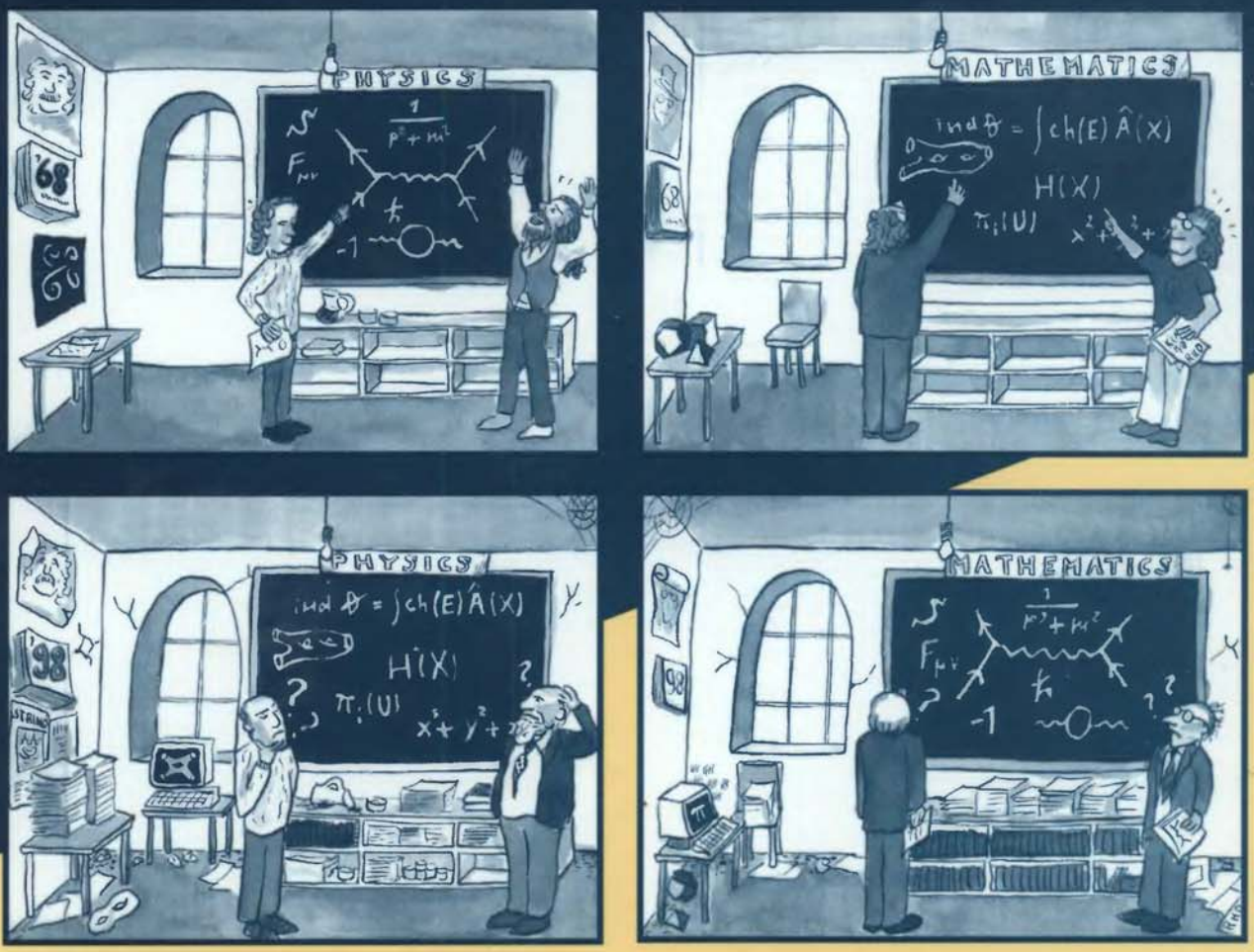
Quantum Fields and Strings: A Course for Mathematicians

VOLUME 1



Pierre Deligne
Pavel Etingof
Daniel S. Freed
Lisa C. Jeffrey
David Kazhdan
John W. Morgan
David R. Morrison
Edward Witten

Editors

American Mathematical Society
Institute for Advanced Study

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ABSTRACT. These two volumes are the written record of the 1996–1997 Special Year in Quantum Field Theory held at the Institute for Advanced Study. The goal of that program was to introduce mathematicians to the basic intuitions and modern developments in quantum field theory and string theory. Included in this publication are lecture notes from courses held during the Special Year, supplementary mathematical essays and texts, and problem sets with solutions.

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Passer de la mécanique de Newton à celle d'Einstein
doit être un peu, pour le mathématicien, comme de passer
du bon vieux dialecte provençal à l'argot parisien dernier cri.
Par contre, passer à la mécanique quantique, j'imagine,
c'est passer du français au chinois.

— ALEXANDRE GROTHENDIECK, *Récoltes et Semailles (1986)*

Preface

The 1996–1997 Special Year at the Institute for Advanced Study was a program in Quantum Field Theory. It was a year of intense activity. There were about four lectures per week, each lecture lasting two hours. In addition, there were weekly problem sessions, as well as supplementary lectures and remedial study meetings at all hours. The students ranged from the obscenely young to those less so.

In these two volumes we share our struggles with a broader audience.

The Special Year would not have been possible without the generous support of the National Science Foundation, the Friends of the Institute for Advanced Study, the J. Seward Johnson Sr. Charitable Trusts, and the Harmon Duncombe Foundation. It is a pleasure to thank these organizations for their support.

We warmly thank Dottie Phares for her infinite patience and skill in dealing with a noxious mixture of TEX flavors. We are indebted to Arthur Greenspoon, who not only proofread the entire manuscript but also prepared the index. The reader will join us in saying: thank you, Arthur. As always, Robbert Dijkgraaf's drawings—enjoy the covers!—liven up the proceedings, and we thank him for his contribution.

Most of all, we thank our teachers. They worked hard to educate us about beautiful ideas in physics, and we hope that they will be rewarded when one day this all becomes a beautiful part of mathematics. We will be pleased if these volumes can aid in that development. Finally, our thanksgiving would be amiss if we did not single out Edward Witten (and so exclude him from the signatories below). He gave generously of his time, energy, and abilities while serving as lecturer, teaching assistant, tutor, grader, organizer, and editor.

