\psi\gamma^0\gamma^2)^T$$

$$= (-1)(-i)\,\bar\psi\gamma^0\gamma^2(\gamma^5)^T\gamma^0\gamma^2\psi$$

$$= +\bar\psi\gamma^5\psi$$

in general

$$C\bar{\psi}\Gamma\psi C = (-1)(-i)^2\,\bar{\psi}\gamma^0\gamma^2(\Gamma)^T\gamma^0\gamma^2\psi$$
$$= +\bar{\psi}\gamma^0\gamma^2(\Gamma)^T\gamma^0\gamma^2\psi \quad \text{as } (\gamma^0\gamma^2)^2 = +1$$

$$C\bar{\psi}\gamma^\mu\psi C = \bar{\psi}\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\psi$$

$\mu = 0,2$ $\quad (\gamma^\mu)^T = +\gamma^\mu$ $\quad \gamma^\mu$ anticommutes w. $\gamma^0\gamma^2$ $\quad = -\bar{\psi}\gamma^\mu\psi$

$\mu = 1,3$ $\quad (\gamma^\mu)^T = -\gamma^\mu$ $\quad \gamma^\mu$ commutes w. $\gamma^0\gamma^2$ $\quad = -\bar{\psi}\gamma^\mu\psi$

$$C\bar{\psi}\gamma^\mu\gamma^5\psi C = \bar{\psi}\gamma^0\gamma^2(\gamma^\mu\gamma^5)^T\gamma^0\gamma^2\psi$$

$(\gamma^5)^T = +\gamma^5$

$$= \bar{\psi}\gamma^0\gamma^2\gamma^5(\gamma^\mu)^T\gamma^0\gamma^2\psi$$
$$= \bar{\psi}\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\gamma^5\times(-1)^3\psi$$
$$= -(-1)^3\bar{\psi}\gamma^\mu\gamma^5\psi = +\bar{\psi}\gamma^\mu\gamma^5\psi$$

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = \bar{\psi}\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\psi$$

$\mu = 01$ $\quad (\sigma^{01})^T = (i\gamma^0\gamma^1)^T = i(-\gamma^1\gamma^0) = +\sigma^{01}$; this anticommutes w. $\gamma^0\gamma^2$

$$= -\bar{\psi}\sigma^{01}\psi$$

$\mu = 03$ is the same story

$\mu = 02$ $\quad (\sigma^{02})^T = (i\gamma^0\gamma^2)^T = i\gamma^2\gamma^0 = -\sigma^{02}$; this commutes w. $\gamma^0\gamma^2$

$$= -\bar{\psi}\sigma^{02}\psi$$

$\mu = 13$ $\quad (\sigma^{13})^T = (i\gamma^1\gamma^3)^T = (i\gamma^3\gamma^1) = -\sigma^{13}$; this commutes w. $\gamma^0\gamma^2$

$$= -\bar{\psi}\sigma^{13}\psi$$

$\mu = 12$ $\quad (\sigma^{12})^T = (i\gamma^1\gamma^2)^T = [i\gamma^2(-\gamma^1)] = +\sigma^{12}$; this anticommutes w. $\gamma^0\gamma^2$

$$= -\bar{\psi}\sigma^{12}\psi$$

$\mu = 32$ σ^{32} is the same story, so in all

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}\sigma^{\mu\nu}\psi$$

≈ all

	$\bar\Psi\psi$	$i\bar\Psi\gamma^5\psi$	$\bar\Psi\gamma^\mu\psi$	$\bar\Psi\gamma^\mu\gamma^5\psi$	$\bar\Psi\sigma^{\mu\nu}\psi$
C	+1	−1	−1	+1	−1

2.) a.) $\langle \Phi^{s't'}(K) \,|\, \Phi^{st}(P)\rangle$

$$= \sqrt{2E_k 2E_P}\int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\,\tilde\phi^*(k)\,\tilde\phi(\vec p)$$

$$\langle 0|\, b^{t'}_{K/2-k}\, a^{s'}_{K/2+k}\, a^{s\,\dagger}_{P/2+p}\, b^{t\,\dagger}_{P/2-p}\,|0\rangle$$

the matrix elem is:

$$(2\pi)^3\delta^{(3)}(k-p+K/2-P/2)\,\delta^{ss'}(2\pi)^3\delta^{(3)}(k-p-K/2+P/2)\,\delta^{tt'}$$

$$= (2\pi)^3\delta^{(3)}(k-p-\tfrac12(K-P))\,(2\pi)^3\delta^{(3)}(-\vec K+\vec P)\,\delta^{ss'}\delta^{tt'}$$

$$= (2\pi)^3\delta(\vec K-\vec P)\,(2\pi)^3\delta(k-p)\,\delta^{ss'}\delta^{tt'}$$

so $\langle \Phi^{s't'}(K)\,|\,\Phi^{st}(P)\rangle = \underbrace{2E_P\,(2\pi)^3\delta(\vec K-\vec P)\,\delta^{ss'}\delta^{tt'}}\ \underbrace{\int\frac{d^3p}{(2\pi)^3}\,|\tilde\phi(\vec p)|^2}_{=1}$

this is our conventional relativistic normalization

for a 2-body bound state

$$\vec r_1 = \vec R + \vec r/2 \qquad \vec r_2 = \vec R - \vec r/2$$

the conjugate momenta are:

$$\vec K = k_1+k_2 \qquad \vec k = \frac{k_1-k_2}{2}$$

CM internal

since then $[R^i, K^j] = [\frac{r_1^i + r_2^i}{2}, k_1^j + k_2^j] = i\delta^{ij}$

$[r^i, k^j] = [r_1^i - r_2^i, \frac{k_1^j - k_2^j}{2}] = i\delta^{ij}$

and $[\vec{R}, \vec{k}] = [\vec{r}, \vec{K}] = 0$

$\vec{K}$ and $\vec{k}$ are the CM and internal momenta of the wavefunction, so the Fourier transform of $\tilde{\phi}(\vec{k})$:

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}} \tilde{\phi}(\vec{k})$$

is the normal Schrödinger wavefunction of the bound state.

b.) We need the part of $\bar{\Psi} \Gamma \Psi(0)$ proportional to $b\,a$.

This is

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \; b_p^{t'} \, \bar{v}^{t'}(p) \, \Gamma \, u^{s'}(q) \, a_q^{s'}$$

and for $\vec{K} = 0$

$$\langle 0 | \, b_p^{t'} a_q^{s'} \; a_k^{+s} b_{-k}^{+t} \, | 0 \rangle = (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{k}) \, (2\pi)^3 \delta^{(3)}(\vec{p} + \vec{k}) \, \delta^{ss'} \delta^{tt'}$$

so

$$\langle 0 | \, \bar{\Psi} \Gamma \Psi(0) \, | \, \bar{\Phi}^{st}(K=0) \rangle = \int \frac{d^3k}{(2\pi)^3} \sqrt{2E_k} \, \frac{1}{2E_k} \, \tilde{\phi}(k) \; \bar{v}^t(-k) \, \Gamma \, u^s(k)$$

now use:

$$u^s(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\, \xi^s \\ \sqrt{k\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}$$

$$v^t(-k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\, \xi^{-t} \\ -\sqrt{k\cdot\bar{\sigma}}\, \xi^{-t} \end{pmatrix} \qquad \bar{v}^t(-k) = \left(-\xi^{-t\dagger} \sqrt{k\cdot\sigma}, \; \xi^{-t\dagger} \sqrt{k\cdot\bar{\sigma}} \right)$$

$$\hat{k}\cdot\sigma = k\cdot\bar{\sigma}$$

so $\bar{v}^\dagger(-k)\,\gamma^\mu\,u(k) = \bar{v}^\dagger(-k)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} u(k)$

$$= \xi^{-s}\left[-\sqrt{k\cdot\sigma}\;\sigma^\mu\sqrt{k\cdot\bar{\sigma}} + \sqrt{k\cdot\bar{\sigma}}\;\bar{\sigma}^\mu\sqrt{k\cdot\sigma}\right]\xi^s$$

In the non relativistic approximate $E_k \approx m \gg |\vec{k}|$

$$\sqrt{k\cdot\sigma} \approx \sqrt{k\cdot\bar{\sigma}} \approx \sqrt{m}$$

$$\bar{v}^\dagger(-\vec{k})\,\gamma^\mu\,u(\vec{k}) = (-m)\,\xi^{-s}\,(0, 2\vec{\sigma})^\mu\,\xi^s$$

similarly $\bar{v}^\dagger(-\vec{k})\,\gamma^\mu\gamma^5\,u(k) = \bar{v}^\dagger(-k)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} u(k)$

$$= \xi^{-s\dagger}\left[-\sqrt{k\sigma}\;\sigma^\mu\sqrt{k\bar{\sigma}} - \sqrt{k\bar{\sigma}}\;\bar{\sigma}^\mu\sqrt{k\sigma}\right]\xi^s$$

in the NR limit $= -m\,\xi^{-s\dagger}\left[(2, \vec{0})^\mu\right]\xi^s$

Now for ξ^s $\quad \xi^{-s} = (-i\sigma^2)\,\xi^{s*} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\xi^{s*}$

$s = \uparrow \quad \xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi^{-s} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$s = \downarrow \quad \xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \xi^{-s} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

in this part, we want $\xi^{-s\dagger} = (0\;1) \quad \xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so $\sigma^1\;\sigma^2$ have nonzero matrix elements, all other matrix elements are zero in the NR limit.

$$\xi^{-s\dagger}\,\sigma^i\,\xi^s = (1, i, 0)^i$$

$$\xi^{-s\dagger}\,1\,\xi^s = 0$$

Puts all of the pieces together:

$$\langle 0|\,\bar\Psi\gamma^1\Psi\,|\,\Phi^{\uparrow\uparrow}(k=0)\rangle = \underbrace{\sqrt{2E_k}}\int\frac{d^3k}{(2\pi)^3}\tilde\phi(k)\underbrace{\left(\frac{-2m}{2E_k}\right)}\cdot 1$$

$$= \sqrt{2M_\Phi}\underbrace{\int\frac{d^3k}{(2\pi)^3}\tilde\phi(k)}_{\phi(0)} \qquad = 1 \simeq \text{the NR approx.}$$

the Schrödinger wavefunction at $\vec r = 0$

again

$$\langle 0|\,\bar\Psi\gamma^1\psi(0)\,|\,\Phi^{\uparrow\uparrow}(k=0)\rangle = -\sqrt{2M_\Phi}\;\phi(0)\cdot 1$$

$$\langle 0|\,\bar\Psi\gamma^2\psi(0)\,|\,\Phi^{\uparrow\uparrow}(k=0)\rangle = -\sqrt{2M_\Phi}\;\phi(0)\cdot i$$

in other words.

$$\langle 0|\,\bar\Psi\gamma^\mu\psi(0)\,|\,\Phi^{\uparrow\uparrow}(k=0)\rangle = -\sqrt{2M_\Phi}\;\phi(0)\sqrt{2}\,\varepsilon^\mu$$

where $\varepsilon^\mu = \frac{1}{\sqrt2}(0,1,i,0)^\mu$

is a polarization vector for $J^3 = +1$ $\quad J=1$

The appearance of $\phi(0)$ also makes sense: the fermion and antifermion must come to $\vec r = 0$ to be annihilated by $\bar\Psi\gamma^\mu\psi(0)$

also $\quad \langle 0|\,\bar\Psi\gamma^\mu\gamma^5\psi(0)\,|\,\Phi^{\uparrow\uparrow}(k=0)\rangle = 0$

this makes sense, because $|\Phi^{\uparrow\uparrow}\rangle$ has

$$P = -1 \quad \text{and} \quad C = -1 \qquad C\, a^\dagger b^\dagger |0\rangle = b^\dagger a^\dagger |0\rangle = -a^\dagger b^\dagger |0\rangle$$

$\bar\Psi \gamma \Psi$ has $P = -1$, $C = -1$

$\bar\Psi \gamma \gamma^5 \Psi$ has $P = +1$, $C = +1$

(c.) Similarly, in the NR limit

$$\langle 0 | \bar\Psi \psi(0) | \bar\Phi^{st}(k=0)\rangle = \sqrt{2M_\Phi}\, \phi(0)\, \frac{1}{2m}\, \bar v^t(\vec 0)\, u^s(\vec 0)$$

$$\langle 0 | \bar\Psi \gamma^5 \psi(0) | \bar\Phi^{st}(k=0)\rangle = \sqrt{2M_\Phi}\, \phi(0)\, \frac{1}{2m}\, \bar v^t(0) \gamma^5 u^s(0)$$

$$\bar v^t_{(k)} u^s(k) = \xi^{-t\,+} \left[-\sqrt{k\cdot\sigma}\sqrt{k\cdot\sigma} + \sqrt{k\cdot\bar\sigma}\sqrt{k\cdot\bar\sigma} \right] \xi^s$$

$$\cong 0 \quad \text{in the NR approx.} \quad \text{(or when integrated over the direction of } \vec k\text{)}$$

$$\bar v^t \gamma^5 u^s = \xi^{-t\,+} \left[-\sqrt{k\sigma}\sqrt{k\sigma} - \sqrt{k\cdot\bar\sigma}\sqrt{k\bar\sigma} \right] \xi^s$$

$$= -2m\, (\xi^{-t\,+} \xi^s)$$

for $s = \uparrow \; t = \downarrow \quad = -2m \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +2m$

$s = \downarrow \; t = \uparrow \quad = -2m \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2m$

& for $\frac{1}{\sqrt 2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad S = 0$

$$\langle 0 | \bar\Psi \psi(0) | \bar\Phi \rangle = 0$$

$$\langle 0 | \bar\Psi \gamma^5 \psi(0) | \bar\Phi \rangle = \sqrt{2M_\Phi}\, \phi(0)\, (-\sqrt{2})$$

This makes sense, because Φ will spin 0 have

$$P = -1 \qquad C = +1$$

check C: $C \frac{1}{\sqrt{2}}(a^{\uparrow +} b^{\downarrow +} - a^{\downarrow +} b^{\uparrow +})|0\rangle = \frac{1}{\sqrt{2}}(b^{\uparrow +} a^{\downarrow +} - b^{\downarrow +} a^{\uparrow +})|0\rangle$

$$= + \frac{1}{\sqrt{2}}(a^{\uparrow +} b^{\downarrow +} - a^{\downarrow +} b^{\uparrow +})|0\rangle$$

$\Psi\Psi$ has $P = +1 \quad C = +1$

$\Psi\gamma^5\Psi$ has $P = -1 \quad C = +1$

3.) a) $$\delta S = \int d^4x \left\{ -\frac{1}{4} \cdot 2 \cdot F^{\mu\nu} \cdot (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) - \delta A_\mu j^\mu \right\}$$

$$= \int d^4x \left\{ + \delta A_\nu \, \partial_\mu F^{\mu\nu} - \delta A_\nu j^\nu \right\}$$

$$= \int d^4x \; \delta A_\nu \left(\partial_\mu F^{\mu\nu} - j^\nu \right)$$

so $$\partial_\mu F^{\mu\nu} = j^\nu$$

b.) $\nu = 0 \qquad \partial_i F^{i0} = j^0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{E} = \rho$

$\nu = i \qquad \partial_0 F^{0i} + \partial_j F^{ji} = j^i$

$$-\frac{\partial E^i}{\partial t} + \partial_j \epsilon^{ijk} B^k = j^i$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

Where are the other two Maxwell equations?

writing $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ gives

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi$$

from which $\vec{\nabla}\cdot\vec{B} = 0$ $\qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ are automatic.

the relativistic version of these equations is the identity

$$\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0$$

c.) $$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \cdots\right)$$

$$= \int d^4x \quad \frac{1}{2} F^{0i}F^{0i} + \cdots$$

$$= \int d^4x \quad \frac{1}{2}\left(\dot{A}^i + \nabla^i\phi\right)^2 + \cdots$$

so $$\frac{\delta \mathcal{L}}{\delta \dot{A}^i} = \dot{A}^i + \nabla^i\phi = -E^i$$

$$\frac{\delta \mathcal{L}}{\delta \dot{A}^0} = 0$$

d.) $$\vec{\nabla}\cdot\vec{E} = \rho \Rightarrow -\nabla^2 A^0 - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho$$

so if $\vec{\nabla}\cdot\vec{A} = 0$ $\qquad -\nabla^2 A^0 = \rho = \mathcal{H}^0$

then

$$A^0(\vec{x}) = \int d^3y \; \frac{1}{4\pi|\vec{x}-\vec{y}|} \; j^0(y)$$

e.) try

$$A^i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p}} \; e^{i\vec{p}\cdot\vec{x}} \sum_a \left(\varepsilon_a^i(\vec{p}) \, a_{pa} + \varepsilon_a^{i*}(-\vec{p}) \, a^\dagger_{-\vec{p}a} \right)$$

$$\Pi^i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p}} (-ip) \; e^{i\vec{p}\cdot\vec{x}} \sum_a \left(\varepsilon_a^{i*}(\vec{p}) \, a_{pa} - \varepsilon_a^{i*}(-\vec{p}) \, a^\dagger_{-\vec{p}a} \right)$$

$$\left[A^i(\vec{k}), \Pi^j(-\vec{k}') \right] = \frac{1}{\sqrt{2k \, 2k'}} \sum_{a,b} (-ik')$$

$$\left[\varepsilon_a^i(\vec{k}) \, a_{ka} + \varepsilon_a^{i*}(-\vec{k}) \, a^\dagger_{-\vec{k}a}, \; \varepsilon_b^j(-\vec{k}') \, a_{-\vec{k}'b} - \varepsilon_b^{j*}(\vec{k}') \, a^\dagger_{\vec{k}'b} \right]$$

$$= \frac{-ik'}{\sqrt{4kk'}} \sum_{a,b} \left(-\varepsilon_a^i(\vec{k}) \, \varepsilon_b^{j*}(\vec{k}') \, (2\pi)^3 \delta(\vec{k}-\vec{k}') \delta_{ab} - \varepsilon_a^{i*}(-\vec{k}) \, \varepsilon_b^j(-\vec{k}') (2\pi)^3 \delta(\vec{k}-\vec{k}') \, \delta_{ab} \right)$$

$$= i \, \frac{1}{2} \sum_{a=1,2} \underbrace{\left[\varepsilon_a^i(\vec{k}) \, \varepsilon_a^{j*}(\vec{k}) + \varepsilon_a^{i*}(-\vec{k}) \, \varepsilon_a^j(-\vec{k}) \right]}_{\text{this is the projector onto transverse directions}} (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$$= i \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right) (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

a similar calculation gives $\left[A^i(\vec{k}), A^j(-\vec{k}') \right] = \left[\Pi^i(\vec{k}), \Pi^j(-\vec{k}') \right] = 0$

f.) $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$

$\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi$; the transverse part of $\vec{E}$ is built from $\vec{\pi}$. Then:

$$\vec{E} = \underbrace{\int d^3y \frac{(\vec{x}-\vec{y})}{4\pi|\vec{x}-\vec{y}|^2} j^0(y)}_{\text{a c-number}} - \vec{\pi}$$

$$\vec{E}(k) = -i\vec{k}\,\frac{+1}{|\vec{k}|^2} j^0(k) - \vec{\pi}(k)$$

cl since $\vec{k}\cdot\vec{\pi}(k) = 0$

$$\int d^3x \ \tfrac{1}{2}E^2 = \int \frac{d^3k}{(2\pi)^3} \tfrac{1}{2}\vec{E}(k)\cdot\vec{E}(-k) = \int \frac{d^3k}{(2\pi)^3} \tfrac{1}{2}|\vec{E}(k)|^2$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left\{ j^0(-k)\frac{1}{k^2} j^0(k) + \vec{\pi}(k)\cdot\vec{\pi}(-k)\right\}$$

g) $\int d^3x \ \tfrac{1}{2}B^2 = \int d^3x \ \tfrac{1}{2}(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A})$

$$= \int d^3x \ \epsilon^{ijk}\epsilon^{ilm}\nabla^j A^k \nabla^l A^m$$

$$= \int d^3x \ A^k(k)\left(-\nabla^2\delta^{km} + \nabla^k\underbrace{\nabla^m}_{=0}\right)A^m$$

$$= \int d^3x \ A^k(x)(-\nabla^2)A^k(x)$$

or $$\int d^3x \; \tfrac{1}{2} B^2 = \int \frac{d^3k}{(2\pi)^3} \; \tfrac{1}{2} \vec{A}(\vec{k}) \; k^2 \; \vec{A}(-\vec{k})$$

then

$$H = \int d^3x \; \tfrac{1}{2}(E^2 + B^2)$$

$$= \int \frac{d^3k}{(2\pi)^3} \left\{ \tfrac{1}{2} j^0(\vec{k}) \frac{1}{|\vec{k}|^2} j^0(-\vec{k}) + \tfrac{1}{2} \vec{\Pi}(-\vec{k}) \vec{\Pi}(\vec{k}) + \tfrac{1}{2} k^2 \vec{A}(\vec{k}) \cdot \vec{A}(\vec{k}) \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \left\{ \tfrac{1}{2} j^0(\vec{k}) \frac{1}{|\vec{k}|^2} j^0(-\vec{k}) \right.$$

$$+ \tfrac{1}{2} \frac{1}{2k} (-k^2) \left(\varepsilon_a^i(-\vec{k}) a_{-\vec{k}a} - \varepsilon_a^{i*}(\vec{k}) a_{\vec{k}a}^\dagger \right) \left(\varepsilon_b^i(\vec{k}) a_{\vec{k}a} - \varepsilon_b^{i*}(-\vec{k}) a_{-\vec{k}b}^\dagger \right)$$

$$\left. + \tfrac{1}{2} \frac{1}{2k} (+k^2) \left(\varepsilon_a^i a_{-\vec{k}a} + \varepsilon_a^{i*} a_{\vec{k}a}^\dagger \right) \left(\varepsilon_b^i a_{\vec{k}a} + \varepsilon_b^{i*} a_{-\vec{k}b}^\dagger \right) \right\}$$

the terms w. aa or $a^\dagger a^\dagger$ cancel. What remains is:

$$H = \int \frac{d^3k}{(2\pi)^3} \left[\tfrac{1}{2} j^0(-\vec{k}) \frac{1}{|\vec{k}|^2} j^0(\vec{k}) + \frac{k}{4} \cdot \left\{ 2\varepsilon_a^i a_{\vec{k}a} a_{\vec{k}b}^\dagger \varepsilon_b^{*i} \right.\right.$$

$$\left.\left. + 2\varepsilon_a^{i*} a_{\vec{k}a}^\dagger a_{\vec{k}b} \varepsilon_b^i \right\} \right]$$

now $\varepsilon_a^i \varepsilon_b^{*i} = \delta_{ab}$ $\quad a,b = 1,2$ so

$$H = \int \frac{d^3k}{(2\pi)^3} \left\{ \tfrac{1}{2} j^0(-\vec{k}) \frac{1}{|\vec{k}|^2} j^0(\vec{k}) + k \; a_{\vec{k}a}^\dagger a_{\vec{k}a} + (\text{const}) \right\}$$

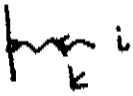
How can we use this Hamiltonian? Write

$$H = H_0 + H_{int}$$

$$H_0 = \int \frac{d^3k}{(2\pi)^3}\, k\, a^+_{ka} a_{ka}$$

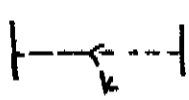
$$H_{int} = \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2}\, j^0(-k)\, \frac{1}{|\vec{k}|^2}\, j^0(k) + \int \frac{d^3k}{(2\pi)^3} (-\vec{j}(-k)\cdot\vec{A}(k))$$

If we do perturbation theory in H_{int}, there are two vertices, a conventional one:



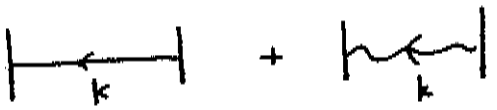
$$= +i\, j^i(-k)$$

and a nonlocal instantaneous interaction



$$= -i\, j^0(-k)\, \frac{1}{|\vec{k}|^2}\, j^0(k)$$

Put these together:



$$= -i\, j^0(-k)\, \frac{1}{|\vec{k}|^2}\, j^0(k) + i j^i(-k)\, \frac{i(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2})}{k^2 + i\varepsilon} (i j^j(k))$$

$$= -i\, j^0(-k)\, \frac{1}{|\vec{k}|^2}\, j^0(k) - i\, \frac{j^i(-k)\, j^i(k)}{k^2} + i\, \frac{\vec{k}\cdot\vec{j}(-k)\ \vec{k}\cdot\vec{j}(k)}{k^2\, |\vec{k}|^2}$$

how use current conservation: $\partial_\mu j^\mu + \vec{\nabla}\cdot\vec{j} = 0$

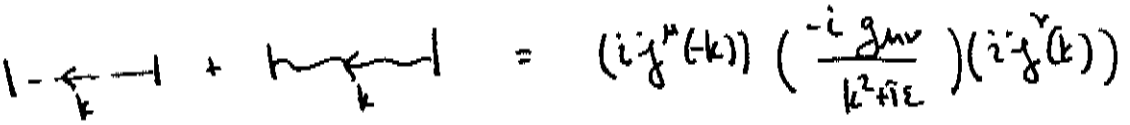
$$\vec{k}\cdot\vec{j}(k) = k^0 j^0(k)$$

so
$$-i\, j^0(-k)\frac{1}{|\vec{k}|^2} j^0(k) + i\,\frac{\vec{k}\cdot\vec{j}(-k)\;\vec{k}\cdot\vec{j}(k)}{k^2\,|\vec{k}|^2}$$

$$= -i\, j^0(-k)\frac{(k^0)^2 - |\vec{k}|^2}{k^2\,|\vec{k}|^2} j^0(k) + i\,\frac{(k^0)^2\, j^0(-k)\, j^0(k)}{k^2\,|\vec{k}|^2}$$

$$= i\,\frac{j^0(-k)\, j_0(k)}{k^2}$$

in all



$$= (i j^\mu(-k))\left(\frac{-i g_{\mu\nu}}{k^2+i\varepsilon}\right)(i j^\nu(k))$$

this is the photon propagator introduced in class.

Physics 330 – Problem Set # 5

(due Wednesday, November 3)

1. Peskin and Schroeder, problem 4.1.

2. Peskin and Schroeder, problem 4.3. In part (d), it is enough to compute the mass of the pion for small a and show that $m_\pi^2 \sim a$.

Physics 330 – Problem Set #5

Solutions

1.) a) Let the source be turned on at a time t_{on} and turned off at t_{off}. Let t_1, t_2 be any times s.t. $t_1 > t_{off} > t_{on} > t_2$

Then for $t_1 \geq t_{off}$ and for $t_2 \leq t_{on}$, the ground state of H is the free-particle vacuum $|0\rangle$.

We can compute $e^{-iH(t_1-t_2)}$ by

$$U(t_1,t_2) = e^{iH_0 t_1} e^{-iH(t_1-t_2)} e^{-iH_0 t_2}$$

$$= T\left\{ e^{-i\int_{t_2}^{t_1} dt\, H_I(t)} \right\}$$

$$= T\left\{ e^{+i\int d^4x\, j(x)\,\phi_I(x)} \right\}$$

Now we can ignore the limits in time, since $j(x)$ is nonzero only for $t_2 < t < t_1$

The probability that no particles are produced while j is turned on is

$$P(0) = \left| \langle 0 | e^{-iH(t_1-t_2)} | 0 \rangle \right|^2 = \left| \langle 0 | T\{ e^{i\int d^4x\, j\,\phi_I} \} | 0 \rangle \right|^2$$

now

$$\langle 0|\, T\{ e^{i\int d^4x\, j(x)\,\phi_I(x)} \}\,|0\rangle$$

$$= \langle 0|\, T\{ 1 + i\int d^4x\, j(x)\phi_I(x) - \tfrac{1}{2}\int d^4x\, d^4y\, j(x)j(y)\phi_I(x)\phi_I(y) + \cdots \}\,|0\rangle$$

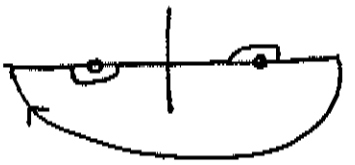
$\langle 0|\phi_I(x)|0\rangle = 0$ $\qquad$ $\langle 0|\, T\{\phi_I(x)\phi_I(y)\}\,|0\rangle = \overbrace{\phi_I(x)\phi_I(y)}$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}$$

$$= 1 - \tfrac{1}{2}\int d^4x\, d^4y\, j(x)j(y) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)} + \cdots$$

$$= 1 - \tfrac{1}{2}\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon}\, \tilde{j}(-p)\,\tilde{j}(p) + \cdots$$

since $j(p)$ has compact support, $\tilde{j}(p)$ is an entire function so we can evaluate the dp^0 integral by contour:



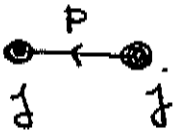
$$= \frac{-2\pi i\, i}{2\pi \cdot 2E_p}\Big|_{p^0 = E_p}$$

$$= 1 - \tfrac{1}{2}\underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2\Big|_{p^0=E_p}}_{\text{call this } = \mathcal{J}} + \cdots$$

$$= 1 - \tfrac{1}{2}\mathcal{J} + \cdots$$

Then

$$P(0) = \left|1 - \tfrac{1}{2}\lambda + \cdots\right|^2 = 1 - \lambda + \cdots$$

c.) Repeat $\int d^4x\, j(x)\, \overbrace{\phi_4'(x) \int d^4y\, j(y)\, \phi_5(y)}$ = 

Then

$$\langle 0|\, T\{ e^{i\int d^4x\, j(x)\phi_5(x)}\}\, |0\rangle$$

$$= 1 + \frac{(-i)^2}{2!} \bullet\!\!-\!\!\bullet + \frac{(-i)^4}{4!}\cdot C_4 \;\; \text{(two lines)} + \frac{(-i)^6}{6!} C_6 \;\; \text{(three lines)}$$

$$+ \cdots$$

$$= 1 - \tfrac{1}{2}\lambda + \tfrac{1}{4!} C_4 \lambda^2 - \tfrac{1}{6!} C_6 \lambda^3 + \cdots$$

C_4, C_6 .. are the numbers of possible contractions

C_4 

$$= 3 \cdot 1 = 3 \qquad \frac{C_4}{4!} = \frac{1}{4\cdot 2} = \frac{1}{2^2}\cdot\frac{1}{2}$$

$$C_6 = 5\cdot 3\cdot 1 \qquad \frac{C_6}{6!} = \frac{1}{6\cdot 4\cdot 2} = \frac{1}{2^3}\cdot\frac{1}{3!}$$

$$C_8 = 7\cdot 5\cdot 3\cdot 1 \qquad \frac{C_8}{8!} = \frac{1}{8\cdot 6\cdot 4\cdot 2} = \frac{1}{2^4}\frac{1}{4!}$$

So $$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n\, n!} = \exp\left[-\frac{\lambda}{2}\right]$$

then $P(0) = e^{-\lambda}$

d.) The probability of producing 1 particle is

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left| \langle k | T\{ e^{i\int J\phi_I} \} | 0 \rangle \right|^2$$

$$\langle k | T\{ e^{i\int J\phi_I} \} | 0 \rangle \cong \overbrace{\langle k | \; i\int d^4x \, J(x)\, \phi_I(x) \; | 0 \rangle} + \cdots$$

$$= i\int d^4x \; e^{ik\cdot x} J(x) + \cdots = i\tilde{J}(k) + \cdots$$

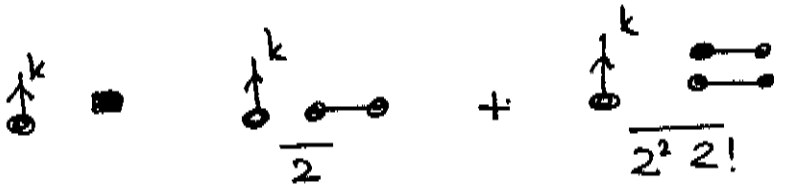
so

$$P_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} |\tilde{J}(k)|^2 + \cdots = \lambda + \cdots$$

represent the matrix element by a Feynman diagram:



The whole series is:



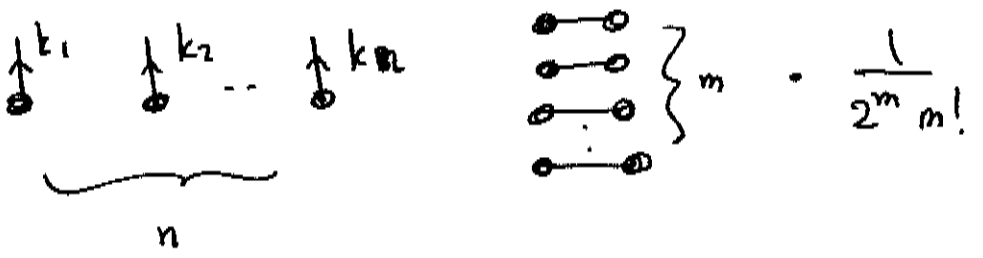
$$= i\tilde{J}(k) \cdot \left[1 - \frac{\lambda}{2} + \frac{\lambda^2}{2^2 2!} + \cdots \right]$$

then

$$I_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left(\tilde{j}(k)\, e^{-\lambda/2} \right)^2$$

$$= \lambda\, e^{-\lambda}$$

e.) A typical diagram for producing n particles is



$$\cdot \frac{1}{2^m\, m!}$$

square this and integrate over phase space

$$\frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_1} \cdots \int \frac{d^3k_n}{(2\pi)^3} \frac{1}{2E_n}$$

The $\frac{1}{n!}$ must be there because these are identical Bose-Einstein particles. We sum over inequivalent configurations only. Then

$$I_n = \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_1} \cdots \int \frac{d^3k_n}{(2\pi)^3} \frac{1}{2E_n} \cdot |\tilde{j}(k_1)|^2 \cdots |\tilde{j}(k_n)|^2 \left(e^{-\lambda/2}\right)^2$$

$$= \frac{\lambda^n}{n!} e^{-\lambda}$$

f:) $$\sum_n P_n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1$$

$$\langle N \rangle = \sum_n n P_n = \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} = \lambda e^{\lambda} e^{-\lambda} = \lambda$$

$$\langle N^2 \rangle = \sum_n n^2 P_n = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \lambda^n e^{-\lambda} = \sum_{n=1}^{\infty} \frac{(n-1)+1}{(n-1)!} \lambda^n e^{-\lambda}$$

$m = (n-1)$

$$= \left(\sum_{m=0}^{\infty} \frac{1}{m!} \lambda^{m+1} + \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \lambda^{m+1} \right) e^{-\lambda}$$

$$= (\lambda e^{\lambda} + \lambda^2 e^{\lambda}) e^{-\lambda}$$

So

$$\langle N^2 \rangle = \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 - 2N\langle N \rangle + \langle N \rangle^2 \rangle$$

$$= \langle N^2 \rangle - (\langle N \rangle)^2$$

$$= \lambda$$

2.) a) $$\mathcal{H} = \int d^3x \left\{ \tfrac{1}{2}\Pi^{i\,2} + \tfrac{1}{2}(\vec{\nabla}\Phi^i)^2 + \tfrac{1}{2}m^2\Phi^{i2} + \frac{\lambda}{4}((\Phi^j)^2)^2 \right\}$$

$$= \sum_{i=1}^{N} H_0^i + \frac{\lambda}{4}\left(\sum_{i=1}^{N}(\Phi^i)^2\right)^2$$

H_0^i is a Klein-Gordon Hamiltonian so

$$\langle 0 | T\{\Phi_I^i(x)\,\Phi_I^j(y)\} | 0\rangle = \delta^{ij} \times \text{Klein-Gordon propagator}$$

$$= \delta^{ij} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}$$

the vertex comes from contracts

$$\langle\ 1\ \ -i\frac{\lambda}{4}\int d^4x\ \Phi^i\Phi^i\,\Phi^j\Phi^j\ 1\rangle \qquad i,j = 1 \text{---} N$$

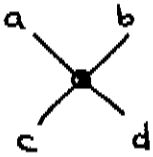
a given contraction is of the form

$$\langle\ a\ b\ 1\ \ -i\frac{\lambda}{4}\int d^4x\ \Phi^i\Phi^i\,\Phi^j\Phi^j\ 1c\ d\ \rangle$$

$$= -i\frac{\lambda}{4}\,\delta^{ai}\delta^{bi}\delta^{cj}\delta^{dj} = -i\frac{\lambda}{4}\,\delta^{ab}\delta^{cd}$$

three different structures are possible. There are $4!$ contractions in all, so each structure has $4!/3 = 8$ possible contractions.

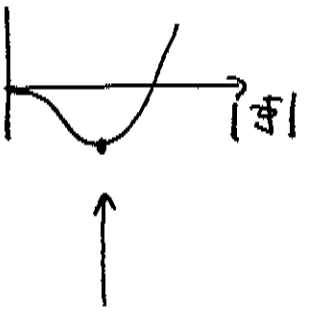
so



$$= -2i\lambda\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)$$

b.) Now let $m^2 = -\mu^2 < 0$

$$V(\Phi^2) = -\frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}(\Phi^2)^2$$



$$\frac{\partial V}{\partial \Phi^i} = -\mu^2\Phi^i + \lambda\,\Phi^2\Phi^i$$

so the minimum occurs at $\Phi^2 = v^2 = \frac{\mu^2}{\lambda}$

Choose the orientation $\Phi = (0 \cdots 0, v)$

so $\Phi^i = \pi^i \qquad i = 1 \text{—} (N-1) \qquad \langle\pi^i\rangle = 0$

$\Phi^N = v + \sigma \qquad \langle\Phi^N\rangle = v$

expanding about this point: $\Phi^2 = \pi^2 + (v+\sigma)^2$

$$V(\Phi^2) = -\frac{\mu^2}{2}\left[\pi^2 + (v+\sigma)^2\right] + \frac{\lambda}{4}\left[\pi^2 + (v+\sigma)^2\right]^2$$

$$= -\mu^2/2\,(\pi^2 + v^2 + 2v\sigma + \sigma^2)$$

$$+ \frac{\lambda}{4}\left(\pi^4 + 2\pi^2v^2 + 4\pi^2v\sigma + 2\pi^2\sigma^2 + v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4\right)$$

now $v^2 = \mu^2/\lambda$ so in the term quadratic in π is:

$$-\frac{\mu^2}{2}\pi^2 + \frac{\lambda}{4}\,2\pi^2 v^2 = -\frac{\mu^2}{2}\pi^2 + \frac{\lambda}{2}\pi^2\frac{\mu^2}{\lambda}$$

$$= 0$$

so there is no π mass term!

$$V(\Phi^2) = -\left(\cancel{\frac{\mu^2}{2}\pi^2} + \frac{\mu^2}{2}v^2 + \cancel{\mu^2\frac{\mu}{\sqrt{\lambda}}\sigma} + \underline{\frac{\mu^2}{2}\sigma^2}\right)$$

$$+ \frac{\lambda}{4}\pi^4 + \cancel{\frac{\lambda}{2}\pi^2\frac{\mu^2}{\lambda}} + \lambda\pi^2\sqrt{\frac{\mu^2}{\lambda}}\sigma + \frac{\lambda}{2}\pi^2\sigma^2$$

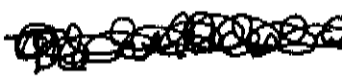
$$+ \frac{\lambda v^4}{4} + \lambda\cancel{\frac{\mu^3}{\lambda^{3/2}}}\sigma + \underline{\frac{3\lambda}{2}\frac{\mu^2}{\lambda}\sigma^2}$$

$$+ \lambda\sqrt{\frac{\mu^2}{\lambda}}\sigma^3 + \frac{\lambda}{4}\sigma^4$$

$$= (\text{const}) + \frac{1}{2}(2\mu^2)\sigma^2 + \sqrt{\lambda}\mu\pi^2\sigma + \sqrt{\lambda}\mu\sigma^3$$

$$+ \frac{\lambda}{2}\pi^2\sigma^2 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{4}(\pi^2)^2$$

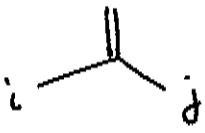
The σ has $(\text{mass})^2 = 2\mu^2$; the π has mass $= 0$

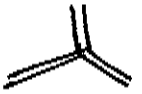


$$\overleftarrow{\sigma}\,\overrightarrow{\sigma} = \frac{i}{p^2 - 2\mu^2}$$

$$\overleftarrow{\pi^i}\,\overrightarrow{\pi^j} = \delta^{ij}\frac{i}{p^2}$$

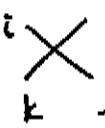
The vertices are:

$\pi^2\sigma \rightarrow$  $= -2i\sqrt{\lambda}\mu\,\delta^{ij}$

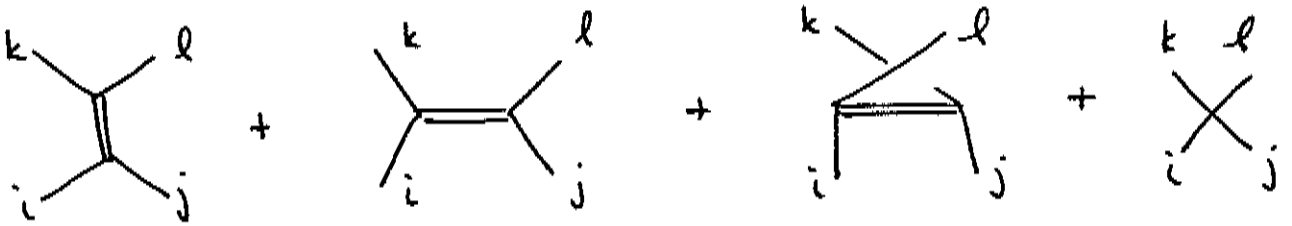
$\sigma^3 \rightarrow$  $= -3!\, i\sqrt{\lambda}\mu = -6i\sqrt{\lambda}\mu$

$\sigma^4 \rightarrow$  $= -6i\lambda$

$\pi^2\sigma^2 \rightarrow$  $= -2i\,\delta^{ij}\lambda$

$(\pi^2)^2 \rightarrow$  $= -2i(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\lambda$

C.) For $\pi^i(p_1) + \pi^j(p_2) \rightarrow \pi^k(p_3) + \pi^l(p_4)$



$$iM = (-2i\sqrt{\lambda}\mu)^2 \delta^{ij}\delta^{kl} \frac{i}{(p_1+p_2)^2 - 2\mu^2}$$

$$+ (-2i\sqrt{\lambda}\mu)^2 \delta^{ik}\delta^{jl} \frac{i}{(p_1-p_3)^2 - 2\mu^2}$$

$$+ (-2i\sqrt{\lambda}\mu)^2 \delta^{il}\delta^{jk} \frac{i}{(p_1-p_4)^2 - 2\mu^2}$$

$$+ (-2i\lambda)(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$$

then

$$iM = \delta^{ij}\delta^{kl}\left\{-4i\lambda\mu^2\frac{1}{(p_1+p_2)^2-2\mu^2} - 2i\lambda\right\}$$

$$+ \delta^{ik}\delta^{jl}\left\{-4i\lambda\mu^2\frac{1}{(p_1-p_3)^2-2\mu^2} - 2i\lambda\right\}$$

$$+ \delta^{il}\delta^{jk}\left\{-4i\lambda\mu^2\frac{1}{(p_1-p_4)^2-2\mu^2} - 2i\lambda\right\}$$

If we set $p_1 = p_2 = p_3 = p_4 = 0$ (this is the threshold for these massless particles)

$$iM = \delta^{ij}\delta^{kl}\left\{\frac{4i\lambda\mu^2}{2\mu^2} - 2i\lambda\right\}$$

$$+ \delta^{ik}\delta^{jl}\left\{\frac{4i\lambda\mu^2}{2\mu^2} - 2i\lambda\right\}$$

$$+ \delta^{il}\delta^{jk}\left\{\frac{4i\lambda\mu^2}{2\mu^2} - 2i\lambda\right\} = 0 !$$

Expand to one further power of momentum

$$\frac{1}{(p_1+p_2)^2-2\mu^2} \simeq -\frac{1}{2\mu^2}\left(1 - \frac{(p_1+p_2)^2}{2\mu^2}\right)^{-1}$$

$$= -\frac{1}{2\mu^2} - \frac{(p_1+p_2)^2}{4\mu^4} + \cdots$$

+2en

$$i\mathcal{M} \cong \delta^{ij}\delta^{kl}\left[-4i\lambda\mu^2\left(-\frac{(p_1+p_2)^2}{4\mu^4}\right)\right] + (2\text{more})$$

$$i\mathcal{M} \cong \frac{i\lambda}{\mu^2}\left[\delta^{ij}\delta^{kl}(p_1+p_2)^2 + \delta^{ik}\delta^{jl}(p_1-p_3)^2 + \delta^{il}\delta^{jk}(p_1-p_4)^2\right]$$

$$+ \cdots$$

for $N=2$ $ijkl = 1$ only. Now use $p_i^2 = 0$

$$(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2$$

$$= 2p_1p_2 - 2p_1p_3 - 2p_1p_4$$

$$= 2p_1(p_2-p_3-p_4) = 2p_1(-p_1) = -2p_1^2 = 0$$

so for $N=2$ $i\mathcal{M} = \mathcal{O}(p^4)$.

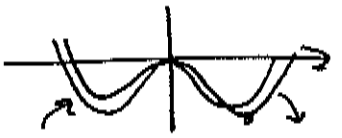
d.) Now add to V $\quad \Delta V = -a\Phi^N$.

$$V = -\frac{\mu^2}{2}(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 - a\Phi^N$$

$i<N$ $\quad \frac{\partial V}{\partial \Phi^i} = -\mu^2\Phi^i + \lambda\Phi^2\Phi^i$

$$\frac{\partial V}{\partial \Phi^N} = -\mu^2\Phi^N + \lambda\Phi^2\Phi^N - a$$

so the potential tilts toward Φ^N



the minimum is at $\Phi^i = 0 \quad i < N$

of Φ^N satisfies:

$$-\mu^2 \Phi^N + \lambda (\Phi^N)^3 - a = 0$$

write $\Phi^N = \frac{\mu}{\sqrt{\lambda}} + \delta\Phi^N$

$$0 = -\mu^2 \left(\frac{\mu}{\sqrt{\lambda}} + \delta\Phi^N\right) + \lambda\left[\left(\frac{\mu}{\sqrt{\lambda}}\right)^3 + 3\frac{\mu^2}{\lambda}\delta\Phi^N + \cdots\right] - a$$

$$0 = -\mu^2 \delta\Phi^N + 3\mu^2 \delta\Phi^N - a$$

$$= 2\mu^2 \delta\Phi^N - a$$

$$\text{or} \quad \delta\Phi^N \cong \frac{a}{2\mu^2} \qquad v = \frac{\mu}{\sqrt{\lambda}} + \frac{a}{2\mu^2} + \cdots$$

Now find the term of order π^2 in V:

$$V = -\frac{\mu^2}{2}(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 - a\Phi^N$$

$$\approx -\frac{\mu^2}{2}(\pi^i)^2 + \frac{\lambda}{2} v^2 \pi^2 + \sigma^2 + \text{nonlinear}$$

$$V = -\frac{\mu^2}{2}\pi^2 + \frac{\partial}{2}\left(\frac{\mu^2}{\partial} + \frac{\mu}{\sqrt{\partial}}\frac{a}{\mu^2} + \cdots\right)\pi^2$$

$$= \frac{\sqrt{\partial}}{2}\frac{a}{\mu}\pi^2$$

so $m_\pi^2 = \frac{\sqrt{\partial}}{\mu} a + \cdots$ for small a.

Physics 330 – Problem Set # 6

(due Wednesday, November 9)

1. Peskin and Schroeder, problem 4.4.

2. Peskin and Schroeder, problem 5.1.

3. Peskin and Schroeder, problem 5.2.

4. In our discussion of $e^+e^- \to \mu^+\mu^-$ in class, we saw that the limit $m_\mu \to 0$ of the cross section was built up in a simple way as a sum of squares of amplitudes for scattering processes with fermions of definite helicity in the initial and final states. We used the following explicit spinors to compute the amplitudes:

$$\begin{aligned}
e^- \text{ with } \vec{p} \parallel \hat{3} \ : \qquad & u_R(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
e^- \text{ in } \theta \text{ direction} : \qquad & u_R(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ c_2 \\ s_2 \end{pmatrix} \quad u_L(p) = \sqrt{2E}\begin{pmatrix} -s_2 \\ c_2 \\ 0 \\ 0 \end{pmatrix} \\
e^+ \text{ with } \vec{p} \parallel -\hat{3} \ : \qquad & v_R(p) = \sqrt{2E}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \\
e^+ \text{ opposite to } \theta : \qquad & v_R(p) = \sqrt{2E}\begin{pmatrix} -c_2 \\ -s_2 \\ 0 \\ 0 \end{pmatrix} \quad v_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ s_2 \\ -c_2 \end{pmatrix} \qquad (1)
\end{aligned}$$

where $c_2 = \cos\theta/2$, $s_2 = \sin\theta/2$, L, R refer to the physical spin direction of the particle or antiparticle, and for antiparticles I have given $v(p)$, not $\overline{v}(p)$. Use these explicit spinors to redo the calculation of the Bhabha scattering cross section.

   (a) There are $2^4 = 16$ possible helicity amplitudes. Show that 10 of these are zero, 4 involve only one Feynman diagram, and 2 require a sum of two Feynman diagrams.

   (b) Compute the nonzero helicity amplitudes. Show that the squares of these amplitudes are equal in pairs. (This is a consequence of C or P invariance.)

   (c) Show that the sum of the three contributions reproduces the result of problem 3.

Physics 330 – Problem Set #6

Solutions

1.) a.) $\langle p'|S|p\rangle = \langle p'|T\{e^{-i\int dt\, H_I(t)}\}|p\rangle$

$= \langle p'|1+iT|p\rangle \cong \langle p'|1|p\rangle + \langle p'|-i\int d^4x\, e\bar{\Psi}_I\gamma^\mu\Psi_I A_\mu(x)|p\rangle + \cdots$

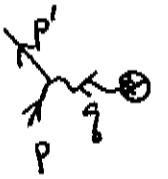
now set $A_\mu(x) = \int\frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}\tilde{A}_\mu(q)$

$\overrightarrow{\Psi_I}|p\rangle = u(p)e^{-ip\cdot x}$

$\langle p'|\overrightarrow{\bar{\Psi}_I} = \bar{u}(p')e^{ip'\cdot x}$

$\int d^4x$ gives $(2\pi)^4\delta(p'-p-q)$ so

$\langle p'|iT|p\rangle \cong -ie\,\bar{u}(p')\gamma^\mu u(p)\,\tilde{A}_\mu(q)\big|_{q=p'-p}$ =



b.) For $A_\mu(x)$ independent of time x^0

$$A_\mu(x) = \int\frac{dq^0}{(2\pi)}\cdot\int\frac{d^3q}{(2\pi)^3}\, e^{-iq\cdot x}\, A_\mu(\vec{q})\,(2\pi)\delta(q^0)$$

this gives

$\langle p'|iT|p\rangle =$  $= -ie\,\bar{u}(p')\gamma^\mu u(p)\,\tilde{A}_\mu(\vec{q})\cdot 2\pi\,\delta(p'^0-p^0)$.

using this formula, we can derive the cross section from initial wave packets.

$$\int d\sigma = \int \frac{d^3\vec{p}_f}{(2\pi)^3}\frac{1}{2E_f} \int d^2b \int \frac{d^3p_1}{(2\pi)^3}\frac{\phi(p_1)}{\sqrt{2E_1}} \int \frac{d^3p_2\ \phi^*(p_2)}{(2\pi)^3\sqrt{2E_2}} e^{i\vec{b}\cdot(\vec{p}_1-\vec{p}_2)_\perp}$$

$$\cdot \langle p'|iT|p_1\rangle\langle p_2|-iT^\dagger|p'\rangle$$

the integral d^2b gives $(2\pi)^2\delta(\vec{p}_1-\vec{p}_2)$ in the $\perp$ direction the energy δ-function in the matrix elements set $p_1^3 = p_2^3$

$$\int \frac{dp_1^3}{(2\pi)}\ 2\pi\,\delta(E_f - E_1) = \frac{1}{\left|\frac{p_1}{E_1}\right|}$$

then

$$\int d\sigma = \int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f} \int \frac{d^3p_1}{(2\pi)^3}\frac{|\phi(p_1)|^2}{2E_1}\,|M|^2\,2\pi\,\delta(E_f-E_1)\frac{1}{\left|\frac{p_1}{E_1}\right|}$$

$$\sigma = \frac{1}{2p}\int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\,|M|^2\,(2\pi)\,\delta(E_f-E)$$

$$\frac{1}{2p} = \frac{1}{v\cdot 2E}$$

now $\int \frac{dp_f\,p_f^2}{(2\pi)}\,2\pi\,\delta(E_f - E) = \frac{p_f^2}{\left|\frac{p_f}{E_f}\right|}$ so

$$\int d\sigma = \frac{1}{4p^2}\int \frac{d\Omega_f}{(2\pi)^2}\,p^2\,|M|^2 = \int d\Omega\,\frac{1}{16\pi^2}\,|M|^2$$

again $$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|M|^2$$

c.) For the Coulomb potential

$$A^0(\vec{q}) = \frac{Ze}{|\vec{q}|^2}$$

$$|M| = -ie\,\bar{u}(p')\gamma^0 u(p)\,A^0(\vec{q})$$

in the non-relativistic limit, $\bar{u}\gamma^0 u \approx 2m\,\delta_{ss'}$. Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\,4m^2\,\frac{Z^2e^4}{(|\vec{q}|^2)^2}$$

$$|\vec{q}|^2 = |\vec{p'}-\vec{p}|^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2\theta/2$$

$$= \frac{e^4}{16\pi^2}Z^2\,\frac{4m^2}{16p^4\sin^4\theta/2} \qquad p = mv$$

so $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4}\,\frac{1}{\sin^4\theta/2}$$

2.) Now redo this in the relativistic case

$$iM = -ie\,\bar{u}(p')\gamma^0 u(p)\,A^0(\vec{q})$$

$$\frac{1}{2}\sum_{spin}|M|^2 = \frac{1}{2}e^2\,\mathrm{tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0]\,|A^0(\vec{q})|^2$$

$$= \frac{1}{2}e^2 \cdot 4 \cdot [p'^0 p^0 \cdot 2 - p'\cdot p + m^2]\,|A^0(\vec{q})|^2$$

$$= 2e^2[2E^2 - (E^2 - p^2\cos\theta) + m^2]\,|A^0(\vec{q})|^2$$

$$= 2e^2[2E^2 - p^2(1-\cos\theta)]\,|A^0(\vec{q})|^2$$

$$= 4e^2E^2\left(1 - \left(\frac{p}{E}\right)^2 \sin^2\theta/2\right)|A^0(\vec{q})|^2$$

For a Coulomb potential, still $A^0(\vec{q}) = \dfrac{Ze}{4p^2\sin^2\theta/2}$ $\qquad \beta = \frac{p}{E} = v$

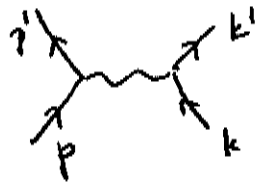
so

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\,4e^4Z^2E^2(1-\beta^2\sin^2\theta/2)\,\frac{1}{16E^4\beta^4\sin^4\theta/2}$$

$$= \frac{\alpha^2Z^2}{4E^2\beta^4}\,\frac{1}{\sin^4\theta/2}\,(1-\beta^2\sin^2\theta/2)$$

or $\qquad \dfrac{d\sigma}{d\Omega} = \dfrac{\alpha^2Z^2}{4p^2\beta^2}\,\dfrac{1}{\sin^4\theta/2}\,(1-\beta^2\sin^2\theta/2)$

Another way: use $e^- \mu^- \to e^- \mu^-$



$$= (-ie)^2 \, \bar{u}(p') \gamma^\mu u(p) \, \frac{-i}{q^2} \, \bar{u}(k') \gamma_\mu u(k)$$

$$\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \, \text{tr}\left[(\not{p}' + m)\gamma^\mu (\not{p} + m)\gamma^\lambda\right] \, \text{tr}\left[(\not{k}' + M)\gamma_\mu (\not{k} + M)\gamma_\lambda\right]$$

now

$$\text{tr}\,(\not{k}' + M)\gamma^\mu(\not{k} + M)\gamma^\lambda$$

$$= 4\left[k'^\mu k^\lambda + k'^\lambda k^\mu - k \cdot k' g^{\mu\lambda} + M^2 g^{\mu\lambda}\right]$$

for $M \to \infty$ $\quad k \approx k' \approx (M, \vec{0})$ $\quad k \cdot k' \approx M^2$

$$= 8M^2 \delta^{\mu 0} \delta^{\lambda 0}$$

$$\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 \approx \frac{e^4}{4q^4} \, \text{tr}\left[(\not{p}' + m)\gamma^0 (\not{p} + m)\gamma^0\right] \cdot 8M^2$$

$$\approx \frac{2e^4 M^2}{q^4} \cdot 4E^2 (1 - \beta^2 \sin^2 \theta/2)$$

$$q^2 = -|\vec{q}|^2 = -4E^2 \beta^2 \sin^2 \theta/2 \qquad q^4 = 16 E^4 \beta^4 \sin^4 \theta/2$$

so $$\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 \approx \frac{e^4 M^2}{E^2 \beta^4 \sin^4 \theta/2} (1 - \beta^2 \sin^2 \theta/2)$$

In the CM frame $M \to \infty$

$$d\sigma = \frac{1}{2E \cdot 2M \cdot v} \frac{1}{8\pi} \frac{2p}{E_{cm}} \cdot \frac{d\Omega}{4\pi} \left(\frac{1}{4}\sum |M|^2\right)$$

$$E_{cm} \approx M$$

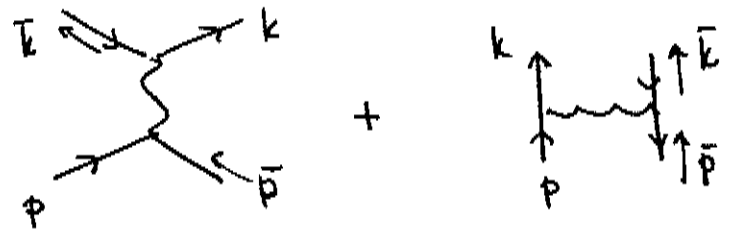
$$= \frac{1}{16\pi^2} \frac{1}{M^2} d\Omega \left(\frac{1}{4}\sum |M|^2\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2 \cdot 4M^2} \cdot \frac{e^4 M^2}{E^2 \beta^4 \sin^4 \theta/2} (1 - \beta^2 \sin^2 \theta/2)$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4E^2 \beta^4 \sin^2 \theta/2} (1 - \beta^2 \sin^2 \theta/2)$$

with $Z = 1$ in the case.

3.) The diagrams are:



 comes from $\langle \bar{k}k| \Phi\psi \quad \Phi\psi \quad |p\bar{p}\rangle \to +1$

 comes from $\langle \bar{k}k| \Phi\psi \quad \Phi\psi \quad |p\bar{p}\rangle \to -1$ from fermion interchange

Treating e^- e^+ masses

so

$$iM = (ie)^2 \left\{ \bar{u}(k)\gamma^\mu v(\bar{k}) \left(\frac{-i}{s}\right) \bar{v}(\bar{p})\gamma_\mu u(p) \right.$$

$$\left. - \bar{u}(k)\gamma^\mu u(p) \left(\frac{-i}{t}\right) \bar{v}(\bar{p})\gamma_\mu v(\bar{k}) \right\}$$

$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{e^4}{4}\sum_{\text{spin}} \left| \frac{1}{s}\bar{u}(k)\gamma^\mu v(\bar{k})\ \bar{v}(\bar{p})\gamma_\mu u(p) \right.$$

$$\left. - \frac{1}{t}\ \bar{u}(k)\gamma^\mu u(p)\ \bar{v}(\bar{p})\gamma_\mu v(\bar{k}) \right|^2$$

$$= \frac{e^4}{4}\left\{ \frac{1}{s^2}\,\mathrm{tr}[\not{k}\gamma^\mu \not{\bar{k}}\gamma^\nu]\,\mathrm{tr}[\not{\bar{k}}\gamma_\nu \not{\bar{p}}\gamma_\mu] + \frac{1}{t^2}\,\mathrm{tr}[\not{k}\gamma^\mu \not{\bar{p}}\gamma^\nu]\,\mathrm{tr}[\not{\bar{k}}\gamma_\nu \not{p}\gamma_\mu] \right.$$

$$- \frac{1}{st}\,\mathrm{tr}[\not{k}\gamma^\mu \not{\bar{k}}\gamma_\alpha \not{\bar{p}}\gamma_\nu \not{p}\gamma^\nu]$$

$$\left. - \frac{1}{st}\,\mathrm{tr}[\not{k}\gamma^\nu \not{\bar{p}}\gamma_\alpha \not{\bar{k}}\gamma_\mu \not{p}\gamma^\alpha] \right\}$$

hw

$$\mathrm{tr}[\not k \gamma^\mu \not{\bar k} \gamma^\lambda]\,\mathrm{tr}[\not{\bar p}\gamma_\nu \not p \gamma_\lambda]$$

$$= 4\cdot 4\,[k^\mu \bar k^\lambda + k^\lambda \bar k^\mu - k\cdot\bar k\, g^{\mu\lambda}]\,[\bar p_\nu p_\lambda + \bar p_\lambda p_\nu - p\cdot\bar p\, g_{\mu\nu}]$$

$$= 16\,[2kp\,\bar k\cdot\bar p + 2k\bar p\,\bar k\cdot p - \cancel{2k\cdot\bar k\,p\cdot\bar p\cdot 2} + \cancel{4k\cdot\bar k\,p\cdot\bar p}]$$

$$= 32\,[kp\,\bar k\bar p + k\bar p\,kp]$$

$$= 8\,[t^2 + u^2]$$

$t = -2kp = -2\bar k\cdot\bar p$

$u = -2k\bar p = -2\bar k\cdot p$

$\mathrm{tr}[\not k\gamma^\mu \not p \gamma^\lambda]\,\mathrm{tr}[\not{\bar p}\gamma_\nu \not{\bar k}\gamma_\lambda]$ $\qquad s = 2p\bar p = 2k\cdot\bar k$

$$= 32\,[k\bar p\; p\cdot\bar k + k\cdot\bar k\; p\cdot\bar p]$$

$$= 8\,[u^2 + s^2]$$

$$\mathrm{tr}[\not k\gamma^\mu \not{\bar k}\gamma_\lambda \not{\bar p}\gamma_\nu \not p \gamma^\lambda] = \mathrm{tr}[\gamma^\mu \not p \gamma_\lambda \not{\bar p}\gamma_\nu \not{\bar k}\gamma^\lambda \not k]$$

(reverse order + interchange $\mu \leftrightarrow \lambda$)

$$= \mathrm{tr}[(-2)\,\not k\gamma^\mu \not{\bar k} \not p \gamma_\nu \not{\bar p}]$$

$$= -2\cdot 4\;\mathrm{tr}\not k \not{\bar k} p \not{\bar p}$$

$$= -2\cdot 4\cdot 4\;\bar k\cdot p\; k\cdot\bar p$$

$$= -8\cdot u^2$$

$$\frac{1}{4}\sum_{spin} |\mathcal{M}|^2 = \frac{e^4}{4} \cdot 8 \left\{ \frac{u^2+t^2}{s^2} + \frac{u^2+s^2}{t^2} + 2\frac{u^2}{st} \right\}$$

$$= 2e^4 \left\{ u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + t^2\frac{1}{s^2} + s^2\frac{1}{t^2} \right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\,\frac{1}{16\pi}\, 2e^4 \{\ \}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2 \right]$$

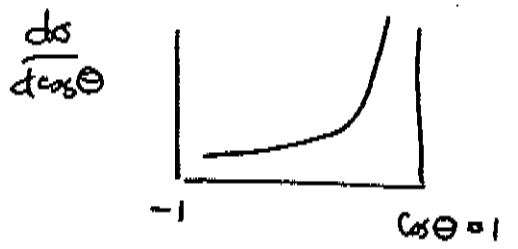
here $s = 4E^2 \quad t = -2E^2(1-\cos\theta) \quad u = -2E^2(1+\cos\theta)$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4E^2}\left[(1+\cos\theta)^2\left[\frac{1}{2} - \frac{1}{(1-\cos\theta)}\right]^2 + \left[\frac{(1-\cos\theta)}{2}\right]^2 + \left(\frac{2}{1-\cos\theta}\right)^2 \right]$$

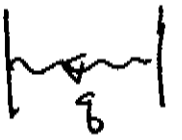
$$= \frac{\pi\alpha^2}{4E^2}\left\{ \frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2} - \frac{(1+\cos\theta)^2}{(1-\cos\theta)} + \frac{1}{4}\cdot(2+2\cos^2\theta) \right\}$$

$$= \frac{\pi\alpha^2}{4E^2}\left\{ \frac{5+2\cos\theta+\cos^2\theta}{(1-\cos\theta)^2} - \frac{(1+\cos\theta)^2}{(1-\cos\theta)} + \frac{1+\cos^2\theta}{2} \right\}$$

this cross section has a strong singularity as $\theta \to 0$



due to the massless vector exchange in the t-channel



4.) a.) Go back to p. 7:



$$= ie^2 \frac{1}{s} \bar{u}(k)\gamma^\mu v(\bar{k})\ \bar{v}(\bar{p})\gamma_\mu u(p)$$



$$= -ie^2 \frac{1}{t} \bar{u}(k)\gamma^\mu u(p)\ \bar{v}(\bar{p})\gamma_\mu v(\bar{k})$$

Now

$$\bar{u}(k)\gamma^\mu u(p) = u^\dagger(k)\gamma^0\gamma^\mu u(p) = u^\dagger(k)\begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} u(p)$$

so this $= 0$ unless both $e(p)$ and $e^-(k)$ are L or R

$$\bar{u}(k)\gamma^\mu v(\bar{k}) = u^\dagger(k)\begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} v(\bar{k})$$

this $= 0$ unless $e^-(k)$ is L, $e^+(\bar{k})$ is R or vice versa

so  is nonzero for

$e_L^- e_R^+ \to e_L^- e_R^+$
$e_L^- e_R^+ \to e_R^- e_L^+$
$e_R^- e_L^+ \to e_L^- e_R^+$
$e_R^- e_L^+ \to e_R^- e_L^+$

only

 is nonzero for

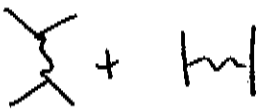
$e_L^- e_L^+ \to e_L^- e_L^+$
$e_L^- e_R^+ \to e_L^- e_R^+$
$e_R^- e_L^+ \to e_R^- e_L^+$
$e_R^- e_R^+ \to e_R^- e_R^+$

only

so

$e_L^- e_R^+ \to e_R^- e_L^+$
$e_R^- e_L^+ \to e_L^- e_R^+$
} are given by  only

$e_L^- e_L^+ \to e_L^- e_L^+$
$e_R^- e_R^+ \to e_R^- e_R^+$
} are given by  only

$\bar{e}_L e_R^+ \to \bar{e}_L e_R^+$
$e_R^- e_L^+ \to e_R^- e_L^+$
} are given by 

Now compute the nonzero spinor products using the explicit spinors in the problem set:

$\bar{u}(\vec{p})\gamma^\mu v(p) = (e_L^- e_R^+)\ 2E \cdot (-1\ 0)\ \underset{\sim}{\bar{\sigma}^\mu}\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= 2E\ (0, 1, -i, 0)^\mu$

$= (e_R^- e_L^+)\ 2E\ (0\ -1)\ (1, \vec{\sigma})^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= -2E\ (0\ 1, i, 0)^\mu$

$$\bar{u}(k)\gamma^\mu v(\bar{k}) = (\bar{e_L} e_R^+) \quad 2E\ (-s_2\ c_2)(1, -\vec{\sigma})\begin{pmatrix} -c_2 \\ -s_2 \end{pmatrix}$$

$$= 2E\,(0,\ c_2^2 - s_2^2,\ i(c_2^2+s_2^2),\ -2c_2s_2)$$

$$= 2E\,(0,\ \cos\theta,\ i,\ -\sin\theta)$$

$$= (\bar{e_R} e_L^+) \quad 2E\,(c_2\ s_2)(1, \vec{\sigma})\begin{pmatrix} s_2 \\ -c_2 \end{pmatrix}$$

$$= -2E\,(0,\ c_2^2 - s_2^2,\ -i(c_2^2+s_2^2),\ -2c_2s_2)$$

$$= -2E\,(0, \cos\theta, -i, -\sin\theta)$$

$$\bar{u}(k)\gamma^\mu u(p) = (\bar{e_L} \to \bar{e_L}) \quad 2E\ (-s_2 c_2)(1, -\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 2E\,(+c_2,\ +s_2,\ -is_2,\ c_2)$$

$$(\bar{e_R} \to \bar{e_R}) \quad 2E\,(c_2\ s_2)(1, \vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 2E\,(c_2,\ s_2,\ is_2,\ c_2)$$

$$\bar{v}(\bar{p})\gamma^\mu v(\bar{k}) = (e_L^+ \to e_L^+) \quad 2E\,(0\ -1)(1, +\vec{\sigma})\begin{pmatrix} s_2 \\ -c_2 \end{pmatrix}$$

$$= 2E\,(c_2,\ -s_2,\ -is_2,\ -c_2)$$

$$(e_R^+ \to e_R^+) \quad 2E\,(-1\ 0)(1, -\vec{\sigma})\begin{pmatrix} -c_2 \\ -s_2 \end{pmatrix}$$

$$= 2E\,(c_2, -s_2, +is_2, -c_2)$$

once again

	$e_L^- e_R^+$	$e_R^- e_L^+$
$\bar{v}(\bar{p})\,\gamma^\mu u(p)$	$2E\,(0\ 1\ -i\ 0)$	$-2E\,(0\ 1\ \ i\ 0)$
$\bar{u}(k)\,\gamma^\mu v(\bar{k})$	$2E\,(0\ \cos\theta\ +i\ -\sin\theta)$	$-2E\,(0\ \cos\theta\ -i\ -\sin\theta)$
	$e_L^- \to e_L^-$	$e_R^- \to e_R^-$
$\bar{u}(k)\,\gamma^\mu u(p)$	$2E\,(c_2\ s_2\ -is_2\ c_2)$	$2E\,(c_2\ s_2\ \ is_2\ c_2)$
$\bar{v}(\bar{p})\,\gamma^\mu v(\bar{k})$	$2E\,(c_2\ -s_2\ -is_2\ -c_2)$	$2E\,(c_2\ -s_2\ +is_2\ -c_2)$
	$e_L^+ \to e_L^+$	$e_R^+ \to e_R^+$

then: cbt w. Lorentz product



$$= \frac{ie^2}{s}\cdot(2E)^2 \begin{cases} -(\cos\theta+1) & e_L^- e_R^+ \to e_L^- e_R^+ \\ +(\cos\theta-1) & e_L^- e_R^+ \to e_R^- e_L^+ \\ +(\cos\theta-1) & e_R^- e_L^+ \to e_L^- e_R^+ \\ -(\cos\theta+1) & e_R^- e_L^+ \to e_R^- e_L^+ \end{cases}$$



$$= -\frac{ie^2}{t}(2E)^2 \begin{cases} 2c_2^2 + 2s_2^2 = 2 & e_L^- e_L^+ \to e_L^- e_L^+ \\ 2c_2^2 = (1+\cos\theta) & e_L^- e_R^+ \to e_L^- e_R^+ \\ 2c_2^2 = (1+\cos\theta) & e_R^- e_L^+ \to e_R^- e_L^+ \\ 2c_2^2 + 2s_2^2 = 2 & e_R^- e_R^+ \to e_R^- e_R^+ \end{cases}$$

& recognize

$$s = 4E^2$$
$$t = -2E^2(1-\cos\theta)$$
$$u = -2E^2(1+\cos\theta)$$

so

$e_L^- e_L^+ \to e_R^- e_R^+$ $= -2ie^2 \frac{s}{t}$

$e_R^- e_L^+ \to e_R^- e_L^+$

$e_L^- e_R^+ \to e_R^- e_L^+$  $= +2ie^2 \frac{t}{s}$

$e_R^- e_L^+ \to e_L^- e_R^+$

$e_L^- e_R^+ \to e_R^- e_R^+$  $+$ $= -2ie^2 u \left(\frac{1}{s} + \frac{1}{t}\right)$

$e_R^- e_L^+ \to e_R^- e_L^+$

then the spin-averaged $\frac{d\sigma}{d\cos\Theta}$ is

$$\frac{d\sigma}{d\cos\Theta} = \frac{1}{2s} \cdot \frac{1}{16\pi} \cdot \frac{1}{4} \underbrace{\sum |m|^2}$$

$$4e^4 \cdot 2 \cdot \left[\left(\frac{s}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2\right]$$

$$= \frac{\pi\alpha^2}{s}\left[\left(\frac{s}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2\right]$$

Physics 330 – Problem Set # 7

(due Wednesday, November 16)

1. Carry out all steps of the Feynman diagram calculation of the Compton scattering cross section, working from the diagrams at the beginning of P & S Section 5.5. In particular, include all steps in the derivation of eq. (5.87). Write the final result as a polarization-averaged cross section $d\sigma/d\cos\theta$ in the CM frame. Take the limit $m/E \to 0$.

2. Just as for Bhabha scattering, we can build up the cross section for Compton scattering as a sum of squares of helicity amplitudes. To do this calculation in the limit $m_e \to 0$, we need the polarization spinors

$$\begin{aligned} e^- \text{ with } \vec{p} \parallel \hat{3} \ : \qquad & u_R(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ e^- \text{ in } \theta \text{ direction} : \qquad & u_R(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ c_2 \\ s_2 \end{pmatrix} \quad u_L(p) = \sqrt{2E}\begin{pmatrix} -s_2 \\ c_2 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{1}$$

   and the following polarization vectors for initial and final photons:

$$\begin{aligned} \gamma \text{ with } \vec{p} \parallel -\hat{3} \ : \qquad & \epsilon_R^\mu(p) = \frac{1}{\sqrt{2}}(0,-1,i,0)^\mu \qquad \epsilon_L^\mu(p) = \frac{1}{\sqrt{2}}(0,-1,-i,0)^\mu \\ \gamma \text{ opposite to } \theta : \qquad & \epsilon_R^\mu(p) = \frac{1}{\sqrt{2}}(0,-c,i,s)^\mu \qquad \epsilon_L^\mu(p) = \frac{1}{\sqrt{2}}(0,-c,-i,s)^\mu \end{aligned} \tag{2}$$

   where $c = \cos\theta$, $s = \sin\theta$, and you should remember to use $\epsilon_\mu^*(p)$ for final-state photons.

   (a) Show that, for zero electron mass, only those helicity amplitudes are nonzero for which the e^- has the same helicity in the initial and final states. This reduces the number of nonzero helicity amplitudes to 8.

   (b) By explicit computation, show that 4 of these helicity amplitudes vanish and the other 4 are equal in pairs.

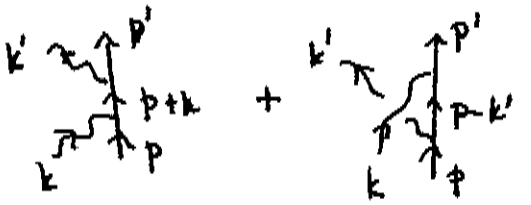
   (c) Show that the sum of the two different nonzero contributions reproduces the final result of Problem 1.

3. Peskin and Schroeder, problem 5.5.

Physics 330 - Problem Set #7

Solutions

1.)



$$i\mathcal{M} = (-ie)^2\,\bar{u}(p')\left\{\gamma^\mu \varepsilon^*_\mu(k')\,\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\,\gamma^\nu \varepsilon_\nu(k)\right.$$

$$\left. + \gamma^\nu \varepsilon_\nu(k)\,\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}\,\gamma^\mu \varepsilon^*_\mu(k')\right\} u(p)$$

now $$(\not{p}+m)\,\gamma^\nu\, u(p) = [2p^\nu - \gamma^\nu(\not{p}-m)]\,u(p)$$

$$= 2p^\nu\, u(p)$$

$$(p+k)^2 - m^2 = m^2 + 2p\cdot k + k^2 - m^2 = 2p\cdot k$$

$$(p-k')^2 - m^2 = m^2 - 2p\cdot k' + k'^2 - m^2 = -2p\cdot k'$$

kinematic relations are:

$$s = (p+k)^2 = (p'+k')^2 \Rightarrow \cancel{p^2} + 2p\cdot k = \cancel{p'^2} + 2p'\cdot k'$$

$$t = (p-p')^2 = (k-k')^2 \Rightarrow 2m^2 - 2p\cdot p' = -2k\cdot k'$$

$$u = (p-k')^2 = (k-p')^2 \Rightarrow m^2 - 2p\cdot k' = m^2 - 2k\cdot p$$

so $$p\cdot k = p'\cdot k' \qquad p\cdot k' = p'\cdot k \qquad p\cdot p' = m^2 + k\cdot k'$$

$$p\cdot(k'+p'-k) = p^2 = m^2 \Rightarrow p\cdot p' = m^2 + p\cdot k - p\cdot k'$$

I will try to systematically eliminate $p'\cdot k'$, $p'\cdot k$, $k\cdot k'$ in favor of $p\cdot k$, $p\cdot k'$, $p\cdot p'$

$$iM = -ie^2\, \varepsilon^{*}_{\mu}(k')\,\varepsilon_{\nu}(k) \cdot \bar{u}(p')\left[\frac{2\gamma^{\mu}p^{\nu} + \gamma^{\mu}\not{k}\gamma^{\nu}}{2p\cdot k} - \frac{2\gamma^{\nu}p^{\mu} - \gamma^{\nu}\not{k}'\gamma^{\mu}}{2p\cdot k'}\right]u(p)$$

then

$$\frac{1}{4}\sum_{\text{spins}}|M|^2 = \frac{e^4}{4}\,\mathrm{tr}\left[(\not{p}'+m)\left\{\frac{2\gamma^{\mu}p^{\nu}+\gamma^{\mu}\not{k}\gamma^{\nu}}{2p\cdot k} + \frac{-2\gamma^{\nu}p^{\mu}+\gamma^{\nu}\not{k}'\gamma^{\mu}}{2p\cdot k'}\right\}\cdot(\not{p}+m)\left\{\frac{2\gamma_{\mu}p_{\nu}+\gamma_{\nu}\not{k}\gamma_{\mu}}{2p\cdot k} + \frac{-2\gamma_{\nu}p_{\mu}+\gamma_{\mu}\not{k}'\gamma_{\nu}}{2p\cdot k'}\right\}\right]$$

$$= \frac{e^4}{4}\left[\frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{2p\cdot k\, 2p\cdot k'} + \frac{\mathrm{III}}{2p\cdot k'\, 2p\cdot k} + \frac{\mathrm{IV}}{(2p\cdot k')^2}\right]$$

now compute I, II, III, IV in turn.

$$\mathrm{I} = \mathrm{tr}\,(\not{p}'+m)(2\gamma^{\mu}p^{\nu}+\gamma^{\mu}\not{k}\gamma^{\nu})(\not{p}+m)(2\gamma_{\mu}p_{\nu}+\gamma_{\nu}\not{k}\gamma_{\mu})$$

$$= \mathrm{tr}\,(\not{p}'+m)\,4\gamma^{\mu}(\not{p}+m)\gamma_{\mu}\,p^2$$

$$+ \mathrm{tr}\,(\not{p}'+m)\cdot 2\gamma^{\mu}(\not{p}+m)\not{p}\not{k}\gamma_{\mu}$$

$$+ \mathrm{tr}\,(\not{p}'+m)\gamma^{\mu}\not{k}\not{p}(\not{p}+m)\gamma_{\mu}\cdot 2$$

$$+ \mathrm{tr}\,(\not{p}'+m)\gamma^{\mu}\not{k}\gamma^{\nu}(\not{p}+m)\gamma_{\nu}\not{k}\gamma_{\mu}$$

Here

$$\text{line 1} = 4m^2\, \mathrm{tr}\,(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma_\mu$$

$$= 4m^2\, \mathrm{tr}\,(\not{p}+m)(-2\not{p}+4m)$$

$$= 4\cdot(-2)\cdot 4\, m^2\, p'\cdot p + 4\cdot 4\cdot 4 m^4$$

$$= -32\, p'\cdot p\, m^2 + 64 m^4$$

$$\text{line 2} = 2\,\mathrm{tr}\,(\not{p}'+m)\gamma^\mu(\not{p}+m)\not{p}\not{k}\gamma_\mu$$

$$= 2\,\mathrm{tr}\,(\not{p}+m)\gamma^\mu(m^2+m\not{p})\not{k}\gamma_\mu$$

$$= 2\,\mathrm{tr}\,(\not{p}'+m)(-2\not{k}m^2 + 4p\cdot k\, m)$$

$$= -4\cdot 4\, m^2(p'\cdot k) + 2\cdot 4\cdot 4\, m^2\, p\cdot k$$

$$= -16 m^2(p\cdot k') + 32 m^2\, p\cdot k$$

$$\text{line 3} = 2\,\mathrm{tr}\,(\not{p}'+m)\gamma^\mu\not{k}\not{p}(\not{p}+m)\gamma_\mu$$

$$= 2\,\mathrm{tr}\,(\not{p}+m)\gamma^\mu\not{k}(m^2+m\not{p})\gamma_\mu$$

$$= 2\,\mathrm{tr}\,(\not{p}'+m)(-2\not{k}m^2 + 4p\cdot k\, m)$$

$$= -4\cdot 4(p'\cdot k)m^2 + 2\cdot 4\cdot 4\, m^2\, p\cdot k$$

$$= -16\, p\cdot k'\, m^2 + 32 m^2\, p\cdot k$$

I have used my trick : $\mathrm{tr}[\text{odd}\#\, q\gamma_\rho] = 0$

$$\text{line 4} = \mathrm{tr}(\not{p}'+m)\gamma^\mu \not{k} \gamma^\nu (\not{p}+m)\gamma_\nu \not{k} \gamma_\mu$$

$$= +\mathrm{tr}\,\gamma_\mu(\not{p}+m)\gamma^\mu \not{k}\gamma^\nu(\not{p}+m)\gamma_\nu \not{k}$$

$$= \mathrm{tr}(-2\not{p}'+4m)\not{k}(-2\not{p}+4m)\not{k}$$

$$= 4\cdot(-2)^2[p'\!\cdot\! k\; p\!\cdot\! k\cdot 2 - p'\!\cdot\! p\underbrace{k^2}_{=0}] + 4\cdot 4\cdot 4m^2\underbrace{k^2}_{=0}$$

$$= 32\, p\!\cdot\! k'\, p\!\cdot\! k$$

$\Rightarrow$ all

$$I = -32\, p'\!\cdot\! p\, m^2 + 64m^4 + 2(-16m^2\, p\!\cdot\! k' + 32m^2\, p\!\cdot\! k) + 32\, p\!\cdot\! k\, p\!\cdot\! k'$$

$$= -32m^2(m^2 + p\!\cdot\! k - p\!\cdot\! k') + 64m^4 - 32\,p\!\cdot\! k'\,m^2 + 64m^2\, p\!\cdot\! k + 32\, p\!\cdot\! k\, p\!\cdot\! k'$$

$$= 32m^4 + 32m^2\, p\!\cdot\! k + 32\, p\!\cdot\! k\, p\!\cdot\! k'$$

In eq. (5.84) of P+S it is claimed:

$$I = 16\left(2m^4 + m^2(s-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\right)$$

$$= 32m^4 + 16m^2\cdot 2p\!\cdot\! k - \tfrac{1}{2}\cdot 16\cdot(2p\!\cdot\! k)(-2p\!\cdot\! k')$$

$$= 32m^4 + 32m^2 p\!\cdot\! k + 32\, p\!\cdot\! k\, p\!\cdot\! k' \quad \checkmark$$

$$\text{IV} = \mathrm{tr}(\not p' + m)[2\gamma^\nu p^\mu - \gamma^\nu \not k' \gamma^\mu](\not p + m)[2\gamma_\nu p_\mu - \gamma_\mu \not k' \gamma_\nu]$$

which is I with $\mu \leftrightarrow \nu$, $k \leftrightarrow -k'$
so we can write the answer.

$$\text{IV} = 32[m^4 - m^2 p\cdot k' + p\cdot k\, p\cdot k']$$

$$\text{II} = \mathrm{tr}(\not p' + m)(2\gamma^\mu p^\nu + \gamma^\mu \not k \gamma^\nu)(\not p + m)(-2\gamma_\nu p_\mu + \gamma_\mu \not k' \gamma_\nu)$$

$$\text{III} = \mathrm{tr}(\not p' + m)(-2\gamma_\nu p_\mu + \gamma_\nu \not k' \gamma_\mu)(\not p + m)(2\gamma^\mu p^\nu + \gamma^\nu \not k \gamma^\mu)$$

$$= \mathrm{tr}(-2\gamma_\nu p_\mu + \gamma_\nu \not k' \gamma_\mu)(\not p + m)(-2\gamma_\nu p_\mu + \gamma_\nu \not k' \gamma_\mu)(\not p' + m)$$

$$= \text{II} \text{ w. order of } \gamma\text{'s in the trace reversed.}$$

$$= \text{II} \qquad \text{by eq. (5.7)}$$

$$\text{II} = \mathrm{tr}(\not p' + m)\, 2\gamma^\mu p^\nu (\not p + m)(-2\gamma_\nu p_\mu)$$
$$+ \mathrm{tr}(\not p' + m)\, 2\gamma^\mu p^\nu (\not p + m)(\gamma_\mu \not k' \gamma_\nu)$$
$$+ \mathrm{tr}(\not p' + m)\, \gamma^\mu \not k \gamma^\nu (\not p + m)(-2\gamma_\nu p_\mu)$$
$$+ \mathrm{tr}(\not p + m)\gamma^\mu \not k \gamma^\nu (\not p + m)\gamma_\mu \not k' \gamma_\nu$$

$$\text{line 1} = -4\,\mathrm{tr}(\not p' + m)\not p (\not p + m)\not p$$

$$= -4\,\mathrm{tr}(\not p' + m)(m^2 \not p + m^3)$$

$$= -4 \cdot 4(p\cdot p' m^2 + m^4)$$

$$= -16(p\cdot p' m^2 + m^4)$$

$$\text{line 2} = 2\,\mathrm{tr}(\not p' + m)\gamma^\mu(\not p + m)\gamma_\mu \not k' \not p$$

$$= 2\,\mathrm{tr}(\not p + m)(-2\not p + 4m)\not k' \not p$$

$$= -4\cdot 4\,[p'\cdot p\; k'\cdot p\cdot 2 - p'\cdot k' m^2] + 2\cdot 4\cdot 4 m^2 k'\cdot p$$

$$= -32\, p'\cdot p\; k'\cdot p + 16 m^2 p\cdot k + 32 m^2 p\cdot k'$$

$$\text{line 3} = -2\,\mathrm{tr}(\not p' + m)\not p \not k \gamma^\nu(\not p + m)\gamma_\nu$$

$$= -2\,\mathrm{tr}(\not p' + m)\not p \not k(-2\not p + 4m)$$

$$= 4\cdot 4\,[p'\cdot p\; k\cdot p\cdot 2 - p'\cdot k\, m^2] - 2\cdot 4\cdot 4 m^2 p\cdot k$$

$$= 32\, p'\cdot p\; k\cdot p - 16 m^2 p'\cdot k - 32 m^2 p\cdot k$$

$$\text{line 4} = \mathrm{tr}(\not p' + m)\gamma^\mu \not k \gamma^\nu(\not p + m)\gamma_\mu \not k' \gamma_\nu$$

$$= \mathrm{tr}(\not p + m)\gamma^\mu \not k\,(-2k'\gamma_\mu \not p + 4m k'_\mu)$$

$$= -2\cdot 4\, k\cdot k'\,\mathrm{tr}(\not p' + m)\not p + 4m^2\,\mathrm{tr}\,\not k' \not k$$

$$= -8\, k\cdot k'\cdot 4\, p\cdot p' + 4\cdot 4 m^2 k\cdot k'$$

$$= -32\, k\cdot k'\, p\cdot p' + 16 m^2 k\cdot k'$$

~ all

$$\overline{|\mathcal{M}|^2} = -16m^4 - 16 p\cdot p' m^2 - 32 p'\cdot p\; p\cdot k' + 16 m^2 p\cdot k + 32 m^2 p\cdot k'$$
$$+ 32 p'\cdot p\; k\cdot p - 16 m^2 p\cdot k' - 32 m^2 p\cdot k$$
$$- 32 p\cdot p'\; k\cdot k' + 16 m^2 k\cdot k'$$

$$= 16 p\cdot p'\{-m^2 - 2p\cdot k' + 2p\cdot k - 2k\cdot k'\}$$

$$+ 16\{-m^4 - m^2 p\cdot k + m^2 p\cdot k' + m^2 k\cdot k'\}$$

$$= 16\, p\cdot p' \{ -m^2 - 2\underbrace{(p+k)}_{p'+k'}\cdot k' + 2p\cdot k \}$$

$$+ 16 \{ -m^4 - m^2 p\cdot k + m^2 p\cdot k' + m^2 (p\cdot p' - m^2) \}$$

$$= 16\, p\cdot p' \{ -m^2 - 2p'\cdot k' + 2p\cdot k \} + 16m^2\, p\cdot p'$$

$$- 32 m^4 - 16m^2 p\cdot k + 16m^2 p\cdot k'$$

$$\text{II} = -32m^4 - 16m^2 p\cdot k + 16m^2 p\cdot k'$$

this does agree with $\text{II} = -8(4m^4 + m^2(s-m^2) + m^2(u-m^2))$

in all

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4}\left[\frac{32}{4}\right]$$

$$\cdot \left\{ \frac{1}{(p\cdot k)^2}(p\cdot k\, p\cdot k' + m^2 p\cdot k + m^4) \right.$$

$$+ \frac{1}{(p\cdot k')^2}(p\cdot k\, p\cdot k' - m^2 p\cdot k' + m^4)$$

$$\left. + \frac{1}{(p\cdot k)(p\cdot k')}(-2m^4 - m^2 p\cdot k + m^2 p\cdot k') \right\}$$

$$= 2e^4 \left\{ \frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) \right.$$

$$\left. + m^4\left(\frac{1}{(p\cdot k)} - \frac{1}{(p\cdot k')}\right)^2 \right\}$$

– the CM frame

$p = (E, 0 \; 0 \; p)$ $\qquad k = (p \; 0 \; 0 \; -p)$

$p' = (E, p\sin\theta, 0 \; p\cos\theta)$ $\qquad k' = (p, -p\sin\theta, 0, -p\cos\theta)$

$p\cdot k = p(E+p)$ $\qquad E+p = E_{cm}$

$p\cdot k' = p(E+p\cos\theta)$ $\qquad p = \dfrac{s-m_e^2}{2\sqrt{s}}$

$$\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} = \frac{1}{p}\left(\frac{1}{E+p} - \frac{1}{E+p\cos\theta}\right)$$

$$= -\frac{1}{p}\,\frac{p\cos\theta}{(E+p)(E+p\cos\theta)}$$

$$= -\frac{\cos\theta}{(E+p)(E+p\cos\theta)}$$

so

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\left\{\left(\frac{E+p\cos\theta}{E+p} + \frac{E+p}{E+p\cos\theta}\right) - 2m^2\,\frac{\cos\theta}{(E+p)(E+p\cos\theta)} + \frac{m^4\cos^2\theta}{(E+p)^2\,(E+p\cos\theta)^2}\right\}$$

$$d\Pi = \frac{1}{16\pi}\,d\cos\theta\cdot\left(\frac{2p}{E+p}\right)$$

then

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{4Ep\left(\frac{p}{E}+1\right)} \cdot \frac{1}{16\pi} \cdot \frac{2p}{(E+p)} \cdot 2e^4 \left\{ \cdots \right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{E_{cm}^2} \left\{ \frac{2E^2 + 2Ep(1+\cos\theta) + p^2(1+\cos^2\theta)}{(E+p)(E+p\cos\theta)} \right.$$

$$\left. - \frac{2m^2\cos\theta}{(E+p)(E+p\cos\theta)} + \frac{m^4\cos^2\theta}{(E+p)^2(E+p\cos\theta)^2} \right\}$$

for $E \gg m$

$$\frac{d\sigma}{d\cos\theta} \sim \frac{\pi\alpha^2}{4E^2} \cdot \frac{5 + 2\cos\theta + \cos^2\theta}{2(1+\cos\theta)}$$

2.) a.) In the limit of $m_e \to 0$

$$i\mathcal{M} = -ie^2\, \bar{u}(p') \left\{ \gamma\cdot\varepsilon^*(k') \frac{\not{p}+\not{k}}{2p\cdot k} \gamma\cdot\varepsilon(k) + \gamma\cdot\varepsilon(k) \frac{\not{p}-\not{k}'}{-2p\cdot k'} \gamma\cdot\varepsilon^*(k') \right\} u(p)$$

Note that

$$\bar{u}(p')\gamma^\mu\gamma^\nu\gamma^\lambda u(p)$$

$$= u^\dagger(p') \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\lambda \\ \bar{\sigma}^\lambda & 0 \end{pmatrix} u(p)$$

$$= u^\dagger(p') \begin{bmatrix} \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\lambda & 0 \\ 0 & \sigma^\mu\bar{\sigma}^\nu\sigma^\lambda \end{bmatrix} u(p)$$

so Left-handed spinor $u_L(p) = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$ and right-handed spinor $u_R(p) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$ do not mix.

so the possible nonzero helicity amplitudes are

$e^-_L \gamma_L \to e^-_L \gamma_L$ $\quad$ $e^-_R \gamma_L \to e^-_R \gamma_L$

$e^-_L \gamma_L \to e^-_L \gamma_R$ $\quad$ $e^-_R \gamma_L \to e^-_R \gamma_R$

$e^-_L \gamma_R \to e^-_R \gamma_L$ $\quad$ $e^-_R \gamma_R \to e^-_R \gamma_L$

$e^-_L \gamma_R \to e^-_L \gamma_R$ $\quad$ $e^-_R \gamma_R \to e^-_R \gamma_R$

first compute $e_L^- \gamma \to e_L^- \gamma$ amplitudes. Then



$$= -ie^2 \sqrt{2E}\,(-s_2 c_2)\; \bar\sigma\cdot\varepsilon^*(k')\; \frac{\sigma\cdot(p+k)}{2p\cdot k}\; \bar\sigma\cdot\varepsilon(k) \begin{pmatrix}0\\1\end{pmatrix} \sqrt{2E}$$



$$= +ie^2 \sqrt{2E}\,(-s_2 c_2)\; \bar\sigma\cdot\varepsilon(k)\; \frac{\sigma\cdot(p-k')}{2p\cdot k'}\; \bar\sigma\cdot\varepsilon^*(k') \begin{pmatrix}0\\1\end{pmatrix} \sqrt{2E}$$

$s_2 = \sin\theta/2 \quad c_2 = \cos\theta/2$

now $\quad 2p\cdot k = 4E^2 \qquad 2p\cdot k' = 2E^2(1+\cos\theta)$

so $\qquad s = \sin\theta, \quad c = \cos\theta$



$$= -ie^2\, \frac{1}{2E}\, (-s_2 c_2)\; \bar\sigma\cdot\varepsilon^*(k')\, \sigma\cdot(p+k)\; \bar\sigma\cdot\varepsilon(k) \begin{pmatrix}0\\1\end{pmatrix}$$



$$= +ie^2\, \frac{1}{E(1+c)}\, (-s_2 c_2)\; \bar\sigma\cdot\varepsilon(k)\; \sigma\cdot(p-k')\; \bar\sigma\cdot\varepsilon^*(k') \begin{pmatrix}0\\1\end{pmatrix}$$

$p+k = (2E, \vec 0)$ so $\quad \sigma\cdot(p+k) = 2E\cdot \underline{1}$

$p-k' = (E\ 0\ 0\ E) - (E, -Es, 0, -Ec) = (0, +Es, 0, E(1+c))$

so $\qquad \sigma\cdot(p-k') = -E \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix}$

so



$$= -ie^2\, (-s_2 c_2)\; \bar\sigma\cdot\varepsilon^*(k')\; \underline{1}\; \bar\sigma\cdot\varepsilon(k) \begin{pmatrix}0\\1\end{pmatrix}$$



$$= \frac{-ie^2}{(1+c)}\, (-s_2 c_2)\; \bar\sigma\cdot\varepsilon(k) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \bar\sigma\cdot\varepsilon^*(k') \begin{pmatrix}0\\1\end{pmatrix}$$

forall we need

$$\bar{\sigma}\cdot\varepsilon(k) = \quad L: \quad \frac{1}{\sqrt{2}}(-\sigma^1 - i\sigma^2) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = -\sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R: \quad \frac{1}{\sqrt{2}}(-\sigma^1 + i\sigma^2) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = -\sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\bar{\sigma}\cdot\varepsilon^*(k') = \quad L: \quad \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -c \\ i & 0 \end{pmatrix} + s\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} s & 1-c \\ -(1+c) & -s \end{pmatrix}$$

$$R: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -c \\ i & 0 \end{pmatrix} + s\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} s & -(1+c) \\ (1-c) & -s \end{pmatrix}$$

again:

$\bar{\sigma}\cdot\varepsilon_L(k)$	$\bar{\sigma}\cdot\varepsilon_R(k)$	$\bar{\sigma}\cdot\varepsilon^*_L(k')$	$\bar{\sigma}\cdot\varepsilon^+_R(k')$
$-\sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$-\sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} s & 1-c \\ -(1+c) & -s \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} s & -(1+c) \\ 1-c & -s \end{pmatrix}$

Begin with } . note that $\bar{\sigma}\cdot\varepsilon_R(k)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ so

$$\mathcal{M}(\bar{e}_L\gamma_R \to \bar{e}_L\gamma_{L\text{or}R}) = 0$$

$$i\mathcal{M}(\bar{e}_L\gamma_L \to \bar{e}_L\gamma_L) = -ie^2(-s_2c_2)\frac{1}{\sqrt{2}}\begin{pmatrix} s & (1-c) \\ -(1+c) & -s \end{pmatrix}(-\sqrt{2})\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= ie^2(-s_2c_2)\begin{pmatrix} s & (1-c) \\ -(1+c) & -s \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= ie^2[-s_2s - c_2(1+c)]$$

$$= +ie^2\left[-\cos\theta/2 - (\cos\Theta\cos\theta/2 + \sin\Theta\sin\theta/2)\right]$$

$$= -2ie^2\cos\theta/2$$

$$i\mathcal{M}(e_L^- \gamma_L \to e_L^- \gamma_R) = -ie^2 (-s_2 c_2) \frac{1}{\sqrt{2}} \begin{pmatrix} s & -(1+c) \\ 1-c & -s \end{pmatrix} (-\sqrt{2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= ie^2 (-s_2 c_2) \begin{pmatrix} s & -(1+c) \\ 1-c & -s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= ie^2 (-s_2 s + c_2 - c_2 c)$$

$$= ie^2 (c_2 - (c_2 c + s_2 s)) = 0$$

so for



$$i\mathcal{M}(e_L \gamma_L \to e_L \gamma_L) = -2ie^2 c_2$$

all other ~~helicity~~ amplitudes are 0.

for  :

$$e_L^- \gamma_L \to e_L^- \gamma_L : = \frac{-ie^2}{1+c} (-s_2 c_2)(-\sqrt{2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} s & 1-c \\ -(1+c) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{+ie^2}{1+c} (0, -s_2) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix}$$

$$= \frac{+ie^2}{1+c} [-s_2 s(1-c) - s_2 s(1+c)]$$

$$= \frac{-2ie^2}{1+c} s_2 s$$

$e_L^- \gamma_L \to e_L^- \gamma_R$:

$$\frac{-ie^2}{1+c}(-s_2 c_2)\begin{pmatrix}0&1\\0&0\end{pmatrix}(-\sqrt{2})\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{ie^2}{1+c}(0\ -s_2)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}-(1+c)\\ -s\end{pmatrix}$$

$$= \frac{ie^2}{1+c}\left[+s_2 s(1+c) - s_2 s(1+c)\right] = 0$$

$e_L^- \gamma_R \to e_L^- \gamma_L$

$$\frac{-ie^2}{1+c}(-s_2 c_2)\,-\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{+ie^2}{1+c}(c_2, 0)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}1-c\\ -s\end{pmatrix}$$

$$= \frac{+ie^2}{1+c}\left[c_2\underbrace{(1+c)(1-c)}_{s^2\theta} - c_2 s^2\right] = 0$$

$e_L^- \gamma_R \to e_L^- \gamma_R$

$$\frac{-ie^2}{1+c}(-s_2 c_2)(-\sqrt{2})\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{+ie^2}{1+c}(+c_2, 0)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}-(1+c)\\ -s\end{pmatrix}$$

$$= \frac{ie^2}{1+c}\left[-c_2(1+c)^2 - c_2 s^2\right]$$

$$= \frac{ie^2}{1+c}\left[-c_2(1+c) - c_2(1-c)\right](1+c)$$

$$= -2ie^2 c_2$$

now add the contributions



$e_L^- \gamma_L \to e_L^- \gamma_L \qquad -2ie^2 \left[c_2 + \frac{ss_2}{(1+c_2)} \right]$

$$= -2ie^2 \frac{c_2 + (c_2 c + s_2 s)}{(1+c)}$$

$$= -4ie^2 \frac{c_2}{1+c}$$

$$= -2ie^2 \frac{1}{c_2}$$

$c c_2 + s s_2 = \cos\theta/2 = c_2$

$c_2^2 = \frac{1+c}{2}$

$e_L^- \gamma_L \to e_L^- \gamma_R \qquad = 0$

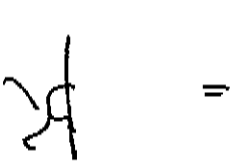
$e_L^- \gamma_R \to e_R^- \gamma_L \qquad = 0$

$e_L^- \gamma_R \to e_L^- \gamma_R \qquad = 0 - 2ie^2 c_2$

$$= -2ie^2 c_2$$

The amplitudes for $e_R^- \gamma \to e_R^- \gamma$ should follow the same pattern. The basic ingredients are:

$$= -ie^2 \sqrt{2E}\,(c_2 s_2)\, \sigma\cdot\varepsilon^*(k') \frac{\bar{\sigma}\cdot p + k}{2p\cdot k} \sigma\cdot\varepsilon(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(\sqrt{2E}\right)^*$$



$$= +ie^2 \sqrt{2E}\,(c_2 s_2)\, \sigma\cdot\varepsilon(k) \frac{\bar{\sigma}\cdot p - k'}{2p\cdot k'} \sigma\cdot\varepsilon^*(k') \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sqrt{2E}$$

now $\sigma\cdot(p+k) = 2E\cdot\underline{1}$ $\quad \sigma\cdot(p-k') = +E\begin{pmatrix} 1+c & s \\ s & -(1+c)\end{pmatrix}$

so



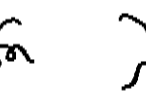
$= -ie^2\,(c_2\ s_2)\ \sigma\cdot\varepsilon^*(k')\ 1\ \sigma\cdot\varepsilon(k)\ \begin{pmatrix}1\\0\end{pmatrix}$



$= \dfrac{+ie^2}{1+c}\,(c_2\ s_2)\ \sigma\cdot\varepsilon(k)\begin{pmatrix} 1+c & s \\ s & -(1+c)\end{pmatrix}\sigma\cdot\varepsilon^*(k')\ \begin{pmatrix}1\\0\end{pmatrix}$

Evaluate these vectors:

$\sigma\cdot\varepsilon_L(k)$	$\sigma\cdot\varepsilon_R(k)$	$\sigma\cdot\varepsilon_L^*(k')$	$\sigma\cdot\varepsilon_R^*(k')$
$+\sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$+\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$-\frac{1}{\sqrt{2}}\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}$	$-\frac{1}{\sqrt{2}}\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}$

for  $\sigma\cdot\varepsilon_L(k)\begin{pmatrix}1\\0\end{pmatrix} = 0$, so the possibly nonzero amplitudes are:

$e_R^-\gamma_R \to e_R^-\gamma_L$: $\quad -ie^2\,(c_2 s_2)\left(-\frac{1}{\sqrt{2}}\right)\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= ie^2\,(c_2 s_2)\begin{pmatrix}s & 1-c\\ -(1+c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$= ie^2\left[c_2(1-c) - s_2 s\right] = ie^2\left[c_2 - (cc_2 + ss_2)\right]$

$= 0$

$e_R^-\gamma_R \to e_R^-\gamma_R$: $\quad -ie^2\,(c_2 s_2)\left(-\frac{1}{\sqrt{2}}\right)\begin{pmatrix}s & -(1+c)\\ 1-c & -s\end{pmatrix}\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= ie^2(c_2 s_2)\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$= ie^2\left[-c_2(1+c) - ss_2\right] = -2ie^2\,c_2$

for χ_1

$e_R^- \gamma_L \to e_R^- \gamma_L$: $\frac{+ie^2}{1+c}(c_2 s_2)\sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\left(\frac{-1}{\sqrt{2}}\right)\begin{pmatrix}s & (1+c)\\ -(1+c) & -s\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= \frac{-ie^2}{1+c}(0, c_2)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}s\\ -(1+c)\end{pmatrix}$

$= \frac{ie^2}{1+c}\left[c_2 s^2 + c_2(1+c)^2\right]$

$= \frac{-ie^2}{1+c}(1+c)\left[c_2(1-c) + c_2(1+c)\right] = 2ie^2 c_2$

$e_R^- \gamma_L \to e_R^- \gamma_R$: $\frac{+ie^2}{1+c}(c_2 s_2)\sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\left(\frac{-1}{\sqrt{2}}\right)\begin{pmatrix}s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= \frac{-ie^2}{1+c}(c_2 s_2)\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}s\\ (1-c)\end{pmatrix}$

$= \frac{-ie^2}{1+c}\left[c_2 s^2 - c_2 \underbrace{(1+c)(1-c)}_{s^2}\right] = 0$

$e_R^- \gamma_R \to e_R^- \gamma_L$ $\frac{+ie^2}{1+c}(c_2 s_2)\sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\left(\frac{-1}{\sqrt{2}}\right)\begin{pmatrix}s & (1-c)\\ -(1+c) & -s\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= \frac{-ie^2}{1+c}(s_2, 0)\begin{pmatrix}1+c & s\\ s & -(1+c)\end{pmatrix}\begin{pmatrix}s\\ -(1+c)\end{pmatrix}$

$= \frac{-ie^2}{1+c}\left[s_2 s(1+c) - s_2 s(1+c)\right] = 0$

$$e_R^- \gamma_R \to e_R^- \gamma_R : \quad \frac{+ie^2}{1+c} (c_1 s_2) \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} s & -(1+c) \\ (1-c) & -s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{-ie^2}{1+c} (s_2, \blacksquare) \begin{pmatrix} 1+c & s \\ s & -(1+c) \end{pmatrix} \begin{pmatrix} s \\ (1-c) \end{pmatrix}$$

$$= \frac{-ie^2}{1+c} [s_2(1+c)s + s_2 s(1-c)]$$

$$= \frac{-2ie^2}{1+c} s s_2$$

the complete amplitude for $e_R^- \gamma_R \to e_R^- \gamma_R$ is:

$$-2ie^2 c_2 - \frac{2ie^2}{1+c} s s_2 = -\frac{2ie^2}{1+c} [c_2 + c c_2 + s s_2]$$

$$= -2ie^2 \frac{2c_2}{1+c} = -2ie^2 \frac{1}{c_2}$$

so!

$$i\mathcal{M}(e_L^- \gamma_L \to e_L^- \gamma_L) = i\mathcal{M}(e_R^- \gamma_R \to e_R^- \gamma_R) = -2ie^2 \frac{1}{c_2}$$

$$i\mathcal{M}(e_L^- \gamma_R \to e_L^- \gamma_R) = i\mathcal{M}(e_R^- \gamma_L \to e_R^- \gamma_L) = -2ie^2 c_2$$

all other helicity amplitudes vanish.

then the polarization summed/averaged cross section is

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\cdot\frac{1}{16\pi}\cdot\frac{1}{4}\sum|\mathcal{M}|^2$$

$$= \frac{1}{2s}\;\frac{1}{16\pi}\;\frac{1}{4}\cdot 4e^4\cdot 2\cdot\left[\frac{1}{c_2^2}+c_2^2\right]$$

$$= \frac{\pi\alpha^2}{E_{cm}^2}\left[\frac{2}{1+\cos\theta}+\frac{1+\cos\theta}{2}\right]$$

$$= \frac{\pi\alpha^2}{4E^2}\;\frac{4+(1+\cos\theta)^2}{2(1+\cos\theta)}$$

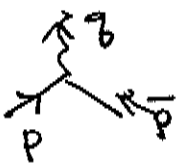
which reproduces the result on p. 9.

3.) The vertex for this problem is

$$\mu\sim\sim = -ig\gamma^\mu$$

I will ignore the electron mass except where it is essential in part (c).

a) $\sigma(e^+e^-\to B)$ is given by



$$= -ig\,\bar{v}(\bar{p})\gamma^\mu u(p)\cdot\varepsilon_\mu^*(q)$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{g^2}{4}\,\mathrm{tr}\left[\bar{\not{p}}\,\gamma^\mu \not{p}\,\gamma^\nu\right](-g_{\mu\nu})$$

$$= -\frac{g^2}{4}\,\mathrm{tr}\,\bar{\not{p}}\,(-2\not{p})$$

$$= \frac{g^2}{4}\cdot 2\cdot 4\,\bar{p}\cdot p = \frac{1}{2}g^2 2\bar{p}\cdot p = g^2(\bar{p}+p)^2$$

$$= \frac{1}{2}g^2 q^2 = M^2\cdot g^2$$

$$\sigma(e^+e^-\to B) = \frac{1}{2s}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}(2\pi)^4\delta(p+\bar{p}-q)\left(\frac{1}{4}|\mathcal{M}|^2\right)$$

$$= \frac{1}{2s}\,2\pi\,\frac{1}{2M}\,\delta(p^0+\bar{p}^0-M)\cdot g^2M^2$$

CM frame.

$$= \frac{1}{2s}\cdot 2\pi\,\delta\left((p+\bar{p})^2-M^2\right)g^2M^2$$

$s=M^2$

$$= \pi g^2\,\delta(s-M^2)$$

$$\Gamma(B\to e^+e^-) = \frac{1}{2M}\int d\Pi_2\left(\frac{1}{3}\sum|\mathcal{M}|^2\right)$$

↑ spin average for spin-1

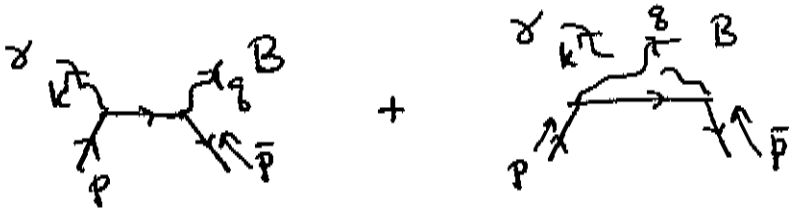
$\sum|\mathcal{M}^2|$ is the same as above $= 4g^2M^2$

$$= \frac{1}{2M}\cdot\frac{1}{8\pi}\cdot\frac{4}{3}\cdot g^2M^2 = \frac{g^2M}{12\pi}$$

so

$$\left.\begin{aligned}\sigma(e^+e^-\to B) &= \pi g^2\,\delta(s-M^2)\\ \Gamma_B &= \frac{g^2M}{12\pi}\end{aligned}\right\}\quad \sigma(e^+e^-\to B) = \frac{12\pi^2}{M}\Gamma_B\,\delta(s-M^2)$$

b.) for $e^+e^- \to \gamma + B$:



$$= (-ig)(ie)\bar{v}(\bar{p}) \left\{ \gamma\cdot\varepsilon^*(q) \frac{i(\not{p}-\not{k})}{(p-k)^2} \gamma\cdot\varepsilon^*(k) + \gamma\cdot\varepsilon^*(k) \frac{i\not{K}-\not{\bar{p}}}{(\bar{p}-k)^2} \gamma\cdot\varepsilon^*(q) \right\} u(p)$$

$$(p-k)^2 = p^2 - 2p\cdot k + k^2 = -2p\cdot k$$
$$(\bar{p}-k)^2 = -2\bar{p}\cdot k$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{e^2g^2}{4} \mathrm{tr}\left\{ \not{\bar{p}} \left[\frac{\gamma^\mu(\not{p}-\not{k})\gamma^\nu}{2p\cdot k} + \frac{\gamma^\nu(\not{k}-\not{\bar{p}})\gamma^\mu}{2\bar{p}\cdot k} \right] \not{p} \left[\frac{\gamma_\nu(\not{p}-\not{k})\gamma_\mu}{2p\cdot k} + \frac{\gamma_\mu(\not{k}-\not{\bar{p}})\gamma_\nu}{2\bar{p}\cdot k} \right] \right\}$$

$$= \frac{e^2g^2}{4} \left\{ \frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{2p\cdot k\, 2\bar{p}\cdot k} + \frac{\mathrm{III}}{2\bar{p}\cdot k\, 2p\cdot k} + \frac{\mathrm{IV}}{(2\bar{p}\cdot k)^2} \right\}$$

$p^2 = \bar{p}^2 = k^2 = 0$
$q^2 = M^2$

$$I = \mathrm{tr}\ \bar{\not{p}}\ \gamma^\mu (\bar{\not{p}} - \not{k})\gamma^\nu \not{p}\ \gamma_\nu (\not{p} - \not{k})\gamma_\mu$$

$$= \mathrm{tr}\ \gamma_\mu \bar{\not{p}}\ \gamma^\nu (\not{p} - \not{k})\gamma^\nu \not{p}\ \gamma_\nu (\not{p} - \not{k})$$

$$= (-2)^2\ \mathrm{tr}\ \bar{\not{p}}\ (\not{p} - \not{k})\ \not{p}\ (\not{p} - \not{k})$$

$$= 4 \cdot 4 \cdot [\ \bar{p}\cdot(p-k)\ p\cdot(p-k)\cdot 2 \ - p\cdot\bar{p}\ (p-k)^2\ ]$$

$$= 16 \cdot [-2\bar{p}\cdot p\ p\cdot k + 2\bar{p}\cdot k\ p\cdot k + 2p\cdot\bar{p}\ p\cdot k\ ]$$

$$= 32\ [p\cdot k]\ [\ -\cancel{\bar{p}\cdot p} + \bar{p}\cdot k + \cancel{\bar{p}\cdot p}\ ]$$

$$= 32\ p\cdot k\ \bar{p}\cdot k$$

$$IV = \mathrm{tr}\ \bar{\not{p}}\ \gamma^\nu (\bar{\not{p}} - \not{k})\gamma^\mu \not{p}\ \gamma_\mu (\bar{\not{p}} - \not{k})\gamma_\nu$$

$$= (-2)^2\ \mathrm{tr}\ \bar{\not{p}}\ (\bar{\not{p}} - \not{k})\ \not{p}\ (\bar{\not{p}} - \not{k})$$

$$= 4 \cdot 4 \cdot \{\ \bar{p}\cdot(\bar{p}-k)\ p\cdot(\bar{p}-k)\cdot 2 \ - \ p\cdot\bar{p}\ (\bar{p}-k)^2\ \}$$

$$= 16\ \{\ -\bar{p}\cdot k\ (\cancel{2p\bar{p}} - 2p\cdot k) \ - (\cancel{p\cdot\bar{p}} - 2\bar{p}\cdot k)\ \}$$

$$= 32\ \bar{p}\cdot k\ p\cdot k$$

$$II = \mathrm{tr}\ \bar{\not{p}}\ \gamma^\mu (\not{p} - \not{k})\gamma^\nu \not{p}\ \gamma_\mu (\bar{\not{p}} - \not{k})\cdot(-1)\gamma_\nu$$

$$= (-1)\ \mathrm{tr}\ \bar{\not{p}}\ \gamma^\mu (p-k)(-2)(\bar{p}-k)\gamma_\mu \not{p}$$

$$= 2\ \mathrm{tr}\ \bar{\not{p}}\ 4(p-k)(\bar{p}-k)\not{p}$$

$$= 2 \cdot 4 \cdot 4\ (p\cdot\bar{p})\ (\ p\cdot\bar{p} - p\cdot k - \bar{p}\cdot k)$$

now $\quad s = 2p\cdot\bar{p} \qquad t = -2p\cdot k \qquad u = -2\bar{p}\cdot k$

$s + t + u = M^2$ so $\quad p\cdot\bar{p} - p\cdot k - \bar{p}\cdot k = M^2/2$

$$II = 16\, p\cdot\bar{p}\, M^2$$

$$III = \mathrm{tr}\, \not{\bar{p}}\, \gamma^\nu (\not{\bar{p}} - \not{k})(-1)\gamma^\mu \not{p}\, \gamma_\nu (\not{p} - \not{k})\gamma_\mu$$

$$= \mathrm{tr}\, (-1)(-2)\, \not{\bar{p}}\, \gamma^\nu (\not{\bar{p}} - \not{k})(\not{p} - \not{k})\gamma_\nu \not{p}$$

$$= 2\cdot 4\, \mathrm{tr}\, \not{\bar{p}}\, (\not{\bar{p}} - \not{k})(\not{p} - \not{k})\not{p}$$

$$= 2\cdot 4\cdot 4\, \bar{p}\cdot p\, (\bar{p} - k)(p - k) = 16\, p\bar{p}\, M^2$$

$\sim$ all:

$$\frac{1}{4}\sum |M|^2 = \frac{e^2 g^2}{4\cdot 4}\left\{ \frac{32\, p\cdot k\, \bar{p}\cdot k}{(p\cdot k)^2} + \frac{32\, p\cdot k\, \bar{p}\cdot k}{(\bar{p}\cdot k)^2} + 2\,\frac{16\, p\cdot\bar{p}\, M^2}{(p\cdot k)(\bar{p}\cdot k)} \right\}$$

$$= 2 g^2 e^2 \left\{ \frac{\bar{p}\cdot k}{p\cdot k} + \frac{p\cdot k}{\bar{p}\cdot k} + \frac{p\cdot\bar{p}\, M^2}{p\cdot k\ \bar{p}\cdot k} \right\}$$

In the CM system:

$p = (E, 0, 0, E)$ $\qquad k = (\mathcal{E}, p\sin\theta, 0, p\cos\theta)$

$\bar{p} = (E, 0, 0, -E)$ $\qquad q = (\mathcal{E}, -p\sin\theta, 0, -p\cos\theta)$

s.t. $\quad \mathcal{E}^2 - p^2 = M^2 \qquad \mathcal{E} + p = 2E$

$$p = \frac{4E^2 - M^2}{2E} \qquad \mathcal{E} = \frac{4E^2 + M^2}{2E}$$

$$p\cdot k = Ep(1-\cos\theta) \qquad \bar{p}\cdot k = Ep(1+\cos\theta)$$

$$p\cdot\bar{p} = 2E^2$$

$$\tfrac{1}{4}\Sigma|\mathcal{M}|^2 = 2e^2g^2\left\{\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} + \frac{2E^2M^2}{E^2p^2(1-\cos^2\theta)}\right\}$$

$$= 4e^2g^2\left\{\frac{1+\cos^2\theta}{1-\cos^2\theta} + \frac{(M^2/p^2)}{(1-\cos^2\theta)}\right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{8E^2}\,\frac{1}{16\pi}\,\frac{2p}{2E}\,4e^2g^2\left\{\frac{M^2/p^2 + (1+\cos^2\theta)}{1-\cos^2\theta}\right\}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2E^3p}\,\pi\alpha\cdot\frac{g^2}{4\pi}\cdot\left(\frac{M^2 + p^2(1+\cos^2\theta)}{1-\cos^2\theta}\right)$$

where, again $p = \dfrac{4E^2-M^2}{2E}$

c.) Restoring the electron mass in the t-channel propagator, the denominator is

$$(p-k)^2 - m_e^2 = p^2 - 2p\cdot k - m_e^2 = -2p\cdot k$$

but not $\quad P = (E, 0, 0, p_e) \qquad p_e = (E^2-m_e^2)^{1/2}$

$$\cong E - \frac{m_e^2}{2E} + \cdots$$

$$p\cdot k = (E - p_e\cos\theta)P$$

Using $p_e \cong E - \frac{m_e^2}{2E} + \cdots$

$$p\cdot k = \cdots pE\left\{(1-\cos\theta) + \frac{m_e^2}{2E^2}\cos\theta + \cdots\right\}$$

$$\cong pE\left(1-\cos\theta + \frac{m_e^2}{2E^2}\right) \quad \text{as} \quad \cos\theta \to 1$$

then

$$\int_0 d\cos\theta \; \frac{1}{1-\cos\theta} \to \int_0 d\cos\theta \; \frac{1}{\left(1-\cos\theta + \frac{m_e^2}{2E^2}\right)}$$

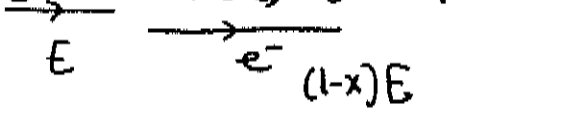
$$= \ln \frac{2E^2}{m_e^2}$$

~~integrate~~ $\frac{d\sigma}{d\cos\theta}$ over forward γ's, we find

$$\sigma(e^+e^- \to \text{forward } \gamma + B)$$

$$\cong \frac{\pi\alpha}{2E^3 p} \frac{g^2}{4\pi} \frac{M^2 + p^2\cdot 2}{2} \ln\left(\frac{2E^2}{m_e^2}\right)$$

Let $p = xE$



x is fixed by the condition that



should add to a mass M

$$M^2 = ((1-x)p + \bar{p})^2 = 2(1-x)\, p\cdot\bar{p}$$

$$\text{or} \qquad (1-x) = M^2/s$$

$$p^2 = x^2 E^2 = \frac{x^2 s}{4}$$

and replace $\ln \frac{2E^2}{m_e^2} \to \ln \frac{s}{m_e^2}$ with small error.

then

$$\sigma(e^+e^- \to \text{formal } \gamma + B)$$

$$= \frac{\pi\alpha}{2E^4 x} \cdot \frac{g^2}{4\pi} \cdot \frac{1}{2} \left[(1-x) + x^2/2\right] \cdot s \cdot \ln \frac{s}{m_e^2}$$

$$= \int_0^1 dx \cdot s \cdot \pi g^2\, \delta(M^2 - (1-x)s) \cdot \frac{\alpha}{s^2 x} \cdot \frac{1}{2\pi} (2 - 2x + x^2) \cdot s \cdot \ln \frac{s}{m_e^2}$$

$$= \int_0^1 dx\, \pi g^2\, \delta(M^2 - (1-x)s) \cdot \frac{\alpha}{2\pi} \cdot \frac{1}{x} \left[1 + (1-x)^2\right] \ln \frac{s}{m_e^2}$$

$$= \int_0^1 dx \left\{ \frac{\alpha}{2\pi} \frac{1}{x} \left[1 + (1-x)^2\right] \ln \frac{s}{m_e^2} \right\} \left[\sigma(e^+e^- \to B)\right]_{E_{cm}^2 = (1-x)s}$$

Physics 330 – Problem Set # 8

(due Wednesday, November 30)

1. Simplify the expression for 3-body phase space

$$\int d\Pi_3 = \int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\,(2\pi)^4\delta^{(4)}(P-p_1-p_2-p_3) \tag{1}$$

in the CM frame: $P^\mu = (P, \vec{0})$. Do not assume any special values of the masses of the final particles; just take these to be m_1, m_2, m_3.

(a) Let $s = P^2$. Let

$$x_1 = \frac{2E_1}{\sqrt{s}} \qquad x_2 = \frac{2E_2}{\sqrt{s}} \qquad x_3 = \frac{2E_3}{\sqrt{s}} \tag{2}$$

Show that $x_1+x_2+x_3 = 2$. We can take any two of these variables as independent.

(b) Show that the invariants $m_{ij}^2 = (p_i + p_j)^2$ are uniquely determined as functions of the x_i: For example,

$$m_{12}^2 = s(1-x_3) + m_3^2 \tag{3}$$

(c) Show that, in the CM system, $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$ lie in a plane (the 'event plane'). Show that the cosine of the angle between any two particles in this plane is uniquely determined as a function of the x_i. The final state then has 5 degrees of freedom– two of the x_i and three Euler angles for the orientation of the event plane. Since $9-4=5$, this number is right.

(d) Integrate over the orientation of the event plane and derive an expression for 3-body phase space as an integral over x_1 and x_2. You should find

$$\int d\Pi_3 = \frac{s}{128\pi^3}\int dx_1 dx_2 \tag{4}$$

The region of integration is somewhat subtle to determine. You worked out the cases of 3 massless and 1 massive and 2 massless particles in Problem Set 1.

(e) Show that the result (4) implies the statement, 'Three body phase space is flat as a function of m_{12}^2 and m_{13}^2.'

This result suggests a way to study reactions with 3-body final states, for example $B^+ \to K^+K^-\pi^+$: Perform a scatter plot of the data in the plane of $m^2(K^+K^-)$ *vs.* $m^2(\pi^+K^-)$. This is called the 'Dalitz plot'. An accumulation of points in this plot on a line parallel to one of the axes indicates a 2-particle resonance in the final state.

2. As a simple model of the strong interactions, assume that quarks interact through exchange of a massless vector boson G with the coupling

$$\Delta\mathcal{L} = g\overline{q}\gamma^\mu q G_\mu \tag{5}$$

This is called the 'Abelian vector gluon' theory. Compute the cross section of gluon emission in e^+e^- annihilation,

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\overline{q}G) \tag{6}$$

where x_1, x_2 are the variables (6) for the quark and antiquark respectively. Assume that the quark has electric charge Q_f and 3 colors. Define $\alpha_g = g^2/4\pi$. Ignore the mass of the electron and the mass of the quark. G couples only to quarks and antiquarks, not to electrons.

(a) Draw the Feynman diagrams for $e^+e^- \to q\overline{q}G$ and evaluate the scattering amplitudes. Square and find the polarization sum and average as a product of two traces.

(b) Now we have to integrate over angles. A sneaky but very convenient way to do this is to hold the event plane fixed and average over the orientation of the initial state. Let the electron and positron momenta be p^μ, $\overline{p}^\mu$, respectively. By writing out the components of the tensor and averaging over angles (or by any other method you can think of), show that

$$\langle p^\mu \overline{p}^\nu + p^\nu \overline{p}^\mu - g^{\mu\nu} p\cdot\overline{p}\rangle = -\frac{1}{3}\left(g^{\mu\nu}P^2 - P^\mu P^\nu\right) \tag{7}$$

where $P = p + \overline{p}$.

(c) Carry out the remaining trace computation. Current conservation might be useful to simplify the computation. You might recognize the trace as (a crossing of) one that you encountered already in the last problem of the previous problem set. Derive the result

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\overline{q}G) = \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2 \cdot \frac{\alpha_g}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \tag{8}$$

This same result is correct (to leading order) in the real theory of the strong interactions, QCD, with the replacement

$$\alpha_g \to \frac{4}{3}\alpha_s \tag{9}$$

Physics 330 – Problem Set #8

Solutions

1.) a.) In the CM frame $P^\mu = (P, \vec{0})$

$$E_1 + E_2 + E_3 = P \qquad s = P^2$$

Then

$$x_1 + x_2 + x_3 = \frac{2}{\sqrt{s}}(E_1 + E_2 + E_3) = \frac{2}{P} \cdot P = 2$$

b.) $$m_{12}^2 = (p_1 + p_2)^2 = (P - p_3)^2 = P^2 - 2P \cdot p_3 + m_3^2$$

$$= s - 2P \cdot E_3 + m_3^2$$

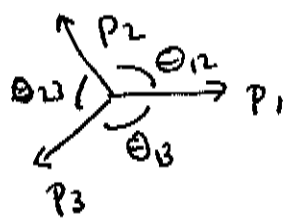
$$= s - 2\sqrt{s} \cdot \frac{\sqrt{s}}{2} x_3 + m_3^2$$

so $$m_{12}^2 = s(1 - x_3) + m_3^2$$

similarly $$m_{23}^2 = s(1 - x_1) + m_1^2$$

$$m_{31}^2 = s(1 - x_2) + m_2^2$$

c.) In the CM frame $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$; the $\vec{p}_3$ lies in the plane determined by $\vec{p}_1, \vec{p}_2$.



$\cos\theta_{12}$ is given by

$$m_{12}^2 = m_1^2 + m_2^2 + 2p_1p_2$$

$$= m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta_{12}$$

Let $x_i = \frac{2}{\sqrt{s}}E_i$ $\quad y_i = \frac{2}{\sqrt{s}}p_i = \frac{2}{\sqrt{s}}\left[E_i^2 - m_i^2\right]^{\frac{1}{2}} = \left(x_i^2 - \frac{4}{s}m_i^2\right)^{\frac{1}{2}}$

$$m_{12}^2 = s(1-x_3) + m_3^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta_{12}$$

$$= m_1^2 + m_2^2 + s\left(\tfrac{1}{2}x_1x_2 - y_1y_2\cos\theta_{12}\right)$$

so

$$\cos\theta_{12} = \frac{x_1x_2 + x_3 - 1 + (m_1^2 + m_2^2 - m_3^2)/s}{y_1y_2}$$

similarly

$$\cos\theta_{23} = \frac{x_2x_3 + x_1 - 1 + (m_2^2 + m_3^2 - m_1^2)/s}{y_2y_3}$$

$$\cos\theta_{31} = \frac{x_3x_1 + x_2 - 1 + (m_3^2 + m_1^2 - m_2^2)/s}{y_3y_1}$$

d.) Now choose coordinates for the event plane so ~~that~~ the $\hat{3}$ axis points along $\vec{p}_1$ and $\vec{p}_1, \vec{p}_2, \vec{p}_3$ lie in the $\hat{3}$-$\hat{1}$ plane.

To reorient to a general orientation, we integrate over $d\Omega_1$ and over the azimuthal angle of $\vec{p}_2$:

$$\int d\Omega_1\, d\varphi_2 = 4\pi \cdot 2\pi = 8\pi^2$$

Then

$$\int d\Pi_3 = \int \frac{d^3p_1\, d^3p_2\, d^3p_3}{(2\pi)^9\, 2E_1 2E_2 2E_3}\, (2\pi)^4\, \delta(P - p_1 - p_2 - p_3)$$

$$= \frac{1}{(2\pi)^6} \int \frac{d^3p_1\, d^3p_2}{2E_1 2E_2 2E_3}\, 2\pi\, \delta(P - E_1 - E_2 - E_3)$$

$$= \frac{1}{(2\pi)^6} \int \frac{dp_1\, p_1^2\, d\Omega_1\, dp_2\, p_2^2\, d\cos\theta_{12}\, d\varphi_2}{2E_1\, 2E_2\, 2E_3}\, 2\pi\, \delta(P - E_1 - E_2 - E_3)$$

$$= \frac{8\pi^2}{(2\pi)^6} \int \frac{dp_1\, p_1^2\, dp_2\, p_2^2\, d\cos\theta_{12}}{2E_1\, 2E_2\, 2E_3}\, 2\pi\, \delta(P - E_1 - E_2 - E_3)$$

now E_3 is determined by $\vec{p}_3 = -\vec{p}_1 - \vec{p}_2$

$$E_3 = \left[p_1^2 + p_1 p_2 \cos\theta + p_2^2 + m_3^2\right]^{\frac{1}{2}}$$

we can eliminate $d\cos\theta_{12}$ by integrating over the energy δ-function

$$\int d\cos\theta_{12}\, \delta(P - E_1 - E_2 - E_3) = \frac{1}{\left(\frac{p_1 p_2}{E_3}\right)}$$

in the region where $-1 \le \cos\theta_{12} \le 1$.

Finally, we $dp_1\, p_1 = \frac{1}{2} dp_1^2 = \frac{1}{2} dE_1^2 = dE_1 E_1$ similarly for E_2

$$\int d\Pi_3 = \frac{8\pi^2}{(2\pi)^6}\, 2\pi\, \frac{1}{8} \int \frac{dE_1\, E_1 p_1\, dE_2 E_2 p_2}{E_1 E_2 E_3\, \left(\frac{p_1 p_2}{E_3}\right)}$$

$$= \frac{1}{32\pi^3} \int dE_1\, dE_2 = \frac{S}{128\pi^3} \int dx_1\, dx_2$$

e.) Using the result of part (b), this can also be written

$$dm_{12}^2 = -S\, dx_3 \quad \text{etc.}$$

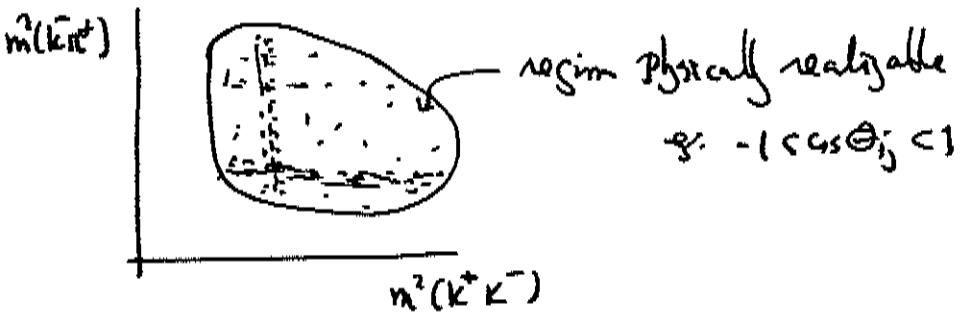
$$\int d\Pi_3 = \frac{1}{128\pi^3 S} \int dm_{23}^2\, dm_{13}^2$$

so

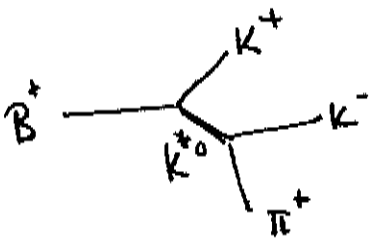
$$\frac{d^2\Pi_3}{dm_{12}^2\, dm_{13}^2} \leftarrow \text{any two } m_{ij}^2 \text{'s here}$$

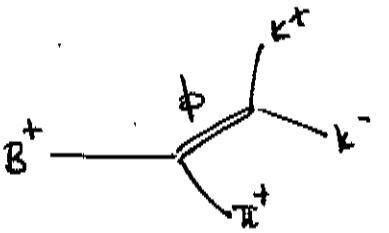
$$= \underline{\text{(const)}}$$

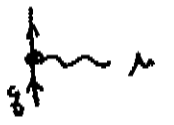
consider eg. $B^+ \to K^+K^-\pi^+$. Point accumulate along lines in the $m^2(K^+\pi^-)$ – $m^2(K^-\pi^+)$ plane:



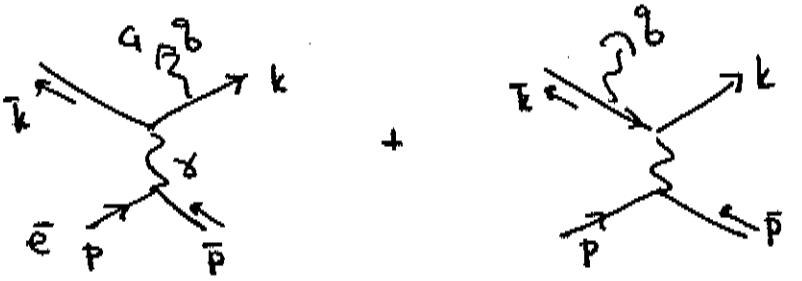
This accumulation cannot be due to phase space, They must be resonances:





2.) a.) The G_μ vertex is  $= +ig\gamma^\mu$

The process $q\bar{q} \to G$ emission is given by



$$= (-ie)(+iQ_f e)(+ig) \quad \bar{v}(\bar{p})\gamma^\mu u(p) \frac{-i}{(p+\bar{p})^2}$$

$$\cdot \bar{v}(k)\left[\not{\varepsilon}^*(q) \frac{i(\not{k}+\not{q})}{(k+q)^2}\gamma_\mu + \gamma_\mu \frac{i(-\not{\bar{k}}+\not{q})}{(\bar{k}+q)^2} \not{\varepsilon}^*(q) \right] v(\bar{k})$$

$$= iQ_f e^2 g \frac{1}{s} \quad \bar{v}(\bar{p})\gamma^\mu u(p) \cdot \bar{u}(k)\left[\frac{\not{\varepsilon}^*(q)(\not{k}+\not{q})\gamma_\mu}{2k\cdot q} - \frac{\gamma_\mu(\not{\bar{k}}+\not{q})\not{\varepsilon}^*(q)}{2\bar{k}\cdot q} \right] v(\bar{k})$$

$$\frac{1}{4}\sum |M|^2 = Q_f^2 \frac{e^4 g^2}{4} \frac{1}{s^2} \quad \mathrm{tr}[\not{\bar{p}}\gamma^\mu \not{p}\gamma^\nu](-g_{\nu\sigma})$$

$$\cdot \mathrm{tr}\left[\not{k}\left\{ \frac{\gamma_\nu(\not{k}+\not{q})\gamma_\mu}{2k\cdot q} - \frac{\gamma_\mu(\not{\bar{k}}+\not{q})\gamma_\nu}{2\bar{k}\cdot q} \right\} \right.$$

$$\left. \cdot \not{\bar{k}}\left\{ \frac{\gamma_\sigma \not{k}\not{q}\gamma_\rho}{2k\cdot q} - \frac{\gamma_\rho(\not{\bar{k}}+\not{q})\gamma_\sigma}{2\bar{k}\cdot q} \right\} \right]$$

$\times$ 3 for color

b.) The electron-positron trace is

$$\mathrm{tr}(\bar{\not{p}}\gamma^\mu \not{p}\gamma^\lambda) = 4\left[\bar{p}^\mu p^\lambda + p^\mu \bar{p}^\lambda - p\cdot\bar{p}\, g^{\mu\lambda}\right]$$

Instead of averaging the final state over orientations, average this over orientations

$$p = (E, E\hat{n}) \qquad \bar{p} = (E, -E\hat{n}) \qquad p\cdot\bar{p} = 2E^2$$

$$(\text{above}) = 4 \begin{array}{c} \mu= \\ 0 \\ i \end{array} \begin{array}{c} \lambda = 0 \quad , \quad j \\ \left(\begin{array}{c|c} E^2+E^2-2E^2 & E\cdot E\hat{n}^j + E\cdot E(-\hat{n}) \\ \hline E(E\hat{n}) + E\cdot E\hat{n} & -E^2\hat{n}^i\hat{n}^j = E^2\hat{n}^i\hat{n}^j + 2E^2\delta^{ij} \end{array}\right) \end{array}$$

$$= 4\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 2E^2(\delta^{ij} - \hat{n}^i\hat{n}^j) \end{array}\right)$$

average over orientation of $\hat{n}$

$$\langle \text{above} \rangle = 4\cdot 2E^2 \cdot \left(\begin{array}{c|c} 0 & \\ \hline & \frac{2}{3}\delta^{ij} \end{array}\right)$$

$$= 4\cdot\left[\frac{4E^2}{3}\left(\begin{array}{c|c} -1 & \\ \hline & \delta^{ij} \end{array}\right) + \frac{4E^3}{3}\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)\right]$$

$$= 4\cdot\left\{P^2\,\frac{1}{3}\left(-g^{\mu\nu}\right) + \frac{1}{3}P^\mu P^\nu\right\}$$

where $P = p+\bar{p} = (2E, \vec{0})$

again $\langle \mathrm{tr}(\bar{p}\gamma^\mu \not{P} \gamma^\nu)\rangle = -\frac{4}{3}(P^2 g^{\mu\nu} - P^\mu P^\nu)$

We can check this by the following properties:

(1) this object must be a function of P^μ only

(2) dotting in P^μ gives zero, since $P^\mu \bar{u}(p)\gamma_\mu u(p)$
$= \bar{v}(p)(\not{\bar{p}} + \not{p})u(p) = 0$

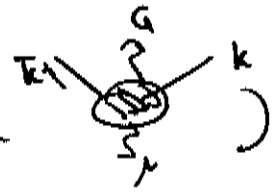
(3) dotting w. $g_{\mu\nu}$ gives $g_{\mu\nu}\,\mathrm{tr}(\bar{p}\gamma^\mu \not{p}\gamma^\nu) = \mathrm{tr}(\bar{p}\gamma^\mu p \gamma_\mu)$
$= \mathrm{tr}(\not{\bar{p}}(4 - 2\not{p})$
$= -8 p\bar{p} = -4\cdot(2p\cdot\bar{p})$
$= -4P^2$

to satisfy (2) we must have

$\langle \mathrm{tr}(\bar{p}\gamma^\mu \not{P}\gamma^\nu)\rangle = A(P^2 g^{\mu\nu} - P^\mu P^\nu)$

and (3) gives $A = -\frac{4}{3}$.

c.) Notice that P^μ dotted into the second trace gives zero:



$P_\mu \cdot (\quad) = 0$

more explicitly, dotting with P_μ gives

$= \bar{u}(k)\left[\gamma\cdot\varepsilon^*(q)\frac{(k+\not{q})\not{P}}{2k\cdot q} - \not{P}\frac{(k+\not{q})\gamma\cdot\varepsilon^*(q)}{k\cdot q}\right] v(k)$

then $(\not{k}+\not{q})\not{\epsilon}^*(\bar{k}) = (\not{k}+\not{q})(K+\not{p}_0+\bar{K})\,v(k)$

now $\bar{K}v(\bar{k}) = 0 \quad (k+q)^2 = (k+q)^2 = 2k\cdot q$

$\bar{u}(k)\not{\epsilon}(\bar{k}+q) = \bar{u}(k)(k+\bar{k}+q)(\bar{k}+q) = \bar{u}(k)(\bar{k}+q)^2$

$= \bar{u}(2\bar{k}\cdot q)$

so the above is

$$\bar{u}(k)\left\{\gamma\cdot\epsilon^*(q)\frac{2k\cdot q}{2k\cdot q} - \frac{2\bar{k}\cdot q}{2k\cdot q}\gamma\cdot\epsilon^*(q)\right\}v(\bar{k})$$

$$= 0$$

now $\frac{1}{4}\sum|M|^2$ reduces to:

$$\frac{1}{4}\sum|M|^2 = 3\cdot\frac{Q_f^2 e^4 g^2}{4}\frac{1}{s^2}(-1)\cdot\left(-\frac{4}{3}s\right)$$

$$\cdot\,\mathrm{tr}\left[\not{k}\left\{\frac{\gamma_\nu\not{k}\not{q}\gamma_\mu}{2k\cdot q} - \frac{\gamma_\mu\bar{\not{k}}\not{q}\gamma_\nu}{2\bar{k}\cdot q}\right\}\right.$$

$$\left.\bar{\not{k}}\left\{\frac{\gamma_\mu(\not{k}\not{q})\gamma_\nu}{2k\cdot q} - \frac{\gamma_\nu(\bar{\not{k}}\not{q})\gamma_\mu}{2\bar{k}\cdot q}\right\}\right]$$

Now we have to bite the bullet and do this trace.

$$\mathrm{tr}[\,] = \frac{\mathrm{I}}{(2k\cdot q)^2} - \frac{\mathrm{II}}{2k_2\,2k\cdot q} - \frac{\mathrm{III}}{2\bar{k}\cdot q\,2k\cdot q} + \frac{\mathrm{IV}}{(2\bar{k}\cdot q)^2}$$

$$I = \mathrm{tr}\, \not{k} \gamma_\nu (\not{k}+\not{q}) \gamma_\mu \not{\bar{k}} \gamma^\mu (\not{k}+\not{q}) \gamma^\nu$$

$$= \mathrm{tr}\, \gamma^\nu \not{k} \gamma_\nu (\not{k}+\not{q}) \gamma^\mu \not{\bar{k}} \gamma_\mu (\not{k}+\not{q})$$

$$= \mathrm{tr}\, (-2\not{k})(\not{k}+\not{q})(-2\not{\bar{k}})(\not{k}+\not{q})$$

$$= (-2)^2 \cdot 4 \left[k\cdot(k+q)\, \bar{k}\cdot(k+q)\cdot 2 - k\cdot\bar{k}\,(k+q)^2 \right]$$

$$= 16 \left[2k\cdot q (2\bar{k}\cdot k + \bar{k}\cdot q) - k\cdot\bar{k}\cdot 2k\cdot q \right]$$

$$= 32\, k\cdot q\, \bar{k}\cdot q$$

$$IV = \mathrm{tr}\, \not{k} \gamma_\mu (\not{\bar{k}}+\not{q}) \gamma_\nu \not{k} \gamma^\nu (\not{\bar{k}}+\not{q}) \gamma^\mu$$

$$= \mathrm{tr}\, (-2)^2 \not{k} (\not{\bar{k}}+\not{q}) \not{k} (\not{\bar{k}}+\not{q})$$

$$= 16 \cdot \left\{ 2k\cdot(\bar{k}+q)\, \bar{k}\cdot(\bar{k}+q) - k\cdot\bar{k}\,(\bar{k}+q)^2 \right\}$$

$$= 16 \left\{ (2\bar{k}\cdot k + 2k\cdot q)\, \bar{k}\cdot q - k\cdot\bar{k}\cdot 2\bar{k}\cdot q \right\}$$

$$= 32\, k\cdot q\, \bar{k}\cdot q$$

$$II = \mathrm{tr}\left[\not{k} \gamma_\nu (\not{k}+\not{q}) \gamma_\mu \not{\bar{k}} \gamma^\nu (\not{\bar{k}}+\not{q}) \gamma^\mu \right]$$

$$= \mathrm{tr}\, \not{k} \gamma_\nu (\not{k}+\not{q}) (-2)(\not{\bar{k}}+\not{q}) \gamma^\nu \not{\bar{k}}$$

$$= \mathrm{tr}\, (-2)\cdot \left[\not{k} \cdot 4\, (k+q)\cdot(\bar{k}+q)\, \not{\bar{k}} \right]$$

$$= -8\cdot 4 \cdot k\cdot\bar{k} \cdot (k\cdot\bar{k} + q\cdot\bar{k} + q\cdot k)$$

now $\quad 2(k\cdot\bar{k} + q\cdot\bar{k} + q\cdot k) = (k+\bar{k}+q)^2 = S$

$$= -16\, k\cdot\bar{k}\cdot S$$

$$\text{III} = \mathrm{tr}[\not k \gamma_\mu (\not k - \not q) \gamma_\nu \not{\bar k} \gamma^\mu (\not k - \not q) \gamma^\nu]$$

$$= \mathrm{tr}\, \not k \not\epsilon(2) \not{\bar k} \gamma_\nu (\not k - \not q)(\not k - \not q) \gamma^\nu$$

$$= (-2)\cdot 4\, \mathrm{tr}\, \not k \not{\bar k} (\bar k - q)\cdot(k - q)$$

$$= -16\, k\cdot\bar k \cdot s \qquad \text{as} \sim \text{II}$$

$$\mathrm{tr}[\,] = 8\cdot\left\{ \frac{(2k\cdot q)(2\bar k\cdot q)}{(2k\cdot q)^2} + 2\,\frac{2k\cdot\bar k\cdot s}{2k\cdot q\; 2\bar k\cdot q} + \frac{2k\cdot q\; 2\bar k\cdot q}{(2\bar k\cdot q)^2}\right\}$$

$$= 8\left\{ \frac{2\bar k\cdot q}{2k\cdot q} + \frac{2k\cdot q}{2\bar k\cdot q} + 2\,\frac{2k\cdot\bar k\, s}{2k\cdot q\; 2\bar k\cdot q}\right\}$$

now $\quad 2\bar k\cdot q = (q+\bar k)^2 = (P-k)^2 = s\cdot(1-x_k)$

$2k\cdot q = (q+k)^2 = (P-\bar k)^2 = s(1-x_{\bar k})$

$2k\cdot\bar k = (k+\bar k)^2 = (P-q)^2 = s(1-x_q) = s(x_k + x_{\bar k} - 1)$

set $\quad x_k = \frac{2}{P}k^0 \qquad x_{\bar k} = \frac{2}{P}\bar k^0 \qquad x_q = \frac{2}{P}q^0$

$$x_k + x_{\bar k} + x_q = 2$$

$$\mathrm{tr}[\,] = 8\left\{ \frac{(1-x_k)}{(1-x_{\bar k})} + \frac{(1-x_{\bar k})}{(1-x_k)} + 2\,\frac{(x_k + x_{\bar k} - 1)}{(1-x_k)(1-x_{\bar k})}\right\}$$

$$= 8\left(\frac{(1-x_k)^2 + (1-x_{\bar k})^2 + 2x_k + 2x_{\bar k} - 2}{(1-x_k)(1-x_{\bar k})}\right)$$

$$= 8\,\frac{x_k^2 + x_{\bar k}^2}{(1-x_k)(1-x_{\bar k})}$$

so!

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 3\cdot\frac{Q_f^2 e^4 g^2}{3s}\cdot 8\cdot\frac{x_k^2+x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$\int d\sigma = \frac{1}{2s}\cdot\int d\Pi_3\ \frac{1}{4}|\mathcal{M}|^2$$

$$= \frac{1}{2s}\cdot\frac{s}{128\pi^3}\int dx_k dx_{\bar{k}}\,3\frac{Q_f^2 e^4 g^2}{3s}\cdot 8\cdot\frac{x_k^2+x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$= \int dx_k dx_{\bar{k}}\frac{3Q_f^2 e^4 g^2}{3\cdot 32\pi^3 s}\ \frac{x_k^2+x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$= \int dx_k dx_{\bar{k}}\frac{2\cdot 3}{3s}Q_f^2\ \alpha^2\ \alpha_g\ \frac{x_k^2+x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

$$\frac{d^2\sigma}{dx_k dx_{\bar{k}}} = \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\frac{x_k^2+x_{\bar{k}}^2}{(1-x_k)(1-x_{\bar{k}})}$$

Physics 330 – Problem Set # 9

(due Wednesday, December 7)

1. This problem concerns non-standard corrections to the electron and muon $(g-2)$.

   (a) Consider adding to QED a new neutral scalar boson ϕ with mass M and a coupling to the electron and the muon of the form

   $$\Delta\mathcal{H} = \lambda\phi\overline{\psi}\psi \ . \tag{1}$$

   Compute the contribution to the lepton $(g-2)$ from the 1-loop diagram that involves this boson.

