Physics 443 Homework #1
Due Thursday, October 7, 2010

1.) Peskin & Schroeder, problem 2.1

2.) Peskin & Schroeder, problem 2.2

3.) Peskin & Schroeder, problem 2.3

4.) The classical limit of a harmonic oscillator can be described in terms of *coherent states*

$$|\alpha\rangle = \exp\left[\alpha a^\dagger\right] \ |0\rangle \ .$$

When α is large, the oscillator state is semiclassical. Proceeding similarly for the Fourier modes of the quantum Klein-Gordon field,

$$\begin{aligned} |f\rangle =& N_f \exp\left[i\int \frac{d^3p}{(2\pi)^3} f(\mathbf{p}) a^\dagger_{\mathbf{p}}\right] \ |0\rangle \\ N_f =& \exp\left[-\tfrac{1}{2}\int \frac{d^3p}{(2\pi)^3}|f(\mathbf{p})|^2\right] \ . \end{aligned}$$

(a) Evaluate the expectation value of the field operator

$$\langle f|\, \phi(x)\, |f\rangle \ ,$$

and show that it satisfies the Klein-Gordon equation.

(b) Evaluate the relative mean square fluctuation of the occupation number of the mode with momentum $\mathbf{p}$, and the relative mean square fluctuation in the total energy

$$\begin{aligned} &\frac{\langle \hat{n}^2_{\mathbf{p}}\rangle - \langle \hat{n}_{\mathbf{p}}\rangle^2}{\langle \hat{n}_{\mathbf{p}}\rangle^2} \\ &\frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2} \ . \end{aligned}$$

Is either of these a good measure of the degree to which the field is classical? Justify your answer.

(c) Take $\Delta(x-y) = \langle 0|\, \phi(\mathbf{x})\phi(\mathbf{y})\, |0\rangle$ (equal times) as a measure of the fluctuations or correlations of the field amplitude. Use your result for problem 2.3 (P&S) to evaluate this quantity. What is the meaning of the divergence as $\mathbf{x} \to \mathbf{y}$?

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QFT 1 : Problem Set 1

1.) Peskin & Schroeder 2.1

We begin with the action for the classical electromagnetic field :

$$S[A] = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}\right) \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(a)

homogeneous Maxwell equations

To derive the homogeneous Maxwell equations we form the Poincare dual of the Faraday tensor.

$$G^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \qquad F^{\alpha\beta} = -\tfrac{1}{2}\epsilon^{\alpha\beta\mu\nu} G_{\mu\nu}$$

G is divergenceless from the definition of F :

$$\partial_\alpha G^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\partial_\alpha\partial_\mu A_\nu = 0$$

From the expressions for the fields $(\mathbf{E}, \mathbf{B})$ in terms of the potentials $A^\alpha = (\Phi, \mathbf{A})^\alpha$:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Phi \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

We find (using Latin letters for spatial indices and Greek for spacetime indices) :

$$E^i = -\partial^0 A^i + \partial^i A^0 = F^{i0}$$

$$B^i = -\epsilon^{0ijk}\partial_j A_k = -\tfrac{1}{2}\epsilon^{0ijk} F_{jk} = G^{i0}$$

Thus,

$$\partial_\alpha G^{\alpha 0} = \partial_i G^{i0} = \partial_i B^i = 0$$

$$\partial_\alpha G^{\alpha i} = \partial_0 G^{0i} + \partial_j \epsilon^{ji0k} F_{0k} = -\partial_0 B^i - \epsilon^{0ijk}\partial_j E^k = 0$$

Or,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{E} = 0$$

inhomogeneous Maxwell equations

We begin with the Euler-Lagrange equation :

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Now,

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\tfrac{1}{2}\left(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha\right)F^{\alpha\beta} = -F^{\mu\nu}$$

Thus,

$$\partial_\alpha F^{\alpha 0} = \partial_i F^{i0} = \partial_i E^i = 0$$

$$\partial_\alpha F^{\alpha i} = \partial_0 F^{0i} - \partial_j \epsilon^{ji0k} G_{0k} = -\partial_0 E^i + \epsilon^{0ijk}\partial_j B^k = 0$$

Or,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0 \qquad \frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times\mathbf{B} = 0$$

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(b)

We derive the Noether current associated with an infinitesimal translation $x \to x+a$. Using the equations of motion :

$$\delta\mathcal{L} \;=\; \frac{\partial\mathcal{L}}{\partial A_\nu}\,\delta A_\nu \;+\; \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\,\partial_\mu\delta A_\nu \;=\; \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\,\delta A_\nu\right)$$

From $\delta\mathcal{L} \;=\; a^\alpha\partial_\alpha\mathcal{L}$ and $\delta A_\nu \;=\; a^\alpha\partial_\alpha A_\nu$ we find $\partial_\alpha T^\alpha{}_\beta \;=\; 0$ where :

$$T^\alpha{}_\beta \;=\; \frac{\partial\mathcal{L}}{\partial\left(\partial_\alpha A_\nu\right)}\,\partial_\beta A_\nu \;-\delta^\alpha{}_\beta\,\mathcal{L}$$

Or,

$$T^\alpha{}_\beta \;=\; -\,F^{\alpha\nu}\partial_\beta A_\nu \;+\; \tfrac{1}{4}\,\delta^\alpha{}_\beta\,F_{\mu\nu}F^{\mu\nu}$$

Note that this tensor is not symmetric. It is also neither gauge invariant nor traceless. To remedy these problems we construct :

$$\widehat{T}^\alpha{}_\beta \;=\; T^\alpha{}_\beta \;+\; \partial_\lambda\left(F^{\alpha\lambda}A_\beta\right)$$

Using the equations of motion :

$$\widehat{T}_{\alpha\beta} \;=\; -\,F_\alpha{}^\nu\partial_\beta A_\nu \;+\; \tfrac{1}{4}\,\eta_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu} \;+\; F_\alpha{}^\nu\partial_\nu A_\beta$$

Or,

$$\widehat{T}_{\alpha\beta} \;=\; F_{\alpha\mu}F_{\nu\beta}\,\eta^{\mu\nu} \;+\; \tfrac{1}{4}\,\eta_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu}$$

We compute the energy and momentum densities in terms of the fields. Now,

$$F_{\mu\nu}F^{\mu\nu} \;=\; 2\,F_{0j}F^{0j} \;+\; F_{ij}F^{ij}$$

From above :

$$F^{0j} \;=\; -\,F_{0j} \;=\; -\,E^j \qquad \text{and} \qquad F^{ij} \;=\; F_{ij} \;=\; -\,\epsilon^{ij0k}\,G_{0k} \;=\; -\,\epsilon^{0ijk}\,B^k$$

Using $\epsilon^{0ijk}\,\epsilon^{0ijq} \;=\; 2\,\delta^{kq}$, we have :

$$F_{\mu\nu}F^{\mu\nu} \;=\; -\,2\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

We consider,

$$\mathcal{E} \;=\; \widehat{T}^{00} \;=\; F^{0\mu}F^{\nu0}\,\eta_{\mu\nu} \;+\; \tfrac{1}{4}\,\eta^{00}\,F_{\mu\nu}F^{\mu\nu}$$

Or,

$$\mathcal{E} \;=\; \tfrac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

Also,

$$S^i \;=\; \widehat{T}^{0i} \;=\; F^{0j}F^{ki}\,\eta_{jk} \;=\; E^j\,\epsilon^{ji0k}\,G_{0k} \;=\; \epsilon^{0ijk}\,E^j\,B^k$$

Or,

$$\mathbf{S} \;=\; \mathbf{E}\times\mathbf{B}$$

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alternate derivation

We may also derive the symmetric, traceless and gauge invariant energy-momentum tensor from the action on a curved spacetime. The action is :

$$S\,[A,g] \;=\; \int d^4x\,\sqrt{g}\,\left(-\tfrac{1}{4}\,F_{\mu\nu}F_{\alpha\beta}\,g^{\mu\alpha}g^{\nu\beta}\,\right) \qquad \text{where} \qquad \sqrt{g} \;=\; \sqrt{|\det\left[\,g_{\mu\nu}\,\right]|}$$

Here the matrix of the components of a tensor in a particular coordinate system is represented by braces. For example :

$$\left[\,g_{\mu\nu}\,\right]^{-1} \;=\; \left[\,g^{\mu\nu}\,\right]$$

To compute the energy-momentum tensor we vary the action with respect to the metric with the definition :

$$T_{\alpha\beta}(x) \;\equiv\; \frac{2}{\sqrt{g}}\,\frac{\delta S\,[A,g]}{\delta g^{\alpha\beta}(x)}$$

Where,

$$S\,[A,g+\delta g] \;-S\,[A,g] \;\simeq\; \int d^4x\,\frac{\delta S\,[A,g]}{\delta g^{\alpha\beta}(x)}\;\delta g^{\alpha\beta}(x)$$

Now,

$$\det\left[\,g_{\mu\nu}\,\right] \;=\; \exp\left[\mathrm{tr}\,(\ln\left[\,g_{\mu\nu}\,\right])\right]$$

Thus,

$$\begin{aligned}
\det\left[\,g_{\mu\nu}+\delta g_{\mu\nu}\,\right] \;&=\; \det\left[\,g_{\mu\nu}\,\right]\,\det\left(\mathbf{1}+\left[\,g_{\mu\nu}\,\right]^{-1}\left[\,\delta g_{\mu\nu}\,\right]\right) \\
&=\; \det\left[\,g_{\mu\nu}\,\right]\exp\left[\mathrm{tr}\left(\ln\left(\mathbf{1}+\left[\,g_{\mu\nu}\,\right]^{-1}\left[\,\delta g_{\mu\nu}\,\right]\right)\right)\right] \\
&\simeq\; \det\left[\,g_{\mu\nu}\,\right]\left(1+\mathrm{tr}\left(\left[\,g_{\mu\nu}\,\right]^{-1}\left[\,\delta g_{\mu\nu}\,\right]\right)\right)
\end{aligned}$$

Since,

$$\mathrm{tr}\left(\left[\,g_{\mu\nu}\,\right]^{-1}\left[\,\delta g_{\mu\nu}\,\right]\right) \;=\; g^{\mu\nu}\,\delta g_{\mu\nu} \;=\; -\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

We find :

$$\sqrt{g+\delta g} \;\simeq\; \sqrt{g}\,\left(1-\tfrac{1}{2}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,\right)$$

This leads to:

$$S\,[A,g+\delta g] \;-S\,[A,g] \;\simeq\; -\,\frac{1}{2}\int d^4x\,\sqrt{g}\,\left(F_{\mu\nu}F_{\alpha\beta}\,g^{\mu\alpha} \;-\;\tfrac{1}{4}\,g_{\nu\beta}\,F_{\sigma\rho}F^{\sigma\rho}\right)\delta g^{\nu\beta}$$

Thus we have :

$$T_{\alpha\beta} \;=\; F_{\alpha\mu}F_{\nu\beta}\,g^{\mu\nu} \;+\; \tfrac{1}{4}\,g_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu}$$

This clearly reduces to the above result (derived via Noethers' theorem) when restricted to flat spacetime.

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2.) Peskin & Schroeder 2.2 : The complex scalar field

We begin with the action for the complex scalar field :

$$S\left[\phi\right] \;=\; \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi \;-\; m^2 \phi^* \phi\right)$$

(a)

We compute the Hamiltonian density associated with this action:

$$\mathcal{H} \;=\; \pi_\phi \dot{\phi} \,+\, \pi_{\phi^*} \dot{\phi}^* \;-\; \mathcal{L}$$

Where,

$$\pi \;\equiv\; \pi_\phi \;\equiv\; \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \;=\; \dot{\phi}^* \qquad \text{and} \qquad \pi_{\phi^*} \;\equiv\; \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} \;=\; \dot{\phi} \;=\; \pi^*$$

Thus since,

$$\mathcal{L} \;=\; \dot{\phi}^* \dot{\phi} \;-\; \nabla \phi^* \cdot \nabla \phi \;-\; m^2 \phi^* \phi$$

We find :

$$\mathcal{H} \;=\; \pi^* \pi \,+\, \nabla \phi^* \cdot \nabla \phi \,+\, m^2 \phi^* \phi$$

We now impose the canonical commutation relations :

$$\left[\,\phi(\mathbf{x}, t)\,,\, \pi(\mathbf{y}, t)\,\right] \;=\; i\,\delta^3(\mathbf{x} \,-\, \mathbf{y})$$

All other commutators (except the one given by hermitian conjugation) vanish. We use these and the Heisenberg equations of motion to verify that $\dot{\phi} = \pi^*$:

$$\begin{aligned} i\,\dot{\phi}(x) \;&=\; \left[\,\phi(x)\,,\, H\,\right] \;=\; \int d^3y \left[\,\phi(\mathbf{x}, t)\,,\, \mathcal{H}(\mathbf{y}, t)\,\right] \\ &=\; i\,\dot{\phi}(x) \;=\; \int d^3y\, \pi^*(\mathbf{y}, t) \left[\,\phi(\mathbf{x}, t)\,,\, \pi(\mathbf{y}, t)\,\right] \;=\; i\,\pi^*(x) \end{aligned}$$

We now consider $\dot{\pi} = \ddot{\phi}^*$:

$$\begin{aligned} i\,\dot{\pi}(x) \;&=\; \left[\pi(x)\,,\, H\,\right] = \int d^3y \left[\,\pi(\mathbf{x}, t)\,,\, \mathcal{H}(\mathbf{y}, t)\,\right] \\ &=\; \int d^3y \left[\,\pi(\mathbf{x}, t)\,,\, \left(\nabla \phi^* \cdot \nabla \phi \,+\, m^2 \phi^* \phi\right)(\mathbf{y}, t)\,\right] \\ &=\; -i \int d^3y \left(\nabla \phi^*(\mathbf{y}, t) \cdot \nabla_{\mathbf{y}} \,+\, m^2 \phi^*(\mathbf{y}, t)\right) \delta^3(\mathbf{y} \,-\, \mathbf{x}) \end{aligned}$$

Integrating by parts we find :

$$\dot{\pi}(x) \;=\; \nabla \cdot \nabla \phi^*(x) \;-\; m^2 \phi^*$$

Thus ϕ satisfies the Klein-Gordon equation :

$$\ddot{\phi} \;=\; \dot{\pi}^* \;=\; \nabla^2 \phi \;-\; m^2 \phi \qquad \text{or} \qquad \partial_\mu \partial^\mu \phi \,+\, m^2 \phi \;=\; 0$$

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(b)

By analogy with the real scalar field we postulate the following form for $\phi(x)$:

$$\phi(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,E_{\mathbf{p}}}}\,\left(\,a_{\mathbf{p}}\,e^{-ip\cdot x} \;+\; b^{\dagger}_{\mathbf{p}}\,e^{ip\cdot x}\,\right)$$

Thus,

$$\pi(x) \;=\; -i\int \frac{d^3p}{(2\pi)^3}\,\sqrt{\frac{E_{\mathbf{p}}}{2}}\,\left(\,b_{\mathbf{p}}\,e^{-ip\cdot x} \;-\; a^{\dagger}_{\mathbf{p}}\,e^{ip\cdot x}\,\right)$$

Again by analogy with the case of the real Klein-Gordon field we postulate :

$$\left[\,a_{\mathbf{p}}\,,\,a^{\dagger}_{\mathbf{k}}\,\right] \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k}) \qquad \text{and} \qquad \left[\,b_{\mathbf{p}}\,,\,b^{\dagger}_{\mathbf{k}}\,\right] \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})$$

We assume all other commutators vanish and verify these relations by computing :

$$\begin{aligned}
[\,\phi(\mathbf{x},t)\,,\,\pi(\mathbf{y},t)\,] \;&=\; \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\,\sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}}\,\left[\,a_{\mathbf{p}}\,,\,a^{\dagger}_{\mathbf{k}}\,\right]\,e^{-ip\cdot x}\,e^{ik\cdot y}\\
&+\; \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\,\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{k}}}}\,\left[\,b_{\mathbf{p}}\,,\,b^{\dagger}_{\mathbf{k}}\,\right]\,e^{ik\cdot x}\,e^{-ip\cdot y}\\
&=\; i\,\delta^3(\mathbf{x}-\mathbf{y})
\end{aligned}$$

We now show that H is diagonal when written in terms of these creation and annihilation operators. We begin with :

$$H \;=\; \int d^3x\,(\pi^{\dagger}\pi \;+\; \nabla\phi^{\dagger}\cdot\nabla\phi \;+\; m^2\phi^{\dagger}\phi)$$

We consider each of the terms that make up H in turn. Since H is a Noether charge, we may evaluate it at $t=0$) without loss of generality :

$$\begin{aligned}
&\int d^3x\;\pi^{\dagger}(\mathbf{x},0)\pi(\mathbf{x},0)\\
&=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\,\sqrt{E_{\mathbf{p}}E_{\mathbf{k}}}\int d^3x\,\left(b^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} - a^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\\
&=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(b^{\dagger}_{\mathbf{p}}b_{\mathbf{p}} \;+\; a_{\mathbf{p}}a^{\dagger}_{\mathbf{p}} \;-\; b^{\dagger}_{\mathbf{p}}a^{\dagger}_{-\mathbf{p}} \;-\; a_{\mathbf{p}}b_{-\mathbf{p}}\right)
\end{aligned}$$

Now,

$$\nabla\phi \;=\; i\int\frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}}{\sqrt{2\,E_{\mathbf{p}}}}\,\left(\,a_{\mathbf{p}}\,e^{i\mathbf{p}\cdot\mathbf{x}} \;-\; b^{\dagger}_{\mathbf{p}}\,e^{-i\mathbf{p}\cdot\mathbf{x}}\,\right)$$

Thus,

$$\begin{aligned}
&\int d^3x\;\nabla\phi^{\dagger}(\mathbf{x},0)\;\cdot\;\nabla\phi(\mathbf{x},0)\\
&=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\,\frac{\mathbf{p}\cdot\mathbf{k}}{\sqrt{E_{\mathbf{p}}E_{\mathbf{k}}}}\int d^3x\,\left(a^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} - b^{\dagger}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\\
&=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}^2}{E_{\mathbf{p}}}\left(a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}} \;+\; b_{\mathbf{p}}b^{\dagger}_{\mathbf{p}} \;+\; a^{\dagger}_{\mathbf{p}}b^{\dagger}_{-\mathbf{p}} \;+\; b_{\mathbf{p}}a_{-\mathbf{p}}\right)
\end{aligned}$$

Also,

$$\begin{aligned}
&\int d^3x\, m^2\, \phi^\dagger(\mathbf{x},0)\, \phi(\mathbf{x},0) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,\frac{m^2}{\sqrt{E_{\mathbf{p}}E_{\mathbf{k}}}}\int d^3x\,\left(a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\frac{m^2}{E_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}} + b_{\mathbf{p}}a_{-\mathbf{p}}\right)
\end{aligned}$$

Combining terms, taking $\mathbf{p}\to-\mathbf{p}$ for cross terms and using $E^2_{\mathbf{p}} = \mathbf{p}^2 + m^2$:

$$\begin{aligned}
&\int d^3x\,\left(\nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi\right)(\mathbf{x},0) \\
&\qquad = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}} - b^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}} - a_{\mathbf{p}}b_{-\mathbf{p}}\right)
\end{aligned}$$

Thus,

$$H = \int d^3x\,\mathcal{H}(\mathbf{x},0) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right)$$

Finally, we normal order to remove the infinite energy of the so-called Dirac sea.

$$:H: = \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right)$$

(c)

We now express the $U(1)$ Noether charge in terms of creation and annihilation operators :

$$Q = \frac{i}{2}\int d^3x\,\left(\phi^\dagger\pi^\dagger - \pi\phi\right)$$

Again the charge is conserved so we may evaluate it at $t = 0$:

$$\begin{aligned}
&\int d^3x\,\pi(\mathbf{x},0)\,\phi(\mathbf{x},0) \\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{k}}}}\int d^3x\,\left(a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}} - b_{\mathbf{p}}a_{-\mathbf{p}}\right)
\end{aligned}$$

Since $(\pi\phi)^\dagger = \phi^\dagger\pi^\dagger$:

$$\begin{aligned}
&\int d^3x\,\phi^\dagger(\mathbf{x},0)\,\pi^\dagger(\mathbf{x},0) \\
&\qquad = \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\left(b_{\mathbf{p}}b^\dagger_{\mathbf{p}} - a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}} - b_{\mathbf{p}}a_{-\mathbf{p}}\right)
\end{aligned}$$

Thus,

$$Q = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right)$$

Upon normal ordering, we see that the (a,b) particles have charge $\left(+\frac{1}{2},-\frac{1}{2}\right)$:

$$:Q: = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right)$$

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(d)

We consider the Lagrangian :

$$\mathcal{L} \;=\; \partial_\mu \phi_a^* \, \partial^\mu \phi_a \;-\; m^2 \phi_a^* \phi_a$$

We will first consider the general case $a = 1 \ldots N$ and then take $N = 2$. We rewrite the Lagrangian in terms of an N dimensional complex vector ϕ and its hermitian conjugate $\phi^\dagger$:

$$\mathcal{L} \;=\; \partial_\mu \phi^\dagger \, \partial^\mu \phi \;-\; m^2 \phi^\dagger \phi$$

This is invariant under a global $U(N)$ transformation :

$$\phi \;\to\; U\phi \qquad\qquad \text{where} \qquad\qquad U^\dagger U \;=\; \mathbf{1}$$

We may decompose any $U(N)$ transformation into a $U(1)$ and an $SU(N)$ transformation. Given $U \in U(N)$ such that $\det U = e^{i\varphi}$ we may form $M = e^{-i\varphi/N} U \in SU(N)$. Thus we may consider these invariances separately. For $U(1)$ we consider $\phi \to \tilde{\phi}(\alpha) = e^{-i\alpha/2}\phi$. Since the Lagrangian itself, rather than merely the action, is invariant under the $U(1)$, we have the conserved current :

$$J^\mu \;=\; \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \left.\frac{\partial \tilde{\phi}}{\partial \alpha}\right|_{\alpha=0} \;+\; \left.\frac{\partial \tilde{\phi}^\dagger}{\partial \alpha}\right|_{\alpha=0} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^\dagger\right)}$$

Now,

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \;=\; \partial^\mu \phi^\dagger \qquad \text{and} \qquad \left.\frac{\partial \tilde{\phi}}{\partial \alpha}\right|_{\alpha=0} \;=\; -\tfrac{i}{2}\,\phi$$

Thus,

$$J^\mu \;=\; -\tfrac{i}{2}\left(\partial^\mu \phi^\dagger \phi \;-\; \phi^\dagger \partial^\mu \phi\right)$$

And,

$$Q \;\equiv\; \int d^3x \, J^0 \;=\; \frac{i}{2} \int d^3x \left(\phi^\dagger \pi^\dagger \;-\; \pi \phi\right)$$

Here we have :

$$\pi \;\equiv\; \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \;=\; \dot{\phi}^\dagger$$

For $SU(N)$ we express each element of the group (here we work in the vector representation and its complex conjugate) in terms of the exponentiation of elements of the Lie algebra $su(N)$. That is if $M \in SU(N)$ then it can be expressed as $M = e^{-i\alpha^j g^j}$ where$g^j \in su(N)$ and $\alpha^j \in \mathbb{R}$. Thus since $U^\dagger U = 1$ we have $\left(g^j\right)^\dagger = g^j$ and since,

$$\ln\left(\det M\right) \;=\; \ln\left(\det\left(e^{-i\alpha^j g^j}\right)\right) \;=\; \mathrm{tr}\left(-i\alpha^j g^j\right) \;=\; 0$$

we see that g^j is traceless. Thus we are looking for a set of linearly independent traceless hermitian matrices in N dimensions. The dimensionality of this space is $\left(N^2 - 1\right)$.
Note: The true underlying invariance of the lagrangian is $O(2N)$ not $U(N)$. There are thus actually $N(2N-1)$, not N^2, symmetry generators.
For SU(N) we have the commutation relations :

$$\left[\, g^j \,,\, g^k \,\right] \;=\; i \, f^{jkl} \, g^l$$

We use the conventional normalizations :

$$\mathrm{tr}\left(g^j g^k\right) \;=\; \tfrac{1}{2}\,\delta^{jk} \qquad\qquad \text{and} \qquad\qquad f^{jmn} \, f^{kmn} \;=\; N \, \delta^{jk}$$

We consider the symmetry $\phi \rightarrow \tilde{\phi}(\boldsymbol{\alpha}) = e^{-i\alpha^j\, g^j}\,\phi$. This leads to the conserved currents :

$$\left(J^k\right)^\mu \;=\; \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\;\frac{\partial\tilde{\phi}}{\partial\alpha^k}\bigg|_{\alpha^k=0} \;+\; \frac{\partial\tilde{\phi}^\dagger}{\partial\alpha^k}\bigg|_{\alpha^k=0}\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^\dagger\right)}$$

Now,

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} \;=\; \partial^\mu\phi^\dagger \qquad \text{and} \qquad \frac{\partial\tilde{\phi}}{\partial\alpha^k}\bigg|_{\alpha^k=0} \;=\; -i\,g^k\,\phi$$

Thus,

$$\left(J^k\right)^\mu \;=\; -i\left(\partial^\mu\phi^\dagger g^k\phi \;-\; \phi^\dagger g^k\partial^\mu\phi\right)$$

And,

$$Q^k \;\equiv\; \int d^3x\;\left(J^k\right)^0 \;=\; i\int d^3x\left(\,\phi^\dagger\,g^k\,\pi^\dagger \;-\; \pi\,g^k\,\phi\,\right)$$

We now show that the charges satisfy the same commutation relations in their action on the Hilbert space as the generators of the Lie algebra satisfy on $\mathbb{C}^N$. As above, since the charges are conserved, we may evaluate the fields that go into their construction at any time. We will therefore suppress time labels on the fields in what follows. We first rewrite the charges with explicit $\mathbb{C}^N$ indices.

$$Q^j \;=\; i\int d^3x\left(\,\phi_a^\dagger(\mathbf{x})\;\left(g^j\right)_{ab}\,\pi_b^\dagger(\mathbf{x}) \;-\; \pi_a(\mathbf{x})\;\left(g^j\right)_{ab}\,\phi_b(\mathbf{x})\,\right)$$

We now evaluate the commutator :

$$\begin{aligned}\left[\,Q^j\,,\,Q^k\,\right] \;=\; -\int d^3x\int d^3y\;\;\Big(\Big[\,\phi_a^\dagger(\mathbf{x})\;\left(g^j\right)_{ab}\,\pi_b^\dagger(\mathbf{x})\,,\,\phi_c^\dagger(\mathbf{y})\;\left(g^k\right)_{cd}\,\pi_d^\dagger(\mathbf{y})\Big]\\ +\;\Big[\,\pi_a(\mathbf{x})\;\left(g^j\right)_{ab}\,\phi_b(\mathbf{x})\,,\,\pi_c(\mathbf{y})\;\left(g^k\right)_{cd}\,\phi_d(\mathbf{y})\Big]\Big)\end{aligned}$$

Now,

$$\left(\,\left[\,\phi^\dagger(\mathbf{x})\;\left(g^j\right)\,\pi^\dagger(\mathbf{x})\,,\,\phi^\dagger(\mathbf{y})\;\left(g^k\right)\,\pi^\dagger(\mathbf{y})\,\right]\,\right) \;=\; -\;\left(\,\left[\,\pi(\mathbf{x})\;\left(g^j\right)\,\phi(\mathbf{x})\,,\,\pi(\mathbf{y})\;\left(g^k\right)\,\phi(\mathbf{y})\,\right]\,\right)^\dagger$$

Thus we only need to evaluate :

$$\begin{aligned}&\left[\,\pi_a(\mathbf{x})\;\left(g^j\right)_{ab}\,\phi_b(\mathbf{x})\,,\,\pi_c(\mathbf{y})\;\left(g^k\right)_{cd}\,\phi_d(\mathbf{y})\,\right]\\ &\qquad=\quad \left(g^j\right)_{ab}\,\left(g^k\right)_{cd}\,\left(\,\pi_a(\mathbf{x})\left[\,\phi_b(\mathbf{x})\,,\,\pi_c(\mathbf{y})\,\right]\,\phi_d(\mathbf{y}) \;+\; \pi_c(\mathbf{y})\,\left[\,\pi_a(\mathbf{x})\,,\,\phi_d(\mathbf{y})\,\right]\,\phi_b(\mathbf{x})\,\right)\end{aligned}$$

Using,

$$\left[\,\phi_a(\mathbf{x})\,,\,\pi_b(\mathbf{y})\,\right] \;=\; i\,\delta_{ab}\,\delta^3(\mathbf{x}-\mathbf{y})$$

We find :

$$\left[\,\pi_a(\mathbf{x})\;\left(g^j\right)_{ab}\,\phi_b(\mathbf{x})\,,\,\pi_c(\mathbf{y})\;\left(g^k\right)_{cd}\,\phi_d(\mathbf{y})\,\right] \;=\; i\,\delta^3(\mathbf{x}-\mathbf{y})\;\left(\,\pi(\mathbf{x})\left[\,g^j\,,\,g^k\,\right]\phi(\mathbf{y})\,\right)$$

Thus,

$$\left[\,Q^j\,,\,Q^k\,\right] \;=\; -\,\epsilon^{jkl}\int d^3x\;\left(\,\phi^\dagger(\mathbf{x})\,g^l\,\pi^\dagger(\mathbf{x}) \;-\; \pi(\mathbf{x})\,g^l\,\phi(\mathbf{x})\,\right) \;=\; i\,f^{jkl}\,Q^l$$

For the case of $SU(2)$ we make the replacements :

$$g^j \;=\; \frac{\sigma^j}{2} \qquad \text{and} \qquad f^{jkl} \;=\; \epsilon^{jkl}$$

Where σ^j are the Pauli matrices and ϵ^{jkl} is the completely anti-symmetric tensor in 3 dimensions $\left(\epsilon^{123}=1\right)$.