Pierre Deligne
Pavel Etingof
Dan Freed
Lisa Jeffrey
David Kazhdan
John Morgan
David Morrison

January, 1999

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Introduction

Until recently the interplay between physics and mathematics followed a familiar pattern: physics provides problems and mathematics provides solutions to these problems. Of course at times this relationship has led to the development of new mathematics. In fact, sometimes this new development was initiated by physicists (the discovery of Fourier series is one of the success stories). But physicists did not traditionally attack problems of "pure" mathematics.

The situation has drastically changed during the last 15 years. Physicists have formulated a number of striking conjectures (such as the existence of mirror symmetry) and concepts (for example, the existence of "fusion" for representations of Kac-Moody algebras, i.e., the structure of a braided tensor category). In many cases mathematicians have been able to verify the conjectures of physicists, but the proofs have dealt with each individual case and ignore the bigger picture which governs the physicists' intuition. The basis of the physicists' intuition is their belief that underlying quantum field theory and string theory is a (as yet undiscovered) self-consistent mathematical framework. After all, physicists believe in the "reality" of these physical theories, and hence in the existence of mathematical foundations for them. A number of mathematical objects and notions, especially geometric ones, can be formulated in terms of quantum field theory and strings. Then the non-rigorous techniques of physics can be brought to bear on these objects, often providing new insight. The great advances come when this insight is not readily apparent from the purely mathematical perspective.

The mathematicians and the physicists at the Institute for Advanced Study decided that it was worthwhile to organize a special year-long program to teach a small group of mathematicians the basic physical concepts—renormalization group, infrared limit, dimensional transmutation, etc.—which are in constant use. The conception for the Special Year at the IAS was eloquently described in a letter by Robert MacPherson:

> The goal is to create and convey an understanding, in terms congenial to mathematicians, of some fundamental notions of physics, such as quantum field theory, supersymmetry and string theory. The emphasis will be on developing the intuition stemming from functional integrals.
>
> One way to define the goals of the program is by negation, excluding certain important subjects commonly pursued by mathematicians whose work is motivated by physics. In this spirit, it is not planned to treat except peripherally the magnificent new applications of field theory, such as Seiberg-Witten equations to Donaldson theory. Nor is the plan to consider fundamental new constructions within mathematics that were inspired by physics, such as quantum groups or vertex operator algebras. Nor is the aim to discuss how to

> provide mathematical rigor for physical theories. Rather, the goal is to develop the sort of intuition common among physicists for those who are used to thought processes stemming from geometry and algebra.

This book is a written record of that attempt. The extent to which the program, and by extension these volumes, meet the goals set down in MacPherson's letter is left for the reader to judge.

The Special Year consisted of many short—and some long—courses covering various aspects of quantum field theory and perturbative string theory. Notes from these lectures form the bulk of this volume. We have fairly accurately kept the chronology of the program: lectures from the fall appear in Volume I and lectures from the spring in Volume II. (There is one exception: Gawędzki's lectures were given in the fall.) During the year there were many problem sets and even one exam. They also appear in these volumes together with solutions to most of the problems. There are also a few mathematical notes which supplement the lectures. Krzystof Gawędzki and Eric D'Hoker wrote their own notes, but otherwise the lecture notes in these volumes were written by mathematicians who participated in the program. In some cases they stick very closely to the content and style of the actual lecture, while in other cases the material is exposed with a mathematical spin.

Part 1 of these volumes is somewhat different in nature from the parts which follow. It is a mathematical text which is meant to be rigorous, fairly self-contained, and consistent in its notation and sign conventions. The topics covered are "classical" in the sense of "non-quantum," and so well within reach of standard notions in algebra and geometry. For the most part it is supplementary material not based on lectures from the Special Year. The exception is *Introduction to Supersymmetry*, which covers parts of lectures given by Joseph Bernstein, but the material has been substantially reworked by Pierre Deligne and John Morgan. This text lays the foundations of supersymmetry, or "$\mathbb{Z}/2$-graded mathematics." The algebraic manifestation is more well-known than the geometric one, where Grothendieck's "functor of points" is used to formalize the notorious "anticommuting variables" of classical supersymmetry. Pierre Deligne's treatment of *Spinors* is tailored to the needs of the rest of these volumes. For example, special facts relevant to Lorentz signature are covered in great detail. *Classical Field Theory*, by Pierre Deligne and Dan Freed, develops a mathematical framework for lagrangian field theory. Despite the word "classical" in the title, these ideas play an important role in quantum field theory as well. (For example, one often describes a quantum theory by an *effective lagrangian*, which one then manipulates according to the rules of classical field theory.) The basic examples are also described here, while more sophisticated examples appear throughout the rest of the volumes. Most modern developments in field theory and string theory are for supersymmetric theories, hence a thorough understanding of the classical supersymmetric lagrangians is a prerequisite. That motivation lay behind Witten's *Superhomework* (which appears here at the end of Part 2). Answers to the problems posed there are incorporated into the text *Supersolutions*, by Pierre Deligne and Dan Freed, which is a systematic development of classical supersymmetric field theories in various dimensions. The flavor of the subject changes with the dimension, in part due to the dimension-dependent behavior of spinors. An appendix, *Sign Manifesto*, collects and explains many of the sign conventions used in Part 1. We emphasize that there are many different sign

choices made in the physics literature, and the lecture notes in subsequent parts of these volumes reflect the sign conventions of the particular lecturer.