   (b) Alternatively, consider adding to QED a new neutral pseudoscalar boson χ with mass M and a coupling to the electron and the muon of the form

   $$\Delta\mathcal{H} = i\lambda\chi\overline{\psi}\gamma^5\psi \ . \tag{2}$$

   Compute the contribution to the lepton $(g-2)$ from the 1-loop diagram that involves this boson.

   (c) At the moment, there seems to be an anomaly in the muon $(g-2)$, such that a contribution is needed from sources outside the known interactions of particle physics, to give a correction

   $$\Delta(\frac{g-2}{2}) \approx +2\times 10^{-9} \ . \tag{3}$$

   [Caution: This is, at best, a 2.5 σ effect.] In each of the two models above, compute the condition on M and $\alpha_\lambda = \lambda^2/4\pi$ to give an effect of this size.

   (d) For the electron $(g-2)$, there is spectacular agreement between theory and experiment:

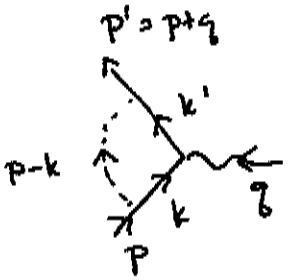
   $$\left|\Delta(\frac{g-2}{2})\right| < 5\times 10^{-11} \ . \tag{4}$$

   This includes the uncertainty in α, which must be determined in another high-precision experiment, for example, the measurement of the quantum unit of resistance in the quantum Hall effect. Are the models of the anomaly in the muon $(g-2)$ that you considered in part (c) consistent with this constraint?

Physics 330 – Problem Set #9

Solutions

1.) a) Using $\Delta H = a \phi \bar{\psi} \psi$



$= \frac{i}{q^2 - M^2}$

$= -ia$

$$= (-ie)(-ia)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k+q)^2 - m^2} \frac{i}{k^2 - m^2} \frac{i}{(k-p)^2 - M^2}$$

$$\bar{u}(p') \; (\not{k}+\not{q}+m) \; \gamma^\mu \; (\not{k}+m) \; u(p)$$

$$= (-ie)(+ia^2) \int dx\,dy\,dz \; \delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4} \frac{2}{[k^2 - \Delta]^3}$$

$$\cdot \; \bar{u}(p') \; (\not{k}+\not{q}+m) \; \gamma^\mu (\not{k}+m) \; u(p)$$

where

$$k^2 - \Delta = x((k+q)^2 - m^2) + y(k^2 - m^2) + z((k-p)^2 - M^2)$$

$$= k^2 + 2k \cdot (xq - zp) + xq^2 + zp^2 - (1-z)m^2 - zM^2$$

Let $\ell = k + xq - zp$

$$k^2 - \Delta = k^2 - (xq - zp)^2 + xq^2 + zp^2 - (1-z)m^2 - zM^2$$

$$= k^2 + \underbrace{x(1-x)}_{y+z} q^2 + 2xz\, p\cdot q + \underbrace{z(1-z)}_{x+y} p^2 - (1-z)m^2 - zM^2$$

$$= k^2 + xy\, q^2 + xz(p+q)^2 + yz\, p^2 - (1-z)m^2 - zM^2$$

$$= k^2 + xy q^2 + \underbrace{(xz + yz)}_{z(1-z)} m^2 - (1-z)m^2 - zM^2$$

$$= k^2 + xy q^2 - (1-z)^2 m^2 - zM^2$$

recap:

$$\Delta = (1-z)^2 m^2 + zM^2 - xy\, q^2 \qquad (> 0)$$

$$k = k - xq + zp$$

$$k+q = k + (1-x)q + zp$$

$$\bar{u}(p')(\not{k} + \not{q} + m)\,\gamma^\mu\,(\not{k} + m)\,u(p)$$

$$= \bar{u}(p')\,[(\not{k} - x\not{q} + z\not{p}) + m]\,\gamma^\mu\,[(\not{k} - x\not{q} + z\not{p}) + m]\,u(p)$$

drop terms linear in k

$$= \bar{u}(p')\,\not{k}\,\gamma^\mu\,\not{k}\,u(p) + \bar{u}(p')\,[\underbrace{(1-x)}_{y+z}\not{q} + z\not{p} + m]\,\gamma^\mu\,[-x\not{q} + z\not{p} + m]\,u(p)$$

now $z\not{p}\,u(p) = zm\,u(p)$

$$\bar{u}(p')\,z(\not{p} + \not{q}) = \bar{u}(p')\cdot zm$$

$$\bar{u}(p')\,\not{k}\gamma^\mu\not{k}\,u(p) = \tfrac{1}{4}k^2\,\bar{u}(p')\,\gamma^\alpha\gamma^\mu\gamma_\alpha\,u(p)$$

$$= -\tfrac{1}{2}k^2\,\bar{u}(p')\,\gamma^\mu\,u(p)$$

contributes only to $F_1(q^2)$; drop this.

$$= \bar{u}(p')\,[y\not{q} + (1+z)m]\,\gamma^\mu\,[-x\not{q} + (1+z)m]\,u(p)$$

$$= \bar{u}(p')\,[\,(1+z)^2m^2\gamma^\mu + (-x\,\gamma^\mu\not{q} + y\not{q}\gamma^\mu)(1+z)m$$

$$- xy\not{q}\gamma^\mu\not{q}\,]\,u(p)$$

now $(1+z)^2m^2\gamma^\mu$ contributes only to F_1

$-xy\not{q}\gamma^\mu\not{q} = -xy\,[2q^\mu\underbrace{\not{q}}_{=0} - \gamma^\mu q^2]$ contributes only to F_1

$-x\gamma^\mu\not{q} + y\not{q}\gamma^\mu = (-x)\,[\gamma^\mu,\not{q}]$ since $\langle x\rangle = \langle y\rangle$

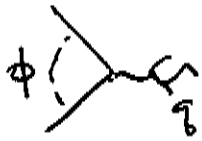
$= (2ix)\,\frac{i}{2}[\gamma^\mu,\gamma^\nu]\,q_\nu$

$= (2ix)\cdot 2m\cdot\frac{1}{2m}\sigma^{\mu\nu}q_\nu$

if $\langle x\rangle = \langle y\rangle = \frac{1}{2}\langle x+y\rangle$
$= \frac{1}{2}(1-z)$

$= 2m(1-z)\,\frac{i}{2m}\sigma^{\mu\nu}q_\nu$

in all $\Longrightarrow$



$$= (-ie)(+i\lambda^2)\int dx\,dy\,dz\;\delta(1-x-y-z)$$

$$\cdot \int \frac{d^4k}{(2\pi)^4}\,\frac{2}{[k^2-\Delta]^3}\cdot \bar{u}(p')\,2m(1-z)\cdot m(1+z)\left(\frac{i}{2m}\delta^{\mu\nu}\gamma_\nu\right)u(p)$$

$$+ \text{(terms contributing to } F_1(q^2)\text{)}$$

$$F_2(q^2) = +i\lambda^2\int dx\,dy\,dz\,\delta(1-x-y-z)\int\frac{d^4k}{(2\pi)^4}\,\frac{2}{[k^2-\Delta]^3}\cdot 2m^2(1-z^2)$$

$$= +i\lambda^2\int dx\,dy\,dz\,\delta(1-x-y-z)\,\frac{-i}{(4\pi)^2}\,\frac{2}{2\cdot\Delta}\,2m^2(1-z)(1+z)$$

$$F_2(0) = +\frac{\lambda^2}{(4\pi)^2}\underbrace{\int dx\,dy\,dz\,\delta(1-x-y-z)}_{\int_0^1 dz\int_0^{1-z}dx}\,\frac{2m^2(1-z)(1+z)}{m^2(1-z)^2+zM^2}$$

$$= +\frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\,\frac{(1-z)^2(1+z)\cdot 2m^2}{m^2(1-z)^2+zM^2}$$

since the correction to $(g-2)$ is small, we are interested in the integral for $M^2 \gg m^2$. It is tempting to approximate it as

$$\int_0^\infty dz\,\frac{(1-z)^2(1+z)}{zM^2}$$

but this is divergent at $z\to 0$. To control this, we need

to keep the m^2 term in the denominator, but only in the divergent term, we only mean $z \to 0$. That is, expand

$$(1-z)^2(1+z) = (1-z^2)(1-z) = 1 - z - z^2 + z^3$$

$$F_1(q^2=0) = + \frac{\lambda^2}{(4\pi)^2} \int_0^1 dz \; \frac{2m^2}{M^2} \left\{ \frac{1}{(z + m^2/M^2)} + \frac{(-z - z^2 + z^3)}{z} \right\}$$

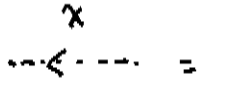
$$= + \frac{\lambda^2}{(4\pi)^2} \frac{2m^2}{M^2} \left\{ \log\left(\frac{M^2}{m^2}\right) - 1 - \frac{1}{2} + \frac{1}{3} \right\}$$

$$= + \frac{\lambda^2}{(4\pi)^2} \frac{2m^2}{M^2} \left(\log \frac{M^2}{m^2} - \frac{7}{6} \right) + O\left(\frac{m^4}{M^4}\right)$$

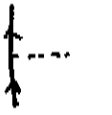
a.

$$\boxed{\delta\left(\frac{g-2}{2}\right)_\phi = + \frac{\alpha_\lambda}{4\pi} \cdot \frac{2m^2}{M^2} \left(\log M^2/m^2 - 7/6 \right)}$$

b.) Redo this for $\Delta H = i\lambda \chi \bar{\psi} \gamma^5 \psi$



$$= \frac{i}{q^2 - M^2}$$



$$= + \lambda \gamma^5$$

c)



$$= (-ie)^2 \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k+q)^2 - m^2} \frac{i}{k^2 - m^2} \frac{i}{(k-q)^2 - M^2}$$

$$\cdot \bar{u}(p') \gamma^5 (\not{k} + \not{q} + m) \gamma^\mu (\not{k} + m) \gamma^5 u(p)$$

ok, moving the γ^5's: $\gamma^5 \not{k} = -\not{k}\gamma^5$, $(\gamma^5)^2 = 1$



$$= (-ie)(+i\partial^2) \int \frac{d^4k}{(2\pi)^4} \int dx\,dy\,dz\,\delta(1-x-y-z)\,\frac{2}{[k^2-\Delta]^3}$$

$$\cdot\ \bar{u}(p')(\not{k}+\not{q}-m)\gamma^\mu(\not{k}-m)u(p)$$

now, the denominator algebra is the same as before

the numerator changes to:

$$\bar{u}(p')(\not{k}+\not{q}-m)\gamma^\mu(\not{k}-m)u(p)$$

$$= \bar{u}(p')\left[\not{k} + \underbrace{(1-x)}_{z}\not{q} + z\not{p} - m\right]\gamma^\mu\left[\not{k} - x\not{q} + z\not{p} - m\right]u(p)$$

$$= \bar{u}(p')\not{k}\gamma^\mu\not{k}\,u(p) \quad \rightarrow \quad F_1 \text{ only}$$

$$+\ \bar{u}(p')\left([z(p+q)-m] + y\not{q}\right)\gamma^\mu(z\not{p} - m - x\not{q})\,u(p)$$

$$= (F_1) + \bar{u}(p')(-(1-z)m + y\not{q})\gamma^\mu(-(1-z)m - x\not{q})\,u(p)$$

$$= (F_1 \text{ term}) - m(1-z)\cdot(-x)\,\bar{u}(p')[\gamma^\mu, \not{q}]\,u(p)$$

$$= (F_1 \text{ terms}) - m(1-z)\cdot 2m(1-z)\left(\bar{u}\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,u\right)$$

so

$$F_2(q^2) = (+i\partial^2)\int dx\,dy\,dz\,\delta(1-x-y-z)\int\frac{d^4k}{(2\pi)^4}\,\frac{2}{[k^2-\Delta]^3}$$

$$\cdot\ \left[-2m^2(1-z)^2\right]$$

$$F_2(0) = +ie^2 \int dx\,dy\,dz\,\delta(1-x-y-z)\,\frac{-i}{(4\pi)^2}\,\frac{2}{2\cdot\Delta}\,(-2m^2(1-z)^2)$$

$$= -\frac{e^2}{(4\pi)^2}\,2m^2 \int_0^1 dz \int_0^{1-z} dx\,\frac{(1-z)^2}{(1-z)^2 m^2 + z M^2}$$

$$= -\frac{e^2}{(4\pi)^2}\left(\frac{2m^2}{M^2}\right)\int_0^1 dz\,\frac{(1-z)^3}{z + (1-z)^2 m^2/M^2}$$

$M^2 \gg m^2$ $$\cong -\frac{\alpha_2}{4\pi}\,\frac{2m^2}{M^2}\int_0^1 dz\left(\frac{1}{z+m^2/M^2} + \frac{-3z+3z^2-z^3}{z}\right)$$

$$= -\frac{\alpha_2}{4\pi}\,\frac{2m^2}{M^2}\left[\ln M^2/m^2 - 3 + \frac{3}{2} - \frac{1}{3}\right]$$

$$F_2(0) = -\frac{\alpha_2}{4\pi}\,\frac{2m^2}{M^2}\left[\ln M^2/m^2 - \frac{11}{6}\right]$$

so

$$\delta\left(\frac{g-2}{2}\right)_\phi = +\frac{\alpha_2}{2\pi}\,\frac{m^2}{M^2}\left[\ln M^2/m^2 - 7/6\right]$$

$$\delta\left(\frac{g-2}{2}\right)_\chi = -\frac{\alpha_2}{2\pi}\,\frac{m^2}{M^2}\left[\ln M^2/m^2 - \frac{11}{6}\right] \quad (*)$$

(*) thanks to Zdanybek Altpishev.

c.) Note that the scalar gives a positive contribution to $(g-2)$. The pseudoscalar gives a negative contribution.

So, to obtain $\Delta\left(\frac{g-2}{2}\right) = 2\times10^{-9}$

we need to take the scalar model.

for $\alpha_\chi = \frac{1}{100}$ we obtain this value for $\frac{m}{M} = 2.9\times10^{-4}$

i.e. $m_\mu = 106\ \text{MeV} \rightarrow M = 370\ \text{GeV}$

more generally $\ln M^2/m^2 - \frac{7}{6} \approx 15$

$$\frac{m}{M} = 3\times10^{-5}\frac{1}{\sqrt{\alpha_\chi}} \quad \text{or} \quad M \approx 3700\ \text{GeV}\cdot\sqrt{\alpha_\chi}$$

d.) For the electron, the effect is smaller by

$$\left(\frac{m_e}{m_\mu}\right)^2 = \left(\frac{0.51\ \text{MeV}}{106\ \text{MeV}}\right)^2 = 2\times10^{-5}$$

so if the effect is there for the μ, it is invisible for the e^-.

Physics 330 – Final Exam

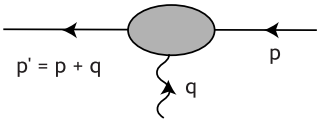
This exam is due on Wednesday, December 10. Please return the exam to Mariel Haag in Varian 336 by 3:00 pm sharp. If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed.

The exam is worth a total of 100 points (plus a possible 5 points extra credit). The distribution of points is indicated below.

The problem deals with the lepton number violating processes $\mu \to e\gamma$, $\tau \to \mu\gamma$, $\tau \to e\gamma$. These decay processes have never been observed. In the 'Standard Model' of particle physics, each species of lepton carries a conserved quantum number and cannot convert to another species. However, most models of physics beyond the Standard Model allow some mechanism for these flavor-changing lepton decays, and so it is interesting to look for these decays experimentally.

a. (20 points) We will see below that a flavor-changing lepton decay such as $\mu \to e\gamma$ is mediated by the 'transition dipole matrix element':



$$= -ie\overline{u}(p')\left[i\sigma^{\mu\nu}q_\nu \overline{F}_2(q^2)\right]u(p) \tag{1}$$

Compute the decay rate for $\mu \to e\gamma$ in terms of the coefficient $\overline{F}_2(0)$. You may set $m_e = 0$ throughout this problem set, except in part (j).

b. (5 points) The typical decay of the muon is by the weak-interaction process

$$\mu^- \to \nu_\mu e^- \overline{\nu}_e \tag{2}$$

The rate of this decay is:

$$\Gamma(\mu) = \frac{\alpha_w^2 m_\mu^5}{384\pi m_W^4} = (2.2 \times 10^{-6}\ \text{sec})^{-1} \tag{3}$$

where $\alpha_w = 1/29.6$ is the coupling strength of the weak interaction, $m_\mu = 0.1057$ GeV is the mass of the muon, and $m_W = 80.4$ GeV is the mass of the W boson. Similarly, the rate of weak-interaction decay of the τ is given by

$$\Gamma(\tau) = 5.3\frac{\alpha_w^2 m_\tau^5}{384\pi m_W^4} = (2.9 \times 10^{-13}\ \text{sec})^{-1} \tag{4}$$

where $m_\tau = 1.777$ GeV and the coefficient in front equals $(1 + 1 + 3.3)$ to account fir the decay of the τ^- to e^-, to μ^-, and to quarks with 3 colors and a 10% enhancement from QCD corrections. (These formulae are not hard to derive using methods from Physics 330, but you may just take them as given.) To compare to these formulae, parametrize the form factor $\overline{F}_2$ in the transition dipole moment as

$$\overline{F}_2(0) = G\frac{\alpha}{4\pi}\frac{m_\ell}{M^2} \tag{5}$$

where m_ℓ is the mass of the decaying lepton, M is the mass of some new particle associated with the flavor violation, and G is a dimensionless constant. The current limits on the branching ratios for flavor changing decays are (at the 90% CL):

$$\begin{aligned} BR(\mu \to e\gamma) &< 1.2 \times 10^{-11} \\ BR(\tau \to e\gamma) &< 2.7 \times 10^{-6} \\ BR(\tau \to \mu\gamma) &< 3.2 \times 10^{-7} \end{aligned} \tag{6}$$

What are the corresponding limits on the values of G, assuming that $M = 200$ GeV (beyond the reach of current experiments)? Which is the strongest constraint if the predictions for G are the same for the three processes? What if the predictions for G depend on the lepton mass and are proportional to m_ℓ^2?

c. (10 points) The last result in (6) is a new result from the BELLE B-factory experiment. The analysis made use of the reaction $e^+e^- \to \tau^+\tau^-$ using 86.3 fb^{-1} of data at $E_{cm} = 10.58$ GeV. The definition of 1 fb^{-1} is that it is the number of collisions needed to produce 1 event for a process whose cross section is 1 fb $= 1 \times 10^{-15}$ barn. How many $\tau^+\tau^-$ pairs were produced in this experiment? (The BaBar and BELLE experiments should eventually each record more than 1000 fb^{-1} of data, leading to very strong constraints on the radiative τ decays.)

d. (5 points) With this introduction, let's compute the transition dipole moment in a simple model. Write down the Lagrangian for the quantum electrodynamics of electrons and muons (with fields e, μ, A^μ). Add to this:

$$\begin{aligned} \mathcal{L} &= (D_\nu\phi_e)^* D^\nu\phi_e - m^2\phi_e^*\phi_e + (D_\nu\phi_\mu)^* D^\nu\phi_\mu - m^2\phi_\mu^*\phi_\mu - \delta m^2(\phi_e^*\phi_\mu + \phi_\mu^*\phi_e) \\ &\quad + \overline{b}(i\,\partial\!\!\!/ - m_b)b + g\left(\overline{e}b\phi_e + \overline{\mu}b\phi_\mu + \phi_e^*\overline{b}e + \phi_\mu^*\overline{b}\mu\right) \end{aligned} \tag{7}$$

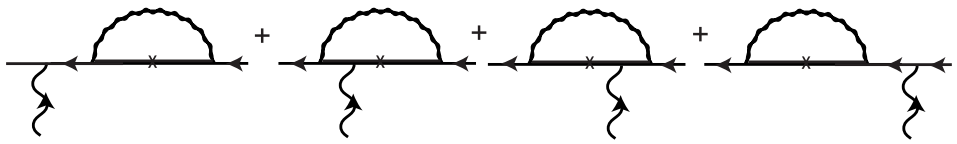
In this Lagrangian, ϕ_e and ϕ_μ are complex scalar fields carrying electron and muon number, respectively, and b is an electrically neutral Dirac fermion. The coupling constant of the new fields is taken to be g, which you should consider to be of the order of e. The two scalar fields are given the same mass for simplicity. The covariant derivative is $D_\nu = \partial_\nu + ieA_\nu$. In thinking about this Lagrangian, imagine that m and m_b are both much larger than 100 GeV, while $m_\mu = 0.106$ GeV.

This Lagrangian is a simplified version of a supersymmetric theory of new particle physics interactions. The scalar fields ϕ_e and ϕ_μ represent the supersymmetric partners

of the electron and muon. The fermion b is the supersymmetric partner of the photon. In a supersymmetric theory, $g = \sqrt{2}e$.

Write the Feynman rules for this theory, treating the δm^2 term as a perturbation that gives rise to a 2-point vertex.

e. (10 points) In the theory (7), the matrix element for the process $\mu \to e\gamma$ is given by the following set of Feynman diagrams:



(8)

with μ on the right, e on the left, and b and $\phi_\mu \to \phi_e$ in the loop in each diagram. Write the values of these diagrams, without doing any of the integrals or Dirac algebra.

Notice that, since $m_\mu \neq m_e$, the diagrams with self-energy-type integrals are not infinite and should be treated as ordinary contributions to the S-matrix element. On the other hand, there are no contributions to $(Z_2 - 1)$ proportional to δm^2, so the sum of diagrams shown in (8) is the whole calculation.

f. (10 points) Show that the expression you wrote in part (e) satisfies the Ward identity:

$$q_\mu (\text{sum of diagrams})^\mu = 0 \ . \tag{9}$$

Use the fact that the initial μ and the final e are on shell, so that $(\not{p} - m_\mu)u(p) = 0$, $(\not{p}' - m_e)u(p') = 0$.

g. (5 points) Show that the Gordon identity for this kinematic situation is

$$(m_e + m_\mu)\overline{u}(p')\gamma^\nu u(p) = \overline{u}(p')\left[(p'+p)^\nu + i\sigma^{\nu\sigma}q_\sigma\right]u(p) \tag{10}$$

Using this identity, the sum of diagrams in (e) can be gathered into the form

$$-ie\overline{u}(p')\left[\gamma^\mu F_1(q^2) + q^\mu F_3(q^2) + i\sigma^{\mu\nu}q_\nu \overline{F}_2(q^2)\right]u(p) \ . \tag{11}$$

Using the Ward identity, find a relation between the form factors F_1 and F_3. In particular, show that $F_1(q^2)$ vanishes at $q^2 = 0$ if $F_3(q^2)$ has a smooth and nonsigular limit as $q^2 \to 0$.

h. (15 points) Compute the Feynman diagrams shown in (8). Notice that all integrals are UV-finite. Write expressions for $F_1(q^2)$, $F_3(q^2)$, and $\overline{F}_2(q^2)$ as integrals over Feynman parameters. Show that all three form factors have nonsingular limits as $q^2 \to 0$ and as $m_\mu \to 0$. (Remember that $m_\mu \ll m, m_b$.) You may ignore m_μ in the rest of the calculation (except in part (j)).

i. (5 points) Show explicitly that $F_1(q^2) = 0$ at $q^2 = 0$.

j. (5 points extra credit) Show the relation between F_1 and F_3 found in (g) holds explicitly without first taking the limits $m_\mu \to 0$, $m_e \to 0$, $q^2 \to 0$.

k. (5 points) Show that F_1 and F_3 do not contribute to the amplitude for $\mu \to e\gamma$.

l. (10 points) Compute $\overline{F}_2(0)$ in the limit $m_\mu \to 0$. Notice that it is of the general form of (5), except that m_ℓ is replaced by m_b. In a realistic theory of supersymmetry, this defect would be remedied. In a typical model $\delta m^2/m^2$ of the order of $\alpha/4\pi$, but in some models this quantity is parametrically larger for decays of the τ.

References: This exam is open-book. If you make strong use of any reference other than the class textbook and notes, please give the reference. Be warned: There are many papers on $\mu \to e\gamma$, but none are simple to follow. It is probably easier just to do the problem from scratch.

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (e). Prospective theorists should slog through to the end.

Physics 330 – Final Exam

Solutions

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a.) $i\mathcal{M} =$ 

$$= -ie\,\bar{u}(p')\, i\sigma^{\mu\nu}(-q_\nu)\,\bar{F}_2(q^2)\,u(p)\;\Big|_{q^2=0}$$

$$\sum_{\text{spin}} |\mathcal{M}|^2 = e^2|\bar{F}_2|^2 \mathrm{tr}\{(\not{p}'+m_e)[-i\sigma^{\mu\nu}q_\nu](\not{p}+m_e)[+i\sigma^{\lambda\sigma}q_\sigma]\}(-g_{\mu\lambda})$$

(p: spinors; q: photon)

$$= e^2|\bar{F}_2(0)|^2 \,\mathrm{tr}\left(\not{p}'\,\tfrac{1}{2}[\gamma^\mu,\not{q}](\not{p}+m_e)\tfrac{1}{2}[\gamma^\lambda,\not{q}]\right)(-g_{\mu\lambda})$$

the traces with m_e are 0. The other terms were:

$$= e^2\frac{|\bar{F}_2(0)|^2}{4}\,\mathrm{tr}\left[\not{p}'(\gamma^\mu\not{q}-\not{q}\gamma^\mu)\not{p}\,(\gamma_\mu\not{q}-\not{q}\gamma_\mu)\right](+1)$$

$$= e^2\frac{|\bar{F}_2(0)|^2}{4}\mathrm{tr}\{\not{p}'\gamma^\mu\not{q}\not{p}\gamma_\mu\not{q} - \not{p}'\gamma^\mu\not{q}\not{p}\not{q}\gamma_\mu - \not{p}'\not{q}\gamma^\mu\not{p}\gamma_\mu\not{q} + \not{p}'\not{q}\gamma^\mu\not{p}\not{q}\gamma_\mu\}$$

$$= \frac{e^4|\bar{F}_2(0)|^2}{4}\mathrm{tr}\{4\not{p}'\,q\cdot p\,\not{q} + 2\not{p}'\not{q}\not{p}\not{q} + 2\not{p}'\not{q}\not{p}\not{q} + 4\not{p}'\not{q}\,p\cdot q\}$$

$$= e^2 \frac{|\bar{F}(\omega)|^2}{4} \cdot 4 \cdot \{ 4 \cdot 2\, p'\cdot q\; p\cdot q + 2\cdot 2\cdot [2\cdot p'\cdot q\; p\cdot q - p'\cdot p\, \underbrace{q^2}_{=0}) \}$$

$$= e^2 |\bar{F}(\omega)|^2 \; 16 \; p'\cdot q\; p\cdot q$$

now $$m_\mu^2 = p^2 = (p'+q)^2 = 2p'\cdot q = 2(p+q)\cdot q = 2p\cdot q$$

so $$2p'\cdot q = 2p\cdot q = m_\mu^2$$

$$\sum |M|^2 = 4e^2 |\bar{F}_2(\omega)|^2 m_\mu^4$$

$$\Gamma(\mu \to e\gamma) = \frac{1}{2m_\mu} \cdot \underbrace{\frac{1}{8\pi}}_{\text{phase space}} \cdot \frac{1}{2} \sum_{\text{spin}} |M|^2$$

$$= \frac{1}{32\pi m_\mu} \cdot 4e^2 |\bar{F}_2(\omega)|^2 m_\mu^4$$

$$\Gamma(\mu \to e\gamma) = \frac{\alpha}{2} |\bar{F}_2(\omega)|^2 m_\mu^3$$

b.) with $\bar{F}_2(0) = G \frac{\alpha}{4\pi} \frac{m_\mu}{M^2}$

$$\Gamma(\mu \to e\gamma) = \frac{\alpha^3}{32\pi^2} G^2 \frac{m_\mu^5}{M^4}$$

compare this to

$$\Gamma(\mu) = \frac{\alpha_w^2 m_\mu^5}{384\pi m_W^4} \qquad 384 = 32 \cdot 12$$

$$BR(\mu \to e\gamma) = \frac{12\alpha^3}{\pi \alpha_w^2} G^2 \left(\frac{m_W}{M}\right)^4$$

for $\frac{m_W}{M} = \frac{80\,GeV}{200\,GeV}$ $= G^2 \cdot (3.3 \times 10^{-5})$

so $BR(\tau \to e/\mu\,\gamma) = G^2 \cdot (6.3 \times 10^{-6})$ $\left(\begin{smallmatrix}\text{divide by}\\ 5.3\end{smallmatrix}\right)$

$BR(\mu \to e\gamma) < 1.2 \times 10^{-11} \Rightarrow G_{\mu e}^2 < 3.6 \times 10^{-7}$ or $G_{\mu e} < 6 \times 10^{-4}$

$BR(\tau \to e\gamma) < 2.7 \times 10^{-6} \Rightarrow G_{\tau e}^2 < 0.43$ or $G_{\tau e} < 0.66$

$BR(\tau \to \mu\gamma) < 3.2 \times 10^{-7} \Rightarrow G_{\tau\mu}^2 < 5.1 \times 10^{-2}$ or $G_{\tau\mu} < 0.23$

In a theory in which $G_{\mu e} \sim G_{\tau\mu} \sim G_{\tau e}$, the $\mu \to e\gamma$ constraint is obviously the strongest.

Since $(m_\tau/m_\mu)^2 = 280$, the $\mu \to e\gamma$ constraint is stronger even if $G_{l \to \mu/e} \sim m_l^2$.

c.) As we learned,

$$\sigma(e^+e^- \to \tau^+\tau^-) \underset{s \gg m_\tau^2}{\sim} \frac{4\pi\alpha^2}{3s} = \frac{86.8\ \text{nb}}{[E_{cm}(\text{GeV})]^2}$$

$$= 0.78\ \text{nb} \qquad \text{at } E_{cm} = 10.58\,\text{GeV}$$

then $86.3\ \text{fb}^{-1}$ gives

$$\#\text{ of events} = (0.78 \times 10^{-9})(86.3 \times 10^{15})$$

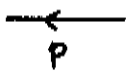
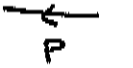
$$= 7.0 \times 10^7 \quad \tau^+\tau^- \text{ pairs}$$

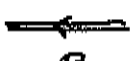
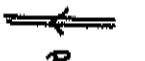
If we could record all of these events & saw no $\tau \to \mu\gamma$ decays. (< 2.3 decays at the 90% CL) we would have a limit of 1.5×10^{-8} (ideal limit)

In practice, the BELLE result is quite an efficient use of the data.

BABAR also has a huge collaboring data sample of τ decays — this could be your thesis !

d) The propagators are

e:  $\frac{i}{\not p - m_e}$ μ:  $\frac{i}{\not p - m_\mu}$

ϕ_e  $\frac{i}{p^2 - m^2}$ ϕ_μ:  $\frac{i}{p^2 - m^2}$

b:  $\frac{i}{\not p - m_b}$

The vertices are:

photon vertices for e μ ϕ_e ϕ_μ:

e, μ — q : $= -ie\gamma^\mu$ $\qquad$ ϕ_e, ϕ_μ — p', p, q : $= -ie(p+p')^\mu$

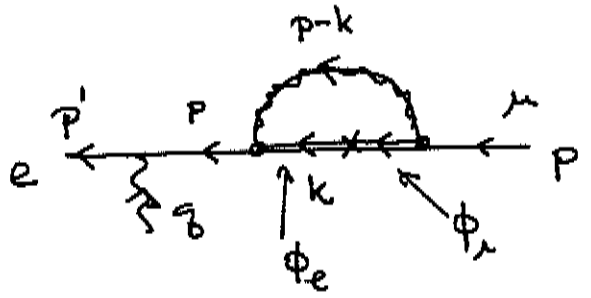
e b ϕ_e : $= ig$ $\qquad$: $= ig$

similarly for μ b ϕ_μ

δm^2 : $\phi_\mu \Longleftarrow\!\!\times\!\!\Longrightarrow \phi_e = -i\delta m^2$

$\phi_e \Longleftarrow\!\!\times\!\!\Longrightarrow \phi_\mu = -i\delta m^2$

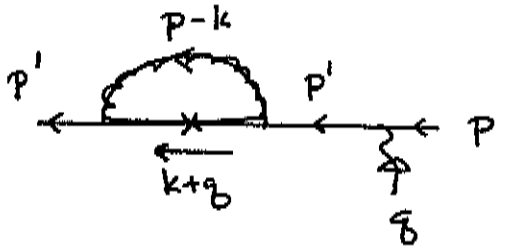
e.)



$$= \bar{u}(p')(-ie\gamma^\mu)\frac{i(\not{p}+m_e)}{(p^2-m_e^2)}(ig)\int\frac{d^4k}{(2\pi)^4}\frac{i(\not{p}-\not{k}+m_b)}{(p-k)^2-m_b^2}ig\,u(p)$$

$$\cdot\frac{i}{k^2-m^2}(-i\delta m^2)\frac{i}{k^2-m^2}$$

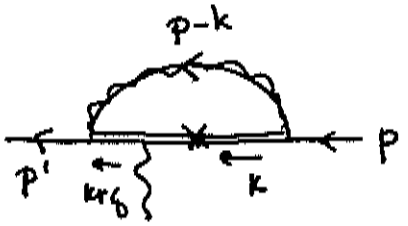
$$= eg^2\delta m^2\int\frac{d^4k}{(2\pi)^4}\bar{u}(p')\gamma^\mu\frac{\not{p}+m_e}{p^2-m_e^2}\frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2}\frac{1}{[k^2-m^2]^2}u(p)$$



$$= \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} (ig) \frac{i(\not{p}-\not{k}+m_b)}{(p-k)^2-m_b^2} (ig) \frac{i(\not{p}'+m_\mu)}{(p')^2-m_\mu^2} (-ie\gamma^\mu) u(p)$$

$$\cdot \frac{i}{(k+q)^2-m^2} (-i\delta m^2) \frac{i}{(k+q)^2-m^2}$$

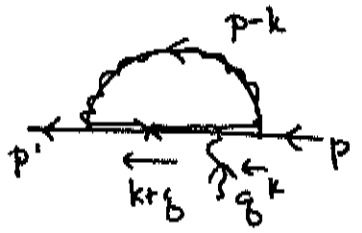
$$= eg^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} \frac{1}{[(k+q)^2-m^2]^2} \frac{\not{p}'+m_\mu}{(p')^2-m_\mu^2} \gamma^\mu u(p)$$



$$= \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} (ig) \frac{i[\not{p}-\not{k}+m_b]}{(p-k)^2-m_b^2} ig\, u(p)$$

$$\cdot \frac{i}{(k+q)^2-m^2} [-ie(2k+q)^\mu] \frac{i}{k^2-m^2} (-i\delta m^2) \frac{i}{k^2-m^2}$$

$$= eg^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} \frac{(2k+q)^\mu}{(k+q)^2-m^2 (k^2-m^2)^2} u(p)$$



$$= \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} (ig) \frac{i\,\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} (ig)\, u(p)$$

$$\cdot \frac{i}{(k+q)^2-m^2} (-i\delta m^2) \frac{i}{(k+q)^2-m^2} -ie(2k+q)^\mu \frac{i}{k^2-m^2}$$

$$= eg^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} \frac{(2k+q)^\mu}{[(k+q)^2-m^2]^2(k^2-m^2)} u(p)$$

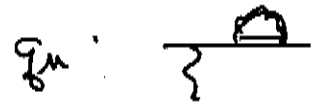
f.) In the first diagram, we: $q_b = p'-p$

$$q_\mu \cdot \gamma^\mu \frac{\not{p}+m_e}{p^2-m_e^2} = (\not{p}'-\not{p}) \frac{\not{p}+m_e}{p^2-m_e^2}$$

$$= [(\not{p}'-m_e) - (\not{p}-m_e)] \frac{1}{\not{p}-m_e}$$

now $\bar{u}(p')(\not{p}'-m_e) = 0$

so

$$q_\mu \cdot \text{} = -eg^2\delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2} \frac{1}{(k^2-m^2)^2} u(p)$$

In the second diagram in (e), use

$$\frac{(\not{p}'+m_\lambda)}{(p')^2-m_\lambda^2}\,\gamma^\lambda\,\not{q}_\nu\,u(p)$$

$$= \frac{1}{\not{p}'-m_\lambda}\left[(\not{p}'-m_\lambda)-(\not{p}-m_\lambda)\right]u(p) = u(p)$$

using $(\not{p}-m_\lambda)\,u(p)=0$

Then

$q_\nu \cdot$  $= + eg^2\delta m^2 \int \frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2}\left[\frac{1}{(k+q)^2-m^2}\right]^2 u(p)$

In the last two diagrams, use:

$$q^\mu\,(2k+q)_\mu = 2k\cdot q + q^2 = (k+q)^2-k^2$$
$$= \left[(k+q)^2-m^2\right]-\left[k^2-m^2\right]$$

So

q_μ ( + )

$$= eg^2\delta m^2 \int \frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\,\frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2}\,\left[(k+q)^2-m^2\right]-\left[k^2-m^2\right]\,u(p)$$

$$\left(\frac{1}{\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]^2} + \frac{1}{\left[(k+q)^2-m^2\right]^2\left(k^2-m^2\right)}\right)$$

$$= eg^2\delta m^2 \int \frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\,\frac{\not{p}-\not{k}+m_b}{(p-k)^2-m_b^2}\,u(p)$$

$$\left\{\frac{1}{\left[k^2-m^2\right]^2} - \frac{1}{\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]} + \frac{1}{\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]} - \frac{1}{\left[(k+q)^2-m^2\right]^2}\right\}$$

or

q_ν ()

$$= e g^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not p - \not k + m_b}{(p-k)^2 - m_b^2} \left(\frac{1}{[k^2-m^2]^2} - \frac{1}{(k+q)^2 - m^2} \right) u(p)$$

which cancels the contribution from the first two diagrams.

2.) The Gordon identity is: ($\not p\, u(p) = m_\mu u(p) \quad \bar{u}(p')\not p' = \bar{u}(p') m_e$)

$$\bar{u}(p')[\not p' \gamma^\mu + \gamma^\mu \not p]\, u(p) = (m_e + m_\mu)\, \bar{u}(p') \gamma^\mu u(p)$$

$$= \bar{u}(p') \left(\tfrac{1}{2}\{\not p', \gamma^\mu\} + \tfrac{1}{2}\{\gamma^\mu, \not p'\} - \tfrac{1}{2}[\gamma^\mu, \not p' - \not p] \right) u(p)$$

$$= \bar{u}(p') \left((p+p')^\mu + i\sigma^{\mu\nu} q_\nu \right) u(p)$$

later we'll use this in the form

$$\bar{u}(p')\,(p+p')^\mu\, u(p) = (m_e + m_\mu)\, \bar{u}(p') \gamma^\mu u(p) - \bar{u}(p')\, i\sigma^{\mu\nu} q_\nu\, u(p)$$

this will allow us to put the sum of diagrams into the form:

$$-ie\, \bar{u}(p')[\gamma^\mu F_1(q^2) + q^\mu F_3(q^2) + i\sigma^{\mu\nu} q_\nu F_2(q^2)]\, u(p)$$

q^μ(this) $= 0 \Rightarrow$ $\bar{u}(p')[\not q F_1(q^2) + \not q^2 F_3(q^2) + 0]\, u(p) = 0$

$$\bar{u}(p')[(\not p' - \not p) F_1(q^2) + \not q^2 F_3(q^2)]\, u(p) = 0$$

$$(m_e - m_\mu) F_1(q^2) + q^2 F_3(q^2) = 0$$

or

$$F_1(q^2) = \frac{q^2}{m_\mu - m_e} F_3(q^2)$$

if $F_3(q^2)$ has a smooth limit as $q^2 \to 0$, $F_1(q^2) \propto q^2 \to 0$ as $q^2 \to 0$.

h.) Now we have to get to work computing diagrams. A little table of integrals will be useful:

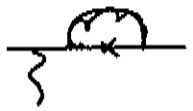
$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-\Delta]^3} = \frac{-i}{(4\pi)^2} \cdot \frac{1}{2} \frac{1}{\Delta}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-\Delta]^4} = \frac{i}{(4\pi)^2} \cdot \frac{1}{3\cdot 2} \frac{1}{\Delta^2}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{[k^2-\Delta]^4} = \frac{-i}{(4\pi)^2} \cdot \frac{2}{3\cdot 2} \frac{1}{\Delta}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{[k^2-\Delta]^4} = \frac{-i}{(4\pi)^2} \frac{1}{3\cdot 2} \frac{1}{\Delta} \cdot \frac{1}{2} g^{\mu\nu}$$

then



$$= e g^2 \delta m^2 \; \bar{u}(p') \gamma^\mu \frac{\not p + m_e}{p^2 - m_e^2} \int \frac{d^4k}{(2\pi)^4} \frac{\not p - \not k + m_b}{[(p-k)^2 - m_b^2][k^2 - m^2]^2} u(p)$$

combine denominators using:

$$\frac{1}{[(p-k)^2 - m_b^2][k^2 - m^2]^2} = \int_0^1 dz \; \frac{2(1-z)}{[z((p-k)^2 - m_b^2) + (1-z)k^2 - m^2]^3}$$

the denominator is:

$$z(k-p)^2 - z m_b^2 + (1-z)k^2 - (1-z)m^2$$

$$= k^2 - 2k \cdot zp + z p^2 - (z m_b^2 + (1-z)m^2)$$

$$= k^2 + \underbrace{z(1-z)p^2 - (z m_b^2 + (1-z)m^2)}_{-\Delta}$$

$$\ell = k - zp$$

$$k = \ell + zp$$

$$k - p = \ell - (1-z)p$$

integrate to 0 ↓

so

$$= e g^2 \delta m^2 \; \bar{u}(p') \gamma^\mu \frac{\not p + m_e}{p^2 - m_e^2} \int_0^1 dz \; 2(1-z) \int \frac{d^4\ell}{(2\pi)^4} \frac{((1-z)\not p - \not\ell + m_b)}{[\ell^2 - \Delta]^3} u(p)$$

$$= -i \frac{e g^2}{(4\pi)^2} \delta m^2 \; \bar{u}(p') \gamma^\mu \frac{\not p + m_e}{p^2 - m_e^2} \int_0^1 dz \; \frac{(1-z)[(1-z)\not p + m_b]}{[z m_b^2 + (1-z)m^2 - z(1-z)p^2]} u(p)$$

now use $\not p \, u(p) = m_\lambda \, u(p) \qquad p^2 = m_\lambda^2$

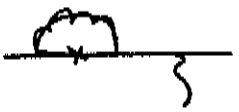
$$\frac{\not p + m_e}{p^2 - m_e^2} = \frac{1}{m_\lambda - m_e}$$

so



$$= (-ie)\ \frac{g^2 \delta m^2}{(4\pi)^2}\ \frac{1}{m_\mu - m_e}\ \bar{u}\gamma^\mu u$$

$$\cdot \int_0^1 dz\ \frac{(1-z)\ ((1-z)m_\mu + m_b)}{[z m_b^2 + (1-z)\, m_e^2 - z(1-z)\, m_\mu^2]}$$



$$= e^* g^2 \delta m^2 \int \frac{d^4k}{(2\pi)^4}\ \bar{u}(p')\ \frac{(\not{p}-\not{k}+m_b)}{(p-k)^2 - m_b^2}\ \frac{1}{[(k+q)^2 - m_e^2]^2}$$

$$\cdot \frac{\not{p}'+m_\mu}{p'^2 - m_\mu^2}\gamma^\mu\ u(p)$$

combine denominators with:

$$\frac{1}{[(p-k)^2 - m_b^2][(k+q)^2 - m_e^2]^2} = \int_0^1 dz\ \frac{2(1-z)}{[z[(p-k)^2 - m_b^2] + (1-z)((k+q)^2 - m_e^2)]^3}$$

the denominator is: $k^2 + 2k\cdot(-zp + (1-z)q) + z p^2 + (1-z)q^2 - (z m_b^2 + (1-z) m_e^2)$

$$= k^2 - (zp - (1-z)q)^2 + z p^2 + (1-z)\, q^2 - (z m_b^2 + (1-z)\, m_e^2)$$

$$= k^2 + (z - z^2)\, p^2 + 2p\cdot q\, z(1-z) + (1-z) z q^2 - (z m_b^2 + (1-z) m_e^2)$$

$$= k^2 + z(1-z)(p+q)^2 - z m_b^2 + (1-z) m_e^2$$

$$\ell k = k - zp + (1-z)q$$

$$k = \ell k + zp - (1-z)q$$

$$k - p = \ell k - (1-z)(p+q) = \ell k - (1-z)\, p'$$

So



$$= eg^2 \delta m^2 \int_0^1 dz\; 2(1-z) \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}[p'(1-z) - \not{k} + m_b]}{[k^2 - D]^3}$$

$$\cdot \frac{\not{p}' + m_\mu}{p'^2 - m_\mu^2} u(p)$$

again the $\not{k}$ ten integrate away. Then we can set $\bar{u}\not{p}' = \bar{u} m_e$

and $$\frac{\not{p}' + m_\mu}{p'^2 - m_\mu^2} = \frac{1}{m_e - m_\mu}$$



$$= (-ie) \frac{g^2 \delta m^2}{(4\pi)^2} \left(\frac{-1}{m_\mu - m_e}\right) \bar{u} \gamma^\mu u$$

$$\cdot \int_0^1 dz \; (1-z) \frac{((1-z)m_e + m_b)}{[z m_b^2 + (1-z)m^2 - z(1-z)m_e^2]}$$

For the remaining two integrals, we need to combine denominators according to:

$$\frac{1}{[(p-k)^2 - m_b^2][(k+\ell)^2 - m^2][k^2 - m^2]} = \int dx\,dy\,dz\,\delta(x+y+z-1) \frac{2}{[z((p-k)^2 - m_b) + x((k+\ell)^2 - m^2) + y(k^2 - m^2)]^3}$$

$$\frac{1}{((p-k)^2 - m_b^2)((k+\ell)^2 - m^2)(k^2 - m^2)^2} = \int dx\,dy\,dz\,\delta(x+y+z-1) \frac{2 \cdot 3y}{[\quad]^4}$$

$$\frac{1}{((p-k)^2 - m_b^2)((k+\ell)^2 - m^2)^2(k^2 - m^2)} = \int dx\,dy\,dz\,\delta(x+y+z-1) \frac{2 \cdot 3x}{[\quad]^4}$$

the denominator in all cases is:

$$z((p-k)^2-m_b^2) + x((k+q)^2-m^2) + y(k^2-m^2)$$

$$= k^2 + 2k(-zp+xq) + zp^2 + xq^2 - [zm_b^2 + (x+y)m^2]$$

$$= \ell^2 - (zp-xq)^2 + zp^2 + xq^2 - [zm_b^2 + (1-z)m^2]$$

$$= \ell^2 + \underbrace{z(1-z)}_{x+y} p^2 + 2p\cdot q\, xz + \underbrace{x(1-x)}_{y+z} q^2 - [z m_b^2 + (1-z)m^2]$$

$$= \ell^2 + yz\,p^2 + xz(p+q)^2 + xy\,q^2 - (zm_b^2 + (1-z)m^2)$$

$$= \ell^2 - \{ z m_b^2 + (1-z)m^2 - xy\,q^2 - xz\,m_e^2 - yz\,m_\mu^2 \}$$

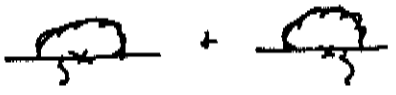
$$\ell = k - zp + xq$$

$$k = \ell + zp - xq$$

$$k-p = \ell - \underbrace{(1-z)}_{x+y} p - xq = \ell - xp' - yp$$

$$2k+q = 2\ell + 2zp + \underbrace{(1-2x)}_{y+z-x} q = 2\ell + z(p+p') + (y-x)q$$

then



$$= e^2 g^2 m^2 \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4}$$

$$\cdot 6(x+y) \cdot \bar{u}(p') \frac{(x\not{p}' + y\not{p} - \not{\ell} + m_b)[2\not{\ell} + z(\not{p}+\not{p}') + (y-x)\not{q}]^2}{[\ell^2 - \Delta]^4}$$

terms linear in ℓ integrate to 0.

$$= (-ie)\frac{g^2\delta m^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)\cdot\underbrace{(x+y)}_{=(1-z)}$$

$$\Big\{\;\bar{u}(p')(-\gamma^\nu)\left(\frac{2}{2}g^{\mu\nu}\right)u(p)\;\frac{1}{\Delta}$$

$$-\frac{1}{\Delta^2}\,\bar{u}(p')\left[x m_e + y m_\lambda + m_b\right]u(p)\;\left[2(p'+p)^\nu + (y-x)\gamma^\nu\right]\Big\}$$

$$= (-ie)\frac{g^2\delta m^2}{(4\pi)^2}\int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\Big\{\;\bar{u}\gamma^\nu u\cdot\left(-\frac{1}{\Delta}\right)(1-z)$$

$$+\;\bar{u}\,q^\mu u\cdot(x-y)\left(x m_e + y m_\lambda + m_b\right)\frac{1}{\Delta^2}(1-z)$$

$$+\;\bar{u}(p+p')^\nu u\cdot(-z)\left(x m_e + y m_\lambda + m_b\right)\frac{(1-z)}{\Delta^2}\Big\}$$

$$= (-ie)\frac{g^2\delta m^2}{(4\pi)^2}\int dx\,dy\,dz\;\delta(x+y+z-1)\;(1-z)$$

$$\cdot\Big\{\;\bar{u}\gamma^\mu u\left(-\frac{1}{\Delta}+\frac{(m_e+m_\lambda)(-z)(x m_e+y m_\lambda+m_b)}{\Delta^2}\right)$$

$$+\;\bar{u}\,q^\mu u\;\frac{(x-y)(x m_e+y m_\lambda+m_b)}{\Delta^2}$$

$$+\;\bar{u}\,i\sigma^{\mu\nu}q_\nu u\;\frac{z(x m_e+y m_\lambda+m_b)}{\Delta^2}\Big\}$$

$$\Delta = z m_b^2 + (1-z)m^2 - xy\,q^2 - xz\,m_e^2 - yz\,m_\lambda^2$$

Now combine the three contributions:



$$= (-ie)\,\bar{u}\left[\gamma^\mu F_1 + \frac{q^\mu}{2m_b} F_3 + i\sigma^{\mu\nu} q_\nu F_2\right] u$$

where

$$F_1 = \frac{g^2 \delta m^2}{(4\pi)^2}\left\{\int dz\, \frac{(1-z)}{m_\mu - m_e}\left(\frac{(1-z)m_\mu + m_b}{[z m_b^2 + (1-z)m^2 - z(1-z)m_\mu^2]}\right.\right.$$

$$\left.\left. - \frac{(1-z)m_e + m_b}{[z m_b^2 + (1-z)m^2 - z(1-z)m_e^2]}\right)\right.$$

$$\left. - \int dx\,dy\,dz\, \delta(x+y+z-1)\left(\frac{(1-z)}{\Delta} + \frac{z(1-z)(m_\mu + m_e)(x m_e + y m_\mu + m_b)}{\Delta^2}\right)\right\}$$

$$F_3 = \frac{g^2 \delta m^2}{(4\pi)^2}\int dx\,dy\,dz\, \delta(x+y+z-1)\, \frac{(x-y)(x m_e + y m_\mu + m_b)(1-z)}{\Delta^2}$$

$$\bar{F}_2 = \frac{g^2 \delta m^2}{(4\pi)^2}\int dx\,dy\,dz\, \delta(x+y+z-1)\, \frac{z(1-z)(x m_e + y m_\mu + m_b)}{\Delta^2}$$

<u>with</u> $\Delta = z m_b^2 + (1-z)m^2 - xy q^2 - xz m_e^2 - yz m_\mu^2$

since Δ is <u>positive</u> in the region we are considering

$m_b^2, m^2 \gg m_\mu^2, m_e^2$

these expressions are nonsingular as $q^2 \to 0$

i.) Now compute $F_1(0)$ in the limit $m_e \to 0$ then $m_\mu \to 0$

$$F_1(0) = \frac{g^2 \delta m^2}{(4\pi)^2} \left\{ \int_0^1 dz \; \frac{(1-z)}{m_\mu} \left[\frac{(1-z)\, m_\mu}{[z m_b^2 + (1-z) m^2]} + \mathcal{O}(m_\mu^2) \right] \right.$$

$$- \int_0^1 dz \int_0^{1-z} dx \; \frac{1}{[z m_b^2 + (1-z) m^2]} *$$

$$\left. - \mathcal{O}(m_\mu) \right\}$$

$$= \frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dz \left\{ \frac{(1-z)^2}{[z m_b^2 + (1-z) m^2]} - \frac{(1-z)\cdot(1-z)}{[z m_b^2 + (1-z) m^2]} \right\}$$

$$= 0$$

j.) (See below)

k.) For $\mu \to e\gamma$, the photon has $q^2 = 0$. But $F_1(q^2) = 0$ at $q^2 = 0$. F_3 multiplies q^μ, and $q^\mu \cdot \Sigma^A(q) = 0$. So neither term contributes to $\mu \to e\gamma$.

l.) In the limit $m_e \to 0$ $m_\nu \to 0$ $q^2 \to 0$

$$\bar{F}_2(0) = \frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dx \frac{z(1-z)\, m_b}{[z m_b^2 + (1-z) m^2]^2}$$

$$= \frac{g^2 \delta m^2}{(4\pi)^2} \int_0^1 dz \frac{z(1-z)^2\, m_b}{[z m_b^2 + (1-z) m^2]^2}$$

Now this is a ugly integral:

$$\text{II} = \int_0^1 dz \frac{z(1-z)^2}{[z m_b^2 + (1-z) m^2]^2} = \frac{m^4 + 4m^2 m_b^2 - 5 m_b^4 + (2 m_b^4 + 4 m^2 m_b^2) \ln \frac{m_b^2}{m^2}}{2(m^2 - m_b^2)^4}$$

so

$$\bar{F}_2(0) = \frac{1}{4\pi} \cdot \frac{g^2}{(4\pi)} \cdot m_b \cdot \left(\frac{\delta m^2}{m^2} \right)$$

$$\cdot \frac{1}{2(1 - m_b^2/m^2)^4} \left(1 + 4\frac{m_b^2}{m^2} - 5\frac{m_b^4}{m^4} + 2\left(\frac{m_b^4}{m^4} + 2\frac{m_b^2}{m^2} \right) \log \frac{m_b^2}{m^2} \right)$$

which is of the form

$$\bar{F}_2(0) = G_2 \frac{\alpha}{4\pi} \cdot m_b \frac{1}{m^2}$$

with $G_2 \sim \left(\frac{\delta m^2}{m^2} \right)$

now there is one unfinished piece of business:

the extra-credit part (j)

We need to simplify the expression for $F_1(q^2)$ on p. 16

First, let $\quad x = (1-z)\xi \qquad y = (1-z)(1-\xi)$

$$\int dx\,dy\,dz\;\delta(x+y+z-1) = \int_0^1 dz \int_0^{1-z} dx = \int_0^1 dz \int_0^1 d\xi\;(1-z)$$

now take the term

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\;\frac{(1-z)}{\Delta}$$

$$= \int_0^1 dz \int_0^1 d\xi\;\frac{(1-z)^2}{\left[z m_b^2 + (1-z)m^2 - z(1-z)(\xi m_e^2 + (1-\xi)m_\mu^2) - (1-z)^2\xi(1-\xi)q^2\right]}$$

ol integrate by parts in ξ:

$$= \int_0^1 dz \left\{ \frac{(1-z)\left[(1-z)(\xi m_e + (1-\xi)m_\mu) + m_b\right]/(m_e - m_\mu)}{\left[z m_b^2 + (1-z)m^2 - z(1-z)(\xi m_e^2 + (1-\xi)m_\mu^2) - (1-z)^2\xi(1-\xi)q^2\right]} \Bigg|_{\xi=0}^{\xi=1} \right.$$

$$\left. - \int_0^1 d\xi\;\frac{(1-z)}{m_e - m_\mu}\;\frac{\left[(1-z)(\xi m_e + (1-\xi)m_\mu) + m_b\right]\left[z(1-z)(m_e^2 - m_\mu^2) + (1-z)^2(1-2\xi)q^2\right]}{\left[z m_b^2 + (1-z)m^2 - z(1-z)(\xi m_e^2 + (1-\xi)m_\mu^2) - (1-z)^2\xi(1-\xi)q^2\right]^2} \right\}$$

then give a nice identity:

$$\int_0^1 dz \int_0^{1-z} dx \quad \frac{(1-z)}{\Delta}$$

$$= \int_0^1 dz \quad \frac{(1-z)}{(m_\mu - m_e)} \left\{ \frac{(1-z)m_\mu + m_b}{[z m_b^2 + (1-z)m^2 - z(1-z)m_\mu^2]} - \frac{((1-z)m_e + m_b)}{[z m_b^2 + (1-z)m^2 - z(1-z)m_e^2]} \right\}$$

cancels the first term → F_1

$$- \int_0^1 dz \int_0^{1-z} dx \quad (1-z) \left\{ \frac{(x m_e + y m_\mu + m_b)}{\Delta^2} \right\}$$

$$\cdot \left\{ \frac{1}{m_\mu - m_e} [z(1-z)(m_\mu^2 - m_e^2) - (1-z)(y-x) q^2] \right\}$$

$$= z(1-z)(m_\mu + m_e)$$

so!

$$\frac{F_1(q^2)}{[g^2 \delta m^2 (4\pi)^2]} = \int_0^1 dz \int_0^{1-z} dx \quad (1-z) \quad \frac{q^2}{(m_\mu - m_e)} \quad \frac{(x-y)(x m_e + y m_\mu + m_b)}{\Delta^2}$$

$$= \frac{q^2}{m_\mu - m_e} \int dx\, dy\, dz\, \delta(x+y+z-1) \quad \frac{(x-y)(x m_e + y m_\mu + m_b)}{\Delta^2}$$

or $$F_1(q^2) = \frac{q^2}{m_\mu - m_e} F_3(q^2)$$ Q.E.D.

Physics 330 – Final Exam

This exam is due at noon on Thursday, December 9. Please hand it in to Alex Giryavets in Varian 361. If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Alex Giryavets has blue books, if you need one.

The exam is worth a total of 100 points. The distribution of points is indicated below.

The problem deals with some aspects of the phenomenology of the Higgs boson. Even though the Higgs boson has not yet been discovered, there is a large literature on its properties. Fortunately, most of this literature is too sophisticated to be useful to you for the purpose of solving this exam. I recommend that you simply try to answer the questions using methods from Physics 330. However, the exam is open-book. If you find a useful reference other than the class textbook and notes, feel free to use it, as long as you cite the reference in your solution.

a. (20 points) Let f be the Dirac field representing a quark or lepton. The Higgs boson is a scalar particle, associated with a field h that couples to each fermion field according to

$$\Delta H = \frac{m_f}{v} \overline{f} f h \tag{1}$$

where m_f is the fermion mass and $v = 246$ GeV. Draw the Feynman diagram for Higgs decay into $f\overline{f}$. Compute the partial width using the formula

$$\Gamma(h \to f\overline{f}) = \frac{1}{2m_h} \int d\Pi \left| \mathcal{M}(h \to f\overline{f}) \right|^2 \tag{2}$$

b. (5 points) Taking the masses of the heaviest quarks and leptons to be:

$$\begin{aligned} \text{charge } \tfrac{2}{3} \text{ quarks} &: \quad m_c = 1.2 \text{ GeV} \qquad m_t = 178 \text{ GeV} \\ \text{charge } -\tfrac{1}{3} \text{ quarks} &: \quad m_b = 4.2 \text{ GeV} \\ \text{charge } -1 \text{ leptons} &: \quad m_\mu = 0.105 \text{ GeV} \qquad m_\tau = 1.77 \text{ GeV} \end{aligned} \tag{3}$$

compute the total width and decay rate of a Higgs boson of mass 120 GeV and its branching fractions to each of these species. Do not forget the color factor of 3. (The Higgs boson is also supposed to decay to W and Z bosons; you should ignore those contributions in this exam.)

c. (20 points) Higgs bosons can be produced in quark-antiquark (or proton-antiproton) collisions through the process

$$q + \overline{q} \to h + \gamma \tag{4}$$

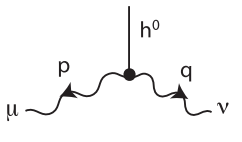
This process is not very much discussed in the literature because it is not a very effective way to produce Higgs bosons. Nevertheless, compute the appropriately polarization summed and averaged differential cross section for this process in the CM frame. You may ignore the mass of the quark, except, of course, in the coupling in eq. (1). Take care with the color factor: Assume that quarks and antiquarks of all three colors are equally probable in the proton wavefunction. Express the cross section as

$$\frac{d^2\sigma}{d^2p_\perp} \tag{5}$$

where $\vec{p}_\perp$ is the photon momentum perpendicular to the collision axis. This expression is invariant to longitudinal boosts. Notice that the cross section depends on the quark species and is much larger for heavier quarks.

d. (5 points) Assuming that the proton contains a b quark or antiquark with probability about 0.1, estimate the order of magnitude of the cross section for production of a Higgs boson in association with a photon with $p_\perp > 100$ GeV. The momentum of the b quark is determined by the proton wavefunction. It might be anything from near 0 to about 1% of the proton energy, so just choose the value of s to maximize the cross section. You do not have to do a detailed calculation; I just want the order of magnitude. If the Large Hadron Collider at CERN will produce enough proton-proton collisions per year to give 100 events of a process whose cross section is 1 fb (1×10^{-39} cm^2), roughly how many events of this type will be produced? (We will discuss the quark structure of the proton in some detail in Physics 331; then you will know how to get a more precise prediction.)

e. (5 points) If m_f is small, the cross section computed in part (c) is absurdly small. However, we could obtain a larger answer if the Higgs boson can couple directly to 2 photons. This vertex is generated by 1-loop Feynman diagrams. It takes the form:



$$= -i\frac{\alpha}{4\pi}C(g^{\mu\nu}p\cdot q - p^\nu q^\mu) \tag{6}$$

Show that this vertex satisfies the Ward Identity.

f. (15 points) Using this vertex, draw a new Feynman diagram for the process $q\overline{q} \to h\gamma$. Assuming that m_f is so small that the previous diagrams can be neglected, compute the differential cross section for $h\gamma$ production. Your answer should be proportional to $|C|^2$.

g. (20 points) Draw the Feynman diagrams by which a loop containing a heavy quark generates the vertex in part (e). For definiteness, use the heavy t quark. Compute these diagrams in the limit $m_t^2 \gg p\cdot q, m_h^2$; that is, ignore all terms that vanish in the

limit $m_t \to \infty$. Use dimensional regularization. You will find a peculiar feature: A crucial term will be proportional to $(d-4)$ times a divergent diagram. However, this term will give a finite and definite value in dimensional regularization, since

$$\lim_{d\to 4} (d-4)\Gamma(2-d/2) = -2 \ . \tag{7}$$

(In Physics 332, if you are still around, I will explain what is going on here and why it is actually correct.) Show that the amplitude actually takes the form of eq. (6). Find the explicit expression for C. (Again, there is a second contribution from W bosons in the loop. Please ignore it for the purpose of this exam.)

h. (10 points) Compute the partial width of the Higgs boson to $\gamma\gamma$. Using the results of parts (b) and (g), obtain a numerical answer for the branching ratio of the Higgs to $\gamma\gamma$.

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (f). Prospective theorists should slog through to the end.

Physics 330 – Final Exam

Solutions

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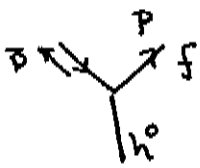
a.) 

$= -i\frac{m_f}{v}$

$$i\mathcal{M} = -i\frac{m_f}{v}\,\bar{v}(\bar{p})\,u(p)$$

$$\sum_{\text{spin}} |\mathcal{M}|^2 = \left(\frac{m_f}{v}\right)^2 \mathrm{tr}\,(\not{p}+m_f)(\not{\bar{p}}-m_f)$$

$$= 4\left(\frac{m_f}{v}\right)^2 (p\cdot\bar{p} - m_f^2)$$

now $(p+\bar{p})^2 = m_h^2 = 2m_f^2 + 2p\cdot\bar{p}$

$$= 2\left(\frac{m_f}{v}\right)^2 (m_h^2 - 4m_f^2)$$

$$\Gamma(h\to f\bar{f}) = \frac{1}{2m_h}\,\frac{(2p/m_h)}{8\pi}\cdot 2\left(\frac{m_f}{v}\right)^2 m_h^2\left(1-\frac{4m_f^2}{m_h^2}\right)$$

$$= \frac{m_h}{8\pi}\left(\frac{m_f}{v}\right)^2\left(1-\frac{4m_f^2}{m_h^2}\right)^{3/2}$$

for quarks there is an extra factor $\underline{3}$ for color.

note: 1. Because h^0 is a scalar, there is no spin-averaging. We just sum over spins in the final state.

2. Because h^0 has $P = +1$ but an $f\bar{f}$ state has $P = (-1)\cdot(-1)^L$, h^0 decays into a state with $L = 1$. This explains the extra factor of $|\vec{p}|^2 = \left(\frac{m_h}{2}\right)^2\left(1 - \frac{4m_f^2}{m_h^2}\right)$

b.) For $m_h = 120$ GeV $\quad \frac{m_h}{8\pi}\left(\frac{1\,\text{GeV}}{v}\right)^2 = 79$ keV

then

$h^0 \to c\bar{c}$ $\quad \times (1.2)^2 \times \left(1 - \frac{4m_c^2}{m_h^2}\right)^{3/2} \cdot 3 = 340$ keV

$h^0 \to t\bar{t}$ $\quad$ forbidden by energy conservation

$h^0 \to b\bar{b}$ $\quad \times (4.2)^2 \cdot \left(1 - \frac{4m_b^2}{m_h^2}\right)^{3/2} \cdot 3 = 4150$ keV

$h^0 \to \mu^+\mu^-$ $\quad \times (0.1)^2 \cdot \left(1 - \frac{4m_\mu^2}{m_h^2}\right)^{3/2} = 0.87$ keV

$h^0 \to \tau^+\tau^-$ $\quad \times (1.77)^2 \cdot \left(1 - \frac{4m_\tau^2}{m_h^2}\right)^{3/2} = 247.$ keV

$\uparrow$ color factor for quarks only

the sum of these values is:

$$\Gamma(h^0) = 4.7 \text{ MeV}$$

decay rate $(\tau)^{-1} = (\hbar/\Gamma)^{-1} = (1.4 \times 10^{-22}\,\text{sec})^{-1} = 7.1 \times 10^{+21}/\text{sec}$

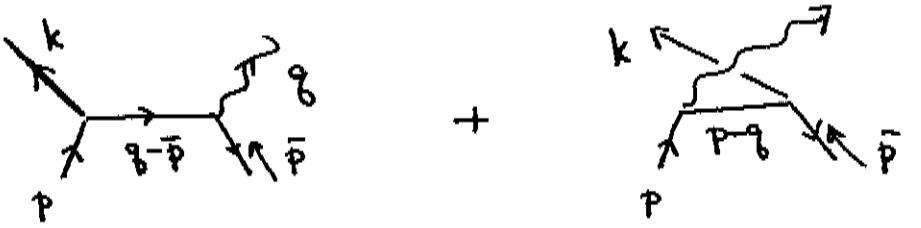
the branching fraction to $f\bar{f}$ is

$$BR(h \to f\bar{f}) = \frac{\Gamma(h \to f\bar{f})}{\Gamma(h)}$$

$c\bar{c}$	$t\bar{t}$	$b\bar{b}$	$\mu^+\mu^-$	$\tau^+\tau^-$
7%	0	88%	1.8×10^{-4}	5%

A more complete thg of h^0 decay would include additional modes $h^0 \to WW^*$, $h^0 \to ZZ^*$, $h^0 \to gg$ ad a better accounty of QCD corrections. See The Higgs Hunter's Guide by Gunions, Haber, Kane, & Dawson.

c.)



$$iM = (+iQ_f e)(-i\frac{m_f}{v})\,\varepsilon^{*\mu}(q)$$

$$\cdot\, \bar{u}(\bar{p})\left[\gamma_\mu \frac{i(\not{q}-\not{p})}{(q-p)^2} + \frac{i(\not{p}-\not{q})}{(p-q)^2}\gamma_\mu\right] u(p)$$

$q^2 = 0$ $\quad k^2 = m_h^2$ $\quad p^2 = \bar{p}^2 = 0$ since we ignore the quark mass

$$(q-\bar{p})^2 = -2q\bar{p} \qquad (q-p)^2 = -2q\cdot p$$

$$\frac{1}{4}\sum_{spin} |M|^2 = \frac{1}{4} e^2 Q_f^2 \left(\frac{m_f}{v}\right)^2 (-g_{\mu\nu})$$

$$\cdot \mathrm{tr}\left[\not{\bar{p}}\left(\frac{\gamma_\mu \not{\bar{p}} \not{q}}{2\bar{p}q} - \frac{\not{p}\gamma_\mu}{2p\cdot q}\right)\not{p}\left(\frac{\not{\bar{p}}\gamma_\nu \not{q}}{2\bar{p}\cdot q} - \frac{\gamma_\nu \not{p}\not{q}}{2p\cdot q}\right)\right]$$

$$= -\frac{1}{4} e^2 Q_f^2 \left(\frac{m_f}{v}\right)^2 \cdot \left\{ \frac{I}{(2\bar{p}\cdot q)^2} - \frac{II}{2\bar{p}\cdot q\, 2p\cdot q} - \frac{III}{2p\cdot q\, 2\bar{p}\cdot q} + \frac{IV}{(2p\cdot q)^2} \right\}$$

$$I = \mathrm{tr}\, \not{\bar{p}}\gamma_\mu(\not{\bar{p}}-\not{q})\not{p}(\not{\bar{p}}-\not{q})\gamma^\mu$$

$$= -2\,\mathrm{tr}\, \not{\bar{p}}(\not{p}-\not{q})\not{p}(\not{\bar{p}}-\not{q})$$

$$= -2\cdot 4\,(\bar{p}\cdot(\bar{p}-q)\, p\cdot(\bar{p}-q)\cdot 2 - \bar{p}\cdot p\,(\bar{p}-q)^2)$$

$$= -2\cdot 4\cdot(-2\bar{p}\cdot q\, p\cdot\bar{p} + 2\bar{p}\cdot q\, p\cdot q - \bar{p}\cdot p(-2\bar{p}\cdot q))$$

$$= -4\cdot 2\bar{p}\cdot q\cdot 2p\cdot q$$

$$II = \mathrm{tr}\, \not{\bar{p}}\gamma_\mu(\not{p}-\not{q})\not{p}\gamma^\mu(\not{p}-\not{q})$$

$$= 4\,\mathrm{tr}\, \not{\bar{p}}(\not{\bar{p}}-\not{q})\cdot p\,(\not{p}-\not{q})$$

$$= 4\cdot 4\cdot \bar{p}\cdot(p-q)\; p\cdot(\bar{p}-q)$$

$$= 4\,(2p\cdot\bar{p} - 2\bar{p}\cdot q)(2p\cdot\bar{p} - 2p\cdot q)$$

$$III = \mathrm{tr}\, \bar{\not p} (\not p - \not q) \gamma^\mu \not p (\bar{\not p} - \not q) \gamma_\mu$$

$$= 4\, \mathrm{tr}\, \bar{\not p} \cdot (\not p - \not q) \quad p \cdot (\bar{p} - q)$$

$$= 4\, (2\bar{p}\cdot p - 2\bar{p}\cdot q)(2\bar{p}p - 2p\cdot q)$$

$$IV = \mathrm{tr}\, [\bar{\not p} (\not p - \not q) \gamma^\mu \not p \gamma_\mu (\not p - \not q)]$$

$$= (-2)\, \mathrm{tr}\, \bar{\not p} (\not p - \not q) \not p (\not p - \not q)$$

$$= -2 \cdot 4 [\bar{p} \cdot (p-q)(-p\cdot q) \cdot 2 - \bar{p}\cdot p\, (p-q)^2]$$

$$= -2 \cdot 4 [\bar{p}\cdot p(-2p\cdot q) + 2\bar{p}\cdot q\; p\cdot q + \bar{p}\cdot p \cdot 2p\cdot q]$$

$$= -4 \quad 2p\cdot q\, 2\bar{p}\cdot q$$

$$\frac{1}{4}\sum_{spin} |M|^2 = -\frac{1}{4} Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \cdot (-4)$$

$$\cdot \left\{ \frac{2\bar{p}\cdot q\, 2p\cdot q}{(2\bar{p}\cdot q)^2} + 2 \cdot \frac{(2\bar{p}\cdot p - 2\bar{p}\cdot q)(2\bar{p}\cdot p - 2p\cdot q)}{2p\cdot q\, 2\bar{p}\cdot q} \right.$$

$$\left. + \frac{2\bar{p}\cdot q\, 2p\cdot q}{(2p\cdot q)^2} \right\}$$

$$= Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left\{ \frac{2p\cdot q}{2\bar{p}\cdot q} + \frac{2\bar{p}\cdot q}{2p\cdot q} + 2 - 2\,\frac{2\bar{p}\cdot p}{2p\cdot q} - 2\,\frac{2\bar{p}\cdot p}{2\bar{p}\cdot q} \right.$$

$$\left. + 2\,\frac{(2\bar{p}\cdot p)^2}{2p\cdot q\, 2\bar{p}\cdot q} \right\}$$

In term of s, t, u:

$$s = (p+\bar{p})^2 = 2p\bar{p} \qquad t = (\bar{p}-q)^2 = -2\bar{p}\cdot q \qquad u = (p-q)^2 = -2p\cdot q$$

$$\frac{1}{4}\sum|M|^2 = Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left\{ \frac{u}{t} + \frac{t}{u} + 2 + 2\frac{s}{u} + 2\frac{s}{t} + 2\frac{s^2}{ut} \right\}$$

$$= Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left\{ \frac{1}{ut} \right\} \left\{ \underbrace{u^2 + t^2 + 2ut + 2st + 2su + 2s^2}_{(s+t+u)^2 + s^2} \right\}$$

$$= (m_h^2)^2 + s^2$$

$$= Q_f^2 e^2 \left(\frac{m_f}{v}\right)^2 \left(\frac{s^2 + m_h^4}{ut}\right)$$

In the CM frame:

$$p = (E, 0\ 0\ E) \qquad k = (\Sigma\ q\sin\theta\ 0\ q\cos\theta)$$

$$\bar{p} = (E, 0\ 0\ -E) \qquad \bar{q} = (q, -q\sin\theta\ 0\ -q\cos\theta)$$

$$\Sigma + q = 2E = \sqrt{s} \qquad \Sigma^2 - q^2 = m_h^2 \rightarrow q = \frac{s - m_h^2}{2\sqrt{s}}$$

$$\text{Phase space} = \frac{1}{16\pi} d\cos\theta \frac{d\phi}{2\pi} \left(\frac{2q}{\sqrt{s}}\right)$$

$$= \frac{1}{16\pi \cdot 2\pi} d\cos\theta\, d\phi \left(1 - \frac{m_h^2}{s}\right)$$

$$s = 4E^2 \qquad t = -2Eq(1-\cos\theta) \qquad u = -2Eq(1+\cos\theta)$$

$$ut = 4E^2 q^2 (1 - \cos^2\theta)$$

Finally, the color factor: the quark & antiquark must have the same color to annihilate. The probability of this is $\frac{1}{3}$

$$\left.\frac{d\sigma}{d\cos\theta\, d\phi}\right|_{cm} = \frac{1}{3}\cdot\frac{1}{2s}\cdot\frac{1}{16\pi\cdot 2\pi}\,\frac{2q}{\sqrt{s}}\,Q_f^2 e^2\left(\frac{m_f}{v}\right)^2\frac{s^2+m_h^4}{s\cdot q^2(1-\cos^2\theta)}$$

$$\left.\frac{d\sigma}{d\cos\theta\, d\phi}\right|_{cm} = \frac{\alpha}{4q\pi\, s}\,Q_f^2\left(\frac{m_f}{v}\right)^2\left(\frac{2q}{\sqrt{s}}\right)\left(1+\frac{m_h^4}{s^2}\right)\cdot\frac{s}{q^2\sin^2\theta}$$

with $q = \frac{s-m_h^2}{2\sqrt{s}}$

now $q^2\sin^2\theta = q_\perp^2$ the γ transverse momentum

we need to convert $d\cos\theta\, d\phi = \sin\theta\, d\theta\, d\phi$ to $d^2q_\perp$

$$q^1 = q\sin\theta\cos\phi \qquad q^2 = q\sin\theta\sin\phi$$

$$\frac{\partial(q^1q^2)}{\partial(\theta\phi)} = \begin{vmatrix} q\cos\theta\cos\phi & q\cos\theta\sin\phi \\ -q\sin\theta\sin\phi & q\sin\theta\cos\phi \end{vmatrix} = q^2\cos\theta\sin\theta$$

so $$\frac{d\theta\sin\theta\, d\phi}{dq_\perp^1\, dq_\perp^2} = \frac{1}{q^2\cos\theta} = \frac{1}{q\,(q^2-q_\perp^2)^{\frac{1}{2}}}$$

then for $q + \bar{b} \to h^0 + \gamma$

$$\frac{d^2\sigma}{dq_\perp^2} = \frac{\alpha}{24\pi} Q_f^2 \left(\frac{m_f}{v}\right)^2 \left(1 + \frac{m_h^4}{s^2}\right) \left(\frac{s}{q^2 - q_\perp^2}\right)^2 \frac{1}{q_\perp^2}$$

d.) To estimate this, write

$$\sigma \sim \frac{\alpha}{24\pi} Q_f^2 \left(\frac{m_f}{v}\right)^2 \frac{2\pi}{q_\perp^2} \cdot P_f^2$$

put $Q_f = -\frac{1}{3}$ $m_f = 4.2\,\text{GeV}$ for the b quark

P_f = prob of b in proton ~ 0.1

$q_\perp \sim 100\,\text{GeV}$

$$\sigma \sim \frac{\alpha}{12} \cdot \frac{1}{9} \cdot \left(\frac{4.2}{246}\right)^2 \cdot 10^{-2} \cdot \frac{1}{(100\,\text{GeV})^2}$$

$$\cdot [0.197\ \text{GeV} \times 10^{13}\,\text{cm}]^2 \qquad 1\,\text{fm} = 10^{-13}\,\text{cm}$$

$$\sim 0.76 \times 10^{-15} \times (10^{-26}\,\text{cm}^2)$$

now $1\ \text{barn} = 10^{-24}\,\text{cm}^2$ so this is

$$\sim .0076\ \underline{\text{fb}}$$

so if the LHC produces enough collisions to give 100 events for $\sigma = 1\,\text{fb}$

we will have <u>1 event</u> !

e.)

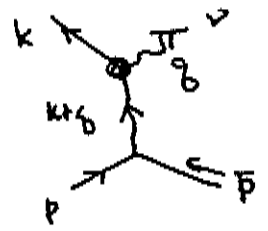
$$p_\mu \left(\overset{p}{\sim}\!\!\!\!\!\!\bigcirc\!\!\!\!\!\!\sim \overset{q}{} \right)^{\mu\nu} = p_\mu \left(-i \frac{\alpha}{4\pi} C \left(g^{\mu\nu} p\cdot q - p^\nu q^\mu \right) \right)$$

$$= -i \frac{\alpha}{4\pi} \left(p^\nu p\cdot q - p^\nu p\cdot q \right)$$

$$= 0 \qquad \checkmark$$

similarly $\left(\overset{p}{\sim}\!\!\!\!\!\!\bigcirc\!\!\!\!\!\!\sim \overset{q}{} \right)^{\mu\nu} q_\nu = 0$

f.)



$$iM = (+ie Q_f)\cdot(-i\frac{\alpha}{4\pi} C)\ \bar{v}(\bar{p}) \gamma_\mu u(p) \ \frac{-i}{s}$$

$$\cdot \left[g^{\mu\nu} q\cdot(-k-q) - (-k-q)^\mu q^\nu \right] \varepsilon^*_\nu(q)$$

$$2q\cdot k = (q+k)^2 - m_h^2 = s - m_h^2 \qquad 2p\cdot\bar{p} = s$$

$$q^2 = 0$$

$$q\cdot\varepsilon^*(q) = 0$$

$$t = -2\bar{p}\cdot q = m_h^2 - 2p\cdot k$$

$$u = -2p\cdot q = m_h^2 - 2\bar{p}\cdot k$$

$$= i\frac{e\alpha}{4\pi} Q_f C \ \frac{1}{s} \ \bar{v}(\bar{p}) \gamma^\nu u(p) \left[g_{\mu\nu} q\cdot k\, \varepsilon^*(q) - q_\mu k\cdot\varepsilon^*(q) \right]$$

$$= i\frac{e\alpha}{4\pi} Q_f C \ \frac{1}{s} \ \bar{v}(\bar{p}) \gamma^\nu u(p) \left[\varepsilon^*_\mu(q)\, q\cdot k - q_\mu k\cdot\varepsilon^*(q) \right]$$

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{1}{4}\frac{e^2\alpha^2}{(4\pi)^2} Q_f^2 |C|^2 \frac{1}{s^2} k[\bar{p}\gamma_\mu \not{q}\gamma_\nu]$$

$$\cdot \sum_{\text{spin}} [\varepsilon^\mu(q)^* q\cdot k - q^\mu k\cdot\varepsilon^*(q)][\varepsilon^\lambda(q) q\cdot k - q^\lambda k\cdot\varepsilon(q)]$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2} \cdot 4[\bar{p}_\mu p_\alpha + \bar{p}_\alpha p_\mu - g_{\mu\nu}\bar{p}\cdot p]$$

$$\cdot [-g^{\mu\lambda}(q\cdot k)^2 + (q^\mu k^\lambda + g^\lambda k^\mu) q\cdot k - q^\mu q^\lambda k^2]$$

$$= \frac{1}{4}\frac{\alpha^3}{(4\pi)} Q_f^2 |C|^2 \frac{1}{s^2} \cdot$$

$$\cdot 4\{(q\cdot k)^2[-2\bar{p}\cdot p + 4\bar{p}\cdot p]$$

$$+ 2q\cdot k[q\bar{p}\, k\cdot p + q\cdot p\, k\bar{p} - q\cdot k\,\bar{p}\cdot p]$$

$$- [2 q\bar{p}\, q\cdot p\, k^2 - \underbrace{q^2}_{0} k^2 p\cdot\bar{p}]\}$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2} \cdot \{(s-m_h^2)^2 \cdot s$$

$$+ (s-m_h^2)[(-t)(m_h^2-t) + (-u)(m_h^2-u) - (s-m_h^2)s]$$

$$-2m_h^2 ut\}$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2}\{[u^2+t^2-(u+t)m_h^2](s-m_h^2) - 2m_h^2 ut\}$$

$$u^2+t^2 = (u+t)^2 - 2ut = (s-m_h^2)^2 - 2ut \qquad u+t = m_h^2 - s$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2}\{(s-m_h^2)^3 - 2ut(s-m_h^2) + (s-m_h^2)^2 m_h^2 - 2ut m_h^2\}$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{s^2}\{(s-m_h^2)^2\cdot s - 2stu\}$$

$$= \frac{1}{4}\frac{\alpha^3}{(4\pi)} Q_f^2 |C|^2 \frac{1}{s}\left[(s-m_h^2)^2 - 2ut\right]$$

in the CM frame, from p.6

$$(s-m_h^2) = 2\sqrt{s}\, q = 4Eq$$

$$ut = 4E^2q^2(1-\cos\theta)(1+\cos\theta) = 4E^2q^2\sin^2\theta$$

$$= \frac{1}{4}\frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \frac{1}{(2E)^2} 4E^2q^2\left[4-2\sin^2\theta\right]$$

$$= \frac{\alpha^3}{4\pi} Q_f^2 |C|^2 \cdot 2q^2(2-\sin^2\theta) = \frac{\alpha^3}{2\pi} Q_f^2 |C|^2 q^2(1+\cos^2\theta)$$

$$\left.\frac{d\sigma}{d\cos\theta\, d\phi}\right|_{cm} = \frac{1}{3}\cdot\frac{1}{2s}\cdot\frac{1}{16\pi\cdot 2\pi}\cdot\frac{2q}{\sqrt{s}}\cdot\frac{\alpha^3}{2\pi} Q_f^2 |C|^2 q^2(1+\cos^2\theta)$$

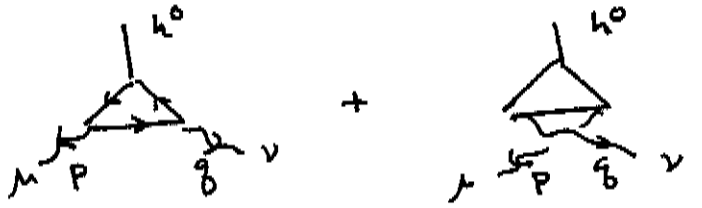
$$= \frac{\alpha^3 Q_f^2 |C|^2}{192\pi^3}\left(\frac{q^3}{s^{3/2}}\right)(1+\cos^2\theta)$$

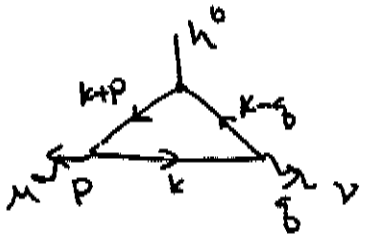
$$= \frac{\alpha^3 Q_f^2 |C|^2}{1536\pi^3}\left(\frac{2q}{\sqrt{s}}\right)^3(1+\cos^2\theta)$$

$$\frac{d\sigma}{dq_t^2} = \frac{\alpha^3 Q_f^2 |C|^2}{1536\pi^3}\left(\frac{2q}{\sqrt{s}}\right)^3 \frac{1}{q(q^2-q_t^2)^{1/2}}(1+\cos^2\theta)_{cm}$$

$$= \frac{\alpha^3 Q_f^2 |C|^2}{1536\pi^3}\left(\frac{2q}{\sqrt{s}}\right)^3 \frac{2q^2-q_t^2}{q^3(q^2-q_t^2)^{1/2}}$$

g.)





fermion loop cola

$$= (+i\tfrac{2}{3}e)^2 (-i\tfrac{m_b}{v}) \int \frac{d^4k}{(2\pi)^4} \cdot (-1) \cdot 3$$

$$\cdot \operatorname{tr}\left[i\frac{(\not k - \not q + m_t)}{(k-q)^2 - m_t^2}\gamma^\nu \; i\frac{(\not k + m_t)}{k^2 - m_t^2}\gamma^\mu \; i\frac{(\not k + \not p + m_t)}{(k+p)^2 - m_t^2}\right]$$

$$= -\frac{4}{9}\cdot 3\, e^2 \frac{m_t}{v} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k-q)^2 - m_t^2][k^2 - m_t^2][(k+p)^2 - m_t^2]}$$

$$\cdot \operatorname{tr}\left[(\not k - \not q + m_t)\gamma^\nu (\not k + m_t)\gamma^\mu (\not k + \not p + m_t)\right]$$

Completar denominadores $\quad x+y+z = 1$

$$\text{Denom} = x((k-q)^2 - m_t^2) + y((k+p)^2 - m_t^2) + z(k^2 - m_t^2)$$

$$= k^2 + 2k\cdot(-xq + yp) + xq^2 + yp^2 - m_t^2$$

$$= \ell^2 - (-xq+yp)^2 + xq^2 + yp^2 - m_t^2$$

$$= \ell^2 + x(1-x)q^2 + y(1-y)p^2 + 2xy\, p\cdot q - m_t^2$$

$$= \ell^2 - \Delta \qquad \text{with}$$

$$\Delta = m_t^2 - x(1-x)q^2 - y(1-y)p^2 - 2xy\, p\cdot q$$

$\ell = k - xq + yp$ so

$$k = \ell + xq - yp$$
$$k - q = \ell - (1-x)q - yp$$
$$k + p = \ell + xq + (1-y)p$$

then

$$\mathrm{tr}\left[(\not{k}+\not{q}+m_t)\,\gamma^\nu(\not{k}+m_t)\gamma^\mu(\not{k}+\not{p}+m_t)\right]$$

(keep terms w. 2 + 4 's inside the trace.)

$$= 4\Big[\, m_t^3 g^{\mu\nu} + m_t\big(k^\nu(k+p)^\mu + k^\mu(k+p)^\nu - g^{\mu\nu}\,k\cdot(k+p)\big)$$
$$+ m_t\big((k-q)^\nu k^\mu + (k-q)^\mu k^\nu - g^{\mu\nu}\,k\cdot(k-q)\big)$$
$$+ m_t\big((k-q)^\nu(k+p)^\mu - (k-q)^\mu(k+p)^\nu + g^{\mu\nu}(k-q)\cdot(k+p)\big)\Big]$$

$$= 4\Big[\, m_t^3 g^{\mu\nu} + m_t\big(k^\mu k^\nu \cdot 4 - g^{\mu\nu}k^2\cdot 1\big)$$

$$+ m_t\Big([x_q+(1-y)p]^\mu[\delta_q-y_p]^\nu + [\delta_q-y_p]^\mu[x_q+(1-y)p]^\nu$$
$$- g^{\mu\nu}(x_q-y_p)\cdot(\delta_q+(1-y)p)$$
$$+ [\delta_q-y_p]^\mu[-\delta_q+(x-1)]^\nu + [-\delta_q+(x-1)]^\mu[\delta_q-y_p]^\nu$$
$$- g^{\mu\nu}(-\delta_q+(x-1))\cdot(x_q-y_p)$$
$$+ [x_q+(1-y)p]^\mu[-\delta_q+(x-1)]^\nu - [-\delta_q+(x-1)]^\mu[\delta_q+(1-y)p]^\nu$$
$$+ g^{\mu\nu}(-\delta_q+(x-1))\cdot(\delta_q+(1-y)p)\Big)\Big]$$

(drop term linear in k.)

$$= 4\Big[m_t^3 g^{\mu\nu} + m_t (4k^\mu k^\nu - k^2 g^{\mu\nu})$$

$$+ m_t \Big(g^\mu g^\nu \cdot [x^2 \cdot 2 - x(1-x)\cdot 2 - \cancel{x(1-x)} + \cancel{x(1-x)}]$$

$$+ p^\mu p^\nu \; [-2y(1-y) + 2y^2 - \cancel{y(1-y)} + \cancel{y(1-y)}]$$

$$+ g^\mu p^\nu \; [-xy + x(1-y) - xy + y(1-x) - xy + (1-x)(1-y)]$$

$$+ p^\mu g^\nu \; [x(1-y) - xy + y(1-x) - xy - (1-x)(1-y)]$$

$$- g^{\mu\nu} g^2 \; [x^2 - x(1-x) + \cancel{x(1-x)}]$$

$$- g^{\mu\nu} p^2 \; [-\cancel{y(1-y)} + y^2 + \cancel{y(1-y)}]$$

$$- g^{\mu\nu} p\cdot g \; [x(1-y) - xy - xy + y(1-x) + xy + (1-x)(1-y)]\Big)\Big]$$

$$= 4\Big[m_t^3 g^{\mu\nu} + m_t (4k^\mu k^\nu - k^2 g^{\mu\nu})$$

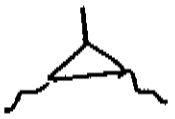
$$+ m_t \Big(g^\mu g^\nu (4x^2 - 2x) + p^\mu p^\nu (4y^2 - 2y)$$

$$+ g^\mu p^\nu (1 - 4xy) + p^\mu g^\nu (-1 + 2(x+y) - 4xy)$$

$$- g^{\mu\nu} g^2 \; x^2 - g^{\mu\nu} p^2 \; y^2$$

$$- g^{\mu\nu} p\cdot g \; [1 - 2xy]\Big)\Big]$$

so



$$= -\frac{4}{3}\cdot e^2\frac{m_t}{v}\cdot 4m_t\int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\cdot\int\frac{d^4k}{(2\pi)^4}\cdot\frac{2}{[k^2-\Delta]^3}$$

$$\Big[\,(4k^\mu k^\nu - g^{\mu\nu}k^2) \;+\; g^{\mu\nu}m_t^2$$

$$+\; q^\mu q^\nu(4x^2-2x) \;+\; p^\mu p^\nu(4y^2-2y) \;+\; g^{\mu\nu}(1-4xy)$$

$$+\; p^\mu q^\nu(-1+2(x+y)-4xy)$$

$$-\,g^{\mu\nu}q^2x^2 \;-\,g^{\mu\nu}p^2y^2 \;-\,g^{\mu\nu}p\cdot q\,[1-2xy]\;\Big]$$

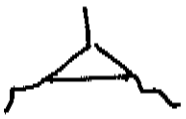
now do the integrals using dimensional regularization:

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-\Delta]^3} = \frac{-i}{(4\pi)^{d/2}}\frac{\Gamma(3-d/2)}{\Gamma(3)}\frac{1}{\Delta^{3-d/2}}$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-\Delta)^3} = \frac{+i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(3)}\frac{1}{\Delta^{2-d/2}}\cdot\frac{d}{2}$$

$$\Gamma(3) = 2$$

so that →



$$= -\frac{16}{3} e^2 \frac{m_t^2}{v} \int dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\cdot \frac{i}{(4\pi)^{d_4}} d_4 \cdot \left\{ \frac{\Gamma(2-d_4)}{\Delta^{2-d_4}} \left(4 \cdot \tfrac{1}{2} g^{\mu\nu} - g^{\mu\nu} \frac{d}{2}\right) \right.$$

$$- \frac{\Gamma(3-d_4)}{\Delta^{3-d_4}} m_t^2$$

$$\left. - \frac{\Gamma(3-d_4)}{\Delta^{3-d_4}} \left(q^\mu q^\nu (4x^2 - 2x) + \cdots - g^{\mu\nu} p\cdot q (1-2xy) \right) \right\}$$

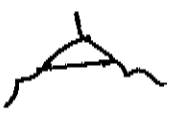
the first term is proportional to

$$\frac{\Gamma(2-d_4)}{\Delta^{2-d_4}} (2 - d_4) g^{\mu\nu} \xrightarrow[d\to 4]{} \frac{1}{\Delta^0} g^{\mu\nu} = \frac{\Delta}{\Delta} g^{\mu\nu}$$

the remaining term involves

$$\frac{\Gamma(3-d_4)}{\Delta^{3-d_4}} \xrightarrow[d\to 4]{} \frac{1}{\Delta}$$

so



$$= -\frac{i}{(4\pi)^2} \frac{16}{3} e^2 \frac{m_t^2}{v} \int dx\, dy\, dz\, \delta(x+y+z-1) \cdot \frac{1}{\Delta}$$

$$\left\{ \left[\cancel{m_t^2} - x(1-x) q^2 - y(1-y) p^2 - 2xy\, p\cdot q \right] g^{\mu\nu} - \cancel{m_t^2} g^{\mu\nu} \right.$$

$$- q^\mu q^\nu (4x^2 - 2x) - p^\mu p^\nu (4y^2 - 2y) - q^\mu p^\nu (1-4x) - p^\mu q^\nu (-1 + 2(x+y) - 4xy)$$

$$\left. + g^{\mu\nu} q^2 x^2 + g^{\mu\nu} p^2 y^2 + g^{\mu\nu} p\cdot q\, [1 - 2xy] \right\}$$

how we can take the limit $m_t \to \infty$. In this limit

$$\frac{m_t^2}{\Delta} = \frac{m_t^2}{[m_t^2 - x(1-x)k_0^2 - y(1-y)p^2 - 2xyp\cdot q]} \to \frac{m_t^2}{m_t^2} = 1$$

this makes the integrals over x, y, z easy to do

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;1 = \int_0^1 dx \int_0^{1-x} dy = \int_0^1 dx\,(1-x) = \frac{1}{2}$$

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;x = \int_0^1 dx\; x\,(1-x) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;x^2 = \int_0^1 dx\; x^2(1-x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\;xy = \int_0^1 dx\; x \int_0^{1-x} dy\; y$$

$$= \int_0^1 dx\; x\,\frac{(1-x)^2}{2} = \int_0^1 dx\; \frac{x}{2}(1-2x+x^2) = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{24}$$

to repeat, let $\langle \cdot \rangle = \int dx\,dy\,dz\,\delta(x+y+z-1)\cdot$

then

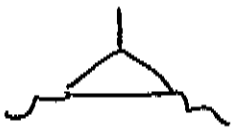
$$\langle 1 \rangle = \frac{1}{2}$$

$$\langle x \rangle = \langle y \rangle = \langle z \rangle = \frac{1}{6} \qquad \langle (x+y+z) \rangle = \frac{1}{2} \;\checkmark$$

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{12}$$

$$\langle xy \rangle = \langle yz \rangle = \langle xz \rangle = \frac{1}{24}$$

$$\langle (x+y+z)^2 \rangle = \langle x^2+y^2+z^2+2xy+2xz+2yz \rangle = 3\cdot\frac{1}{12} + 3\cdot 2\cdot\frac{1}{24} = \frac{1}{2} \;\checkmark$$



$$= -\frac{i}{4\pi}\,\frac{16}{3}\,\alpha\,\frac{1}{v}$$

$$\cdot\Big\{ g^{\mu\nu}\, q^2 \langle 2x^2 - x\rangle + g^{\mu\nu} p^2 \langle 2y^2 - y\rangle$$

$$+ g^{\mu\nu} p\cdot q \langle 1 - 4xy\rangle$$

$$- q^\mu q^\nu \langle 4x^2 - 2x\rangle - p^\mu p^\nu \langle 4y^2 - 2y\rangle - q^\mu p^\nu \langle 1 - 4xy\rangle$$

$$- p^\mu q^\nu \langle -1 + 2(x+y) - 4xy\rangle \Big\}$$

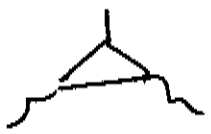
now $\quad \langle 2x^2 - x\rangle = \langle 2y^2 - y\rangle = 2\cdot\frac{1}{12} - \frac{1}{6} = 0$

$$\langle 4x^2 - 2x\rangle = \langle 4y^2 - 2y\rangle = 4\cdot\frac{1}{12} - 2\frac{1}{6} = 0$$

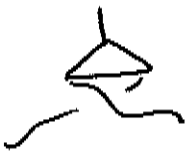
$$\langle -1 + 2(x+y) - 4xy\rangle = -\frac{1}{2} + 2\cdot 2\cdot\frac{1}{6} - 4\frac{1}{24} = 0$$

$$\langle 1 - 4xy\rangle = \frac{1}{2} - 4\frac{1}{24} = \frac{1}{3}$$

so!

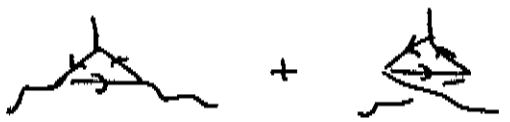


$$= \frac{-i\alpha}{4\pi}\cdot\frac{16}{3}\cdot\frac{1}{v}\cdot\frac{1}{3}\cdot\left(g^{\mu\nu}\, p\cdot q - p^\nu q^\mu\right)$$



$$=$$ the above, with $\begin{pmatrix} p \\ \mu \end{pmatrix} \leftrightarrow \begin{pmatrix} q \\ \nu \end{pmatrix} =$ the above.

so



$$= +i\,\frac{\alpha}{4\pi}\left(-\frac{32}{9}\frac{1}{v}\right)\left(g^{\mu\nu}p\cdot q - p^{\nu}q^{\mu}\right)$$

so, from the top quark loop, $C = -\frac{32}{9}\frac{1}{v}$

[It turns out that W boson loops give a larger contribution of the opposite sign!]

h.) $iM(h^0 \to \gamma\gamma)$ =  $= -i\,\frac{\alpha}{4\pi}\,C\left(g^{\mu\nu}p\cdot q - p^{\nu}q^{\mu}\right)$

$\cdot\ \varepsilon_{\mu}^{*}(p)\,\varepsilon_{\nu}^{*}(q)$

$$\sum_{\text{spin}} |M|^2 = \left(\frac{\alpha}{4\pi}\right)^2 |C|^2 \left(g^{\mu\nu}p\cdot q - p^{\nu}q^{\mu}\right)\left(g_{\mu\nu}p\cdot q - p_{\nu}q_{\mu}\right)$$

$$= \left(\frac{\alpha}{4\pi}\right)^2 |C|^2 \cdot \left[4(p\cdot q)^2 - 2(pq)^2 + \underbrace{p^2 q^2}_{=0}\right]$$

$$= \left(\frac{\alpha}{4\pi}\right)^2 |C|^2 \cdot 2(p\cdot q)^2$$

and $m_h^2 = (p+q)^2 = 2p\cdot q$

$$= \frac{1}{2}\left(\frac{\alpha}{4\pi}\right)^2 |C|^2 \, m_h^4$$

$$\Gamma(h^0 \to \gamma\gamma) = \frac{1}{2m_h} \cdot \frac{1}{8\pi} \cdot \underbrace{\frac{1}{2}}_{\text{id. particles}} \cdot \frac{1}{2}\left(\frac{\alpha^2}{4\pi}\right) |C|^2 m_h^4$$

$$= \frac{\alpha^2}{8 \cdot 8 \cdot 16\pi^3} |C|^2 m_h^3$$

$$= \frac{\alpha^2}{(32)^2 \pi^3} \left(-\frac{32}{9v}\right)^2 m_h^3$$

$$\Gamma(h^0 \to \gamma\gamma) = \frac{\alpha^2}{81\pi^3} \frac{m_h^3}{v^2}$$

numerically:

$$\Gamma(h^0 \to \gamma\gamma) = \underbrace{\frac{m_h}{8\pi}\left(\frac{1\,\text{GeV}}{v}\right)^2}_{79\ \text{keV}} \cdot \underbrace{\frac{8\alpha^2}{81\pi^2}\left(\frac{m_h}{\text{GeV}}\right)^2}_{7.7 \times 10^{-3}}$$

$$= 0.61\ \text{keV}$$

$$BR(h^0 \to \gamma\gamma) = (\text{above}) / 4.7\ \text{MeV} = 1.3 \times 10^{-4}$$

[For the real Higgs boson of mass 120 GeV, $BR(h^0 \to \gamma\gamma)$ is enhanced by the W boson loop to about 10^{-3}.]

Physics 330 – Final Exam

This exam is due at noon on Friday, December 16. Please hand it in to Alex Giryavets in Varian 361. If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Alex Giryavets has blue books, if you need some.

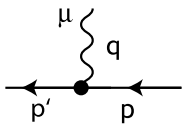
The exam is worth a total of 100 points. The distribution of points is indicated below.

This problem has some historical interest, because it concerns a very interesting exercise in quantum field theory that Feynman solved under circumstances that clarified for him the power and general applicability of the methods taught in this course. I attach the story at the end of the exam.

Since the problem is of historical interest only, I doubt that you will find a clear treatment of it in the literature. I recommend that you simply try to answer the questions using methods from Physics 330. However, the exam is open-book. If you find a useful reference other than the class textbook and notes, feel free to use it, as long as you cite the reference in your solution.

This problem concerns a simple field theory description of the interactions of pions and nucleons.

a. (10 points) To begin, we will need to know something about the electromagnetic interactions of scalar particles. I claim that, if ϕ is a scalar field of charge $(-e)$, like the electron, then the Feynman rule for the coupling of ϕ to a photon is :



$$= -ie(p + p')^\mu \tag{1}$$

This interaction can be derived from the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi \tag{2}$$

with $D_\mu = \partial_\mu + ieA_\mu$. The term linear in A_μ is

$$\delta\mathcal{L} = \partial_\mu \phi^* (ieA^\mu)\phi - ieA^\mu \phi^* \partial_\mu \phi \tag{3}$$

Since the incoming ϕ has a wavefunction $e^{-ip\cdot x}$, we can replace $\partial_\mu \phi$ by $(-ip_\mu)$, and treat $\partial_\mu \phi^*$ similarly, to obtain the Feynman rule.

Now, here is the question: Verify that this vertex satisfies the following consistency checks:

(1) Show that the nonrelativistic electromagnetic scattering of a charged scalar particle ϕ from an electron gives a Coulomb potential with the correct sign. Show this also for the positively charged antiparticle ϕ^*.

(2) Show that the vertex $\gamma \to \phi\phi^*$ for pair production, which would be used to compute $e^+e^- \to \phi\phi^*$, satisfies current conservation for ϕ, ϕ^* on-shell.

b. (10 points) Compute the cross section for $e^+e^- \to \phi\phi^*$, averaged over electron spins. You may ignore the mass of the electron, but not the mass of the ϕ. How does the total cross section compare with that for $e^+e^- \to \mu^+\mu^-$ as $E_{cm} \to \infty$?

c. (5 points) Now turn to the description of pion-nucleon interactions. Let N represent a doublet of Dirac fermion fields

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \tag{4}$$

where p, n represent the Dirac fermion fields of the proton and neutron. Let π^i, $i = 1, 2, 3$ represent three pseudoscalar pion fields

$$\pi^0 = \pi^3 \qquad \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2) \tag{5}$$

A coupling of the pions to the nucleons can be written

$$\Delta\mathcal{H} = ig_{\pi NN}\pi^i\overline{N}\gamma^5\sigma^i N \tag{6}$$

The notation is slightly tricky; the γ^5 acts on the Dirac indices to create a pseudoscalar coupling. The σ^i acts on the 2-component indices that the distinguish the p and the n. The π^- field *destroys* a π^- and *creates* a π^+. Work out the Feynman rules. Write explicitly the vertex by which a proton emits a π^+ and converts to a neutron. Show that the electric charge assignments work out correctly for this process.

For simplicity, let the three pions have equal masses m_π and let the proton and neutron have equal masses m_N.

d. (10 points) Consider proton-proton scattering in the leading order of pertubation theory. Work out the leading term in the scattering amplitude in the limit of nonrelativistic velocities. Show that this term represents a spin-spin interaction, mediated by a potential related to the Yukawa potential. Write the similar expressions for neutron-neutron and proton-neutron scattering.

Since we do not obtain a universally attractive potential, the interaction (6) does not, by itself, explain the binding of protons and neutrons into nuclei. For this, we need a more complicated model, such as the linear sigma model considered in Problem Set 5.

e. (10 points) The coupling (6) is called the 'pseudoscalar' interaction. It is also possible to write a 'pseudovector' interaction:

$$\Delta\mathcal{H} = -\frac{g_{\pi NN}}{2m_N}\partial_\mu\pi^i\overline{N}\gamma^\mu\gamma^5\sigma^i N \tag{7}$$

Work out the leading term for nonrelativistic proton-proton scattering in the pseudovector model, and show that it is identical to the result for the pseudoscalar model.

f. (10 points) Write the full relativistic amplitudes for proton-proton scattering in the leading order of perturbation theory in the pseudoscalar and pseudovector theories. Show that these are identical. Does the trick work also for other scattering processes, considered in leading order?

g. (5 points) In classical physics, one might not expect the neutron to interact with electromagnetic fields. However, in quantum field theory, the neutron can fluctuate to a proton plus a pion. Thus, there are nonzero contributions to the electromagnetic form factors of the neutron at one-loop order. Draw the diagrams that contribution to the F_1 and F_2 form factors of the neutron in order $g_{\pi NN}^2$.

h. (10 points) Compute these diagrams, up to the integrals over Feynman parameters. Use dimensional regularization. Show that the results for F_1 and F_2 are finite both in the UV and in the IR.

i. (10 points) Show explicitly that $F_1(0) = 0$, as required by general principles.

j. (10 points) The charge radius r of the neutron can be defined by

$$F_1(q^2) = 6q^2r^2 + \mathcal{O}(q^4) \tag{8}$$

Compute r^2 in terms of the parameters of this theory.

k. (10 points) Show that the pseudovector theory gives a different expression for the form factors of the neutron from that of the pseudoscalar theory, one which is UV divergent. Is it only F_1 or also F_2 that is divergent?

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (g). Prospective theorists should slog through to the end.

Finally, here is the story about Feynman. It is one of my favorite Feynman stories, not least because probably happened more or less as it is told. I quote extensively from Jagdish Mehra's biography, *The Beat of a Different Drum*, Oxford U. Press, 1994:

At the 1949 APS meeting, Murray Slotnik presented new results, which he had obtained after two years of calculation, concerning the interaction between an electron and a nuetron. He had found that the answers for the pseudoscalar theory and the pesudovector theory were different. In fact, in the case of the pseudovector theory, the answer was logarithmically divergent, but, in the pseudoscalar theory, it was convergent and gave well-defined answers. [J. Robert] Oppenheimer, who was in the audience, asked Slotnik: 'Well, what about Case's

theorem?' By this he meant the new result of Kenneth Case (who, at that time, was a postdoctoral fellow at Institute for Advanced Study in Princeton), which was going to be reported the next day. [Oppenheimer was, at that time, the Director of the Institute.] Case had announced that he had proved that the results for both the pseudoscalar and pseudovector theories were the same. Slotnick answered: 'I never heard of Case's theorem!'

Feynman had missed Slotnick's talk, but somebody asked him about the discussion between Oppenheimer and Slotnick. He went to Slotnick and said: 'Look, I am very anxious to try out if I understand what these things mean. So just tell me what you did.' He replied, 'I scattered the electron off the neutron and I have a correction due to the mesons.'

Feynman described what happened next as follows: 'This was a welcome opportunity to test my guesses as to whether I really understood what these two couplings were. So I went home, and during the evening I worked out the electron-neutron scattering for the pseudoscalar and pseudovector couplings, saw that they were not equal and subtracted them, and worked out the difference in detail. The next day, at the meeting, I saw Slotnick and said, "Slotnick, I worked it out last night, I wanted to see if I got the same answers you do. I got a different answer for each coupling—but I would like to check in detail with you because I want to be sure of my methods." And he said, "What do you mean you worked it out last night? It took me six months!" And, when we compared the answers, he looked at mine and he asked, "What is that Q in there, that variable Q?" ... I said, "That's the momentum transferred by the electron, the electron deflected by different angles." "Oh," he said, "No, I only have the limiting value as Q approaches [0 for] the forward scattering." Well, it was easy enough to just substitute Q equals zero in my answers as he did. But it took him six months to do the case of zero momentum transfer, whereas, during one evening, I had done the finite and arbitrary momentum transfer. That was a thrilling moment for me, like receiving the Nobel Prize, because that convinced me, at last, I did have some kind of method and technique and understood how to do something that other people did not know how to do. That was my moment of triumph in which I realized I really had succeeeded in working out something worthwhile.' ...

As Feynman further recalled, next day Case reported his theorem at the APS meeting. 'And, just to be annoying, when Case finished, I said, "Yeah, but what about Slotnick's calculation?" You know, I mean Oppenheimer was imperious. If Case proved the theorem, it must be true. And I argued, "What about Slotnick's calculation? That theorem can't be true." And everybody laughed becuase it was perfectly logical to suppose the theorem is at fault rather than the calculation, you know. So [Oppenheimer] said: "Well, maybe Slotnick's calculation is wrong." I said, "No, I checked it last night and it's all right. I believe it's right." '

Physics 330 - Final Exam

Solutions

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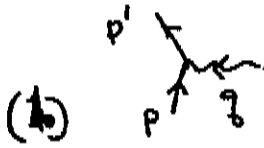
a.) 

$$= (-ie)\,\bar{u}(p')\gamma^{\mu}u(p)\;\frac{-ig_{\mu\nu}}{q^2}\;(-ie)(k+k')^{\nu}$$

$$iM = ie^2\,\bar{u}(p')\gamma^{\mu}u(p)\,(k+k')_{\mu}\,\frac{1}{q^2}$$

in the limit of nonrelativistic scattering

$$\bar{u}^{(s')}(p')\gamma^{\mu}u^{(s)}(p) \cong (2m_e\,\xi^{s'\dagger}\xi^{s},\ \vec{0})$$

$\mu = 0$ $\quad$ $\mu = i$

$$(k+k')^{\mu} \cong (2m_\phi\,,\ \vec{0})$$

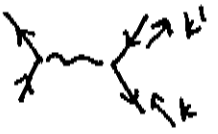
$$q^2 = -|\vec{q}|^2$$

$$iM = -i\frac{e^2}{|\vec{q}|^2}\,\delta^{s's}\,2m_e\,2m_\phi$$

which corresponds to the Born approximation with

$$\tilde{V}(\vec{q}) = +\frac{e^2}{|\vec{q}|} \qquad V(r) = +\frac{e^2}{4\pi r}$$ repulsive

for $e^- \phi^*$ scatt'g



$$iM = -ie\,\bar{u}\gamma^\mu u \left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)(-k-k')^\nu$$

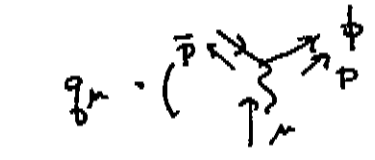
$$= (-1) \cdot \text{previous}$$

so

$$iM = +i\frac{e^2}{|\vec{q}|^2}\,\delta^{s's}\,2m_e\,2m_\phi \quad \Rightarrow \quad V(r) = -\frac{e^2}{4\pi r}$$

attractive !

(b.)



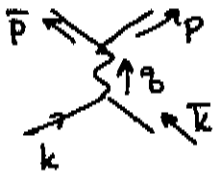
$$q_\mu \cdot (\;) = -ie\,(p-\bar{p})\cdot q$$

In this kinematics $q = p+\bar{p}$ so the above

$$= -ie\,(p-\bar{p})\cdot(p+\bar{p}) = -ie\,(p^2-\bar{p}^2) = -ie(m_\phi^2 - m_\phi^2)$$

$$= 0 \;\checkmark$$

b.)



$$iM = -ie\,\bar{v}(k)\gamma^\mu u(k)\cdot\frac{-i}{q^2}\cdot -ie(p-\bar{p})_\mu$$

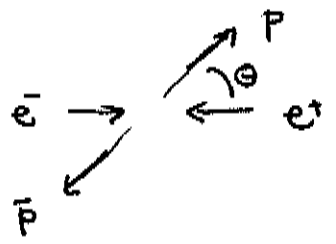
$$= \frac{ie^2}{q^2}\,\bar{v}(k)\gamma^\mu u(k)\,(p-\bar{p})_\mu$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{q^4}\,\mathrm{tr}[\bar{k}\gamma^\mu k\gamma^\nu]\,(p-\bar{p})_\mu (p-\bar{p})_\nu$$

$$= \frac{e^4}{q^4}\left(\bar{k}^\mu k^\nu + \bar{k}^\nu k^\mu - g^{\mu\nu}\, k\cdot\bar{k}\right)(p-\bar{p})_\mu (p-\bar{p})_\nu$$

$$= \frac{e^4}{q^4}\left[2\, k\cdot(p-\bar{p})\;\bar{k}\cdot(p-\bar{p}) - (p-\bar{p})^2\, k\cdot\bar{k}\right]$$

In the kinematics



$k = (E,0,0,E)$ $\qquad p = (E,\ p\sin\theta,\ 0\ p\cos\theta)$

$\bar{k} = (E,0,0,-E)$ $\qquad \bar{p} = (E,-p\sin\theta,\ 0,-p\cos\theta)$

$$p-\bar{p} = 2p\,(0,\ \sin\theta,\ 0,\ \cos\theta)$$

$$k\cdot(p-\bar{p}) = -E\cdot 2p\cdot\cos\theta$$

$$\bar{k}\cdot(p-\bar{p}) = +E\cdot 2p\cdot\cos\theta$$

$$(p-\bar{p})^2 = -4p^2$$

$$k\cdot\bar{k} = 2E^2 \qquad q^2 = 4E^2$$

$$= \frac{e^4}{(4E^2)^2}\left\{-2\cdot 4E^2p^2\cos^2\theta + 4p^2\cdot 2E^2\right\}$$

$$= \frac{2e^4}{4E^2}\,p^2(1-\cos^2\theta) = \frac{1}{2}\,e^4\left(\frac{p}{E}\right)^2\sin^2\theta$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{p}{E}\,\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

$$= \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{1}{2}e^4\left(\frac{p}{E}\right)^3 \sin^2\theta$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4s}\left(\frac{p}{E}\right)^3 \sin^2\theta$$

the total cross section is:

$$\int_{-1}^{1} d\cos\theta\, \sin^2\theta = \int_{-1}^{1} d\cos\theta\,(1-\cos^2\theta)$$

$$= 2 - \tfrac{2}{3} = \tfrac{4}{3}$$

$$\sigma = \frac{\pi\alpha^2}{3s}\left(\frac{p}{E}\right)^3$$

as $E/m \to \infty$ $\quad \frac{p}{E} \to 1$

$$\sigma(e^+e^- \to \phi\phi^*) \to \frac{\pi\alpha^2}{3s}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) \to \frac{4\pi\alpha^2}{3s}$$

so $\quad \sigma(e^+e^- \to \phi\phi^*) \Rightarrow \frac{1}{4}\left(\sigma(e^+e^- \to \mu^+\mu^-)\right)$

c.) Write out $\Delta\mathcal{H}$ in detail:

$$\Delta\mathcal{H} = i g_{\pi NN}\, \pi^i \bar{N} \gamma^5 \sigma^i N$$

$$= i g_{\pi NN} \left\{ \pi^1 (\bar{p}\bar{n}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma^5 \begin{pmatrix} p \\ n \end{pmatrix} + \pi^2 (\bar{p}\bar{n}) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \gamma^5 \begin{pmatrix} p \\ n \end{pmatrix} \right.$$

$$\left. + \pi^3 (\bar{p}\bar{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma^5 \begin{pmatrix} p \\ n \end{pmatrix} \right\}$$

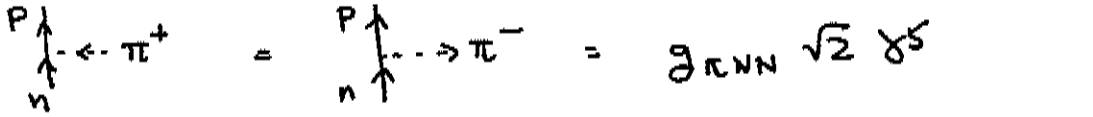
$$= i g_{\pi NN} \left\{ \bar{p} (\pi^1 - i\pi^2) \gamma^5 n + \bar{n} (\pi^1 + i\pi^2) \gamma^5 p \right.$$

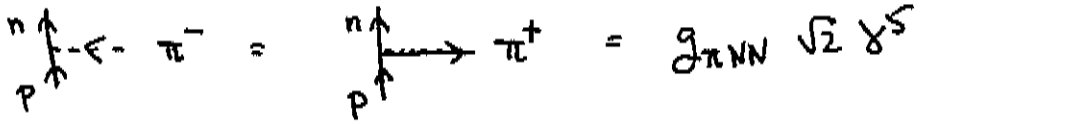
$$\left. + \pi^3 (\bar{p}\gamma^5 p - \bar{n}\gamma^5 n) \right\}$$

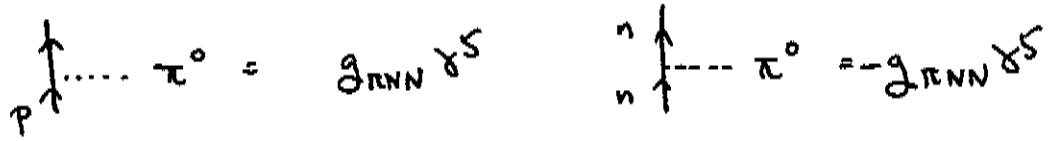
$$= i g_{\pi NN} \left\{ \sqrt{2}\, \pi^+ \bar{p}\gamma^5 n + \sqrt{2}\, \pi^- \bar{n}\gamma^5 p \right.$$

$$\left. + \pi^0 (\bar{p}\gamma^5 p - \bar{n}\gamma^5 n) \right\}$$

Then the Feynman rules are:

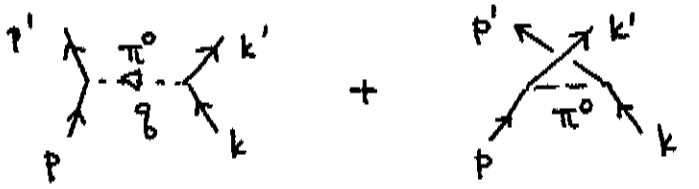


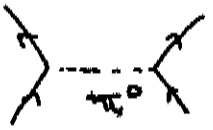




and all charge assignments are consistent.

d.) for proton-proton scatter:





$$iM = (g_{\pi NN})^2 \, \bar{u}(p') \gamma^5 u(p) \, \frac{i}{q^2 - m_\pi^2} \, \bar{u}(k') \gamma^5 u(k)$$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{m_N - \vec{\sigma}\cdot\vec{p} + \Theta(p^2)}\;\xi \\ \sqrt{m_N + \vec{\sigma}\cdot\vec{p} + \Theta(p^2)}\;\xi \end{pmatrix}$$

$$= \sqrt{m_N} \begin{pmatrix} (1 - \tfrac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{m_N})\,\xi \\ (1 + \tfrac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{m_N})\,\xi \end{pmatrix} + \Theta(p^2)$$

$$\bar{u}(p') \gamma^5 u(p) = m_N \; \xi'^{\dagger} \Big((1 - \tfrac{1}{2}\tfrac{\vec{\sigma}\cdot\vec{p}'}{m_N}), (1 + \tfrac{1}{2}\tfrac{\vec{\sigma}\cdot\vec{p}'}{m_N}) \Big) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (1 - \frac{1}{2m_N}\vec{\sigma}\cdot\vec{p}) \\ (1 + \frac{1}{2m_N}\vec{\sigma}\cdot\vec{p}) \end{pmatrix} \xi + \Theta(p^2)$$

(γ^0, γ^5)

$$= m_N \; \xi'^{\dagger} \Big[(1 - \tfrac{\vec{\sigma}\cdot\vec{p}'}{2m_N})(+1)(1 + \tfrac{\vec{\sigma}\cdot\vec{p}}{2m_N}) + (1 + \tfrac{\vec{\sigma}\cdot\vec{p}'}{2m_N})(-1)(1 - \tfrac{\vec{\sigma}\cdot\vec{p}}{2m_N}) \Big] \xi$$

$$= m_N \; \xi'^{\dagger} \Big[-\big(\tfrac{\vec{\sigma}\cdot p' - \vec{\sigma}\cdot\vec{p}}{2m_N}\big) \cdot 2 + \Theta(p^2) \Big] \xi$$

so

$$\bar{u}(p')\gamma^5 u(p) = -\xi'^{\dagger}\vec{\sigma}\cdot(\vec{p}'-\vec{p})\xi = -\xi'^{\dagger}\vec{\sigma}\xi\cdot\vec{q}$$

where $\vec{q} = \vec{p}'-\vec{p}$

Now $\xi'^{\dagger}\vec{\sigma}\xi$ is related to the matrix element of the proton spin

$$\langle\xi'|\,\vec{S}\,|\xi\rangle = \xi'^{\dagger}\left(\frac{\hbar\vec{\sigma}}{2}\right)\xi$$

so

$$\bar{u}(p')\gamma^5 u(p) = -2\,\vec{S}\cdot\vec{q}$$

$$iM = i(2m_N)^2 \cdot g_{\pi NN}^2 \cdot \frac{\vec{S}_p\cdot\vec{q}}{m_N}\,\frac{1}{q^2-m^2}\,\frac{(-\vec{S}_k\cdot\vec{q})}{m_N}$$

for elastic scatt: $q^\mu = (0,\vec{q})$ $\quad q^2 = -|\vec{q}|^2$

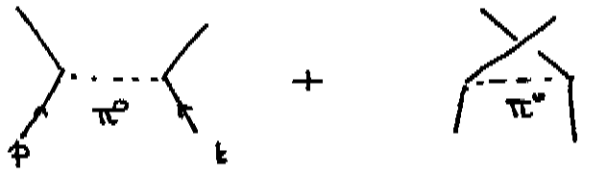
$$iM = -i(2m_N)^2\, g_{\pi NN}^2\, \vec{S}_p\cdot(-i\vec{q})\;\vec{S}_k\cdot(-i\vec{q})\,\frac{1}{|\vec{q}|^2+m_\pi^2}$$

in coordinate space

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\vec{r}}\frac{1}{|\vec{q}|^2+m_\pi^2} = V_Y(\vec{r}) = \frac{e^{-m_\pi r}}{4\pi r}$$ Yukawa potential

$$iM = -i(2m_N)^2\int d^3r\;\frac{\vec{S}_p\cdot\vec{\nabla}}{m_N}\,\frac{\vec{S}_k\cdot\vec{\nabla}}{m_N}\left[g_{\pi NN}^2 V_Y(\vec{r})\right]$$

that is:



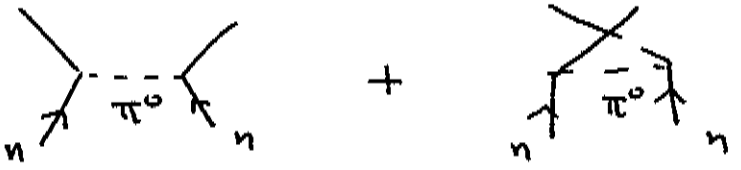
$$iM = -i\,(2m_N)^2\; s_p^i\, s_k^j \left(\frac{-iq^i}{m_N}\right)\left(\frac{-iq^j}{m_N}\right) \frac{g^2_{\pi NN}}{|\vec{q}|^2 + m_\pi^2}$$

$$- \text{(exchg term.)}$$

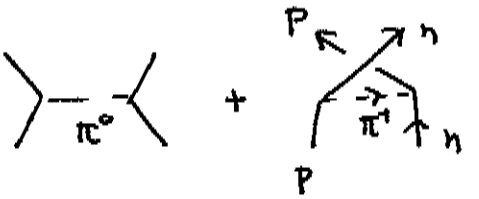
$$\Updownarrow$$

$$p' \leftrightarrow k' \,,\quad \xi_p' \leftrightarrow \xi_k'$$

for nn scattering we have.



which leads to the same expression

for pn scattering, we have 

the direct term is

$$M = +i\,(2m_N)^2\; s_p^i\, s_n^j \left(\frac{-iq^i}{m_N}\right)\left(\frac{-iq^j}{m_N}\right) \frac{g^2_{\pi NN}}{|\vec{q}|^2 + m_\pi^2}$$

the exchge term is

$$iM = (-1)\cdot(-i)\ \xi^+_{k'}\frac{\sigma^i}{2}\xi_p\ \ \xi^+_{p'}\frac{\sigma^j}{2}\xi_k\ \left(\frac{-iq^i_b}{m_N}\right)\left(\frac{-iq^j_b}{m_N}\right)\frac{g^2_{\pi NN}}{|\vec{q}_b|+m^2_\pi}$$

e.) For the pseudovecta interaction, if q^μ is the incoming π momentum

$$\partial_\mu \rightarrow -iq^\mu$$

$$\Delta\mathcal{H} = +i\frac{q^\mu_b}{2m_N}g_{\pi NN}\bar{N}\gamma^\mu\gamma^5\sigma^i N\ \pi^i$$

repeat the analysis on p. 5:

$$= i\frac{q^\mu_b g_{\pi NN}}{2m_N}\Big\{\sqrt{2}\,\pi^+\,\bar{p}\,\gamma^\mu\gamma^5 n + \sqrt{2}\,\pi^-\,\bar{n}\gamma^\mu\gamma^5 p + \pi^0(\bar{p}\gamma^\mu\gamma^5 p - \bar{n}\gamma^\mu\gamma^5 n)\Big\}$$

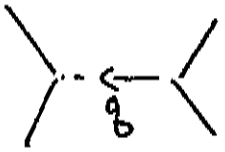
so

$$\begin{matrix} p\uparrow \\ n\uparrow \end{matrix}\!\!\leftarrow^{q_b}\ \pi^+ = g_{\pi NN}\sqrt{2}\,\frac{q_{b\mu}\gamma^\mu\gamma^5}{2m_N} = -\ \begin{matrix} p\uparrow \\ n\uparrow \end{matrix}\!\!\rightarrow^{q_b}\ \pi^-$$

$$\begin{matrix} n\uparrow \\ p\uparrow \end{matrix}\!\!\leftarrow^{q_b}\ \pi^- = g_{\pi NN}\sqrt{2}\,\frac{q_{b\mu}\gamma^\mu\gamma^5}{2m_N} = -\ \begin{matrix} n\uparrow \\ p\uparrow \end{matrix}\!\!\rightarrow^{q_b}\ \pi^+$$

$$\begin{matrix} p\uparrow \\ p\uparrow \end{matrix}\!\!\leftarrow^{q_b}\ \pi^0 = g_{\pi NN}\frac{q_{b\mu}\gamma^\mu\gamma^5}{2m_N} \qquad \begin{matrix} n\uparrow \\ n\uparrow \end{matrix}\!\!\leftarrow^{q_b} = -\,g_{\pi NN}\frac{q_{b\mu}\gamma^\mu\gamma^5}{2m_N}$$

f.) For proton proton scattering in the pseudovector model, consider first



$$= g_{\pi NN}^2 \; \bar{u}(p') \frac{q_\mu \gamma^\mu \gamma^5}{2m_N} u(p) \; \frac{i}{q^2 - m_\pi^2} \; \bar{u}(k') \frac{[-q_\mu]\gamma^\mu \gamma^5}{2m_N} u(k)$$

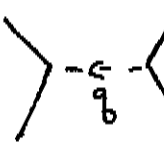
$$\bar{u}(p')\gamma^\mu\gamma^5 u(p) \cong \sqrt{m_N}\,(\xi'^\dagger, \xi'^\dagger \sigma^\alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix}\sqrt{m_N}$$

$$= m_N\, \xi'^\dagger(-\sigma^\alpha + \sigma^\alpha)\xi$$

$$= 2m_N\,(0,\; \xi'^\dagger \vec{\sigma} \xi)$$

$$q_\mu\, \bar{u}(p')\gamma^\mu\gamma^5 u(p) = (2m_N)\cdot\; -\vec{q}\cdot(\xi'^\dagger \vec{\sigma}\cdot\xi)$$

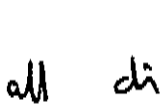
so we get the same expression as in the middle of p.7:



$$= i\,(2m_N)^2\, g_{\pi NN}^2 \left(\frac{-1}{q^2 - m_\pi^2}\right)(\vec{q}\cdot\xi'^\dagger\vec{\sigma}\xi_p)(\vec{q}\cdot\xi'^\dagger\vec{\sigma}\cdot\xi_k)$$

more generally,

$$\frac{q_\mu}{2m_N}\bar{u}(p')\gamma^\mu\gamma^5 u(p) = -\vec{q}\cdot\xi'^\dagger\vec{\sigma}\cdot\xi = \bar{u}(p')\gamma^5 u(p)$$

so all diagrams of the form  or 

reduce to the same expressions as in the pseudoscalar theory.

f.) Actually, the equations above is exact in the full relativistic theory:

$$\frac{1}{2m_N} q_\mu \bar{u}(p') \gamma^\mu \gamma^5 u(p)$$

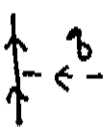
$$= \frac{1}{2m_N} \bar{u}(p') \not{q} \gamma^5 u(p)$$

$$= \frac{1}{2m_N} \bar{u}(p') (\not{p}' - \not{p}) \gamma^5 u(p)$$

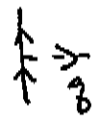
$$= \frac{1}{2m_N} \underbrace{\bar{u}(p') \not{p}'}_{\bar{u}(p') \cdot m_N} \gamma^5 u(p) + \bar{u}(p') \gamma^5 \underbrace{\not{p} u(p)}_{m_N \cdot u(p)}$$

$$= \bar{u}(p') \gamma^5 u(p)$$

then if all $\underline{p}, \underline{n}$ $\underline{\text{are}}$ $\underline{\text{on-shell}}$, the Feynman rules on p. 9 can be rewritten by replacing.



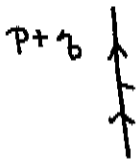
$$= g_{\pi NN} \sqrt{2} \frac{\not{q} \gamma^5}{2m_N} = g_{\pi NN} \sqrt{2} \gamma^5$$



$$= g_{\pi NN} \sqrt{2} \frac{(-\not{q}) \gamma^5}{2m_N} = g_{\pi NN} \sqrt{2} \gamma^5$$

Then the Feynman rules on p.9 reduce precisely to those on p.8.

In diagrams in which there is a fermion line off shell, we find extra terms. If $(p+q)$ is off-shell:



$$= \frac{i}{\not{p}+\not{q}-m_N} \frac{\not{q}}{2m_N} \gamma^5 u(p)\; g_{\pi NN}$$

$$= \frac{i}{\not{p}+\not{q}-m_N} \frac{(\not{p}+\not{q}-\not{p})}{2m_N} \gamma^5 u(p) \cdot g_{\pi NN}$$

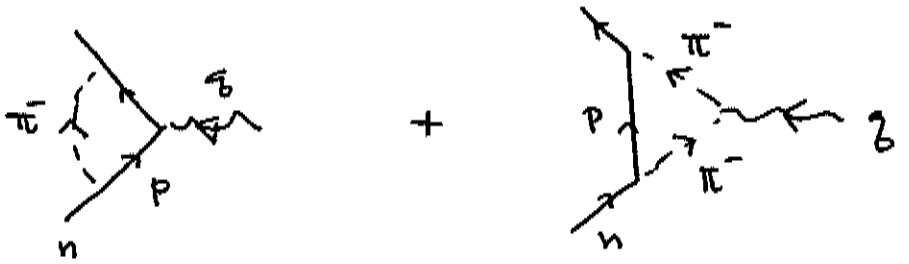
$$= \frac{i}{\not{p}+\not{q}-m_N} \frac{(\not{p}+\not{q}+m_N)}{2m_N} \gamma^5 u(p)\; g_{\pi NN}$$

$$= \frac{i}{\not{p}+\not{q}-m_N} \left(\frac{\not{p}+\not{q}-m_N+2m_N}{2m_N}\right) \gamma^5 u(p)\; g_{\pi NN}$$

$$= \frac{i}{\not{p}+\not{q}-m_N} \gamma^5 u(p)\; g_{\pi NN} \quad \Big] \text{ result of pseudoscalar theory}$$

$$+ \frac{i}{2m_N} \gamma^5 u(p)\; g_{\pi NN} \quad \Big] \text{ extra term.}$$

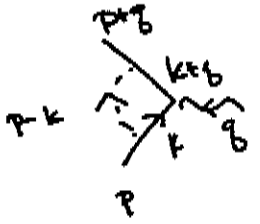
g.) Now consider the electromagnetic form factors of the neutron. These come from:



Since $F_1, F_2 = 0$ in leading order $\rightarrow$ because the neutron has zero charge, there is no contribution from Z_2 terms.

h.) Now compute the diagram in the pseudoscalar theory. I will use dimensional regularization assuming $\{\gamma^5, \gamma^\mu\} = 0$ for all μ. (It is not obvious that this is permitted, but it is. See Chapter 19 of Peskin + Schroeder for the explanation.)

Then $\rightarrow$



$$= (+ie) \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (\sqrt{2} g_{\pi NN} \gamma^5) \frac{i(\not{k}+\not{q}+m_N)}{(k+q)^2 - m_N^2} \gamma^\mu$$

proton has a + charge.

$$\cdot \frac{i(\not{k}+m_N)}{k^2 - m_N^2} \cdot (\sqrt{2} g_{\pi NN} \gamma^5) u(p) \frac{i}{(p-k)^2 - m_\pi^2}$$

(anticommute γ^5 to the right)

$$= (+ie) \, 2i g_{\pi NN}^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{((k+q)^2 - m_N^2)(k^2 - m_N^2)((p-k)^2 - m_\pi^2)}$$

$$\cdot \bar{u}(p') (\not{k}+\not{q} - m_N) \gamma^\mu (\not{k} - m_N) u(p)$$

$$= (+ie) \, 2i g_{\pi NN}^2 \int dx\,dy\,dz \, \delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - \Delta]^3}$$

$$\cdot \bar{u}(p') (\not{k}+\not{q} - m_N) \gamma^\mu (\not{k} - m_N) u(p)$$

where the denominator is

$$x((k+q)^2 - m_N^2) + y(k^2 - m_N^2) + z((k-p)^2 - m_\pi^2)$$

$$= k^2 + 2k \cdot (xq - zp) + xq^2 + zp^2 - (1-z) m_N^2 - z m_\pi^2$$

$$= k^2 - (xq - zp)^2 + xq^2 + zp^2 - (1-z) m_N^2 - z m_\pi^2$$

$$= k^2 + x(1-x) q^2 + z(1-z) p^2 + 2xz\, p \cdot q - (1-z) m_N^2 - z m_\pi^2$$

$$= k^2 + xy q^2 + xz (p+q)^2 + yz\, p^2 - (1-z) m_N^2 - z m_\pi^2$$

$$= k^2 + x q^2 - (1-z)^2 m_N^2 - z m_\pi^2$$

so $\Delta = (1-z)^2 m_N^2 + z m_\pi^2 - xy q^2$

$k = \ell - xq + zp$

$k+q = \ell + (1-x) q + zp$

so

$$= (+ie)\quad (2ig^2_{\pi NN}) \int dx\,dy\,dz\;\delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4}\;\frac{2}{[k^2-\Delta]^3}$$

$$\cdot\ \bar{u}(p')\Big[\ \not{k}\gamma^\mu\not{k} \;+\; [(1-\underbrace{x}_{z+y})\not{p}' + z\not{p} - m_N]\gamma^\mu[-x\not{p}' + z\not{p} - m_N]\Big]u(p)$$

$$= (+ie)\,(2ig^2_{\pi NN}) \int dx\,dy\,dz\;\delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4}\;\frac{2}{[k^2-\Delta]^3}$$

$$\bar{u}(p')\ \Big[\not{k}\gamma^\mu\not{k} \;+\; [m_N(1-z) + y\not{p}']\,\gamma^\mu\,[-m_N(1-z) - x\not{p}']\Big]\,u(p)$$

$$= (+ie)\,(2ig^2_{\pi NN}) \int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\cdot\Big\{\ \frac{-i}{(4\pi)^{d_k}}\,\frac{\Gamma(2-d_k)}{\Delta^{2-d_k}}\quad \bar{u}(p')\,\gamma^\alpha\gamma^\mu\gamma_\alpha\,u(p)\cdot\frac{1}{2}$$

$$+\ \frac{i}{(4\pi)^{d_k}}\,\frac{\Gamma(3-d_k)}{\Delta^{3-d_k}}\quad \bar{u}(p')\,[m_N(1-z) + y\not{p}']\,\gamma^\mu\,[m_N(1-z) - x\not{p}']\,u(p)\Big\}$$

$$\gamma^\alpha\gamma^\mu\gamma_\alpha \;=\; 2g^{\mu\alpha}\gamma_\alpha - \gamma^\mu\gamma^\alpha\gamma_\alpha \;=\; (2-d)\,\gamma^\mu$$

$$= (+ie)\,(2ig^2_{\pi NN})\,\frac{i}{(4\pi)^{d_k}}\,\Gamma(3-d_k) \int_0^1 dx\,dy\,dz\quad \delta(x+y+z-1)$$

$$\cdot\Big\{\ -\frac{2-d}{2}\,\frac{1}{2-d_k}\quad \bar{u}(p')\,\gamma^\mu\,u(p)\quad \frac{1}{\Delta^{2-d_k}}$$

$$+\ \bar{u}(p')\,[m_N(1-z) + y\not{p}']\,\gamma^\mu\,[m_N(1-z) + x\not{p}']\,u(p)\quad \frac{1}{\Delta^{3-d_k}}\Big\}$$

The second line simplifies as:

$$\bar{u}(p')\,[\;]\,\gamma^\mu\,[\;]\,u(p)$$

$$= \bar{u}(p')\, m_N^2(1-z)^2\,\gamma^\mu\, u(p) \;-\; y\, m_N(1-z)\,\bar{u}(p')\,\not{q}\,\gamma^\mu\, u(p)$$

$$+ x\, m_N(1-z)\,\bar{u}(p')\,\gamma^\mu\,\not{q}\, u(p) \;-\; xy\,\bar{u}(p')\,\not{q}\gamma^\mu\not{q}\, u(p)$$

Using $\langle x\rangle = \langle y\rangle$, $\not{q}\gamma^\mu\not{q} = 2q^\mu\not{q} - \gamma^\mu\not{q}\not{q} = (0) - \gamma^\mu q^2$

$$= \bar{u}(p')\,\gamma^\mu\, u(p)\,\left[m_N^2(1-z)^2 + xy\,q^2\right]$$

$$+ x\, m_N(1-z)\;\bar{u}(p')\,[\gamma^\mu, \not{q}]\, u(p)$$

In all:



$$= (+ie)\;\frac{-2g_{\pi NN}^2}{(4\pi)^{d_k}}\,\Gamma(3-d_k)\int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\Big\{ -\frac{2-d}{2}\;\frac{1}{\Delta^{2-d_k}}\;\frac{1}{2-d_k}\;\;\bar{u}\gamma^\mu u$$

$$+\;\frac{1}{\Delta^{3-d_k}}\left(\bar{u}\gamma^\mu u\,\left[m_N^2(1-z)^2 + xyq^2\right] + x(1-z)\,m_N\,\bar{u}\,[\gamma^\mu,\not{q}]\,u\right)\Big\}$$

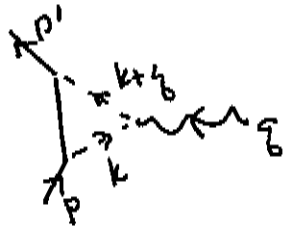
$$= (+ie)\;\bar{u}(p')\left[\gamma^\mu \delta F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_N}\,\delta F_2(q^2)\right] u(p)$$

$$\delta F_1 = \frac{2g_{\pi NN}^2}{(4\pi)^{d_k}}\,\Gamma(3-d_k)\int dx\,dy\,dz\;\delta(x+y+z-1)\left\{\frac{2-d}{2(2-d_k)}\;\frac{1}{\Delta^{2-d_k}}\right.$$

$$\left. -\;\frac{1}{\Delta^{3-d_k}}\left(m_N^2(1-z)^2 + xyq^2\right)\right\}$$

$$\delta_a F_2 = +\frac{2g_{\pi NN}^2}{(4\pi)^{d_k}}\,\Gamma(3-d_k)\int dx\,dy\,dz\;\delta(x+y+z-1)\;\frac{1}{\Delta^{3-d_k}}\;4x(1-z)\,m_N^2$$

Next:



$$= \left(\sqrt{2}\, g_{\pi NN}\right)^2 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p')\, \gamma^5\, \frac{i(\not{p}-\not{k}+m_N)}{(p-k)^2-m_N^2}\, \gamma^5\, u(p)$$

$$\cdot \quad \frac{i}{(k+q)^2-m_\pi^2} \quad \underbrace{[-ie\,(k+q+k)^\mu]}_{\text{Feynman rule from part (a)}} \quad \frac{i}{k^2-m_N^2}$$

$$= (+ie)\,(-2ig_{\pi NN}^2) \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4}\, \frac{2}{[k^2-\hat{\Delta}]^3}$$

$$\cdot\, (k+k+q)^\mu \quad \bar{u}(p')\,[(\not{p}-\not{k})-m_N]\,u(p)$$

where the denominator is:

$$x\left((k+q)^2-m_\pi^2\right) + y\left(k^2-m_\pi^2\right) + z\left((k-p)^2-m_N^2\right)$$

$$= \quad \ldots \quad \text{(as on p. 14)}$$

$$= k^2 + xyq^2 + \underbrace{xz(p+q)^2 + yz\,p^2}_{z(1-z)\,m_N^2} - (1-z)m_\pi^2 - zm_N^2$$

$$= k^2 + xyq^2 - z^2 m_N^2 - (1-z)m_\pi^2$$

$$\hat{\Delta} = z^2 m_N^2 + (1-z)m_\pi^2 - xyq^2$$

$$k = \ell k - xq + zp$$

$$k+q = \ell k + (1-x)q + zp$$

$$k-p = \ell k - xq - (1-z)p$$

$$= (+ie)(-2ig_{\pi NN}^2) \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4} \frac{2}{[k^2-\Delta]^3}$$

$$\cdot \Big\{ -2k^\mu k_\alpha \quad \bar{u}(p')\gamma^\alpha u(p)$$

$$+ ((1-2x)q + 2zp)^\mu \quad \bar{u}(p')\,[x\not{q} + (1-z)\not{p} - m_N]\,u(p) \Big\}$$

$\bar{u}\not{q}u = 0 \qquad \bar{u}(p')\not{p}\,u(p) = m_N\ \bar{u}u$

$$(1-2x)q + 2zp = 2zp + (z+y-x)q$$

$$= z[p+(p+q)] + \underbrace{(y-x)q}_{\langle\rangle = 0}$$

$$= (+ie)(-2ig_{\pi NN}^2) \int dx\,dy\,dz\,\delta(x+y+z-1)$$

$$\Big\{ \frac{-i}{(4\pi)^{d_k}} \frac{\Gamma(2-d_k)}{\hat{\Delta}^{2-d_k}} (-2)\,\frac{1}{2}\ \bar{u}\gamma^\mu u$$

$$+ \frac{i}{(4\pi)^{d_k}} \frac{\Gamma(3-d_k)}{\hat{\Delta}^{3-d_k}} \cdot z \cdot (p+p')^\mu\,(-z)\,m_N\ \bar{u}u \Big\}$$

→ the Gordon id: $\frac{(p+p')^\nu}{2m_N}\bar{u}u = \bar{u}\gamma^\mu u - \bar{u}\,\frac{i\sigma^{\mu\nu}q_\nu}{2m_N}u$

$$= (+ie)(-2ig_{\pi NN}) \frac{i}{(4\pi)^{d_k}} \Gamma(3-d_k) \int dx\,dy\,dz\,\delta(x+y+z-1)$$

$$\cdot \Big\{ \frac{1}{2-d_k}\ \bar{u}\gamma^\mu u\ \frac{1}{\hat{\Delta}^{2-d_k}}$$

$$- 2z^2 m_N^2\ \frac{1}{\hat{\Delta}^{3-d_k}} \Big[\bar{u}\gamma^\mu u - \bar{u}\ \frac{i\sigma^{\mu\nu}q_\nu}{2m_N}u \Big] \Big\}$$

adding the two diagrams:

$$F_1(\hat{q}^2) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_4}} \Gamma(3-d_4) \int dx\,dy\,dz\, \delta(x+y+z-1)$$

$$\left\{ \frac{1}{2-d_4} \left(\frac{1}{\Delta^{2-d_4}} \frac{2-d}{2} + \frac{1}{\hat{\Delta}^{2-d_4}} \right) \right.$$

$$\left. + \frac{1}{\Delta^{3-d_4}} \left(-m_N^2 (1-z)^2 + xy q^2 \right) + \frac{1}{\hat{\Delta}^{3-d_4}} (-2z^2 m_N^2) \right\}$$

$$F_2(\hat{q}^2) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_4}} \Gamma(3-d_4) \int dx\,dy\,dz\, \delta(x+y+z-1)$$

$$\left\{ - \frac{1}{\Delta^{3-d_4}} (+4x(1-z) m_N^2) + \frac{1}{\hat{\Delta}^{3-d_4}} (2z^2 m_N^2) \right\}$$

$$\therefore F_2 \qquad \langle 4x \rangle = \langle 2(x+y) \rangle = 2(1-z)$$

$$F_2(\hat{q}^2) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_4}} \Gamma(3-d_4) \int_0^1 dz \int_0^{1-z} dx$$

$$\left\{ + \frac{2(1-z)^2 m_N^2}{\Delta^{3-d_4}} + \frac{2z^2 m_N^2}{\hat{\Delta}^{3-d_4}} \right\}$$

we can take the limit $d \to 4$

$$F_2(\hat{q}^2) = \frac{2g_{\pi NN}^2}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dx$$

$$\left\{ + \frac{2(1-z)^2 m_N^2}{(1-z)^2 m_N^2 + z m_\pi^2 - xy q^2} + \frac{2z^2 m_N^2}{z^2 m_N^2 + (1-z) m_\pi^2 - xy q^2} \right\}$$

this is UV and IR finite. $F_1(q^2)$ is IR finite, because Δ, $\hat{\Delta}$ are positive for all x, y, z, $q^2 \leq 0$.

However, there are UV divergent terms:

$$F_1(q^2) = \frac{2g_{\pi NN}^2}{(4\pi)^2} \int dx\,dy\,dz\, \delta(x+y+z-1) \left\{ \cdot \frac{1}{2-d_4} \left(\frac{2-d}{2} + 1 \right) + \text{finite as } d \to 4 \right\}$$

$$= \ldots \left\{ \frac{1}{2-d_4} \cdot \frac{4-d}{2} + \ldots \right.$$

$$= \ldots \left\{ 1 + \text{finite as } d \to 4 \right\}$$

So $F_1(q^2)$ is UV finite.