3.) Peskin & Schroeder 2.3

We evaluate the function

$$\langle 0|\,\phi(x)\phi(y)\,|0\rangle \;=\; D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,e^{-ip\cdot(x-y)}$$

for spacelike $(x-y)$, such that $(x-y)^2 \;=\; -r^2$, explicitly in terms of Bessel functions. Since $D(x)$ is invariant under Lorentz transformations, $D(x) = D(\Lambda x)$ $(\Lambda \in SO\,(3,1))$, we may choose $x^0 \;=\; y^0$. Thus, denoting $\mathbf{x}-\mathbf{y}=\mathbf{r}$,

$$D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,e^{i\mathbf{p}\cdot\mathbf{r}} \;=\; \frac{1}{(2\pi)^2}\int_0^{\pi} d\theta\,\sin\theta\int_0^{\infty} dp\,\frac{p^2}{2\left(p^2+m^2\right)^{1/2}}\,e^{ipr\cos\theta}$$

Where we have introduced a sperical coordinate system (p,θ,ϕ) such that $\mathbf{p}\cdot\mathbf{r} \;=\; pr\cos\theta$. Thus,

$$D(x-y) \;=\; \frac{1}{(2\pi)^2}\int_{-1}^{1} dx\int_0^{\infty} dp\,\frac{p^2}{2\left(p^2+m^2\right)^{1/2}}\,e^{iprx} \;=\; \frac{1}{(2\pi)^2}\,\frac{1}{r}\int_0^{\infty} dp\,\frac{p\,\sin(pr)}{\left(p^2+m^2\right)^{1/2}}$$

From,

$$K_1(x) \;=\; -\,\pi/2\,\left(J_1(ix)+iN_1(ix)\right) \;=\; \int_0^{\infty} du\,\frac{u}{\left(1+u^2\right)^{1/2}}\,\sin(xu)$$

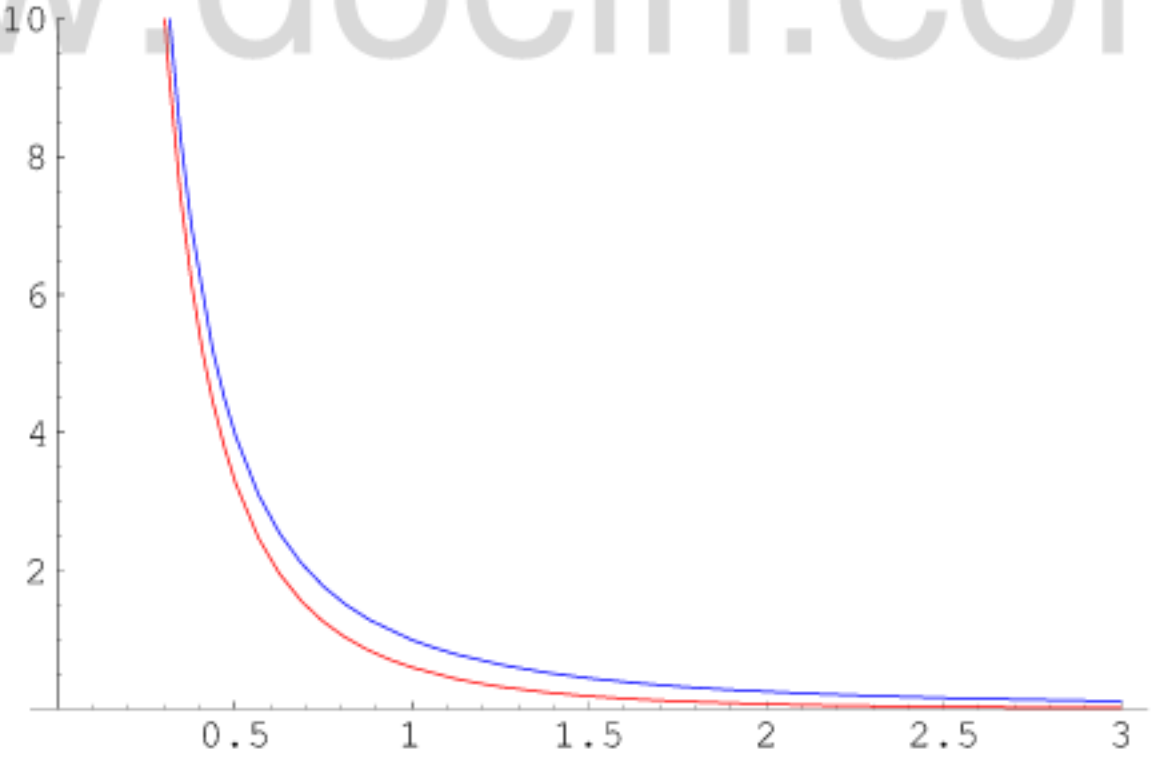
Defining $u \;=\; p/m$,

$$D(x-y) \;=\; \frac{1}{(2\pi)^2}\,\frac{m}{r}\int_0^{\infty} du\,\frac{u}{\left(1+u^2\right)^{1/2}}\,\sin(mru) \;=\; \frac{1}{(2\pi)^2}\,\frac{m}{r}\,K_1(mr)$$

From the properties of $K_1(x)$, we find that for (x,y) spacelike separated :

$$D(x-y) \;\sim\; \frac{1}{(2\pi)^2\;r^2} \qquad \text{as} \qquad x \to y$$

The following is a plot of $K_1(x)/x$ in red and $1/x^2$ in blue :



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4.)

We consider coherent states for the real Klein-Gordon field :

$$|f\rangle \;=\; N_f \,\exp\left[\, i \int \frac{d^3p}{(2\pi)^3}\, f(\mathbf{p})\, a^\dagger_{\mathbf{p}} \right] |0\rangle$$

Where,

$$N_f \;=\; \exp\left[-\tfrac{1}{2} \int \frac{d^3p}{(2\pi)^3}\, |f(\mathbf{p})|^2 \right]$$

(a)

We evaluate the expectation value of the field operator :

$$\langle f |\, \phi(x)\, | f \rangle$$

Where,

$$\phi(x) \;=\; \phi_+(x) \,+\, \phi_-(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2\,E_{\mathbf{p}}}}\, \left(a_{\mathbf{p}}\, e^{-ip\cdot x} \,+\, a^\dagger_{\mathbf{p}}\, e^{ip\cdot x} \right)$$

Since, $\phi_-(x) = \phi_+^\dagger(x)$ we need only evaluate :

$$\langle f |\, \phi_+(x)\, | f \rangle \;=\; |N_f|^2 \,\langle 0|\, e^{-i\Gamma^\dagger}\, \phi_+(x)\, e^{i\Gamma}\, |0\rangle$$

Where,

$$\Gamma \;=\; \int \frac{d^3p}{(2\pi)^3}\, f(\mathbf{p})\, a^\dagger_{\mathbf{p}}$$

Thus,

$$\langle f |\, \phi_+(x)\, | f \rangle \;=\; |N_f|^2 \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2\,E_{\mathbf{p}}}}\, e^{-ip\cdot x}\, \langle 0|\, e^{-i\Gamma^\dagger}\, a_{\mathbf{p}}\, e^{i\Gamma}\, |0\rangle$$

We now show that $|f\rangle$ is normalised so that :

$$\langle f |\, f \rangle \;=\; |N_f|^2 \,\langle 0|\, e^{-i\Gamma^\dagger}\, e^{i\Gamma}\, |0\rangle \;=\; 1$$

Now, if $[A, B] \in \mathbb{C}$ then:

$$e^A\, e^B \;=\; e^{A+B+\frac{1}{2}[A,B]} \;=\; e^B\, e^A e^{[A,B]}$$

Also,

$$\Gamma^\dagger\, |0\rangle \;=\; 0 \qquad \text{and} \qquad \left[\Gamma^\dagger, \Gamma\right] \;=\; \int \frac{d^3p}{(2\pi)^3}\, |f(\mathbf{p})|^2$$

Thus,

$$\langle f |\, f \rangle \;=\; |N_f|^2 \,\exp\left[\int \frac{d^3p}{(2\pi)^3}\, |f(\mathbf{p})|^2 \right] \langle 0|\, e^{i\Gamma}\, e^{-i\Gamma^\dagger}\, |0\rangle \;=\; 1$$

Now,

$$\left[\, a_{\mathbf{p}}\,,\, \Gamma \,\right] \;=\; f(\mathbf{p}) \qquad \text{since} \qquad \left[a_{\mathbf{p}}\,,\, a^\dagger_{\mathbf{k}} \right] \;=\; (2\pi)^3\, \delta^3(\mathbf{p} - \mathbf{k})$$

Thus,

$$a_{\mathbf{p}}\, e^{i\Gamma}\, |0\rangle \;=\; \sum_{n=1}^{\infty} \frac{i^n}{n!}\, n\, \left[\, a_{\mathbf{p}}\,,\, \Gamma \,\right] \Gamma^{n-1}\, |0\rangle \;=\; i\, f(\mathbf{p})\, e^{i\Gamma}\, |0\rangle$$

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This leads to :

$$\langle f|\,\phi_+(x)\,|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{i}{\sqrt{2\,E_{\mathbf{p}}}}\,f(\mathbf{p})\,e^{-ip\cdot x}$$

Thus the expectation value of the field operator is :

$$\langle f|\,\phi(x)\,|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{i}{\sqrt{2\,E_{\mathbf{p}}}}\,\left(\,f(\mathbf{p})\,e^{-ip\cdot x} \;-\; f^*(\mathbf{p})\,e^{ip\cdot x}\,\right)$$

The expectation value trivially satisfies the Klein-Gordon equation since the field operator satisfies it and $|f\rangle$ is a Heisenberg state vector.

(b)

We evaluate the expectation value of the number density operator in momentum space for the coherent state $|f\rangle$:

$$\langle f|\;n_{\mathbf{p}}\;|f\rangle \;=\; \langle f|\;a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\;|f\rangle \;=\; \left|N_f\,a_{\mathbf{p}}\,e^{i\Gamma}\;|0\rangle\right|^2 \;=\; |f(\mathbf{p})|^2$$

Also,

$$\begin{aligned}\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{p}}\;|f\rangle \;&=\; |N_f|^2\;\langle 0|\;e^{-i\Gamma^\dagger}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\,e^{i\Gamma}\;|0\rangle \\ &=\; |N_f|^2\;|f(\mathbf{p})|^2\;\langle 0|\;e^{-i\Gamma^\dagger}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{p}}\,e^{i\Gamma}\;|0\rangle\end{aligned}$$

From the commutation relations we see that this is a divergent quantity.

$$\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{p}}\;|f\rangle \;=\; |f(\mathbf{p})|^2\;\left(|f(\mathbf{p})|^2 \;+\; (2\pi)^3\,\delta^3(\mathbf{0})\right)$$

Thus,

$$\frac{\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{p}}\;|f\rangle \;-\; \left(\langle f|\;n_{\mathbf{p}}\;|f\rangle\right)^2}{\left(\langle f|\;n_{\mathbf{p}}\;|f\rangle\right)^2} \;=\; \frac{(2\pi)^3\,\delta^3(\mathbf{0})}{|f(\mathbf{p})|^2}$$

This divergence arises since $n_{\mathbf{p}}$ is an operator-valued distribution and must be integrated before a well-defined product with another operator valued distribution may be taken.
We now evaluate the expectation value of the Hamiltonian for the coherent state $|f\rangle$:

$$\langle f|\;H\;|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\;\langle f|\;n_{\mathbf{p}}\;|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\;|f(\mathbf{p})|^2$$

Also,

$$\begin{aligned}\langle f|\;H^2\;|f\rangle \;&=\; \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,E_{\mathbf{p}}\,E_{\mathbf{k}}\;\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{k}}\;|f\rangle \\ &=\; |N_f|^2\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,E_{\mathbf{p}}\,E_{\mathbf{k}}\;\langle 0|\;e^{-i\Gamma^\dagger}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{k}}\,a_{\mathbf{k}}\,e^{i\Gamma}\;|0\rangle \\ &=\; |N_f|^2\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,E_{\mathbf{p}}\,E_{\mathbf{k}}\,f^*(\mathbf{p})\,f(\mathbf{k})\;\langle 0|\;e^{-i\Gamma^\dagger}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{k}}\,e^{i\Gamma}\;|0\rangle\end{aligned}$$

Now,

$$\langle 0|\;e^{-i\Gamma^\dagger}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{k}}\,e^{i\Gamma}\;|0\rangle \;=\; |N_f|^{-2}\;\left(\,f(\mathbf{p})f^*(\mathbf{k}) \;+\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})\right)$$

Thus,

$$\langle f|\;H^2\;|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,E^2_{\mathbf{p}}\;|f(\mathbf{p})|^2 \;+\; \left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\;|f(\mathbf{p})|^2\right)^2$$

Finally,

$$\frac{\langle f|\ H^2\ |f\rangle\ -\ (\langle f|\ H\ |f\rangle)^2}{(\langle f|\ H\ |f\rangle)^2}\ =\ \frac{\int \frac{d^3p}{(2\pi)^3}\ E_{\mathbf{p}}^2\ |f(\mathbf{p})|^2}{\left(\int \frac{d^3p}{(2\pi)^3}\ E_{\mathbf{p}}\ |f(\mathbf{p})|^2\right)^2}$$

(c)

As we found above, for (x, y) spacelike separated :

$$\langle 0|\,\phi(x)\phi(y)\,|0\rangle\ =\ D(x-y)\ \sim\ \frac{1}{(2\pi)^2\ r^2}\qquad\text{as}\qquad x\to y$$

Here, as above, $(x-y)^2 = -r^2$. Again, as for $n_{\mathbf{p}}$ above, this divergence arises since $\phi(x)$ is an operator-valued distribution. This is also a sign that local quantities that are quadratic in $\phi(x)$, such as the energy-momentum tensor, do not have well defined values and must be renormalized. Note that the divergence is independent of the mass of the particle. This is an indication that all particles behave as massless particles at sufficiently high energies (short distances).

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Physics 443 Homework #2

Due Thursday, October 14, 2010

1.) Consider the path integral for a single point particle, with the action

$$S = \int_0^1 dt\big[p_\mu(t)\dot{x}^\mu(t) + \tfrac{1}{2}N(t)(p^2(t) - m^2 + i\epsilon)\big] \ .$$

This represents the quantization of the coordinates and momenta of the particle, subject to the mass shell constraint $p^2 = m^2$ (together with the $i\epsilon$ prescription) imposed by the Lagrange multiplier N. This action admits the reparametrization symmetry $\delta x = \alpha p$, $\delta p = 0$, $\delta N = -\partial_t \alpha$ where $\alpha(t)$ is any function. This symmetry allows us to fix the gauge condition $N(t) = T$; the constant T must still be integrated over, however.

a) Path integrate over $x(t)$, subject to the boundary conditions $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$, yielding a delta function $\delta(\dot{p})$ along the path. Solve this constraint (find the set of functions that solve it) and path integrate over those $p(t)$ to find the quantum mechanical propagation amplitude

$$\langle y|x\rangle = D_F(x-y) = \int_0^\infty dT(2\pi i T)^{-d/2}\exp\left[\frac{-i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \ ,$$

where d is the number of spacetime dimensions.

b) Use this integral representation to show that D_F satisfies

$$(\partial^2 + m^2)D_F = i\delta^{(d)}(x-y) \ .$$

c) Evaluate the T integral in terms of Bessel functions.

2.) Peskin and Schroeder 9.2a-c

Hints: For 9.2a, it is sufficient to *formulate* the partition function in terms of a path integral; you are going to evaluate it in part (b). For 9.2c, first show that the partition function can be formulated as a path integral over fields in Euclidean 4-space that are periodic in the imaginary time direction. The spatial field modes are harmonic oscillators; take the log of the partition function to get the free energy as a sum over modes of the free energy of each oscillator, and use your result for (b) to evaluate it.

A second approach to (c) uses the methods of problem (1). Use the representation

$$\begin{aligned}\log[Z] = \log[\det(-\partial_E^2 + m^2)] &= \mathrm{tr}\log(-\partial_E^2 + m^2) \\ &= \int d^4x \int_0^\infty \frac{dT}{T}\langle x|e^{-T(-\partial_E^2 + m^2)}|x\rangle \ ,\end{aligned}$$

together with the representation of the matrix element derived in problem (1), to evaluate the functional determinant and hence the partition function. You may want to take $\partial/\partial m$ of the above expression to remove an m-independent divergence and render the integral finite.

3.) Write a field theory action describing *nonrelativistic* scalar particles interacting via a potential $U(\vec{x} - \vec{y})$ (this action-at-a-distance form of interaction is permissible in a nonrelativistic setting, but not in relativistic field theory, where it would break Lorentz invariance by selecting a preferred surface of simultaneity). Find the corresponding Hamiltonian for the field. Use your expression for the energy in terms of fields and evaluate it to show that the expectation value of the Hamiltonian in the noninteracting ground state of a system of N particles in a volume V is, to first order in perturbation theory,

$$\frac{E^{(1)}}{N} = \frac{N-1}{2V}\tilde{U}(0) \ ,$$

where

$$\bar{U}(\vec{q}) = \int d^3x U(\vec{x})\exp[-i\vec{q}\cdot\vec{x}] \ .$$

Use of 'first quantized' methods to derive this answer is not acceptable (the point of the exercise is to gain familiarity with quantized fields; you may find it useful, however, to compare the two approaches).

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QFT 1 : Problem Set 2

1.)

We begin by attempting to motivate the action for a relativistic point particle appearing in the homework set. Perhaps the more familiar action is that given by the invariant length of the worldline:

$$S_1[x] \;=\; -\,m \int ds\; (\,\eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu)^{1/2}$$

Where $\dot{x}^\mu \equiv \partial_s x^\mu \equiv \partial x^\mu/\partial s$.
In addition to being Poincare invariant, this action is invariant under arbitrary reparameterizations of the worldline coordinate s . The coordinates x^μ of course transform like scalars under this transformation. The equation of motion for $x(s)$ is the familiar:

$$\frac{\partial}{\partial s}\left(\,\dot{x}^\nu\;(\,\eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu)^{-1/2}\right) \;=\; 0$$

We now introduce a new action which incorporates a field which transforms like a metric under reparameterizations of the worldline coordinate s. We will see that it leads to the same equations of motion for $x(s)$.

$$S_2[x\,,\,g] \;=\; -\,\frac{m}{2}\int ds\,\sqrt{g_{ss}}\;(\,g^{ss}\,\partial_s x\,\cdot\,\partial_s x\;+\;1\,)$$

Where $\partial_s x\cdot\partial_s x \;\equiv\; \eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu$ and $g^{ss}g_{ss} = 1$. The equation of motion for g is found to be:

$$g_{ss} \;=\; \partial_s x\cdot\partial_s x$$

Thus, if the equations of motion are satisfied, g coincides with the metric along the worldline inherited (through its imbedding in spacetime) from η . From this we find:

$$S_2[x\,,\,\partial_s x\,\cdot\,\partial_s x\,] \;=\; S_1[x]$$

Thus the action S_2 leads to the same classical equations of motion for x as does S_1 . We now make a change of field variables by taking advantage of the fact that, in one dimension, we may replace the metric by a one-form. Thus we choose $N = \sqrt{g_{ss}}/m$. This leads to the action :

$$S_3[x\,,\,N] \;=\; -\,\frac{1}{2}\int ds\;\left(\,N^{-1}\,\dot{x}\cdot\dot{x}\;+\;N\,m^2\,\right)$$

From this action we derive the Hamiltonian:

$$\mathcal{H} \;=\; p_\mu\dot{x}^\mu \;-\; \mathcal{L} \;=\; -\,\tfrac{1}{2}\,N\,(p\cdot p\;-\;m^2)$$

Here $p_\mu = \partial\mathcal{L}/\partial\dot{x}^\mu \;=\; -\,N^{-1}\,\dot{x}_\mu$ and the canonical momentum of the N variable vanishes. We may define the path integral associated with S_3 as follows :

$$D_F(y-x) \;=\; \int \mathcal{D}N \int_{x(s_0)=x}^{x(s_1)=y} \mathcal{D}x\; e^{\,i\,S_3[x,N]}$$

Perhaps more fundamentally, we may consider the path integral to be defined through the use of the following action :

$$S_4[x\,,\,p\,,\,N] \;=\; \int ds\;(\,p_\mu\dot{x}^\mu\;-\;\mathcal{H}(p,x)\,) \;=\; \int ds\;\left(\,p_\mu\dot{x}^\mu\;+\;\tfrac{1}{2}\,N\,(p\cdot p\;-\;m^2)\,\right)$$

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We write the path integral as :

$$D_F(y-x) \;=\; \int \mathcal{D}N \int \mathcal{D}p \int_{x(s_0)=x}^{x(s_1)=y} \mathcal{D}x \; e^{i\,S_4[x\,,\,p\,,\,N]}$$

This may be seen to give the same result as the path integral involving S_3 since the action S_4 is quadratic in p. Note that we do not integrate over the canonical momentum for N since it is identically zero. To ensure that this path integral converges we substitute $m^2 \to \left(m^2 - i\epsilon\right)$ and constrain the path integral to positive values of N .

In the discussion of gauge fixing we are going to diverge a bit from the statement of the problem. The problem presents a symmetry under canonical transformations induced by the Hamiltonian where N is treated as a Lagrange Multiplier for a first-class constraint. The symmetry is $\delta x = \alpha p$, $\delta p = 0$, $\delta N = \; -\partial_s \alpha$ for arbitrary $\alpha(s)$. I am more comfortable discussing fixing the symmetry of the action under diffeomorphisms; that is reparameterizations of the time parameter s . This symmetry treats x and p as scalars and N as a one-form so that $\delta x = \; -\beta \partial_s x$, $\delta p = \; -\beta \partial_s p$, $\delta N = \; -\,\partial_s \left(\beta N\right)$ for arbitrary $\beta(s)$. I strongly suspect that the symmetries are equivalent and certainly lead to the same result. This symmetry allows us to transform N subject to the condition that $\int ds\, N(s)$ is preserved as it must be under diffeomorphisms. The finite transformation of N is just the tensor transformation law:

$$\tilde{N}(\tilde{s})\,\frac{\partial \tilde{s}}{\partial s} \;=\; N(s)$$

We may use this freedom to transform any $N(s)$ to $\tilde{N}(\tilde{s}) = 1$. Then we have:

$$\int_{s_0}^{s_1} ds\, N(s) \;=\; \int_{\tilde{s}_0}^{\tilde{s}_1} d\tilde{s} \;=\; \left(\tilde{s}_1 \;-\; \tilde{s}_0\right) \;\equiv\; T$$

We may now do a further transformation to set the limits of the integral to $s_0 = 0$ and $s_1 = 1$ with $N(s) = T$. We cannot gauge away N entirely and T must be integrated over in the path integral. The principal reason for avoiding the tranformation in the homework is that I am not sure what the analog of the $\int ds\, N(s)$ constraint is. We are thus lead to the following gauge fixed path integral:

$$D_F(y-x) = \int_0^\infty dT \int \mathcal{D}p \int_{x(0)=x}^{x(1)=y} \mathcal{D}x \, \exp\left(i \int_0^1 ds \, \left[\, p\cdot\dot{x} \,+\, \tfrac{1}{2}\,T\left(p^2 - m^2 + i\epsilon\right)\,\right]\right)$$

(a)

We will respect the apparent time-honored tradition in theoretical physics of treating the solution of the path integral somewhat loosely. But first we present an expression that may be worked with to provide a perhaps more careful solution (here $\Delta = 1/n$) :

$$D_F(y-x) \;=\; \lim_{n\to\infty} \quad \int_0^\infty dT \left(\prod_{j=0}^{n} \int d^d x_j\right) \left(\prod_{k=1}^{n} \int \frac{d^d p_k}{(2\pi)^d}\right) \delta^d(x_0 - x)\, \delta^d(x_n - y)$$
$$\times \exp\left(i\,\Delta \sum_{k=1}^{n} \left[\, p_k \cdot \left(x_k - x_{k-1}\right)/\Delta \;-\; T/2\,\left(p_k\cdot p_k - m^2 + i\epsilon\right)\,\right]\right)$$

We treat the x path integral, following an integration by parts in the action, as a functional Fourier transform :

$$\int_{x(0)=x}^{x(1)=y} \mathcal{D}x \, \exp\left(\, i \int_0^1 ds \, \left(p\cdot\dot{x}\right)\right) \;=\; \delta\left[\,\dot{p}\,\right] \, \exp\left(\, i\,\left[\, p(1)\cdot y \;-\; p(0)\cdot x\,\right]\right)$$

Inserting this into the path integral we blithely convert the p functional integral into an ordinary integral since $\dot{p} = 0$:

$$D_F(y-x) \;=\; \lambda_d \int_0^\infty dT \int \frac{d^d p}{(2\pi)^d} \exp\left(i \left[p \cdot (y - x) + \tfrac{1}{2} T \left(p^2 - m^2 + i\epsilon \right) \right] \right)$$

Note that the integral over s in the action produces its integrand since $\dot{p} = 0$. Also note that a factor λ_d has been inserted to provide the normalization to be determined below. We now use the following formula for the integral of a gaussian :

$$\int d^d p \, e^{-i\,p\cdot\alpha} \, e^{-a\,p^2} \;=\; e^{-\alpha^2/(4a)} \left(\frac{\pi}{a} \right)^{d/2}$$

Setting $a = -i\,T/2$ and $\alpha = (x - y)$ we find :

$$D_F(y-x) \;=\; \lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \int_0^\infty dT \, T^{-\,d/2} \exp\left(-\,i/2 \left[T \left(m^2 - i\epsilon \right) + T^{-1} (y-x)^2 \right] \right)$$

(b)

We now show that D_F is a Green function for the Klein-Gordon equation. We find that :

$$\begin{aligned}
&\left(\partial^2 + m^2 - i\epsilon \right) \, D_F(x) \\
&\quad = \;\; \lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \int_0^\infty dT \, T^{-\,d/2} \, \left(m^2 - i\epsilon - x^2/T^2 - id/T \right) \\
&\qquad\qquad \times \exp\left(-\,i/2 \left[T \left(m^2 - i\epsilon \right) + T^{-1} x^2 \right] \right) \\
&\quad = \;\; 2i\,\lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \int_0^\infty dT \, \frac{d}{dT} \left(T^{-\,d/2} \exp\left(-\,i/2 \left[T \left(m^2 - i\epsilon \right) + T^{-1} x^2 \right] \right) \right) \\
&\quad = \;\; -\,2i\,\lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \lim_{T \to 0} \left[T^{-\,d/2} \exp\left(-\,i\,x^2/(2\,T) \right) \right] \;=\; -\,2i\,\lambda_d \, \delta^d(x)
\end{aligned}$$

To see that the distribution given here is a delta function we may integrate it against a test function. We will find that the integral oscillates wildly in the $T \to 0$ limit except near $x = 0$. We remove the test function (evaluated at $x = 0$) and the distribution integrates to 1 since it is a normalized gaussian for all T . This is of course pretty loose language but is essentially correct. To verify this result we return to the (normalized) expression for D_F prior to performing the momentum integral. To conform with the definition in Peskin and Schroeder we choose the normalization $\lambda_d = 1/2$ and take $p \to -p$ in the integral :

$$D_F(x) \;=\; \frac{1}{2} \int_0^\infty dT \int \frac{d^d p}{(2\pi)^d} \exp\left(-\,i \left[p \cdot x - \tfrac{1}{2} T \left(p^2 - m^2 + i\epsilon \right) \right] \right)$$

We perform the T integral to find ;

$$D_F(x) \;=\; \int \frac{d^d p}{(2\pi)^d} \, \frac{i\,e^{-\,i\,p\cdot x}}{(p^2 - m^2 + i\epsilon)}$$

Thus,

$$\left(\partial^2 + m^2 - i\epsilon \right) \, D_F(x) \;=\; -\,i\,\delta^d(x)$$

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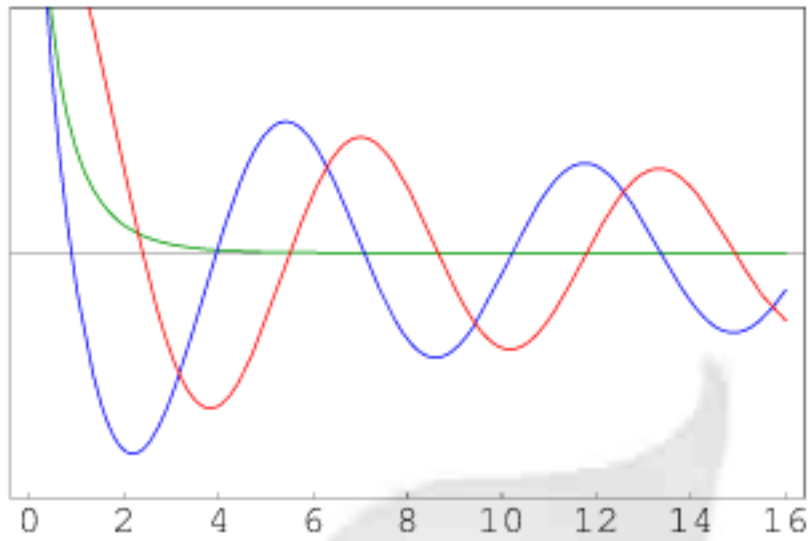
(c)

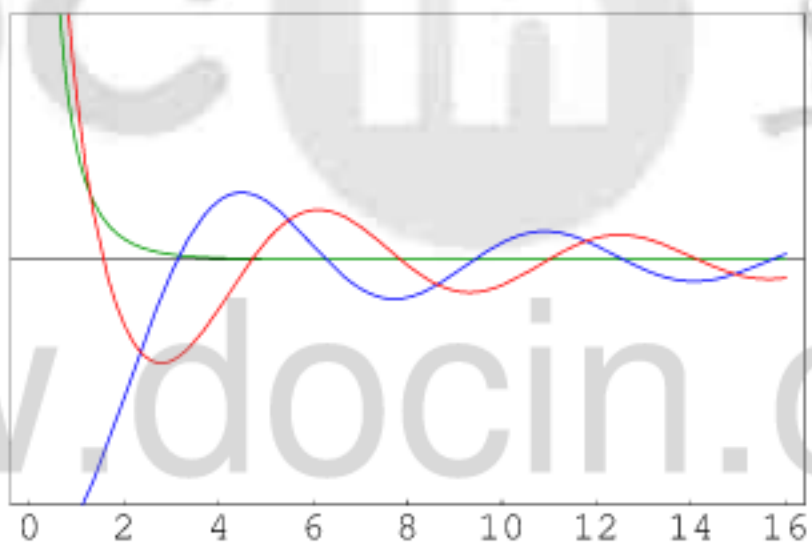
The most straightforward way to approach this problem is to use a table of integrals or plug the expression for D_F as an integral over T into a program like mathematica. The result is:

$$D_F(x) = \frac{1}{2\pi}\left(\frac{i\,m}{2\pi\sqrt{x^2}}\right)^{(d/2-1)} K_{(d/2-1)}(-i\,m\sqrt{x^2})$$

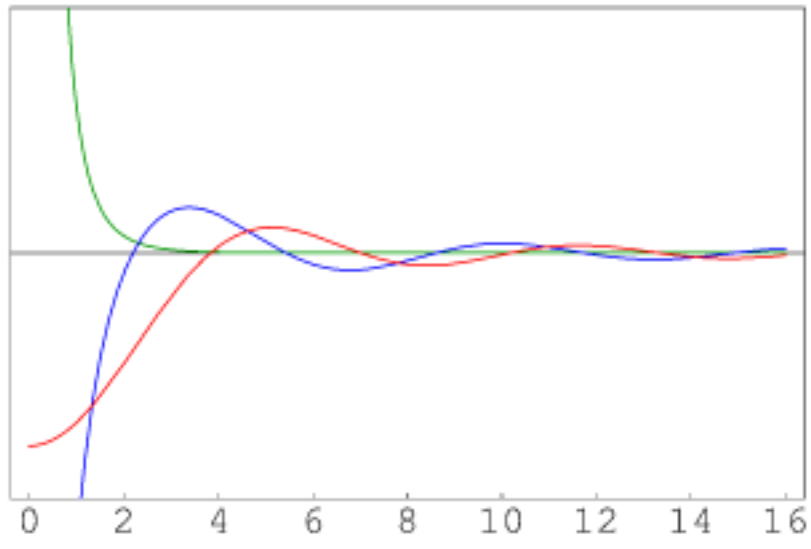
The following are graphs of $x^{(1-d/2)}\,K_{d/2-1}(x)$ (spacelike) in green and the real and imaginary parts of $(-ix)^{(1-d/2)}\,K_{d/2-1}(-ix)$ (timelike) in blue and red respectively.
For $d = 2$:



For $d = 3$:



For $d = 4$:



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2.) Peskin & Schroeder 9.2 (a-c)

(a)

We want to express the quantum statistical partition function in terms of a functional integral. For notational clarity we will consider a one-dimensional single particle quantum system. The extension to a more complicated system is trivial. Inserting a complete set of position eigenstates we have:

$$Z(\beta) \;=\; \mathrm{tr}\left[e^{-\beta\,H}\right] \;=\; \int dx\;\langle x|\,e^{-\beta H}\,|x\rangle$$

Rather than evaluating the propagator for complex time and facing relatively delicate issues related to analytic continuation, we derive the path integral directly. Defining $\epsilon = \beta/N$ and inserting complete sets of position and momentum eigenstates, we have:

$$Z(\beta) \;=\; \lim_{N\to\infty}\int dx\,\left(\prod_{j=0}^{N}\int dx_j\right)\left(\prod_{k=1}^{N}\int dp_k\right)$$

$$\times\,\delta(x_N - x)\,\delta(x_0 - x)\,\prod_{n=1}^{N}\left[\,\langle x_n|\,e^{-\epsilon H}\,|p_n\rangle\;\langle p_n\,|x_{n-1}\rangle\,\right]$$

Now,

$$\prod_{n=1}^{N}\left(\langle x_n\,|p_n\rangle\,\langle p_n|\,e^{-\epsilon H}\,|x_{n-1}\rangle\right) \;=\; \exp\left[-\epsilon\,\sum_{n=1}^{N}\left(\,-ip_n\,(x_n - x_{n-1})\,/\epsilon \;+\; H\,(p_n, x_n)\,\right)\right]$$

Thus, using the definition of the phase space path integral appearing in P&S, we may write the partition function as:

$$Z(\beta) \;=\; \int dx\,\int_{q(0)=x}^{q(\beta)=x}\mathcal{D}q\int\mathcal{D}p\;\exp\left[-\int_0^{\beta} ds\;\tilde{L}_E(q,\dot{q},p)\right]$$

Where we are using the hybrid Euclidean Lagrangian:

$$\tilde{L}_E(q,\dot{q},p) \;=\; -i\,p\,\dot{q} \;+\; H(p,q)$$

Note that the path integral is over all periodic paths that have period β. If $H(p,q)$ can be written as $H = p^2/2m + V(q)$, we may evaluate the p integrals explicitly:

$$Z(\beta) \;=\; \lim_{N\to\infty}\int dx\,\left(\prod_{j=0}^{N}\int dx_j\right)\;\delta(x_N - x)\,\delta(x_0 - x)\;\exp\left[-\epsilon\,\sum_{n=1}^{N}\,V\,(x_n)\right]$$

$$\times\,\left(\prod_{k=1}^{N}\int dp_k\right)\exp\left[-\epsilon\,\sum_{n=1}^{N}\,\left(\,-ip_n\,(x_n - x_{n-1})\,/\epsilon \;+\; p_n^2/2m\,\right)\right]$$

Thus,

$$Z(\beta) \;=\; \lim_{N\to\infty}\int dx\,\left(\prod_{j=0}^{N}\int dx_j\right)\;\left(\frac{m}{2\pi\epsilon}\right)^{N/2}\;\delta(x_N - x)\,\delta(x_0 - x)$$

$$\times\;\exp\left[-\epsilon\,\sum_{n=1}^{N}\,\left(\,\frac{m}{2}\,(x_n - x_{n-1})^2\,/\epsilon^2 \;+\; V\,(x_n)\,\right)\right]$$

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Again, using the definition of the configuration space path integral appearing in P&S, we may write the partition function as:

$$Z(\beta) \;=\; \int dx \int_{q(0)=x}^{q(\beta)=x} \tilde{\mathcal{D}}q \;\exp\left[-\int_0^\beta ds\, L_E(q,\dot{q}) \right]$$

Where we are using the Euclidean Lagrangian:

$$L_E(q,\dot{q}) \;=\; \frac{m}{2}\,\dot{q}^2 \;+\; V(q)$$

The measure in the configuration space path integral is written as $\tilde{\mathcal{D}}q$ to reflect the additional factors in the measure that do not appear in the phase space path integral.