Part 2 covers disparate topics in quantum field theory which we can loosely collect under the rubrics "axiomatics" and "perturbation theory." They are linked in that both are more amenable to current mathematics than the dynamical ideas of Part 4. Pierre Deligne's short note *Quantization* summarizes general mathematical notions which apply in both mechanics and field theory. *Introduction to QFT*, by David Kazhdan, begins with the Wightman axioms, a mathematical framework for rigorous quantum field theory. He treats scattering theory within that framework and also gives an exposition of Gaussian integrals and Feynman diagrams. A major component of Part 2 is Edward Witten's lectures *Perturbative Quantum Field Theory*. He begins with the relationship between the spectrum of the hamiltonian of a Wightman quantum field theory and the analytic behavior of its 2-point function. In the simplest example of ϕ^3 theory this quickly leads to a 1-loop Feynman diagram, and so to a general discussion about ultraviolet divergences and renormalization theory. Here the reader finds an explanation of basic concepts and tools—Feynman diagrams, quantum operators, operator product expansion, renormalizability—as well as discussions of many general issues related to other lectures in Part 2. Edward Witten's two lectures *Index of Dirac Operators* give applications of the path integral to Dirac operators, both on finite dimensional manifolds where he derives the index formula, and on the infinite dimensional loop space where he constructs the elliptic genus and derives some topological consequences of the field theoretic viewpoint. A somewhat different emphasis is given by Ludwig Faddeev in his *Elementary Introduction to Quantum Field Theory*. In particular, the first lecture explains very clearly how the path integral naturally arises from the usual formalism of quantum mechanics. After some comments about scattering, he goes on to describe the Faddeev-Popov method of quantization of gauge theories. Another important component of Part 2 is David Gross' *Renormalization Groups*. The renormalization group flow in quantum field theory explains how information from theories that describe the microscopic world is integrated into a few macroscopic parameters. There is a rigorous definition of the renormalization group flow in perturbation theory; a detailed treatment of the renormalization group differential equation, both for lagrangians and operators; the definition of anomalous dimension; a discussion of asymptotic freedom; and a first example of a well-defined nontrivial renormalizable quantum field theory, the Gross-Neveu model. Pavel Etingof's note *Dimensional Regularization* explains the mathematical structure behind a commonly used regularization technique. Edward Witten's *Homework* problems are a useful supplement to all of these lectures. We provide solutions to most of the problems.

Part 3 treats the related subjects of conformal field theory and perturbative string theory. Gawedzki's *Lectures on Conformal Field Theory* begins with a discussion of free fields, including circle- and torus-valued free fields. Then he presents a general axiomatic approach to two-dimensional conformal models. Perturbative aspects of the two-dimensional sigma model—regularization and renormalization—are treated in his third lecture. Finally, he takes up the Wess-Zumino-Witten sigma model (the sigma model with target a compact group) which can be described nonperturbatively using harmonic analysis on the loop group of the compact group. D'Hoker's *String Theory* is a comprehensive introduction to the subject. He begins with a treatment of the Hilbert space associated to the free (non-interacting)

bosonic string, showing among other things that $D = 26$ is the critical dimension in which the physical "Hilbert" space has no negative norm states. Next come interacting bosonic strings, where the topology of Riemann surfaces, the moduli space of conformal structures, and the space of Weyl rescalings all play a role. The bosonic discussion concludes with Faddeev-Popov ghosts and BRST quantization, which are used to properly treat the diffeomorphism group of the surface in the theory. Then the lectures turn to superstrings, introducing the various species: Type I, Type IIA, Type IIB, and the heterotic strings. There is a detailed study of the perturbative properties of these various theories to one-loop. Finally, the lectures end with a general introductory discussion of supersymmetry and supergravity. Two mathematical texts follow D'Hoker's lectures. Pierre Deligne's brief *Super Gravity* explains the two-dimensional example important in perturbative string theory. *Notes on 2D Conformal Field Theory and String Theory*, by Dennis Gaitsgory, grew out of an effort to record solutions to the Exercises in D'Hoker's lectures. It explains many aspects of conformal field theory in the language of D-modules. Part 3 ends with *Kaluza-Klein Compactifications, Supersymmetry, and Calabi-Yau Spaces* by Andrew Strominger. These lectures discuss compactifying a theory of gravity on a higher dimensional spacetime along the fibers of a fibration. As a warmup he presents the classical example of Kaluza-Klein, where the fibers are circles, and then moves to the more recent supersymmetric theories compactified along Calabi-Yau complex three-folds.

Although non-perturbative aspects enter in several places to this point, it is only in Part 4 that they are the focus. The two lecture series presented here include results that were discovered in the mid '90s. *Dynamics of Quantum Field Theory*, by Edward Witten, is a systematic semester-length course which presents important old examples as well as new ones. The first lectures cover some formal aspects—spontaneous symmetry breaking and Goldstone's theorem, gauge symmetry breaking, BRST quantization of gauge theories—which for the most part do not appear elsewhere. The non-perturbative discussion begins with two dimensional bosonic sigma models, particularly in the large N limit. Next come various aspects of two- and three-dimensional theories: the Bose-Fermi correspondence in dimension two, gauge theories with Wilson and 't Hooft lines, confinement, abelian duality (including four dimensions), and solitons. There follows a mathematical application of two-dimensional gauge theories to certain moduli spaces attached to surfaces. Here the Cartan model of equivariant cohomology is related to path integrals in gauge theory. At this point supersymmetric field theories are introduced. Several lectures are devoted to two-dimensional supersymmetric gauge theories, particularly with $N = 2$ supersymmetry. Here we encounter topological twisting, the chiral ring of operators, quantum cohomology, and Landau Ginzburg models. The last part of the course deals with gauge theories in four dimensions. After a general discussion, the focus is again $N = 2$ supersymmetric theories, culminating with the Seiberg-Witten relationship between $SU(2)$-gauge theory (including Donaldson theory of four-manifolds) and $N = 2$ supersymmetric $U(1)$-gauge theory with matter (including the Seiberg-Witten invariants of four-manifolds). At the end of the lectures are Exercises and some solutions by Dan Freed; the Superhomework was also given as an accompaniment to the course. The other text in Part 4 is a series of three lectures by Nathan Seiberg called *Dynamics of $N = 1$ Supersymmetric Field Theories*. He reviews some general features—effective actions and nonrenormalization theorems, anomalies, confinement—for Wess-Zumino models and gauge

theories. The main subject is supersymmetric quantum chromodynamics, with varying amount of matter. There is a wide variety of nonperturbative behavior exhibited by these theories, which provide a nice body of examples for many of the concepts quantum field theory. One notable feature is a nonabelian version of electric-magnetic duality. In the lectures the gauge group is the unitary group. In his solution to the Exercises, Siye Wu explains the theories with symplectic and orthogonal gauge groups.

The *Glossary* collects terms from physics which may be unfamiliar to the mathematical reader. Our capsule definitions should help bridge the gap between mathematics and physics vocabulary.

Quantum field theory and string theory are vast subjects, and it did not make sense to collect a comprehensive bibliography for these volumes. Rather, at the end of each lecture we often give a few references which are most relevant, or most accessible, and which are an entry into the literature. To our colleagues who feel slighted by omission we offer an apology: your work has become so standard that an explicit reference is superfluous.

As we did not set out to write a textbook, which would have been inappropriate for a myriad of reasons, a diagram of dependencies of various sections would not embed in a space of less than 10 dimensions. To help the reader make his way we give cross references at the end of each lecture and in many places in the text. These cross-references include the volume number. A list of cross-reference codes is printed facing the back cover of each volume.