2.) $$F_1(q^2=0) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_4}} \Gamma(3-d_4) \int_0^1 dz \int_0^{1-z} dx$$

$$\left\{ \frac{1}{2-d/2} \left(\frac{2-d}{2} \frac{1}{\left[(1-z)^2 m_N^2 + z m_\pi^2\right]^{2-d_4}} + \frac{1}{\left[z^2 m_N^2 + (1-z) m_\pi^2\right]^{2-d_4}} \right) \right.$$

$$\left. - \left(\frac{m_N^2 (1-z)^2}{\left[(1-z)^2 m_N^2 + z m_\pi^2\right]^{3-d_4}} + \frac{2z^2 m_N^2}{\left[z^2 m_N^2 + (1-z) m_\pi^2\right]^{3-d_4}} \right) \right\}$$

put $\omega = z$ in these terms ↑ $\omega = (1-z)$ in these terms ↑

$$F_1(q^2=0) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_2}} \Gamma(3-d_2) \int_0^1 dw$$

$$\left\{ \frac{1}{2-d_2} \left[\frac{(2-d)}{2}(1-w) + w \right] \frac{1}{\left[(1-w)^2 m_N^2 + w m_\pi^2\right]^{2-d_2}} \right.$$

$$\left. - \frac{m_N^2 \left[(1-w)^3 + 2(1-w)^2 w\right]}{\left[(1-w)^2 m_N^2 + w m_\pi^2\right]^{3-d_2}} \right\}$$

$$= \frac{2g_{\pi NN}^2}{(4\pi)^{d_2}} \Gamma(3-d_2) \int_0^1 dw$$

$$\left\{ \frac{(1-w)}{\left[(1-w)^2 m_N^2 + w m_\pi^2\right]^{2-d_2}} + \frac{(-(1-2w))}{(2-d_2)} \frac{1}{\left[(1-w)^2 m_N^2 + w m_\pi^2\right]^{2-d_2}} \right.$$

$$\left. - \frac{m_N^2 (1-w)^2 (1+w)}{\left[(1+w)^2 m_N^2 + w m_\pi^2\right]^{3-d_2}} \right\}$$

Integrate the middle term by parts:

$$\int_0^1 dw \; -(1-2w) \frac{1}{2-d_2} \frac{1}{[\;]^{2-d_2}}$$

$$= - \frac{w(1-w)}{2-d_2 \; [\;]^{2-d_2}} \Bigg|_{w=0}^{w=1} - \int_0^1 dw \; \frac{-w(1-w)\left[2(1-w)m_N^2 - m_\pi^2\right]}{\left[(1-w)^2 m_N^2 + w m_\pi^2\right]^{3-d_2}}$$

$$= 0 + \int_0^1 dw \; \frac{2w(1-w)^2 m_N^2 - w(1-w) m_\pi^2}{\left[(1-w)^2 m_N^2 + w m_\pi^2\right]^{3-d_2}}$$

reassemble the pieces:

$$F_1(q^2=0) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_L}} \Gamma(3-d_L) \int_0^1 d\omega \; \frac{1}{[(1-\omega)^2 m_N^2 + \omega m_\pi^2]^{3-d_L}}$$

$$\{ (1-\omega)^3 m_N^2 + (1-\omega)\omega m_\pi^2 + 2\omega(1-\omega)^2 m_N^2 - \omega(1-\omega) m_\pi^2 - m_N^2 (1-\omega)^2(1+\omega) \}$$

$$= 0 \; !$$ (before taking $d \to 4$)

j.) Now expand $F_1(q^2)$ to order q^2. The coefficient should naturally be finite.

$$F_1(q^2) = \frac{2g_{\pi NN}^2}{(4\pi)^{d_L}} \Gamma(3-d_L) \int dx\,dy\,dz\; \delta(x+y+z-1)$$

$$\{ \frac{1}{2-d_L} \left[\frac{1}{(1-z)^2 m_N^2 + z m_\pi^2 - xy q^2}\right]^{2-d_L} \quad \frac{2-d}{2}$$

$$+ \frac{1}{2-d_L} \left[\frac{1}{z^2 m_N^2 + (1-z) m_\pi^2 - xy q^2}\right]^{2-d_L}$$

$$+ \frac{1}{[(1-z)^2 m_N^2 + z m_\pi^2 - xy q^2]^{2-d_L}} \left(-m_N^2 (1-z)^2 + xy q^2\right)$$

$$+ \frac{1}{[z^2 m_N^2 + (1-z) m_\pi^2 - xy q^2]^{3-d_L}} 2z^2 m_N^2 \; \}$$

= the first two term, we can write

$$\frac{1}{2-d_2}\,\frac{1}{[D-xyq^2]^{2-d_2}} = (\text{const}) - \frac{2-d_2}{2-d_2}\,\frac{-xyq^2}{[D]^{3-d_2}}$$

$$\xrightarrow[d\to 4]{} (\text{const}) + \frac{xyq^2}{D}$$

the last two term are finite as $d\to 4$; we just expand.

Then the $O(\hat{q})$ term is

$$F_1(\hat{q}) = \frac{2g_{\pi NN}^2}{(4\pi)^2}\cdot 1\cdot \int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\Big\{ -1\cdot\frac{xyq^2}{(1-z)^2 m_N^2 + z m_\pi^2} + \frac{xyq^2}{z^2 m_N^2 + (1-z) m_\pi^2}$$

$$+ \frac{xyq^2}{(1-z)^2 m_N^2 + z m_\pi^2} + \frac{(-m_N^2(1-z)^2)\,xyq^2}{[(1-z)^2 m_N^2 + z m_\pi^2]^2}$$

$$+ \frac{(-2z^2 m_N^2)\,xyq^2}{[z^2 m_N^2 + (1-z) m_\pi^2]^2} \Big\}$$

$$= \frac{2g_{\pi NN}^2}{(4\pi)^2}\int dx\,dy\,dz\;\delta(x+y+z-1)\;(xyq^2)$$

$$\cdot\Big[\frac{1}{(1-z)^2 m_N^2 + z m_\pi^2}\Big\{-1+1-1+\frac{z m_\pi^2}{(1-z)^2 m_N^2 + z m_\pi^2}\Big\}$$

$$+\frac{1}{z^2 m_N^2 + (1-z) m_\pi^2}\Big\{1-2+\frac{2(1-z) m_\pi^2}{z^2 m_N^2 + (1-z) m_\pi^2}\Big\}\Big]$$

$$F_1(q^2) = \frac{2g_{\pi NN}^2}{(4\pi)^2}\, q^2 \cdot \int dx\,dy\,dz\,\delta(x+y+z-1)\cdot xy$$

$$\cdot(-1)\left\{ \frac{1}{(1-z)^2 m_N^2 + z m_\pi^2} - \frac{z m_\pi^2}{[\quad]^2} \right.$$

$$\left. + \frac{1}{z^2 m_N^2 + (1-z) m_\pi^2} - \frac{2(1-z) m_\pi^2}{[\quad]^2} \right\}$$

Now we just have to do the ~~integrals~~.

$$\int_0^{1-z} dx\,dy\;\delta(x+y+z-1)\;xy$$

$$= \int_0^{1-z} dx\; x(1-z-x) = \frac{(1-z)^2}{2}(1-z) - \frac{(1-z)^3}{3} = \frac{(1-z)^3}{6}$$

Now let $\omega = (1-z)$ in the first line above

$\omega = z$ in the second line.

$$= -\frac{2g_{\pi NN}^2}{(4\pi)^2}\,\frac{q^2}{6}\int_0^1 d\omega$$

$$\left\{ \frac{\omega^3}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} + m_\pi^2 \frac{\partial}{\partial m_\pi^2}(\text{this}) \right.$$

$$\left. + \frac{(1-\omega)^3}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} + 2 m_\pi^2 \frac{\partial}{\partial m_\pi^2}(\text{this}) \right\}$$

These are ugly integrals. I'll evaluate them in the limit $m_N \gg m_\pi$. That is, I will drop terms of order $m_\pi/m_N \sim 0.14$.

$$\int_0^1 d\omega \frac{\omega^3}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} \cong \int_0^1 d\omega \; \omega^3 \left\{ \frac{1}{\omega^2 m_N^2} - \frac{m_\pi^2 (1-\omega)}{\omega^4 m_N^4} + \cdots \right\}$$

$$= \frac{1}{2m_N^2} - \mathcal{O}\left(\frac{m_\pi^2}{m_N^4} \ln m_N^2/m_\pi^2 \right)$$

acts $m_\pi^2 \frac{\partial}{\partial m_\pi^2}$ on this gives $0 - \mathcal{O}\left(\frac{m_\pi^2}{m_N^4}\right)$

$$\int_0^1 d\omega \frac{(1-\omega)^3}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} = \int_0^1 d\omega \frac{1}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} (1 - 3\omega + 3\omega^2 - \omega^3)$$

Proceed systematically

$$\int_0^1 d\omega \frac{1}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} = \int_0^1 d\omega \frac{1}{\omega^2 m_N^2 + m_\pi^2} + \frac{\omega m_\pi^2}{(\omega^2 m_N^2 + m_\pi^2)^2} + \cdots$$

$$= \frac{1}{m_\pi m_N} \tan^{-1}\left(\frac{m_N}{m_\pi}\right) + \int_0^1 \tfrac{1}{2} d\omega^2 \frac{m_\pi^2}{[\omega^2 m_N^2 + m_\pi^2]^2} + \cdots$$

$$= \frac{1}{m_\pi m_N} \left(\frac{\pi}{2} - \frac{m_\pi}{m_N} + \cdots \right) + \frac{1}{2m_N^2} m_\pi^2 \left(\frac{1}{m_\pi^2} - \frac{1}{m_N^2 + m_\pi^2} \right) + \cdots$$

$$= \frac{\pi/2}{m_\pi m_N} - \frac{1}{m_N^2} + \frac{1}{2m_N^2} + \mathcal{O}\left(\frac{m_\pi}{m_N}\right) \qquad \mathcal{O}\left(\frac{m_\pi}{m_N}\right)$$

$$= \frac{\pi/2}{m_\pi m_N} - \frac{1}{2m_N^2}$$

but note that

$$\left(1 + 2m_\pi^2 \frac{\partial}{\partial m_\pi^2}\right)\left(\frac{\pi/2}{m_\pi m_N}\right) = \left(1 + m_\pi \frac{\partial}{\partial m_\pi}\right) \frac{\pi/2}{m_\pi m_N} = 0$$

$$\int_0^1 d\omega \; \frac{\omega}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} \simeq \frac{1}{2}\int_0^1 d\omega^2 \; \frac{1}{\omega^2 m_N^2 + m_\pi^2}$$

$$= \frac{1}{2m_N^2} \ln \frac{m_N^2 + m_\pi^2}{m_\pi^2} \simeq \frac{1}{2m_N^2} \ln \frac{m_N^2}{m_\pi^2}$$

$$\left(1 + 2m_\pi^2 \frac{\partial}{\partial m_\pi^2}\right) \frac{1}{2m_N^2} \ln \frac{m_N^2}{m_\pi^2} = \frac{1}{2m_N^2} \ln \frac{m_N^2}{m_\pi^2} - \frac{1}{2m_N^2} + \cdots$$

$$\int_0^1 d\omega \; \frac{\omega^2}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} \approx \int_0^1 d\omega \; \frac{1}{m_N^2} = \frac{1}{m_N^2}$$

$$\int_0^1 d\omega \; \frac{\omega^3}{\omega^3 m_N^2 + (1-\omega) m_\pi^2} \simeq \frac{1}{2m_N^2}$$

∴ all:

$$\left(1 + 2m_\pi^2 \frac{\partial}{\partial m_\pi^2}\right) \int d\omega \; \frac{(1-\omega)^3}{\omega^2 m_N^2 + (1-\omega) m_\pi^2} =$$

$$- \frac{1}{2m_N^2} - \frac{3}{2m_N^2} \ln \frac{m_N^2}{m_\pi^2} + \frac{3}{2m_N^2} + \frac{3}{m_N^2} - \frac{1}{2m_N^2}$$

$$= - \frac{3}{2m_N^2} \ln \frac{m_N^2}{m_\pi^2} + \frac{2}{m_N^2} - \frac{1}{2m_N^2}$$

and the whole bracket on p. 24 becomes

$$\int_0^1 d\omega \{ \} = - \frac{3}{2m_N^2} \ln \frac{m_N^2}{m_\pi^2} + \frac{2}{m_N^2}$$

then

$$F_1(q^2) = \frac{1}{6} q^2 \cdot \left(-\frac{2 g_{\pi NN}^2}{(4\pi)^2}\right)\left(-\frac{3}{2}\frac{1}{m_N^2}\right)$$

$$\cdot \left(\ln \frac{m_N^2}{m_\pi^2} - \frac{4}{3}\right) \quad \left(1 + O\left(\frac{m_\pi}{m_N}\right)\right)$$

compare to

$$F_1(q^2) = \frac{1}{6} q^2 r^2$$

we find

$$r^2 = \frac{3 g_{\pi NN}^2}{(4\pi)^2} \frac{1}{m_N^2} \left(\ln \frac{m_N^2}{m_\pi^2} - \frac{4}{3}\right) \quad \left(1 + O\left(\frac{m_\pi}{m_N}\right)\right)$$

put in $\frac{g_{\pi NN}^2}{4\pi} = 14.$ from measurements of

$NN \to NN$

$$r^2 = \frac{8.5}{m_N^2} \qquad \text{or} \qquad r = 0.61 \text{ fm}$$

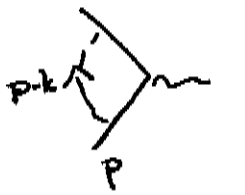
This badly overestimates the experimental result of $r \approx 0.11$ fm

On the other hand, the pion-nucleon model underestimates the neutron magnetic moment

$$g_n = -3.83 \qquad \text{(all "anomalous")}$$

(k) Compare this to the result of the pseudo-vector theory.

I will collect only the divergent contributions to F_1 and F_2:



$$= (+ie)\int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\sqrt{2}g_{\pi NN}\frac{(p\!\!\!/-k\!\!\!/)}{2m_N}\gamma^5\,\frac{i(k\!\!\!/+q\!\!\!/+m_N)}{(k+q)^2-m_N^2}\gamma^\mu$$

$$\frac{i(k\!\!\!/+m_N)}{k^2-m_N^2}\sqrt{2}g_{\pi NN}\frac{(k\!\!\!/-p\!\!\!/)}{2m_N}\gamma^5\,u(p)\,\frac{i}{(p-k)^2-m_\pi^2}$$

$$= (+ie)(2ig_{\pi NN}^2)\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(k+q)^2-m_N^2][k^2-m_N^2][(k-p)^2-m_\pi^2]}$$

$$\bar{u}(p')\frac{(p\!\!\!/-k\!\!\!/)}{2m_N}(k\!\!\!/+q\!\!\!/-m_N)\gamma^\mu(k\!\!\!/-m_N)\frac{(p\!\!\!/-k\!\!\!/)}{2m_N}u(p)$$

now write

$$\left(\frac{p\!\!\!/-k\!\!\!/}{2m_N}\right)u(p) = \frac{m_N-k\!\!\!/}{2m_N}u(p) = \frac{2m_N-(k\!\!\!/+m_N)}{2m_N}u(p)$$

$$= \left(\underbrace{1}_{\text{result in pseudo-scalar theory}} - \underbrace{\frac{k\!\!\!/+m_N}{2m_N}}_{\text{extra term.}}\right)u(p)$$

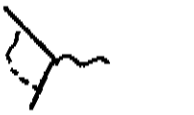
with the extra term we can write $\dfrac{k\!\!\!/-m_N}{k^2-m_N^2}\,k\!\!\!/+m_N = 1$,

cancelling a propagator as on p. 12

Similarly:

$$\bar{u}(p')\left(\frac{p\!\!\!/-k\!\!\!/}{2m_N}\right) = \bar{u}(p')\frac{m_N-k\!\!\!/-q\!\!\!/}{2m_N} = \bar{u}(p')\left[1-\frac{(k\!\!\!/+q\!\!\!/+m_N)}{2m_N}\right]$$

then



$= (+ie)(2 I g_{\pi NN}^2)$

$\cdot \Big\{$ (P-S result) $+ \int \frac{d^4k}{(2\pi)^4} \left(-\frac{1}{2m_N}\right) \frac{\bar{u}(p')(k\!\!\!/ + q\!\!\!/ - m_N)\gamma^\mu u(p)}{[(k+q)^2 - m_N^2][(p-k)^2 - m_\pi^2]}$

$+ \int \frac{d^4k}{(2\pi)^4} \left(-\frac{1}{2m_N}\right) \frac{\bar{u}(p')\gamma^\mu (k\!\!\!/ - m_N) u(p)}{(k^2 - m_N^2)((p-k)^2 - m_\pi^2)}$

$+ \int \frac{d^4k}{(2\pi)^4} \frac{1}{4m_N^2} \frac{\bar{u}(p')\gamma^\mu u(p)}{(p-k)^2 - m_\pi^2} \Big\}$

These integrals are not so hard, especially if we only need the divergent parts:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 - m_\pi^2} = \frac{-i}{(4\pi)^{d_2}} \frac{\Gamma(1-d_2)}{[m_\pi^2]^{1-d_2}}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k\!\!\!/ - m_N}{(k^2 - m_N^2)((k-p)^2 - m_\pi^2)}$$

$\ell = k - zp$

$k = \ell + zp$

$$= \int_0^1 dz \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell\!\!\!/ + zp\!\!\!/ - m_N}{[\ell^2 - \Delta]^2} \qquad p\!\!\!/\, u(p) = m_N u(p)$$

$$= \int_0^1 dz \frac{i}{(4\pi)^{d_2}} \frac{\Gamma(2-d_2)}{\Delta^{2-d_2}} \cdot [-(1-z)m_N]$$

$$\Delta = z m_\pi^2 + (1-z)m_N^2 - z(1-z)p^2$$

$$= z m_\pi^2 + (1-z)^2 m_N^2$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{(k+q - m_N)}{[(k+q)^2 - m_N^2][(k-p)^2 - m_\pi^2]} \qquad k' = k+q \quad p' = p+q$$

$$= \int \frac{d^4k'}{(2\pi)^4} \frac{(k' - m_N)}{[k'^2 - m_N^2][(k'-p')^2 - m_\pi^2]} = \text{previous integral}$$

$$= \int_0^1 dz \; \frac{i}{(4\pi)^{d_2}} \frac{\Gamma(2-d_2)}{\Delta^{2-d_2}} (z\not{p}' - m_N)$$

and, using $\bar{u}(p')\not{p}' = \bar{u}(p')\, m_N$ $= \int_0^1 dz \; \frac{i}{(4\pi)^{d_2}} \frac{\Gamma(2-d_2)}{\Delta^{2-d_2}} [-(1-z)m_N]$

here $\Delta = z m_\pi^2 + (1-z) m_N^2 - z(1-z)p'^2 = z m_\pi^2 + (1-z)^2 m_N^2$ as before.

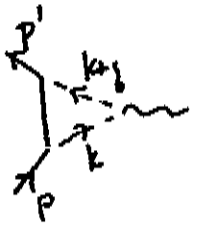
in all



$$= \text{(Pseudoscalar result)}$$

$$+ \quad (+ie)(2 i g_{\pi NN}^2) \frac{i}{(4\pi)^{d_2}} \bar{u}\gamma^\mu u$$

$$\int_0^1 dz \cdot \left\{ - \frac{1}{4m_N^2} \frac{\Gamma(1-d_2)}{[m_\pi^2]^{1-d_2}} - \frac{2}{2m_N} \frac{\Gamma(2-d_2)}{\Delta^{2-d_2}} [-(1-z)m_N] \right\}$$



$$= (\sqrt{2} g_{\pi NN})^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k+q)^2 - m_\pi^2} [-ie(2k+q)^\mu] \frac{i}{k^2 - m_\pi^2}$$

$$\bar{u}(p') \left(\frac{\not{k}+\not{q}}{2m_N}\right) \gamma^5 \frac{i(\not{p}-\not{k}+m_N)}{(p-k)^2 - m_N^2} \left(\frac{-\not{k}}{2m_N}\right) \gamma^5 u(p)$$

$$= (+ie)(-2ig_{\pi NN}^2) \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+q)^2 - m_\pi^2][k^2 - m_\pi^2]\,[(k-p)^2 - m_N^2]}$$

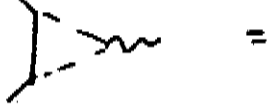
$$\cdot (2k+q)^\mu \; \bar{u}(p') \left(\frac{\not{k}+\not{q}}{2m_N}\right) (\not{p}-\not{k}-m_N) \left(\frac{\not{k}}{2m_N}\right) u(p)$$

Write $\frac{\not{k}}{2m_N} u(p) = \left[1 - \frac{\not{p}-\not{k}+m_N}{2m_N}\right] u(p)$

$$\bar{u}(p') \frac{(\not{k}+\not{q})}{2m_N} = \bar{u}(p) \left[1 - \frac{(\not{p}-\not{k}+m_N)}{2m_N}\right]$$

P-S result; extra term.

So



$$= (+ie)(-2ig_{\pi NN}^2) \cdot \Big\{ (\text{Pseudo-scalar result})$$

$$+ \int \frac{d^4k}{(2\pi)^4} \frac{(2k+q)^\mu}{[(k+q)^2 - m_\pi^2][k^2 - m_\pi^2]} \left[-\frac{1}{2m_N} - \frac{1}{2m_N} + \frac{1}{4m_N^2} (\not{p}-\not{k}+m_N) \right] \Big\}$$

Again, the integrals are easy.

$$\int \frac{d^4k}{(2\pi)^4} \frac{(2k+q)^\mu}{[(k+q)^2 - m_{\pi^0}^2][k^2 - m_{\pi^\pm}^2]} \qquad \ell = k + zq$$
$$2k+q = \ell + (1-2z)q$$

$$= \int_0^1 dz \int \frac{d^4\ell}{(2\pi)^4} \frac{[2\ell^\mu + (1-2z)q^\mu]}{[\ell^2 - \Delta]^2}$$

$$= \int_0^1 dz \; \frac{i}{(4\pi)^{d_L}} \frac{\Gamma(2-d_L)}{[\Delta]^{2-d_L}} [(1-z) - z] q^\mu$$

$\Delta = m_\pi^2 - z(1-z)q^2$ so the integral is antisymmetric under $z \leftrightarrow (1-z)$. So

$$= 0$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{(2k+q)^\mu}{[(k+q)^2 - m_\pi^2][k^2 - m_\pi^2]} (\not{p} - \not{k} + m_N) \cdot \qquad \not{p}\, u(p) = m_N u(p)$$

$$= \int_0^1 dz \int \frac{d^4\ell}{(2\pi)^4} \frac{[2\ell^\mu + (1-2z)q^\mu][2m_N - \not{\ell} + z\not{q}]}{[\ell^2 - \Delta]^2}$$

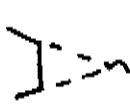
$\bar{u}(p')\not{q}\,u(p) = 0$

$$= \int_0^1 dz \; \frac{-i}{(4\pi)^{d_L}} \cdot (-2) \cdot \frac{\Gamma(1-d_L)}{[\Delta]^{1-d_L}} \frac{1}{2} \gamma^\mu$$

$$+ \quad (\text{terms that are } 0 \text{ as above})$$

$$= \int_0^1 dz \; \frac{i}{(4\pi)^{d_L}} \frac{\Gamma(1-d_L)}{[\Delta]^{1-d_L}} \gamma^\mu$$

so



$= (ie)(-2ig_{\pi NN}^2)\left\{ \text{(Pseudo-scalar result)} \right.$

$\left. + \int_0^1 dz \; \frac{1}{4m_N^2} \; \frac{i}{(4\pi)^{d_4}} \; \frac{\Gamma(1-d_4)}{[\Delta]^{1-d_4}} \; \bar{u}\gamma^\mu u \right\}$

$\simeq$ all



$=$ (Pseudo scalar result)

$+ (ie) \; \frac{2g_{\pi NN}^2}{(4\pi)^{d_4}} \; \bar{u}(p')\gamma^\mu u(p)$

$\int_0^1 dz \; \left\{ \frac{1}{4m_N^2} \; \frac{\Gamma(1-d_4)}{[m_\pi^2]^{1-d_4}} + \frac{1}{4m_N^2} \; \frac{\Gamma(1-d_4)}{[m_\pi^2 - z(1-z)q^2]^{1-d_4}} \right.$

$\left. - \frac{\Gamma(2-d_4)}{[zm_\pi^2 + (1-z)^2 m_N^2]^{2-d_4}} \cdot (1-z) \right\}$

Note that:

(1) only F_1 is affected, not F_2

(2) this expression is Λ^2 - divergent (pole at $d=2$) at $q^2=0$

(3) the coefficient of q^2 in this expression is $\log \Lambda^2$ divergent.

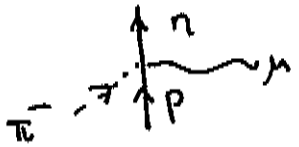
Now, why doesn't $F_1(0) = 0$ in the pseudovector theory? The answer (which is somewhat beyond the scope of Physics 330) is that the pseudovector theory as I have defined it is not gauge invariant. To make the interaction (7) gauge-invariant, we should have written:

$$\Delta\mathcal{H} = -\frac{g_{\pi NN}}{2m_N} D_\mu \pi^i \, \bar{N}\gamma^\mu\gamma^5 \tau^i N$$

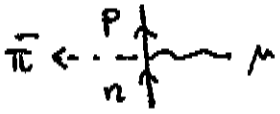
where $D_\mu\pi^- = (\partial_\mu + ie A_\mu)\pi^-$ $\qquad$ $D_\mu\pi^+ = (\partial_\mu - ie A_\mu)\pi^+$

This generates some additional vertices (only in the pseudovector theory)

$$\Delta\mathcal{H} = (\nabla\cdot\vec{q}) - ie A_\mu \frac{\sqrt{2}\, g_{\pi NN}}{2m_N} \pi^- \bar{n}\gamma^\mu\gamma^5 p + ie A_\mu \frac{\sqrt{2}\, g_{\pi NN}}{2m_N} \pi^+ \bar{p}\gamma^\mu\gamma^5 n$$



$$= -e \frac{\sqrt{2} g_{\pi NN}}{2m_N} \gamma^\mu\gamma^5$$

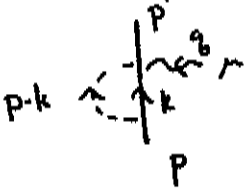


$$= +e \frac{\sqrt{2} g_{\pi NN}}{2m_N} \gamma^\mu\gamma^5$$

and vertices w. π^+ emission or absorption.

from there, we can build two new diagrams:





$$= \int \frac{d^4k}{(2\pi)^4}\ \bar{u}(p')\left(-e\,\frac{\sqrt{2}g_{\pi NN}}{2m_N}\,\gamma^\mu\gamma^5\right)\frac{i(\not{k}+m_N)}{k^2-m_N^2}$$

$$\cdot\,(\sqrt{2}g_{\pi NN})\left(-\frac{(\not{p}-\not{k})}{2m_N}\right)\gamma^5\,u(p)\cdot\frac{i}{(p-k)^2-m_\pi^2}$$

$$= -e\cdot\frac{2g_{\pi NN}^2}{2m_N}\cdot\int\frac{d^4k}{(2\pi)^4}\ \frac{1}{(k^2-m_N^2)((p-k)^2-m_\pi^2)}$$

$$\cdot\,\bar{u}(p')\,\gamma^\mu(\not{k}-m_N)\,\frac{(\not{p}-\not{k})}{2m_N}\,u(p)$$

$$= (ie)(2ig_{\pi NN}^2)\,\frac{1}{2m_N}\int\frac{d^4k}{(2\pi)^4}\ \frac{1}{(k^2-m_N^2)((k-p)^2-m_\pi^2)}$$

$$\cdot\,\bar{u}(p')\,\gamma^\mu(\not{k}-m_N)\left(1-\frac{\not{k}+m_N}{2m_N}\right)u(p)$$

In the second term, a propagator is cancelled. We read the integrals:

$$\int\frac{d^4k}{(2\pi)^4}\ \frac{1}{(k-p)^2-m_\pi^2}$$

$$\int\frac{d^4k}{(2\pi)^4}\ \frac{\not{k}-m_N}{[k^2-m_N^2][(k-p)^2-m_\pi^2]}$$

But, these are already given on p. 29



$$= (ie)(2ig_{\pi NN}^2)\ \bar{u}\gamma^\mu u\ \frac{i}{(4\pi)^{d_k}}$$

$$\cdot \int_0^1 dz \left\{ \left[-\frac{(1-z)m_N}{2m_N} \right] \frac{\Gamma(2-d_k)}{[\Delta]^{2-d_k}} + \frac{1}{2m_N}\ \frac{-1}{2m_N}(-1)\ \frac{\Gamma(1-d_k)}{[m_\pi]^{1-d_k}} \right\}$$

similarly:



$$= \int \frac{d^4k}{(2\pi)^4}\ \bar{u}(p')\ \sqrt{2}\, g_{\pi NN} \left(\frac{\not{p}-\not{k}}{2m_N}\right)\gamma^5 \frac{i(\not{k}+\not{q}-m_N)}{(k+q)^2-m_N^2}$$

$$\cdot \left(+e\ \frac{\sqrt{2}\, g_{\pi NN}}{2m_N}\ \gamma^\mu\gamma^5\right) u(p)\ \frac{i}{(p-k)^2-m_\pi^2}$$

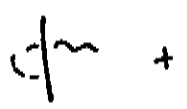
$$= (ie)\ 2ig_{\pi NN}^2\ \frac{1}{2m_N} \int \frac{d^4k}{(2\pi)^4}\ \frac{1}{(k+q)^2-m_N^2}\ \frac{1}{((k-p)^2-m_\pi^2)}$$

$$\bar{u}(p')\left(1 - \frac{\not{k}+\not{q}+m_N}{2m_N}\right)(\not{k}+\not{q}-m_N)\gamma^\mu u(p)$$

$$= (ie)\ (2ig_{\pi NN}^2)\ \bar{u}\gamma^\mu u \cdot \frac{i}{(4\pi)^{d_k}}$$

$$\cdot \int_0^1 dz \left\{ \left(-\frac{(1-z)m_N}{2m_N} \right) \frac{\Gamma(2-d_k)}{\Delta^{2-d_k}} + \frac{-1}{(2m_N)^2}(-1)\ \frac{\Gamma(1-d_k)}{[m_\pi^2]^{1-d_k}} \right\}$$

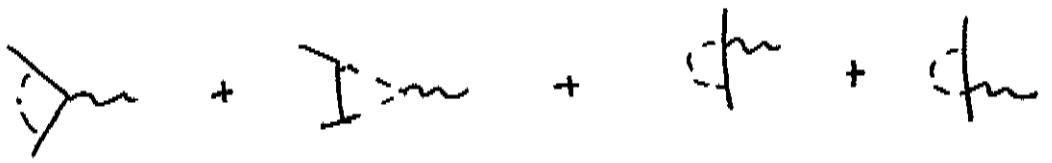
in all



$$= (ie)\ \frac{2g_{\pi NN}^2}{(4\pi)^{d_k}}\ \bar{u}(p')\gamma^\mu u(p)$$

$$\cdot \int_0^1 dz \left\{ -\frac{2}{4m_N^2}\ \frac{\Gamma(1-d_k)}{[m_\pi^2]^{1-d_k}} + \frac{\Gamma(2-d_k)}{[\Delta]^{2-d_k}}(1-z) \right\}$$

add this to p. 33:



$$= \text{(Pseudoscalar result.)}$$

$$+ (ie)\ (\bar{u}\gamma^\mu u)\ \frac{2g_{\pi NN}^2}{(4\pi)^{d_k}}$$

$$\cdot \int_0^1 dz\ \frac{1}{4m_N^2}\ \Gamma(1-d_k)\left[\frac{1}{[m_\pi^2 - z(1-z)q^2]^{1-d_k}} - \frac{1}{[m_\pi^2]^{1-d_k}}\right]$$

Excellent! This <u>vanishes</u> at $q^2 = 0$, as required.

The expression is given for $d<4$. As $d \to 4$, there is a pole:

$$\Delta F_1(q^2) = \frac{2g^2_{\pi NN}}{(4\pi)^{d/2}} \frac{1}{(1-d/2)} \frac{1}{(4m_N^2)} \Gamma(2-d/2)$$

$$\int_0^1 dz \cdot \left\{ \frac{m_\pi^2 - z(1-z)q^2}{[m_\pi^2 - z(1-z)q^2]^{2-d/2}} - \frac{m_\pi^2}{[m_\pi^2]^{2-d/2}} \right\}$$

$$\xrightarrow[d\to 4]{} \frac{2g^2_{\pi NN}}{(4\pi)^2} \frac{1}{(-1)} \frac{1}{4m_N^2} \left(\frac{1}{2-d/2}\right) \int_0^1 dz \, (-z(1-z)q^2)$$

$$\Delta F_1(q^2) \sim \frac{1}{6} \cdot \frac{g^2_{\pi NN}}{32\pi^2} \frac{q^2}{m_N^2} \ln \Lambda^2/m_\pi^2$$

so, even after the cancellation, $F_1(q^2)$ is divergent in the pseudo-vector theory.

The calculation of $F_1(q^2)|_{PV} - F_1(q^2)|_{PS}$ involves no difficult integrals. So it is not unlikely that Feynman could do this calculation in one night !

Physics 331 – Problem Set # 1

(due Wednesday, January 25)

1. Consider the set of $SU(3)$ generators in the **3** representation given in Peskin and Schroeder, Problem 15.1. Compute the structure constants f^{abc}. You may use the fact that f^{abc} is totally antisymmetric. (Why?). Show explicitly that

$$f^{abc}f^{abd} = C_2(G)\delta^{cd} = 3\delta^{cd} \tag{1}$$

as claimed in class.

2. Review the discussion from Physics 330 of the nonrelativistic potential between heavy fermions due to photon exchange. Then:

   (a) Show that the nonrelativistic potential between two electrons is

$$V(R) = +\frac{e^2}{4\pi R} \tag{2}$$

   (b) A field of charge Q transforms under local $U(1)$ transformations as

$$\Psi(x) \to e^{-iQ\alpha(x)}\Psi(x) \tag{3}$$

   Show that this implies that the covariant derivative on this field is

$$D_\mu = \partial_\mu - ieQA_\mu \tag{4}$$

   and that the coupling to the photon field is the photon-electron coupling multiplied by $(-Q)$. (The electron has charge $Q = -1$.)

   (c) Show that the nonrelativistic potential between two heavy fermions with charges Q_1 and Q_2 is

$$V(R) = +\frac{e^2Q_1Q_2}{4\pi R} \tag{5}$$

   (d) Now generalize this result to Yang-Mills theory. Assume that the propagator for a Yang-Mills field is that given in Peskin and Schroeder, problem 15.3(c). Show that the nonrelativistic potential between two heavy fermions, one in the representation r_1 of the gauge group G and the other in the representation r_2, has the form

$$V(R) = +\frac{g^2}{4\pi R} \cdot t^a_{r_1} \otimes t^a_{r_2} \tag{6}$$

(e) The matrix $t^a_{r_1} \otimes t^a_{r_2}$ is a matrix of dimension $d_{r_1} d_{r_2} \times d_{r_1} d_{r_2}$. To diagonalize this matrix, represent the product representation $r_1 \times r_2$ in terms of its irreducible components

$$r_1 \times r_2 = \sum_i r_i \ . \tag{7}$$

A state in which r_1 and r_2 are combined into a given irreducible representation r_I is an eigenstate of this matrix. Show this, and show that the eigenvalue is given by a simple combination of $C_2(r_1)$, $C_2(r_2)$, and $C_2(r_I)$.

(f) Show that, if $r_1 = r$, $r_2 = \overline{r}$, the complex conjugate representation, and the two fermions are in a state of zero (singlet) total charge ($r_I = 1$), then

$$V(R) = -\frac{g^2 C_2(r)}{4\pi R} \tag{8}$$

3. Peskin and Schroeder, Problem 15.5.

Physics 331 – Problem Set #1

Solutions

1.) The t^a as given satisfy $\mathrm{tr}\,[t^a t^b] = \frac{1}{2}\delta^{ab}$

(note $\mathrm{tr}\,(t^8)^2 = \frac{1}{12}[1+1+4] = \frac{1}{2}$).

Then

$$\mathrm{tr}\; t^c [t^b, t^c] = \mathrm{tr}\; t^a\, i f^{bcd} t^d = \frac{i}{2} f^{bca}$$

but $= \mathrm{tr}\,[t^a t^b t^c - t^a t^c t^b]$ which is totally antisymmetric by the cyclic invariance of the trace.

So we need only compute one of each triple eg. $f^{123}, f^{231}, f^{312}$

$$[t^1, t^2] = \left(\tfrac{1}{2}\right)^2 \left[\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\right] = \frac{1}{4}\left(\begin{smallmatrix} 2i & -2i & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) = \frac{i}{2}\left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$$

$$[t^1, t^3] = \left(\tfrac{1}{2}\right)^2 \left[\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right)\right] = \frac{1}{4}\left(\begin{smallmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \end{smallmatrix}\right) = -\frac{i}{2}\left(\begin{smallmatrix} 0 & -i & 0 \\ i & 0 & 0 \end{smallmatrix}\right)$$

so $[t^1, t^2] = i t^3 \quad [t^1, t^3] = -i t^2$

or: $f^{123} = f^{312} = f^{231} = 1 = -f^{132} = -f^{321} = -f^{213}$

$$[t^1,t^4] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right) = \tfrac{i}{2}t^7$$

$$[t^1,t^5] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}-i&0\\0&i\end{smallmatrix}\right) = -\tfrac{i}{2}t^6$$

$$[t^1,t^6] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) = \tfrac{i}{2}t^5$$

$$[t^1,t^7] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}-i&0\\0&i\end{smallmatrix}\right) = -\tfrac{i}{2}t^4$$

$$[t^2,t^4] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}i&0\\0&i\end{smallmatrix}\right) = \tfrac{i}{2}t^6$$

$$[t^2,t^5] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&i\\-i&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) = \tfrac{i}{2}t^7$$

$$[t^2,t^6] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}-i&0\\0&i\end{smallmatrix}\right) = -\tfrac{i}{2}t^4$$

$$[t^2,t^7] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right),\left(\begin{smallmatrix}-i&0\\0&i\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) = -\tfrac{i}{2}t^5$$

$$[t^3,t^4] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right) = \tfrac{i}{2}t^5$$

$$[t^3,t^5] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}-i&0\\0&-i\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}0&-i\\-i&0\end{smallmatrix}\right) = -\tfrac{i}{2}t^4$$

$$[t^3,t^6] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = -\tfrac{i}{2}t^7$$

$$[t^3,t^7] = \tfrac{1}{4}\left[\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)\right] = \tfrac{1}{4}\left(\begin{smallmatrix}0&i\\i&0\end{smallmatrix}\right) = \tfrac{i}{2}t^6$$

in this group, the nonzero components of f^{abc} are:

$$f^{147} = +\tfrac{1}{2} = -f^{174} \qquad f^{156} = -\tfrac{1}{2} = -f^{165}$$

$$f^{246} = \tfrac{1}{2} = -f^{264} \qquad f^{257} = +\tfrac{1}{2} = -f^{275}$$

and $f^{345} = \frac{1}{2} = -f^{354}$ $\quad f^{367} = -\frac{1}{2} = -f^{376}$

and cyclic permutations of these formulae.

$$[t^4,t^5] = \frac{1}{4}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= i\cdot\left[\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1&1\\1&-2\end{pmatrix}\right]$$

$$= i\left[\frac{1}{2}t^3 + \frac{\sqrt{3}}{2}t^8\right]$$

this confirms $f^{453} = \frac{1}{2} = f^{345}$ and sives $f^{458} = \frac{\sqrt{3}}{2}$

$$[t^4,t^6] = \frac{1}{4}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{i}{2}t^2$$

this confirms $f^{462} = f^{246} = \frac{1}{2}$

$$[t^4,t^7] = \frac{1}{4}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}0&i\\-i&0\end{pmatrix} = +\frac{i}{2}t^{\blacksquare}$$

$$[t^5,t^6] = \frac{1}{4}\left[\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}i&-i\\-i&i\end{pmatrix} = -\frac{i}{2}t^1$$

$$[t^5,t^7] = \frac{1}{4}\left[\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}-1&1\\1&-1\end{pmatrix} = \frac{i}{2}t^2$$

these confirm:

$f^{47\blacksquare} = f^{\blacksquare 47} = +\frac{1}{2}$ $\qquad f^{561} = f^{156} = -\frac{1}{2}$

$f^{572} = f^{257} = +\frac{1}{2}$

$$[t^6,t^7] = \frac{1}{4}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= i\left[-\frac{1}{2}t^3 + \frac{\sqrt{3}}{2}t^8\right]$$

confirming $f^{673} = f^{367} = -\frac{1}{2}$ and sing $f^{678} = \frac{\sqrt{3}}{2}$

we could stop here, since cyclic symmetry gives:

$$f^{845} = \frac{\sqrt{3}}{2} \qquad f^{867} = \frac{\sqrt{3}}{2}$$

but we could also check these results!

$$[t^8, t^4] = \frac{1}{2\sqrt{3}} \cdot \frac{1}{2} \left[\left(\begin{array}{cc|c} 1 & 1 & \\ \hline & & -2 \end{array}\right), \left(\begin{array}{cc|c} & & 1 \\ \hline 1 & 0 & \end{array}\right) \right] = \frac{1}{4}\frac{1}{\sqrt{3}} \left(\begin{array}{cc|c} & & 1+2 \\ \hline -2 & -1 & 0 \end{array}\right)$$

$$= \frac{1}{\sqrt{3}\cdot 4} \left(\begin{array}{cc|c} & & 3 \\ \hline -3 & 0 & \end{array}\right) = \frac{i\sqrt{3}}{2} t^5 \quad \checkmark$$

$$[t^8, t^6] = \frac{1}{2\sqrt{3}} \frac{1}{2} \left[\left(\begin{array}{cc|c} 1 & 1 & \\ \hline & & -2 \end{array}\right), \left(\begin{array}{cc|c} & & 0 \\ \hline 0 & 1 & \end{array}\right) \right] = \frac{1}{4}\frac{1}{\sqrt{3}} \left(\begin{array}{cc|c} & & 0 \\ \hline 0 & -1 & 2 \end{array}\right)$$

$$= i \frac{\sqrt{3}}{2} t^7$$

In all, the nonzero elements of f^{abc} are

$$f^{123} = 1$$

$$f^{147} = \tfrac{1}{2} \qquad f^{156} = -\tfrac{1}{2} \qquad f^{246} = \tfrac{1}{2} \qquad f^{257} = -\tfrac{1}{2}$$

$$f^{345} = \tfrac{1}{2} \qquad f^{367} = -\tfrac{1}{2} \qquad f^{845} = \frac{\sqrt{3}}{2} \qquad f^{867} = \frac{\sqrt{3}}{2}$$

and elements related by total antisymmetry.

Now compute $f^{abc} f^{abd}$.

First consider cases $c = d$:

$c = d = 1$: $(f^{231})^2 + (f^{321})^2 + (f^{471})^2 + (f^{561})^2 + (f^{741})^2 + (f^{651})^2$

$$= 2 \cdot \left[(f^{231})^2 + (f^{471})^2 + (f^{561})^2 \right]$$

$$= 2 \cdot \left[1^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 \right] = 3$$

$c = d = 2$ $\quad 2\left[(f^{132})^2 + (f^{462})^2 + (f^{572})^2\right] = 2(1+\frac{1}{4}+\frac{1}{4}) = 3$

$c = d = 3$ $\quad 2\left[(f^{123})^2 + (f^{453})^2 + (f^{693})^2\right] = 2(1+\frac{1}{4}+\frac{1}{4}) = 3$

$c = d = 4$ $\quad 2\left[(f^{714})^2 + (f^{624})^2 + (f^{534})^2 + (f^{584})^2\right]$

$= 2\cdot\left[\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{3}{4}\right] = 3$

$c = d = 5$ $\quad 2\left[(f^{615})^2 + (f^{725})^2 + (f^{345})^2 + (f^{485})^2\right] = 3$

$c = d = 6$ $\quad 2\left[(f^{156})^2 + (f^{246})^2 + (f^{736})^2 + (f^{786})^2\right] = 3$

$c = d = 7$ $\quad 2\left[(f^{147})^2 + (f^{257})^2 + (f^{367})^2 + (f^{867})^2\right] = 3$

$c = d = 8$ $\quad 2\left[(f^{458})^2 + (f^{678})^2\right] = 2\cdot(\frac{3}{4}+\frac{3}{4}) = 3$

For $c \neq d$, it is not possible in most cases to find nonzero contributions for $f^{abc}f^{abd}$. For example, for $c=1$ $d=2$, there is no a, b st.

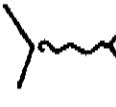
$f^{ab1}f^{ab2}$ is nonzero.

The only exception is $c=3$ $d=8$:

$$f^{ab3}f^{ab8} = 2\left[f^{453}f^{458} + f^{673}f^{678}\right]$$

$$= 2\cdot\left[\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2}\right] = 0$$

So indeed $f^{abc}f^{abd} = 3\,\delta^{cd}$ for $SU(3)$

2.) a)  $= (-ie)^2 \bar{u}\gamma^\mu u \; \frac{-i}{q^2} \; \bar{u}\gamma_\mu u$

for nonrelativistic fermions, this is. $\bar{u}\gamma^\mu u \approx 2m\,\delta_{ss'}\delta^{\mu 0}$

$$\approx +ie^2 \frac{1}{-|\vec{q}|^2} (2m)^2 = -i \left(\frac{+e^2}{|\vec{q}|^2}\right)(2m)^2$$

This implies $\tilde{V}(q) = + \frac{e^2}{|\vec{q}|^2}$ or $V(r) = + \frac{e^2}{4\pi r}$

b.) If $\Psi(x) \to e^{-i\alpha(x)Q}\Psi(x)$, we want to find

$D_\mu\Psi$ s.t. $D_\mu\Psi \to e^{-i\alpha(x)\cdot Q} D_\mu\Psi$ with $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha$

For $D_\mu = (\partial_\mu - ieQA_\mu)$:

$$D_\mu\Psi \to (\partial_\mu - ieQA_\mu + iQ\partial_\mu\alpha)\, e^{-iQ\alpha(x)}\Psi$$

$$= e^{-iQ\alpha(x)}\left[\partial_\mu - iQ\partial_\mu\alpha(x) - ieQA_\mu + iQ\partial_\mu\alpha(x)\right]\Psi(x)$$

$$= e^{-iQ\alpha(x)}\left[\partial_\mu - ieQA_\mu\right]\Psi(x)$$

So this form of D_μ satisfies the requirement.

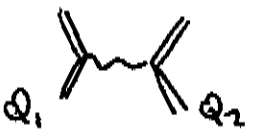
The locally gauge invariant Dirac Lagrangian is then

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi + \bar{\Psi}eQA\!\!\!/\Psi$$

so $\Delta H = -eQ\bar{\Psi}\gamma^\mu\Psi A_\mu$

the corresponding Feynman rule is

 $= +ieQ\gamma^\mu$

c.) 

$= (+ieQ_1)(+ieQ_2)\,\bar{u}\gamma^\mu u\,\frac{-1}{q^2}\,\bar{u}\gamma_\mu u$

$= +ie^2Q_1Q_2(2m)^2\frac{1}{-|\vec{q}|^2} = -i(2m)^2\frac{e^2Q_1Q_2}{|\vec{q}|^2}$

So $\tilde{V}(q) = +\frac{Q_1Q_2e^2}{|\vec{q}|^2}$ or $V(r) = +\frac{Q_1Q_2e^2}{4\pi r}$

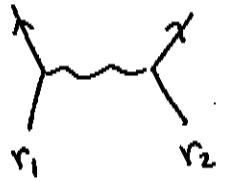
d.) In Yang-Mills theory, the gauge invariant Dirac equation is

$\mathcal{L} = \bar{\Psi}_r(i\not{D} - m)\Psi_r \qquad D_\mu = \partial_\mu - igA^a_\mu t^a_r$

for Ψ in representation r

$= \bar{\Psi}_r(i\not{\partial} - m)\Psi_r + gA^a_\mu\,\bar{\Psi}\gamma^\mu t^a_r\Psi$

so  $= ig\gamma^\mu t^a_r$

then 

$= (ig)^2\,\bar{u}\gamma^\mu u\,\frac{-i}{q^2}\,\bar{u}\gamma_\mu u\cdot t^a_{r_1}\otimes t^a_{r_2}$

$= -ig^2(2m)^2\frac{1}{|\vec{q}|^2}\cdot(t^a_{r_1}\otimes t^a_{r_2})$

so $\tilde{V}(q) = + \frac{g^2}{|\vec{q}|^2} t^a_{r_1} \otimes t^a_{r_2}$

or $V(r) = + \frac{g^2}{4\pi|\vec{r}|} (t^a_{r_1} \otimes t^a_{r_2})$

e.) Now, in the product representation $r_1 \times r_2$

$$t^a = t^a_{r_1} \otimes 1 + 1 \otimes t^a_{r_2}$$

$$t^2 = (t^a_{r_1} t^a_{r_1}) \otimes 1 + 2\, t^a_{r_1} \otimes t^a_{r_2} + 1 \otimes t^a_{r_2} t^a_{r_2}$$

$$= C_2(r_1) \cdot \underset{\sim}{1} + 2\, t^a_{r_1} \otimes t^a_{r_2} + C_2(r_2) \cdot \underset{\sim}{1}$$

The states of the irreducible representation r_I are those eigenstates of t^2 for which $t^2 = C_2(r_I)$

so in the irreducible representation r_I:

$$t^a_{r_1} \otimes t^a_{r_2} = \frac{1}{2} \left[C_2(r_I) - C_2(r_1) - C_2(r_2) \right]$$

In these states

$$V = \frac{g^2}{4\pi r} \cdot \frac{1}{2} \left[C_2(r_I) - C_2(r_1) - C_2(r_2) \right]$$

f) If we take $r_I = \underline{1}$ $\quad C_2(r_I) = 0$. Then

for $r \times \bar{r} \to 1$ $\quad V = - \frac{g^2 C_2(r)}{4\pi r}$

There is a nice way to check this: In SU(N)

$$N \times \bar{N} = \underbrace{1}_{\text{singlet}} + \underbrace{(N^2-1)}_{\text{adjoint}}$$

In the singlet state $t^a_N \otimes t^a_{\bar{N}} = \frac{1}{2}[-2C_2(N)] = -\left(\frac{N^2-1}{2N}\right)$

In the adjoint state $t^a_N \otimes t^a_{\bar{N}} = \frac{1}{2}[C_2(G) - 2C_2(N)]$

$$= \frac{1}{2}\left[N - 2\cdot\frac{N^2-1}{2N}\right] = +\frac{1}{2N}$$

but there are N^2-1 states in G. so

$$\text{tr}\left[t^a_N \otimes t^a_{\bar{N}}\right] = -\frac{N^2-1}{2N} + (N^2-1)\cdot\frac{1}{2N} = 0$$

This is as required: $\text{tr}[t^a_N \otimes t^a_{\bar{N}}] = (\text{tr}\, t^a_N)\cdot(\text{tr}\, t^a_{\bar{N}}) = 0\cdot 0 = 0$ ✓

3.) Every non-Abelian Lie grp G has an $SU(2)$ subgrp.

Let $a,b = 1,2,3$ be the generators of this $SU(2)$. Now consider $\text{tr}\, t^a_r t^b_r = C(r)\,\delta^{ab}$

a.) set $a=b$ and sum over $a=b=1,2,3$

$$\text{tr}\left[\sum_{a=1,2,3}(t^a_r)^2\right] = 3\,C(r)$$

but $\sum_{a=1,2,3} (t_r^a)^2 = J^2$, the quadratic Casimir operator in the SU(2). If r decomposes under SU(2) as

$$r \longrightarrow \sum_i j_i$$

then $\mathrm{tr}\, J^2 = \sum_i \dim(j_i)\, C_2(j_i)$

$$= \sum_i (2j_i+1)\, j_i(j_i+1)$$

then

$$3\, C(r) = \sum_i (2j_i+1)\, j_i(j_i+1)$$

b.) In SU(N), let a basis vector of the N representation be $\begin{pmatrix} \uparrow \\ \downarrow \\ \vdots \\ N \end{pmatrix}$.

Let $t^1 = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right)$ $t^2 = \frac{1}{2}\left(\begin{array}{cc|c} 0 & -i & \\ i & 0 & \\ \hline & & 0 \end{array}\right)$ $t^3 = \frac{1}{2}\left(\begin{array}{cc|c} +1 & 0 & \\ 0 & -1 & \\ \hline & & 0 \end{array}\right)$

t^1, t^2, t^3 generate an SU(2) subgroup of SU(N).

under this subgroup, the N transforms as

$$1\ (\text{spin } \tfrac{1}{2}) + \quad (N-2)\ \text{singlets.}\ (\text{spin } 0)$$

$$3\,C(N) = \underbrace{2 \cdot \tfrac{3}{4}}_{\text{spin } \frac{1}{2}} + 0 = \frac{3}{2}$$

so $\quad C(N) = \frac{1}{2}$

the adjoint representation transforms as

1 spin 1 $\quad \frac{1}{2}\left(\begin{smallmatrix} a & 1 \\ 1 & a \end{smallmatrix}\right), \frac{1}{2}\left(\begin{smallmatrix} a & -i \\ i & a \end{smallmatrix}\right), \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

$2(N-2)$ spin $\frac{1}{2}$ $\quad \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta \\ \alpha \end{smallmatrix}\right)$

and singlets

[another way to see this: $\quad N = \left(\frac{1}{2}\right) + (N-2)(0)$

$\bar{N} = \left(\frac{1}{2}\right) + (N-2)(0)$

$$N \times \bar{N} = \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + (N-2) \cdot \tfrac{1}{2} \times 0 + (N-2)\left(0 \times \tfrac{1}{2}\right) + (0 \times 0) + 0.$$

$$= 1 + 0 + 2(N-2) \cdot \tfrac{1}{2} + 0 + 0.$$

but $\quad N \times \bar{N} = (\text{adj}) + 0$]

then

$$3\,C(G) = \frac{3\cdot 2}{1} + 2(N-2)\cdot 2\cdot \tfrac{3}{4} + 0$$

$$= 3\cdot[\,2+(N-2)\,] = 3N$$

so $\quad C(G) = N \quad \rightarrow \quad C_2(G) = N$

c.) $\quad (N\times N)_{\text{symmetric}} = (\tfrac{1}{2}\times\tfrac{1}{2})_{\text{symmetric}} + (N-2)(\tfrac{1}{2}\times 0) + 0'0.$

$$= (1) + (N-2)\cdot(\tfrac{1}{2}) + 0'0$$

$$(N\times N)_{\text{antisymmetric}} = (\tfrac{1}{2}\times\tfrac{1}{2})_{\text{antisymmetric}} + (N-2)(\tfrac{1}{2}\times 0) + 0'0.$$

$$= (0) + (N-2)\cdot\tfrac{1}{2}\cdot + 0'0.$$

so

$$3\,C(\text{anti}) = (N-2)\cdot 2\cdot\frac{3}{4} = 3\cdot\frac{(N-2)}{2}$$

$$3\,C(\text{symm.}) = 3\cdot 2 + (N-2)\cdot 2\cdot\frac{3}{4} = 3\cdot\left(\frac{N+2}{2}\right)$$

$$\text{Now} \quad d(\text{anti}) = \frac{N(N-1)}{2} \qquad d(\text{symm}) = \frac{N(N+1)}{2}$$

$$C_2(\text{anti}) = \frac{d(G)}{d(\text{anti})}\, C(\text{anti}) = \frac{N^2-1}{\frac{N(N-1)}{2}}\,\frac{N-2}{2} = \frac{(N+1)(N-2)}{N}$$

$$C_2(\text{sym}) = \frac{d(G)}{d(\text{sym})}\, C(\text{sym}) = \frac{N^2-1}{\frac{N(N+1)}{2}}\,\frac{N+2}{2} = \frac{(N-1)(N+2)}{N}$$

now $\quad \text{tr}\,(t^2_{r_1 \times r_2}) = [C_2(r_1) + C_2(r_2)]\, d(r_1)\, d(r_2)$

$$= \sum_i C_2(r_i)\, d(r_i) \qquad \text{for } r_1 \times r_2 = \Sigma r_i$$

Check this will $\quad N \times N = (\text{anti}) + (\text{sym.})$

$$[C_2(N) + C_2(N)]\; d(N) \cdot d(N) = \frac{N^2-1}{2N} \cdot 2 \cdot N^2 = N(N^2-1)$$

$$\sum_i C_2(r_i)\, d(r_i) = C_2(\text{anti}) \cdot d(\text{anti}) + C_2(\text{sym}) \cdot d(\text{sym})$$

$$= \frac{(N+1)(N-2)}{N}\,\frac{N(N-1)}{2} + \frac{(N-1)(N+2)}{N} \cdot \frac{N(N+1)}{2}$$

$$= \frac{(N^2-1)\cdot N}{N} \cdot \left[\frac{(N-2)}{2} + \frac{N+2}{2}\right]$$

$$= N(N^2-1) = (\text{above}) \;\checkmark$$

Physics 331 – Problem Set # 2

(due Wednesday, February 1)

1. Peskin and Schroeder, Problem 9.1.

2. Let Φ be a linear combination of free fields: $\Phi = \int d^4x g(x)\phi(x)$, where $g(x)$ is a fixed function and $\phi(x)$ is a free Klein-Gordon field.

   (a) First, look at the evaluation of products of Φ's in canonical quantization. Time-ordered expectation values of Φ are evaulated as sums of contractions. Show that

   $$\left\langle \Phi^4 \right\rangle = 3 \cdot \left\langle \Phi^2 \right\rangle^2 \quad \left\langle \Phi^4 \right\rangle = 5 \cdot 3 \cdot \left\langle \Phi^2 \right\rangle^3 \quad etc. \tag{1}$$

   where $\langle \cdots \rangle$ denotes the time-ordered expectation value and $\langle \Phi^2 \rangle$ is the contraction. Using these results, show that

   $$\langle \exp[\Phi] \rangle = \exp[\left\langle \Phi^2 \right\rangle /2] \tag{2}$$

   (b) Rederive (2) using the functional integral to define the expectation values of $\phi(x)$.

3. Peskin and Schroeder, Problem 15.3. Compare these results to Problem 2 of the previous problem set.

4. Peskin and Schroeder, Problem 15.4. The formula at the top of p. 504 should read:

   $$D_F(x,y) = \int_0^\infty dT \int \mathcal{D}x \ \exp\left[i \int dt \, \frac{1}{2}(-(\frac{dx^\mu}{dt})^2 - m^2) - ie \int dt \, \frac{dx^\mu}{dt} A_\mu(x) \right] \tag{3}$$

   which is correct, because $(d\vec{x}/dt)^2$ (the square of the space components of x^μ) should have a positive coefficient.

Physics 331 – Problem Set #2

Solutions

1.) a.) $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi$

where $D_\mu = \partial_\mu + ieA_\mu$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \partial_\mu\phi^*(ieA_\mu\phi) - ieA_\mu\phi^*\partial^\mu\phi + e^2A_\mu A^\mu\phi^*\phi$$

We need to show, first, that the propagator of the ϕ field is

$$\overline{\phi\phi^*} = \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - m^2 + i\varepsilon}$$

Using $\mathcal{L}_0 = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi = \phi^*(-\partial^2 - m^2)\phi$

$$Z[J, J^*] = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left[i\int d^4x (\mathcal{L}_0 + J^*\phi + \phi^*J)\right]$$

$$\langle \phi(x)\phi^*(y)\rangle = \left(-i\frac{\delta}{\delta J^*(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right) Z \;/\; Z\Big|_{J=J^*=0}$$

now

$$Z[J, J^*] = \int \mathcal{D}\phi\,\mathcal{D}\phi^* \exp\Big[i\int d^4x \left[\phi^*(x) + \int d^4z\, J^*(z)\, G(z,x)\right](-\partial_x^2 - m^2) \cdot \left[\phi(x) + \int d^4y\, G(x,y) J(y)\right] - i\int d^4x\, d^4y\, \phi^*(x)\, G(x,y)\, \phi(y)\Big]$$

where $(-\partial_x^2 - m^2)G(x,y) = \delta(x-y)$

then, write $\phi'(x) = \phi(x) + \int d^4y\, G(x,y)\, J(y)$

$\phi^{*\prime}(x) = \phi^{\prime *}(x) + \int d^4z\, J^*(z)\, G(z,x)$

$\mathcal{D}\phi' \mathcal{D}\phi^{*\prime} = \mathcal{D}\phi\, \mathcal{D}\phi^*$

$$Z[J,J^*] = Z[0,0]\ \exp\left[-i \int d^4x\, d^4y\ \phi^*(x)\, G(x,y)\, \phi(y)\right]$$

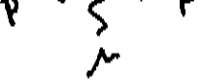
$$\langle \phi(x) \phi^*(y) \rangle = (-i)^2(-i)\ G(x,y) = i\, G(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - m^2}$$

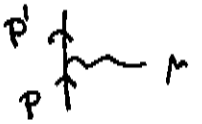
as promised.

The ~~interaction~~ vertices follow from

$$i\Delta\mathcal{L} = i\left[\partial_\mu \phi^*\, ieA_\mu \phi \;-\; ieA_\mu \phi^* \partial^\mu \phi \;+\; e^2 A_\mu A^\mu \phi^* \phi\right]$$

$$= eA_\mu \underbrace{(-\partial_\mu \phi^*)}_{-ip'^\mu} \phi \;+\; eA_\mu \phi^* \underbrace{\partial^\mu \phi}_{-ip^\mu} \;+\; ie^2 A_\mu A^\mu \phi^* \phi$$

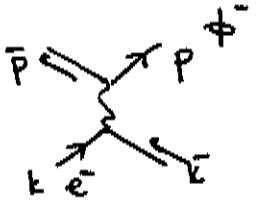
for 

so 

$= -ie(p+p')^\mu$ 

$= 2ie\, g^{\mu\nu}$

b.)



$$i\mathcal{M} = [-ie\,\bar{v}(\bar{k})\gamma_\mu u(k)]\frac{-i}{q^2}[-ie(p-\bar{p})_\mu]$$

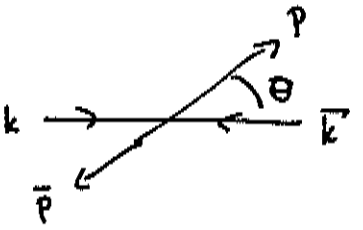
$$= \frac{ie^2}{q^2}\,\bar{v}(\bar{k})\gamma_\mu u(k)\,(p-\bar{p})_\mu$$

$$\frac{1}{4}\sum_{pols}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{(q^2)^2}\,\mathrm{tr}[\bar{k}\gamma_\mu k\gamma_\nu](p-\bar{p})^\mu(p-\bar{p})^\nu$$

$$= \frac{1}{4}\frac{e^4}{(q^2)^2}\cdot 4\left[\bar{k}^\mu k^\nu + \bar{k}^\nu k^\nu - g^{\mu\nu}\,\bar{k}\cdot k\right](p-\bar{p})_\mu(p-\bar{p})_\nu$$

$$= \frac{e^4}{(q^2)^2}\left[2\left[\bar{k}\cdot(p-\bar{p})\;k\cdot(p-\bar{p})\right] - (p-\bar{p})^2\,\bar{k}\cdot k\right]$$

kinematics :



$k = (E\,0,0,E)$ $\qquad p = (E,\ p\sin\theta,\ 0\ p\cos\theta)$

$\bar{k} = (E,0,0,-E)$ $\qquad \bar{p} = (E,\ -p\sin\theta,\ 0,\ -p\cos\theta)$

$$p-\bar{p} = 2p\,(0\ ,\sin\theta,\ 0\ \cos\theta)$$

$k\cdot(p-\bar{p}) = -2pE\cos\theta$ $\qquad k\cdot\bar{k} = 2E^2$

$\bar{k}\,(p-\bar{p}) = +2pE\cos\theta$ $\qquad (p-\bar{p})^2 = -4p^2$

$q^2 = 4E^2$

so $\qquad (\text{stuff}) = \dfrac{e^4}{(q^2)^2}\cdot 2\cdot 4E^2p^2\,(-\cos^2\theta + 1)$

$$\frac{1}{4}\sum_{pols}|M|^2 = \frac{e^4}{2}\frac{p^2}{E^2}\sin^2\theta$$

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to \phi^+\phi^-) = \frac{1}{2s}\left[\frac{1}{16\pi}\frac{p}{E}\right]\frac{e^4}{2}\frac{p^2}{E^2}\sin^2\theta$$

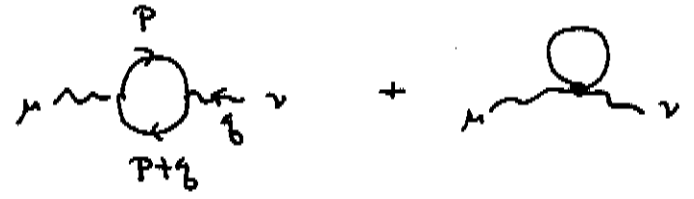
$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to \phi^+\phi^-) = \frac{\pi\alpha^2}{4s}\left(\frac{p}{E}\right)^3\sin^2\theta$$

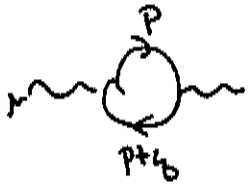
$$\int_{-1}^{1} d\cos\theta \sin^2\theta = \frac{4}{3}$$

$$\sigma(e^+e^- \to \phi^+\phi^-) = \frac{\pi\alpha^2}{3s}\left(\frac{p}{E}\right)^3$$

for $E \gg m$	$d\sigma/d\cos\theta$	$\sigma_{tot.}$
$e^+e^- \to \mu^+\mu^-$	$\sim(1+\cos^2\theta)$	$\to \frac{4\pi\alpha^2}{3s}$
$e^+e^- \to \phi^+\phi^-$	$\sim(1-\cos^2\theta)$	$\to \frac{\pi\alpha^2}{3s}$

c.)





$$= (-ie)^2 \int \frac{d^4p}{(2\pi)^4} (2p+q)^\mu \frac{i}{p^2-m^2} (2p+q)^\nu \frac{i}{(p+q)^2-m^2}$$

$$= e^2 \int_0^1 dx \int \frac{d^4P}{(2\pi)^4} \frac{1}{[P^2+x(1-x)q^2-m^2]^2} (2P+(1-2x)q)^\mu (2P+(1-2x)q)^\nu$$

$$P = p+xq$$

$$p = P-xq \qquad (2p+q) = 2P+(1-2x)q$$

$$p+q = P+(1-x)q$$

$$= e^2 \int_0^1 dx \cdot \frac{i}{(4\pi)^{d_k}} \left\{ - \frac{\Gamma(1-d_k)}{[m^2 - x(1-x)q^2]^{1-d_k}} \cdot 4 \cdot \frac{g^{\mu\nu}}{2} \right.$$

$$\left. + \frac{\Gamma(2-d_k)}{[m^2 - x(1-x)q^2]^{2-d_k}} (1-2x)^2 q^\mu q^\nu \right\}$$



$$= 2ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} g^{\mu\nu}$$

$$= 2ie^2 \cdot ig^{\mu\nu} \int \frac{d^4p}{(2\pi)^4} \frac{(p+q)^2 - m^2}{(p^2-m^2)((p+q)^2-m^2)}$$

$$= -2e^2 g^{\mu\nu} \int_0^1 dx \int \frac{d^4P}{(2\pi)^4} \frac{1}{[P^2 + x(1-x)q^2 - m^2]^2} \left[(P + (1-x)q)^2 - m^2 \right]$$

$$= -2e^2 g^{\mu\nu} \int_0^1 dx \frac{i}{(4\pi)^{d_k}} \left\{ - \frac{\Gamma(1-d_k)}{[m^2 - x(1-x)q^2]^{1-d_k}} \frac{d}{2} \right.$$

$$\left. + \frac{\Gamma(2-d_k)}{[m^2 - x(1-x)q^2]^{2-d_k}} \left[(1-x)^2 q^2 - m^2 \right] \right\}$$

Φ

actually, we can symmetrize:

$$(1-x)^2 \longrightarrow \frac{1}{2}\left[(1-x)^2 + x^2\right] = \frac{1}{2}\left[1 - 2x + 2x^2\right]$$

the sum of the diagrams is:



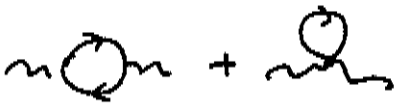
$$= \frac{ie^2}{(4\pi)^{d_k}} \int_0^1 dx \left\{ - \frac{\Gamma(1-d_k)}{[m^2 - x(1-x)q^2]^{1-d_k}} g^{\mu\nu} (2-d) \right.$$

$$\left. + \frac{\Gamma(2-d_k)}{[m^2 - x(1-x)q^2]^{2-d_k}} \left((1-2x)^2 q^\mu q^\nu - (1 - 2x + 2x^2) q^2 g^{\mu\nu} + 2m^2 g^{\mu\nu} \right) \right]$$

the first term is:

$$\frac{\Gamma(1-d_4)}{\left[m^2-x(1-x)q^2\right]^{1-d_4}}\,2-d \;=\; \frac{2\cdot\Gamma(2-d_4)}{\left[m^2-x(1-x)q^2\right]^{2-d_4}}\cdot\left[m^2-x(1-x)q^2\right]$$

so



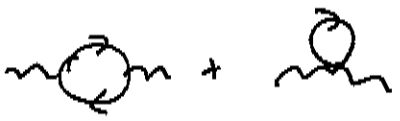
$$= \frac{ie^2}{(4\pi)^{d_4}}\,\Gamma(2-d_4)\int_0^1 dx\,\frac{1}{\left[m^2-x(1-x)q^2\right]^{2-d_4}}$$

$$\{\; -2g^{\mu\nu}(m^2-x(1-x)q^2) + 2m^2 g^{\mu\nu} - (1-2x+2x^2)\,q^2 g^{\mu\nu} + (1-2x)^2 q^\mu q^\nu \;\}$$

$$\{\} = (-1)\left[(1-4x+4x^2)\,q^2 g^{\mu\nu} - (1-2x)^2 q^\mu q^\nu\right]$$

$$= (-1)\,(1-2x)^2\; q^\mu g^{\mu\nu} - q^\mu q^\nu$$

so



$$= \frac{-ie^2}{(4\pi)^{d_4}}\,\Gamma(2-d_4)\int_0^1 dx\,\frac{(1-2x)^2}{\left[m^2-x(1-x)q^2\right]^{2-d_4}}\cdot\left(q^2 g^{\mu\nu}-q^\mu q^\nu\right)$$

$d\to 4$

$$= \frac{-i\,e^2}{(4\pi)^2}\left(q^2 g^{\mu\nu}-q^\mu q^\nu\right)\int_0^1 dx\,(1-2x)^2\,\ln\left(\frac{\Lambda^2}{m^2-x(1-x)q^2}\right)$$

$$\int_0^1 dx\,(1-2x)^2 = 1-4\cdot\tfrac{1}{2}+4\tfrac{1}{3} = \tfrac{1}{3}$$

for an electron positron or $\mu^+\mu^-$ loop, we found ~ 330:



$$= -\frac{8ie^2}{(4\pi)^2}(q^2 g^{\mu\nu} - q^\mu q^\nu)\int_0^1 dx\; x(1-x)\,\ln\frac{\Lambda^2}{m^2 - x(1-x)q^2}$$

$$\int dx\; x(1-x) = \frac{1}{6}$$

for $q^2 \gg m^2$



$$= -i\frac{e^2}{(4\pi)^2}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot C \cdot \ln\frac{\Lambda^2}{q^2}$$

$$C = \begin{cases} \frac{4}{3} & \mu^+\mu^- \\ \frac{1}{3} & \phi^*\phi \end{cases}$$

2.) * Let Φ be a linear combination of free fields:

$$\Phi = \int dx\; f(x)\,\phi(x)$$

a.) $\langle \Phi^2 \rangle = \int dx\,dy\; f(x) f(y)\; \overbrace{\phi(x)\,\phi(y)} \equiv \overbrace{\Phi\,\Phi}$

then

$$\langle \Phi^4 \rangle = \langle \Phi\,\Phi\,\Phi\,\Phi \rangle$$

$$= \overbrace{\Phi\,\Phi}\,\overbrace{\Phi\,\Phi} + \overbrace{\Phi\,\overbrace{\Phi\,\Phi}\,\Phi} + \overbrace{\overbrace{\Phi\,\Phi}\,\overbrace{\Phi\,\Phi}}$$

$$= 3\left(\langle \overbrace{\Phi\,\Phi} \rangle\right)^2$$

~~simple~~

$$\langle \Phi^{2n} \rangle = (\#\text{ of contractions}) \cdot \overbrace{\Phi\Phi}^{n}$$

$$= (2n-1)(2n-3) \cdots 1$$

then

$$\langle e^{\Phi} \rangle = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n \right\rangle$$

terms w. n = odd vanish

$$= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\overbrace{\Phi^2}\right)^m (2m-1)(2m-3) \cdots 1$$

$$= \sum_{m=0}^{\infty} \frac{1}{2m\,(2m-2)(2m-4)\cdots} \left(\overbrace{\Phi^2}\right)^m$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{1}{m!} \left(\overbrace{\Phi^2}\right)^m$$

$$= e^{\overbrace{\Phi^2}/2}$$

b) With functional integration:

$$\langle e^{\Phi} \rangle = \frac{\int \mathcal{D}\phi \; e^{i\int d^4x \, \mathcal{L}_0(\phi)} \; e^{\int d^4x \, f(x)\,\phi(x)}}{\int \mathcal{D}\phi \; e^{i\int \mathcal{L}_0}}$$

if $\mathcal{L}_0 = \frac{1}{2}\phi \Delta \phi$ where eg. $\Delta = (-\partial^2 - m^2)$

$$\overbrace{\phi\phi} = i\Delta^{-1}(x,y)$$

the exponent of the numerator above is

$$i\int d^4x \left[\tfrac{1}{2}\phi\,\Delta\,\phi + (-if)\cdot\phi\right]$$

$$= i\int d^4x \left(\tfrac{1}{2}(\phi - i f\Delta^{-1})\,\Delta\,(\phi - i\Delta^{-1}f) + \tfrac{1}{2} f\,\Delta^{-1} f\right)$$

then

$$\langle e^{\Phi}\rangle = \exp\left[\tfrac{1}{2}\int dx\,dy\; f(x)\, i\Delta^{-1}(x,y)\, f(y)\right]$$

$$= \exp\left[\tfrac{1}{2}\int dx\,dy\; f(x)\,\overbrace{\phi(x)\,\phi(y)}\, f(y)\right]$$

$$= \exp\left[\tfrac{1}{2}\langle\Phi^2\rangle\right]$$

3.) a.) Using the result

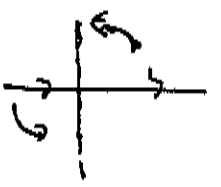
$$\langle e^{-ie\oint_P dx_\mu A^\mu}\rangle = \exp\left[-\frac{e^2}{2}\oint_P dx_\mu \oint_P dy_\nu \langle A_\mu(x)\, A_\nu(y)\rangle\right]$$

Now

$$\langle A_\mu(x)\, A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i g_{\mu\nu}}{k^2 + i\varepsilon}$$

to evaluate the integral, rotate to Euclidean space.

$$x^0, y^0 = -ix^0_E,\ -iy^0_E \qquad k^0 = ik^0_E$$



then

$$\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\varepsilon} e^{-ik\cdot(x-y)} \longrightarrow \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{-k_E^2} e^{-ik_E(x_E-y_E)}$$

this is the solution of

$$\nabla^2 \Delta_E(x_E, y_E) = \delta^{(4)}(x_E - y_E) \quad \text{4-d Euclidean space}$$

let $\vec{E} = \vec{\nabla} \Delta_E(r)$ then

$$\int d^4x \; \vec{\nabla}\cdot\vec{E} = \int d^3S \; \hat{r}\cdot\vec{E} = 1$$

so $$\vec{E} = \frac{\hat{r}}{(\text{Area of sphere})} = \frac{\hat{r}}{2\pi^2 r^3}$$

$$\Delta_E = -\frac{1}{4\pi^2 r^2}$$

rotate back to Minkowski space

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon} = \frac{1}{4\pi^2 (x-y)^2}$$

then

$$\left\langle e^{-ie\oint_P dx_\mu A^\mu} \right\rangle = \exp\left[-e^2 \oint_P dx_\mu \oint_P dy_\mu \frac{1}{8\pi^2 (x-y)^2}\right]$$

$$\| \quad \langle U_P \rangle$$

b.) For a loop of the form:



$$\langle U_p \rangle = (\text{R-independent factor}) \cdot \exp\left[-e^2 \cdot 2 \int_{-T/2}^{T/2} dx^0 \underbrace{\int_{T/2}^{-T/2} dy^0}_{\text{opposite direction}} \frac{1}{8\pi^2 [(x^0-y^0)^2 - R^2 - i\varepsilon]} \right]$$

$$\cong (\text{R-indep}) \cdot \exp\left[+ \frac{e^2}{4\pi^2} T \int_{-\infty}^{\infty} dx^0 \frac{1}{[(x^0)^2 - R^2 - i\varepsilon]} \right]$$

to evaluate the integral, rotate the contour:

$$\int dx^0 \frac{1}{(x^0)^2 - R^2} = -i \int dx_E^0 \frac{1}{-((x_E^0)^2 + R^2)} = i \cdot \frac{\pi}{R}$$

so

$$\langle U_p \rangle = (\text{const}) \cdot \exp\left[+ i \frac{e^2}{4\pi R} T \right]$$

comparing to $\langle U_p \rangle = (\text{const}) \cdot e^{-iV(R)T}$

$$V(R) = -\frac{e^2}{4\pi R}$$

c.) In a non-Abelian Sym theory

$$\langle U_P \rangle = \langle \mathrm{tr}\left(\mathcal{P} \exp\left[+ig \oint dx_\lambda A_\lambda^a t_r^a \right] \right) \rangle$$

$$= \mathrm{tr}\left(1 - g^2 \oint dx_\lambda dy_\nu A_\lambda^a A_\nu^b t_r^a t_r^b + \cdots \right)$$

$\mathrm{tr}\, 1 = d_r$

$A_\lambda^a A_\nu^b$ is a δ^{ab}, so the second term includes $\mathrm{tr}\, t_r^a t_r^a = C_2(r) d_r$

so $$= d_r \left(1 + \frac{i g^2 C_2(r)}{4\pi R} T + \cdots \right)$$

If this exponentiates (it does)

$$\langle U_P \rangle = (\text{const.}) \cdot \exp\left[+ i \frac{g^2 C_2(r)}{4\pi R} T \right]$$

or $$V(R) = - \frac{g^2 C_2(r)}{4\pi R}$$

Both answers for $V(R)$ agree with Problem 2 of the previous problem set.

3) a.) Let $D(x,y,T)$ satisfy:

$$\left[i\frac{\partial}{\partial T} - (\partial^2 + m^2) \right] D(x,y,T) = i\,\delta(T)\,\delta^{(4)}(x-y)$$

$D(x,y,T)$ is the propagator of the Schrödinger equation

$$\langle x | e^{-iHT} | y \rangle \quad \text{with} \quad H = [-\nabla^2 + \partial_0^2 + m^2]$$

for $\hat{H} = (-\nabla^2 + m^2)$ we would have

$$\langle x | e^{-i\hat{H}T} | y \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} e^{-i(|\vec{k}|^2+m^2)T}$$

simply, in 4 dimensions

$$\langle x | e^{-iHT} | y \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} e^{-i(|\vec{k}|^2-(k^0)^2+m^2)T}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} e^{i(k^2-m^2)T}$$

$$\int_0^\infty dT \; D(x,y,T) = \int_0^\infty dT \; \langle x | e^{-iHT} | y \rangle$$

$$= \int_0^\infty dT \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} e^{i(k^2-m^2)T} e^{-\epsilon T}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{-i(k^2-m^2)+\epsilon}$$

so that, finally,

$$\int_0^\infty dT\ D(x,y,T) = \int \frac{d^4k}{(2\pi)^4}\ e^{-ik\cdot(x-y)}\ \frac{i}{k^2 - m^2 + i\varepsilon}$$

Now, the solution of the Schrödinger eq has a functional integral representation:

$$\langle x|\ e^{-i\hat{H}T}\ |y\rangle = \int_{\substack{X(T)=x^\mu \\ X(0)=y^\mu}} \mathcal{D}X(t)\ e^{i\int_0^T dt\ [\frac{1}{2}m|\dot{\vec{x}}|^2 - V(x)]}$$

to represent $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$

if we want $H = -\nabla^2 + \partial_0^2 + [\text{illegible}]\ m^2$

we set $2m = 1$ $\quad m/2 = \frac{1}{4}$ $\quad V(x) = m^2$

$$= \int \mathcal{D}X(t)\ e^{i\int_0^T dt\ [\frac{1}{4}(-\dot{X}^2) - m^2]}$$

where $-\dot{X}^2 = (\dot{\vec{X}})^2 - (\dot{X}^0)^2$ $\quad$ all $\dot{} = \frac{d}{dT}$

If we rescale time $t = t'/2$ $\quad \frac{dx^\mu}{dt} = 2\frac{dx^\mu}{dt'}$

$$= \int \mathcal{D}X(t)\ e^{i\int_0^{2T} dt'\ \frac{1}{2}(-\dot{X}^2 - m^2)}$$

where now $\dot{X} = \frac{d}{dt'}X^\mu$

then (dropping primes)

$$D_F(x,y) = \int_0^\infty dT \int_{\substack{X(T)=x' \\ X(0)=y'}} \mathcal{D}X \; e^{i\int_0^{2T} dt \; \frac{1}{2}(-\dot{x}^2 - m^2)}$$

b.) Now add a background electromagnetic field. We need to replace the ~~functional~~ integral above by one ~~that~~ solves

$$i\frac{\partial}{\partial T} D(x,y,T) = (D^2 + m^2) \; D(x,y,T)$$

I claim ~~that~~ this is done by adding a Wilson line.

$$D(x,y,T) = \int \mathcal{D}X \; e^{i\int_0^T dt \; [\frac{1}{4}(-\dot{x}^2) - m^2]} \; e^{-ie\int_0^T dt \; \frac{dx}{dt}\cdot A(x)}$$

To show this, discretize

$$D(x,y,T+\varepsilon) = \int_{-\infty}^{\infty} \frac{dx'}{C} \; e^{i\frac{1}{4}\left[-\frac{(x-x')^2}{\epsilon}\right]} \; e^{-i\epsilon m^2} \; e^{ie(x-x')\cdot \underset{\sim}{A}\left(\frac{x+x'}{2}\right)} \; D(x',y,T)$$

Expanding to $O(\epsilon)$ with $(x-x') \sim \sqrt{\epsilon}$

$$D(x,y,T) + \epsilon \frac{\partial}{\partial T} D(x,y,T) = \int \frac{dx'}{C} \; e^{-i \frac{1}{4\epsilon}(x-x')^2}$$

$$\cdot (1 - i\epsilon m^2)$$

$$\cdot \left(1 - ie\,(x-x')^\mu A_\mu(x) - \frac{e^2}{2} (x-x')^\mu A_\mu(x) (x-x')^\nu A_\nu(x) \right.$$

$$\left. -ie\,(x-x')^\mu \cdot \tfrac{1}{2}(x'-x)^\nu \partial_\nu A_\mu(x) + \cdots \right)$$

$$\left(D(x,y,T) + (x'-x)^\nu \partial_\nu D(x,y,T) + \tfrac{1}{2}(x'-x)^\mu (x'-x)^\nu \partial_\mu \partial_\nu D(x,y,T) \right.$$

$$\left. + \cdots \right)$$

now $$\int \frac{dx'}{C} \; e^{-i \frac{1}{4\epsilon}(x-x')^2} = \frac{1}{C} \left[\frac{4\epsilon}{i} \right]^{4/2}$$

set C so that this $= 1$

then $$\langle (x-x')^\mu (x-x')^\nu \rangle = g^{\mu\nu} \left(\frac{2\epsilon}{i} \right)$$

Using these relations, the above integral

becomes

$\longrightarrow$

$$D(x,y,T) + \epsilon \frac{\partial}{\partial T} D(x,y,T)$$

$$= D(x,y,T) - i\epsilon m^2 D(x,y,T)$$

$$+ \frac{1}{2}(-i2\epsilon) g^{\mu\nu} \partial_\mu \partial_\nu D(x,y,T)$$

$$- \frac{e^2}{2}(-2i\epsilon) g^{\mu\nu} A_\mu A_\nu D(x,y,T)$$

$$- i\frac{e}{2}(+2i\epsilon) g^{\mu\nu} \partial_\nu A_\mu(x) D(x,y,T)$$

$$-ie(+2i\epsilon) g^{\mu\nu} A_\mu \partial_\nu D(x,y,T)$$

$$= D(x,y,T) - i\epsilon \{ m^2 D(x,y,T) + \partial_\mu \partial^\mu D(x,y,T)$$

$$- e^2 A_\mu A^\mu D(x,y,T) + ie(\partial_\mu A^\mu) D(x,y,T)$$

$$+ ie A_\mu \partial^\mu D(x,y,T) \}$$

$$= D(x,y,T) - i\epsilon \{ (\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) D(x,y,T)$$

$$+ m^2 D(x,y,T) \}$$

so indeed.

$$i\frac{\partial}{\partial T} D(x,y,T) = (D^2 + m^2) D(x,y,T)$$

c.) In a non-Abelian gg theory, this calculation can be repeated as long as the factors of $A_\mu(x) \to A_\mu^a(x) t^a$ come to the left of all other elements. Then, in the non-Abelian case:

$$D(x,y;T) = \int \mathcal{D}X \; e^{i\int_0^T dt \left(\frac{1}{4}(-\dot{x}^2)-m^2\right)} \cdot \mathcal{P}\left(e^{+ig\int dt\, \dot{x}^\mu A_\mu^a t^a}\right)$$

$\mathcal{P}$: path ordering

yields a solution of

$$i\frac{\partial}{\partial T} D(x,y;T) = \left[(\partial_\mu - igA_\mu^a t^a)^2 + m^2\right] D(x,y;T)$$

Physics 331 – Problem Set # 3

(due Wednesday, February 8)

1. Peskin and Schroeder, Problem 9.2.

2. In class, we computed the matrix element for the process $q\overline{q} \to gg$, where q is a massless fermion in the representation r of a Yang-Mills gauge group G and g is the Yang-Mills gauge boson. Our result had the form:

$$i\mathcal{M} = \overline{v}(\overline{p})\gamma \cdot \epsilon^*(k_1)t^a \cdots \gamma \cdot \epsilon^*(k_2)t^b u(p) + \cdots \tag{1}$$

   In QED, we were typically interested in cross sections summed over final spins and averaged over initial spins. In Yang-Mills theory, we might also wish to sum over final gauge indices ('colors') and average over initial colors.

   (a) For the term written out in (1), show that the group theory factor corresponding to this color average and sum of the squared matrix elements is

$$\frac{1}{d_r^2}\mathrm{tr}[t^a t^b t^b t^a] \tag{2}$$

   (b) Show that this factor evaluates to

$$\frac{1}{d_r}[C_2(r)]^2 \tag{3}$$

   (c) Evalute

$$\frac{1}{d_r^2}\mathrm{tr}[t^a t^b t^a t^b] \tag{4}$$

3. Compute the differential cross section $d\sigma/d\cos\theta$ for $q\overline{q} \to gg$, averaged over initial spins and colors and summed over final spins and colors. Use the method of Peskin and Schroeder, Problem 17.3(a). Use the same explicit spinors and polarization vectors that appeared in the Problem Sets 6 and 7 of Physics 330. Compute the color averages and sums using the results of Problem 2.

Physics 331 – Problem Set #3

Solutions

1.) a) Evaluate $\langle q | e^{-\beta H} | q' \rangle$

$$\prod_{-\infty}^{\infty} \int dq_i \; \langle q | e^{-\epsilon H} | q_1 \rangle \langle q_1 | e^{-\epsilon H} | q_2 \rangle \cdots \langle q_{N-1} | e^{-\epsilon H} | q' \rangle$$

Using the method in the text.

$$\langle q_i | e^{-\epsilon H} | q_{i+1} \rangle = \int \frac{dp_i}{2\pi} \; e^{i p_i (q_i - q_{i+1})} \; e^{-\epsilon H\left(p, \frac{q_i + q_{i+1}}{2}\right)}$$

assembling the pieces

$$\langle q | e^{-\beta H} | q' \rangle = \int \mathcal{D}q \, \mathcal{D}p \; e^{\int_0^\beta dt \, [\, i p \cdot \dot{q} - H(p,q)\,]}$$

for $H = \frac{p^2}{2} + V(q)$ we can eliminate p by doing the integral

$$\int dp_i \; e^{i p_i (q_i - q_{i+1}) - \epsilon \frac{p_i^2}{2}}$$ complete the square

$$= (\text{const}) \cdot \exp\left[-\frac{1}{2\epsilon} (q_i - q_{i+1})^2 \right]$$

then

$$\langle q | e^{-\beta H} | q' \rangle = \int_{\substack{q(\beta)=q \\ q(0)=q'}} \mathcal{D}q \; e^{-\int_0^\beta dt \, (\frac{1}{2}\dot{q}^2 + V(q))}$$

To compute the trace, set $q' = q$ and integrate $\int dq$

$$\text{tr } e^{-\beta H} = \int_{q(0)=q(\beta)} \mathcal{D}q \; \exp\left[- \int_0^\beta dt \, [\tfrac{1}{2}\dot{q}^2 + V(q)] \right]$$

b.) Let $q \to x$ $\quad V(q) = \frac{1}{2}\omega^2 x^2$

$$\text{tr } e^{-\beta H} = \int \mathcal{D}x(t) \; \exp\left[- \int_0^\beta dt \, [\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2] \right]$$

$x(t)$ is periodic, so write

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}$$

$x(t)$ is <u>real</u> so $x_{-n} = x_n^*$

the functional integral becomes:

$$\mathrm{tr}\, e^{-\beta H} = C(\beta) \cdot \int dx_0 \prod_{n=1}^{\infty} \int d(\mathrm{Re}\, x_n)\, d(\mathrm{Im}\, x_n)\; e^{-\int_0^\beta dt\, L_E}$$

where

$$\int_0^\beta dt\, L_E = \int_0^\beta dt \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2\right)$$

$$= \tfrac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^{\infty} \tfrac{1}{2}\left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \omega^2\right] |x_n|^2 \cdot 2$$

$$= \tfrac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^{\infty} \left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \omega^2\right] \left[(\mathrm{Re}\, x_n)^2 + (\mathrm{Im}\, x_n)^2\right]$$

so, using $\int dx\, e^{-\frac{1}{2}bx^2} = \sqrt{2\pi} \cdot \frac{1}{b}$

$$\mathrm{tr}[e^{-\beta H}] = C'(\beta) \cdot \frac{1}{\omega} \prod_{n=1}^{\infty} \left(\frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]^{1/2}}\right)^2$$

$$= C''(\beta) \left(\omega \prod_{n=1}^{\infty} \left[1 + \frac{\omega^2\beta^2/4}{(\pi n)^2}\right]\right)^{-1}$$

$$= C'''(\beta) \left(\sinh \omega\beta/2\right)^{-1}$$

$$= C''''(\beta) \left[e^{\omega\beta/2}\left(1 - e^{-\beta\omega}\right)\right]^{-1}$$

$$= \underbrace{C^{(5)}(\beta)}\; e^{-\beta\omega/2} \sum_{m=0}^{\infty} e^{-\beta\omega m}$$

various ω-independent factors

so indeed:

$$\mathrm{tr}\, e^{-\beta H} = \sum_{m=0}^{\infty} e^{-\beta E_m} \qquad E_m = \omega(m+\tfrac{1}{2})$$

up to an overall constant independent of ω.

c.) The Hamiltonian of the Klein-Gordon equation is

$$\mathcal{H} = \sum_k h(\omega_k)$$

where h is a Harmonic oscillator Hamiltonian with frequency $\omega_k = (\vec{k}^2 + m^2)^{1/2}$. Then it follows from the previous section that

$$\mathrm{tr}\, e^{-\beta H} = \int \mathcal{D}\phi \; e^{-\int d^4x \, \mathcal{L}_E}$$

where $\int d^4x \, \mathcal{L}_E = \int d^4x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right]$

and $\phi(t,\vec{x})$ is integrated over configurations periodic in t with period β:

$$\phi(t+\beta, \vec{x}) = \phi(t, \vec{x})$$

try this out Fourier decompose:

$$\phi(t,\vec{x}) = \sum_{n,k} \frac{1}{\sqrt{\beta}\sqrt{V}} e^{i\frac{2\pi n}{\beta}} e^{i\vec{k}\cdot\vec{x}} \phi_{n,k}$$

$$\int d^4x \, \mathcal{L}_E = \sum_{n,k} \tfrac{1}{2} \phi_{-n,-k} \phi_{n,k} \left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \vec{k}^2 + m^2\right]$$

$$= \sum_{n>0} \sum_{k} |\phi_{n,k}|^2 \left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \omega_k^2\right] + \sum_k |\phi_{0,k}|^2 \omega_k^2$$

$$\int \mathcal{D}\phi \, e^{-\int d^4x \mathcal{L}_E} = \prod_k \cdot \left(\omega_k \prod_{n>0} \left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \omega^2\right]\right)^{-1} \cdot C(\beta)$$

$$= \prod_k \left(e^{-\beta\omega_k/2} \sum_m e^{-\beta m \omega_k}\right) \cdot C'(\beta)$$

by the manipulation on the previous page.

For future reference, I will say that this result defines

$$\left[\det(-\partial^2 + m^2)\right]^{-\frac{1}{2}}(\beta)$$

d.) Now generalize to fermions. There is an issue with the boundary condition. Consider

$$\mathrm{tr}\left[\psi(\beta)\bar{\psi}(0) e^{-\beta H}\right] = \mathrm{tr}\left(\left[e^{\beta H}\psi(0)e^{-\beta H}\right]\bar{\psi}(0) e^{-\beta H}\right)$$

$$= \mathrm{tr}\left(\psi(0) e^{-\beta H} \bar{\psi}(0)\right)$$

this should equal $= \mathrm{tr}\ \bar{\psi}(0)\psi(0)\, e^{-\beta H}$

However, now $\psi, \bar{\psi}$ are in the opposite order:

$$\mathrm{tr}\ \psi(\beta)\bar{\psi}(0)\, e^{-\beta H} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int \mathcal{L}_E}\, \psi(\beta)\bar{\psi}(0)$$

$$\mathrm{tr}\ \bar{\psi}(0)\psi(0)\, e^{-\beta H} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int \mathcal{L}_E}\, \bar{\psi}(0)\psi(0)$$

$$= -\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int \mathcal{L}_E}\, \psi(0)\bar{\psi}(0)$$

these are consistent only if

$$\psi(t+\beta) = -\psi(t)$$

Assuming this, evaluate:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-\int dt\,(\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi)}$$

$$\psi(t) = \sum_{p=n+\frac{1}{2}} e^{i\frac{2\pi}{\beta}pt}\psi_p \qquad \bar{\psi}(t) = \sum_{p=n+\frac{1}{2}} e^{i\frac{2\pi}{\beta}pt}\bar{\psi}_p$$

$$\int dt\, \mathcal{L}_E = \sum_p \bar{\psi}_{-p}\left(\frac{2\pi i p}{\beta} + \omega\right)\psi_p$$

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-\int \mathcal{L}_E} = \prod_p \left(\frac{2\pi i p}{\beta} + \omega\right) \cdot C(\beta)$$

then

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\; e^{-\int \mathcal{L}_E} = \prod_{p>0} \left(\frac{2\pi i p}{\beta} + \omega\right)\left(\frac{-2\pi i p}{\beta} + \omega\right) \cdot C(\beta)$$

$$= \prod_{n=0}^{\infty} \left[\left[\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]^2 + \omega^2\right] \cdot C'(\beta)$$

$$= C''(\beta) \prod_{n=0}^{\infty} \left(1 + \frac{(\beta\omega/2)^2}{(\pi(n+\tfrac{1}{2}))^2}\right)$$

$$= C''(\beta) \cosh\left(\frac{\beta\omega}{2}\right)$$

$$= C''(\beta)\left(e^{\beta\omega/2} + e^{-\beta\omega/2}\right)$$

then

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\; e^{-\int \mathcal{L}_E} = \sum_{m=0,1} e^{-\beta E_m}$$

with $E_0 = -\omega/2 \qquad E_1 = +\omega/2$

a 2-level system with splitting ω.

e.) Finally, apply this to electrodynamics

$$\int \mathcal{D}A \; e^{-\int d^4x \; + \frac{1}{4}(F_{\mu\nu})^2} \qquad \leftarrow \text{Euclidean action}$$

gauge fix: $\xi = 1$

$$= (\text{const}) \cdot \int \mathcal{D}A \; e^{-\int d^4x \; \frac{1}{2} A_\mu(-\partial^2)A^\mu} \; \underbrace{\det[-\partial^2]}$$

Faddev.-Popov det.

$$= (\text{const}) \cdot \left((\det[-\partial^2])^{-\frac{1}{2}} \right)^4 \cdot \det(-\partial^2)$$

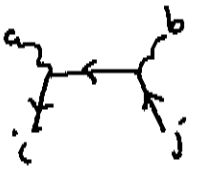
in both cases, over scalar functions periodic w. period β.

$$= (\text{const}) \cdot \left([\det(-\partial^2)]^{-\frac{1}{2}} \right)^2$$

This is the partition function for 2 massless Bose fields.

(The ghosts act as 2 negative degrees of freedom.)

2.) a.) A term contributing to $\bar{q}_i q_j \to g^a g^b$ (G indices; color indices)



can be of the form

$$iM = (iM) \cdot (t^a t^b)_{ij}$$

then

$$\sum_{ijab} |iM|^2 = |iM|^2 \; (t^a t^b)_{ij} \; (t^b t^a)_{ji}$$

$$= |M|^2 \; \mathrm{tr}\,(t^a t^b t^b t^a)$$

If we average over initial colors, we need:

$$\left(\frac{1}{d_r}\right)^2 \sum_{ij} \sum_{ab} |iM|^2 = |M|^2 \cdot \frac{1}{d_r^2} \mathrm{tr}(t^a t^b t^b t^a)$$

b.) $$\frac{1}{d_r^2} \mathrm{tr}\, t^a t^b t^b t^a = \frac{1}{d_r^2} \mathrm{tr}\, t^a t^a t^b t^b = \frac{1}{d_r^2} \mathrm{tr}\,(C_2(r))^2 \cdot 1$$

$$= \frac{1}{d_r} C_2^2(r)$$

c.) $\frac{1}{d_r^2}\, tr[t^a t^b t^a t^b]$

$= \frac{1}{d_r^2}\left(tr[t^a t^b t^b t^a] + tr\,[t^a t^b\,[t^a, t^b]]\right)$

$= \frac{(C_2(r))^2}{d_r} + \frac{1}{d_r^2}\, tr\,[t^a t^b\,(i f^{abc} t^c)]$

$= \frac{C_2^2(r)}{d_r} + \frac{1}{d_r^2}\, tr\,[\tfrac{1}{2}[t^a, t^b] \cdot i f^{abc} t^c]$

$= \frac{(C_2(r))^2}{d_r} + \frac{1}{d_r^2}\, tr\,[\frac{i}{2} f^{abd}\, i f^{abc}\, t^d t^c]$

now $\quad f^{abd} f^{abc} = C_2(G)\,\delta^{cd}$

$= \frac{(C_2(r))^2}{d_r} + \frac{1}{d_r^2} \cdot (\tfrac{-1}{2})\, C_2(G)\, tr\, t^c t^c$

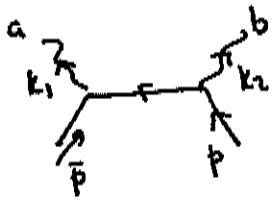
$= \frac{(C_2(r))^2}{d_r} - \frac{1}{2}\,\frac{1}{d_r}\, C_2(G)\, C_2(r)$

$= \frac{1}{d_r}\, C_2(r)\left(C_2(r) - \tfrac{1}{2} C_2(G)\right)$

3.) Now compute the cross section for $q_L \bar{q}_R \to gg$

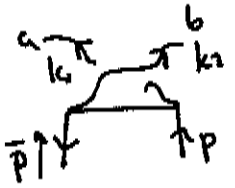
with <u>massless</u> quarks and gluons.

$i\mathcal{M}$ = 



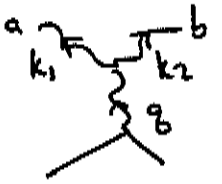
$$= (ig)^2\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_1)\, t^a\, \frac{i\,\gamma\cdot(p-k_2)}{(p-k_2)^2}\, \gamma\cdot\varepsilon^*(k_2)\, t^b\, u(p)$$

$$= +ig^2\, \frac{1}{2p\cdot k_2}\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_1)\, \gamma\cdot(p-k_2)\, \gamma\cdot\varepsilon^*(k_2)\, u(p)\,(t^a t^b)$$



$$= (ig)^2\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_2)\, t^b\, \frac{i\,\gamma\cdot(p-k_1)}{(p-k_1)^2}\, \gamma\cdot\varepsilon^*(k_1)\, t^a\, u(p)$$

$$= +ig^2\, \frac{1}{2p\cdot k_1}\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_2)\, \gamma\cdot(\not{p}-\not{k}_1)\, \gamma\cdot\varepsilon^*(k_1)\, u(p)\,(t^b t^a)$$



$$= (ig)\, \bar{v}(\bar{p})\, \gamma^\lambda t^c\, u(p)\, \frac{-i}{q^2}\, \varepsilon^*_\lambda(k_1)\, \varepsilon^*_\nu(k_2)$$

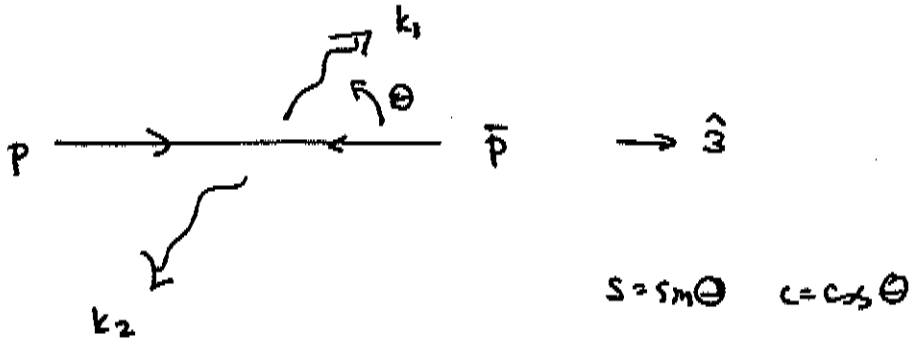
$$g f^{abc}\left[(-k_1+k_2)^\lambda g^{\mu\nu} + g^{\nu\lambda}\underbrace{(-k_2-q)^\mu}_{k_1+k_2} + g^{\lambda\mu}(q+k_1)^\nu\right]$$

$$= g^2 f^{abc}\, \bar{v}(\bar{p})\, \gamma^\lambda t^c\, u(p)\, \frac{1}{q^2}$$

$$\left[(k_2-k_1)^\lambda\, \varepsilon^*_\mu(k)\varepsilon^\mu(k_2) - 2\,k_2\cdot\varepsilon^*(k_1)\,\varepsilon^{*\lambda}(k_2) + 2\,k_1\cdot\varepsilon^*(k_2)\,\varepsilon^{\lambda *}(k_1)\right]$$

using $\quad k_1\cdot\varepsilon^*(k_1) = k_2\cdot\varepsilon^*(k_2) = 0$

set up the kinematics as



$p = (E, 0, 0, E)$ $\quad k_1 = (E, Es, 0, Ec)$

$\bar{p} = (E, 0, 0, -E)$ $\quad k_2 = (E, -Es, 0, -Ec)$

$2p\cdot k_2 = 2E^2(1+c)$ $\quad q^2 = 4E^2$

$2p\cdot k_1 = 2E^2(1-c)$

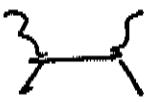
$(p-k_2) = (0, Es, 0, E(1+c))$

$(p-k_1) = (0, -Es, 0, E(1-c))$

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ $\quad \sigma^\mu = (1, \vec{\sigma})$ $\quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$

Evaluate the diagrams for $q_L \bar{q}_R \to gg$

$u_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \bar{v}_R(\bar{p}) = \sqrt{2E}\,(0\,0\,1\,-1\,0)$



$$= ig^2 \frac{1}{2E^2(1+c)} \sqrt{2E}\,(00|-10)\left(\overline{\bar\sigma\cdot\varepsilon_1^*}\,\middle|\,\overline{\sigma\cdot\varepsilon_1^*}\right)$$

$$\cdot\left(\overline{\bar\sigma\cdot(p-k_2)}\,\middle|\,\overline{\sigma\cdot(k_2-p)}\right)\left(\overline{\bar\sigma\cdot\varepsilon_2^*}\,\middle|\,\overline{\sigma\cdot\varepsilon_2^*}\right)\sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\; t^a t^b \cdot$$



$$= \frac{ig^2}{E(1+c)}\,(-10)\;\bar\sigma\cdot\varepsilon_1^*\;\sigma\cdot(p-k_2)\;\bar\sigma\cdot\varepsilon_2^*\begin{pmatrix}0\\1\end{pmatrix}\; t^a t^b$$

$$= \frac{ig^2}{E(1-c)}\,(-10)\;\bar\sigma\cdot\varepsilon_2^*\;\sigma\cdot(p-k_1)\;\bar\sigma\cdot\varepsilon_1^*\begin{pmatrix}0\\1\end{pmatrix}\; t^b t^a$$



$$= g^2 f^{abc}\,\frac{2E}{4E^2}\,(-10)\;\bar\sigma^\lambda\begin{pmatrix}0\\1\end{pmatrix}\; t^c$$

$$\cdot\left[\,(k_2-k_1)^\lambda\,\varepsilon_1^*\cdot\varepsilon_2^* - 2k_2\cdot\varepsilon_2^*(k_1)\,\varepsilon_2^{*\lambda} + 2k_1\cdot\varepsilon_2^*\,\varepsilon_1^{*\lambda}\,\right]$$

$$= \frac{g^2 f^{abc}}{2E}\,t^c\;(-10)\left[\;\bar\sigma\cdot(k_2-k_1)\,\varepsilon_1^*\cdot\varepsilon_2^* - 2k_2\cdot\varepsilon_1^*\,\varepsilon_2^{*\bar\sigma} + 2k_1\cdot\varepsilon_2^*\,\varepsilon_1^{*\bar\sigma}\;\right]\begin{pmatrix}0\\1\end{pmatrix}$$

to go further we need the explicit forms for

$$\sigma\cdot(p-k_2) = \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix}$$

$$\sigma\cdot(p-k_1) = \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix}$$

$$\bar{\sigma}\cdot(k_2-k_1) = \begin{pmatrix} -2Ec & -2Es \\ -2Es & 2Ec \end{pmatrix}$$

and for the polarization vectors:

$$\varepsilon_{1R}^* = \frac{1}{\sqrt{2}}(0, c, -i, -s) \qquad \bar{\sigma}\cdot\varepsilon_{1R}^* = \frac{1}{\sqrt{2}}\begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix}$$

$$\varepsilon_{1L}^* = \frac{1}{\sqrt{2}}(0\ c, +i, -s) \qquad \bar{\sigma}\cdot\varepsilon_{1L}^* = \frac{1}{\sqrt{2}}\begin{pmatrix} -s & c+1 \\ c-1 & s \end{pmatrix}$$

$$\varepsilon_{2R}^* = \frac{1}{\sqrt{2}}(0, -c, -i, s) \qquad \bar{\sigma}\cdot\varepsilon_{2R}^* = \frac{1}{\sqrt{2}}\begin{pmatrix} s & -c-1 \\ -c+1 & -s \end{pmatrix}$$

$$\varepsilon_{2L}^* = \frac{1}{\sqrt{2}}(0, -c, +i, s) \qquad \bar{\sigma}\cdot\varepsilon_{2L}^* = \frac{1}{\sqrt{2}}\begin{pmatrix} s & -c+1 \\ -c-1 & -s \end{pmatrix}$$

Note that $k_1\cdot\varepsilon_1^* = 0 = k_2\cdot\varepsilon_2^*$

and also ~~that~~ $k_2\cdot\varepsilon_1^* = 0 = k_1\cdot\varepsilon_2^*$

$$\varepsilon_{1R}^*\cdot\varepsilon_{2R}^* = 1 = \varepsilon_{1L}^*\cdot\varepsilon_{2L}^*$$

$$\varepsilon_{1R}^*\cdot\varepsilon_{2L}^* = 0 = \varepsilon_{1L}^*\cdot\varepsilon_{2R}^*$$

Now evaluate the diagrams on p. 13 for the case

$$q_L \bar{q}_R \to g_R g_R$$



$$= \frac{ig^2}{E(1+c)} (-1\ 0)\ \frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} s & -(c+1) \\ 1-c & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad t^a t^b$$

$$= \frac{ig^2}{(1+c)} \frac{1}{2E} (s \quad 1-c) \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix} t^a t^b$$

$$[(1-c)(1+c) = s^2]$$

$$= \frac{ig^2}{1+c} \cdot \frac{E}{2E} \left\{ s(1+c)^2 + s^3 + s^3 - s^3 \right\} t^a t^b$$

$$= \frac{ig^2}{1+c} \frac{1}{2} s (1+c)\left[(1+c) + (1-c)\right] t^a t^b$$

$$= ig^2 s \frac{(1+c)}{(1+c)} t^a t^b = ig^2 s\ t^a t^b$$



$$= \frac{ig^2}{E(1-c)} (-1\ 0)\ \frac{1}{\sqrt{2}} \begin{pmatrix} s & -(c+1) \\ 1-c & -s \end{pmatrix} \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad t^b t^a$$

$$= \frac{ig^2}{2(1-c)E} (-s \quad c+1) \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix} \begin{pmatrix} c-1 \\ s \end{pmatrix} t^b t^a$$

$$= \frac{ig^2}{2(1-c)} \cdot \frac{E}{2E} \cdot \{ -s(1-c)^2 - s^3 + \underbrace{s(c+1)(c-1)}_{-s^3} + s(s^2) \} \cdot t^b t^a$$

$$= \frac{ig^2}{2(1-c)} \frac{1}{2} \{ -s(1-c)^2 - s(1-c)(1+c) \} t^b t^a$$

$$= + \frac{ig^2}{(1-c)} [-s(1-c)] \left[\frac{(1-c)+(1+c)}{2}\right] t^b t^a$$

$$= \underline{-ig^2 s} \; t^b t^a$$



$$= \frac{g^2 f^{abc}}{2E} t^c (-10) \begin{pmatrix} -2Ec & -2Es \\ -2Es & 2Ec \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \varepsilon_{1R}^* \cdot \varepsilon_{2R}^*$$

$$= \frac{g^2 f^{abc}}{2E} t^c \, 2Es \cdot 1$$

$$= g^2 f^{abc} t^c s$$

adding the three pieces

$$iM(q_L \bar{q}_R \to g_R g_R) = +ig^2 s\, t^a t^b - ig^2 s\, t^b t^a + g^2 f^{abc} t^c s$$

$$= ig^2 s \underbrace{[t^a, t^b]}_{if^{abc}t^c} + g^2 s\, f^{abc} t^c$$

$$= 0$$

the amplitude for $q_L \bar{q}_R \to g_L g_L$ has a similar cancellation. (by CP)

$$iM(q_L \bar{q}_R \to g_L g_L) = 0$$

Finally, consider $q_L \bar{q}_R \to g_R(k_1)\, g_L(k_2)$



$$= \frac{ig^2}{E(1+c)} (-1\ 0)\, \frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} s & -c+1 \\ -(c+1) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^a t^b$$

$$= \frac{ig^2}{E(1+c)} \frac{1}{2} (s\ \ (1-c)) \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix} t^a t^b$$

$$= \frac{ig^2}{1+c}\,\frac{E}{2E}\left\{ -s(1+c)(1-c) + s^3 - s(1-c)^2 - s(s^2)\right\} t^a t^b$$

$$= \frac{ig^2}{1+c}\cdot\frac{1}{2}\left\{ -s(1-c)(1+c) - s(1-c)^2\right\} t^a t^b$$

$$= \frac{ig^2}{1+c}\cdot\frac{1}{2}\; -s(1-c)\left[(1+c)+(1-c)\right]\; t^a t^b$$

$$= \frac{-ig^2}{1+c}\; s(1-c)\; t^a t^b$$



$$= \frac{ig^2}{E(1-c)}\,(-1\;0)\,\frac{1}{\sqrt{2}}\begin{pmatrix} s & -c+1 \\ -(c+1) & -s\end{pmatrix}\begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c)\end{pmatrix}$$

$$\cdot\,\frac{1}{\sqrt{2}}\begin{pmatrix} -s & c-1 \\ c+1 & s\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix}\; t^b t^a$$

$$= \frac{ig^2}{E(1-c)}\,\frac{1}{2}\,(-s\;-(1-c))\begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c)\end{pmatrix}\begin{pmatrix} -(1-c) \\ s\end{pmatrix}\; t^b t^a$$

$$= \frac{ig^2}{1-c}\,\frac{E}{2E}\left\{ -s(1-c)^2 - s^3 + s(1-c)^2 - s(1-c)^2\right\} t^b t^a$$

$$= \frac{ig^2}{1-c}\,\frac{1}{2}\left\{ -s(1-c)\left[(1-c)+(1+c)\right]\right\} t^b t^a$$

$$= \frac{ig^2}{1-c}\,[-s(1-c)]\,t^b t^a = -ig^2 s\; t^b t^a$$

and



$= 0$ since $\varepsilon^*_{1R}\cdot\varepsilon^*_{2L} = 0$

then

$$\cdot i\mathcal{M}(q_L\bar{q}_R \to g_{1R}g_{2L}) = (-ig^2 s)\left(\frac{1-c}{1+c}\,t^a t^b + t^b t^a\right)$$

now square and sum over colors

$$\frac{1}{d_r^2}\sum_{\text{colors}} |i\mathcal{M}|^2 = g^4 s^2 \left\{ \frac{(1-c)^2}{(1+c)^2}\cdot\frac{1}{d_r}(C_2(r))^2 \right.$$

$$+ \quad 1 \cdot \frac{1}{d_r}(C_2(r))^2$$

$$\left. + \quad 2\,\frac{(1-c)}{(1+c)}\cdot\frac{1}{d_r}\,C_2(r)\left(C_2(r) - \frac{C_2(G)}{2}\right)\right\}$$

$$= g^4 s^2 \cdot\left[\frac{1}{d_r}C_2(r)\right]\cdot\left\{\left[\frac{1-c}{1+c} + 1\right]^2 C_2(r)\right.$$

$$\left. - \quad \frac{(1-c)}{1+c}\;C_2(G)\right\}$$

$$= g^4 s^2 \left(\frac{1}{dr} C_2(r)\right) \left\{ \frac{4}{(1+c)^2} C_2(r) \right.$$

$s^2 = (1+c)(1-c)$

$$\left. - \frac{(1-c)}{(1+c)} C_2(G) \right\}$$

$$= g^4 \frac{1}{dr} C_2(r) \cdot \left\{ 4 C_2(r) \cdot \left(\frac{1-c}{1+c}\right) \right.$$

$$\left. - (1-c)^2 C_2(G) \right\}$$

put :-

$$s = 4E^2 \qquad t = -2E^2(1-c) \qquad u = -2E(1+c)$$

$$= g^4 \frac{1}{dr} C_2(r) \left\{ 4 C_2(r) \frac{t}{u} - 4 C_2(G) \frac{t^2}{s^2} \right\}$$

$$\frac{d\sigma}{d\cos\theta} \left(q_L \bar{q}_R \to g_R(k_1) g_L(k_2)\right)$$

$$= \frac{1}{2s} \frac{1}{16\pi} g^4 \frac{C_2(r)}{dr} \cdot 4 \left(C_2(r) \frac{t}{u} - C_2(G) \frac{t^2}{s^2} \right)$$

$$= \frac{2\pi (g^2/4\pi)^2}{s} \frac{C_2(r)}{dr} \left(C_2(r) \frac{t}{u} - C_2(G) \frac{t^2}{s^2} \right)$$

to include $g_L(k_1)\, g_R(k_2)$

add the formula with $t \leftrightarrow u$

to average over initial spins multiply by $2 \times \frac{1}{4}$

$$\frac{d\sigma}{d\cos\theta}(q\bar{q} \to gg) = \frac{\pi}{s}\left(\frac{g^2}{4\pi}\right)^2 \frac{C_2(r)}{dr}$$

$$\cdot\left[C_2(r)\left(\frac{t}{u}+\frac{u}{t}\right) - C_2(G)\,\frac{t^2+u^2}{s^2}\right]$$

$$= \frac{\pi}{s}\left(\frac{g^2}{4\pi}\right)^2 \frac{C_2(r)}{dr}$$

$$\cdot\left[C_2(r)\left(\frac{2}{1-\cos^2\theta}\right) - \frac{C_2(G)}{2}(1+\cos^2\theta)\right]$$

this formula is to be integrated over

$\cos\theta > 0$ only

(identical particles)

Physics 331 – Problem Set # 4

(due Wednesday, February 15)

1. In the previous problem set, you computed the differential cross section for $q\overline{q} \to gg$ in a general Yang-Mills theory with massless fermions. Specialize your answer to QCD and show that it produces eq. (17.75) of Peskin and Schroeder. Using crossing appropriately (being careful to average over initial state colors but sum over final state colors), derive eqs. (17.76) and (17.77).

2. The various fermion-fermion scatting cross sections in QCD can be derived from QED results by multiplying by appropriate color factors. Using this strategy, derive eqs. (17.64), (17.65), (17.70), and (17.71) of Peskin and Schroeder.

3. Compute the differential cross section $d\sigma/d\cos\theta$ for $gg \to gg$, averaged over initial spins and colors and summed over final spins and colors. Use the method of Peskin and Schroeder, Problem 17.3(b). Derive eq. (17.78) of Peskin and Schroeder. This completes the set of $2 \to 2$ parton cross sections needed to compute the cross sections for hard-scattering processes at hadron colliders.

Physics 331 – Problem Set #4

Solutions

1.) Start from the final formula of the previous solution set

$$\frac{d\sigma}{d\cos\Theta_{cm}}(q\bar{q}\to gg) = \frac{\pi}{s}\left(\frac{g^2}{4\pi}\right)^2 \frac{C_2(r)}{d_r}\left[C_2(r)\left(\frac{t}{u}+\frac{u}{t}\right) - C_2(G)\,\frac{t^2+u^2}{s^2}\right]$$

set $\frac{g^2}{4\pi} = \alpha_s \qquad C_2(r) = \frac{4}{3} \qquad d_r = 3 \qquad C_2(G) = 3$

$$\frac{(C_2(r))^2}{d_r} = \frac{16}{27}$$

$$d\cos\Theta_{cm} = \frac{dt}{\frac{1}{2}s} \qquad \text{since} \quad t = -\tfrac{1}{2}s(1-\cos\Theta)$$

in all

$$\frac{d\sigma}{dt} = \frac{\pi}{s}\,\frac{1}{(s/2)}\,\alpha_s^2 \cdot \frac{16}{27}\left[\left(\frac{t}{u}+\frac{u}{t}\right) - \left(\frac{3}{4/3}\right)\frac{t^2+u^2}{s^2}\right]$$

that is:

$$\frac{d\sigma}{dt}(q\bar{q}\to gg) = \frac{32}{27}\frac{\pi\alpha_s^2}{s^2}\left[\frac{t}{u}+\frac{u}{t}-\frac{9}{4}\left(\frac{t^2+u^2}{s^2}\right)\right]$$

as required

for $gg\to q\bar{q}$ we simply exchange the initial and final states. The colour and spin sum are:

$q\bar{q}\to gg$: $\frac{1}{4}\sum_{spin}\cdot\left(\frac{1}{3}\right)^2\sum_{colour}$

$gg\to q\bar{q}$ $\frac{1}{4}\sum_{spin}\left(\frac{1}{8}\right)^2\sum_{colour}$

so

$$\frac{d\sigma}{dt}(gg\to q\bar{q}) = \frac{9}{64}\times(\text{above})$$

$$\frac{d\sigma}{dt}(gg\to q\bar{q}) = \frac{1}{6}\frac{\pi\alpha_s^2}{s^2}\left[\frac{t}{u}+\frac{u}{t}-\frac{9}{4}\frac{t^2+u^2}{s^2}\right]$$

for this is $gg\to gg$ $\frac{1}{4}\sum_{spin}\frac{1}{38}\sum_{colour}$

i.e. $\frac{1}{6}\times\frac{8}{3}\to\frac{4}{9}$

ed we must cross

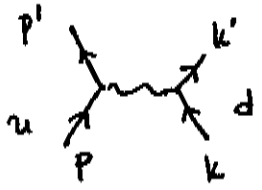


ie. $s \leftrightarrow t$ $\quad u \leftrightarrow u$

There is also a (-1) when we cross a fermion into the final state. (recall Peskin + Schroeder p. 168)

$$\frac{d\sigma}{dt}(qg \to qg) = \frac{4}{9}\frac{\pi\alpha_s^2}{s^2}\left[-\frac{s}{u} - \frac{u}{s} + \frac{9}{4}\left(\frac{u^2+s^2}{t^2}\right)\right]$$

2.) Next, consider quark-quark scattering. First, let the quarks be of different flavors eg. $ud \to ud$.



$$= (+ig)^2\, \bar{u}(p')\gamma^\mu u(p) t^a \left(\frac{-i}{t}\right) t^a \bar{u}(k')\gamma_\mu u(k)$$

$$\sum_{\substack{\text{spin}\\ \text{color}}} |\mathcal{M}|^2 = \frac{g^4}{t^2}\, \mathrm{tr}[\not{p}'\gamma^\mu \not{p}\gamma^\nu]\, \mathrm{tr}[\not{k}'\gamma_\mu \not{k}\gamma_\nu]\, \mathrm{tr}\, t^a t^b\, \mathrm{tr}\, t^a t^b$$

$$= \frac{g^4}{t^2} \cdot 4\,(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu} p\cdot p') \cdot 4\,(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu} k\cdot k') \cdot \blacksquare\, (C(r))^2 \delta^{aa}$$

$$= \frac{g^4}{t^2} \cdot 16 \cdot 2 \cdot (p'\cdot k'\, p\cdot k + p'\cdot k\; p\cdot k')\; d_G (C(r))^2$$

$$= 16\,\frac{g^4}{t^2}\cdot\frac{1}{2}(s^2+u^2)\cdot 8\cdot\left(\frac{1}{2}\right)^2$$

$$= \frac{16\,g^4}{t^2}(s^2+u^2)$$

To average over initial spins & colors, multiply this by $\left(\frac{1}{2}\right)^2\cdot\left(\frac{1}{3}\right)^2$

$$\frac{1}{4}\cdot\frac{1}{9}\sum_{\substack{\text{spin}\\ \text{color}}}|M|^2 = \frac{4}{9}\,g^4\left(\frac{s^2+u^2}{t^2}\right)$$

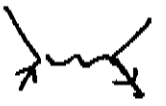
$$\frac{d\sigma}{d\cos\Theta} = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{4}{9}\,g^4\,\frac{s^2+u^2}{t^2} = \frac{4}{9}\,\frac{1}{2s}\,\pi\alpha_s^2\left(\frac{s^2+u^2}{t^2}\right)$$

$$dt = \tfrac{s}{2}d\cos\Theta$$

$$\frac{d\sigma}{dt}(ud\to ud) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left(\frac{s^2+u^2}{t^2}\right)$$

To compute $u\bar{d}\to u\bar{d}$ cross $s\leftrightarrow u$ $t\leftrightarrow t$

$$\frac{d\sigma}{dt}(u\bar{d}\to u\bar{d}) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left(\frac{s^2+u^2}{t^2}\right)$$



To compute $u\bar{u}\to d\bar{d}$ cross $s\leftrightarrow t$ $u\leftrightarrow u$

$$\frac{d\sigma}{dt}(u\bar{u}\to d\bar{d}) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left(\frac{t^2+u^2}{s^2}\right)$$

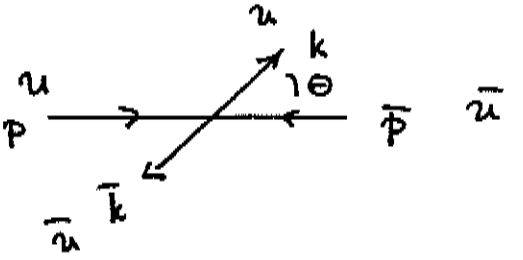


The spin/color averaging factors are the same for all three cases.

Finally we need $u\bar{u} \to u\bar{u}$ 

In Peskin + Schroeder, this is done by some tricks. But it is not hard to do the calculation by straightforward evaluation of amplitudes.

Use the kinematics:



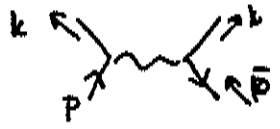
$$u_R(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_R(\bar{p}) = \sqrt{2E}\begin{pmatrix}-1\\0\\0\\0\end{pmatrix}$$

$$u_L(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad v_L(\bar{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$$

$$u_R(k) = \sqrt{2E}\begin{pmatrix}0\\0\\c_2\\s_2\end{pmatrix} \qquad v_R(\bar{k}) = \sqrt{2E}\begin{pmatrix}-c_2\\-s_2\\0\\0\end{pmatrix}$$

$$u_L(k) = \sqrt{2E}\begin{pmatrix}-s_2\\c_2\\0\\0\end{pmatrix} \qquad v_L(\bar{k}) = \sqrt{2E}\begin{pmatrix}0\\0\\s_2\\-c_2\end{pmatrix}$$

$$c_2 = \cos\theta/2 \qquad s_2 = \sin\theta/2$$



$$= (ig)^2\, \bar{u}(k)\gamma^\mu u(p)\, t^a \left(\frac{-i}{t}\right) t^a\, \bar{v}(\bar{p})\gamma_\mu v(\bar{k})$$

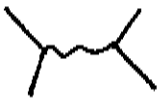


$$= (ig)^2\, \bar{u}(k)\gamma^\mu v(\bar{k})\, t^a \left(\frac{-i}{s}\right) t^a\, \bar{v}(\bar{p})\gamma_\mu u(p)$$

$$\times (-1)$$

↑ from changing the order of fermions relative to the 1st diagram

By helicity conservation



is nonzero only for $u_R \bar{u}_R \to u_R \bar{u}_R$

$u_R \bar{u}_L \to u_R \bar{u}_L$

and R↔L

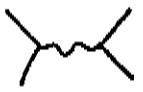


is nonzero only for $u_R \bar{u}_L \to u_R \bar{u}_L$

$u_R \bar{u}_L \to u_L \bar{u}_R$

and R↔L

Evaluate this 4 amplitudes. The R↔L amplitudes are equal by CP.



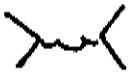
$u_R \bar{u}_R \longrightarrow u_R \bar{u}_R$

$$i\mathcal{M} = \frac{ig^2}{t}(2E)^2 (c_2 s_2) \sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (-1\ 0)\, \bar{\sigma}_\mu \begin{pmatrix} -c_2 \\ -s_2 \end{pmatrix} t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 (c_2, s_2, is_2, c_2)^\mu g_{\mu\nu} (c_2, -s_2, is_2, -c_2)^\nu t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 \cdot (c_2^2 + s_2^2 + s_2^2 + c_2^2)\, t^a \otimes t^a$$

$$= 2ig^2 \frac{s}{t} \quad t^a \otimes t^a$$



$u_R \bar{u}_L \to u_R \bar{u}_L$

$$i\mathcal{M} = \frac{ig^2}{t}(2E)^2 (c_2 s_2)\sigma^\mu \begin{pmatrix}1\\0\end{pmatrix} \cdot (0\ -1)\sigma_\nu \begin{pmatrix}s_2\\-c_2\end{pmatrix} \quad t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 (c_2, s_2, is_2, c_2)^\mu g_{\mu\nu} (c_2, -s_2, -is_2, -c_2)^\nu t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 (c_2^2 + s_2^2 - s_2^2 + c_2^2)\ t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 (2c_2^2)\ t^a \otimes t^a \qquad 2\cos^2\tfrac{\theta}{2} = 1 + \cos\theta$$

$$u = -\tfrac{1}{2}s(1+\cos\theta)$$

$$= -2ig^2 \frac{u}{t} \quad t^a \otimes t^a$$



$u_R \bar{u}_L \to u_R \bar{u}_L$

$$i\mathcal{M} = -i\frac{g^2}{s}(2E)^2 (c_2 s_2)\sigma^\mu \begin{pmatrix}s_2\\-c_2\end{pmatrix} g_{\mu\nu} (0\ -1)\sigma^\nu \begin{pmatrix}1\\0\end{pmatrix} \quad t^a \otimes t^a$$

$$= -i\frac{g^2}{s} 4E^2 (0, (-c_2^2 + s_2^2), i(c_2^2 + s_2^2), 2c_2 s_2)^\mu$$

$$\cdot g_{\mu\nu} (0, -1, -i, 0)^\nu \quad t^a \otimes t^a$$

$$= -i\frac{g^2}{s} 4E^2 \cdot [(-c_2^2 + s_2^2) - (c_2^2 + s_2^2)]\ t^a \otimes t^a$$

$$= -i\frac{g^2}{s} \cdot 2\,(-2E^2(1+\cos\theta))\ t^a \otimes t^a$$

$$= -2i\frac{g^2}{s} u \cdot t^a \otimes t^a$$



$u_R \bar{u}_L \to u_L \bar{u}_R$

$$iM = -i\frac{g^2}{s}(2E)^2(-s_2\, c_2)\,\bar{\sigma}^\mu \begin{pmatrix} -c_2 \\ -s_2 \end{pmatrix} g_{\mu\nu}\,(0,-1)\sigma^\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; t^a \otimes t^a$$

$$= -i\frac{g^2}{s}\,4E^2\,(0,\, -s_2^2+c_2^2,\, i(s_2^2+c_2^2),\, -2c_2 s_2)^\mu \cdot g_{\mu\nu}\,(0,-1,-i,0)^\nu \; t^a \otimes t^a$$

$$= -i\frac{g^2}{s}\,4E^2\,\left[(c_2^2-s_2^2) - (c_2^2+s_2^2)\right] \; t^a \otimes t^a$$

$$= -2i\frac{g^2}{s}\,(-2E^2)\,(1-\cos\Theta) \; t^a \otimes t^a$$

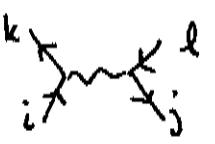
$$= -2i\frac{g^2}{s}\,t \; t^a \otimes t^a$$

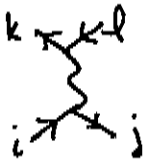
Now square the amplitudes & sum over colors. For the amplitudes with 1 diagram we can use

$$|t^a \otimes t^a|^2 = \mathrm{tr}\, t^a t^b \;\; \mathrm{tr}\, t^a t^b = C(r)^2 \cdot d_A = \frac{8}{2\cdot 2}$$

For $u_R \bar{u}_L \to u_R \bar{u}_L$ we must be more careful:

$$M(u_{Ri}\,\bar{u}_{Lj} \to u_{Rk}\,\bar{u}_{Ll}) = (-2ig^2)\cdot\left[\frac{u}{t}\,(t^a)_{ki}(t^a)_{jl} + \frac{u}{s}\,(t^a)_{ji}(t^a)_{kl}\right]$$





$$\sum_{\text{colors}} |\mathcal{M}|^2 = (4g^4)\left[\left(\frac{u}{t}\right)^2 \mathrm{tr}(t^a t^b)\,\mathrm{tr}(t^a t^b) + \left(\frac{u}{s}\right)^2 \mathrm{tr}(t^a t^b)\,\mathrm{tr}(t^a t^b) + 2\,\frac{u^2}{st}\,\mathrm{tr}[t^a t^b t^a t^b] \right]$$

$$= 4g^4 \left\{ \left[\left(\frac{u}{t}\right)^2 + \left(\frac{u}{s}\right)^2\right] d_G\, C(r)^2 + 2\,\frac{u^2}{st}\cdot d_r\, C_2(r)\left(C_2(r) - \tfrac{1}{2}C_2(G)\right) \right\}$$

$$= 4g^4 \left\{ \left[\left(\frac{u}{t}\right)^2 + \left(\frac{u}{s}\right)^2\right] \cdot 2 + 2\,\frac{u^2}{st}\cdot 3 \cdot \underbrace{\frac{4}{3}\cdot\left(\frac{4}{3} - \frac{1}{2}\cdot 3\right)}_{(-4/6)} \right\}$$

$$= 8g^4 \left[\frac{u^2}{t^2} + \frac{u^2}{s^2} + \frac{u^2}{st}\left(-\frac{2}{3}\right) \right]$$

then

$$\sum_{\text{colors}} \left\{ |\mathcal{M}|^2(u_R\bar{u}_L \to u_R\bar{u}_L) + |\mathcal{M}|^2(u_R\bar{u}_R \to u_R\bar{u}_R) + |\mathcal{M}|^2(u_R\bar{u}_L \to u_L\bar{u}_R) \right\}$$

$$= 8g^4 \left[\frac{u^2}{t^2} + \frac{u^2}{s^2} - \frac{2}{3}\,\frac{u^2}{st} + \frac{s^2}{t^2} + \frac{t^2}{s^2} \right]$$

$$\frac{1}{4}\frac{1}{9}\sum_{\text{spin colors}} |\mathcal{M}|^2 = \frac{1}{4}\cdot\frac{1}{9}\cdot 2 \times (\text{above})$$

$$= \frac{4}{9}g^4\left[\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2} - \frac{2}{3}\,\frac{u^2}{st} \right]$$

Finally,

$$\frac{d\sigma}{d\cos\theta}(u\bar{u}\to u\bar{u}) = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{4}{9}\,g^4\,[\cdots]$$

$$\frac{d\sigma}{dt}(u\bar{u}\to u\bar{u}) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left[\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2} - \frac{2}{3}\,\frac{u^2}{st}\right]$$

to obtain $uu\to uu$ or $\bar{u}\bar{u}\to\bar{u}\bar{u}$,

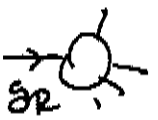
cross $s\leftrightarrow u$ $t\leftrightarrow t$

$$\frac{d\sigma}{dt}(uu\to uu) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left[\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} - \frac{2}{3}\,\frac{s^2}{ut}\right]$$

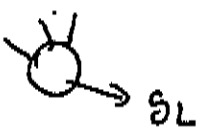
3.) Finally, we need to compute $gg\to gg$. Begin by computing the amplitudes for

$$g_Rg_R\to g_Lg_L$$
$$g_Rg_R\to g_Rg_L$$
$$g_Rg_R\to g_Rg_R$$

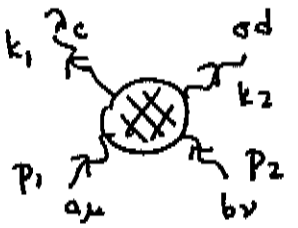
The other needed helicity amplitudes can be obtained from these by P and crossing

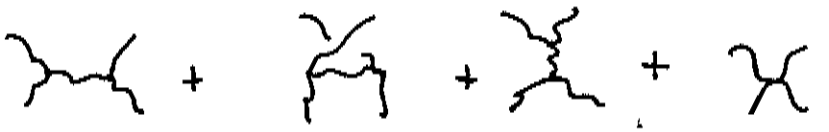


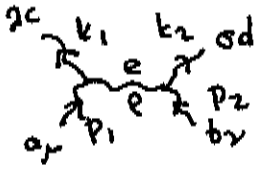
crosses to



Setup the calculation as



There are 4 diagrams: 



$$= (g f^{ace}) \left[g^{\mu\lambda}(p_1+k_1)^\rho + g^{\lambda\rho}(-k_1-(k_1-p_1))^\mu + g^{\rho\mu}(k_1-p_1-p_1)^\lambda \right]$$

$$\cdot \frac{-i}{t} (g f^{bde}) \left[g^{\nu\sigma}(p_2+k_2)^\rho + g^{\sigma\rho}(-k_2-(k_2-p_2))^\nu + g^{\rho\nu}(k_2-p_2-p_2)^\sigma \right]$$

$$\varepsilon_\lambda^*(k_1)\ \varepsilon_\sigma^*(k_2)\ \varepsilon_\mu(p_1)\ \varepsilon_\nu(p_2)$$

Using $p\cdot\varepsilon(p) = 0$:

$$= g f^{ace} \left[\varepsilon_1\cdot\varepsilon_1^* (p_1+k_1)^\rho - 2k_1\cdot\varepsilon_1\ \varepsilon_1^{*\rho} + (-2) p_1\cdot\varepsilon_1^*\ \varepsilon_1^\rho \right] \frac{-i}{t}$$

$$\cdot g f^{bde} \left[\varepsilon_2\cdot\varepsilon_2^* (p_2+k_2)^\rho - 2k_2\cdot\varepsilon_2\ \varepsilon_2^{*\rho} - 2 p_2\cdot\varepsilon_2^*\ \varepsilon_2^\rho \right]$$

$$= -i\frac{g^2 f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1\cdot\varepsilon_1^*\ \varepsilon_2\cdot\varepsilon_2^*\ (p_1+k_1)\cdot(p_2+k_2)$$

$$+ \varepsilon_1\cdot\varepsilon_1^* (-2)\cdot \left[\varepsilon_2\cdot k_2\ \varepsilon_2^*\cdot(p_1+k_2) + \varepsilon_2^*\cdot p_2\ \varepsilon_2\cdot(p_1+k_1) \right]$$

$$+ \varepsilon_2\cdot\varepsilon_2^* (-2) \left[\varepsilon_1\cdot k_1\ \varepsilon_1^*(p_2+k_2) + \varepsilon_1^*\cdot p_1\ \varepsilon_1\cdot(p_2+k_2) \right]$$

$$+ 4 \left[\varepsilon_1^*\cdot\varepsilon_2^*\ k_1\cdot\varepsilon_1\ k_2\cdot\varepsilon_2 + \varepsilon_1^*\cdot\varepsilon_2\ k_1\cdot\varepsilon_1\ p_2\cdot\varepsilon_2^* \right.$$

$$\left. + \varepsilon_1\cdot\varepsilon_2^*\ p_1\cdot\varepsilon_1^*\ k_2\cdot\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2\ p_1\cdot\varepsilon_1^*\ p_2\cdot\varepsilon_2^* \right] \Big\}$$


$= \begin{pmatrix} c \\ k_1 \\ \varepsilon_1^* \end{pmatrix} \leftrightarrow \begin{pmatrix} d \\ k_2 \\ \varepsilon_2^* \end{pmatrix}$ + the above

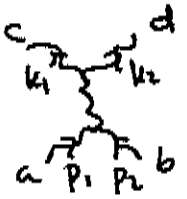
$$= \frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \; (p_1+k_2)\cdot(p_2+k_1)$$

$$- 2\,\varepsilon_1\varepsilon_2^* \left[\varepsilon_2\cdot k_1 \; \varepsilon_1^*\cdot(p_1+k_2) + \varepsilon_1^*\cdot p_2 \; \varepsilon_2\cdot(p_1+k_2) \right]$$

$$- 2\,\varepsilon_2\cdot\varepsilon_1^* \left[\varepsilon_1\cdot k_2 \; \varepsilon_2^*\cdot(p_2+k_1) + \varepsilon_2^*\cdot p_1 \; \varepsilon_1\cdot(p_2+k_1) \right]$$

$$+ 4 \left[\varepsilon_1^*\cdot\varepsilon_2^* \; k_2\cdot\varepsilon_1 \; k_1\cdot\varepsilon_2 + \varepsilon_2^*\cdot\varepsilon_2 \; \varepsilon_1\cdot k_2 \; \varepsilon_1^*\cdot p_2 \right.$$

$$\left. + \varepsilon_1\cdot\varepsilon_1^* \; p_1\cdot\varepsilon_2^* \; k_1\cdot\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2 \; p_1\cdot\varepsilon_2^* \; p_2\cdot\varepsilon_1^* \right] \Big\}$$



$$= g f^{abe} \left[g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2+(p_1+p_2))^\mu + g^{\rho\mu}(-p_1-p_2-p_1)^\nu \right]$$

$$\cdot \frac{-i}{s} g f^{cde} \left[g^{\lambda\sigma}(-k_1+k_2)^\rho + g^{\sigma\rho}(-k_2-(k_1+k_2))^\lambda + g^{\rho\lambda}((k_1+k_2)+k_1)^\sigma \right]$$

$$\cdot \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_1^{*\lambda} \varepsilon_2^{*\sigma}$$

$$= \frac{-ig^2 f^{abe} f^{cde}}{s} \left[\varepsilon_1\varepsilon_2 (p_1-p_2)^\rho + 2\varepsilon_1\cdot p_2 \, \varepsilon_2^\rho - 2\,\varepsilon_2\cdot p_1 \, \varepsilon_1^\rho \right]$$

$$\left[-\varepsilon_1^*\varepsilon_2^* (k_1-k_2)^\rho - 2\,\varepsilon_1^*\cdot k_2 \, \varepsilon_2^{*\rho} + 2\,\varepsilon_2^*\cdot k_1 \, \varepsilon_1^{*\rho} \right]$$

$$= +i\frac{g^2 f^{abe} f^{cde}}{s} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* \; (p_1-p_2)\cdot(k_1-k_2)$$

$$+ 2\,\varepsilon_1\varepsilon_2 \left[\varepsilon_1^*\cdot k_2 \; \varepsilon_2^*\cdot(p_1-p_2) - \varepsilon_2^*\cdot k_1 \; \varepsilon_1^*\cdot(p_1-p_2) \right]$$

$$+ 2\,\varepsilon_1^*\varepsilon_2^* \left[\varepsilon_1\cdot p_2 \; \varepsilon_2\cdot(k_1-k_2) - \varepsilon_2\cdot p_1 \; \varepsilon_1\cdot(k_1-k_2) \right]$$

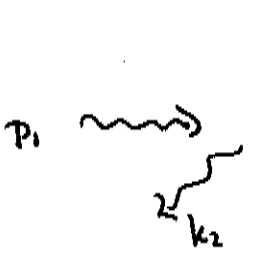
$$+ \text{(more)} \longrightarrow$$

$$+ 4\left[\ \varepsilon_2\cdot\varepsilon_2^*\ \ \varepsilon_1\cdot p_2\ \ \varepsilon_1^*\cdot k_2 \;-\; \varepsilon_2\cdot\varepsilon_1^*\ \ \varepsilon_2\cdot p_1\ \ \varepsilon_2^*\cdot k_1 \right.$$
$$\left. -\ \varepsilon_1\cdot\varepsilon_2^*\ \ \varepsilon_2\cdot p_1\ \ \varepsilon_1^*\cdot k_2 \;+\; \varepsilon_1\cdot\varepsilon_1^*\ \ \varepsilon_2\cdot p_1\ \ \varepsilon_2^*\cdot k_1\right] \Big\}$$



$$= -ig^2\Big[\ f^{abe}f^{cde}\ \big[\ \varepsilon_1\cdot\varepsilon_1^*\ \ \varepsilon_2\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^*\ \ \varepsilon_2\cdot\varepsilon_1^*\big)$$
$$+\ f^{ace}f^{bde}\ \big[\ \varepsilon_1\cdot\varepsilon_2\ \ \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^*\ \ \varepsilon_2\cdot\varepsilon_1^*\big)$$
$$+\ f^{ade}f^{bce}\ \big(\ \varepsilon_1\cdot\varepsilon_2\ \ \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_1^*\ \ \varepsilon_2\cdot\varepsilon_2^*\big)\Big]$$

Now evaluate these with explicit ε's.