(b)

We consider the Euclidean action for the unit mass harmonic oscillator:

$$\mathcal{L} \;=\; \tfrac{1}{2}\left(\,\dot{x}^2 \,+\, \omega^2 x^2\,\right)$$

Since we are considering a path integral over periodic functions, we expand $x(t)$ in a Fourier series:

$$x(t) \;=\; \frac{1}{\sqrt{\beta}} \sum_{n\in\mathbb{Z}} x_n\, e^{2\pi i n t/\beta} \qquad \text{and} \qquad x_n \;=\; \frac{1}{\sqrt{\beta}} \int_0^\beta dt\, x(t)\, e^{-2\pi i n t/\beta}$$

The reality of $x(t)$ imposes $x_n^* = x_{-n}$. We will proceed in a cavalier manner and simply define the path integral measure to be:

$$\int dx \int_{q(0)=x}^{q(\beta)=x} \tilde{\mathcal{D}}q \;=\; \prod_{n=0}^{\infty} \prod_{m=1}^{\infty} \int da_n db_m$$

Where $x_n = a_n + i b_n$. Note that b_0 is absent due to the reality condition. We will pay dearly below for this choice of measure in the form of infinite β-dependent prefactors. It is possible to avoid these infinities through a more careful definition of the discrete form of the path integral (see Itzykson and Zuber 9-1). We will proceed as P&S intends and neglect the divergent pieces. We first compute the action in terms of the Fourier modes.

$$S_E \;=\; \int_0^\beta dt\, \tfrac{1}{2}\left(\,\dot{x}^2 \,+\, \omega^2 x^2\,\right)$$

Now,

$$\begin{aligned} \int_0^\beta dt\, x^2(t) \;&=\; \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{\beta}\, x_n\, x_m \int_0^\beta dt\, e^{2\pi i (n+m) t/\beta} \\ &=\; \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} x_n\, x_m\, \delta(m+n) \;=\; a_0^2 \,+\, 2\sum_{n=1}^{\infty} \left(\, a_n^2 \,+\, b_n^2 \,\right) \end{aligned}$$

And,

$$\begin{aligned} \int_0^\beta dt\, \dot{x}^2(t) \;&=\; \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{\beta}\,(-mn) \left(\frac{2\pi}{\beta}\right)^2 x_n\, x_m \int_0^\beta dt\, e^{2\pi i (n+m) t/\beta} \\ &=\; \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} (-mn) \left(\frac{2\pi}{\beta}\right)^2 x_n\, x_m\, \delta(m+n) \;=\; 2\left(\frac{2\pi}{\beta}\right)^2 \sum_{n=1}^{\infty} n^2 \left(\, a_n^2 \,+\, b_n^2 \,\right) \end{aligned}$$

Thus,

$$S_E \;=\; \tfrac{1}{2}\omega^2 a_0^2 \;+\; \sum_{n=1}^{\infty}\left(\left(\frac{2\pi n}{\beta}\right)^2 \;+\; \omega^2\right)\left(\,a_n^2 \;+\; b_n^2\,\right)$$

This leads to:

$$Z(\beta) \;=\; \int da_0\, e^{-\,\frac{1}{2}\omega^2 a_0^2}\,\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\int da_n db_m\, e^{-\left(\left(\frac{2\pi n}{\beta}\right)^2+\omega^2\right)\left(\,a_n^2+b_n^2\,\right)}$$

Or,

$$Z(\beta) \;=\; \left(\frac{2\pi}{\omega^2}\right)^{1/2}\,\prod_{n=1}^{\infty}\,\pi\left(\left(\frac{2\pi n}{\beta}\right)^2 \;+\; \omega^2\right)^{-1}$$

We may write this as :

$$Z(\beta) \;=\; \left[\,2\,\beta\,\sqrt{\frac{\pi}{2}}\,\prod_{n=1}^{\infty}\,\pi\left(\frac{2\pi n}{\beta}\right)^2\right]\;(\beta\omega)^{-1}\,\prod_{n=1}^{\infty}\left(\,1\;+\;\frac{(\beta\omega/2)^2}{(\pi n)^2}\,\right)^{-1}$$

Neglecting the ω-independent factor in backets and using the product representation for sinh appearing in P&S we find:

$$Z(\beta) \;=\; \left(\,2\,\sinh\left(\beta\omega/2\right)\,\right)^{-1}$$

You are invited to feel troubled by this derivation.

(c)

We formulate the partition function for a real scalar field by first considering the following matrix element.

$$U(\phi_a,\phi_b|-i\gamma) \;\equiv\; \langle\,\phi_b\,|\,e^{-\gamma H}\,|\,\phi_a\,\rangle$$

We are working in the Schroedinger picture with Hamiltonian:

$$H\left[\,\pi\,,\,\phi\,\right] = \int d^3x\,\mathcal{H}(\pi,\nabla\phi,\phi) \;=\; \frac{1}{2}\int d^3x\left(\,\pi^2+(\nabla\phi)^2+m^2\phi^2\,\right)$$

Rather than treating the problem of real and imaginary time separately, with time or temperature as a continuous parameter in the path integral, it is more straightforward to introduce a continuous parameter which indexes the insertion of an infinite number of complete sets of states. Defining $\epsilon = 1/N$ and inserting complete sets of field and momentum eigenstates:

$$U(\phi_a,\phi_b|\,i\gamma) \;=\; \lim_{N\to\infty}\left(\prod_{j=0}^{N}\int\mathcal{D}\phi_j(\mathbf{x}_j)\right)\left(\prod_{k=1}^{N}\int\mathcal{D}\pi_k(\mathbf{x}_k)\right)$$

$$\times\,\delta\left[\phi_0-\phi_a\right]\,\delta\left[\phi_N-\phi_b\right]\,\prod_{n=1}^{N}\,\left[\,\langle\phi_n|\,e^{-\,\epsilon\,\gamma H}\,|\pi_n\rangle\;\langle\pi_n\,|\phi_{n-1}\rangle\,\right]$$

Where we have introduced the functional delta function:

$$\mathcal{F}[\tilde{\phi}] \;=\; \int\mathcal{D}\phi\,\delta[\phi-\tilde{\phi}]\,\mathcal{F}[\phi]$$

From the canonical commutation relations:

$$\exp\left[\,-i\Gamma\left[\,\pi\,,\,\phi\,\right]\,\right] \;\equiv\; \langle\pi\,|\phi\rangle \;=\; \exp\left[-i\int d^3x\,\pi(\mathbf{x})\,\phi(\mathbf{x})\right]$$

Thus,

$$\prod_{n=1}^{N} \left[\langle \phi_n | \, e^{-\epsilon \gamma H} \, | \pi_n \rangle \, \langle \pi_n | \phi_{n-1} \rangle \right]$$

$$= \exp \left[\epsilon \sum_{n=1}^{N} \left[i\Gamma \left[\pi_n, \left(\phi_n - \phi_{n-1} \right) / \epsilon \right] - \gamma H \left[\pi_n, \phi_n \right] \right] \right]$$

We now introduce a continuous parameter σ which indexes the complete set of states and define a hybrid Lagrangian:

$$\tilde{L}_\gamma[\dot{\phi}, \phi, \pi] = \Gamma[\pi, \dot{\phi}] + i\,\gamma H\,[\pi, \phi] \qquad \text{where} \qquad \dot{\phi} \equiv \frac{\partial \phi}{\partial \sigma}$$

This leads to the path integral form of the matrix element:

$$U(\phi_a, \phi_b | -i\gamma) = \int_{\phi(0)=\phi_a}^{\phi(1)=\phi_b} \mathcal{D}\phi(\sigma) \int \mathcal{D}\pi(\sigma) \, \exp \left[i \int_0^1 d\sigma \, \tilde{L}_\gamma[\dot{\phi}, \phi, \pi] \right]$$

Substituting $\gamma = it$ and $\phi(\sigma) \to \phi(\sigma t)$ and changing variables to $s = \sigma t$ we find:

$$U(\phi_a, \phi_b | \, t) = \int_{\phi(0)=\phi_a}^{\phi(t)=\phi_b} \mathcal{D}\phi(s) \int \mathcal{D}\pi(s) \, \exp \left[i \int_0^t ds \int d^3x \, \tilde{\mathcal{L}}(\dot{\phi}, \nabla\phi, \phi, \pi) \right]$$

Where,

$$\tilde{\mathcal{L}}(\dot{\phi}, \nabla\phi, \phi, \pi) = \pi\dot{\phi} - \mathcal{H}(\pi, \nabla\phi, \phi) \qquad \text{where} \qquad \dot{\phi} \equiv \frac{\partial \phi}{\partial s}$$

The partition function is defined as:

$$Z(\beta) = \int \mathcal{D}\phi_0 \, U(\phi_0, \phi_0 | -i\beta)$$

Substituting $\gamma = \beta$ and $\phi(\sigma) \to \phi(\sigma\beta)$ and changing variables to $s = \sigma\beta$ we find:

$$Z(\beta) = \int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \mathcal{D}\phi(s) \int \mathcal{D}\pi(s) \, \exp \left[- \int_0^\beta ds \int d^3x \, \tilde{\mathcal{L}}_E(\dot{\phi}, \nabla\phi, \phi, \pi) \right]$$

Where,

$$\tilde{\mathcal{L}}_E(\dot{\phi}, \nabla\phi, \phi, \pi) = -i\,\pi\dot{\phi} + \mathcal{H}(\pi, \nabla\phi, \phi) \qquad \text{where} \qquad \dot{\phi} \equiv \frac{\partial \phi}{\partial s}$$

We now take advantage of the fact that the Hamiltonian is quadratic in π and integrate out the momentum variables. We write the path integral as:

$$U(\phi_a, \phi_b | -i\gamma) = \int_{\phi(0)=\phi_a}^{\phi(1)=\phi_b} \mathcal{D}\phi(\sigma) \exp \left[-\frac{\gamma}{2} \int_0^1 d\sigma \int d^3x \left((\nabla\phi)^2 + m^2\phi^2 \right) \right]$$

$$\times \int \mathcal{D}\pi(\sigma) \, \exp \left[- \int_0^1 d\sigma \int d^3x \left(\frac{\gamma}{2}\pi^2 - i\,\pi\,\dot{\phi} \right) \right]$$

Performing the gaussian integral and absorbing γ dependent terms into the measure:

$$U(\phi_a, \phi_b | -i\gamma) = \int_{\phi(0)=\phi_a}^{\phi(1)=\phi_b} \tilde{\mathcal{D}}\phi(\sigma) \exp \left[-\frac{\gamma}{2} \int_0^1 d\sigma \int d^3x \left(\gamma^{-2}\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \right) \right]$$

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Substituting $\gamma = it$ and $\phi(\sigma) \to \phi(\sigma t)$ and changing variables to $s = \sigma t$ we find:

$$U(\phi_a, \phi_b | \, t) \;=\; \int_{\phi(0)=\phi_a}^{\phi(t)=\phi_b} \tilde{\mathcal{D}}\phi(s) \;\exp\left[\, i \int_0^t ds \;\int d^3x \, \mathcal{L}(\dot{\phi}, \nabla\phi, \phi)\right]$$

Where,

$$\mathcal{L}(\dot{\phi}, \nabla\phi, \phi) \;=\; \frac{1}{2}\left(\, \dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \,\right) \qquad \text{where} \qquad \dot{\phi} \equiv \frac{\partial\phi}{\partial s}$$

Substituting $\gamma = \beta$ and $\phi(\sigma) \to \phi(\sigma\beta)$ and changing variables to $s = \sigma\beta$ we find:

$$Z(\beta) \;=\; \int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \tilde{\mathcal{D}}\phi(s) \;\exp\left[\, - \int_0^\beta ds \;\int d^3x \, \mathcal{L}_E(\dot{\phi}, \nabla\phi, \phi)\right]$$

Where,

$$\mathcal{L}_E(\dot{\phi}, \nabla\phi, \phi) \;=\; \frac{1}{2}\left(\, \dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \,\right) \qquad \text{where} \qquad \dot{\phi} \equiv \frac{\partial\phi}{\partial s}$$

Integrating by parts we have:

$$Z(\beta) \;=\; \int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \tilde{\mathcal{D}}\phi(s) \;\exp\left[\, -\frac{1}{2} \int_0^\beta ds \;\int d^3x \, \phi\left(\, -\partial_s^2 - \nabla^2 + m^2 \,\right)\phi\right]$$

Or,

$$Z(\beta) \;=\; \left(\det(-\partial_E^2 + m^2)\right)^{-1/2}$$

We will compute this path integral in a manner analogous to that used for the partition function for the harmonic oscillator. We introduce periodic boundary conditions on $\mathbb{R}^3$ and Fourier decompose $\phi(\mathbf{x}, s)$ $(\, V = L^3 \,)$:

$$\phi(\mathbf{x}, s) \;=\; \frac{1}{\sqrt{\beta V}} \sum_{n\in\mathbb{Z}} \sum_{\mathbf{n}\in\mathbb{Z}^3} \phi_{(n,\mathbf{n})} \, e^{2\pi i \, ns/\beta} \, e^{2\pi i \, \mathbf{n}\cdot\mathbf{x}/L}$$

Since ϕ is real, if we define $\phi_{(n,\mathbf{n})} = A_{(n,\mathbf{n})} + iB_{(n,\mathbf{n})}$, we find :

$$A_{(n,\mathbf{n})} \;=\; A_{(-n,-\mathbf{n})} \qquad \text{and} \qquad B_{(n,\mathbf{n})} \;=\; -B_{(-n,-\mathbf{n})}$$

This allows us to define the functional measure as:

$$\int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \tilde{\mathcal{D}}\phi(s) \;=\; \left[\prod_{n\geq 0} \prod_{\mathbf{n}} \int dA_{(n,\mathbf{n})}\right] \left[\prod_{m>0} \prod_{\mathbf{m}} \int dB_{(m,\mathbf{m})}\right]$$

With some algebra we find:

$$\frac{1}{2} \int_0^\beta ds \;\int d^3x \, \phi\left(\, -\partial_s^2 - \nabla^2 + m^2 \,\right)\phi \;=\; \sum_{\mathbf{n}} \frac{1}{2} \, A_{(0,\mathbf{n})}^2 \left(\, m^2 + (2\pi\mathbf{n}/L)^2 \,\right)$$

$$+ \sum_{\mathbf{n}} \sum_{n>0} \left(A_{(n,\mathbf{n})}^2 + B_{(n,\mathbf{n})}^2\right) \left(\, m^2 + (2\pi n/\beta)^2 + (2\pi\mathbf{n}/L)^2 \,\right)$$

Defining,

$$I_{(n,\mathbf{n})} \;=\; \int d\alpha \, \exp\left[\, -\alpha^2 \;\left(\, m^2 + (2\pi n/\beta)^2 + (2\pi\mathbf{n}/L)^2 \,\right)\,\right]$$

We have:

$$Z(\beta) \;=\; \prod_{\mathbf{n}} \left[\sqrt{2}\, I(0,\mathbf{n}) \prod_{n>0} \left(I(n,\mathbf{n}) \right)^2 \right]$$

Defining,

$$\omega_{\mathbf{n}}^2 \;=\; \left(m^2 + (2\pi \mathbf{n}/L)^2 \right)$$

We find:

$$I_{(n,\mathbf{n})} \;=\; \sqrt{\pi}\, \left((2\pi n/\beta)^2 + \omega_{\mathbf{n}}^2 \right)^{-1/2}$$

Thus,

$$Z(\beta) \;=\; \prod_{\mathbf{n}} \left[\frac{\sqrt{2\pi}}{\omega_{\mathbf{n}}} \prod_{n>0} \pi \left((2\pi n/\beta)^2 + \omega_{\mathbf{n}}^2 \right)^{-1} \right]$$

Dropping $\omega_{\mathbf{n}}$-independent factors as in part (**b**) above, we may write this as :

$$Z(\beta) \;=\; \prod \left(2 \sinh(\beta\, \omega_{\mathbf{n}}/2) \right)^{-1}$$

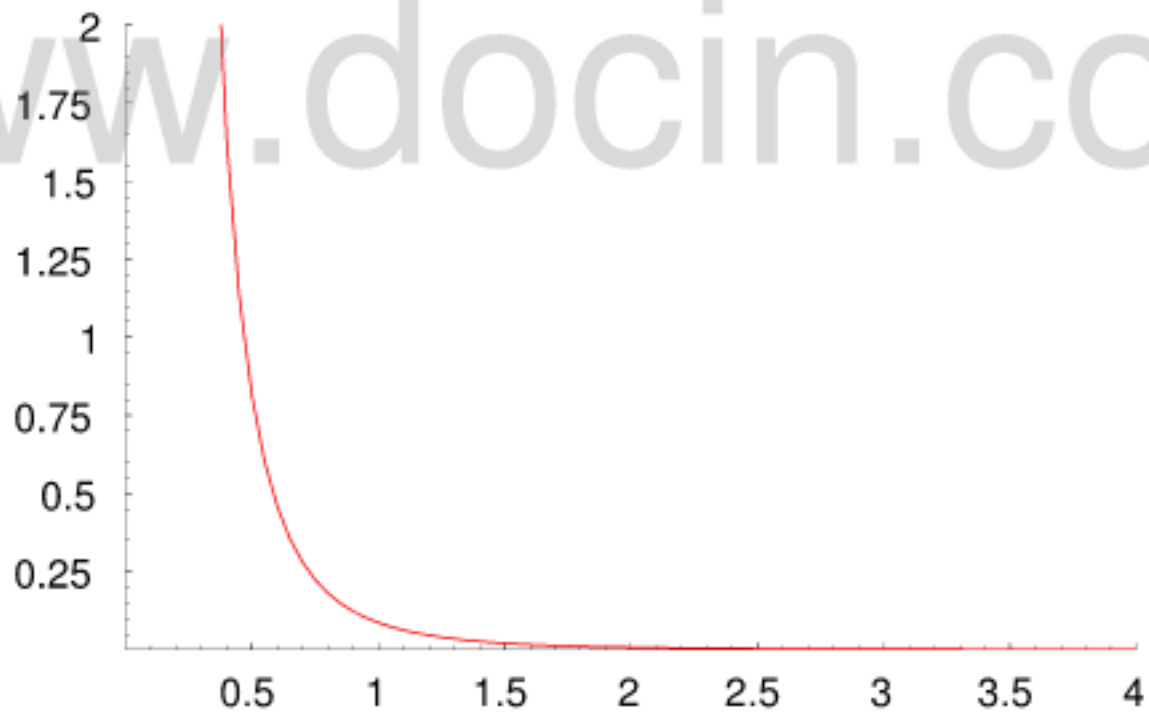
We rewrite this as:

$$\ln Z(\beta) \;=\; -\sum_{\mathbf{n}} (\beta\, \omega_{\mathbf{n}}/2) \;-\; \sum_{\mathbf{n}} \ln \left(1 - \exp(-\beta\, \omega_{\mathbf{n}}) \right)$$

Dropping the first term, which amounts to the normal ordering prescription, and writing the sum as an integral over $\mathbf{k}$ we find:

$$\ln Z(\beta) \;=\; -V \int \frac{d^3k}{(2\pi)^3} \ln \left(1 - \exp \left(-\beta \sqrt{\mathbf{k}^2 + m^2} \right) \right)$$

As for the harmonic oscillator, this derivation is much simpler using operator methods. Please see Itzykson and Zuber 3-1-5. The following is a plot of $\ln Z(\beta)/(Vm^3)$ as a function of βm obtained through numerical integration.



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3.)

We begin with the following Lagrangian:

$$\mathcal{L}_0 \;=\; \psi^*\dot{\psi} \;-\; \frac{1}{2m}\,\nabla\psi^*\cdot\nabla\psi$$

This leads to the Schroedinger equation as the wave equation obeyed by the field (that is, the creation/annihilation operators will create/destroy particles in Schroedinger wave modes):

$$i\,\dot{\psi} \;=\; -\,\frac{1}{2m}\,\nabla^2\psi$$

We find that the canonical momenta are:

$$\pi_\psi \;\equiv\; \frac{\partial\mathcal{L}_0}{\partial\dot{\psi}} \;=\; i\,\psi^* \qquad \text{and} \qquad \pi_{\psi^*} \;\equiv\; \frac{\partial\mathcal{L}_0}{\partial\dot{\psi}^*} \;=\; 0$$

The Hamiltonian (density) is:

$$\mathcal{H}_0 \;\equiv\; \pi_\psi\dot{\psi} + \pi_{\psi^*}\dot{\psi}^* - \mathcal{L}_0 \;=\; \frac{1}{2m}\,\nabla\psi^*\cdot\nabla\psi \;=\; \frac{-i}{2m}\,\nabla\pi\cdot\nabla\psi$$

Essentially, because the Lagrangian is linear in $\dot{\psi}$, the phase space for each eigenmode of the wave operator (labelled by the spatial momentum $\mathbf{p}$) is two dimensional rather than four dimensional. We could have placed ψ and ψ^* on a more equal footing but this would have required us to implement constraints to define the Hamiltonian. In what follows we treat ψ and its momentum π as the only phase space variables. Note that the equation of motion for ψ^* follows from the Hamilton equation for $\dot{\pi}$. This is similar to how the Dirac equation is generally treated in the Hamiltonian formalism.
We now implement the canonical commutation relations:

$$[\,\psi(\mathbf{x},t)\,,\,\pi(\mathbf{y},t)\,] \;=\; i\,\delta^3(\mathbf{x}-\mathbf{y})$$

These may be implemented by defining ψ as follows:

$$\psi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\,a_{\mathbf{p}}\,e^{i\mathbf{p}\cdot\mathbf{x}\,-\,i\,\omega(\mathbf{p})\,t} \qquad \text{where} \qquad \omega(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}$$

where, as usual,

$$\left[\,a_{\mathbf{p}}\,,\,a^\dagger_{\mathbf{k}}\,\right] \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})$$

Note that one does not need to include a term $b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}+i\omega(\mathbf{p})t}$ in the mode expansion, as there are no negative frequency solutions to the nonrelativistic dispersion relation $\omega = \mathbf{p}^2/2m$, in contrast to the relativistic version $\omega = \pm\sqrt{\mathbf{p}^2+m^2}$. The time dependence of this choice is consistent with the Hamiltonian:

$$H_0 \;=\; \int d^3x\,\mathcal{H}_0 \;=\; \int d^3x\,\frac{1}{2m}\,\nabla\psi^*\cdot\nabla\psi \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}^2}{2m}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}$$

We now introduce an interaction of the form:

$$H_1(t) \;=\; \frac{1}{2}\int d^3x\int d^3y\;|\psi(\mathbf{x},t)|^2\;U(\mathbf{x}-\mathbf{y})\;|\psi(\mathbf{y},t)|^2$$

This is exactly the same expression as the interaction energy of two particles in nonrelativistic QM interacting via the potential $U(\mathbf{x}-\mathbf{y})$. The only difference is that ψ is a quantum field

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rather than a wavefunction.[1] Nevertheless, it serves much the same purpose, since $\psi^\dagger\psi$ is the operator that measures the particle density. Note that, in the Heisenberg picture, this operator is not time independent since it does not commute with H_0 . However, importantly, it may be shown that it does commute with the number operator:

$$N \;=\; \int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

Thus, as expected, the number of particles is conserved in the non-relativistic interaction. To see that this interaction is the correct one we consider an n-particle state consisting of position eigenfunctions localized at time $t = 0$:

$$|\, x_1 \ldots x_n \,\rangle \;=\; \psi^\dagger(\mathbf{x}_1, 0) \,\ldots\, \psi^\dagger(\mathbf{x}_n, 0)\; |0\rangle \;=\; \left[\prod_{j=1}^{n} \int \frac{d^3p_j}{(2\pi)^3}\, e^{-i\mathbf{p}_j\cdot\mathbf{x}_j}\; a^\dagger_{\mathbf{p}_j} \right]\; |0\rangle$$

Denoting $\psi_{\mathbf{x}} = \psi(\mathbf{x}, 0)$, we compute:

$$:H_1(0):\; |\, x_1 \ldots x_n \,\rangle \;=\; \frac{1}{2} \int d^3x \int d^3y\, \psi^\dagger_{\mathbf{x}}\, \psi^\dagger_{\mathbf{y}}\, U(\mathbf{x}-\mathbf{y})\, \psi_{\mathbf{x}}\, \psi_{\mathbf{y}}\; \psi^\dagger_{\mathbf{x}_1} \;\ldots\; \psi^\dagger_{\mathbf{x}_n}\; |0\rangle$$

Using the canonical commutation relations we find:

$$:H_1(0):\; |\, x_1 \ldots x_n \,\rangle \;=\; \frac{1}{2} \sum_{j=1}^{n} \sum_{k\neq j}^{n} U(\mathbf{x}_k - \mathbf{y}_j)\; |\, x_1 \ldots x_n \,\rangle$$

Note that the normal ordering removes the self-interactions. We now consider the system in a large cubic box of volume $V = L^3$. We replace the field expansion $(t = 0)$ by :

$$\psi_{\mathbf{x}} \;=\; \frac{1}{V} \sum_{\mathbf{n}\in\mathbb{Z}} a_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}} \qquad \text{where} \qquad \mathbf{p} \;=\; \frac{2\pi\mathbf{n}}{L}$$

And the commutation relations are:

$$\left[\psi_{\mathbf{x}}\,,\, \psi^\dagger_{\mathbf{y}}\right] \;=\; \delta^3(\mathbf{x}-\mathbf{y}) \qquad \text{and} \qquad \left[a_{\mathbf{p}}\,,\, a^\dagger_{\mathbf{k}}\right] \;=\; V\,\delta_{\mathbf{pk}}$$

We now compute the expectation value of the non-interacting N-particle ground state $|\Omega_N\rangle$ to first order in perturbation theory. That is we compute :

$$E^{(1)} \;=\; \langle\Omega_N|\; :H_1(0):\; |\Omega_N\rangle$$

Where,

$$|\Omega_N\rangle \;=\; \frac{V^{-N/2}}{\sqrt{N!}} \left(a^\dagger_{\mathbf{0}}\right)^N\; |0\rangle$$

Thus,

$$E^{(1)} \;=\; \frac{V^{-N}}{N!}\, \frac{1}{2} \int d^3x \int d^3y\, U(\mathbf{x}-\mathbf{y})\; \langle 0|\left(a_{\mathbf{0}}\right)^N \psi^\dagger_{\mathbf{x}}\, \psi^\dagger_{\mathbf{y}} \psi_{\mathbf{x}}\, \psi_{\mathbf{y}} \left(a^\dagger_{\mathbf{0}}\right)^N |0\rangle$$

[1] By the way, the terminology "second quantization" refers to this distinction – ordinary quantum mechanics ("first quantization") has $\mathbf{x}$ as the dynamical variable, and ψ as the wavefunction. Here we have ψ as the dynamical variable, and the states are formally described by wavefunctions $\Psi(\psi)$. In a garbled sense, one is quantizing the wavefunction, hence the terminology "second quantization". But this abuse of terminology arises from overlooking the fact that in field theory, the field always was the dynamical object, not $\mathbf{x}$ which simply parametrizes the space on which the field is defined.