For the most part this is physics written by mathematicians for mathematicians (read: the blind leading the blind). For that reason, throughout these volumes we have assumed without reference standard mathematical facts which are covered in textbooks and other literature. On the other hand, some very basic concepts in physics are explained. Certainly we did not attempt to explain everything, and the reader is well-advised to learn/review standard topics like special relativity, classical electromagnetism, etc. One basic prerequisite is a familiarity with at least the general framework of quantum mechanics.

There are many topics not covered in the Special Year which would have been natural to include. For example, the perturbative quantization of gauge theories is not treated systematically. Our mathematical discussion of supersymmetry in Part 1 contains nothing about supergravity (though there is a short piece about it in Part 3). Nor is there anything here about statistical mechanics. Although the lectures on quantum field theory reached quite modern developments, the exciting work over the past few years in nonperturbative string theory and beyond are outside the scope of these volumes. We offer no apologies for these omissions, though; the books are weighty enough as they are and should stimulate ample exercise of one sort or another.

A casual browse through these pages confirms what the reader must have gleaned from our blow-by-blow description: there is a great diversity of material here. The diversity of subject matter reflects the nature of the field; the rich mix of mathematical topics is quite typical. On the other hand, the diversity of quality is the predictable outcome of writing a book while learning the material. Most of all there is a diversity in the level of mathematical rigor and understanding. The task of absorbing physical intuition into mainstream mathematics is very much a work in progress. We offer this collection as a contribution to that effort.

Glossary

σ-model

Field theory in which the **elementary field**[1] is a map ϕ from **spacetime** to an auxiliary space M, usually a Riemannian manifold. In superspace[2] formulations, the source of ϕ (spacetime) is a supermanifold, and ϕ can be subject to constraints.

There are also σ-models coupled with a **gauge** theory. Let the gauge group be a compact group G and let P be a principal G-bundle over spacetime. Also suppose that G acts by isometries on M. In this case, the **lagrangian** is written in terms of two elementary fields: (i) a section ϕ of the fiber bundle $M^P = P \times_G M$ with fiber M, and (ii) a connection on P.

θ-term

A special term in a classical **lagrangian**: a differential form constructed from the **elementary fields** which represents a cohomology class on **spacetime**. For example, in gauge theory a Chern-Weil form. In a σ-model with elementary field $\phi\colon \Sigma \to M$ the pullback of a closed differential form on M. The term is usually written as a multiple of θ, hence the name. These are often called "topological terms," though we reserve that phrase for something different. See **topological terms**.

action

Often denoted by S, the action is the integral over **spacetime** of the **lagrangian density**, and hence is a functional of the **elementary fields** in the theory. Many, but not all, field theories are defined by giving an action. The classical version of the theory is the study of field configurations of least action (the minima of the action functional) or possibly the extrema of the action functional (solutions of the Euler-Lagrange equations). See also **lagrangian, lagrangian framework, Feynman diagram, path integral, expectation value, correlation function, partition function**.

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[1] Words and phrases appearing in bold are defined elsewhere in the glossary.

[2] We use freely notions from superalgebra and supergeometry. A reference for these is [I-**Supersymmetry**].

anomaly

Failure of a classical symmetry to be a symmetry of the quantum theory (i.e. of the algebra of quantum observables).

Common types of anomalies:

1) anomaly in a global symmetry – failure of a global symmetry of the lagrangian to preserve the measure in the **path integral**. This failure is due to the fact that the index of a related operator is non-zero. An example is a **chiral** anomaly — a chiral symmetry multiplies chiral spinors by $e^{i\theta}$ and antichiral ones by $e^{-i\theta}$. If the index of the Dirac operator between these spinors is nonzero, then the "number" of chiral degrees of freedom is not be the same as the "number" of antichiral ones. Since the measure of integration includes a measure on spaces of both types of fermions, the action of the chiral symmetry will be non-trivial on the measure, and in fact given by the character which is the index of the Dirac operator.

2) local gauge anomaly—failure of gauge symmetry (or, equivalently, of the **BRST differential**) to preserve the measure of integration in a gauge theory path integral.

A local gauge anomaly arises, for instance, in four-dimensional gauge theories with chiral spinor fields in a non-real representation of the gauge group, and is related to the fact that the fermionic path integral is not a function of the gauge class of the gauge field but rather a section of some nontrivial line bundle and thus cannot be integrated. Often, a gauge theory with a local gauge anomaly arises by **gauging** an anomalous chiral symmetry.

An anomalous theory does not make sense quantum mechanically, so a fundamental condition for such a theory to make sense quantum mechanically is cancellation of anomalies.

ansatz

An "assumed form" for a solution; a simplifying assumption.

asymptotic freedom

Property of a quantum field theory which says that interactions are weak at high energies (momenta). Mathematically this means that suitably normalized **correlation functions** in momentum space tend to correlation functions of a **free theory** when the momenta go to infinity. For a renormalizable theory defined by a **lagrangian**, with renormalizable couplings $g_1, ..., g_n$, asymptotic freedom can be checked by looking at the **beta function** $\beta = (\beta_1, ..., \beta_n)$. The theory is asymptotically free if the **renormalization group** differential equation $g_i' = \beta_i(g_1, .., g_n)$ has an attracting fixed point at $g_i = 0$ and not asymptotically free if the equation has a repelling fixed point at $g_i = 0$. Thus a sufficient condition for asymptotic freedom is that all eigenvalues of $d\beta(0)$ are negative, and for its absence is that they are positive. In the case of one coupling, negative $\beta'(0)$ implies asymptotic freedom and positive $\beta'(0)$ implies no asymptotic freedom.

Renormalizable theories defined by a lagrangian are believed to be well defined if and only if they are asymptotically free, if they have no other pathologies (like energy unbounded from below). In 2 dimensions, there are many asymptotically free, well defined theories (**σ-model** into a positively curved space, the Gross-Neveu model), but in four dimensions only nonabelian gauge theories with a limited number of other (matter) fields (e.g. the standard model) are asymptotically free and believed to be well defined.

background field

A field that is held fixed. For example, in a **σ-model** of maps from a space Σ to a target space X, in which the metric or conformal structure of Σ is arbitrary but is held fixed in doing the quantum theory, that metric or conformal structure might be regarded as a background field. One can study physical systems with a background electric field or a background gravitation field; these fields are held fixed and are not subject to the equations of motion, though of course they will influence these equations. Compare **dynamical variable**.