$\longrightarrow \hat{3}$

$c = \cos\theta$
$s = \sin\theta$

$p_1 = (E, 0, 0, E)$

$$\varepsilon_{1R}^{\mu} = \tfrac{1}{\sqrt{2}}\,(0, 1, i, 0)^{\mu}$$
$$\varepsilon_{1L}^{\mu} = \tfrac{1}{\sqrt{2}}\,(0, 1, -i, 0)^{\mu}$$

$p_2 = (E, 0, 0, -E)$

$$\varepsilon_{2R}^{\nu} = \tfrac{1}{\sqrt{2}}\,(0, -1, i, 0)^{\nu}$$
$$\varepsilon_{2L}^{\nu} = \tfrac{1}{\sqrt{2}}\,(0, -1, -i, 0)^{\nu}$$

$k_1 = (E, Es, 0, Ec)$

$$\varepsilon_{1R}^{*\lambda} = \tfrac{1}{\sqrt{2}}\,(0, c, -i, -s)^{\lambda}$$
$$\varepsilon_{1L}^{*\lambda} = \tfrac{1}{\sqrt{2}}\,(0, c, +i, -s)^{\lambda}$$

$k_2 = (E, -Es, 0, -Ec)$

$$\varepsilon_{2R}^{*\sigma} = \tfrac{1}{\sqrt{2}}\,(0, -c, -i, s)^{\sigma}$$
$$\varepsilon_{2L}^{*\sigma} = \tfrac{1}{\sqrt{2}}\,(0, -c, +i, s)^{\sigma}$$

It is useful to make some tables of dot products:

	ε_{1R}	ε_{1L}	ε_{2R}	ε_{2L}	ε_{1R}^*	ε_{1L}^*	ε_{2R}^*	ε_{2L}^*
ε_{1R}	0	−1	1	0	$-\left(\frac{1+c}{2}\right)$	$\frac{1-c}{2}$	$-\frac{(1-c)}{2}$	$\left(\frac{1+c}{2}\right)$
ε_{1L}	−1	0	0	1	$\frac{1-c}{2}$	$-\left(\frac{1+c}{2}\right)$	$\left(\frac{1+c}{2}\right)$	$-\frac{(1-c)}{2}$
ε_{2R}			0	−1	$-\left(\frac{1-c}{2}\right)$	$\left(\frac{1+c}{2}\right)$	$-\frac{(1+c)}{2}$	$\frac{(1-c)}{2}$
ε_{2L}			−1	0	$\left(\frac{1+c}{2}\right)$	$-\left(\frac{1-c}{2}\right)$	$\frac{(1-c)}{2}$	$-\frac{(1+c)}{2}$
ε_{1R}^*					0	−1	1	0
ε_{1L}^*					−1	0	0	1
ε_{2R}^*							0	−1
ε_{2L}^*							−1	0

	ε_{1R} ε_{1L}	ε_{2R} ε_{2L}	ε_{1R}^* ε_{1L}^*	ε_{2R}^* ε_{2L}^*
p_1	0	0	$\frac{E_S}{\sqrt{2}}$	$-\frac{E_S}{\sqrt{2}}$
p_2	0	0	$-\frac{E_S}{\sqrt{2}}$	$\frac{E_S}{\sqrt{2}}$
k_1	$-\frac{E_S}{\sqrt{2}}$	$\frac{E_S}{\sqrt{2}}$	0	0
k_2	$\frac{E_S}{\sqrt{2}}$	$-\frac{E_S}{\sqrt{2}}$	0	0

Since the dot products $p_i \cdot \varepsilon^*$ $k_i \cdot \varepsilon$ are independent of chromaticity, we might as well put in their values and simplify the diagrams:



$$= \frac{-ig^2 f^{abe} f^{bde}}{t} \Big\{ \varepsilon_1 \cdot \varepsilon_1^* \; \varepsilon_2 \cdot \varepsilon_2^* \; (s-u)$$

$$- 2\, \varepsilon_1 \cdot \varepsilon_1^* \left[\left(-\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) + \left(\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) \right]$$

$$- 2\, \varepsilon_2 \cdot \varepsilon_2^* \left[\left(-\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) + \left(\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) \right]$$

$$+ 4 \left[\varepsilon_1^* \cdot \varepsilon_2^* \left(-\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) + \varepsilon_1^* \varepsilon_2 \left(-\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) \right.$$

$$\left. + \varepsilon_1 \cdot \varepsilon_2^* \left(\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) + \varepsilon_1 \cdot \varepsilon_2 \left(\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) \right] \Big\}$$

$$= \frac{-ig^2 f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1 \cdot \varepsilon_1^* \; \varepsilon_2 \cdot \varepsilon_2^* \; (s-u)$$

$$- 2 (E_s)^2 \left[\varepsilon_1 \cdot \varepsilon_1^* + \varepsilon_2 \cdot \varepsilon_2^* + \varepsilon_1^* \cdot \varepsilon_2 + \varepsilon_1 \cdot \varepsilon_2^* \right.$$

$$\left. - \varepsilon_1^* \cdot \varepsilon_2^* - \varepsilon_1 \cdot \varepsilon_2 \right] \Big\}$$



$$= \frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1 \varepsilon_2^* \; \varepsilon_2 \cdot \varepsilon_1^* \; (s-t)$$

$$- 2\, \varepsilon_1 \varepsilon_2^* \left[\left(+\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) + \left(-\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) \right]$$

$$- 2\, \varepsilon_2 \cdot \varepsilon_1^* \left[\left(\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) + \left(-\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) \right]$$

$$+ 4 \left[\varepsilon_1^* \varepsilon_2^* \left(\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) + \varepsilon_2 \cdot \varepsilon_2^* \left(\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) \right.$$

$$\left. + \varepsilon_1 \varepsilon_1^* \left(-\tfrac{E_s}{\sqrt{2}}\right)\left(\tfrac{E_s}{\sqrt{2}}\right) + \varepsilon_1 \cdot \varepsilon_2 \left(-\tfrac{E_s}{\sqrt{2}}\right)\left(-\tfrac{E_s}{\sqrt{2}}\right) \right]$$



$$= \frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \; (s-t)$$

$$- 2(Es)^2 \Big[\varepsilon_1\cdot\varepsilon_2^* + \varepsilon_2\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1\cdot\varepsilon_1^* + \varepsilon_1\cdot\varepsilon_2 + \varepsilon_1^*\cdot\varepsilon_2^* \Big] \Big\}$$



$$= \frac{+ig^2 f^{abe} f^{cde}}{s} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* \; (u-t)$$

$$+ 2\,\varepsilon_1\cdot\varepsilon_2 \,[0] + 2\,\varepsilon_1^*\cdot\varepsilon_2^* \,[0] + 4\,[0] \Big\}$$



$$= -ig^2 f^{abe} f^{cde} \Big\{ \varepsilon_1\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \Big\}$$

$$-ig^2 f^{ace} f^{bde} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\varepsilon_2^* - \varepsilon_1\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \Big\}$$

$$-ig^2 f^{ade} f^{bce} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\varepsilon_2^* - \varepsilon_1\varepsilon_1^* \; \varepsilon_2\varepsilon_2^* \Big\}$$

in all $\longrightarrow$



$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* \; (t-u) + \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* \cdot s - \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \; s \Big\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* \; (s-u) + \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* \cdot t - \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \cdot t - 2(Es)^2 \left[\varepsilon_1\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1^*\cdot\varepsilon_2 + \varepsilon_2^*\cdot\varepsilon_1 - \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2 \right] \Big\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \; (s-t) + \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* \cdot u - \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* \cdot u - 2(Es)^2 \left[\varepsilon_1\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1^*\cdot\varepsilon_2 + \varepsilon_2^*\cdot\varepsilon_1 - \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2 \right] \Big\}$$

Now it is not so hard to evaluate this for all 16 helicity amplitudes. However, <u>these</u> will suffice.

for $g_R g_R \longrightarrow g_L g_L$

$$\varepsilon_1 \varepsilon_2 = \varepsilon_1^* \cdot \varepsilon_2^* = 1 \qquad \varepsilon_1 \cdot \varepsilon_1^* = \frac{1-c}{2} = \varepsilon_2 \cdot \varepsilon_2^*$$

$$\varepsilon_1 \cdot \varepsilon_2^* = \frac{1+c}{2} = \varepsilon_2 \cdot \varepsilon_1^*$$

$$\varepsilon_1 \cdot \varepsilon_1^* + \varepsilon_2 \cdot \varepsilon_2^* + \varepsilon_1^* \varepsilon_2 + \varepsilon_2^* \varepsilon_1 - \varepsilon_1^* \cdot \varepsilon_2^* - \varepsilon_1 \cdot \varepsilon_2$$

$$= (1-c) + (1+c) - 1 - 1 = 0$$

then



$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ t-u + \left(\frac{1-c}{2}\right)^2 s - \left(\frac{1+c}{2}\right)^2 s \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ \left(\frac{1-c}{2}\right)^2 (s-u) + t - \left(\frac{1+c}{2}\right)^2 t \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ \left(\frac{1+c}{2}\right)^2 (s-t) + u - \left(\frac{1-c}{2}\right)^2 u \right\}$$

now $\quad s = 4E^2 \qquad t = -2E^2(1-c) \qquad u = -2E^2(1+c)$

$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ 4E^2 c - \frac{1}{4} \cdot 4c \cdot 4E^2 \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ \frac{(1-c)^2}{4} (2E^2)(3+c) - \frac{2E^2}{4} (1-c)\left[4 - (1+c)^2\right] \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ \frac{(1+c)^2}{4} (2E^2)(3-c) - \frac{2E^2}{4} (1+c)\left[4 - (1-c)^2\right] \right\}$$

All the lines are zero !

coeff of $\frac{1}{s}$: $\{ 4E^2 C - \frac{1}{4} C \, 4E^2 \} = 0$

coeff of $\frac{1}{t}$: $\{ \frac{(1-c)}{4} 2E^2 \cdot (1-c)(3+c) - \frac{(1-c)\,2E^2}{4} [3 - 2c - c^2] \} = 0$

coeff of $\frac{1}{u}$: $\{ \frac{(1+c)}{4} 2E^2 \, (1+c)(3-c) - \frac{(1+c)\,2E^2}{4} [3 + 2c - c^2] \} = 0$

So $iM(g_R g_R \to g_L g_L) = 0$ and also (by P)

$iM(g_L g_L \to g_R g_R) = 0$

So $g_R g_R \longrightarrow g_R g_L$

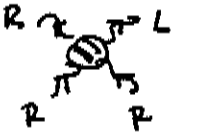
$\varepsilon_1 \varepsilon_2 = 1 \qquad \varepsilon_1^* \cdot \varepsilon_2^* = 0 \qquad \varepsilon_1 \cdot \varepsilon_1^* = -\left(\frac{1+c}{2}\right) \qquad \varepsilon_2 \cdot \varepsilon_2^* = \frac{1-c}{2}$

$\varepsilon_1 \cdot \varepsilon_2^* = \frac{1+c}{2} \qquad \varepsilon_2 \cdot \varepsilon_1^* = -\frac{(1-c)}{2}$

$\varepsilon_1 \cdot \varepsilon_1^* + \varepsilon_2 \cdot \varepsilon_2^* + \varepsilon_1 \cdot \varepsilon_2^* + \varepsilon_2 \cdot \varepsilon_1^* - \varepsilon_1^* \varepsilon_2^* - \varepsilon_1 \cdot \varepsilon_2$

$= -\left(\frac{1+c}{2}\right) + \left(\frac{1-c}{2}\right) + \left(\frac{1+c}{2}\right) - \frac{(1-c)}{2} - 0 - 1 = -1$

then



$= -ig^2 \frac{f^{abe} f^{cde}}{s} \{ 0 + [-\frac{(1+c)(1-c)}{4}] s - [-\frac{(1+c)(1-c)}{4}] s \}$

$-ig^2 \frac{f^{ace} f^{bde}}{t} \{ -\left(\frac{1-c^2}{4}\right)(s-u) + 0 - [-\left(\frac{1-c^2}{4}\right)] t + 2E^2 s^2 \}$

$-ig^2 \frac{f^{ade} f^{bce}}{u} \{ -\frac{(1-c^2)}{4}(s-t) + 0 - [-\left(\frac{1-c^2}{4}\right)] u + 2E^2 s^2 \}$

The coefficient of $\frac{1}{s}$ is $\left[-\frac{s^2}{4}\cdot s + \frac{s^2}{4}\cdot s\right] = 0$

$\uparrow \quad \uparrow$

$\sin\theta \quad 4E^2$

The coefficient of $\frac{1}{t}$ is:

$$-\frac{s^2}{4}\,2E^2(3+c) + \frac{s^2}{4}(-2E^2(1-c)) + 2E^2\cdot s^2$$

$$= 2E^2\cdot s^2\cdot\left[-\frac{4}{4}+1\right] = 0$$

The coefficient of $\frac{1}{u}$ is:

$$-\frac{s^2}{4}\,2E^2(3-c) + \frac{s^2}{4}(-2E^2(1+c)) + 2E^2s^2 = 0$$

so

$$iM(g_Rg_R \to g_Rg_L) = 0$$

by crossing, also $iM(g_Rg_L \to g_Rg_R) = 0$

We have now seen that, of the 16 cases, the 10 cases.

$RR \to LL$	$LL \to RR$
$RR \to LR, RL$	$LL \to RL, LR$
$LR, RL \to RR$	$LR, RL \to LL$

have $iM(gg \to gg) = 0$

for $gRgR \to gRgR$

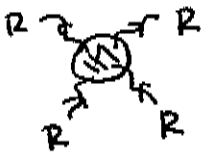
$$\varepsilon_1\cdot\varepsilon_2 = 1 = \varepsilon_1^*\varepsilon_2^* \qquad \varepsilon_1\cdot\varepsilon_1^* = -\frac{(1+c)}{2} = \varepsilon_2\cdot\varepsilon_2^*$$

$$\varepsilon_2\cdot\varepsilon_1^* = -\frac{(1-c)}{2} = \varepsilon_1\cdot\varepsilon_2^*$$

$$\varepsilon_1\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_2\cdot\varepsilon_1^* + \varepsilon_1\cdot\varepsilon_2^* - \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2$$

$$= -(1+c) - (1-c) - 1 - 1 = -4$$

Then



$$= \frac{-ig^2 f^{abc} f^{cde}}{s}\left\{ t-u + \left(\frac{1+c}{2}\right)^2 s - \left(\frac{1-c}{2}\right)^2 s \right\}$$

$$- \frac{ig^2 f^{ace} f^{bde}}{t}\left\{ \left(\frac{1+c}{2}\right)^2 (s-u) + t - \left(\frac{1-c}{2}\right)^2 t + 8E^2 s^2 \right\}$$

$$- \frac{ig^2 f^{ade} f^{bce}}{u}\left\{ \left(\frac{1-c}{2}\right)^2 (s-t) + u - \left(\frac{1-c}{2}\right)^2 u + 8E^2 s^2 \right\}$$

Coeff of $\frac{1}{s}$:

$$t-u + s\left[\left(\frac{1+c}{2}\right)^2 - \left(\frac{1-c}{2}\right)^2\right]$$

$$= -2E^2(1-c) - 2E^2(1+c) + 4E^2\cdot c$$

$$= 4E^2 c + 4E^2 c$$

$$= 8E^2 c$$

coeff of $\frac{1}{t}$: $\left(\frac{1+c}{2}\right)^2(s-u) + t\left(1-\left(\frac{1-c}{2}\right)^2\right) + 8E^2s^2$

$= \frac{2E^2}{4}\left\{(1+c)^2(3+c) - (1-c)[4-(1-c)^2] + 16s^2\right\}$

$= \frac{E^2}{2}\left\{(1+c)^2(3+c) - (1-c)\underbrace{(3+2c-c^2)}_{(1+c)(3-c)} + 16(1-c^2)\right\}$

$= \frac{E^2}{2}\left\{(1+c)\right\}\left\{(3+4c+c^2) - (3-4c+c^2) + 16(1-c)\right\}$

$= \frac{E^2}{2}(1+c)\left\{8c + 16(1-c)\right\}$

$= E^2(1+c)\left\{8-4c\right\}$

$= E^2(8+4c-4c^2)$

$= E^2(8+4c(1-c))$

coeff of $\frac{1}{u}$: $\left(\frac{1-c}{2}\right)(s-t) + u\left(1-\left(\frac{1+c}{2}\right)^2\right) + 8E^2s^2$

= above w. $(c \to -c)$

$= E^2(8-4c(1+c))$

~ all:



$= -ig^2 f^{abe} f^{cde}\,(2c)$

$-ig^2 f^{ace} f^{bde}\left[-\frac{4}{1-c} - 2c\right]$

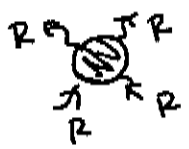
$-ig^2 f^{ade} f^{bce}\left[-\frac{4}{1+c} + 2c\right]$

Now, the terms with (2c) form:

$$(2c)\left[f^{abe}f^{cde} - f^{ace}f^{bde} + f^{ade}f^{bce}\right]$$

$$= (2c)\left[f^{abe}f^{ecd} + f^{bce}f^{ead} + f^{cae}f^{ebd}\right]$$

$$= (2c) \cdot 0 \quad \text{by the Jacobi identity.}$$



$$= +ig^2 f^{ace}f^{bde}\frac{4}{(1-c)} + ig^2 f^{ade}f^{bce}\frac{4}{(1+c)}$$

$$= (-2ig^2)\left(f^{ace}f^{bde}\frac{s}{t} + f^{ade}f^{bce}\frac{s}{u}\right)$$

to compute $g_R g_L \to g_R g_L$, cross $s \leftrightarrow u$, $b \leftrightarrow d$



$$= +2ig^2\left(f^{ace}f^{bde}\frac{u}{t} + f^{abe}f^{cde}\frac{u}{s}\right)$$

Let's check this explicitly: for $g_R g_L \to g_R g_L$

$$\varepsilon_1\cdot\varepsilon_2 = 0 \qquad \varepsilon_1^*\varepsilon_2^* = 0 \qquad \varepsilon_1\varepsilon_1^* = -\left(\tfrac{1+c}{2}\right) = \varepsilon_2\cdot\varepsilon_2^*$$

$$\varepsilon_1\cdot\varepsilon_2^* = \left(\tfrac{1+c}{2}\right) = \varepsilon_2\cdot\varepsilon_1^*$$

$$\varepsilon_1\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1\cdot\varepsilon_2^* + \varepsilon_2\cdot\varepsilon_1^* - \varepsilon_1\cdot\varepsilon_2 - \varepsilon_1^*\cdot\varepsilon_2^* = 0$$



$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ 0 + \left(\tfrac{1+c}{2}\right)^2 s - \left(\tfrac{1+c}{2}\right)^2 s \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ \left(\tfrac{1+c}{2}\right)^2 (s-u-t) \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ \left(\tfrac{1+c}{2}\right)^2 (s-t-u) \right\}$$

$$= -ig^2 f^{ace} f^{bde} \left(\tfrac{1+c}{2}\right)^2 \frac{2s}{t} \quad -ig^2 f^{ade} f^{bce} \left(\tfrac{1+c}{2}\right)^2 \left(\frac{2s}{u}\right)$$

$$= 2ig \left\{ f^{ace} f^{bde} \frac{(1+c)^2}{2} \frac{1}{1-c} + f^{ade} f^{bce} \frac{(1+c)^2}{2} \frac{1}{(1+c)} \right\}$$

the Jacobi identity implies:

$$f^{bce} f^{ade} = f^{ace} f^{bde} - f^{abe} f^{cde}$$

$$= 2ig^2 \left\{ f^{ace} f^{bde} \tfrac{1}{2}(1+c)^2 \underbrace{\left(\frac{1}{1-c} + \frac{1}{1+c}\right)}_{\frac{2}{(1-c)(1+c)}} - f^{abe} f^{cde} \frac{(1+c)^2}{2(1+c)} \right\}$$

$$= 2ig^2 \left\{ f^{ace} f^{bde} \frac{(1+c)}{(1-c)} + f^{abe} f^{cde} \left[\frac{-\frac{1}{2}(1+c)}{1} \right] \right\}$$

$$= 2ig^2 \left\{ f^{ace} f^{bde} \frac{u}{t} + f^{abe} f^{cde} \frac{u}{s} \right\}$$

as required.

to finish, compute

$$\sum_{\text{colors}} |i\mathcal{M}(g_R g_R \to g_R g_R)|^2$$

$$= 4g^4 \left[\left(\frac{s}{t}\right)^2 f^{ace} f^{bde} f^{ace'} f^{bde'} + \left(\frac{s}{u}\right)^2 f^{ade} f^{bce} f^{ade'} f^{bde'} + 2\frac{s^2}{tu} f^{ace} f^{bde} f^{ade'} f^{bce'} \right]$$

$$f^{ace} f^{bde} f^{ace'} f^{bde'} = (C_2(G_1))^2 \delta^{ee'} \delta^{ee'} = d_G (C_2(G))^2 = 8(3)^2$$

$$f^{ace} f^{bde} f^{adf} f^{bcf} = f^{ace} f^{bde} (t_G^d)_{af} (t_G^b)_{cf} \qquad (t_G^b)_{cf} = i f^{cbf}$$

$$= f^{ace} f^{dbe} (t_G^d)_{af} (t_G^b)_{fc}$$

$$= \tfrac{1}{2} f^{ace} f^{dbe} ([t_G^d, t_G^b])_{ac}$$

$$= \tfrac{1}{2} f^{ace} f^{dbe} i f^{dbg} (t_G^g)_{ac}$$

$$= \tfrac{1}{2} C_2(G_1) f^{acg} \cdot i (i f^{agc})$$

$$= \tfrac{1}{2} C_2(G_1) \cdot C_2(G) \cdot d_G = \tfrac{1}{2} d_G (C_2(G))^2$$

$$= 4 \cdot (72) g^4 \left[\left(\frac{s}{t}\right)^2 + \left(\frac{s}{u}\right)^2 + \frac{s^2}{tu} \right]$$

$$\left(\frac{1}{8}\right)^2 \sum_{\text{colors}} |\mathcal{M}|^2 = \frac{9}{2} g^4 \left[\left(\frac{s}{t}\right)^2 + \left(\frac{s}{u}\right)^2 + \frac{s^2}{tu} \right]$$

$$\frac{d\sigma}{dt}(g_R g_R \to g_R g_R) = \frac{1}{2s}\,\frac{1}{s/2}\,\frac{1}{16\pi}\cdot\frac{9}{2}\,g^4\left[\left(\frac{s}{t}\right)^2+\left(\frac{s}{u}\right)^2+\frac{s^2}{tu}\right]$$

$$= \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{s}{u}\right)^2+\left(\frac{s}{t}\right)^2+\frac{s^2}{tu}\right]$$

similarly $$\frac{d\sigma}{dt}(g_L g_L \to g_L g_L) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{s}{t}\right)^2+\left(\frac{s}{u}\right)^2+\frac{s^2}{tu}\right]$$

$$\frac{d\sigma}{dt}(g_R g_L \to g_R g_L) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{u}{t}\right)^2+\left(\frac{u}{s}\right)^2+\frac{u^2}{st}\right]$$

$$= \frac{d\sigma}{dt}(g_L g_R \to g_L g_R)$$

$$\frac{d\sigma}{dt}(g_R g_L \to g_L g_R) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{t}{u}\right)^2+\left(\frac{t}{s}\right)^2+\frac{t^2}{su}\right]$$

$$= \frac{d\sigma}{dt}(g_L g_R \to g_R g_L)$$

for the spin-averaged cross section, add all of these formulae & divide by $\to \frac{1}{2}\cdot\frac{1}{2}$

$$\frac{d\sigma}{dt}(gg \to gg) = \frac{9}{2}\cdot\frac{1}{4}\cdot 2\cdot\frac{\pi\alpha_s^2}{s^2}$$

$$\times\left[\left(\frac{s}{t}\right)^2+\left(\frac{s}{u}\right)^2+\left(\frac{t}{s}\right)^2+\left(\frac{t}{u}\right)^2+\left(\frac{u}{s}\right)^2+\left(\frac{u}{t}\right)^2\right.$$

$$\left.+\frac{s^2}{tu}+\frac{t^2}{su}+\frac{u^2}{st}\right]$$

Now, $$\frac{s^2}{tu} = \frac{(-t-u)^2}{tu} = \frac{t}{u} + \frac{u}{t} + 2$$

$$\frac{s^2}{tu} + \frac{t^2}{su} + \frac{u^2}{st} = \frac{t}{u} + \frac{u}{t} + \frac{t}{s} + \frac{s}{t} + \frac{u}{s} + \frac{s}{u} + 6$$

$$= \frac{t+s}{u} + \frac{u+s}{t} + \frac{t+u}{s} + 6$$

$$= -1 - 1 - 1 + 6 = 3$$

$$\frac{s^2}{t^2} + \frac{u^2}{t^2} = \frac{(s+u)^2}{t^2} - \frac{2su}{t^2} = 1 - \frac{2su}{t^2}$$

Similarly, $$\frac{s^2}{u^2} + \frac{t^2}{u^2} = 1 - \frac{2st}{u^2} \qquad \frac{t^2}{s^2} + \frac{u^2}{s^2} = 1 - \frac{2tu}{s^2}$$

in all

$$[\ ] = 3 + 3 - \frac{2su}{t^2} - \frac{2st}{u^2} - \frac{2tu}{s^2}$$

$$\frac{d\sigma}{dt}(gg \to gg) = \frac{9}{2}\frac{\pi\alpha_s^2}{s^2}\left[3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2}\right]$$

Physics 331 – Problem Set # 5

(due Wednesday, February 22)

1. The effective interaction used in class to compute the cross section for neutrino deep inelastic scattering can be tested in purely leptonic processes, in particular, in muon decay $\mu^- \to e^- \overline{\nu}_e \nu_\mu$. From the vertex

$$\Delta\mathcal{L} = \frac{4G_F}{\sqrt{2}} \overline{e}\gamma^\mu P_L \nu_e \; \overline{\nu}_\mu \gamma_\mu P_L \mu \tag{1}$$

where $P_L = (1-\gamma^5)/2$, and ignoring the masses of the electron and the neutrinos:

   (a) Compute the muon decay rate Γ_μ. The measured muon lifetime, $\tau_\mu = 2.19703(4) \times 10^{-6}$ sec, gives the most accurate determination of G_F. Compute G_F (2 significant figures suffice).

   (b) Compute the electron energy distribution $d\Gamma/dE(e^-)$ in the muon rest frame.

   (c) (extra credit) For a muon at rest with spin oriented along the $+\hat{z}$ axis, compute the electron energy and angular distribution. When this distribution is averaged with that for a muon with spin oriented in the $-\hat{z}$ direction, you should find an angle-independent result that agrees with the answer in (b).

   Parts (a) and (b) are quite straightforward with the use of the tricks for 3-body phase space described in Problem Set # 8 of Physics 330. Part (c) is more difficult; I have made it optional. It might be useful to use the identity for integrating over the phase space of massless vectors k and q such that $(k+q) = P$:

$$\int \frac{d^3k}{(2\pi)^3 2k} \int \frac{d^3q}{(2\pi)^3 2q} (2\pi)^4 \delta^{(4)}(k+q-P) \cdot k^\alpha q^\beta = \frac{1}{96\pi}(2P^\alpha P^\beta + g^{\alpha\beta}P^2) \tag{2}$$

2. Peskin and Schroeder, Problem 17.4.

3. Peskin and Schroeder, Problem 17.5. Work out both the total cross section and the differential cross section

$$E\frac{d\sigma}{d^3p} \tag{3}$$

   where p, E are the energy and momentum of the heavy quark Q. You can work in the γ-p CM frame, though the quantity in (3) is actually invariant to longitudinal boosts.

Physics 331 – Problem Set #5

Solutions

1.) $\Delta\mathcal{L} = \frac{4G_F}{\sqrt{2}}\, \bar{e}_L \gamma^\nu \nu_{eL}\, \bar{\nu}_{\mu L} \gamma_\nu \mu_L$

then

$$i\mathcal{M} = i\frac{4G_F}{\sqrt{2}}\, \bar{u}(p_e)\gamma^\nu P_L v(p_{\bar{\nu}})\; \bar{u}(p_\nu)\gamma_\nu P_L u(p_\mu)$$

We can do parts (a) and (b) with spin sums and projectors.

$$\frac{1}{2}\sum_{spin} |\mathcal{M}|^2 = \left(\frac{4G_F}{\sqrt{2}}\right)^2 \cdot \frac{1}{2} \cdot \mathrm{tr}\left[\not{p}_e \gamma^\mu P_L \not{p}_{\bar{\nu}} \gamma^\nu P_L\right]$$

↑ spin avg for μ

$$\cdot\, \mathrm{tr}\left[\not{p}_\nu \gamma_\mu P_L (\not{p}_\mu + m)\gamma_\nu P_L\right]$$

$P_L = \left(\frac{1-\gamma^5}{2}\right)$ $\quad P_L\gamma_\mu = \gamma_\mu P_R$ $\quad P_R\gamma_\nu = \gamma_\nu P_L$

$P_R = \left(\frac{1+\gamma^5}{2}\right)$ $\quad P_L^2 = P_L$ $\quad P_R^2 = P_R$ $\quad P_L \cdot P_R = 0$

$$= 4G_F^2\, \mathrm{tr}\left[\not{p}_e \gamma^\mu \not{p}_{\bar{\nu}} \gamma^\nu P_L\right] \mathrm{tr}\left[\not{p}_\nu \gamma_\mu \not{p}_\mu \gamma_\nu P_L\right]$$

$$= 4G_F^2 \cdot 4 \cdot \frac{1}{2} \cdot \left[p_e^\mu p_{\bar{\nu}}^\nu + p_e^\nu p_{\bar{\nu}}^\mu - g^{\mu\nu} p_e \cdot p_{\bar{\nu}} + i\epsilon^{\alpha\mu\beta\nu}(p_e)_\alpha (p_{\bar{\nu}})_\beta\right]$$

$$\cdot\, 4 \cdot \frac{1}{2}\left[(p_\nu)_\mu (p_\mu)_\nu + (p_\nu)_\nu (p_\mu)_\mu - g_{\mu\nu} p_\nu \cdot p_\mu + i\epsilon^{\gamma}{}_{\mu}{}^{\delta}{}_{\nu} (p_\nu)_\gamma (p_\mu)_\delta\right]$$

$$= 16 G_F^2 \cdot [\, 2\, p_e \cdot p_\nu\; p_{\bar\nu} \cdot p_\mu + 2\, p_e \cdot p_\mu\; p_{\bar\nu} \cdot p_\nu$$

$$+ (4 - 4)\; p_e \cdot p_{\bar\nu}\; p_\nu \cdot p_\mu$$

$$+ 2\,(p_e \cdot p_\nu\; p_{\bar\nu} \cdot p_\mu - p_e \cdot p_\mu\; p_{\bar\nu} \cdot p_\nu)\,] \quad \leftarrow \varepsilon \cdot \varepsilon$$

$$= 64 G_F^2 \quad p_e \cdot p_\nu\; p_{\bar\nu} \cdot p_\mu$$

then

$$\Gamma_\mu = \frac{1}{2m_\mu} \cdot \int d\Pi_3 \quad 64 G_F^2 \; [p_e \cdot p_\nu\; p_{\bar\nu} \cdot p_\mu]$$

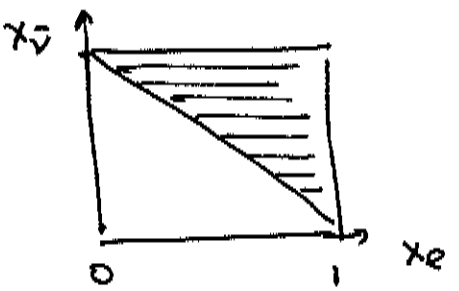
Evaluate this using tricks discussed in 330: Let $p_\mu = (m, \vec{0})$

$$x_e = \frac{2p_e^0}{m} \qquad x_\nu = \frac{2p_\nu^0}{m} \qquad x_{\bar\nu} = \frac{2p_{\bar\nu}^0}{m}$$

$$x_e + x_\nu + x_{\bar\nu} = 2 \qquad 0 < x_i < 1$$

$$\int d\Pi_3 = \frac{m_\mu^2}{128\pi^3} \int dx_e\, dx_{\bar\nu}$$

over the region:



$$p_{\bar\nu}\cdot p_\mu = m p^*_{\bar\nu} = \frac{1}{2} x_{\bar\nu} m^2$$

$$p_e \cdot p_\nu = \frac{1}{2}(p_e+p_\nu)^2 = \frac{1}{2}(p_\mu - p_{\bar\nu})^2 = \frac{1}{2}[m^2 - 2p_\mu\cdot p_\nu]$$

$$= \frac{1}{2} m^2 (1-x_{\bar\nu})$$

so

$$\Gamma_\mu = \frac{1}{2m_\mu}\frac{m_\mu^2}{128\pi^3}\int dx_e\, dx_{\bar\nu}\quad 64 G_F^2 \cdot \frac{1}{4} x_{\bar\nu}(1-x_{\bar\nu})\cdot m_\mu^4$$

$$= \frac{G_F^2 m^5}{16\pi^3}\int dx_e\, dx_{\bar\nu}\quad x_{\bar\nu}(1-x_{\bar\nu})$$

To get the total rate, integrate over x_e

$$\int_{1-x_{\bar\nu}}^{1} dx_e = x_{\bar\nu}$$

$$\Gamma_\mu = \frac{G_F^2 m^5}{16\pi^3}\int_0^1 dx_{\bar\nu}\; x_{\bar\nu}^2(1-x_{\bar\nu})$$

$$= \frac{G_F^2 m^5}{16\pi^3}\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{G_F^2 m^5}{192\pi^3}$$

now $\tau_\mu = (2.20 \times 10^{-6}\ \text{sec}) \cdot \dfrac{1}{6.582 \times 10^{-25}\ \text{GeV sec}}$

$= (3.0 \times 10^{-19}\ \text{GeV})^{-1} = \Gamma_\mu^{-1}$

then

$$G_F = \left(\frac{192\pi^3 \Gamma_\mu}{m^5}\right)^{\frac{1}{2}} = 1.16 \times 10^{-5}\ \text{GeV}^{-2}$$

b.) From the previous page:

$$\frac{1}{\Gamma_\mu}\frac{d^2\Gamma_\mu}{dx_e\, dx_{\bar\nu}} = 12\, x_\nu (1 - x_{\bar\nu})$$

Integrate over $x_{\bar\nu}$:

$$\int_{1-x_e}^{1} dx_{\bar\nu}\ x_{\bar\nu}(1-x_{\bar\nu}) = \tfrac{1}{2}\left[1 - (1-x_e)^2\right] - \tfrac{1}{3}\left[1 - (1-x_e)^3\right]$$

$$= x_e - \tfrac{1}{2}x_e^2 - x_e + x_e^2 - \tfrac{1}{3}x_e^3$$

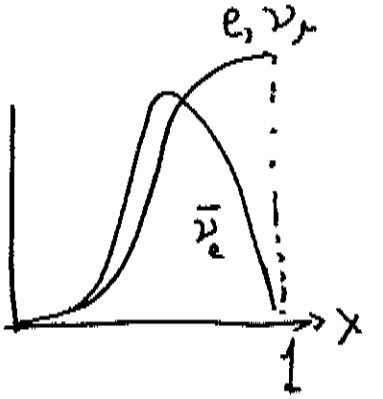
$$= \tfrac{1}{2}x_e^2 - \tfrac{1}{3}x_e^3$$

$$= \tfrac{1}{2}x_e^2\left(1 - \tfrac{2}{3}x_e\right)$$

so

$$\frac{1}{\Gamma_\mu}\frac{d\Gamma_\mu}{dx_e} = 6\, x_e^2\left(1 - \tfrac{2}{3}x_e\right)$$

$$\frac{1}{\Gamma_\mu}\frac{d\Gamma_\mu}{dx_{\bar\nu}} = 12\, x_{\bar\nu}^2 (1 - x_{\bar\nu})$$



of course, only the e^- energy distribution is directly observable. The e^- energy distribution is in excellent agreement with experiment!

C.) Redo the calculation on p. 1 keeping the non spin $u(p_\nu)$ ambiguity:

It makes things easier to go back to

$$iM = i \frac{4G_F}{\sqrt{2}} \bar{u}(p_e)_L \gamma^\mu v_L(p_{\bar{\nu}}) \; \bar{u}_L(p_\nu) \gamma_\mu u_L(p_\nu)$$

and perform a Fierz transformation (Peskin + Schroeder (3.79))

$$= -i \frac{4G_F}{\sqrt{2}} \bar{u}_L(p_\nu) \gamma^\mu v_L(p_{\bar{\nu}}) \; \bar{u}_L(p_e) \gamma_\mu u_L(p_\nu)$$

[this is why we found the same energy distribution for e and ν_μ.]

$\sum_{\text{spin}} |M|^2$ contains:

$$\sum_{\text{spin}} \text{tr} \left[\bar{u}_L(p_\nu) \gamma^\alpha v_L(p_{\bar{\nu}}) \right] \left[\bar{v}_L(p_{\bar{\nu}}) \gamma^\beta u_L(p_\nu) \right]$$

$$= 2 \left[p_\nu^\alpha p_{\bar{\nu}}^\beta + p_\nu^\beta p_{\bar{\nu}}^\alpha - g^{\alpha\beta} p_\nu \cdot p_{\bar{\nu}} + i \varepsilon^{\alpha\gamma\beta\delta} p_{\nu\gamma} p_{\bar{\nu}\delta} \right]$$

Now write the 3-body phase space as:

$$\int d\Pi_3 = \int \frac{d^3 p_e}{(2\pi)^3} \frac{1}{2p_e} \underline{\int \frac{d^3 p_\nu}{(2\pi)^3} \frac{1}{2p_\nu} \int \frac{d^3 p_{\bar{\nu}}}{(2\pi)^3} \frac{1}{2p_{\bar{\nu}}} (2\pi)^4 \delta(p_\nu + p_{\bar{\nu}} - (p_\mu - p_e))}$$

and integrate over the 2-body phase space of ν and $\bar{\nu}$.

To do this you can use the identity in the problem set:

for $k+q = P$ $\quad$ k, q massless

$$\int d\Pi_2 \; k^\alpha q^\beta = \frac{1}{96\pi}\left[2P^\alpha P^\beta + g^{\alpha\beta} P^2\right]$$

Here is a proof: Go to a frame where $P = (M, \vec{0})$.

In the frame $k = (\frac{M}{2}, \frac{M}{2}\hat{n})$ $\quad$ $q = (\frac{M}{2}, -\frac{M}{2}\hat{n})$

$$\int d\Pi_2 = \frac{1}{8\pi}\int \frac{d\Omega_{\hat{n}}}{4\pi}$$

$$\int d\Pi_2 \; k^\alpha q^\beta = \frac{1}{8\pi}\int \frac{d\Omega_{\hat{n}}}{4\pi} \quad \therefore \quad \begin{array}{c} \\ \alpha=0 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} \beta=0 & \quad j \end{array} \\ \left[\begin{array}{c|c} (\frac{M}{2})^2 & -(\frac{M}{2})^2 \hat{n}^j \\ \hline (\frac{M}{2})^2 \hat{n}^i & -(\frac{M}{2})^2 \hat{n}^i \hat{n}^j \end{array}\right] \end{array}$$

$$= \frac{1}{8\pi}\cdot\left(\frac{M}{2}\right)^2 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{1}{3}\delta^{ij} \end{array}\right]$$

$$= \frac{1}{96\pi} M^2 \left(\left[\begin{array}{c|c} 2 & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 1 & \\ \hline & -\delta^{ij} \end{array}\right]\right)$$

$$= \frac{1}{96\pi}\left[2P^\alpha P^\beta + g^{\alpha\beta} P^2\right]$$

If the identity is true in this frame, it is true in any frame.

then

$$\int d\Pi_2 \quad 2\left[p_\nu^\alpha p_{\bar\nu}^\beta + p_\nu^\beta p_{\bar\nu}^\alpha - g^{\alpha\beta} p_\nu \cdot p_{\bar\nu} + i\varepsilon^{\alpha\gamma\beta\delta} p_{\nu\gamma} p_{\bar\nu\delta}\right]$$

$$= \frac{1}{96\pi} \cdot 2 \cdot \left[\left[2P^\alpha P^\beta + g^{\alpha\beta} P^2\right] \cdot 2 - g^{\alpha\beta}[2+4]P^2 + i \cdot 0\right]$$

$$= \frac{1}{48\pi} \cdot \left[4P^\alpha P^\beta - 4 g^{\alpha\beta} P^2\right] \quad \text{with} \quad P = (p_\lambda - p_e)$$

then

$$\int d\Pi_2 \sum_{\substack{\text{spins of} \\ e\nu\bar\nu}} |\mathcal{M}|^2 = \frac{1}{48\pi} \cdot 4\left[P^\alpha P^\beta - g^{\alpha\beta} P^2\right] \cdot \left(\frac{4G_F}{\sqrt{2}}\right)^2$$

$$\cdot \bar u(p_\lambda)\, \gamma_\alpha \not{p}_e \gamma_\beta P_L\, u(p_\lambda)$$

$$= \frac{8G_F^2}{12\pi} \bar u(p_\lambda)\left[\not{P} \not{p}_e \not{P} - \gamma^\alpha \not{p}_e \gamma_\alpha P^2\right] P_L\, u(p_\lambda)$$

$$\not{P} \not{p}_e \not{P} = 2P\cdot p_e \not{P} - \not{p}_e P^2$$

$$-\gamma^\alpha \not{p}_e \gamma_\alpha P^2 = 2\not{p}_e P^2$$

$$= \frac{2G_F^2}{3\pi} \bar u(p_\lambda)\left[2(\not{p}_\lambda - \not{p}_e)(p_e\cdot P) + \not{p}_e P^2\right] P_L\, u(p_\lambda)$$

$$P^2 = (p_\lambda - p_e)^2 = m^2 - 2p_\lambda p_e = m^2(1-x_e)$$

$$2p_e\cdot P = 2p_e\cdot p_\lambda = m^2 x_e$$

$$\bar u(p)\not{p}_\lambda = \bar u(p_\lambda)\cdot m$$

so

$$\int d\Pi_2 \sum_{spin} |M|^2 = \frac{2G_F^2}{3\pi}\, \bar{u}(p_\nu)\left[m_\mu^3 x_e + m^2 \not{p}_e (1-2x_e)\right] P_L u(p_\nu)$$

for a muon at rest $\quad u(p_\nu) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \qquad \bar{u}(p_\nu) = \sqrt{m}\,(\xi^\dagger, \xi^\dagger)$

$$P_L\, u(p_\nu) = \sqrt{m}\begin{pmatrix}\xi\\ 0\end{pmatrix}$$

$$= \frac{1}{3\pi}\, m^2 \cdot m \quad \xi^\dagger\left[x_e + \frac{\vec{\sigma}\cdot\vec{p}_e}{m}(1-2x_e)\right]\xi$$

now let $\quad p_e = \left(\frac{mx_e}{2},\ \frac{mx_e}{2}\sin\Theta,\ 0,\ \frac{mx_e}{2}\cos\Theta\right)$

$$\vec{\sigma}\cdot\vec{p}_e = \frac{mx_e}{2}\begin{pmatrix} 1+\cos\Theta & \sin\Theta \\ \sin\Theta & (1-\cos\Theta)\end{pmatrix}$$

for $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad \xi^\dagger \vec{\sigma}\cdot\vec{p}_e\,\xi = \frac{mx_e}{2}(1+\cos\Theta)$

$\xi = \begin{pmatrix}0\\1\end{pmatrix}$ $\qquad \xi^\dagger \vec{\sigma}\cdot\vec{p}_e\,\xi = \frac{mx_e}{2}(1-\cos\Theta)$

so

$$= \frac{2G_F^2}{3\pi}\, m^4 x_e \left[1 + \frac{1}{2}(1-2x_e)(1 \pm \cos\Theta)\right]$$

now we can construct the total width:

$$\Gamma_\mu = \frac{1}{2m}\int d\Pi_3 \sum_{e\nu\bar{\nu}\ spins} |\mathcal{M}|^2$$

$$= \frac{1}{2m}\int \frac{d^3p_e}{(2\pi)^3}\frac{1}{2p_e}\cdot\frac{2G_F^2 m^4 x_e}{3\pi}\left[1+\frac{(1-2x_e)}{2}(1\pm\cos\theta)\right]$$

$$= \frac{1}{2m}\int \frac{dp_e\ p_e}{16\pi^3}\ 2\pi\int d\cos\theta\ \frac{2G_F^2 m^4 x_e}{2\cdot 3\pi}\left[(3-2x_e)\pm(1-2x_e)\cos\theta\right]$$

$$= \int dx_e\ x_e\left(\frac{m}{2}\right)^2\cdot\frac{1}{2m}\cdot\frac{2\pi}{16\pi^3}\int d\cos\theta\ \frac{G_F^2 m^4 x_e}{3\pi}\left[3-2x_e\pm\cdots\right]$$

$$= \frac{G_F^2 m^5}{192\pi^3}\int_0^1 dx_e\int_{-1}^{1} d\cos\theta\ x_e^2\left[3-2x_e\pm(1-2x_e)\cos\theta\right]$$

Integrating over $\cos\theta$ we get

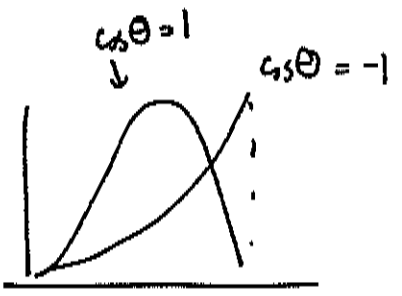
$$\Gamma_\mu = \frac{G_F^2 m^5}{192\pi^3}\cdot\int_0^1 dx_e\ 6\ x_e^2\left(1-\tfrac{2}{3}x_e\right)$$

as required. Varying $\cos\theta$:

$$\frac{d\Gamma}{dx_e\, d\cos\theta} \sim x_e^2\left[(3-2x_e)+(1-2x_e)\cos\theta\right]$$

$\cos\theta = 1 \quad \sim \cdot\ 4(1-x_e)x_e^2$

$\cos\theta = -1 \quad \sim \quad 2x_e^2$



What is going on ?

Consider a μ^- with spin $\parallel \hat{3}$ decays in an $\bar{e}_L$,

a ν_L and a $(\bar{\nu})_R$

If $x_e = 1$, the momentum of the electron is maximal :

$\cos\Theta = 1$



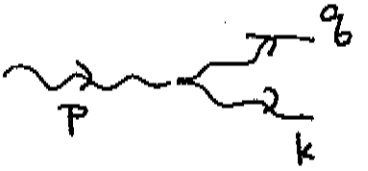
forbidden by angular momentum

$\cos\Theta = -1$



allowed.

The vanishing of $\dfrac{d\Gamma}{dx_e\, d\cos\Theta}$ at x_e in the direction of the muon spin has been verified in an elegant Berkeley-TRIUMF experiment.

2.) form the vertex 

we can derive the short splitting fmcts.

let $p = (p, 0, 0, p)$ $\quad q = (zp, p_\perp, 0, zp)$

$k = ((1-z)p, -p_\perp, 0, (1-z)p)$

$$\varepsilon_{R\atop L}(p) = \frac{1}{\sqrt{2}}(0\ 1, \pm i, 0)$$

$$\varepsilon_{R\atop L}(q) = \frac{1}{\sqrt{2}}\left(0, 1, \pm i, -\frac{p_\perp}{zp}\right)$$

$$\varepsilon_{R\atop L}(k) = \frac{1}{\sqrt{2}}\left(0\ \ 1, \pm i, +\frac{p_\perp}{(1-z)p}\right)$$

$$iM = g f^{abc}\Big[\varepsilon(p)\cdot\varepsilon^*(q)\,(p+q)\cdot\varepsilon^*(k) + \varepsilon^*(q)\cdot\varepsilon^*(k)\,(-q+k)\cdot\varepsilon^*(p) + \varepsilon^*(k)\cdot\varepsilon(p)\,(-k-p)\cdot\varepsilon^*(q)\Big]$$

$$= g f^{abc}\Big[\varepsilon(p)\cdot\varepsilon^*(q)\,[2p\cdot\varepsilon^*(k)] + \varepsilon^*(q)\varepsilon^*(k)\,[2k\cdot\varepsilon^*(p)] - \varepsilon^*(k)\cdot\varepsilon(p)\ \ 2p\cdot\varepsilon^*(q)\Big]$$

using $k\cdot\varepsilon^*(k) = 0$ etc.

Now evaluate this for the four cases

$g_R \to g_R g_L$
$g_R \to g_L g_R$
$g_R \to g_R g_L$
$g_R \to g_R g_R$

We need to work only to $\mathcal{O}(p_\perp^1)$

The other four cases are equal by $\mathcal{P}$

$g_R \to g_L g_L$

$$\varepsilon_R(p)\cdot\varepsilon_L^*(q)=0 \qquad \varepsilon_{q_L}^*\cdot\varepsilon_{k_L}^*=0 \qquad \varepsilon_{k_L}^*\cdot\varepsilon_{p_R}=0$$

so $iM = 0$

$g_R \to g_L g_R$

$$\varepsilon_R(p)\cdot\varepsilon_{q_L}^* = 0 \qquad \varepsilon_{q_L}^*\cdot\varepsilon_{k_R}^* = -1 \qquad \varepsilon_{k_R}^*\cdot\varepsilon_{p_R} = -1$$

$$iM = g f^{abc}\left[\, 0 + (-1)\left(\frac{2p_\perp}{\sqrt{z}}\right) - (-1)\;\; 2p\cdot\frac{p_\perp}{\sqrt{z}\,zp}\,\right]$$

$$= g f^{abc}\,(-\sqrt{2}p_\perp)\left[\,1-\frac{1}{z}\,\right] = g f^{abc}\sqrt{2}p_\perp\,\frac{(1-z)}{z}$$

$g_R \to g_R g_L$

$$\varepsilon_{p_R}\cdot\varepsilon_{q_R}^* = -1 \qquad \varepsilon_{q_R}^*\cdot\varepsilon_{k_L}^* = -1 \qquad \varepsilon_{k_L}^*\cdot\varepsilon_{p_R} = 0$$

$$iM = g f^{abc}\left[(-1)\left(-\frac{2p}{\sqrt{z}}\cdot\frac{p_\perp}{d(z-1)p}\right) + (-1)\,\frac{2p_\perp}{\sqrt{z}} - 0\,\right]$$

$$= g f^{abc}\,(\sqrt{2}p_\perp)\left(\frac{1}{(1-z)} - 1\right) = g f^{abc}(\sqrt{2}p_\perp)\,\frac{z}{(z-1)}$$

$g_R \to g_R g_R$ $\quad \varepsilon_{p_R}\cdot\varepsilon_{q_R}^* = -1 \qquad \varepsilon_{q_R}^*\cdot\varepsilon_{k_R}^* = 0 \qquad \varepsilon_{k_R}^*\cdot\varepsilon_{p_R} = -1$

$$iM = g f^{abc}\left[(-1)\left(-\frac{2p}{\sqrt{z}}\frac{p_\perp}{d(z-1)p}\right) + 0 - (-1)\,\frac{2p}{\sqrt{z}}\cdot\frac{p_\perp}{zp}\,\right]$$

$$= g f^{abc}\,(\sqrt{2}p_\perp)\left(\frac{1}{(1-z)} + \frac{1}{z}\right) = g f^{abc}(\sqrt{2}p_\perp)\,\frac{1}{z(1-z)}$$

$\sim$ all

$$\frac{1}{8}\sum_{\text{colors}}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{8}(f^{abc}f^{abc})\ \frac{1}{2}\ g^2\cdot(2p_\perp^2)\cdot 2$$

$$\cdot\left[\left(\frac{1-z}{z}\right)^2 + \left(\frac{z}{(1-z)}\right)^2 + \frac{1}{(z(1-z))^2}\right]$$

$$= \frac{1}{8}\cdot 8\cdot 3\ g^2\cdot\ 2p_\perp^2\ \frac{1}{z^2(1-z)^2}\ [(1-z)^4 + z^4 + 1]$$

$$= \frac{2g^2p_\perp^2}{z^2(1-z)^2}\cdot 3\cdot[(1-z)^4 + z^4 + 1]$$

to get from $\frac{1}{2}\sum|\mathcal{M}|^2$ to the splitting function, divide by $\frac{2g^2p_\perp^2}{z(1-z)}$

$$P_{g\to g}(z) = 3\ \frac{1}{z(1-z)}\ [(1-z)^4 + z^4 + 1]$$

$$= 3\left[\frac{(1-z)^3}{z} + \frac{z^3}{(1-z)} + \frac{1}{z(1-z)}\right]$$

$$= 3\left[\frac{1}{z} - 3 + 3z - z^2 + \frac{1}{1-z} - 3 + 3(1-z) - (1-z)^2 + \frac{1}{z} + \frac{1}{1-z}\right]$$

$$= 3\left[\frac{2}{z} + \frac{2}{1-z} + [-3+3z-z^2-3+3-3z-1+2z-z^2]\right]$$

$$= 3\left[\frac{2}{z} + \frac{2}{1-z} + [-4+2z-2z^2]\right]$$

$$= 3\left[\frac{2}{z} - 2 + \frac{2}{1-z} - 2 + 2z(1-z)\right]$$

so!

$$P_{g\leftarrow g}(z) = 6\left[\frac{(1-z)}{z} + \frac{z}{1-z} + z(1-z)\right]$$

Now, this analysis does not include the δ-function at $z=1$.

We should write more exactly:

$$P_{g\leftarrow g}(z) = P^{(z)}_{\substack{\text{no}\\ i\to g}} + A\,\delta(z-1) + B\,\delta(z-1)$$

where $B\,\delta(z-1)$ removes gluon to conserve longitudinal momentum when a gluon splits to $q\bar{q}$

$$\int dz \cdot n_f\left[z P_{q\leftarrow g}(z) + z P_{\bar{q}\leftarrow g}(z)\right] + \int dz\, z B\,\delta(z-1) = 0$$

[This is equivalent to the condition that 1 gluon is removed for each splitting:

$$\int dz\; n_f\, P_{q\leftarrow g}(z) + B\,\delta(z-1) = 0,$$

since $\int dz\,[z\,P_{q\leftarrow g}(z) + z\,P_{\bar{q}\leftarrow g}]$

$= \int dz\,[z\,P_{q\leftarrow g}(z) + (1-z)\,P_{\bar{q}\leftarrow g}]$ since $P_{\bar{q}\leftarrow g}$ is symmetric under $\frac{z}{1-z} \to (1-z)$

$= \int dz\; P_{q\leftarrow g}(z)$

since this is $= \int dz\; 4[z^2+(1-z)^2] = \frac{1}{3}$, $\quad B = -\frac{n_f}{3}$

the constraint for A is

$$\int dz\, z \left[P_{0,SFG}(z) + A\delta(z-1) \right] = 0$$

Define $P_{0,SFG}$ with $\frac{1}{(1-z)} \to \frac{1}{(1-z)_+}$, then

$$0 = \int dz\, z \left[\frac{(1-z)}{z} + \frac{z}{(1-z)_+} + z(1-z) + 6A\delta(z-1) \right]$$

$$= \int dz \left[(1-z) + \frac{z^2}{(1-z)_+} + z^2 - z^3 + 6A\delta(z-1) \right]$$

$$= \frac{1}{2} + \int_0^1 dz \left(\frac{z^2-1}{1-z} \right) + \frac{1}{3} - \frac{1}{4} + 6A$$

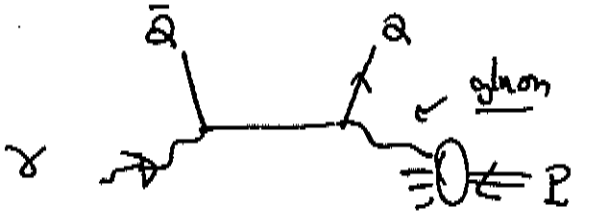
$$= \frac{1}{2} - \frac{3}{2} + \frac{1}{12} + 6A$$

$$= -\frac{11}{12} + 6A \qquad 6A = \frac{11}{12}$$

in all:

$$P_{SFG}(z) = 6 \left[\frac{(1-z)}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18} \right) \delta(z-1) \right]$$

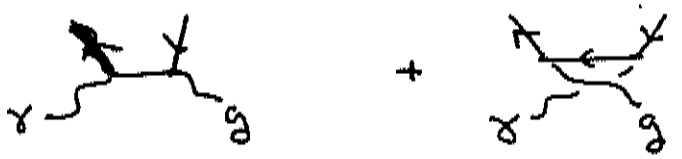
3.) For $\gamma p \to Q\bar{Q}$, the leading parton process is:



so that

$$\sigma(\gamma p \to Q\bar{Q} + X) = \int_0^1 dx \; f_g(x) \; \sigma(\gamma + g(xp) \to Q\bar{Q})$$

For this, we need $\mathcal{M}(\gamma g \to Q\bar{Q})$



There is no γgg vertex, and in fact the cross section is just that of an <u>Abelian</u> gauge theory

In QED, the matrix element for $\gamma\gamma \to Q\bar{Q}$ is given by: (Peskin + Schroeder (5.101))

$$\frac{1}{4}\sum_{spin} |\mathcal{M}|^2 = 2e^4 \left[\frac{p_1 k_1}{p_1 k_2} + \frac{p_1 k_2}{p_1 k_1} + 2m^2 \left(\frac{1}{p_1 k_1} + \frac{1}{p_1 k_2} \right) - m^4 \left(\frac{1}{p_1 k_1} + \frac{1}{p_1 k_2} \right)^2 \right]$$

where k_1, k_2 are the photon momenta of p_1 is the Q momentum.

to go to QED replace $e \to g$

and multiply by

$$Q_q^2 \cdot \frac{1}{8} \mathrm{tr}\, t^a t^a = Q_q^2 \cdot \frac{1}{8} \cdot C_2(r) \cdot 3 = \frac{Q_q^2}{2}$$

so

$$\frac{1}{8}\sum_{\text{color}} \frac{1}{4}\sum_{\text{spin}} |M|^2 (\gamma g \to Q\bar{Q}) = \frac{2e^2 g^2 Q_q^2}{2} \quad [\text{above}]$$

In the CM frame

$$\gamma: \; k_1 = (E, 0\,0, E) \qquad Q: \; p_1 = (E, p\,s, 0\; p\,c)$$
$$g: \; k_2 = (E, 0\,0, -E)$$

$$p_1 \cdot k_1 = E(E - pc) \qquad p_1 \cdot k_2 = E(E + pc)$$

$$\frac{1}{8}\sum_{\text{color}} \frac{1}{4}\sum_{\text{spin}} |M|^2 = e^2 g^2 Q_q^2 \left[\frac{E - pc}{E + pc} + \frac{E + pc}{E - pc} + \frac{2m^2}{E^2}\left(\frac{2E^2}{E^2 - p^2c^2}\right) \right.$$
$$\left. - \frac{m^4}{E^4}\left(\frac{2E^2}{E^2 - p^2c^2}\right)^2 \right]$$

$$= e^2 g^2 Q_q^2 \left[\frac{2(E^2 + p^2c^2)}{E^2 - p^2c^2} + 4m^2 \frac{1}{E^2 - p^2c^2} - \frac{4m^4}{(E^2 - p^2c^2)^2} \right]$$

$$\frac{d\sigma}{d\cos\theta_{cm}} = \frac{1}{2s} \frac{1}{16\pi} \frac{p}{E} \cdot (\text{above})$$

$$\frac{d\sigma}{d\cos\theta_{cm}} = \frac{\pi\alpha\,\alpha_s\,Q_q^2}{s}\,\frac{p}{E}\left[\frac{E^2+p^2c^2}{E^2-p^2c^2} + \frac{2m^2}{E^2-p^2c^2} - \frac{2m^4}{(E^2-p^2c^2)^2}\right]$$

to obtain the total cross section, integrate over $\cos\theta$. We need:

$$\int_{-1}^{1} d\cos\theta\ \frac{E^2}{E^2-p^2c^2} = \int_{-1}^{1} d\cos\theta\ \frac{E^2}{2E}\left[\frac{1}{E+pc\cos\theta} + \frac{1}{E-pc\cos\theta}\right]$$

$$= \frac{E^2}{2Ep}\left(\ln\frac{E+p}{E-p} + \ln\frac{E+p}{E-p}\right)$$

$$= \frac{E}{p}\ln\frac{E+p}{E-p} = \frac{E}{p}\ln\frac{(E+p)^2}{(E+p)(E-p)} = \frac{E}{p}\ln\frac{(E+p)^2}{m^2}$$

$$= \frac{2E}{p}\ln\frac{E+p}{m}$$

$$\int_{-1}^{1} d\cos\theta\ \frac{E^2+p^2c^2}{E^2-p^2c^2} = \int_{-1}^{1} d\cos\theta\ \frac{2E^2-(E^2-p^2c^2)}{E^2-p^2c^2}$$

$$= \frac{4E}{p}\ln\frac{E+p}{m} - 2$$

$$\int_{-1}^{1} d\cos\theta\ \left(\frac{1}{E^2-p^2c^2}\right)^2 = \int d\cos\theta\ \left(\frac{1}{2E}\left[\frac{1}{(E+pc)} + \frac{1}{(E-pc)}\right]\right)^2$$

$$= \frac{1}{4E^2}\int_{-1}^{1} d\cos\theta\left[\frac{1}{(E+pc)^2} + \frac{2}{E^2-p^2c^2} + \frac{1}{(E-pc)^2}\right]$$

$$= \frac{1}{4E^2}\cdot\left\{\frac{1}{p}\left(\frac{1}{E-p} - \frac{1}{E+p}\right)\cdot 2 + \frac{4}{Ep}\ln\frac{E+p}{m}\right\}$$

$$= \frac{1}{4E^2}\left\{\frac{4p}{p}\,\frac{1}{E^2-p^2} + \frac{4}{Ep}\ln\frac{E+p}{m}\right\}$$

$$= \frac{1}{E^2 m^2} + \frac{1}{E^3 p} \ln \frac{E+p}{m}$$

$$\simeq \text{all}$$

$$\sigma = \frac{\pi \alpha \alpha_s Q_q^2}{s} \frac{P}{E} \left\{ \frac{4E}{p} \ln \frac{E+p}{m} - 2 + \frac{4m^2}{Ep} \ln \frac{E+p}{m} \right.$$

$$\left. - \frac{2m^4}{E^3 p} \ln \frac{E+p}{m} - \frac{2m^4}{E^2 m^2} \right\}$$

$$= \frac{\pi \alpha \alpha_s Q_q^2}{4E^2} \frac{P}{E} \left\{ \frac{E}{p} \ln \frac{E+p}{m} \left\{ 4 + 4\frac{m^2}{E^2} - \frac{2m^4}{E^4} \right\} \right.$$

$$\left. - 2 - \frac{2m^2}{E^2} \right\}$$

$$\sigma(\gamma g \to Q\bar{Q}) = \frac{\pi \alpha \alpha_s Q_q^2}{4E^2} \left\{ \left(4 + 4\frac{m^2}{E^2} - 2\frac{m^4}{E^4}\right) \ln \frac{E+p}{m} \right.$$

$$\left. - 2\frac{P}{E}\left(1 + \frac{m^2}{E^2}\right) \right\}$$

to go to a general frame, write

$$4E^2 = \hat{s} \qquad 4p^2 = \hat{s} - 4m^2$$

$$\hat{s} = (k_1 + k_2)^2 = 2k_1 k_2 = 2 k_1 \cdot x P = x s$$

So $\longrightarrow$

$$\sigma(\gamma p \to Q\bar{Q} + X) = \int dx \; f_g(x)$$

$$\cdot \frac{\pi \alpha \alpha_s}{xs} Q_Q^2 \left\{ \left(4 + \frac{16m^2}{\hat{s}} - \frac{32m^4}{\hat{s}^2}\right) \ln\left(\frac{\hat{E}+\hat{p}}{m}\right) \right.$$

$$\left. - 2\left(1 - \frac{4m^2}{\hat{s}}\right)^{1/2} \left(1 + \frac{4m^2}{\hat{s}}\right) \right\}$$

where $\hat{s} = xs$ $\qquad \frac{\hat{E}+\hat{p}}{m} = \frac{\sqrt{\hat{s}} + \sqrt{\hat{s} - 4m^2}}{2m}$

x must be ≤ 1. $\qquad xs - 4m^2 > 0 \quad$ or $\quad x > 4m^2/s$

Now compute the cross section differential in the momenta of the Q. We have from p. 18

$$\sigma(\gamma p \to Q\bar{Q}X) = \int dx \int d\cos\theta_{cm} \; f_g(x) \cdot \frac{\pi \alpha \alpha_s}{\hat{s}} Q_Q^2 \cdot \frac{p}{E}$$

$$\cdot \left[\frac{E^2 + p^2c^2}{E^2 - p^2c^2} + \frac{2m^2}{E^2 - p^2c^2} - \frac{2m^4}{(E^2 - p^2c^2)^2} \right]$$

We must change variables from $(x, \cos\theta_{cm})$ to $(p_\parallel, p_\perp)$ where $p^* = p_1^*$ is the momentum of the Q

Begin by finding x. In the CM frame of the γp collision

$$k_1 = (\mathcal{E}, 0, 0, \mathcal{E}) \qquad P = (\mathcal{E}, 0, 0, -\mathcal{E})$$

$$s = 4\mathcal{E}^2 \qquad k_2 = (x\mathcal{E}, 0, 0, -x\mathcal{E})$$

$$p_1 = (E, p_\perp, 0, p_{\parallel}) \qquad E^2 - p_\perp^2 - p_\parallel^2 = m^2$$

$$p_2^2 = m^2 = (k_1 + k_2 - p_1)^2 = (k_1 + k_2)^2 - 2p_1(k_1 + k_2) + p_1^2$$

$$m^2 = xs - 2E\cdot\mathcal{E}\cdot(1+x) + 2p_\parallel\cdot\mathcal{E}(1-x) + m^2$$

$$0 = x(s - 2\mathcal{E}E - 2\mathcal{E}p_\parallel) - 2\mathcal{E}(E - p_\parallel)$$

$$x = \frac{(E - p_\parallel)}{2\mathcal{E} - (E + p_\parallel)}$$

Next find an equation for θ_{cm}:

$$p_\perp = p_{cm}\sin\theta_{cm} \qquad p_{cm} = \frac{[xs - 4m^2]^{1/2}}{2}$$

$$\cos\theta_{cm} = \left[1 - \frac{p_\perp^2}{p_{cm}^2}\right]^{1/2}$$

we need

$$\frac{\partial(x, \cos\theta_{cm})}{\partial(p_{\parallel}, p_{\perp})} = \left| \det \begin{bmatrix} \frac{\partial x}{\partial p_{\parallel}} & \frac{\partial x}{\partial p_{\perp}} \\ \frac{\partial \cos\theta_{cm}}{\partial p_{\parallel}} & \frac{\partial \cos\theta_{cm}}{\partial p_{\perp}} \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} & \left.\frac{\partial x}{\partial p_{\perp}}\right|_{p_{\parallel}} \\ \left.\frac{\partial \cos\theta_{cm}}{\partial x}\right|_{p_{\perp}} \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} & \left.\frac{\partial \cos\theta_{cm}}{\partial x}\right|_{p_{\perp}} \left.\frac{\partial x}{\partial p_{\perp}}\right|_{p_{\parallel}} + \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} & \left.\frac{\partial x}{\partial p_{\perp}}\right|_{p_{\parallel}} \\ 0 & \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \end{bmatrix} \right|$$

$$= \left| \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \right|$$

$$\left| \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \right| = \frac{p_{\perp}/p_{cm}^2}{\left[1 - \frac{p_{\perp}^2}{p_{cm}^2}\right]^{1/2}} = \frac{p_{\perp}}{p_{\parallel} p_{cm}}$$

$$\left| \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} \right| = \left| \frac{\left(\frac{p_{\parallel}}{E} - 1\right)}{[2\mathcal{E} - (E + p_{\parallel})]} + \frac{(E - p_{\parallel})\left(\frac{p_{\parallel}}{E} + 1\right)}{[2\mathcal{E} - (E + p_{\parallel})]^2} \right|$$

$$= \left| x \cdot \left[\left(\frac{E + p_{\parallel}}{E}\right) - \frac{2\mathcal{E} - (E + p_{\parallel})}{E}\right] \frac{1}{2\mathcal{E} - E + p_{\parallel}} \right|$$

$$= \frac{2\mathcal{E} - 2(E + p_{\parallel})}{2\mathcal{E} - (E + p_{\parallel})} \frac{x}{E} = \left(1 - \frac{E + p_{\parallel}}{E - p_{\parallel}} x\right) \frac{x}{E}$$

so

$$dx\, d\cos\Theta_{cm} = dp_{\parallel}\, dp_{\perp} \cdot \frac{p_{\perp}}{p_{\parallel}\, p_{cm}} \left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}} x\right) \frac{x}{E}$$

$$= \frac{d^3p}{E} \frac{1}{2\pi} \frac{1}{p_{\parallel}\, p_{cm}} \left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}} x\right) \cdot x$$

then

$$E \frac{d\sigma}{d^3p}(\gamma p \to Q(p)\, \bar{Q} + \bar{X})$$

$$= x f_g(x) \cdot \frac{\alpha \alpha_s Q_Q^2}{2\hat{s}} \frac{1}{\hat{E}} \frac{\left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}} x\right)}{p_{\parallel}}$$

$$\cdot \left[\frac{\hat{E}^2 + \hat{p}^2 \cos^2\Theta_{cm}}{\hat{E}^2 - \hat{p}^2 \cos^2\Theta_{cm}} + \frac{2m^2}{\hat{E}^2 - \hat{p}^2 \cos^2\Theta_{cm}} - \frac{2m^4}{\hat{E}^2 - \hat{p}^2 \cos^2\Theta_{cm}} \right]$$

where $x = \dfrac{E - p_{\parallel}}{2E - (E+p_{\parallel})}$

$\hat{s} = x s \qquad \hat{E} = \frac{1}{2}\sqrt{\hat{s}} \qquad \hat{p} = (\hat{E}^2 - m^2)^{1/2}$

Physics 331 – Problem Set # 6

(due Wednesday, March 8)

1. Peskin and Schroeder, Problem 20.1.

2. Peskin and Schroeder, Problem 20.2.

3. Download and unpack the file `partons.tar` from the course Web page. Information on how to do this is given in the information sheet on the programs which you will also find on the course Web site. Compile and run the program `plotxf`. This will produce a plot of the evolution of the up quark parton distribution in the proton, similar to Peskin and Schroeder, Fig. 17.21, and also a plot of the evolution of the gluon distribution in the proton. (The up quark plot differs in detail from Fig. 17.21 because a different fit to the deep inelastic scattering data was used.) Modify this program to make similar plots for the down quark, anti-down quark, and bottom quark distributions.

4. Compile and run the program `ppcollider` contained in the `partons.tar` distribution. This will produce a plot of the p_T distribution of 2-jet events produced by gg scatting in $p\overline{p}$ collisions at 1.8 TeV in the center of mass, and also a plot of the distribution of 2-jet invariant mass in these events. The latter plot will be similar to the curve gg in Peskin and Schroeder, Fig. 17.13. Modify this program to produce the similar curves for quark-gluon and quark-quark scattering. (Here 'quark' refers to both quark and antiquark.)

Physics 331 - Problem Set #6

Solutions

1.) For a scalar field in the adjoint representation of SU(N)

$$D_\mu \Phi^a = \partial_\mu \Phi^a + g f^{abc} A_\mu^b \Phi^c$$

if we write

$$\underset{\sim}{\Phi} = \Phi^a t^a \qquad \text{(an } N \times N \text{ matrix)}$$

we can multiply the above by t^a & write this as:

$$D_\mu \underset{\sim}{\Phi} = \partial_\mu \underset{\sim}{\Phi} - i g [A_\mu^b t^b, \underset{\sim}{\Phi}]$$

$$= i f^{abc} t^a A^b \Phi^c$$

The kinetic term for Φ is then:

$$\mathcal{L} = \tfrac{1}{2}(D_\mu \Phi^a)^2 = \mathrm{tr}\,(D_\mu \underset{\sim}{\Phi})^2$$

The gauge boson masses come from the piece of the $\mathcal{L}$ that is quadratic in A_μ:

$$\mathcal{L} = \mathrm{tr}\left(-ig\,[A_\mu^b t^b, \langle\Phi\rangle]\right)^2 = -g^2 A_\mu^a A^b\, \mathrm{tr}\,[t^a,\langle\Phi\rangle][t^b,\Phi]$$

If $[t^a, \langle\Phi\rangle] = 0$, the corresponding A_μ^a is massless. These t^a generate the unbroken symmetries.

Now consider the two cases in the problem set:

SU(5) with $\langle\Phi\rangle = A\,\mathrm{diag}(1,1,1,|\,-4)$

the SU(5) generators can be written with the basis

$$\begin{pmatrix} t^a & \\ & \end{pmatrix} \qquad \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -4 & \end{pmatrix}$$

t^a = 4×4 Hermitian traceless

$$\frac{1}{2}\begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} & & & -i \\ & & & \\ & & & \\ i & & & \end{pmatrix}$$

The first two structures commute with $\langle\Phi\rangle$. These matrices generate $SU(4)\times U(1)$

The other two structures have nonzero commutators with $\langle \underset{\sim}{\Phi} \rangle$. For example:

$$\left[\tfrac{1}{2}\left(\begin{array}{c|c} & 1 \\ \hline 1 & \end{array}\right), \langle \underset{\sim}{\Phi} \rangle\right] = \frac{A}{2}\left(\begin{array}{c|c} & 1 \\ \hline +4+1 & 1 \end{array}\right)^{1\,-4-1}$$

$$= (-5A) \cdot \tfrac{1}{2}\left(\begin{array}{c|c} & 1 \\ \hline -1 & \end{array}\right)$$

then $$-\mathrm{tr}\left([t^9, \langle \underset{\sim}{\Phi} \rangle]\,[t^9, \underset{\sim}{\Phi}]\right) = \frac{25A^2}{2}$$

$$\tfrac{1}{2} m_a^2 = -g^2\,\mathrm{tr}\left([t^9, \langle \underset{\sim}{\Phi} \rangle]\,[t^9, \langle \underset{\sim}{\Phi} \rangle]\right)$$

$$= \frac{25A^2}{2} \cdot g^2 \quad \Rightarrow \quad m_a = \underline{5gA}$$

$$\left[\tfrac{1}{2}\left(\begin{array}{c|c} & -i \\ \hline i & \end{array}\right), \langle \underset{\sim}{\Phi} \rangle\right] = \frac{A}{2}\left(\begin{array}{c|c} & -i(-4-1) \\ \hline i(4+1) & \end{array}\right)$$

and again

$$\tfrac{1}{2} m_a^2 = -g^2\,\mathrm{tr}\left([t^9, \langle \underset{\sim}{\Phi} \rangle]\right)^2 = \frac{25A^2}{2} g^2$$

So all broken generators have the mass $m = \underline{5gA}$

SU(5) with $\langle \Phi \rangle = B \left(\begin{array}{ccc|cc} 2 & 2 & 2 & & \\ & & & & \\ \hline & & & -3 & \\ & & & & -3 \end{array} \right)$

the generators that commute with $\langle \Phi \rangle$ have the basis:

$\left(\begin{array}{c|c} t^a & \\ \hline & \end{array} \right)$ $\left(\begin{array}{c|c} & \\ \hline & t^a \end{array} \right)$ $\frac{1}{\sqrt{15}} \left(\begin{array}{ccc|cc} 2 & 2 & 2 & & \\ & & & & \\ \hline & & & -3 & \\ & & & & -3 \end{array} \right)$

generate SU(3) generate SU(2) generates U(1)

The remaining generators have the structure

$\frac{1}{2} \left(\begin{array}{c|c} & 1 \\ \hline 1 & \end{array} \right)$ or $\frac{1}{2} \left(\begin{array}{c|c} & -i \\ \hline i & \end{array} \right)$

$$\left[\frac{1}{2} \left(\begin{array}{c|c} & 1 \\ \hline 1 & \end{array} \right), \langle \Phi \rangle \right] = \frac{B}{2} \left(\begin{array}{c|c} & -3-2 \\ \hline -2+3 & \end{array} \right)$$

$$= 5 \frac{B}{2} \left(\begin{array}{c|c} & -1 \\ \hline 1 & \end{array} \right)$$

ajo

$$\frac{1}{2} m_a^2 = -g^2 \, tr\left([t^a, \langle \Phi \rangle] \right)^2 = \frac{25 B^2}{2} g^2$$

$$m_a = 5 g B$$

so:

$$\langle \underset{\sim}{\Phi} \rangle = A \left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ \hline & & & -4 \end{array} \right)$$

the unbroken symmetry is $SU(4) \times U(1)$ (16 generators)

the 8 broken generators have $m = 5g\,A$

$$16 + 8 = 24 \checkmark$$

$$\langle \underset{\sim}{\Phi} \rangle = B \left(\begin{array}{ccc|c} 2 & 2 & 2 & \\ \hline & & & -3 \quad -3 \end{array} \right)$$

the unbroken symmetry is $SU(3) \times SU(2) \times U(1)$

$$8 + 3 + 1 = 12 \text{ generators}$$

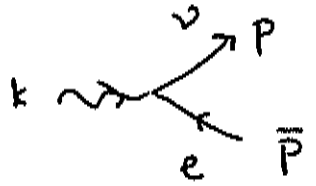
the 12 broken generators have $m = 5g\,B$

$$12 + 12 = 24 \checkmark$$

2.) a.) The decay modes of W^+ are:

$$W^+ \to e^+\nu_e \;,\; \mu^+\nu_\mu \;,\; \tau^+\nu_\tau \;,\; u\bar{d} \;,\; c\bar{s}$$

All of the partial width computations are basically the same. For $W^+ \to e^+\nu_e$



$$iM = i\frac{g}{\sqrt{2}}\,\bar{u}(p)\,\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)v(\bar{p})\,\varepsilon_\nu(k)$$

$$\sum_{spins}|M|^2 = \frac{g^2}{2}\,\mathrm{tr}\left[\not{p}\,\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\not{\bar{p}}\,\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\varepsilon_\mu\,\varepsilon_\nu^*$$

$$= \frac{g^2}{2}\,\mathrm{tr}\left[\not{p}\,\gamma^\mu\not{\bar{p}}\,\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_W^2}\right)$$

↑ by symmetry in $\mu\nu$ the γ^5 gives zero

$$= \frac{g^2}{2}\cdot 2\left[p^\mu\bar{p}^\nu + p^\nu\bar{p}^\mu - p\cdot\bar{p}\,g^{\mu\nu}\right]\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_W^2}\right)$$

$$= g^2\left[p\bar{p}\,(4-2) + \frac{k\cdot p\;k\cdot\bar{p}}{m_W^2}\cdot 2 - p\bar{p}\,\frac{k^2}{m_W^2}\right]$$

Now $k = (m_W, \vec{0})$ $\quad p = (E,0,0,E)$ $\quad \bar{p} = (E,0,0,-E)$

$p\cdot\bar{p} = \frac{m_W^2}{2}$ $\quad k\cdot p = k\cdot\bar{p} = \frac{m_W^2}{2}$ $\quad E = \frac{m_W}{2}$

so

$$\sum_{spins} |\mathcal{M}|^2 = g^2 \left[2 \cdot \frac{m_W^2}{2} + 2 \cdot \frac{(m_W^2/2)^2}{m_W^2} - \frac{m_W^2}{2} \right]$$

$$= g^2 m_W^2$$

then spin average is given by dividing by $\underline{3}$

$$\underline{\Gamma}(W^+ \to e^+ \nu) = \frac{1}{2m_W} \cdot \frac{1}{8\pi} \cdot \frac{1}{3} g^2 m_W^2$$

$$= \frac{\alpha_W}{12} m_W$$

$$\alpha_W = \frac{g^2}{4\pi} = \frac{e^2}{4\pi \sin^2\theta_W} \simeq \frac{1}{129} \simeq \frac{1}{30}$$

The computation for the other leptons is identical.
For the quarks, multiply by the color factor 3 and
the QCD correction $\left(1 + \frac{\alpha_s(m_W)}{\pi}\right) \simeq 1 + \underline{4\%}$

$$\Gamma(W^+ \to e^+ \nu_e) = \Gamma(W^+ \to \mu^+ \nu_\mu) = \Gamma(W^+ \to \tau^+ \nu_\tau) = \frac{\alpha_W}{12} m_W$$

$$\Gamma(W^+ \to u\bar{d}) = \Gamma(W^+ \to c\bar{s}) = \frac{\alpha_W}{12} m_W \cdot 3 \cdot \left(1 + \frac{\alpha_s}{\pi}\right)$$

numerically

$$\Gamma(W^+ \to e^+ \nu_e) = 0.22 \text{ GeV}$$

$$\Gamma(W^+ \to u\bar{d}) = 0.69 \text{ GeV}$$

$$\Gamma(W^+) = 2.03 \text{ GeV}$$

branching ratios:

$e^+\nu$, $\mu^+\nu$, $\tau^+\nu$ } 11% each

$u\bar{d}$, $c\bar{s}$ } 34% each

Repeat the calculation for the Z^0

for a chiral R or L-handed flavor

$$iM = i \frac{e}{\cos\theta_W \sin\theta_W} \bar{u}(p) \gamma^\mu \left(\frac{1 \pm \gamma^5}{2}\right) v(\bar{p}) \, \varepsilon_\lambda(k) \cdot \left(I^3 - \sin^2\theta_W Q\right)$$

$$\frac{1}{3} \sum_{\text{spin}} |M|^2 = \frac{2}{3} m_Z^2 \cdot \frac{e^2}{\sin^2\theta_W \cos^2\theta_W} \left(I^3 - \sin^2\theta_W Q\right)^2$$

then

$$\Gamma(Z^0 \to f\bar{f}) = \frac{1}{2m_Z}\,\frac{1}{8\pi}\cdot\frac{e^2}{6\sin^2\theta_W\cos^2\theta_W}\,\frac{2}{3}m_Z^2\left(I^3-\sin^2\theta_W Q\right)^2$$

$$= \frac{\alpha}{6\sin^2\theta_W\cos^2\theta_W}\,m_Z\left(I^3-\sin^2\theta_W Q\right)^2$$

$$= \frac{\alpha_W}{6\cos^2\theta_W}\,m_Z\left(I^3-\sin^2\theta_W Q\right)^2$$

the unit is:

$$\frac{\alpha_W}{6\cos^2\theta_W}\,m_Z = 0.66\ \text{GeV}$$

now account the Z^0 decays into all chiral flavors

	$(I^3-\sin^2\theta_W Q)^2 =$	$\to f_L\bar{f}_R$	$\to f_R\bar{f}_L$
$Z\to\nu\bar{\nu}$	$(\frac{1}{2})^2 = 0.25$	×	
$Z\to e^-e^+$	$(-\frac{1}{2}+\sin^2\theta_W)^2 = 0.073$	$(\sin^2\theta_W)^2 = 0.053$	
$Z\to u\bar{u}$	$(\frac{1}{2}-\frac{2}{3}\sin^2\theta_W)^2 = 0.130$	$(-\frac{2}{3}\sin^2\theta_W)^2 = 0.024$	
$Z\to d\bar{d}$	$(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)^2 = 0.179$	$(\frac{1}{3}\sin^2\theta_W)^2 = .006$	

for the quarks, multiply by

$$3\cdot\left(1+\frac{\alpha_s}{\pi}\right) = 3.12$$

so $$\Gamma(Z^0 \to f\bar{f}) = \frac{\alpha_w m_Z}{6 \cos^2\theta_w} .$$

$\nu_e\ \nu_\mu\ \nu_\tau$	0.25	× 3	=	0.750
$e\ \mu\ \tau$	0.126	×3	=	0.378
$u\ c$	0.480	×2	=	0.960
$d\ s\ b$	0.578	×3	=	1.732
				3.82

$\Gamma(Z) = 2.5$ GeV

the branching fractions are

$\nu\bar{\nu}$ (invisible)	20%
e^+e^-, $\mu^+\mu^-$, $\tau^+\tau^-$	3.3% each
$u\bar{u}$ $c\bar{c}$	12% each
$d\bar{d}$ $s\bar{s}$ $b\bar{b}$	15% each

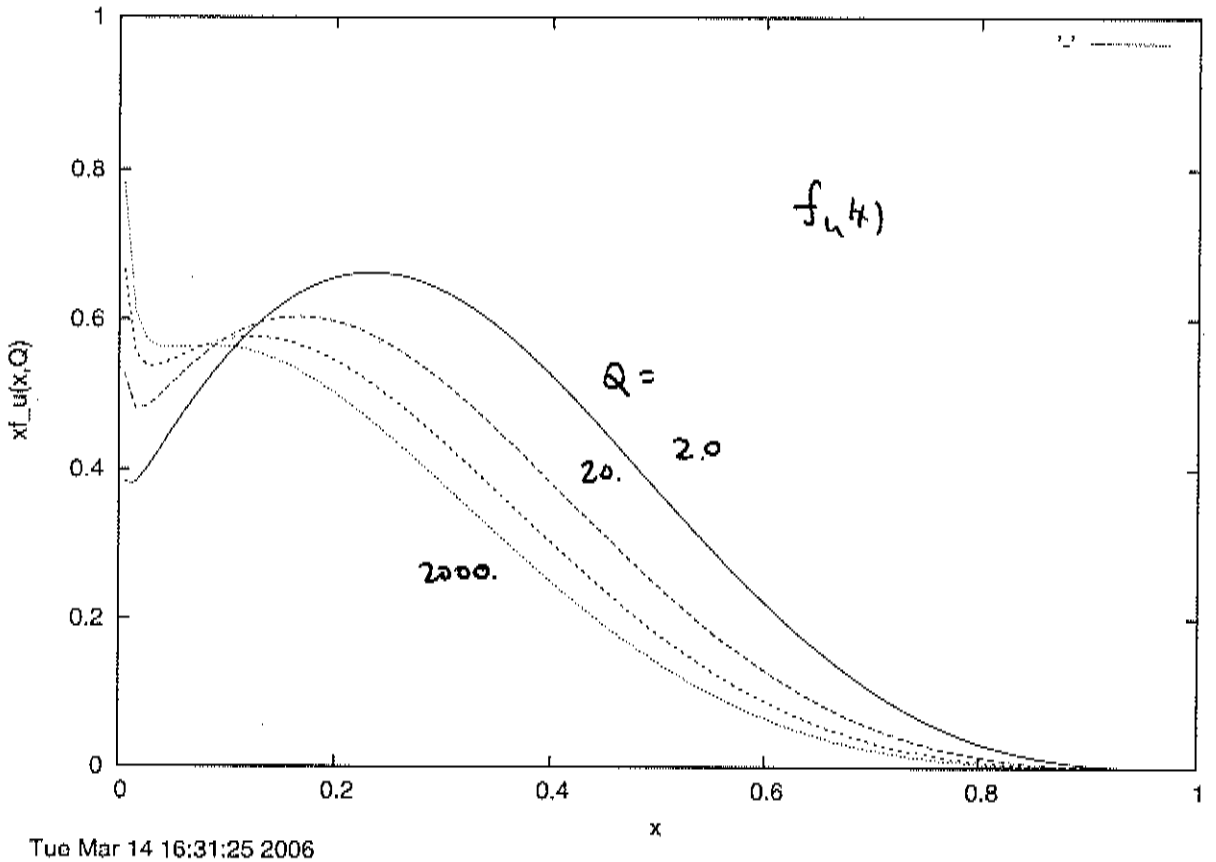
3.) To plot the evolution of d, $\bar{d}$, b quark distributions, it is only necessary to change some obvious things in the program plotxf.cpp.

In particular,

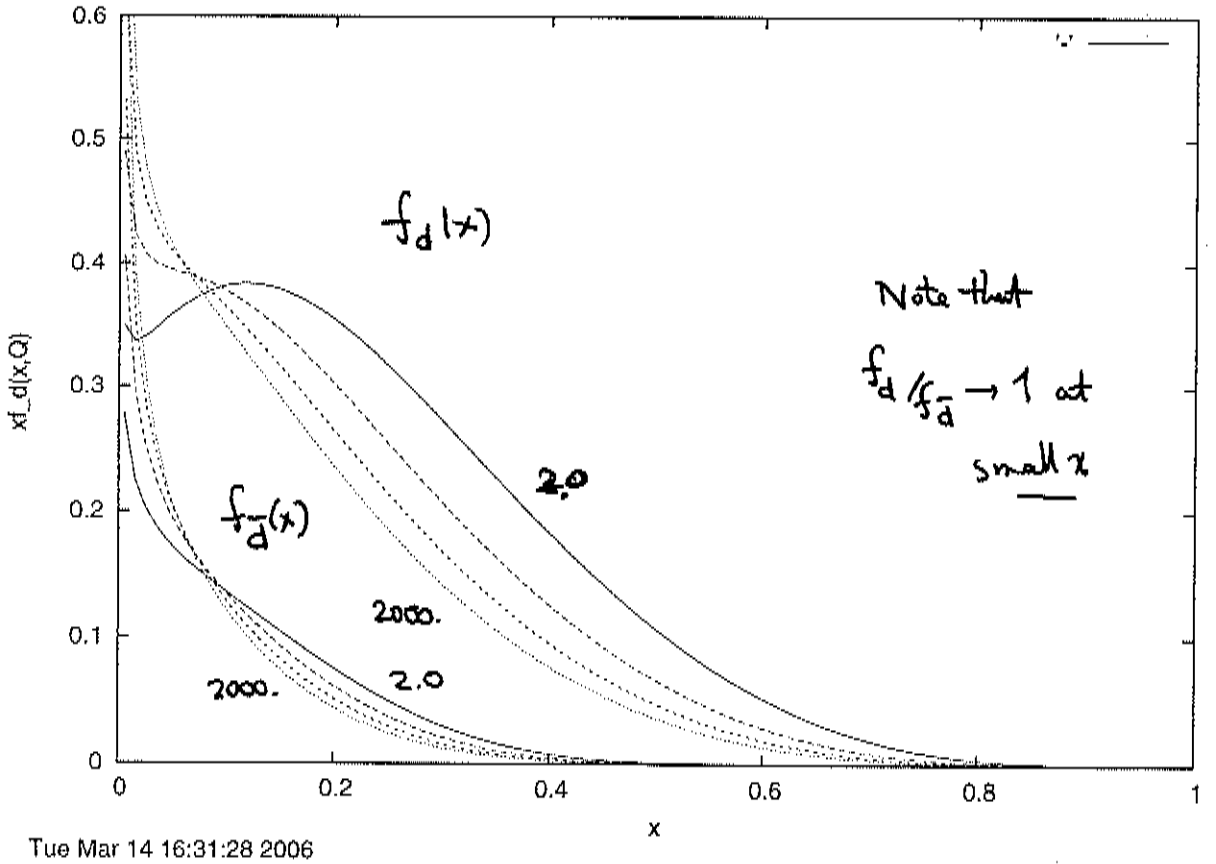
xfu → xfd or xfdbar

I have put a program plotxfmore.cpp that makes all of the required plots on the course web page.

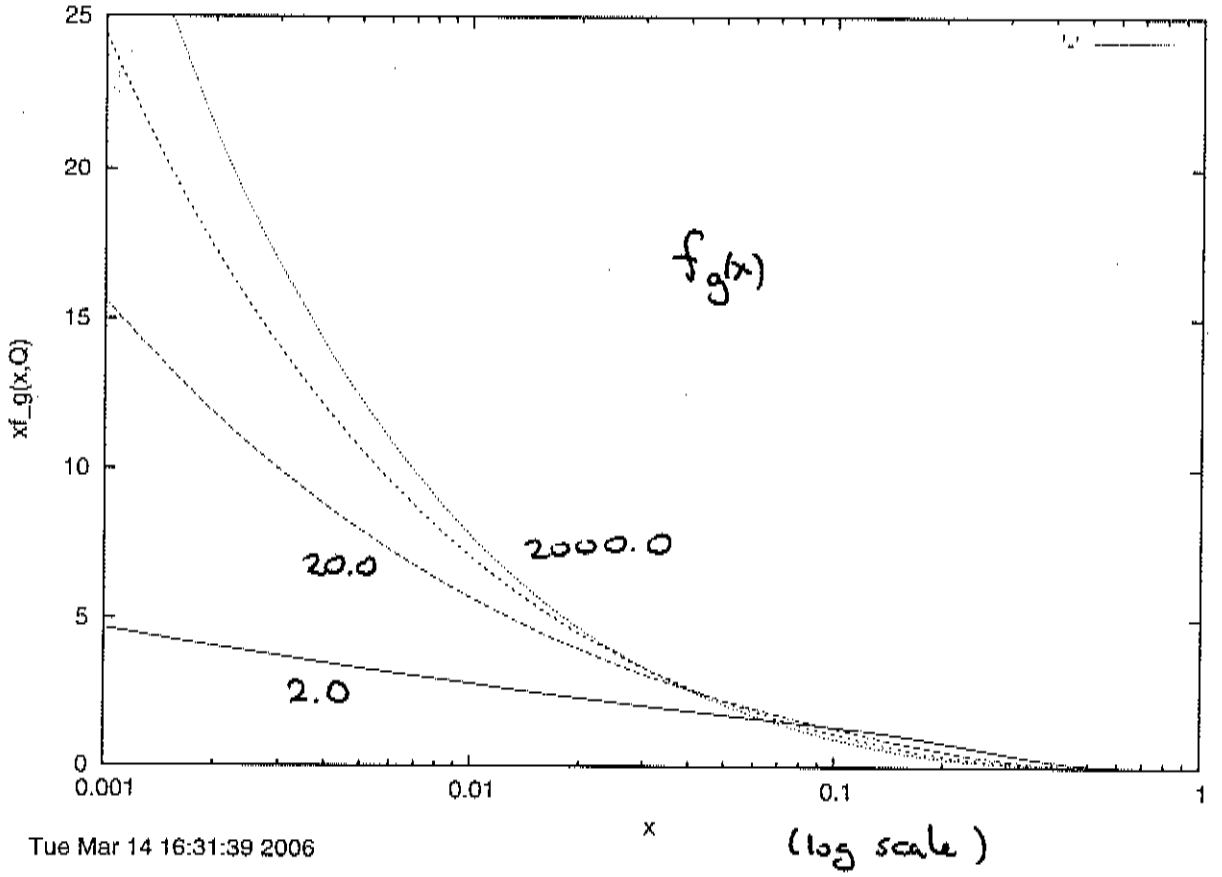
Here are the plots →



Tue Mar 14 16:31:25 2006

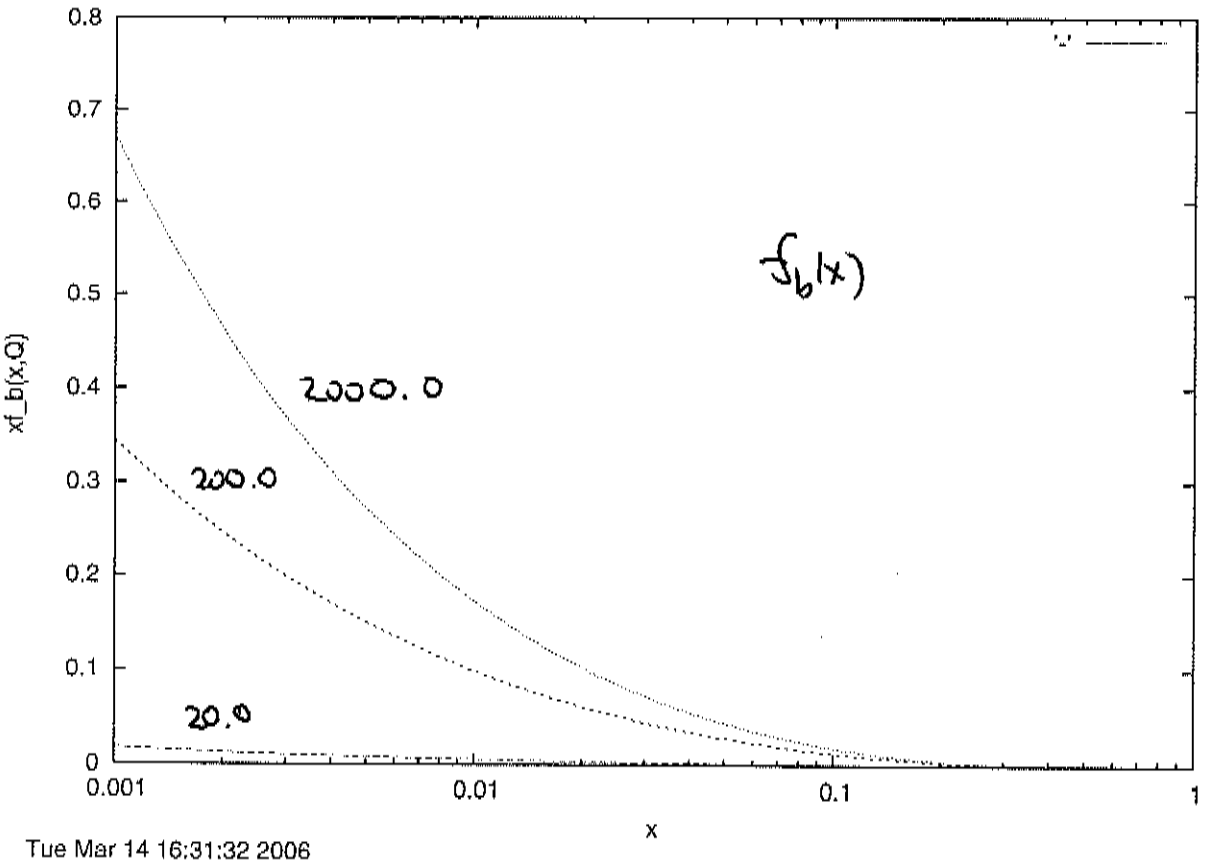


Tue Mar 14 16:31:28 2006



$f_g(x)$

(log scale)



$f_b(x)$

4.) For this problem, it is necessary to implement in ppcollider the following formulas for the total cross section:

$$\sigma = \int dx_1\, dx_2\, d\cos\Theta$$

$$\Big\{ f_g(x_1)\, f_g(x_2) \left[\frac{d\sigma}{d\cos\Theta}(gg \to gg) + n_f \frac{d\sigma}{d\cos\Theta}(gg \to u\bar{u}) \right]$$

$$+ 2 \sum_{f=udscb} \left[f_f(x_1) + f_{\bar{f}}(x_1) \right] f_g(x_2) \frac{d\sigma}{du_1\Theta}(ug \to ug)$$

$$+ \sum_{f=udscb} \left(f_f(x_1)\, f_f(x_2) + f_{\bar{f}}(x_1) f_{\bar{f}}(x_2) \right)$$

$$\left(\frac{d\sigma}{d\cos\Theta}(u\bar{u} \to u\bar{u}) + (n_f - 1) \frac{d\sigma}{d\cos\Theta}(u\bar{u} \to d\bar{d}) \right.$$

$$\left. + \frac{d\sigma}{d\cos\Theta}(u\bar{u} \to gg) \right)$$

$$+ 2 \sum_{f} f_f(x_1) f_{\bar{f}}(x_2) \frac{d\sigma}{d\cos\Theta}(uu \to uu)$$

$$+ \sum_{f \neq f'} \left(f_f(x_1) f_{f'}(x_2) + f_{\bar{f}}(x_1) f_{\bar{f}'}(x_2) \right) \frac{d\sigma}{d\cos\Theta}(u\bar{d} \to u\bar{d})$$

$$+ \sum_{f \neq f'} \left(f_f(x_1) f_{\bar{f}'}(x_2) + f_{\bar{f}}(x_1) f_{f'}(x_2) \right) \frac{d\sigma}{d\cos\Theta}(ud \to ud) \Big\}$$

Note that I have used

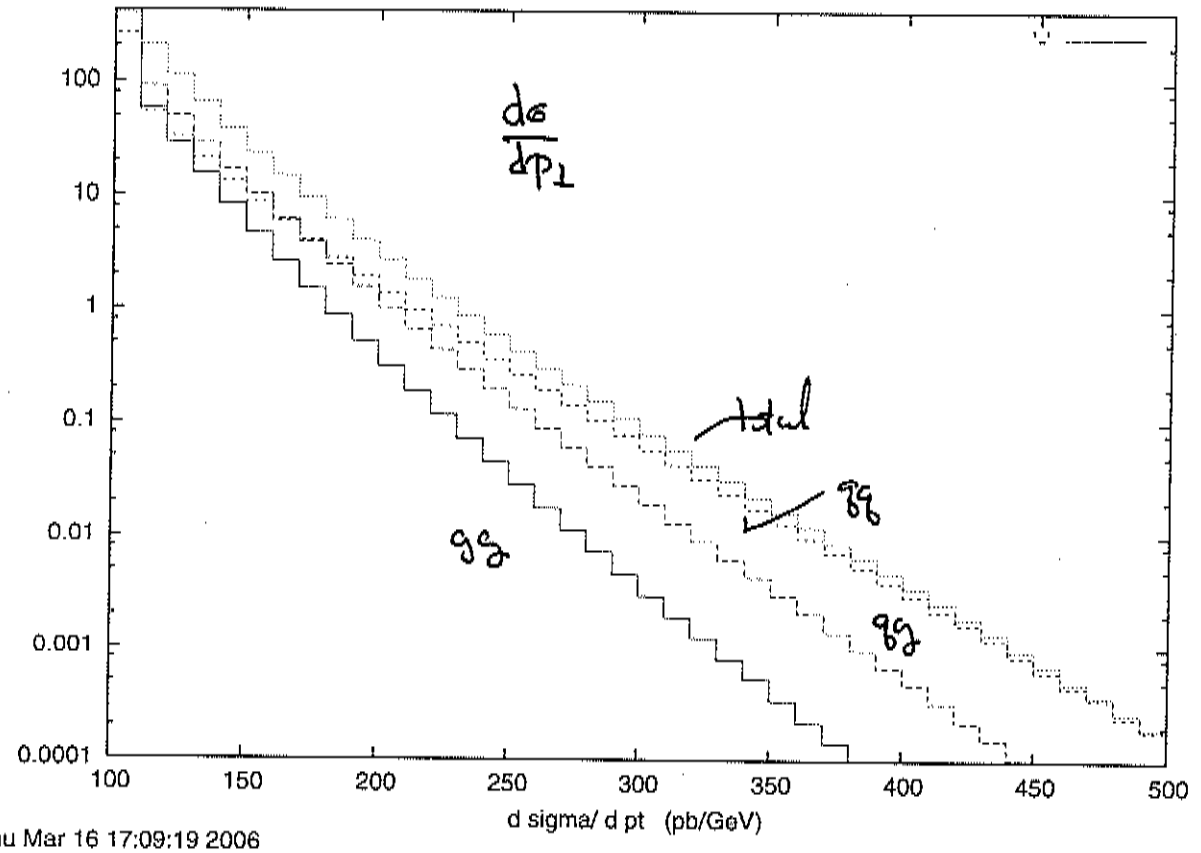
$$f_q(x) \text{ in } \bar{p} = f_{\bar{q}}(x) \text{ in } p$$

$$f_{\bar{q}}(x) \text{ in } \bar{p} = f_q(x) \text{ in } p$$

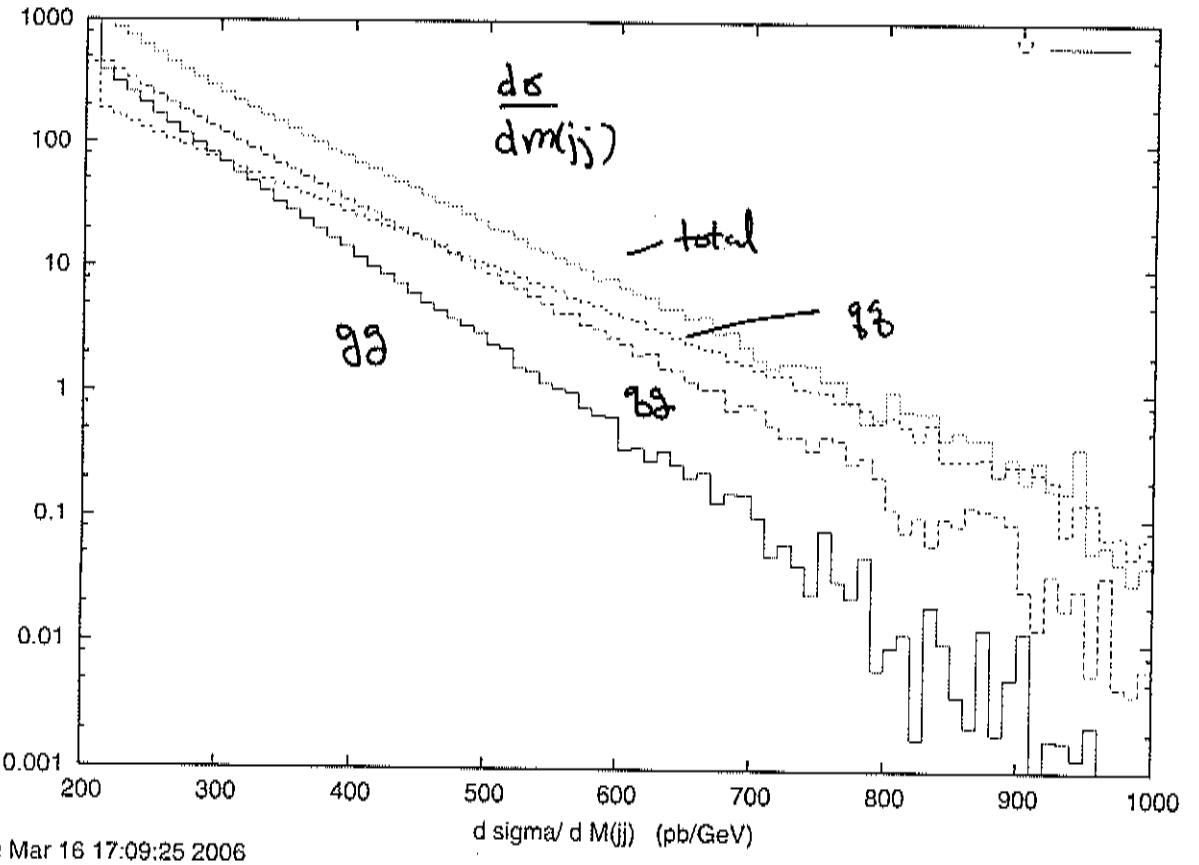
$$f_g(x) \text{ in } \bar{p} = f_g(x) \text{ in } p$$

The plots requested are shown ⟶

I have put a program ppbarcolltheden.cpp that makes these plots on the course Web site.



Thu Mar 16 17:09:19 2006



Thu Mar 16 17:09:25 2006

Physics 331 – Problem Set # 7

(due Wednesday, March 15)

1. Peskin and Schroeder, Problem 20.3.

2. Peskin and Schroeder, Problem 20.4.

3. Using explicit polarization vectors, compute the scattering amplitudes for $e_R^- e_L^+ \to W^- W^+$ and $e_L^- e_R^+ \to W^- W^+$ for the case of longitudinally polarized W bosons. Do the exact calculation, without making the approximations used by Peskin and Schroeder in Section 21.2. (You may ignore the electron mass.) Show that the individual diagrams are of the order of $e^2 s/m_W^2$ but that a power of s/m_W^2 cancels to produce the final answer. It is much easier to see the cancellation in the case of $e_R^- e_L^+$. Show that the final answer in each case is proportional to $e^2 \sin\theta$ at high energy. Show that these results reproduce the amplitudes for $e_R^- e_L^+ \to \phi^- \phi^+$, $e_L^- e_R^+ \to \phi^- \phi^+$, where ϕ^+, ϕ^- are the Goldstone bosons in the Higgs multiplet.

4. Using the computer program `ppcollider` from Problem Set 6 and the result of Problem 20.3, compute the dimuon invariant mass distribution for the Drell-Yan process $p\overline{p} \to \mu^+\mu^- + X$ in $p\overline{p}$ collisions at a center of mass energy of 2.0 TeV.

Physics 331 – Problem Set #7

Solutions

1.) a.) Recall that in QED, all particles massless,

$$\frac{d\sigma}{d\cos\theta}(e^-_L e^+_R \to \mu^-_L \mu^+_R) = \frac{d\sigma}{d\cos\theta}(e^-_R e^+_L \to \mu^-_R \mu^+_L)$$

$$= \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2$$

$$\frac{d\sigma}{d\cos\theta}(e^-_L e^+_R \to \mu^-_R \mu^+_L) = \frac{d\sigma}{d\cos\theta}(e^-_R e^+_L \to \mu^-_L \mu^+_R)$$

$$= \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2$$

In all cases, the total cross section is $\sigma(\text{polarized } e^+e^- \to \text{pol. } \mu^+\mu^-)$

$$= \frac{4}{3}\frac{\pi\alpha^2}{s}$$

Check: $\sigma(e^+e^- \to \mu^+\mu^-) = \frac{1}{4} \times (\text{sum of 4 cross sects})$

$$= \frac{4\pi\alpha^2}{3s}$$

In the electroweak thy, we must compute



$$i\mathcal{M}(\underbrace{e^-e^+ \to f\bar{f}}_{\text{chiral states}}) = (ie)^2\ \bar{v}\gamma^\mu u\ \bar{u}\gamma_\mu v \cdot \frac{-i}{s} \cdot \left[Q_e Q_f + \frac{s}{s-m_Z^2} \frac{(I_e^3 - s_W^2 Q_e)(I_f^3 - s_W^2 Q_f)}{s_W^2 c_W^2} \right]$$

where $s_W^2 = \sin^2\theta_W \qquad c_W^2 = \cos^2\theta_W$

the QED result is modified by the expression in the brackets.

For $e^+e^- \to \mu^+\mu^-$ we then have:

$$\frac{d\sigma}{d\cos\theta}(e_L^- e_R^+ \to \mu_L^- \mu_R^+) = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2 \left[1 + \frac{(\frac{1}{2} - s_W^2)^2}{s_W^2 c_W^2} \frac{s}{s-m_Z^2} \right]^2$$

$$\frac{d\sigma}{d\cos\theta}(e_L^- e_R^+ \to \mu_R^- \mu_L^+) = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2 \left| 1 - \frac{(\frac{1}{2} - s_W^2)\, s_W^2}{s_W^2 c_W^2} \frac{s}{s-m_Z^2} \right|^2$$

$$\frac{d\sigma}{d\cos\theta}(e_R^- e_L^+ \to \mu_L^- \mu_R^+) = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2 \left| 1 - \frac{s_W^2(\frac{1}{2} - s_W^2)}{s_W^2 c_W^2} \frac{s}{s-m_Z^2} \right|^2$$

$$\frac{d\sigma}{d\cos\theta}(e_R^- e_L^+ \to \mu_R^- \mu_L^+) = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2 \left| 1 + \frac{(s_W^2)^2}{s_W^2 c_W^2} \frac{s}{s-m_Z^2} \right|^2$$

the spin-averaged total cross section is then

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{\pi\alpha^2}{3s}$$

$$\cdot \Big\{ \Big| 1 + \frac{(\tfrac{1}{2} - s_w^2)^2}{s_w^2 c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + 2 \Big| 1 - \frac{(\tfrac{1}{2} - s_w^2)}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| 1 + \frac{s_w^2}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 \Big\}$$

the corresponding formulae for quarks are:

$$\sigma(e^+e^- \to u\bar{u}) = \frac{\pi\alpha^2}{3s} \cdot 3 \cdot \Big(1 + \frac{\alpha_s}{\pi}\Big)$$

$$\cdot \Big\{ \Big| \frac{2}{3} + \frac{(\tfrac{1}{2} - s_w^2)(\tfrac{1}{2} - \tfrac{2}{3}s_w^2)}{s_w^2 c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| \frac{2}{3} - \frac{(\tfrac{1}{2} - s_w^2)\cdot \tfrac{2}{3}}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| \frac{2}{3} - \frac{(\tfrac{1}{2} - \tfrac{2}{3}s_w^2)}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| \frac{2}{3} + \frac{\tfrac{2}{3}s_w^2}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 \Big\}$$

$$\sigma(e^+e^- \to d\bar{d}) = \frac{\pi\alpha^2}{3s} \cdot 3 \cdot \Big(1 + \frac{\alpha_s}{\pi}\Big)$$

$$\Big\{ \Big| \frac{1}{3} + \frac{(\tfrac{1}{2} - s_w^2)(\tfrac{1}{2} - \tfrac{1}{3}s_w^2)}{s_w^2 c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| \frac{1}{3} - \frac{(\tfrac{1}{2} - s_w^2)\tfrac{1}{3}}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| \frac{1}{3} - \frac{(\tfrac{1}{2} - \tfrac{1}{3}s_w^2)}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 + \Big| \frac{1}{3} + \frac{\tfrac{1}{3}s_w^2}{c_w^2} \frac{s}{s - m_Z^2} \Big|^2 \Big\}$$

we can write these formulae as

$$\sigma(e^-e^+ \to f\bar{f}) = \frac{\pi\alpha^2}{3s}\left\{ |G_{LL}|^2 + |G_{LR}|^2 + |G_{RL}|^2 + |G_{RR}|^2 \right\}$$

where

G_{LL} is the amplitude correction for $e^-_L \to f_L$

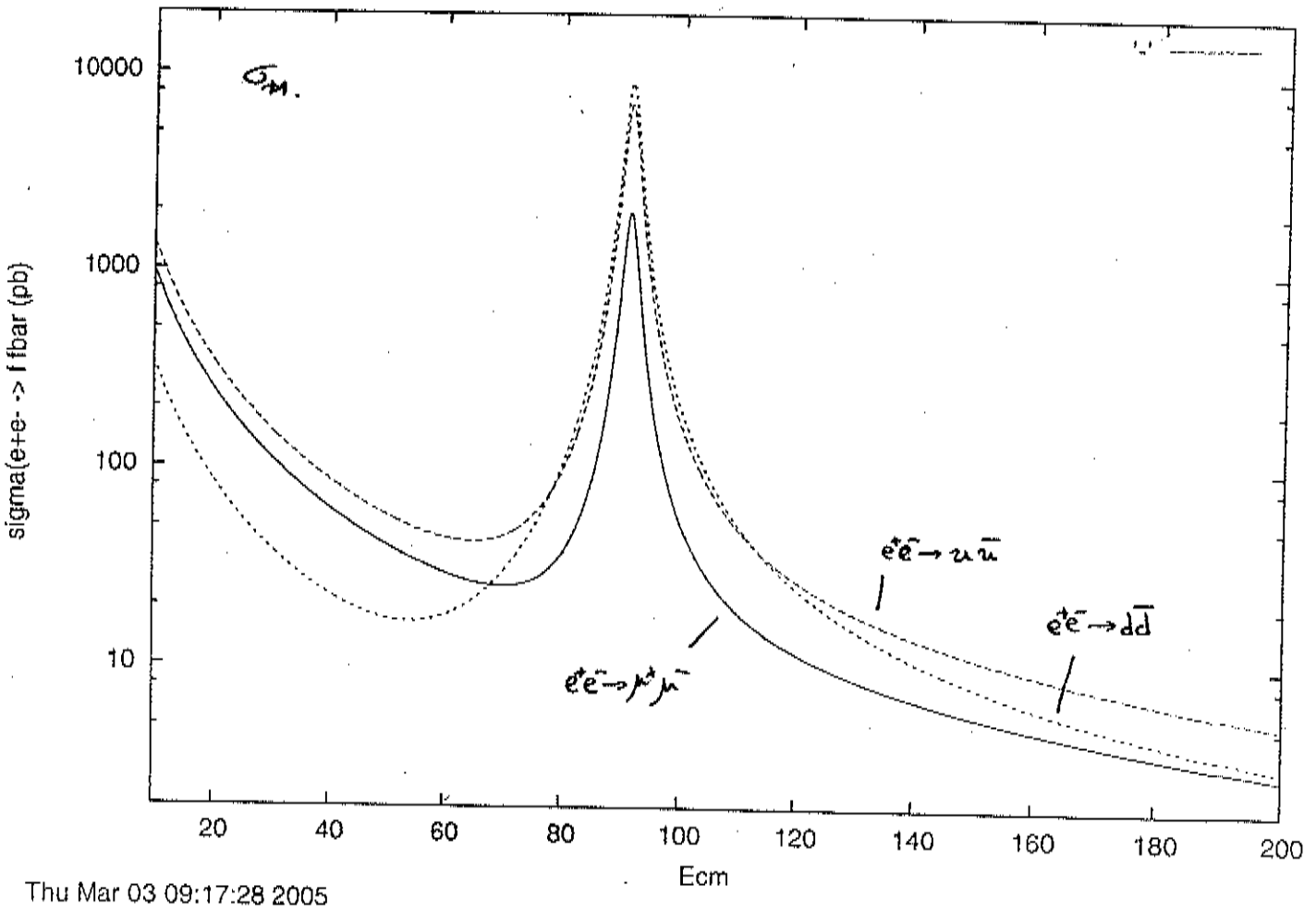
G_{LR} is the amplitude correction for $e^-_L \to f_R$

etc.

to treat the Z^0 resonance more correctly, we should replace

$$\frac{s}{s-m_Z^2} \quad \to \quad \frac{s}{s-m_Z^2+im_Z\Gamma_Z}$$

A plot of the three cross-sections is given on the next page $\longrightarrow$


10000
1000
100
10
sigma(e+e- -> f fbar (pb)
20
40
60
80
100
120
140
160
180
200
Ecm
$\sigma_{TH.}$
$e^+e^- \to u\bar{u}$
$e^+e^- \to d\bar{d}$
$e^+e^- \to \mu^+\mu^-$
Thu Mar 03 09:17:28 2005

b.) To compute forward-backward asymmetries, we need:

$$\int_{-1}^{1} d\cos\Theta \; (1+\cos\Theta)^2 = 2 + 2\cdot 0 + \tfrac{2}{3} = \tfrac{8}{3}$$

$$\int_{0}^{1} d\cos\Theta \; (1+\cos\Theta)^2 = 1 + 2\left(\tfrac{1}{2}\right) + \tfrac{1}{3} = \tfrac{7}{3}$$

$$\int_{-1}^{0} d\cos\Theta \; (1+\cos\Theta)^2 = 1 + 2\left(-\tfrac{1}{2}\right) + \tfrac{1}{3} = \tfrac{1}{3}$$

so

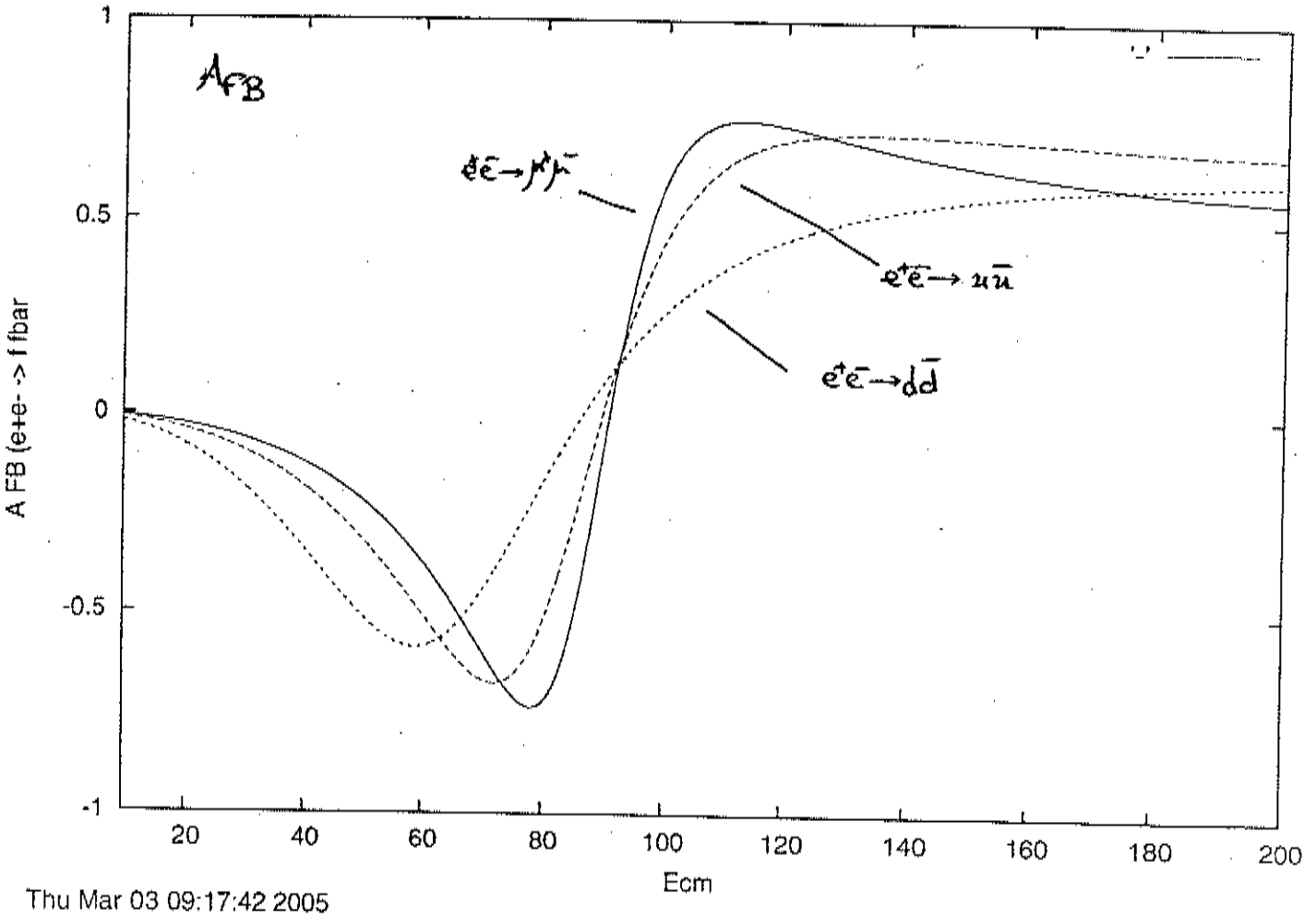
$$\frac{\left(\int_0^1 d\cos\Theta - \int_{-1}^{0} d\cos\Theta\right)(1+\cos\Theta)^2}{\left(\quad + \quad\right)(1+\cos\Theta)^2} = \frac{\tfrac{7}{3} - \tfrac{1}{3}}{\tfrac{8}{3}} = \frac{3}{4}$$

then for any species:

$$A_{FB}(\bar{e}e \to f\bar{f}) = \frac{3}{4}\,\frac{|G_{LL}|^2 + |G_{RR}|^2 - |G_{LR}|^2 - |G_{RL}|^2}{|G_{LL}|^2 + |G_{RR}|^2 + |G_{LR}|^2 + |G_{RL}|^2}$$

where G_{ij} are the factors on pp. 2, 3

Plots of the asymmetries for μ, u, d are given on the next page $\rightarrow$


A_{FB}
$e^+e^- \to \mu^+\mu^-$
$e^+e^- \to u\bar{u}$
$e^+e^- \to d\bar{d}$
A FB (e+e- -> f fbar
Ecm
1
0.5
0
-0.5
-1
20
40
60
80
100
120
140
160
180
200
Thu Mar 03 09:17:42 2005

C.) as $s \to m_Z^2$

$$G_{LL} \to \frac{(-\frac{1}{2}+s_W^2)(I^3 - s_W^2 Q)}{s_W^2 c_W^2} \frac{s}{s-m_Z^2}$$

$$G_{LR} \to \frac{(-\frac{1}{2}+s_W^2)(-s_W^2 Q)}{s_W^2 c_W^2} \frac{s}{s-m_Z^2}$$

$$G_{RL} \to \frac{s_W^2 (I^3 - s_W^2 Q)}{s_W^2 c_W^2} \frac{s}{s-m_Z^2}$$

$$G_{RR} \to \frac{s_W^2 (-s_W^2 Q)}{s_W^2 c_W^2} \frac{s}{s-m_Z^2}$$

then

$$A_{FB} \to \frac{3}{4} \frac{(-\frac{1}{2}+s_W^2)^2 (I^3 - s_W^2 Q)^2 - (-\frac{1}{2}+s_W)^2 (-s_W Q)^2 - (s_W^2)^2 (I^3 - s_W^2 Q)^2 + (s_W^2)(-s_W^2 Q)^2}{(\quad + \quad + \quad + \quad)}$$

$$= \frac{3}{4} \frac{[(-\frac{1}{2}+s_W^2)^2 - (s_W^2)^2][(I^3 - s_W^2 Q)^2 - (-s_W^2 Q)^2]}{[(-\frac{1}{2}+s_W^2)^2 + (s_W^2)^2][(I^3 - s_W^2 Q)^2 + (-s_W^2 Q)^2]}$$

$$= \frac{3}{4} A_{LR}^e A_{LR}^f$$

d.) At the peak of the Z^0

$$\frac{s}{s - m_Z^2 + i m_Z \Gamma_Z} \rightarrow \frac{m_Z^2}{i m_Z \Gamma_Z}$$

$$G_{LL} \rightarrow \frac{(-\tfrac{1}{2} + s_W^2)(I^3 - s_W^2 Q)}{s_W c_W} \cdot \left(\frac{-i m_Z}{\Gamma_Z}\right)$$

$$G_{LR} \rightarrow \frac{(-\tfrac{1}{2} + s_W^2)(-s_W^2 Q)}{s_W c_W} \left(\frac{-i m_Z}{\Gamma_Z}\right)$$

etc.

$$\sigma(e^+e^- \rightarrow f\bar{f})\Big|_{s=m_Z^2} = \frac{\pi\alpha^2}{3s} \cdot \underbrace{3}_{\text{3 quarks}} \cdot \left(1 + \frac{\alpha_s}{\pi}\right) \cdot \frac{m_Z^2}{\Gamma_Z^2}$$

$$\cdot \frac{[(-\tfrac{1}{2} + s_W^2)^2 + s_W^2]}{s_W c_W} \cdot \frac{(I^3 - s_W^2 Q)^2 + (s_W^2 Q)^2}{s_W c_W}$$

now $\Gamma(Z^0 \rightarrow f\bar{f}) = \frac{\alpha m_Z}{6 s_W^2 c_W^2} \left[(I^3 - s_W^2 Q)^2 + (s_W^2 Q)^2\right] \cdot \underbrace{3 \cdot (\ )}_{\text{quarks}}$

so

$$\sigma(e^+e^- \rightarrow f\bar{f}) = \frac{\pi}{3 m_Z^2} \cdot 36 \frac{1}{\Gamma_Z^2} \cdot \Gamma(Z \rightarrow e^+e^-)\,\Gamma(Z \rightarrow f\bar{f})$$

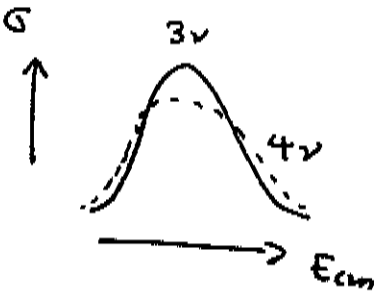
$$= \frac{12\pi}{m_Z^2} \frac{\Gamma(Z \rightarrow e^+e^-)\,\Gamma(Z \rightarrow f\bar{f})}{\Gamma_Z^2}$$

According to the previous problem set

$$\Gamma_Z = 2.5 \text{ GeV} \qquad \Gamma(Z \to \nu\bar{\nu}) = 0.165 \text{ GeV/species}.$$

so an extra neutrino species gives

$$\frac{\Delta\Gamma_Z}{\Gamma_Z} = +6.6\%$$



$$\frac{\Delta\sigma}{\sigma} = -2\left(\frac{\Delta\Gamma_Z}{\Gamma_Z}\right) = -13\%$$

Current measurements give

$$\Gamma_Z = 2.4952 \pm .0023 \text{ GeV}$$

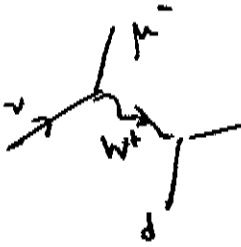
$$\sigma_{had.,Z} = 41.541 \pm .037 \text{ nb}$$

hadronic cross section at the Z^0 peak

both in good agreement w. 3 neutrino species. These measurements exclude contributions to the invisible decays of the Z^0 larger than 1% of the contribution of a ν species.

=

2.) a.) The charged-current weak interaction is mediated by:



$$= \frac{g^2}{2m_W^2} (\bar{\mu}\gamma^\mu P_L \nu)(\bar{u}\gamma_\mu P_L d)$$

$$= \frac{4G_F}{\sqrt{2}} (\bar{\mu}\gamma^\mu P_L \nu)(\bar{u}\gamma_\mu P_L d)$$

we saw that this leads to the formula

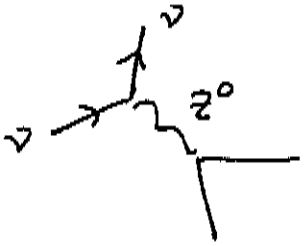
$$\frac{d\hat{\sigma}}{dxdy}(\nu p \to \mu^- X) = \frac{G_F^2 s}{\pi}\left[x f_d(x) + (1-y)^2 x f_{\bar{u}}(x)\right]$$

$$\frac{d\hat{\sigma}}{dxdy}(\bar{\nu} p \to \mu^+ X) = \frac{G_F^2 s}{\pi}\left[x f_u(x)(1-y)^2 + x f_{\bar{d}}(x)\right]$$

The factor $(1-y)^2$ appears when chialities do not match

$\nu_L \bar{u}_R$ or $\bar{\nu}_R u_L$ scatter

The neutral-current interaction is mediated by



$$= \frac{e^2}{c_W^2 s_W^2} \frac{1}{m_Z^2} (\bar{\nu}\gamma^\mu I^3 \nu)$$

$$\cdot \bar{f}\gamma^\mu (I^3 - s_W^2 Q) f$$

$$= \frac{g^2}{2m_W^2}\,\bar{\nu}\gamma^\mu P_L \nu \cdot \bar{f}\gamma^\mu (I^3 - \sin^2\theta_W Q) f$$

using $\frac{e^2}{s_W^2} = g^2 \qquad c_W^2 m_Z^2 = m_W^2$

$$= \frac{4G_F^2}{\sqrt{2}} (\bar{\nu}\gamma^\mu P_L \nu)\;\; \bar{f}\gamma^\mu (I^3 - \sin^2\theta_W Q) f$$

the Z couples to $u_L \;\; u_R \;\; d_L \;\; d_R$

$\bar{u}_L \;\; \bar{u}_R \;\; \bar{d}_L \;\; \bar{d}_R$

adding all of these terms:

$$\frac{d\sigma}{dx\,dy}(\nu p \to \nu X) = \frac{G_F^2 s}{\pi}\Big[x f_u(x)\Big[\underbrace{\left(\tfrac{1}{2} - \tfrac{2}{3}s_W^2\right)^2}_{L} + \underbrace{\left(\tfrac{2}{3}s_W^2\right)^2}_{R}(1-y)^2\Big]$$

$$+ x f_d(x)\Big[\left(-\tfrac{1}{2} + \tfrac{1}{3}s_W^2\right)^2 + \left(\tfrac{1}{3}s_W^2\right)^2(1-y)^2\Big]$$

$$+ x f_{\bar{u}}(x)\Big[\left(\tfrac{2}{3}s_W^2\right)^2 + \left(\tfrac{1}{2} - \tfrac{2}{3}s_W^2\right)^2(1-y)^2\Big]$$

$$+ x f_{\bar{d}}(x)\Big[\left(\tfrac{1}{3}s_W^2\right)^2 + \left(-\tfrac{1}{2} + \tfrac{1}{3}s_W^2\right)(1-y)^2\Big]\Big]$$

for νn scatt, exchge $u \leftrightarrow d$

for $\bar{\nu} p$ scatt, exchange $1 \leftrightarrow (1-y)^2$

b.) If $f_u(x) = f_d(x)$ for a heavy nuclear target A

$$\frac{d^2\sigma}{dx\,dy}(\nu A \to \nu X) = \frac{G_F^2 s}{\pi}\Big[x f_q(x) \Big\{ \left(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W\right)^2 + \left(\tfrac{1}{2} - \tfrac{1}{3}\sin^2\theta_W\right)^2 + \left[\left(\tfrac{2}{3}\sin^2\theta_W\right)^2 + \left(\tfrac{1}{3}\sin^2\theta_W\right)^2\right](1-y)^2 \Big\}$$

$$+ x f_{\bar{q}}(x) \Big\{ \left[\left(\tfrac{2}{3}\sin^2\theta_W\right)^2 + \left(\tfrac{1}{3}\sin^2\theta_W\right)^2\right] + \left[\left(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W\right)^2 + \left(\tfrac{1}{2} - \tfrac{1}{3}\sin^2\theta_W\right)^2\right](1-y)^2 \Big\} \Big]$$

$$= \frac{G_F^2 s}{\pi}\Big[x f_q(x) \left[\left(\tfrac{1}{2} - \sin^2\theta_W + \tfrac{5}{9}\sin^4\theta_W\right) + \tfrac{5}{9}\sin^4\theta_W (1-y)^2\right]$$

$$+ x f_{\bar{q}}(x) \left[\tfrac{5}{9}\sin^4\theta_W (1-y)^2 + \left(\tfrac{1}{2} - \sin^2\theta_W + \tfrac{5}{9}\sin^4\theta_W\right)(1-y)^2\right] \Big]$$

$$\frac{d^2\sigma}{dx\,dy}(\bar{\nu} A \to \bar{\nu} X) = \frac{G_F^2 s}{\pi}\Big[x f_q(x) \left[\left(\tfrac{1}{2} - \sin^2\theta_W + \tfrac{5}{9}\sin^4\theta_W\right)(1-y)^2 + \tfrac{5}{9}\sin^4\theta_W\right]$$

$$+ x f_{\bar{q}}(x) \left[\tfrac{5}{9}\sin^4\theta_W (1-y)^2 + \left(\tfrac{1}{2} - \sin^2\theta_W + \tfrac{5}{9}\sin^4\theta_W\right)\right] \Big]$$

$$\frac{d^2\sigma}{dx\,dy}(\nu A \to \mu^- X) = \frac{G_F^2 s}{\pi}\left[x f_q(x) + (1-y)^2 x f_{\bar{q}}(x)\right]$$

$$\frac{d^2\sigma}{dx\,dy}(\bar{\nu} A \to \mu^+ X) = \frac{G_F^2 s}{\pi}\left[x f_q(x)(1-y)^2 + x f_{\bar{q}}(x)\right]$$

$$r = \frac{\frac{d\sigma}{dxdy}(\bar{\nu} A \to \mu^+ X)}{\frac{d\sigma}{dxdy}(\nu A \to \mu^- X)} = \frac{x f_q(x)(1-y)^2 + x f_{\bar{q}}(x)}{x f_q(x) + (1-y)^2 x f_{\bar{q}}(x)}$$

then

$$R^{\nu} = \frac{\frac{d^2\sigma}{dxdy}(\nu A \to \nu X)}{\frac{d^2\sigma}{dxdy}(\nu A \to \mu^- X)} = \left[\frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4\right] + \frac{5}{9}s_w^4 \left[\frac{x f_q(x)(1-y)^2 + x f_{\bar{q}}(x)}{x f_q(x) + x f_{\bar{q}}(x)(1-y)^2}\right]$$

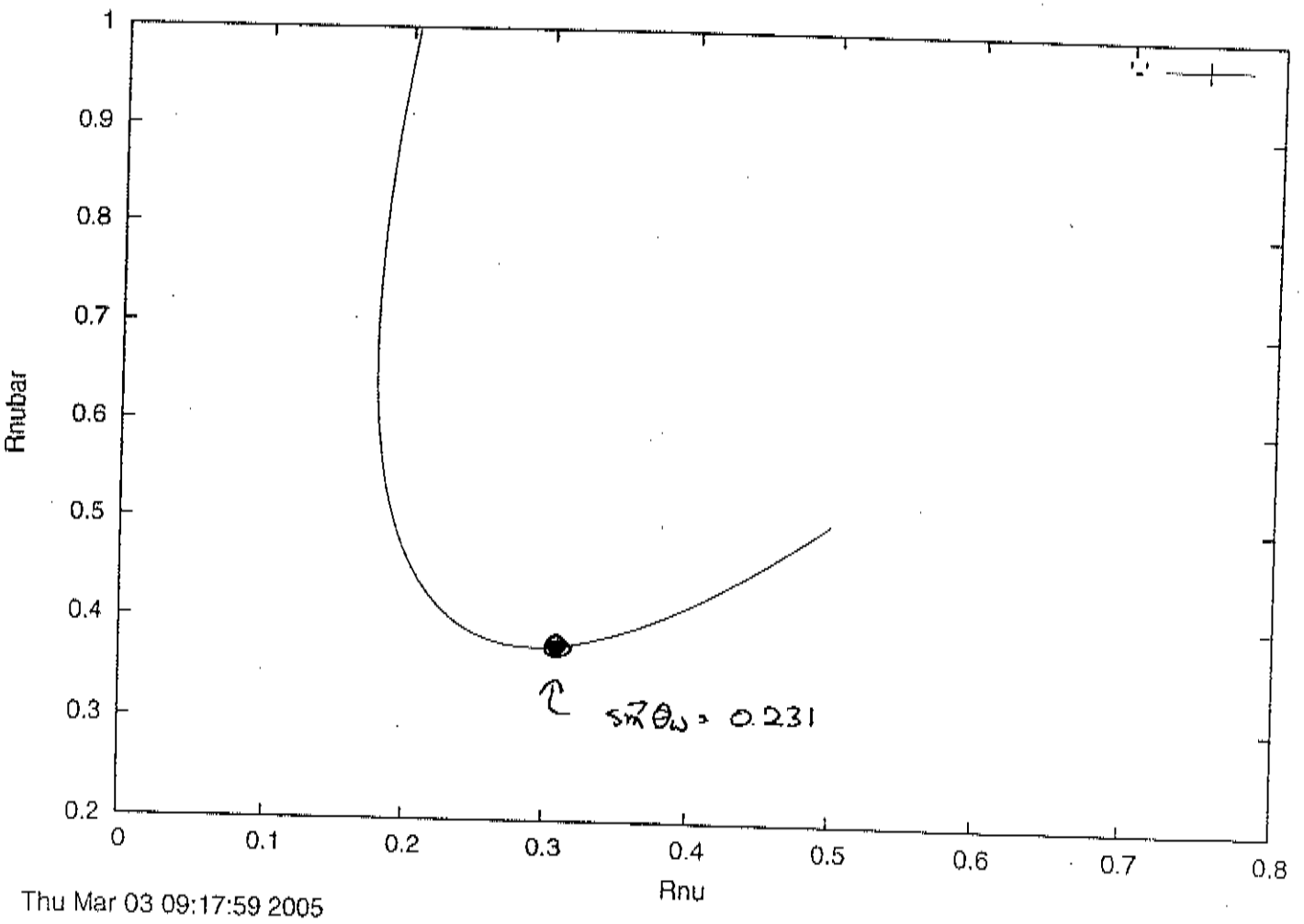
$$= \frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4 + \frac{5}{9}s_w^4 \, r$$

$$R^{\bar{\nu}} = \frac{\frac{d^2\sigma}{dxdy}(\bar{\nu} A \to \bar{\nu} X)}{\frac{d^2\sigma}{dxdy}(\bar{\nu} A \to \mu^+ X)} = \frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4 + \frac{5}{9}s_w^4 \left[\frac{x f_q(x) + x f_{\bar{q}}(x)(1-y)^2}{x f_q(x)(1-y)^2 + x f_{\bar{q}}(x)}\right]$$

$$= \frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4 + \frac{5}{9}s_w^4 \, \frac{1}{r}$$

c.) A sketch of the curve R^{ν} vs. $R^{\bar{\nu}}$ is given on the next page

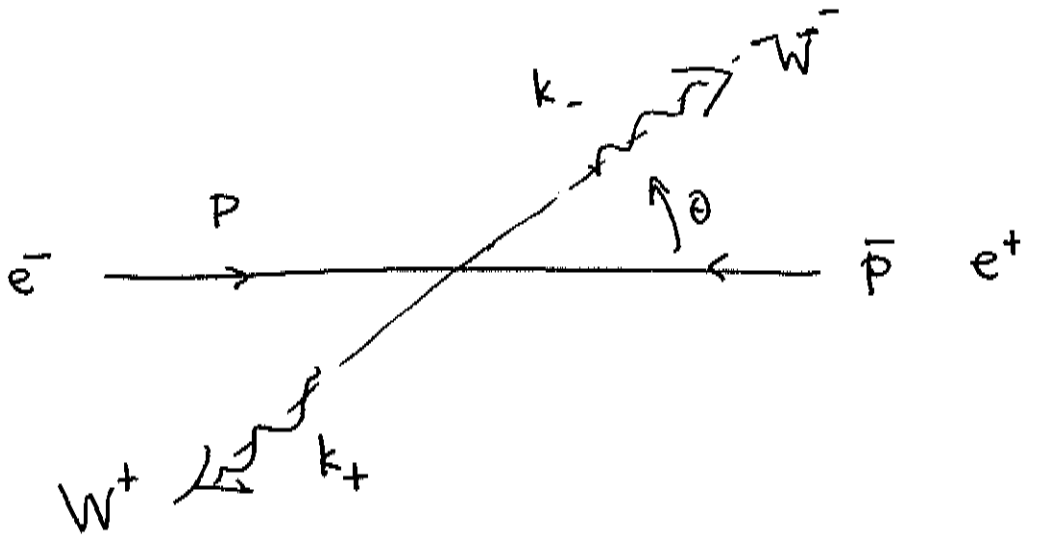



Rnubar
Rnu
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
$\sin^2\theta_W = 0.231$
Thu Mar 03 09:17:59 2005

3.) For a longitudinally polarized W boson moving in the $\hat{3}$ direction

$$\varepsilon_0^\mu = \frac{1}{m_W}(k, 0, 0, E)$$

setting $e^- e^+ \to W^- W^+$ as follows:



$p = (E, 0, 0, E)$ $\qquad$ $\bar{p} = (E, 0, 0, -E)$

$k_- = (E, ks, 0, kc)$ $\qquad$ $k_+ = (E, -ks, 0, -kc)$

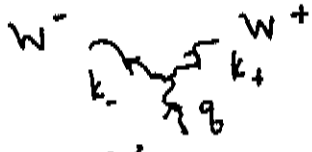
$s = \sin\theta$ $\qquad$ $c = \cos\theta$

$\varepsilon_0^\mu(-) = \frac{1}{m_W}(k, Es, 0, Ec)$ $\qquad$ $\varepsilon_0^\mu(+) = \frac{1}{m_W}(k, -Es, 0, -Ec)$

spinors: $u_R(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $\qquad$ $v_L(\bar{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$

$u_L(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\qquad$ $v_R(\bar{p}) = \sqrt{2E}\begin{pmatrix}-1\\0\\0\\0\end{pmatrix}$

We need:



$$= ie\left[\varepsilon^{*}_{(-)}\varepsilon^{*}(+)\,(k_- - k_+)^{\lambda} + \varepsilon^{\lambda *}_{(-)}\,(-q-k_-)\cdot\varepsilon^{*}(+) + \varepsilon^{\lambda}(+)\,(q+k_+)\cdot\varepsilon^{*}(-)\right]$$

$$= ie\left[\varepsilon^{*}_{(-)}\cdot\varepsilon^{*}(+)\,(k_- - k_+)^{\lambda} - 2\,k_-\cdot\varepsilon^{*}(+)\,\varepsilon^{\lambda *}_{(-)} + 2\,k_+\cdot\varepsilon^{*}_{(-)}\,\varepsilon^{\lambda}(+)\right]$$

since $k_-\cdot\varepsilon^{*}_{(-)} = 0 = k_+\cdot\varepsilon^{*}(+)$

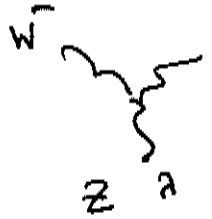
$$= ie\left[\frac{E^2+k^2}{m_W^2}(k_- - k_+)^{\lambda} - 2\,\frac{2Ek}{m_W^2}\,\varepsilon^{\lambda *}_{(-)} + 2\,\frac{2Ek}{m_W^2}\,\varepsilon^{\lambda *}(+)\right]$$

$$= ie\,\frac{1}{m_W^2}\,(0,s,0,c)^{\lambda}\cdot\left[2k\,(E^2+k^2) - 4\,Ek\cdot E\cdot 2\right]$$

$$= ie\,\frac{1}{m_W^2}\,(0\;s\;0\;c)^{\lambda}\left(-\frac{3E^2+k^2}{m_W^2}\right)\cdot 2k$$

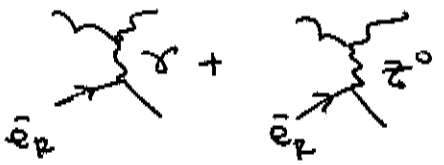
$$= -ie\,\frac{1}{m_W^2}\,(0\;s\;0\;c)^{\lambda}\cdot 2k\cdot\frac{2E^2+m_W^2}{m_W^2}$$

similarly,



$$= -ie\,\frac{c_W}{s_W}\,\frac{1}{m_W^2}\,(0,s,0,c)^{\lambda}\,2k\,\frac{2E^2+m_W^2}{m_W^2}$$

then



$$= (ie)\,\bar{v}\gamma_\lambda u\left[(-1)\frac{-i}{s}(-ie) + \frac{s_w^2}{s_w c_w}\frac{-i}{s-m_Z^2}\frac{-iec_w}{s_w}\right]$$

$$\cdot\,(0,s,0,c)^\lambda \cdot 2k\,\frac{2E^2+m_W^2}{m_W^2}$$

$$= ie^2\,\bar{v}\gamma_\lambda u\,(0,s,0,c)^\lambda \cdot 2k \cdot \frac{2E^2+m_W^2}{m_W^2}$$

$$\cdot\left[\frac{1}{s} - \frac{1}{s-m_Z^2}\right]$$

$$= ie^2\,\underline{\bar{v}\gamma_\lambda u\,(0,s,0,c)^\lambda}\;2k\;\frac{2E^2+m_W^2}{m_W^2}\left(\frac{-m_Z^2}{s(s-m_Z^2)}\right)$$

this spinor element is

$$\bar{v}\gamma_\lambda u\,(0\,s\,0\,c)^\lambda = 2E\,(0\;-1)\,\sigma_\lambda\,(0\,s\,0\,c)^\lambda\begin{pmatrix}1\\0\end{pmatrix}$$

$$= 2E\,(0\;-1)\begin{pmatrix}-c & -s\\ -s & c\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= 2Es$$

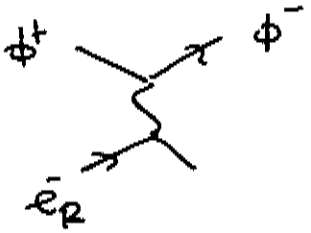
$$= ie^2\,(2E\sin\theta)\,2k\;\frac{2E^2+m_W^2}{m_W^2}\left[\frac{-m_Z^2}{s(s-m_Z^2)}\right]$$

If $s \gg m_W^2 \; m_Z^2$ $\qquad 2E^2 \approx \frac{s}{2}$

$$\mathcal{M}(\bar{e}_R e_L^+ \to W_0^- W_0^+) \longrightarrow -ie^2 (4Ek \sin\theta) \cdot \frac{s}{2} \frac{m_Z^2}{m_W^2} \frac{1}{s^2}$$

$$\longrightarrow -i \frac{e^2}{2c_W^2} \, 4Ek \sin\theta \, \frac{1}{s}$$

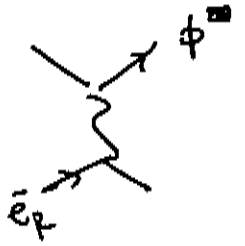
in QED



$$= (-ie)^2 \bar{v} \gamma_\lambda u \, (k_- - k_+)^\lambda \, \frac{-i}{s}$$

$$= ie^2 \; 2E \cdot 2k \sin\theta \; \frac{1}{s}$$

So in unbroken SU(2)×U(1) gauge theory



$$= ig'^2 (4Ek \sin\theta) (-1) (-\tfrac{1}{2}) \; \frac{1}{s}$$
$$Y_{e_R} \; Y_\phi$$

$$= +i \frac{e^2}{2c_W^2} (4Ek \sin\theta) \; \frac{1}{s}$$

in agreement with the result above.

Now consider $e_L^- e_R^+ \to W^- W^+$

to compute the γ and Z diagrams, we need the spinor element

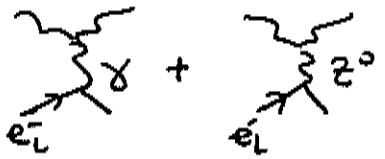
$$\bar{v}\gamma_\beta u \quad (0\,s,0,c)^\lambda = (\sqrt{2E})^2 (-1\,0)\,\bar{s}\cdot(0\,s\,0\,c)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 2E\,(-1\,0)\begin{pmatrix}c & s\\ s & -c\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -2E\sin\theta$$

we also need to put in the Z^0 charge of the e_L: $\left(\frac{-\frac{1}{2}+s_w^2}{s_w c_w}\right)$

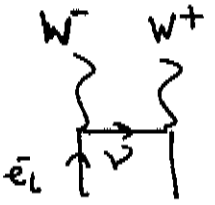
then

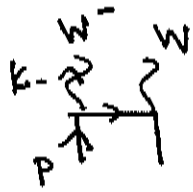


$$= ie^2\,(-2E\sin\theta)\;2k\;\frac{2E^2+m_W^2}{m_W^2}\left[\frac{1}{s} - \frac{(-\frac{1}{2}+s_w^2)}{s_w c_w}\frac{c_w}{s_w}\frac{1}{s-m_Z^2}\right]$$

$$= -ie^2\,(4Ek\sin\theta)\left(\frac{2E^2+m_W^2}{m_W^2}\right)\left[\frac{1}{2s_w^2}\frac{1}{s-m_Z^2} - \frac{m_Z^2}{s(s-m_Z^2)}\right]$$

to this, we must add the diagram:





$$= \left(i\frac{g}{\sqrt{2}}\right)^2 \bar{u}(\bar{p})\gamma\cdot\varepsilon^*(+)\ \frac{i\,\gamma\cdot(p-k_-)}{(p-k_-)^2}\ \gamma\cdot\varepsilon^*(-)\ u(p)$$

$$= -i\frac{e^2}{2s_W^2}\sqrt{2E}\,(-1\ 0)\ \bar{\sigma}\cdot\varepsilon^*(+)\ \frac{\sigma\cdot(p-k_-)}{m_W^2-2p\cdot k_-}\ \bar{\sigma}\cdot\varepsilon^*(-)\ \begin{pmatrix}0\\1\end{pmatrix}\sqrt{2E}$$

$$= -i\frac{e^2}{2s_W^2}\ \frac{2E}{m_W^2}\ \frac{1}{[m_W^2-2E(E-kc)]}$$

$$\cdot\quad (-1\ 0)\begin{pmatrix} k-Ec & -Es \\ -Es & +(k+Ec)\end{pmatrix}\begin{pmatrix} kc-E & ks \\ ks & E-kc\end{pmatrix}\begin{pmatrix} k+Ec & Es \\ Es & k-Ec\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

the 2nd line is:

$$(Ec-k \quad Es)\begin{pmatrix} kc-E & ks \\ ks & E-kc\end{pmatrix}\begin{pmatrix} Es \\ k-Ec\end{pmatrix}$$

$$= (Ec-k)(kc-E)\,Es \quad + Es(E-kc)(k-Ec)$$

$$+ \quad Es\cdot ks\cdot Es \quad + (Ec-k)\,ks\,(k-Ec)$$

$$= \quad 2\,Es\,(k-Ec)(E-kc) \ + \ ks\,(k-Ec)(-k+Ec)$$

$$+ \quad E^2ks^3$$

$$= \quad 2E^2ks \ - \ 2E^3sc \ - \ 2Ek^2sc \ + 2E^2ksc^2$$

$$- k^3s \ + \ 2Ek^2sc \ - \ E^2ksc^2$$

$$+ \ E^2ks - E^2ksc^2$$

$$= \quad (3E^2-k^2)\,ks \ - \ 2E^3sc$$

so



$$= +i\frac{e^2}{2s_w^2}\frac{2E}{m_w^2}(\sin\theta)\left(\frac{[(3E^2-k^2)k - 2E^3c]}{[2E^2 - m_w^2 - 2Ekc]}\right)$$

try to make the last factor look as much like 1 as possible:

$$(\quad) = \frac{E}{k}\frac{(3Ek^2 - k^4/E - 2E^2kc)}{(E^2+k^2-2Ekc)}$$

$$= \frac{E}{k}\frac{1}{(E^2+k^2-2Ekc)}\cdot\left[(E^2+k^2-2Ekc) + [-E^2+2k^2-k^4/E^2]\right]$$

$$= \frac{E}{k}\frac{1}{(E^2+k^2-2Ekc)}\left[(E^2+k^2-2Ekc) + \frac{(E^2-k^2)^2}{E^2}\right]$$

$$= \frac{E}{k} + \frac{1}{(E^2+k^2-2Ekc)}\frac{m_w^4}{E^2}$$

so



$$= +i\frac{e^2}{2s_w^2}\frac{2E}{m_w^2}\sin\theta\,\frac{E}{k}\cdot\left(1+\theta\left(\frac{m_w^4}{E^4}\right)\right)$$

now we can combine the pieces!



$$= -ie^2 (4Ek\sin\theta)\left(\frac{s}{2m_W^2}+1\right)\left[\frac{1}{2s_W^2}\frac{1}{s-m_Z^2} - \frac{m_Z^2}{s(s-m_Z^2)}\right]$$

$$+ ie^2\, 4Ek\sin\theta\ \frac{E^2}{2k^2}\frac{1}{m_W^2}\left(1+\mathcal{O}\left(\frac{m_W^4}{E^4}\right)\right)$$

Now take the limit $E, k \gg m_W$ $\qquad \frac{E^2}{k^2} = 1 + \frac{m_W^2}{E^2} + \dots$

$$= \left[-ie^2\, 4Ek\sin\theta\right]\cdot\left\{-\frac{1}{2c_W^2}\frac{1}{s}\right.$$

$$+ \frac{1}{2s_W^2}\left\{\cancel{\frac{1}{2m_W^2}} + \frac{m_Z^2}{2sm_W^2} + \frac{1}{s}\right.$$

$$\left.\left. \cancel{-\frac{1}{2m_W^2}} = \frac{1}{2m_W^2}\frac{4m_W^2}{s}\right\} + \mathcal{O}\left(\frac{m_W^2}{s}\right)\right\}$$

$$= -ie^2\cdot(4Ek\sin\theta)$$

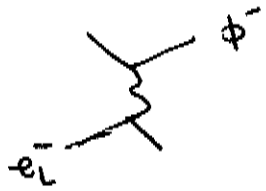
$$\cdot\left\{-\frac{1}{2c_W^2}\frac{1}{s} + \frac{1}{\cancel{2}c_W^2 s_W^2}\frac{1}{s} - \frac{1}{2s_W^2}\frac{1}{s}\right\}$$

$$= +ie^2 (4Ek\sin\theta) \cdot \frac{1}{s}$$

$$\cdot \left\{ \frac{1}{2c_w^2} + \frac{1}{2s_w^2} - \frac{1}{4}\left(\frac{1}{c_w^2} + \frac{1}{s_w^2}\right) \right\}$$

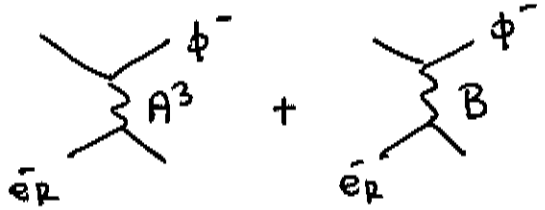
$$= ie^2 (4Ek\sin\theta) \cdot \frac{1}{s} \cdot \left(\frac{1}{4c_w^2} + \frac{1}{4s_w^2} \right)$$

in QED



$$= -ie^2 \; 4Ek\sin\theta \cdot \frac{1}{s}$$

in unbroken SU(2) x U(1) gauge theory



$$= (-ie^2 \; 4Ek\sin\theta \; \frac{1}{s})$$

$$\left(g = \frac{e}{s_w},\ g' = \frac{e}{c_w}\right) \quad \cdot \left\{ \frac{1}{s_w^2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + \frac{1}{c_w^2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) \right\}$$

$I^3_{e_L^-}$ $I^3_{\phi^-}$ $Y_{e_L^-}$ Y_{ϕ^-}

in agreement with the result above!