Now,

$$\psi_{\mathbf{x}}\,\psi_{\mathbf{y}}\left(a_{\mathbf{0}}^{\dagger}\right)^{N}|0\rangle \;=\; N\,\psi_{\mathbf{x}}\left(a_{\mathbf{0}}^{\dagger}\right)^{N-1}|0\rangle \;=\; N\;(N-1)\left(a_{\mathbf{0}}^{\dagger}\right)^{N-2}|0\rangle$$

Thus,

$$\langle 0|\left(a_{\mathbf{0}}\right)^{N}\psi_{\mathbf{x}}^{\dagger}\,\psi_{\mathbf{y}}^{\dagger}\psi_{\mathbf{x}}\,\psi_{\mathbf{y}}\left(a_{\mathbf{0}}^{\dagger}\right)^{N}|0\rangle \;=\; N\,(N-1)\;N!\,V^{N-2}$$

Also,

$$\int d^3x\int d^3y\,U(\mathbf{x}-\mathbf{y}) \;=\; V\int d^3x\,U(\mathbf{x}) \;=\; V\,\tilde{U}(\mathbf{0})$$

Where,

$$\tilde{U}(\mathbf{p}) \;=\; \int d^3x\,U(\mathbf{x})\,e^{i\mathbf{p}\cdot\mathbf{x}}$$

Thus, finally,

$$\frac{E^{(1)}}{N} \;=\; \frac{N-1}{2\,V}\,\tilde{U}(\mathbf{0})$$

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Physics 443 Homework #3

Due Thursday, October 28, 2010

1.)

a) Show by explicit computation the Lorentz invariance of the Dirac Lagrangian, by considering a Lorentz transformation of the fields.

b) Consider the chiral rotation of a massless Dirac field

$$\psi' = \exp[i\alpha\gamma_5]\psi \ ;$$

find the corresponding Noether current. Show that the corresponding Noether charge measures the total helicity of a collection of massless Dirac particles, and that the addition of a mass term to the Lagrangian violates the symmetry; find an equation that expresses the violation of current conservation by the mass.

c) Find the Noether current related to charge conservation by considering a phase rotation of a Dirac field (of arbitrary mass)

$$\psi' = \exp[i\alpha]\psi \ .$$

2.) Peskin and Schroeder 3.1

3.) Peskin and Schroeder 3.4; you may omit part (d) – you did it in problem 1.

4.) Peskin and Schroeder 3.7

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QFT 1 : Problem Set 3

1.)

(a)

We consider the Dirac Lagrangian:

$$\mathcal{L}\left[\psi\right] \,=\, \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi$$

Where we have:

$$\left\{\,\gamma^{\mu}\,,\,\gamma^{\nu}\,\right\} \,=\, 2\,\eta^{\mu\nu} \qquad \text{and} \qquad \bar{\psi} \,=\, \psi^{\dagger}\gamma^{0}$$

And we use the conventions:

$$\partial\!\!\!/\psi \,\equiv\, \gamma^{\mu}\partial_{\mu}\psi \qquad \text{and} \qquad \left(\gamma^{\mu}\right)^{\dagger} \,=\, \gamma^{0}\gamma^{\mu}\gamma^{0}$$

Lorentz invariance means that:

$$\mathcal{L}\left[\psi\right](x) \,=\, \mathcal{L}\left[\psi'\right](\Lambda x) \qquad \text{where} \qquad \psi'(\Lambda x) \,=\, M_{\Lambda}\psi(x)$$

Where $\Lambda \in SO(3,1)$ such that:

$$\Lambda^{\mu}{}_{\nu} \,=\, \left[\,\exp\left(-\tfrac{1}{2}\,i\,\omega_{\alpha\beta}\,L^{\alpha\beta}\right)\right]^{\mu}{}_{\nu}$$

Where,

$$\left[\,L^{\alpha\beta}\,\right]^{\mu}{}_{\nu} \,=\, i\left(\eta^{\alpha\mu}\delta^{\beta}{}_{\nu} \,-\, \eta^{\beta\mu}\delta^{\alpha}{}_{\nu}\right)$$

And where $M_{\Lambda} \in Spin(3,1)$ such that:

$$M_{\Lambda} \,=\, \exp\left(-\tfrac{1}{2}\,i\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right)$$

Where,

$$S^{\alpha\beta} \,=\, \frac{i}{4}\left[\,\gamma^{\alpha}\,,\,\gamma^{\beta}\,\right]$$

Here we have suppressed spinor indices. We now consider

$$\mathcal{L}\left[\psi'\right](\Lambda x) \,=\, \bar{\psi}'(\Lambda x)\left(i\partial\!\!\!/\psi'\right)(\Lambda x) \,-\, m\,\bar{\psi}'(\Lambda x)\psi'(\Lambda x)$$

Now,

$$M_{\Lambda}^{\dagger}\gamma^{0} \,=\, \gamma^{0}M_{\Lambda}^{-1} \qquad \text{since} \qquad \gamma^{0}\left(\,S^{\alpha\beta}\,\right)^{\dagger}\gamma^{0} \,=\, -S^{\alpha\beta}$$

Thus,

$$\bar{\psi}'(\Lambda x) \,=\, \psi'^{\dagger}(\Lambda x)\gamma^{0} \,=\, \psi^{\dagger}(x)M_{\Lambda}^{\dagger}\gamma^{0} \,=\, \bar{\psi}(x)M_{\Lambda}^{-1}$$

Also,

$$\left(i\partial\!\!\!/\psi'\right)(\Lambda x) \,=\, i\,\gamma^{\mu}\,\left.\frac{\partial}{\partial y^{\mu}}\,\psi'(y)\right|_{y=\Lambda x} \,=\, i\,\gamma^{\mu}\,M_{\Lambda}\,\left.\frac{\partial}{\partial y^{\mu}}\,\psi(\Lambda^{-1}y)\right|_{y=\Lambda x}$$

Defining $a = \Lambda^{-1}y$ and using the chain rule,

$$\left.\frac{\partial}{\partial y^{\mu}}\,\psi(\Lambda^{-1}y)\right|_{y=\Lambda x} \,=\, \left.\frac{\partial\psi}{\partial a^{\nu}}(a)\,\frac{\partial a^{\nu}}{\partial y^{\mu}}\right|_{y=\Lambda x} \,=\, \frac{\partial\psi}{\partial x^{\nu}}(x)\,\left(\Lambda^{-1}\right)^{\nu}{}_{\mu}$$

Thus,

$$\left(i\partial\!\!\!/\psi'\right)(\Lambda x) \,=\, i\,\left(\Lambda^{-1}\right)^{\nu}{}_{\mu}\,\gamma^{\mu}\,M_{\Lambda}\,\partial_{\nu}\psi(x)$$

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Thus we have:

$$\mathcal{L}\left[\psi'\right](\Lambda x) \;=\; \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}\,\bar{\psi}(x)\,M_{\Lambda}^{-1}\gamma^{\mu}\,M_{\Lambda}\,i\partial_{\nu}\psi(x) \;-\; m\,\bar{\psi}(x)\psi(x)$$

Now, using the γ matrix algebra, for infinitesimal $\omega_{\alpha\beta}$, one may show:

$$\left(1+i/2\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right)\gamma^{\mu}\left(1-i/2\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right) \;=\; \left(1-i/2\,\omega_{\alpha\beta}\,L^{\alpha\beta}\right)^{\mu}{}_{\nu}\gamma^{\nu}$$

Thus,

$$M_{\Lambda}^{-1}\gamma^{\mu}\,M_{\Lambda} \;=\; \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

And, finally,

$$\mathcal{L}\left[\psi'\right](\Lambda x) \;=\; \mathcal{L}\left[\psi\right](x) \;=\; \bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x)$$

(b)

We consider the Lagrangian of a massless Dirac field:

$$\mathcal{L}_0\left[\psi\right] \;=\; \bar{\psi}\left(i\partial\!\!\!/\right)\psi$$

A chiral rotation is given by:

$$\psi' \;=\; e^{\,i\alpha\gamma^5}\,\psi \qquad \text{where} \qquad \gamma^5 \;=\; i\gamma^0\gamma^1\gamma^2\gamma^3$$

This transformation can be seen to be a symmetry of the Lagrangian and thus of the action:

$$\mathcal{L}_0\left[\psi'\right] \;=\; \bar{\psi}'\left(i\partial\!\!\!/\right)\psi' \;=\; \psi^{\dagger}\,e^{\,-i\alpha(\gamma^5)^{\dagger}}\,\gamma^0\,\left(i\partial\!\!\!/\right)\,e^{\,i\alpha\gamma^5}\,\psi$$

Now, it may be shown,

$$(\gamma^5)^{\dagger} \;=\; \gamma^5 \qquad \text{and} \qquad \left\{\,\gamma^{\mu}\,,\,\gamma^5\,\right\} \;=\; 0$$

Thus,

$$\mathcal{L}_0\left[\psi'\right] \;=\; \psi^{\dagger}\,e^{\,-i\alpha\gamma^5}\,\gamma^0\,e^{\,-i\alpha\gamma^5}\,\left(i\partial\!\!\!/\right)\,\psi \;=\; \bar{\psi}\left(i\partial\!\!\!/\right)\psi \;=\; \mathcal{L}_0\left[\psi\right]$$

We now derive the Noether current. For an infinitesimal variation $\delta\psi$, after integrating by parts and applying the equations of motion, we find:

$$S\left[\psi+\delta\psi\right]-S\left[\psi\right] \simeq \int d^4x\;\partial_{\mu}\left(\,i\bar{\psi}\gamma^{\mu}\delta\psi\,\right)$$

Using $\delta\psi \;=\; i\epsilon\gamma^5\psi$ we find the conserved axial current:

$$J_A^{\mu} \;=\; \bar{\psi}\gamma^{\mu}\gamma^5\psi \qquad \text{where} \qquad \partial_{\mu}J_A^{\mu} \;=\; 0$$

This leads to the Noether charge:

$$Q_A \;=\; \int d^3x\,J_A^0 \;=\; \int d^3x\,\psi^{\dagger}\gamma^0\gamma^0\gamma^5\psi \;=\; \int d^3x\,\psi^{\dagger}\gamma^5\psi$$

Using,

$$\psi \;=\; \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \qquad \text{and} \qquad \gamma^5 \;=\; \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

We find,

$$Q_A \;=\; \int d^3x\,\left(\,{\psi_R}^{\dagger}\psi_R \;-\; {\psi_L}^{\dagger}\psi_L\,\right)$$

To show that this operator measures the helicity of a collection of massless particles we introduce the matrices:

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}) \qquad \text{and} \qquad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma})$$

The γ matrices take the form:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

This allows us to decouple the Dirac equation for massless particles:

$$i\bar{\sigma} \cdot \partial \psi_L = 0 \qquad \text{and} \qquad i\sigma \cdot \partial \psi_R = 0$$

We write the field operators as:

$$\psi_L = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_L(\mathbf{p})\, \alpha_L(\mathbf{p})\, e^{-ip\cdot x} + b^\dagger{}_L(\mathbf{p})\, \beta_L(\mathbf{p})\, e^{ip\cdot x} \right)$$

$$\psi_R = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_R(\mathbf{p})\, \alpha_R(\mathbf{p})\, e^{-ip\cdot x} + b^\dagger{}_R(\mathbf{p})\, \beta_R(\mathbf{p})\, e^{ip\cdot x} \right)$$

Imposing the equations of motion we find:

$$p \cdot \bar{\sigma} \alpha_L(\mathbf{p}) = p \cdot \bar{\sigma} \beta_L(\mathbf{p}) = 0 \qquad \text{where} \qquad p \cdot \bar{\sigma} = E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}$$

And,

$$p \cdot \sigma \alpha_R(\mathbf{p}) = p \cdot \sigma \beta_R(\mathbf{p}) = 0 \qquad \text{where} \qquad p \cdot \sigma = E_{\mathbf{p}} - \mathbf{p} \cdot \boldsymbol{\sigma}$$

Thus, defining the helicity operator,

$$h = \mathbf{p} \cdot \boldsymbol{\sigma} / E_{\mathbf{p}}$$

We find:

$$h\, \alpha_L(\mathbf{p}) = -\alpha_L(\mathbf{p}) \qquad \text{and} \qquad h\, \beta_L(\mathbf{p}) = -\beta_L(\mathbf{p})$$

And,

$$h\, \alpha_R(\mathbf{p}) = \alpha_R(\mathbf{p}) \qquad \text{and} \qquad h\, \beta_R(\mathbf{p}) = \beta_R(\mathbf{p})$$

Using the orthogonality relations (see P+S pg 48):

$$\alpha_{L,R}{}^\dagger(\mathbf{p}) \alpha_{L,R}(\mathbf{p}) = \beta_{L,R}{}^\dagger(\mathbf{p}) \beta_{L,R}(\mathbf{p}) = 2E_{\mathbf{p}}$$

$$\alpha_{L,R}{}^\dagger(\mathbf{p}) \beta_{L,R}(-\mathbf{p}) = \beta_{L,R}{}^\dagger(-\mathbf{p}) \alpha_{L,R}(\mathbf{p}) = 0$$

We find:

$$: Q_A : = \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger{}_R\, a_R + b^\dagger{}_R\, b_R - a^\dagger{}_L\, a_L - b^\dagger{}_L\, b_L \right)$$

Thus we see that the Noether charge measures the helicity of a collection of particles.

We now add a mass term to the Lagrangian and verify that the chiral symmetry is violated.

$$\mathcal{L}[\psi] = \bar{\psi} (i\partial\!\!\!/ - m) \psi = \mathcal{L}_0[\psi] - m\bar{\psi}\psi$$

Thus,

$$\mathcal{L}[\psi'] = \mathcal{L}_0[\psi'] - m\, \psi^\dagger e^{-i\alpha\gamma^5} \gamma^0 e^{i\alpha\gamma^5} \psi = \mathcal{L}_0[\psi] - m\, \bar{\psi}\, e^{2i\alpha\gamma^5} \psi$$

Thus for an infinitesimal variation $\delta\psi = i\epsilon\gamma^5\psi$ we find:

$$S[\psi + \delta\psi] - S[\psi] \simeq -\epsilon \int d^4x \ \left(2im\, \bar{\psi}\gamma^5\psi \right)$$

Thus the chiral symmetry is violated by the mass term. From above:

$$S\left[\psi+\delta\psi\right]-S\left[\psi\right]\;\simeq\;-\epsilon\int d^4x\;\partial_\mu\left(\,\bar{\psi}\gamma^\mu\gamma^5\psi\,\right)$$

Thus we may express the violation of current conservation as:

$$\partial_\mu\left(\,\bar{\psi}\gamma^\mu\gamma^5\psi\,\right)\;=\;2im\,\bar{\psi}\gamma^5\psi$$

It is simpler to derive this by computing $\partial_\mu J^\mu_A$ and making use of the Dirac equation.

(c)

We now consider the symmetry under phase rotations:

$$\psi'\;=\;e^{\,i\alpha}\,\psi$$

Clearly,

$$\mathcal{L}\left[\psi'\right]\;=\;\mathcal{L}\left[\psi\right]$$

For an infinitesimal variation $\delta\psi\;=\;i\epsilon\psi$ we find:

$$S\left[\psi+\delta\psi\right]-S\left[\psi\right]\;\simeq\;-\epsilon\int d^4x\;\partial_\mu\left(\,\bar{\psi}\gamma^\mu\psi\,\right)$$

Thus we find the conserved current:

$$J^\mu\;=\;\bar{\psi}\gamma^\mu\psi\qquad\text{where}\qquad\partial_\mu J^\mu\;=\;0$$

2.) Peskin & Schroeder 3.1 : Lorentz group

We begin with the Lorentz algebra:

$$\left[\,J^{\mu\nu}\,,\,J^{\rho\sigma}\,\right]\;=\;i\left(\,\eta^{\nu\rho}J^{\mu\sigma}-\eta^{\mu\rho}J^{\nu\sigma}-\eta^{\nu\sigma}J^{\mu\rho}+\eta^{\mu\sigma}J^{\nu\rho}\right)$$

(a)

We define the generators of rotations and boosts as:

$$L^i\;=\;\tfrac{1}{2}\,\epsilon^{ijk}\,J^{jk}\qquad\text{and}\qquad K^i\;=\;J^{0i}$$

We compute:

$$\left[\,L^i\,,\,L^j\,\right]\;=\;\tfrac{1}{4}\,\epsilon^{ipq}\,\epsilon^{jmn}\,\left[\,J^{pq}\,,\,J^{mn}\,\right]\;=\;i\,\epsilon^{ipq}\,\epsilon^{jmn}\,\delta^{pm}\,J^{qn}$$

Where we have substituted $\delta^{ij}\;=\;-\,\eta^{ij}$. Now,

$$J^{ij}\;=\;\epsilon^{ijk}\,L^k\qquad\text{and}\qquad\epsilon^{ipq}\,\epsilon^{jmn}\,\delta^{pm}\;=\;\left(\,\delta^{ij}\delta^{qn}-\delta^{in}\delta^{qj}\,\right)$$

Thus,

$$\left[\,L^i\,,\,L^j\,\right]\;=\;i\,\left(\,\delta^{ij}\delta^{qn}-\delta^{in}\delta^{qj}\,\right)\,\epsilon^{qnk}\,L^k\;=\;i\,\epsilon^{ijk}\,L^k$$

We next compute:

$$\left[\,K^i\,,\,K^j\,\right]\;=\;\left[\,J^{0i}\,,\,J^{0j}\,\right]\;=\;-i\,\eta^{00}\,J^{ij}\;=\;-i\,\epsilon^{ijk}\,L^k$$

And, finally we compute:

$$\left[\, L^i \,,\, K^j \,\right] \;=\; \tfrac{1}{2}\,\epsilon^{ipq}\,\left[\, J^{pq} \,,\, J^{0j} \,\right] \;=\; -i\,\epsilon^{ipq}\,\delta^{pj}\,J^{q0} \;=\; i\,\epsilon^{ijk}\,K^k$$

We now define:

$$\mathbf{J}_\pm \;=\; \tfrac{1}{2}\,\left(\mathbf{L} \,\pm\, i\,\mathbf{K}\right)$$

We compute:

$$\begin{aligned}\left[\, J^i_+ \,,\, J^j_+ \,\right] \;&=\; \tfrac{1}{4}\,\left(\left[\, J^i \,,\, J^j \,\right] - \left[\, K^i \,,\, K^j \,\right] + i\left[\, J^i \,,\, K^j \,\right] + i\left[\, K^i \,,\, J^j \,\right]\right)\\ &=\; i\,\epsilon^{ijk}\,\tfrac{1}{2}\,\left(\,L^k \,+\, i\,K^k\,\right) \;=\; i\,\epsilon^{ijk}\,J^k_+\end{aligned}$$

Similarly,

$$\begin{aligned}\left[\, J^i_- \,,\, J^j_- \,\right] \;&=\; \tfrac{1}{4}\,\left(\left[\, J^i \,,\, J^j \,\right] - \left[\, K^i \,,\, K^j \,\right] - i\left[\, J^i \,,\, K^j \,\right] - i\left[\, K^i \,,\, J^j \,\right]\right)\\ &=\; i\,\epsilon^{ijk}\,\tfrac{1}{2}\,\left(\,L^k \,-\, i\,K^k\,\right) \;=\; i\,\epsilon^{ijk}\,J^k_-\end{aligned}$$

And,

$$\left[\, J^i_+ \,,\, J^j_- \,\right] \;=\; \tfrac{1}{4}\,\left(\left[\, J^i \,,\, J^j \,\right] + \left[\, K^i \,,\, K^j \,\right] - i\left[\, J^i \,,\, K^j \,\right] + i\left[\, K^i \,,\, J^j \,\right]\right) \;=\; 0$$

Thus we find that we may put the representations of the Lorentz group in correspondence with the representations of $SU(2) \times SU(2)$. Note that this does not mean that $Spin(3,1) = SL(2,\mathbb{C})$ is the same Lie group as $SU(2) \times SU(2)$ since $SO(3,1)$ and thus $Spin(3,1)$ are non-compact while $SU(2) \times SU(2)$ is compact. $SU(2) \times SU(2)$ is the same manifold as the product $S^3 \times S^3$. $Spin(3,1)$ is topologically $S^3 \times \mathbb{R}^3$.

(b)

We consider the (j_+, j_-) representation of $SU(2)_+ \times SU(2)_-$. A field in this representation transforms under an infinitesimal $\Lambda \in SO(3,1)$ as:

$$\tilde{\Phi}_{(j_+,j_-)}(\Lambda x) = \left(\,\mathbf{1} - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K}\,\right)\Phi_{(j_+,j_-)}(x)$$

For the $(1/2, 0)$ representation we have:

$$\mathbf{J}_+ \;=\; \boldsymbol{\sigma}/2 \;=\; \tfrac{1}{2}\,\left(\mathbf{L} \,+\, i\,\mathbf{K}\right) \qquad \text{and} \qquad \mathbf{J}_- \;=\; 0 \;=\; \tfrac{1}{2}\,\left(\mathbf{L} \,-\, i\,\mathbf{K}\right)$$

Thus $\mathbf{L} = i\mathbf{K} = \boldsymbol{\sigma}/2$ and,

$$\tilde{\Phi}_{(1/2,0)}(\Lambda x) \;=\; \left(\,\mathbf{1} - i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2\,\right)\Phi_{(1/2,0)}(x)$$

For the $(0, 1/2)$ representation we have:

$$\mathbf{J}_+ \;=\; 0 \;=\; \tfrac{1}{2}\,\left(\mathbf{L} \,+\, i\,\mathbf{K}\right) \qquad \text{and} \qquad \mathbf{J}_- \;=\; \boldsymbol{\sigma}/2 \;=\; \tfrac{1}{2}\,\left(\mathbf{L} \,-\, i\,\mathbf{K}\right)$$

Thus $\mathbf{L} = -i\mathbf{K} = \boldsymbol{\sigma}/2$ and,

$$\tilde{\Phi}_{(0,1/2)}(\Lambda x) \;=\; \left(\,\mathbf{1} - i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2\,\right)\Phi_{(0,1/2)}(x)$$

Thus in the notation of P&S we have:

$$\Phi_{(1/2,0)} \;\equiv\; \psi_L \qquad \text{and} \qquad \Phi_{(0,1/2)} \;\equiv\; \psi_R$$

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(c)

We consider the behavior of the matrix,

$$\mathbb{V} = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix} = V_\mu \, \bar{\sigma}^\mu \qquad \text{where} \qquad V^\mu = \tfrac{1}{2} \operatorname{tr} (\sigma^\mu \, \mathbb{V})$$

under the infinitesimal transformation:

$$\tilde{\mathbb{V}} = \left(\mathbf{1} + \tfrac{1}{2}(\boldsymbol{\beta} - i\boldsymbol{\theta}) \cdot \boldsymbol{\sigma} \right) \mathbb{V} \left(\mathbf{1} + \tfrac{1}{2}(\boldsymbol{\beta} + i\boldsymbol{\theta}) \cdot \boldsymbol{\sigma} \right)$$

Or using the cyclicity of the trace and defining $\mathbf{a} = (\boldsymbol{\beta} + i\boldsymbol{\theta})/2$:

$$\tilde{V}^\mu = \tfrac{1}{2} \operatorname{tr} \left[\left(\mathbf{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \right) \sigma^\mu \left(\mathbf{1} + \bar{\mathbf{a}} \cdot \boldsymbol{\sigma} \right) \left(V^0 + \mathbf{V} \cdot \boldsymbol{\sigma} \right) \right]$$

Now, to first order in $\mathbf{a}$:

$$\left(\mathbf{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \right) \sigma^0 \left(\mathbf{1} + \bar{\mathbf{a}} \cdot \boldsymbol{\sigma} \right) = \left(\mathbf{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \right)$$

And, using $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$:

$$\left(\mathbf{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \right) \boldsymbol{\sigma} \left(\mathbf{1} + \bar{\mathbf{a}} \cdot \boldsymbol{\sigma} \right) = \boldsymbol{\sigma} + \boldsymbol{\beta} + \boldsymbol{\theta} \times \boldsymbol{\sigma}$$

Thus,

$$\tilde{V}^0 = V^0 + \boldsymbol{\beta} \cdot \mathbf{V} \qquad \text{and} \qquad \tilde{\mathbf{V}} = \mathbf{V} + \boldsymbol{\beta} V^0 + \boldsymbol{\theta} \times \mathbf{V}$$

This is just the transformation of a Lorentz vector.

3.) Peskin & Schroeder 3.4 : Majorana fermions

(a)

Borrowing notation from problems 1 and 2 above, we define

$$M_\Lambda = \exp \left(-\tfrac{1}{2} i \, \omega_{\alpha\beta} \, S^{\alpha\beta} \right) = \begin{pmatrix} \Omega_L & 0 \\ 0 & \Omega_R \end{pmatrix}$$

Where,

$$\Omega_L = \exp \left(-\tfrac{1}{2} i \, (\boldsymbol{\theta} - i\boldsymbol{\beta}) \cdot \boldsymbol{\sigma} \right)$$

And using $\sigma^2 \boldsymbol{\sigma} \sigma^2 = -\boldsymbol{\sigma}^*$ we have

$$\Omega_R^* = \sigma^2 \, \Omega_L \, \sigma^2$$

And,

$$\bar{\sigma}^\mu = \sigma^2 \, (\sigma^\mu)^* \, \sigma^2$$

From the properties of the γ matrices we have

$$\Omega_R^{-1} \bar{\sigma}^\mu \Omega_L = \Lambda^\mu{}_\nu \, \bar{\sigma}^\nu \qquad \Omega_L^{-1} \sigma^\mu \Omega_R = \Lambda^\mu{}_\nu \, \sigma^\nu$$

And,

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2 \, \eta^{\mu\nu}$$

We consider the equation of motion for left-handed Majorana fermions

$$i\bar{\sigma} \cdot \partial \chi - im\sigma^2 \, \chi^* = 0$$

Where χ transforms as

$$\chi'(\Lambda x) \;=\; \Omega_L \chi(x)$$

Now,

$$-im\sigma^2(\chi')^*(\Lambda x) \;=\; -im\sigma^2\Omega_L^*\chi^*(x) \;=\; -im\Omega_R\sigma^2\chi^*(x)$$

As in problem 1 above,

$$i\bar{\sigma}\cdot\partial\chi'(\Lambda x) \;=\; i\big(\Lambda^{-1}\big)^{\nu}{}_{\mu}\,\bar{\sigma}^{\mu}\,\Omega_L\,\partial_\nu\chi(x)$$

Or,

$$i\bar{\sigma}\cdot\partial\chi'(\Lambda x) \;=\; i\,\Omega_R\,\bar{\sigma}\cdot\partial\chi(x)$$

Thus the equations of motion are Lorentz invariant

$$i\bar{\sigma}\cdot\partial\chi' - im\sigma^2\,(\chi')^* \;=\; 0$$

Complex conjugating the equations of motion

$$-i(\bar{\sigma}^{\mu})^*\partial_\mu\chi^* - im\sigma^2\,\chi \;=\; -i\sigma^2\sigma^{\mu}\sigma^2\partial_\mu\chi^* - im\sigma^2\,\chi \;=\; 0$$

Operating on this equation with $i(\bar{\sigma}\cdot\partial)\sigma^2$ we find

$$\bar{\sigma}^{\mu}\sigma^{\nu}\sigma^2\partial_\nu\partial_\mu\chi^* + m^2\sigma^2\chi^* \;=\; 0$$

Using,

$$\bar{\sigma}^{\mu}\sigma^{\nu}\partial_\mu\partial_\nu \;=\; \tfrac{1}{2}\left(\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\mu}\sigma^{\nu}\right)\partial_\mu\partial_\nu \;=\; \partial^2$$

We find that χ satisfies the Klein-Gordon equation

$$\left(\partial^2 + m^2\right)\chi \;=\; 0$$

(b)

The action for Majorana fermions is

$$S[\chi,\chi^*] \;=\; \int d^4x\left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{1}{2}im\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$$

To show that $S = S^*$ we first compute

$$\left[\tfrac{1}{2}im\left(\chi_a(\sigma^2)_{ab}\chi_b - \chi_a^*(\sigma^2)_{ab}\chi_b^*\right)\right]^* \;=\; \left[\tfrac{1}{2}im\left(\chi_b^*(\sigma^2)_{ab}\chi_a^* - \chi_b(\sigma^2)_{ab}\chi_a\right)\right]$$

Where we have used $(\chi_a\chi_b)^* = \chi_b^*\chi_a^*$ and $(\sigma^2)^* = -\sigma^2$. Using $\chi_a\chi_b = -\chi_b\chi_a$ we find

$$\left[\tfrac{1}{2}im\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]^* \;=\; \left[\tfrac{1}{2}im\left(\chi_a(\sigma^2)_{ab}\chi_b - \chi_a^*(\sigma^2)_{ab}\chi_b^*\right)\right]$$

Using $(\bar{\sigma}^{\mu})^\dagger$ we consider

$$\left[i\,\chi_a^*\bar{\sigma}^{\mu}_{ab}\partial_\mu\chi_b\right]^* \;=\; \left[-i\,\bar{\sigma}^{\mu}_{ba}\partial_\mu\chi_b^*\chi_a\right]$$

Ignoring surface terms we find

$$\left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi\right]^* \;=\; \left[i\,\bar{\sigma}^{\mu}_{ba}\chi_b^*\partial_\mu\chi_a\right]$$

Thus $S = S^*$. Varying S with respect to χ^*

$$S[\chi,\chi^* + (\delta\chi)^*] - S[\chi,\chi^*] \;\simeq\; \int d^4x\left[(\delta\chi)^\dagger i\bar{\sigma}\cdot\partial\chi - im\,(\delta\chi)^\dagger\sigma^2\chi^*\right]$$

This leads directly to the equations of motion

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\,\chi^* \;=\; 0$$

(c)

We write the Dirac Lagrangian as

$$\mathcal{L} = \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} i\bar{\sigma}\cdot\partial & -m \\ -m & i\sigma\cdot\partial \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Setting $\psi_L = \chi_1$ and $\psi_R = i\sigma^2\chi_2^*$

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^T i(\sigma^2\sigma\sigma^2)\cdot\partial\chi_2^* - im\chi_1^\dagger\sigma^2\chi_2^* + im\chi_2^T\sigma^2\chi_1$$

Note that since (χ_1, χ_2) are Grassmann variables

$$\chi_2^T i(\sigma^2\sigma\sigma^2)\cdot\partial\chi_2^* = -\partial\chi_2^\dagger\cdot i\bar{\sigma}\chi_2$$

Thus, integrating by parts

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^\dagger i\bar{\sigma}\cdot\partial\chi_2 - im\chi_1^\dagger\sigma^2\chi_2^* + im\chi_2^T\sigma^2\chi_1$$

If we set $\chi \equiv \chi_1 = \chi_2$ we find (twice) the Majorana Lagrangian. If we make this identification then the $U(1)$ symmetry $\chi_1 \to e^{i\theta}\chi_1$ and $\chi_2 \to e^{-i\theta}\chi_2$ of the Dirac Lagrangian is lost.