BPS state

In classical supersymmetric field theory: A static (time-independent) field configuration which is stable by some (usually half) but not all of the supersymmetry generators. Such a configuration is a **soliton**.

In quantum supersymmetric field theory: A vector (i.e., a state) in the quantum Hilbert space which is invariant under some (again usually half) but not all of the **supersymmetry** generators.

BRST (current, charge, differential, cohomology, quantization)

BRST quantization is a method of quantization of gauge theories. It is a refinement of the **Faddeev-Popov** method that makes explicit the fact that the quantization is independent of a choice of a particular **gauge**. The procedure of BRST quantization consists of two steps. The first step is the introduction of **ghosts**, as in the Faddeev-Popov method. Then the theory with ghosts is quantized in the usual way (with gauge symmetry ignored). Since ghosts violate **spin-statistics**, the state space of the resulting quantum theory has a pseudo-Hermitian rather than Hermitian inner product. The second step both restores positivity and brings back the gauge symmetry. It consists in defining a BRST differential on the algebra of local operators, making it into a differential graded algebra (graded by ghost number). Then one declares the actual algebra of operators to be the 0-th cohomology of this DGA (the BRST cohomology).

Often the BRST differential can be defined as the commutator with the BRST **charge**, which is the integral of the BRST **current** over a space-like cycle. The BRST current is a certain local operator in the ghost theory which can be written explicitly in a way which is independent of the **lagrangian** and involves no choice of a gauge. The BRST charge is itself an operator of square zero on the pseudo-Hilbert space of the ghost theory. Using the BRST charge as a differential makes the pseudo-Hilbert space into a complex graded by ghost number. The 0-th cohomology of this complex is declared to be the actual

space of states of the theory. The restriction of the inner product to the kernel of the BRST charge in degree 0 is degenerate, and its kernel is the image of this charge. The induced inner product on cohomology is positive, and is the Hilbert space structure on the quantum states.

beta function

A function or vector field which governs the behavior of a quantum field theory under renormalization. Mathematically, the beta function is a vector field on the space of dimensionless parameters for a particular quantum theory. The differential equation defined by this vector field is called the **renormalization group equation**. It is conjectured that topologically this vector field is a gradient vector field. Fixed points of this equation (ultraviolet and infrared stable) correspond to **ultraviolet** and **infrared** limits of the theory. The reason for the name "beta function" is that the vector field generating the renormalization flow in the case when there is a single coupling constant g is written

$$\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g},$$

where μ is the scale parameter. See also **asymptotic freedom**.

boson, fermion

A boson is a particle satisfying Bose-Einstein statistics, e.g., a **photon**. In mathematical terms this means that the quantum mechanical Hilbert space of a multiparticle system is the *symmetric* product of the one-particle Hilbert space. A fermion is a particle satisfying Fermi statistics, including the Pauli exclusion principle, e.g., an electron. In mathematical terms this means that the quantum mechanical Hilbert space of a multiparticle system is the exterior *antisymmetric* product of the one-particle Hilbert space. More concisely, using notions of superalgebra we write the one-particle Hilbert space $H = H_0 \oplus H_1$ as a graded space whose even part H_0 consists of bosons and whose odd part H_1 consists of fermions. Then the multiparticle Hilbert space is the graded symmetric tensor product of the one-particle Hilbert space. A bosonic state (or operator) is an even state (or operator) and a fermionic state (or operator) is an odd state (or operator). The *graded* commutator implements the distinction between Bose-Einstein statistics and Fermi statistics.

From the relativistic quantum point of view, particles are irreducible subrepresentations of the **Poincaré group** inside the Hilbert space of states $\mathcal{H}$ of the theory. This Hilbert space of states is split as $\mathcal{H}_0 \oplus \mathcal{H}_1$. (In the **Fock space** example above the grading on the multiple particle states is obtained by counting the number of fermions modulo two.) A boson is a particle associated to an irreducible subrepresentation of $\mathcal{H}_0$, while a fermion is associated to an irreducible subrepresentation of $\mathcal{H}_1$. By the **spin-statistics theorem**, which holds when the dimension of spacetime is at least three, the subspace $\mathcal{H}_0 \subset \mathcal{H}$ is the subspace on which the center $\{\pm 1\}$ of the Poincaré group acts trivially. Thus, bosonic fields are fields in vector representations of the Poincaré group (those having integral spin), while fermion fields are in spinor representations (those having half-integral spin).

In classical field theory, a field is a map from spacetime into a supermanifold (more precisely, a section of a bundle of supermanifolds over spacetime). For simplicity suppose that the supermanifold is a super vector space. Then maps into the even part are called bosons, and maps into the odd part are called fermions. Loosely: bosons are "commuting functions" and fermions are "anticommuting functions". In theories on Minkowski space which obey spin-statistics: a Bose field is a scalar field, **gauge field**, or metric; a Fermi field is a **spinor** field or **Rarita-Schwinger field**.

central charge

A central charge is a **bosonic** operator in a quantum theory which commutes with all operators but can be obtained by commuting other operators. Examples of central charges are as follows.

In conformal field theory: The quantum Hilbert space is a representation of the universal central extension of the diffeomorphism group of the circle; the central charge is the value of the center of the Lie algebra in the representation.

In supersymmetric field theory: In some cases the **super Poincaré algebra** admits a central extension where some brackets of odd generators equal a central element. The value of the central element(s) on a quantum state is the central charge of the state. The quantum operator which corresponds to the central element is also called the central charge.

charge

See **current**.

chiral, chirality

Chirality is an asymmetry between left and right for **fermions**, left and right fermions being sections of two basic real spin bundles over space-time. A chiral symmetry, for instance, transforms left and right fermions differently, and a chiral theory is one that has chiral fields. Fermions transforming under a chiral symmetry are called chiral fermions.