4.) The cross section for $q_f \bar{q}_f \to \mu^+\mu^-$ is

$$\frac{d\sigma}{d\cos\Theta_{cm}} = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{\pi\alpha^2}{2s}$$

$$\cdot\Big\{ (1+\cos\Theta)^2 \left| Q_f + \frac{(\xi - s_w^2)(I_f^3 - s_w^2 Q_f)}{s_w^2 c_w^2}\frac{s}{s-m_z^2}\right|^2$$

$$+ (1-\cos\Theta)^2 \left| Q_f - \frac{s_w^2 (I_f^3 - s_w^2 Q_f)}{s_w^2 c_w^2}\frac{s}{s-m_z^2}\right|^2$$

$$+ (1-\cos\Theta)^2 \left| Q_f - \frac{(\xi - s_w^2)(s_w^2 Q_f)}{s_w^2 c_w^2}\frac{s}{s-m_z^2}\right|^2$$

$$+ (1+\cos\Theta)^2 \left| Q_f + \frac{s_w^2 \cdot s_w^2 Q_f}{s_w^2 c_w^2}\frac{s}{s-m_z^2}\right|^2 \Big\}$$

We need to implement the cross sect. in ppcollider:

$$\sigma = \int dx_1\, dx_2\, d\cos\Theta$$

$$\sum_{f=udscb} \left[f_f(x_1) f_f(x_2) + f_{\bar{f}}(x_1) f_{\bar{f}}(x_2)\right] \frac{d\sigma}{d\cos\Theta}(q_f \bar{q}_f \to \mu^+\mu^-)$$

In the program drellyan.cpp posted on the course Web site, I implent this. I use

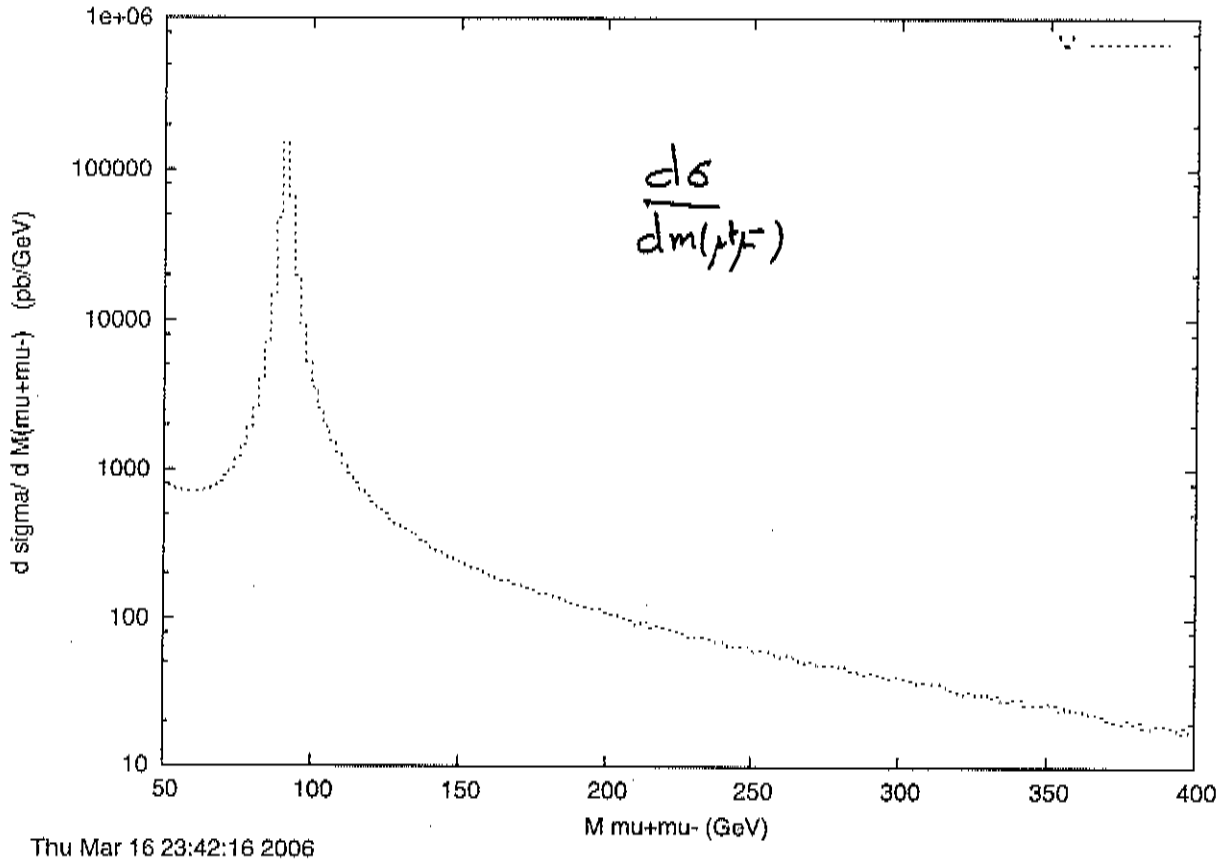
$$(1+\cos\Theta)^2 = 4\frac{u^2}{s^2} \qquad (1-\cos\Theta)^2 = 4\frac{t^2}{s^2}$$

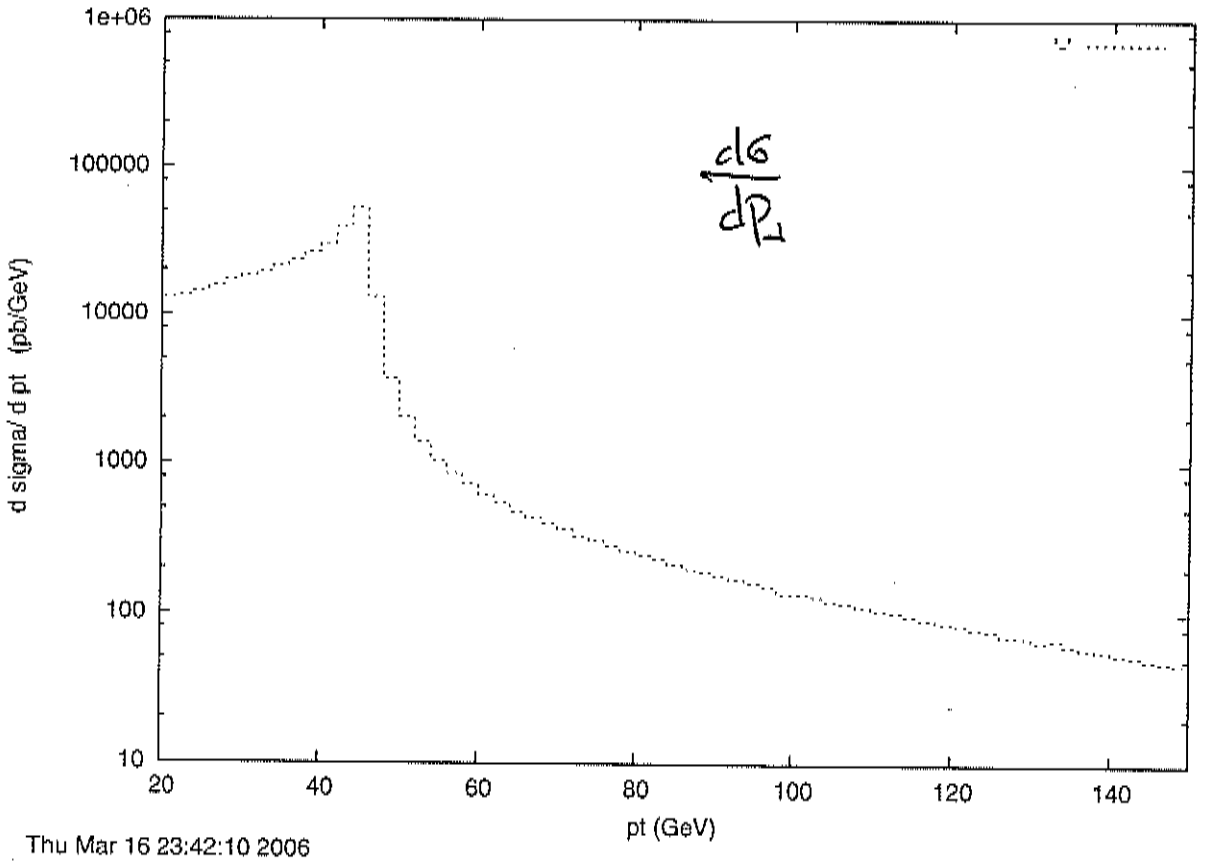
drellyan makes a plot of $d\sigma/dm(\mu^+\mu^-)$ and $d\sigma/dp_\perp$. The Jacobian peak from

$$q\bar{q} \to Z^0 \to \mu^+\mu^-$$

is clearly seen in the latter plot.

Here are the plots ⟶


1e+06
100000
10000
1000
100
10
d sigma/ d M(mu+mu-) (pb/GeV)
50
100
150
200
250
300
350
400
M mu+mu- (GeV)
dσ/dm(μ+μ-)
Thu Mar 16 23:42:16 2006


1e+06
100000
10000
1000
100
10
d sigma/ d pt (pb/GeV)
20
40
60
80
100
120
140
pt (GeV)
dσ/dP_T
Thu Mar 16 23:42:10 2006

Physics 331 – Final Exam

This exam is due at noon on Wednesday, March 16. Please hand it in to Alex Saltman in Varian 363. If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Alex Saltman has blue books, if you need one.

The exam is worth a total of 100 points. The distribution of points is indicated below.

I recommend that you try simply to answer the questions using methods from Physics 331. However, the exam is open-book. If you find a useful reference other than the class textbook and notes, feel free to use it, as long as you cite the reference in your solution.

Consider an $SU(3) \times SU(2) \times U(1)$ gauge theory. Fermions and scalars in this theory are in gauge representations (R, I, Y), where R is the $SU(3)$ representation, I is the $SU(2)$ representation, and Y is the $U(1)$ charge. Consider a Higgs boson boson Σ in the represention $(3, 2, -\frac{1}{2})$. This can be written as a 3×2 matrix of complex fields. The covariant derivative on this multiplet is

$$D_\mu \Sigma = \partial_\mu \Sigma - i g_3 \mathcal{A}_\mu^A t^A \Sigma + i g_2 \mathcal{B}_\mu^a \Sigma \tau^a - i g_1 \frac{1}{2} \mathcal{C}_\mu \Sigma \tag{1}$$

where t^A are the usual $SU(3)$ generators and τ^a are the usual $SU(2)$ generators. Note that the theory has three independent coupling constants g_1, g_2, g_3.

a. (10 points) Show that

$$\langle \Sigma \rangle = V \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

spontaneously breaks $SU(3) \times SU(2) \times U(1)$ to $SU(2) \times U(1)$. Write the unbroken gauge group generators explicitly and show that they annihilate $\langle \Sigma \rangle$.

b. (20 points) Work out the mass matrix for the vectors bosons due to the Higgs mechanism. Write explicitly the linear combinations of the gauge bosons in (1) that are mass eigenstates. Find the quantum numbers of each under the unbroken $SU(2) \times U(1)$. Compute the masses of the massive bosons.

c. (10 points) Introduce quarks and leptons into the model. In the Standard Model, quarks and leptons belong to representations (I, Y) of $SU(2) \times U(1)$. For example, the left-handed quark doublet Q_L is a $(2, \frac{1}{6})$. In this model, we can put quarks and leptons into the corresponding representations $(1, I, Y)$, e.g., $(1, 2, \frac{1}{6})$ for Q_L. Write the covariant derivative acting on quark and lepton multiplets in terms of the mass

eigenstate boson fields. Note that the couplings of the massless $SU(2) \times U(1)$ gauge bosons are functions of g_1, g_2, g_3. Show that, if $g_1, g_2 \gg g_3$, the model predicts $\sin^2 \theta_w = \frac{1}{4}$ at the scale $Q = M_\Sigma$ of the spontaneous breaking.

d. (15 points) Using the equations for the running of the $SU(2)$ and $U(1)$ couplings, compute the $SU(2)$ and $U(1)$ gauge couplings at values of Q lower than M. If

$$\frac{g^2}{4\pi} = \frac{1}{29.6} \qquad \frac{g'^2}{4\pi} = \frac{1}{98.5} \tag{3}$$

at $Q = m_Z$—and still assuming $g_1, g_2 \gg g_3$—find g_3 and M_Σ. For simplicity, treat the top quark as approximately massless in this analysis.

e. (15 points) How many of the heavy gauge bosons are singly produced in the Drell-Yan process $q\overline{q} \to V$? For each of these, write the parton model expression for the total cross section

$$\sigma(pp \to V + X) \tag{4}$$

f. (10 points) Write the parton model formula for the neutrino deep inelastic scatting cross section

$$\frac{d^2\sigma}{dxdy}(\nu p \to \mu^- + X) \tag{5}$$

in this model. How accurately do we have to measure this cross section to be sensitive to the effects of the exchanges of the heavy vector bosons?

g. (20 points) The heavy charged boson $\mathcal{W}^+$ can decay either to a pair of fermions or to a pair of gauge bosons. List the possible decay products in each case, and compute the partial widths in terms of g_1, g_2, g_3. Which set of decays is more important?

Physics 331 - Final Exam

Solutions

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a.) The unbroken symmetries are symmetry transformations that leave $\langle \Sigma \rangle$ invariant : $\delta \langle \Sigma \rangle = 0$

The $SU(3) \times SU(2) \times U(1)$ symmetry acts on $\langle \Sigma \rangle$ as:

$SU(3)$ $\quad \delta \langle \Sigma \rangle = t^A \langle \Sigma \rangle$

$SU(2)$ $\quad \delta \langle \Sigma \rangle = \langle \Sigma \rangle (-\tau^a)$

$U(1)$ $\quad \delta \langle \Sigma \rangle = -\frac{1}{2} \cdot 1 \cdot \langle \Sigma \rangle$

The following combination leave $\langle \Sigma \rangle$ invariant:

$$\delta_a \langle \Sigma \rangle = t^a \langle \Sigma \rangle - \langle \Sigma \rangle \tau^a$$

where $a = 1,2,3$ $\quad t^a = \begin{pmatrix} \frac{\tau^a}{2} & \\ & 0 \end{pmatrix}$

$$\delta_1 \langle \Sigma \rangle = \sqrt{3}\, t^8 \langle \Sigma \rangle - \tfrac{1}{2} \cdot 1 \cdot \langle \Sigma \rangle$$

where $t^8 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ so $\mathrm{tr}\,(t^8)^2 = \frac{1}{2}$

This corresponds to an unbroken $SU(2) \times U(1)$ symmetry. If y is the original $U(1)$, the final $U(1)$ is

$$Y = y + \sqrt{3}\, T^8$$

b.) The kinetic energy term for Σ is $D_\mu \Sigma^\dagger D^\mu \Sigma$

If we write $D_\mu \Sigma = \partial_\mu \Sigma - i\,[A\Sigma]$, the mass term for gauge fields is

$$\frac{1}{2} m_{AB}^2 A_\mu^A A^{\mu B} = \text{tr}\,([A\Sigma])^\dagger [A\Sigma]$$

evaluate with $\Sigma \to \langle\Sigma\rangle = v \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$[A\Sigma] = g_3 A_\mu^A t^A \Sigma - g_2 \Sigma \tau^a B_\mu^a - \tfrac{1}{2} g_1 C_\mu \Sigma$$

$$\text{tr}\,[A\Sigma]^\dagger [A\Sigma]\Big|_{\Sigma \to \langle\Sigma\rangle}$$

$$= \text{tr}\,[(\, g_3 A_\mu^A t^A \langle\Sigma\rangle - g_2 \langle\Sigma\rangle \tau^a B_\mu^a - \tfrac{1}{2} g_1 C_\mu \langle\Sigma\rangle\,)$$
$$(\, g_3 \langle\Sigma\rangle^\dagger t^B A^{\mu B} - g_2 B^{\mu b} \tau^a \langle\Sigma\rangle^\dagger - \tfrac{1}{2} g_1 C_\mu \langle\Sigma\rangle^\dagger\,]$$

$$= \text{tr}\Big\{ g_3^2 A_\mu^A A^{\mu B} \langle\Sigma\rangle\langle\Sigma\rangle^\dagger t^B t^A$$
$$+ g_2^2 B_\mu^a B^{\mu b} \langle\Sigma\rangle^\dagger \langle\Sigma\rangle \tau^a \tau^b + g_1^2 \tfrac{1}{4} C_\mu C^\mu \langle\Sigma\rangle^\dagger \langle\Sigma\rangle$$
$$+ (-g_3 g_2) A_\mu^A B^{\mu a} \underbrace{(t^A \langle\Sigma\rangle \tau^a \langle\Sigma\rangle^\dagger + \langle\Sigma\rangle \tau^a \langle\Sigma\rangle^\dagger t^A)}_{\text{equal!}}$$
$$+ (-g_3 g_1) A_\mu^A C^\mu \,(t^A \langle\Sigma\rangle\langle\Sigma\rangle^\dagger \cdot \tfrac{1}{2} \cdot 2)$$
$$+ g_1 g_2 B_\mu^a C^\mu \,\langle\Sigma\rangle \tau^a \langle\Sigma\rangle^\dagger \cdot \tfrac{1}{2} \cdot 2 \Big\}$$

Let's first see which of the cross terms are nonzero

$$\mathrm{tr}\,\langle\Sigma\rangle\,\tau^a\langle\Sigma\rangle^\dagger = \mathrm{tr}\,\langle\Sigma\rangle^\dagger\langle\Sigma\rangle\tau^a = \mathrm{tr}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\tau^a = 0$$

$$\mathrm{tr}\;\; t^A\langle\Sigma\rangle\langle\Sigma\rangle^\dagger = \mathrm{tr}\; t^A \left(\begin{array}{cc|c}1&0&0\\0&1&0\\\hline 0&0&0\end{array}\right)$$

of the 8 SU(3) generators, this is nonzero only for $t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}$

$$= \delta^{A,8}\,\frac{1}{\sqrt{3}}$$

$$\mathrm{tr}\left(t^A\langle\Sigma\rangle\tau^a\langle\Sigma\rangle^\dagger\right) = \mathrm{tr}\; t^A\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\tau^a\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)$$

if $t^A = \left(\begin{array}{cc|c}\tau^b&&0\\&&0\\\hline 0&0&0\end{array}\right)$ this is $\mathrm{tr}\;\tau^b\cdot 1\cdot\tau^a\cdot 1 = \delta^{ab}\cdot\frac{1}{2}$

$t^A = \left(\begin{array}{cc|c}&&\beta\\&&\\\hline \beta^\dagger&&\end{array}\right)$ $\quad \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\left(\begin{array}{cc|c}&&\beta\\&&\\\hline \beta^\dagger&&\end{array}\right)\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix} = 0$

$t^A = t^8 = \mathrm{tr}\,\frac{1}{2\sqrt{3}}\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\tau^a\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$

$$= \frac{1}{2\sqrt{3}}\,\mathrm{tr}\;1\cdot 1\cdot\tau^a\cdot 1 = 0$$

So t^A mixes with C for $A=8$

mixes with B^a for $A=1,2,3$.

Now for the diagonal terms:

$$\mathrm{tr}\,\langle\Sigma\rangle^\dagger\langle\Sigma\rangle = \mathrm{tr}\begin{pmatrix}1&0\\0&1\end{pmatrix} = 2$$

$$\mathrm{tr}\,\langle\Sigma\rangle^\dagger\langle\Sigma\rangle\,\tau^a\tau^b = \mathrm{tr}\begin{pmatrix}1&0\\0&1\end{pmatrix}\tau^a\tau^b = \frac{1}{2}\delta^{ab}$$

$$\mathrm{tr}\,\langle\Sigma\rangle\langle\Sigma\rangle^\dagger\, t^A t^B = \mathrm{tr}\left(\begin{array}{cc|c}1&0&0\\0&1&0\\\hline 0&0&0\end{array}\right)t^A t^B$$

for $A,B = a,b = 1,2,3$

$$= \mathrm{tr}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\left(\tfrac{\tau^a}{2}\right)\left(\tfrac{\tau^b}{2}\right) = \tfrac{1}{2}\delta^{ab}$$

$A = B = 8$

$$= \mathrm{tr}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\tfrac{1}{12}\left(\begin{smallmatrix}1&&\\&1&\\&&-4\end{smallmatrix}\right) = \frac{1}{6}$$

$$t^A = \tfrac{1}{2}\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&0\end{smallmatrix}\right) = t^B \qquad A = B = 4$$

$$= \mathrm{tr}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\tfrac{1}{4}\left(\begin{smallmatrix}1&0&0\\0&-1&0\\0&0&0\end{smallmatrix}\right) = \frac{1}{4}$$

$$t^5 = \tfrac{1}{2}\left(\begin{smallmatrix}0&0&-i\\0&0&0\\i&0&0\end{smallmatrix}\right) \qquad t^6 = \tfrac{1}{2}\left(\begin{smallmatrix}0&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right) \qquad t^7 = \tfrac{1}{2}\left(\begin{smallmatrix}0&0&0\\0&0&-i\\0&i&0\end{smallmatrix}\right)$$ give the same answer.

So

$$\tfrac{1}{2} m_{AB} A^A A^B = g_3^2 \Big\{ \tfrac{1}{2} A^a_\mu A^{\mu a} + \tfrac{1}{6} A^8_\mu A^{\mu 8} + \tfrac{1}{4} A^i_\mu A^{\mu i} \Big\} V^2 \qquad \begin{matrix} a = 1,2,3 \\ i = 4,5,6,7 \end{matrix}$$

$$+ g_2^2 \Big\{ \tfrac{1}{2} B^a_\mu B^{\mu a} \Big\} V^2$$

$$+ g_1^2 \Big\{ \tfrac{1}{2} C_\mu C^\mu \Big\} V^2$$

$$- 2 g_2 g_3 \left(\tfrac{1}{2} A^a_\mu B^{\mu a}\right) V^2$$

$$- 2 g_1 g_3 \left(\tfrac{1}{2\sqrt{3}} A^8_\mu C^\mu\right) V^2$$

$$= \tfrac{1}{2}\left(g_1 C^\mu - \tfrac{g_3}{\sqrt{3}} A^8_\mu\right)^2 V^2 + \tfrac{1}{2}\left(g_2 B^a_\mu - g_3 A^a_\mu\right)^2$$

$$+ \tfrac{1}{2} \cdot \tfrac{g_3^2}{2} (A^i_\mu)^2 V^2$$

so the massive vector bosons are:

$$Z_\mu = \frac{g_1 C^\mu - g_3 A_\mu^8/\sqrt{3}}{[g_1^2 + g_3^2/3]^{1/2}}$$

$$W_\mu^a = \frac{g_2 B_\mu^a - g_3 A_\mu^a}{[g_2^2 + g_3^2]^{1/2}}$$

$$X_\mu^i = A_\mu^i$$

with masses

$$m^2(Z) = (g_1^2 + g_3^2/3)\, v^2$$

$$m^2(W) = (g_2^2 + g_3^2)\, v^2$$

$$m^2(X) = \frac{g_3^2}{2} v^2$$

the orthogonal vector boson eigenstates are massless:

$$B_\mu = \frac{g_3/\sqrt{3}\, C^\mu + g_1 A_\mu^8}{[g_1^2 + g_3^2/3]^{1/2}}$$

$$A_\mu^a = \frac{g_3 B^{\mu a} + g_2 A^{\mu a}}{[g_2^2 + g_3^2]^{1/2}}$$

It is clear that W_μ^a A_μ^a are SU(2) triplets, Z_μ, B_μ are SU(2) singlets, X_μ^i are two SU(2) 2's.

B_μ^a and C_μ have $Y = 0$ $T^8 = 0$.

A_μ^A have $Y = 0$, and T^8 acts on them as $[t^8, t^A]$

Then $Y = Y + \sqrt{3}\, T^8 = 0 + \frac{1}{2}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & -1 \end{pmatrix}$

this commutes with A_μ^a $A_\mu^8 \rightarrow Y = 0$

the χ_μ^i form pairs with $Y = \pm 3/2$

so under the unbroken SU(2)xU(1):

B_μ	(1, 0)	A_μ^a	(3, 0)
Z_μ	(1, 0)	W_μ^a	(3, 0)
χ_μ^i	(2, 3/2) + (2, −3/2)		

c.) The covariant derivative on quark and lepton fields is

$$D_\mu = \partial_\mu - i g_2 \vec{B}_\mu^a \tau^a - i g_1 C_\mu Y$$

$$= \partial_\mu - i g_2 \left(\frac{g_3 A_\mu^a + g_2 W_\mu^a}{[g_3^2 + g_2^2]^{1/2}} \right) \tau^a - i g_1 \left(\frac{\frac{g_3}{\sqrt{3}} B_\mu + g_1 Z_\mu}{[g_3^2/3 + g_1^2]^{1/2}} \right) Y$$

$$= \partial_\mu - i \left[\frac{g_2 g_3}{[g_2^2 + g_3^2]^{1/2}} \right] A_\mu^a \tau^a - i \left[\frac{g_1 g_3/\sqrt{3}}{[g_3^2 + g_3^2/3]^{1/2}} \right] B_\mu Y$$

$$- i \left(\frac{g_2^2}{[g_2^2 + g_3^2]^{1/2}} \right) W_\mu^a \tau^a - i \left[\frac{g_1^2}{[g_3^2/3 + g_1^2]^{1/2}} \right] Z_\mu Y$$

The second line of this expression contains heavy bosons. The first line contains the $SU(2) \times U(1)$ bosons that survive to lower energies. The $SU(2) \times U(1)$ couplings are:

$$g = \frac{g_2 g_3}{[g_2^2 + g_3^2]^{1/2}} \qquad g' = \frac{g_1 g_3/\sqrt{3}}{[g_1^2 + g_3^2/3]^{1/2}}$$

If $g_1, g_2 \gg g_3$

$$g \approx g_3 \qquad g' \approx g_3/\sqrt{3}$$

$$\sin^2\theta_W = \frac{g'^2}{g^2 + g'^2} \cong \frac{1}{4}$$

d.) Evolve this result to energies Q below the scale of spontaneous breaking of the $SU(2)$ and $U(1)$ symmetry

$$\alpha^{-1}(Q) = \alpha^{-1}(M) + \frac{b}{2\pi} \ln \frac{Q}{M}$$

$$b = \frac{11}{3} C_2(G) - \frac{2}{3} \sum_{\text{dirac fermion}} C(r) - \frac{1}{3} \sum_{\substack{\text{complex} \\ \text{bosons}}} C(r)$$

for $SU(2)$:

$C_2(G) = 2$ $\qquad C(r) = \frac{1}{2}$ for one doublet;

we need to sum over $\quad 3 \times \left\{ \begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \right\} + 1 \cdot \left\{ \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \right\} +$ Higgs.

$$b(SU(2)) \equiv b_2 = \frac{22}{3} - \frac{2}{3}\cdot\frac{1}{2}\cdot(3+1)\cdot 3 - \frac{1}{3}\cdot\frac{1}{2}$$

$$= \frac{22}{3} - 4 - \frac{1}{6} = \frac{19}{6}$$

$$b(U(1))_Y \equiv b_1 = -\frac{2}{3}\cdot\{\mathrm{tr}\, Y^2\} - \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2\cdot 2$$

$$\mathrm{tr}\, Y^2 = \left[3\cdot 2\cdot\left(\frac{1}{6}\right)^2 + 3\left(-\frac{1}{3}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 2\cdot\left(-\frac{1}{2}\right)^2 + 1^2\right]\cdot 3 \quad \leftarrow \text{3 generations}$$

$$= \frac{10}{3}\cdot 3 = 10$$

$$b_1 = -\frac{20}{3} - \frac{1}{6} = -\frac{41}{6}$$

So

$$\alpha_\omega = \frac{g^2}{4\pi} \qquad \alpha'_\omega = \frac{g'^2}{4\pi}$$

$$\alpha_\omega^{-1}(m_z) = \alpha_\omega^{-1}(M) + \frac{19}{12\pi}\ln\frac{m_z}{M}$$

$$\alpha_\omega'^{-1}(m_z) = \alpha_\omega'^{-1}(M) - \frac{41}{12\pi}\ln\frac{m_z}{M}$$

Now, at M

$$\alpha_\omega \equiv \frac{g_3^2}{4\pi} \equiv \alpha_3 \qquad \alpha'_\omega \equiv \frac{g_3^2}{12\pi} \equiv \alpha_3/3$$

So $$\alpha_W^{-1}(m_z) = \alpha_3^{-1}(M) - \frac{19}{12\pi} \ln \frac{M}{m_z}$$

$$\alpha_W'^{-1}(m_z) = 3\alpha_3^{-1}(M) + \frac{41}{12\pi} \ln \frac{M}{m_z}$$

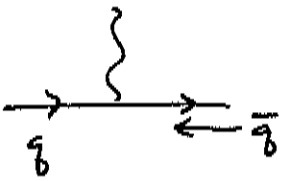
$$-3\alpha_W^{-1}(m_z) + \alpha_W'^{-1}(m_z) = + \frac{98}{12\pi} \ln \frac{M}{m_z}$$

usg $\alpha_W^{(M)} = \frac{1}{29.6}$ $\alpha_W'(m_z) = \frac{1}{98.5}$

$$\ln \frac{M}{m_z} = 3.7 \qquad M = 42 \cdot m_z \cong 3.8 \text{ TeV}$$

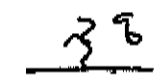
$$\alpha_3 = \frac{g_3^2}{4\pi} = \frac{1}{31.4} \qquad (g_3 \cong 0.63)$$

e.) The heavy vector bosons W_μ^a and Z_μ appear in the covariant derivative on quark fields. So they couple to $q\bar{q}$ and can be produced in the Drell-Yan process.



If the vertex is $\Gamma_\mu = i G \gamma^\mu \left(\frac{1-\gamma^5}{2}\right)$,

we can compute the cross-section:



$$= iM = i\,G\,\bar{v}(\bar{p})\,\gamma^\mu\left(\tfrac{1-\gamma_5}{2}\right)u(p)\cdot\varepsilon^{k}_{\mu}(q)$$

$$\tfrac{1}{4}\sum_{\text{spins}}|M|^2 = \tfrac{1}{4}\,G^2\,\mathrm{tr}\left[\not{\bar{p}}\,\gamma^\mu\left(\tfrac{1-\gamma_5}{2}\right)\not{p}\,\gamma^\nu\left(\tfrac{1-\gamma_5}{2}\right)\right]\varepsilon_\mu(q)\,\varepsilon^{*}_{\nu}(q)$$

$$= \tfrac{1}{4}\,G^2\,\mathrm{tr}\left[\not{\bar{p}}\,\gamma^\mu\not{p}\,\gamma^\nu\left(\tfrac{1-\gamma_5}{2}\right)\right]\left[-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_V^2}\right]$$

$$= \tfrac{1}{4}\,G^2\cdot 2\left[\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu - g^{\mu\nu}\,p\cdot\bar{p}\right]\left[-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_V^2}\right]$$

$$= \tfrac{1}{4}\,G^2\cdot 2\cdot(4-2)\,p\cdot\bar{p}$$

[note that $q^\mu\left[\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu - g^{\mu\nu}\,p\cdot\bar{p}\right] = \left[q\cdot\bar{p}\;p^\nu + q\cdot p\;\bar{p}^\nu - q^\nu\,p\cdot\bar{p}\right]$

$q = p+\bar{p}$ $= \left[p\cdot\bar{p}\;p^\nu + \bar{p}\cdot p\;\bar{p}^\nu - (p+\bar{p})^\nu\,p\cdot\bar{p}\right] = 0\;.$]

$q^2 = m_V^2 = 2p\cdot\bar{p}$

$$= G^2\,p\cdot\bar{p} = \tfrac{1}{2}\,G^2 m_V^2$$

$$\sigma(q+\bar{q}\to V) = \frac{1}{2s}\cdot\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_{\bar{p}}}(2\pi)^4\delta(p+\bar{p}-q)\left(\tfrac{1}{4}\sum|M|^2\right)$$

$$\cdot\left[\tfrac{1}{3}\cdot\tfrac{1}{3}\cdot\sum_{\text{colors}}1\right] \leftarrow \text{color factor } \tfrac{1}{3}$$

$$= \frac{1}{2m_V^2}\cdot\int\frac{d^4q}{(2\pi)^4}\,2\pi\,\delta(q^2-m_V^2)\,(2\pi)^4\delta(p+\bar{p}-q)\cdot\tfrac{1}{2}\,G^2 m_V^2\cdot\tfrac{1}{3}$$

$$= \frac{\pi}{6}\,G^2\,\delta(s-m_V^2)$$

again:

$$\sigma(q\bar{q} \to V) = \frac{\pi}{6} G^2 \delta(s - m_V^2) = \frac{2\pi^2}{3} \frac{G^2}{4\pi} \delta(s - m_V^2)$$

for W_μ^3

$u_L \bar{u}_R \to W_\mu^3$ has $G = \frac{1}{2} g_W$

$d_L \bar{d}_R \to W_\mu^3$ has $G = -\frac{1}{2} g_W$

for W_μ^+ $u_L \bar{d}_R \to W_\mu^+$ has $G = \frac{g_W}{\sqrt{2}}$

for W_μ^- $d_L \bar{u}_R \to W_\mu^-$ has $G = \frac{g_W}{2}$

where, from p. 6, $g_W = \frac{g_2^2}{[g_2^2 + g_3^2]^{\frac{1}{2}}} \approx g_2$

for Z_μ

$u_L \bar{u}_L \to Z_\mu$ has $G = \frac{1}{6} g_Z$

$u_R \bar{u}_R \to Z_\mu$ has $G = \frac{2}{3} g_Z$

these are independent processes which add in σ

$d_L \bar{d}_L \to Z$ has $G = \frac{1}{6} g_Z$

$d_R \bar{d}_R \to Z$ has $G = -\frac{1}{3} g_Z$

where, from p. 6 $g_Z = \frac{g_1^2}{[g_1^2 + g_3^2/3]^{\frac{1}{2}}} \approx g_1$

so! $\alpha_W = \frac{g_2^2}{4\pi}\frac{g_2^2}{[g_2^2+g_3^2]}$ $\alpha_Z = \frac{g_1^2}{4\pi}\frac{g_1^2}{[g_1^2+g_3^2/3]}$

$\sigma(pp \to W^3 + X)$

$$= \int dx_1\,dx_2 \left\{ f_u(x_1) f_{\bar{u}}(x_2) + f_{\bar{u}}(x_1) f_u(x_2) + f_d(x_1) f_{\bar{d}}(x_2) + f_{\bar{d}}(x_1) f_d(x_2) \right\} \cdot \frac{\pi^2}{6} \alpha_W \delta(x_1 x_2 s - m_W^2)$$

$\sigma(pp \to W^+ + X)$

$$= \int dx_1\,dx_2 \left[f_u(x_1) f_{\bar{d}}(x_2) + f_{\bar{d}}(x_1) f_u(x_2) \right] \cdot \frac{\pi^2}{3} \alpha_W \delta(x_1 x_2 s - m_W^2)$$

$\sigma(pp \to W^- + X)$

$$= \int dx_1\,dx_2 \left[f_d(x_1) f_{\bar{u}}(x_2) + f_{\bar{u}}(x_1) f_d(x_2) \right] \frac{\pi^2}{3} \alpha_W \delta(x_1 x_2 s - m_W^2)$$

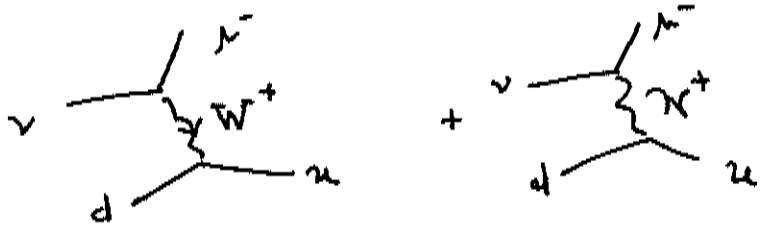
$\sigma(pp \to Z + X)$

$$= \int dx_1\,dx_2 \left\{ \left[f_u(x_1) f_{\bar{u}}(x_2) + f_{\bar{u}}(x_1) f_u(x_2) \right] \cdot \frac{17}{36} + \left[f_d(x_1) f_{\bar{d}}(x_2) + f_{\bar{d}}(x_1) f_d(x_2) \right] \cdot \frac{5}{36} \right\} \cdot \frac{2\pi^2 \alpha_Z}{3} \delta(x_1 x_2 s - m_Z^2)$$

f.) Looking back again at p. 6, we have for the W^+ and W^+ coupling to u, d: $W^+ = \frac{A^1 - iA^2}{\sqrt{2}}$ is the usual W.

$$\delta\mathcal{L} = \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu d_L + \frac{g_W}{\sqrt{2}} \mathcal{W}_\mu^+ \bar{u}_L \gamma^\mu d_L$$

and similarly for (ν_μ, μ). Adding $g_W = \frac{q^2}{[q_1^2 + q_3^2]^{1/2}}$ from p. 11



we find the usual expression for ν deep inelastic scatt

with
$$\frac{g^2}{2m_W^2} \longrightarrow \frac{g^2}{2m_W^2} + \frac{g_W^2}{2m_{\mathcal{W}}^2}$$

$$= \frac{g^2}{2m_W^2}\left[1 + \frac{g_W^2}{g^2}\frac{m_W^2}{m_{\mathcal{W}}^2}\right]$$

so

$$\frac{d^2\sigma}{dx\,dy}(\nu p \to \mu^- + X) = \frac{G_F^2 S}{\pi}\left[x f_d(x) + x(1-y)^2 f_{\bar{u}}(x)\right] \cdot \left[1 + \frac{g_W^2}{g^2}\frac{m_W^2}{m_{\mathcal{W}}^2}\right]^2$$

the factor in the 2nd line is

$$\approx 1 + 2\frac{g_W^2}{g^2}\frac{m_W^2}{m_{\mathcal{W}}^2}$$

where the correction term is

$$2\,\frac{g_W^2}{g^2}\,\frac{m_W^2}{m_W^2} \;\cong\; 2\,\frac{g_2^2}{g^2}\cdot\frac{m_W^2}{m_W^2}$$

$$\sim\; 2\cdot\frac{g_2^2/4\pi}{\alpha_W}\cdot\left(\frac{80\ \text{GeV}}{4000\ \text{GeV}}\right)^2$$

If $g_2^2/4\pi \sim O(1)$

$$\sim\quad 2\% \cdot \left(\frac{g_2^2}{4\pi}\right)$$

so if g_2 is very large, we might be sensitive to this effect if we can measure $\sigma(\nu p \to \mu^- X)$ to the 1% level.
[Unfortunately, uncertainties from the strong interactions are also at this level.]

g.) The decay of the W^+ to $u\bar{d}$ + other quark or lepton pairs is easy to work out from the couplings on p. 13



$$= \; i\,\frac{g_W}{\sqrt{2}}\,\bar{u}_L(p)\,\gamma_\mu\,v_L(\bar{p})\,\varepsilon_\mu(q)$$

In fact, this is just a cross of the matrix element computed on p. 10

$$\sum_{\text{spins}} |M|^2 \;=\; 2\,G^2 m_W^2 \cdot 2\left(\frac{g_W}{\sqrt{2}}\right)^2 m_W^2$$

The spin average is now over the spin state of the W, & the color factor is $\underline{3}$, so

$$\frac{1}{3}\sum_{\substack{\text{spin,}\\ \text{color}}} |\mathcal{M}|^2 = \underbrace{3}_{\text{for quarks}} \cdot \frac{1}{3} \cdot g_W^2 m_W^2$$

then

$$\Gamma(W^+ \to l^+ \nu) = \frac{1}{2m_W}\frac{1}{8\pi} \cdot \frac{1}{3} g_W^2 m_W^2$$

$$= \frac{\alpha_W}{12} m_W$$

$$\Gamma(W^+ \to u\bar{d}) = 3 \cdot \frac{\alpha_W}{12} m_W \qquad \underline{\alpha_W \text{ on p.12}}$$

cf this should be summed over 3 generations (including $W \to t\bar{b}$)

To work out $W^+ \to V_1 V_2$, we need the 3-vector-boson vertex. The terms in W^+ come from both the SU(2) & SU(3) 3-vector terms.

The SU(2) terms come from

$$-\frac{1}{4}(F^a_{\mu\nu})^2 \to -g_2 \epsilon^{abc} \partial_\mu B^a_\nu B^{\mu b} B^{\nu c}$$

$$= -g_2 \{ \partial_\mu B^1_\nu (B^{\mu 2} B^{\nu 3} - B^{\nu 2} B^{\mu 3}) + \partial_\mu B^2_\nu (B^{\mu 3} B^{\nu 1} - B^{\nu 3} B^{\mu 1})$$

$$+ \partial_\mu B^3_\nu (B^{\mu 1} B^{\nu 2} - B^{\nu 1} B^{\mu 2}) \}$$

now write $B_\mu^+ = \frac{1}{\sqrt{2}}(B_\mu^1 - i B_\mu^2)$ $\quad B_\mu^- = \frac{1}{\sqrt{2}}(B_\mu^1 + i B_\mu^2)$

or $\quad B_\mu^1 = \frac{1}{\sqrt{2}}(B_\mu^+ + B_\mu^-)$ $\quad B_\mu^2 = \frac{i}{\sqrt{2}}(B_\mu^+ - B_\mu^-)$

$$= -g_2 \frac{i}{2} \Big\{ \partial_\mu (B_\nu^+ + B_\nu^-)\left[(B^{\mu+} - B^{\mu-}) B^{\nu 3} - (B^{\nu+} - B^{\nu-}) B^{\mu 3} \right]$$
$$+ \partial_\mu (B_\nu^+ - B_\nu^-)\left[B^{\mu 3}(B^{\nu+} + B^{\nu-}) - B^{\nu 3}(B^{\mu+} + B^{\mu-}) \right]$$
$$+ \partial_\mu B_\nu^3 \left[(B^{\mu+} + B^{\mu-})(B^{\nu+} - B^{\nu-}) - (B^{\mu+} - B^{\mu-})(B^{\nu+} + B^{\nu-}) \right] \Big\}$$

$$= -\frac{ig_2}{2} \cdot \Big\{ \partial_\mu B_\nu^+ \left[-2 B^{\mu-} B^{\nu 3} + 2 B^{\nu-} B^{\mu 3} \right]$$
$$+ \partial_\mu B_\nu^- \left[2 B^{\mu+} B^{\nu 3} - 2 B^{\nu+} B^{\mu 3} \right]$$
$$+ \partial_\mu B_\nu^3 \left[2 B^{\mu-} B^{\nu+} - 2 B^{\mu+} B^{\nu-} \right] \Big\}$$

$$= -ig_2 \Big\{ \partial_\mu B_\nu^+ \left[B^{\mu 3} B^{\nu-} - B^{\mu-} B^{\nu 3} \right]$$
$$+ \partial_\mu B_\nu^- \left[B^{\mu+} B^{\nu 3} - B^{\mu 3} B^{\nu+} \right]$$
$$+ \partial_\mu B_\nu^3 \left[B^{\mu-} B^{\nu+} - B^{\mu+} B^{\nu-} \right] \Big\}$$

We need a W^+ field to annihilate the W^+, so let

$$B^{\mu+} \to \frac{g_2}{[g_2^2 + g_3^2]^{1/2}} W^{\mu+}$$

then

$$B^{\mu-} \to \frac{g_3}{[g_2^2 + g_3^2]^{1/2}} W^{\mu-} \quad \text{(creates } W^+\text{)} \qquad B^{\mu 3} \to \frac{g_3 A^3 + g_2 W^3}{[g_2^2 + g_3^2]^{1/2}}$$

It is possible that $m(W^3) < m(W^+)$ but in any event these state are almost degenerate so the the phase space for

$$W^+ \to W^3 + W^+$$

will be very small. I will ignore this decay from here on.

on the other hand, $A^3 = c_\omega Z + s_\omega A$ $\qquad s_\omega = \sin\theta_\omega = \frac{g'}{\sqrt{g^2+g'^2}}$

so!

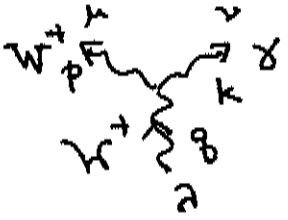
$$= -i\, g_2 \frac{g_2}{[g_2^2+g_2^2]^{1/2}} \cdot \frac{g_3}{[g_2^2+g_3^2]^{1/2}} \cdot \frac{g_3}{[g_3^2+g_3^2]^{1/2}} \cdot s_\omega \cdot$$

$$\cdot \{ \partial_\nu W_\nu^+ [A^\mu W^{\nu-} - W^{\mu-} A^\nu] + \partial_\nu W_\nu^- [W^{\mu+} A^\nu - A^\mu W^{\nu+}]$$

$$+ \partial_\nu A_\nu [W^{\mu-} W^{\nu+} - W^{\mu+} W^{\nu-}] \}$$

$$+ (s_\omega A \to c_\omega Z)$$

this gives the vertex:



$$= + \frac{g_2^2 g_3^2}{[g_2^2+g_3^2]^{3/2}} s_\omega$$

$$\cdot \{ -iq^\nu g^{\mu\lambda} + iq^\mu g^{\nu\lambda} + ip^\lambda g^{\mu\nu} - ip^\nu g^{\mu\lambda}$$

$$+ ik^\mu g^{\nu\lambda} - ik^\lambda g^{\mu\nu} \}$$

$$= +i \frac{g_2^2 g_3^2}{[g_2^2+g_3^2]^{3/2}} s_\omega \left[g^{\mu\nu}(p-k)^\lambda + g^{\nu\lambda}(q+k)^\mu + g^{\lambda\mu}(-q-p)^\nu \right]$$

and similarly for Z with c_W. There is also an $SU(3)$ contribution to the vertex

$$-\tfrac{1}{4}(F^A_{\mu\nu})^2 \to -g_3 f^{ABC}\,\partial_\mu A^A_\nu A^{\mu B} A^{\nu C}$$

For $A,B,C = 1,2,3$ ($t^A = \left(\frac{\sigma^A}{2}\right)$) $f^{ABC} = \varepsilon^{ABC}$ and we set

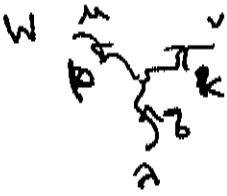
$$= i g_3 \{\partial_\mu A^+ [A^{\mu 3} A^{\nu -} - \cdots]$$

now

$$A^{\mu +} \to \frac{-g_3}{[g_2^2+g_3^2]^{1/2}} W^{\mu +}$$

$$A^{\mu -} \to \frac{g_2}{[g_2^2+g_3^2]^{1/2}} W^{\mu -} \qquad A^{\mu 3} \to \frac{g_2}{[g_2^2+g_3^2]^{1/2}} A^3 = \frac{g_2}{[g_2^2+g_3^2]^{1/2}}(c_W Z + s_W A)$$

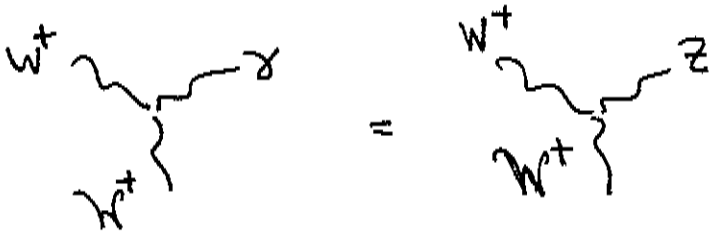
so the $SU(3)$ vertex is



$$= +g_3 \left[\frac{-g_3}{[g_2^2+g_3^2]^{1/2}}\right]\left[\frac{g_2}{[g_2^2+g_3^2]^{1/2}}\right]\left[\frac{g_2}{[g_2^2+g_3^2]^{1/2}}\right] \cdot s_W \cdot \{\}$$

$$= +i\frac{(-g_2^2 g_3^2)}{[g_2^2+g_3^2]^{1/2}} s_W \left[g^{\mu\nu}(p-k)^\lambda + \cdots\right]$$

and exactly cancels the result on p. 17. In all



$= 0$ at tree level.

Why is this? Since the vertex is $\propto f^{abc}$, the $W^+ A^3$ vertex can only involve generators ~ the same subspace — cf this is the unbroken SU(2) subspace of the gauge group.

However, the SU(3) vertex also allows

$$W^+ \to X^{+2}\,\bar{X}$$

Let's work this out. The matrices

$\left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right)$ have $T^3 = +\frac{1}{2}$ $\quad Y = \frac{3}{2}$ $\Rightarrow$ $Q = 2$

$\left(\begin{array}{c|c} - & 0 \\ \hline - & 1 \end{array}\right)$ $\quad T^3 = -\frac{1}{2}$ $\quad Y = \frac{3}{2}$ $\Rightarrow$ $Q = 1$

$\left(\begin{array}{c|c} - & - \\ \hline 1 & 0 \end{array}\right)$ $\quad T^3 = -\frac{1}{2}$ $\quad Y = -\frac{3}{2}$ $\Rightarrow$ $Q = -2$

$\left(\begin{array}{c|c} - & - \\ \hline 0 & 1 \end{array}\right)$ $\quad T^3 = +\frac{1}{2}$ $\quad Y = -\frac{3}{2}$ $\Rightarrow$ $Q = -1$

so write the covariant derivative ≅ SU(3) as $\hat{W}^+_\mu \equiv A^{3+}_\mu = \frac{-g_3}{\sqrt{2}} W^+$

$$D_\mu = \partial_\mu - i\frac{g_3}{\sqrt{2}}\left[\hat{W}^+_\mu \left(\begin{array}{c|c} 0^+ & - \\ \hline - & - \end{array}\right) + X^{+2}_\mu \left(\begin{array}{c|c} - & 1^0 \\ \hline - & - \end{array}\right) + X^{-2}_\mu \left(\begin{array}{c|c} - & - \\ \hline 1^0 & - \end{array}\right)\right.$$

$$\left. + X^+_\mu \left(\begin{array}{c|c} - & 0^0 \\ \hline - & 1 \end{array}\right) + X^-_\mu \left(\begin{array}{c|c} - & - \\ \hline 0^1 & - \end{array}\right)\right]$$

$$-ig_3 F^A_{\mu\nu} t^A = [D_\mu, D_\nu]$$

$$= -ig_3\left[(\partial_\mu A^A_\nu - \partial_\nu A^A_\mu) t^A - ig_3 [A^A_\mu t^A, A^B_\nu t^B]\right]$$

$$= -ig_3 \left\{ \frac{1}{\sqrt{2}} (\partial_\mu \hat{W}_\nu^+ - \partial_\nu \hat{W}_\mu^+) \left(\frac{\sigma^+}{\quad}\right) + \frac{1}{\sqrt{2}} (\partial_\mu X_\nu^{-2} - \partial_\nu X_\mu^{-2}) \left(\frac{}{10}\right) + \dots \right.$$

$$-\frac{ig_3}{2} \Big[(\hat{W}_\mu^+ X_\nu^+ - \hat{W}_\nu^+ X_\mu^+) \left(\frac{}{\quad}\right)$$

$$+ (\hat{W}_\mu^+ X_\nu^{-2} - \hat{W}_\nu^+ X_\mu^{-2}) \left(\frac{}{0-1}\right)$$

$$+ (X_\mu^{-2} X_\nu^+ - X_\nu^{-2} X_\mu^+) \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & \\ 0 & 0 & \end{array}\right) + \dots$$

$$-\frac{1}{4} (F_{\mu\nu}^A)^2 = -\frac{1}{2} \mathrm{tr} (F_{\mu\nu}^A t^A)^2$$

$$= -\mathrm{tr} \left[(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A) t^A \right] \left[A^{\mu B} t^B, A^{\nu C} t^C \right]$$

$$= -\frac{1}{\sqrt{2}} \cdot \left(\frac{-ig_3}{2}\right) \Big\{ (\partial_\mu \hat{W}_\nu^+ - \partial_\nu W_\mu^+)(X_\mu^{-2} X_\nu^+ - X_\nu^{-2} X_\mu^+) \cdot (-1)$$

$$+ (\partial_\mu X_\nu^{-2} - \partial_\nu X_\mu^{-2}) (\hat{W}_\mu^+ X_\nu^+ - X_\mu^+ \hat{W}_\nu^+) \cdot 1$$

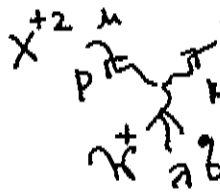
$$+ (\partial_\mu X_\nu^+ - \partial_\nu X_\mu^+) (\hat{W}_\mu^+ X_\nu^{-2} - \hat{W}_\nu^+ X_\mu^{-2}) \cdot (-1) \Big\}$$

$$= -i \frac{g_3}{2\sqrt{2}} \Big\{ (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(X^{\mu -2} X^{\nu +} - X^{\mu +} X^{\nu -2})$$

$$+ (\partial_\mu X_\nu^{-2} - \partial_\nu X_\mu^{-2})(X^{\mu +} W^{\nu +} - W^{\mu +} X^{\nu +})$$

$$+ (\partial_\mu X_\nu^+ - \partial_\nu X_\mu^+)(W^{\mu +} X^{\nu -2} - X^{\mu -2} W^{\nu +}) \Big\}$$

compare to p. 17, this gives:



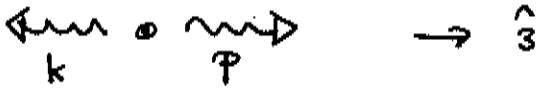
$$= \frac{-ig_3}{2\sqrt{2}} \left(\frac{-g_3}{[q_3^2 + S_3^2]^{1/2}} \right) \left[g^{\mu\nu} (p-k)^\lambda + g^{\mu\lambda} (q+k)^\nu + g^{\nu\lambda} (k-p)^\mu \right]$$

Now evaluate this as a matrix element between on-shell vector boson states:

$$iM = \frac{i}{2\sqrt{2}}\frac{g_3^2}{[q_2^2+\zeta_3^2]^{\frac{1}{2}}}\Big\{ \varepsilon^*(p)\cdot\varepsilon^*(k)\ (p-k)\cdot\varepsilon(q) + 2q\cdot\varepsilon^*(p)\ \varepsilon^*(k)\cdot\varepsilon(q) - 2q\cdot\varepsilon^*(k)\ \varepsilon^*(p)\cdot\varepsilon(q)\Big\}$$

$q\cdot\varepsilon^*(p) = k\cdot\varepsilon^*(k) = 0$

we can evaluate this for



If we average over the three polarization states of $\varepsilon(q)$

So write $\varepsilon^\mu(q) = (0, \hat{n})$ $\qquad q^\mu = (m_W, \vec{0})$

$$\varepsilon^{*\mu}(p) = \frac{1}{\sqrt{2}}(0\ 1-i\ 0)\ R \qquad \frac{1}{\sqrt{2}}(0\ 1+i\ 0)\ L \qquad \frac{1}{m_W}(p,0,0,E)\ 0$$

$$\varepsilon^{*\mu}(k) = \frac{1}{\sqrt{2}}(0\ -1,-i,0)\ R \qquad \varepsilon^{*\mu}(k) = \frac{1}{\sqrt{2}}(0\ -1\ i\ 0)\ L \qquad \frac{1}{m_W}(p,0\ 0\ -E)\ 0$$

$$2q\cdot\varepsilon^*(p) = \begin{cases} 0 & R \\ 0 & L \\ 2\frac{m_W}{m_W}p & 0 \end{cases} \qquad 2q\cdot\varepsilon^*(k) = \begin{cases} 0 & R \\ 0 & L \\ 2\frac{m_W}{m_W}p & 0 \end{cases}$$

$$\varepsilon^*(p)\cdot\varepsilon^*(k) = \begin{cases} 1 & RR \\ 1 & LL \\ \dfrac{p^2+E^2}{m_X^2} & 00 \\ 0 & \text{all other cases.} \end{cases}$$

$$(p-k)\cdot\varepsilon(q) = -2p\,\hat{3}\cdot\hat{n}$$

and if $\quad \vec{e}_+ = \dfrac{\hat{1}+i\hat{2}}{\sqrt{2}} \qquad \vec{e}_- = \dfrac{\hat{1}-i\hat{2}}{\sqrt{2}}$

$$\varepsilon^*(p)\cdot\varepsilon(q) = \begin{cases} -\frac{1}{\sqrt{2}}\,\vec{e}_-\cdot\hat{n} & R \\ -\frac{1}{\sqrt{2}}\,\vec{e}_+\cdot\hat{n} & L \\ -\frac{E}{m_X}\,\hat{3}\cdot\hat{n} & 0 \end{cases} \qquad \varepsilon^*(k)\cdot\varepsilon(q) = \begin{cases} \frac{1}{\sqrt{2}}\,\vec{e}_+\cdot\hat{n} \\ \frac{1}{\sqrt{2}}\,\vec{e}_-\cdot\hat{n} \\ \frac{E}{m_X}\,\hat{3}\cdot\hat{n} \end{cases}$$

with these ingredients:

$$i\mathcal{M}(W\to RR) = \frac{i g_3^2}{\sqrt{8(\zeta_2^2+\zeta_3^2)}}\;1\cdot(-2p\,\hat{3}\cdot\hat{n})$$

$$i\mathcal{M}(W\to RL) = \mathcal{M}(W\to LR) = 0$$

$$i\mathcal{M}(W\to LL) = \frac{i g_3^2}{\sqrt{8(\zeta_2^2+\zeta_3^2)}}\cdot 1\,[-2p(\hat{3}\cdot\hat{n})]$$

$$i\mathcal{M}(W \to R0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \left\{ 0 + 0 - 2\frac{m_W}{m_E} p \cdot \left(-\frac{1}{\sqrt{2}}\vec{e}_- \cdot \hat{n}\right)\right.$$

$$= \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_E} p\, \vec{e}_- \cdot \hat{n}$$

$$i\mathcal{M}(W \to L0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_E} p\, \vec{e}_+ \cdot \hat{n}$$

$$i\mathcal{M}(W \to R0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_E} p\, \vec{e}_- \cdot \hat{n}$$

$$i\mathcal{M}(W \to L0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_E} p\, \vec{e}_+ \cdot \hat{n}$$

Longitudinal

$$i\mathcal{M}(W \to 00) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \left\{ \frac{E^2+p^2}{m_E^2} \cdot (-2p\,\hat{3}\cdot\hat{n}) + \left(2\frac{m_W}{m_E}p\right)\left(\frac{E}{m_E}\hat{3}\cdot\hat{n}\right)\right.$$
$$\left. - \left(2\frac{m_W}{m_E}p\right)\left(-\frac{E}{m_E}\hat{3}\cdot\hat{n}\right)\right\}$$

Now $E = \frac{m_W}{2}$ or $m_W = 2E$ $\qquad p^2 = E^2 - m_X^2$

$$= \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} p\cdot \hat{3}\cdot\hat{n}\, \frac{1}{m_E^2}\left\{ 8E^2 - 2E^2 - 2p^2\right\}$$

$$= \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} p\, \frac{\hat{3}\cdot\hat{n}}{m_E^2}\left\{ 4E^2 + 2m_E^2\right\}$$

in all

$$\sum_{\text{final spin}} |\mathcal{M}(W^+ \to X^{++} X^-)|^2 = \frac{g_3^4}{8(g_2^2+g_3^2)} \cdot \{(2p)^2\}$$

$$\cdot \{ (\hat{3}\cdot\hat{n})^2 \cdot 2 + \left(\frac{m_W}{m_X} \frac{\vec{e}_-\cdot\hat{n}}{\sqrt{2}}\right)^2 \cdot 2 + \left(\frac{m_W}{m_X} \frac{\vec{e}_+\cdot\hat{n}}{\sqrt{2}}\right)^2 \cdot 2$$

$$+ (\hat{3}\cdot\hat{n})^2 \frac{1}{m_X^4} (4m_W^2 + m_X^2)^2 \}$$

now avg over $\hat{n}$

$$\langle(\hat{3}\cdot\hat{n})^2\rangle = \frac{1}{3} = \langle(\hat{1}\cdot\hat{n})^2\rangle = \langle(\hat{2}\cdot\hat{n})^2\rangle$$

$$\langle|\vec{e}_-\cdot\hat{n}|^2\rangle = \frac{1}{2}\left[(\hat{1}\cdot\hat{n})^2 + (\hat{2}\cdot\hat{n})^2\right] = \frac{1}{3} \quad \text{etc.}$$

$$= \frac{1}{3} \frac{g_3^4}{8(g_2^2+g_3^2)} 4p^2 \left\{ 2 + \frac{4}{2}\frac{m_W^2}{m_X^2} + \frac{(m_W^2+m_X^2)^2}{m_X^4} \right\}$$

$$= \frac{g_3^4}{24(g_2^2+g_3^2)} (m_W^2 + 4m_X^2) \left\{ \frac{m_W^4}{m_X^4} + 4\frac{m_W^2}{m_X^2} + 3 \right\}$$

$$= \frac{g_3^4}{24(g_2^2+g_3^2)} \cdot \frac{m_W^6}{m_X^4} \left(1 - \frac{4m_X^2}{m_W^2}\right) \left\{ 1 + 4\frac{m_X^2}{m_W^2} + 3\frac{m_X^4}{m_W^4} \right\}$$

then

$$\Gamma(W^+ \to X^{++} X^-) = \frac{1}{2m_W} \frac{1}{8\pi} \langle \sum |\mathcal{M}|^2 \rangle \left(1 - \frac{4m_X^2}{m_W^2}\right)^{1/2}$$

so

$$\Gamma(W^+ \to X^{+2} X^-) = \frac{\alpha_W}{96} m_W \cdot \frac{g_3^4}{g_2^4} \frac{m_W^4}{m_X^4} \cdot \left(1 - \frac{4m_X^2}{m_W^2}\right)^{3/2}$$

$$\cdot \left(1 + 4\frac{m_X^2}{m_W^2} + 3\frac{m_X^4}{m_W^4}\right)$$

finally (from p. 5)

$$\frac{m_W^2}{m_X^2} = \frac{g_2^2 + g_3^2}{\frac{1}{2} g_3^2}$$

so

$$\Gamma(W^+ \to X^{+2} X^-) = \frac{\alpha_W}{24} m_W \left(1 + \frac{g_3^2}{g_2^2}\right)^2 \left(1 - \frac{4m_X^2}{m_W^2}\right)^{3/2}$$

$$\cdot \left(1 + 4\frac{m_X^2}{m_W^2} + 3\frac{m_X^4}{m_W^4}\right)$$

coming to (p. 15)

$$\Gamma(W^+ \to l^+ \nu) = \frac{\alpha_W}{12} m_W$$

we see that decays to fermions are numerically - but not parametrically - more important if $g_2 \gg g_3$

for more ~~interesting~~ analysis of this model, see

S. Dimopoulos + D.E. Kaplan

Phys. Lett. B 531, 127 (2002) hep-ph/0201148

Phys. Lett. B 534, 124 (2002) hep-ph/0202136

[extra-dimensional version]

Physics 331 – Final Exam

This exam is due at noon on Friday, March 24. Please hand it in to Wu-Yen Chuang (the officemate of Alex Giryavets) in Varian 363. If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Wu-Yen has blue books, if you need one.

The exam is worth a total of 100 points. The distribution of points is indicated below.

It will quickly be apparent that the model described in this exam is not easy to restructure as a true model of Nature. So you will probably not find a good reference to help you do the calculations. However, the exam is open-book. If you find a useful reference other than the class textbook and notes, feel free to use it, as long as you cite the reference in your solution.

Consider an $SU(4)$ gauge theory with fermions in the two-index antisymmetric tensor representation, which is 6 dimensional. It is possible to break the gauge symmetry down to $SU(3)$; then the representation 6 of $SU(4)$ becomes the representation $3 + \overline{3}$ of $SU(3)$. We can identify this with the QCD representation of a quark. The theory contains some new particles, whose properties are explored in this exam.

There is a problem including electroweak interactions in this model. It will become obvious that the heavy vector bosons do not have a consistent assignment of electric charge, and that it is very difficult to assign the quarks the correct weak interaction quantum numbers. Fortunately, this is just an exam, not a serious proposal, so we do not have worry too much about this. Just ignore electroweak interactions and proceed.

For the purpose of this exam, ignore the running of the QCD coupling. When a value of α_s is required, set $\alpha_s = 0.12$.

a. (5 points) Let ϕ be a Higgs field in the fundamental representation of $SU(4)$:

$$V = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \tag{1}$$

Show that the solution

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} \tag{2}$$

minimizes this potential, and find v in terms of μ, λ.

b. (5 points) The Lie algebra of $SU(4)$ has 15 generators. These can be represented as 4×4 traceless Hermitian matrices. Show that 8 generators, corresponding to the Lie algebra of $SU(3)$, annihilate $\langle \phi \rangle$. The other 7 generators correspond to spontaneously

broken symmetries. Show that these can be grouped into a multiplet that transforms as a 3 of $SU(3)$, a multiplet that transforms as a $\overline{3}$ of $SU(3)$, and a singlet of $SU(3)$. Show that the Goldstone bosons transform under $SU(3)$ in the same way, so that when we gauge the $SU(4)$ symmetry, each broken gauge boson can eat a Goldstone boson.

c. (5 points) Couple ϕ to an $SU(4)$ gauge theory

$$\mathcal{L} = |D_\mu \phi|^2 + \cdots \tag{3}$$

and compute the masses obtained by the gauge bosons through the Higgs mechanism in terms of v and the $SU(4)$ gauge coupling g. It is convenient to group the massive bosons into two groups of three C, $\overline{C}$ which transform as $\overline{3}$ and 3 of $SU(3)$, respectively. and one $SU(3)$-singlet boson D. You should find $m(D)/2 < m(C) < m(D)$.

d. (5 points) Now couple the $SU(4)$ gauge theory to two massless left-handed fermions U, D in the 6-dimensional antisymmetric tensor representation of $SU(4)$. The antisymmetric tensor representation is the representation of antisymmetric matrices A_{ij} where each index is in the 4 of $SU(4)$. If ξ_i is a complex vector in the 4, then the transformation law is:

$$\xi_i \to U_{ij}\xi_j \qquad A_{ij} \to U_{ik}U_{jl}A_{kl} \tag{4}$$

(We met antisymmetric tensor representations in $SU(3)$, where we saw that the antisymmetric matrices with two indices in the 3 is a 3-dimensional representation equivalent to the $\overline{3}$.)

Show that each fermion can be considered under the unbroken $SU(3)$ symmetry as a left-handed fermion in the 3 and a left-handed fermion in the $\overline{3}$ of $SU(3)$. That is, three components of the U can be identified with u_L and three components with $\overline{u}_L$, the antiparticle of u_R. Similarly D can be recast as $(d_L, \overline{d}_L)$. We thus have a theory of QCD with 2 massless flavors, with some extra massive gauge bosons and scalars. From here on, ignore the scalars, and ignore all heavier quarks. Show that the $SU(3)$ (QCD) coupling g_s is equal to the original $SU(4)$ coupling g.

e. (10 points) Compute the partial width for a C boson to decay to quark pairs. Note that C decays to qq, not $q\overline{q}$. Compute the partial width for a D boson to decay to $q\overline{q}$ pairs.

f. (5 points) Show that neither the C nor the D boson decays to gluon pairs at the leading order in g_s.

g. (5 points) For the case $m(C) = 700$ GeV, compute the total widths and branching ratios of the C and D bosons.

h. (5 points) Compute the cross sections for C and D bosons to be produced singly in proton-proton collisions—$\sigma(p\overline{p} \to C + X)$, $\sigma(p\overline{p} \to \overline{C} + X)$, $\sigma(p\overline{p} \to D + X)$—in terms of the proton's parton distributions.

i. (10 points) For the case of $m(C) = 700$ GeV, compute the total cross sections numerically. If the Tevatron $p\overline{p}$ collider ($E_{CM} = 2000$ GeV) provides 1 fb^{-1} of luminosity (that is, the luminosity such that a cross section of 1 fb $= 10^{-12}$ mbarns gives 1 event), how many C. $\overline{C}$, and D production events would be expected?

j. (5 points) Make a plot of the distribution of produced C, $\overline{C}$, and D bosons as a function of rapidity for $p\overline{p}$ collisions for the parameters of part (i).

k. (5 points) We might be interested in observing a C or D boson at large transverse momentum. To compute the cross section for this process, we would need to compute the parton-level cross sections for

$$\text{parton} + \text{parton} \rightarrow \text{parton} + D \tag{5}$$

Make a list of the required cross sections, and draw the corresponding Feynman diagrams.

l. (5 points) Repeat this analysis for

$$\text{parton} + \text{parton} \rightarrow \text{parton} + C \tag{6}$$

m. (10 points) Compute the cross sections listed in part (k). The method of computing the helicity amplitudes is probably the most efficient.

n. (10 points) Compute the cross sections listed in part (l).

o. (10 points) Compute numerically the cross section $d\sigma/dp_T$ for C and D boson production for $p_T > 100$ GeV using the parameters of part (i).

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (j). Prospective theorists should slog through to the end.

Physics 331 - Final Exam

Solutions

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a.) $$V = -\mu^2 |\phi|^2 + \lambda (|\phi|^2)^2$$

$$\frac{\partial V}{\partial |\phi|^2} = -\mu^2 + 2\lambda |\phi|^2$$

so V is minimized for any ϕ s.t. $|\phi|^2 = \frac{\mu^2}{2\lambda}$

one such solution is

$$\langle \phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad v = \mu/\sqrt{\lambda}$$

All other solutions are given by

$$\langle \phi \rangle = \frac{v}{\sqrt{2}} \underset{\sim}{S} \qquad \text{where } \underset{\sim}{S} \text{ is a 4-d complex unit vector}$$

$$= U \cdot \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

for some SU(4) transformation U

b.) The 15 generators of SU(4) can be represented by 4×4 traceless Hermitian matrices. A convenient basis is

$$\left(\begin{array}{ccc|c} & t^a & & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad t^a \text{ } 3\times 3 \text{ traceless Hermitian}$$

$$\mathrm{tr}\, t^a t^b = \tfrac{1}{2}\delta^{ab}$$

$$\frac{1}{2}\left(\begin{array}{ccc|c} & & & 1 \\ & & & 0 \\ & & & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right), \quad \frac{1}{2}\left(\begin{array}{ccc|c} & & & 0 \\ & & & 1 \\ & & & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}\right), \quad \frac{1}{2}\left(\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right)$$

$$\frac{1}{2}\left(\begin{array}{ccc|c} & & & -i \\ & & & 0 \\ & & & 0 \\ \hline i & 0 & 0 & 0 \end{array}\right), \quad \frac{1}{2}\left(\begin{array}{ccc|c} & & & 0 \\ & & & -i \\ & & & 0 \\ \hline 0 & i & 0 & 0 \end{array}\right), \quad \frac{1}{2}\left(\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & i \\ \hline 0 & 0 & i & 0 \end{array}\right)$$

$$T_4 = \frac{1}{\sqrt{24}}\left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & 1 & \\ \hline & & & -3 \end{array}\right)$$

$$8 + 3 + 3 + 1 = 15 \checkmark$$

The 8 generators with t^a give $T^A \langle \phi \rangle = 0$

These generate an unbroken SU(3) subgroup of SU(4).

The other four generators are spontaneously broken.

We can describe these as:

$\left(\begin{array}{c|c} 0 & \xi \\ \hline 0\,0\,0 & q \end{array}\right)$ 3 of SU(3)

$\left(\begin{array}{c|c} 0 & 0\,0\,0 \\ \hline \eta & 0 \end{array}\right)$ $\bar{3}$ of SU(3)

T_4 1 of SU(3)

The multiplet ϕ contains 8 fields 4 complex fields, which we can write as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\pi \\ \hline v + h + i\chi \end{pmatrix} \qquad \phi^\dagger = \frac{1}{\sqrt{2}} \left(\sqrt{2}\pi^+,\ v + h - i\chi\right)$$

π is a 3, π^+ is a $\bar{3}$ χ is a 1

of Goldstone bosons.

These 7 fields are eaten by the 7 susy fields which become massive.

c) $D_\mu \phi = \partial_\mu \phi - ig A_\mu^A T^A \phi$

$$\mathcal{L} = |D_\mu \phi|^2 + \dots = g^2 |A_\mu^A T^A \langle \phi \rangle|^2 + \dots$$

this term gives the A_μ^A mass terms.

$A_\mu^A T^A =$ A_μ^a (t^a —|) + $\frac{1}{2} A_\mu^{\dot{a}}$ (—| $\xi^{\dot{a}}$) $\xi^{\dot{a}\dagger}$

+ $\frac{1}{2} \hat{A}_\mu^i$ (—| $-i\xi^j$) $i\xi^j$ + $A_\mu^4 T_4$

$\xi^j = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

write

$$C_\mu^j = \frac{1}{\sqrt{2}} (A_\mu^j + i \hat{A}_\mu^j)$$

$$\bar{C}_\mu^j = \frac{1}{\sqrt{2}} (A_\mu^j - i \hat{A}_\mu^j)$$

then

$A_\mu^A T^A =$ A_μ^a (t^a —|) + $\frac{1}{\sqrt{2}}$ (—| $\bar{C}_m^j$) + $\frac{1}{\sqrt{2}}$ (—| C_m^j)

+ $A_\mu^4 T_4$

from here on, I will write $A_\mu^4 = D_\mu$ (i.e. the D boson)

C_μ is a $\bar{3}$ $\quad$ $\bar{C}_\mu$ is a 3 of SU(3)

$$A_\mu^A T^A \langle\Phi\rangle = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \cdot & \bar{C}_\mu \\ \hline \cdot & \end{array}\right) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+ \frac{1}{\sqrt{24}} D_\mu \left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & -3 & \end{array}\right) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+ \text{(term} = 0)$$

$$= \frac{v}{2} \begin{pmatrix} \bar{C}_\mu^1 \\ \bar{C}_\mu^2 \\ \bar{C}_\mu^3 \\ \hline \sqrt{\frac{3}{4}} D_\mu \end{pmatrix}$$

so

$$\mathcal{L} = \left| g A_\mu^A T^A \langle\Phi\rangle \right|^2 = \frac{g^2 v^2}{4} C_\mu^i \bar{C}^{\mu i}$$

$$+ \frac{1}{2} \cdot \left(\frac{3}{8} g^2 v^2\right) D_\mu D^\mu$$

so $\quad m_C^2 = \frac{g^2 v^2}{4} \qquad m_D^2 = \frac{3}{8} g^2 v^2$

indeed

$$\frac{m_D}{2} < m_C < m_D$$

$$m_C = 0.5\, gv \qquad m_D = 0.612\, gv$$

so the decay $D \to C\bar{C}$ is kinematically forbidden.

d) Now consider adding the fermion U in the 6 of SU(4).
(The fermion D will work identically.)

U has the form:

$$\frac{1}{\sqrt{2}}\left(\begin{array}{ccc|c} 0 & a & b & d \\ -a & 0 & c & e \\ -b & -c & 0 & f \\ \hline -d & -e & -f & 0 \end{array}\right)$$

$\frac{1}{\sqrt{2}}$ to give the correct normalization

$+ t\bar{U} i\not{\partial} U$
$= \bar{u} i\not{\partial} u$

(d,e,f) transform as a 3 of SU(3)

(a,b,c) transform as a $(3\times3)_{\text{antisym}} = \bar{3}$

to be more clear, write:

$$d = u_L^1 \qquad e = u_L^2 \qquad f = u_L^3$$

$$a = \epsilon^{123}\bar{u}_L^3 = \bar{u}_L^3 \qquad b = \epsilon^{132}\bar{u}_L^2 = -\bar{u}_L^2 \qquad c = \epsilon^{231}\bar{u}_L^1 = +\bar{u}_L^1$$

again:

$$U_L = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|c} 0 & \bar{u}_L^3 & -\bar{u}_L^2 & u_L^1 \\ -\bar{u}_L^3 & 0 & \bar{u}_L^1 & u_L^2 \\ +\bar{u}_L^2 & -\bar{u}_L^1 & 0 & u_L^3 \\ \hline -u_L^1 & u_L^2 & u_L^3 & 0 \end{array}\right)$$

and similarly for d_L $\bar{d}_L$

The action of T^A on U is

$$\phi \to \phi + i\alpha^A T^A \phi$$

$$U_{ij} \to U_{ij} + i(\alpha^A T^A)_{ik} U_{kj} + i(\alpha^A T^A)_{jk} U_{ik}$$

$$= [U + i\alpha^A T^A U + i U (\alpha^A T^A)^T]_{ij}$$

so

$$D_\mu U = \partial_\mu U - ig (A_\mu^A T^A) U - ig U (A_\mu^A T^A)^T$$

$$A_\mu^A T^A = \left(\begin{array}{c|c} A_\mu^a t^a & \\ \hline & + \end{array} \right)$$

$$+ \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} & \bar{C}_\mu \\ \hline C_\mu & \end{array} \right) + \frac{D_\mu}{\sqrt{24}} \left(\begin{array}{c|c} 1\ 1\ 1 & \\ \hline & -3 \end{array} \right)$$

A_μ^a are the gluons. The gluon couplings comes from

$$D_\mu U = \partial_\mu U - ig\, t_R \left[\begin{array}{c|c} (A_\mu^a t^a) \begin{pmatrix} 0 & r & \bar{u} \\ \bar{r} & 0 & \bar{u} \\ u & u & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{u} & u \\ u & 0 & \bar{u} \\ \bar{u} & u & 0 \end{pmatrix} (A_\mu^c t^a)^T & (A_\mu^a t^a) \begin{pmatrix} y \\ y \\ w \end{pmatrix} \\ \hline (r\ r\ w)(A_\mu^a t^a)^T & 0 \end{array} \right]$$

It might be clearer to write:

$$\bar{u}_L^{ij} = \epsilon^{ijk} \bar{u}^k$$

$\bar{u}^k$ is a vector with one index in the $\bar{3}$

$\bar{u}^{ij}$ is a matrix with two indices in the 3

$$\mathrm{tr}\, \bar{U}_L\, i\bar{\sigma}\cdot\partial U_L = \mathrm{tr}\, U_L^\dagger\, i\bar{\sigma}\cdot\partial U_L$$

$$= \frac{1}{2}\,\mathrm{tr}\, \bar{u}_L^\dagger\, i\sigma\cdot\partial\, \bar{u}_L + \frac{1}{2}\, u_L^\dagger\, i\bar{\sigma}\cdot\partial u_L \cdot 2$$

$$= \bar{u}_L^{k\dagger}\, i\sigma\cdot\partial\, \bar{u}_L^k + u_L^{i\dagger} i\bar{\sigma}\cdot\partial u_L^i$$

correct!

the couplings to A_μ^a are:

$$g\,\mathrm{tr}\, U_L^\dagger\, i\bar{\sigma}^\mu A_\mu^a \left[\left(\frac{t^a}{+}\right) U + U \left(\frac{t^{aT}}{+}\right)\right]$$

$$= 2\times\frac{1}{2}\, g\, u_L^\dagger\, \bar{\sigma}\cdot A_\mu^a t^a u_L + \frac{g}{2}\,\mathrm{tr}\, \bar{u}_L^\dagger\, i\sigma\cdot A^a\, (t^a \bar{u}_L + \bar{u}_L t^a)$$

$$= g\, u_L^\dagger (\sigma\cdot A_\mu^a t_3^a)\, u_L + g\, \bar{u}_L\, (i\sigma\cdot A^a\, t_{\bar{3}}^a)\, u_L$$

which is the standard QCD coupling if we set

$$g = g_s.$$

e.) Next, compute the couplings of C and $\bar{C}$ bosons.

$$\Delta\mathcal{L} = \frac{g}{\sqrt{2}} \operatorname{tr}\, U_L^\dagger \bar{\sigma}^\mu \left\{ \left(\begin{array}{c|c} & \bar{C}_\mu \\ \hline C_\mu & \end{array} \right) U_L + U_L \left(\begin{array}{c|c} & C_\mu^T \\ \hline \bar{C}_\mu^T & \end{array} \right) \right\}$$

$$= \frac{g}{2\sqrt{2}} \operatorname{tr} \left(\begin{array}{c|c} \bar{u}^\dagger & -u^\dagger \\ \hline u^\dagger & 0 \end{array} \right) \bar{\sigma}^\mu \left\{ \left(\begin{array}{c|c} 0 & \bar{C} \\ \hline C & \end{array} \right) \left(\begin{array}{c|c} \bar{u} & u \\ \hline -u^T & 0 \end{array} \right) + \left(\begin{array}{c|c} \bar{u} & u \\ \hline -u^T & 0 \end{array} \right) \left(\begin{array}{c|c} & C^T \\ \hline \bar{C}^T & \end{array} \right) \right\}$$

$$= \frac{g}{2\sqrt{2}} \operatorname{tr} \left(\begin{array}{c|c} \bar{u}^\dagger & -u^\dagger \\ \hline u^\dagger & 0 \end{array} \right) \bar{\sigma}^\mu \left(\begin{array}{c|c} -\bar{C} u^T + u \bar{C}^T & \bar{u} C^T \\ \hline C \bar{u} & C u - u^T C^T \end{array} \right)$$

$$= \frac{g}{2\sqrt{2}} \left\{ \operatorname{tr} \left[\bar{u}^\dagger \bar{\sigma}^\mu \left(-\bar{C} u^T + u \bar{C}^T \right) \right] + u^\dagger \bar{\sigma}^\mu \bar{u} C^T - u^\dagger \bar{\sigma}^\mu C \bar{u} \right\}$$

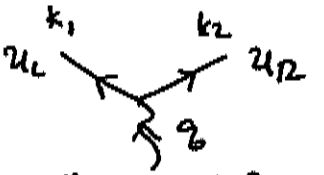
$$= \frac{g}{2\sqrt{2}} \left\{ \epsilon^{ijk} \bar{u}^{\dagger k} \bar{\sigma}^\mu \left[-\bar{C}_\mu^j u^i + u^j \bar{C}_\mu^i \right] + u^{\dagger i} \bar{\sigma}^\mu \bar{u}^k \epsilon^{ijk} C^j - u^{\dagger i} \bar{\sigma}^\mu C^j \bar{u}^k \epsilon^{ijk} \right\}$$

$$= \frac{g}{\sqrt{2}} \left\{ \epsilon^{ijk} \bar{u}^{\dagger k} \bar{\sigma}^\mu u^i \bar{C}^j + \epsilon^{ijk} u^{\dagger i} \bar{\sigma}^\mu C^j \bar{u}^k \right\}$$

this is an SU(3) - invariant coupling.

This C decay to $u_L (\bar{u}_L)^+ = u_L u_R$

The decay width is given by:



$$i\mathcal{M} = i \frac{g}{\sqrt{2}} \epsilon^{ijk} \bar{u}(k_1) \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) v(k_2) \varepsilon_\mu(q)$$

for $C^j \to u_L^i u_R^k$

$$\sum_{\text{color}} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{g^2}{2} \epsilon^{ijk} \epsilon^{ijk} \, \text{tr}\left[\not{k}_1 \gamma^\mu \not{k}_2 \gamma^\nu \left(\frac{1-\gamma^5}{2}\right)\right] \cdot \left(-g_{\mu\nu} + \frac{2q_\mu q_\nu}{m_C^2}\right)$$

$$= \frac{g^2}{2} \cdot 6 \cdot \frac{4}{2} \left[k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1 k_2 g^{\mu\nu} \right] (-g_{\mu\nu})$$

since $q^\mu [\,] = 0$

$$= \frac{g^2}{2} \cdot 6 \cdot 2 \cdot (4-2) \, k_1 k_2$$

$$= 6 g^2 m_C^2 \qquad \text{since } 2k_1 k_2 = m_C^2$$

the color and spin averaged quantity is

$$\frac{1}{3} \sum_{\text{color}} \frac{1}{3} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{2}{3} g^2 m_C^2$$

$$\Gamma = \frac{1}{2m_C} \frac{1}{8\pi} \cdot \frac{2}{3} g_s^2 m_C^2 = \frac{\alpha_s}{6} m_C$$

$$\Gamma(C \to u_L u_R) = \Gamma(C \to d_L d_R) = \frac{\alpha_s}{6} m_C$$

the coupling of D_λ is easier to understand.

$$D_\lambda [T_4 U + U T_4^T]$$

$$= \frac{1}{\sqrt{24}} \left[\left(\begin{array}{c|c} 1 & \\ \hline & -3 \end{array}\right) \left(\begin{array}{c|c} \bar{u} & u \\ \hline -u^T & 0 \end{array}\right) + \left(\begin{array}{c|c} \bar{u} & u \\ \hline -u^T & 0 \end{array}\right) \left(\begin{array}{c|c} 1 & \\ \hline & -3 \end{array}\right) \right]$$

$$= \frac{1}{\sqrt{24}} \left(\begin{array}{c|c} 2\bar{u} & -2u \\ \hline (-2)(-u^T) & 0 \end{array} \right)$$

so u_L has a D_λ charge $-\frac{g}{\sqrt{6}}$

$\bar{u}_L$ has a D_λ charge $+\frac{g}{\sqrt{6}}$

for $D \to u_L \bar{u}_R$

$$iM = i \frac{g}{\sqrt{6}} \bar{u} \gamma^\mu v \, \varepsilon_\lambda(q) \, \delta^{ii}$$

$$\sum_{\text{spins}} \sum_{\text{colors}} |M|^2 = \frac{g^2}{6} \cdot \text{tr}\left[k_1 \gamma^\mu k_2 \gamma^\nu \left(\frac{1-\gamma^5}{2}\right) \right] (-g_{\mu\nu}) \, \delta^{ii}$$

$$= \frac{g^2}{6} \cdot 2 \cdot 2 k_1 \cdot k_2 \cdot 3 = \frac{g^2}{3} m_D^2 \cdot 3 = g^2 m_D^2$$

the correct color and spin average is:

$$\sum_{\text{colors}} \frac{1}{3} \sum_{\text{spins}} |M|^2 = \frac{g^2}{3} m_c^2$$

$$\Gamma(D \to u_L \bar{u}_R) = \frac{1}{2m_D} \frac{1}{8\pi} \frac{g^2}{3} m_c^2$$

$$= \frac{\alpha_s}{12} m_c$$

$$= \Gamma(D \to u_R \bar{u}_L) = \Gamma(D \to d_L \bar{d}_R) = \Gamma(D \to d_R \bar{d}_L)$$

$$= \text{all}$$

$$\Gamma(C) \cong \Gamma(D) = \frac{\alpha_s}{3}(m_c/m_D)$$

f.) The vertex



is ~~proportional~~ to $f^{ABC} \propto \mathrm{tr}\, T^A [T^B, T^C]$

take T^B, T^C to be these.

$$[T^B, T^C] = \left(\begin{array}{c|c} [t^B, t^C] & 0 \\ \hline 0 & 0 \end{array}\right) = i f^{dbc} \left(\begin{array}{c|c} t^d & 0 \\ \hline 0 & 0 \end{array}\right)$$

where f^{dbc} is the SU(3) structure constant.

This has zero trace with the C and D generators

e.g.

$$\mathrm{tr}\, T_4 \left(\begin{array}{c|c} t^d & 0 \\ \hline 0 & 0 \end{array}\right) = 0$$

so the vertex $C \to gg$ or $D \to gg$ vanishes!

g.) $m(C) = 700$ GeV

$\Rightarrow m(D) = \sqrt{\frac{3}{2}}\, m(C) = 857.$ GeV

$\Gamma(C) = \frac{\alpha_s}{3} m_C = 28.$ GeV

50% into $u_L u_R$ 50% into $d_L d_R$

$\Gamma(D) = \frac{\alpha_s}{3} m_D = 34$ GeV

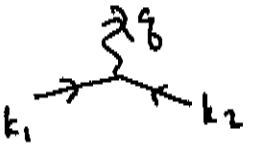
50% into $u\bar{u}$ 50% into $d\bar{d}$

Note that $\Gamma(C) + \Gamma(D) \ll m(D) - m(C)$

the resonances are well separated.

h.) We can also use the matrix elements in part (e) to compute the parton-level production cross sections.

To begin, recall that 1-body phase space is.



$$\int \frac{d^3 q}{(2\pi)^3} \frac{1}{2E_q} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q)$$

$$= \int \frac{d^4 q}{(2\pi)^4} 2\pi \delta(q^2 - (k_1+k_2)^2)\, (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q)$$

$$= 2\pi \delta(m_B^2 - \hat{s})$$

so

$$\sigma(q_1+q_2 \to C) = \frac{1}{2\hat{s}}\, 2\pi\, \delta(m_c^2-\hat{s}) \cdot \underbrace{(|m|^2)}$$

appropriately spin & color summed and averaged.

for $u + u \to C$ ↓ p.10

$$(|m|^2) = \underbrace{\frac{1}{4}\cdot 2}_{\text{spin}} \cdot \underbrace{\frac{1}{3}\cdot\frac{1}{3}}_{\text{color}} \cdot (6 g^2 m_c^2)$$

$$= \frac{1}{3}\, g^2 m_c^2$$

the same for $d+d \to C$ $\quad \bar{u}+\bar{u} \to \bar{C}$ $\quad \bar{d}+\bar{d} \to \bar{C}$

for $u+\bar{u} \to D$

$$(|m|^2) = \frac{1}{4}\cdot 2 \cdot \frac{1}{3}\cdot\frac{1}{3} \cdot g^2 m_D^2$$

$$= \frac{1}{18}\, g^2 m_D^2$$

the same for $d+\bar{d} \to D$

now $\quad \frac{1}{2\hat{s}}\, 2\pi\, \delta(m_c^2-\hat{s}) \cdot \frac{1}{3} g^2 m_c^2$

$$= \frac{4}{3}\pi^2 \alpha_s\, \delta(m_c^2-\hat{s})$$

$$\sigma(p\bar{p} \to C + \bar{X})$$

$$= \int dx_1 dx_2 \quad [f_u(x_1) f_{\bar{u}}(x_2) + f_d(x_1) f_{\bar{d}}(x_2)]$$

$$\cdot \frac{4}{3}\pi^2 \alpha_s \quad \delta(m_c^2 - x_1 x_2 s)$$

$$\sigma(p\bar{p} \to \bar{C} + \bar{X})$$

$$= \int dx_1 dx_2 \quad [f_{\bar{u}}(x_1) f_u(x_2) + f_{\bar{d}}(x_1) f_d(x_2)]$$

$$\cdot \frac{4}{3}\pi^2 \alpha_s \quad \delta(m_c^2 - x_1 x_2 s)$$

$$\sigma(p\bar{p} \to D + \bar{X})$$

$$= \int dx_1 dx_2 \quad [f_u(x_1) f_u(x_2) + f_{\bar{u}}(x_1) f_{\bar{u}}(x_2) + f_d(x_1) f_d(x_2)$$

$$+ f_{\bar{d}}(x_1) f_{\bar{d}}(x_2)] \cdot \frac{2}{9}\pi^2 \alpha_s \, \delta(m_D^2 - x_1 x_2 s)$$

(using $f_u|_{\bar{p}} = f_{\bar{u}}|_p$ and vice versa)

Note that $\sigma(p\bar{p} \to C + \bar{X}) = \sigma(p\bar{p} \to \bar{C} + X)$

Actually, the widths of the C and D bosons smear out the delta functions into Breit-Wigner functions:

$$\delta(m_c^2 - \hat{s}) \rightarrow \frac{\eta}{(\hat{s} - m_c^2)^2 + (m_c \Gamma_c)^2}$$

η should be chosen s.t. $\int_{-\infty}^{\infty} d\hat{s}$ [above] = 1

$$\int_{-\infty}^{\infty} d\hat{s} \frac{1}{(\hat{s} - m_c^2)^2 + (m_c \Gamma_c)^2} = \frac{\pi}{m_c \Gamma_c}$$

so

$$\delta(m_c^2 - \hat{s}) \rightarrow \frac{m_c \Gamma_c / \pi}{(\hat{s} - m_c^2)^2 + (m_c \Gamma_c)^2}$$

and similarly for $\delta(m_D^2 - \hat{s}^2)$

2.) An implementation of the total cross section in the program pp collider cpp is given on the next page.

I find:

$$\sigma(p\bar{p} \rightarrow C + \bar{X}) = 197.6 \pm 0.3$$
$$\sigma(p\bar{p} \rightarrow \bar{C} + X) = 197.4 \pm 0.3 \text{ pb}$$
$$\sigma(p\bar{p} \rightarrow D + \bar{X}) = 52.8 \pm 0.1$$

```
#include "jets.h"
#include "gnuplot.h"


class CDjets: public jets {

  /* computes the rate for heavy boson production at a ppbar collider  */

 public:

  CDjets(double ecm, int Which, double Mass, double gamma): jets(ecm, 5.0),
                    P("mrst/vnvalf1155.dat"), which(Which), M(Mass), Gamma(gamm
a){}

  mrst P;
  int which;  /*  set which = 0 for D , -1 for Cbar, 1 for C  */
  double M, Gamma;

   /* at the time crosssection is called, x1, x2, etc. are defined */

   /* crosssection is supposed to return
              (d sigma/ dx1 dx2 d cos theta_CM ) * (2/shat))
    for this program, compute   d sigma /dx1 dx2  (independent of cos theta).
      Then the  integral over cos theta will give a factor 2.  If we multiply
          by   1/shat, the integral will then be the correctly normalized
                       cross section   */

  double BW(double mys){
     return   (M * Gamma/PI)/(SQR(mys - M*M) + SQR(M*Gamma));
  }

  double crosssection(){
    if (shat < SQR(500.0)) return 0.0;
    const double alphas = 0.12;
    double fu1 = P.xfu(x1,Q)/x1;
    double fd1 = P.xfd(x1,Q)/x1;
    double fub1 = P.xfubar(x1,Q)/x1;
    double fdb1 = P.xfdbar(x1,Q)/x1;
    double fu2 = P.xfu(x2,Q)/x2;
    double fd2 = P.xfd(x2,Q)/x2;
    double fub2 = P.xfubar(x2,Q)/x2;
    double fdb2 = P.xfdbar(x2,Q)/x2;
    double pref =  4.0 * PI * PI * alphas/3.0;
    double cs = 0.0;
    if (which == 1 ) {
      cs = (fu1 * fub2 + fd1 * fdb2)
            * (4.0/3.0)* PI * PI * alphas * BW(shat) * pb;
    }   else if (which == -1) {
      cs = (fub1 * fu2 + fdb1 * fd2)
            * (4.0/3.0)* PI * PI * alphas * BW(shat) * pb;
    }   else if (which == 0) {
      cs = (fu1 * fu2  + fd1 * fd2  +  fub1 * fub2 + fdb1 * fdb2)
           *(2.0/9.0)*PI *PI*alphas*BW(shat) * pb;
    }
    return cs/shat;
  }

};
```

but, most of the cross section is at low mass, the Breit-Wigner does not cut off fast enough.

Restricts to $\sqrt{s} > 500$ GeV

$p\bar{p} \to C + \not{E}$ $\quad \sigma = 2.05$ pb $\longrightarrow$ 2000 events $\sim 1\,fb^{-1}$

$p\bar{p} \to \bar{C} + \not{E}$ $\quad \sigma = 2.05$ pb $\longrightarrow$ 2000 events $\sim 1\,fb^{-1}$

$p\bar{p} \to D + \not{E}$ $\quad \sigma = 2.77$ pb $\longrightarrow$ 2800 events $\sim 1\,fb^{-1}$

Anyway, this is enough that present Tevatron data already exclude the model.

j.) The program generates the rapidity distribution of the bosons

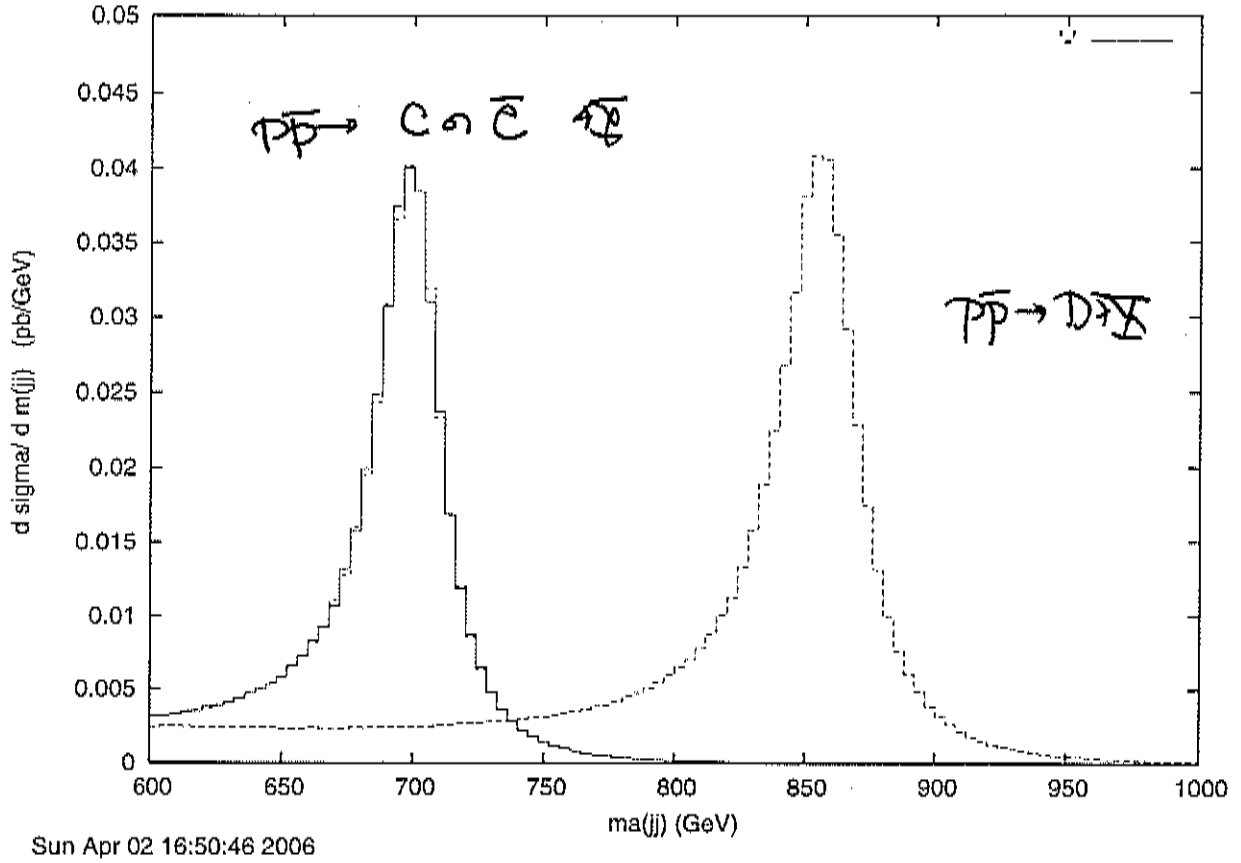
$$Y = \frac{1}{2} \log\left(\frac{x_1}{x_2}\right)$$

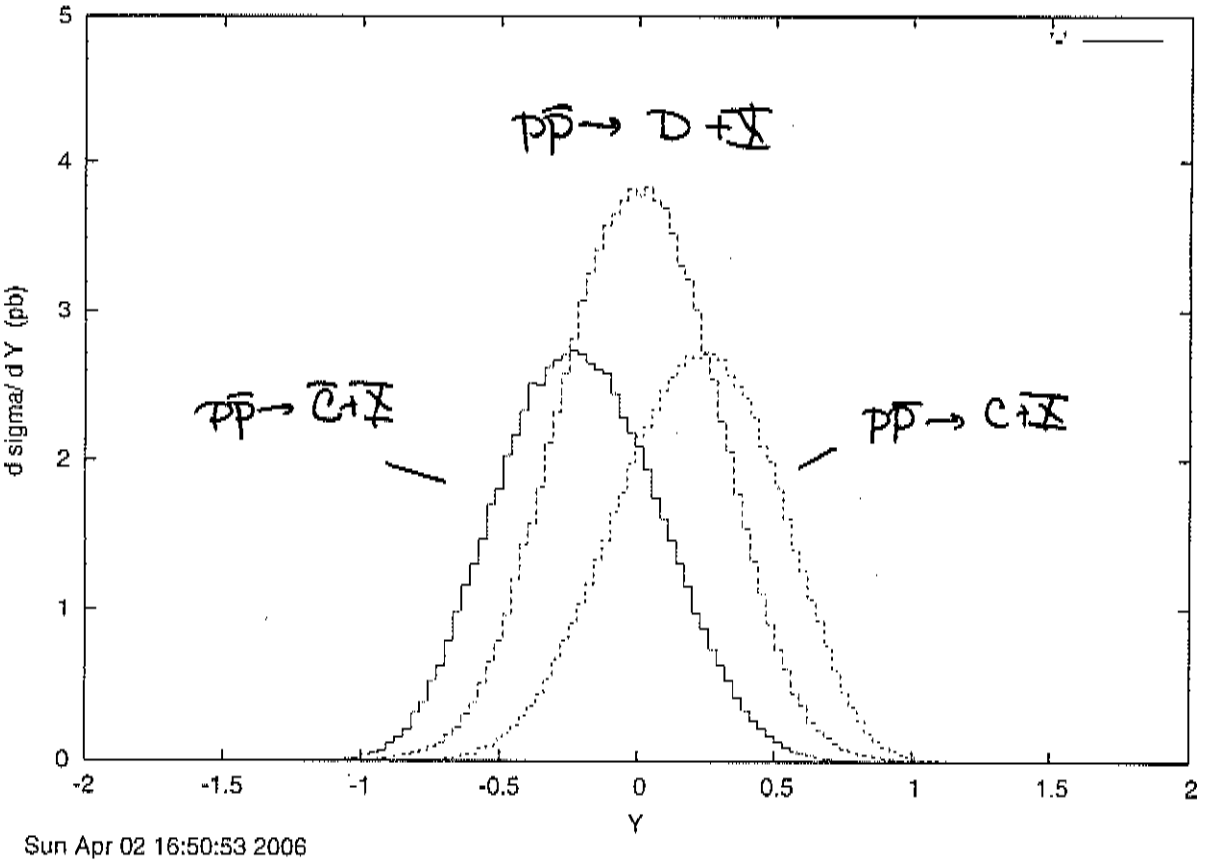
Note that, in $p\bar{p}$ collisions, since f_u, f_d are much larger at large x than $f_{\bar{u}}, f_{\bar{d}}$

$p\bar{p} \to C + \not{E}$ has $\langle Y \rangle > 0$ p side

$p\bar{p} \to \bar{C} + \not{E}$ has $\langle Y \rangle < 0$ $\bar{p}$ side

$p\bar{p} \to D + \not{E}$ has $\langle Y \rangle = 0$ symmetric


0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
d sigma/ d m(jj) (pb/GeV)
600
650
700
750
800
850
900
950
1000
ma(jj) (GeV)
Sun Apr 02 16:50:46 2006


5
4
3
2
1
0
d sigma/ d Y (pb)
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
Y
Sun Apr 02 16:50:53 2006

k.) The processes we need are:

$u\bar{u} \to D + g$

$ug \to D + u$ and similar processes w. $u \to d$

$\bar{u}g \to D + \bar{u}$

For $u\bar{u} \to D$ we need to compute



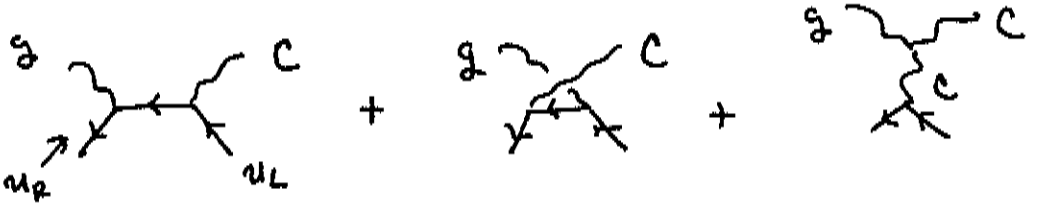
The other two processes use the same diagrams after crossing g into the initial state, u or ū into the final state.

l.) The processes we need are:

$uu \to C + g$

$ug \to C + \bar{u}$ and similar processes w. $u \to d$

The diagrams for $uu \to C + g$ are:

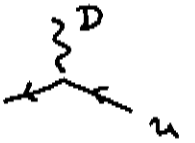


Again, we can obtain $ug \to C + \bar{u}$ by crossing.

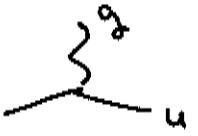
m.) It suffices to compute the cross sect. for

$$u\bar{u} \to g + D.$$

all of the other processes can be obtained from this one by crossing. The relevant Feynman rules are:



$= -i \frac{g_s}{\sqrt{6}} \gamma^\mu$ (u_L, u_R have charge $-\frac{g_s}{\sqrt{6}}$)



$= +i g_s t^a \gamma^\mu$ see p. 11

so the computation is very straightforward, similar to P+S problem 5.5



$$i\mathcal{M} = \left(-i\frac{g_s}{\sqrt{6}}\right)(i g_s t^a)\,\bar{v}(k)\left[\gamma^\mu \frac{i(\not{k}-\not{q})}{(k-q)^2}\gamma^\nu + \gamma^\nu \frac{i\not{k}-\not{p}}{(k-p)^2}\gamma^\mu\right]u(k)$$

$$\cdot \varepsilon^*_\mu(p)\,\varepsilon^*_\nu(q)$$

$$= i\frac{g_s^2 t^a}{\sqrt{6}}\,\bar{v}(k)\left[\gamma^\mu \underbrace{\frac{(\not{k}-\not{q})}{t}}\gamma^\nu + \gamma^\nu \underbrace{\frac{(\not{k}-\not{p})}{u}}\gamma^\mu\right]u(k)\;\varepsilon^*_\mu(p)\,\varepsilon^*_\nu(q)$$

where

$$u = (k-p)^2 = -2k\cdot p$$

$$t = (k-q)^2 = (\bar{k}-p)^2 = -2\bar{k}\cdot p$$

To do the sum of D polarizations sums, we can write

$$\sum \varepsilon_\nu^* \varepsilon_\lambda(q) \sim \left(-g_{\nu\lambda} + \frac{q_\nu q_\lambda}{m_D^2}\right)$$

but q_ν or p_μ dotted with the fermi matrix element gives zero., so

$$\sum \varepsilon_\nu^* \varepsilon_\lambda = -g_{\nu\lambda}$$

will suffice.

$$\sum_{\text{color}}\sum_{\text{spin}} |M|^2 = \left[\frac{g_s^4 \, tr\, t^a t^a}{6}\right] tr\left[\bar{k}\left\{\gamma^\mu \frac{(k-q)}{t}\gamma^\nu + \gamma^\nu \frac{(k-p)}{u}\gamma^\mu\right\} \bar{k}\left\{\gamma_\nu \frac{(k-q)}{t}\gamma_\mu + \gamma_\mu \frac{(k-p)}{u}\gamma_\nu\right\}\right]$$

$$= \frac{g_s^4}{6}\cdot\frac{1}{2}\cdot 8 \cdot \left(\frac{I}{t^2} + \frac{II + III}{tu} + \frac{IV}{u^2}\right)$$

with $\rightarrow$

$$s = 2k\cdot\bar{k} = 2p\cdot q + m_D^2$$

$$t = -2\bar{k}\cdot p = -2k\cdot q + m_D^2$$

$$u = -2k\cdot p = -2\bar{k}\cdot q + m_D^2$$

$$I = \mathrm{tr}[\bar{k}\gamma^\mu(k\!\!\!/-q\!\!\!/)\gamma^\nu k\!\!\!/\,\gamma_\nu\, k\!\!\!/\,\gamma_\mu]$$

$$= (-2)^2\,\mathrm{tr}\,\bar{k}(k\!\!\!/-q\!\!\!/)\,k\!\!\!/\,(k\!\!\!/-q\!\!\!/)$$

$$= 4\cdot 4\cdot[\,2\,\bar{k}\cdot\underbrace{(k-q)}_{p-\bar{k}}\,k\cdot(k-q) - k\cdot\bar{k}\,(k-q)^2]$$

$$= 8\,[(2\bar{k}\cdot p)(-2k\cdot q) - 2k\bar{k}\,(k-q)^2]$$

$$= 8\,[(-t)(t-m_D^2) - st]$$

$$= -8t(t+s-m_D^2) \qquad s+t+u=m_D^2$$

$$= -8tu$$

$$IV = +\,\mathrm{tr}\,\bar{k}\gamma^\nu(k\!\!\!/-p\!\!\!/)\gamma^\mu k\!\!\!/\,(k\!\!\!/-p\!\!\!/)\gamma_\nu$$

$$= (-2)^2\,\mathrm{tr}\,\bar{k}(k\!\!\!/-p\!\!\!/)\,k\!\!\!/\,(\bar{k}\!\!\!/-p\!\!\!/)$$

$$= 4\cdot 4\cdot[\,2\,\bar{k}\cdot(k-p)\;k\cdot(k-p) - \bar{k}\cdot k\,(k-p)^2]$$

$$= 8\,[(2\bar{k}\cdot q)(-2k\cdot p) - su]$$

$$= 8\,[-(t-m_D^2)\,u - su]$$

$$= -8ut$$

$$II = +\,\mathrm{tr}\,\bar{k}\gamma^\mu(k-q)\gamma^\lambda k\!\!\!/\,(k-p)\gamma_\nu \qquad = III$$

$$= (-2)\,\mathrm{tr}\,\bar{k}\gamma^\mu(k\!\!\!/-q\!\!\!/)(\bar{k}\!\!\!/-p\!\!\!/)\gamma_\nu\,k\!\!\!/$$

$$= (-2)\cdot 4\,\mathrm{tr}\,\bar{k}\bar{k}\;(k-q)\cdot(k-p)$$

$$= -2\cdot 4\cdot 4\,(\bar{k}\cdot k)\,(k-q)\cdot(k-p)$$

$$= -8 \cdot s \cdot 2 \cdot [p\cdot q - k\cdot p - k\cdot q]$$

$$= -8s \cdot [2pq - 2k\cdot k]$$

$$= -8s\,[(s - m_D^2) - s] = 8s m_D^2$$

so

$$\sum_{\text{colors}} \sum_{\text{spins}} |M|^2 = \frac{2}{3} g_s^4 \cdot 8 \cdot \left[\frac{u}{t} + 2\frac{s m_D^2}{ut} + \frac{t}{u} \right]$$

$$= \frac{16}{3} g_s^4 \left[\frac{u}{t} + \frac{t}{u} + 2\frac{s m_D^2}{ut} \right]$$

for $u\bar{u} \to g + D$ we need

$$\left(\tfrac{1}{3}\right)^2 \sum_{\text{colors}} \frac{1}{4} \sum_{\text{spin}} |M|^2 = \frac{4}{27} g_s^4 \left[\frac{u}{t} + \frac{t}{u} + 2\frac{s m_D^2}{ut} \right]$$

for $ug \to uD$ cross $s \leftrightarrow u$ and write

$$\frac{1}{3}\cdot\frac{1}{8} \sum_{\text{colors}} \frac{1}{4} \sum_{\text{spins}} |M|^2 = -\frac{1}{18} g_s^4 \left[\frac{s}{t} + \frac{t}{s} + 2\frac{u m_D^2}{st} \right]$$

the cross section for $\bar{u}g \to \bar{u}D$ is the same.

so $$\frac{d\sigma}{d\cos\theta_{cm}}(u\bar{u} \to Dg) = \frac{1}{2s} \frac{1}{16\pi} \underbrace{\frac{p}{E}}_{\downarrow \left(\frac{s - m_D^2}{s}\right)} \cdot \frac{4}{27} g_s^4 \left[\frac{u}{t} + \frac{t}{u} + 2\frac{s m_D^2}{ut} \right]$$

$$\frac{d\sigma}{d\cos\theta_{cm}}(u\bar{u} \to D + g)$$

$$= \frac{2}{27}\frac{\pi\alpha_s^2}{s}\left(1 - \frac{m_D^2}{s}\right)\left[\frac{u}{t} + \frac{t}{u} + 2\frac{sm_D^2}{ut}\right]$$

$$\frac{d\sigma}{d\cos\theta_{cm}}(ug \to Du) = \frac{d\sigma}{d\cos\theta_{cm}}(\bar{u}g \to D\bar{u})$$

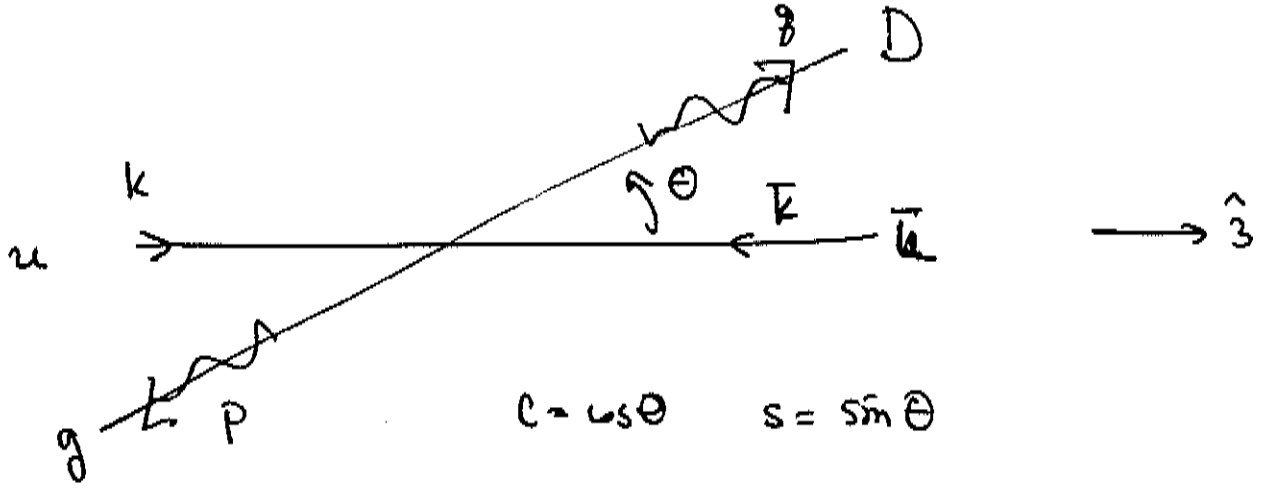
$$= \frac{1}{36}\frac{\pi\alpha_s^2}{s}\left(1 - \frac{m_D^2}{s}\right)\left[-\frac{s}{t} - \frac{t}{s} - \frac{2um_D^2}{st}\right]$$

with $t = (k-q)^2 = [(\text{mom. of } u) - (\text{mom. of } D)]^2$

in all cases.

In preparation for part (n), I'd like to redo the calculation of $u\bar{u} \to D+g$ with helicity amplitudes.

The kinematics is



$k = (E, 0, 0, E)$ $\qquad \bar{k} = (E, 0, 0, -E)$

$q = (\mathcal{E}, ps, 0, pc)$ $\qquad p = (p, -ps, 0, -pc)$

$$\mathcal{E} = \frac{4E^2 + m_D^2}{4E} \qquad p = \frac{4E^2 - m_D^2}{4E} \qquad \mathcal{E}^2 - p^2 = m_D^2$$

It suffices to consider $u_L \bar{u}_R \to D + g$

$$u(k) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad v(k) = \sqrt{2E}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

the D, g polarization vectors are:

D: $\varepsilon_+^\mu = \frac{1}{\sqrt{2}}(0\ c\ i\ -s)$ $\qquad$ g: $\varepsilon_+^\mu = \frac{1}{\sqrt{2}}(0\ -c, i, s)$

$\varepsilon_0^\mu = \frac{1}{m_D}(p, \mathcal{E}s, 0\ \mathcal{E}c)$

$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0\ +c, -i, -s)$ $\qquad \varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0\ -c, -i, s)$

we will need:

$$\bar{\sigma}\cdot\varepsilon_+^*(q) = \frac{1}{\sqrt{2}}\begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \qquad \bar{\sigma}\cdot\varepsilon_+^*(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} s & -(1+c) \\ 1-c & -s \end{pmatrix}$$

$$\bar{\sigma}\cdot\varepsilon_0^*(q) = \frac{1}{m_D}\begin{pmatrix} p+\mathcal{E}c & \mathcal{E}s \\ \mathcal{E}s & p-\mathcal{E}c \end{pmatrix}$$

$$\bar{\sigma}\cdot\varepsilon_-^*(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} s & 1-c \\ -(1+c) & -s \end{pmatrix}$$

$$\bar{\sigma}\cdot\varepsilon_-^*(q) = \frac{1}{\sqrt{2}}\begin{pmatrix} -s & c+1 \\ c-1 & s \end{pmatrix}$$

we will also need

$$\sigma\cdot(k-q) = E-\varepsilon - \vec{\sigma}\cdot(\vec{k}-\vec{q})$$

$$= \begin{pmatrix} (E-\varepsilon-E+pc) & ps \\ ps & \underbrace{E-\varepsilon+E-pc}_{p} \end{pmatrix}$$

$$\sigma\cdot(k-q) = \begin{pmatrix} -(\varepsilon-pc) & ps \\ ps & p(1-c) \end{pmatrix}$$

similarly:

$$\sigma\cdot(k-p) = \begin{pmatrix} -p(1+c) & -ps \\ -ps & \varepsilon+pc \end{pmatrix}$$

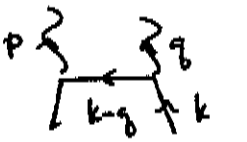
as above:

$$t = -2E\cdot p = -2Ep(1-c)$$

$$u = -2k\cdot p = -2Ep(1+c)$$

$$s = 4E^2$$

Now compute the two diagrams $\longrightarrow$



$$= \frac{-ig_s}{\sqrt{6}}\; ig_s t^a\; v^\dagger\, \bar\sigma\cdot\varepsilon^*(p)\, \frac{i\sigma\cdot k-q}{(k-q)^2}\, \bar\sigma\cdot\varepsilon^*(q)\, u(k)$$

$$= \frac{ig_s^2 t^a}{\sqrt{6}}\, \frac{(\sqrt{2E})^2\,(-1\;0)}{-2Ep(1-c)}\, \bar\sigma\cdot\varepsilon^*(p)\, \sigma\cdot(k-q)\, \bar\sigma\cdot\varepsilon^*(q) \begin{pmatrix}0\\1\end{pmatrix}$$

$$= -i\,\frac{g_s^2 t^a}{\sqrt{6}}\,\frac{1}{p(1-c)}\,(-1\;0)\,\bar\sigma\cdot\varepsilon^*(p)\,\sigma(k-q)\,\bar\sigma\cdot\varepsilon^*(q)\begin{pmatrix}0\\1\end{pmatrix}$$

$u_L \bar u_R \to D(+)\, g(+)$

$$\frac{-ig_s^2 t^a}{\sqrt{6}}\,\frac{1}{p(1-c)}\,\frac{1}{\sqrt2}\,(-1\;0)\begin{pmatrix} s & -(1+c)\\ (1-c) & -s\end{pmatrix}\begin{pmatrix}-(E-pc) & ps\\ ps & p(1-c)\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}-s & -(1-c)\\ 1+c & s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}}\,\frac{1}{p(1-c)}\,(-s\;\;(1+c))\begin{bmatrix}-(E-pc) & ps\\ ps & p(1-c)\end{bmatrix}\begin{pmatrix}-(1-c)\\ s\end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}}\,\frac{1}{p(1-c)}\,\big[-s(1-c)(E-pc) - ps^3 - \underbrace{ps^3 + ps^3}_{=0}\big]$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}}\,\frac{1}{p(1-c)}\,(-s(1-c))\big[E-pc+p(1+c)\big]$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}}\,\frac{1}{p(1-c)}\,(-s(1-c))\,\underbrace{(E+p)}_{2E}$$

$$= \frac{ig_s^2 t^a}{2\sqrt{6}}\;\;\frac{2Es}{p}$$

$s^2 = (1-c)(1+c)$

$E+p = 2E$

$u_L \bar{u}_R \rightarrow D(+)\, g(-)$

$$\frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} (-1\ 0) \begin{pmatrix} s & (1-c) \\ -(1+c) & -s \end{pmatrix} \begin{pmatrix} -(\varepsilon - pc) & ps \\ ps & \not{p}(1+c) \end{pmatrix} \begin{pmatrix} -(1-c) \\ s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} (-s\ \ -(1-c)) \begin{pmatrix} -(\varepsilon - pc) & ps \\ ps & p(1-c) \end{pmatrix} \begin{pmatrix} -(1-c) \\ s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \left\{ -s(1-c)(\varepsilon - pc) - ps^3 + ps(1-c)^2 - ps(1-c)^2 \right\}$$

$$= \frac{ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} s(1-c)\left[\varepsilon - pc + p(1+c)\right]$$

$$= \frac{ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} s(1-c)\, 2\varepsilon = \frac{ig_s^2 t^a}{2\sqrt{6}} \cdot \frac{2\varepsilon s}{p}$$

$u_L \bar{u}_R \rightarrow D(-)\, g(+)$

$$\frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} (-s\ \ (1+c)) \begin{bmatrix} -(\varepsilon - pc) & ps \\ ps & p(1-c) \end{bmatrix} \begin{pmatrix} c+1 \\ s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \left\{ s(\varepsilon - pc)(1+c) - ps^3 + ps(1+c)^2 + ps^3 \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} s(1+c)\left[\varepsilon - pc + p(1+c)\right]$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \frac{2\varepsilon s}{p} \left(\frac{1+c}{1-c}\right)$$

$u_L \bar{u}_R \to D(-)\, g(-)$

$$\frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \begin{pmatrix} -s & -(1-c) \end{pmatrix} \begin{pmatrix} -(\varepsilon - pc) & ps \\ ps & p(1-c) \end{pmatrix} \begin{pmatrix} c+1 \\ s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \left[s(1+c)(\varepsilon - pc) - ps^3 - ps^3 - ps(1-c)^2 \right]$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \left[s(1+c)\left[\varepsilon - pc - p(1-c)\right] - p(1-c)s\left[(1+c)+(1-c)\right] \right]$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1-c)} \left[s(1+c)(\varepsilon - p) - 2ps(1-c) \right]$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \left[\left(\frac{m_D^2}{2E_p}\right) s \frac{(1+c)}{(1-c)} - 2s \right]$$

$u_L \bar{u}_R \to D(\sigma)\, g(+)$

$$\frac{-ig_s^2 t^a}{\sqrt{2}\sqrt{6}\, m_D} \frac{1}{p(1-c)} \begin{pmatrix} -s & (1+c) \end{pmatrix} \begin{pmatrix} -(\varepsilon - pc) & ps \\ ps & p(1-c) \end{pmatrix} \begin{pmatrix} \varepsilon s \\ p - \varepsilon c \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1-c)} \Big\{ \varepsilon s^2(\varepsilon - pc) - ps^2(p - \varepsilon c) + \varepsilon ps^2(1+c) + (1+c)\, p(1-c)(p - \varepsilon c) \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1-c)} \cdot s^2 \Big\{ \varepsilon^2 - \varepsilon pc - p^2 + \varepsilon pc + \varepsilon p + \varepsilon pc + p^2 - \varepsilon pc \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1-c)} \cdot s^2 \cdot 2E \cdot \varepsilon = \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \left(\frac{1+c}{\quad}\right) \frac{2E\varepsilon}{p}$$

$u_L \bar{u}_R \to D(\circ)\, g(-)$

$$\frac{-ig_s^2 t^a}{\sqrt{2}\sqrt{6}} \frac{1}{m_D} \frac{1}{p(1-c)} \begin{pmatrix} -s & -(1-c) \end{pmatrix} \begin{pmatrix} -(\varepsilon - pc) & ps \\ ps & p(1-c) \end{pmatrix} \begin{pmatrix} \varepsilon s \\ p - \varepsilon c \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1-c)} \Big\{ \varepsilon s^2 (\varepsilon - pc) - ps^2 (p - \varepsilon c) - \varepsilon p s^2 (1-c) - p(1-c)^2 (p - \varepsilon c) \Big\}$$

$$= \frac{-ig^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1-c)} \Big\{ s^2 (\varepsilon^2 - \varepsilon pc - p^2 + \varepsilon pc) - (1-c)^2 [\varepsilon p (1+c) + (p^2 - \varepsilon pc)] \Big\}$$

$$= \frac{-ig^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1-c)} \Big\{ s^2 m_D^2 - (1-c)^2 \, p \cdot 2E \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \Big\{ \Big(\frac{1+c}{\text{[illegible]}}\Big) \frac{m_D^2}{p} - (1-c)\, 2E \Big\}$$



$$= \frac{-ig_s}{\sqrt{6}} ig_s t^a \; v^+ \, \bar{\sigma}\cdot\varepsilon^*(q) \, \frac{i\sigma\cdot(k-p)}{(k-p)^2} \, \bar{\sigma}\cdot\varepsilon^*(p) \, u(k)$$

$$= \frac{ig_s^2 t^a}{\sqrt{6}} \frac{(\sqrt{2E})^2}{(-2Ep(1+c))} (-1\,0)\, \bar{\sigma}\cdot\varepsilon^*(q)\, \sigma\cdot(k-p)\, \bar{\sigma}\cdot\varepsilon^*(p) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{\sqrt{6}} \frac{1}{p(1+c)} (-1\,0)\, \bar{\sigma}\cdot\varepsilon^*(q)\, \sigma\cdot(k-p)\, \bar{\sigma}\cdot\varepsilon^*(p) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$u_L \bar{u}_R \rightarrow D(+)\, g(+)$

$$-\frac{ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left(s \;\; +(1+c)\right) \begin{bmatrix} -p(1+c) & -ps \\ -ps & \Sigma+pc \end{bmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ ps(1+c)^2 + ps^3 + ps^3 - s(1-c)(\Sigma+pc) \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ ps(1+c)\left[(1+c)+(1-c)\right] + s(1-c)\left[p(1+c) - \Sigma - pc\right] \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ 2ps(1+c) - (\Sigma - p)\, s(1-c) \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ 2ps(1+c) - \frac{m_D^2}{2E} s(1-c) \right\}$$

$$= -\frac{ig_s^2 t^a}{2\sqrt{6}} \left\{ 2s - \frac{m_D^2}{2Ep} s \frac{(1-c)}{(1+c)} \right\}$$

$u_L \bar{u}_R \rightarrow D(+)\, g(-)$

$$-\frac{ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left(s \;\; (1-c)\right) \begin{bmatrix} -p(1+c) & -ps \\ -ps & (\Sigma+pc) \end{bmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ \cancel{-ps^3} + \cancel{ps^3} - ps(1-c)^2 - (\Sigma+pc)\, s(1-c) \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \; -s(1-c)\left(p(1-c) + \Sigma + pc\right)$$

$$= \frac{ig_s^2 t^a}{2\sqrt{6}} \; \frac{2E}{p}\, s \left(\frac{(1-c)}{1+c}\right)$$

$u_L \bar{u}_R \to D(-)\, g(+)$

$$\frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \begin{pmatrix} s & -(1+c) \end{pmatrix} \begin{bmatrix} -p(1+c) & -ps \\ -ps & \Sigma+pc \end{bmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ ps(1+c)^2 + ps^3 - ps(1+c)^2 + s(1+c)(\Sigma+pc) \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ s(1+c) \right\} \left\{ p(1-c) + \Sigma + pc \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \quad \frac{2Es}{p}$$

$u_L \bar{u}_R \to D(-)\, g(-)$

$$\frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \begin{pmatrix} s & -(1+c) \end{pmatrix} \begin{bmatrix} -p(1+c) & -ps \\ -ps & \Sigma+pc \end{bmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ -ps^3 + ps^3 + ps^3 + (\Sigma+pc)s(1+c) \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{1}{p(1+c)} \left\{ s(1+c) \right\} \left\{ p(1-c) + \Sigma + pc \right\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \quad \frac{2Es}{p}$$

$u_L u_R \to D(0)\, g(+)$

$$\frac{-ig_s^2 t^a}{\sqrt{2}\sqrt{6}\, m_D} \frac{1}{p(1+c)} \left(-(p+\Sigma c)-\Sigma s\right) \begin{bmatrix} -p(1+c)-ps \\ -ps \quad \Sigma+pc \end{bmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1+c)} \Big\{ -p(p+\Sigma c)(1+c)^2 - p(p+\Sigma c)s^2 - \Sigma p s^2(1+c) + \Sigma s^2(\Sigma+pc) \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1+c)} \Big\{ (\Sigma^2 + \cancel{\Sigma_R c} - p^2 - \cancel{\Sigma_R c})\, s^2 - (1+c)^2 \left[\Sigma p(1-c) + p(p+\Sigma c)\right] \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1+c)} \Big\{ m_D^2 s^2 - (1+c)^2\, p\cdot 2E \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \Big\{ \frac{m_D^2}{p}(1-c) - 2E(1+c) \Big\}$$

$u_L \bar{u}_L \to D(0)\, g(-)$

$$= \frac{-ig_s^2 t^a}{\sqrt{2}\sqrt{6}\, m_D} \frac{1}{p(1+c)} \left(-(p+\Sigma c)-\Sigma s\right) \begin{bmatrix} -p(1+c) & -ps \\ -ps & \Sigma+pc \end{bmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1+c)} \Big\{ (p+\Sigma c)ps^2 - \cancel{(p+\Sigma c)ps^2} + \Sigma ps^2(1-c) + \Sigma(\Sigma+pc)s^2 \Big\}$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{1}{p(1+c)} s^2 \left(\Sigma p(1-c) + \Sigma^2 + pc\right)$$

$$= \frac{-ig_s^2 t^a}{2\sqrt{6}} \frac{\sqrt{2}}{m_D} \frac{2E\Sigma}{p}(1-c)$$

assemble the pieces:

 $= \frac{i g_s^2 t^a}{2\sqrt{6}} \times$

	D(+)	D(0)	D(−)
g(+)	$\frac{2Es}{p}$	$\left(-\frac{\sqrt{2}}{m_D}\frac{2E\mathcal{E}}{p}(1+c)\right)$	$\left(-\frac{2Es}{p}\frac{1+c}{1-c}\right)$
g(−)	$\frac{2Es}{p}$	$\left(\frac{\sqrt{2}}{m_D}\right)\left[2E(1+c)-\frac{m_D^2}{p}(1+c)\right]$	$\left(2s-\frac{m_D^2 s}{2Ep}\frac{1+c}{1-c}\right)$

 $= \frac{i g_s^2 t^a}{2\sqrt{6}} \times$

	D(+)	D(0)	D(−)
g(+)	$-\left[2s-\frac{m_D^2}{2Ep}s\frac{(1-c)}{(1+c)}\right]$	$\frac{\sqrt{2}}{m_D}\left[2E(1+c)-\frac{m_D^2}{p}(1-c)\right]$	$\left(-\frac{2Es}{p}\right)$
g(−)	$\frac{2Es}{p}\left(\frac{1-c}{1+c}\right)$	$-\frac{\sqrt{2}}{m_D}\left[\frac{2E\mathcal{E}}{p}(1-c)\right]$	$\left(-\frac{2Es}{p}\right)$

then

 $= \dfrac{ig_s^2 t^a}{2\sqrt{6}}$.

$D(+)g(+)$: $\dfrac{2Es}{p} - 2s + \dfrac{m_D^2}{2Ep}\, s\, \dfrac{(1-c)}{(1+c)} = 2\dfrac{(E-p)s}{p} + \dfrac{m_D^2}{2Ep}\, s\, \dfrac{1-c}{1+c}$

$= \dfrac{m_D^2}{2Ep}\, s \left(1 + \dfrac{1-c}{1+c}\right) = \dfrac{m_D^2}{Ep} \cdot \dfrac{s}{1+c}$

$D(+)\,g(-)$: $\dfrac{2Es}{p} + \dfrac{2Es}{p}\left(\dfrac{1-c}{1+c}\right) = \dfrac{4Es}{p}\dfrac{1}{1+c}$

$D(-)\,g(+)$: $-\dfrac{2Es}{p}\dfrac{1+c}{1-c} - \dfrac{2Es}{p} = -\dfrac{4Es}{p}\dfrac{1}{1-c}$

$D(-)\,g(-)$: $2s - \dfrac{m_D^2 s}{2Ep}\dfrac{1+c}{1-c} - 2s\,\dfrac{E}{p} = -2s\,\dfrac{E-p}{p} - \dfrac{m_D^2 s}{2Ep}\dfrac{1+c}{1-c}$

$= -\dfrac{m_D^2}{2Ep}\, s\, \left(1 + \dfrac{1+c}{1-c}\right) = -\dfrac{m_D^2}{Ep}\dfrac{s}{1-c}$

$D(0)\,g(+)$: $-\dfrac{\sqrt{2}}{m_D}\left[\dfrac{2E\Sigma}{p}(1+c) + \dfrac{m_D^2}{p}(1-c) - 2E(1+c)\right]$

$= -\dfrac{\sqrt{2}}{m_D}\left[2E\,\dfrac{\Sigma - p}{p}(1+c) + \dfrac{m_D^2}{p}(1-c)\right]$

$= -\dfrac{\sqrt{2}}{m_D}\left[\dfrac{2E}{p}\,\dfrac{m_D^2}{2E}(1+c) + \dfrac{m_D^2}{p}(1-c)\right] = -\dfrac{2\sqrt{2}}{m_D}\,\dfrac{m_D^2}{p}$

$D(0)\,g(-)$: $\dfrac{\sqrt{2}}{m_D}\left[2E(1-c) - \dfrac{m_D^2}{p}(1+c) - 2E\,\dfrac{\Sigma}{p}(1-c)\right]$

$= -\dfrac{2\sqrt{2}}{m_D}\,\dfrac{m_D^2}{p}$

then $u_L \bar{u}_R + u_R \bar{u}_L$

$$\sum_{\text{colors}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_s^4}{4 \cdot 6} \underbrace{[\text{tr}\, t^a t^c]}_{\frac{1}{4}} \cdot 2$$

$$\cdot \left\{ \left(\frac{m_D^2}{Ep}\frac{s}{1+c}\right)^2 + \left(\frac{4Es}{p}\frac{s}{1+c}\right)^2 + \left(\frac{4Es}{p}\frac{s}{1-c}\right)^2 + \left(\frac{m_D^2}{Ep}\frac{s}{1-c}\right)^2 + 2 \cdot \frac{8m_D^4}{m_D^2 p^2} \right\}$$

$$= \frac{g_s^4}{3} \left\{ \frac{16E^2 + m_D^2}{E^2 p^2} \frac{(1-c)}{(1+c)} + \frac{16E^2 + m_D^2}{E^2 p^2}\left(\frac{1+c}{1-c}\right) + \frac{16 m_D^2 E^2}{E^2 p^2} \right\}$$

$$= \frac{16}{3} g_s^4 \left\{ \left[\frac{(E^2 - \frac{1}{4}m_D^2)^2 + \frac{1}{2}E^2 m_D^2}{E^2 p^2} \right] \left[\left(\frac{1-c}{1+c}\right) + \left(\frac{1+c}{1-c}\right) \right] + \frac{E^2 m_D^2}{E^2 p^2} \right.$$

$$= \frac{16}{3} g_s^4 \left\{ \frac{1-c}{1+c} + \frac{1+c}{1-c} + \frac{E^2 m_D^2}{E^2 p^2} \left[\frac{1+c^2}{(1+c)(1-c)} + 1 \right] \right\}$$

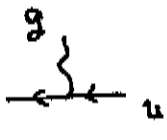
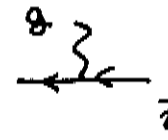
$$= \frac{16}{3} g_s^4 \left\{ \frac{1-c}{1+c} + \frac{1+c}{1-c} + \frac{2E^2 m_D^2}{E^2 p^2 (1+c)(1-c)} \right\}$$

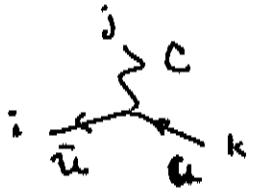
$$= \frac{16}{3} g_s^4 \left\{ \frac{t}{u} + \frac{u}{t} + \frac{2sm_D^2}{tu} \right\} \quad \checkmark$$

n.) Now compute $u\bar{u} \to c + g$ by this same method.

compute $u_L u_R \to c + g$

treat u_R as an antifermion of $\bar{u}_L$ in $\bar{3}$ SU(3)

so  $= i g_s t^a \bar{\sigma}^\mu$  $= -i g_s (t^a)^T \bar{\sigma}^\mu$

 $= -i \frac{g}{\sqrt{2}} \epsilon^{ijk} \bar{\sigma}^\mu \qquad (p.9)$

Finally, we need



$$= -g f^{man} \left[\varepsilon^*(q)\cdot\varepsilon^*(p)\,(q-p)^\lambda + \varepsilon^*(p)^\lambda\,(p+l)\cdot\varepsilon^*(q) + \varepsilon^{*\lambda}(q)\,(-l-q)\cdot\varepsilon^*(p) \right]$$

where

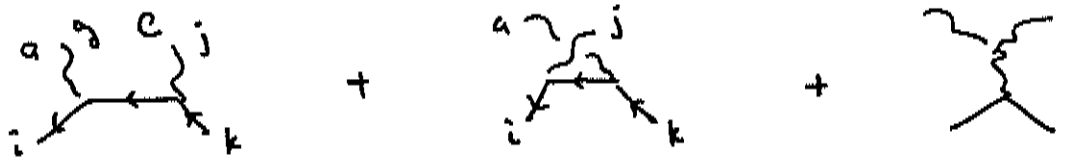
$$f^{man} = \frac{-2i}{(\sqrt{2})^2} tr\left\{ \left(\underline{\quad\quad} \right)^{\leftarrow m} \left[\left(\underline{\quad t^a \quad} \right) , \left(\underline{\quad\quad} \right) \right] \right\}$$

$$= +i\, tr \left(\underline{\quad} \right) \left(\underline{\quad t^a \quad} \right) = i\,(t^a)_{nm}$$

$\uparrow$ (m,n)

$$= -i g_s (t^a)_{nm} \left[\varepsilon^*(q)\cdot\varepsilon^*(p)\,(q-p)^\lambda + 2p\cdot\varepsilon^*(q)\,\varepsilon^{*\lambda}(p) - 2q\cdot\varepsilon^*(p)\,\varepsilon^{*\lambda}(q) \right]$$

then: for $u_L u_R \to C\, g$



$$= -ig\,[(t^a)^T]_{im}\left(\frac{-ig}{\sqrt{2}}\right)\epsilon^{mjk}\;\bar{v}(k)\,\bar{\sigma}\cdot\varepsilon^*(q)\,\frac{i\sigma\cdot(k-q)}{(k-q)^2}\,\bar{\sigma}\cdot\varepsilon^*(q)\,u(k)$$

$$+ig\,(t^a)_{mk}\left(\frac{-ig}{\sqrt{2}}\right)\epsilon^{ijm}\;\bar{v}\,\bar{\sigma}\cdot\varepsilon^*(q)\,\frac{i\sigma(k-p)}{(k-p)^2}\,\bar{\sigma}\cdot\varepsilon^*(p)\,u$$

$$-i\frac{g}{\sqrt{2}}\,\epsilon^{imk}\;\bar{v}\,\bar{\sigma}^\lambda u\;\left(\frac{-i}{l^2-m_c^2}\right)(-ig\,(t^a)_{mj})$$

$$[\varepsilon^*(q)\cdot\varepsilon^*(q)\,(q-p)^\lambda + 2p\cdot\varepsilon^*(q)\,\varepsilon^\lambda(p) - 2q\cdot\varepsilon^*(p)\,\varepsilon^\lambda(q)]$$

Check the relative sign by checking the Ward identity: $\varepsilon^*(p)\to p$

$$\to\quad -i\frac{g^2}{\sqrt{2}}\,\epsilon^{mjk}(t^a)_{mi}\;\bar{v}\,\underbrace{\bar{\sigma}\cdot p}_{\bar{\sigma}\cdot(p-k)}\,\frac{\sigma\cdot(p-k)}{(p-k)^2}\,\bar{\sigma}\cdot\varepsilon^*(q)\,u(k)$$

$$+i\frac{g^2}{\sqrt{2}}\,\epsilon^{ijm}(t^a)_{mk}\;\bar{v}\,\bar{\sigma}\cdot\varepsilon^*(q)\frac{\sigma(k-p)}{(k-p)^2}\,\underbrace{\bar{\sigma}\cdot p}_{\bar{\sigma}\cdot(p-k)}\,u(k)$$

$$+i\frac{g^2}{\sqrt{2}}\,\epsilon^{imk}(t^a)_{mj}\;\bar{v}\,\bar{\sigma}^\lambda u\;\frac{1}{l^2-m_c^2}$$

$$[\;\underbrace{p\cdot\varepsilon^*(q)}_{\varepsilon^*(q)\cdot p}\;\underbrace{(q-p)^\lambda}_{(p+q)^\lambda} + 2p\cdot\varepsilon^*(q)\,p^\lambda - \underbrace{2p\cdot q}_{l^2-m_c^2}\,\varepsilon^\lambda(q)]$$

$$= -i\frac{g^2}{\sqrt{2}}\,\epsilon^{mjk}(t^a)_{mi}\;\bar{v}\,\bar{\sigma}\cdot\varepsilon^*(q)\,u \;-\; i\frac{g^2}{\sqrt{2}}\,\epsilon^{ijm}(t^a)_{mk}\,\bar{v}\,\bar{\sigma}\cdot\varepsilon^*(q)\,u$$

$$-i\frac{g^2}{\sqrt{2}}\,\epsilon^{imk}(t^a)_{mj}\;\bar{v}\,\bar{\sigma}^\lambda u\cdot\frac{1}{(l^2-m_c^2)}\cdot l^2-m_c^2\;\;\varepsilon_\lambda(q)$$

which is $= 0$ by virtue of the identity

$$\epsilon^{mjk}(t^a)_{mi} + \epsilon^{imk}(t^a)_{mj} + \epsilon^{ijm}(t^a)_{mk} = 0 \quad !$$

It will be convenient to replace

$$\epsilon^{imk}(t^a)_{mj} = -\epsilon^{mjk}(t^a)_{mi} - \epsilon^{ijm}(t^a)_{mk}$$

in the 3-vector vertex diagram.