(e)

Consider the following form of the Weyl spinor components of a solution to the Dirac equation

$$\psi_L(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \sqrt{p\cdot\sigma}\left(\alpha_s(\mathbf{p})\,\xi_s\,e^{-ip\cdot x} + \beta_s^*(\mathbf{p})\,(-i\sigma^2\xi_s^*)\,e^{ip\cdot x}\right)$$

$$\psi_R(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \sqrt{p\cdot\bar{\sigma}}\left(\alpha_s(\mathbf{p})\,\xi_s\,e^{-ip\cdot x} - \beta_s^*(\mathbf{p})\,(-i\sigma^2\xi_s^*)\,e^{ip\cdot x}\right)$$

Here we have used the conventions of Peskin and Schroeder. The Majorana fermions satisfy the condition

$$\chi = \psi_L(x) = -i\sigma^2\,\psi_R^*(x)$$

Imposing this condition, after some algebra we find

$$\alpha_s(\mathbf{p}) = \beta_s(\mathbf{p})$$

Promoting $\alpha_s(\mathbf{p})$ to an operator $\hat{a}_s(\mathbf{p})$, χ may be written as

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \sqrt{p\cdot\sigma}\left(\hat{a}_s(\mathbf{p})\,\xi_s\,e^{-ip\cdot x} + \hat{a}_s^\dagger(\mathbf{p})\,(-i\sigma^2\xi_s^*)\,e^{ip\cdot x}\right)$$

It may be verified that this satisfies the Majorana equation. Furthermore it may be seen that the canonical anti-commutation relations

$$\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\} = \delta_{ab}\,\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

follow from the anti-commutation relations

$$\{\hat{a}_s(\mathbf{p}), \hat{a}_r^\dagger(\mathbf{k})\} = (2\pi)^3\,\delta_{sr}\,\delta^{(3)}(\mathbf{p}-\mathbf{k})$$

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To see this we write

$$\chi(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\sum_s \left(\hat{a}_s(\mathbf{p})\,A_s(p)\,e^{-ip\cdot x} + \hat{a}_s^\dagger(\mathbf{p})\,B_s(p)\,e^{ip\cdot x}\right)$$

Where

$$A_s(p) \;=\; \sqrt{p\cdot\sigma}\,\xi_s \qquad\qquad B_s(p) \;=\; \sqrt{p\cdot\sigma}\,(-i\sigma^2)\,\xi_s^*$$

Setting $x^0 = y^0 = 0$ and imposing the anti-commuation relations

$$\{\chi_a(\mathbf{x}),\chi_b^\dagger(\mathbf{y})\} \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\sum_s \Big((A_s(p))_a\,(A_s(p))_b^* e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + (B_s(p))_a\,(B_s(p))_b^* e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big)$$

It may be verified that

$$\sum_s (A_s(p))_a\,(A_s(p))_b^* \;=\; \sum_s (B_s(p))_a\,(B_s(p))_b^* \;=\; (\,p\cdot\sigma\,)_{ab}$$

This leads directly to the canonical anti-commutation relations. The Hamiltonian may be constructed from the Lagrangian for Majorana fermions. However it is straightforward to see that the Hamiltonian takes the form expected from the projection of the Dirac equation

$$H \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}} \sum_s \hat{a}_s^\dagger(\mathbf{p})\,\hat{a}_s(\mathbf{p})$$

That is we simply identify the $\hat{a}_s(\mathbf{p})$ and $\hat{b}_s(\mathbf{p})$ operators and, by convention, divide the action and thus the Hamiltonian by two.

4.) Peskin & Schroeder 3.7 : PCT

(a)

We consider the discrete transformation properties of:

$$\bar{\psi}\,\sigma^{\mu\nu}\,\psi \qquad \text{where} \qquad \sigma^{\mu\nu} \;=\; \frac{i}{2}\,[\,\gamma^\mu\,,\,\gamma^\nu\,]$$

We begin with the parity operator $\mathcal{P} = \mathcal{P}^\dagger = \mathcal{P}^{-1}$:

$$\mathcal{P}\,\psi(t,\mathbf{x})\,\mathcal{P} \;=\; \gamma^0\,\psi(t,-\mathbf{x})$$

Now, since $(\gamma^0)^\dagger = \gamma^0$,

$$\mathcal{P}\,\bar{\psi}(t,\mathbf{x})\,\mathcal{P} \;=\; (\mathcal{P}\,\psi(t,\mathbf{x})\,\mathcal{P})^\dagger\;\gamma^0 \;=\; \bar{\psi}(t,-\mathbf{x})\,\gamma^0$$

Thus,

$$\left(\mathcal{P}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{P}\right)(t,\mathbf{x}) \;=\; \left(\mathcal{P}\,\bar{\psi}\,\mathcal{P}\,\sigma^{\mu\nu}\,\mathcal{P}\,\psi\,\mathcal{P}\right)(t,\mathbf{x}) \;=\; \left(\bar{\psi}\,\gamma^0\,\sigma^{\mu\nu}\,\gamma^0\,\psi\right)(t,-\mathbf{x})$$

Now,

$$\gamma^0\,\sigma^{0j}\,\gamma^0 \;=\; -\,\sigma^{0j} \qquad \text{and} \qquad \gamma^0\,\sigma^{jk}\,\gamma^0 \;=\; \sigma^{jk}$$

Thus,

$$\left(\mathcal{P}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{P}\right)(t,\mathbf{x}) \;=\; \eta^{\mu\mu}\,\eta^{\nu\nu}\;\left(\bar{\psi}\,\sigma^{\mu\nu}\,\psi\right)(t,-\mathbf{x})$$

Where no summation is implied.

Next we consider the time reversal operator $\mathcal{T} = \mathcal{T}^\dagger = \mathcal{T}^{-1}$:

$$\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T} \;=\; -\,\gamma^1\,\gamma^3\,\psi(-t,\mathbf{x})$$

$\mathcal{T}$ is an anti-linear operator so that:

$$\mathcal{T}\,\bar{\psi}(t,\mathbf{x})\,\mathcal{T} \;=\; (\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T})^\dagger\;(\gamma^0)^* \;=\; \left(-\,\gamma^1\,\gamma^3\,\psi(-t,\mathbf{x})\right)^\dagger\;\gamma^0$$

Since $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ we find:

$$\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T} \;=\; -\,\psi^\dagger(-t,\mathbf{x})\,(\gamma^1)^\dagger\,(\gamma^3)^\dagger\,\gamma^0 \;=\; \bar{\psi}(-t,\mathbf{x})\,\gamma^1\,\gamma^3$$

Thus,

$$\begin{aligned}\left(\mathcal{T}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{T}\right)(t,\mathbf{x}) \;&=\; \left(\mathcal{T}\,\bar{\psi}\,\mathcal{T}\,(\,\sigma^{\mu\nu}\,)^*\,\mathcal{T}\,\psi\,\mathcal{T}\right)(t,\mathbf{x})\\ &=\; \left(\bar{\psi}\,\gamma^1\,\gamma^3\,(\,\sigma^{\mu\nu}\,)^*\,\gamma^3\,\gamma^1\,\psi\right)(-t,\mathbf{x})\end{aligned}$$

Now, in the Weyl representation:

$$(\gamma^0)^* \;=\; \gamma^0 \qquad (\gamma^1)^* \;=\; \gamma^1 \qquad (\gamma^2)^* \;=\; -\,\gamma^2 \qquad (\gamma^3)^* \;=\; \gamma^3$$

And,

$$(\,\sigma^{\mu\nu}\,)^* \;=\; -\,\frac{i}{2}\,[\,(\gamma^\mu)^*\,,\,(\gamma^\nu)^*\,]$$

Thus, using the γ matrix algebra:

$$\begin{aligned}\gamma^1\,\gamma^3\left(\sigma^{12}\right)^*\gamma^3\,\gamma^1 \;&=\; \gamma^1\,\gamma^3\,\sigma^{12}\,\gamma^3\,\gamma^1 \;=\; -\,\sigma^{12}\\ \gamma^1\,\gamma^3\left(\sigma^{23}\right)^*\gamma^3\,\gamma^1 \;&=\; \gamma^1\,\gamma^3\,\sigma^{23}\,\gamma^3\,\gamma^1 \;=\; -\,\sigma^{23}\\ \gamma^1\,\gamma^3\left(\sigma^{31}\right)^*\gamma^3\,\gamma^1 \;&=\; -\,\gamma^1\,\gamma^3\,\sigma^{31}\,\gamma^3\,\gamma^1 \;=\; -\,\sigma^{31}\end{aligned}$$

And,

$$\begin{aligned}\gamma^1\,\gamma^3\left(\sigma^{01}\right)^*\gamma^3\,\gamma^1 \;&=\; -\,\gamma^1\,\gamma^3\,\sigma^{01}\,\gamma^3\,\gamma^1 \;=\; \sigma^{01}\\ \gamma^1\,\gamma^3\left(\sigma^{02}\right)^*\gamma^3\,\gamma^1 \;&=\; \gamma^1\,\gamma^3\,\sigma^{02}\,\gamma^3\,\gamma^1 \;=\; \sigma^{02}\\ \gamma^1\,\gamma^3\left(\sigma^{03}\right)^*\gamma^3\,\gamma^1 \;&=\; -\,\gamma^1\,\gamma^3\,\sigma^{03}\,\gamma^3\,\gamma^1 \;=\; \sigma^{03}\end{aligned}$$

Thus, again with no summation implied:

$$\left(\mathcal{T}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{T}\right)(t,\mathbf{x}) \;=\; -\,\eta^{\mu\mu}\,\eta^{\nu\nu}\;\left(\bar{\psi}\,\sigma^{\mu\nu}\,\psi\right)(-t,\mathbf{x})$$

We now consider the charge conjugation operator: $\mathcal{C} = \mathcal{C}^\dagger = \mathcal{C}^{-1}$:

$$\mathcal{C}\,\psi\,\mathcal{C} \;=\; -i\,\left(\bar{\psi}\,\gamma^0\,\gamma^2\right)^T$$

Thus,

$$\mathcal{C}\,\bar{\psi}\,\mathcal{C} \;=\; (\mathcal{C}\,\psi\,\mathcal{C})^\dagger\;\gamma^0 \;=\; -i\,\left(\left(\bar{\psi}\,\gamma^0\,\gamma^2\right)^\dagger\right)^T\;\gamma^0 \;=\; i\,\left(\gamma^2\,\psi\right)^T\;\gamma^0$$

And,

$$\mathcal{C}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{C} \;=\; \mathcal{C}\,\bar{\psi}\,\mathcal{C}\,\sigma^{\mu\nu}\,\mathcal{C}\,\psi\,\mathcal{C} \;=\; \left(\gamma^2\,\psi\right)^T\,\gamma^0\,\sigma^{\mu\nu}\,\left(\bar{\psi}\,\gamma^0\,\gamma^2\right)^T$$

Now, in the Weyl representation:

$$(\gamma^0)^T \;=\; \gamma^0 \qquad (\gamma^1)^T \;=\; -\gamma^1 \qquad (\gamma^2)^T \;=\; \gamma^2 \qquad (\gamma^3)^T \;=\; -\gamma^3$$

Thus,

$$\mathcal{C}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{C} \;=\; \psi^T\,\gamma^2\,\gamma^0\,\sigma^{\mu\nu}\,\gamma^2\,\gamma^0\,\bar{\psi}^T \;=\; \bar{\psi}\,\gamma^0\,\gamma^2\,\left(\sigma^{\mu\nu}\right)^T\,\gamma^0\,\gamma^2\,\psi$$

Now,

$$\left(\sigma^{\mu\nu}\right)^T \;=\; -\,\frac{i}{2}\,\left[\,(\gamma^\mu)^T\,,\,(\gamma^\nu)^T\,\right]$$

Thus, using the γ matrix algebra:

$$\begin{aligned}
\gamma^0\,\gamma^2\left(\sigma^{12}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{12}\,\gamma^0\,\gamma^2 \;&=\; -\,\sigma^{12}\\
\gamma^0\,\gamma^2\left(\sigma^{23}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{23}\,\gamma^0\,\gamma^2 \;&=\; -\,\sigma^{23}\\
\gamma^0\,\gamma^2\left(\sigma^{31}\right)^T\gamma^0\,\gamma^2 \;&=\; -\,\gamma^0\,\gamma^2\,\sigma^{31}\,\gamma^0\,\gamma^2 \;&=\; -\,\sigma^{31}
\end{aligned}$$

And,

$$\begin{aligned}
\gamma^0\,\gamma^2\left(\sigma^{01}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{01}\,\gamma^0\,\gamma^2 \;&=\; -\,\sigma^{01}\\
\gamma^0\,\gamma^2\left(\sigma^{02}\right)^T\gamma^0\,\gamma^2 \;&=\; -\,\gamma^0\,\gamma^2\,\sigma^{02}\,\gamma^0\,\gamma^2 \;&=\; -\,\sigma^{02}\\
\gamma^0\,\gamma^2\left(\sigma^{03}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{03}\,\gamma^0\,\gamma^2 \;&=\; -\,\sigma^{03}
\end{aligned}$$

Thus,

$$\mathcal{C}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{C} \;=\; -\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi$$

(b)

We now consider the complex scalar field:

$$\phi(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{\mathbf{p}}\,e^{-ip\cdot x} \;+\; b^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\right)$$

The parity operator acts as:

$$\mathcal{P}\,a_{\mathbf{p}}\,\mathcal{P} \;=\; a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{P}\,b_{\mathbf{p}}\,\mathcal{P} \;=\; b_{-\mathbf{p}}$$

Thus,

$$\begin{aligned}
\mathcal{P}\,\phi(t,\mathbf{x})\,\mathcal{P} \;&=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{-\mathbf{p}}\,e^{-ip\cdot x} \;+\; b^\dagger_{-\mathbf{p}}\,e^{ip\cdot x}\right)\\
&=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{\mathbf{p}}\,e^{-iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}} \;+\; b^\dagger_{\mathbf{p}}\,e^{iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\right)\\
&=\; \phi(t,-\mathbf{x})
\end{aligned}$$

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The anti-linear time reversal operator acts as:

$$\mathcal{T}\, a_{\mathbf{p}}\, \mathcal{T} \;=\; a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{T}\, b_{\mathbf{p}}\, \mathcal{T} \;=\; b_{-\mathbf{p}}$$

Thus,

$$\begin{aligned}
\mathcal{T}\, \phi(t,\mathbf{x})\, \mathcal{T} &= \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, \left(a_{-\mathbf{p}}\, e^{ip\cdot x} \,+\, b^{\dagger}_{-\mathbf{p}}\, e^{-ip\cdot x} \right) \\
&= \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, \left(a_{\mathbf{p}}\, e^{iE_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}} \,+\, b^{\dagger}_{\mathbf{p}}\, e^{-iE_{\mathbf{p}}t - i\mathbf{p}\cdot\mathbf{x}} \right) \\
&= \phi(-t,\mathbf{x})
\end{aligned}$$

The charge conjugation operator acts as:

$$\mathcal{C}\, a_{\mathbf{p}}\, \mathcal{C} \;=\; b_{\mathbf{p}} \qquad \text{and} \qquad \mathcal{C}\, b_{\mathbf{p}}\, \mathcal{C} \;=\; a_{\mathbf{p}}$$

Thus,

$$\mathcal{C}\, \phi(x)\, \mathcal{C} \;=\; \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, \left(b_{\mathbf{p}}\, e^{-ip\cdot x} \,+\, a^{\dagger}_{\mathbf{p}}\, e^{ip\cdot x} \right) \;=\; \phi(x)^{\dagger}$$

We now consider the transformation properties under $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$ of the current associated with the $U(1)$ symmetry:

$$J^{\mu} \;=\; i\, \left(\phi^{\dagger}\, (\partial^{\mu}\phi) \,-\, (\partial^{\mu}\phi^{\dagger})\, \phi \right)$$

We begin with the parity operator:

$$\mathcal{P}\, J^{\mu}(t,\mathbf{x})\, \mathcal{P} \;=\; i\, \left(\phi^{\dagger}(t,-\mathbf{x})\, \partial^{\mu}(\phi(t,-\mathbf{x})) \,-\, \partial^{\mu}(\phi^{\dagger}(t,-\mathbf{x}))\, \phi(t,-\mathbf{x}) \right)$$

Now,

$$\partial^{\mu}(\phi(t,-\mathbf{x})) \;=\; \eta^{\mu\mu}\, (\partial^{\mu}\phi)(t,-\mathbf{x})$$

Thus,

$$\mathcal{P}\, J^{\mu}(t,\mathbf{x})\, \mathcal{P} \;=\; \eta^{\mu\mu}\, J^{\mu}(t,-\mathbf{x})$$

For time reversal:

$$\mathcal{T}\, J^{\mu}(t,\mathbf{x})\, \mathcal{T} \;=\; -i\, \left(\phi^{\dagger}(-t,\mathbf{x})\, \partial^{\mu}(\phi(-t,\mathbf{x})) \,-\, \partial^{\mu}(\phi^{\dagger}(-t,\mathbf{x}))\, \phi(-t,\mathbf{x}) \right)$$

Now,

$$\partial^{\mu}(\phi(-t,\mathbf{x})) \;=\; -\eta^{\mu\mu}\, (\partial^{\mu}\phi)(t,-\mathbf{x})$$

Thus,

$$\mathcal{T}\, J^{\mu}(t,\mathbf{x})\, \mathcal{T} \;=\; \eta^{\mu\mu}\, J^{\mu}(-t,\mathbf{x})$$

For charge conjugation:

$$\mathcal{C}\, J^{\mu}\, \mathcal{C} \;=\; i\, \left(\phi\, (\partial^{\mu}\phi^{\dagger}) \,-\, (\partial^{\mu}\phi)\, \phi^{\dagger} \right) \;=\; -\, J^{\mu}$$

This may be seen since J^{μ} is normal ordered and it may be seen that:

$$:\phi^{\dagger}(x)\, \phi(y): \;=\; :\phi(y)\, \phi^{\dagger}(x):$$

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(c)

Any Hermitian Lorentz scalar built from ψ and ϕ may be written as a sum of products of Hermitian tensors, and their derivatives, which are quadratic in the fields. We consider the transformation properties of such tensors for the complex scalar and Dirac spinor fields under $\Theta = \mathcal{CPT}$. For the Dirac field we consider:

$$\bar{\psi}\psi \qquad i\bar{\psi}\gamma^5\psi \qquad \bar{\psi}\gamma^\mu\psi \qquad \bar{\psi}\gamma^\mu\gamma^5\psi \qquad \bar{\psi}\sigma^{\mu\nu}\psi$$

for the complex scalar field we consider:

$$\phi^\dagger\phi \qquad J^\mu = i\left(\phi^\dagger\left(\partial^\mu\phi\right) - \left(\partial^\mu\phi^\dagger\right)\phi\right) \qquad \partial^\mu\phi^\dagger\,\partial^\nu\phi$$

As can be seen from the results of part (b) and the table on pg 71 of P+S, all tensors with an even(odd) number of indices are even(odd) under Θ. Furthermore the derivative operator ∂_μ is odd under Θ. Thus, since all scalars are a contraction of an even number of indices, all Hermitian Lorentz scalars are invariant under Θ.

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Physics 443 Homework #4
Due Tuesday, November 9, 2010

1.) Peskin and Schroeder 3.5a-b (not c)

2.) Among the uses of Grassmann numbers is to geometrize the supersymmetry between bosons and fermions. Consider a space parametrized by the ordinary spacetime coordinates x^μ, and also the Grassmann coordinates θ^α (anticommuting numbers that transform as a Weyl (chiral, or (1/2,0)) spinor under Lorentz transformations) as well as their conjugate ((0,1/2) or antichiral) coordinates $\bar{\theta}^{\dot{\alpha}}$.

a) Show that the derivative operators

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$$
$$\bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

satisfy the algebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$$

Thus the Q's are like the 'square root' of a translation. Show similarly that the derivative operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$$
$$\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

anticommute with the Q's.

b) Let's organize the fields ϕ, χ_α, F of the previous problem (P&S 3.5) into a *superfield* $\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ satisfying the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

Find the general solution to this constraint by performing theTaylor expansion of Φ in θ and $\bar{\theta}$. Show that this solution contains the requisite functions ϕ, χ, F, and show that the supersymmetry transformations of these fields are generated by the commutator of Φ and Q. Thus supersymmetry can be thought of as an extension of the geometry of spacetime to include Grassman coordinates as well as ordinary coordinates, and "supertranslations" that square to ordinary translations.

It is currently thought that supersymmetry is an approximate symmetry of nature above energy scales of about 1 TeV.

3.) Peskin and Schroeder 9.2d

Those who are looking for a little more reading on supersymmetry, Grassmann numbers, etc., may consult

J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd ed., Princeton U. Press (1992).
P. Argyres, *An Introduction to Global Supersymmetry*, http://www.physics.uc.edu/ argyres/661/susy2001.pdf
B. DeWitt, *Supermanifolds*, Cambridge U. Press (1992).
F.A. Berezin, *Mathematical Base of Supersymmetrical Field Theories* Yad. Fiz. **29** (1979) 1670-1687 (Russian); translated in Sov. J. Nucl. Phys.

QFT 1 : Problem Set 4

1.) Peskin & Schroeder 3.5 : Supersymmetry

In the interest of notational consistency with problem 2, we will adopt conventions that are common in treatments of supersymmetry. This will require us to rewrite expressions appearing in P&S. The following conventions are similar to those appearing in Ryder and in Bailin and Love. Wess and Bagger also use similar conventions except for the sign of the metric. There is a review by Lykken (hep-th/9612114) that uses somewhat different conventions. I highly recommend it for aspiring string theorists since it discusses dimensions other than four and places emphasis on formalism rather than phenomenology.

We represent Dirac spinors as:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

The γ matrices are given by:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Thus the σ matrices have the following index structure:

$$(\sigma^\mu)_{\beta\dot{\alpha}} \qquad\qquad (\bar{\sigma}^\mu)^{\dot{\beta}\alpha}$$

We raise undotted indices and lower dotted indices as follows:

$$\psi^\alpha = \epsilon^{\alpha\beta}\,\psi_\beta \qquad\qquad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}$$

Note that we act with the second index of the ϵ tensor. We introduce inverses to raise dotted indices and lower undotted indices:

$$\epsilon^{\alpha\beta}\,\epsilon_{\beta\gamma} = \delta^\alpha{}_\gamma \qquad\qquad \epsilon_{\dot{\alpha}\dot{\beta}}\,\epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}{}^{\dot{\gamma}}$$

Note that it is not consistent to raise or lower indices on the ϵ tensor itself since:

$$\epsilon^{\alpha\beta}\,\epsilon^{\gamma\delta}\,\epsilon_{\beta\delta} = \epsilon^{\gamma\alpha} = -\epsilon^{\alpha\gamma}$$

This problem can be easily resolved (see Wald (1984)) but we will stick with the conventions as they appear in most of the literature and avoid raising and lowering indices on the ϵ tensor.

We use the following explicit forms for the σ and ϵ tensors:

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}) \qquad\qquad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma})$$

And,

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -i\sigma^2 \qquad\qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma^2$$

We introduce the following conventions for Lorentz invariant quantities:

$$\chi\,\psi \equiv \chi^\alpha\,\psi_\alpha = -\,\psi_\alpha\,\chi^\alpha = \psi^\alpha\,\chi_\alpha = \psi\,\chi$$

Where we have used the Grassmann character of the fields and the antisymmetry of ϵ.
Similarly,

$$\bar{\chi}\,\bar{\psi} \equiv \bar{\chi}_{\dot{\alpha}}\,\bar{\psi}^{\dot{\alpha}} = -\,\bar{\psi}^{\dot{\alpha}}\,\bar{\chi}_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\,\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\,\bar{\chi}$$

We define complex conjugation to act as:

$$(\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}} \qquad\qquad \left(\bar{\chi}^{\dot{\alpha}}\right)^* \equiv \chi^\alpha$$

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With the following action on products of Grassmann variables:

$$(\,\psi^\alpha\,\chi_\beta\,)^* \;\equiv\; \bar\chi_{\dot\beta}\,\bar\psi^{\dot\alpha}$$

We also define the Lorentz vector quantities:

$$\chi\,\sigma^\mu\,\bar\psi \;\equiv\; \chi^\alpha\,(\sigma^\mu)_{\alpha\dot\beta}\;\bar\psi^{\dot\beta} \;=\; -\,\bar\psi_{\dot\beta}\,(\bar\sigma^\mu)^{\dot\beta\alpha}\,\chi_\alpha \;=\; -\,\bar\psi\,\bar\sigma^\mu\,\chi$$

Where we have used:

$$\epsilon^{\dot\beta\dot\alpha}\,(\sigma^\mu)_{\beta\dot\alpha}\;\epsilon^{\beta\alpha} \;=\; -\,(\bar\sigma^\mu)^{\dot\beta\alpha} \qquad \text{or} \qquad \sigma^2\,(\sigma^\mu)^T\,\sigma^2 \;=\; \bar\sigma^\mu$$

I will not provide a detailed description of the Lorentz transformation properties of the various spinor types. The reader is encouraged to consult any of the references mentioned above or in the homework statement. Further conventions and identities will be introduced as needed.

(a)

We write the Lagrangian appearing in P&S in the above conventions:

$$\mathcal{L} \;=\; \partial\phi^*\cdot\partial\phi \;+\; i\,\bar\chi\,\bar\sigma\cdot\partial\chi \;+\; F^*\,F$$

P&S writes the supersymmetry transformation as:

$$\begin{aligned}\delta\phi &= -i\,\epsilon^T\,\sigma^2\,\chi\\ \delta\chi &= \epsilon\,F \;+\; \sigma\cdot\partial\phi\,\sigma^2\,\epsilon^*\\ \delta F &= -i\epsilon^\dagger\bar\sigma\cdot\partial\chi\end{aligned}$$

These are written in the above conventions as:

$$\begin{aligned}\delta\phi &= \epsilon\,\chi\\ \delta\chi_\alpha &= \epsilon_\alpha\,F \;-\; i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\epsilon^{\dot\alpha}\,\partial_\mu\phi\\ \delta F &= -i\,\bar\epsilon\,\bar\sigma\cdot\partial\chi\end{aligned}$$

We consider each term in the Lagrangian separately. We begin with:

$$\mathcal{L}_\phi \;=\; \partial\phi^*\cdot\partial\phi$$

Under the supersymmetry transformation:

$$\delta\mathcal{L}_\phi \;=\; \partial(\delta\phi)^*\cdot\partial\phi \;+\; \partial\phi^*\cdot\partial(\delta\phi)$$

Now,

$$(\delta\phi)^* \;=\; (\epsilon^\alpha\,\chi_\alpha)^* \;=\; \bar\chi_{\dot\alpha}\,\bar\epsilon^{\dot\alpha} \;=\; \bar\epsilon\,\bar\chi$$

Thus,

$$\delta\mathcal{L}_\phi \;=\; \bar\epsilon\,\partial\bar\chi\cdot\partial\phi \;+\; \partial\phi^*\cdot\partial\chi\,\epsilon$$

We now consider the supersymmetry transformation of:

$$\mathcal{L}_\chi \;=\; i\,\bar\chi\,\bar\sigma\cdot\partial\chi \;=\; i\,\bar\chi_{\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\;\partial_\mu\chi_\beta$$

We have:

$$\delta\mathcal{L}_\chi \;=\; i\,\delta\bar\chi_{\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\;\partial_\mu\chi_\beta \;+\; i\,\bar\chi_{\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\;\partial_\mu(\delta\chi_\beta)$$

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Now,

$$\delta\bar{\chi}_{\dot{\alpha}} \;=\; \bar{\epsilon}_{\dot{\alpha}}\, F^* \;+\; i\,\left(\,(\sigma^{\mu})_{\dot{\alpha}\alpha}\,\right)^*\,\epsilon^{\alpha}\,\partial_{\mu}\phi^*$$

Where we have renamed indices on σ^{μ} to comport with the convention for the complex conjugate. Now, since $(\sigma^{\mu})^{\dagger} = \sigma^{\mu}$:

$$\left(\,(\sigma^{\mu})_{\dot{\alpha}\alpha}\,\right)^* \;=\; (\sigma^{\mu})_{\alpha\dot{\alpha}}$$

Thus,

$$\delta\bar{\chi}_{\dot{\alpha}} \;=\; \bar{\epsilon}_{\dot{\alpha}}\, F^* \;+\; i\,\epsilon^{\alpha}\,(\sigma^{\mu})_{\alpha\dot{\alpha}}\,\partial_{\mu}\phi^*$$

And,

$$\delta\mathcal{L}_{\chi} \;=\; i\,\bar{\epsilon}\,(\bar{\sigma}\cdot\partial\chi)\,F^* \;-\; \epsilon\,\sigma^{\mu}\,(\bar{\sigma}\cdot\partial\chi)\,\partial_{\mu}\phi^* \;+\; i\,\bar{\chi}\,\bar{\sigma}^{\mu}\,\epsilon\,\partial_{\mu}F \;+\; \bar{\chi}\,\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\epsilon}\,\partial_{\mu}\partial_{\nu}\phi$$

We now consider:

$$\mathcal{L}_F \;=\; F^*\,F \qquad \text{so that} \qquad \delta\mathcal{L}_F \;=\; \delta F^*\,F \;+\; F^*\,\delta F$$

Now,

$$\delta F^* \;=\; i\,(\partial_{\mu}\bar{\chi}_{\dot{\alpha}})\,(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\,\epsilon_{\alpha} \;=\; i\,(\partial_{\mu}\bar{\chi})\,\bar{\sigma}^{\mu}\,\epsilon$$

Thus,

$$\delta\mathcal{L}_F \;=\; i\,(\partial_{\mu}\bar{\chi})\,\bar{\sigma}^{\mu}\,\epsilon\,F \;-\; i\,F^*\,\bar{\epsilon}\,\bar{\sigma}\cdot\partial\chi$$

Returning to $\delta\mathcal{L}_{\chi}$, we integrate the second term by parts and use the identities,

$$(\,\bar{\sigma}^{\mu}\sigma^{\nu} \;+\; \bar{\sigma}^{\nu}\sigma^{\mu}\,) \;=\; (\,\sigma^{\mu}\bar{\sigma}^{\nu} \;+\; \sigma^{\nu}\bar{\sigma}^{\mu}\,) \;=\; 2\,\eta^{\mu\nu}$$

to find:

$$\delta\mathcal{L}_{\chi} \;=\; -\,\partial_{\nu}\,(\,\epsilon\,\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\chi\,\partial_{\mu}\phi^*\,) \;+\; i\,F^*\,\bar{\epsilon}\,\bar{\sigma}\cdot\partial\chi \;+\; i\,\bar{\chi}\,\bar{\sigma}^{\mu}\,\epsilon\,\partial_{\mu}F \;+\; \epsilon\,\chi\,\partial^2\phi^* \;+\; \bar{\chi}\,\bar{\epsilon}\,\partial^2\phi$$

Considering $\delta\mathcal{L}_{\phi}$, we integrate both terms by parts:

$$\delta\mathcal{L}_{\phi} \;=\; \partial_{\mu}\,(\,\bar{\chi}\,\bar{\epsilon}\,\partial^{\mu}\phi \;+\; \epsilon\,\chi\,\partial^{\mu}\phi^*\,) \;-\; \epsilon\,\chi\,\partial^2\phi^* \;-\; \bar{\chi}\,\bar{\epsilon}\,\partial^2\phi$$

We integrate the first term of $\delta\mathcal{L}_F$ by parts:

$$\delta\mathcal{L}_F \;=\; \partial_{\mu}\,(\,i\,\bar{\chi}\,\bar{\sigma}^{\mu}\,\epsilon\,F\,) \;-\; i\,\bar{\chi}\,\bar{\sigma}^{\mu}\,\epsilon\,\partial_{\mu}F \;-\; i\,F^*\,\bar{\epsilon}\,\bar{\sigma}\cdot\partial\chi$$

Thus, collecting terms, we find that $\delta\mathcal{L}$ is a total divergence:

$$\delta\mathcal{L} \;=\; \partial_{\nu}\,(\,\bar{\chi}\,\bar{\epsilon}\,\partial^{\nu}\phi \;+\; \epsilon\,\chi\,\partial^{\nu}\phi^* \;+\; i\,\bar{\chi}\,\bar{\sigma}^{\nu}\,\epsilon\,F \;-\; \epsilon\,\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\chi\,\partial_{\mu}\phi^*\,)$$

(b)

We now introduce the following additional term to the Lagrangian:

$$\Delta\mathcal{L} \;=\; m\,(\,\phi\,F \;+\; \phi^*F^*\,) \;-\; \frac{m}{2}\,(\,\chi\,\chi \;+\; \bar{\chi}\,\bar{\chi}\,)$$

Where we have transcribed the notation of P&S into that introduced above.