Chirality is an asymmetry between left and right that is "built in" to a theory because it depends on a discrete choice (like a choice of fermion representation), not a continuous choice (like a choice of **coupling**).

classical field, elementary field

A **dynamical variable** of a classical field theory. A classical field is local data of some kind on **spacetime** (for example: a function, a section of a bundle, a metric, a connection on some bundle, or a map to some target space) which, in a classical field theory, assigns some information to each point (or formal neighborhood of a point) in spacetime. In the **lagrangian framework**, the elementary fields are the classical fields that appear explicitly in the lagrangian. See also **field, lagrangian**.

classical limit

In the limit as $\hbar$ goes to zero, quantum mechanics reduces to classical mechanics. This is called the classical limit of quantum mechanics. See also **semi-classical approximation**.

compactification

A process whereby the **effective** or **low energy** observable dimension of **spacetime** is reduced. One considers spacetime as being a product (or more generally a fibration) with compact manifolds as fibers. If the diameters of these fibers are sufficiently small (usually they are on the order of the Planck length, which is 10^{-33} cm), then they cannot be probed by particles of energy scales that we can produce experimentally and hence they will not be directly observable by us. Mathematically, one considers fields on the higher dimensional spacetime and expands in Fourier modes of the compact factor, so as to write everything in terms of fields in the non-compact directions. Since the compact factor is assumed to be very small, the nonconstant Fourier modes of the compact factor are of very high energy and thus effectively at low energies dimensional reduction occurs. The effective theory is then written as a a theory on the non-compact, lower dimensional manifold. Certain characteristics of the compact factor remain in this low energy effective theory.

component field

See **supersymmetry**.

confinement

Confinement is the appearance of forces which prevent particles from separation. In this case the **effective potential** describing the particles grows when particles are separated until it becomes so big that new particles are born. The usual example of confinement in nature is confinement of quarks in the case of the strong interactions. The usual mathematical criterion for confinement is expressed in terms of the **expectation value** of the **Wilson loop** operator $W(\ell)$; confinement arises if its expectation value decays as $\exp(-A(\ell))$, where $A(\ell)$ is the area of a disc bounded by the loop ℓ.

conserved (current):

A closed, or divergence-free **current**.

correlation function

Correlation functions are the main output information of a quantum field theory. In operator language, a correlation function is the diagonal matrix element of a product of local operators corresponding to the vacuum vector. (More generally, one can consider correlation functions with respect to any pair of states, which means computing arbitrary matrix elements of the product of local operators.) This is a (generalized) function of the positions of local operators. A correlation function in momentum space is the Fourier transform of the usual correlation function, in theories with linear spacetime. If $\phi(x)$ is a local field then the n-point function of $\phi(x)$ is the vacuum-to-vacuum matrix element of

$\phi(x_1)\cdots\phi(x_n)$. These functions are also called Wightman functions. They exist in Lorentzian signature. The Wick rotation of these to Euclidean signature are called Schwinger functions. In the **path integral** language, the correlation function of local fields $\phi_1(x_1),\ldots,\phi_n(x_n)$ is given as the integral

$$\int \mathcal{D}\Phi\phi_1(x_1)\cdots\phi_n(x_n)\exp(iS)$$

where $\mathcal{D}\Phi$ is the path integral measure (which is not rigorously defined) on the space of configurations of the elementary fields and S is the **action**. In Euclidean signature the last term is replaced by $\exp(-S)$. The coincidence of the Hamiltonian and path integral definitions is the Feynman-Kac formula. This formula holds on the nose in quantum mechanics, but in field theory it only holds if both sides have been **renormalized** in a consistent manner. See also **expectation value**.

Coulomb phase

A Coulomb phase is a phase of **gauge** theory in which there is no mass gap, and both **'t Hooft** and **Wilson loops** show perimeter rather than area law. In four dimensions, such a phase is invariant under possible electric-magnetic duality.

counterterm

A **cutoff** dependent term in the **lagrangian** which tends to infinity as the cutoff goes to infinity, and which is introduced to compensate **ultraviolet** divergences in **renormalization** theory. A renormalizable theory is a theory where one needs only finitely many counterterms to cancel all divergences. See also **renormalization**.

coupling

A coefficient in a **lagrangian**, measuring the strength of a particular interaction among elementary fields.

current

In classical field theory, a current is a codimension-one form on spacetime or equivalently, in the presence of a volume form on spacetime, a vector field. A *supercurrent* on a super spacetime is similarly a type of codimension-one form. Usually, "supercurrent" is reserved for the current associated to odd translations in **super Minkowski space**. The **charge** associated to a current is the integral of the current over a codimension-one space-like slice. In quantum theory, the operator with values in codimension-one forms associated to a classical current is a quantum current. Its integral over a space-like slice is the quantum charge, which is an operator in the theory.

cutoff

The integrals that arise in studying **expectation values** for **Feynman diagrams** are often divergent at large momenta. To avoid these divergences one wishes to cut off the integration in some manner. A cutoff might be made simply by carrying the momentum integrals only up to an upper bound Λ, though

more sophisticated cutoffs are usually used in practice. Having decided on a cutoff scheme to render the integral finite one must then study the nature of the divergence as $\Lambda \to \infty$.

dimension of an operator: scaling, engineering, quantum, anomalous

Classical scaling dimension of a local field ϕ in classical field theory—an integer d which tells how the field scales under dilation of spacetime (i.e. $\phi(x) \to a^d \phi(a^{-1}x)$). The classical scaling dimension of an **elementary field** is determined from the condition that the kinetic term for it (as a density) in the **lagrangian** is invariant under dilations. For a composite field which is a product of elementary fields, the scaling dimension is the sum of the scaling dimensions of the factors.

Engineering dimension of a field ϕ is the integer d such that ϕ is measured in (length unit)$^{-d}$. The difference from scaling dimension is that dimensionful constants have zero scaling dimensions but nonzero engineering dimension.

Quantum dimension of a local quantum field ϕ is the real number d (not necessarily an integer) which tells how the operator scales. One way to define it is: the quantum dimension is d if the 2-point function $\langle \phi(x)\phi^*(y) \rangle$ is asymptotic to $|x-y|^{-2d}$ as $x \to y$ (here ϕ^* is the Hermitian conjugate to ϕ and ϕ is a scalar field). The anomalous dimension of ϕ is the difference between the quantum dimension and the classical scaling dimension of the underlying classical functional. The anomalous dimension of a field can be computed perturbatively to any order.

dimensional reduction

Given a classical **lagrangian** $\mathcal{L}$ on a **spacetime** of dimension d, dimensional reduction is a way to deduce from $\mathcal{L}$ interesting lagrangians $\mathcal{L}'$ on spacetimes of dimension $d' < d$. One writes the spacetime M^d as $M^{d'} \times \mathbb{R}^{d-d'}$, and one considers fields "independent of the last $d-d'$ coordinates". They can be viewed as fields on $M^{d'}$, and $\mathcal{L}$ is the product of the wished for $\mathcal{L}'$ with the volume form on $\mathbb{R}^{d-d'}$. For a gauge field, "independence of the last $d-d'$ coordinates" means that one considers $\mathbb{R}^{d-d'}$-equivariant bundles (for the action by translations) with a connection invariant by $\mathbb{R}^{d-d'}$. See also **compactification**.