For the kinematic part of the first two diagrams, we can use the expressions on p. 33. For the third diagram, we need:

$$\bar{v}\bar{\epsilon}^{\lambda}u = (\sqrt{2E})^2 (-i\sigma)\,\bar{\epsilon}^{\lambda}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}$$

$$= 2E\,(0,-1,i,0)^{\lambda}$$

$$\bar{v}\bar{\epsilon}^{\lambda}u\cdot(q-p)_{\lambda} = 2E\,(0,-1,i,0)\cdot(\Sigma-p,\,2ps,\,0,\,2pc)$$

$$= 2E\cdot 2ps$$

$$\bar{v}\bar{\epsilon}^{\lambda}u\cdot\epsilon^{*}_{\lambda}(q) = \begin{cases} 2E\cdot\frac{1}{\sqrt{2}}(c-is) & + \\ 2E\cdot\frac{\Sigma s}{m_D} & 0 \\ 2E\cdot\frac{1}{\sqrt{2}}(c+is) & - \end{cases}$$

$$\bar{v}\bar{\epsilon}^{\lambda}u\cdot\epsilon^{*}_{\lambda}(p) = \begin{cases} 2E\,\frac{1}{\sqrt{2}}(-c-is) & + \\ 2E\,\frac{1}{\sqrt{2}}(-c+is) & - \end{cases}$$



$$= -i\frac{g^2}{\sqrt{2}}\left[\epsilon^{mjk}(t^a)_{mi} + \epsilon^{ijm}(t^a)_{mk}\right] \cdot \frac{1}{(2Ep)}$$

$$\times .$$

$u_L u_R \to C(+)\, g(+)$ $\quad$ $=0$ unless $C(-)$ $\quad$ $=0$ for all cases

$$[\bar{v}\bar{\sigma}^\lambda u][\,\varepsilon^*(q)\varepsilon^*(p)(q-p)^\lambda + 2p\cdot\varepsilon^*(q)\cdot\varepsilon^{*\lambda}(p) - 2\,q\cdot\varepsilon^*(p)\,\varepsilon^{*\lambda}(q)\,]$$

$$= \quad 2E\cdot 2ps\cdot 1$$

$u_L u_R \to C(+)\, g(-)$

$$= 0$$

$u_L u_R \to C(-)\, g(+)$

$$= 0$$

$u_L u_R \to C(-)\, g(-)$

$$= \quad 2E\cdot 2ps\cdot 1$$

$u_L u_R \to C(0)\, g(+)$

$$= \quad 2E\cdot 2ps\cdot 0 \quad - \frac{2E}{\sqrt{2}}(1+c)\cdot 2\,\frac{p^2+pE}{m_D}$$

$$= \quad -2E\cdot 2p\cdot\frac{2E}{\sqrt{2}m_D}(1-c)$$

$u_L u_R \to C(0)\, g(-)$

$$= \quad +2E\;2p\;\frac{2E}{\sqrt{2}m_D}(1+c)$$

Now compute:



$$= +i\frac{g^2}{2\sqrt{2}}\,\epsilon^{mjk}(t^a)_{mi}$$

	$\mathcal{G}(+)$	$\mathcal{G}(0)$	$\mathcal{G}(-)$
$g(+)$	$\frac{2Es}{p}$	$(-\frac{\sqrt{2}}{m_D})[\frac{2E\mathcal{E}}{p}(1+c)]$	$[-\frac{2Es}{p}(\frac{1+c}{1-c})]$
$g(-)$	$\frac{2Es}{p}$	$(+\frac{\sqrt{2}}{m_D})[2E(1-c)-\frac{m_g^2}{p}(1+c)]$	$[2s-\frac{m_g^2 s}{2Ep}\frac{1+c}{1-c}]$



$$= +i\frac{g^2}{2\sqrt{2}}\,\epsilon^{ijm}(t^a)_{mk}$$

	$\mathcal{G}(+)$	$\mathcal{G}(0)$	$\mathcal{G}(-)$
$g(+)$	$[2s-\frac{m_g^2}{2Ep}s\frac{(1-c)}{(1+c)}]$	$(-\frac{\sqrt{2}}{m_D})[2E(1+c)-\frac{m_g^2}{p}(1-c)]$	$\frac{2Es}{p}$
$g(-)$	$[-\frac{2Es}{p}(\frac{1-c}{1+c})]$	$(\frac{\sqrt{2}}{m_D})[\frac{2E\mathcal{E}}{p}(1-c)]$	$\frac{2Es}{p}$



$$= +i\frac{g^2}{2\sqrt{2}}\left[\epsilon^{mjk}(t^a)_{mi}+\epsilon^{ijm}(t^a)_{mk}\right]$$

	$\mathcal{G}(+)$	$\mathcal{G}(0)$	$\mathcal{G}(-)$
$g(+)$	$(-2s)$	$(\frac{\sqrt{2}}{m_D})\,2E(1+c)$	0
$g(-)$	0	$-(\frac{\sqrt{2}}{m_D})\,2E(1+c)$	$(-2s)$

So



$= \left(\frac{ig^2}{2\sqrt{2}}\right) \times$

$q/D(+)g(+)$: $\left[\epsilon^{mjk}(t^a)_{mi}\right] \cdot \left[\frac{2(E-p)}{p} s\right] + \left[\epsilon^{ijm}(t^a)_{mk}\right]\left(-\frac{m_q^2}{2Ep} s \frac{(1-c)}{(1+c)}\right)$

$q/D(+)g(-)$: $\left[\epsilon^{mjk}(t^a)_{mi}\right] \cdot \left(\frac{2Es}{p}\right) + \left[\epsilon^{ijm}(t^a)_{mk}\right]\left[-\frac{2Es}{p}\frac{(1-c)}{(1+c)}\right]$

$q/D(-)g(+)$: $\left[\epsilon^{mjk}(t^a)_{mi}\right]\left[-\frac{2Es}{p}\frac{(1+c)}{(1-c)}\right] + \left[\epsilon^{ijm}(t^a)_{mk}\right]\left[\frac{2Es}{p}\right]$

$q/D(-)g(-)$: $\left[\epsilon^{mjk}(t^a)_{mi}\right]\left[-\frac{m_D^2 s}{2Ep}\frac{(1+c)}{(1-c)}\right] + \left[\epsilon^{ijm}(t^a)_{mk}\right] \underbrace{\frac{2(E-p)s}{p}}_{\frac{m_D^2}{2Ep}\cdot s}$

$q/D(0)g(+)$: $\left[\epsilon^{mjk}(t^a)_{mi}\right]\left[-\sqrt{\frac{2}{m_D^2}}\,\frac{2E(E-p)}{p}/(1+c)\right]$

$\frac{m_q^2}{p}$ →

$+ \left[\epsilon^{ijm}(t^a)_{mk}\right]\left[\frac{\sqrt{2}}{m_D}\frac{m_q^2}{p}(1-c)\right]$

$q/D(0)g(-)$: $\left[\epsilon^{mjk}(t^a)_{mi}\right]\frac{\sqrt{2}}{m_D}\left[-\frac{m_q^2}{p}(1+c)\right]$

$+ \left[\epsilon^{ijm}(t^a)_{mk}\right]\frac{2E(E-p)}{p}(1-c)$

to square the amplitude we need:

$$\left|\epsilon^{mjk}(t^a)_{mi}\right|^2 = 2\,\mathrm{tr}(t^a t^a) = 8 = 6\cdot\frac{4}{3}$$

$$\left|\epsilon^{ijm}(t^a)_{mk}\right|^2 = 2\,(t^a)_{mk}(t^a)_{km} = 8$$

$$\epsilon^{mjk}(t^a)_{mi}\,\epsilon^{ijl}(t^a)^*_{lk} = (\delta^{im}\delta^{kl} - \delta^{ml}\delta^{ki})\,t^a_{mi}(t^a)_{kl}$$

$$= 0 - \mathrm{tr}\,t^a t^a = -4$$

then ← $u_1 u_2 + u_2 u_1$

$$\sum_{\text{color}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^4}{8} \cdot 2 \cdot$$

$$\not{D}(+)\not{g}(+) \cdot \Big\{ 8 \cdot \left(\frac{m_{\text{ex}}^2}{2Ep} s\right)^2 + 8 \left(\frac{m_{\text{ex}}^2}{2Ep} s\right)^2 \left(\frac{1-c}{1+c}\right)^2 - 4 \cdot 2 \left[-\left(\frac{m_{\text{ex}}^2 s}{2Ep}\right)^2 \left(\frac{1-c}{1+c}\right) \right]$$

$$\not{D}(+)\not{g}(-) + 8 \left(\frac{2Es}{p}\right)^2 + 8 \left(\frac{2Es}{p}\right)^2 \left(\frac{1-c}{1+c}\right)^2 - 4 \cdot 2 \left[-\left(\frac{2Es}{p}\right)^2 \left(\frac{1-c}{1+c}\right) \right]$$

$$\not{D}(-)\not{g}(+) + \not{D}(-)\not{g}(-) + (1-c) \leftrightarrow (1+c)$$

$$\not{D}(0)\not{g}(+) + \left(\frac{q}{m_D^2}\right) \Big[8 \left[\frac{m_{\text{ex}}^2}{p} (1+c) \right]^2 + 8 \left[\frac{m_{\text{ex}}^2}{p} (1-c) \right]^2 - 4 \cdot 2 \left[-\left(\frac{m_{\text{ex}}^2}{p}\right)^2 (1+c)(1-c) \right]$$

$$\not{D}(0)\not{g}(-) \times 2 \Big\}$$

$$= \frac{g^4}{8} \cdot 2 \cdot 8 \cdot \Big\{ \frac{m_{\text{ex}}^4 s^2}{(2Ep)^2} \left[\left(1 + \frac{1-c}{1+c}\right)^2 - \left(\frac{1-c}{1+c}\right) \right]$$

$$+ \left(\frac{2Es}{p}\right)^2 \left[\left(1 + \frac{1-c}{1+c}\right)^2 - \frac{1-c}{1+c} \right]$$

$$+ (1-c) \leftrightarrow (1+c)$$

$$+ \frac{2}{m_D^2} \left(\frac{m_{\text{ex}}^2}{p}\right)^2 \left[[(1+c) + (1-c)]^2 - (1+c)(1-c) \right] \cdot 2 \Big\}$$

$$= 2g^2 \left\{ \frac{16E^4+m_c^4}{(Ep)^2} \frac{s^2}{(1+c)^2} - \frac{16E^4+m_c^4}{(Ep)^2} \frac{1}{4} \frac{(1-c)}{(1+c)} s^2 \right.$$

$$+ (1-c) \leftrightarrow (1+c)$$

$$\left. + \frac{4m_c^2}{p^2} [4 - (1-c^2)] \right\}$$

we recognize the structure

$$\frac{16E^4+m_c^4}{(Ep)^2} \left(\frac{1-c}{1+c} + \frac{1+c}{1-c} \right) + \frac{16m_c^2}{p^2}$$

$$= 16 \left(\frac{t}{u} + \frac{u}{t} + \frac{2sm_c^2}{ut} \right)$$

the rest is:

$$- \left[\frac{16E^4+m_c^4}{(Ep)^2} \frac{1}{4} \left(\frac{1-c}{1+c} s^2 + \frac{1+c}{1-c} s^2 \right) + \frac{4m_c^2}{p^2} (1-c^2) \right]$$

$$= - \left[4 \frac{\left[\left(E^2 - \frac{1}{4}m_c^2 \right)^2 + \frac{1}{2}E^2 m_c^2 \right]}{E^2 p^2} \left[(1+c)^2 + (1-c)^2 \right] + \frac{4m_c}{p^2} (1-c^2) \right]$$

$$= - 4 \cdot 2 (1+c^2) - 4 \frac{m_c^2}{p^2} \left[(1+c^2) + (1-c^2) \right]$$

$$= - 2 \cdot \frac{[2Ep(1-c)]^2 + [2Ep(1+c)]^2}{2E^2p^2} - \frac{4m_c^2 E^2}{E^2p^2}$$

$$= - \frac{u^2 + t^2 + m_c^2 s}{E^2p^2} = -16 \frac{u^2+t^2+m_c^2 s}{s(s-m_c^2)}$$

$\approx$ all

$$\sum_{\text{colors}} \sum_{\text{spins}} |\mathcal{M}|^2 = 32 g^2 \left[\frac{u}{t} + \frac{t}{u} + \frac{2sm_c^2}{ut} - \frac{u^2+t^2+m_c^2 s}{s(s-m_c^2)} \right]$$

$$\left[\text{cf.} \quad q\bar{q} \to gg \propto \left(\frac{u}{t} + \frac{t}{u} - \frac{9}{4} \frac{u^2+t^2}{s^2} \right) \right]$$

$$\frac{d\sigma}{d\cos\theta_{cm}} (uu \to c+g) = \frac{1}{2s} \frac{1}{16\pi} \left(1 - \frac{m_c^2}{s}\right) \frac{1}{9} \frac{1}{4} \sum_{\text{colors}} \sum_{\text{spins}} |\mathcal{M}|^2$$

$$\frac{d\sigma}{d\cos\theta_{cm}} (uu \to cg) = \frac{4}{9} \frac{\pi \alpha_s^2}{s} \left(1 - \frac{m_c^2}{s}\right)$$

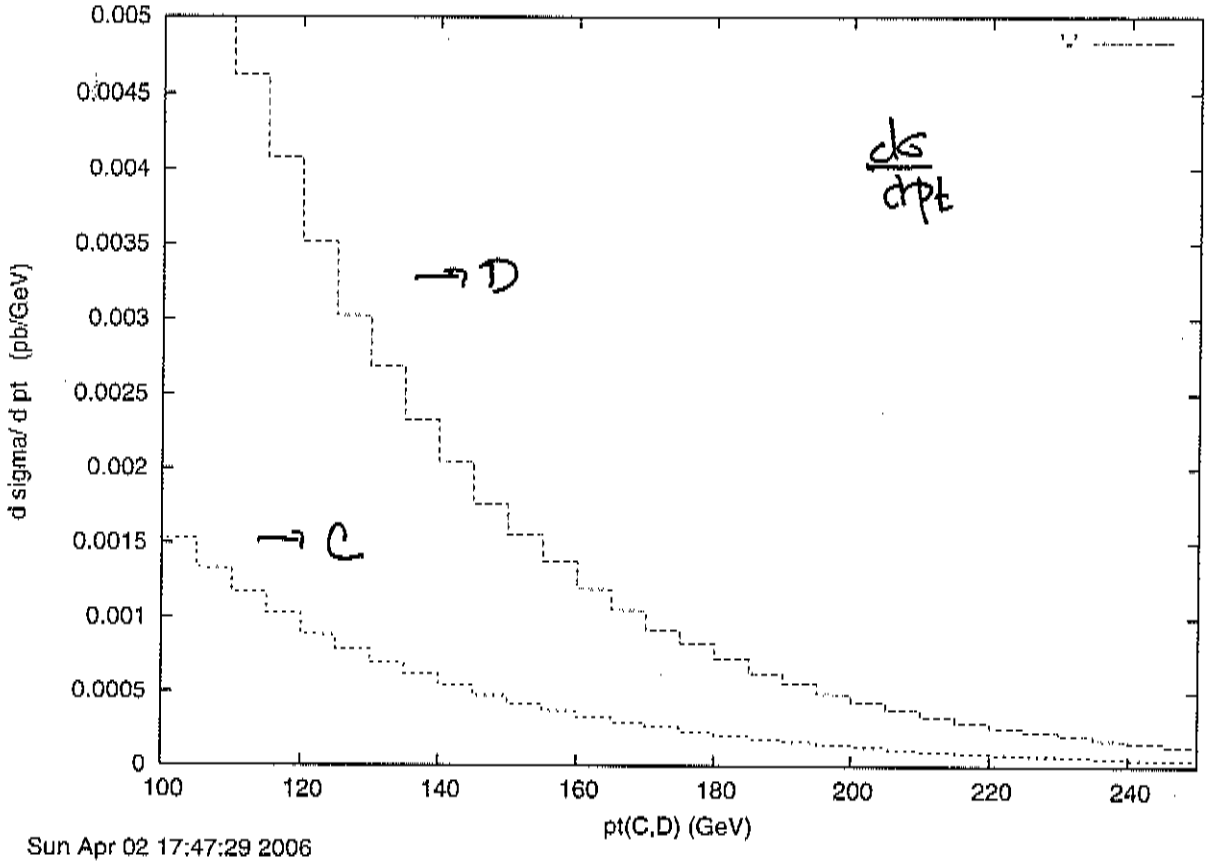
$$\cdot \left[\frac{u}{t} + \frac{t}{u} + \frac{2sm_c^2}{ut} - \frac{u^2+t^2+m_c^2 s}{s(s-m_c^2)} \right]$$

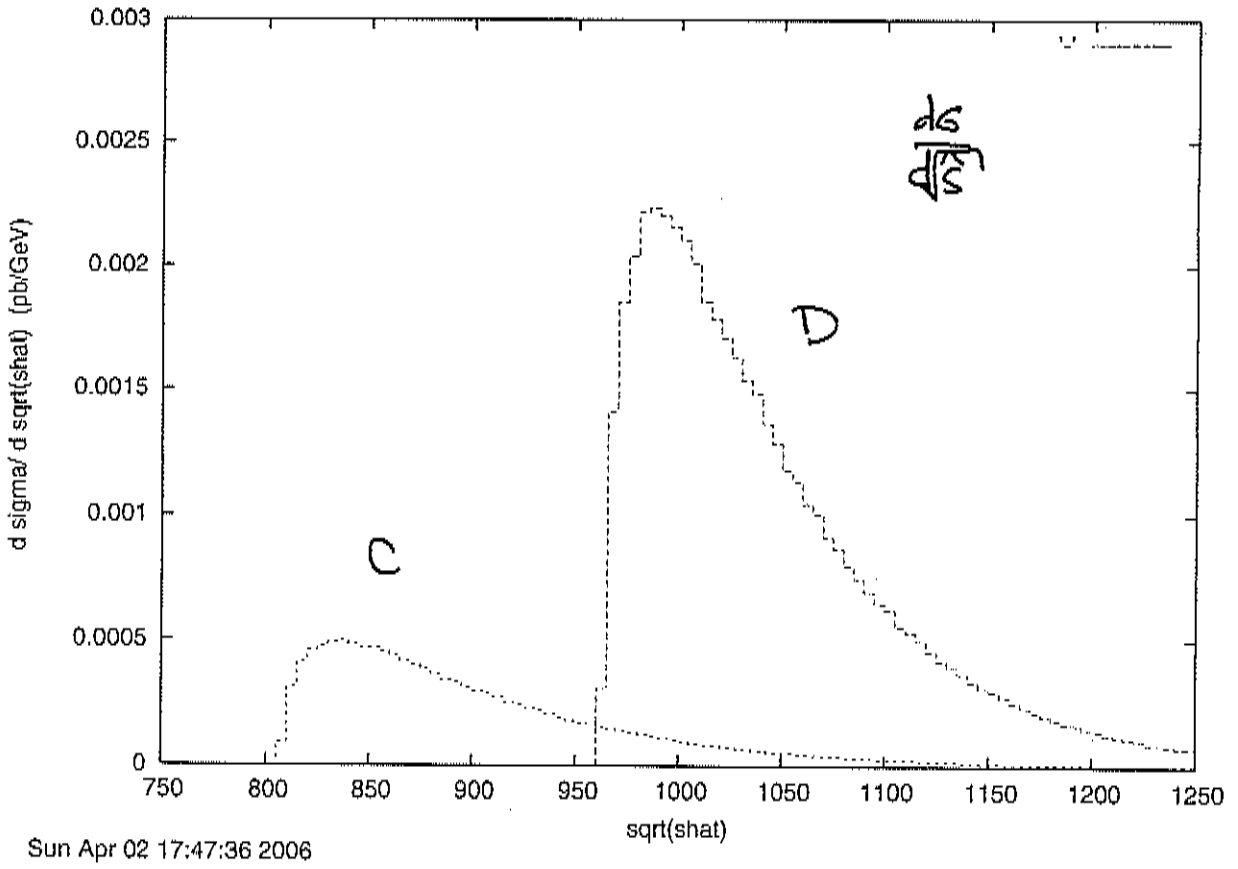
$$\frac{d\sigma}{d\cos\theta_{cm}} (ug \to cu) = \frac{4}{9} \frac{\pi \alpha_s^2}{s} \left(1 - \frac{m_c^2}{s}\right)$$

$$\cdot \left[-\frac{s}{t} - \frac{t}{s} - \frac{2um_c^2}{st} + \frac{s^2+t^2+m_c^2 u}{u(u-m_c^2)} \right]$$

```
#include "jets.h"
#include "gnuplot.h"


class CDjets: public jets {

  /* computes the rate for heavy boson production at a ppbar collider  */

 public:

  CDjets(double ecm, int Which, double Mass, double gamma): jets(ecm, 100.0),
      P("mrst/vnvalf1155.dat"), which(Which), M(Mass), Gamma(gamma){}

  mrst P;
  int which;  /*  set which = 0 for D , -1 for Cbar, 1 for C  */
  double M, Gamma;
  double prefactor;

   /* at the time crosssection is called, x1, x2, etc. are defined */

   /* crosssection is supposed to return
               (d sigma/ dx1 dx2 d cos theta_CM ) * (2/shat))
    for this program, compute   d sigma /dx1 dx2 d cos thata_CM
       and multiply by   2/shat at the end   */

  int redokinematics(){
       /*   redo  kinematics for a massive particle in the final state  */
    if (shat <=   M*M) return 0;     /* validity check: this is important ! */
    double E = sqrt(shat)/2.0;
    p =   (shat - M*M)/(4.0*E);
    that = - 2.0 * E * p * (1.0 - cost);
    uhat = - 2.0 * E * p * (1.0 + cost);
    pt = p*sint;
    if (pt < ptmin) return 0;   /*   redo the pt cut on the integral */
    const double alphas = 0.12;
    prefactor = (PI * alphas * alphas/shat) * ( 1.0 - M*M/shat) * pb ;
    return 1;
  }

  double uubartoDg(){
    int valid = redokinematics();
    if (valid == 0 ) return 0.0;
    return  (2.0/27.0) * prefactor * (uhat/that + that/uhat
                                      + 2.0 * shat * M*M/(uhat*that));
  }

  double ugtoDu(){
    int valid = redokinematics();
    if (valid == 0 ) return 0.0;
    return  - (1.0/36.0) * prefactor * (shat/that + that/shat
                                      + 2.0 * uhat * M*M/(shat*that));
  }

  double uutoCg(){
    int valid = redokinematics();
    if (valid == 0 ) return 0.0;
    return  (4.0/9.0) * prefactor * (uhat/that + that/uhat
                  + 2.0 * shat * M*M/(uhat*that)
          - (uhat*uhat + that*that + shat * M *M)/(shat * (shat - M*M)));
  }
```

```
  double ugtoCu(){
    int valid = redokinematics();
    if (valid == 0 ) return 0.0;
    return  - (4.0/9.0) * prefactor * (shat/that + that/shat
                + 2.0 * uhat * M*M/(shat*that)
          - (shat*shat + that*that + uhat * M *M)/(uhat * (uhat - M*M)));
  }

  double crosssection(){
    if (shat < SQR(500.0)) return 0.0;
    const double alphas = 0.12;
    double fu1 = P.xfu(x1,Q)/x1;
    double fd1 = P.xfd(x1,Q)/x1;
    double fub1 = P.xfubar(x1,Q)/x1;
    double fdb1 = P.xfdbar(x1,Q)/x1;
    double fu2 = P.xfu(x2,Q)/x2;
    double fd2 = P.xfd(x2,Q)/x2;
    double fub2 = P.xfubar(x2,Q)/x2;
    double fdb2 = P.xfdbar(x2,Q)/x2;
    double fg1 = P.xfg(x1,Q)/x1;
    double fg2 = P.xfg(x2,Q)/x2;
    double cs = 0.0;
    if (which == 1 ) {
      cs = (fu1 * fub2 + fd1 * fdb2) * uutoCg()
        + (fu1 * fg2 + fg1 * fub2  + fd1 * fg2 * fg1 * fdb2) * ugtoCu();
    }   else if (which == -1) {
      cs = (fub1 * fu2 + fdb1 * fd2) * uutoCg()
        + (fub1 * fg2 + fg1 * fu2  + fdb1 * fg2 * fg1 * fd2) * ugtoCu();
    }   else if (which == 0) {
      cs = (fu1 * fu2  + fd1 * fd2  +  fub1 * fub2 + fdb1 * fdb2)* uutoCg()
     +  ((fu1 + fub1 + fd1 + fdb1) * fg2 +  fg1* (fu2 + fub2 + fd2 + fdb2))
            * ugtoDu();
    }
    return cs * (2.0/shat);
  }

};

main(){
```


0.005
0.0045
0.004
0.0035
0.003
0.0025
0.002
0.0015
0.001
0.0005
0
d sigma/ d pt (pb/GeV)
100
120
140
160
180
200
220
240
pt(C,D) (GeV)
dσ/dpt
D
C
Sun Apr 02 17:47:29 2006


0.003
0.0025
0.002
0.0015
0.001
0.0005
0
d sigma/ d sqrt(shat) (pb/GeV)
750
800
850
900
950
1000
1050
1100
1150
1200
1250
sqrt(shat)
dσ/dŝ
D
C
Sun Apr 02 17:47:36 2006

Physics 332 – Problem Set # 1

(due Wednesday, April 19)

1. Peskin and Schroeder, Problem 10.2.

2. Peskin and Schroeder, Problem 10.3.

3. Peskin and Schroeder, Problem 10.4. The correct answer to this problem should be:

$$i\mathcal{M} \sim -i\lambda - i\frac{\lambda^2}{(4\pi)^2}\log s - i\frac{3}{2}\frac{\lambda^3}{(4\pi)^4}\log^2 s + \cdots \tag{1}$$

   [Thanks to Dean Lee.]

Physics 332 – Problem Set #1

Solutions

1.) a) $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\not{\partial} - M)\psi$

$$-ig\bar{\psi}\gamma^5\psi\phi$$

the theory has a parity symmetry under which $\phi \to -\phi$

g is a dimensionless coupling constant

The degree of divergence of a diagram with N_ψ external ψ lines, N_ϕ external ϕ lines, P_ψ P_ϕ propagators, V vertices, L loops is

$$D = 4L - P_\psi - 2P_\phi$$

$$L = P_\psi + P_\phi - V + 1$$

so

$$D = 3P_\psi + 2P_\phi - 4V + 4$$

$$V = 2P_\phi + N_\phi \qquad 2V = 2P_\psi + N_\psi$$

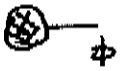
$$3V = 3P_\psi + \tfrac{3}{2}N_\psi$$

so finally

$$D = 4 - N_\phi - \tfrac{3}{2}N_\psi$$

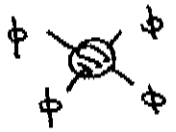
The superficially divergent amplitudes are:

$D = 4$ vacuum energy → ignore

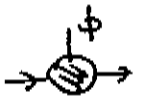

$D = 3$ $= 0$ by parity


$D = 2$ $= A p^2 + B$ 2 divergent terms


$D = 1$ $= 0$ by parity


$D = 0$ $= C$ 1 divergent term


$D = 1$ $= D\not{p} + E$ 2 divergent terms.


$D = 0$ $= F\gamma^5$ 1 divergent term

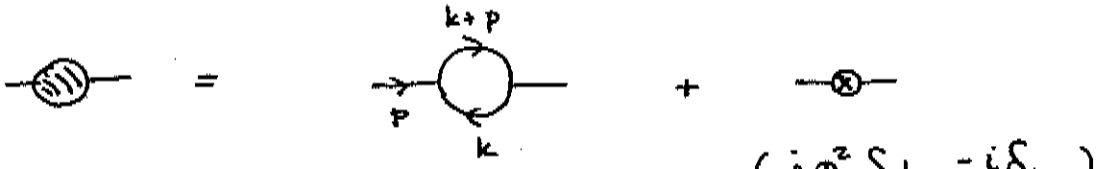
We need a counterterm for each divergent amplitude:

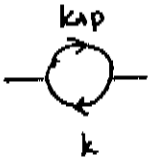
$$\delta\mathcal{L} = \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{1}{4!}\delta_\lambda\phi^4$$

$$+ \delta_\psi\,\bar{\Psi}\, i\not{\partial}\psi - \delta_M\,\bar{\Psi}\psi - i\,\delta_g\,\bar{\Psi}\gamma^5\psi\,\phi$$

Notice that, even though we omitted the ϕ^4 term in $\mathcal{L}$, we need the counterterm in $\delta\mathcal{L}$. The ϕ^4 interaction is generated by quantum corrections.

b.) Now compute the diagrams that diverge in 1 loop and determine the counterterms:





$$= g^2 (-1) \int \frac{d^d k}{(2\pi)^d} \, \mathrm{tr}\left[\gamma^5 \, i\frac{(\not{k}+\not{p}+M)}{(k+p)^2 - M^2} \, \gamma^5 \, i\frac{(\not{k}+M)}{k^2 - M^2} \right]$$

(5 min loop)

$$= g^2 \int \frac{d^d k}{(2\pi)^d} \, \frac{\mathrm{tr}\, \gamma^5 (\not{k}+\not{p}+M) \gamma^5 (\not{k}+M)}{[(k+p)^2 - M^2][k^2 - M^2]}$$

$$\frac{1}{((k+p)^2 - M^2)(k^2 - M^2)} = \int_0^1 dx \, \frac{1}{[\ell k^2 + x(1-x)p^2 - M^2]^2}$$

where $\ell k = k + xp$ $\qquad k = \ell k - xp$

$k+p = \ell k + (1-x)p$

$$\mathrm{tr}\, \gamma^5 (\not{k}+\not{p}+M) \gamma^5 (\not{k}+M) = \mathrm{tr}\,[(\not{\ell k} + (1-x)\not{p}) + M][-(\not{\ell k} - x\not{p}) + M]$$

$$= -4\,[\ell k + (1-x)p]\cdot[\ell k - xp] + 4M^2$$

$$= -4\ell k^2 + 4x(1-x)p^2 + 4M^2$$

So

$$= 4g^2 \int_0^1 dx \int \frac{d^d \ell k}{(2\pi)^d} \, \frac{[-\ell k^2 + x(1-x)p^2 + M^2]}{[\ell k^2 + x(1-x)p^2 - M^2]^2}$$

Now $$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2-\Delta]^2} = \frac{i}{(4\pi)^{d_2}} \frac{\Gamma(2-d_2)}{\Delta^{2-d_2}} \sim \frac{i}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{-k^2}{[k^2-\Delta]^2} = \frac{i}{(4\pi)^{d_2}} \frac{\Gamma(1-d_2)}{\Delta^{1-d_2}} \cdot \frac{d}{2}$$

$$= \frac{i}{(4\pi)^{d_2}} \frac{d}{2} \frac{1}{1-d_2} \Delta \frac{\Gamma(2-d_2)}{\Delta^{2-d_2}}$$

$$\sim \frac{i}{(4\pi)^2} (-2\Delta) \frac{2}{\epsilon} \qquad \text{near } d=4$$

so the divergent part of this graph is:

$$= 4g^2 \int_0^1 dx \; \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \left\{ -2[M^2 - x(1-x)p^2] + [x(1-x)p^2 + M^2] \right\}$$

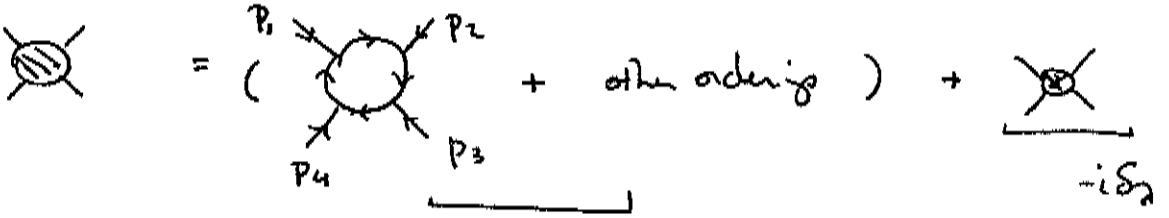
$$\int_0^1 dx \; x(1-x) = \frac{1}{6}$$

$$= \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \; 4g^2 \left(-M^2 + \tfrac{1}{2} p^2\right)$$

then, to cancel this:

$$\delta_\phi = -\frac{4g^2}{(4\pi)^2} \frac{1}{\epsilon} \qquad \delta_m = -\frac{8g^2}{(4\pi)^2} M^2 \frac{1}{\epsilon}$$

if we had included a $\frac{\lambda \phi^4}{4!}$ term in $\mathcal{L}$, as we should have, we would have had also the diagram Ω, so that $\delta_m = \left(-\frac{8g^2 M^2}{(4\pi)^2} - \frac{\lambda m^2}{(4\pi)^2}\right) \frac{1}{\epsilon}$



there are $3! = 6$ diagrams in which the external lines appear in different orders.

To evaluate the divergent part, we can set all external momenta equal to zero

$$= 6 \cdot g^4 (-1) \int \frac{d^d k}{(2\pi)^d} \, \mathrm{tr}\left[\frac{i(\not{k}+M)}{(k^2-M^2)} \gamma^5 \right]^4$$

only the $\not{k}$ term gives a divergence

$$= 6 \cdot g^4 (-1) \cdot \int \frac{d^d k}{(2\pi)^d} (i^4) \, \mathrm{tr} \frac{\not{k}\gamma^5 \not{k}\gamma^5 \not{k}\gamma^5 \not{k}\gamma^5}{[k^2-M^2]^4}$$

$$= -6 g^4 \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{tr}\, \not{k}(k) \not{k}(-k)}{[k^2-M^2]^4}$$

$$= -6 g^4 \int \frac{d^4 k}{(2\pi)^4} \frac{4(k^2)^2}{[k^2-M^2]^4}$$

this has the same divergence as

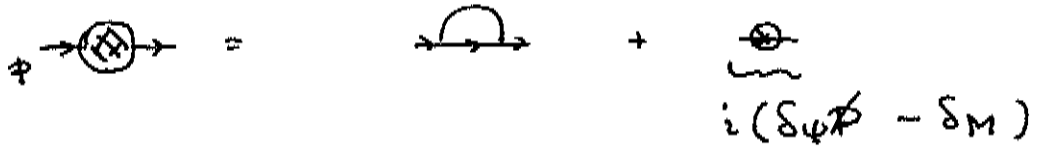
$$= -24 g^4 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2)^2}$$

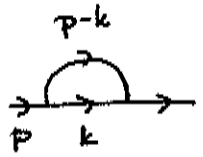
$$= -24 g^4 \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \qquad \text{so} \qquad \delta_2 = -\frac{48}{(4\pi)^2} g^4 \frac{1}{\epsilon}$$

again, if the $\lambda\phi^4$ vertex had been included, we would have found additional diagrams 

so that

$$\delta_\lambda = \left[-\frac{48 g^4}{(4\pi)^2} + \frac{3\lambda^2}{(4\pi)^2} \right] \frac{1}{\epsilon}$$





$$= g^2 \int \frac{d^d k}{(2\pi)^d} \gamma^5 \frac{i(\not{k} + M)}{k^2 - M^2} \gamma^5 \frac{i}{(p-k)^2 - m^2}$$

$$= -g^2 \int \frac{d^d k}{(2\pi)^d} \frac{(-\not{k} + M)}{(k^2 - M^2)((p-k)^2 - m^2)}$$

continue w. a Feynman parameter

$$\ell k = k - xp$$
$$k = \ell k + xp$$
$$k - p = \ell k - (1-x)p$$

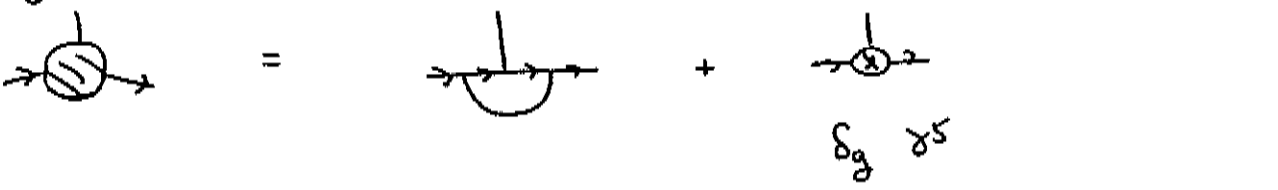
$$= -g^2 \int_0^1 dx \int \frac{d^d \ell k}{(2\pi)^d} \frac{-\ell\!\!\!/ + x\not{p} + M}{[\ell k^2 + x(1-x)p^2 - M^2]^2}$$

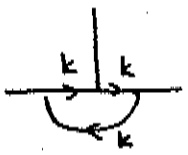
$$= -\frac{i g^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2 - d/2)}{[M^2 - x(1-x)p^2]^{2-d/2}} (x\not{p} + M)$$

$$= -\frac{i g^2}{(4\pi)^2} \left(\frac{1}{2}\not{p} + M\right) \frac{2}{\epsilon} + \cdots$$

so $\delta_\psi = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}$ $\qquad \delta_M = -\frac{2g^2}{(4\pi)^2} M \frac{1}{\epsilon}$

Finally





$$= g^3 \int \frac{d^dk}{(2\pi)^d} \gamma_5 \frac{i\not{k}}{k^2} \gamma_5 \frac{i\not{k}}{k^2} \gamma_5 \frac{i}{k^2}$$

set ext. mom = 0

$$= -i g^3 \int \frac{d^dk}{(2\pi)^d} \frac{(-k^2)\gamma_5}{(k^2)^3}$$

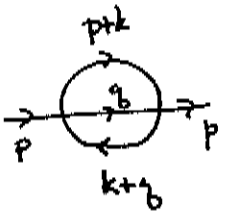
$$= i g^3 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2} \gamma_5$$

$$= i g^3 \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \gamma_5$$

$$= -\frac{g^3}{(4\pi)^2}\frac{2}{\epsilon}\gamma_5$$

so $\delta_g = \frac{2g^3}{(4\pi)^2}\frac{1}{\epsilon}$

2.)



$$= \frac{(-i\lambda)^3}{3!}\int \frac{d^dk\, d^dq}{(2\pi)^{2d}} \frac{i}{(q+k)^2-m^2}\frac{i}{q^2-m^2}\frac{i}{(p+k)^2-m^2}$$

$$= \frac{-i\lambda^2}{6}\int \frac{d^dk\, d^dq}{(2\pi)^{2d}} \frac{1}{m^2-(q+k)^2}\frac{1}{m^2-q^2}\frac{1}{m^2-(p+k)^2}$$

It is easier to rotate to Euclidian space. At the end of the calculation, we will return to Minkowski space by setting $p^2 \to -p^2$

$$= \frac{-i\lambda^2 (i)^2}{6}\int \frac{d^dk\, d^dq}{(2\pi)^{2d}} \frac{1}{m^2+(q+k)^2}\frac{1}{m^2+q^2}\frac{1}{m^2+(p+k)^2}$$

$$= \frac{i\lambda^2}{6}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx \frac{1}{(4\pi)^{d_k}} \frac{\Gamma(2-d_k)}{[m^2+x(1-x)k^2]^{2-d_k}}\frac{1}{m^2+(p+k)^2}$$

$$= \frac{i\lambda^2}{6}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx\,[x(1-x)]^{d_k-2}\int_0^1 dy \frac{\Gamma(2-d_k)}{(4\pi)^{d_k}}$$

$$\cdot\, y^{d_k-1}\frac{\Gamma(3-d_k)}{\Gamma(2-d_k)\underbrace{\Gamma(1)}_{1}}\left[\frac{1}{k^2+y(1-y)p^2+\left[y\frac{m^2}{x(1-x)}+(1-y)m^2\right]}\right]^{3-d_k}$$

$$= \frac{i\lambda^2}{6}\int_0^1 dx\,[x(1-x)]^{-\varepsilon/2}\int_0^1 dy\; y^{\varepsilon/2-1}\frac{\Gamma(4-d_k)}{(4\pi)^{d_k}}\cdot\frac{1}{(4\pi)^{d_k}}\frac{\Gamma(3-d)}{\Gamma(4-d_k)}$$

$$\cdot\frac{1}{\left[y(1-y)p^2+y\frac{m^2}{x(1-x)}+(1-y)m^2\right]^{3-d}}$$

At this point, we could in fact send $m^2 \to 0$

$$= \frac{i\partial^2}{6} \int_0^1 dx\, [x(1-x)]^{-\varepsilon/2} \int_0^1 dy\, y^{\varepsilon/2-1} [y(1-y)p^2]^{1-\varepsilon} \frac{\Gamma(3-d)}{(4\pi)^d}$$

$$= i\frac{\partial^2}{6} \int_0^1 dx\, [x(1-x)]^{-\varepsilon/2} \int_0^1 dy\, y^{-\varepsilon/2} (1-y)^{1-\varepsilon} [p^2]^{1-\varepsilon} \frac{\Gamma(3-d)}{(4\pi)^d}$$

$$\Gamma(3-d) = \frac{1}{3-d}\Gamma(4-d) \sim -\frac{1}{\epsilon} \quad \text{as} \quad \epsilon \to 0$$

$$\int_0^1 dy\, y^{-\varepsilon/2}(1-y)^{1-\varepsilon} \approx \int_0^1 dy\,(1-y) = \frac{1}{2}$$

$$= -i\frac{\partial^2}{12} [p^2]^{1-\varepsilon} \frac{1}{(4\pi)^4} \frac{1}{\epsilon} + \cdots$$

$$\cong -i\frac{\partial^2}{12} p^2 \frac{1}{(4\pi)^4} \left(\frac{1}{\epsilon} - \ln p^2 + \cdots\right)$$

retin to Minkowski space:

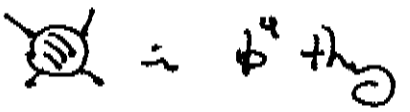


$$= -i\frac{\partial^2}{12}\frac{p^2}{(4\pi)^4} \cdot \left(-\frac{1}{\epsilon} + \ln(-p^2) + \cdots\right)$$



is manifestly indepent of p thus

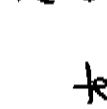
$$= -i\,(\text{const})\, m^2 \to 0 \quad \text{as} \quad m^2 \to 0$$

3.) We would like to evaluate  in ϕ^4 theory for $s \to \infty$, t fixed. This implies

$$u = 4m^2 - s - t \sim -s \quad \text{as} \quad s \to \infty$$

 $= -i\lambda$

-  $= i\frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx \ \ln\left[\frac{M^2}{m^2 - x(1-x)s}\right]$ + divergent constant

where m is the boson mass, M is the $\overline{MS}$ regulator mass. The divergence is cancelled by a part of . The leading term for large s is

$$= -i\frac{\lambda^2}{2(4\pi)^2}\ln s$$

 $\sim \ln t/M^2$ + divergent which is $\theta(1) + \ln s$

 $\sim -i\frac{\lambda}{2(4\pi)^2}\ln u \sim -i\frac{\lambda}{2(4\pi)^2}\ln s$

so



$$= -i\frac{\lambda}{(4\pi)^2}\ln s + \ldots$$

Now evaluate the 2-loop diagrams. I will start with the s-channel diagrams:



Additional contributions come from the t- and u-channel diagrams.

$s=P^2$ 

$$= (-i\partial)^3 \left[\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{(k+P)^2-m^2} \right]^2$$

$$= i\partial^3 \left[\frac{1}{2} \frac{i}{(4\pi)^{d_4}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \frac{1}{[m^2-x(1-x)P^2]^{2-d_4}} \right]^2$$

$$= i\partial^3 \left[\frac{1}{2} \frac{i}{(4\pi)^2} \left\{ \frac{2}{\epsilon} + \cdots \right\} \right.$$

$$\left. \cdot \left\{ 1 - \tfrac{\epsilon}{2} \ln s + \tfrac{1}{2}(\tfrac{\epsilon}{2})^2 \ln^2 s + \cdots \right\} \right]^2$$

The second line comes from $\frac{1}{s^{\epsilon/2}} = \exp[-\tfrac{\epsilon}{2}\ln s]$

$$= i\partial^3 \frac{1}{4} \frac{(-1)^2}{(4\pi)^4} \left(\frac{2}{\epsilon}\right)^2 \left\{ 1 - \epsilon \ln s + \tfrac{\epsilon^2}{2} \ln^2 s + \cdots \right\}$$

The finite term with the strongest s dependence is

$$\sim -i \frac{\partial^3}{4(4\pi)^2} \cdot 2 \cdot \ln^2 s$$

Something is odd here. If we had made finite by adding the counterterm before squaring, we would have found:

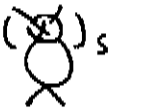
$$(-i\lambda)^3 \left[\frac{1}{2} \frac{1}{(4\pi)^2} \log \frac{M^2}{s} + \cdots \right]^2$$

$$\sim \quad \frac{-i\lambda^3}{4(4\pi)^2} \log^2 s$$

The difference is accounted for by the contribution from the counterterm diagrams. Recall that

 $= -i\delta_\lambda = -i \frac{3\lambda^2}{2(4\pi)^2} \cdot \frac{2}{\epsilon}$

of this, $\frac{1}{3}$ cancels the divergence of the s-channel subdiagram. Then



$$= \left(-i \frac{\lambda^2}{2(4\pi)^2} \frac{2}{\epsilon}\right)(-i\lambda) \cdot \underbrace{2}_{\text{2 diagrams}} \cdot \underbrace{\frac{1}{2}}_{\text{sym factor of loop}} \int \frac{d^d k}{(4\pi)^d} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

$$\cong \quad \frac{\lambda^3}{(4\pi)^2} \frac{1}{\epsilon} \frac{i}{(4\pi)^2} \Gamma(2) \cdot \frac{1}{s^{2\epsilon}}$$

$$\cong \quad \frac{+i\lambda^3}{(4\pi)^4} \frac{2}{\epsilon^2} \left[1 - \epsilon \log s + \frac{\epsilon^2}{8} \log^2 s + \cdots \right]$$

So



$$= -i\frac{\lambda^3}{(4\pi)^4}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ell_S + \frac{1}{2}\ell_S^2 + \cdots\right]$$

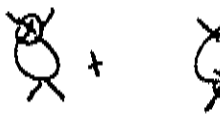
$$-i\frac{\lambda^3}{(4\pi)^4}\left[-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\ell_S - \frac{1}{4}\ell_S^2 + \cdots\right]$$

$$= -i\frac{\lambda^3}{(4\pi)^4}\left[\text{(divergent contact)} + \frac{1}{4}\ell_S^2\right]$$

$\uparrow$ cancelled by $(\quad)_{\Theta(\lambda^3)}$

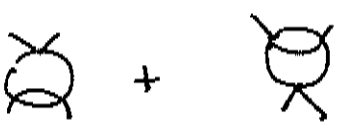
Note that the $\frac{1}{\epsilon}\ell_S$ terms cancel automatically, as they must.

The full contribution from the counterterm diagrams is



$$= -i\frac{3\lambda^3}{(4\pi)^2}\left[\text{(const)} + \frac{1}{\epsilon}\ell_S - \frac{1}{4}\ell_S^2\right]$$

Now we must evaluate





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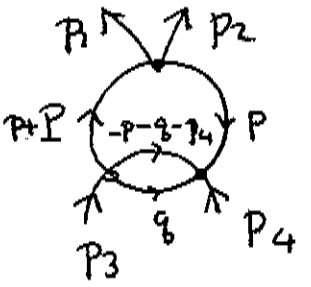


$P^2 = (p_1+p_2)^2 = s$

$$= (-i\partial)^3 \cdot \frac{1}{2} \cdot \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{i}{(p+P)^2-m^2} \frac{i}{p^2-m^2} \frac{i}{q^2-m^2} \frac{i}{(p+q+p_4)^2-m^2}$$

$$= \frac{i\partial^3}{2} \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \int_0^1 dz \frac{1}{[p^2+2zp\cdot P+zP^2-m^2]^2} \int_0^1 dx \frac{1}{[q^2+x(1-x)(p+p_4)^2-m^2]^2}$$

$$= \frac{i\partial^3}{2} \int \frac{d^dp}{(2\pi)^d} \int_0^1 dz\, dx \frac{1}{[p^2+2zp\cdot P+zP^2-m^2]^2} \frac{i}{(4\pi)^{d_4}} \frac{\Gamma(2-d_4)}{[m^2-x(1-x)(p+p_4)^2]^{2-d_4}}$$

It will be useful to rotate to Euclidean space, rotate back at the end

$$= \frac{i\partial^3}{2} \cdot \frac{i}{(4\pi)^{d_4}} \Gamma(2-d_4) \cdot i \int_0^1 dz \int_0^1 dx\, [x(1-x)]^{-\varepsilon_4}$$

$$\cdot \int \frac{d^dp_{(E)}}{(2\pi)^d} \frac{1}{[p^2+2zp\cdot P+zP^2+m^2]^2} \frac{1}{\left[\frac{m^2+x(1-x)(p+p_4)^2}{x(1-x)}\right]^{2-d_4}}$$

↑ $(1-y)$ ↑ ∂

combine a Feynman parameter

the denominator is:

$$\text{Denom} = y\left((p+p_4)^2 + \frac{m^2}{x(1-x)}\right) + (1-y)\left(p^2 + 2zp\cdot P + zP^2 + m^2\right)$$

$$= p^2 + 2p\cdot(yp_4 + (1-y)zP) + yp_4^2 + (1-y)zP^2 + \left[y\frac{1}{x(1-x)} + (1-y)\right]m^2$$

$$= \mathbb{P}^2 - (yp_4 + (1-y)zP)^2 + yp_4^2 + (1-y)zP^2 + m^2\left[\frac{y}{x(1-x)} + 1 - y\right]$$

$$= \mathbb{P}^2 + y(1-y)p_4^2 + (1-y)zP^2 - (1-y)^2z^2P^2 - 2y(1-y)z\,p_4\cdot P + m^2\left[\frac{y}{x(1-x)} + (1-y)\right]$$

$$= \mathbb{P}^2 + y(1-y)z(P-p_4)^2 + y(1-y)(1-z)p_4^2 + P^2\left[(1-y)z - y(1-y)z - (1-y)^2z^2\right] + m^2\left[\frac{y}{x(1-x)} + (1-y)\right]$$

$$= \mathbb{P}^2 + y(1-y)z\,p_3^2 + y(1-y)(1-z)p_4^2 + m^2\left[\frac{y}{x(1-x)} + (1-y)\right] + P^2\left[(1-y)^2z(1-z)\right]$$

$$\equiv \mathbb{P}^2 + \Delta$$

then

$$= -\frac{i\partial^3}{2}\frac{1}{(4\pi)^{d_2}}\Gamma(2-d_2)\int_0^1 dx\,dz\,[x(1-x)]^{-\varepsilon_2}$$

$$\int_0^1 dy\,(1-y)\,y^{\varepsilon_2-1}\,\frac{\Gamma(4-d_2)}{\Gamma(2-d_2)\Gamma(2)}\int\frac{d^d\mathbb{P}}{(2\pi)^d}\,\frac{1}{[\mathbb{P}^2+\Delta]^{4-d_2}}$$

$$= -i \frac{\partial^3}{2} \frac{1}{(4\pi)^{d_2}} \Gamma(2-d_2) \int_0^1 dx\, dz\, [x(1-x)]^{-\varepsilon/2} \int dy\, (1-y)\, y^{\varepsilon/2-1}$$

$$\frac{\Gamma(4-d_2)}{\Gamma(2-d_2)} \frac{1}{(4\pi)^{d_2}} \frac{\Gamma(4-d)}{\Gamma(4-d_2)} \frac{1}{[\Delta]^{4-d}}$$

$$= -i \frac{\partial^3}{2} \frac{1}{(4\pi)^d} \Gamma(4-d) \int_0^1 dx\, dz\, [x(1-x)]^{-\varepsilon/2} \int_0^1 dy\, (1-y)\, y^{\varepsilon/2-1} \frac{1}{[\Delta]^{4-d}}$$

now $p_3^2 = p_4^2 = m^2$ for an on-shell scatt amplitude

for $s \to \infty$ we can ignore m^2. then

$$\Delta \cong P^2 (1-y)^2 z(1-z) = -s\,(1-y)^2 z(1-z)$$

P from Euclidean space.

we have

$$\frac{1}{[s]^\varepsilon} = 1 - \varepsilon \ln(-s) + \frac{\varepsilon^2}{2} \ln^2(-s) + \dots$$

to get a $\ln^2(-s)$, we need to expd to $O(\varepsilon)$, so we need a coefficient $\frac{1}{\varepsilon^2}$. this comes from

$$\Gamma(4-d) \cong \frac{1}{\epsilon} \qquad \int_0^1 dy\, y^{\varepsilon/2-1} = \frac{2}{\epsilon}$$

so

$$= -i \frac{\partial^3}{2} \frac{1}{(4\pi)^4} \frac{1}{\epsilon} \cdot 1 \cdot \frac{2}{\epsilon} \cdot \left[1 - \epsilon \ln(-s) + \varepsilon^2/2 \ln^2 s\right]$$

$$= -i \frac{\partial^3}{2} \frac{1}{(4\pi)^4} \left[\ln^2 s - \frac{2}{\epsilon} \ln s + \dots\right]$$

so

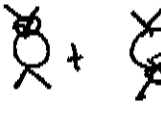
 $= -i \frac{\lambda^3}{(4\pi)^4}\left[-\frac{1}{\epsilon} l_{fs} + \frac{1}{2} l_{fs}^2\right]$

similarly

 $= -i \frac{\lambda^3}{(4\pi)^4}\left[-\frac{1}{\epsilon} l_{fs} + \frac{1}{2} l_{fs}^2\right]$

the contributions computed earlier were:

 $= -i \frac{\lambda^3}{(4\pi)^4}\left[-\frac{1}{\epsilon} l_{fs} + \frac{1}{2} l_{fs}^2\right]$

 $= -i \frac{\lambda^3}{(4\pi)^4}\left[+\frac{3}{\epsilon} l_{fs} - \frac{3}{4} l_{fs}^2\right]$

in all, the s-channel diagrams give

$$-i \frac{\lambda^3}{(4\pi)^4} \cdot \frac{3}{4} l_{fs}^2 s$$

the t-channel diagrams give $-\frac{i\lambda^3}{(4\pi)^4} \cdot \frac{3}{4} l_{fs}^2 t \sim O(1)$

 + ...

the u-channel diagrams give $-i \frac{\lambda^3}{(4\pi)^4} \frac{3}{4} l_{fs}^2 u \sim -i \frac{\lambda^3}{(4\pi)^4} \frac{3}{4} l_{fs}^2 s$

 + ...

in all


$\sim \quad -i\frac{\lambda^3}{(4\pi)^4}\ \frac{3}{2}\ l_s^2 S$


$\sim \quad -i\lambda \quad -i\frac{\lambda^2}{(4\pi)^2} l_s S \quad -i\ \frac{3}{2}\frac{\lambda^3}{(4\pi)^4} l_s^2 S$

Physics 332 – Problem Set # 2

(due Wednesday, April 26)

1. Peskin and Schroeder, Problem 11.1.
2. Peskin and Schroeder, Problem 11.2.
3. Peskin and Schroeder, Problem 11.3.

Physics 332 - Problem Set #2

Solutions

1.) a.) We discussed the formula for $\langle e^{i\phi} \rangle$ in Physics 331, problem set #2. But here is the derivation again: Work with a Euclidean field, with eq. of motion

$$\Delta_x \phi = 0 \qquad \text{eg.} \qquad \Delta_x = (-\partial^2 + m)$$

Let $D(x)$ be the Green's function for Δ

$$\Delta_x D(x-y) = \delta^{(4)}(x-y)$$

then

$$Z = \int \mathcal{D}\phi \; e^{-\int dx \; \frac{1}{2} \phi \Delta_x \phi}$$

$$\langle \phi(x) \phi(y) \rangle = \frac{\int \mathcal{D}\phi \; e^{-\int dx \; \frac{1}{2} \phi \Delta_x \phi} \; \phi(x) \phi(y)}{\int \mathcal{D}\phi \; e^{\int dx \; \frac{1}{2} \phi \Delta_x \phi}}$$

$$= D(x-y)$$

To derive this, add a source J :

$$Z[J] = \int \mathcal{D}\phi \; e^{-\int dx \, \frac{1}{2}\phi \Delta_x \phi + J\phi}$$

→ evaluate

$$Z[J] = \int \mathcal{D}\phi \; e^{-\int \frac{1}{2}(\phi + \Delta^{-1}J)\,\Delta\,(\phi + \Delta^{-1}J) - J\Delta^{-1}J}$$

let $\phi' = \phi + \Delta^{-1}J$ $\qquad \Delta^{-1}J = \int dz \; D(x-z)\,J(z)$

$$Z[J] = (\text{const}) \cdot \exp\left[\frac{1}{2}\int dy\,dz \; J(y)\,D(y-z)\,J(z)\right]$$

for $J_0(y) = i[\delta(y-x) - \delta(y)]$

$$\langle e^{i[\phi(x) - \phi(0)]} \rangle = \frac{Z[J_0]}{Z[0]}$$

$$= \exp\left[+\frac{i}{2}\int dy\,dz \; [\delta(y-x)-\delta(y)]\,D(y-z)\,[\delta(z-x)-\delta(z)]\right]$$

$$= \exp\left[-D(0) + D(x)\right] \quad \text{as required}$$

b.) Since ϕ has positive mass dimension $(\text{mass})^{d-2}$

$(\vec{\nabla}\phi)$ has dimension > 1

$(\vec{\nabla}\phi)^4$ has dimension > 4, and so is

non-renormalizable in $d > 2$

so take $\mathcal{H} = \frac{1}{2}\rho(\vec{\nabla}\phi)^2$

$$\langle \phi(x)\,\phi(0)\rangle = \int \frac{d^d k}{(2\pi)^d}\,\frac{1}{\rho}\,\frac{1}{k^2}\,e^{ik\cdot x}$$

the right-hand side is the Green's function of the laplace operator. If $(-\nabla^2)\,G(x-y) = \delta(x-y)$

$$\vec{E} = -\vec{\nabla} G$$

satisfies $\int d\hat{n}\cdot\vec{E} = 1$

so $\vec{E} = \dfrac{\hat{x}}{x^{d-1}}\cdot\dfrac{1}{A(d)}$ $\quad A(d)$ = area of unit sphere

$$G = \frac{1}{|x|^{d-2}}\,\frac{1}{A(d)}\,\frac{1}{d-2}$$

so

$$G = \begin{cases} d=3 & \dfrac{1}{4\pi|x|} \\ d=2 & -\dfrac{1}{2\pi}\ln|x| \\ d=1 & -\dfrac{1}{2}|x| \end{cases}$$

$$\langle \phi(x)\,\phi(y)\rangle = \frac{1}{\rho} \times (\text{the above})$$

then Let $S = A e^{i\phi(x)}$

$\underline{d=1}$ $\langle S(x) S^*(0)\rangle = A^2 \; e^{\langle \phi(x)\phi(0)\rangle - \langle \phi^2(0)\rangle}$

$= A \; e^{-\frac{1}{2\rho}|x|}$

$\langle S(x) S^*(0)\rangle \longrightarrow 0$ as $|x| \to \infty$, no spontaneous symmetry breaking

$\underline{d=2}$

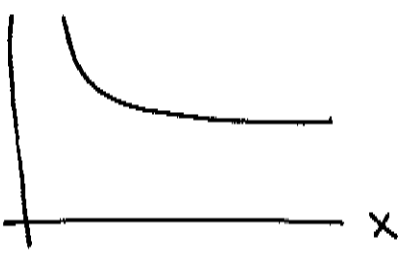
$\langle S(x) S^*(0)\rangle = A^2 \; e^{-\frac{1}{2\pi\rho} \ln|x| - (\text{const})}$

$\sim \frac{B}{|x|^{\frac{1}{2\pi\rho}}}$

again $\langle S(x) S^*(0)\rangle \to 0$ no spontaneous symmetry breaking.

$\underline{d=3}$ $\langle S(x) S^*(0)\rangle = A^2 \; e^{\frac{1}{4\pi|x|\cdot\rho} - (\text{const})}$

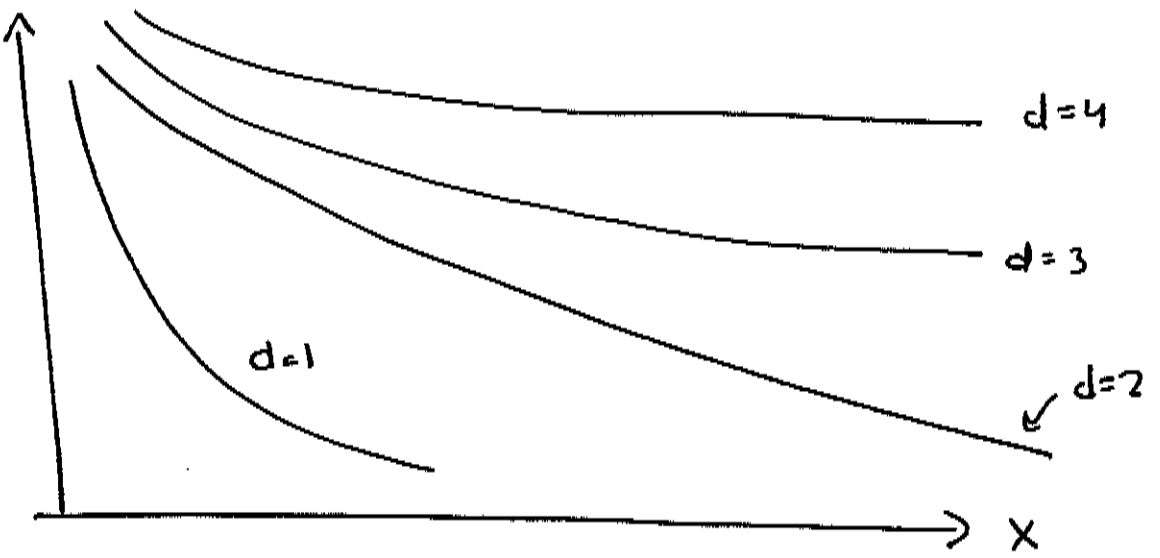
$= B \; e^{\frac{1}{4\pi\rho}\frac{1}{x}}$



$\to$ const as $x \to \infty$

me generally,

$\langle s(x)\, s^*(y) \rangle$



d=2 is the boundary dimensionality which fluctuations of spin waves destroy spontaneous symmetry breaking.

2.) a.) $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2$

$+ \bar{\psi} i \not{\partial} \psi - g \bar{\psi}(\phi^1 + i\gamma^5 \phi^2)\psi \qquad i = 1,2$

under $\phi^1 \to \cos\alpha\, \phi^1 - \sin\alpha\, \phi^2 \qquad \phi^2 \to \sin\alpha\, \phi^1 + \cos\alpha\, \phi^2$

$(\phi^1)^2 + (\phi^2)^2 \to (\phi^1)^2 + (\phi^2)^2$ so the first line of $\mathcal{L}$ is invariant

also, since $(\gamma^5)^2 = 1$

$(\phi^1 + i\gamma^5\phi^2) \to (\cos\alpha\, \phi^1 - \sin\alpha\, \phi^2) + i\gamma^5(\sin\alpha\, \phi^1 + \cos\alpha\, \phi^2)$

$= (\cos\alpha + i\gamma^5 \sin\alpha)\phi^1 + i\gamma^5(\cos\alpha + i\sin\alpha\, \gamma^5)\phi^2$

$= e^{i\alpha\gamma^5}(\phi^1 + i\gamma^5\phi^2)$

so if also $\psi \to e^{i\alpha\gamma^5/2}\psi$

$\bar{\psi} \to \psi^\dagger e^{+i\alpha\gamma^5/2}\gamma^0$

$= \bar{\psi}\, e^{-i\alpha\gamma^5/2}$

then $\bar{\psi} i\not{\partial}\psi \to \bar{\psi} i \not{\partial}\psi$

$\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi \to \bar{\psi}\, e^{-i\alpha\gamma^5/2}\, e^{i\alpha\gamma^5}(\phi^1 + i\gamma^5\phi^2) e^{-i\alpha\gamma^5/2}\psi$

$= \bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi$

so the entire Lagrangian is invariant to the U(1) symmetry!

The equation of motion for ϕ^i is

$$0 = -\partial^2\phi^i + \mu^2\phi^i - \lambda\phi^i(\phi^j)^2 - g\begin{cases}\bar{\psi}\psi & i=1\\ i\bar{\psi}\gamma^5\psi & i=2\end{cases}$$

If $\psi, \bar{\psi} = 0$ in the vacuum configuration, $\phi^i = \text{const}$, we can solve

$$\mu^2 - \lambda(\phi^j)^2 = 0 \quad \text{or} \quad v^2 = \langle|\phi^i|^2\rangle = \frac{\mu^2}{\lambda}$$

It is not convenient to choose coordinates s.t.

$$\langle\phi^1\rangle = v \qquad \langle\phi^2\rangle = 0$$

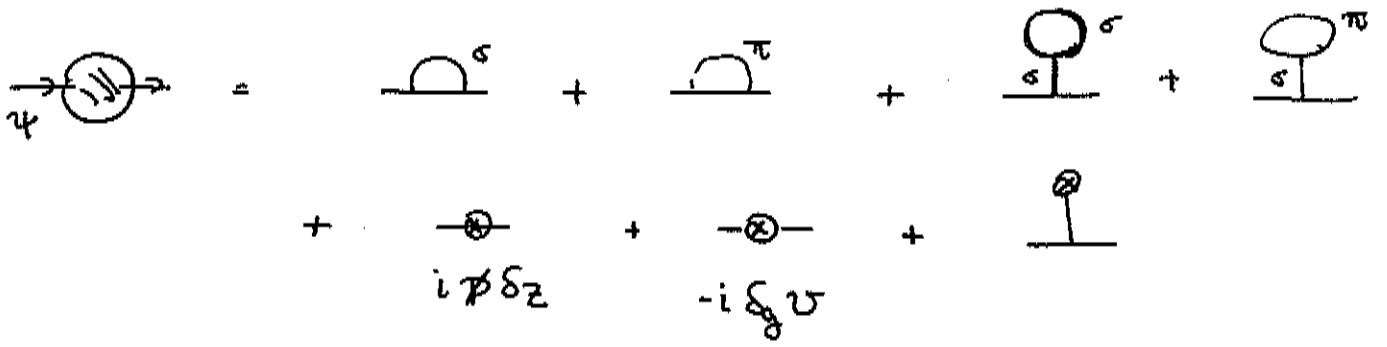
Any other choice can be rotated into this one by the U(1) symmetry.

b.) Set $\phi^1 = v+\sigma \qquad \phi^2 = \pi$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 - \mu^2\sigma^2 - \lambda v\sigma^3 - \lambda v\sigma\pi^2$$
$$- \tfrac{\lambda}{4}\sigma^4 - \tfrac{\lambda}{2}\sigma^2\pi^2 - \tfrac{\lambda}{4}\pi^4$$
$$+ \bar{\psi}(i\not{\partial} - gv)\psi - g\sigma\bar{\psi}\psi - ig\bar{\psi}\gamma^5\psi\pi$$

Indeed $m_f = gv$.

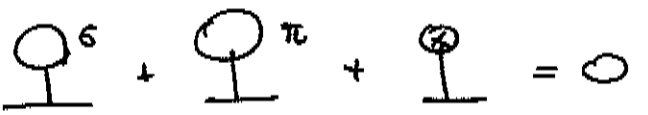
c.) The correction to the fermion self-energy is



the last diagram is

$$-ig\left(\frac{+i}{-m_\sigma^2}\right)\cdot[\text{contraction of }\sigma]$$

Fortunately, if we insist that $\langle\phi\rangle = v$ as a renormalization condition then



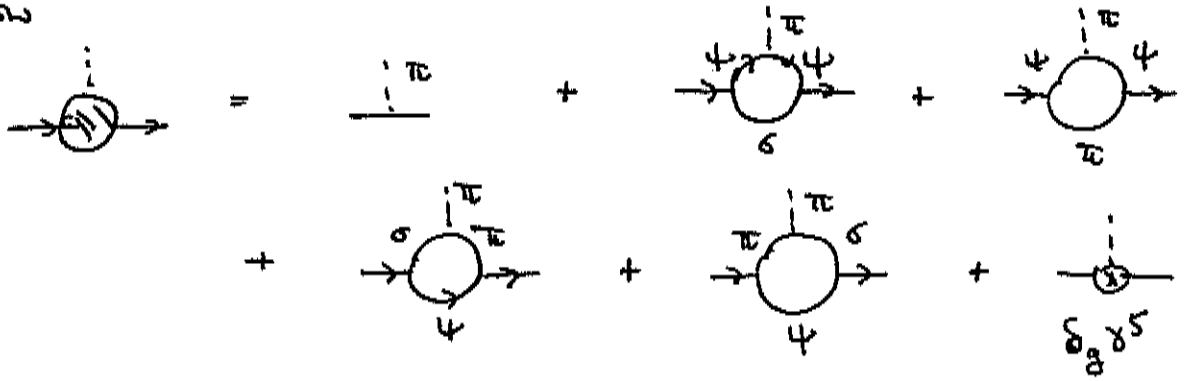
δ_Z is determined by the condition $\left.\frac{\partial}{\partial \not{p}}\right|_{\not{p}=m_f}$  $= 0$

δ_g is determined [according to the problem set] by the condition



$= g\gamma^5$ at $k=0$ ψ on shell

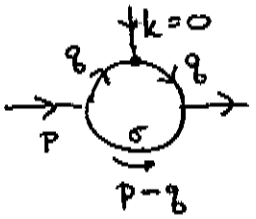
now



This assumes that $Z_\psi = Z_\pi = 1$; otherwise there are additional factors from $\sqrt{Z}$'s. The condition $Z_\pi = 1$ assumes we have imposed the additional renormalization condition

$$\frac{d}{dp^2}\left[\pi\text{—}\bigcirc\text{—}\right]\Big|_{p^2=0} = 0$$

Anyway, with this understood, let's compute δ_g:



$$= (-ig)^2 g \int \frac{d^dq}{(2\pi)^d} \frac{i(\not{q}+m_f)\,\gamma^5\, i(\not{q}+m_f)}{(q^2-m_f^2)^2} \frac{i}{(p-q)^2-m_\sigma^2}$$

$$= ig^3\gamma^5 \int \frac{d^dq}{(2\pi)^d} \frac{(-\not{q}+m_f)(\not{q}+m_f)}{(q^2-m_f^2)^2\,(p-q)^2-m_\sigma^2}$$

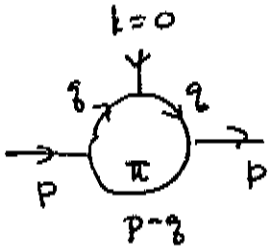
$(\not{q}-m_f)(\not{q}+m_f) = q^2-m_f^2$

$$= -ig^3\gamma^5 \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2-m_f^2)((p-q)^2-m_\sigma^2)}$$

$$= -ig^3\gamma^5 \int_0^1 dx \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \frac{1}{\left[x m_\sigma^2 + (1-x)m_f^2 - x(1-x)p^2\right]^{2-d_2}}$$

at set $p^2 = m_f^2$

$$= -ig^3\gamma^5 \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \int_0^1 dx \frac{1}{\left[x m_\sigma^2 + (1-x)^2 m_f^2\right]^{2-d_2}}$$

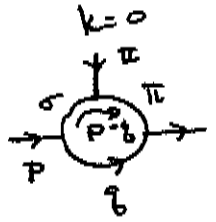


$$= g^3 \int \frac{d^dq}{(2\pi)^d} \gamma^5 \frac{i(\not{q}+m_f)\,\gamma^5\, i(\not{q}+m_f)}{(q^2-m_f^2)^2} \gamma^5 \frac{i}{(p-q)^2}$$

$$= -ig^3 \int \frac{d^dq}{(2\pi)^d} \frac{(\not{q}+m_f)(-\not{q}+m_f)}{(q^2-m_f^2)\,(p-q)^2} \quad \leftarrow m_\pi = 0$$

$$= +ig^3 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m_f^2)\left((p-q)^2\right)}$$

$$= ig^3 \gamma^5 \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \int_0^1 dx \left[\frac{1}{(1-x)^2 m_f^2}\right]^{2-d_2}$$



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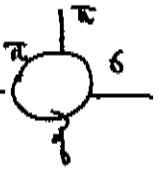


$= (-ig)(g)(-2i\lambda v)$

$$\cdot \int \frac{d^d q}{(2\pi)^d} \left[\gamma^5 \frac{i(\not{q}+m_f)}{q^2 - m_f^2} + \frac{i(\not{q}+m_f)}{q^2 - m_f^2} \gamma^5\right] \frac{i}{(p-q)^2 - m_\sigma^2} \left[\frac{i}{(p-q)^2}\right]$$

$$= 2 \cdot 2i \lambda v g^2 \gamma^5 m_f \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m_f^2} \frac{1}{(p-q)^2 - m_\sigma^2} \frac{1}{(p-q)^2}$$

$$= 4i \lambda v g^2 \gamma^5 m_f \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m_f^2} \int_0^1 dy \left[\frac{1}{(p-q)^2 - y m_\sigma^2}\right]^2$$

$$= 4i \lambda v g^2 \gamma^5 m_f \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \; x \cdot 2 \int_0^1 dy \left[\frac{1}{q^2 + x(1-x)p^2 - xy m_\sigma^2 - (1-x) m_f^2}\right]^3$$

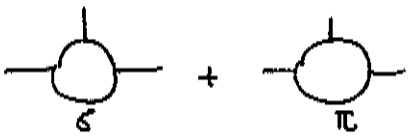
$$= 4i \lambda v g^2 \gamma^5 m_f \frac{-i}{(4\pi)^{d_2}} \frac{\Gamma(3-d_2)}{\Gamma(3)} \int_0^1 dx \int_0^1 dy \; 2x \left[\frac{1}{xy m_\sigma^2 + (1-x)^2 m_f^2}\right]^{3-d_2}$$

we can set $d=4$ ~~el~~ in the ~~y integral~~

$$= 4 \lambda v g^2 \gamma^5 m_f \frac{1}{(4\pi)^2} \int dx \int dy \left[x \frac{x}{y m_\sigma^2 + (1-x)^2 m_f^2}\right]$$

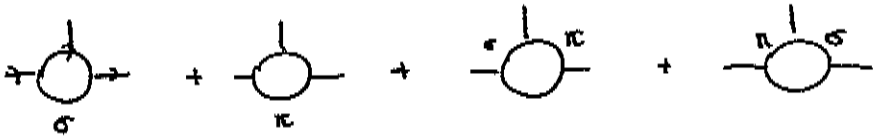
$$= 4 \lambda v g^2 \gamma^5 \frac{m_f}{m_\sigma^2} \frac{1}{(4\pi)^2} \int_0^1 dx \; \ln\left[\frac{x m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right]$$

simply the divergences of the first two diagrams cancel [this is something of an accident] and we find



$$= -g^3 \gamma^5 \frac{1}{(4\pi)^2} \int_0^1 dx \; \log\left[x \frac{m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2} \right]$$

= all



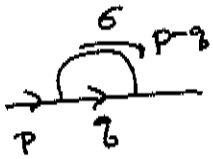
$$= (-\gamma^5) \frac{1}{(4\pi)^2} \left[g^3 - 4 \partial v \, g^2 \frac{g v}{m_\sigma^2} \right] \int_0^1 dx \; \log\left(\frac{x m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2} \right)$$

$$m_\sigma^2 = 2\mu^2 = 2 \partial v^2$$

$$= + \gamma^5 \frac{1}{(4\pi)^2} g^3 \left[\int_0^1 dx \; \log \frac{x m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2} \right]$$

$$= - \gamma^5 \delta_g$$

Now turn to:  = 



$$= (-ig)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i(\not{q} + m_f)}{(q^2 - m_f^2)} \frac{i}{(p-q)^2 - m_\sigma^2}$$

Feynman parameters $\longrightarrow$ $(1-x)$ x

$$\text{Denom} = q^2 - 2xp + xp^2 - x m_\sigma^2 + (1-x) m_f^2$$
$$= q_\parallel^2 + x(1-x) p^2 - (x m_\sigma^2 + (1-x) m_f^2)$$

$$q = \ell + xp$$



$$= g^2 \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 dx \frac{\not\ell + x\not p + m_f}{[\ell^2 + x(1-x)p^2 - (x m_b^2 + (1-x)m_f^2)]^2}$$

$$= g^2 \frac{1}{(4\pi)^{d_2}} \Gamma(2-d_2) \int_0^1 dx \frac{x\not p + m_f}{[x m_b^2 + (1-x)m_f^2 - x(1-x)p^2]^{2-d_2}}$$

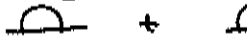


$$= g^2 \int \frac{d^d\ell}{(2\pi)^d} \gamma^\rho \frac{i(\not\ell + m_f)}{\ell^2 - m_f^2} \gamma^\rho \frac{i}{(p-\ell)^2}$$

$$= g^2 \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 dx \frac{(x\not p - m_f)}{[\ell^2 + x(1-x)p^2 - (1-x)m_f^2]^2}$$

$$= g^2 \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \int_0^1 dx \frac{x\not p - m_f}{[(1-x)m_f^2 - x(1-x)p^2]^{2-d_2}}$$

in all



$$= -i \frac{g^2}{(4\pi)^2} \left\{ -\frac{2}{\epsilon} \not p \right.$$

$$+ \int_0^1 dx \left[x\not p \left\{ \ln \frac{x m_b^2 + (1-x)m_f^2 - x(1-x)p^2}{M^2} + \ln \frac{(1-x)m_f^2 - x(1-x)p^2}{M^2} \right\} \right.$$

$$\left. \left. + m_f \ln \left(\frac{x m_b^2 + (1-x)m_f^2 - x(1-x)p^2}{(1-x)m_f^2 - x(1-x)p^2} \right) \right] \right\}$$

write this as

$$= i \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \not p \quad - i \frac{g^2}{(4\pi)^2} \hat{\Sigma}(\not p)$$

then, to all, the inverse fermion propagator is

$$(\not{p} - m_f) + \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\not{p} - \frac{g^2}{(4\pi)^2}\hat{\Sigma}(\not{p}) + \not{p}\,\delta_Z - \delta_g v$$

where δ_g is given on p. 6. The fermion mass is given by

$$0 = \not{p} - m_f + \left(\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} + \delta_Z\right)\not{p} - \frac{g^2}{(4\pi)^2}\hat{\Sigma}(p) - \delta_g v$$

where δ_Z is determined by.

$$0 = \left(\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} + \delta_Z\right) - \frac{g^2}{(4\pi)^2}\frac{d\hat{\Sigma}}{d\not{p}}\bigg|_{\not{p}=m_f}$$

This removes the last ∞ in the expression for M_f!

so the solution to the equation for m_f is

$$\not{p} = m_f \equiv \not{p}\left(\frac{g^2}{(4\pi)^2}\frac{d\hat{\Sigma}}{d\not{p}}\bigg|_{\not{p}=m_f}\right) + \frac{g^2}{(4\pi)^2}\hat{\Sigma}(\not{p}) + \delta_g v$$

is the exact fermion mass, thru order g^2, is

$$M_f = m_f + \frac{g^2}{(4\pi)^2}\hat{\Sigma}(m_f) + \delta_g v - \frac{g^2}{(4\pi)^2} m_f \frac{d\Sigma}{d\not{p}}\bigg|_{m_f}$$

$$= m_f + \frac{g^2}{(4\pi)^2}\int_0^1 dx \left[m_f \ln\left[\frac{x m_b^2 + (1-x)m_f^2 - x(1-x)m_f^2}{(1-x)m_f^2 - x(1-x)m_f^2}\right]\right.$$

$$\left. - 2x(1-x)m_f^2\left(\frac{1}{x m_b^2 + (1-x)m_f^2 - x(1-x)m_f^2} - \frac{1}{(1-x)m_f^2 - x(1-x)m_f^2}\right)\right]$$

$m_f = gv$

$$- x m_f \cdot 2x(1-x)m_f\left(\frac{1}{x m_b^2 + (1-x)m_f^2 - x(1-x)m_f^2} + \frac{1}{(\quad)}\right)$$

$$- \frac{g^3 v}{(4\pi)^2}\int_0^1 dx \ln\left[\frac{x m_b^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right]$$

finally

$$M_f - gv = -\frac{g^2}{(4\pi)^2}\int_0^1 dx \left\{ 2x(1-x)\, m_f^2 (1+x) \quad \frac{1}{x m_\sigma^2 + (1-x)^2 m_f^2} \right.$$

$$\left. - 2x(1-x)^2 m_f^2 \quad \frac{1}{(1-x)^2 m_f^2} \right\}$$

or

$$M_f - gv = -\frac{g^2}{(4\pi)^2}\left(\int_0^1 dx \quad \frac{2x(1-x^2)\, m_f^2}{x m_\sigma^2 + (1-x)^2 m_f^2} \quad - 1 \right)$$

2.) a) $\mathcal{L} = \bar{\Psi}_i \not{\partial} \Psi_i + \frac{1}{2} g^2 (\bar{\Psi}_i \Psi_i)^2$

$\gamma^5 = \gamma^0\gamma^1$ $\quad \{\gamma^5, \gamma^\mu\} = 0$ $\quad (\gamma^5)^2 = 1$

under $\Psi_i \to \gamma^5 \Psi_i$ $\quad \bar{\Psi}_i \to \bar{\Psi}_i(-\gamma^5)$

$\bar{\Psi}_i \not{\partial} \Psi_i \to +\bar{\Psi}_i \not{\partial} \Psi_i$

$\bar{\Psi}_i \Psi_i \to -\bar{\Psi}_i \Psi_i$

so $\mathcal{L} \to \mathcal{L}$ by $\delta\mathcal{L} = -m\bar{\Psi}_i\Psi_i \to -\delta\mathcal{L}$

The mass term for Ψ_i does not respect the discrete symmetry

b.) In 2-D, $\partial_x \sim (\text{mass})^1$

$$\bar{\Psi} \not{\partial} \Psi \sim (\text{mass})^d \cdot (\text{mass})^2$$

so $\Psi \sim (\text{mass})^{\frac{1}{2}}$

then $\bar{\Psi}\Psi \sim (\text{mass})$

$$(\bar{\Psi}\Psi)^2 \sim (\text{mass})^2 = (\text{mass})^d$$

since $\mathcal{L} \sim (\text{mass})^d$ [$\int d^4x \, \mathcal{L}$ is dim-less]

g must be dimensionless $\rightarrow$ the thy is renormalizable.

c.) Since $\int \mathcal{D}\sigma \; e^{i \int \left(-\frac{1}{2g^2}\sigma^2 - \sigma \bar{\Psi}_i \Psi_i\right) d^2x}$

$$= \int \mathcal{D}\sigma \; e^{i \int d^2x \left\{ -\frac{1}{2g^2}\left(\sigma + g^2 \bar{\Psi}_i \Psi_i\right)^2 + \frac{g^2}{2}\left(\bar{\Psi}_i \Psi_i\right)^2 \right.}$$

$$= (\text{const}) \; e^{i \int d^4x \; \frac{g^2}{2} \left(\bar{\Psi}_i \Psi_i\right)^2}$$

$$Z = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}\sigma \; e^{i \int d^2x \left(\bar{\Psi}_i i \not{\partial} \Psi_i - \sigma \bar{\Psi}_i \Psi_i - \frac{1}{2g^2}\sigma^2\right)}$$

$$= \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \; e^{i \int d^2x \left(\bar{\Psi}_i i \not{\partial} \Psi_i + \frac{g^2}{2}\left(\bar{\Psi}_i \Psi_i\right)^2\right)}$$

Note that $\sigma \rightarrow -\sigma$ to preserve the discrete symmetry

d.) Anticipate that we will need to renormalize the coupling g^2, write

$$\mathcal{L} = \bar{\psi}_i (i\not{\partial} - \sigma)\psi_i - \frac{1}{2g^2}\sigma^2 - \underbrace{\delta_g \sigma^2}_{\text{counterterm}}$$

Integrating over ψ_i $\bar{\psi}_i$ gives

$$Z = \int \mathcal{D}\sigma \, [\det(i\not{\partial} - \sigma)]^N \, e^{i\int\left(-\frac{1}{2g^2}\sigma^2 - \delta_g\sigma^2\right) d^2x}$$

Now evaluate the determinant for $\sigma(x) \to \sigma = \text{const}$. In a style Fourier mode

$$\begin{aligned}
\det(i\not{\partial} - \sigma) &= \det(\not{k} - \sigma) \\
&= \det(\not{k} - \sigma) \qquad \text{for } k = (k, 0) \\
&= \det\begin{pmatrix} -\sigma & -ik \\ ik & -\sigma \end{pmatrix} \qquad \sigma^0 = \sigma^2 \\
&= \sigma^2 - k^2 = \sigma^2 + k_E^2
\end{aligned}$$

then

$$\begin{aligned}
\log\det(i\not{\partial} - \sigma) &= \sum_k \log(\sigma^2 + \hat{k}_E^2) \\
&= (VT)\int \frac{d^d k}{(2\pi)^d} \log(\sigma^2 + k^2) \\
&= (VT)\,(-i)\,\frac{\Gamma(-d_2)}{(4\pi)^{d_2}}(\sigma^2)^{d_2}
\end{aligned}$$

Near $d=2$ $\quad \Gamma(-d/2) = \frac{1}{1-d/2}\,\Gamma(1-d/2)$

$d = 2-\epsilon$ $\quad = \frac{1}{-1+\epsilon/2}\,\Gamma(\epsilon/2)$

$= -(1+\epsilon/2)\left(\frac{2}{\epsilon} - \gamma + \cdots\right)$

$$\log\det(i\partial\!\!\!/ - \sigma) = (VT)\,\frac{i}{4\pi}\,\sigma^2\left(\frac{2}{\epsilon} + 1 + \log\frac{M^2}{\sigma^2}\right)$$

when M^2 is the $\overline{MS}$ mass.

$$Z = \int \mathcal{D}\sigma\, \exp\Big[\, i\int d^2x \Big\{ -\frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\frac{2}{\epsilon}+1+\log\frac{M^2}{\sigma^2}\right) - \delta_g\sigma^2 \Big\}\,\Big]$$

δ_g will the renormalization, cancel the $\frac{2}{\epsilon}$ term

We then find an effective Lagrangian for σ

$$\mathcal{L}_{eff} = -\frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\log\frac{M^2}{\sigma^2} + 1\right)$$

e.) $\quad \frac{\delta\mathcal{L}}{\delta\sigma} = 0 \Rightarrow \quad -\frac{1}{2g^2} + \frac{N}{4\pi}\left(\log\frac{M^2}{\sigma^2} + 1\right) - \frac{N}{4\pi} = 0$

$$\log\frac{M^2}{\sigma^2} = \frac{1}{2g^2}\cdot\frac{4\pi}{N}$$

$$\sigma^2 = M^2 \exp\left[-\frac{2\pi}{g^2 N}\right]$$

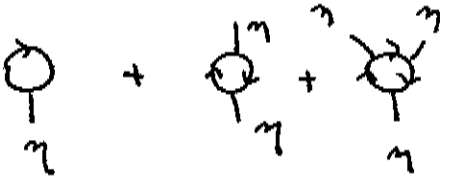
so the potential is minimized at

$$\sigma_0 = \pm M e^{-\pi/g^2 N}$$

For $g^2 N \ll 1$, $|\sigma| \ll M$

A change in the renormalization condition changes M to some other constant, but it does not change the dependence on $g^2 N$.

8.) Now let $\sigma \to \underbrace{\sigma}_{\text{constant}} + \underbrace{\eta(x)}_{\text{vanish with } \eta(k=0)=0}$

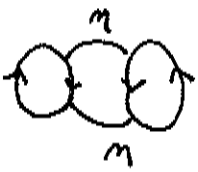
$$\log\det(i\partial\!\!\!/ - \sigma - \eta) = \log\det(i\partial\!\!\!/ - \sigma) +$$ 

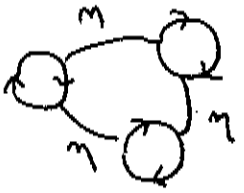
with the propagator $\longrightarrow \cdot \frac{1}{k\!\!\!/ - \sigma}$

since η has no $k=0$ mode $\quad \circ\!\!-\eta = 0$

then the perturbative corrections to the effective potential come from integrating over η. The propagator for η is

$$\mathcal{L}_0 = i\int d^2x \, \frac{-1}{2g^2}\eta^2 \quad \to \quad \overbrace{\eta\,\eta} = g^2$$

then 
$\sim N^2 \cdot (g^2)^2 = g^2 \cdot N \cdot (g^2 N)$



$= (g^2)^3 N^3 = (g^2 N)^3$



$= (g^2)^3 N^2 = g^2 (g^2 N)^2$

The result of part (e) gave $V_{eff} \sim N^2 f(g^2 N)$

The corrections give g^2 $(g^2)^2$ etc. terms this.

So for $N \to \infty$, $g^2 N$ fixed, the result of part (e) becomes <u>exact</u>, even though it is non-perturbative.

Physics 332 – Problem Set # 3

(due Wednesday, May 3)

1. Peskin and Schroeder, Problem 12.1.

2. Peskin and Schroeder, Problem 12.2. You should show that, with this β function, the mass m_ψ of the ψ field satisfies the Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} \right] m_\psi(g, M) = 0 \tag{1}$$

3. Using the result from the two-loop calculation of the scalar self-energy in ϕ^4-theory that you derived in Problem Set 1 (Peskin and Schroeder, Problem 10.3), show that the Callan-Symanzik γ function in ϕ^4-theory is given by

$$\gamma = \frac{\lambda^2}{12(4\pi)^4} + \mathcal{O}(\lambda^3) \tag{2}$$

   Show that, in the N-component linear sigma model (with its different normalization for λ),

$$\gamma = (N+2)\frac{\lambda^2}{(4\pi)^4} + \mathcal{O}(\lambda^3) \tag{3}$$

Physics 332 – Problem Set # 3

Solutions

1.) In problem set 1, problem 1, we computed the ultraviolet divergence of the Yukawa th. For

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 + \bar\psi i\not\partial\psi - ig\bar\psi\gamma^5\psi\phi$$

$$+ \tfrac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{4!}\delta_\lambda\phi^4 + \delta_\psi\bar\psi i\not\partial\psi - i\delta_g\bar\psi\gamma^5\psi\phi$$

we found:

$$\delta_\phi = -\frac{4g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta_\lambda = -\frac{48g^4}{(4\pi)^2}\frac{1}{\epsilon} + \frac{3\lambda^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta_\psi = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta_g = \frac{2g^3}{(4\pi)^2}\frac{1}{\epsilon}$$

With a momt. cutoff $\frac{2}{\epsilon} \to \ln\frac{\Lambda^2}{M^2}$

so $M\frac{\partial}{\partial M}\left(\frac{1}{\epsilon}\right) = -1$

Then, to 1-loop order

$$\beta_\lambda = M\frac{\partial}{\partial M}(-\delta_\lambda + 2\lambda\delta_\phi)$$

$$= M\frac{\partial}{\partial M}\left(-\frac{3\lambda^2}{(4\pi)^2}\frac{1}{\epsilon} + \frac{24g^4}{(4\pi)^2}\frac{1}{\epsilon} - \frac{8g^2\lambda}{(4\pi)^2}\frac{1}{\epsilon}\right)$$

$$= \frac{1}{(4\pi)^2}(3\lambda^2 + 8\lambda g^2 - 48g^4)$$

$$\beta_g = M\frac{\partial}{\partial M}(-\delta_g + g\delta_\psi + \tfrac{1}{2}g\delta_\phi)$$

$$= M\frac{\partial}{\partial M}\left(-\frac{2g^3}{(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon} - \frac{2g^2}{(4\pi)^2}\frac{1}{\epsilon}\right)$$

$$= \frac{3g^3}{(4\pi)^2}$$

again:

$$\beta_g = \frac{3g^3}{(4\pi)^2} \qquad \beta_\lambda = \frac{1}{(4\pi)^2}(3\lambda^2 + 8\lambda g^2 - 48g^4)$$

β_g is always positive. But β_λ can be zero, at

$$0 = 3\lambda^2 + 8\lambda g^2 - 48g^4$$

$$= (\lambda/g^2)^2 + 2\cdot\tfrac{4}{3}(\lambda/g^2) - 16$$

$$= (\lambda/g^2 + \tfrac{4}{3})^2 - 16 - \tfrac{16}{9}$$

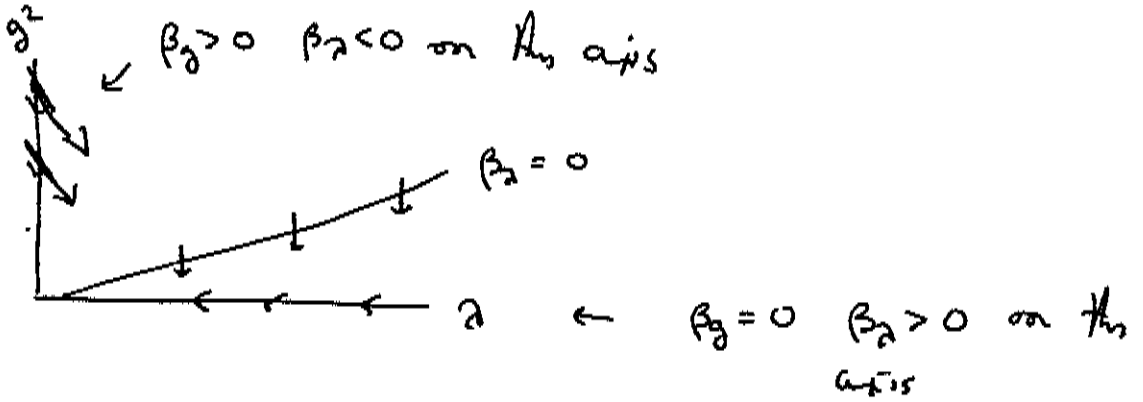
or $\frac{\lambda}{g^2} = -\frac{4}{3} \pm \frac{4}{3}\sqrt{10}$

so

$\beta_\lambda = 0$ on $\lambda = g^2 \cdot \frac{4}{3}(\sqrt{10}-1)$

$\beta_\lambda > 0$ for $\lambda >$, < 0 for $\lambda <$

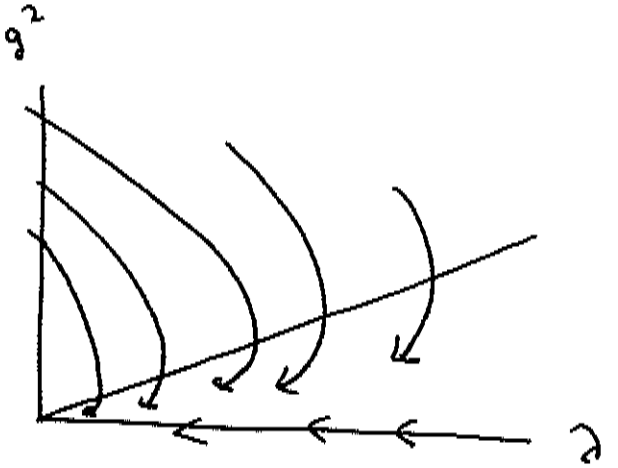
then flows to IR or $M \to 0$



the flow diagram for

$$\frac{d}{d\ln M}\begin{pmatrix} \lambda(M) \\ g^2(M) \end{pmatrix} = \begin{pmatrix} \beta_\lambda \\ 2g\,\beta_g \end{pmatrix}$$

is

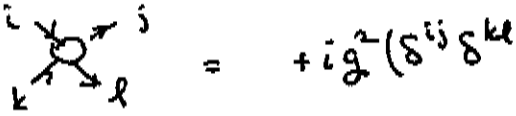


with $g^2, \lambda \to 0$ as $M \to 0$

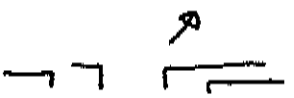
2.) In the Gross-Neveu model

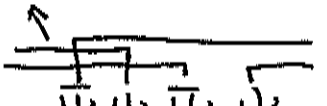
$$\mathcal{L} = \bar{\Psi}_i i\not{\partial} \Psi_i + \tfrac{1}{2} g^2 (\bar{\Psi}_i \Psi_i)^2 \quad \sim \quad d > 2$$

the Feynman rule is



$$= +ig^2(\delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk}) = -ig^2 (\quad - \quad)$$

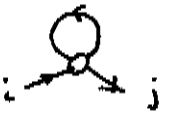




$= (-1)$ fermion minus sign



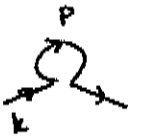
$=$

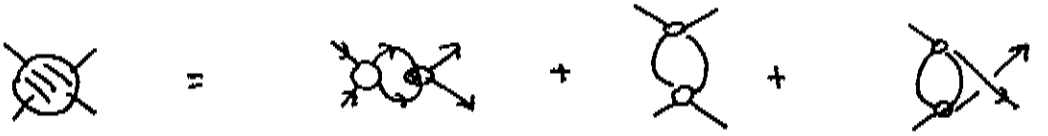


$$= -ig^2 \left[\overbrace{\bar{\Psi}_i \psi_i}\ \overbrace{\bar{\Psi}_j \psi_j} + \overbrace{\bar{\Psi}_i \psi_i}\ \overbrace{\bar{\Psi}_j \psi_j} \right]$$

$$\overbrace{\bar{\Psi}_j \psi_\sigma} = \int \frac{d^2p}{(2\pi)^2} (-1)\, \mathrm{tr}\, \frac{i\not{p}}{p^2} = 0 \quad \text{massless flavor}$$

$$= \delta_{ij} \int \frac{d^2p}{(2\pi)^2} (+ig^2) \frac{i\not{p}}{p^2} = 0$$



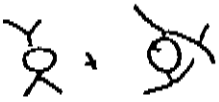




$$= (+ig)^2 (\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj}) \cdot 2 \cdot \tfrac{1}{2} \cdot \int \frac{d^2p}{(2\pi)^2} \frac{i\not{p}}{p^2} \frac{i\not{p}}{p^2}$$

Note that this is a term of $O(1)$. The other diagrams have terms of $O(N)$



(sum) = 

$$= (+ig^2)^2 (\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj})(-1)\cdot N \cdot \int \frac{d^2p}{(2\pi)^2} \operatorname{tr}\left[\frac{i\not{p}}{p^2}\frac{i\not{p}}{p^2}\right]$$

$$= + g^4 N (\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj}) \int \frac{d^2p}{(2\pi)^2} \cdot 2 \cdot \frac{1}{p^2} \qquad \operatorname{tr} \underline{1} = 2 \text{ in } d=2$$

$$= + g^4 N (\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj})\, 2 \cdot \frac{-i}{(4\pi)^{d_2}} \Gamma(1-d_2)$$

$$= -ig^4 N (\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj}) \frac{2}{4\pi} \ln \frac{\Lambda^2}{M^2}$$

so

$$\delta_{g^2} = -\frac{1}{2\pi} g^4 N \ln \Lambda^2/M^2$$

$$\beta_{g^2} = \frac{dg^2}{d\ln M} = -\frac{g^4 N}{\pi} + O(1)$$

$$\beta_g = \frac{dg}{d\ln M} = -\frac{g^3 N}{2\pi} + O(1)$$

In ~~target~~ this equation, we find

$$g^2(M) = \frac{g_0^2}{1 + \frac{g_0^2 N}{\pi} \ln M_0^2/M^2} \to 0 \text{ as } M \to \infty$$

In the previous problem set, we found

$$m_\psi = M e^{-\pi/g^2 N}$$

Since m_ψ is physical (measurable), its normalization should be unambiguous. If we change M (definition convention), but change g^2 accordingly, the value of m_ψ should be unchanged.

Check this:

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} \right] m_\psi$$

$$= m_\psi + \beta(g) \cdot \frac{2\pi}{g^3 N} \cdot m_\psi = 0 \quad \checkmark$$

3.) In problem set #1, problem 2 we found in ϕ^4 theory

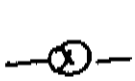


$$= -i \frac{\lambda^2}{12} \frac{p^2}{(4\pi)^4} \left(-\frac{1}{\epsilon} + \log p^2 + \cdots \right)$$

$$\left(-\frac{\Gamma(4-d)}{[p^2]^{4-d}} \right)$$

$$= -i \frac{\lambda^2}{12} \frac{p^2}{(4\pi)^4} \left(-\log \frac{\Lambda^2}{p^2} + \cdots \right)$$

then



$$= i \delta_z p^2 \qquad \delta_z = -\frac{\lambda^2}{12} \frac{1}{(4\pi)^4} \log \frac{\Lambda^2}{M^2}$$

this is the first correction to Z, so

$$\gamma = \tfrac{1}{2} M \frac{\partial}{\partial M} \ln Z = \tfrac{1}{2} M \frac{\partial}{\partial M} \delta_Z$$

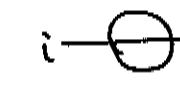
$$= \frac{\lambda^2}{12} \frac{1}{(4\pi)^4}$$

Now repeat for the O(N) linear sigma model.

$$\times = -i\lambda \quad \Rightarrow \quad \overset{i}{\underset{k}{\times}}\overset{j}{\underset{l}{}} = -2i\lambda \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)$$

then

ϕ^4 theory  $= \frac{(-i\lambda)^2}{3!} \cdot \int \frac{d^d k\, d^d r}{(2\pi)^{2d}} \cdots$

O(N) linear s.m.  $= \frac{(-2i\lambda)^2}{3!} \left(\delta^{ia}\delta^{bc} + \delta^{ib}\delta^{ac} + \delta^{ic}\delta^{ab}\right)$

$$\cdot \left(\delta^{ja}\delta^{bc} + \delta^{jb}\delta^{ac} + \delta^{jc}\delta^{ab}\right) \cdot \int \frac{d^d q\, d^d r}{(2\pi)^{2d}} \cdots$$

that is

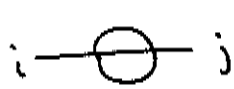
$$\frac{(-i\lambda)^2}{3!} \to \frac{(i\lambda)^2}{3!} \cdot 4 \cdot \left(\delta^{ia}\delta^{bc} + \delta^{ib}\delta^{ac} + \delta^{ic}\delta^{ab}\right)\left(\delta^{ja}\delta^{bc} + \delta^{jb}\delta^{ac} + \delta^{jc}\delta^{ab}\right)$$

$$= \frac{(-i\lambda)^2}{3!} \cdot 4 \cdot \left(3 \cdot N\, \delta^{ij} + 6\delta^{ij}\right)$$

with $\delta^{ab}\delta^{ab} = N$

$$= \frac{(-i\partial)^2}{3!} \cdot 12 \cdot (N+2)\,\delta^{ij}$$

so



$$= -i(N+2)\,\partial^2 \frac{p^2}{(4\pi)^4}\left(-\ell_\delta \Lambda^2/p^2\right)$$

$$\delta_Z = -\frac{\partial^2 (N+2)}{(4\pi)^4}\,\ell_\delta \Lambda^2/M^2$$

$$\gamma = \frac{(N+2)}{(4\pi)^4}\,\partial^2$$

Physics 332 – Problem Set # 4

(due Wednesday, May 10)

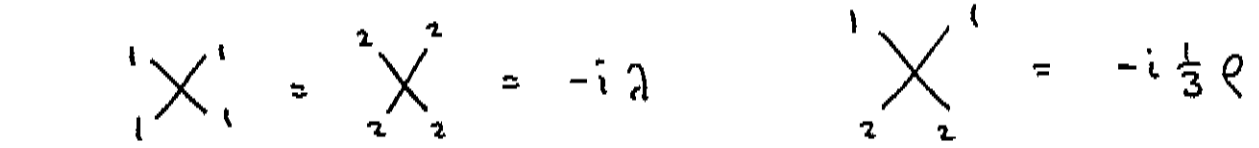
1. Peskin and Schroeder, Problem 12.3.
2. Peskin and Schroeder, Problem 13.1.

Physics 332 – Problem Set #4

Solutions

1.) a.) $\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\rho}{4!}\phi_1^2\phi_2^2$

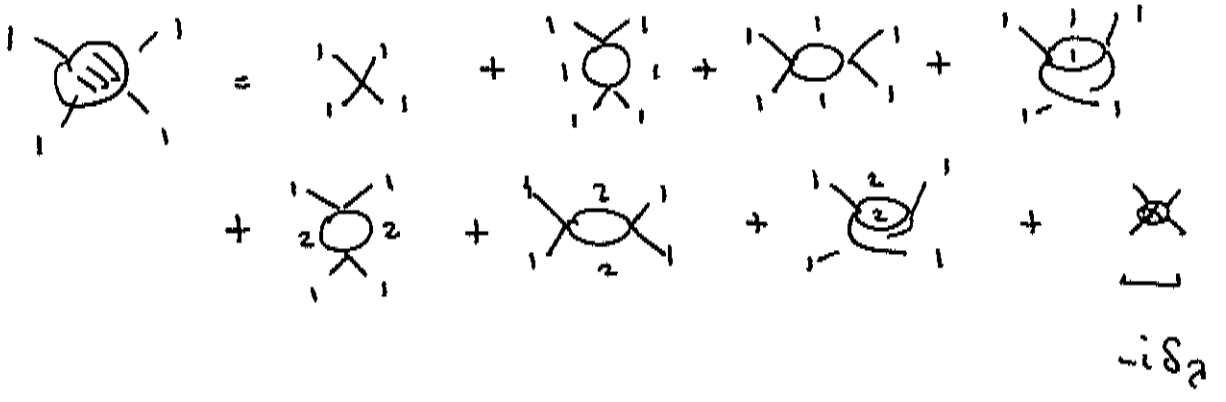
The Feynman rules are:



As usual in scalar field thy, $\delta_Z = 0$ in one-loop order. So, to 1-loop order

$$\beta_\lambda = M\frac{\partial}{\partial M}(-\delta_\lambda) \qquad \beta_\rho = M\frac{\partial}{\partial M}(-\delta_\rho)$$

Let's compute:





$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^dp}{(2\pi)^d} \frac{i}{p^2}\frac{i}{p^2}$$

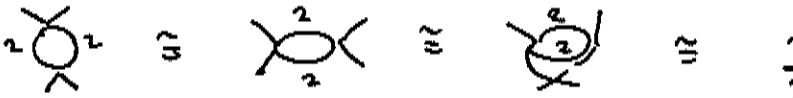
$$= \frac{1}{2}\lambda^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2)^2} = \frac{1}{2}\frac{i\lambda^2}{(4\pi)^{d_2}}\Gamma(2-d_2)$$

$$= \frac{i\lambda^2}{2(4\pi)^2} \ell_J \Lambda^2$$

similarly



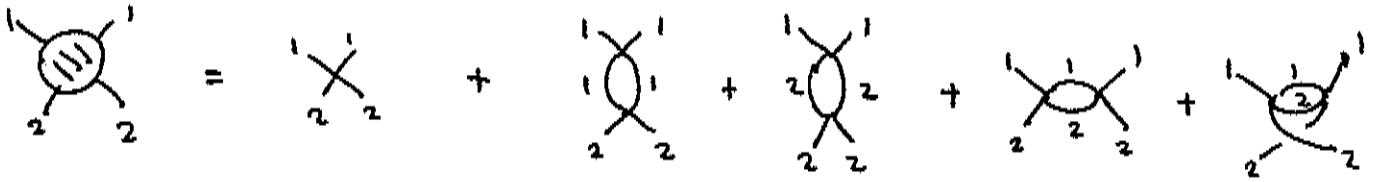
$$\cong \frac{i\lambda^2}{2(4\pi)^2} \ell_J \Lambda^2$$



$$\cong \frac{i}{2}\left(\frac{\rho}{3}\right)^2 \frac{1}{(4\pi)^2} \ell_J \Lambda^2$$

then  $= -i\delta_\rho = (-i)\cdot 3\left(\frac{\lambda^2}{2} + \frac{\rho^2}{18}\right)\frac{1}{(4\pi)^2}\ell_J \Lambda^2/M^2$

so $\beta_\rho = \dfrac{3\lambda^2 + \rho^2/3}{(4\pi)^2}$



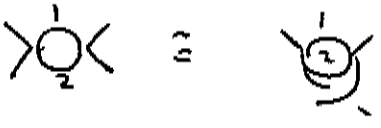


$$-i\frac{\delta\rho}{3}$$



$$= \frac{1}{2}(-i\lambda)\left(-i\frac{\rho}{3}\right)\int\frac{d^dp}{(2\pi)^d}\,\frac{i}{p^2}\,\frac{i}{p^2}$$

$$= \frac{1}{2}\cdot\frac{1}{3}\,\lambda\rho\cdot\frac{i}{(4\pi)^2}\log\Lambda^2$$



$$= \left(-i\frac{\rho}{3}\right)^2\int\frac{d^dp}{(2\pi)^d}\,\frac{i}{p^2}\,\frac{i}{p^2}$$

$$= \frac{\rho^2}{9}\,\frac{i}{(4\pi)^2}\log\Lambda^2$$

So

$$-i\,\frac{\delta\rho}{3} = (-i)\,\frac{1}{(4\pi)^2}\left(\frac{\lambda\rho}{3}+\frac{2\rho^2}{9}\right)\log\Lambda^2/M^2$$

$$\beta_\rho = \frac{2\lambda\rho+4\rho^2/3}{(4\pi)^2}$$

again

$$\beta_\lambda = \frac{1}{(4\pi)^2}\left(3\lambda^2+\frac{\rho^2}{3}\right)$$

$$\beta_\rho = \frac{1}{(4\pi)^2}\left(2\lambda\rho+\frac{4}{3}\rho^2\right)$$

b.) $$\frac{d}{d\ln p}\left(\frac{\rho}{\lambda}\right) = \frac{1}{\lambda}\frac{d\rho}{d\ln p} - \frac{1}{\lambda^2}\rho\frac{d\lambda}{d\ln p}$$

$$= \frac{1}{\lambda}\beta_\rho - \frac{\rho}{\lambda^2}\beta_\lambda$$

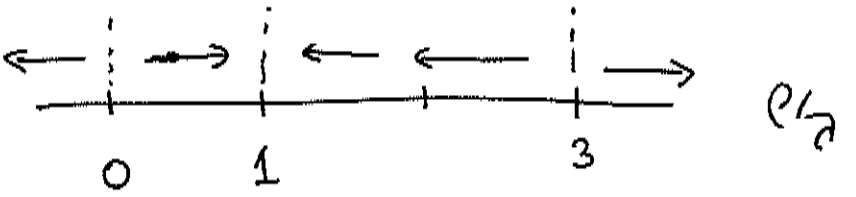
$$= \frac{1}{(4\pi)^2}\left\{ 2\rho + \frac{4}{3}\frac{\rho^2}{\lambda} - 3\rho - \frac{\rho^3}{3\lambda^2} \right\}$$

$$= \frac{1}{(4\pi)^2}\rho\left(-1 + \frac{4}{3}\frac{\rho}{\lambda} - \frac{1}{3}\frac{\rho^2}{\lambda^2} \right)$$

$$\frac{d}{d\ln p}\left(\frac{\rho}{\lambda}\right) = \frac{-1}{(4\pi)^2}\rho\left(1 - \frac{\rho}{\lambda}\right)\left(1 - \frac{1}{3}\frac{\rho}{\lambda}\right)$$

so:

∞ $p \to 0$
∞ $p \to$ IR



for $0 < \rho/\lambda < 3$, $\rho/\lambda \to 1$ ∞ $p \to 0$

Note that both ρ and λ tend to 0 ∞ $\frac{1}{\ln M/p}$

c.) In $d = 4 - \epsilon$

$$\beta_\lambda = -\epsilon\lambda + \frac{1}{(4\pi)^2}\left(3\lambda^2 + \frac{\rho^2}{3}\right)$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{(4\pi)^2}\left(2\lambda\rho + \frac{4}{3}\rho^2\right)$$

A fixed point is a simultaneous zero of β_λ and β_ρ

Note that

$$\frac{d}{d\ln p}\left(\frac{\rho}{\lambda}\right) = -\frac{1}{(4\pi)^2}\rho\left(1 - \frac{\rho}{\lambda}\right)\left(1 - \frac{1}{3}\frac{\rho}{\lambda}\right)$$

just as in the previous part, the ϵ terms cancel. So we can have fixed points only at $\rho/\lambda = 0, 1, 3$

For $\underline{\rho = 0}$,

$$\beta_\rho = 0 \qquad \beta_\lambda = -\epsilon\lambda + \frac{3\lambda^2}{(4\pi)^2}$$

$$= 0 \quad \text{at} \quad \lambda_* = \frac{(4\pi)^2}{3}\epsilon$$

For $\underline{\rho = \lambda}$.

$$\beta_\lambda = -\epsilon\lambda + \frac{1}{(4\pi)^2}\frac{10}{3}\lambda^2$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{(4\pi)^2}\frac{10}{3}\rho^2$$

both $= 0$ at $\lambda_* = \rho_* = (4\pi)^2 \cdot \epsilon \cdot \frac{3}{10}$

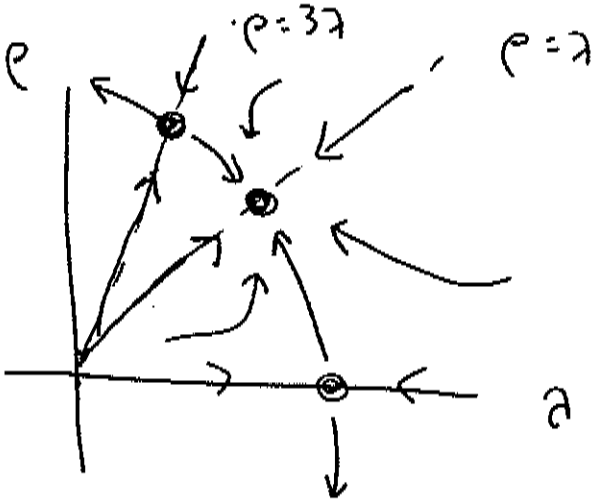
For $\rho = 3\lambda$

$$\beta_\lambda = -\epsilon\lambda + \frac{1}{(4\pi)^2} 6\lambda^2$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{(4\pi)^2} 2\rho^2$$

both $= 0$ at $\lambda_* = (4\pi)^2\epsilon \cdot \frac{1}{6}$ $\rho_* = (4\pi)^2\epsilon \cdot \frac{1}{2}$

The behavior of β_λ determines the flow pattern:



For $0 < \frac{\rho}{\lambda} < 3$, the most stable point is

$$\rho_* = \lambda_* = \frac{3}{10}(4\pi)^2\epsilon,$$

an $O(2)$ - symmetric theory.

2.) The Gibbs free eng obeys

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial g_i} - \gamma M \frac{\partial}{\partial M} \right] G_2 = 0$$

where M is magnetization, μ is the renormalization scale, g_i are coupling constants. We can specialize to the coupling constants g_m (relevant) and λ (slowest irrelevant) couplings. For $\lambda = \lambda_*$, the trajectory that emerges from the fixed pint

$$G_*(M,t) = t^{2-\alpha} f(Mt^{-\beta})$$

If $\lambda \neq \lambda_*$, then there are subleading terms proportional to $\bar{\lambda} - \lambda_*$. Let m be a mass scale; then

$$\frac{d}{d \ln m/\mu} \bar{\lambda}(m) = \beta(\bar{\lambda}) = \underbrace{\beta(\lambda_*)}_{=0} + \underbrace{\left.\frac{d\beta}{d\lambda}\right|_{\lambda_*}}_{\equiv \omega} \cdot (\bar{\lambda} - \lambda_*) + \cdots$$

$$\frac{d}{d \ln m/\mu} \bar{\lambda}(m) = \omega (\bar{\lambda} - \lambda_*) + \cdots$$

$$\bar{\lambda}(m) = \lambda_* + c \left(\frac{m}{\mu}\right)^{\omega} \qquad c \sim \lambda - \lambda_*$$

then

$$G(M,t) = G_*(M,t) + \delta G(M,t) \cdot \left(\frac{1}{\mu}\right)^{\omega}$$

Now, as t varies from 0 (T from T_c)

$$\xi^{-1} \sim (\text{mass}) \sim t^{\nu} \qquad \text{so} \qquad \text{mass scale} \sim t^{\nu}$$

so

$$G = G_+ + \delta G \cdot (t^{\nu}/\mu)^{\omega}$$

or

$$G = t^{2-\alpha}\left[f(Mt^{-\beta}) + (\lambda - \lambda_+)\, t^{\omega\nu}\, g(Mt^{-\beta}) + \cdots \right]$$

Here is another, equivalent derivation. Use Pestin + Schroeder, eq. (13.31), with $\varphi_i = \lambda$, M = magnetization

$$\frac{d}{d\ell_M}\bar{\lambda} = \frac{2}{d-2+2\gamma(\bar{\lambda})}\,\beta(\bar{\lambda})$$

$$\approx \frac{2}{d-2+2\gamma(\lambda_*)}\,\omega\,(\bar{\lambda}-\lambda_+)$$

so

$$\bar{\lambda} = \lambda_* + (\lambda - \lambda_+)\cdot(\text{const})\cdot M^{\frac{2\omega}{d-2+2\gamma_*}}$$

now $\dfrac{2}{d-2+2\gamma_*} = \dfrac{1}{2-\gamma_{\phi^2 *}}\cdot\dfrac{1}{\beta} = \nu/\beta$

so the correction term is proportional to

$$M^{\omega\nu/\beta}\, h(Mt^{-\beta}) = M^{\omega\nu/\beta}\left[M^{-1}t^{\beta}\right]^{(\nu/\beta)\cdot\omega} g(Mt^{-\beta})$$

$$= t^{\omega\nu}\, g(Mt^{-\beta})$$

Physics 332 – Problem Set # 5

(due Wednesday, May 17)

1. Consider scalar electrodynamics:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{6}(\phi^\dagger\phi)^2 \tag{1}$$

where ϕ is a complex-valued field and $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$.

(a) Remind yourself that, classically, the A_μ field acquires a mass when ϕ acquires a classical vacuum expectation value.

(b) Write the functional integral for A_μ and ϕ in the R_ξ gauge for a fixed background $\varphi_{cl} = \langle\phi\rangle$. Notice that the dependence on φ_{cl} simplifies for $\xi = 0$ (Lorenz gauge).

(c) Compute the 1-loop quantum correction to the effective potential for φ_{cl}. Renormalize by minimal subtraction, introducing a renormalization scale M. For minimal credit, perform this calculation at $\xi = 0$. For extra credit, show that, to first order in perturbation theory, the answer is independent of ξ. Anyway, from here on, it is simplest to set $\xi = 0$.

(d) Take the limit in which the renormalized mass m^2 goes to zero. Consider this theory for small values of λ, such that, parametrically, $\lambda \sim e^4$. Show that there are parameters in this region such that the $m^2 = 0$ theory has spontaneous symmetry breaking; thus, there is symmetry breaking due to quantum corrections.

(e) Compute the renormalization group β functions for this theory. Since the β functions are gauge-invariant, you could do this in Feynman gauge ($\xi = 1$), but it is somewhat easier to use $\xi = 0$. Either way, you should find:

$$\beta_e = \frac{e^3}{48\pi^2} \qquad \beta_\lambda = \frac{1}{24\pi^2}(5\lambda^2 - 18e^2\lambda + 54e^4) \tag{2}$$

Sketch the renormalization group flows (toward the infrared) in the (λ, e^2) plane. Show that every renormalization group trajectory actually passes through $\lambda = 0$ and goes into the region $\lambda < 0$.

(f) Consider the approximation to the $m^2 = 0$ effective potential given by

$$V_{eff} = \frac{1}{6}\overline{\lambda}(\varphi_{cl})(Z(\varphi_{cl})^2\varphi_{cl}^4 \tag{3}$$

Show that this expression has spontaneous symmetry breaking, just as we found in part(d).

(g) Now consider $m^2 \neq 0$. Show that the theory has a first-order phase transition as a function of m^2.

2. Apply the methods of this problem to the Glashow-Salam-Weinberg model of weak interactions.

   (a) Compute the effective potential for the Higgs field to 1-loop order, ignoring all effects of quark masses but including the contributions of gauge fields.

   (b) Show that the theory has a first-order phase transition as a function of the renormalized Higgs mass parameter μ^2.

   (c) Show that this result implies a lower bound on the physical mass of the Higgs boson (the 'Linde-Weinberg bound'). Compute the bound to leading order in coupling constants.

   (d) Now add in the contribution of the top quark. Show that, when the top quark mass is sufficiently heavy, the symmetry-breaking effect found in part (b) goes away. However, another pathology develops, in which, when m_t is sufficiently large, the effective potential becomes negative at very large field values and causes an instability of the model. Estimate the value of the top quark mass, as a function of the Higgs boson and W boson masses, at which this instability takes place.

Physics 332 – Problem Set #5

Solutions

1.) a.) Write the terms in the Lagrangian that are quadratic in A_μ:

$$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \phi^*(eA^\mu)(eA_\mu)\phi$$

$$= \frac{1}{2}A_\nu\left(\partial^2 g^{\nu\lambda} - \partial^\nu\partial^\lambda + g^{\nu\lambda}e^2 2|\phi|^2\right)A_\lambda$$

so if we ignore $\partial^\lambda A_\lambda$

$$= A_\nu\left(\partial^2 + m_A^2\right)g^{\nu\lambda}A_\lambda \qquad \text{where } m_A^2 = 2e^2\langle|\phi|^2\rangle$$

b.) Now compute the contribution of A_μ to the effective potential for ϕ. Write

$$\phi = \varphi_{cl.} + \frac{1}{\sqrt{2}}(\sigma + i\pi)$$

φ assume real

$$= V/\sqrt{2} + \frac{1}{\sqrt{2}}(\sigma + i\pi) \qquad m_A^2 = e^2V^2$$

then

$$\mathcal{L} = A_\mu\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu + \left|\left(\frac{1}{\sqrt{2}}\right)\partial_\mu(\sigma+i\pi) + ieA_\mu(V+\sigma+i\pi)\right|^2$$

$$- \frac{m^2}{2}|V+\sigma+i\pi|^2 - \frac{\partial}{4!}\left(|V+\sigma+i\pi|^2\right)^2$$

$$= A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu + \tfrac{1}{2}(\partial_\mu \sigma)^2$$

$$+ \tfrac{1}{2}(\partial \pi + e v A_\mu)^2$$

$$- \left(\frac{m^2}{2} v^2 + \frac{\lambda}{24} v^4\right) - (\text{linear in } \sigma)$$

$$- \frac{m^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda}{4!} \cdot (6 v^2 \sigma^2 + 2 v^2 \pi^2)$$

$$- (\text{cubic, quartic in fields})$$

We would like to quantize so as to cancel the term $\partial \pi \cdot e v A_\mu$

To do this, introduce the R_ξ gauge [see P+S section 21.1]

$$\int \mathcal{D}A = \int \mathcal{D}A \int \mathcal{D}\alpha \int d\omega \; e^{-i \int d^4x \, \omega^2/2} \, \delta(\omega - G_\alpha) \det \frac{\delta G_\alpha}{\delta \alpha}$$

where $$G_\alpha = \frac{1}{\sqrt{\xi}} (\partial_\mu A^\mu - \xi e v \pi)$$

$$\frac{\delta G}{\delta \alpha} = \frac{1}{\sqrt{\xi}} \left(\frac{1}{e}\partial^2 + \xi e v^2\right) = \frac{1}{\sqrt{\xi} e} (\partial^2 + \xi e^2 v^2)$$

$$\int \mathcal{D}A = \int \mathcal{D}\alpha \cdot e^{i \int d^4x \left\{ -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \partial_\mu A^\mu \cdot e v \pi - \frac{1}{2}\xi (e v)^2 \pi^2 \right\}}$$

$$\cdot \det (\partial^2 + \xi (e v)^2)$$

Notice that the overall constant terms in the measure do not depend on v !

Then

$$\int \mathcal{D}A \int d\Phi \quad e^{i\int d^4x\, \mathcal{L}}$$

$$= \int \mathcal{D}A\, \mathcal{D}\sigma\, \mathcal{D}\pi \quad \exp\Big[i \int d^4x \Big\{ \tfrac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) A_\nu$$

$$+ \tfrac{1}{2} (ev)^2 A_\mu g^{\mu\nu} A_\nu + \tfrac{1}{2}\tfrac{1}{\xi} A_\mu \partial^\mu\partial^\nu A_\nu$$

$$+ \tfrac{1}{2} (\partial\pi)^2 - \tfrac{1}{2}\xi (ev)^2 \pi^2 - \tfrac{1}{2}\left(\frac{\partial v^2}{\partial} + m^2\right)\pi^2$$

$$+ \tfrac{1}{2} (\partial\sigma)^2 - \tfrac{1}{2}\left(\frac{3\lambda}{6} v^2 + m^2\right)\sigma^2 - \left(\frac{m^2}{2} v^2 + \frac{\lambda}{24} v^4\right)$$

$$+ (\text{linear} \approx 0) + \text{cubic, quartic} \Big\} \Big]$$

$$\cdot \det\left(\partial^2 + \xi (ev)^2\right)$$

c) We might as well do the integral. Note that the transverse & longitudinal components of A_μ separate:

$$\tfrac{1}{2} (ev)^2 A_\mu g^{\mu\nu} A_\nu = \tfrac{1}{2} m_A^2 \left[A_\mu \left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right) A_\nu + A_\mu \frac{\partial^\mu\partial^\nu}{\partial^2} A_\nu\right]$$

Then the quadratic terms in A are:

$$\tfrac{1}{2} A_\mu \left[\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right)(\partial^2 + m_A^2) + \frac{\partial^\mu\partial^\nu}{\partial^2} \cdot \frac{1}{\xi}(\partial^2 + \xi m_A^2)\right] A_\nu$$

So, the value of the integral is:

$$\int \mathcal{D}A \int d\phi \; e^{i\int \mathcal{L}}$$

$$= e^{-i\int d^4x \left(\frac{m^2}{2}v^2 + \frac{\lambda}{24}v^4\right) + (\text{ln}\,\text{sum} \to \sigma)}$$

$$\cdot \left[\det\left(\partial^2 + m_A^2\right)\right]^{-3/2} \left[\det\left(\partial^2 + \xi m_A^2\right)\right]^{-1/2} \left[\det\left(\partial^2 + \xi m_A^2\right)\right]^{+1}$$

from $A_{\log}$

$$\cdot \left(\det\left[\partial^2 + \xi m_A^2 + \left(\frac{\lambda v^2}{6} + m^2\right)\right]\right)^{-1/2} \quad \text{from } \pi$$

$$\det\left[\partial^2 + \left(\frac{3\lambda v^2}{6} + m^2\right)\right]^{-1/2} \quad \text{from } \sigma$$

From here, we see ~~that~~

$$V_{\text{eff}}(v) = \frac{m^2}{2}v^2 + \frac{\lambda}{24}v^4$$

$$+ i\,\frac{1}{\text{Vol.}} \cdot \left[-\frac{3}{2}\log\det\left(\partial^2 + m_A^2\right) + \frac{1}{2}\log\det\left(\partial^2 + \xi m_A^2\right)\right.$$

$$- \frac{1}{2}\log\det\left(\partial^2 + \xi m_A^2 + \left(\frac{\lambda v^2}{6} + m^2\right)\right)$$

$$\left. - \frac{1}{2}\log\det\left(\partial^2 + \left(\frac{3\lambda v^2}{6} + m^2\right)\right)\right]$$

In perturbation thy, at the zeroth order minim of V_{eff} for $m^2 < 0$

$$-(-m^2)v + \frac{\lambda}{6}v^3 = 0 \quad \text{or} \quad \left(\frac{\lambda v^2}{6} + m^2\right) = 0$$

then the ξ dependence of this expression cancels.

In fact, the energy at the minim and the mass of the σ and A are ξ-invariant to all orders.

For simplicity, I will set $\xi = 0$ from here on. The functional determinants have the values:

$$\frac{1}{\text{Vol}} \ln \det(\partial^2 + m^2) = -i \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} (m^2)^{d/2}$$

$$d = 4 - \varepsilon \qquad = -i \frac{(m^2)^2}{2 \cdot (4\pi)^2} \cdot \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln m^2 + \tfrac{3}{2} \right)$$

the divergent terms are:

$$V_{\text{eff}} = + \frac{1}{2(4\pi)^2} \cdot \frac{2}{\epsilon} \left\{ -\frac{3}{2} (eU)^4 - \frac{1}{2} \left(\frac{\partial U^2}{6} + m^2 \right)^2 - \frac{1}{2} \left(\frac{\partial U^2}{2} + m^2 \right)^2 \right\}$$

$$= \frac{1}{2(4\pi)^2} \frac{2}{\epsilon} \cdot \left\{ (\text{const}) + U^2 \left[-\frac{\partial}{6} - \frac{\partial}{2} \right] m^2 + U^4 \left[-\frac{3}{2} e^4 - \frac{\partial^2}{72} - \frac{\partial^2}{8} \right] \right\}$$

$$= \frac{1}{2(4\pi)^2} \frac{2}{\epsilon} \left\{ (\text{const}) + U^2 \left[-\frac{\partial}{3} m^2 \right] + U^4 \left[-\frac{3}{2} e^4 - \frac{5}{36} \partial^2 \right] \right\}$$

these divergences are to be cancelled by the counterterms.

$$V_{eff} = \ldots + \frac{1}{2}\delta_m v^2 + \frac{1}{24}\delta_\lambda v^4$$

$$\delta_m = \frac{\partial m^2}{3(4\pi)^2}\cdot\frac{2}{\epsilon} \qquad \delta_\lambda = \frac{1}{(4\pi)^2}\left[18e^4 + \frac{5}{3}\lambda^2\right]\cdot\frac{2}{\epsilon}$$

Making the subtraction (eliminating at the same time $(-\gamma + \ln 4\pi)$'s), we find

$$V_{eff} = \frac{m^2}{2}v^2 + \frac{\lambda}{24}v^4$$

$$+ \frac{1}{4(4\pi)^2}\cdot\Big\{ 3(ev)^4\left(\ln\frac{(ev)^2}{M^2} - \tfrac{3}{2}\right)$$

$$+ \left(\frac{\lambda v}{6} + m^2\right)^2\left(\ln\frac{\lambda v/6 + m^2}{M^2} - \tfrac{3}{2}\right)$$

$$+ \left(\frac{\lambda v}{2} + m^2\right)^2\left(\ln\frac{\lambda v/2 + m^2}{M^2} - \tfrac{3}{2}\right)\Big\}$$

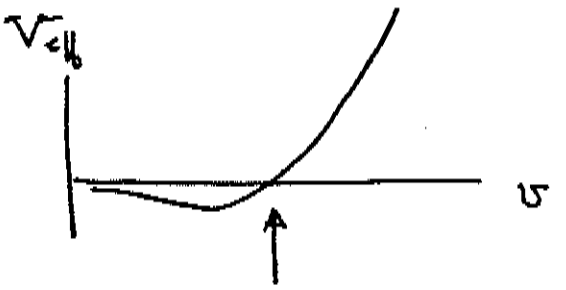
d) Let $m^2 \to 0$

$$V_{eff} = \frac{\lambda v^4}{24} - \frac{3}{4(4\pi)^2} e^4 v^4 \left(\ln \frac{M^2}{(ev)^2} + \frac{3}{2} \right)$$

$$+ \text{ small if } \lambda \sim e^4$$

for $\ln M^2/(ev)^2 > -3/2$ $\qquad ev < M e^{3/4}$

the second term is negative. For sufficiently small v, the coefficient of v^4 becomes negative, so the potential takes the form



$$\frac{\lambda}{24} - \frac{3}{4(4\pi)^2} e^4 \left(\ln \frac{M^2}{(ev)^2} + \frac{3}{2} \right) = 0$$

$$\ln M^2/(ev)^2 = -\frac{3}{2} + \frac{(4\pi)^2}{18} \frac{\lambda}{e^4}$$

$$ev = M \exp\left[-\frac{4\pi^2}{9} \frac{\lambda}{e^4} + \frac{3}{4} \right]$$

e.) Now compute the renormalization group β-function for this theory. To do this, we need

δ_2 scalar field strength counterterm

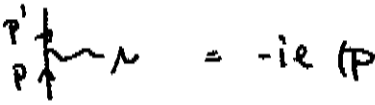
δ_3 A_μ field strength counterterm

δ_λ

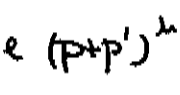
δ_2 and δ_λ are gauge-dependent. I will compute them for general ξ, but it is OK to compute them just for $\xi=0$ or $\xi=1$.

The β functions are gauge-independent.

I will use complex fields $\phi, \phi^\dagger$. The Feynman rules are



$= -ie(p+p')^\mu$



$= 2ie^2 g^{\mu\nu}$



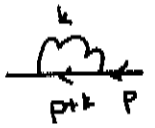
$= -i\frac{2}{3}\lambda$

Begin with



$= \delta_2 + \delta_3 + \delta_\lambda + i p^2 \delta_2$

δ_2 and δ_3 are mass renormalizations only with no contribution to δ_2



$$= (-ie)^2 \int \frac{d^dk}{(2\pi)^d} (2p+k)^\mu \frac{i}{(p+k)^2} (2p+k)^\nu \frac{-i}{k^2}$$

$$\cdot \left[g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2} (1-\xi) \right]$$

$$= (-ie)^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+k)^2 (k^2)^2} \left[(2p+k)^2 k^2 - [k\cdot(2p+k)](1-\xi) \right]$$

$$= -e^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+k)^2 (k^2)^2} \left[k^4 + 4k\cdot p\, k^2 + 4p^2 k^2 - (1-\xi)(k^4 + 4k\cdot p\, k^2 + 4(k\cdot p)^2) \right]$$

Combine denominator

$$\mathbb{k} = k + xp \qquad k = \mathbb{k} - xp$$

$$k^2 = \mathbb{k}^2 - 2xp\cdot\mathbb{k} + x^2p^2$$

$$k\cdot p = \mathbb{k}\cdot p - xp^2$$

$$= -e^2 \int \frac{d^d\mathbb{k}}{(2\pi)^d} \int dx \frac{2(1-x)}{[\mathbb{k}^2 + x(1+x)p^2]^3}$$

$$\cdot \left\{ \xi \left[k^2 (k^2 + 4k\cdot p) \right] + 4\left[p^2 k^2 - (1-\xi)(k\cdot p)^2 \right] \right\}$$

$$= -e^2 \int \frac{d^d\mathbb{k}}{(2\pi)^d} \frac{2(1-x)}{[\mathbb{k}^2 + x(1-x)p^2]^3}$$

$$\left\{ \xi \left((\mathbb{k}^2)^2 - 4x\, p\cdot\mathbb{k}\,\mathbb{k}^2 + 4x^2 (p\cdot\mathbb{k})^2 + 2x^2 \mathbb{k}^2 p^2 + \cdots \right.\right.$$

$$\left. + 4\mathbb{k}^2 \mathbb{k}\cdot p - 4x\, p\cdot\mathbb{k}\, \mathbb{k}\cdot p - 4x\mathbb{k}^2 p^2 + \cdots \right]$$

$$\left. + 4\left[p^2\mathbb{k}^2 - (1-\xi)(\mathbb{k}\cdot p)^2 \right] \right\}$$

$$= -e^2 \int_0^1 dx \quad 2(1-x) \quad \frac{1}{(4\pi)^{d_4}} d_4 \quad \frac{1}{[-x(1-x)p^2]^{2-d_4}} \quad \frac{1}{\Gamma(3)}$$

$$\{ \quad \xi \left[-\frac{d(d+2)}{4} \Gamma(1-d_4) (-x(1-x)p^2) \right.$$

$$+ \quad \Gamma(2-d_4) \left[4x^2 \cdot \tfrac{1}{2} \cdot p^2 + 2x^2 \tfrac{d}{2} p^2 \right.$$

$$\left. - 8x \tfrac{1}{2} p^2 - 4x \tfrac{d}{2} p^2 \right]$$

$$\left. + 4 \Gamma(2-d_4) \left[\tfrac{d}{2} p^2 - (1-\xi) \tfrac{1}{2} p^2 \right] \right\}$$

mit $d = 4-\varepsilon$ $\qquad \Gamma(2-d_4) \cong \frac{2}{\epsilon}$ $\qquad \Gamma(1-d_4) \cong \frac{\Gamma(2-d_4)}{1-d_4} \cong -\frac{2}{\epsilon}$

$$= -\frac{ie^2}{(4\pi)^{d_4}} d_4 \frac{2}{\epsilon} \int_0^1 dx \quad 2(1-x) \cdot \frac{1}{2} \cdot p^2$$

$$\cdot \{ \quad \xi \left[-6x(1-x) + 2x^2 + 4x^2 - 4x - 8x \right]$$

$$+ \quad 4 \left[2 - \tfrac{1}{2} + \xi/2 \right] \}$$

$$= -\frac{ie^2}{(4\pi)^{d_4}} d_4 \frac{2}{\epsilon} p^2 \int_0^1 dx \quad (1-x)$$

$$\cdot \{ \quad 6 + \xi \left[2 - 18x + 12x^2 \right] \}$$

$$= -\frac{ie^2}{(4\pi)^2} \frac{2}{\epsilon} p^2 \left(3 + \xi \left[\underbrace{1 - \frac{18}{6} + \frac{12}{12}}_{-1} \right] \right)$$

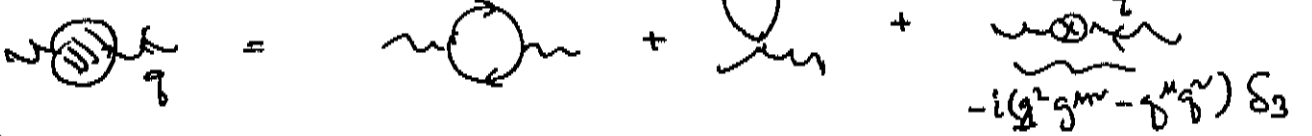
So

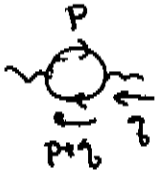


$$= \frac{-ie^2}{(4\pi)^2} p^2 \cdot (3-\xi) \cdot \frac{2}{\epsilon} \qquad \frac{2}{\epsilon} \sim \ln \Lambda^2$$

$$+ \; ip^2 \delta_2$$

so $$\delta_2 = \frac{e^2}{(4\pi)^2} (3-\xi) \ln \Lambda^2/M^2$$





$$= (-ie)^2 \int \frac{d^d p}{(2\pi)^d} (2p+q)^\mu \frac{i}{(p+q)^2} (2p+q)^\nu \frac{i}{p^2}$$

$$= e^2 \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d} \frac{(2\mathbb{P} + (1-2x)q)^\mu (2\mathbb{P} + (1-2x)q)^\nu}{[\mathbb{P}^2 + x(1-x)q^2]^2}$$

using $\mathbb{P} = p + xq$

$$2p+q = 2\mathbb{P} + (1-2x)q$$

$$p+q = \mathbb{P} + (1-x)q$$

$$= e^2 \frac{i}{(4\pi)^{d_2}} \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d_2}}$$

$$\Big\{ -4 \cdot \tfrac{1}{2} g^{\mu\nu} \Gamma(1-d_2) (-x(1-x)q^2)$$

$$+ \; \Gamma(2-d_2) q^\mu q^\nu (1-2x)^2 \Big\}$$

 $= 2ie^2 g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d} \frac{+i}{p^2}$

$= -2ie^2 g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 (p+q)^2} (p+q)^2$

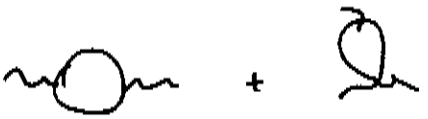
$= -2ie^2 g^{\mu\nu} \int \frac{d^dP}{(2\pi)^d} \int dx \frac{[P + (1-x)q]^2}{[P^2 + x(1-x)q^2]^2}$

or symmetry the integrand : $x \leftrightarrow (1-x)$

$= -2e^2 g^{\mu\nu} \int dx \frac{i}{(4\pi)^{d_4}} \left\{ -\Gamma(1-d_4)\, d_2\, [-x(1-x)q^2] \right.$

$\left. + \tfrac{1}{2}[(1-x)^2 + x^2]\, q^2\, \Gamma(2-d_4) \right\} \frac{1}{[-x(1-x)q^2]^{2-d_4}}$

$=$ all



$= \frac{ie^2}{(4\pi)^{d_4}} \int_0^1 dx \frac{1}{[-x(1-x)q^2]^{2-d_4}}$

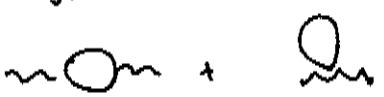
$\cdot \left\{ \Gamma(1-d_4)\, (-x(1-x)q^2)\, (-2+d) \right.$

$+ \Gamma(2-d_4)\, (1-2x)^2\, q^\mu q^\nu$

$\left. - \Gamma(2-d_4)\, q^2 g^{\mu\nu}\, (1-2x+2x^2) \right\}$

$\Gamma(1-d_4)(-2+d) = -2\,\Gamma(2-d_4)$

so



$$= \frac{ie^2}{(4\pi)^{d_L}} \Gamma(2-d_L) \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d_L}}$$

$$\{ - q^2 g^{\mu\nu} (1 - 4x + 4x^2) + q^\mu q^\nu (1-2x)^2 \}$$

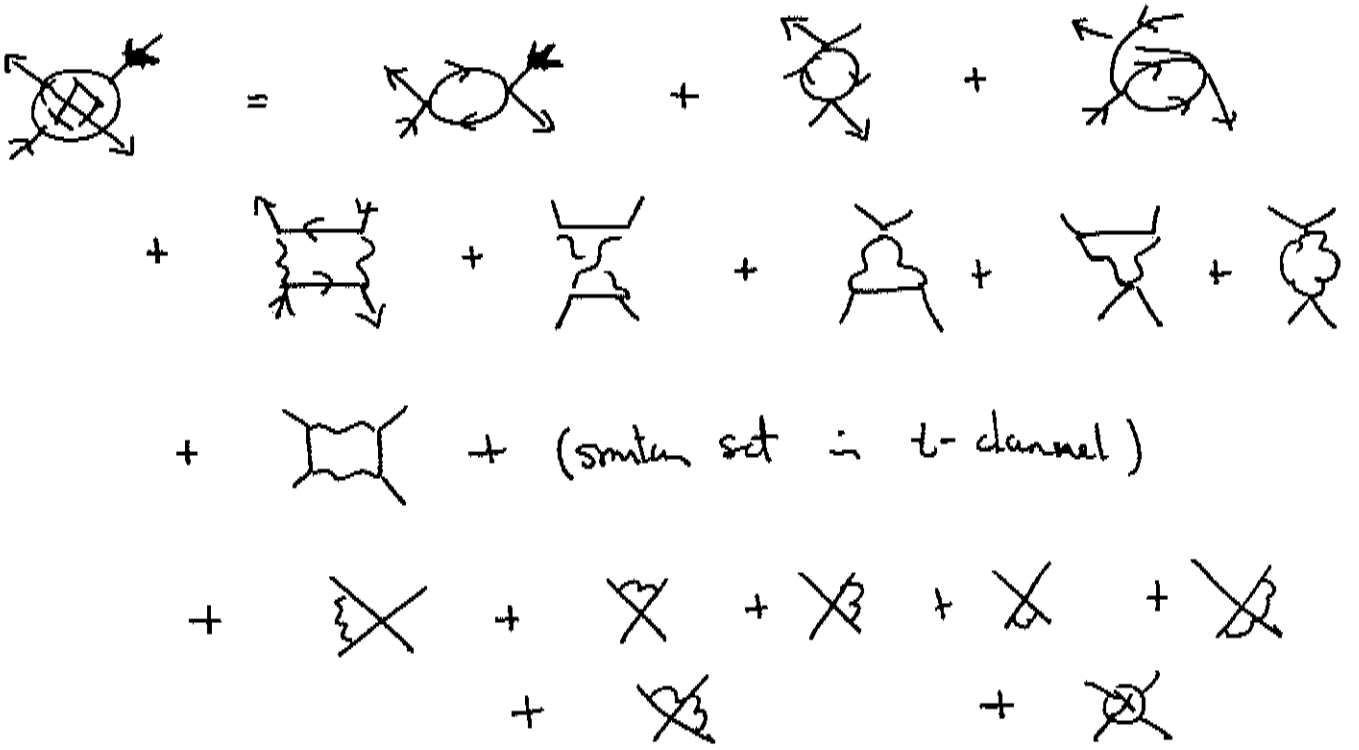
$$\simeq - \frac{ie^2}{(4\pi)^2} \cdot \frac{2}{\epsilon} (q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx \; (1 - 4x + 4x^2)$$

$$1 - \frac{4}{2} + \frac{4}{3} = \frac{1}{3}$$

so

$$\delta_3 = - \frac{1}{3} \frac{e^2}{(4\pi)^2} \ln \Lambda^2/M^2$$

Finally, we need to compute δ_2. There are 3 sets of loop diagrams:



We can compute all of these diagrams with all external momenta set = 0



$$= (-i\tfrac{2}{3}\lambda)^2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2} \frac{i}{p^2}$$

$$= \frac{4}{9}\lambda^2 \frac{i}{(4\pi)^{d_4}} \Gamma(2-d_4) = i\frac{4}{9}\lambda^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

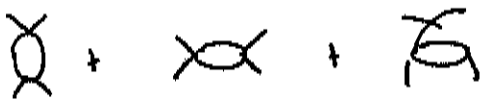


$$= (-i\tfrac{2}{3}\lambda)^2 \int \frac{d^d p}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 = i\frac{4}{9}\lambda^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$



$$= (-i\tfrac{2}{3}\lambda)^2 \cdot \frac{1}{2} \cdot \int \frac{d^d p}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 = i\frac{2}{9}\lambda^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

$\sim$ all



$$= i\frac{10}{9}\lambda^2 \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$



$$= (-ie)^4 \int \frac{d^d p}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)^2$$

$$\cdot\, p^\mu p^\nu \left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right) p^\lambda p^\sigma$$

$$= (-ie)^4 \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^4} \cdot \xi^2 (p^2)^2$$

$$= \frac{ie^4}{(4\pi)^2} \Gamma(2-d_4) \cdot \xi^2$$



$$= -ie^4 \int \frac{d^d p}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)^2 p^\mu p^\nu \left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right)(-p^\beta)(-p^\sigma)$$

$$= \frac{ie^4}{(4\pi)^2} I(2-d_4)\ \xi^2$$



$$= 2ie^2 g^{\mu\nu} (-ie)^2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2}\left(\frac{-i}{p^2}\right)^2 \cdot \left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right) p^\beta p^\sigma$$

$$= -2e^4 \frac{1}{(4\pi)^2} I(2-d_4)\ \xi^2$$

$$=$$





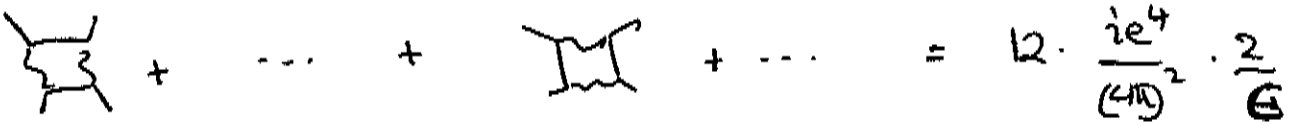
$$= \frac{1}{2} \cdot 2ie^2 g^{\mu\nu}\ 2ie^2 g^{\lambda\sigma} \int \frac{d^d p}{(2\pi)^d} \left(\frac{-i}{p^2}\right)^2 \left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right)$$

$$= 2e^4 \cdot (3+\xi^2)\ \frac{i}{(4\pi)^2} \cdot I(2-d_4)$$

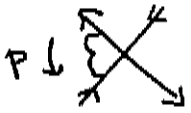


$$= e^4 \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \cdot (2\xi^2 - 4\xi^2 + 6 + 2\xi^2)$$

the t channel works similarly, so



$$= 12 \cdot \frac{ie^4}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$



$$= (-i\tfrac{2}{3}\lambda)(-ie)^2 \int \frac{d^dp}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)$$

$$\cdot p^\mu p^\nu \left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\lambda}{p^2} \right]$$

$$= -\frac{2}{3} e^2 \lambda \cdot \xi \cdot \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k)$$



However

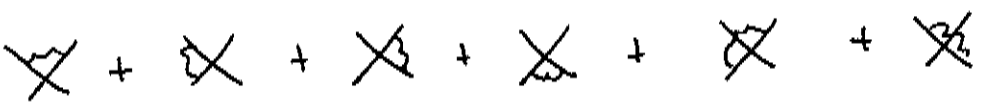


$$= (-i\tfrac{2}{3}\lambda)(-ie)^2 \int \frac{d^dp}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)$$

$$\cdot p^\mu \cdot (-p^\nu) \cdot \left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2} \right]$$

$$= +\frac{2}{3} e^2 \lambda \cdot \xi \cdot \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k)$$

so



$$= -2\xi \cdot \frac{2}{3} e^2 \lambda \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

$\Rightarrow$ all: $\cancel{\otimes} = -\frac{2}{3} i \delta_\lambda$

$$\frac{2}{3}\delta_\lambda = \left[\frac{10}{9}\frac{\lambda^2}{(4\pi)^2} - \frac{2}{3}\cdot 2\xi\frac{e^2\lambda}{(4\pi)^2} + 12\frac{e^4}{(4\pi)^2}\right] f^{1/M^2}$$

$$\text{then} \quad \delta_\lambda = \left[\frac{5}{3}\frac{\lambda^2}{4\pi^2} - 2\xi\frac{e^2\lambda}{(4\pi)^2} + 18\frac{e^4}{(4\pi)^2}\right] f^{1/M^2}$$

$$\beta_e = M\frac{\partial}{\partial M}(-e\delta_1 + e\delta_2 + \tfrac{1}{2}e\delta_3)$$

$$= (\text{Abelian gauge theory}) = M\frac{\partial}{\partial M}(\tfrac{1}{2}e\delta_3)$$

$$= \frac{1}{3}\frac{e^3}{(4\pi)^2} = \frac{e^3}{48\pi^2}$$

$$\beta_\lambda = M\frac{\partial}{\partial M}[-\delta_\lambda + 2\delta_2\lambda]$$

$$= \frac{1}{(4\pi)^2}\left\{\frac{10}{3}\lambda^2 - 4\xi e^2\lambda + 36e^4 - 4(3-\xi)e^2\lambda\right\}$$

$$= \frac{1}{(4\pi)^2}\left\{\frac{10}{3}\lambda^2 - 12e^2\lambda + 36e^4\right\}$$

$$= \frac{1}{24\pi^2}\left\{5\lambda^2 - 18e^2\lambda + 54e^4\right\}$$

Note that this expression is independent of ξ. If we set $\xi = 0$ in δ_λ, we find:

$$\delta_\lambda\ (\xi=0) = \left[\frac{5}{3}\lambda^2 + 18\ e^4\right] \frac{1}{(4\pi)^2}\ \ell_J\ \Lambda^2/M^2$$

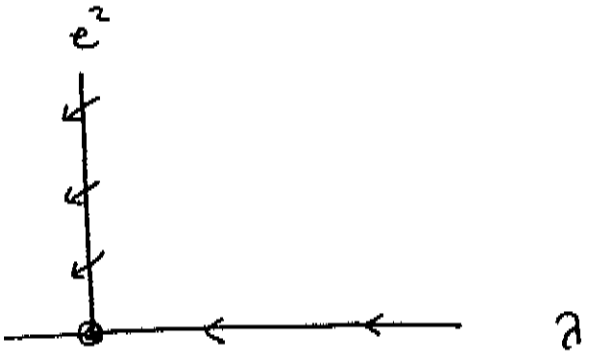
$\doteq$ agrees w. p.6

Now look at flows of (λ, e^2) to the IR

e^2 always flows to 0 smoothly.

λ flows to 0 for $e^2 = 0$; however, on the line

$\lambda = 0$, λ flows to <u>negative</u> values:



$$\frac{d}{d\ln q}\left(\frac{\lambda}{e^2}\right) = \frac{1}{e^2}\beta_\lambda - 2\frac{\lambda}{e^3}\beta_e$$

$$= \frac{1}{(4\pi)^2}\left\{\frac{10}{3}\lambda^2/e^2 - 12\lambda + 36\,e^2 - \frac{2}{3}\lambda\right\}$$

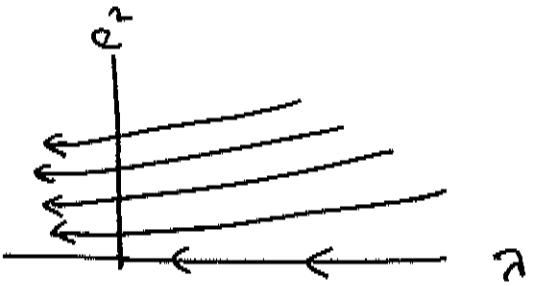
$$= \frac{e^2}{(4\pi)^2}\left\{\frac{10}{3}\left(\lambda/e^2\right)^2 - \frac{38}{3}\left(\lambda/e^2\right) + 36\right\}$$

$$= \frac{e^2}{(4\pi)^2}\left\{\frac{10}{3}\left(\lambda/e^2 - \frac{19}{10}\right)^2 + \underbrace{36 - \frac{10}{3}\cdot\left(\frac{19}{10}\right)^2}_{>0}\right\}$$

> 0

so as $q \to 0$ λ/e^2 decreases, so we cannot set to $e^2 = 0$ $\lambda > 0$ from $e^2 > 0$

then, the flows are:



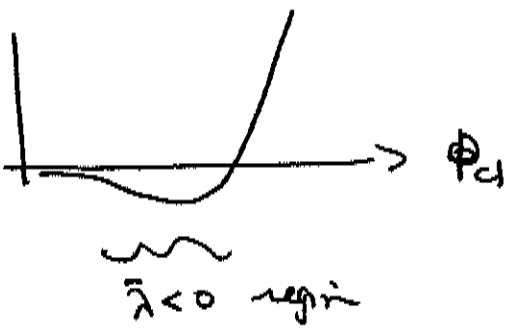
into the region $\lambda < 0$

f.) Then $\bar{\lambda}(\varphi_{cl})$ becomes negative for some φ_{cl}.