We define $\Delta\mathcal{L} = \Omega + \Omega^*$ where:

$$\Omega \;=\; m\,\phi\,F \;-\; \frac{m}{2}\,\chi\,\chi$$

Under the supersymmetry transformation we have:

$$\delta\Omega \;=\; m\,\phi\,\delta F \;+\; m\,\delta\phi\,F \;-\; m\,\chi\,\delta\chi$$

Thus,

$$\delta\Omega \;=\; -i\,m\,\phi\,\bar{\epsilon}\,\bar{\sigma}\cdot\partial\chi \;+\; i\,m\,\chi\,\sigma^{\mu}\,\bar{\epsilon}\,\partial_{\mu}\phi$$

Or, using $\chi\,\sigma^\mu\,\bar\epsilon = -\,\bar\epsilon\,\bar\sigma^\mu\,\chi$:

$$\delta\Omega = \partial_\mu\,(-i\,m\,\phi\,\bar\epsilon\,\bar\sigma^\mu\,\chi)$$

Thus, up to a total divergence, $\Delta\mathcal{L}$ is invariant under the supersymmetry transformation. We now write out the augmented Lagrangian:

$$\mathcal{L} + \Delta\mathcal{L} = \partial\phi^*\cdot\partial\phi + i\,\bar\chi\,\bar\sigma\cdot\partial\chi + F^*\,F + m\,(\phi\,F + \phi^*F^*) - \frac{m}{2}\,(\chi\,\chi + \bar\chi\,\bar\chi)$$

Using the equations of motion for F ,

$$F^* + m\,\phi = 0$$

We rewrite the Lagrangian as:

$$\mathcal{L} + \Delta\mathcal{L} = \partial\phi^*\cdot\partial\phi - m^2\,\phi^*\phi + i\,\bar\chi\,\bar\sigma\cdot\partial\chi - \frac{m}{2}\,(\chi\,\chi + \bar\chi\,\bar\chi)$$

This is the Lagrangian for a complex scalar field and a Majorana spinor field of equal mass. Please see P&S problem 3.4 .

(c)

We now consider a Lagrangian which contains the fields (ϕ_i, χ_i, F_i) for $i = 1\ldots n$:

$$\mathcal{L} = \partial\phi_i^*\cdot\partial\phi_i + i\,\bar\chi_i\,\bar\sigma\cdot\partial\chi_i + F_i^*\,F_i + \Gamma + \Gamma^*$$

Where,

$$\Gamma = F_i\,W_i - \frac{1}{2}\,W_{ij}\,\chi_i\,\chi_j$$

Where we introduce a superpotential $W[\phi]$ and define:

$$W_{i_1\ldots i_p} \equiv \frac{\partial W[\phi]}{\partial\phi_{i_1}\ldots\phi_{i_p}}$$

For instance, the Lagrangian in part (b) corresponds to $W[\phi] = m\phi^2/2$. We have already shown in part (a) that the first three terms yield a total derivative under the supersymmetry transformation:

$$\begin{aligned}\delta\phi_i &= \epsilon\,\chi_i\\ (\delta\chi_i)_\alpha &= \epsilon_\alpha\,F_i - i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\epsilon^{\dot\alpha}\,\partial_\mu\phi_i\\ \delta F_i &= -i\,\bar\epsilon\,\bar\sigma\cdot\partial\chi_i\end{aligned}$$

Note that we use the same Grassmann parameter ϵ for each component of the respective fields. Thus to prove that $\mathcal{L}$ is supersymmetric we examine:

$$\delta\Gamma = \delta F_i\,W_i + F_i\,W_{ij}\,\delta\phi_j - W_{ij}\,\chi_i\,\delta\chi_j - \frac{1}{2}\,W_{ijk}\,\chi_i\,\chi_j\,\delta\phi_k$$

Or,

$$\begin{aligned}\delta\Gamma &= -i\,\bar\epsilon\,\bar\sigma\cdot\partial\chi_i\,W_i + F_i\,W_{ij}\,\epsilon\,\chi_j - W_{ij}\,\chi_i\,\epsilon\,F_j\\ &\quad + i\,W_{ij}\,\chi_i\,\sigma^\mu\,\bar\epsilon\,\partial_\mu\phi_j - \frac{1}{2}\,W_{ijk}\,\chi_i\,\chi_j\,\epsilon\,\chi_k\end{aligned}$$

From the symmetry of W_{ij} :

$$W_{ij}\,(F_i\,\chi_j\,\epsilon - F_j\,\chi_i\,\epsilon) = 0$$

Also, using $\chi_i\,\sigma^\mu\,\bar\epsilon = -\,\bar\epsilon\,\bar\sigma^\mu\,\chi_i$:

$$-i\,\bar\epsilon\,\bar\sigma\cdot\partial\chi_i\,W_i = i\,(\partial_\mu\chi_i)\,\sigma^\mu\,\bar\epsilon\,W_i = \partial_\mu\,(i\,\chi_i\,\sigma^\mu\,\bar\epsilon\,W_i) - i\,W_{ij}\,\chi_i\,\sigma^\mu\,\bar\epsilon\,\partial_\mu\phi_j$$

Thus,

$$\delta\Gamma \;=\; \partial_\mu \left(i\,\chi_i\,\sigma^\mu\,\bar{\epsilon}\,W_i \right) \;-\; \frac{1}{2}\,W_{ijk}\,\chi_i\,\chi_j\,\epsilon\,\chi_k$$

We could now introduce a Fierz identity but it is simpler to define $\chi_i = (a_i, b_i)$ and write:

$$\chi_i\,\chi_j\,\epsilon\,\chi_k \;=\; (\,b_i\,a_j \;-\; a_i\,b_j\,)\,(\,\epsilon_2\,a_k \;-\; \epsilon_1\,b_k\,)$$

Upon expansion, each element in this product is anti-symmetric under exchange of one pair of indices. Since W_{ijk} is totally symmetric, the second term in $\delta\Gamma$ vanishes and we have:

$$\delta\Gamma \;=\; \partial_\mu \left(i\,\chi_i\,\sigma^\mu\,\bar{\epsilon}\,W_i \right)$$

Thus $\mathcal{L}$ is supersymmetric.

We now consider $n = 1$ and $W[\phi] = g\phi^3/3$:

$$\mathcal{L} \;=\; \partial\phi^* \cdot \partial\phi \;+\; i\,\bar{\chi}\,\bar{\sigma} \cdot \partial\chi \;+\; F^*\,F \;+\; \Gamma \;+\; \Gamma^*$$

Where,

$$\Gamma \;=\; F\,g\,\phi^2 \;-\; g\,\phi\,\chi\,\chi$$

The equation of motion for F yields:

$$F^* \;+\; g\,\phi^2 \;=\; 0$$

Thus, after some algebra:

$$\mathcal{L} \;=\; \partial\phi^* \cdot \partial\phi \;-\; g^2\,(\phi^*\phi)^2 \;+\; i\,\bar{\chi}\,\bar{\sigma} \cdot \partial\chi \;-\; g\,(\,\phi\,\chi\,\chi \;+\; \phi^*\bar{\chi}\,\bar{\chi}\,)$$

This leads to the following equation of motion for ϕ:

$$\partial^2\phi \;+\; 2\,g^2\,\phi\,(\phi^*\phi) \;+\; g\,\phi^*\bar{\chi}\,\bar{\chi} \;=\; 0$$

And for χ:

$$i\,\bar{\sigma} \cdot \partial\chi \;-\; 2\,g\,\phi^*\bar{\chi} \;=\; 0$$

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2.)

(a)

We introduce the superspace representation of the supersymmetry generators:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\,\partial_\mu \qquad \text{and} \qquad \bar Q_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\,\theta^\alpha\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu$$

To find the anticommutator algebra of Q_α and $\bar Q_{\dot\alpha}$ we must compute a total of 10 terms. First we have, trivially:

$$\{\frac{\partial}{\partial\theta^\alpha},\frac{\partial}{\partial\theta^\beta}\} = \{\frac{\partial}{\partial\theta^\alpha},\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\} = \{\frac{\partial}{\partial\bar\theta^{\dot\alpha}},\frac{\partial}{\partial\bar\theta^{\dot\beta}}\} = 0$$

Defining,

$$A_\alpha = i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\,\partial_\mu \qquad \text{and} \qquad \bar A_{\dot\alpha} = i\,\theta^\alpha\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu$$

And using,

$$\{\theta^\alpha,\theta^\beta\} = \{\theta^\alpha,\bar\theta^{\dot\alpha}\} = \{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\} = 0$$

We find:

$$\{A_\alpha,A_\beta\} = \{A_\alpha,\bar A_{\dot\alpha}\} = \{\bar A_{\dot\alpha},\bar A_{\dot\beta}\} = 0$$

Also since,

$$\{\frac{\partial}{\partial\theta^\alpha},\bar\theta^{\dot\alpha}\} = \{\frac{\partial}{\partial\bar\theta^{\dot\alpha}},\theta^\alpha\} = 0$$

We find:

$$\{\frac{\partial}{\partial\theta^\alpha},A_\beta\} = \{\frac{\partial}{\partial\bar\theta^{\dot\alpha}},\bar A_{\dot\beta}\} = 0$$

Finally, since,

$$\{\frac{\partial}{\partial\theta^\alpha},\theta^\beta\} = \delta_\alpha{}^\beta \qquad \text{and} \qquad \{\frac{\partial}{\partial\bar\theta^{\dot\alpha}},\bar\theta^{\dot\beta}\} = \delta^{\dot\beta}{}_{\dot\alpha}$$

We find:

$$\{\frac{\partial}{\partial\theta^\alpha},\bar A_{\dot\beta}\} = \{\frac{\partial}{\partial\bar\theta^{\dot\alpha}},A_\beta\} = i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu$$

The above relations lead to:

$$\{Q_\alpha,Q_\beta\} = \{\bar Q_{\dot\alpha},\bar Q_{\dot\beta}\} = 0 \qquad \text{and} \qquad \{Q_\alpha,\bar Q_{\dot\alpha}\} = -2i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu$$

Introducing the derivative operators:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + A_\alpha \qquad \text{and} \qquad \bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \bar A_{\dot\alpha}$$

We find:

$$\{D_\alpha,D_\beta\} = \{\bar D_{\dot\alpha},\bar D_{\dot\beta}\} = 0 \qquad \text{and} \qquad \{D_\alpha,\bar D_{\dot\alpha}\} = 2i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu$$

And,

$$\{Q_\alpha,D_\beta\} = \{\bar Q_{\dot\alpha},\bar D_{\dot\beta}\} = \{Q_\alpha,\bar D_{\dot\alpha}\} = \{D_\alpha,\bar Q_{\dot\alpha}\} = 0$$

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(b)

We define the variable $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then $\bar{D}_{\dot{\alpha}} y^\mu = 0$, so a chiral superfield Φ that satisfies $\bar{D}_{\dot{\alpha}}\Phi = 0$ can be expressed as $\Phi(y^\mu, \theta)$ and expanded in θ:

$$\begin{aligned}\Phi(y^\mu, \theta) &= \phi(y) + \theta\chi(y) + \theta\theta F(y) \qquad (1)\\ &= \phi(x) + \theta\chi(x) + \theta\theta F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x)\\ &\quad + \frac{i}{2}\theta\theta\partial_\mu\chi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x)\,. \qquad (2)\end{aligned}$$

Let us re-express the supersymmetry charges in terms of the new variables $y, \theta, \bar{\theta}$:

$$\begin{aligned}Q_\alpha(y, \theta, \bar{\theta}) &= \frac{\partial y^\nu}{\partial\theta^\alpha}\frac{\partial}{\partial y^\nu} + \frac{\partial\theta^\beta}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\\ &= \frac{\partial}{\partial\theta^\alpha}\,, \qquad (3)\\ \bar{Q}_{\dot{\alpha}}(y, \theta, \bar{\theta}) &= \frac{\partial y^\nu}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial y^\nu} + \frac{\partial\bar{\theta}^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\\ &= \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - 2i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\frac{\partial}{\partial y^\mu}\,. \qquad (4)\end{aligned}$$

We apply these supercharges to Φ. The derivative with respect to $\bar{\theta}$ has no effect.

$$\begin{aligned}(\epsilon Q + \bar{\epsilon}\bar{Q})\Phi &= \left(\epsilon\frac{\partial}{\partial\theta} - 2i\theta\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\right)\Phi\\ &= \epsilon\chi + 2\epsilon\theta F - 2i\theta\sigma^\mu\bar{\epsilon}\left(\frac{\partial}{\partial y^\mu}\phi + \theta\frac{\partial}{\partial y^\mu}\chi\right)\\ &= \epsilon\chi + 2\theta\left(\epsilon F - i\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\phi\right) - 2i\theta\sigma^\mu\bar{\epsilon}\theta\frac{\partial}{\partial y^\mu}\chi\\ &= \epsilon\chi + 2\theta\left(\epsilon F - i\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\phi\right) - \theta\theta\frac{\partial}{\partial y^\mu}\chi\sigma^\mu\bar{\epsilon}\,. \qquad (5)\end{aligned}$$

These are the desired supersymmetry transformations.
NB: depending on conventions, the above expressions change by negative signs and constant factors ($\sqrt{2}$).

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3.) Peskin & Schroeder 9.2 (d)

We begin with the following Euclidean Lagrangian for a Fermionic oscillator where $\psi(t)$, $\bar{\psi}(t)$ are Grassmann fields:

$$L_E \;=\; \bar{\psi}\,\dot{\psi} \,+\, \omega\,\bar{\psi}\,\psi$$

We compute the partition function:

$$Z(\beta) \;=\; \int \mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0 \int_{\psi(0)=\psi_0}^{\psi(\beta)=-\psi_0} \int \mathcal{D}\bar{\psi}(t)\mathcal{D}\psi(t)\; \exp\left[-\int_0^\beta dt\, L_E[\,\bar{\psi}\,,\,\psi\,] \right]$$

We first introduce an expansion for ψ in terms of anti-periodic functions:

$$\psi(t) \;=\; \frac{1}{\sqrt{\beta}} \sum_n x_n\, e^{2\pi i(n+1/2)t/\beta} \qquad \text{and} \qquad x_n \;=\; \frac{1}{\sqrt{\beta}} \int_0^\beta \psi(t) e^{-2\pi i(n+1/2)t/\beta}$$

Computing the Euclidean action, we find:

$$\int_0^\beta dt\, L_E[\,\bar{\psi}\,,\,\psi\,] \;=\; \sum_n \Omega_n\, \bar{x}_n\, x_n \qquad \text{where} \qquad \Omega_n \;=\; (\,2\pi i(n+1/2)/\beta \,+\, \omega\,)$$

We make the substitution:

$$\int \mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0 \int_{\psi(0)=\psi_0}^{\psi(\beta)=-\psi_0} \int \mathcal{D}\bar{\psi}(t)\mathcal{D}\psi(t) \;=\; \prod_n \int d\bar{x}_n\, dx_n$$

Thus,

$$Z \;=\; \prod_n \int d\bar{x}_n\, dx_n\, e^{-\,\Omega_n\,\bar{x}_n\,x_n} \;=\; \prod_n \int d\bar{x}_n\, dx_n\; (\,1 \,-\, \Omega_n\,\bar{x}_n\,x_n\,) \;=\; \prod_n \Omega_n$$

Where we have used the fact that x_n , $\bar{x}_n$ are Grassmann variables so that:

$$\{x_n, x_m\} = \{x_n, \bar{x}_m\} = \{\bar{x}_n, \bar{x}_m\} = 0$$

And,

$$\int d\bar{x}_n\, dx_n \;=\; 0 \qquad \text{and} \qquad \int d\bar{x}_n\, dx_n\, x_n\, \bar{x}_n \;=\; 1$$

Defining $\alpha = (i\pi/2 + \beta\omega/2)$ so that $\Omega_n = 2/\beta\,(i\pi n + \alpha)$, we find:

$$Z \;=\; \Omega_n \prod_{n=1}^{\infty} \Omega_n \Omega_{-n} \;=\; \left[\prod_n (2/\beta) \right]\; \alpha \prod_{n=1}^{\infty} \left((\pi n)^2 \,+\, \alpha^2 \right)$$

Using the product representation of $\sinh(z)$ given in P+S :

$$Z \;=\; \left[(2i/\beta) \prod_{n=1}^{\infty} (2\pi n\beta)^2 \right]\; (\,-i\,\sinh(\alpha)\,)$$

Thus, dropping the infinite ω-independent factor in brackets:

$$Z(\beta) \;=\; -i\,\sinh(\alpha) \;=\; \cosh(\beta\omega/2)$$

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Physics 443 Homework #5
Due Thursday, November 18, 2010

1.) Peskin and Schroeder 4.1

2.) Peskin and Schroeder 4.2

3.) *Aspects of scaling behavior in QFT:*

a) As discussed in class, massless free scalar field theory is invariant under scale transformations, if we assign the appropriate scale dimension to the field ϕ. The action of dilations on coordinates is implemented by the operator $D = ix^\mu \partial_\mu$. Find the commutation relations of this operator with the operators that implement the Poincaré algebra of translations $P_\mu = i\partial_\mu$ and Lorentz transformations $J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Find the canonical Noether current and Noether charge for this symmetry in the free field theory, and find the commutation relations of the charge with the field ϕ to show that it implements the infinitesimal form of the symmetry on the fields. Check also that this current has the right commutation relations with the field theory Hamiltonian and momentum generators.

b) The sound waves in a solid obey a wave equation, which at low enough wavelengths/energies is described by the action (for simplicity, consider one spatial dimension)

$$S_0 = \int dt dx [\partial_t \phi \partial_t \phi - c_s^2 \partial_x \phi \partial_x \phi]$$

where c_s is the speed of sound in the material. The field ϕ describes the distortion of the lattice away from equilibrium. Model this situation by a lattice of atoms (mass m, lattice spacing a) interacting through harmonic nearest-neighbor forces (spring constant k). Find the leading term in the action which corrects the above kinetic energy in a power series in $a \times$ *derivatives* (this can be done for instance by expanding the exact dispersion relation for the chain of masses and springs). Show that this correction term is an *irrelevant* perturbation of the action, so that all traces of the lattice structure disappear in the continuum limit $a \to 0$ (*e.g.* all traces of the structure of interactions at the atomic level, such as a general potential $V(r)$ between the atoms rather than harmonic forces, disappear, being summarized in the constant c_s – and the effective action S_0 is independent of the cutoff a). Estimate the momentum scale at which the irrelevant corrections amount to 10% of the total energy of a phonon.

(In three dimensions, we have to generalize the gradient kinetic energy to

$$S_0 = \int dt d^3x [\partial_t \phi_i \partial_t \phi_i - D^{ijkl} \partial_i \phi_j \partial_k \phi_l]$$

where ϕ_i, $i = 1, 2, 3$, describe the displacements in the three spatial directions. Some features of the lattice structure survive in the effective action in that the symmetry properties of D^{ijkl} reflect the crystallographic symmetries of the lattice; for instance the speed of sound in different directions, or for longitudinally versus transversely polarized phonons, typically differ. However these features are orientational in nature; the effective action is still independent of the lattice spacing at leading order.)

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QFT 1 : Problem Set 5

1.) Peskin & Schroeder 4.1

We begin with the Shroedinger picture Hamiltonian for a real Klein-Gordon field interacting with an external source:

$$H_S(t) \,=\, H_0 \,-\, \int d^3x\, j(\mathbf{x},t)\, \phi(\mathbf{x})$$

Here H_0 is the normal-ordered free Klein-Gordon Hamiltonian and $\phi(\mathbf{x})$ is the field operator in the Shroedinger picture. That is,

$$\phi(\mathbf{x}) \,=\, \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \,+\, a^{\dagger}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \right)$$

And,

$$H_0 \,=\, \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}\, a^{\dagger}_{\mathbf{p}}\, a_{\mathbf{p}}$$

We consider a state vector in the interaction picture:

$$|\Psi_I(t)\rangle \,=\, e^{iH_0 t}\, |\Psi_S(t)\rangle$$

From the Schroedinger equation we find:

$$i\,\partial_t\, |\Psi_I(t)\rangle \,=\, H_I(t)\, |\Psi_I(t)\rangle$$

Where,

$$H_I(t) \,=\, e^{iH_0 t}\, (\, H_S(t) \,-\, H_0\,)\, e^{-iH_0 t} \,=\, -\int d^3x\, j(\mathbf{x},t)\, \phi_I(\mathbf{x},t)$$

And,

$$\phi_I(x) \,=\, \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, \left(a_{\mathbf{p}} e^{-ip\cdot x} \,+\, a^{\dagger}_{\mathbf{p}} e^{ip\cdot x} \right)$$

The equation of motion has the formal solution:

$$|\Psi_I(t_2)\rangle \,=\, U(t_2,t_1)\, |\Psi_I(t_1)\rangle$$

Where, introducing time-ordering, we have:

$$U(t_2,t_1) \,=\, T\left[\exp\left(-i \int_{t_1}^{t_2} dt\, H_I(t) \right) \right]$$

(a)

We consider the amplitude for the system to evolve from the Schroedinger picture vacuum $|\,\Omega_S(-\tau)\rangle$ at time $-\tau$ to the Schroedinger picture vacuum $|\,\Omega_S(\tau)\rangle$ at time τ in the limit $\tau \to \infty$. In the interaction picture this is given by:

$$A(0) \,=\, \lim_{\tau\to\infty}\, \langle\, \Omega_I(\tau)|\; U(\tau,-\tau)\; |\,\Omega_I(-\tau)\rangle$$

The Schroedinger picture vacuum is defined by:

$$H_S(t)\,|\,\Omega_S(t)\rangle \,=\, 0 \qquad \text{and} \qquad \langle\, \Omega_S(t)|\; \Omega_S(t)\rangle \,=\, 1$$

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We assume that $j(\mathbf{x},t)=0$ for $|t|\geq\tau$. Thus,

$$H_S(\pm\tau)=H_0 \qquad \text{and} \qquad |\,\Omega_S(\pm\tau)\rangle=|0\rangle$$

Here $|0\rangle$ is the free Klein-Gordon vacuum defined by:

$$H_0\,|0\rangle\;=\;0 \qquad \text{and} \qquad \langle 0|\,0\rangle\;=\;1$$

Thus,

$$|\,\Omega_I(\pm\tau)\rangle\;=\;e^{\pm iH_0t}\,|0\rangle\;=\;|0\rangle$$

This leads to:

$$A(0)\;=\;\langle 0|\;T\left[\exp\left(\,i\int d^4x\,j(x)\,\phi_I(x)\right)\right]|0\rangle$$

Thus the probability for vacuum to evolve to the vacuum is:

$$P(0)\;=\;|A(0)|^2\;=\;\left|\langle 0|\;T\left[\exp\left(\,i\int d^4x\,j(x)\,\phi_I(x)\right)\right]|0\rangle\right|^2$$

In what follows we drop the I subscript and write $\phi_I(x)$ as $\phi(x)$. Also, where convenient, we write $\phi(x)$ as ϕ_x or $\phi(x_i)$ as ϕ_i. We also abbreviate $\int d^4x$ as $\int dx$.

(b)

We consider the expansion of $A(0)$ to $\mathcal{O}(j^2)$:

$$A_2(0)\;=\;\langle 0|\;T\left[1+i\int dx\,j_x\,\phi_x\;-\;\tfrac{1}{2}\int dx\,dy\,j_x\,j_y\,\phi_x\,\phi_y\right]|0\rangle$$

Since $\langle 0|\,\phi_x\,|0\rangle=0$, we have $A_2=1-\lambda/2$. Where we define:

$$\lambda\;=\;\int dx\,dy\,j_x\,j_y\,\langle 0|\,T\,(\phi_x\,\phi_y)\,|0\rangle$$

Since,

$$\langle 0|\,T\,(\phi_x\,\phi_y)\,|0\rangle\;=\;D_F(x-y)\;=\int\frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

We have:

$$\lambda\;=\;\int\frac{d^4p}{(2\pi)^4}\,\frac{i\,\tilde{\jmath}(p)\,\tilde{\jmath}(-p)}{p^2-m^2+i\epsilon} \qquad \text{where} \qquad \tilde{\jmath}(p)\;=\;\int d^4x\,e^{ip\cdot x}\,j(x)$$

To $\mathcal{O}(\epsilon)$, writing $\omega=p^0$ and $E_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$, we have:

$$p^2-m^2+i\epsilon\;=\;(\omega-\gamma_\epsilon)\,(\omega+\gamma_\epsilon) \qquad \text{where} \qquad \gamma_\epsilon\;=\;E_{\mathbf{p}}\;-\;i\epsilon/(2E_{\mathbf{p}})$$

Thus,

$$\lambda\;=\;\frac{i}{2\pi}\int\frac{d^3p}{(2\pi)^3}\int d\omega\,\frac{|\tilde{\jmath}(\omega,\mathbf{p})|^2}{(\omega-\gamma_\epsilon)\,(\omega+\gamma_\epsilon)}$$

Where we have used the reality condition $\tilde{\jmath}(-p)=(\tilde{\jmath}(p))^*$. Closing the ω contour in either half-plane, we find:

$$\lambda\;=\;\lambda^*\;=\;\int\frac{d^3p}{(2\pi)^3}\,\frac{|\tilde{\jmath}(p)|^2}{2E_{\mathbf{p}}}$$

Thus, to $\mathcal{O}(j^2)$:

$$P_2(0)\;=\;|A_2(0)|^2\;=\;1\;-\;(\lambda+\lambda^*)\,/2\;=\;1\;-\;\lambda$$

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(c)

We represent λ as a Feynman diagram with external sources:

$$\lambda \;=\; \quad = \int dx\, dy\, j_x\, j_y\, \langle 0|\, T\,(\phi_x\, \phi_y)\, |0\rangle$$

We now demonstrate that the vacuum to vacuum amplitude may be written as:

$$A(0) \;=\; e^{-\lambda/2} \;=\; \exp\left[-\tfrac{1}{2}\right.$$

$$\left.\right]$$

We first expand the exponential in $A(0)$:

$$\begin{aligned} A(0) \;&=\; \langle 0|\, T\left[\exp\left(i\int d^4x\, j(x)\,\phi(x)\right)\right]|0\rangle \\ &=\; \sum_{n=0}^{\infty}\frac{i^n}{n!}\left[\prod_{k=1}^{n}\int dx_k\, j_k\right]\langle 0|\, T\,(\phi_1\ldots\phi_n)\,|0\rangle \end{aligned}$$

We now expand the time-ordered product using Wicks theorem. Terms in which n is odd vanish. For a given n there are $(n-1)!!$ terms, each of which is a product of $n/2$ Feynman propagators. Since we are integrating with products of the sources $j_1 \ldots j_n$, we may change variables to obtain $(n-1)!!$ identical terms:

$$\begin{aligned} A(0) \;&=\; \sum_{\substack{n=0\\ even}}^{\infty}\frac{i^n}{n!}\,(n-1)!!\left(\int dx\, dy\, j_x\, j_y\,\langle 0|\, T\,(\phi_x\,\phi_y)\,|0\rangle\right)^{n/2} \\ &=\; \sum_{\substack{n=0\\ even}}^{\infty}\frac{i^n}{n!}\,(n-1)!!\,\lambda^{n/2} \end{aligned}$$

Now, for $n = 2m$:

$$\frac{(n-1)!!}{n!} \;=\; \frac{1}{2^m\, m!}$$

Thus, as above,

$$A(0) \;=\; \sum_{m=0}^{\infty}\frac{(i)^{2m}}{m!}(\lambda/2)^m \;=\; e^{-\lambda/2}$$

(d)

We now consider:

$$A(\{\mathbf{k}\}) \;=\; \langle \mathbf{k}|\, T\left[\exp\left(i\int d^4x\, j(x)\,\phi(x)\right)\right]|0\rangle$$

We first compute this amplitude to $\mathcal{O}(j)$. Now, from $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}}\, a^{\dagger}_{\mathbf{k}}\,|0\rangle$, we find:

$$\langle \mathbf{k}|\, 0\rangle \;=\; 0 \qquad \text{and} \qquad \langle \mathbf{k}|\;\phi(x)\;|0\rangle \;=\; e^{ik\cdot x}$$

Thus,

$$A_1(\{\mathbf{k}\}) \;=\; i\int d^4x\, j(x)\, e^{ik\cdot x} \;=\; \tilde{j}(k)$$

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We might consider the probabilty density in momentum space to be the square of this amplitude but this would treat the integration measure too naively. Rather we should first introduce the projection operator onto the single particle subspace of the full asymptotic Fock space:

$$\mathbf{1}_1 \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;|\mathbf{p}\rangle\,\langle\mathbf{p}|$$

The probability of finding a single particle in the outgoing state is:

$$P(1) \;=\; \int d^3p\,P(\{\mathbf{p}\}) \;=\; \mathrm{tr}\,(\,\mathbf{1}_1\;|\mathrm{out}\rangle\,\langle\mathrm{out}|\,)$$

Here,

$$|\mathrm{out}\rangle \;\equiv\; S\,|0\rangle \qquad \text{where} \qquad S \;\equiv\; \lim_{\tau\to\infty} U(\tau,-\tau) \;=\; T\,e^{\,i\int dx\,j(x)\,\phi(x)}$$

Thus,

$$P(1) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;\left|\langle\mathbf{p}|\,S\,|0\rangle\right|^2$$

Or,

$$P(\{\mathbf{p}\}) \;=\; \frac{\left|\langle\mathbf{p}|\,S\,|0\rangle\right|^2}{(2\pi)^3\,2E_{\mathbf{p}}} \;=\; \frac{\left|A(\{\mathbf{p}\})\right|^2}{(2\pi)^3\,2E_{\mathbf{p}}}$$

Thus to $\mathcal{O}(j)$:

$$P_1(\{\mathbf{p}\}) \;=\; \frac{\left|\tilde{j}(p)\right|^2}{(2\pi)^3\,2E_{\mathbf{p}}}$$

We now consider this quantity to all orders in j:

$$A(\{\mathbf{p}\}) \;=\; \langle\mathbf{p}|\,S\,|0\rangle \;=\; \sum_{n=0}^{\infty}\frac{i^n}{n!}\left[\prod_{k=1}^{n}\int dx_k\,j_k\right]\langle\mathbf{p}|\,T\,(\phi_1\ldots\phi_n)\,|0\rangle$$

At a particular point in the $4n$ dimensional space being integrated over at order n, we have some permutation of the fields due to time-ordering:

$$\begin{aligned}
\langle\mathbf{p}|\,T\,(\phi_1\ldots\phi_n)\,|0\rangle \;&=\; \sqrt{E_{\mathbf{p}}}\;\langle 0|\,a_{\mathbf{p}}\,\phi_{\pi_1}\ldots\phi_{\pi_n}\,|0\rangle \\
&=\; \sqrt{E_{\mathbf{p}}}\;\langle 0|\,[a_{\mathbf{p}},\phi_{\pi_1}\ldots\phi_{\pi_n}]\,|0\rangle \\
&=\; \sqrt{E_{\mathbf{p}}}\sum_{j=1}^{n}\langle 0|\,\phi_{\pi_1}\ldots\phi_{\pi_{j-1}}[a_{\mathbf{p}},\phi_{\pi_j}]\phi_{\pi_{j+1}}\ldots\phi_{\pi_n}\,|0\rangle \\
&=\; \sqrt{E_{\mathbf{p}}}\sum_{j=1}^{n}\langle 0|\,T\left(\phi_{\pi_1}\ldots\phi_{\pi_{j-1}}\,\phi_{\pi_{j+1}}\ldots\phi_{\pi_n}\right)|0\rangle\;[a_{\mathbf{p}},\phi_{\pi_j}]
\end{aligned}$$

Here we have re-introduced the time-ordering since the ordering is preserved and removed the commutator since it is a c-number:

$$[a_{\mathbf{p}},\phi(x)] \;=\; \frac{1}{\sqrt{E_{\mathbf{p}}}}\,e^{ip\cdot x}$$