Dirac spinor

See **spinor**.

duality

In quantum field theory: Two or more descriptions of the same theory. Or, in some cases, a symmetry of a quantum theory which is not usually a symmetry of any **classical limit**.

The simplest examples occur in **free field theory**. In the massless case an irreducible representation of the Poincaré group in n dimensions is determined by a representation of the **little group** $\mathrm{Spin}(n-2)$. The pth exterior power representation is equivalent to the $(n-p-2)^{\text{th}}$ exterior power representation. The corresponding field theory has different descriptions as a free theory of a

p-form field or as a free theory of an $(n-p-2)$-form field. When $n = 2p+2$ we obtain a nontrivial symmetry of a single theory.

dynamical variable

An **elementary field** that is allowed to vary and whose derivatives enter into the classical equations of motion. Classically, the value of a dynamical field is determined by the equations of motion and boundary conditions. Quantum mechanically, the space of all possibilities for the field is one of the infinite dimensional spaces over which path integration is performed.

effective action, effective lagrangian, effective theory, effective potential

Effective theory: A theory obtained by integrating out (=averaging over) part of the variables in an original, more detailed theory. Effective action is an **action** describing the effective theory.

One situation where this notion is used is the theory of **renormalization group flow**. In this case, if $\Lambda > \Lambda'$ are two momentum **cutoffs**, and $\mathcal{L}$ a **lagrangian**, then the effective theory at Λ' for the theory defined by $\mathcal{L}$ at Λ has the lagrangian $\mathcal{L}'$ given by the formula

$$e^{-\int_M \mathcal{L}'(\phi)} = \int e^{-\int_M \mathcal{L}(\phi+\eta)} d\eta$$

where the η-integration takes place over the space of η with Fourier modes having momentum between Λ and Λ' and where ϕ has Fourier modes with momenta below Λ'. Thus the effective action is obtained by integrating out degrees of freedom between Λ and Λ'.

Another usage: In a quantum field theory with lagrangian $\mathcal{L}$ the sum of amplitudes of **Feynman diagrams** which are 1-particle irreducible is called the 1-particle irreducible effective action; the classical field theory for this action is equivalent to the original quantum field theory, in the sense that the tree approximation of the correlation functions for the new action gives the exact quantum correlation functions of the original theory. Effective potential (for a scalar field) is the effective action restricted to constant values of fields.

See also **low energy effective theory**.

electric charge

In classical field theory: In a $U(1)$-gauge theory on **Minkowski space** the integral of (a suitable multiple of) the Hodge star of the curvature over a sphere at spatial infinity. On curved n-manifolds this integral can be taken over any $(n-2)$-cycle. More generally, in a gauge theory with gauge group G, there is an electric charge defined for each trace on the Lie algebra of G as the integral of (a suitable multiple of) the Hodge star of the trace of the curvature.

In quantum field theory: The quantum operators which correspond to the classical observables described above.

electron

A particle, discovered at the end of the nineteenth century, of spin $1/2$, electric charge -1, and mass 511 KeV. In quantum field theory, it is described by a charged Fermi field in a $U(1)$-gauge theory.

elementary field

See **classical field.**

energy-momentum tensor

In classical field theory: For a **Poincaré**-invariant field theory, the **Noether currents** associated to (minus) infinitesimal translations fit together into the energy-momentum tensor $T_{\mu\nu}$—the Noether current of the infinitesimal translation $a^\mu \partial_\mu$ is $-T_{\mu\nu} a^\mu * dx^\nu$. If the theory couples to a **background** metric (i.e., if the theory is defined on a curved Riemannian or Lorentzian manifold), then the energy-momentum tensor is twice the derivative of the **lagrangian** with respect to the inverse metric. Evaluated at the standard metric on **Minkowski space**, it agrees with the first definition up to correction by the *Belinfante term*, at least if there are no **higher derivative terms** in the lagrangian.

The time-time component $T_{00} * dx^0$ integrates over a space-like slice to the *energy* of a field configuration. The space-time components $T_{i0} * dx^0$ integrate to the *momentum*. The space-space components T_{ij} comprise the *stress tensor.*

Euler-Lagrange equations

The equations arising from the first-order variation of the **action** functional on the space of **elementary fields**. Classical solutions are critical points of the action functional, i.e., solutions to these equations. (Sometimes minima or local minima of the action functional are of special importance.)

expectation value

In quantum field theory, the expectation value of a local operator $\phi(x)$ is the diagonal matrix element of this operator corresponding to the vacuum vector. In **lagrangian** language, the expectation value of this operator is the integral of the corresponding local functional against the path integral measure, $\int \mathcal{D}\phi \, e^{iS(\phi)} \phi(x)$. See also **correlation function.**

Faddeev-Popov method

The Faddeev-Popov method is a method of quantization of a system with symmetries. If one thinks about quantization of a system with an infinite-dimensional group $\mathcal{G}$ of symmetries as a computation of a **path integral** then one has to integrate over the quotient space $\mathcal{F}/\mathcal{G}$, where $\mathcal{F}$ is the space of field configurations. The usual way to integrate over a quotient space consists of a choice of a slice S for the $\mathcal{G}$-action and an integration over S with an appropriate measure. Usually this measure differs from a "natural" one by multiplication by a determinant of an elliptic operator. Faddeev-Popov's suggestion is to replace this determinant by a **Gaussian integral** over a corresponding linear **fermionic** space of **ghosts**. Then one can treat ghosts as one

treats other **fields** and apply the standard technique of **Feynman diagrams**. Ghost fields violate the **spin-statistics theorem** since they are even fields but fermions. Of course, the state space with ghosts included is not a Hilbert space because the inner product is not positive definite. By the time we take **BRST cohomology** to produce the physical (positive definite) Hilbert space the spin-statistics theorem is re-instated.

fermion

See **boson**.