The resummed effective potential for $m^2 = 0$

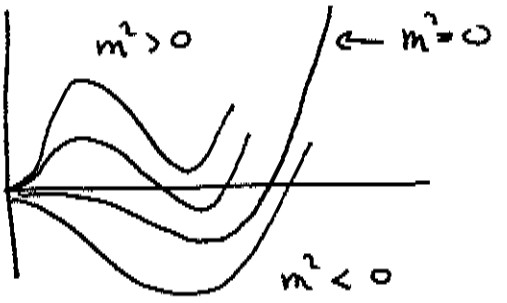
$$V_{eff} = \frac{1}{6}\,\bar{\lambda}(\varphi_{cl}) \cdot Z^2(\varphi_{cl}) \cdot \varphi_{cl}^4 + (\text{small correction})$$

has the shape

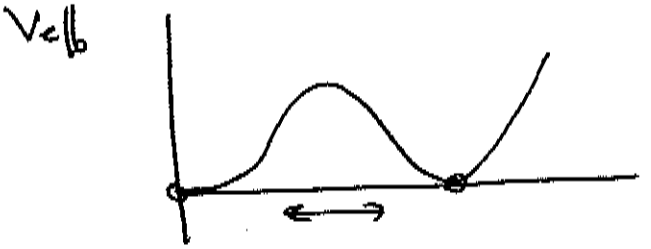


cl has a symmetry-breaking minimum.

g.) Add back $m^2 \neq 0$. The successive effective potentials for $m^2 \neq 0$ have the form



at the m^2 st.



we have a 1st order phase transition

For more details of this they, see

S. Coleman and E. Weinberg PRD 7 1888 (1973)

2.) a.) It is not so hard to extend the analysis to the GSW weak-interaction model. Using the gauge $\xi = 0$ and ignoring contributions from Higgs boson loops (which are proportional to ∂^2), we find

$$V_{eff} = -\frac{\mu^2}{2}v^2 + \frac{\lambda}{4}v^4$$

$$+ i\frac{1}{(Vol)}\cdot\left[-\frac{3}{2}\cdot 2\cdot \log\det(\partial^2+m_W^2) - \frac{3}{2}\log\det(\partial^2+m_Z^2)\right]$$

where $m_W^2 = \frac{g^2v^2}{4}$ $\qquad m_Z^2 = \frac{(g^2+g'^2)v^2}{4}$

$$= \text{(after minimal subtraction)}$$

$$-\mu^2\frac{v^2}{2} + \frac{\lambda}{4}v^4$$

$$+ \frac{3}{4}\frac{1}{(4\pi)^2}\cdot\left\{2\left(\frac{g^2v^2}{4}\right)^2\left(\log\frac{g^2v^2}{4M^2} - 3/2\right)\right.$$

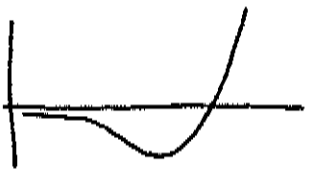
$$\left. + \left(\frac{(g^2+g'^2)v^2}{4}\right)^2\left(\log\frac{(g^2+g'^2)v^2}{4M^2} - 3/2\right)\right\}$$

b.) This effective potential is of the form

$$V_{eff} = -\mu^2 \frac{v^2}{2} + \frac{a}{4} \frac{g^4}{(4\pi)^2} v^4 \left(\ln \frac{g^2 v^2}{M^2} + C \right)$$

ed, to simplify the analysis, we could absorb C into the definition of M. For $\mu^2 = 0$, we have then

$$V_{eff} = \frac{a}{4} \frac{g^4}{(4\pi)^2} v^4 \ln \frac{g^2 v^2}{M^2}$$



ed, tuning on μ^2, we have a first-order phase transition,
we on p. 20.

c.) The minimum of the potential is given by

$$\frac{\partial V_{eff}}{\partial v} = 0 \qquad \text{let} \quad -\mu^2 = m^2$$

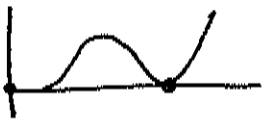
$$0 = m^2 v + a \frac{g^4}{(4\pi)^2} v^3 \ln \frac{g^2 v^2}{M^2}$$

$$+ \frac{a}{2} \frac{g^4}{(4\pi)^2} v^4 \cdot \frac{1}{v}$$

or

$$0 = m^2 + a\frac{g^4}{(4\pi)^2}v^2\left(\ln\frac{g^2v^2}{M^2} + \tfrac{1}{2}\right)$$

The condition for a 1st order phase transition is that the symmetry-breaking minimum should be at $V_{eff} = 0$



$$0 = m^2\frac{v^2}{2} + \frac{a}{4}\frac{g^4}{(4\pi)^2}v^4\ln\frac{g^2v^2}{M^2}$$

$$m^2 = -\frac{a}{2}\frac{g^4}{(4\pi)^2}v^2\ln\frac{g^2v^2}{M^2} > 0$$

Then the condition at the top of the page becomes.

$$0 = a\frac{g^4}{(4\pi)^2}v^2\left(\frac{1}{2}\ln\frac{g^2v^2}{M^2} + \frac{1}{2}\right)$$

or $$\ln\frac{g^2v^2}{M^2} = -1$$

at this point:

$$\frac{\partial^2 V_{eff}}{\partial v^2} = m_h^2 = m^2 + 3a\frac{g^4}{(4\pi)^2}v^2\left(\ln\frac{g^2v^2}{M^2} + \tfrac{1}{2}\right) + 2a\frac{g^4}{(4\pi)^2}v^2$$

$$m_h^2 = + \frac{a}{2} \frac{g^4 v^2}{(4\pi)^2} + 3a \frac{g^4}{(4\pi)^2} v^2 \left(-1 + \tfrac{1}{2}\right)$$

$$+ 2a \frac{g^4}{(4\pi)^2} v^2$$

$$= a \frac{g^4 v^2}{(4\pi)^2} \cdot \left(\frac{1}{2} - \frac{3}{2} + 2\right) = a \frac{g^4 v^2}{(4\pi)^2}$$

from p. 21

$$\frac{a}{4} \frac{g^4}{(4\pi)^2} = \frac{1}{(4\pi)^2} \cdot \frac{1}{4} \cdot \left\{ \frac{6}{16} g^4 + \frac{3}{16} (g^2 + g'^2)^2 \right\}$$

$$\left(\frac{g'}{g}\right)^2 = \tan^2 \theta_w$$

$$\frac{g^2 + g'^2}{g^2} = \frac{1}{\cos^2 \theta_w}$$

$$\frac{a g^4}{(4\pi)^2} = \frac{1}{(4\pi)^2} \frac{3}{16} \left(2 + \frac{1}{\cos^2 \theta_w}\right) g^4$$

so

$$m_h^2 = \frac{g^2}{(4\pi)^2} \cdot \frac{3}{4} \left(2 + \frac{1}{\cos^2 \theta_w}\right) \left(\frac{g^2 v^2}{4}\right)$$

$$= \frac{3}{16\pi} \left(2 + \frac{1}{\cos^2 \theta_w}\right) a_w m_w^2$$

with $\alpha_w = \frac{g^2}{4\pi} = \frac{1}{30}$, $m_W = 80$ GeV, $\cos\theta_w = .77$

$$m_h^2 = \frac{3}{16\pi} \alpha_w \left(2 + \frac{1}{\cos^2\theta_w}\right) m_W^2$$

evaluates to

$$m_h = 6.8 \text{ GeV}$$

This is the Linde-Weinberg bound. For $m^2 = 0$, the minimum is at $b_3 \frac{g^2 v^2}{M^2} = -\frac{1}{2}$ and

$$m_h^2 = 2a \frac{g^4}{(4\pi)^2} v^2 = \frac{3}{8\pi} \alpha_w \left(2 + \frac{1}{\cos^2\theta_w}\right) m_W^2$$

or $m_h = 9.2$ GeV

the "Weinberg-Gildener value".

In fact, $m_h > 114$ GeV.

d.) Now add the contributions of the top quark to the Higgs bosons.

$$V_{eff} = \quad (9.21)$$

$$+ i\left(\frac{1}{Vol}\right)\Big[-\frac{1}{2}\log\det\left(\partial^2 + (3\lambda v^2 - \mu^2)\right)$$

$$-\frac{3}{2}\log\det\left(\partial^2 + (\lambda v^2 - \mu^2)\right)$$

$$+ 3\log\det\left(i\not{\partial} - m_t\right)\Big]$$

where $m_t = \frac{\lambda_t v}{\sqrt{2}}$, 3 is a color factor,

and the + sign is for fermions.

$$\log\det\left(i\not{\partial} - m_t\right) = 2\log\det\left(\partial^2 + m_t^2\right)$$

evaluating these determinants in the same way as before and setting $\lambda v^2 \ll \mu^2$, we find a $\lambda v^2/M^2$ term:

$$V_{eff} = \cdots + \frac{1}{4}\frac{1}{(4\pi)^2}\Big[\, 6m_W^4 + 3m_Z^4 + m_h^4$$

$$- 12\, m_t^4 \,\Big]\left(\lambda \frac{g^2 v^2}{M^2} + \cdots\right)$$

This decreases the coefficient of the $\ln v^2$ and brings us back to the "normal" situation in which there is no radiative symmetry breaking. However, when m_t becomes large, the asymptotic behavior of the potential is

$$V_{eff} = - b \frac{v^4}{(4\pi)^2} \ln v^2/M^2$$

and there is an instability to escape to $v^2 \to \infty$.

To analyze this situation, look at the leading pieces of each term of V_{eff} for $v \to \infty$

$$V_{eff} = \cdots + \frac{1}{4}\frac{1}{(4\pi)^2}\left\{ 6\left(\frac{g^2 v^2}{4}\right)^2 + 3\left(\frac{(g^2+g'^2)v^2}{4}\right)^2 \right.$$

σ and π contributions $\to$

$$+ (3\partial v^2)^2 + 3(\partial v^2)^2$$

$$\left. - 12\left(\frac{\partial_t^2 v^2}{2}\right)^2 \right\} \ln \frac{v^2}{M^2}$$

the term in brackets is $\qquad (m_h^2 = 2\partial v^2)$

$$\left\{ 6m_W^4 + 3m_Z^4 - 12 m_t^4 + 3 m_h^4 \right\}$$

this term is negative if

$$m_h^4 < \frac{1}{3}\left(12 m_t^4 - 6 m_W^4 - 3 m_Z^4\right)$$

$$= (240 \text{ GeV})^4$$

Actually omitted finite corrections are important, and one actually finds the bound

$$m_h > 140 \text{ GeV}$$

in the Standard Model. For more details, see

M. Sher, Phys. Repts. 179, 273 (1989).

Physics 332 – Problem Set # 6

(due Wednesday, May 24)

1. Peskin and Schroeder, Problem 17.1.

2. In class, I sketched the derivation of the β function of a non-Abelian gauge theory from the renormalization counterterms δ_1, δ_2, and δ_3. Work through this calculation in full detail. Use the Feynman-'t Hooft gauge.

3. Peskin and Schroeder, Problem 16.3.

   Please note: This is a *long* calculation. The solution set for this problem set is 50 pages long, of which 35 pages are devoted to this problem. I do assure you that you will learn a considerable amount about how to calculation in gauge theories by doing this problem to the end. (Of course, it might be true that these are things that you never wanted to know ...)

Physics 332 - Problem Set # 6

Solutions

1.) a.) $$\frac{dg}{d\log q} = -\frac{b_0 g^3}{(4\pi)^2} - \frac{b_1 g^5}{(4\pi)^4} + \cdots$$

the solution to

$$\frac{dg^{(0)}}{d\log q} = -\frac{b_0 g^3}{(4\pi)^2} \quad \text{is}$$

$$\frac{d\alpha_s^{(0)}}{d\log q} = -\frac{b_0}{2\pi}\alpha_s^2$$

$$\alpha_s^{(0)} = \frac{2\pi}{b_0}\frac{1}{\log q/\Lambda}$$

now write

$$\alpha_s = \alpha_s^{(0)} + \alpha_s^{(1)}$$

$$\frac{d\alpha_s}{d\log q} = -\frac{b_0}{2\pi}\alpha_s^2 - \frac{b_1}{8\pi^2}\alpha_s^3$$

$$\frac{d\alpha_s^{(1)}}{d\log q} = -\frac{b_0}{2\pi}\cdot 2\alpha_s^{(0)}\alpha_s^{(1)} - \frac{b_1}{8\pi^2}(\alpha_s^{(0)})^3$$

$$\frac{d\alpha_s^{(1)}}{d\ln q^2/\Lambda^2} = -2\,\frac{1}{\ln q^2/\Lambda^2}\,\alpha_s^{(1)} - \pi\frac{b_1}{b_0^3}\,\frac{1}{(\ln q^2/\Lambda^2)^3}$$

the solution of the homogeneous equation

$$\frac{d\alpha_s^{(1,h)}}{d\ln q^2/\Lambda^2} = -2\,\frac{1}{\ln q^2/\Lambda^2}\,\alpha_s^{(1,h)}$$

is

$$\alpha_s^{(1,h)} = \text{const}\cdot\frac{1}{(\ln q^2/\Lambda^2)^2}$$

so we cannot solve the full equation with a function of this form.

try

$$\alpha_s^{(1)} = A\,\frac{\log(\log q^2/\Lambda^2) + B}{(\ln q^2/\Lambda^2)^2}$$

$$\frac{d\alpha_s^{(1)}}{d\ln q^2/\Lambda^2} = -\frac{2}{\log q^2/\Lambda^2}\,\alpha_s^{(1)} + A\,\frac{1}{(\ln q^2/\Lambda^2)^3}$$

this solves the equation for $A = -\frac{\pi b_1}{b_0^3}$

then

$$\alpha_s(q) = \frac{2\pi}{b_0}\,\frac{1}{\log q^2/\Lambda^2} - \frac{\pi b_1}{b_0^3}\,\frac{\ln\ln q^2/\Lambda^2 + (\text{const})}{(\ln q^2/\Lambda^2)^2}$$

$$= \frac{4\pi}{b_0}\left\{\frac{1}{\log q^2/\lambda^2} - \frac{b_1}{b_0^2}\,\frac{\log\ln q^2/\lambda^2 + (\text{const})}{\ln q^2/\lambda^2}\right\} + \cdots$$

b.) Now

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \cdot 3\Sigma Q_f^2 \cdot \left(1 + \frac{\alpha_s(q)}{\pi} + a_2\left(\frac{\alpha_s(q)}{\pi}\right)^2 + \ldots\right)$$

the factor in parenthesis is

$$1 + \frac{4\pi}{\pi b_0}\left(\frac{1}{\ln q^2/\Lambda^2} - \frac{b_1}{b_0^2}\frac{\ln\ln q^2/\Lambda^2 + C}{(\ln q^2/\Lambda^2)^2} + \ldots\right)$$

$$+ a_2\left(\frac{4}{b_0}\right)^2 \frac{1}{\ln^2 q^2/\Lambda^2} + \ldots$$

$$= 1 + \frac{4}{b_0}\frac{1}{\ln q^2/\Lambda^2} - \frac{4b_1}{b_0^3}\frac{\ln\ln q^2/\Lambda^2}{(\ln q^2/\Lambda^2)^2}$$

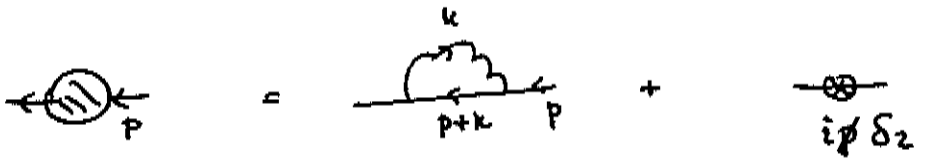
$$+ O\left(\frac{1}{(\ln q^2/\Lambda^2)^2}\right)$$

Since $\sigma(e^+e^- \to \text{hadrons})$ is an observable quantity, the factors $4/b_0$ and $-4b_1/b_0^3$ should be independent of the scheme of calculation. Thus, both $\underline{b_0}$ and $\underline{b_1}$ should be independent of scheme.

2.) $\beta(g) = M \frac{\partial}{\partial M} (-\delta_1 + \delta_2 + \frac{1}{2}\delta_3)$

so we need to compute these counterterms. Use Feynman-'tHooft gauge.

$\underline{\delta_2}$:



$$= (ig)^2 t^a t^a \int \frac{dk}{(2\pi)^D} \gamma_\mu \frac{i(\not{p}+\not{k})}{(p+k)^2} \gamma^\mu \left(\frac{-i}{k^2}\right) \qquad \begin{aligned} \ell k &= k + xp \\ \ell + p &= \ell k + (1-x)p \end{aligned}$$

$$= -g^2 C_2(r) \int_0^1 dx \int \frac{d\ell k}{(2\pi)^D} \frac{1}{[\ell k^2 - \Delta]^2} \gamma_\mu (\ell \!\!\!/ k + (1-x)\not{p}) \gamma^\mu$$

$$= -g^2 C_2(r) \int_0^1 dx \int \frac{d\ell k}{(2\pi)^D} \frac{1}{[\ell k^2 - \Delta]^2} \cdot (2-d)(1-x)\not{p}$$

$$= (d-2) g^2 C_2(r) \frac{i}{(4\pi)^2} \Gamma(2 - d/2) \int_0^1 dx\, (1-x) \not{p}$$

$\Gamma(\epsilon)$, $\epsilon = 2 - d/2$

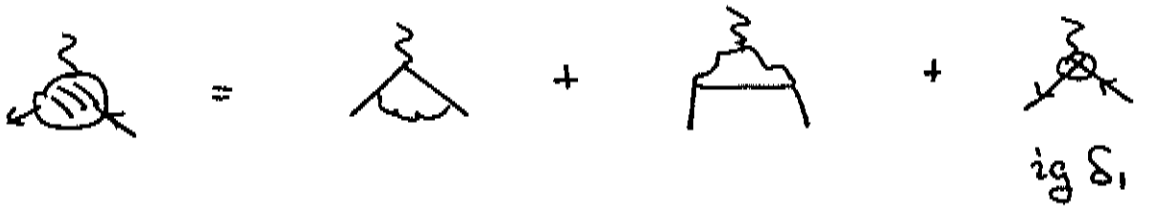
$d \to 4$ $$= \frac{i}{(4\pi)^2} g^2 C_2(r) \cdot \left(\frac{2}{\epsilon}\right)$$

so

$$\delta_2 = -\frac{1}{(4\pi)^2} g^2 C_2(r) \left(\frac{2}{\epsilon}\right)$$

with $\frac{2}{\epsilon} \sim \ln \Lambda^2/m^2$; $\left(-M \frac{\partial}{\partial M}\right)\left(\frac{2}{\epsilon}\right) = 2$

δ_1:



we can evaluate the diagrams at zero external momentum



$$= (ig)^3 \int \frac{dk}{2\pi} \gamma^\lambda t^b \frac{ik\!\!\!/}{k^2} \gamma^\mu t^a \frac{ik\!\!\!/}{k^2} \gamma_\lambda t^b \frac{-i}{k^2}$$

$$= g^3 (t^b t^a t^b) \frac{1}{d} \int \frac{dk}{(2\pi)^D} \frac{1}{(k^2)^3} \gamma^\lambda \gamma^\sigma \gamma^\mu \gamma_\sigma \gamma_\lambda$$

I have used $k^\alpha k^\beta = \frac{1}{d} k^2 g^{\alpha\beta}$ under the symmetric integral.

$$t^b t^a t^b = t^a t^b t^b - [t^a, t^b] t^b$$

$$= t^a C_2(r) - i f^{abc} t^c t^b$$

$$= t^a C_2(r) - i (-\tfrac{1}{2}) f^{abc} [t^b, t^c]$$

$$= t^a C_2(r) + \tfrac{i}{2} i f^{abc} f^{bcd} t^d$$

$$= t^a \left[C_2(r) - \tfrac{1}{2} C_2(G) \right]$$

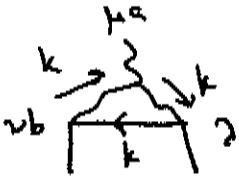
$$\gamma^\lambda \gamma^\sigma \gamma^\mu \gamma_\sigma \gamma_\lambda = (2-d)^2 \gamma^\mu = 4 \gamma^\mu \text{ in } \underline{d=4}$$

In all



$$= g^3 \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] \cdot \frac{4}{4} \frac{i}{(4\pi)^2} \Gamma(2-d_k)\, t^a \gamma^\mu$$

$$= i \frac{g^3}{(4\pi)^2} \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] \cdot \frac{2}{\epsilon} \cdot t^a \gamma^\mu$$



$$= (ig)^2 \cdot g f^{abc} \cdot \int \frac{dk}{2\pi} \gamma^\nu t^b \frac{i\not{k}}{k^2} \gamma^\lambda t^c \left(\frac{-i}{k^2}\right)^2$$

$$\cdot \left[g^{\mu\nu}(-k)^\lambda + g^{\nu\lambda}(2k)^\mu + g^{\lambda\mu}(-k)^\nu \right]$$

$$= i g^3 \left(f^{abc} t^b t^c\right) \int \frac{dk}{2\pi} \frac{1}{(k^2)^3}$$

$$\cdot \left\{ -\gamma^\mu \not{k}\not{k} + 2\gamma^\nu \not{k} \gamma_\nu k^\mu - \not{k}\not{k}\gamma^\mu \right\}$$

$$= i g^3 \left(\frac{i}{2} C_2(G) t^a\right) \int \frac{dk}{2\pi} \frac{1}{(k^2)^3}$$

$$\left\{ -\gamma^\mu k^2 + \frac{2}{d}(2-d) k^2 \gamma^\mu - k^2 \gamma^\mu \right\}$$

$$= -\frac{g^3}{2} C_2(G) t^a \gamma^\mu \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k) \cdot \left(\frac{4}{d} - 4\right)$$

$$= i \frac{3}{2} g^3 C_2(r) \gamma^\mu t^a \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

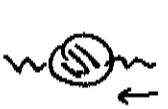
then

$$i\frac{g^3}{(4\pi)^2}\left(C_2(r) - \tfrac{1}{2}C_2(G)\right) t^a \gamma^\mu \cdot \frac{2}{\epsilon}$$

$$+ \; i\frac{g^3}{(4\pi)^2}\, \tfrac{3}{2} C_2(G)\, t^a \gamma^\mu \cdot \frac{2}{\epsilon} + i g \delta_1 = (\text{no divergences})$$

$$\delta_1 = -\frac{g^2}{(4\pi)^2}\left(C_2(r) + C_2(G)\right) \cdot \frac{2}{\epsilon}$$

$\underline{\delta_3}$:



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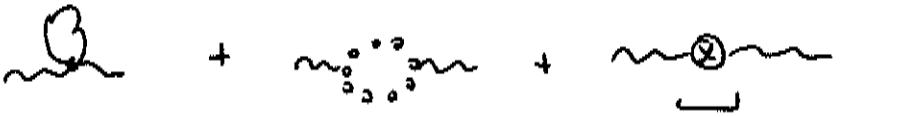
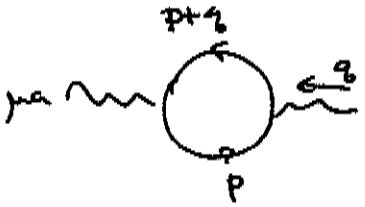
$+$



$$-i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\delta_3$$



$=$ (Fermions!) $(ig)^2(-1)\,\mathrm{tr}\, t^a t^b \int \frac{dp}{(2\pi)}$

$$\cdot \mathrm{tr}\left[\gamma^\mu \frac{i(\not{p}+\not{q})}{(p+q)^2}\gamma^\nu \frac{i\not{p}}{p^2}\right]$$

In all of these graphs, we will combine denominators with

$$\mathbb{P} = p + x q$$

$$p = \mathbb{P} - x q \qquad p+q = \mathbb{P} + (1-x) q$$

$$2p+q = 2\mathbb{P} + (2-x) q$$

$$= -g^2 n_f C(r)\,\delta^{ab} \int \frac{d\mathbb{P}}{2\pi} \int dx \; \frac{1}{\left[\mathbb{P}^2 + x(1-x)q^2\right]^2}$$

$$\cdot \operatorname{tr}\left[\gamma^\lambda \left(\not{\mathbb{P}} + (1-x)\not{q}\right) \gamma^\nu \left(\not{\mathbb{P}} - x\not{q}\right)\right]$$

$$= -g^2 C(r) n_f \delta^{ab} \int_0^1 dx \int \frac{d\mathbb{P}}{2\pi} \frac{1}{\left[\mathbb{P}^2 + x(1-x)q^2\right]^2}$$

$$\cdot 4\left[2\mathbb{P}^\mu \mathbb{P}^\nu - g^{\mu\nu}\mathbb{P}^2 - 2x(1-x)q^\mu q^\nu + x(1-x)g^{\mu\nu}q^2\right]$$

$$= -4g^2 n_f C(r)\,\delta^{ab} \int_0^1 dx \; \frac{i}{(4\pi)^{d_4}} \frac{1}{\left[-x(1-x)q^2\right]^{2-d_4}}$$

$$\Big\{ -\Gamma(1-d_4)\cdot\left[-x(1-x)q^2\right]\cdot\left(2\cdot\frac{1}{2}g^{\mu\nu} - \frac{d}{2}g^{\mu\nu}\right)$$

$$+ \Gamma(2-d_4)\left(-2x(1-x)q^\mu q^\nu + x(1-x)g^{\mu\nu}q^2\right)\Big\}$$

now $\quad \Gamma(1-d_4)\cdot(1-d_4) = \Gamma(2-d_4)$

$$= -4g^2 n_f C(r)\,\delta^{ab} \int_0^1 dx \; \frac{i}{(4\pi)^{d_4}} \frac{1}{\left[-x(1-x)q^2\right]^{2-d_4}} \cdot \Gamma(2-d_4)$$

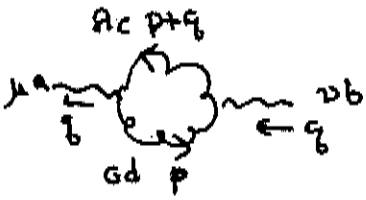
$$\cdot \Big\{ x(1-x)q^2 g^{\mu\nu}\cdot 2 - 2x(1-x)q^\mu q^\nu \Big\}$$

$$= -8i\, g^2 n_f C(r)\,\delta^{ab} \frac{1}{(4\pi)^{d_4}} \Gamma(2-d_4)$$

$$\cdot \int_0^1 dx \; x(1-x) \frac{1}{\left[-x(1-x)q^2\right]^{2-d_4}}$$

$$= -8i\, g^2 n_f C(r)\, \delta^{ab}\, \frac{1}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot\frac{1}{6}$$

$$= i g^2 \frac{C(r) n_f \delta^{ab}}{(4\pi)^2}\cdot\left(-\frac{4}{3}\right)\cdot\frac{2}{\epsilon}$$



$$g f^{acd}\, g f^{bcd}\cdot\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\,\frac{-i}{(p+q)^2}\,\frac{-i}{p^2}$$

(symmetry factor)

$$\cdot\Big[\, g^{\mu\rho}(-q-(p+q))^\sigma + g^{\rho\sigma}[(p+q)+p]^\mu + g^{\sigma\mu}[-p+q]^\rho \,\Big]$$

$$\Big[\, g^{\nu\rho}[q+(p+q)]^\sigma + g^{\rho\sigma}[-(p+q)-p]^\nu + g^{\sigma\nu}[p-q]^\rho \,\Big]$$

$$= -\frac{g^2}{2} C_2(G)\,\delta^{ab}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{\left[\Delta+x(1-x)q^2\right]^2}$$

$$\Big\{\, g^{\mu\nu}\left[-(2q+p)^2-(p-q)^2\right] + (-d)\,(2p+q)^\mu(2p+q)^\nu$$

$$+ (p+2q)^\mu(2p+q)^\nu + (p+2q)^\nu(2p+q)^\mu - (p-q)^\nu(2p+q)^\mu -$$

$$- (p-q)^\mu(2p+q)^\nu + (2p+q)^\mu(p-q)^\nu + (p-q)^\mu(2p+q)^\nu \,\Big\}$$

$$= -\frac{g^2}{2} C_2(G)\,\delta^{ab}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{\left[\Delta+x(1-x)q^2\right]^2}$$

$$\Big\{\, g^{\mu\nu}\left[-2p^2-3p\cdot q-5q^2\right] - d\,(2p+q)^\mu(2p+q)^\nu$$

$$+ p^\mu p^\nu\{2+2-1-1+2+2\} + (p^\mu q^\nu + q^\mu p^\nu)\left[5-1-1\right]$$

$$+ q^\mu q^\nu\{2+2+2+2-1-1\}\,\Big\}$$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d} \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\{ - g^{\mu\nu} [(p+xq)^2 + (p-q)^2] - d\, (2p+q)^\mu (2p+q)^\nu$$

$$+ 6 p^\mu p^\nu + 3 (p^\mu q^\nu + p^\nu q^\mu) + 6\, q^\mu q^\nu \}$$

substitute $p = \mathbb{P} - xq$ and drop terms linear in $\mathbb{P}$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d} \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\{ - g^{\mu\nu} [\mathbb{P}^2 + (2-x)^2 q^2 + \mathbb{P}^2 + (1+x)^2 q^2]$$

$$- d\, [4 \mathbb{P}^\mu \mathbb{P}^\nu + (1-2x)^2 q^\mu q^\nu]$$

$$+ 6 (\mathbb{P}^\mu \mathbb{P}^\nu + x^2 q^\mu q^\nu) + 3(-x) \cdot 2\, q^\mu q^\nu$$

$$+ 6 q^\mu q^\nu \}$$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \left[\frac{i}{\epsilon - x(1-x) q^2}\right]^{2-d_4} \frac{1}{(4\pi)^{d_4}}$$

$$\{ - \Gamma(1-d_4) [-x(1-x) q^2] \left(- g^{\mu\nu} \frac{d}{2} \cdot 2 - 4d \cdot \tfrac{1}{2} g^{\mu\nu} + 6 \cdot \tfrac{1}{2} \cdot g^{\mu\nu} \right)$$

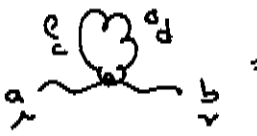
$$+ \Gamma(2-d_4) \left(- g^{\mu\nu} [(2-x)^2 q^2 + (1+x)^2 q^2] \right.$$

$$\left. + q^\mu q^\nu \left(- d (1-2x)^2 + 6x^2 - 6x + 6 \right) \right)$$

$$= -i \frac{g^2}{2} \frac{1}{(4\pi)^{d_4}} C_2(G)\, \delta^{ab} \int_0^1 dx \left(\frac{1}{-x(1-x) q^2} \right)^{2-d_4}$$

$$\{ \Gamma(1-d_4)\, x(1-x) q^2\, g^{\mu\nu} [-3d + 3]$$

$$+ \Gamma(2-d_4) \left[- g^{\mu\nu} q^2 (5 - 2x + 2x^2) + q^\mu q^\nu (-d (1-2x)^2 + 6(x^2 - x + 1)) \right] \}$$



$$= -ig^2 \cdot \frac{1}{2} \cdot \int \frac{d^dp}{(2\pi)^d} \frac{-i}{p^2} g^{\rho\sigma} \delta^{cd}$$

$$\Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big]$$

$$= -\frac{g^2}{2} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} \cdot \frac{(p+q)^2}{(p^2+q^2}$$

$$[\, 0 + g^{\mu\nu} C_2(A)\, \delta^{ab} (d-1) \cdot 2 \,]$$

$\swarrow \; k + (1-x)q$, $\longleftarrow (k - xq)$

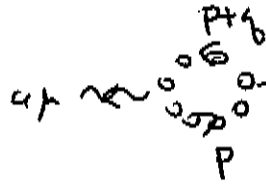
$$= -g^2 C_2(A)\, \delta^{ab} (d-1) \int \frac{d^dk}{(2\pi)^d} g^{\mu\nu}_{(d-1)} \frac{[(k+q)^2 + k^2]}{k^2 (k+q)^2} \cdot \frac{1}{2}$$

$$= -g^2 C_2(A)\, \delta^{ab} (d-1)\, g^{\mu\nu} \frac{i^{(d-1)}}{(4\pi)^{d_2}} \cdot \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d_2}}$$

$$\Big\{ -\Gamma(1-d_2)(-x(1-x)q^2) \cdot 2 \, \frac{d}{2} \cdot \frac{1}{2}$$
$$+ \Gamma(2-d_2) \cdot \frac{1}{2} [(1-x)^2 q^2 + x^2 q^2] \Big\}$$

$$= -\frac{ig^2}{2} C_2(A)\, \delta^{ab} \frac{1}{(4\pi)^{d_2}} \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d_2}}$$

$$\Big\{ \Gamma(1-d_2) \cdot x(1-x)q^2 \cdot g^{\mu\nu} \cdot d(d-1)$$
$$+ \Gamma(2-d_2)\; g^{\mu\nu} q^2 (4)(1 - 2x + 2x^2) \Big\}$$



$$= (-1)(-g)^2 f^{cad} f^{dbc} \int \frac{d^d p}{(2\pi)^d} \frac{i}{(p+q)^2} \frac{1}{p^2} \cdot (p+q)^\nu p^\mu$$

$$= -g^2 C_2(G) \delta^{ab} \int_0^1 dx \int \frac{d^d P}{(2\pi)^d} \frac{1}{[P^2 + x(1-x)q^2]^2} \left(P^\mu P^\nu - x(1-x) g^\mu q^\nu \right)$$

$$= -g^2 C_2(G) \delta^{ab} \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\left(-\Gamma(1-d/2)[-x(1-x)q^2] \tfrac{1}{2} g^{\mu\nu} + \Gamma(2-d/2)(-x(1-x) q^\mu q^\nu) \right)$$

$$= -i \frac{g^2}{2} \frac{1}{(4\pi)^{d/2}} C_2(G) \delta^{ab} \int_0^1 dx \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\left(\Gamma(1-d/2)\, x(1-x) q^2 g^{\mu\nu} + \Gamma(2-d/2)(-2x(1-x) q^\mu q^\nu) \right)$$

= all



$$= -i \frac{g^2}{2} \frac{1}{(4\pi)^{d/2}} C_2(G) \delta^{ab} \int_0^1 dx \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\Big\{ \Gamma(1-d/2)\, x(1-x) q^2 g^{\mu\nu} (-3d + 3 + d^2 - d + 1)$$

$$+ \Gamma(2-d/2) q^2 g^{\mu\nu} (-5 + 2x - 2x^2 + (d-1)(1 - 2x + 2x^2))$$

$$+ \Gamma(2-d/2) q^\mu q^\nu (-d(1-2x)^2 + (1-2x)^2 + (5 - 2x + 2x^2) - 2x(1-x)) \Big\}$$

how $\Gamma(1-d/2)(d^2 - 4d + 4) = \Gamma(1-d/2) \cdot 4(1-d/2)^2 = \Gamma(2-d/2)(4-2d)$

so

$$\{\} = \Gamma(2-d_4)\Big([2-2(d-1)]\,x(1-x)\,q^2\,g^{\mu\nu} + [(d-1)(1-2x+2x^2) - 5 + 2x - 2x^2]\,q^2 g^{\mu\nu}\Big)$$

$$+ \Gamma(2-d_4)(-g^\mu q^\nu)\Big[(d-1)(1-2x)^2 \bullet + (-5+2x-2x^2) + 2x(1-x)\Big]$$

$$= \Gamma(2-d_4)(q^2 g^{\mu\nu} - q^\mu q^\nu)\{(d-1)(1-2x)^2 - 5 + 4x - 4x^2\}$$

the result is transverse; doing the $\int dx$ we find

$$\int_0^1 dx\ \{\} = \frac{2}{\epsilon}\cdot(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\{3\cdot(1-\tfrac{4}{2}+\tfrac{4}{3}) - 5 + \tfrac{4}{2} - \tfrac{4}{3}\}$$

$$= \frac{2}{\epsilon}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\{1 - 3 - \tfrac{4}{3}\}$$

$$= \frac{2}{\epsilon}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\{-\tfrac{10}{3}\}$$

so



$$= i\frac{g^2}{(4\pi)^2} C_2(G)\,\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\frac{5}{3}\cdot\frac{2}{\epsilon}$$

adding this to the result at the top of p. 9 and cancelling w. δ_3, we find:

$$\delta_3 = \frac{g^2}{(4\pi)^2}\left[\frac{5}{3}C_2(G) - \frac{4}{3}C(r)n_f\right]\cdot\frac{2}{\epsilon}$$

Then, finally

$$\beta(g) = M\frac{\partial}{\partial M}\left(-g\delta_1 + g\delta_2 + \frac{1}{2}g\delta_3\right)$$

using $M\frac{\partial}{\partial M}\frac{2}{\epsilon} = -2$

$$\beta = -2\frac{g^3}{(4\pi)^2}\cancel{[C_2(r) + C_2(G)]} + 2\frac{g^3}{(4\pi)^2}\cancel{C_2(r)}$$

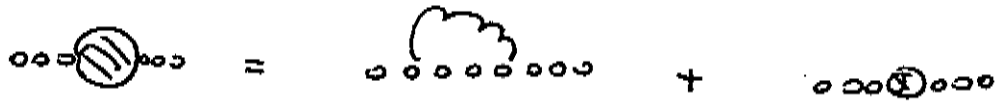
$$-\frac{1}{2}\cdot 2\frac{g^3}{(4\pi)^2}\left[\frac{5}{3}C_2(G) - \frac{4}{3}C(r)n_f\right]$$

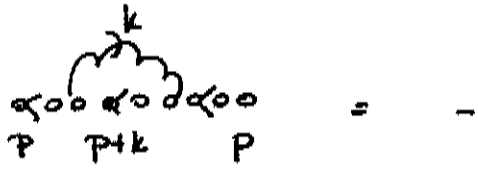
$$\beta = -\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]\frac{g^3}{(4\pi)^2}$$

For reference in the next problem:

$$\delta_1 - \delta_2 = -\frac{g^2}{(4\pi)^2}C_2(G)\cdot\frac{2}{\epsilon}$$

3.) a.) The ghost field strength counterterm is given by





$$= -g f^{acd} f^{dcb} \int \frac{d^d k}{(2\pi)^d} p^\mu \frac{i}{(p+k)^2} (p+k)_\mu \frac{-1}{k^2}$$

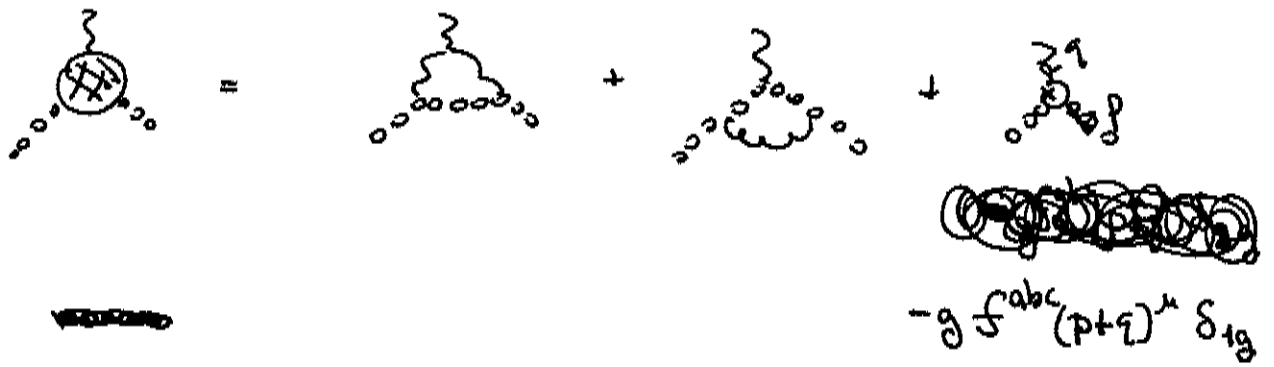
$$= (-g)^2 (-C_2(G)) \delta^{ab} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + x(1-x)p^2]^2} \cdot \qquad k = k - xp, \quad p+k = k + (1-x)p$$

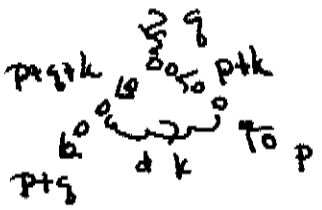
$$\cdot p^\mu (k + (1-x)p)_\mu$$

$$= \frac{-i g^2}{(4\pi)^2} C_2(G) \delta^{ab} \cdot \frac{2}{\epsilon} \cdot \underbrace{\int dx \, (1-x)}_{\frac{1}{2}} p^2$$

so $$\delta_{2g} = \frac{1}{2} C_2(G) \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon}$$

The ghost vertex counterterm is given by





$$= (-g f^{ade})(-g f^{ebf})(-g f^{fdc})$$

$$\cdot \int \frac{d^d k}{(2\pi)^d} \frac{i}{(p+k)^2} \frac{i}{(p+q+k)^2} \frac{-i}{k^2} (p+q)^\beta (p+q+k)^\mu (p+k)_\beta$$

$$= -i g^3 (f^{ade} f^{ebf} f^{fdc}) \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2 (p+q+k)^2 k^2}$$

$$\cdot (p+q)^\beta (p+q+k)^\mu (p+k)_\beta$$

To simplify the product of f's we use:

$$[f^{ade} f^{ebf} + f^{dbe} f^{eaf} + f^{bae} f^{edf}] f^{fdc} = 0 \quad \text{by the Jacobi id.}$$

$$f^{ade} f^{ebf} f^{fdc} + f^{afe}(-f^{ebd})(-f^{dfc}) + (-f^{abe}) f^{edf}(-f^{dfc}) = 0$$

$$2 f^{ade} f^{ebf} f^{fdc} + f^{abc} C_2(G) = 0$$

so

$$f^{ade} f^{ebf} f^{fdc} = (-\tfrac{1}{2} C_2(G)) f^{abc}$$

To simplify the numerator, combine denominators with Feynman parameters



$$\text{Denom} = k^2 + x k\cdot p + x p^2 + y k\cdot(p+q) + y (p+q)^2$$

$$= k^2 - [xp + y(p+q)]^2 + x p^2 + y (p+q)^2$$

$$= k^2 + \underbrace{x(1-x)}_{y+z} p^2 + y \underbrace{(1-y)}_{x+z} (p+q)^2 - 2xy\, p\cdot(p+q)$$

Denom $= k^2 + xz p^2 + yz (p+q)^2 + xy q^2$

$= k^2 - \Delta \qquad -\Delta = xz p^2 + yz (p+q)^2 + xy q^2$

$k = k - xp - y(p+q) = k - (1-z) p - y q$

$k+p = k + zp - yq$

$k+p+q = k + zp + (1-y) q$

then

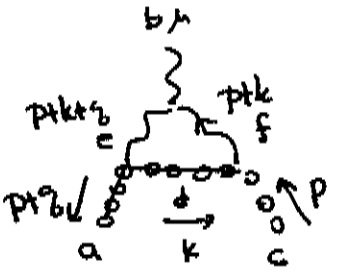


$= -ig^3 \left(-\tfrac{1}{2} C_2(G) f^{abc}\right) \int dx dy dz\, \delta(\,) \cdot 2 \cdot \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-\Delta]^3}$

$\cdot \{ (p+q)^\lambda k^\mu k^\lambda + (\text{finite}) \}$

$= + i\frac{g^3}{2} C_2(G) f^{abc} (p+q)^\lambda \cdot \frac{i}{(4\pi)^{d/2}} \int dx dy dz\, \delta(\,) \cdot \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \cdot \frac{1}{2} g^{\mu\lambda}$

$= -\frac{g^3}{8} (p+q)^\mu f^{abc} C_2(G) \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon}$



$= (-g f^{aed})(-g f^{dfc})(g f^{ebf})$

$\cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{-i}{(p+k)^2} \frac{-i}{(p+k+q)^2} (p+q)^\lambda (-k)^\eta$

$[\, g^{\lambda\mu}(-(p+k+q) - q)^\eta + g^{\mu\eta}(q - (p+k))^\lambda + g^{\lambda\eta}(p+k + p+k+q)^\mu \,]$

$$= \; i g^3 \left(f^{ade} f^{ebf} f^{fdc}\right) \int dx\,dy\,dz\,\delta(\cdot) \cdot 2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - \Delta]^3}$$

$$\cdot \; (p+q)^\lambda \, k^\eta \left[g^{\lambda\mu}(-k)^\eta + g^{\mu\eta}(-k)^\lambda + g^{\lambda\eta}\, 2k^\mu \right]$$

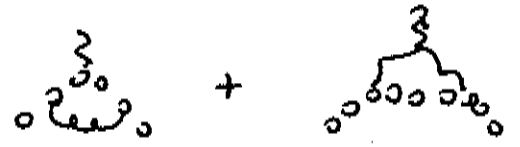
$$+ \text{finite}$$

$$= \; i g^3 \left(f^{ade} f^{ebf} f^{fdc}\right) \cdot \frac{1}{2} \cdot \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \tfrac{d}{2}\right)$$

$$\left[(p+q)^\mu \left(-\tfrac{d}{2}\right) + (p+q)^\mu \left(-\tfrac{1}{2}\right) + (p+q)^\mu \cdot 2 \cdot \tfrac{1}{2} \right]$$

$$= \; + \frac{(-g^3)}{(4\pi)^2} \cdot \frac{1}{2} \left(-\frac{1}{2} C_1(G) f^{abc} \right) (p+q)^\mu \, \frac{2}{\epsilon} \cdot \left[-\frac{(d-1)}{2} \right]$$

$=$



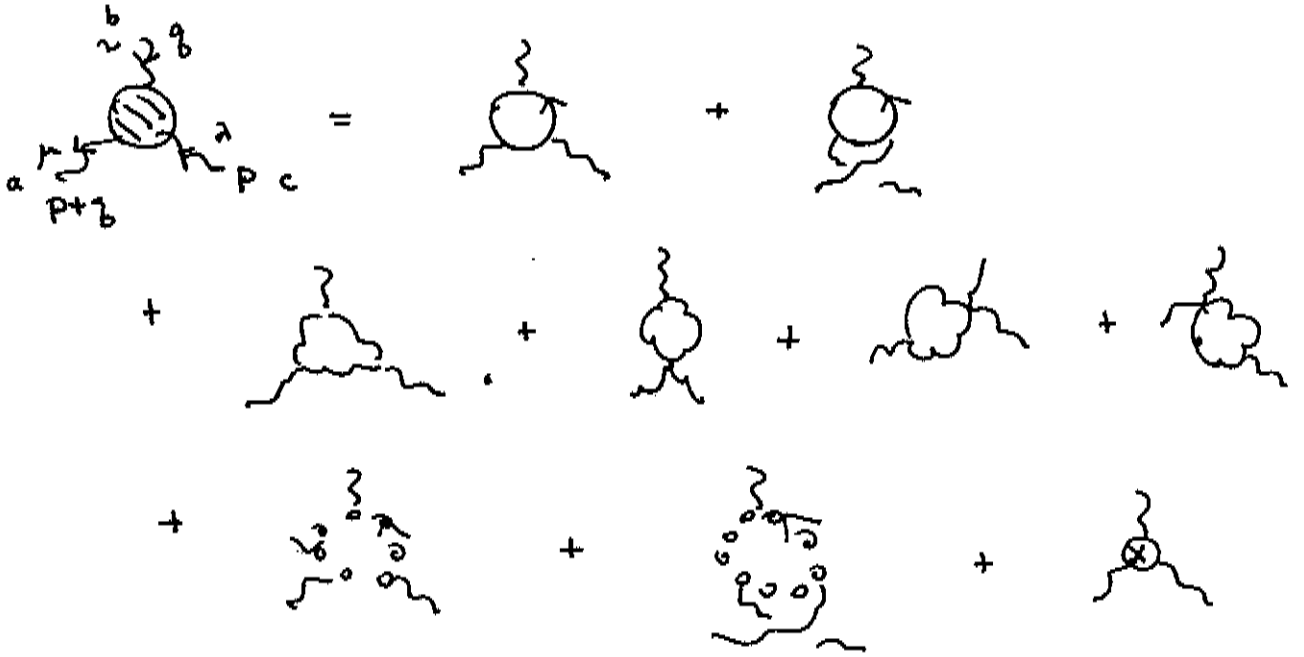
$$= \left[\frac{g^3 C_2(G)}{(4\pi)^2} (p+q)^\mu f^{abc} \cdot \frac{2}{\epsilon} \right] \underbrace{\left(-\frac{1}{8} - \frac{3}{8} \right)}_{-\frac{1}{2}}$$

so $$\delta_{1g} = \; - \frac{1}{2} \frac{g^2}{(4\pi)^2} C_2(G) \cdot \frac{2}{\epsilon}$$

$$\delta_{1g} - \delta_{2g} = \; - \frac{g^2}{(4\pi)^2} C_2(G) \cdot \frac{2}{\epsilon}$$

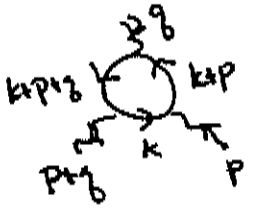
as required !

b.) The 3-gluon vertex correction is given by



with

$$= g f^{abc}\left[g^{\mu\nu}(-p-2q)^{\lambda} + g^{\nu\lambda}(q-p)^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right] \times \delta_1^{3g}$$



$$= (ig)^3 \mathrm{tr}[t^a t^b t^c] \int \frac{d^d k}{(2\pi)^d} \frac{i}{(k+p)^2} \frac{i}{(k+p+q)^2} \frac{i}{k^2}$$

$$\cdot (-1) \cdot \mathrm{tr}\left[\gamma^{\mu}(k\!\!\!/+p\!\!\!/+q\!\!\!/)\,\gamma^{\nu}(k\!\!\!/+p\!\!\!/)\,\gamma^{\lambda}\,k\!\!\!/\right]$$

fermions

$$= + g^3\,\mathrm{tr}[t^a t^b t^c] \int dx\,dy\,dz\,\delta(x+y+z-1)\cdot 2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2-\Delta]^3}$$

$$\cdot \mathrm{tr}\left[\gamma^{\mu}(k\!\!\!/ + z p\!\!\!/ + y q\!\!\!/)\,\gamma^{\nu}(k\!\!\!/ + z p\!\!\!/ - q\!\!\!/ + y q\!\!\!/)\,\gamma^{\lambda}(k\!\!\!/ - (1-z)p\!\!\!/ - q\!\!\!/ + y q\!\!\!/)\right]$$

to get a divergence, we must take two factors of k in the numerator. Then we find (using expressions on p. 197)

$$= g^3 \, \mathrm{tr}(t^a t^b t^c) \int dx\, dy\, dz\, \delta(\,) \cdot \frac{i}{(4\pi)^{d_k}} \frac{1}{\Delta^{2-d_k}} \Gamma(2-d_k) \cdot \frac{1}{2}$$

$$\Big\{ \mathrm{tr}\big[\gamma^\mu [\not{z}p + (1-y)\not{q}] \gamma^\nu \gamma^a \gamma^\lambda \gamma_a\big] \quad \leftarrow \text{evaluate in } d=4, \; \gamma^a\gamma^\lambda\gamma_a = -2\gamma^\lambda$$

$$+ \mathrm{tr}\big[\gamma^\mu \gamma^a \gamma^\nu (\not{z}p - y\not{q}) \gamma^\lambda \gamma_a\big] +$$

$$+ \mathrm{tr}\big[\gamma^\mu \gamma^a \gamma^\nu \gamma_a \gamma^\lambda \in (1-z)\, p - y\not{q})\big] \Big\}$$

$$= \frac{ig^3}{2(4\pi)^2} \mathrm{tr}(t^a t^b t^c) \int dx\, dy\, dz\, \delta(\,) \cdot \frac{\Gamma(2-d_k)}{\Delta^{2-d_k}}$$

$$\Big\{ (-8) \Big\} \cdot \Big\{ (\not{z}p + (1-y)\not{q}) \, g^{\nu\lambda} - (\gamma^\nu q (y-1) + \gamma^\nu d z) \Big\}$$

$$g^{\mu\nu} (\not{q}(y-1) + \not{d}z) +$$

$$\gamma^\nu (\gamma q (y-1) + \gamma d z) +$$

$$+ (-(\not{z}p - y\not{q}) g^{\mu\nu} + \gamma^\mu (y\not{q} - \not{d}z) -) +$$

$$+ \gamma^\nu (\not{q}y - \not{d}z) +$$

$$g^{\mu\nu} \gamma (\not{q}y - d(z-1)) + \gamma^\mu (-\not{q}y - d(z-1)) +$$

$$+ \gamma^\nu (\not{q}y + (z-1)) +$$

$$\Big\{ \Big\} = g^{\mu\nu} \big[-(1-z)p + d(z-1)\big] \gamma^\lambda + g^{\mu\lambda} \big[\not{q}(y-1) + d(z-1)\big] \gamma^\nu$$

$$+ \gamma^\lambda (\not{q}(y-1) + d(z+1)) \gamma^\nu +$$

w.z. $\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot 1 = \frac{1}{2}$

$\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot x = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

$$\int dx\,dy\,dz\;\delta(1-x-y-z)\{\} = \frac{1}{3}\left(g^{\mu\nu}[-p-2q]^{\lambda} + g^{\nu\lambda}[q-p]^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right)$$

= all



$$= \frac{g^3}{2}\,\mathrm{tr}\,t^a t^b t^c \cdot \frac{i}{(4\pi)^{d_4}}\,\Gamma(2-d_4)\,(-8)\cdot\left(\frac{1}{3}\right)$$
$$\cdot\left[g^{\mu\nu}(-p-2q)^{\lambda} + g^{\nu\lambda}(q-p)^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right]$$

symbolically, for



we exchange $a \leftrightarrow c$, $p \leftrightarrow -(p+q)$, $\mu \leftrightarrow \lambda$

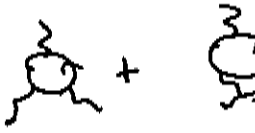
$(-p-2q) \to p-q$

$(q-p) \to p+2q$

$(2p+q) \to -(2p+q)$

$$= \frac{g^3}{2}\,\mathrm{tr}\,t^c t^b t^a\,\frac{i}{(4\pi)^{d_4}}\,\Gamma(2-d_4)\left(-\frac{8}{3}\right)$$
$$\left[g^{\lambda\nu}(p-q)^{\mu} + g^{\nu\mu}(p+2q)^{\lambda} + g^{\mu\lambda}(-(2p+q)^{\nu})\right]$$

so



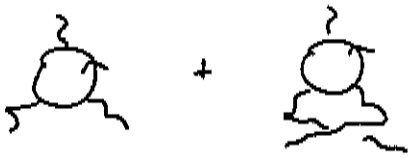
$$= \frac{-i\,g^3}{(4\pi)^2}\cdot\frac{4}{3}\,\Gamma(2-d_4)$$
$$\left[g^{\mu\nu}(-p-2q)^{\lambda} + g^{\nu\lambda}(q-p)^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right]$$
$$\cdot\;\mathrm{tr}\left[t^a t^b t^c - t^c t^b t^a\right]$$

with, finally,

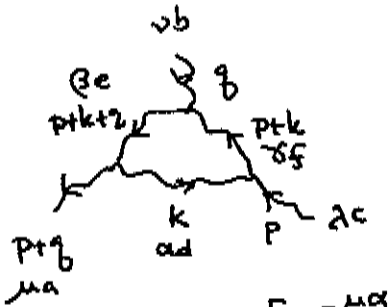
$$\mathrm{tr}\,[t^a t^b t^c - t^c t^b t^a] = \mathrm{tr}\,t^a [t^b, t^c] = \mathrm{tr}\,[t^a\ i f^{bcd} t^d]$$

$$= i f^{dbc}\, n_f\, C(r)$$

in all



$$= \frac{g^3}{(4\pi)^2} f^{dbc} \left(\frac{4}{3} n_f C(r)\right) \cdot \frac{2}{\epsilon} \cdot \left[g^{\mu\nu}(-p-2q)^\lambda + \ldots\right]$$



$$= g^3 f^{ade} f^{ebf} f^{fdc} \int \frac{d^d k}{(2\pi)^d} \frac{-1}{(k+p)^2} \frac{-i}{(k+p+q)^2} \frac{-1}{k^2}$$

$$\left[\ g^{\mu\alpha}[-(p+q)+k]^\beta + g^{\alpha\beta}[-k-(p+k+q)]^\mu + g^{\beta\mu}[p+k+q+p+q]^\alpha\ \right]$$

$$\cdot \left[\ g^{\beta\nu}[-(p+k+q)-q]^\gamma + g^{\nu\gamma}[q-(p+k)]^\beta + g^{\gamma\beta}[p+k+p+k+q]^\nu\ \right]$$

$$\cdot \left[\ g^{\gamma\alpha}[-(p+k)-k]^\lambda + g^{\alpha\lambda}[k-p]^\gamma + g^{\lambda\gamma}[p+p+k]^\alpha\ \right]$$

Evaluate this as above: combine denominators, use the formulae on p. 17 to represent momenta in the numerators, extract the term proportional to $I(2-d\epsilon)$. These terms contain one power of finite momenta in the numerator or two powers of k. The finite term can come from either the first, the second, or the third line above. Using

$$k^\alpha k^\beta = \frac{1}{d} g^{\alpha\beta} k^2$$

we find $\underbrace{\int dx\,dy\,dz\,\delta(\ldots)}_{\int d\Gamma \leftarrow \text{(Feynman parameters)}} \cdot [\,][\,][\,] = ① + ② + ③$

$$① = \int dF \Big[g^{\mu\alpha}(-p-q-d(z-1)p-yq)^{\beta} + g^{\alpha\beta}(-p-q+z(1-z)p+2yq)^{\mu} + g^{\beta\mu}(2p+2q-(1-z)p-yq)^{\alpha} \Big]$$

$$\cdot \Big[g^{\beta\nu}(-k^{\gamma}) + g^{\nu\gamma}(-k)^{\beta} + g^{\gamma\beta}(2k)^{\nu} \Big] \cdot$$

$$\cdot \Big[g^{\gamma\alpha}(-2k)^{\rho} + g^{\alpha\rho}(k^{\beta}) + g^{\rho\gamma}(k^{\alpha}) \Big]$$

$$\rightsquigarrow \quad k^{\gamma}k^{\rho} = \frac{1}{d}k^2 g^{\gamma\rho} \qquad \int dF \cdot 1 = \frac{1}{2} \qquad \int dF\,(x,y,z) = \frac{1}{3}\cdot\frac{1}{2}$$

$$= \frac{1}{2}\frac{1}{d}k^2 \Big[g^{\mu\alpha}\left(-\tfrac{5}{3}p - \tfrac{4}{3}q\right)^{\beta} + g^{\alpha\beta}\left(\tfrac{1}{3}p - \tfrac{1}{3}q\right)^{\mu} + g^{\beta\mu}\left(\tfrac{4}{3}p + \tfrac{5}{3}q\right)^{\alpha} \Big]$$

$$\Big[2g^{\beta\nu}g^{\gamma\rho}g^{\gamma\alpha} - g^{\beta\gamma}g^{\rho\alpha}g^{\rho\gamma} - g^{\beta\nu}g^{\rho\alpha}g^{\rho\gamma} \Big] \leftarrow \text{set } g^{\rho\rho},\ 4 \overset{!}{=} d=4$$

$$+ 2g^{\nu\rho}g^{\beta\gamma}g^{\rho\alpha} - g^{\nu\rho}g^{\beta\gamma}g^{\rho\alpha} - g^{\nu\rho}g^{\alpha\beta}g^{\rho\gamma}$$

$$- 4g^{\rho\beta}g^{\nu\rho}g^{\rho\gamma} + 2g^{\rho\beta}g^{\alpha\nu}g^{\rho\gamma} + 2g^{\rho\beta}g^{\nu\alpha}g^{\rho\gamma} \Big]$$

$$= \frac{1}{9}\frac{1}{d}k^2 \Big[g^{\beta\nu}g^{\alpha\rho}\,[2-4-1-1+2] + g^{\beta\rho}g^{\alpha\nu}\,[2+2] + g^{\alpha\beta}g^{\rho\nu}\,[-1-4] \Big]$$

$$\cdot \Big[g^{\mu\alpha}(-5p-4q)^{\beta} + g^{\alpha\beta}(p-q)^{\mu} + g^{\beta\mu}(4p+5q)^{\alpha} \Big]$$

$$= \frac{1}{9}\cdot\frac{1}{d}k^2 \Big[(-2)\cdot\left(g^{\mu\nu}(-5p-4q)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\mu\rho}(4p+5q)^{\nu} \right)$$

$$+ 4\cdot\left(g^{\mu\nu}(-5p-4q)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\mu\rho}(4p+5q)^{\nu} \right)$$

$$- 5\left(g^{\rho\nu}(-5p-4q)^{\mu} + g^{\rho\mu}(p-q)^{\nu} + g^{\mu\nu}(4p+5q)^{\rho} \right) \Big]$$

$$= \frac{1}{6}\frac{1}{d}k^2\Big\{ g^{\mu\nu}\left[-8p-10q-20p-16q\right]^\alpha + g^{\nu\alpha}\left[(p-q)\underbrace{(-2+4-20)}_{-18}+25p+20q-20p-25q\right]^\mu + g^{\alpha\mu}\left[+10p+8q+16p+20q\right]^\nu \Big\}$$

$$① = \frac{1}{6}\frac{1}{d}k^2\Big\{ g^{\mu\nu}\left[-28p-26q\right]^\alpha + g^{\nu\alpha}\left[-13p+13q\right]^\mu + g^{\alpha\mu}\left[26p+28q\right]^\nu \Big\}$$

$$② = \int dF \Big[g^{\beta\nu}\left[-p-2q+(1-z)p+yq\right]^\alpha + g^{\nu\alpha}\left[q-p+(1-z)p+yq\right]^\beta + g^{\alpha\beta}\left[2p+q-2(1-z)p-2yq\right]^\nu \Big]$$

$$\Big[g^{\mu\alpha}k^\beta - 2g^{\alpha\beta}k^\mu + g^{\beta\mu}k^\alpha \Big]$$

$$\Big[-2g^{\nu\alpha}k^\beta + g^{\alpha\beta}k^\nu + g^{\beta\nu}k^\alpha \Big]$$

$$= \frac{1}{2}\cdot\frac{1}{d}k^2\Big[g^{\beta\nu}\left[-\tfrac{1}{3}p-\tfrac{5}{3}q\right]^\alpha + g^{\nu\alpha}\left[-\tfrac{1}{3}p+\tfrac{4}{3}q\right]^\beta + g^{\alpha\beta}\left[\tfrac{2}{3}p+\tfrac{1}{3}q\right]^\nu \Big]$$

$$\Big[g^{\mu\alpha}(-2)g^{\beta\lambda}g^{\nu\alpha} + g^{\mu\alpha}g^{\beta\nu}g^{\alpha\lambda} + g^{\mu\alpha}g^{\alpha\beta}g^{\lambda\nu} + 4\,g^{\alpha\beta}g^{\mu\lambda}g^{\nu\alpha} - 2\,g^{\alpha\beta}g^{\mu\nu}g^{\alpha\lambda} = 2\,g^{\alpha\beta}g^{\mu\alpha}g^{\lambda\nu} - 2\,g^{\beta\mu}g^{\alpha\lambda}g^{\nu\alpha} + g^{\beta\mu}g^{\alpha\nu}g^{\alpha\lambda} + g^{\beta\nu}g^{\alpha\alpha}g^{\lambda\nu} \Big]$$

$$= \frac{1}{6d} k^2 \left[g^{\beta\nu}(-p-5q)^\alpha + g^{\nu\alpha}[-p+4q]^\beta + g^{\alpha\beta}[2p+q]^\nu \right]$$

$$\left[g^{\mu\alpha} g^{\beta\lambda} [-2-2] \right.$$

$$+ g^{\mu\lambda} g^{\beta\alpha} [1+4] +$$

$$\left. + g^{\mu\beta} g^{\alpha\nu} [1-2-2+1+4] \right]$$

$$= \frac{1}{6d} k^2 \left[g^{\lambda\nu}(-p-5q)^\mu \cdot (-4) + g^{\mu\nu}(-p+4q)^\lambda (-4) \right.$$

$$+ g^{\mu\lambda}(2p+q)^\nu (-4)$$

$$+ 5 \cdot (g^{\mu\nu})(-p-5q-p+4q+4\cdot(2p+q))^\nu$$

$$\left. + 2 \cdot (g^{\mu\nu}(-p-5q)^\lambda + g^{\nu\lambda}(-p+4q)^\mu + g^{\mu\lambda}(2p+q)^\nu \right]$$

$$= \frac{1}{6d} k^2 \left[g^{\mu\nu}(+4p-16q-2p-10q)^\lambda \right.$$

$$+ g^{\nu\lambda}(4p+20q-2p+8q)^\mu$$

$$\left. + g^{\lambda\mu}(-8p-4q+56p+3q)+4p+2q) \right)$$

$$(2) = \frac{1}{6}\frac{1}{d} k^2 \left[g^{\mu\nu}(2p-26q)^\lambda + g^{\nu\lambda}(2p+28q)^\mu \right.$$

$$\left. + g^{\lambda\mu}(26p+13q)^\nu \right]$$

$$\textcircled{3} = \int d^d p \; \Big[g^{\rho\alpha}(-p + 2(1-z)p + 2q_2) \; {}^{\nu}(q_2 + d(z-1)2 + d-) \; g^{\rho}$$

$$+ g^{\alpha\nu}(d - 2q - d(z-1)) \; {}^{\rho}(q_2 + d(z-1)2 + d-) \; g^{\rho}$$

$$+ g^{\nu\rho}(2p - d(z-1) - d2) \; {}^{\nu}(2q - d(z-1) - d2) \; g^{\rho} \Big]$$

$$\cdot \Big[g^{\mu\alpha} k^{\beta} - 2 g^{\alpha\beta} k^{\mu} + g^{\mu\beta} k^{\alpha} \Big] \cdot$$

$$\Big[{}^{\mu}(k^{\lambda}) g^{\beta\nu} - {}^{\beta}(k^{\lambda}) g^{\mu\nu} + 2 g^{\lambda\beta} k^{\mu} \Big] \cdot$$

$$= \frac{1}{2} \frac{P}{1} k^2 \Big[g^{\rho\nu} \big(\tfrac{1}{3} d + \tfrac{2}{3} \big) \; {}^{\nu}(q_2 + d) \; g^{\rho} + g^{\nu\rho} \big(-\tfrac{4}{3} d - \tfrac{1}{3} q \big) \; {}^{\rho}(q_2 + d) \; g^{\nu} \Big]$$

$$\Big[\big(\tfrac{1}{3} q - \tfrac{4}{3} d \big) g^{\rho\nu} g^{\nu\rho} + g^{\rho\nu} g^{\nu\rho} \big(\tfrac{2}{3} q + d \big) \Big]$$

$$= \frac{1}{1} \frac{P}{9} k^2 \Big[g^{\rho\nu} \big(-4p + q \big) \; {}^{\nu}(q_2 + d) \; g^{\rho} + g^{\rho\nu} \big(-5p + q \big) \; {}^{\nu}(q_2 + d) \; g^{\rho} + g^{\rho\nu} \big(-4p + q \big) \; {}^{\rho}(q_2 + d) \; g^{\nu} \Big]$$

$$\Big[g^{\rho\nu} g^{\rho\nu} \big[-1 - 4 + 2 + 2 - 1 \big] + g^{\rho\nu} g^{\rho\nu} \big[2 + 2 \big] + g^{\rho\nu} g^{\rho\nu} \big[-4 - 1 \big] \Big]$$

$$= \frac{1}{1} \frac{P}{9} k^2 \Big[(-2) \cdot \big[g^{\rho\nu} \; {}^{\nu}(q_2 + d) \; g^{\rho} + g^{\rho\nu} \; {}^{\rho}(q_2 + d) \; g^{\nu} \big] \cdot (2-) \Big]$$

$$+ 4 \Big[g^{\rho\nu} \; {}^{\nu}(q_2 + d) \; g^{\rho} + g^{\rho\nu} \; {}^{\rho}(q_2 + d) \; g^{\nu} \Big] \; 4 +$$

$$- 5 \Big[g^{\rho\nu} \Big] \Big[4(q_2 + d) - {}^{\nu}(q_2 + d) \; 5p - q^2 + 4p - q^2 \Big] \Big]$$

$$= \frac{P}{9} \frac{1}{7} k^2 \Big[g^{\rho\nu} \Big[-2p + 4p - 15p - 4q + 8q - 3 \textcircled{9} q \Big] \; {}^{\nu}(q_2 + d) \; g^{\rho}$$

$$+ g^{\rho\nu} \Big[+10p + 2q + 16p - 4q - 20p + 2q - 8p + d \blacksquare 4q \Big] \; {}^{\rho}(q_2 + d) \; g^{\nu} +$$

$$③ = \frac{1}{6}\frac{1}{d}k^2\left[g^{\mu\nu}(-13p-26q)^\lambda + g^{\nu\lambda}(-28p-2q)^\mu + g^{\lambda\mu}(26p-2q)^\nu\right]$$

$$①+②+③ = \frac{1}{6}\frac{1}{d}k^2\left[g^{\mu\nu}(-39p-78q)^\lambda + g^{\nu\lambda}(-39p+39q)^\mu + g^{\lambda\mu}(78p+39q)^\nu\right]$$

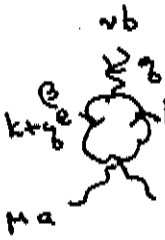
$$= \frac{13}{2}\frac{1}{d}k^2\left[g^{\mu\nu}(-p-2q)^\lambda + g^{\nu\lambda}(-p+q)^\mu + g^{\lambda\mu}(2p+q)^\nu\right]$$

cont'd from ≈ p.22



$$= ig^3\left(-\tfrac{1}{2}C_2(G)f^{abc}\right)\cdot\frac{i}{(4\pi)^{d_k}}I\left(2-\frac{d_k}{2}\right)\cdot\frac{d}{2}\cdot 2 \cdot\frac{13}{2}\cdot\frac{1}{d}\left[g^{\mu\nu}(-p-2q)^\lambda + \cdots\right]$$

$$= +\frac{13}{8}g^3 C_2(G) f^{abc}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\left[g^{\mu\nu}(-p-2q)^\lambda + \cdots\right]$$



$$= g f^{ebf}\cdot\underbrace{\frac{1}{2}}_{\text{symmetry factor}}\cdot(-ig^2)\Big[f^{aed}f^{efd}(g^{\mu\beta}g^{\rho\lambda}-g^{\mu\lambda}g^{\rho\beta}) + f^{aed}f^{efd}(g^{\rho\lambda}g^{\mu\beta}-g^{\mu\rho}g^{\lambda\beta}) + f^{afd}f^{ced}(g^{\mu\lambda}g^{\rho\beta}-g^{\rho\beta}g^{\mu\lambda})\Big]$$

$$\cdot\int\frac{d^dk}{(2\pi)^d}\;\frac{-i}{k^2}\;\frac{-i}{(k+q)^2}$$

$$\cdot\left[g^{\beta\nu}(-(k+q)-q)^\alpha + g^{\nu\alpha}(q-k)^\beta + g^{\alpha\beta}(k+k+q)^\nu\right]\cdot$$

combine denominators w. Feynman parameters. $\ell = k + xq$

$k = \ell - xq \qquad (k+q) = \ell + (1-x)q$

To obtain a divergence, keep the finite part of η [ ].

$$= \frac{ig^3}{2} \int_0^1 dx \; \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2)$$

$$\cdot \Big\{ f^{ebf} f^{acd} f^{efd} (g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta})$$

$$+ f^{ebf} f^{aed} f^{cfd} (g^{\mu\nu} g^{\beta\alpha} - g^{\mu\alpha} g^{\nu\beta})$$

$$+ f^{ebf} f^{afd} f^{ced} (g^{\mu\nu} g^{\beta\alpha} - g^{\mu\beta} g^{\nu\alpha}) \Big\}$$

$$\cdot \Big[g^{\beta\nu} (-2q + xq)^{\alpha} + g^{\nu\alpha} (q + xq)^{\beta} + g^{\alpha\beta} (-2xq + q)^{\nu} \Big]$$

$$f^{ebf} f^{acd} f^{efd} = -f^{acb} = + C_2(G) f^{abc}$$

$$f^{ebf} f^{aed} f^{cfd} = (-f^{ade}) f^{ebf} f^{fdc} = + \tfrac{1}{2} C_2(G) f^{abc}$$

$$f^{ebf} f^{afd} f^{ced} = (-f^{adf})(-f^{fbe})(f^{edc}) = - \tfrac{1}{2} C_2(G) f^{abc}$$

$$= \frac{ig^3}{2} \frac{1}{(4\pi)^{d_2}} \Gamma(2-d_2) \; C_2(G) f^{abc}$$

$$\Big\{ (g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta}) + \tfrac{1}{2}(g^{\nu\beta} g^{\mu\alpha} - g^{\mu\nu} g^{\alpha\beta}) - \tfrac{1}{2}(g^{\mu\nu} g^{\beta\alpha} - g^{\mu\beta} g^{\nu\alpha}) \Big\}$$

$$\cdot \Big[g^{\beta\nu} (-\tfrac{3}{2}) q^{\alpha} + g^{\nu\alpha} \tfrac{3}{2} q^{\beta} + 0 \Big]$$

$$= -\frac{g^3}{2} \frac{1}{(4\pi)^{d_2}} \Gamma(2-d_2) \; C_2(G) f^{abc} \; \tfrac{3}{2} (g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta}) (-\tfrac{3}{2})$$

$$\cdot (g^{\beta\nu} q^{\alpha} - g^{\nu\alpha} q^{\beta})$$

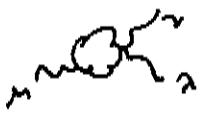
$$= + \frac{9}{8} g^3 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} C_1(G) f^{abc}$$

$$\cdot \{ g^{\mu\nu} q^\lambda - g^{\nu\lambda} q^\mu - g^{\nu\lambda} q^\mu + g^{\mu\nu} q^\lambda \}$$



$$= \frac{9}{4} g^3 C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ g^{\mu\nu} q^\lambda - g^{\nu\lambda} q^\mu \}$$

how cross



$$\begin{pmatrix} a \\ \mu \\ p+q \end{pmatrix} \leftrightarrow \begin{pmatrix} b \\ \nu \\ -q \end{pmatrix}$$

$$= \frac{9}{4} g^3 C_2(G) (-f^{abc}) \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ g^{\mu\nu} (-p-q)^\lambda + g^{\mu\lambda} (p+q)^\nu \}$$



$$\begin{pmatrix} c \\ \lambda \\ -p \end{pmatrix} \leftrightarrow \begin{pmatrix} b \\ \nu \\ -q \end{pmatrix}$$

$$= \frac{9}{4} g^3 C_2(G) (-f^{abc}) \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ g^{\nu\lambda} p^\nu - g^{\nu\lambda} p^\mu \}$$

so



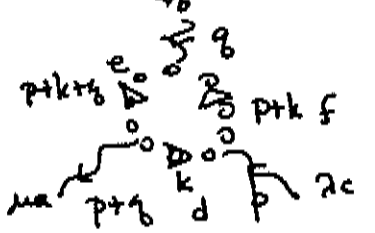
$$= \frac{9}{4} g^3 C_2(G) f^{abc} \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon} \cdot [g^{\mu\nu} (q^\lambda + (p+q)^\lambda)$$

$$+ g^{\nu\lambda} (-q^\mu + p^\mu) + g^{\lambda\mu} (-p-q-p)^\nu]$$

$$= -\frac{9}{4} g^3 C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} [g^{\mu\nu} (-p-2q)^\nu + g^{\nu\lambda} (q-p)^\mu$$

$$+ g^{\lambda\mu} (2p+q)^\nu]$$

finally, the ghost loops:



$$= (-g)^3 f^{dae} f^{ebf} f^{fcd} \; (-1) \quad \leftarrow \text{fermion}$$

$$\cdot \int \frac{d^dk}{(2\pi)^d} \; \frac{i}{k^2} \; \frac{i}{(p+k)^2} \; \frac{i}{(p+k+q)^2} \; k^\mu (p+k+q)^\nu (p+k)^\lambda$$

$$= (-i)\, g^3 (f^{ade} f^{ebf} f^{fdc}) \cdot \int dx\, dy\, dz\, \delta(1-x-y-z) \cdot 2 \cdot \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2-\Delta]^3}$$

(p. 17) $\rightarrow$ $\cdot (k-(1-z)p-yq)^\mu (k+zp+(1-y)q)^\nu (k+zp-yq)^\lambda$

For the divergent term, picking up $2\,k^1$'s of one finite term.

$$= (-i)\, g^3 \left(-\tfrac{1}{2} C_2(G)\, f^{abc}\right) \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \cdot \frac{1}{2}$$

$$\cdot \int dF \; \{ g^{\mu\nu} (zp-yq)^\lambda + g^{\mu\lambda} (zp+(1-y)q)^\nu + g^{\nu\lambda} (-(1-z)p-yq)^\mu \}$$

$$= -\frac{g^3}{2\cdot 2} C_2(G)\, f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \cdot \frac{1}{6}$$

$$\cdot \{ g^{\mu\nu} (p-q)^\lambda + g^{\nu\lambda} (-2p-q)^\mu + g^{\lambda\mu} (p+2q)^\nu \}$$

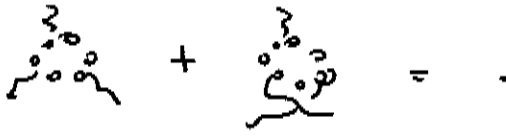
$$= -\frac{g^3}{24} C_2(G)\, f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\cdot \{ g^{\mu\nu} (p-q)^\lambda + g^{\nu\lambda} (-2p-q)^\mu + g^{\lambda\mu} (p+2q)^\nu \}$$

for  exchg. $\mu \leftrightarrow \lambda$, $a \leftrightarrow c$, $p+q \leftrightarrow -p$

 $= -\frac{g^3}{24} C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$

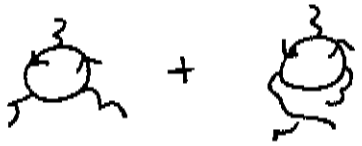
$$\cdot (-1) \cdot \{ g^{\lambda\nu}(-p-2q)^\mu + g^{\mu\nu}(2p+q)^\lambda + g^{\mu\lambda}(q-p)^\nu \}$$



$$= -\frac{g^3}{24} C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ g^{\mu\nu}(p-q-2p-q)^\lambda + g^{\nu\lambda}(-2p-q+p+2q)^\mu + g^{\lambda\mu}(p+2q-q+p)^\nu \}$$

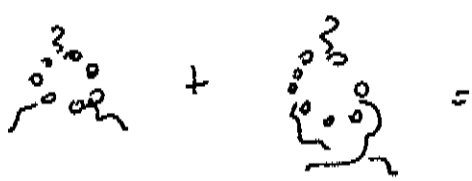
$$= -\frac{g^3}{24} C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ g^{\mu\nu}(-p-2q)^\lambda + g^{\nu\lambda}(q-p)^\mu + g^{\lambda\mu}(p+2q)^\nu \}$$

so!

 $= (\frac{4}{3} n_f C(r)) \cdot \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} [g^{\mu\nu}(-p-2q)^\lambda + \dots]$

 $= \frac{13}{8} C_2(G) \cdot \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} [g^{\mu\nu}(-p-2q)^\lambda + \dots]$

 $= -\frac{9}{4} C_2(G) \cdot \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} [g^{\mu\nu}(-p-2q)^\lambda + \dots]$

 $= -\frac{1}{24} C_2(G) \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} [g^{\mu\nu}(-p-2q)^\lambda + \dots]$

to cancel these:

$$\delta_1^{3g} = \left(-\frac{4}{3} n_f C(r) + \left(\underbrace{-\frac{13}{8} + \frac{9}{4} + \frac{1}{24}}\right) C_2(G)\right) \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

$$\frac{5}{8} + \frac{1}{24} = 2/3$$

$$\delta_1^{3g} = \left(2/3 C_2(G) - 4/3 n_f C(r)\right) \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

from p. 14

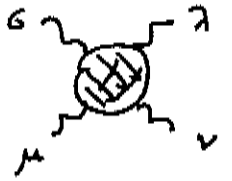
$$\delta_3 = \left(5/3 C_2(G) - 4/3 n_f C(r)\right) \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

so

$$\delta_1^{3g} - \delta_3 = -\frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon} \cdot C_2(G)$$

as required.

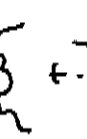
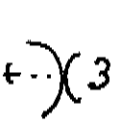
c.) The 4-gluon vertex is given by:



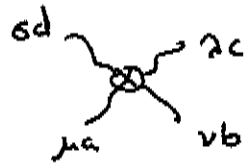
$=$ ( $+ \cdots$) (3! diagrams)

$+$ ( $+ \cdots$) ($3!/2$ diagrams)

$+$ ( $+ \cdots$) ($\frac{4 \cdot 3}{2}$ diagrams)

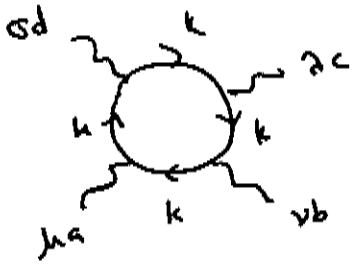
$+$ ( $+\cdots$) (3 diagrams) $+$ ( $+\cdots$) (3! diagrams) $+$

the contraction is



$$= -ig^2 \delta_1^{4g} \left[f^{abe} f^{cde} (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda}) \right.$$
$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\nu\lambda})$$
$$\left. + f^{ade} f^{bce} (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma}) \right]$$

This has no powers of external momenta, so we can set the external momenta in the diagrams = 0. Also we can set $d=4$ in the divergent term, e.g. $\gamma^\alpha \gamma^\mu \gamma_\alpha = (2-d)\gamma^\mu = -2\gamma^\mu$.



$$= (ig)^4 (-1) \int \frac{d^dk}{(2\pi)^d} \, tr\left[\gamma^\mu \frac{ik}{k^2} \gamma^\nu \frac{ik}{k^2} \gamma^\lambda \frac{ik}{k^2} \gamma^\sigma \frac{ik}{k^2} \right]$$

fermion

$$\cdot \, tr[t^a t^b t^c t^d]$$

we need

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\alpha k^\beta k^\gamma k^\delta}{[k^2]^4} = \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2) \frac{1}{3!} \cdot \frac{1}{4} \left[g^{\alpha\beta} g^{\gamma\delta} + g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\gamma} g^{\beta\delta} \right]$$

$$= -\frac{g^4}{24} \cdot \frac{i}{(4\pi)^2} \cdot \frac{2}{\epsilon} \cdot tr[t^a t^b t^c t^d]$$

$$\cdot \, tr\left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma_\alpha \gamma^\lambda \gamma^\sigma \gamma^\alpha \gamma_\gamma \right.$$
$$+ \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\lambda \gamma_\beta \gamma^\sigma \gamma_\alpha$$
$$\left. + \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\lambda \gamma_\alpha \gamma^\sigma \gamma_\beta \right]$$

$$= -i\frac{g^4}{24}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}(t^a t^b t^c t^d)$$

$$\cdot\,\mathrm{tr}\left[(-2)^2(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)\cdot 2 + (-2)\,\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\sigma\gamma_\beta\right]$$

$$= -i\frac{g^4}{24}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}(t^a t^b t^c t^d)$$

$$\cdot\,\mathrm{tr}\left[8\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma - 2\cdot 4\,\gamma^\mu\gamma^\lambda g^{\nu\sigma}\right]$$

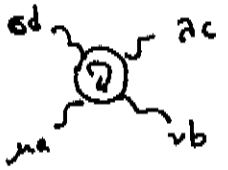
$$= -i\frac{g^4}{24}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}(t^a t^b t^c t^d)\cdot 32$$

$$\cdot\,(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

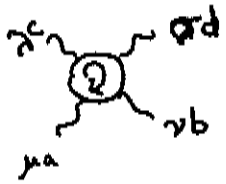
$$= -i\frac{4}{3}\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}(t^a t^b t^c t^d)$$

$$(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\sigma})$$

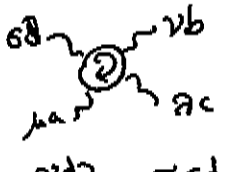
now look at the coefficient of $g^{\mu\lambda}g^{\nu\sigma}$ in all 6 diagrams.



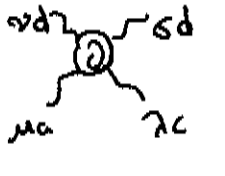
$$= +i\frac{4}{3}\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\left[2\,\mathrm{tr}(t^a t^b t^c t^d)\right] g^{\mu\lambda}g^{\nu\sigma}$$



$$= +i\frac{4}{3}\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\left[-\mathrm{tr}(t^a t^b t^d t^c)\right] g^{\mu\lambda}g^{\nu\sigma}$$



$$= \;\;''\;\;\left[-\mathrm{tr}(t^a t^c t^b t^d)\right]\;\;''$$



$$= \;\;''\;\;\left[-\mathrm{tr}(t^a t^c t^d t^b)\right]\;\;''$$



$= \quad " \; [- tr [t^a t^d t^b t^c]] \; "$



$= \quad " \; [\; 2 tr [t^c \, t^d \, t^c \, t^b] \;] \; "$

$\sim$ all



$$= + \frac{4}{3} i \frac{g^4}{(4\pi)^2} \frac{2}{\epsilon} \cdot g^{\mu\lambda} g^{\nu\sigma}$$

$$\cdot tr \left[\; t^a t^b [t^c, t^d] + t^a [t^b, t^c] t^d + t^a t^d [t^c, t^b] \right.$$

$$\left. + \; t^a [t^d, t^c] t^b \; \right]$$

$$+ \text{(2 more tensor structures)}$$

the trace is:

$$tr \left\{ t^a t^b \, i f^{cde} t^e + t^a \, i f^{bce} t^d + t^a t^d (i f^{bce} t^e) \right.$$

$$\left. - i \, t^a f^{cde} t^e t^b \right\}$$

$$= tr \left(t^a \, i f^{cde} [t^b, t^e] - i f^{bce} \, t^a [t^d, t^e] \right)$$

$$= tr \; t^a \left(i f^{cde} \, i f^{bef} t^f - i f^{bce} \, i f^{def} t^f \right)$$

$$= - n_f C(r) \left[[f^{abe} f^{cde}] - [f^{ade} f^{bce}] \right]$$

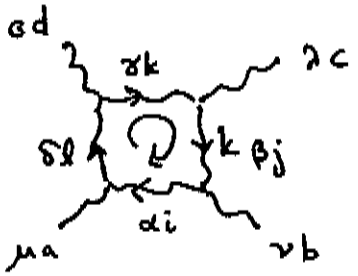
so!



$$= - \frac{4}{3} i \frac{g^4}{(4\pi)^2} \cdot \frac{2}{\epsilon} \; g^{\mu\lambda} g^{\nu\sigma} \; n_f C(r) \left(f^{abe} f^{cde} - f^{ade} f^{bce} \right)$$

$$+ \text{ 2 more tensor structures.}$$

cycling. The other two tensor structures give the remaining term in the expression at the top of p. 33.



$$= g^4 f^{lai} f^{ibj} f^{jck} f^{kdl}$$

$$\cdot \int \frac{d^dk}{(2\pi)^d} \left(\frac{-i}{k^2}\right)^4 \cdot \left[g^{\delta\mu}(-k)^\alpha + g^{\mu\alpha}(-k)^\delta + g^{\alpha\delta}(k+k)^\mu \right]$$

$$\cdot \left[g^{\alpha\nu}(-k)^\beta + g^{\nu\beta}(-k)^\alpha + g^{\beta\alpha}(2k)^\nu \right]$$

$$\cdot \left[g^{\beta\lambda}(-k)^\gamma + g^{\lambda\gamma}(-k)^\beta + g^{\gamma\beta}(2k)^\lambda \right]$$

$$\cdot \left[g^{\gamma\sigma}(-k)^\delta + g^{\sigma\delta}(-k)^\gamma + g^{\delta\gamma}(2k)^\sigma \right]$$

Let $(T^a)_{ij} = i f^{iaj}$ then $f^{lai} f^{ibj} f^{jck} f^{kdl} = \mathrm{tr}\left[T^a T^b T^c T^d\right]$

$$= \mathrm{tr}\left[T^d T^c T^b T^a\right]$$

$$= g^4 \, \mathrm{tr}\left[T^a T^b T^c T^d\right] \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2]^4}$$

$$\cdot \{ g^{\mu\delta} k^\nu k^\beta + g^{\mu\delta} k^2 g^{\nu\beta} + (-2) g^{\mu\delta} k^\beta k^\nu$$

$$+ g^{\mu\nu} k^\delta k^\beta + g^{\nu\beta} k^\delta k^\mu - 2 g^{\mu\beta} k^\delta k^\nu$$

$$- 2 g^{\nu\delta} k^\mu k^\beta - 2 g^{\nu\beta} k^\delta k^\mu + 4 g^{\delta\beta} k^\mu k^\nu \}$$

$$\{ g^{\beta\lambda} k^\delta k^\sigma + g^{\beta\lambda} k^2 g^{\sigma\delta} - 2 g^{\beta\lambda} k^\delta k^\sigma$$

$$+ g^{\lambda\sigma} k^\beta k^\delta + g^{\sigma\delta} k^\lambda k^\beta - 2 g^{\lambda\delta} k^\beta k^\sigma$$

$$- 2 g^{\beta\delta} k^\lambda k^\delta - 2 g^{\sigma\delta} k^\lambda k^\beta + 4 g^{\beta\delta} k^\lambda k^\sigma \}$$

$$= g^4 \, \mathrm{tr}[T^a T^b T^c T^d] \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2]^4}$$

$$\{ g^{\mu\delta} g^{\nu\beta} k^2 - g^{\mu\delta} k^\beta k^\nu - g^{\nu\beta} k^\mu k^\delta - 2 g^{\mu\beta} k^\nu k^\delta - 2 g^{\nu\delta} k^\mu k^\beta + g^{\mu\nu} k^\beta k^\delta + 4 g^{\beta\delta} k^\mu k^\nu \}$$

$$\{ g^{\beta\lambda} g^{\sigma\delta} k^2 - g^{\beta\lambda} k^\sigma k^\delta - g^{\delta\sigma} k^\beta k^\lambda - 2 g^{\lambda\delta} k^\beta k^\sigma - 2 g^{\beta\sigma} k^\lambda k^\delta + g^{\lambda\sigma} k^\beta k^\delta + 4 g^{\beta\delta} k^\lambda k^\sigma \}$$

$$= g^4 \, \mathrm{tr}[T^a T^b T^c T^d] \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2]^4}$$

$$\{ g^{\mu\sigma} g^{\nu\lambda} (k^2)^2 - g^{\nu\lambda} k^\mu k^\sigma (k^2) - g^{\mu\sigma} k^2 k^\nu k^\lambda - 2 g^{\mu\lambda} k^\nu k^\sigma k^2 - 2 g^{\nu\sigma} k^\lambda k^\mu k^2 + g^{\lambda\sigma} k^\mu k^\nu k^2 + 4 g^{\mu\nu} k^\lambda k^\sigma k^2$$

$$- g^{\mu\sigma} k^2 k^\nu (k^2) + k^\mu k^\nu k^\lambda k^\sigma + g^{\mu\sigma} k^2 k^\nu k^\lambda + 2 g^{\mu\lambda} k^2 k^\nu k^\sigma + 2 k^\mu k^\nu k^\lambda k^\sigma - g^{\lambda\sigma} k^\mu k^\nu k^2 - 4 k^\mu k^\nu k^\lambda k^\sigma$$

$$- g^{\nu\lambda} k^\mu k^\sigma k^2 + g^{\lambda\nu} k^\mu k^\sigma k^2 + k^\mu k^\nu k^\lambda k^\sigma + 2 k^\mu k^\nu k^\lambda k^\sigma + 2 g^{\nu\sigma} k^\mu k^\lambda k^2 - g^{\lambda\sigma} k^\mu k^\nu k^2 - 4 k^\mu k^\nu k^\lambda k^\sigma$$

$$- 2 g^{\mu\lambda} k^\nu k^\sigma k^2 + 2 g^{\mu\lambda} k^\nu k^\sigma k^2 + 2 k^\mu k^\nu k^\lambda k^\sigma + 4 k^\mu k^\nu k^\lambda k^\sigma + 4 g^{\mu\sigma} k^\nu k^\lambda k^2 - 2 g^{\lambda\sigma} k^\mu k^\nu k^2 - 8 k^\mu k^\nu k^\lambda k^\sigma$$

$$- 2 g^{\nu\sigma} k^\mu k^\lambda k^2 + 2 k^\mu k^\nu k^\lambda k^\sigma + 2 g^{\nu\sigma} k^2 k^\mu k^\lambda + 4 g^{\nu\lambda} k^2 k^\mu k^\sigma + 4 k^\mu k^\nu k^\lambda k^\sigma - 2 g^{\lambda\sigma} k^\mu k^\nu k^2 - 8 k^\mu k^\nu k^\lambda k^\sigma$$

+ (more) ⟶

$$+ g^{\mu\nu} k^2 k^\lambda k^\sigma - g^{\mu\nu} k^2 k^\lambda k^\sigma - g^{\mu\nu} k^2 k^\lambda k^\sigma$$
$$- 2 g^{\mu\nu} k^2 k^\lambda k^\sigma - 2 g^{\mu\nu} k^2 k^\lambda k^\sigma + g^{\mu\nu} g^{\lambda\sigma} (k^2)^2 + 4 g^{\mu\nu} k^\lambda k^\sigma k^2$$
$$+ 4 g^{\lambda\sigma} k^2 k^\mu k^\nu - 4 k^\mu k^\nu k^\lambda k^\sigma - 4 k^\mu k^\nu k^\lambda k^\sigma$$
$$- 8 k^\mu k^\nu k^\lambda k^\sigma - 8 k^\mu k^\nu k^\lambda k^\sigma + 4 k^\mu k^\nu g^{\lambda\sigma} k^2 + 16 k^\mu k^\nu k^\lambda k^\sigma \Big\}$$

$$= g^4 \, tr(T^a T^b T^c T^d) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^4}$$

$$\Big\{ g^{\mu\sigma} g^{\lambda\nu} (k^2)^2 \cdot 1 + g^{\mu\nu} g^{\lambda\sigma} (k^2)^2 \cdot 1$$
$$+ g^{\mu\sigma} k^2 k^\nu k^\lambda [-1 - 1 + 1 + 4]$$
$$+ g^{\nu\lambda} k^2 k^\mu k^\sigma [-1 - 1 + 1 + 4]$$
$$+ g^{\mu\nu} k^2 k^\lambda k^\sigma [+4 + 1 - 1 - 1 - 2 - 2 + 4]$$
$$+ g^{\lambda\sigma} k^2 k^\mu k^\nu [+1 - 1 - 1 - 2 - 2 + 4 + 4]$$
$$+ g^{\mu\lambda} k^2 k^\nu k^\sigma [-2 + 2 - 2 + 2]$$
$$+ g^{\nu\sigma} k^2 k^\mu k^\lambda [-2 + 2 - 2 + 2]$$
$$+ k^\mu k^\nu k^\lambda k^\sigma [1 + 2 - 4 + 1 + 2 - 4 + 2 + 4 - 8$$
$$+ 2 + 4 - 8 - 4 - 4 - 8 - 8 + 64] \Big\}$$

$$= g^4 \, tr(T^a T^b T^c T^d) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^4}$$

$$\cdot \Big\{ g^{\mu\sigma} g^{\lambda\nu} (k^2)^2 + g^{\mu\nu} g^{\lambda\sigma} (k^2)^2 + (g^{\mu\sigma} k^\nu k^\lambda + g^{\nu\lambda} k^\mu k^\sigma) k^2 \cdot 3$$
$$+ (g^{\mu\nu} k^\lambda k^\sigma + g^{\lambda\sigma} k^\mu k^\nu) \cdot 3$$
$$+ k^\mu k^\nu k^\lambda k^\sigma \cdot 34 \Big\}$$

now $$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2} = \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k)$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^3} k^\mu k^\nu = \frac{i}{(4\pi)^{d_k}} \Gamma\left(\frac{2-d_k}{2}\right) \cdot \frac{1}{2} g^{\mu\nu} = \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k) \cdot \frac{1}{4} g^{\mu\nu}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^4} k^\mu k^\nu k^\lambda k^\sigma = \frac{i}{(4\pi)^{d_k}} \frac{\Gamma(2-d_k)}{6} \cdot \frac{1}{4}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma} + g^{\nu\sigma}g^{\mu\sigma})$$

$$= \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k) \frac{1}{24} (g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma} + g^{\nu\sigma}g^{\mu\sigma})$$

so

$$= g^4 \, \mathrm{tr}\, T^a T^b T^c T^d \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k)$$

$$\cdot \{ (g^{\mu\sigma}g^{\lambda\nu} + g^{\mu\nu}g^{\lambda\sigma}) + (g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\nu}g^{\lambda\sigma}) \cdot 2 \cdot 3 \cdot \frac{1}{4}$$

$$+ 34 \frac{1}{24} (g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} + g^{\nu\sigma}g^{\mu\lambda}) \}$$

$$= \frac{ig^4}{12} \, \mathrm{tr}\, T^a T^b T^c T^d \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\cdot \{ (g^{\mu\sigma}g^{\lambda\nu} + g^{\mu\nu}g^{\lambda\sigma}) \cdot [12 + 18 + 17]$$

$$+ g^{\mu\lambda}g^{\nu\sigma} \cdot 17 \}$$

$$= \frac{ig^4}{12} \, \mathrm{tr}\, T^a T^b T^c T^d \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\cdot \{ (g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\lambda\nu}) \, 47 + g^{\mu\lambda}g^{\nu\sigma} \cdot 17 \}$$

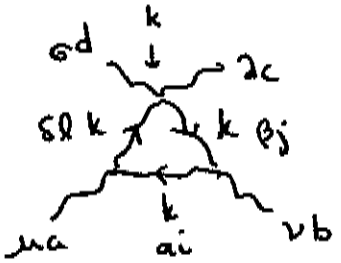
then, summing over 3 diagrams:



$$= \frac{ig^4}{12}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\cdot g^{\mu\lambda}g^{\nu\sigma}$$

$$\cdot\Big\{ 47\ \mathrm{tr}\,T^aT^cT^bT^d + 47\ \mathrm{tr}\,T^aT^bT^dT^c + 17\ \mathrm{tr}\,T^aT^bT^cT^d \Big\}$$

$$+ (\text{cyclic in } a,b,c)$$



$$= -ig^2\Big[f^{ldk}f^{jck}(g^{\delta\beta}g^{\sigma\lambda} - g^{\delta\lambda}g^{\sigma\beta}) + f^{ljk}f^{dck}(g^{\delta\sigma}g^{\beta\lambda} - g^{\delta\lambda}g^{\beta\sigma}) + f^{lck}f^{djk}(g^{\delta\sigma}g^{\beta\lambda} - g^{\delta\beta}g^{\sigma\lambda})\Big]$$

$$\cdot \int \frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)^2 \cdot g^2 f^{lai}f^{ibj}$$

$$\cdot\ [-g^{\delta\alpha}k^\sigma - g^{\alpha\sigma}k^\delta + 2k^\alpha g^{\sigma\delta}]\,[-g^{\alpha\nu}k^\beta - g^{\nu\beta}k^\alpha + 2k^\nu g^{\alpha\beta}]$$

Let's write the f's as traces of T's.

$$f^{ldk}f^{jck}f^{lai}f^{ibj} = f^{lai}f^{ibj}f^{jck}(-f^{kdl}) = -\mathrm{tr}\,T^aT^bT^cT^d$$

$$f^{lai}f^{lbj}f^{ljk}f^{dck} = f^{lai}f^{ibj}f^{jkl}(-f^{cdk}) = -if^{cdk}\,\mathrm{tr}\,T^aT^bT^k$$

$$= -\mathrm{tr}\,T^aT^b[T^c,T^d]$$

$$f^{lai}f^{ibj}f^{lck}f^{djk} = f^{lai}f^{ibj}(-f^{jdk})(-f^{kcl}) = +\mathrm{tr}\,T^aT^bT^dT^c$$

so

$$= g^4 \Big\{ \mathrm{tr}(T^a T^b T^c T^d)\,[\,(p\!\!\!/ L)(c\!\!\!/ L)(b\!\!\!/ L)(a\!\!\!/ L)\,] + \mathrm{tr}(T^a T^b T^d T^c)\,[\,(c\!\!\!/ L)(b\!\!\!/ L)(d\!\!\!/ L)(a\!\!\!/ L)\,] \Big\}$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2}\right)^3 \left[\frac{k^2}{4}\right] \Big\{ g^{\mu\delta} g^{\alpha\beta} g^{\nu\lambda} + g^{\nu\delta} g^{\mu\alpha} g^{\lambda\beta} - 2 g^{\mu\delta} g^{\nu\lambda} g^{\alpha\beta} + g^{\mu\alpha} g^{\delta\beta} g^{\nu\lambda} + g^{\nu\alpha} g^{\delta\alpha} g^{\lambda\beta} - 2 g^{\mu\nu} g^{\delta\lambda} g^{\alpha\beta} - 2 g^{\alpha\delta} g^{\mu\beta} g^{\nu\lambda} - 2 g^{\alpha\delta} g^{\mu\nu} g^{\beta\lambda} + 4 g^{\alpha\delta} g^{\mu\nu} g^{\beta\lambda} \Big\}$$

$$= g^4 \Big\{ \mathrm{tr}(T^a T^b T^c T^d)\,[\,(p\!\!\!/ L)(c\!\!\!/ L)(b\!\!\!/ L)(a\!\!\!/ L)\,] + \mathrm{tr}(T^a T^b T^d T^c)\,[\,(c\!\!\!/ L)(b\!\!\!/ L)(d\!\!\!/ L)(a\!\!\!/ L)\,] \Big\}$$

$$\cdot \Big\{ g^{\mu\delta} g^{\nu\beta} \underbrace{(1 + 4 - 2 + 1 - 2)}_{2} + g^{\mu\nu} g^{\beta\delta} \underbrace{(1 + 4)}_{5} + g^{\mu\beta} g^{\nu\delta} \underbrace{(-2 - 2)}_{-4} \Big\} \cdot \left(\frac{i}{(4\pi)^2} \Gamma(2 - d/2) \cdot \frac{1}{4}\right)$$

$$= \frac{ig^4}{4} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \Big\{ \mathrm{tr}(T^a T^b T^c T^d)\,[\,4 g^{\mu\nu} g^{\rho\lambda} + 10 g^{\mu\nu} g^{\lambda\rho} - 8 g^{\mu\rho} g^{\nu\lambda} - 2 g^{\mu\nu} g^{\lambda\rho} - 20 g^{\mu\lambda} g^{\rho\nu} + 4 g^{\mu\rho} g^{\lambda\nu} - 2 g^{\mu\rho} g^{\nu\lambda} - 5 g^{\mu\nu} g^{\lambda\rho} + 4 g^{\mu\lambda} g^{\nu\rho}\,]$$

$$+ \mathrm{tr}(T^a T^b T^d T^c)\,[\,4 g^{\mu\rho} g^{\nu\lambda} + 10 g^{\mu\nu} g^{\rho\lambda} - 8 g^{\mu\lambda} g^{\nu\rho} - 2 g^{\mu\nu} g^{\rho\lambda} - 5 g^{\mu\nu} g^{\lambda\rho} + 4 g^{\mu\rho} g^{\nu\lambda} - 2 g^{\mu\nu} g^{\lambda\rho} - 20 g^{\mu\nu} g^{\rho\lambda} + 4 g^{\mu\nu} g^{\rho\lambda}\,] \Big\}$$

$$= \frac{ig^4}{4}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\cdot$$

$$\{ \mathrm{tr}(T^aT^bT^cT^d)\cdot(-13\,g^{\mu\nu}g^{\lambda\sigma} + 8\,g^{\mu\lambda}g^{\nu\sigma} - 10\,g^{\mu\sigma}g^{\nu\lambda})$$

$$+ \mathrm{tr}(T^aT^bT^dT^c)(-13\,g^{\mu\nu}g^{\lambda\sigma} - 10\,g^{\mu\lambda}g^{\nu\sigma} + 8\,g^{\mu\sigma}g^{\nu\lambda})\}$$

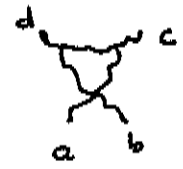
as before, focus on the structure $g^{\mu\lambda}g^{\nu\sigma}$ and sum over the 6 diagrams.



$$= \frac{ig^4}{4}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\; g^{\mu\lambda}g^{\nu\sigma}$$



$$\{ \mathrm{tr}(T^aT^bT^cT^d)\cdot 8 + \mathrm{tr}\,T^aT^bT^dT^c\cdot(-10)$$

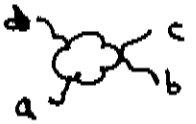


$$+ \mathrm{tr}(T^cT^dT^aT^b)\cdot 8 + \mathrm{tr}\,T^cT^dT^bT^a(-10)$$

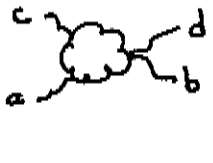


$$+ \mathrm{tr}(T^aT^bT^cT^d)\cdot 8 + \mathrm{tr}\,T^aT^dT^bT^c(-10)$$

$$+ \mathrm{tr}(T^aT^bT^cT^d)\cdot 8 + \mathrm{tr}\,T^aT^cT^bT^d(-10)$$



$$+ \mathrm{tr}[T^aT^bT^dT^c](-13) + \mathrm{tr}\,T^aT^dT^bT^c(-13)$$



$$+ \mathrm{tr}(T^aT^bT^dT^c)(-13) + \mathrm{tr}(T^aT^cT^bT^d)(-13)\}$$

now, since $\mathrm{tr}\,T^aT^bT^cT^d = \mathrm{tr}\,T^dT^cT^bT^a$ all we have cyclic invariance, there are only 3 structures, & we may write the above as:



$$= \frac{ig^4}{4}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}$$

$$\cdot\Big\{ \mathrm{tr}(T^c T^b T^c T^d)\,[8+8+8+8]$$

$$+ \mathrm{tr}(T^a T^b T^d T^c)\,[-10-10-13-13]$$

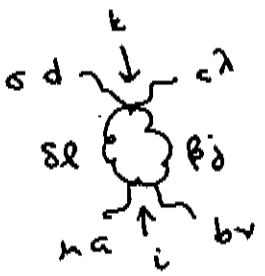
$$+ \mathrm{tr}(T^a T^c T^b T^d)\,[-10-10-13-13]\Big\}$$

$$= \frac{ig^3}{4}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}$$

$$\Big\{ \mathrm{tr}(T^a T^b T^c T^d)\cdot 32$$

$$+ \mathrm{tr}(T^a T^b T^d T^c + T^a T^c T^b T^d)(-46)\Big\}$$

$$+ \text{cyclic.}$$



$$= (-ig^2)^2\Big[f^{\ell dk}f^{jck}\,(g^{\delta\beta}g^{\lambda\sigma} - g^{\delta\lambda}g^{\beta\sigma})$$

$$+ f^{\ell jk}f^{dck}\,(g^{\delta\sigma}g^{\beta\lambda} - g^{\delta\lambda}g^{\beta\sigma})$$

$$+ f^{\ell ck}f^{djk}\,(g^{\delta\sigma}g^{\beta\lambda} - g^{\delta\beta}g^{\lambda\sigma})\Big]$$

$$\Big[f^{a\ell i}f^{bji}\,(g^{\mu\nu}g^{\beta\delta} - g^{\mu\beta}g^{\nu\delta})$$

$$+ f^{abi}f^{\ell ji}\,(g^{\mu\delta}g^{\nu\beta} - g^{\mu\beta}g^{\nu\delta})$$

$$+ f^{aji}f^{\ell bi}\,(g^{\mu\delta}g^{\nu\beta} - g^{\mu\nu}g^{\beta\delta})\Big]$$

$$\cdot\ \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{-i}{k^2}\right)^2$$

$$= \left(\frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\right)\Big\{ (f^{ali}f^{bji}f^{ldk}f^{jck}) \cdot [\,4g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\lambda\nu} + g^{\mu\sigma}g^{\nu\lambda}\,]$$

$$+ (f^{ali}f^{bji}f^{ljk}f^{dck})\,[\,\cancel{g^{\mu\nu}g^{\lambda\sigma}} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\,]$$

$$+ (f^{ali}f^{bji}f^{lck}f^{djk})\,[\,g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\nu}g^{\lambda\sigma}\,]$$

$$+ (f^{abi}f^{lji}f^{ldk}f^{jck})\,[\,\cancel{g^{\mu\nu}g^{\lambda\sigma}} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\,]$$

$$+ (f^{abi}f^{lji}f^{ljk}f^{dck})\,[\,g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\,]$$

$$+ (f^{abi}f^{lji}f^{lik}f^{djk})\,[\,g^{\mu\sigma}g^{\nu\lambda} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} - \cancel{g^{\mu\lambda}g^{\nu\sigma}} + \cancel{g^{\mu\nu}g^{\lambda\sigma}}\,]$$

$$+ (f^{aji}f^{lbi}f^{ldk}f^{jck})\,[\,g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\nu}g^{\lambda\sigma}\,]$$

$$+ (f^{aji}f^{lbi}f^{ljk}f^{dck})\,[\,g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} + \cancel{g^{\mu\nu}g^{\lambda\sigma}}\,]$$

$$+ (f^{aji}f^{lbi}f^{lik}f^{djk})\,[\,g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\nu}g^{\lambda\sigma} + 4g^{\mu\nu}g^{\lambda\sigma}\,]\Big\}$$

$$= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\Big\{ (-f^{lai})f^{ibj}f^{jck}(-f^{kdl})\;(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$+ (-f^{lai})(f^{ibj})(f^{jkl})(-f^{cdk})\;(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ (-f^{lai})(f^{ibj})(-f^{jdk})(-f^{kcl})\;(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ (f^{abi})(-f^{lij})(f^{jck})(-f^{kdl})\;(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ f^{abi}(-f^{lij})(+f^{jkl})(-f^{cdk})\;(2g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\sigma})$$

$$+ f^{abi}(-f^{lij})(-f^{jdk})(-f^{kcl})\;(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

+ (more →)

$$+ (-f^{jai})(-f^{ibl})(f^{ldk})(-f^{kcj})(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ (-f^{jai})(-f^{ibl})(-f^{lkj})(-f^{cdk})(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ (-f^{jai})(-f^{ibl})(f^{lck})(f^{kdj})(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\Big\{ \mathrm{tr}(T^aT^bT^cT^d)(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$+ \mathrm{tr}(T^aT^b[T^c,T^d])(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$- \mathrm{tr}(T^aT^bT^dT^c)(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ \mathrm{tr}[T^a,T^b]T^cT^d(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ \mathrm{tr}[T^a,T^b][T^c,T^d](2g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\sigma})$$

$$- \mathrm{tr}[T^a,T^b]T^dT^c(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$- \mathrm{tr}\,T^aT^bT^dT^c(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ \mathrm{tr}\,T^aT^b[T^c,T^d](g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ \mathrm{tr}\,T^aT^bT^cT^d(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})\Big\}$$

$\mathrm{tr}\,T^aT^bT^cT^d - \mathrm{tr}\,T^aT^bT^dT^c - \mathrm{tr}\,T^bT^aT^cT^d + \mathrm{tr}\,T^bT^aT^dT^c = 0$ (cyclic perm).

$$= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\cdot$$

$$\Big\{ \mathrm{tr}(T^aT^bT^cT^d)[2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma}$$

$$+ g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} + \tfrac{1}{4}g^{\mu\sigma}g^{\nu\lambda} + \tfrac{1}{4}g^{\mu\lambda}g^{\nu\sigma}$$

$$+ g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma}$$

$$+ 2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda}]$$

$$+ \quad \text{more} \longrightarrow$$

$$+ \mathrm{tr}(T^a T^b T^d T^c)\,(-g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\sigma} + 2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma}$$
$$- g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\sigma} - 4g^{\mu\sigma}g^{\nu\lambda} + 4g^{\mu\lambda}g^{\nu\sigma}$$
$$+ (-g^{\mu\sigma}g^{\nu\lambda}) + g^{\mu\lambda}g^{\nu\sigma} + 2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma}$$
$$- g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\sigma})\}$$

$$= \frac{ig^4}{\mu^{4-d}} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \Big\{ \mathrm{tr}(T^a T^b T^c T^d)$$
$$\cdot (4g^{\mu\nu}g^{\lambda\sigma} + 10\, g^{\mu\sigma}g^{\nu\lambda} - 8\, g^{\mu\lambda}g^{\nu\sigma})$$
$$+ \mathrm{tr}(T^a T^b T^d T^c)$$
$$(4g^{\mu\nu}g^{\lambda\sigma} + 10 g^{\mu\lambda}g^{\nu\sigma} - 8\, g^{\mu\sigma}g^{\nu\lambda}) \Big\}$$

again, track the coefficient of $g^{\mu\lambda}g^{\nu\sigma}$ through the 3 diagrams:



$$= \frac{ig^4}{2} \frac{1}{(4\pi)^2} \frac{2}{\epsilon}\; g^{\mu\lambda}g^{\nu\sigma}\left[-8\,\mathrm{tr}T^aT^bT^cT^d + 10\,\mathrm{tr}T^aT^bT^dT^c\right]$$



$$= \quad " \quad \left[-8\,\mathrm{tr}\,T^aT^bT^cT^d + 10\,\mathrm{tr}\left[T^aT^cT^bT^d\right]\right]$$



$$= \quad " \quad \left[4\,\mathrm{tr}\,T^aT^bT^dT^c + 4\,\mathrm{tr}\left[T^aT^cT^bT^d\right]\right.$$

$=$ all



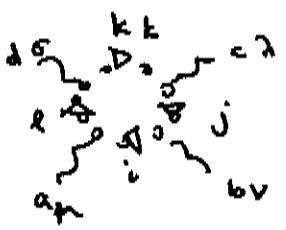
$$= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}$$

$$\cdot\Big[\mathrm{tr}(T^aT^bT^cT^d)(-16) + \mathrm{tr}(T^aT^bT^dT^c)\cdot 14$$

$$+ \mathrm{tr}(T^aT^cT^bT^d)\cdot 14\Big]$$

$$+ \text{ cyclic.}$$

finally



$$= (-g)^4 f^{lai} f^{ibj} f^{jck} f^{kdl} (-1)$$

fermion.

$$\int \frac{d^dk}{(2\pi)^d}\left(\frac{i}{k^2}\right)^4 k^\mu k^\nu k^\lambda k^\sigma$$

$$= -g^4\,\frac{1}{24}\left(\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\right)\left(g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}\right)\mathrm{tr}\,T^aT^bT^cT^d$$

there are 6 digrams. Tracing the coefficient of $g^{\mu\lambda}g^{\nu\sigma}$, we find



$$= -\frac{g^4}{24}\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\; g^{\mu\lambda}g^{\nu\sigma}\Big\{\mathrm{tr}\,T^aT^bT^cT^d + \mathrm{tr}\,T^aT^bT^dT^c + \mathrm{tr}\,T^aT^cT^bT^d\Big\}\cdot 2$$

now add all of the contributions:



$$= i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\left\{\left(f^{abe}f^{cde} - f^{ade}f^{bce}\right)\left[-\frac{4}{3}n_f(r)\right]\right\}$$



$$= i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\left\{ \mathrm{tr}\, T^aT^cT^bT^d\ \frac{47}{12}\right.$$

$$+ \mathrm{tr}\, T^aT^bT^dT^c \cdot \frac{47}{12}$$

$$\left. + \mathrm{tr}\, T^aT^bT^cT^d \cdot \frac{17}{12}\right\}$$



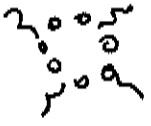
$$= i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\left\{ \mathrm{tr}\, T^aT^bT^cT^d \cdot 8\right.$$

$$\left. + \mathrm{tr}\left(T^aT^bT^dT^c + T^aT^cT^bT^d\right)\left(-\frac{23}{2}\right)\right\}$$



$$= i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\left\{ \mathrm{tr}\, T^aT^bT^cT^d \cdot (-8)\right.$$

$$\left. + \mathrm{tr}\left(T^aT^bT^cT^d + T^aT^cT^bT^d\right)\cdot 7\right\}$$



$$= \frac{ig^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\nu}g^{\lambda\sigma}\left\{ \mathrm{tr}\, T^aT^bT^cT^d\ \left(-\frac{1}{12}\right)\right.$$

$$\left. + \mathrm{tr}\left(T^aT^bT^dT^c + T^{ac}T^bT^d\right)\cdot\left(-\frac{1}{12}\right)\right\}$$

in all



$$= \frac{ig^4}{(4\pi)^2}\,\frac{2}{\epsilon}\,g^{\mu\nu}g^{\nu\sigma}$$

$$\cdot\Big\{\big(f^{abe}f^{cde} - f^{ade}f^{bce}\big)\Big(-\frac{4}{3}\,n_f\,C(r)\Big)$$

$$+\quad \mathrm{tr}\,T^aT^bT^cT^d\,\Big(\frac{17}{12} - \frac{1}{12} + 8 - 8\Big) \quad \Leftarrow\ = \frac{4}{3}$$

$$+\quad \mathrm{tr}\big(T^aT^bT^dT^c + T^aT^cT^bT^d\big)\,\Big(\frac{47}{12} - \frac{1}{12} - \frac{23}{2} + 7\Big)\Big\}$$

$$\frac{23}{6} - \frac{23}{2} + \frac{21}{3} = -\frac{2}{3}\ !$$

now

$$\mathrm{tr}\,\big(2T^aT^bT^cT^d - T^aT^bT^dT^c - T^cT^cT^bT^d\big)$$

$$= \mathrm{tr}\Big\{\big[T^aT^b\,if^{cde}\,T^e\big] + \big[T^a(if^{bce})\,T^e\,T^d\big]\Big\}$$

since $\mathrm{tr}\,T^aT^bT^e = -\mathrm{tr}\,T^eT^bT^a = -\mathrm{tr}\,T^aT^eT^b$

$$= \frac{1}{2}\,\mathrm{tr}\,T^a[T^b, T^e]$$

$$= \frac{i}{2}f^{fbe}\,\mathrm{tr}\,T^aT^f$$

$$= \frac{i}{2}f^{fbe}\,(if^{iaj}f^{jfi})$$

$$= \frac{i}{2}f^{fbe}\,C_2(G)\,\delta^{af} = \frac{i}{2}f^{abe}$$

So



$$= \frac{ig^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda} g^{\nu\sigma}$$

$$\cdot \left(f^{abe} f^{cde} - f^{ade} f^{bce} \right)$$

$$\left(-\frac{4}{3} n_f C(r) - \frac{1}{3} C_2(G) \right)$$

$$+ (\text{cyclic})$$

$$\delta_1^{4g} = \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\left(-\frac{4}{3} n_f C(r) - \frac{1}{3} C_2(G) \right)$$

$$\delta_3 = \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\left(-\frac{4}{3} n_f C(r) + \frac{5}{3} C_2(G) \right)$$

$$\frac{1}{2}\left(\delta_1^{4g} - \delta_3 \right) = \frac{1}{2} \cdot \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} \cdot -2 C_2(G)$$

$$= -\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} C_2(G)$$

as required !!!

Physics 332 – Problem Set # 7

(due Wednesday, May 31)

1. Peskin and Schroeder, Problem 18.1.

2. Compute the anomalous dimension of the twist-2 quark-antiquark operators:

$$\overline{q}_f \gamma^{\{\mu_1} (iD^{\mu_2}) \cdots (iD^{\mu_n\}}) q_f - \text{traces} \tag{1}$$

The calculation is sketched in Peskin and Schroeder, pp. 636-8. You need to track the 1-loop corrections that yield matrix elements of the form of eq. (18.161). Ignore any contributions proportional to $g^{\mu_i \mu_j}$, since these correspond to operators of lower spin. The final result is given in (18.172):

$$\gamma_f^n = \frac{8}{3} \frac{g^2}{(4\pi)^2} \left[1 + 4 \sum_2^n \frac{1}{j} - \frac{2}{n(n+1)} \right] . \tag{2}$$

3. In class, I stated that the expression (2) for γ_f^n is the Mellin transform of the kernel in the Altarelli-Parisi equations. Verify this explicitly as follows: Begin from the equation

$$\frac{d}{d \log Q} f(x,Q) = \frac{\alpha_s(Q)}{\pi} \int_x^1 \frac{dz}{z} P_{q \leftarrow q}(z) \, f(\frac{x}{z}) \tag{3}$$

where

$$P_{q \leftarrow q}(z) = \frac{4}{3} \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(z-1) \right] . \tag{4}$$

Work out the equation for the Mellin transform of $f(x)$

$$M_n = \int dx x^{n-1} f(x) \tag{5}$$

Show explicitly that $M_n(Q)$ obey

$$\frac{d}{d \log Q} M_n(Q) = -\gamma_f^n M_n(Q) \ . \tag{6}$$

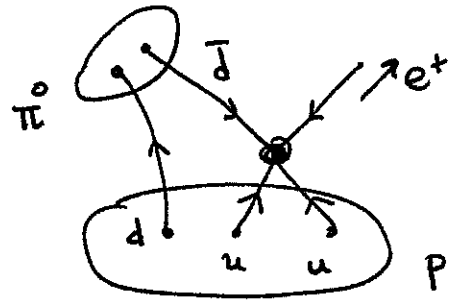
You will find the calculation sketched in Peskin and Schroeder, pp. 644-646.

Physics 332 – Problem Set #7

Solutions

1.) a.) Consider for definiteness

$p \to \pi^0 e^+$



for a $1 \to 2$ deg. iM has dimensions of (mass)^1

so $$iM \sim \frac{2}{m_X^2} \cdot m_p^3$$

we might want to add a factor $\alpha^2 \sim 4\pi\left(\frac{g^2}{4\pi}\right) \sim \frac{1}{2}$

$C \sim \frac{1}{24}$

It makes little difference.

$$\Gamma \sim \frac{1}{2m_p}\,\frac{1}{8\pi} \cdot \left|\frac{2m_p^3}{m_X^2}\right|^2$$

$$\sim \frac{1}{4\pi}\,\frac{m_p^5}{m_X^4} \sim \frac{1}{4\pi}\,\frac{(1\ \text{GeV})^5}{(10^{16}\ \text{GeV})^4}$$

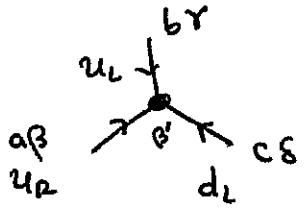
so

$$\Gamma \sim 10^{-65}\,\text{GeV}^{+1} \cdot (6.58\times10^{-25}\,\text{GeV-sec})^{-1}$$

$$\sim (10^{41}\,\text{sec})^{-1} \sim (10^{34}\,\text{yr})^{-1}$$

b.) for $[\mathcal{O}_\beta]_M = \epsilon_{abc}\,\epsilon^{\gamma\delta}\,u_{R a\beta}\,u_{L b\gamma}\,d_{L c\delta}$

a renormalizable coupling is



$= \epsilon_{abc}\,\delta_{\beta\beta'}\,\varepsilon_{\gamma\delta}$ at $Q\sim M$

this vertex is corrected by QCD diagrams:



to compute these diagrams, we need the Fierz identities for

$\sigma^\mu = (1, \vec{\sigma})^\mu \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})^\mu \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

$$(\sigma^\mu)_{\alpha\beta}\,(\sigma_\mu)_{\gamma\delta} = -2\,\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}$$

check: $(\sigma^\mu\,\bar{\sigma}^\nu\,\sigma_\mu)_{\alpha\delta} = -2(\sigma^\nu)_{\alpha\delta} \overset{?}{=} -2\,(i\sigma^2)_{\alpha\gamma}\,(\bar{\sigma}^\nu)^T_{\gamma\beta}\,(i\sigma^2)_{\beta\delta}$

$$= +2\,[\sigma^2\,(\bar{\sigma}^\nu)^T\,\sigma^2]_{\alpha\delta}$$

$$= 2\,(\sigma^\nu)_{\alpha\delta}$$

✓

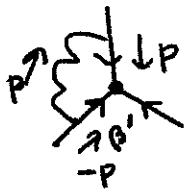
Ref

$$(\bar{\sigma}^\mu)_{\alpha\beta} \quad (\sigma_\mu)_{\gamma\delta} = 2\,\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$

check: $(\bar{\sigma}^\mu \cdot 1 \cdot \sigma_\mu)_{\alpha\delta} \stackrel{?}{=} 2\,\delta_{\alpha\delta}\,\delta_{\gamma\gamma}$

$$= 4\,\delta_{\alpha\beta} \stackrel{\checkmark}{=} 4\,\delta_{\alpha\delta}$$

now



$$= (ig)^2 \int \frac{d^dp}{(2\pi)^d} \frac{-i}{p^2}\, \varepsilon_{a'b'c}\,(t^B)_{c'a}\,(t^A)_{b'b}$$

$$\varepsilon_{\gamma'\delta'}\,\delta_{\beta'\delta}\cdot\left(\left(\frac{-i\,p\cdot\bar{\sigma}}{p^2}\right)\sigma^\mu\right)_{\beta'\beta}$$

$$\cdot\left(\frac{i\,p\cdot\sigma}{p^2}\,\bar{\sigma}_\mu\right)_{\gamma'\gamma}$$

$$= i\,g^2\;\varepsilon_{\gamma'\delta}\;\varepsilon_{a'b'c}\,(t^A)_{a'a}\,(t^A)_{b'b}$$

$$\cdot\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2)^3}\,(p\cdot\bar{\sigma})_{\beta'\theta}\;(p\cdot\sigma)_{\gamma'\eta}\cdot 2\,\delta_{\eta\beta}\,\delta_{\theta\gamma}$$

now

$$\varepsilon_{a'b'c}\,(t^A)_{a'a}\,(t^A)_{b'b} = t^A t^A \text{ for } 3\times 3 \to \bar{3}$$

$$= \tfrac{1}{2}\left[C_2(\bar{3}) - C_2(3) - C(3)\right]$$

$$= \tfrac{1}{2}\left[\tfrac{4}{3} - \tfrac{4}{3} - \tfrac{4}{3}\right]$$

$$= -\tfrac{2}{3}$$

$$= -\frac{2}{3} i g^2 \varepsilon_{\gamma'\delta} \varepsilon_{abc} \cdot \frac{i}{(4\pi)^2} \frac{\Gamma(2-d/2)}{2} \cdot \frac{1}{2} g^{\alpha\beta}$$

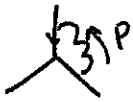
$$\cdot 2 (\bar{\sigma}_\alpha)_{\beta'\gamma} (\sigma_\beta)_{\gamma'\beta}$$

$$= -\frac{2}{3} i g^2 \cdot \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \cdot \frac{1}{4} \cdot 2 \cdot 2 \quad \varepsilon_{abc} \varepsilon_{\gamma\delta} \delta_{\beta\beta'}$$

$$= \frac{2}{3} g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \quad \delta_{\beta\beta'} \varepsilon_{\gamma\delta} \varepsilon_{abc}$$



gives the same result.



$$= \delta_{\beta\beta'} (ig)^2 \int \frac{d^d p}{(2\pi)^d} \frac{-i}{p^2} \varepsilon_{ab'c'} (t^A)_{b'd} (t^A)_{c'c}$$

$$\varepsilon_{\gamma'\delta'} \left(-i \frac{p\cdot\sigma}{p^2} \bar{\sigma}_\nu\right)_{\beta'\beta} \left(\nu \bar{\sigma} \frac{d\cdot\sigma}{d^2} i\right)_{\delta\gamma}$$

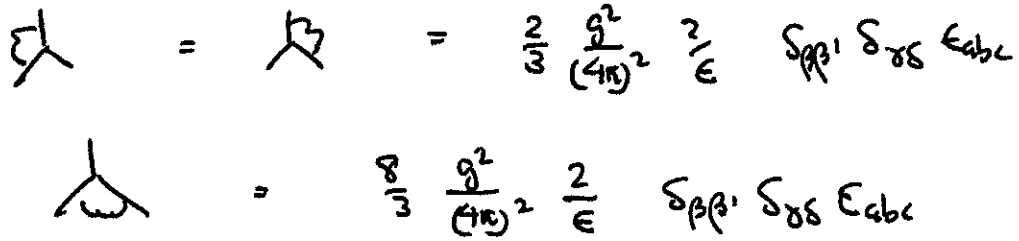
$$= \left(-\frac{2}{3}\right) i g^2 \delta_{\beta\beta} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^3} \varepsilon_{abc} \varepsilon_{\gamma\beta\gamma} (p\cdot\sigma)_{\gamma'\gamma} (\sigma\cdot d)_{\gamma'\delta} \eta_{\gamma\delta} (-2\varepsilon_{\gamma\eta} \varepsilon_{\delta\eta})$$

$$= -i \frac{2}{3} g^2 \delta_{\beta\beta'} \varepsilon_{abc} \varepsilon_{\gamma'\delta'} \frac{i}{(4\pi)^{p/2}} \frac{\Gamma(\frac{p}{2}-2)}{2} \frac{1}{2} g^{\alpha\beta} (\sigma^\alpha)_{\gamma\gamma'} \eta_{\gamma\delta} \varepsilon_{\eta\delta} \varepsilon_{\delta\eta} \quad (-2)$$

$$= \frac{2}{3} g^2 \delta_{\beta\beta'} \varepsilon_{abc} \varepsilon_{\gamma'\delta'} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{4} (-2 \varepsilon_{\gamma\delta} \varepsilon_{\eta\delta}) (-2) \varepsilon_{\gamma\delta} \varepsilon_{\eta\delta}$$

$$= \frac{2}{3} g^2 \delta_{\beta\beta'} \varepsilon_{abc} \varepsilon_{\gamma\delta} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{4} \cdot 4 \cdot 4$$

so



$$= \frac{2}{3}\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\,\delta_{\beta\beta'}\delta_{\gamma\delta}\epsilon_{abc}$$

$$= \frac{8}{3}\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\,\delta_{\beta\beta'}\delta_{\gamma\delta}\epsilon_{abc}$$

$$\delta_O = -\frac{12}{3}\frac{g^2}{(4\pi)^2}\cdot\frac{2}{\epsilon} = -4\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}$$

recall $\quad \delta_2 = -\frac{g^2}{(4\pi)^2}\cdot\frac{4}{3}\cdot\frac{2}{\epsilon}$

with $\quad 2/\epsilon \to \ln \Lambda^2/M^2 \qquad M\frac{\partial}{\partial M}\left(\frac{2}{\epsilon}\right) = -2$

$$\gamma_O = M\frac{\partial}{\partial M}\left(-\delta_O + \tfrac{3}{2}\delta_2\right)$$

$$= \left(-8 + \tfrac{3}{2}\cdot\tfrac{4}{3}\cdot 2\right)\frac{g^2}{(4\pi)^2}$$

$$\gamma_O = -4\frac{g^2}{(4\pi)^2}$$

Using $\quad \gamma_O = \frac{a_O}{(4\pi)^2}g^2$, this operator is enhanced at

large distances by

$$\left[\frac{\ln g(M_x^2/\Lambda^2)}{\ln g(M_p^2/\Lambda^2)}\right]^{a_O/2b_O}$$

$$b_0 = 11 - \frac{2}{3} n_f = 7 \quad \text{for } n_f = 6$$

$m_p = 1\ \text{GeV}$ $\qquad \Lambda = 200\ \text{MeV}$

$M_I = 10^{16}\ \text{GeV}$

the enhancement factor is:

$$\left(\frac{\ln M_I^2/\Lambda^2}{\ln M_p^2/\Lambda^2} \right)^{2/7} = 2.5$$

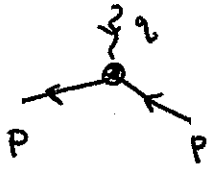
The proton decay rate is increased by the square of this factor

$$\times 6 .$$

2.) Consider the operator

$$O_f^{\mu_1 \cdots \mu_n}{}_m = [\bar{\Psi} \gamma^{\{\mu_1} (iD)^{\mu_2} \cdots (iD)^{\mu_n\}} \Psi]_m$$

at the scale $Q \sim M$. It has matrix element



$$= \gamma^{\{\mu_1} p^{\mu_2} \cdots p^{\mu_n\}}$$

for $q \to 0$ or $|q| \ll |p|$

the vertex obtains QCD correction from





$$= (ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i}{(k-p)^2} \gamma^\lambda t^a \frac{i\not{k}}{k^2} (\gamma^{\mu_1} k^{\mu_2} - k^{\mu_n}) \frac{i\not{k}}{k^2} \gamma_\lambda t^a$$

combine denominators using

$$\left(\frac{1}{k^2}\right)^2 \frac{1}{(k-p)^2} = \int_0^1 dx \frac{2(1-x)}{[\ell^2 + x(1-x)p^2]^3}$$

with $\ell = k - xp$ $\quad k = \ell + xp$

We need to find a divergent term with the structure $\gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}$.

To do this, set

$$\not{k} \to \not{\ell}$$

$$k^{\mu_i} \to x p^{\mu_i}$$

then

$$= -i g^2 \underbrace{(t^a t^a)}_{4/3} \int_0^1 dx \; 2(1-x) \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 + x(1-x)p^2]^3}$$

$$\cdot \gamma^\lambda \not{\ell} \gamma^{\mu_1} \not{\ell} \gamma_\lambda \cdot x^{n-1} p^{\mu_2} \cdots p^{\mu_n}$$

$$= -i g^2 \cdot \frac{4}{3} \int_0^1 dx \; 2(1-x) x^{n-1} \frac{i}{(4\pi)^{d_4}} \Gamma\left(2 - \frac{d_4}{2}\right) \cdot \frac{1}{2} \gamma^\lambda \gamma^\alpha \gamma^{\mu_1} \gamma^\alpha \gamma_\lambda$$

$$\cdot p^{\mu_2} \cdots p^{\mu_n}$$

Evaluate the residue at $\varepsilon=0$ i.e. $d=4$

$$\gamma^\lambda \gamma^\alpha \gamma^{\mu_1} \gamma_\alpha \gamma_\lambda = (-2)^2 \gamma^{\mu_1}$$

$$= \frac{4}{3} \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} \cdot \underbrace{\int_0^1 dx\, 2(1-x)\, x^{n-1}}_{\downarrow}$$

$$2 \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{2}{n(n+1)}$$



$$= \frac{8}{3} \frac{1}{n(n+1)} \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}$$

For the other diagram, use $iD^\mu = i\partial^\mu + g A^{\mu a} t^a$!



$$= \sum_{j=2}^{n} (ig) \cdot g \int \frac{d^d k}{(2\pi)^d} \gamma^\lambda t^a \frac{ik}{k^2} \gamma^{\mu_1} \cdot k^{\mu_2} \cdots k^{\mu_{j-1}} t^a g_\alpha^{\mu_j} p^{\mu_{j+1}} \cdots p^{\mu_n} \cdot \frac{-i}{(k-p)^2}$$

$$= \sum_{j=2}^{n} ig^2 \cdot (t^a t^a) \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \gamma^{\mu_j} x\not{p} \gamma^{\mu_1} \frac{1}{(k^2-\Delta)^2} \cdot (xp)^{\mu_2} \cdots (xp)^{\mu_{j-1}} p^{\mu_{j+1}} \cdots p^{\mu_n}$$

$$\underbrace{\qquad\qquad}_{(j-2 \text{ } x\text{'s.})}$$

now

$$\gamma^{\{\mu_j} \not{p} \gamma^{\mu_1\}} = \frac{1}{2}\left[\gamma^{\mu_j} \not{p} \gamma^{\mu_1} + (\mu_1 \leftrightarrow \mu_j)\right]$$

$$= \frac{1}{2}\left[2p^{\mu_j}\gamma^{\mu_1} - \gamma^{\mu_j}\gamma^{\mu_1} \not{p} + \mu_1 \leftrightarrow \mu_j\right]$$

$$= 2\gamma^{\{\mu_1} p^{\mu_j\}} - (\text{term} \propto g^{\mu_1\mu_j})$$

$$= \sum_{j=2}^{n} ig^2 \cdot \frac{4}{3} \int_0^1 dx \; \frac{i}{(4\pi)^{d_k}} \Gamma(2-d_k) \; x^{j-1} \; 2\gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}$$

$$= -\frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \cdot \sum_{j=2}^{n} \frac{2}{j} \cdot \gamma^{\{\mu_1} p^{\mu_2} \cdots p^{\mu_n\}}$$



$$= \sum_{j=2}^{n} (ig) \cdot g \cdot \int \frac{d^{d_k}k}{(2\pi)^d} \; \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_{j-1}} g^{\mu_j\lambda} t^a$$

$$\cdot k^{\mu_{j+1}} \cdots k^{\mu_n} \; \frac{i\not{k}}{k^2} \gamma_\lambda t^a \; \frac{-i}{(k-p)^2}$$

$$= \sum_{j=2}^{n} ig^2 \cdot \frac{4}{3} \int_0^1 dx \int \frac{d^{d_k}k}{(2\pi)^d} \frac{1}{[k^2-\Delta]^2}$$

$$\cdot \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_{j-1}} \; p^{\mu_{j+1}} \cdots p^{\mu_n} \; \not{p} \gamma^{\mu_j} \; x^{n-j+1}$$

$\ell = n-j+2$

$$= \sum_{\ell=n}^{2} ig^2 \cdot \frac{4}{3} \left(\frac{i}{(4\pi)^2} \frac{2}{\epsilon}\right) \int_0^1 dx \; x^{\ell-1}$$

$$\cdot \; 2\gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}$$

$$= (\text{same result as above})$$

so



$$= \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\cdot\left(-\frac{4}{3}\right)\cdot\left[\sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}\right]$$

$$\delta_{\mathcal{O}f}^{n} = \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\,\frac{4}{3}\sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}$$

$$\delta_2 = -\frac{4}{3}\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}$$

$$\gamma_f^m = M\frac{\partial}{\partial M}\left(-\delta_{\mathcal{O}f}^{n} + \delta_2\right)$$

$$= M\frac{\partial}{\partial M}\left(-\frac{g^2}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot\frac{4}{3}\left[\left(\sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}\right) + 1\right)\right.$$

$$M\frac{\partial}{\partial M}\left(\frac{2}{\epsilon}\right) = -2$$

$$\gamma_f^n = \frac{8}{3}\frac{g^2}{(4\pi)^2}\left[1 + \sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}\right]$$

3.) In the problem set, I asked you to work out the Altarelli-Parisi evolution results from

$$\frac{d}{d\ln Q} f(x;Q) = \frac{\alpha_s(Q)}{\pi} \int_x^1 \frac{dz}{z} P_{q\to q}(z) f\left(\frac{x}{z}\right)$$

This is the evolution of the non-singlet part of the parton distribution function. For example $(f_u - f_d)$ or $(f_u - f_{\bar{u}})$ obey this equation.

Integrate $\int_0^1 dx\, x^{n-1}$. Then

$$\frac{d}{d\ln Q} M_n = \frac{\alpha_s(Q)}{\pi} \int_0^1 dx\, x^{n-1} \int_x^1 \frac{dz}{z} P_{q\to q}(z) f\left(\frac{x}{z}\right)$$

Let $y = \frac{x}{z}$ $\quad \frac{dx}{z} = dy$ $\quad x = yz$

$$= \frac{\alpha_s(Q)}{\pi} \int\int dy\, dz\, (yz)^{n-1} P_{q\to q}(z) f(y)$$

The region of integration is:

$0 < x < 1$

$x < z < 1$ or $y < 1$

so the integral covers $0 < y < 1$

$0 < z < 1$

then

$$\frac{d}{d\ln Q} M_n = \frac{\alpha_s(Q)}{\pi} \left[\int_0^1 dz\; z^{n-1} P_{q\leftarrow q}(z)\right] \left[\int_0^1 dy\; y^{n-1} f(y)\right]$$

$$= \frac{\alpha_s(Q)}{\pi} \cdot P_n \cdot M_n$$

where

$$P_n = \int_0^1 dz\; z^{n-1} P_{q\leftarrow q}(z)$$

$$= \int_0^1 dz\; z^{n-1} \; \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right]$$

$$= \frac{4}{3}\cdot\left\{\frac{3}{2} + \int_0^1 dz\; \frac{z^{n-1}(1+z^2)-2}{(1-z)_+}\right\}$$

$$= \frac{4}{3}\cdot\left\{\frac{3}{2} + \int_0^1 dz\; \frac{(z^{n-1}-1)+(z^{n+1}-1)}{(1-z)}\right\}$$

$$= \frac{4}{3}\left\{\frac{3}{2} + \int_0^1 dz\; (-1)\left[z^{n-2}+z^{n-3}+\cdots+1 + z^n + z^{n-1}+z^{n-2}+\cdots+1\right]\right.$$

$$= \frac{4}{3}\left\{\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n} - 2\left(\frac{1}{n-1}+\cdots+1\right)\right\}$$

$$= \frac{4}{3}\left\{-\frac{1}{2} - 2\sum_2^n \frac{1}{j} + \frac{1}{n} - \frac{1}{n+1}\right\}$$

$$= -\frac{2}{3}\left\{1 + 4\sum_2^n \frac{1}{j} - \frac{2}{n(n+1)}\right\}$$

so

$$\frac{d}{d\ln Q} M_n = -\frac{\alpha_s(Q)}{\pi} \cdot \frac{2}{3} \cdot \left\{ 1 + 4 \sum_2^n \frac{1}{j} - \frac{2}{n(n+1)} \right\} M_n$$

$$= -\frac{8}{3} \frac{g^2}{(4\pi)^2} \left[1 + 4 \sum_2^n \frac{1}{j} - \frac{2}{n(n+1)} \right] M_n$$

or

$$\frac{d}{d\ln Q} M_n = -\gamma_f^n M_n$$

Physics 332 – Problem Set # 8

(due Wednesday, June 7)

1. Peskin and Schroeder, Problem 19.1.
2. Peskin and Schroeder, Problem 19.2.
3. Peskin and Schroeder, Problem 19.3.
4. Peskin and Schroeder, Problem 19.4.

Physics 332 - Problem Set #8

Solutions

1.) a) The Adler-Bell-Jackiw anomaly equation

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$$

can be rewritten as follows.

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L$$

$$= j^\mu_R - j^\mu_L$$

$$\varepsilon^{\alpha\beta\gamma\delta} = 4\cdot\epsilon^{ijk}F_{oi}F_{jk}$$

$$= 8\,F_{oi}\cdot\left(\tfrac{1}{2}\epsilon^{ijk}F_{jk}\right)$$

$$= 8\,E^i(-B^i)$$

If $N_R = \int d^3x\, j^0_R$ $\quad N_L = \int d^3x\, j^0_L$

$\int dt$ of the ABJ equation is

$$\Delta N_R - \Delta N_L = \int dt\,\frac{e^2}{2\pi^2}\,\vec{E}\cdot\vec{B}$$

now consider

$$A^\mu = (0, 0, Bx^1, At)$$

$$B^3 = \partial_1 A^2 = B \qquad E^3 = -\partial_0 A^3 = -\partial_0 A$$

so if A goes adiabatically from 0 to $\frac{2\pi}{eL}$, ABJ implies

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2} \cdot \frac{2\pi}{eL} \cdot L^3 \cdot B$$

$$= -\frac{eB}{\pi} L^2$$

b.) $$H = \int d^3x \; \psi^\dagger \gamma^0 (-i \vec{\gamma} \cdot \vec{D}) \psi$$

$$\gamma^0 \vec{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \text{ on } \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$H = \int d^3x \left(\psi_R^\dagger (-i\vec{\sigma} \cdot \vec{D}) \psi_R - \psi_L^\dagger (-i \vec{\sigma} \cdot \vec{D}) \psi_L \right)$$

$$(\vec{D})^i = D_i = (\partial_i - ieA^i)$$

c.) We now need to diagonalize $-i\vec{\sigma} \cdot \vec{D}$

Look for eigenvectors of the form

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i(k_2 x^2 + k_3 x^3)}$$

$$-i\vec{\sigma}\cdot\vec{D}\,\psi_R = [-i\sigma^1\partial_1 - i\sigma^2(ik_2 - ieBx') - i\sigma^3(ik_3 - ieA)]\psi_R$$

$$= [-i\sigma^1\partial_1 + \sigma^2(k_2 - eBx') + \sigma^3(k_3 - eA)]\psi_R$$

so that $-i\vec{\sigma}\cdot\vec{D}\,\psi_R = E\psi_R$

reads

$$-i\partial_1\phi_2 - i(k_2 - eBx')\phi_2 + (k_3 - eA)\phi_1 = E\phi_1$$

$$-i\partial_1\phi_1 + i(k_2 - eBx')\phi_1 - (k_3 - eA)\phi_2 = E\phi_2$$

so

$$-i[\partial_1 - (k_2 - eBx')]\phi_1 = [E + (k_3 - eA)]\phi_2$$

$$-i[\partial_1 + (k_2 - eBx')]\phi_2 = [E - (k_3 - eA)]\phi_1$$

$$-(\partial_1 + (k_2 - eBx'))(\partial_1 - (k_2 - eBx'))\phi_1$$

$$= [E^2 - (k_3 - eA)^2]\phi_1$$

$$\left[-\partial_1^2 + (eBx' - k_2)^2 + (k_3 - eA)^2 - \partial_1(eBx')\right]\phi_1 = E^2\phi_1$$

better:

$$\frac{1}{2}\left[-\partial_1^2 + (eB)^2\left(x' - \frac{k_2}{eB}\right)^2\right]\phi_1 = \left(eB + [E^2 - (k_3 - eA)^2]\right)\phi_1$$

for ϕ_2 the differential operators are in the other order, so

$$\left\{\left[-\tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}(eB)^2\left(x' - \frac{k_2}{eB}\right)^2\right] - \tfrac{1}{2}eB\right\}\phi_1 = \tfrac{1}{2}\left[E^2 - (k_3 - eA)^2\right]\phi_1$$

$$\left\{\left[-\tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}(eB)^2\left(x' - \frac{k_2}{eB}\right)^2\right] + \tfrac{1}{2}eB\right\}\phi_2 = \tfrac{1}{2}\left[E^2 - (k_3 - eA)^2\right]\phi_2$$

the operator

$$\mathcal{H} = \left[-\tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}(eB)\left(x' - \frac{k_2}{eB}\right)^2\right]$$

is a harmonic oscillator Hamiltonian with eigenvalues

$$\partial_n = (n + \tfrac{1}{2})(eB)$$

so ϕ_1 has $(\mathcal{H} - \tfrac{1}{2}eB)$ acting on it; this has a less eigenvalue

ϕ_2 has $(\mathcal{H} + \tfrac{1}{2}eB)$ acting on it; this has only positive eigenvalues

For $(\mathcal{H} - \tfrac{1}{2}eB) = eB\,n > 0$

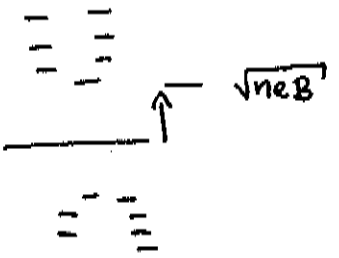
$$E = \pm\left[(k_3 - eA)^2 + n\,eB\right]^{\frac{1}{2}}$$

For $(\mathcal{H} - \tfrac{1}{2}eB) = 0$, $\phi_2 = 0$

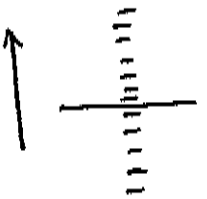
$$\Rightarrow \quad E = (k_3 - eA)$$

d.) If $k^3 = \frac{2\pi m}{L}$, then spectra look like

$n > 0$



$n = 0$



spacing = $\frac{2\pi}{L}$

each level is a Harmonic Oscillator centered at

$$x' = \frac{k_2}{eB}$$

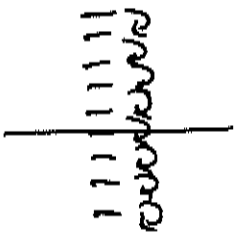
for $0 < x' < L$, $0 < k_2 < eBL$. If k_2 is

quantized: $k_2 = \frac{2\pi m}{L}$

there are $\frac{eBL^2}{2\pi}$ states in each level.

e.) Now increase A adiabatically from 0 to $\frac{2\pi}{eL}$

this causes the $n=0$ state to move down one notch:



$$\Delta N_R = -\frac{eBL^2}{2\pi}$$

For the left-handed spectrum,

$$E\psi_L = +i\vec{\sigma}\cdot\vec{D}\psi_L$$

for $n=0$ $\quad E = -(k_3 - eA)$

so $\quad \Delta N_L = +\frac{eBL^2}{2\pi}$

in all $\quad \Delta N_R - \Delta N_L = -\frac{eBL^2}{\pi}$

as predicted on p. 2.

2.) a) $\quad \bar{u}_L \gamma^\mu d_L = \bar{Q}\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left(\frac{1-\gamma^5}{2}\right) Q$

where $Q = \begin{pmatrix} u \\ d \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}\sigma^1 + i\tfrac{1}{2}\sigma^2 = \tau^1 + i\tau^2$$

so $\quad \bar{u}_L \gamma^\mu d_L = \frac{1}{2}\left(j^{\mu 1} + ij^{\mu 2} - j^{\mu 5 1} - ij^{\mu 5 2}\right)$

$|\pi^+\rangle$ is annihilated by $\bar{d}\gamma^\mu u_L = \frac{1}{2}(g^{\mu 1} - i g^{\mu 2} - g^{\mu 51} + i g^{\mu 52})$

so $|\pi^+\rangle = \frac{1}{\sqrt{2}}(|\pi^1\rangle + i|\pi^2\rangle)$

$$\langle 0|\bar{d}\gamma^\mu u_L|\pi^+\rangle = -ip^\mu f_\pi \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \cdot 2 \cdot (-1)$$

$$= +ip^\mu f_\pi \frac{1}{\sqrt{2}}$$

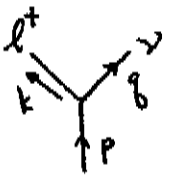
then the weak interaction effective Lagrangian

$$\Delta\mathcal{L} = \frac{4G_F}{\sqrt{2}} \left[\bar{\nu}\gamma^\mu \left(\frac{1}{2}(1-\gamma_5)\right) l\right] \left[\bar{d}\gamma_\mu \left(\frac{1}{2}(1-\gamma_5)\right) u\right]$$

gives

$$iM(\pi^+ \to l^+\nu) = i\frac{4G_F}{\sqrt{2}} \bar{u}\left(\frac{1}{2}(1-\gamma_5)\right)\gamma^\mu v \cdot \frac{1}{\sqrt{2}} f_\pi p_\mu$$

$$= -2G_F \bar{u}\left(\frac{1}{2}(1-\gamma_5)\right) \not{p} v \cdot f_\pi$$



for the ν: $\bar{u}\not{p} = 0$

for the l^+: $\not{p} v = -m_l v$

$$\bar{u} m_l \left(\frac{1}{2}(1-\gamma_5)\right) = \bar{u}\left(\frac{1}{2}(1-\gamma_5)\right)\not{p}$$

b.) $iM(\pi^+ \to l^+\nu) = 2G_F m_l \bar{u}(q)\left(\frac{1}{2}(1+\gamma_5)\right) v(k) \cdot f_\pi$

$$\sum_{spin} |\mathcal{M}|^2 = 4 G_F^2 m_\ell^2 f_\pi^2 \, \mathrm{tr}\, \not{q} \left(\frac{1+\gamma^5}{2}\right)(\not{k} - m_\ell)\left(\frac{1-\gamma^5}{2}\right)$$

$$= 4 G_F^2 m_\ell^2 f_\pi^2 \cdot 2\, q\cdot k$$

$$2q\cdot k = (q+k)^2 - k^2 = m_\pi^2 - m_\ell^2$$

$$= 4 G_F^2 f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2)$$

$\frac{2p}{\sqrt{s}}$

$$\Gamma(\pi^+ \to \ell^+ \nu) = \frac{1}{2m_\pi} \cdot \frac{1}{8\pi} \cdot 4 G_F^2 f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2) \cdot \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)$$

$$= \frac{1}{4\pi} G_F^2 f_\pi^2 m_\pi^3 \frac{m_\ell^2}{m_\pi^2}\left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

note that

$$\frac{\Gamma(\pi^+ \to \ell^+ \gamma e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} = \frac{m_e^2}{m_\mu^2} \frac{(1 - m_e^2/m_\pi^2)^2}{(1 - m_\mu^2/m_\pi^2)^2} = 1.27 \times 10^{-4}$$

indeed! $\quad BR(\pi^+ \to e^+ \nu) = 1.23 \times 10^{-4}$!

now

$$\Gamma(\pi^+ \to \mu^+ \nu) = \frac{1}{4\pi} G_F^2 m_\pi^3 f_\pi^2 \cdot \underbrace{\frac{m_\mu^2}{m_\pi^2} \cdot \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}_{0.104}$$

$$\Gamma(\pi^+ \to \mu^+ \nu) = \frac{1}{4\pi} \cdot (1.166 \times 10^{-5}\ \text{GeV}^{-2})^2 \cdot (0.14\ \text{GeV})^3$$

$$\cdot (0.1\ \text{GeV})^2 \cdot \left(\frac{f_\pi}{100\ \text{MeV}}\right)^2 \cdot 0.104$$

$$= 3.1 \times 10^{-17}\ \text{GeV} \cdot \left(\frac{f_\pi}{100\ \text{MeV}}\right)^2 \times (6.58 \times 10^{-25}\ \text{GeV sec})^{-1}$$

$$= \left[2.1 \times 10^{-8}\ \text{sec} \cdot \left(\frac{100\ \text{MeV}}{f_\pi}\right)^2\right]^{-1}$$

from $\tau_{\pi^+} = 2.6 \times 10^{-8}$ sec $\qquad f_\pi = 90$ MeV.