Since we are integrating with products of the sources $j_1\ldots j_n$, we may change variables to replace the index π_j with n:

$$\left[\prod_{k=1}^{n}\int dx_k\,j_k\right]\langle\mathbf{p}|\,T\,(\phi_1\ldots\phi_n)\,|0\rangle \;=\; n\,\tilde{j}(p)\left[\prod_{k=1}^{n-1}\int dx_k\,j_k\right]\langle 0|\,T\,(\phi_1\ldots\phi_{n-1})\,|0\rangle$$

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Thus, since $\langle \mathbf{k}|\, 0\rangle \,=\, 0$, we have:

$$A(\{\mathbf{p}\}) \,=\, \sum_{n=1}^{\infty} \frac{i^n}{n!}\, n\, \tilde{\jmath}(p) \left[\prod_{k=1}^{n-1} \int dx_k\, j_k \right] \langle 0|\, T\left(\phi_1 \ldots \phi_{n-1}\right) |0\rangle$$

Or, changing the limits of the sum:

$$A(\{\mathbf{p}\}) \,=\, i\, \tilde{\jmath}(p)\, A(0) \,=\, i\, \tilde{\jmath}(p)\, e^{-\lambda/2}$$

Thus, to all orders in j :

$$P(\{\mathbf{p}\}) \,=\, e^{-\lambda/2}\, \frac{\left|\tilde{\jmath}(p)\right|^2}{(2\pi)^3\, 2E_{\mathbf{p}}} \qquad \text{where} \qquad \lambda \,=\, \int \frac{d^3p}{(2\pi)^3}\, \frac{\left|\tilde{\jmath}(p)\right|^2}{2E_{\mathbf{p}}}$$

(e)

We now consider the projection operator onto the n-particle subspace of the Fock space:

$$\mathbf{1}_n \,=\, \frac{1}{n!} \left[\prod_{k=1}^{n} \int \frac{d^3p_k}{(2\pi)^3}\, \frac{1}{2E_{\mathbf{p}_k}} \right] |\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \, \langle \mathbf{p}_1 \ldots \mathbf{p}_n|$$

Here the $n!$ comes from the Bose symmetry of the scalar field. Also,

$$|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \,=\, \left[\prod_{k=1}^{n} \sqrt{E_{\mathbf{p}_k}}\, a^\dagger_{\mathbf{p}_k} \right] |0\rangle$$

The probability to create n particles is:

$$P(n) \,=\, \langle 0|\, S^\dagger\, \mathbf{1}_n\, S\, |0\rangle \,=\, \frac{1}{n!} \left[\prod_{k=1}^{n} \int \frac{d^3p_k}{(2\pi)^3} \right] \left| \langle 0| \left[\textstyle\prod_{k=1}^{n} a_{\mathbf{p}_k} \right] S\, |0\rangle \right|^2$$

Using the result from part (d), we find:

$$\langle 0| \left[\textstyle\prod_{k=1}^{n} a_{\mathbf{p}_k} \right] S\, |0\rangle \,=\, \langle 0|\, S\, |0\rangle \prod_{k=1}^{n} \frac{i\, \tilde{\jmath}(p_k)}{\sqrt{E_{\mathbf{p}_k}}}$$

Thus,

$$P(n) \,=\, \frac{1}{n!}\, \left| \langle 0|\, S\, |0\rangle \right|^2 \left(\int \frac{d^3p}{(2\pi)^3}\, \frac{\left|\tilde{\jmath}(p)\right|^2}{2E_{\mathbf{p}}} \right)^n \,=\, \frac{1}{n!}\, \lambda^n\, e^{-\lambda}$$

(f)

Some facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) \,=\, e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!}\, \lambda^n \,=\, 1$$

$$\langle N\rangle \,=\, \sum_{n=0}^{\infty} n\, P(n) \,=\, e^{-\lambda} \sum_{n=1}^{\infty} n\, \frac{1}{n!}\, \lambda^n \,=\, \lambda\, e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!}\, \lambda^n \,=\, \lambda$$

$$\left\langle N^2 \right\rangle \,=\, \sum_{n=0}^{\infty} n^2\, P(n) \,=\, \lambda\, e^{-\lambda} \sum_{n=1}^{\infty} \frac{n}{(n-1)!}\, \lambda^{n-1} \,=\, \lambda\, e^{-\lambda} \sum_{n=0}^{\infty} \frac{(n+1)}{n!}\, \lambda^n \,=\, \lambda\,(\lambda+1)$$

Thus,

$$\left\langle \left(N - \langle N\rangle\right)^2 \right\rangle \,=\, \left\langle N^2 \right\rangle - \left(\langle N\rangle\right)^2 \,=\, \lambda$$

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2.) Peskin & Schroeder 4.2

We begin with the Lagrangian:

$$\mathcal{L} \;=\; \tfrac{1}{2}\left[\,\partial\Phi\cdot\partial\Phi \;-\; M^2\Phi^2\,\right] \;+\; \tfrac{1}{2}\left[\,\partial\phi\cdot\partial\phi \;-\; m^2\phi^2\,\right] \;-\; \mu\,\Phi\,\phi^2$$

We consider the rate at which a Φ particle decays into two ϕ particles to first order in μ. We have the (connected and amputated) amplitude:

$$(2\pi)^4\;\delta^4\;(\,p_0-(p_1+p_2))\;i\,\mathcal{M} \;=\; \langle\,\mathbf{p}_1\,\mathbf{p}_2\,|\;T\left(e^{-i\int d^4x\,\mathcal{H}_I(x)}\right)\;|\,\mathbf{p}_0\,\rangle$$

Where,

$$\mathcal{H}_I \;=\; \mu\,\Phi\,\phi^2$$

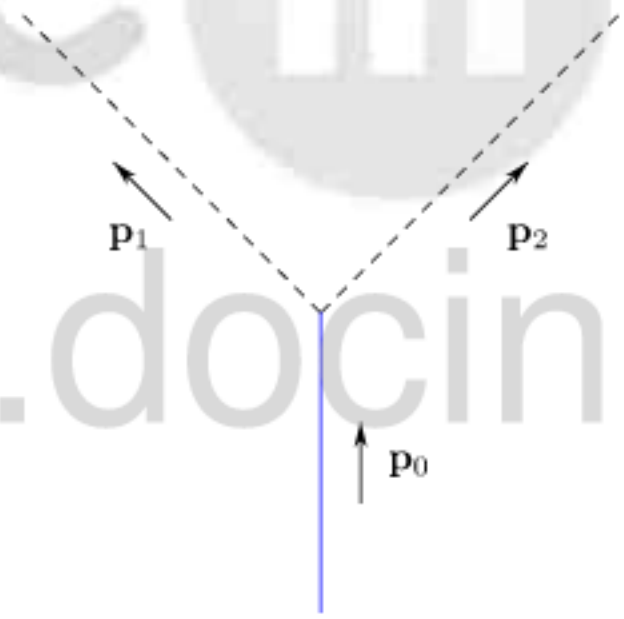
In what follows $p_0=(M,\mathbf{0})$. For this choice of p_0 the decay rate is given by:

$$d\Gamma \;=\; \frac{1}{2}\,(2M)^{-1}\left(\frac{d^3p_1}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_1}}\right)\left(\frac{d^3p_2}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_2}}\right)\,|\mathcal{M}|^2\;(2\pi)^4\;\delta^4\;(\,p_0-(p_1+p_2))$$

Here the factor of $\frac{1}{2}$ comes from the Bose symmetry of the outgoing ϕ particles. Working to first order in μ we consider the following amplitude:

$$(2\pi)^4\;\delta^4\;(\,p_0-(p_1+p_2))\;i\,\mathcal{M} \;=\; -i\,\mu\int d^4x\;\langle\,\mathbf{p}_1\,\mathbf{p}_2\,|\;\Phi(x)\,\phi(x)\,\phi(x)\;|\,\mathbf{p}_0\,\rangle$$

This corresponds to the following Feynman diagram:



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Defining $E_{\mathbf{p}}(\mu) \;=\; \sqrt{\mathbf{p}^2+\mu^2}$, we use the following expansions for the fields:

$$\Phi(x) \;=\; \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,E_{\mathbf{p}}(M)}}\;\left(\,a_{\mathbf{p}}\,e^{-ip\cdot x}\;+\;a^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\,\right)\Big|_{p^0=E_{\mathbf{p}}(M)}$$

And,

$$\phi(x) \;=\; \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,E_{\mathbf{p}}(m)}}\;\left(\,b_{\mathbf{p}}\,e^{-ip\cdot x}\;+\;b^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\,\right)\Big|_{p^0=E_{\mathbf{p}}(m)}$$

We have the states:

$$|\,\mathbf{p}_0\,\rangle \;=\; \sqrt{2\,E_{\mathbf{p}_0}(M)}\;a^\dagger_{\mathbf{p}_0}\;|\,0\,\rangle$$

And,

$$|\,\mathbf{p}_1\,\mathbf{p}_2\,\rangle \;=\; \sqrt{2\,E_{\mathbf{p}_1}(m)}\,\sqrt{2\,E_{\mathbf{p}_2}(m)}\;b^\dagger_{\mathbf{p}_1}\,b^\dagger_{\mathbf{p}_2}\;|\,0\,\rangle$$

From,

$$[\Phi(x),a^\dagger_{\mathbf{p}}] \;=\; \frac{e^{-ip\cdot x}}{\sqrt{2\,E_{\mathbf{p}}(M)}} \qquad \text{and} \qquad [\phi(x),b_{\mathbf{p}}] \;=\; \frac{e^{ip\cdot x}}{\sqrt{2\,E_{\mathbf{p}}(m)}}$$

We find:

$$\langle\,\mathbf{p}_1\,\mathbf{p}_2\,|\;\Phi(x)\,\phi(x)\,\phi(x)\;|\,\mathbf{p}_0\,\rangle \;=\; 2\,e^{-i\,(\,p_0-(p_1+p_2))}$$

Thus $i\mathcal{M} = -2i\mu$ and

$$\Gamma \;=\; \frac{\mu^2}{M}\int\frac{d^3p_1}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_1}}\int\frac{d^3p_2}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_2}}\;(2\pi)^4\;\delta^4\;(\,p_0-(p_1+p_2))$$

Where here and below $E_{\mathbf{p}} = E_{\mathbf{p}}(m)$. Making use of $p_0 = (M,\mathbf{0})$ and

$$\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}}} \;=\; \int\frac{d^4p}{(2\pi)^4}\,(2\pi)\,\delta(p^2-m^2)\,\theta(p^0)$$

We find:

$$\Gamma \;=\; \frac{\mu^2}{M}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}}}\,(2\pi)\,\delta((p_0-p)^2-m^2)\,\theta(M-E_{\mathbf{p}})$$

Using $(p_0-p)^2 = M^2 - 2M\,E_{\mathbf{p}} + m^2$, we have:

$$\delta((p_0-p)^2-m^2)\,\theta(M-E_{\mathbf{p}}) \;=\; \delta(M^2-2M\,E_{\mathbf{p}})\,\theta(M-E_{\mathbf{p}}) \;=\; (2M)^{-1}\,\delta(E_{\mathbf{p}}-M/2)$$

From $d^3p = |\mathbf{p}|\,E_{\mathbf{p}}dE_{\mathbf{p}}d\Omega$:

$$\Gamma \;=\; \frac{1}{4\pi}\,\frac{\mu^2}{M^2}\int_0^\infty dE_{\mathbf{p}}\,\sqrt{E^2_{\mathbf{p}}-m^2}\,\delta(E_{\mathbf{p}}-M/2)$$

Or, finally,

$$\Gamma \;=\; \frac{M}{8\pi}\,\frac{\mu^2}{M^2}\,\sqrt{1-(2m/M)^2}$$

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3.)

We consider the action for a massless scalar field in d dimensions:

$$S\left[\phi,\gamma\right] \;=\; \tfrac{1}{2}\int d^dx\,\sqrt{\gamma}\,\gamma^{ab}\,\partial_a\phi\,\partial_b\phi$$

Here we write the action in a form that is manifestly invariant under diffeomorphisms. However we will only consider metrics that may be brought to the Minkowsi form ($\gamma_{ab} = \eta_{ab}$) by a suitable coordinate transformation. We consider the behavior of the action under a scale transformation:

$$S\left[e^{\alpha\sigma}\phi, e^{2\sigma}\gamma\right] \;=\; \tfrac{1}{2}\int d^dx\,\sqrt{\gamma}\,e^{(d-2)\sigma}\,\gamma^{ab}\,e^{2\alpha\sigma}\,\partial_a\phi\,\partial_b\phi \;=\; e^{(2\alpha+(d-2))\sigma}S\left[\phi,\gamma\right]$$

Thus the action is invariant if $\alpha = -(d-2)/2$. This transformation takes a perhaps more familiar form if we compose it with a diffeomorphism which compensates for the transformation of the metric. Such a transformation amounts to a coordinate change under which the action is trivially invariant. That is we consider:

$$S\left[e^{\alpha\sigma}\phi, e^{2\sigma}\gamma\right] \;=\; S\left[e^{\alpha\sigma}\tilde{\phi}, e^{2\sigma}\tilde{\gamma}\right]$$

Where,

$$\tilde{\phi}(\tilde{x}) \;=\; \phi(x) \qquad \text{and} \qquad \tilde{\gamma}_{ab}(\tilde{x}) \;=\; \frac{\partial x^c}{\partial\tilde{x}^a}\,\frac{\partial x^d}{\partial\tilde{x}^b}\,\gamma_{cd}(x)$$

Choosing the x coordinate system such that $\gamma_{ab} = \eta_{ab}$ and taking $\tilde{x} = e^{\sigma}x$ we find:

$$e^{\alpha\sigma}\tilde{\phi}(x) \;=\; e^{\alpha\sigma}\phi(e^{-\sigma}x) \qquad \text{and} \qquad \tilde{\gamma}_{ab}(x) \;=\; \gamma_{ab}(e^{-\sigma}x) \;=\; \eta_{ab}$$

Thus writing the action as,

$$S\left[\phi\right] \;\equiv\; S\left[\phi,\eta\right] \;=\; \tfrac{1}{2}\int d^dx\,\eta^{ab}\,\partial_a\phi\,\partial_b\phi$$

If we choose $\alpha = (d-2)/2$, we find that:

$$S\left[\phi\right] \;=\; S\left[\phi'_\sigma\right] \qquad \text{where} \qquad \phi'_\sigma(x) \;=\; e^{-\sigma(d-2)/2}\phi(e^{-\sigma}x)$$

Note that this is not in general equivalent to the application of a coordinate transformation on ϕ, leaving the metric alone, as would be true for a Poincare transformation. However for $d = 2$ the transformation is equivalent to a coordinate transformation applied to ϕ. In fact, for $d = 2$, $S\left[\phi,\gamma\right]$ is invariant under the larger group of conformal transformations for which $\sigma = \Lambda(x)$. This is also true of the action for the electromagnetic field in $d = 4$. These considerations have a consequence for the form of the stress tensor. Consider an infinitesimal variation of the action under a conformal transformation:

$$S\left[e^{\alpha\Lambda}\phi, e^{2\Lambda}\gamma\right] \;-\; S\left[\phi,\gamma\right] \;=\; \int d^dx\,\left[\,\alpha\,\frac{\delta S}{\delta\phi}\,\phi \;+\; 2\,\frac{\delta S}{\delta\gamma_{ab}}\,\gamma_{ab}\,\right]\,\Lambda(x)$$

If the action is invariant under a conformal transformation, and thus under a scale transformation, then if $\alpha = 0$ we find:

$$2\,\frac{\delta S}{\delta\gamma_{ab}}\,\gamma_{ab} \;\equiv\; \sqrt{\gamma}\,T^{ab}\,\gamma_{ab} \;=\; 0$$

Thus the stress tensor is traceless for conformally invariant actions when the fields transform trivially. This is true of the massless scalar field in $d = 2$ since, in Minkowski space:

$$T_{ab} \;=\; \partial_a\phi\,\partial_b\phi \;-\; \tfrac{1}{2}\,\eta_{ab}\,\partial_c\phi\,\partial^c\phi$$

It is also true of the electromagnetic field in $d = 4$ where:

$$T_{\alpha\beta} \;=\; F_{\alpha\mu}F_{\nu\beta}\,\eta^{\mu\nu} \;+\; \tfrac{1}{4}\,\eta_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu}$$

It is important to note that nowhere above is mention made of the equations of motion. Current conservation is a consequence of the equations of motion but symmetries do not involve them. It is a nontrivial statement in a quantum field theory to say that a current associated with a symmetry of the action has a divergence with a vanishing expectation value. Sometimes an anomaly arises which causes the expectation value to be non-zero. Such a violation of the above conformal symmetry arises for the bosonic string and avoiding it fixes the dimension of a flat spacetime to be 26 .

(a)

We now consider the group of the Poincare transformations extended by dilations (scale transformations). We consider the action of the Lie algebra of this group acting on functions which have zero scaling dimension (α above). Thus for such a function $f(x)$ we have:

$$\tilde{f}(\bar{x}) \;=\; f(x) \qquad \text{or} \qquad \tilde{f} \;=\; \exp\left[\, i\,\lambda^\alpha\, G_\alpha \,\right] f$$

Here G^α is an element of the Lie algebra with λ_α parameterizing the group manifold. The respective generators are defined as follows:

Translations: $\quad \tilde{x}^\mu = x^\mu + a^\mu$

$$\lambda^\alpha\, G_\alpha \;=\; a^b\, P_b \qquad \text{where} \qquad P_b \;=\; i\,\partial_b$$

Lorentz Transformations: $\quad \tilde{x}^\mu = \exp\left[-i\,\frac{1}{2}\,\omega^{ab}\, L_{ab}\right]\, x^\mu$

$$\lambda^\alpha\, G_\alpha \;=\; \tfrac{1}{2}\,\omega^{ab}\, L_{ab} \qquad \text{where} \qquad L_{ab} \;=\; i\,\left(\, x_a\,\partial_b \;-\; x_b\,\partial_a \,\right)$$

Dilations: $\quad \tilde{x}^\mu = e^\sigma\, x^\mu$

$$\lambda^\alpha\, G_\alpha \;=\; \sigma\, D \qquad \text{where} \qquad D \;=\; i\, x^b\, \partial_b$$

We now compute the algebra. We know that (see P&S):

$$[P_\mu, P_\nu] \;=\; 0 \qquad\qquad [L_{\mu\nu}, P_\alpha] \;=\; i\,\left(\,\eta_{\nu\alpha} P_\mu - \eta_{\mu\alpha} P_\nu\,\right)$$

And,

$$[L_{\mu\nu}, L_{\alpha\beta}] \;=\; i\,\left(\,\eta_{\nu\alpha} L_{\mu\beta} - \eta_{\mu\alpha} L_{\nu\beta} - \eta_{\nu\beta} L_{\mu\alpha} + \eta_{\mu\beta} L_{\nu\alpha}\,\right)$$

We compute:

$$[P_\mu, D] \;=\; -[\partial_\mu, x^\alpha \partial_\alpha] \;=\; -\partial_\mu \;=\; i\, P_\mu$$

Also,

$$[L_{\mu\nu}, D] \;=\; -[x_\mu\,\partial_\nu \;-\; x_\nu\,\partial_\mu, x^\alpha \partial_\alpha]$$

Now,

$$[x_\mu\,\partial_\nu, x^\alpha \partial_\alpha] \;=\; x_\mu [\partial_\nu, x^\alpha]\partial_\alpha \;-\; x^\alpha [\partial_\alpha, x_\mu]\partial_\nu \;=\; 0$$

Thus,

$$[L_{\mu\nu}, D] \;=\; 0$$

We now derive the Noether current associated with the scaling symmetry. We return to the scalar field Lagrangian:

$$S\,[\phi] \;=\; \int d^d x\, \mathcal{L}\,[\phi] \;=\; \tfrac{1}{2} \int d^d x\, \eta^{ab}\, \partial_a \phi\, \partial_b \phi$$

Under the scaling transformation,

$$\phi'_\sigma(x) \;=\; e^{-\sigma(d-2)/2}\, \phi(e^{-\sigma} x)$$

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We find:

$$\mathcal{L}'_\sigma\left[\phi\right](x) \equiv \mathcal{L}\left[\phi'_\sigma\right](x) = e^{-\sigma d}\,\mathcal{L}\left[\phi\right](e^{-\sigma}x)$$

Thus $\mathcal{L}$ transforms as a density. We could have found the scaling dimension of ϕ by imposing this property but it is perhaps more satisfying to have this property manifest and explicitly introduce the scaling of the metric.

We derive the Noether currents as in P&S:

$$\delta\mathcal{L} = \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\,\delta\phi\right)$$

Now,

$$\delta\mathcal{L} = \mathcal{L}'_\epsilon - \mathcal{L} = -\epsilon\,(d + x^a\partial_a)\,\mathcal{L} = -\epsilon\,\partial_a\,(x^a\mathcal{L})$$

And,

$$\delta\phi = \phi'_\epsilon - \phi = -\epsilon\,[(d-2)/2 + x^a\partial_a]\,\phi$$

Thus,

$$\partial_a\,(x^a\mathcal{L}) = \partial_a\left(\,\partial^a\phi\,\left[(d-2)/2 + x^b\partial_b\right]\phi\,\right)$$

From,

$$T^{ab} = \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\,\partial^b\phi - \eta^{ab}\mathcal{L}$$

We find,

$$x^a\mathcal{L} = \partial^a\phi\,x^b\,\partial_b\phi - T^{ab}x_b$$

Thus we have the Noether current:

$$J^a_D = T^{ab}\,x_b + (d/2-1)\,\phi\,\partial^a\phi$$

Note the second term arises since the stress tensor is not in general traceless.

We now consider the Noether charge $\hat{D}$. Setting $\alpha = (1-d/2)$:

$$\hat{D} = \int d^{(d-1)}x\,\left(T^{0a}\,x_a - \alpha\,\phi\,\partial^0\phi\right) = t\,\hat{H} + \int d^{(d-1)}x\,\left(\mathcal{P}_j\,x^j - \alpha\,\phi\,\pi\right)$$

Here we have defined:

$$T^{0a} = \mathcal{P}^a \qquad \text{and} \qquad \hat{P}^a = \int d^{(d-1)}x\,\mathcal{P}^a$$

Note that we use G to denote a generic generator of the action of a symmetry on classical fields and $\hat{G}$ to denote the corresponding Noether charge. The action of the Poincare and scaling symmetries on a Heisenberg operator $O[\phi,\pi](x)$ is given by:

$$O_\lambda = \exp\left[i\lambda^\alpha\hat{G}_\alpha\right]\,O\,\exp\left[-i\lambda^\alpha\hat{G}_\alpha\right]$$

Note that we have included an explicit coordinate dependence in O since this appears in the Lorentz and dilation charges. This leads to the relation:

$$i\,\frac{\partial O_\lambda}{\partial\lambda^\alpha} = [O_\lambda, \hat{G}_\alpha]$$

There is the possibilty of some confusion here since the fields can explicitly depend on the coordinates x which are closely related to the translation symmetry parameters.

For example, the Noether charges themselves are independent of time by definition but do not all commute with the Hamiltonian. Those that do not, the dilation and boost generators, explicitly depend on time. For the spacetime derivative of an operator we compute:

$$i\,\partial_a O(x) = i\,\frac{\partial}{\partial x^a}\,O(x) + [O(x), \hat{P}_a]$$

The commutator serves to implement the derivative on the field variables. This notation is admittedly somewhat somewhat ambiguous but an example involving the dilation charge should add some clarity:

$$i\,\partial_0\,\hat{D} \equiv i\,\frac{d}{dt}\,\hat{D} = i\,\frac{\partial}{\partial t}\,\hat{D} + [\hat{D}, H]$$

Below we will show that, as expected, $[\hat{D}, \hat{H}] = -i\hat{H}$. Thus,

$$i\,\frac{d}{dt}\,\hat{D} = i\,\frac{\partial}{\partial t}\,\hat{D} - i\,\hat{H} = 0$$

We now compute $[\hat{D}, \phi(x)]$. Using $\mathcal{P}_j = \pi\,\partial_j\phi$, we find:

$$\begin{aligned}[\hat{D}, \phi(x)] &= t\,[\hat{H}, \phi(x)] + \int d^3y\,[\pi(y), \phi(x)]\,(y^j\,\partial_j\phi - \alpha\,\phi) \\ &= -i\,(-\alpha + x^a\partial_a)\,\phi(x)\end{aligned}$$

Thus if we define,

$$\phi_\sigma = \exp\left[i\sigma\hat{D}\right]\,\phi\,\exp\left[-i\sigma\hat{D}\right]$$

We have:

$$i\,\frac{\partial\phi_\sigma}{\partial\sigma} = [\phi_\sigma, \hat{D}] = D\,\phi$$

Where for scaling dimension α:

$$D = -i\,\alpha + i\,x^a\,\partial_a$$

Thus,

$$\phi_\sigma = \exp[-i\sigma D]\,\phi$$

When we were considering the dilation symmetry acting on classical fields above we found:

$$\phi_\sigma = \exp[\,i\sigma D]\,\phi \qquad \text{(classical)}$$

It is important to understand how this minus sign arises. The issue is briefly discussed on pages 59-60 of P&S. Essentially it is because a classical field is the expectation value of the corresponding quantum field. Consider the expectation value $V(x)$ of the operator $O(x)$ in the Heisenberg state $|\Psi\rangle$:

$$V(x) = \langle\Psi|\,O(x)\,|\Psi\rangle = \langle\Psi_\lambda|\,O_\lambda(x)\,|\Psi_\lambda\rangle$$

Where, for some Noether charge $\hat{G}$:

$$O_\lambda = \exp\left[i\lambda\hat{G}\right]\,O\,\exp\left[-i\lambda\hat{G}\right] \qquad \text{and} \qquad |\Psi_\lambda\rangle = \exp\left[i\lambda\hat{G}\right]|\Psi\rangle$$

Now, we define:

$$V_\lambda(x) = \exp[\,i\lambda\,G]\,V(x) \equiv \langle\Psi_\lambda|\,O(x)\,|\Psi_\lambda\rangle$$

This serves as a definition of G in terms of $\hat{G}$. Thus,

$$V_\lambda(x) = \langle\Psi|\,O_{-\lambda}(x)\,|\Psi\rangle = \exp[\,i\lambda\,G]\,\langle\Psi|\,O(x)\,|\Psi\rangle$$

Thus we find:

$$O_\lambda(x) = \exp[-i\lambda\,G]\,O(x)$$

We now compute $[\hat{D}, \hat{P}_a]$:

$$[\hat{D}, \hat{P}_a] = \int d^{(d-1)}x\left([\mathcal{P}_j(x), \hat{P}_a]\,x^j - \alpha\,[\phi(x)\,\pi(x), \hat{P}_a]\right)$$

Where we have used $[\hat{H}, \hat{P}_a] = 0$. Now,

$$\hat{P}_0 \;=\; \hat{H} \;=\; \int d^{(d-1)}x \, \tfrac{1}{2} \left(\pi^2 \,+\, (\nabla\phi)^2 \right) \qquad\qquad \text{and} \qquad\qquad \hat{P}_j \;=\; \int d^{(d-1)}x \, \pi \, \partial_j \phi$$

For operators without explicit time dependence:

$$i \, \partial_a O(x) \;=\; [O(x), \hat{P}_a]$$

Thus,

$$[\hat{D}, \hat{P}_k] \;=\; i \int d^{(d-1)}x \left[\left(\partial_k \mathcal{P}_j \right) \, x^j \;-\; \alpha \, \partial_k \left(\phi \, \pi \right) \right] \;=\; -i \, \hat{P}_k$$

Where we have integrated by parts. Also,

$$[\hat{D}, \hat{H}] \;=\; i \int d^{(d-1)}x \left[\left(\nabla^2 \phi \, \partial_j \phi \,+\, \pi \, \partial_j \pi \right) \, x^j \;-\; \alpha \, \left(\phi \, \nabla^2 \phi \,+\, \pi \, \pi \right) \right]$$

Where we have used $\dot{\phi} = \pi$ and $\dot{\pi} = \ddot{\phi} = \nabla^2 \phi$. Integrating by parts,

$$\begin{aligned} & i \int d^{(d-1)}x \left(\nabla^2 \phi \, \partial_j \phi \,+\, \pi \, \partial_j \pi \right) \, x^j \\ &= \quad i \int d^{(d-1)}x \left[\tfrac{1}{2} \, \partial_j \left(\pi^2 - (\nabla\phi)^2 \right) \,-\, \partial_k \left(\partial^k \phi \, \partial_j \phi \right) \right] \, x^j \\ &= \quad -i \int d^{(d-1)}x \left[\tfrac{1}{2} \, (d-1) \left(\pi^2 - (\nabla\phi)^2 \right) \,+\, (\nabla\phi)^2 \right] \end{aligned}$$

And,

$$-i \, \alpha \int d^{(d-1)}x \left(\phi \, \nabla^2 \phi \,+\, \pi \, \pi \right) \;=\; -i \, \alpha \int d^{(d-1)}x \left(\pi^2 - (\nabla\phi)^2 \right)$$

Thus, since $\alpha = (1 - d/2)$, we find:

$$[\hat{D}, \hat{H}] \;=\; -i \int d^{(d-1)}x \, \tfrac{1}{2} \left(\pi^2 \,+\, (\nabla\phi)^2 \right) \;=\; -i \, \hat{H}$$

Thus, as expected:

$$[\hat{D}, \hat{P}_a] \;=\; -i \, \hat{P}_a$$

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(b)

We consider the action for a one-dimensional lattice of particles interacting through nearest neighbor harmonic potentials:

$$S[q] \;=\; \int dt \sum_i \tfrac{1}{2}\left(m\,(\dot{q}_i)^2 - k\,(q_{i+1}-q_i)^2 \right)$$

We now define a continuum field $\phi(x)$ such that on a lattice $x_{i+1} = x_i + a$ we have:

$$\phi(x_i) \;=\; \sqrt{m/a}\, q_i \qquad\qquad \text{and} \qquad\qquad k = a^{-2}\, m\, c_s^2$$

This leads to:

$$S[q] \;=\; \tfrac{1}{2} \int dt\, a \sum_i \left[(\dot{\phi}(x_i))^2 \;-\; c_s^2\, a^{-2}\, (\phi(x_i + a) - \phi(x_i))^2 \right]$$

Taking a to be small we write:

$$S[\phi] \;=\; \tfrac{1}{2} \int dt\, dx \;\left[(\partial_t \phi)^2 \;-\; c_s^2\, a^{-2}\, (\phi(x+a) - \phi(x))^2 \right]$$

Strictly speaking the limit necessary to define the integral will lead to a free theory. We will simply take $S[\phi]$ to define the physical system. Expanding $\phi(x+a)$ and retaining terms quadratic in a we find:

$$a^{-2}\,(\phi(x+a) - \phi(x))^2 \;=\; (\partial_x \phi)^2 \;+\; a\,(\partial_x \phi)\,(\partial_x^2 \phi) \;-\; \tfrac{1}{12}\, a^2\, (\partial_x^2 \phi)^2$$

We neglect the second term since it is a total derivative. This leads to,

$$S[\phi] \;=\; S_0[\phi] \;+\; S_I[\phi] \;=\; S_0[\phi] \;+\; \frac{a^2 c_s^2}{24} \int dt\, dx\, (\partial_x^2 \phi)^2$$

Where $S_0[\phi]$ is the conformally invariant free Lagrangian:

$$S_0[\phi] \;=\; \tfrac{1}{2} \int dt\, dx\, \left((\partial_t \phi)^2 - c_s^2\, (\partial_x \phi)^2 \right)$$

The Lagrangian is invariant under the dilation:

$$\phi'_\sigma(t, x) \;=\; \phi(e^{-\sigma} t,\, e^{-\sigma} x)$$

The associated free Hamiltonian scales as:

$$H_0[\phi'_\sigma] \;=\; e^{-\sigma}\, H_0[\phi]$$

Thus the dilation maps higher energy solutions to lower energy solutions. The action of the dilation on the interaction is:

$$S_I[\phi'_\sigma] \;=\; e^{-2\sigma}\, S_I[\phi]$$

Thus the interaction becomes irrelevant at low energies.

For a plane wave solution of momentum p we find:

$$H_I / H_0 \;\simeq\; p^2\, a^2/12$$

Thus when 10% of the energy is in the interaction $p \simeq \sqrt{6}/(a\sqrt{5})$.

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QFT 1 : Problem Set 6

1.) Peskin & Schroeder 4.3

We consider N real scalar fields interacting according to the Hamiltonian:

$$H = \int d^3x \, \left(\tfrac{1}{2} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \tfrac{1}{2} \nabla \boldsymbol{\Phi} \cdot \nabla \boldsymbol{\Phi} + V(\boldsymbol{\Phi} \cdot \boldsymbol{\Phi}) \right)$$

Where $\boldsymbol{\Phi} = (\Phi_1 \dots \Phi_N)$, $\boldsymbol{\Pi}$ is the conjugate momentum and

$$V(\boldsymbol{\Phi} \cdot \boldsymbol{\Phi}) = \tfrac{1}{2} m^2 \, \boldsymbol{\Phi} \cdot \boldsymbol{\Phi} + \frac{\lambda}{4} (\boldsymbol{\Phi} \cdot \boldsymbol{\Phi})^2$$

(a)

For $\lambda = 0$ we have N decoupled Klein-Gordon fields of equal mass. This leads directly to the following definition of the field in the interaction picture:

$$\Phi_j(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{j\mathbf{p}} \, e^{-ip\cdot x} + a^\dagger_{j\mathbf{p}} \, e^{ip\cdot x} \right)$$

Where $E^2_{\mathbf{p}} = \mathbf{p}^2 + m^2$. The canonical commutation relations lead to:

$$[a_{i\mathbf{p}}, a^\dagger_{j\mathbf{k}}] = \delta_{ij} \, (2\pi)^3 \, \delta(\mathbf{p} - \mathbf{k}) \qquad [a_{i\mathbf{p}}, a_{j\mathbf{k}}] = [a^\dagger_{i\mathbf{p}}, a^\dagger_{j\mathbf{k}}] = 0$$

This leads to the spacetime propagator:

$$\overbrace{\Phi_i(x) \, \Phi_j(y)} = \langle 0 | \, T \left(\Phi_i(x) \, \Phi_j(y) \right) | 0 \rangle = \delta_{ij} \, D_F(x - y)$$

This leads to the momentum-space Feynman rule:

$$a \xrightarrow{\quad p \quad} b \quad \Rightarrow \quad \frac{i \, \delta_{ab}}{p^2 - m^2 + i\epsilon}$$

We compute the following amplitude to $\mathcal{O}(\lambda)$:

$$i \, \mathcal{M}(p'_k, q'_l \mid p_i, q_j) \, (2\pi)^4 \, \delta(p_i + p_j - p_k - p_l) = \langle p'_k, q'_l | \, i \, \mathcal{T} \, | p_i, q_j \rangle$$

Where,

$$S = 1 + i \, \mathcal{T} = T \left(e^{-i \int d^4x \, \mathcal{H}_I(x)} \right)$$

And,

$$|p_i, q_j\rangle = \sqrt{2E_{\mathbf{p}_i}} \, \sqrt{2E_{\mathbf{q}_j}} \, a^\dagger_{i\mathbf{p}_i} \, a^\dagger_{j\mathbf{q}_j} \, |0\rangle$$

Of course the interaction Hamiltonian is:

$$\mathcal{H}_I(x) = \frac{\lambda}{4} (\boldsymbol{\Phi} \cdot \boldsymbol{\Phi})^2$$

To $\mathcal{O}(\lambda)$:

$$\langle p'_k, q'_l | \, i \, \mathcal{T} \, | p_i, q_j \rangle = -i \, \frac{\lambda}{4} \int d^4x \, \langle p'_k, q'_l | \, \Phi_{ax} \Phi_{ax} \Phi_{bx} \Phi_{bx} \, | p_i, q_j \rangle$$

Here summation over a and b is implied and we have defined $\Phi_{ax} \equiv \Phi_a(x)$.

Ignoring vacuum bubbles and taking account of the symmetry under exchange of a and b as well as under exchange of fields within both of the pairings we find:

$$\begin{aligned}
\langle p'_k, q'_l |\, \Phi_{ax}\Phi_{ax}\Phi_{bx}\Phi_{bx}\, | p_i, q_j \rangle \;\; &= \;\; 8\, \langle p'_k, q'_l |\, \overbrace{\Phi_{ax}\Phi_{ax}}\, \overbrace{\Phi_{bx}\Phi_{bx}}\, | p_i, q_j \rangle \\
&+ \;\; 8\, \langle p'_k, q'_l |\, \overbrace{\Phi_{ax}\Phi_{ax}\Phi_{bx}}\, \Phi_{bx}\, | p_i, q_j \rangle \\
&+ \;\; 8\, \langle p'_k, q'_l |\, \Phi_{ax}\, \overbrace{\Phi_{ax}\Phi_{bx}\Phi_{bx}}\, | p_i, q_j \rangle
\end{aligned}$$

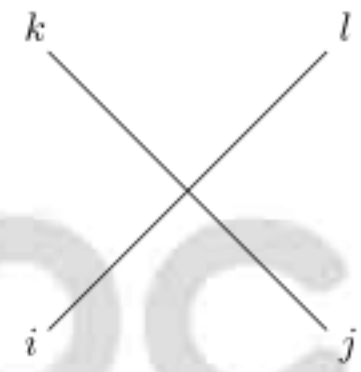
Now,

$$\overbrace{\Phi_a(x)\, |p_b\rangle} \;=\; \delta_{ab}\, e^{-ip\cdot x}\; |0\rangle \qquad \text{and} \qquad \langle p_b |\, \overbrace{\Phi_a(x)} \;=\; \langle 0 |\; e^{ip\cdot x}\, \delta_{ab}$$

Thus,

$$i\,\mathcal{M}(k,l\,|\,i,j) \;\equiv\; i\,\mathcal{M}(p'_k, q'_l\,|\,p_i, q_j) \;=\; -2i\,\lambda\;(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,)$$

This leads to the vertex Feynman rule:



$$\Rightarrow \qquad -2i\,\lambda\;(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,)$$

We now consider scattering processes in the center of mass frame. Since the masses are identical we may use the formula (eq. 4.85) in P&S for the differential cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2} \qquad \text{where} \qquad E_{CM}^2 \;=\; (p_i + q_j)^2$$

For $\Phi_1\Phi_2 \to \Phi_1\Phi_2$ scattering, $i\mathcal{M} = -2i\lambda$. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{16\pi^2 E_{CM}^2} \qquad \text{for} \qquad \Phi_1\Phi_2 \to \Phi_1\Phi_2$$

For $\Phi_1\Phi_1 \to \Phi_2\Phi_2$ scattering, $i\mathcal{M} = -2i\lambda$. We also divide by a Bose factor of 2. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{32\pi^2 E_{CM}^2} \qquad \text{for} \qquad \Phi_1\Phi_1 \to \Phi_2\Phi_2$$

For $\Phi_1\Phi_1 \to \Phi_1\Phi_1$ scattering, $i\mathcal{M} = -6i\lambda$. We again divide by a Bose factor of 2. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{9\,\lambda^2}{32\pi^2 E_{CM}^2} \qquad \text{for} \qquad \Phi_1\Phi_1 \to \Phi_1\Phi_1$$

(b)

We now consider the potential:

$$V(\mathbf{\Phi}\cdot\mathbf{\Phi}) \;=\; -\tfrac{1}{2}\,\mu^2\,\mathbf{\Phi}\cdot\mathbf{\Phi} \;+\; \frac{\lambda}{4}\,(\mathbf{\Phi}\cdot\mathbf{\Phi})^2$$

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We compute the minimum of the potential:

$$\frac{\partial V}{\partial \Phi_j} \;=\; -\,\mu^2\,\Phi_j \;+\; \lambda\,\Phi_j\,\mathbf{\Phi}\cdot\mathbf{\Phi}$$

Thus there is a minimum at:

$$v^2 \;=\; \mathbf{\Phi}\cdot\mathbf{\Phi} \;=\; \mu^2/\lambda$$

We make the field redefinitions:

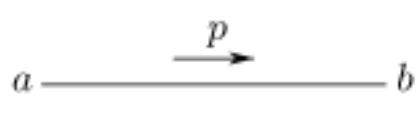
$$\Phi_j(x) \;=\; \pi_j(x) \qquad i = 1\ldots N-1 \qquad\qquad \text{and} \qquad\qquad \Phi_N(x) \;=\; v \;+\; \sigma(x)$$

Ignoring constant terms, this leads to the potential:

$$V(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;=\; \tfrac{1}{2}\,m_\sigma\,\sigma^2 \;+\; \frac{\lambda}{4}\,\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 \;+\; \mu\,\sqrt{\lambda}\,\sigma\,\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)$$

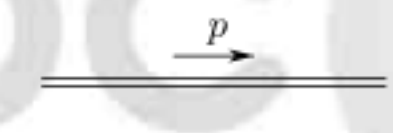
Where $m_\sigma = 2\mu^2$. Since the field displacement v does not affect the kinetic terms, we may read off the Feynman rules for the propagators and the quartic interactions from the results of part (a) above.

The π_j fields are massless:



$$\Rightarrow \qquad \frac{i\,\delta_{ab}}{p^2 + i\epsilon}$$

There is also the single massive field σ :



$$\Rightarrow \qquad \frac{i}{p^2 + m_\sigma^2 + i\epsilon}$$

The four π vertex is as found above:



$$\Rightarrow \qquad -2i\,\lambda\,(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,)$$

The four σ vertex is:



$$\Rightarrow \qquad -6i\,\lambda$$

The two σ two π vertex is:



$$\Rightarrow \qquad -2i\,\lambda\,\delta_{ij}$$

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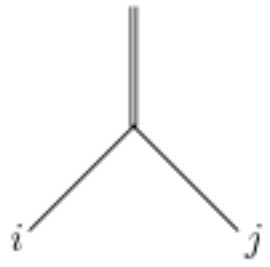
For the one σ two π vertex we compute:

$$i\mathcal{M}\,(\sigma(q)|\;\pi_i(p)\pi_j(k))\;(2\pi)^4\,\delta^4(p+k-q)\;=\;\langle\sigma(q)|\,i\,\mathcal{T}\,|\pi_i(p)\pi_j(k)\rangle$$

Where,

$$\begin{aligned}
\langle\sigma(q)|\,i\,\mathcal{T}\,|\pi_i(p)\pi_j(k)\rangle &= -i\,\mu\,\sqrt{\lambda}\int d^4x\;\langle\sigma(q)|\,\sigma_x\pi_{ax}\pi_{ax}\,|\pi_i(p)\pi_j(k)\rangle \\
&= -2i\,\mu\,\sqrt{\lambda}\int d^4x\;\langle\overbrace{\sigma(q)|\,\sigma_x}\overbrace{\pi_{ax}\pi_{ax}\,|\pi_i(p)}\pi_j(k)\rangle \\
&= -2i\,\mu\,\sqrt{\lambda}\,\delta_{ij}\;(2\pi)^4\,\delta^4(p+k-q)
\end{aligned}$$

Thus we find the vertex rule:



$$\Rightarrow \qquad -2i\,\mu\,\sqrt{\lambda}\,\delta_{ij}$$

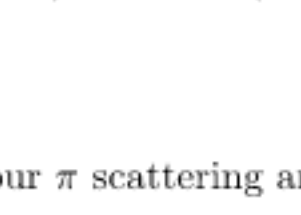
For the three σ vertex we compute:

$$i\mathcal{M}\,(\sigma(q)|\;\sigma(p)\sigma(k))\;(2\pi)^4\,\delta^4(p+k-q)\;=\;\langle\sigma(q)|\,i\,\mathcal{T}\,|\sigma(p)\sigma(k)\rangle$$

Where,

$$\begin{aligned}
\langle\sigma(q)|\,i\,\mathcal{T}\,|\sigma(p)\sigma(k)\rangle &= -i\,\mu\,\sqrt{\lambda}\int d^4x\;\langle\sigma(q)|\,\sigma_x\sigma_x\sigma_x\,|\sigma(p)\sigma(k)\rangle \\
&= -6i\,\mu\,\sqrt{\lambda}\int d^4x\;\langle\overbrace{\sigma(q)|\,\sigma_x}\overbrace{\sigma_x\sigma_x\,|\sigma(p)}\sigma(k)\rangle \\
&= -6i\,\mu\,\sqrt{\lambda}\,(2\pi)^4\,\delta^4(p+k-q)
\end{aligned}$$

Thus we find the vertex rule:

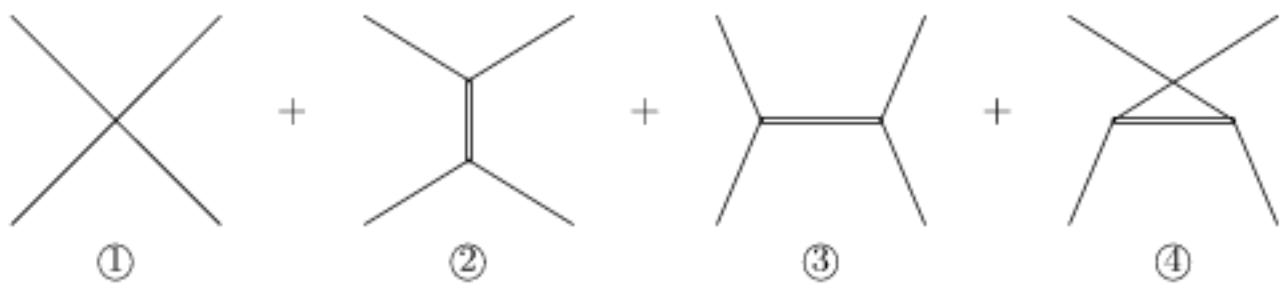


$$\Rightarrow \qquad -6i\,\mu\,\sqrt{\lambda}$$

(c)

We now compute the four π scattering amplitude to $\mathcal{O}(\lambda)$:

$$i\,\mathcal{M}\,(\,\pi_k(p_3)\pi_l(p_4)|\;\pi_i(p_1)\pi_j(p_2)\,)\;=$$



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Using the Feynman rules derived above:

$$\begin{aligned}
① &= -2i\,\lambda\,(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,) \\
② &= -2i\,\lambda\,\delta_{ij}\,\delta_{kl}\,\frac{m_\sigma^2}{(p_1+p_2)^2 - m_\sigma^2} \\
③ &= -2i\,\lambda\,\delta_{ik}\,\delta_{jl}\,\frac{m_\sigma^2}{(p_1-p_3)^2 - m_\sigma^2} \\
④ &= -2i\,\lambda\,\delta_{il}\,\delta_{jk}\,\frac{m_\sigma^2}{(p_1-p_4)^2 - m_\sigma^2}
\end{aligned}$$

Here we have used $m_\sigma^2 = 2\,\mu^2$. Thus,

$$i\,\mathcal{M} \;=\; -2i\,\lambda\,\left(\,\frac{\delta_{ij}\,\delta_{kl}\,(p_1+p_2)^2}{(p_1+p_2)^2 - m_\sigma^2} \;+\; \frac{\delta_{ik}\,\delta_{jl}\,(p_1-p_3)^2}{(p_1-p_3)^2 - m_\sigma^2} \;+\; \frac{\delta_{il}\,\delta_{jk}\,(p_1-p_4)^2}{(p_1-p_4)^2 - m_\sigma^2}\,\right)$$

Since the pions are massless, clearly at threshold ($\mathbf{p}_a = 0$) this amplitude vanishes.

For $N = 2$ we have $\delta_{ij} = 1$ since there is only one species of pion. To $\mathcal{O}(p_a^2)$:

$$\begin{aligned}
i\,\mathcal{M} &= \frac{2i\,\lambda}{m_\sigma^2}\,\left(\,(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2\,\right) \\
&= \frac{4i\,\lambda}{m_\sigma^2}\,(\,p_1\cdot p_2 \;-\; p_1\cdot p_3 \;-\; p_1\cdot p_4\,) \\
&= -\,\frac{4i\,\lambda}{m_\sigma^2}\,(\,p_1\cdot p_1\,) \;=\; 0
\end{aligned}$$

Where we have used $p_1 + p_2 = p_3 + p_4$.

(d)

We now add a symmetry breaking term to the Lagrangian:

$$\Delta V = -\,a\,\Phi_N$$

We express the Lagrangian in terms of the fields:

$$\Phi_j(x) \;=\; \pi_j(x) \qquad i = 1 \ldots N-1 \qquad\qquad \text{and} \qquad\qquad \Phi_N(x) \;=\; v \;+\; \sigma(x) \;+\; \delta$$

Here δ is small and we expect a minimum of the potential at $\boldsymbol{\pi} = \sigma = 0$. We derived an expression for the unperturbed potential in terms of $\boldsymbol{\pi}$ and σ in part (c) above. We simply substitute $\sigma(x) \to \sigma(x) + \delta$ into this expression and find the minimum of the potential by varying δ . Thus,

$$\tilde{V}(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;=\; V(\sigma+\delta, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;-\; a\,(\sigma + \delta + v)$$

Where, as above:

$$V(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;=\; \mu^2\,\sigma^2 \;+\; \frac{\lambda}{4}\,\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 \;+\; \mu\,\sqrt{\lambda}\,\sigma\,\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)$$

Now, ignoring terms quadratic and higher in δ :

$$\left.\frac{\partial \tilde{V}}{\partial \delta}\right|_{\boldsymbol{\pi}=\sigma=0} \;=\; 2\,\mu^2\,\delta \;-\; a$$

Thus $\delta = a/(2\mu^2)$. Working to $\mathcal{O}(a)$ and ignoring constant terms:

$$\tilde{V}(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) = \tfrac{1}{2}\, m_\sigma^2\, \sigma^2 + \tfrac{1}{2}\, m_\pi^2\, \boldsymbol{\pi}\cdot\boldsymbol{\pi} + \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 + \gamma\,\mu\,\sqrt{\lambda}\,\sigma\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)$$

Where,

$$m_\sigma^2 = 2\,\mu^2 + 3\,\sqrt{\lambda}\,\frac{a}{\mu} \qquad \text{and} \qquad m_\pi^2 = \sqrt{\lambda}\,\frac{a}{\mu}$$

And,

$$\gamma = 1 + \tfrac{1}{2}\,\sqrt{\lambda}\,\frac{a}{\mu^3}$$

The Feynman rules are given by substituting the modified masses into the π and σ propagators and introducing a factor of γ into the cubic vertex expressions.

We now compute the four pion interaction considered above. The first diagram is unmodified; however the last three terms take the form:

$$② = -2i\,\lambda\,\delta_{ij}\,\delta_{kl}\,\frac{2\,\gamma^2\,\mu^2}{(p_1+p_2)^2 - m_\sigma^2}$$

$$③ = -2i\,\lambda\,\delta_{ik}\,\delta_{jl}\,\frac{2\,\gamma^2\,\mu^2}{(p_1-p_3)^2 - m_\sigma^2}$$

$$④ = -2i\,\lambda\,\delta_{il}\,\delta_{jk}\,\frac{2\,\gamma^2\,\mu^2}{(p_1-p_4)^2 - m_\sigma^2}$$

At threshold ($\mathbf{p}_a = 0$):

$$(p_1+p_2)^2 = 4\,m_\pi^2 \qquad \text{and} \qquad (p_1-p_3)^2 = (p_1-p_4)^2 = 0$$

After a little algebra we find:

$$i\,\mathcal{M} = i\,(\lambda)^{3/2}\,\frac{a}{\mu^3}\,\left(3\,\delta_{ij}\,\delta_{kl} - (\,\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\,)\right)$$

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2.) Peskin & Schroeder 4.4

We consider the quantized Dirac field interacting with a classical potential $A_\mu(x)$:

$$H_I \;=\; \int d^3x\, e\, \bar{\psi}\gamma^\mu\psi\, A_\mu$$

(a)

We consider the $(\mathcal{O}(e))$ matrix element:

$$\langle \mathbf{p}'s' |\, i\,\mathcal{T}\, |\mathbf{p}s\rangle \;=\; -ie \int d^4x\, A_\mu(x)\; \langle \mathbf{p}'s' |\, \bar{\psi}(x)\gamma^\mu\psi(x)\, |\mathbf{p}s\rangle$$

Here $S = 1 + i\mathcal{T}$. Now, ignoring vacuum bubbles:

$$\langle \mathbf{p}'s' |\, \bar{\psi}(x)\gamma^\mu\psi(x)\, |\mathbf{p}\, s\rangle \;=\; \langle \mathbf{p}'\overbrace{s' |\, \bar{\psi}}(x)\gamma^\mu\overbrace{\psi(x)\, |\mathbf{p}}\, s\rangle$$

Where,

$$\overbrace{\psi(x)\, |\mathbf{p}\, s\rangle} \;=\; e^{-ip\cdot x}\, u^s(p)\, |0\rangle \qquad \text{and} \qquad \langle \mathbf{p}'\overbrace{s' |\, \bar{\psi}}(x) \;=\; \langle 0 |\, \bar{u}^{s'}(p')\, e^{ip'\cdot x}$$

Thus,

$$\langle \mathbf{p}'s' |\, i\,\mathcal{T}\, |\mathbf{p}\, s\rangle \;=\; -ie\, \bar{u}^{s'}(p')\gamma^\mu u^s(p)\, \tilde{A}_\mu(p' - p)$$

Where,

$$\tilde{A}_\mu(q) \;=\; \int d^4x\, A_\mu(x)\, e^{iq\cdot x}$$

(b)

If $A_\mu(x)$ is time independent:

$$\tilde{A}_\mu(q) \;=\; \int dt\, e^{iq^0 t} \int d^3x\, A_\mu(\mathbf{x})\, e^{-i\mathbf{q}\cdot\mathbf{x}} \;=\; (2\pi)\, \delta(q^0)\, \tilde{A}_\mu(\mathbf{q})$$

Where it is to be understood that $\tilde{A}_\mu(\mathbf{q})$ is the spatial Fourier transform of $A_\mu(\mathbf{x})$.

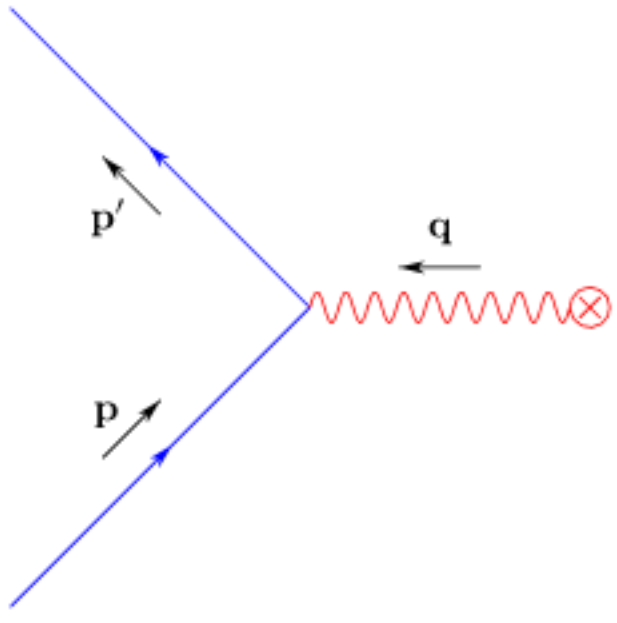
With the following definition of $\mathcal{M}$:

$$\langle \mathbf{p}'s' |\, i\,\mathcal{T}\, |\mathbf{p}\, s\rangle = i\mathcal{M}\,(p's' \mid p\, s)\; (2\pi)\, \delta(E_{\mathbf{p}'} - E_{\mathbf{p}})$$

We find:

$$i\mathcal{M}\,(p's' \mid p\, s) \;=\; -ie\, \bar{u}^{s'}(p')\gamma^\mu u^s(p)\, \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})$$

Thus, with the usual momentum space Feynman rules for fermions, we find the following vertex rule:



$$\Rightarrow \qquad i\, e\, \gamma^\mu\, \tilde{A}_\mu(\mathbf{q}) \qquad (q^0 = 0)$$

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We consider the following expression for $d\sigma$:

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int d^2b\,\left|\langle p's'|\,i\,\mathcal{T}\,e^{i\mathbf{b}\cdot\mathbf{P}}\,|\phi\,s\rangle\right|^2$$

Here $e^{i\mathbf{b}\cdot\mathbf{P}}$ is the translation operator perpendicular to the beam. Also,

$$|\phi\,s\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\,\phi(\mathbf{k})\,|k\,s\rangle \qquad \text{where} \qquad \langle\phi\,s|\;\phi\,s\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\,|\phi(\mathbf{k})|^2 \;=\; 1$$

Note that $\phi(\mathbf{k})$ has support in a highly localized region centered at $\mathbf{p} = p_z\hat{\mathbf{z}}$.

We are essentially considering a mixed incoming state found by integrating over all impact parameters $\mathbf{b}$. We could also include a mixture of incoming spin states but we will defer this for the time being.

Now,

$$\langle p's'|\,i\,\mathcal{T}\,e^{i\mathbf{b}\cdot\mathbf{P}}\,|\phi\,s\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\,\phi(\mathbf{k})\,e^{i\mathbf{b}\cdot\mathbf{k}}\;\langle p's'|\,i\,\mathcal{T}\,|k\,s\rangle$$

Using the definition of $\mathcal{M}$ from above:

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\int \frac{d^3q}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{q}}}}\,(2\pi)^2\,\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\mathbf{p}'}-E_{\mathbf{q}})$$

$$\times\,\phi(\mathbf{k})\,\phi^*(\mathbf{q})\,\mathcal{M}\,(p's'\mid k\,s)\;\mathcal{M}^*\,(p's'\mid q\,s)\int d^2b\,e^{i\mathbf{b}\cdot(\mathbf{k}-\mathbf{q})}$$

Using,

$$\int d^2b\,e^{i\mathbf{b}\cdot(\mathbf{k}-\mathbf{q})} \;=\; (2\pi)^2\,\delta(k_x-q_x)\,\delta(k_y-q_y)$$

And,

$$\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\mathbf{p}'}-E_{\mathbf{q}}) \;=\; \delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\mathbf{q}}-E_{\mathbf{k}})$$

And defining $\tilde{\mathbf{q}} = q_z\hat{\mathbf{z}} + k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$, we find:

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\int \frac{dq_z}{(2\pi)}\,\frac{1}{\sqrt{2E_{\tilde{\mathbf{q}}}}}\,(2\pi)^2\,\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\tilde{\mathbf{q}}}-E_{\mathbf{k}})$$

$$\times\,\phi(\mathbf{k})\,\phi^*(\tilde{\mathbf{q}})\,\mathcal{M}\,(p's'\mid k\,s)\;\mathcal{M}^*\,(p's'\mid \tilde{q}\,s)$$

Now,

$$\delta(E_{\tilde{\mathbf{q}}}-E_{\mathbf{k}}) \;=\; \frac{E_{\mathbf{k}}}{|k_z|}\,[\,\delta(q_z-k_z)+\delta(q_z+k_z)\,]$$

Since $\phi(\mathbf{k})$ is highly localized around $\mathbf{p} = p_z\hat{\mathbf{z}}$, the term produced by $\delta(q_z+k_z)$ vanishes since $\phi(k_z\hat{\mathbf{z}})\,\phi^*(-k_z\hat{\mathbf{z}}) = 0$. Thus,

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\,|k_z|}\,(2\pi)\,\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\;|\phi(\mathbf{k})|^2\;\left|\mathcal{M}\,(p's'\mid k\,s)\right|^2$$

We now consider $\phi(\mathbf{k})$ to be so highly localized that we may take:

$$|\phi(\mathbf{k})|^2 \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})$$

Thus we find:

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\,\frac{1}{2\,|p_z|}\,(2\pi)\,\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})\;\left|\mathcal{M}\,(p's'\mid p\,s)\right|^2$$

We know integrate over $|\mathbf{p}'|$ to compute $d\sigma/d\Omega$. Since $E_{\mathbf{p}'}^2 = |\mathbf{p}'|^2 + m^2$, we have:

$$d^3p' \;=\; |\mathbf{p}'|^2\;d\,|\mathbf{p}'|\;d\Omega \;=\; |\mathbf{p}'|\;E_{\mathbf{p}'}\,dE_{\mathbf{p}'}\,d\Omega$$

Thus,

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{4\,|p_z|}\,\int_0^\infty dE_{\mathbf{p}'}\,\frac{|\mathbf{p}'|}{(2\pi)^2}\,\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})\;\left|\mathcal{M}\left(p's'\mid p\,s\right)\right|^2$$

Finally, since the delta function enforces $|\mathbf{p}'|=|\mathbf{p}|=|p_z|$, we find:

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{(4\pi)^2}\;\left|\mathcal{M}\left(p's'\mid p\,s\right)\right|^2$$

(c)

We consider the Coulomb potential:

$$A_\mu(\mathbf{x}) \;=\; \frac{Ze}{4\pi\,|\mathbf{x}|}\,\delta_{\mu 0} \qquad \text{or} \qquad \tilde{A}_\mu(\mathbf{q}) \;=\; \frac{Ze}{|\mathbf{q}|^2}\,\delta_{\mu 0}$$

Thus,

$$i\mathcal{M}\left(p's'\mid p\,s\right) \;=\; -ie\,\bar{u}^{s'}(p')\gamma^\mu u^s(p)\,\tilde{A}_\mu(\mathbf{p}'-\mathbf{p}) \;=\; \frac{-iZe^2}{|\mathbf{p}'-\mathbf{p}|^2}\,\bar{u}^{s'}(p')\gamma^0 u^s(p)$$

In the non-relativistic limit:

$$u^s(p) \;=\; \sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}$$

Thus,

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) \;=\; u^{s'\dagger}(p')\,u^s(p) \;=\; 2\,m\,\delta^{ss'}$$

Since $|\mathbf{p}'|=|\mathbf{p}|=mv$:

$$|\mathbf{p}'-\mathbf{p}|^2 \;=\; |\mathbf{p}|^2\;2\,(1-\cos(\theta)) \;=\; m^2\,v^2\,4\,(\sin(\theta/2))^2$$

Thus,

$$i\mathcal{M}\left(p's'\mid p\,s\right) \;=\; \frac{-iZe^2}{2mv^2}\;(\sin(\theta/2))^{-2}\;\delta^{ss'}$$

This leads to:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2}\;\left|\mathcal{M}\left(p's'\mid p\,s\right)\right|^2 = \left(\frac{Ze^2}{4\pi}\right)^2\frac{\delta^{ss'}}{4m^2v^4}\;(\sin(\theta/2))^{-4}$$

We now sum over outgoing and average over incoming spins. Using $\alpha=e^2/(4\pi)$ we find:

$$\frac{d\sigma}{d\Omega} \;=\; \frac{Z^2\alpha^2}{4m^2v^4}\;(\sin(\theta/2))^{-4}$$

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