3.) The anomaly coefficient A of representation r of SU(N) is defined by

$$\text{tr}\left[t^a_r \{t^b_r, t^c_r\}\right] = \frac{1}{2} A(r)\, d^{abc}$$

so $\quad A(N) = 1$

a) In $r_1 \times r_2$ $\qquad t^a_{r_1 \times r_2} = t^a_{r_1} \otimes 1 + 1 \otimes t^a_{r_2}$

$$\mathrm{tr}\left[t^a_{r_1\times r_2}\left\{t^b_{r_1\times r_2},\ t^c_{r_1\times r_2}\right\}\right]$$

$$= \mathrm{tr}\left(t^a_{r_1}\otimes 1 + 1\otimes t^a_{r_2}\right)\left\{\left(t^b_{r_1}\otimes 1 + 1\otimes t^b_{r_2}\right),\left(t^c_{r_1}\otimes 1 + 1\otimes t^c_{r_2}\right)\right\}$$

$$= \mathrm{tr}_{r_1}\left[t^a_{r_1}\left\{t^b_{r_1}, t^c_{r_1}\right\}\right]\mathrm{tr}_{r_2}[1]$$

$$+ \mathrm{tr}_{r_1}[1]\cdot\mathrm{tr}\left[t^a_{r_2}\left\{t^b_{r_2}, t^c_{r_2}\right\}\right]$$

$$+ [\text{term} \propto \mathrm{tr}(t^a_{r_1})] + [\text{term} \propto \mathrm{tr}(t^a_{r_2})]$$

$$= \frac{1}{2}A(r_1)\,d^{abc}\cdot d_{r_2} + \frac{1}{2}d_{r_1}A(r_2)\,d^{abc}$$

$$+ 0 + 0$$

since $\mathrm{tr}(t^a) = 0$

on the other hand, if we write $r_1\times r_2 = \bigoplus r_j$ where r_j are irreducible,

$$= \sum_j \mathrm{tr}\left[t^a_{r_j}\left\{t^b_{r_j}, t^c_{r_j}\right\}\right] = \frac{1}{2}\sum_j A(r_j)\,d^{abc}$$

so

$$d_1 A(r_2) + d_2 A(r_1) = \sum_j A(r_j)$$

b.f) We can compute the anomaly for any SU(N) rep. by looking in an SU(3) subgroup. Under SU(3)

$$N \to \underset{\sim}{3} + (N-3)\,\underset{\sim}{1}$$

$$(N\times N)_a \to (3\times 3)_a + (N-3)\,\underset{\sim}{3} + \underline{1}\cdot 0$$

$$\underset{\sim}{\bar{3}} + (N-3)\,\underset{\sim}{3} + 1\cdot 0.$$

$$A(a) = A(\bar{3}) + (N-3)\,A(3)$$

$$= -1 + (N-3)\cdot 1 = N-4$$

check: in SU(4) $a = 6$ is real and so has $A(6) = 0$

then

$$N\times N = a + s$$

$$d(N)\cdot A(N)\cdot 2 = A(a) + A(s)$$

$$2N = (N-4) + A(s)$$

$$A(a) = N-4$$

$$A(s) = N+4$$

In SU(3) $s = 6$. A chiral spectrum with $\underset{\sim}{6} + 7\,\underset{\sim}{\bar{3}}$ is anomaly-free.

alternately

$$(N\times N)_S = (3\times 3)_S + (N-3)\,\underset{\sim}{3} + 1^{\prime\prime}$$

$$= \underset{\sim}{6} + (N-3)\,\underset{S}{3} + 1^{\prime\prime}$$

In SU(3) $\quad (3\times 3)_a = \bar{3} \qquad A(\bar{3}) = -1$

$$3\cdot \underbrace{A(3)}_{1}\cdot 2 = A(6) + \underbrace{A(\bar{3})}_{-1} \qquad \text{so} \qquad A(6) = 7$$

then

$$(N\times N)_S \Rightarrow \underset{S}{6} + (N-3)\,\underset{\sim}{3} + \cdots$$

has anomaly coefficient $\quad A(6) + (N-3)\,A(3) =$

$$7 + (N-3) = N+4$$

by the same method

$$(N\times N\times N)_a = (3\times 3\times 3)_a + (N-3)\,(3\times 3)_a + \frac{(N-3)(N-4)}{2}\,3 + 1^{\prime\prime\prime}$$

$$A\left((N\times N\times N)_a\right) = A(1) + (N-3)\,A(\bar{3}) + \frac{(N-3)(N-4)}{2}\,A(3)$$

$$= \frac{(N-3)(N-4)}{2} - (N-3)$$

$$= \frac{(N-3)(N-6)}{2}$$

$$A((N\times N\times N\times N)_a) = A\Big\{(N-3)\,(3\times 3\times 3)_a + \frac{(N-3)(N-4)}{2}(3\times 3)_a + \frac{(N-3)(N-4)(N-5)}{3!}(3)_a + \text{l.o.}\Big\}$$

$$= \frac{(N-3)(N-4)(N-5)}{3!} - \frac{(N-3)(N-4)}{2}$$

$$= \frac{(N-3)(N-4)(N-8)}{6}$$

similarly

$$A(\underbrace{N\times\cdots\times N}_{a})_a = \frac{(N-3)(N-4)\cdots(N-j-1)}{(j-1)!} - \frac{(N-3)\cdots(N-j)}{(j-2)!}$$

$$= \frac{(N-3)(N-4)\cdots(N-j)\cdot(N-2j)}{(j-1)!}$$

this must vanish for $N=3, N=4, \cdots N=j$, since

$$(\overbrace{N\times N\times\cdots\times N}^{j})_a$$

has at most singlets for $N\le j$

it must vanish for $N=2j$ because $\simeq SU(2j)$

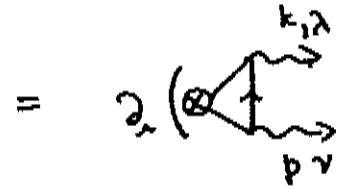
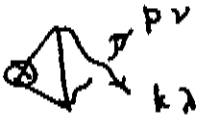
$$A^{\overbrace{ab\cdots k}^{j}} = \epsilon_{ab\cdots kl\cdots m}\,A_{l\cdots m}$$

is a real representation.

4) a.) The Adler-Bell-Jackiw anomaly corresponds to

$$\langle p,k| \partial_i j^{MS}_i |0\rangle = + \frac{e^2}{2\pi^2} \varepsilon^{\alpha\nu\beta\lambda} p_\alpha \varepsilon^*_\nu(p) k_\beta \varepsilon^*_\lambda(k)$$

If we regularize the current matrix elem by Pauli Villars regularization:

$\langle p,k| \partial_i j^{MS}_i |0\rangle =$ ∂_μ ( + 



e
heavy fermion of mass $M \to \infty$

the divergence goes 0 on the bp diagram, but in the bottom line we should we

$$\partial_i j^{MS}_M = \partial_\mu (\bar\Psi \gamma^\mu \gamma^5 \Psi) = 2i M \bar\Psi \gamma^5 \Psi$$

then we should have

$$\lim_{M\to\infty} \langle p,k| 2iM \Psi\gamma^5\Psi(0) |0\rangle = - \frac{e^2}{2\pi^2} \varepsilon^{\alpha\nu\beta\lambda} p_\alpha \varepsilon^*_\nu k_\beta \varepsilon^*_\lambda$$

b.) I did not discuss the trace anomaly in class, but it is discussed in Sect. 19.5 of Peskin & Schroeder.

In QED,

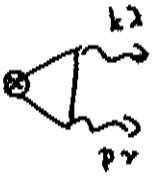
$$\Theta^{\mu}{}_{\mu} = \frac{\beta(e)}{2e^3}(eF_{\lambda\sigma})^2 = \frac{e^2}{24\pi^2}(F_{\lambda\sigma})^2$$

this corresponds to the matrix element

$$\langle p,k|\,\Theta^{\mu}{}_{\mu}\,|0\rangle = -\frac{e^2}{6\pi^2}\left[p\cdot k\;\varepsilon^{*}_{\lambda}(p)\varepsilon^{\nu *}(k) - p\cdot\varepsilon^{*}(k)\,k\cdot\varepsilon^{*}(p)\right]$$

If we define $\Theta^{\mu}{}_{\mu}$ by Pauli-Villars regularization,

$\langle p,k|\,\Theta^{\mu}{}_{\mu}\,|0\rangle =$ (



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)

↑ keeping terms of mass $M \to \infty$

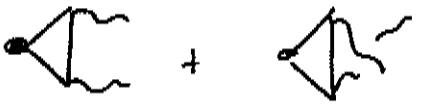
In the bottom line, we can use

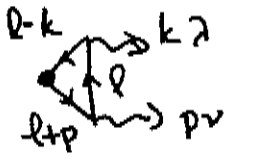
$$(\Theta^{\mu}{}_{\mu})_M = M\,\bar{\Psi}\Psi$$

so we should have

$$\lim_{M\to\infty}\langle p,k|\,M\,\bar{\Psi}\Psi(0)\,|0\rangle = +\frac{e^2}{6\pi^2}\left[p\cdot k\;\varepsilon^{*}(p)\cdot\varepsilon^{*}(k) - p\cdot\varepsilon^{*}(k)\;k\cdot\varepsilon^{*}(p)\right]$$

c.) Now check the prediction explicitly:

$\langle p,k|\, 2iM\,\Phi\gamma^5\psi(0)\,|0\rangle$ = 



$$= (-1)(-ie)^2 \int \frac{d^4l}{(2\pi)^4}\, \mathrm{tr}\Big\{(2iM\gamma^5)\, i\frac{(\not{l}-\not{k}+M)}{(l-k)^2-M^2}\,\gamma^\lambda\, i\frac{(\not{l}+M)}{l^2-M^2}\,\gamma^\nu\, i\frac{(\not{l}+\not{p})+M}{(l+p)^2-M^2}\Big\}$$

Combine denominators:

$$\text{Den} = x\,((l-k)^2-M^2) + y\,((l+p)^2-M^2) + z\,(l^2-M^2)$$

$$= l^2 + 2l\cdot(-xk+yp) + xk^2 + yp^2 - M^2$$

$$= \Pi^2 + xk^2 + yp^2 - x^2k^2 - y^2p^2 + 2k\cdot p\,xy - M^2$$

$$= \Pi^2 + xy\cdot(k+p)^2 + xz\,k^2 + yz\,p^2 - M^2$$

$$= \Pi^2 - \Delta$$

where $\Delta = M^2 - xy\,(k+p)^2 - xz\,k^2 - yz\,p^2$

$\Pi = l - xk + yp$

so $l = \Pi + xk - yp$

$l-k = \Pi - (1-x)k - yp$

$l+p = \Pi + xk + (1-y)p$

to get a non zero trace, we need

$$\mathrm{tr}\ \gamma^5\ \gamma^\lambda\ \gamma^\nu \cdot (2(k')_\rho + 1M)$$

if both k's are $\not{k}$, we get 0, so both must be finite momenta. Then the integral is finite and equals

$$= -ie^2 \int dx\,dy\,dz\ \delta(x+y+z-1)\cdot 2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2-\Delta]^3} \cdot (2iM)$$

$$\cdot \mathrm{tr}\{\gamma^5\ M\ \gamma^\rho\ (\not{p}+y\not{q}+x\not{k})\ \gamma_\rho\ (x\not{k}+(1-x)\not{q})$$
$$(\not{q}(x-1)+\not{k}x)\ \gamma_\nu\ M\ \gamma^\nu\ (x\not{k}+(1-x)\not{q})$$
$$+\gamma^5(-(1-x)\not{k}-y\not{q})\ \gamma^\nu\ M\ \gamma_\nu\ (\not{p}+y\not{q}+x\not{k})$$
$$+\gamma^5\ (-(1-x)\not{k}-\not{p})\ \gamma^\nu\ (\not{p}+y\not{q}+x\not{k})\ \gamma_\nu\ M\}$$

$$= +2Me^2 \int dx\,dy\,dz\ \delta(x+y+z-1)\cdot 2\int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2-\Delta]^3}$$

$$\cdot(-4iM)\Big[\varepsilon^{\alpha\nu\beta\rho}\ k_\alpha p_\beta\ [(+x)(1-y)+xy]$$
$$+\varepsilon^{\alpha\rho\nu\beta}\ k_\alpha p_\beta\ [-(1-x)(1-y)+xy]$$
$$+\varepsilon^{\alpha\beta\rho\nu}\ k_\alpha p_\beta\ [xy+x(x-1)+xy]\Big]$$

$$= -8iM^2e^2 \int dx\,dy\,dz\ \delta(x+y+z-1)\cdot 2\int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2-\Delta]^3}$$

$$(\varepsilon^{\alpha\nu\beta\rho}\ p_\alpha k_\beta)\{+x(1-y)+xy+(1-x)(1-y)-xy$$
$$+xy+x(x-1)+(1-x)y+xy\}$$

$$\{\} = x - \cancel{xz} + \cancel{xz} + 1 - x - y + \cancel{xz} - \cancel{xz}$$
$$+ y - \cancel{xy} + \cancel{xy} = 1$$

so

$$= -8i M^2 e^2 \int dx dy dz\, \delta(x+y+z-1) \cdot \left[\frac{-i}{(4\pi)^2} \frac{1}{\Delta}\right] \varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta$$

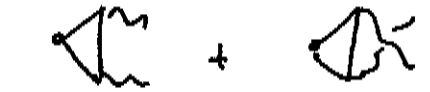
$$\lim_{M\to\infty} \frac{M^2}{\Delta} = 1 \quad \text{then} \quad \int dx dy dz\, \delta() \cdot 1 = \frac{1}{2}$$

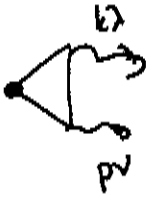
$$= -\frac{1}{4\pi^2} e^2 \varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta$$

adding the second diagram with $p \leftrightarrow k$ $\alpha \leftrightarrow \lambda$, we have

$$\langle \beta k | \, 2i M \bar{\Psi} \gamma^5 \Psi \, | 0 \rangle = -\frac{e^2}{2\pi^2} \varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta$$

d.) similarly

$\langle p,k | \, M \bar{\Psi} \Psi \, | 0 \rangle =$  +



$$= (-1)(-ie)^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{tr}\Big\{ M \; \frac{i(\cancel{l}\cancel{k} + M)}{(l-k)^2 - M^2} \gamma^\lambda$$

$$\frac{i}{l^2 - M^2} \frac{\cancel{l} + M}{} \gamma^\nu \; \frac{i}{l^2 - M^2} \frac{(\cancel{l}\cancel{p}) + M}{} \Big\}$$

$$= -ie^2 M \int \frac{d^4 l}{(2\pi)^4} \, tr \frac{[(\not{l}-\not{k})+M]\gamma^\lambda (\not{l}+M)\gamma^\nu (\not{l}+\not{p}+M)}{((l-k)^2-M^2)(l^2-M^2)((l+p)^2-M^2)}$$

$$= -ie^2 M \int dx dy dz \, \delta(x+y+z-1) \cdot 2 \int \frac{d^4 \Pi}{(2\pi)^4} \frac{1}{[\Pi^2-\Delta]^3}$$

$$tr\{ [(\not{l}+\not{k})+M]\gamma^\lambda (\not{l}+M)\gamma^\nu (\not{l}+\not{p}+M) \}$$

To obtain a term with $p^\lambda k^\nu$, we need to keep 2 powers of finite momenta and one M in the trace. This gives a finite integral. By gauge invariance, the full structure should be

$$t^{\lambda\nu} = (p\cdot k\, g^{\lambda\nu} - p^\lambda k^\nu)$$

since this automatically satisfies $k_\lambda t^{\lambda\nu} = p_\nu t^{\lambda\nu} = 0$

Better, let's evaluate the full integral by dimensional regularization.

$$tr\{\} = tr\{ M\gamma^\lambda \not{k}\gamma^\nu \not{k} + \not{k}\gamma^\lambda M\gamma^\nu \not{k} + \not{k}\gamma^\lambda \not{k}\gamma^\nu M + M^3\gamma^\lambda\gamma^\nu$$
$$+ (-(1-x)\not{k}-y\not{p})\gamma^\lambda (x\not{k}-y\not{p})\gamma^\nu M$$
$$+ (-(1-x)\not{k}-y\not{p})\gamma^\lambda M\gamma^\nu (x\not{k}+(1-y)\not{p})$$
$$+ M\gamma^\lambda (x\not{k}-y\not{p})\gamma^\nu (x\not{k}+(1-y)\not{p}) \}$$

$$= 4M \{ (\Box^2 \Box^\nu + \Box^\nu \Box^2 - g^{\nu\lambda} \Box^2) + \Box^2 g^{\nu\lambda}$$

$$+ (\Box^2 \Box^\nu + \Box^\nu \Box^\lambda - g^{\nu\lambda} \Box^2) + M^2 g^{\nu\lambda}$$

$$+ (2k^2 k^\nu - k^2 g^{\nu\lambda})(-x(1-x) + x^2) - x(1-x) k^2 g^{\nu\lambda}$$

$$+ (2p^2 p^\nu - p^2 g^{\nu\lambda})(y^2 - y(1-y)) - y(1-y) p^2 g^{\nu\lambda}$$

$$+ [k^2 p^\nu + k^\nu p^2 - g^{\nu\lambda} k\cdot p] (y(1-x) - xy + x(1-y) - xy)$$

$$+ (k^2 p^\nu + g^{\mu\nu} k\cdot p - k^\nu p^2)[-(1-x)(1-y)]$$

$$+ (p^2 k^\nu + g^{\mu\nu} k\cdot p - p^\nu k^\lambda)(-xy) \}$$

$$= 4M \{ (4\Box^2 \Box^\nu - g^{\nu\lambda} \Box^2) + M^2 g^{\nu\lambda}$$

$$+ k^2 k^\nu (-2x + 4x^2) - k^2 g^{\nu\lambda} x^2$$

$$+ p^2 p^\nu (-2y + 4y^2) - p^2 g^{\nu\lambda} y^2$$

$$+ k^2 p^\nu (y - 2xy + x - 2xy - 1 + x + y - xy + xy)$$

$$+ k^\nu p^2 (y - 2xy + x - 2xy + 1 - x - y + xy - xy)$$

$$+ g^{\mu\nu} k\cdot p (-y + 2xy - x + 2xy - 1 + x + y - xy - xy) \}$$

$$= 4M \{ (4\Box^2 \Box^\nu - g^{\nu\lambda} \Box^2) + M^2 g^{\nu\lambda}$$

$$+ g^{\mu\nu} (-x^2 k^2 - y^2 p^2 - (1 - 2xy) k\cdot p)$$

$$+ k^2 p^\nu (-1 + 2x + 2y - 4xy) + k^\nu p^2 (1 - 4xy) \}$$

$$2\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\Delta]^3} \; tr\{\}$$

$$= \frac{-i}{(4\pi)^{d/2}} \frac{1}{\Delta^{3-d/2}} \cdot 4M\Big\{ - \Gamma(2-d/2)\Big(4\,\underbrace{\tfrac{1}{2} g^{\nu\lambda}}_{2-\frac{d}{2}\ g^{\nu\lambda}} - \tfrac{d}{2} g^{\nu\lambda}\Big)\Delta$$

$$+ \Gamma(3-d/2) M^2 g^{\nu\lambda} + \Gamma(3-d/2)\Big[g^{\nu\lambda}(-x^2k^2 - y^2p^2 - (1-2xy)\,k\cdot p$$

$$+ k^\lambda p^\nu (-1 + 2x + 2y - 4xy) + k^\nu p^\lambda (1-4xy)\Big]\Big\}$$

$$= \frac{-i\,4M}{(4\pi)^{d/2}} \frac{\Gamma(3-d/2)}{\Delta^{3-d/2}} \Big\{ - g^{\nu\lambda}\cancel{(M^2} - (x-x^2)k^2 - (y-y^2)p^2 - 2xy\,k\cdot p)$$

$$\cancel{+ g^{\nu\lambda} M^2}$$

$$+ g^{\nu\lambda}(-x^2k^2 - y^2p^2 - (1-2xy)\,k\cdot p)$$

$$+ k^\lambda p^\nu(-(1-2x)(1-2y)) + k^\nu p^\lambda (1-4xy)\Big\}$$

take $d \to 4$; this is finite. take $M \to \infty$

$$= -4i \frac{1}{(4\pi)^2} \frac{1}{M} \Big\{ g^{\nu\lambda}\big((x-2x^2)k^2 + (y-2y^2)p^2$$

$$+ [-(1-4xy)\,k\cdot p]\big)$$

$$+ k^\lambda p^\nu(-(1-2x)(1-2y)) + k^\nu p^\lambda (1-4xy)\Big\}$$

now

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot 1 = \tfrac{1}{2}$$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot(x,y,z) = \tfrac{1}{2}\cdot\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)$$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot x^2 = \int_0^1 dx\; x^2(1-x) = \frac{1}{12}$$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\;xy = \int_0^1 dx\; x \int_0^{1-x} dy\; y$$

$$= \int_0^1 dx\; x\,\frac{(1-x)^2}{2} = \frac{1}{24}$$

so $\lim_{M\to\infty}$ $\lim_{d\to 4}$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot 2\int \frac{d^dl}{(2\pi)^d}\,\frac{1}{[l^2-\Delta]^3}\,N\{\}$$

$$= -4i\,\frac{1}{(4\pi)^2}\cdot\frac{1}{M}\cdot\Big\{ g^{\nu\lambda}\Big[\Big(\overbrace{\tfrac{1}{6}-\tfrac{2}{12}}^{0}\Big)k^2+\Big(\overbrace{\tfrac{1}{6}-\tfrac{2}{12}}^{0}\Big)p^2 - \left(\tfrac{1}{2}-4\,\tfrac{1}{24}\right)k\cdot p\Big]$$

$$+ k^2p^\nu\left(-\tfrac{1}{2}+2\cdot\tfrac{1}{6}+2\cdot\tfrac{1}{6}-\tfrac{4}{24}\right) \leftarrow 0$$

$$+ k^\nu p^\lambda\left(\tfrac{1}{2}-\tfrac{4}{24}\right)\Big\}$$

$$= -4i\,\frac{1}{(4\pi)^2}\,\frac{1}{M}\left(-\tfrac{1}{3}\right)\left(g^{\nu\lambda}k\cdot p - p^\lambda k^\nu\right)$$

so finally



$$= -ie^2 M \left(-\frac{4i}{(4\pi)^2 M}\right)\left(-\frac{1}{3}\right)\left(g^{\nu\lambda} k\cdot p - p^\lambda k^\nu\right)$$

$$= + \frac{e^2}{12\pi^2}\left(g^{\nu\lambda} k\cdot p - p^\lambda k^\nu\right)$$

adding the second diagram with $p \leftrightarrow k$ $\nu \leftrightarrow \lambda$, we have

$$\langle p,k|\ M_{\mathrm{TH}}|0\rangle = + \frac{e^2}{6\pi^2}\left(p\cdot k\, g^{\nu\lambda} - p^\lambda k^\nu\right)$$

as required !

Physics 332 – Final Exam

This exam is due at noon on Wednesday, June 8. Please hand it in to Alex Giryavets in Varian 361.

If you have any questions about the exam, please contact me at mpeskin@slac.stanford.edu or 926-3250. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Alex Giryavets has blue books, if you need one.

The exam is worth a total of 100 points. The distribution of points is indicated below.

I recommend that you try simply to answer the questions using methods from Physics 332. However, the exam is open-book. If you find a useful reference other than the class textbook and notes, feel free to use it, as long as you cite the reference in your solution.

The exam concerns various aspects of quark masses in the Standard Model.

1. (20 points) Consider the heavy top quark. For $Q \gg m_t$, the running QCD coupling constant has the form
$$\alpha_s(Q) = \frac{2\pi}{b_0(6)\log(Q/\Lambda_6)} \tag{1}$$
where
$$b_0(n_f) = 11 - \frac{2}{3}n_f \tag{2}$$
For $Q \ll m_t$, the running coupling constant has the form
$$\alpha_s(Q) = \frac{2\pi}{b_0(5)\log(Q/\Lambda_5)} \tag{3}$$
Find Λ_5 in terms of Λ_6 and m_t. You will need to use the value of the top quark vacuum polarization diagram to get the numerical coefficient. Using the same method, compute Λ_4 for α_s below the b quark threshold.

2. (20 points) Compute the one-loop QCD correction to the top quark mass. Regularize the diagram using $\overline{MS}$ subtraction at the scale $m_t(\overline{MS})$. Compute the position of the pole in the top quark propagator, *i.e.*, the kinematic mass of the top quark. In this way, find the relation between the top quark "pole" mass and the $\overline{MS}$ mass. The pole mass is measured to be 175 GeV. What is the numerical value of the $\overline{MS}$ mass, assuming $\alpha_s = 0.11$ at m_t?

3. (20 points) The $\overline{MS}$ masses of the lighter quarks can be found from QCD sum rules and the values of the pseudoscalar meson masses. The results are: For u, d, s, $m_q(2\text{ GeV}) = 3$ MeV, 5 MeV, 100 MeV. For c, $m_c(m_c) = 1.2$ GeV. For b, $m_b(m_b) = 4.2$ GeV. Express all six quark masses at a common scale $Q = m_t(m_t)$. What are the true ratios of the quark masses?

4. (20 points) Compute the β function for the t quark Yukawa coupling. Use the interaction

$$\Delta\mathcal{L} = -\lambda_t \epsilon^{ab} \overline{Q}_{La} \phi_b^\dagger t_R + h.c. \,, \tag{4}$$

where $Q_L = (t_L, b_L)$. This coupling is invariant to all of the Standard Model gauge symmetries. Show that

$$\beta_{\lambda_t} = \frac{\lambda_t}{(4\pi)^2} \left[\frac{9}{2}\lambda_t^2 - 8g_s^2 - \frac{9}{4}g^2 - \frac{17}{12}g'^2 \right] \tag{5}$$

If λ_t has a very large value at a high scale (*e.g.*, at $Q = 10^{16}$ GeV), what value of m_t would be observed in experiments?

5. (20 points) Because of the answer to question 1, the value of the proton mass depends on the value of the top quark mass. Compute

$$\frac{\partial}{\partial m_t} m_p \tag{6}$$

holding fixed the parameters of QCD at very high Q. The Higgs boson coupling to the proton should be given by

$$\frac{\lambda_{hpp}}{\sqrt{2}} = \frac{\partial}{\partial v} m_p \tag{7}$$

where $m_q = \lambda_q v/\sqrt{2}$. Compute λ_{hpp}, including contributions from t, b, c. Compare these contributions to the direct contributions from the mass of the valence quarks u and d.

Physics 332 – Trial Exam

Solutions

1.) If we only consider the top quark effect on the running of α_s, we have, as in QED

$$g^{-2}(Q) = g^{-2}(M)\left(1 - [\Pi(Q) - \Pi(M)]\right)$$

where $\Pi(Q) - \Pi(M) = -\frac{2\alpha_s}{\pi} \cdot \frac{1}{2} \cdot \int_0^1 dx\, x(1-x) \ln\left(\frac{m_t^2 - x(1-x)M^2}{m_t^2 - x(1-x)Q^2}\right)$

$\uparrow_{C(K)}$

for $Q^2 \gg m_t^2$

$$\Pi(Q) - \Pi(M) = +\frac{\alpha_s}{\pi} \cdot \frac{1}{6} \ln Q^2/M^2$$

$$\alpha_s^{-1}(Q) - \alpha_s^{-1}(M) = \frac{1}{6\pi} \ln Q^2/M^2$$

with all 6 quarks of the sheet, we have $Q^2 \gg m_t^2$

$$\alpha_s^{-1}(Q) = \alpha_s^{-1}(M) + \frac{1}{4\pi} b_{0,6} \ln Q^2/M^2$$

$$b_{0,n_f} = \left(11 - \frac{2}{3} \cdot n_f\right)$$

Thus the formula for $\alpha_s^{-1}(Q)$, with full dependence on the

top quark mass, should be

$$\alpha_s^{-1}(Q) = \alpha_s^{-1}(M) + \frac{b_{05}}{4\pi} \ln Q^2/M^2 + \frac{1}{\pi} \int_0^1 dx\, x(1-x) \ln\left(\frac{m_t^2 - x(1-x)M^2}{m_t^2 - x(1-x)Q^2}\right)$$

work in NL $\quad Q^2 < 0 \qquad Q^2 = -|Q^2|$

for $|Q^2| \gg m_t^2$

$$\int dx\, x(1-x) \ln(m_t^2 + x(1-x)|Q^2|)$$

$$= \int dx\, x(1-x) \left[\ln|Q^2| + \ln x(1-x) + O(m_t^2/Q^2)\right]$$

$$= \frac{1}{6}\ln|Q^2| - \frac{5}{18} + O(m_t^2/Q^2)$$

from $\quad \int_0^1 dx\, x(1-x) \ln x(1-x) = 2\int dx\, x(1-x) \ln x$

$$= 2\int_0^1 dx\, x \ln x - 2\int_0^1 dx\, x^2 \ln x$$

$$= 2\left(-\frac{1}{4} + \frac{1}{9}\right)$$

$$= \frac{1}{6}\left(\ln|Q^2| - \frac{5}{3}\right)$$

for $|Q^2| \ll m_t^2$

$$\int dx\, x(1-x) \ln(m_t^2 + x(1-x)|Q^2|) = \int dx\, x(1-x)\left[\ln m_t^2 + O\left(\frac{Q^2}{m_t^2}\right)\right]$$

$$= \frac{1}{6}\ln m_t^2 + \cdots$$

So

$$\alpha_s^{-1}(Q) = \begin{cases} C + \frac{b_{05}}{4\pi} \ln|Q^2| - \frac{1}{6\pi}\left(\ln Q^2 - \frac{5}{3}\right) & |Q| \gg m_t^2 \\ C + \frac{b_{05}}{4\pi} \ln|Q^2| - \frac{1}{6\pi} \ln m_t^2 & |Q^2| \ll m_t^2 \end{cases}$$

for $|Q|^2 \gg m_t^2$

$$\alpha_s^{-1}(Q) = \frac{b_{06}}{4\pi} \ln|Q^2| + \left(C + \frac{5}{18\pi}\right)$$

$$\equiv \frac{b_{06}}{4\pi} \ln|Q^2|/\Lambda_6^2$$

~~so that~~ $\ln \Lambda_6^2 = -\frac{4\pi}{b_{06}}\left(C + \frac{5}{18\pi}\right)$

or $C + \frac{5}{18\pi} = -\frac{b_{06}}{4\pi} \ln \Lambda_6^2$

then for $|Q^2| \ll m_t^2$

$$\alpha_s^{-1}(Q) = \frac{b_{05}}{4\pi} \ln|Q^2| - \frac{b_{06}}{4\pi} \ln \Lambda_6^2 - \frac{5}{18\pi} - \frac{1}{6\pi} \ln m_t^2$$

$$\equiv \frac{b_{05}}{4\pi} \ln |Q^2|/\Lambda_5^2$$

where

$$\frac{b_{05}}{4\pi} \ln \Lambda_5^2 = \frac{b_{06}}{4\pi} \ln \Lambda_6^2 + \frac{1}{6\pi} \ln m_t^2 + \frac{5}{18\pi}$$

$$\lg \Lambda_5 = \frac{b_{06}}{b_{05}} \lg \Lambda_6 + \frac{2/3}{b_{05}} \lg m_t + \frac{5/9}{b_{05}}$$

$$\Lambda_5 = (\Lambda_6)^{b_{06}/b_{05}} (m_t)^{(2/3)/b_{05}} e^{\frac{5}{9 b_{05}}}$$

Similarly

$$\Lambda_4 = (\Lambda_5)^{b_{05}/b_{04}} (m_b)^{2/3/b_{04}} e^{\frac{5}{9 b_{04}}}$$

$$\Lambda_3 = (\Lambda_4)^{b_{04}/b_{03}} (m_c)^{2/3/b_{03}} e^{\frac{5}{9 b_{03}}}$$

now

$$b_{06} = \frac{21}{3} \qquad b_{05} = \frac{23}{3} \qquad b_{04} = \frac{25}{3} \qquad b_{03} = \frac{27}{3}$$

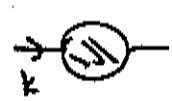
$$\Lambda_5 = (\Lambda_6)^{21/23} (m_t)^{2/23} e^{5/69}$$

$$\Lambda_4 = (\Lambda_6)^{21/25} (m_t)^{2/25} (m_b)^{2/25} e^{10/75}$$

$$\Lambda_3 = (\Lambda_6)^{21/27} (m_t m_b m_c)^{2/27} e^{15/81}$$

we'll need later

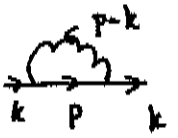
$$\Lambda_4 = (\Lambda_5)^{23/25} (m_b)^{2/25} e^{5/75}$$

2.) Let  $= -i\Sigma(\not k)$

then the pole $\simeq$ the top quark propagator is located at

$$\not k - m_t - \Sigma(\not k) = 0 \qquad \not k = m_{t,\text{pole}}$$

$$m_{t\,\text{pole}} = m_t + \Sigma(m_{t\,\text{pole}})$$
$$\cong m_t + \Sigma(m_t) \qquad \text{if } \Sigma(m_t) \text{ is } \mathcal{O}(\alpha_s)$$

$\Sigma(\not k) =$ i 

$$= i\,(ig)^2\, t^a t^a \int \frac{d^dp}{(2\pi)^d} \frac{-i}{(p-k)^2} \gamma_\mu \frac{i(\not p + m_t)}{p^2 - m_t^2} \gamma^\mu$$

$$= -ig^2 \cdot \frac{4}{3} \cdot \int_0^1 dx \int \frac{d^dP}{(2\pi)^d} \frac{\gamma_\mu(\not p + m_t)\gamma^\mu}{[P^2 + x(1-x)k^2 - (1-x)m_t^2]^2}$$

where $P = p - xk \qquad p = P + xk$

$$\gamma^\mu(\not p + m_t)\gamma_\mu = (2-d)\not p + d\, m_t$$
$$= (-2+\varepsilon)\not p + (4-\varepsilon) m_t$$
$$= (4m_t - 2\not p) + \varepsilon(-m_t + \not p)$$
$$= (\text{Irrelev} \simeq P) + (4m_t - 2x\not k) + \varepsilon(-m_t + x\not k)$$

$$= -ig^2 \cdot \frac{4}{3} \int_0^1 dx \; \frac{i}{(4\pi)^{d_2}} \Gamma\left(2 - \tfrac{d}{2}\right) \frac{1}{[(1-x)m_t^2 - x(1-x)k^2]^{2-\frac{d}{2}}}$$

$$\cdot \{(4m_t - 2x\not k) - \varepsilon(m_t - x\not k)\}$$

$$\Sigma(\not k) = \frac{4}{3} g^2 \int_0^1 dx \; \frac{1}{(4\pi)^2} \left[\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi \right) - \log\left[(1-x) m_t^2 - x(1-x) k^2 \right] \right]$$

$$\cdot \left[(4 m_t - 2x \not k) - \varepsilon (m_t - x \not k) \right]$$

apply the $\overline{MS}$ subtraction and let $\varepsilon \to 0$

$$\Sigma(\not k)\Big|_{\overline{MS}, M} = \frac{4}{3} \frac{g^2}{(4\pi)^2} \int_0^1 dx \left\{ \log \left[\frac{M^2}{(1-x) m_t^2 - x(1-x) k^2} \right] (4 m_t - 2x \not k) \right.$$

$$\left. - 2 (m_t - x \not k) \right\}$$

now set $M = m_t \qquad k^2 = m_t^2 \qquad \not k = m_t$

$$\Sigma(m_t)\Big|_{\overline{MS} \text{ at } m_t} = \frac{4}{3} \frac{g^2}{(4\pi)^2} \int_0^1 dx \left\{ \log \left[\frac{1}{(1-x)^2} \right] (4 m_t - 2x m_t) \right.$$

$$\left. - 2 (m_t - x m_t) \right\}$$

$$= \frac{4}{3} \frac{\alpha_s}{4\pi} m_t \left\{ \int_0^1 dx \; [-2 \log(1-x)] [4 - 2x] - 2 \left(1 - \tfrac{1}{2}\right) \right\}$$

$$\int dx \, \ln(1-x)(4-2x) = \int_0^1 dy \, \ln y \, (2 + 2y)$$

$$= 2 \left(-1 - \frac{1}{4} \right) = -\frac{5}{2}$$

$$= \frac{4}{3} \frac{\alpha_s}{4\pi} m_t \cdot \left[(-2)\left(-\tfrac{5}{2}\right) - 2 \left(\tfrac{1}{2}\right) \right]$$

$$= \frac{4}{3} \frac{\alpha_s}{\pi} m_t$$

so if m_t is defined by $\overline{MS}$ subtraction at $M = m_t$, the pole in the t quark propagator is located at

$$m_{t,pole} = m_t \left(1 + \frac{4\alpha_s}{3\pi}\right)$$

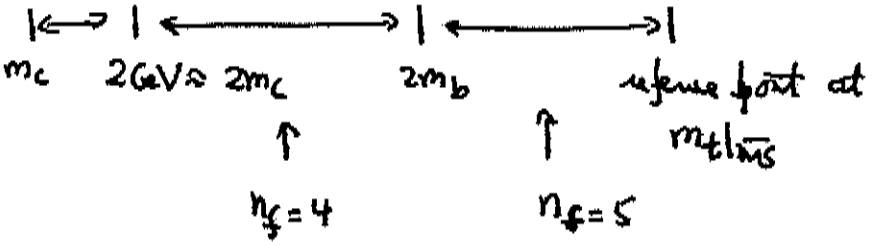
with $m_{t,pole} = 175.\ GeV$ $\quad \alpha_s(m_t) = 0.11$ $\quad \frac{4\alpha_s}{3\pi} = 4.7\%$

$$m_t(m_t)_{\overline{MS}} = 167.\ GeV$$

3.) To scale quark masses from one value of Q to another, use

$$\frac{\overline{m}(Q)}{\overline{m}(M)} = \left[\frac{\log(M/\Lambda)}{\log(Q/\Lambda)}\right]^{4/b_0}$$

Evaluate this formula for the transitions between quark thresholds.



First, we need the Λ's and b_0's.

$n_f = 5 \qquad b_0 = \frac{23}{3}$

$$\alpha_s(m_t) = 0.11 = \frac{2\pi}{b_0 \ln m_t/\Lambda_5} \Rightarrow \Lambda_5 = 97\ \text{MeV}$$

from p. 4 $\qquad n_f = 4 \rightarrow b_0 = \frac{25}{3}$

$$\Lambda_4 = 140\ \text{MeV}$$

then

$$\left(\frac{\log[2m_b/\Lambda_4]}{\log[2\text{GeV}/\Lambda_4]}\right)^{4\cdot\frac{2}{25}} = 1.2$$

$$\left(\frac{\log[m_t/\Lambda_5]}{\log[2m_b/\Lambda_5]}\right)^{4\cdot\frac{3}{23}} = 1.3$$

$$\left[\frac{\log(2m_b/\Lambda_4)}{\log(m_c/\Lambda_4)}\right]^{4\cdot\frac{3}{25}} = 1.4$$

so

$m_u(m_t) = (3\ \text{MeV})/(1.2)\cdot(1.3) = 2\ \text{MeV}$

$m_d(m_t) = (5\ \text{MeV})/(1.2)\cdot(1.3) = 3\ \text{MeV}$

$m_s(m_t) = (100\ \text{MeV})/(1.2)(1.3) = 62\ \text{MeV}$

$m_c(m_t) = (1.2\ \text{GeV})/(1.4)(1.3) = 670\ \text{MeV}$

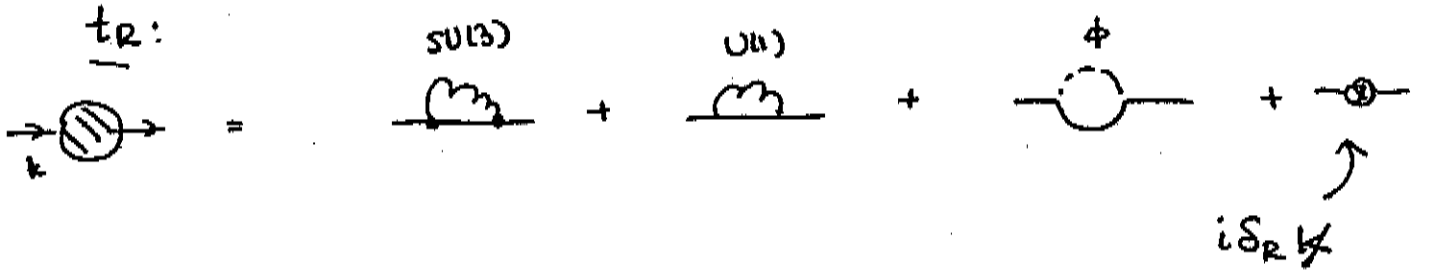
$m_b(m_t) = (4.2\ \text{GeV})/1.4 = 2900\ \text{MeV}$

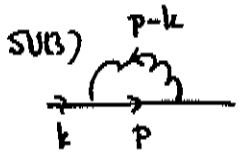
$m_t(m_t) = 167.\ \text{GeV} = 167\ \text{GeV}$

so the more intrinsic quark mass ratios are, eg.

$$\frac{m_b}{m_t} = \frac{1}{58.} \qquad \frac{m_c}{m_b} = \frac{1}{4.3}$$

4.) To compute the β-function, work in an SU(3)×SU(2)×U(1) gauge theory with all symmetries restored. We need the t_L, t_R, ϕ self-energies and the $t_L t_R \phi$ vertex corrections. Use Feynman-'t Hooft gauge.





$$= (ig_s)^2 t^a t^a \int \frac{d^d p}{(2\pi)^d} \frac{-i}{(p-k)^2} \gamma_\lambda \frac{i\not{p}}{p^2} \gamma^\lambda$$

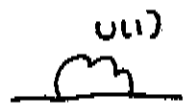
$$= -g_s^2 \frac{4}{3} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^4} \frac{1}{[\mathbb{P}^2 - \Delta]^2} (2-d) \not{p}$$

$$\mathbb{P} = p - xk \qquad p = \mathbb{P} + xk$$

$$= -g_s^2 \frac{4}{3} \int_0^1 dx \; \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) (-2-\varepsilon) \times \not{k}$$

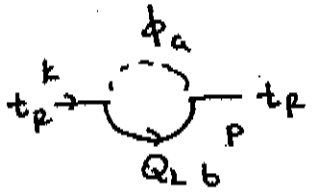
$$= -g_s^2 \frac{4}{3} (-2) \frac{1}{2} \not{k} \; \frac{i}{(4\pi)^2} \cdot \frac{2}{\epsilon} = i \frac{g_s^2}{(4\pi)} \cdot \frac{4}{3} \cdot \frac{2}{\epsilon} \not{k}$$

similarly



$$= i \frac{g'^2}{(4\pi)} \cdot Y^2 \cdot \frac{2}{\epsilon} \not{k}$$

$$= i \frac{g'^2}{4\pi} \cdot \left(\frac{2}{3}\right)^2 \frac{2}{\epsilon} \not{k}$$



$$= (-i\partial_t \epsilon^{ab})(-i\partial_t \epsilon^{ab}) \int \frac{d^d p}{(2\pi)^d} \frac{i \not{p}}{p^2} \frac{i}{(k-p)^2}$$

$$= \partial_t^2 \cdot 2 \cdot \int_0^1 dx \int \frac{d^4 P}{(2\pi)^4} \frac{1}{[P-\Delta]^2} \times \not{k}$$

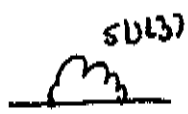
$$= \partial_t^2 \cdot 2 \cdot \frac{1}{2} \not{k} \left(\frac{i}{(4\pi)^2} i \frac{2}{\epsilon}\right)$$

$$= i \frac{\partial_t^2}{(4\pi)^2} \not{k} \frac{2}{\epsilon}$$

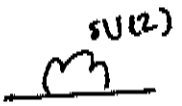
so for t_R

$$\delta_R = \left[- \frac{g_s^2}{(4\pi)^2} \cdot \frac{4}{3} - \frac{g'^2}{(4\pi)^2} \cdot \frac{4}{9} - \frac{\partial_t^2}{(4\pi)^2} \right] \frac{2}{\epsilon}$$

t_L:



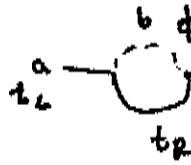
$$= i \frac{g_s^2}{(4\pi)^2} \cdot \frac{4}{3} \frac{2}{\epsilon} \not{k}$$



$$= i \frac{g^2}{(4\pi)^2} C_2(r) \frac{2}{\epsilon} \not{k} = i \frac{g^2}{(4\pi)^2} \frac{3}{4} \frac{2}{\epsilon} \not{k}$$



$$= i \frac{g'^2}{(4\pi)^2} Y^2 \frac{2}{\epsilon} \not{k} = i \frac{g'^2}{(4\pi)^2} \left(\frac{1}{6}\right)^2 \frac{2}{\epsilon} \not{k}$$

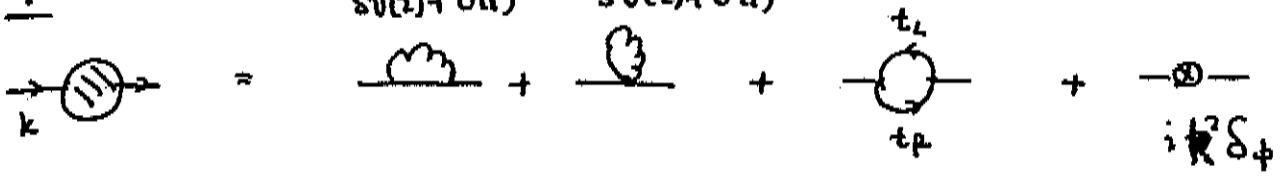


$$= (-i\partial_t \epsilon^{ab})(-i\partial_t \epsilon^{cb}) \int \frac{d^d p}{(2\pi)^d} \frac{i \not{p}}{p^2} \frac{i}{(k-p)^2}$$

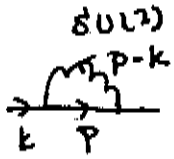
$$= i \frac{\partial_t^2}{(4\pi)^2} \underbrace{\left(\frac{\delta^{ac}}{2}\right)}_{\text{only charge from } t_R} \not{k} \frac{2}{\epsilon}$$

$$\delta_L = \left[- \frac{g_s^2}{(4\pi)^2} \cdot \frac{4}{3} - \frac{g^2}{(4\pi)^2} \frac{3}{4} - \frac{g'^2}{(4\pi)^2} \cdot \frac{1}{36} - \frac{\partial_t^2}{(4\pi)^2} \cdot \frac{1}{2} \right] \frac{2}{\epsilon}$$

$\underline{\Phi}$:



$\underline{\quad}$ has no term $\propto k^2$



$$= (+ig)^2 t^a t^a \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2} (p+k)^\mu (p+k)_\mu \frac{-1}{(p-k)^2}$$

$$= -g^2 C_2(r) \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{(p+k)^2}{[P^2 + x(1-x)k^2]^2} \qquad \begin{matrix} p+k \\ = P + (1+x)k \end{matrix}$$

$$= -g^2 C_2(r) \int_0^1 dx \frac{i}{(4\pi)^{d_4}} \Gamma \left\{ \frac{-\Gamma(1-d_4)}{[-x(1-x)k^2]^{1-d_4}} \cdot \frac{d}{2} \right.$$

$$\left. + \frac{\Gamma(2-d_4)}{[-x(1-x)k^2]^{2-d_4}} (1+x)^2 k^2 \right\}$$

$$= -g^2 C_2(r) \int_0^1 dx \frac{i}{(4\pi)^{d_4}} \frac{1}{[-x(1-x)k^2]^{2-d_4}} \Gamma(2-d_4)$$

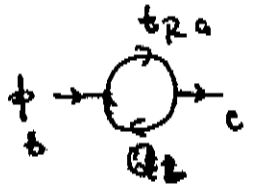
$$\cdot \left\{ - \frac{1}{1-d_4} \frac{d}{2} (-x(1-x)k^2) + (1+x)^2 k^2 \right\}$$

$$= -i\frac{g^2 C_2(r)}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\,\left[(-2x+2x^2)k^2 + (1+2x+x^2)k^2\right]$$

poles at $\epsilon = 0$

$$= -i\frac{g^2 C_2(r)}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot 2\cdot k^2$$

$$= -i\frac{2g^2 C_2(r)}{(4\pi)^2}\frac{2}{\epsilon}k^2 \qquad C_2(r) = \begin{cases} SU(2) & 3/4 \\ & T^2 = \frac{1}{4} \\ U(1) & \end{cases}$$



$$= (-i\partial_t \varepsilon^{ac})(-i\partial_t \Sigma^{ab})\,(-1)\,\underbrace{\mathrm{tr}[1]}_{\text{fermions}\ \text{color}}$$

$$\cdot\int\frac{d^d p}{(2\pi)^d}\,\mathrm{tr}\left[\left(\frac{1+\gamma^5}{2}\right)\frac{i\not{p}}{p^2}\left(\frac{1-\gamma^5}{2}\right)\frac{i(\not{p}+\not{k})}{(p+k)^2}\right]$$

$$= -\partial_t^2\,\delta^{bc}\cdot 3\int_0^1 dx\int\frac{d^d P}{(2\pi)^d}\,\frac{1}{\left[P+x(1-x)k^2\right]^2}\,2\,p\cdot(p+k)$$

$$P = p+xk \qquad p = P-xk \qquad (p+k) = P + (1-x)k$$

$$= -\partial_t^2\,\delta^{bc}\cdot 3\cdot\int_0^1 dx\;\frac{i}{(4\pi)^{d_4}}\cdot 2\cdot\left\{-\frac{\Gamma(1-d_4)}{\left[-x(1-x)k^2\right]^{1-d_4}}\cdot\frac{d}{2}\right.$$

$$\left. + \frac{\Gamma(2-d_4)}{\left[-x(1-x)k^2\right]^{2-d_4}}\left[-x(1-x)k^2\right]\right\}$$

$$= -3\partial_t^2\,\delta^{bc}\,\frac{i}{(4\pi)^{d_4}}\Gamma(2-d_4)\cdot 2\int_0^1 dx\;\frac{1}{\left[-x(1-x)k^2\right]^{2-d_4}}$$

$$\cdot\left\{-\frac{1}{1-d_4/2}\cdot d_4/2\cdot(-x(1-x)k^2) - x(1-x)k^2\right\}$$

$$\text{poleat}_{d=4} = -i\,6\,\lambda_t^2\,\delta^{bc}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\int_0^1 dx\,\left[-2x(1-x)\,k^2 - x(1-x)\,k^2\right]$$

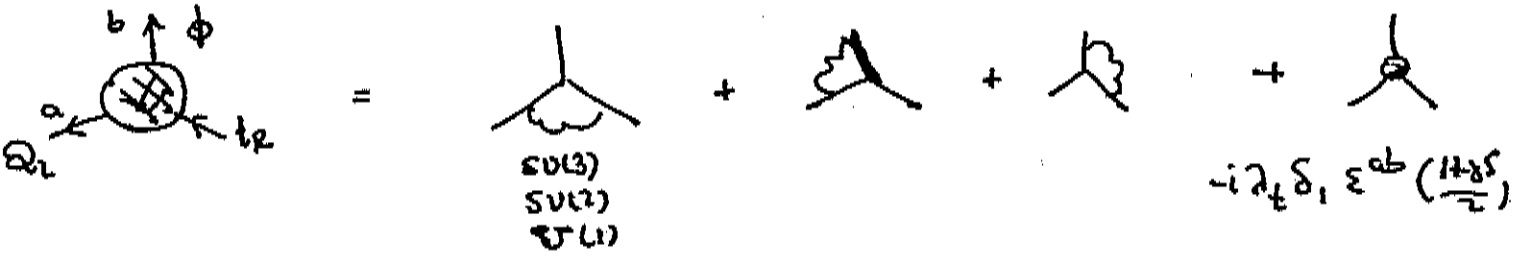
$$= +i\,6\,\lambda_t^2\,\delta^{bc}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\cdot 3\cdot\frac{1}{6}\cdot k^2$$

$$= +3\lambda_t^2\,\delta^{bc}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\,k^2$$

so

$$\delta_\phi = \frac{1}{(4\pi)^2}\left[2\cdot\left(\frac{3}{4}g^2 + \frac{1}{4}g'^2\right) - 3\lambda_t^2\right]\frac{2}{\epsilon}$$

finally, we need the radiative corrections to the vertex:

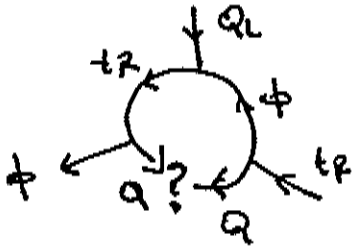


note that there is no vertex correction $\propto \lambda_t^3$

the vertices are

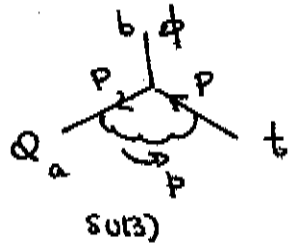


so



is inconsistent.

We can evaluate the gauge dependence w. zero external momenta.



$$= (-i\lambda_t)\,\varepsilon^{ab}\,(t^A t^A)\,(ig_s)^2 \int \frac{d^dp}{(2\pi)^d}\,\gamma^\mu\,\frac{i\not{p}}{p^2}\,\frac{1+\gamma^5}{2}\,\frac{i\not{p}}{p^2}\,\gamma_\mu\,\frac{-i}{p^2}$$

$$= -i\lambda_t\,\varepsilon^{ab}\,\frac{4}{3}\,(-ig_s^2)\int \frac{d^dp}{(2\pi)^d}\,\gamma^\mu\,\frac{p^2}{[p^2]^3}\,\gamma_\mu\,\left(\frac{1+\gamma^5}{2}\right)$$

$$= -i\lambda_t\,\varepsilon^{ab}\,\frac{4}{3}\,(-ig_s^2)\cdot d\cdot\frac{i}{(4\pi)^{d_4}}\,\Gamma(2-d_4)\,\left(\frac{1+\gamma^5}{2}\right)$$

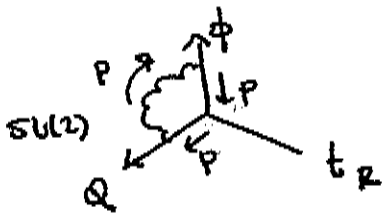
$$= (-i\lambda_t\,\varepsilon^{ab})\,\frac{4}{3}\,\frac{4g_s^2}{(4\pi)^2}\,\frac{2}{\epsilon}\,\left(\frac{1+\gamma^5}{2}\right)$$

for SU(2), the diagram is zero

for U(1) $(t^A t^A) \to \frac{1}{6}\cdot\frac{3}{3} = \frac{1}{9}$



$$= -i\lambda_t\,\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)\cdot\frac{1}{9}\,\frac{4g'^2}{(4\pi)^2}\,\frac{2}{\epsilon}$$



$$= (-i\lambda_t)\,(ig)^2\,(\tau^i)_{aa'}\,(\tau^i)_{bb'}\,\varepsilon^{a'b'}$$

$$\cdot\int\frac{d^dp}{(2\pi)^d}\,\gamma_\mu\,\frac{i\not{p}}{p^2}\,\left(\frac{1+\gamma^5}{2}\right)\cdot\frac{i}{p^2}\,\frac{-i}{p^2}\cdot(-p^\mu)$$

$$= (-i\lambda_t)\;ig^2\int\frac{d^dp}{(2\pi)^d}\,\frac{p\!\!\!/\,p\!\!\!/}{[p^2]^3}\,(\tau^i)_{aa'}\,\varepsilon^{a'b'}[L(\tau^i)]_{b'b}\,\left(\frac{1+\gamma^5}{2}\right)$$

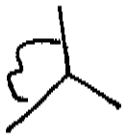
$$= (-i\lambda_t)\;ig^2\left(\frac{i}{(4\pi)^2}\,\frac{2}{\epsilon}\right)\;\left[3\,(\tau^i\tau^i-\tau^i\tau^i)\right]^{ab}\left(\frac{1+\gamma^5}{2}\right)$$

$$= (-i\lambda_t)\;\left(-\frac{g^2}{(4\pi)^2}\,\frac{2}{\epsilon}\right)\cdot\left(-\frac{3}{4}\right)\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)$$

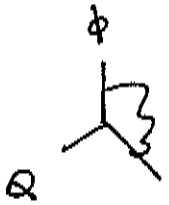
$$= (-i\lambda_t)\,\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)\left(\frac{3}{4}\,\frac{g^2}{(4\pi)^2}\,\frac{2}{\epsilon}\right)$$

for SU(3), this diagram is zero

for U(1) $(\tau^i)_{aa'}(\tau^i)_{bb'}\,\varepsilon^{a'b'} \longrightarrow \varepsilon^{ab}\cdot\frac{1}{6}\cdot\frac{1}{2}$



$$= \left[-i\partial_t\,\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)\right]\cdot\left(-\frac{1}{12}\,\frac{g'^2}{(4\pi)^2}\,\frac{2}{\epsilon}\right)$$



is nonzero for U(1) only:

$$= \left[-i\partial_t\,\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)\right](ig')^2\int\frac{d^dp}{(2\pi)^d}\left(\frac{i\not{p}}{p^2}\gamma_\mu\right)\frac{-i}{p^2}\,\frac{i}{p^2}\,p^\mu$$

$$\cdot\,\frac{1}{2}\cdot\frac{2}{3}$$

$$= \left[-i\partial_t\,\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)\right](-ig'^2)\,\frac{i}{(4\pi)^2}\,\frac{2}{\epsilon}\cdot\frac{1}{3}$$

$$= \left(-i\partial_t\,\varepsilon^{ab}\left(\frac{1+\gamma^5}{2}\right)\right)\,\frac{1}{3}\,\frac{g'^2}{(4\pi)^2}\,\frac{2}{\epsilon}$$

in all

$$\delta_1 = -\frac{16}{3}\,\frac{g_s^2}{(4\pi)^2}\,\frac{2}{\epsilon} - \frac{3}{4}\,\frac{g^2}{(4\pi)^2}\,\frac{2}{\epsilon}$$

$$-\left[\underbrace{\frac{4}{9}-\frac{1}{12}+\frac{1}{3}}_{\frac{25}{36}}\right]\frac{g'^2}{(4\pi)^2}\,\frac{2}{\epsilon}$$

Now combine all of these results:

$$\beta = M\frac{\partial}{\partial M}\left[-\partial_t \delta_1 + \partial_t \cdot \tfrac{1}{2}(\delta_R + \delta_L + \delta_\phi)\right]$$

$$= -\partial_t M\frac{\partial}{\partial M}\left\{\left[\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\right]\right.$$

$$\cdot\left[\left(-\frac{16}{3}g_s^2 - \frac{3}{4}g^2 - \frac{25}{36}g'^2\right)\right.$$

$$-\frac{1}{2}\left(-\frac{4}{3}g_s^2 - \frac{4}{9}g'^2 - \partial_t^2\right)$$

$$-\frac{1}{2}\left(-\frac{4}{3}g_s^2 - \frac{3}{4}g^2 - \frac{1}{36}g'^2 - \frac{1}{2}\partial_t^2\right)$$

$$\left.\left.-\frac{1}{2}\left(\frac{9}{2}g^2 + \frac{1}{2}g'^2 - 3\partial_t^2\right)\right]\right\}$$

$2/\epsilon \Rightarrow \ln \Lambda^2/M^2$ so $M\frac{\partial}{\partial M}\left(\frac{2}{\epsilon}\right) = -2$

$$\beta = 2\partial_t \cdot \frac{1}{(4\pi)^2}\left\{g_s^2\left(-\frac{16}{3} + \frac{4}{3}\right)\right.$$

$$+ g^2\left(-\frac{3}{4} + \frac{3}{8} - \frac{3}{4}\right)$$

$$+ g'^2\left(-\frac{25}{36} + \frac{2}{9} + \frac{1}{12} - \frac{1}{4}\right)$$

$$\left.+ \partial_t^2\left(\frac{1}{2} + \frac{1}{4} + \frac{3}{2}\right)\right\}$$

$$= \frac{2\partial_t}{(4\pi)^2}\left\{-4g_s^2 - \frac{9}{8}g^2 - \frac{51}{72}g'^2 + \frac{9}{4}\partial_t^2\right\}$$

$$+ \frac{17}{24}$$

$$\beta = \frac{\lambda_t}{(4\pi)^2}\left\{\frac{9}{2}\lambda_t^2 - 8g_s^2 - \frac{9}{4}g^2 - \frac{17}{12}g'^2\right\}$$

as requested.

If we consider only the λ_t and g_s contributions, the λ_t^3 term makes λ_t decrease toward the infrared; the $\lambda_t g_s^2$ term makes λ_t increase toward the infrared.

Balance is found at

$$\frac{9}{2}\lambda_t^2 \cong 8g_s^2$$

$$\frac{9}{8\pi}\lambda_t^2 \cong 8\alpha_s$$

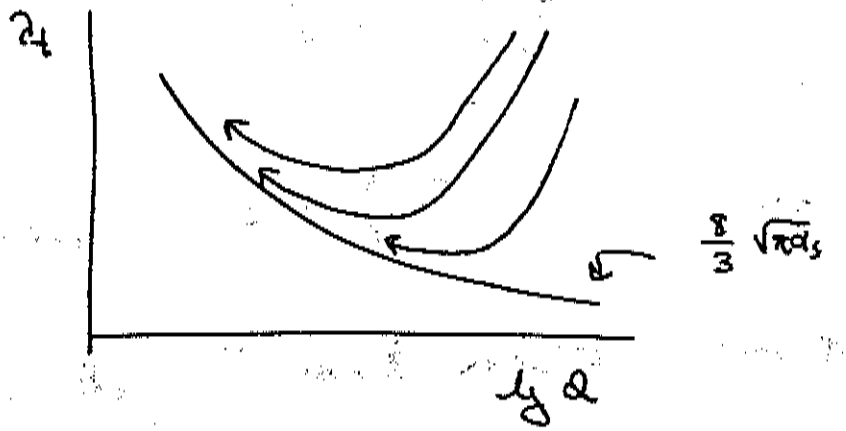
$$\lambda_t^2 \approx \frac{64\pi}{9}\alpha_s$$

Of course, α_s runs, but if it runs sufficiently slowly

$$\lambda_t(m_t) \approx \frac{8}{3}\sqrt{\pi\alpha_s(m_t)} \approx 1.6 \qquad \alpha_s(m_t) = 0.11$$

$$m_t \approx \frac{\lambda_t}{\sqrt{2}}v \approx 270\ \text{GeV} \qquad v = 246\ \text{GeV}$$

Integrate the differential equation for α_s, λ_t, one finds that, for λ_t large at $Q \gg 1000$ GeV, this asymptote is approximately reached



5.) Using $m_p = (\text{const}) \cdot \Lambda_3$

treating u, d, s as light quarks, c b t as heavy quarks at the proton scale

$$m_p \propto (m_t m_b m_c)^{2/27}$$

$$\frac{\partial m_p}{\partial m_t} = \frac{2}{27}\frac{m_p}{m_t}$$

since all three of m_b, m_b, m_c are $\propto v$

$$\frac{\partial m_p}{\partial v} = 3 \cdot \frac{2}{27} \cdot \frac{m_p}{v}$$

$$= \frac{2}{9} \cdot \frac{0.938}{246.}$$

$$\partial_{hpp} = \sqrt{2} \cdot \frac{2}{9} \frac{m_p}{v} = 1.2 \times 10^{-3}$$

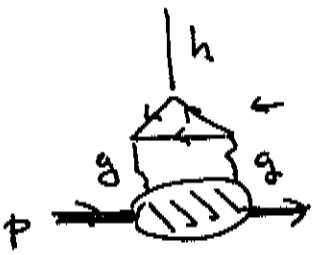
Compare this to the direct contribution from the valence quarks u and d

$$\Delta m_p = 2m_u + m_d$$

$$\frac{\partial}{\partial v} \Delta m_p = \frac{2m_u + m_d}{v} = \frac{(2 \cdot 3 + 5) \cdot 10^{-3}}{246}$$

$$\Delta \partial_{hpp} = \sqrt{2} \cdot (\text{above}) = 6 \times 10^{-5}$$

the large contribution to ∂_{hpp} can be thought of as coming from the diagram



this diagram is constant as $m_q \to \infty$!

Physics 332 – Final Exam

This exam is due at 3:30 pm on Friday, June 16. Please hand it in to Nav Sivanandam in Varian 365. If you have any questions about the exam, please contact me by email at mpeskin@slac.stanford.edu. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed.

The exam is worth a total of 100 points. The distribution of points is indicated below.

Many of the calculations on this exam can be found in the literature. But in order to get the answers from the literature, you will need to spend a lot of time translating notation. What fun is that? If you make strong use of a reference other than the class textbook and notes, please cite the reference in your solutions.

Quantum field theory models with supersymmetry (fermion-boson symmetry) have many interesting renormalization properties. In this exam, you will work out several of these.

a. (20 points) Here is a simple model of massless scalars and fermions with supersymmetry: Let $\phi(x)$ be a complex scalar field, and let $\psi(x)$ be a 2-component left-handed fermion. Consider the Lagrangian

$$\begin{aligned}\mathcal{L} &= \partial_\mu \phi^* \partial^\mu \phi + \psi^\dagger i\sigma\cdot\partial\psi \\ &\quad + \frac{1}{2}(2y)(\psi^T\epsilon\psi\phi - \psi^\dagger\epsilon\psi^*\phi^*) - z(\phi^*\phi)^2 \end{aligned} \tag{1}$$

where y and z are real-valued coupling constants. Notice that the fermions are contracted in the second line in a Lorentz-invariant way:

$$\psi^T\epsilon\psi = \epsilon_{\alpha\beta}\psi_\alpha\psi_\beta \ ; \qquad (\psi^T\epsilon\psi)^\dagger = -\psi^\dagger\epsilon\psi^* = \epsilon_{\alpha\beta}\psi^\dagger_\alpha\psi^\dagger_\beta \tag{2}$$

According to Peskin and Schroeder, Problem 3.5, this model is supersymmetric if $z = y^2$.

Compute the β functions of y and z in this model. Show that the relation $z = y^2$ is preserved by the renormalization group.

b. (10 points) Compute the scalar self-energy in this model. Show that it is not quadratically divergent—and that it even vanishes identically—when $z = y^2$.

c. (15 points) Here is a supersymmetric version of Yang-Mills theory: Let ψ_i be left-handed 2-component fermions in the representation r of the gauge group G, and let ϕ_i be complex-valued scalars in the same representation. Let A^a_μ be the Yang-Mills

fields, and let λ^a be left-handed 2-component fermions in the adjoint representation of G. Let

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F^a_{\mu\nu})^2 + \lambda^\dagger i\sigma\cdot D\lambda + D_\mu\phi_i^* D^\mu\phi_i + \psi_i^\dagger i\sigma\cdot D\psi_i \\ &\quad +\sqrt{2}h(\phi_i^*\lambda^{aT}t^a\epsilon\psi_i - \psi_i^\dagger\epsilon\lambda^{a*}t^a\phi_i) - \frac{1}{2}k(\phi_i^* t^a\phi_i)^2 \ , \end{aligned} \tag{3}$$

where t^a are representation matrices of G in the representation r. The label $i = 1,\ldots,n_f$; remember that the fermions are Weyl, not Dirac. In the last term, the sum over i is done before squaring. There are three coupling constants: g (the Yang-Mills coupling), h, and k. The theory is supersymmetric if $h = g$, $k = g^2$.

Compute the scalar self-energy in this model. Use Feynman gauge. Show that the ϕ self-energy is not quadratically divergent when the supersymmetry relations of coupling constants are satisfied. Show that this self-energy does not vanish identically in dimensional regularization, but that it does vanish if we set

$$g_{\mu\nu}g^{\mu\nu} = 4 \tag{4}$$

as we continue in dimensionality. This is a different conventional regularization prescription called 'dimensional reduction'. Only the latter prescription for the regulator preserves supersymmetry.

d. (15 points) Compute the β function of g using formulae derived in the course. Compute the β functions of h, k explicitly. Show that the relations $h = g$, $k^2 = g$ are preserved by the renormalization group.

e. (15 points) Next, build a supersymmetric theory with one quark coupled to a Higgs boson. Let ψ_q be a left-handed fermion in the (3,2) of $SU(3)\times SU(2)$, and let ϕ_q be the corresponding complex scalar. Let $\psi_{\overline{q}}$ be a left-handed fermion in the ($\overline{3}$,1) of $SU(3))\times SU(2)$ and let $\phi_{\overline{q}}$ be the corresponding complex scalar, and let ψ_h, ϕ_h be the Higgs fermion and scalar. It is useful to visualize the $SU(3)$ and $SU(2)$ indices by thinking of $\phi_{\overline{q}}$, $\psi_{\overline{q}}$ as 3-dimensional row vectors, ϕ_h, ψ_h as 2-dimensional column vectors, and ϕ_q, ψ_q as 3×2 matrices. In the following, t^a is the color representation matrix in the 3 of $SU(3)$; in the $\overline{3}$, $t^a_{\overline{3}} = -t^{aT}$.

The supersymmetric interaction terms for this model (including QCD but not $SU(2)$ gauge interactions) are

$$\begin{aligned}\mathcal{L}_{int} &= +\sqrt{2}g(\phi_q^*\lambda^{aT}t^a\epsilon\psi_q - \psi_{\overline{q}}^T t^a\epsilon\lambda^a\phi_{\overline{q}}^* - h.c.) - \frac{1}{2}g^2(\phi_q^* t^a\phi_q - \phi_{\overline{q}}t^a\phi_{\overline{q}}^*)^2 \\ &\quad +y(\psi_{\overline{q}}\epsilon\psi_q\phi_h + \psi_{\overline{q}}\epsilon\phi_q\psi_h + \phi_{\overline{q}}\psi_q\epsilon\psi_h + h.c.) \\ &\quad -y^2(|\phi_{\overline{q}}\phi_q|^2 + |\phi_{\overline{q}}\phi_h|^2 + |\phi_q\phi_h|^2)\end{aligned} \tag{5}$$

The coupling y is the Higgs Yukawa coupling, which would be large, for example, for the top quark.

Compute the β functions due to QCD (terms of order $g^2 y$) for the three different 2-fermion-1-scalar interaction terms in the second line. There is no obvious reason why the terms with one Higgs scalar should have the same renormalization as the terms with one Higgs fermion. But, show that it is so.

f. (15 points) Add to this theory supersymmetry-violating mass terms for the quarks and the Higgs:

$$\Delta\mathcal{L} = -M_q^2|\phi_q|^2 - M_{\overline{q}}^2|\phi_{\overline{q}}|^2 - M_h^2|\phi_h|^2 \tag{6}$$

where the M_i^2 are positive and are equal at the renormalization scale μ. We have already seen that, in a supersymmetric model, these operators do not receive additive radiative corrections. However, they do have γ functions and so the coefficients undergo a renormalization group evolution. Compute the renormalization group equations for the M_i^2 parameters. Show that, if the 'gauginos' λ^a are massless, the only effect in one-loop order comes from the interactions in the third line of (5). Show that this effect makes the M_i^2 *decrease* as we move toward the infrared.

g. (10 points) Describe the qualitative nature of this theory in the infrared. Is there symmetry breaking? Which field gets a vacuum expectation value?

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (c). Prospective supersymmetry theorists should slog through to the end.

Physics 332 – Final Exam
Solutions

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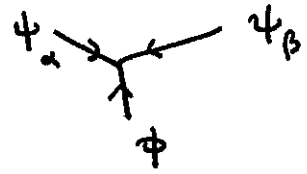
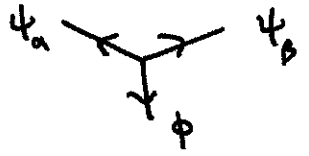
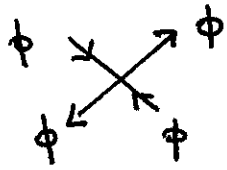
a.) the Feynman rules for this theory are



$= 2iy\,\varepsilon_{\alpha\beta}$



$= -2iy\,\varepsilon_{\alpha\beta}$



$= -4iz$

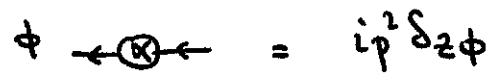
The signs in the fermion terms are tricky. To be safe, I'll work them out by making explicit contractions. The propagators are:

$$\psi_\alpha(p)\,\psi^\dagger_\beta(-p) = \frac{i(\sigma\cdot p)_{\alpha\beta}}{p^2} = -\overbrace{\psi^\dagger_\beta(-p)\,\psi_\alpha(p)} = +\overbrace{\psi_\beta(p)\,\psi^\dagger_\alpha(-p)}$$

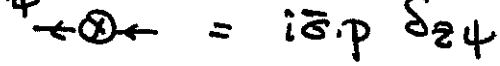
(odd under $p \to -p$)

$$\overbrace{\phi(p)\,\phi^*(-p)} = \frac{i}{p^2}$$

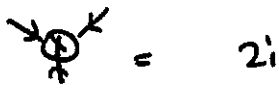
The counterterms are:



$= ip^2\delta_z\phi$



$= i\bar{\sigma}\cdot p\,\delta_z\psi$



$= 2i\delta_y\,\varepsilon_{\alpha\beta}$



$= -2i\delta_y\,\varepsilon_{\alpha\beta}$



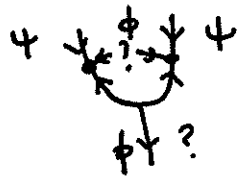
$= -4i\delta_z$

First, compute the counterterms δ_y, δ_z:

Actually, there are no 1-loop divergences for the vertex

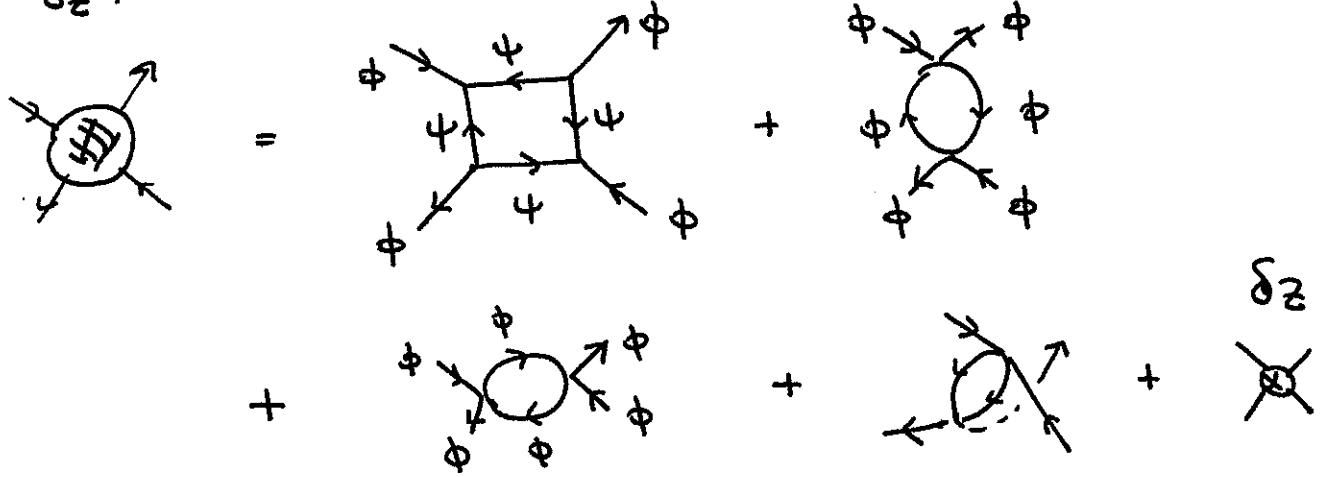


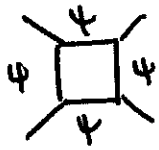
The arrows do not work!



so $\delta_y = 0$

for δ_z:





$$= (2iy)(-2iy)(2iy)(-2iy) \cdot \overbrace{(\psi^T \epsilon \psi\phi)(\psi^\dagger \epsilon \psi^* \phi^* \chi \psi^T \epsilon \psi\phi)(\psi^\dagger \epsilon \psi^* \phi^*)}$$

$$\psi(p)\psi^\dagger(p) = \frac{i\sigma\cdot p}{p^2} \qquad \overbrace{\psi^*(p)\,\psi^T(-p)} = (-1)\cdot\left[\frac{i\sigma\cdot(-p)}{p^2}\right]^T$$

$$= 16y^4 \int \frac{d^dp}{(2\pi)^d}\, tr\left[\epsilon \frac{i\sigma\cdot p}{p^2} \epsilon \left(\frac{i\sigma\cdot p}{p^2}\right)^T \epsilon \frac{i\sigma\cdot p}{p^2} \epsilon \underbrace{(-1)\left(\frac{i\sigma\cdot p}{p^2}\right)^T}\right]$$

the last propagator is in the order $\overbrace{\psi^T(-p)\,\psi^*(p)}$

now $\epsilon\,(\sigma\cdot p)^T\epsilon = \epsilon^2\,\bar\sigma\cdot p = (-1)\cdot\bar\sigma\cdot p$

$\sigma\cdot p\;\bar\sigma\cdot p = p^2$

so

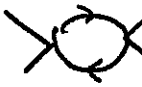
$$= 16y^4 \int\frac{d^dp}{(2\pi)^d}\,\mathrm{tr}\left[(-1)\,\frac{p^2}{p^2p^2}\cdot(-1)(-1)\,\frac{p^2}{p^2\cdot p^2}\right]$$

$$= -16\cdot 2y^4 \int\frac{d^dp}{(2\pi)^d}\,\frac{1}{[p^2]^2} = (-32y^4)\,\frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2)$$



$$= (-4iz)^2 \int\frac{d^dp}{(2\pi)^d}\left(\frac{i}{p^2}\right)^2$$

$$= 16z^2 \int\frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2)^2} = 16z^2\cdot\frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2)$$



$$= 16z^2\,\frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2)$$



$$= \frac{1}{2}\cdot 16z^2\,\frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2)$$

Σ all:

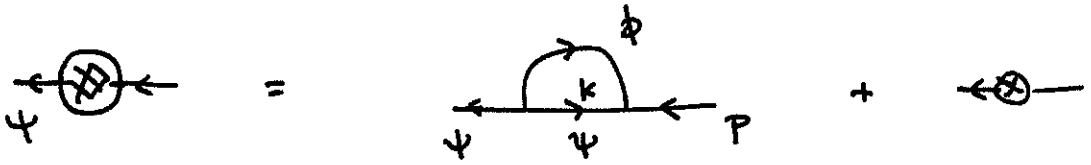


$$= (-32y^4 + \tfrac{5}{2}\cdot 16z^2)\,\frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2) - 4i\delta_z$$

$$= (-32y^4 + 40z^2)\,\frac{i}{(4\pi)^2}\,\ln\Lambda^2/M^2 - 4i\delta_z$$

so $\delta_y = 0$ $\quad \delta_z = (10z^2 - 8y^4)\,\frac{1}{(4\pi)^2}\ln\Lambda^2/M^2$

Now compute the self-energy counterterms:



$$\psi^\dagger \epsilon \overbrace{\psi^* \psi^T} \epsilon \psi$$

$$= (-2iy)(2iy) \int \frac{d^dk}{(2\pi)^d} \epsilon (-1) \left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon \frac{i}{(p+k)^2} + i\bar{\sigma}\cdot p\, \delta_{Z\psi}$$

$$= 4y^2 \int \frac{d^dk}{(2\pi)^d} \epsilon (\sigma\cdot k)^T \epsilon \frac{1}{k^2(p+k)^2} + i\bar{\sigma}\cdot p\, \delta_{Z\psi}$$

$$= 4y^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} (-\bar{\sigma}\cdot k) \frac{1}{[k^2 + xp\cdot k + xp^2]^2} + \cdots$$

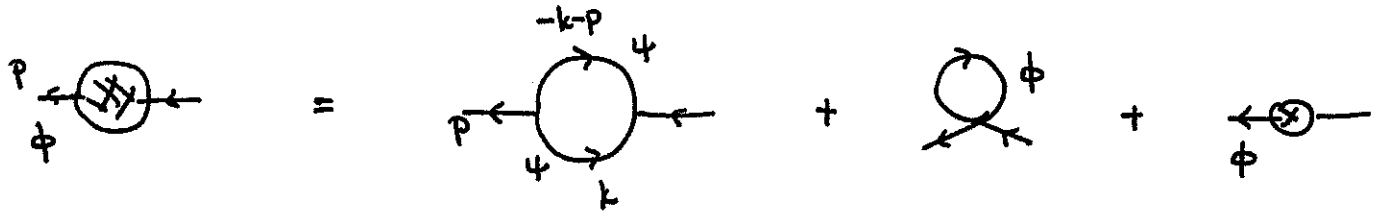
$$\ell = k + xp \qquad k = \ell - xp$$

$$= (-4y^2) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\bar{\sigma}\cdot(\ell - xp)}{[\ell^2 - \Delta]^2} + \cdots$$

$$= -4y^2 \int_0^1 dx\, [-x(\bar{\sigma}\cdot p)] \cdot \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2)$$

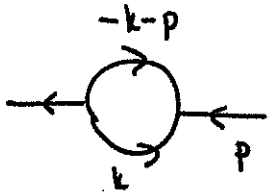
$$= +2y^2 \frac{i}{(4\pi)^2} \ln \Lambda^2/M^2 \cdot \bar{\sigma}\cdot p + i\bar{\sigma}\cdot p\, \delta_{Z\psi}$$

$$\delta_{Z\psi} = -2y^2 \frac{1}{(4\pi)^2} \ln \Lambda^2/M^2$$



to compute $\delta_{Z\phi}$, we need the term of order p^2 from the diagrams. However, for part (b), we also need the term of order $(p^2)^0$.

so, why not compute everything now?



$$\bar{\phi}^* \psi^\dagger \epsilon \psi^* \psi^T \epsilon \psi \phi$$

$$= \frac{1}{2}(-2iy)(2iy)\int \frac{d^dk}{(2\pi)^d} \operatorname{tr} \epsilon (-1)\left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon (-1) \frac{i\sigma\cdot(k-p)}{(k+p)^2}$$

$$= 2y^2 \int \frac{d^dk}{(2\pi)^d} \operatorname{tr} \frac{\epsilon (\sigma\cdot k)^T \epsilon \, \sigma\cdot(k+p)}{k^2 (k+p)^2}$$

$$= 2y^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \, -2\, k(k+p) \frac{1}{[k^2 - x(1-x)p^2]^2}$$

$$\ell k = k + xp \qquad k = \ell k - xp \qquad k+p = \ell k + (1-x)p$$

$$= -4y^2 \int_0^1 dx \int \frac{d^d\ell k}{(2\pi)^d} \frac{[\ell k^2 - x(1-x)p^2]}{[\ell k^2 - x(1-x)p^2]^2}$$

$$= -4y^2 \int_0^1 dx \, \frac{i}{(4\pi)^{d_4}} \left\{ - \frac{\Gamma(1-d_4)\, d_4 [-x(1-x)p^2] - x(1-x)p^2 \Gamma(2-d_4)}{[-x(1-x)p^2]^{2-d_4}} \right.$$

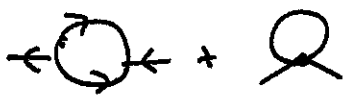
$$= -4y^2 \frac{i}{(4\pi)^{d_4}} \int_0^1 dx \, \frac{(\Gamma(1-d_4)\cdot d_4 - \Gamma(2-d_4))}{[-x(1-x)p^2]^{2-d_4}} [x(1-x)p^2]$$



$$= (-4iz)\int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2}$$

$$= 4z\cdot\frac{-i}{(4\pi)^{d_k}}\,\Gamma(1-d_k)$$

the quadratic divergence of the scalar mass is:



$$\sim -4y^2\int\frac{d^dk}{(2\pi)^d}\,\frac{k^2}{k^4} + 4z\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

which vanishes when $z = y^2$!

Now continue to $d=4$ to find the term of order p^2.  gives no contribution.



$$= -4y^2\,\frac{i}{(4\pi)^{d_k}}\int_0^1 dx\;\frac{\Gamma(2-d_k)}{[-x(1-x)p^2]^{2-d_k}}\,[x(1-x)p^2]$$

$$\cdot\left(\underbrace{\frac{d/2}{1-d/2}}-1\right)$$

$$= -4y^2\cdot(-3)\cdot\frac{i}{(4\pi)^2}\int_0^1 dx\;\underbrace{x(1-x)}_{1/6}\;p^2\,\ell n\,\Lambda^2/M^2$$



$$= i p^2\,\delta_{Z\phi}$$

so $$\delta_{Z\phi} = -2y^2\,\frac{1}{(4\pi)^2}\,\ell n\,\Lambda^2/M^2 = \delta_{Z4}$$

Now compute the β functions for y and z:

$$\beta_y = M\frac{\partial}{\partial M}\left[-\delta_y + \frac{y}{2}(\delta_{z\phi} + 2\delta_{z4})\right]$$

$$= M\frac{\partial}{\partial M}\left[0 + \frac{y}{2}\cdot 3\cdot\left(-2y^2\left(\frac{1}{4\pi}\right)^2 \ln \Lambda^2/M^2\right)\right]$$

$$\beta_y = \frac{6y^3}{(4\pi)^2}$$

$$\beta_z = M\frac{\partial}{\partial M}\left[-\delta_z + \frac{z}{2}\cdot(4\delta_{z\phi})\right]$$

$$= M\frac{\partial}{\partial M}\left[\frac{8y^4 - 10z^2}{(4\pi)^2}\ln \Lambda^2/M^2 + (2z)\left(-2y^2\frac{1}{(4\pi)^2}\ln \Lambda^2/M^2\right)\right.$$

$$= (-2)\frac{8(y^4 - z^2)}{(4\pi)^2} + \frac{2\cdot 2\, z^2}{(4\pi)^2} + \frac{2\cdot 4\, y^2 z}{(4\pi)^2}$$

$$\beta_z = \frac{1}{(4\pi)^2}\left[16(z^2 - y^4) + 4z^2 + 8y^2 z\right]$$

for $z = y^2$ $\quad \beta_z = \frac{1}{(4\pi)^2}\, 12\, y^4 = 2y\,\beta_y$

this is just what is required so that

$$\frac{dy}{d\ln Q} = \beta_y \qquad \frac{dz}{d\ln Q} = \beta_z$$ preserves $z = y^2$

something seems odd here. If we have supersymmetry, shouldn't we have $\delta_y = 0 = \delta_z$, $\delta_{z\phi} = \delta_{z\psi}$?

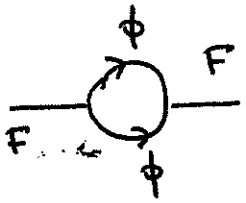
For the explanation, please look at P+S problem 3.5. In that problem, the Lagrangian is written:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \psi^\dagger i\bar{\sigma}\cdot\partial\psi + F^\dagger F + y(\psi^T \epsilon \psi \phi + h.c.) + yF\phi^2 + h.c.$$

The supersymmetry multiplet is actually (ϕ, ψ, F). The propagator for F is

$$\overline{F F^*} = i$$

F has a self-energy:



$$+ \quad \otimes$$

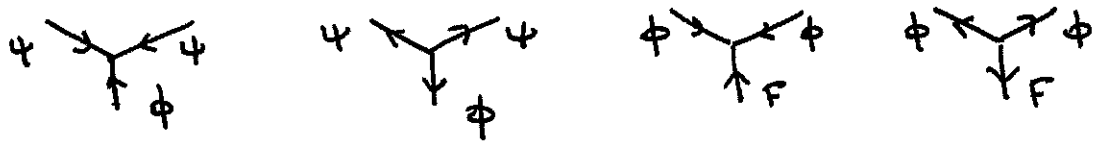


$$= \frac{1}{2}\cdot(2iy)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2}\frac{i}{(k+p)^2} + i\delta_{ZF}$$

$$= 2y^2 \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) + i\delta_{ZF}$$

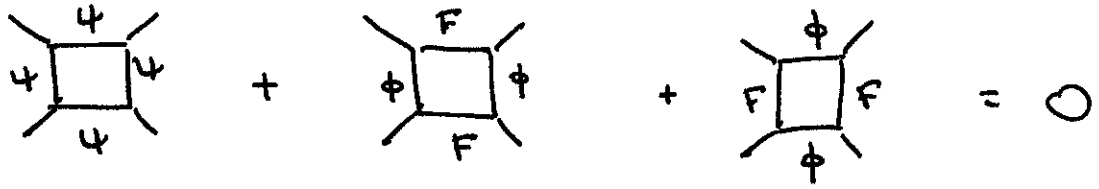
so $\delta_{ZF} = -2y^2 \frac{1}{(4\pi)^2} \ln \Lambda^2/M^2 = \delta_{Z\phi} = \delta_{Z\psi}$

the vertices of this thy are :

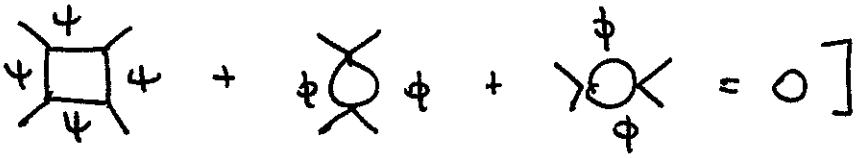


there are no 1-loop corrections to the $\phi\psi\psi$ vertex or to the $F\phi\phi$ vertex. It seems that we can generate a $\phi^2\phi^{*2}$ vertex, but

if $z=y^2$



[as we saw on p. 3:

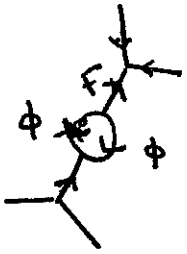


]

On p. 3, there is one more graph that does not cancel.



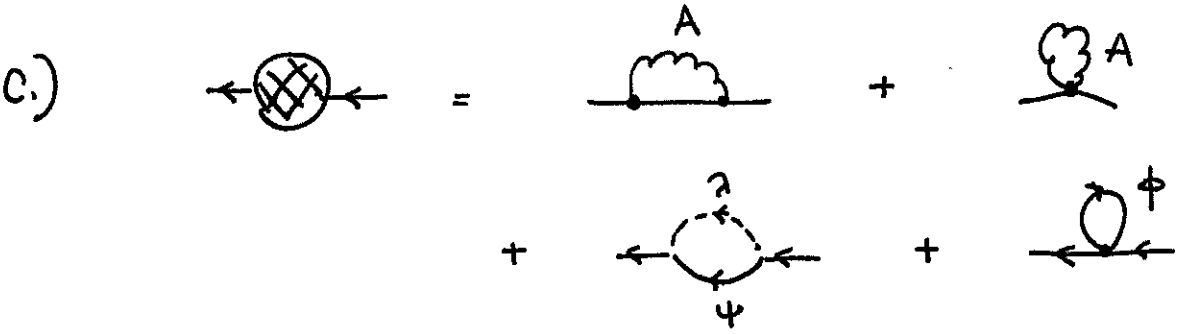
In the thy with F, this is

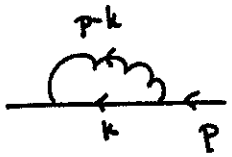


i.e., it is the effect of the F self energy when F is integrated out.

b.) [this was shown on p.6]

c.)





$$= (ig)^2 \int \frac{d^dk}{(2\pi)^d}\, t^a (k+p)_\mu \frac{i}{k^2} (k+p)^\mu t^a \frac{-i}{(p-k)^2}$$

$$= -g^2 C_2(r) \int \frac{d^dk}{(2\pi)^d} \frac{(k+p)^2}{k^2 (k-p)^2} \qquad \begin{array}{l} \ell k = k - xp \\ k = \ell k + xp \end{array}$$

$$= -g^2 C_2(r) \int dx \int \frac{d^d\ell k}{(2\pi)^d} \frac{[\ell k + (1+x)p]^2}{[\ell k^2 - x(1-x)p^2]^2}$$

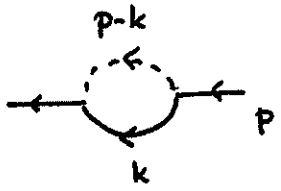
$$= -g^2 C_2(r) \int_0^1 dx \; \frac{i}{(4\pi)^{d_2}} \left\{ - \frac{\Gamma(1-d_2)}{[-x(1-x)p^2]^{1-d_2}} \frac{d}{2} + \frac{\Gamma(2-d_2)(1+x)^2 p^2}{[-x(1-x)p^2]^{2-d_2}} \right.$$

$$= +g^2 C_2(r) \int_0^1 dx \; \frac{i}{(4\pi)^{d_2}} \left\{ \frac{d_2 \Gamma(1-d_2)}{[-x(1-x)p^2]^{1-d_2}} - \frac{\Gamma(2-d_2)(1+x)^2 p^2}{[-x(1-x)p^2]^{2-d_2}} \right\}$$



$$= \frac{1}{2} 2ig^2 \int \frac{d^dk}{(2\pi)^d} g_\mu^\mu \frac{-i}{k^2} t^a t^a$$

$$= \left(\tfrac{1}{2}\right) \cdot 2 \cdot 4 \; g^2 C_2(r) \left[\frac{-i}{(4\pi)^{d_2}} \Gamma(1-d_2) \right] \qquad \left(\text{in DR} \simeq \; g_\mu^\mu = 4 \right)$$



$\phi^* \overline{\partial^T \epsilon \psi} \; \overline{\psi^+ \epsilon \partial^* \phi}$

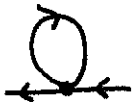
$$= (\sqrt{2} i h)(-\sqrt{2} i h) \int \frac{d^d k}{(2\pi)^d} \, tr\, \epsilon \left(\frac{i\sigma\cdot k}{k^2}\right) \epsilon \left[\frac{i\sigma(p-k)}{(p-k)^2}\right]^T t^a t^a$$

$$= 2h^2 C_2(r) \int \frac{d^d k}{(2\pi)^d} \frac{2k\cdot(p-k)}{k^2 (p-k)^2}$$

$$\ell k = k - xp$$
$$k = \ell k + xp$$
$$(k-p) = \ell k - (1-x)p$$

$$= 2h^2 C_2(r) \int_0^1 dx \int \frac{d^d \ell k}{(2\pi)^d} \, 2(\ell k + xp)\cdot(-\ell k + (1-x)p) \left[\frac{1}{\ell k^2 - x(1-x)p^2}\right]^2$$

$$= -4h^2 C_2(r) \int_0^1 dx \, \frac{i}{(4\pi)^{d_k}} \left\{ - \frac{\Gamma(1-d_k)\cdot d_k}{[-x(1-x)p^2]^{1-d_k}} - \frac{\Gamma(2-d_k)\, x(1-x)p^2}{[-x(1-x)p^2]^{2-d_k}} \right\}$$



$$= \quad \blacksquare \quad - ik \int \frac{d^d k}{(2\pi)^d} \, t^a \frac{i}{k^2} t^a + t^a \frac{i}{k^2} \underbrace{tr[t^a]}_{=0}$$

$$= +k \, C_2(r) \left[\frac{-i}{(4\pi)^{d_k}} \Gamma(1-d_k)\right]$$

The terms at $p^2 = 0$ are:

$$\left[\frac{-i}{(4\pi)^{d_k}} \Gamma(1-d_k)\right] \left\{ C_2(r) \right\} \left\{ -g^2 \frac{d}{2} + 4g^2 - 4h^2 \frac{d}{2} + k \right\}$$

$$\underset{\text{at } d=2}{=} \frac{-i}{(4\pi)^{d_k}} \Gamma(1-d_k) \, C_2(r) \left\{ -g^2 + 4g^2 - 4h^2 + k \right\}$$

$$= 0 \quad \text{if} \quad h = g \,, \; k = g^2$$

d.) For a gauge theory with matter fermions and scalars,

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{\substack{\text{Dirac}\\\text{fermions}}} C(r_i) - \frac{1}{3}\sum_{\substack{\text{complex}\\\text{scalars}}} C(r_j)\right]$$

a Weyl fermion counts $\frac{1}{2}$ of a Dirac fermion. In the theory we have

Weyl fermions λ $\quad C(r_\lambda) = C(G) = C_2(G)$

ψ_i $\quad C(r)$

and scalars ϕ_i $\quad C(r)$

so

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{2}{3}C_2(G) - \frac{2}{3}n_f C_2(G) - \frac{1}{3}n_f C(r)\right]$$

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right]$$

to compute the β function for theory k, we need the self-energy contributions for ϕ, ψ, λ

For ϕ, look back at pp. 10-11



$$= g^2 C_2(r)\int_0^1 dx\ \frac{i}{(4\pi)^{d_k}}\left\{\frac{\frac{d_k}{2}\Gamma(1-\frac{d_k}{2})}{[-x(1-x)p^2]^{1-\frac{d_k}{2}}} - \frac{(q+x)^2 p^2\,\Gamma(2-\frac{d_k}{2})}{[\quad]^{2-\frac{d_k}{2}}}\right\}$$



$$= -4h^2 C_2(r)\int_0^1 dx\ \frac{i}{(4\pi)^{d_k}}\left\{\frac{\frac{d_k}{2}\Gamma(1-\frac{d_k}{2})}{[-x(1-x)p^2]^{1-\frac{d_k}{2}}} + \frac{x(1-x)p^2\,\Gamma(2-\frac{d_k}{2})}{[\quad]}\right\}$$

 do not contribute to δ_{2p}.



$$= \frac{i}{(4\pi)^{d_4}} \int_0^1 dx \; \frac{\Gamma(2-d_4)}{[-x(1-x)p^2]^{2-d_4}} \Big\{ g^2 C_2(r) \left[-\left(\frac{d_4/2}{1-d_4}\right) x(1-x)p^2 - (1+x)^2 p^2 \right] + 4h^2 C_2(r) \left[-\left(\frac{d_4/2}{1-d_4}\right) x(1-x)p^2 + x(1-x)p^2 \right] \Big\}$$

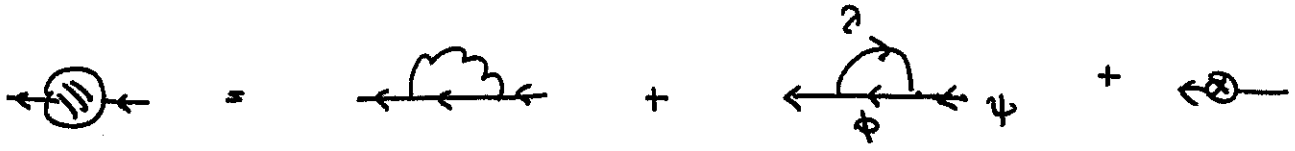
$$= \frac{i}{(4\pi)^{d_4}} \int_0^1 dx \; (d_4 \Lambda^2/M^2) \; p^2 \Big\{ g^2 C_2(r) \cdot [2x(1-x) - (1+x)^2] + 4h^2 C_2(r) \left[2x(1-x) + x(1-x) \right] \Big\}$$

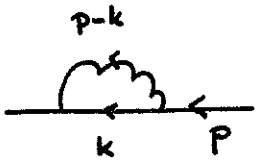
$$= i p^2 \left(\frac{1}{(4\pi)^2} d_4 \Lambda^2/M^2\right) \left[g^2 C_2(r) \left(\tfrac{1}{3} - \tfrac{7}{3} \right) + 4h^2 C_2(r) \cdot \tfrac{3}{6} \right]$$

$$= i p^2 \; \frac{1}{(4\pi)^2} d_4 \Lambda^2/M^2 \; \left[-2g^2 C_2(r) + 2h^2 C_2(r) \right]$$

so $$\delta_{Z\phi} = (2g^2 - 2h^2) C_2(r) \frac{1}{(4\pi)^2} d_4 \Lambda^2/M^2$$

for ψ:



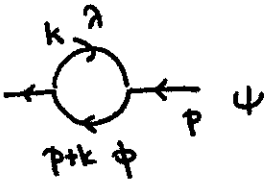


$$= (ig)^2 C_2(r) \int \frac{d^dk}{(2\pi)^d} \bar{\sigma}^\mu \frac{i\sigma\cdot k}{k^2} \bar{\sigma}_\mu \frac{-i}{(p-k)^2}$$

$$= -g^2 C_2(r) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{-2\,\bar{\sigma}\cdot k}{[\ell^2 - x(1-x)p^2]^2} \qquad \ell = k - xp \quad k = \ell + xp$$

$$= 2g^2 C_2(r) \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2) \frac{\bar{\sigma}\cdot xp}{[-x(1-x)p^2]^{2-d/2}}$$

$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) g^2 C_2(r)\, \bar{\sigma}\cdot p$$



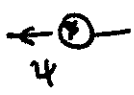
$$\psi^+ \epsilon \overleftarrow{\partial^*} \phi \quad \phi^* \overrightarrow{\partial^T} \epsilon \psi$$

$$= (-\sqrt{2} i h)(\sqrt{2} i h) \int \frac{d^dk}{(2\pi)^d} t^a \epsilon (-1) \left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon t^a \frac{i}{(p+k)^2}$$

$$= 2h^2 (-1) \int \frac{d^dk}{(2\pi)^d} C_2(r) \frac{\bar{\sigma}\cdot k}{k^2 (p+k)^2}$$

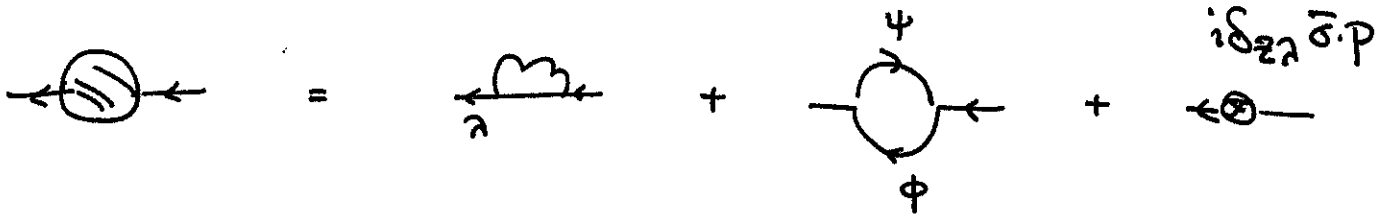
$$= -2h^2 C_2(r) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\bar{\sigma}\cdot \ell - xp}{[\ell^2 - x(1-x)p^2]^2}$$

$$= \frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \cdot [+h^2 C_2(r)]\, \bar{\sigma}\cdot p$$

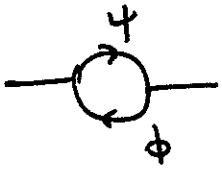


$$= i\bar{\sigma}\cdot p\, \delta_{Z\psi} \qquad \text{so} \qquad \delta_{Z\psi} = -(g^2 + h^2) C_2(r) \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

for λ:



$$= \left(\frac{i}{(4\pi)^2} b_f \Lambda^2/M^2\right) g^2 C_2(G)\ \bar{\sigma}\cdot p$$

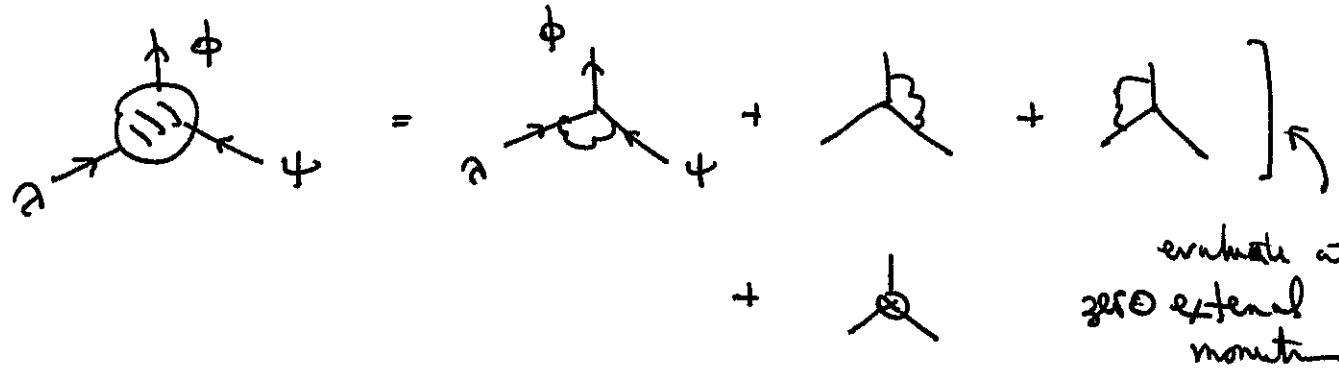


$$= \frac{i}{(4\pi)^2} b_f \Lambda^2/M^2 \cdot h^2 \cdot \underbrace{n_f C(r)}\ \bar{\sigma}\cdot p$$

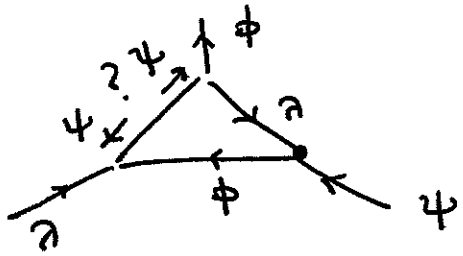
$t^a t^a$ is replaced by $\mathrm{tr}[t^a t^b]$

$$\delta_{Z\lambda} = -\left[g^2 C_2(G) + h^2 n_f C(r)\right] \frac{1}{(4\pi)^2} b_f \Lambda^2/M^2$$

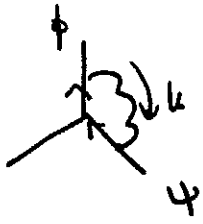
Now look at the vertex diagrams. Start with h:



Note that there is no h^3 diagram:



the arrows do not work out.



$$= (\sqrt{2}ih)(ig)^2 \int \frac{d^dk}{(2\pi)^d} (k^\mu t^b) \frac{1}{k^2} t^a \epsilon \frac{i\sigma\cdot k}{k^2} \bar{\sigma}_\mu t^b \cdot \frac{-i}{k^2}$$

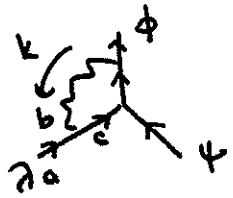
$$= (\sqrt{2}hg^2)(+1)[t^b t^a t^b \epsilon] \int \frac{d^dk}{(2\pi)^d} \frac{\sigma\cdot k\, \bar{\sigma}\cdot k}{(k^2)^3}$$

$$= +\sqrt{2}hg^2 \frac{+i}{(4\pi)^2} \int \Lambda^2/M^2 \cdot t^b t^a t^b \epsilon$$

$$(-1)^2 \overbrace{\lambda^c f^{cba} \lambda^a}^{t} \lambda^\tau \epsilon \psi$$

$$\frac{b}{4} = igt^b$$

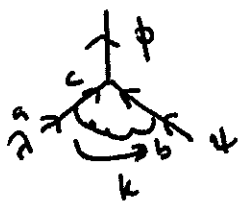
$$\frac{c}{a} = igt^b_a = -gf^{abc}$$



$$= \sqrt{2}ih \cdot ig(-g) \int \frac{d^dk}{(2\pi)^d} k^\mu t^b \frac{i}{k^2} \cdot (\bar{\sigma}^\mu)^T \left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon\, f^{cba} t^c \frac{-i}{k^2}$$

$$= +i\sqrt{2}hg^2 \int \frac{d^dk}{(2\pi)^d} (k\cdot\bar{\sigma})^T (\sigma\cdot k)^T \frac{1}{(k^2)^3} \epsilon\, f^{cba} t^b t^c$$

$$= \sqrt{2}hg^2 \frac{i}{(4\pi)^2} \int \Lambda^2/M^2 \cdot \epsilon \cdot (+i f^{cba} t^b t^c)$$



$$= \sqrt{2}ih(-g)ig \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2} (\bar{\sigma}^\mu)^T \left(\frac{i\sigma\cdot(-k)}{k^2}\right)^T \epsilon \frac{i\sigma\cdot k}{k^2} \bar{\sigma}_\mu f^{cba} t^c t^b$$

$$= \sqrt{2}hg^2(-i) \cdot \int \frac{d^dk}{(2\pi)^d} \epsilon\, \sigma_\mu \frac{\bar{\sigma}\cdot k\, \sigma\cdot k\, \bar{\sigma}^\mu}{(k^2)^3} \cdot f^{cba} t^c t^b$$

$$= \sqrt{2}\, h g^2 \, \frac{i}{(4\pi)^2} \ln \Lambda^2/M^2 \cdot 4\epsilon \cdot (-i f^{cba} t^c t^b)$$

now $$t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b]$$

$$= C_2(r)\, t^a + i f^{abc} t^b t^c$$

$$= C_2(r)\, t^a + i f^{abc} \tfrac{1}{2} i f^{bcd} t^d$$

$$= \left[C_2(r) - \tfrac{1}{2} C_2(G) \right] t^a$$

$$-i f^{cba} t^c t^b = -i f^{cba} \tfrac{1}{2} i f^{cbd} t^d = + \tfrac{1}{2} C_2(G)\, t^a$$

$$+i f^{cba} t^b t^c = + \tfrac{1}{2} C_2(G)\, t^a$$

so



$$= \sqrt{2}\, h g^2 \cdot i \cdot \frac{1}{(4\pi)^2} \ln \Lambda^2/M^2 \cdot \epsilon\, t^a$$

$$\cdot \Big\{ + \left[C_2(r) - \tfrac{1}{2} C_2(G) \right] + \tfrac{1}{2} C_2(G) + \tfrac{4}{2} C_2(G) \Big\}$$

$$= \sqrt{2}\, i h\, g^2 \, \frac{\log \Lambda^2/M^2}{(4\pi)^2} \epsilon\, t^a$$

$$\cdot \Big\{ + C_2(r) + 2\, C_2(G) \Big\}$$

$$\text{a} \qquad \delta_h = -g^2 h \left(\frac{1}{(4\pi)^2} I_J \Lambda^2/M^2\right) [C_2(r) + 2C_2(G)]$$

then

$$\beta_h = M \frac{\partial}{\partial M} \left[-\delta_h + h/2 \left[\delta_{2a} + \delta_{2\phi} + \delta_{24} \right] \right]$$

$$= M \frac{\partial}{\partial M} \Big[+ g^2 h \frac{1}{(4\pi)^2} [C_2(r) + 2C_2(G)] I_J \Lambda^2/M^2$$

$$+ h/2 \Big\{ - g^2 [C_2(G) + h^2 n_f C(r)] + (2g^2 - 2h^2) C_2(r)$$

$$- (g^2 + h^2) C_2(r) \Big\} \frac{1}{(4\pi)^2} I_J \Lambda^2/M^2 \Big]$$

$$= \frac{1}{(4\pi)^2} \Big\{ -2g^2 h [C_2(r) + 2 C_2(G)]$$

$$+ h [g^2 C_2(G) + h^2 n_f C(r) + 2(h^2 - g^2) C_2(r)$$

$$+ (g^2 + h^2) C_2(r)] \Big\}$$

$$= + \frac{1}{(4\pi)^2} \Big\{ C_2(G) [-4g^2 h + g^2 h]$$

$$+ C_2(r) [-2g^2 h + 2h(h^2 - g^2) + h(g^2 + h^2)]$$

$$+ h^3 n_f C(r) \Big\}$$

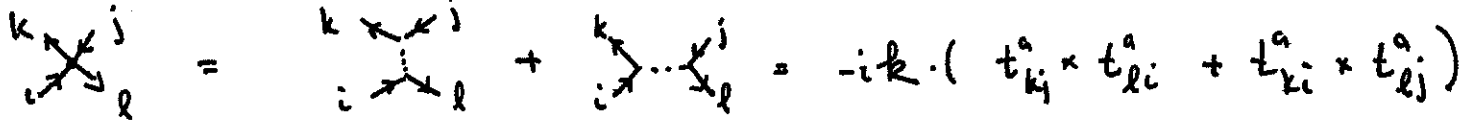
so

$$\beta_h = -\frac{1}{(4\pi)^2}\left\{ 3g^2 h\, C_2(G) - 3(h^2-g^2)h\, C_2(r) - h^3 n_f C(r) \right\}$$

$$= -\frac{1}{(4\pi)^2} g^3 \left[3C_2(G) - n_f C(r) \right] \quad \text{for } g = h$$

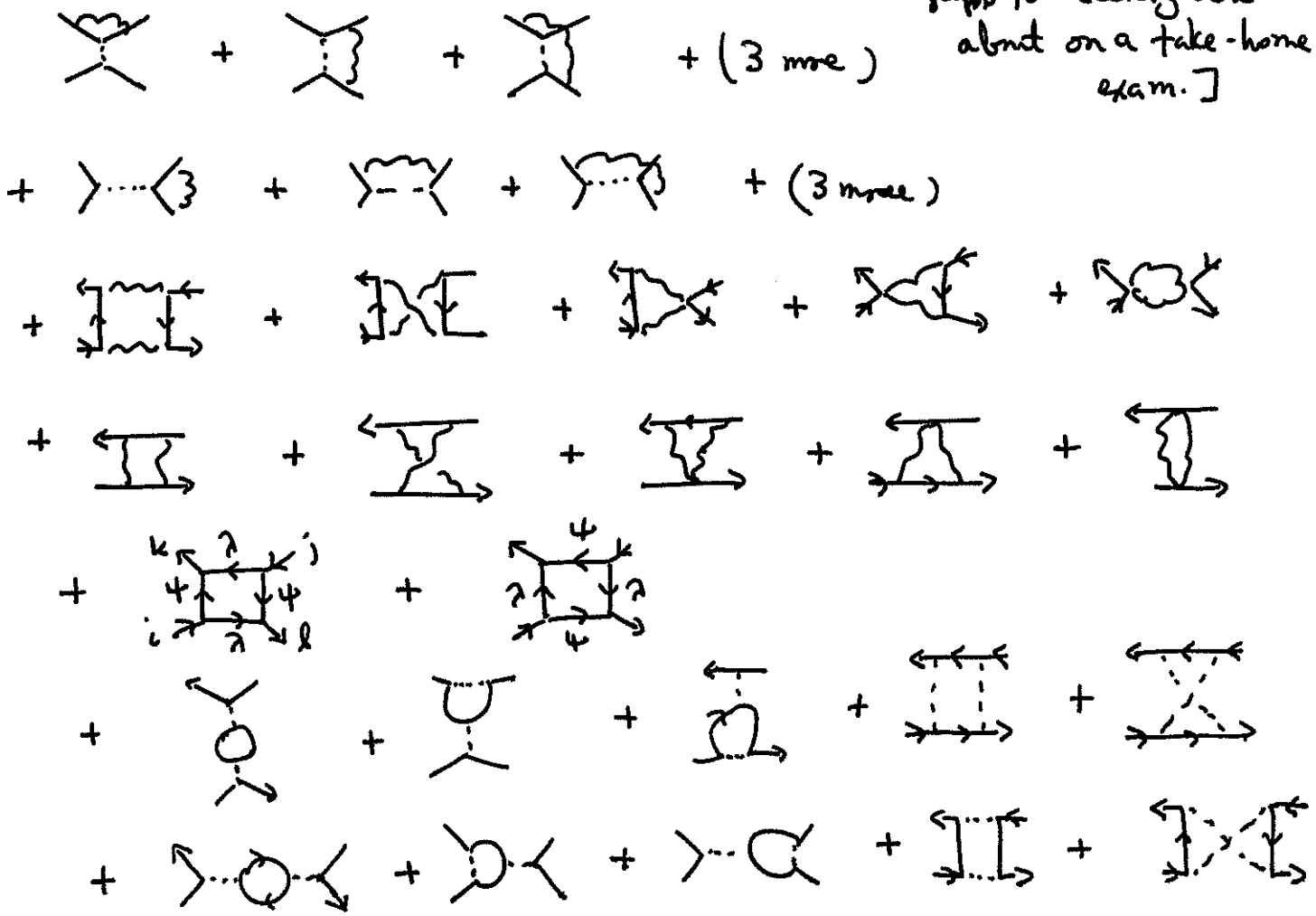
Next, k:

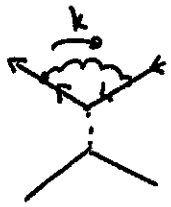
to write the graphs, it is convenient to use the representation:



The vertex corrections are then:

[This is probably too many graphs to decently ask about on a take-home exam.]





$$= -ik \cdot (ig)^2 \cdot \int \frac{d^dk}{(2\pi)^d} \; k_\mu \frac{i}{k^2} t^b \; t^a \frac{i}{k^2} t^b k^\mu$$

$$t^a \cdot \frac{-i}{k^2}$$

$$= -kg^2 \cdot t^b t^a t^b \times -t^a \cdot \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2)^3}$$

$$= -kg^2 \left[C_2(S) - \tfrac{1}{2} C_2(A)\right] (t^a \times t^a) \frac{i}{(4\pi)^2} \ln \Lambda^2/M^2$$



$$= -ik \cdot (ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-1}{k^2}$$

$$\cdot (t^a t^b) \; k_\mu \frac{i}{k^2} \times t^b t^a \; k_\mu \frac{i}{k^2}$$

$$= -kg^2 \left[(t^a t^b) \times (t^b t^a)\right] \cdot \frac{i}{(4\pi)^2} \ln \Lambda^2/M^2$$

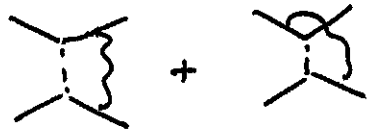


$$= -ik \; (ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2}$$

$$\left[\frac{i}{k^2}(k_\mu t^b t^a)\right] \times \left(\frac{i}{k^2}(-k_\mu) t^b t^a\right)$$

$$= +kg^2 \left[t^b t^a \otimes t^b t^a\right] \cdot \frac{i}{(4\pi)^2} \ln \Lambda^2/M^2$$

so



$$= -kg^2\ [t^a, t^b] \times t^b t^a\ \frac{i}{(4\pi)^2} I_J\, \Lambda^2/M^2$$

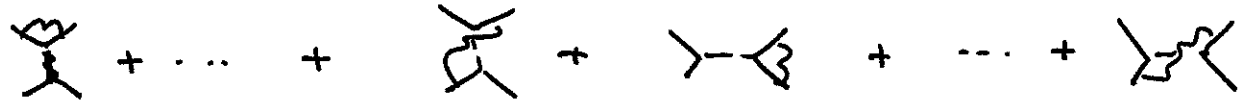
$$= kg^2\ i f^{abc} (t^c \times t^b t^a)\ \frac{i}{(4\pi)^2} I_J\, \Lambda^2/M^2$$

$$= kg^2\ \tfrac{1}{2}\, i f^{abc}\, i f^{bad}\ t^c \times t^d\ \frac{i}{(4\pi)^2} I_J\, \Lambda^2/M^2$$

$$= -kg^2\ \tfrac{1}{2} C_2(G)\ (t^a \times t^a)\ \frac{i}{(4\pi)^2} I_J\, \Lambda^2/M^2$$

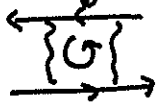
the other 3 diagrams of this type are equal to these.

the 6 diagrams in the 2nd line on p. 19 are equal to these 6 with k→l. So



$$= -\frac{i}{(4\pi)^2}\ 2kg^2\ C_2(r)\ (t^a_{kj} \times t^a_{li} + t^a_{ki} \times t^a_{lj})\ I_J\, \Lambda^2/M^2$$

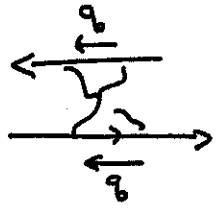
next, the graphs with gauge exchange only:



$$= (ig)^4 \int \frac{d^dq}{(2\pi)^d} \left(g^\mu \frac{i}{q^2} g^\nu\right) \frac{-i}{q^2} \left(g_\nu \frac{i}{q^2} g_\mu\right) \frac{-i}{q^2} \times (t^a t^b)_{kj} (t^b t^a)_{li}$$

$$= + g^4 \int \frac{d^dq}{(2\pi)^d} \frac{(q^2)^2}{(q^2)^4} (t^a t^b)_{kj} \times (t^b t^a)_{li}$$

$$= \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) \cdot g^4 (t^a t^b)_{kj} \times (t^b t^a)_{li}$$



$$= (ig)^4 \int \frac{d^dq}{(2\pi)^d} \left(g^\mu \frac{i}{q^2} g^\nu\right) \frac{-i}{q^2} \left((-g^\mu) \frac{i}{q^2} (-g^\nu)\right) \frac{-i}{q^2} \times (t^a t^b)_{kj} (t^a t^b)_{li}$$

$$= \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) g^4 (t^a t^b) \times (t^a t^b)$$



$$= (ig)^2 (ig^2) \int \frac{d^dq}{(2\pi)^d} \left(g^\mu \frac{i}{q^2} g^\nu\right) \left(\frac{-i}{q^2}\right)^2 g_{\mu\nu} \times (t^a t^b)_{kj} (\{t^a, t^b\})_{li}$$

$$= (-g^4) \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) (t^a t^b) \times (\{t^a, t^b\})$$



$$= -g^4 \frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2 (\{t^a, t^b\}) \times (t^a t^b)$$



$$= \frac{1}{2}(ig^2)^2 \int \frac{d^dq}{(2\pi)^d} \left(\frac{-i}{q^2}\right)^2 g^{\mu\nu} g_{\mu\nu} \{t^a, t^b\}\{t^a, t^b\}$$

$$= +g^4 \cdot \left(\frac{i}{(4\pi)^2} b_g \Lambda^2/M^2\right) \cdot 4 \cdot \frac{1}{2} \{t^a, t^b\}\{t^a, t^b\}$$

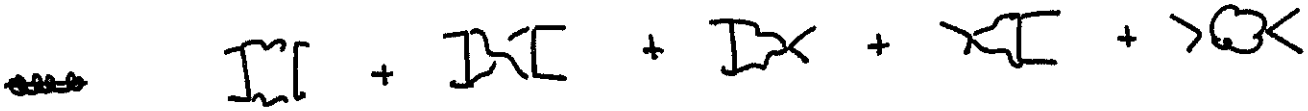
then



$$= \frac{i}{(4\pi)^2}(b_g \Lambda^2/M^2)\, g^4 \,(t^a t^b)\cdot(\{t^a, t^b\})$$

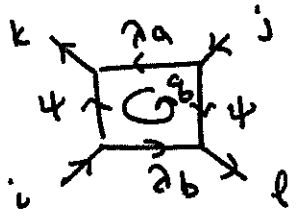
$$\cdot (1 - 2 + 4)$$

$$= \left(\frac{i}{(4\pi)^2} b_g \Lambda^2/M^2\right)(+3g^4)(t^a t^b)_{kj} \cdot (\{t^a, t^b\})_{li}$$



$$= \left(\frac{i}{(4\pi)^2} b_g \Lambda^2/M^2\right)(+3g^4)\,(t^a t^b)_{ki} \cdot (\{t^a, t^b\})_{lj}$$

next, the graph with fermion loops:



$=$ $\phi_k^* \overbrace{\partial^{aT} t^a \epsilon \psi_k \; \bar\psi_i \epsilon \partial^{bT} t^b \phi_i \; \phi_l^* \partial^{T b} t^b \psi_l \; \bar\psi_j \epsilon t^a \partial^{*a} \phi_j}$

correct order: $\partial \partial^*$

(-1) for wrong order

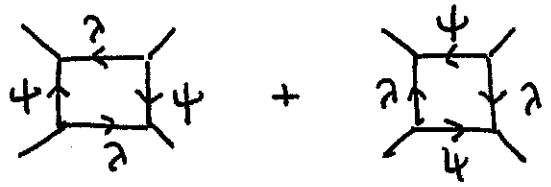
$$= (\sqrt{2} i h)(-i\sqrt{2} h)(\sqrt{2} i h)(-i\sqrt{2} h) \int \frac{d^d q}{(2\pi)^d} \; (t^a t^b)_{ki} \, (t^b t^a)_{lj}$$

$$\mathrm{tr}\left[\epsilon \; \frac{i\sigma\cdot(-q)}{q^2} \; \epsilon \, (-1) \left(\frac{i\sigma\cdot q}{q^2}\right)^T \epsilon \left(\frac{i\sigma\cdot(-q)}{q^2}\right) \epsilon \left(\frac{i\sigma\cdot q}{q^2}\right)^T \right]$$

$$= 4h^4 \cdot (-1) \cdot \int \frac{d^d q}{(2\pi)^d} \, \mathrm{tr}\left(\frac{\sigma\cdot q \; \bar\sigma\cdot q \; \sigma\cdot q \; \bar\sigma\cdot q}{(q^2)^4}\right) (t^a t^b) \times (t^b t^a)$$

$$= -8h^4 \left(\frac{i}{(4\pi)^2} \ln \Lambda^2/M^2\right) \; (t^a t^b) \times (t^b t^a)$$

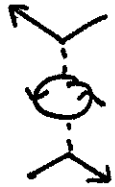
the other diagram gives the same result in the other channel.



$$= (-8h^4) \left(\frac{i}{(4\pi)^2} \ln \Lambda^2/M^2\right) \left[(t^a t^b)_{kj} \times (t^b t^a)_{li} \right.$$

$$\left. + \, (t^a t^b)_{ki} \times (t^b t^a)_{lj} \right]$$

Finally, the diagrams with 2 scalar vertices:



$$= (-ik)^2 \int \frac{d^d q}{(2\pi)^d} \left(\frac{i}{q^2}\right)^2 (t^a)_{kj} \cdot \mathrm{tr}(t^a t^b) \cdot (t^b)_{li}$$

$$= k^2 \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) t^a_{kj}\, t^a_{li} \cdot n_f C(r)$$

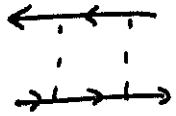


$$= (-ik)^2 \int \frac{d^d q}{(2\pi)^d} \left(\frac{i}{q^2}\right)^2 (t^b t^a t^b)_{kj} \cdot (t^a)_{li}$$

$$= k^2 \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] t^a_{kj} \cdot t^a_{li}$$

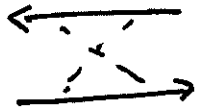


$$= k^2 \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] t^a_{kj}\, t^a_{li}$$



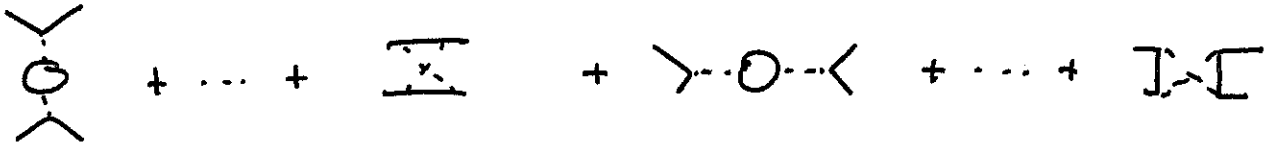
$$= (-ik)^2 \int \frac{d^d q}{(2\pi)^d} d\left(\frac{i}{q^2}\right) (t^a t^b)_{kj}\, (t^b t^a)_{li}$$

$$= k^2 \frac{i}{(4\pi)^2} (\ell_J \Lambda^2/M^2)\, (t^a t^b)_{kj}\, (t^b t^a)_{li}$$



$$= k^2 \left(\frac{i}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) (t^a t^b)_{kj}\, (t^a t^b)_{li}$$

the five diagrams in the last line of p. 19 give the same result in the other channel.



$$= \left(\frac{i}{(4\pi)^2} g_f \Lambda^2/M^2\right) \cdot k^2 \cdot$$

$$\{ \left[(t^a)_{ij}(t^a)_{li} + (t^a)_{ki}(t^a)_{lj}\right]\left[2C_2(r) - C_2(G) + n_f C(r)\right]$$

$$+ \quad (t^a t^b)_{kj}(\{t^a, t^b\})_{li} + (t^a t^b)_{ki}(\{t^a, t^b\})_{lj} \}$$

Now add all of the contributions:



$$= \left(\frac{i}{(4\pi)^2} g_f \Lambda^2/M^2\right)$$

$$\cdot \Big[\quad -2g^2 k\, C_2(r) \quad [t^a \cdot t^a + t^a \cdot t^a]$$

$$+ 3g^4 \quad [(t^a t^b) \cdot (\{t^a, t^b\}) + (t^a t^b) \cdot (\{t^a, t^b\})]$$

$$- 8h^4 \quad [(t^a t^b) \cdot (t^b t^a) + (t^a t^b)(t^b t^a)]$$

$$+ k^2 \quad [2C_2(r) - C_2(G) + n_f C(r)]$$

$$+ -k^2 \quad [(t^a t^b)(\{t^a, t^b\}) + (t^a t^b)(\{t^a, t^b\})] \Big]$$

this is a very complicated expression in the general case. However, it has a nice simplification for the supersymmetric situation: $h = g \quad k = g^2$. From here on, I will restrict myself to that case.



$$= \frac{i}{(4\pi)^2}\left(\ln \Lambda^2/M^2\right)$$

$$\cdot \Big[\left(t^a \cdot t^a + t^a \cdot t^a\right) \cdot \Big[-2 g^4 C_2(r) + g^4 \left[2C_2(r) - C_2(G) + n_f C(r) \right] \Big]$$

$$+ \left[(t^a t^b) \cdot (t^a t^b + t^b t^a) + (t^a t^b) \cdot (t^a t^b + t^b t^a) \right] \cdot \left[3g^4 + g^4 \right]$$

$$+ \left[(t^a t^b)\,(t^b t^a) + (t^a t^b)\,(t^b t^a) \right] \left[-8 g^4 \right] \Big]$$

$$= \left(\frac{i}{(4\pi)^2} \ln \Lambda^2/M^2 \right) g^4$$

$$\Big[\left(t^a \cdot t^a + t^a \cdot t^a\right) \left[-C_2(G) + n_f C(r) \right]$$

$$+ 4 \left[(t^a t^b)\,(t^a t^b - t^b t^a) + (t^a t^b)(t^a t^b - t^b t^a) \right] \Big]$$

now $t^a t^b \; [t^a, t^b] = i f^{abc} (t^a t^b)(t^c)$

$= i f^{abc} \cdot \frac{1}{2} i f^{abd} (t^d)(t^c) = -\frac{1}{2} C_2(G_1)\, t^a \cdot t^a$

then



$= \frac{i}{(4\pi)^2} g_3^2 \Lambda^2/M^2 \quad (t^a \cdot t^a + t^a \cdot t^a)$

$\cdot [- C_2(G_1) + n_f C(r) + 4(-\tfrac{1}{2} C_2(G_1))]$

$= \frac{i}{(4\pi)^2} g_3^2 \Lambda^2/M^2 \quad (t^a \cdot t^a + t^a \cdot t^a)\, [- 3 C_2(G_1) + n_f C(r)]$

this is cancelled by



$= -i \delta_k \quad [t^a \cdot t^a + t^a \cdot t^a]$

so $\delta_k = (\frac{-1}{(4\pi)^2} g_3^2 \Lambda^2/M^2)\, [3 C_2(G_1) - n_f C(r)]$

from p. 13, for $g = h$, $\delta_{Z\phi} = 0$

then

$\beta_k = M \frac{\partial}{\partial M} [- \delta_k + 4 \cdot \tfrac{1}{2} \delta_{Z\phi}]$

$= \frac{-2}{(4\pi)^2} [3 C_2(G_1) - n_f C(r)]\, g^4$

we have now seen that, for $g = h$ $k = g^2$:

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right]$$

$$\beta_h = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right] = \beta_g$$

$$\beta_k = -2\frac{g^4}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right] = 2g\beta_g$$

This is just what is needed for the relation

$$h = g \qquad k = g^2$$

to be preserved by the renormalization group.

e.) To compute the QCD contributions to the β function of the three fermion-fermion-boson interaction in (5) we need the QCD corrections to these vertices and the QCD contributions to $\delta_{Z\phi}$ $\delta_{Z\psi}$:

From pp. 13, 14

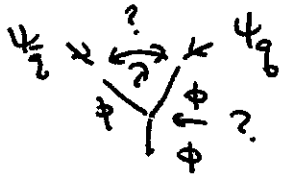
$$\delta_{Z\phi_q} = 0 \qquad \delta_{Z\psi_q} = -2g^2 C_2(r)\frac{1}{(4\pi)^2}\int \frac{1}{M^2}$$

for $q, \bar{q}$

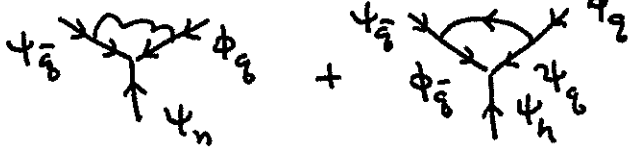
of course, for h: $\delta_{Z\phi_h} = \delta_{Z\psi_h} = 0$ since h is a color singlet.

for the vertex $\delta\mathcal{L} = y\, \psi_{\bar{q}} \epsilon \psi_q \phi_n$

the only correction is 

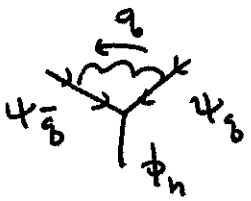
since there is no possible diagram with ∂ : 

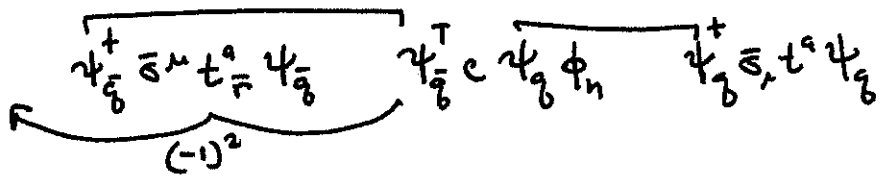
However, for $\delta\mathcal{L} = y\, \psi_{\bar{q}} \epsilon \phi_q \psi_n$

there are corrections 

The third vertex $\delta\mathcal{L} = y\, \phi_{\bar{q}}\, \psi_q \epsilon \psi_n$ has the same situation as this.

So, let's compute these three diagrams :





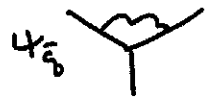
$$= (iy)(ig)^2 \int \frac{d^d q}{(2\pi)^d} \; (\bar{\sigma}^\mu)^T (t^a_{\bar{r}})^T \left(\frac{i\sigma q}{q^2}\right)^T \epsilon \; \frac{i\sigma\cdot(-q)}{q^2}\, \bar{\sigma}_\nu\, t^a_r \left(\frac{-i}{q^2}\right.$$

$$= -yg^2 \left(t^{a\,T}_{\bar{r}} t^a_r\right) \int \frac{d^d q}{(2\pi)^d} \; \sigma^\mu \frac{\bar{\sigma}\cdot q\, \sigma\cdot q\, \bar{\sigma}_\nu}{(q^2)^3}$$

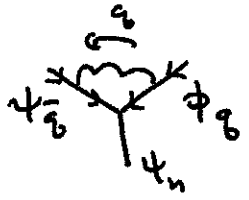
$$= -yg^2 \left[(t^a_{\bar{r}})^T t^a_r\right] \int \frac{d^d q}{(2\pi)^d} \; 4\, \frac{1}{(q^2)^2}$$

now $t^a_{\bar r} = -(t^a_r)^T$ $\quad (t^a_{\bar r})^T t^a_r = - t^a_r t^a_r = -C_2(r)$

so



$$= +4 C_2(r)\, y g^2 \epsilon \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right)$$

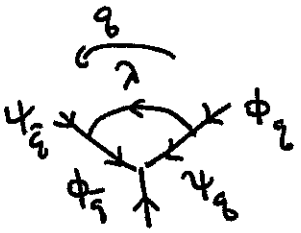


$$= (iy)(ig)^2 \int \frac{d^d q}{(2\pi)^d} (\bar\sigma^\mu)^T (t^a_{\bar r})^T \left(\frac{i\sigma\cdot q}{q^2}\right)^T \epsilon$$

$$\left(\frac{+i}{q^2}\right)(-q_\mu)\, t^a_r \left(\frac{-1}{q^2}\right)$$

$$= -y g^2 (t^a_r)^T t^a_r \epsilon \int \frac{d^d q}{(2\pi)^d} \frac{\sigma\cdot q\ \bar\sigma\cdot q}{(q^2)^3}$$

$$= +y g^2 C_2(r) \epsilon \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right)$$



$$(\psi^T_{\bar q} \epsilon (-t^a_{\bar r}) \partial^a \phi^*_{\bar q})(\phi_{\bar q}\, \psi^T_{\bar q} \epsilon\, \psi_n)\underbrace{(\psi^\dagger_{\bar q} \epsilon\, \partial^{*a} t^a_r \phi_q)}_{(-1)^2}$$

$$= (\sqrt{2} ig)(iy)(-\sqrt{2} ig) \int \frac{d^d q}{(2\pi)^d} (-t^a_r t^a_r)\ \frac{i}{q^2}$$

$$\cdot \epsilon\ \frac{i\sigma\cdot q}{q^2}\ \epsilon^T \left(\frac{i\sigma\cdot(-q)}{q^2}\right)^T \epsilon \qquad \epsilon^T = -\epsilon,\ \epsilon^2 = -1$$

$$= 2 y g^2\ C_2(r) \int \frac{d^d q}{(2\pi)^d} \epsilon \left(\frac{\sigma q}{q^2} \frac{\bar\sigma\cdot q}{q^2} \frac{1}{q^2}\right)$$

$$= 2 y g^2 C_2(r) \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

cancelling the corrections with  $= i\delta_y \epsilon$

$$\delta_y(\psi_{\bar{q}}\psi_q\phi_n) = -4yg^2\frac{C_2(r)}{(4\pi)^2}\ell y\,\Lambda^2/M^2$$

$$\delta_y(\psi_{\bar{q}}\phi_q\psi_n) = -3yg^2\frac{C_2(r)}{(4\pi)^2}\ell y\,\Lambda^2/M^2$$

then the QCD contributions to the β functions are:

$$\beta_y(\psi_{\bar{q}}\psi_q\phi_n) = M\frac{\partial}{\partial M}\left[-\delta_y(\psi_{\bar{q}}\psi_q\phi_n) + \frac{2}{2}y\,\delta_2\psi_q\right]$$

$$= -8yg^2C_2(r)\frac{1}{(4\pi)^2} + y\,4g^2C_2(r)\frac{1}{(4\pi)^2}$$

$$= -4\frac{yg^2C(r)}{(4\pi)^2}$$

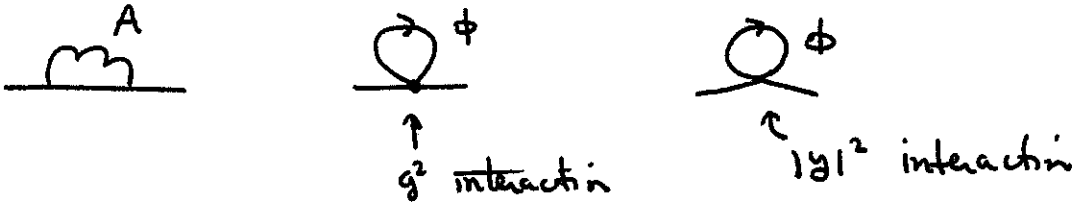
$$\beta_y(\psi_{\bar{q}}\phi_q\psi_n) = M\frac{\partial}{\partial M}\left[-\delta_y(\psi_{\bar{q}}\phi_q\psi_n) + \tfrac{1}{2}y(\delta_2\psi_q + \delta_2\phi_q)\right]$$

$$= -6yg^2C_2(r)\frac{1}{(4\pi)^2} + y\cdot(2+0)g^2\frac{C_2(r)}{(4\pi)^2}$$

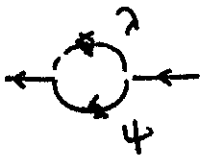
$$= -4\frac{yg^2C(r)}{(4\pi)^2}$$

these β functions are <u>equal</u>!

f.) The corrections to the scalar masses come from the diagrams on p. 5 & p. 10. The supersymmetric contributions cancel, but if there are supersymmetry-breaking perturbations such as eq. (6), we can get a nonzero contribution. The scalar masses potentially contribute to



There is also a contribution from



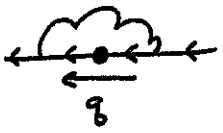
if the λ are massive.

I would first like to show that the contribution from the g^2 diagrams cancel. Expand in the Φ mass

$$\frac{i}{q^2-m^2} = \frac{i}{q^2} + \frac{i}{q^2}(-im^2)\frac{i}{q^2} + \cdots$$



The $g^2 m^2$ terms are then: (at zero external momentum)



$$= (ig)^2 \int \frac{d^d q}{(2\pi)^d}\, t^a q^\mu \frac{i}{q^2}(-im^2)\frac{i}{q^2} q_\mu t^a \frac{-i}{q^2}$$

$$= -g^2 C_2(r) \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(q^2)^3}$$

$$= -i\, g^2 C_2(r) \frac{1}{(4\pi)^2} \ln \Lambda^2/M^2$$



$$= -ik \int \frac{d^d q}{(2\pi)^d}\, t^a \frac{i}{q^2}(-im^2)\frac{i}{q^2} t^a$$

$$= +\, g^2 C_2(r) \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{q^2}\right)^2 \qquad \text{w/ } k = g^2$$

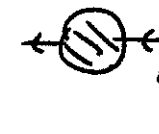
$$= +\, i g^2 C_2(r) \frac{1}{(4\pi)^2} \ln \Lambda^2/M^2$$

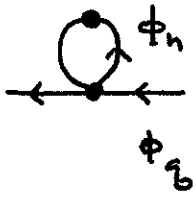
[Note: this cancellation is specifically in Feynman gauge; otherwise the cancellation involves these two diagrams and $\delta_{Z\phi}$.]

The remaining contributions come from the interaction

$$\delta\mathcal{L} = -y^2 |\phi\phi|^2$$

In the specific model of eq. (5), there are three vertices of this type, and these contribute to the ϕ_h, ϕ_q, and $\phi_{\bar{q}}$ masses. Let's begin by working out one of these contributions quite explicitly:

the contribution to  from $\delta\mathcal{L} = -y\,|\phi_q \phi_h|^2$



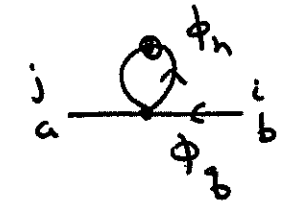
$$= (-iy^2) \int \frac{d^d q}{(2\pi)^d} \frac{i}{q^2} (-i m_h^2) \frac{i}{q^2}$$

$$= y^2 \left(\frac{i}{(4\pi)^2} \ln \Lambda^2/M^2\right) \cdot M_h^2$$

there is also a flow of indices. Let $ij = 123$ be color indices, $a,b = 1,2$ be $SU(2)$ indices. The vertex is, explicitly

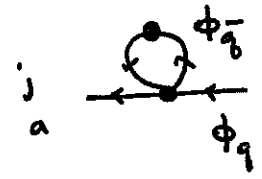
$$\delta\mathcal{L} = -y^2 \phi_h^{a*} \phi_{q\,a}^{*i} \phi_{q\,i}^{b} \phi_{h\,b}$$

contracting ϕ_h^*, ϕ_h gives δ_{ab} so

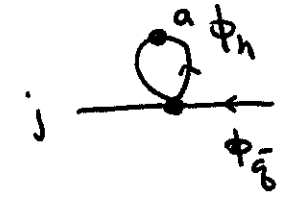


$$= \left(\frac{i}{(4\pi)^2} \ln \Lambda^2/M^2\right)(y^2 M_h^2)\, \delta_{ij}\delta_{ab}$$

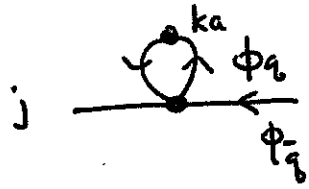
similarly :



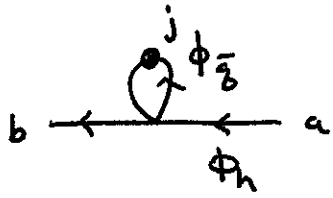
$$= \left(\frac{i}{(4\pi)^2} I_J \Lambda^2/M^2\right) \; y^2 M_{\bar{q}}^2 \quad \delta_{ij}\delta_{ab}$$



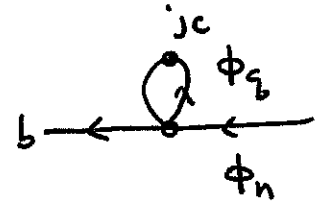
$$= \left(\frac{i}{(4\pi)^2} I_J \Lambda^2/M^2\right) \; y^2 M_h^2 \quad \delta_{ij}\,(\delta_{aa})$$



$$= \left(\frac{i}{(4\pi)^2} I_J \Lambda^2/M^2\right) \; y^2 M_q^2 \quad \delta_{ij}(\delta_{aa})$$



$$= \left(\frac{i}{(4\pi)^2} I_J \Lambda^2/M^2\right) \; y^2 M_{\bar{q}}^2 \,(\delta_{jj})\,\delta_{ab}$$



$$= \left(\frac{i}{(4\pi)^2} I_J \Lambda^2/M^2\right) \; y^2 M_q^2 \,(\delta_{jj})\,\delta_{ab}$$

$$\delta_{aa} = 2 \qquad \delta_{jj} = 3$$

from p. 6, there is also a $\delta_{Z\phi}$ proportional to y^2:

$$\delta_{Z\phi_q} = -2y^2 \frac{1}{(4\pi)^2} I_J \Lambda^2/M^2$$

$$\delta_{Z\phi_{\bar{q}}} = -2y^2 \frac{1}{(4\pi)^2} I_J \Lambda^2/M^2 \cdot (\delta_{aa})$$

$$\delta_{Z\phi_h} = -2y^2 \frac{1}{(4\pi)^2} I_J \Lambda^2/M^2) \cdot (\delta_{jj})$$

the contributions to  on p. 36 are

cancelled by $-i\,\delta_{M^2}$, so

$$\delta M_q^2 = \left(\frac{1}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) \cdot y^2 \left(M_{\tilde{q}}^2 + M_h^2\right)$$

$$\delta M_{\tilde{q}}^2 = \left(\frac{1}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) \cdot y^2 \left(M_q^2 + M_h^2\right) \cdot 2$$

$$\delta M_h^2 = \left(\frac{1}{(4\pi)^2} \ell_J \Lambda^2/M^2\right) \cdot y^2 \left(M_q^2 + M_{\tilde{q}}^2\right) \cdot 3$$

then

$$\beta(M_q^2) = M \frac{\partial}{\partial M}\left(-\delta M_q^2 + \frac{z}{2} M_q^2 \delta z \phi_q\right)$$

$$= + \frac{2}{(4\pi)^2} y^2 \left(M_{\tilde{q}}^2 + M_h^2\right) + M_q^2 y^2 \frac{2}{(4\pi)^2}$$

$$= \frac{2}{(4\pi)^2} y^2 \left(M_{\tilde{q}}^2 + M_h^2 + M_q^2\right)$$

$$\beta(M_{\tilde{q}}^2) = \frac{2}{(4\pi)^2} y^2 \left(M_q^2 + M_h^2 + M_{\tilde{q}}^2\right) \cdot 2$$

$$\beta(M_h^2) = \frac{2}{(4\pi)^2} y^2 \left(M_q^2 + M_{\tilde{q}}^2 + M_h^2\right) \cdot 3$$

so

$$\frac{d}{d\ln Q} M_Q^2 = + \frac{2}{(4\pi)^2} y^2 \left(M_h^2 + M_Q^2 + M_{\bar{Q}}^2\right)$$

$$\frac{d}{d\ln Q} M_{\bar{Q}}^2 = + 2 \cdot \frac{2}{(4\pi)^2} y^2 \left(M_h^2 + M_Q^2 + M_{\bar{Q}}^2\right)$$

$$\frac{d}{d\ln Q} M_h^2 = + 3 \cdot \frac{2}{(4\pi)^2} y^2 \left(M_h^2 + M_Q^2 + M_{\bar{Q}}^2\right)$$

the sign is such that the M^2 increase at large Q, decrease at small Q.

g.) The qualitative nature of the flow depends on the initial conditions on the RG flow. But

assume ~~that~~ $M_h^2 = M_Q^2 = M_{\bar{Q}}^2 > 0$ at a high scale Q_0

Then, at smaller Q, $M_h^2 \to 0$ faster than $M_Q^2, M_{\bar{Q}}^2$. Going further M_h^2 becomes negative, ϕ_h acquires a VEV, & $SU(2)$ is spontaneously broken. This is a concrete mechanism for $SU(2) \times U(1)$ electroweak symmetry breaking.