Feynman diagram (graph)

For a mathematician, a Feynman diagram is a graph which defines a term in the perturbation expansion of an integral over a vector space V of a function e^{-S}, where S has a quadratic term and higher terms. Vertices of a Feynman diagram correspond to cubic and higher terms of S (they are tensors on V^*), and edges correspond to contraction using the propagator—the inverse to the quadratic part of S. The answer obtained after contraction is called the *amplitude* of the Feynman diagram. The perturbation expansion of the integral is the sum of amplitudes of all Feynman diagrams.

For a physicist, a Feynman diagram is a scenario of particle interaction. The vertices of Feynman diagrams correspond to interactions of particles, interactions which are governed by the terms appearing in the **lagrangian**. The edges correspond to particles moving freely in the **spacetime**. Feynman's idea was that **expectation values** of quantities in quantum theory should be obtained by summing the values of these quantities over all histories, i.e., over all diagrams and for each diagram over all positions in spacetime for the particle interactions at the vertices. Each process is to be counted with weight e^{-S} (in the case when spacetime has a metric of Euclidean signature) or weight e^{iS} (in the case when spacetime has Lorentzian signature), where S is the **action**. This is how **path integrals** came into being. See also **expectation value, correlation function, path integral**.

field

Some local data on spacetime. In classical field theory, a typical example of a field is a section of some fiber bundle over **spacetime**. In quantum theory, an example of a field is an operator-valued distribution on spacetime, or more generally, a linear map from the space of smooth sections of some vector bundle over spacetime to operators. For example, the electromagnetic field strength is an operator-valued 2-form $F_{\mu\nu}$.

See also, **σ-model, classical field, gauge field**.

field strength

Synonym for the curvature of a connection, in the case of a **gauge** field; of a metric, in the case of gravity.

Fock space

If W is a space of one-particle states, then the corresponding Fock space is (an appropriate completion of) the symmetric algebra $\mathrm{Sym}^*(W)$. See also **free field theory**.

free field theory

In the **Hamiltonian framework**, a quantum field theory is free if the quantum Hilbert space (and representation of the **Poincaré group**) is the completed symmetric product of the one-particle Hilbert space. In particular, this means that the **S-matrix** vanishes. On any **spacetime** the **correlation functions** of a free quantum field theory have simple expressions in terms of the inverse of the Laplacian or wave operator.

A classical field theory—defined to be a space of fields together with a subspace of classical solutions endowed with a symplectic structure—is free if the space of fields is linear and the subspace of classical solutions is also linear with a linear symplectic structure. For theories defined by **lagrangians**, this restricts the lagrangian to be quadratic in the fields so that the **Euler-Lagrange equations** are linear. See also **Fock space**.

gauge

The choice of a trivialization or a local trivialization of a principal bundle.

gauge boson (vector boson)

The gauge particle in a gauge theory; it has spin 1 and transforms in the adjoint representation of the **gauge group**. It is massless if the gauge symmetry is unbroken and unconfined and realized in a "Coulomb phase." In nature, the gauge particles are the **photon**, associated with the $U(1)$-gauge theory of electromagnetism (it is massless as electromagnetism is in a Coulomb phase), the W and Z bosons of the weak interactions (which are massive because of a **Higgs mechanism**), and the **gluons** of the strong interactions. See **symmetry breaking**.

gauge field (vector potential)

Synonym for a classical elementary field which is a connection on a principal bundle. The gauge field is also called the vector potential; sometimes, the spatial part of the connection alone is called the vector potential.

gauge group

Synonym for the structure group of a principal bundle.

gauge transformation, change of gauge

Synonym for an automorphism of a principal bundle which covers the identity on the base manifold.

gauging a symmetry

In classical field theory: Suppose that we have a theory with a scalar **elementary field** ϕ: $M \to X$ and a group G which acts on X and preserves the **lagrangian**. Gauging the symmetry means introducing a new elementary field which is a connection on a principal bundle P with structure group G and now taking the elementary field ϕ to be a section of the associated bundle X^P with fiber X. The new lagrangian is *minimally coupled* in the sense that it is derived from the original lagrangian by replacing ordinary derivatives of ϕ by covariant derivatives.

Gaussian integral

On $\mathbb{R}^n$, if B is a positive defined symmetric bilinear form,

$$\int \exp(-B(x,x)/2)dx = \det(B)^{-1/2}(2\pi)^{n/2},$$

and the integral of $\exp(-B(x,x)/2)$ times a polynomial P is given by Wick's formula: it vanishes for P odd, and if $B^\vee$ is the inverse bilinear form, for $\ell_1, \ldots, \ell_{2k}$ linear forms, one has

$$\frac{\int \exp(-B(x,x)/2)\ell_1(x)\cdots\ell_{2k}(x)dx}{\int \exp(-B(x,x)/2)dx} = \sum B^\vee(\ell_{\imath_1},\ell_{\jmath_1})\cdots B^\vee(\ell_{\imath_k},\ell_{\jmath_k}),$$

the sum running over all partitions of $\{1,\ldots,2k\}$ into two-element subsets. Such a partition is conveniently represented by the graph with an edge for each of its two-element subsets. If some ℓ's coincide, the sum is similarly indexed by graphs with vertices the ℓ's and valence at ℓ its multiplicity. For $B = \hbar^{-1}B_0$ and more general P, the same formula, applied formally to the Taylor expansion of P, sometimes provides an asymptotic expansion for $\hbar \to 0$.

In the **fermionic** case, i.e. on $\mathbb{R}^{0|n}$, similar formulas hold. The quadratic function B is now a symplectic form on $\mathbb{R}^n$, and $(2\pi)^{n/2}\det(B)^{-1/2}$ is replaced by the Pfaffian of B.

The Wick's formula is taken as the definition of **perturbative expansions** of quantum field theory, where **correlation functions** are formally written as integrals over infinite dimensional spaces (**path integrals**). The indexing graphs are **Feynman diagrams**. The inverse bilinear form $B^\vee$ is given by a classical Green's function, and the terms corresponding to the Feynman diagrams are given by usually divergent integrals, which have to be **renormalized**.

ghost

Ghosts are **fermions** (or more generally, in the case of local supersymmetry, particles of opposite statistics from the symmetry generators) which are introduced in the **Faddeev-Popov method** of quantization of systems with symmetries and its generalizations. Ghosts violate the **spin-statistics theorem** because they have integral spin (i.e., transform trivially under the center -1 of the **Poincaré group**) but they are fermions in the sense that they are odd states in the (super) state space. See also **Faddeev-Popov method, BRST**.

gluon

The **gauge bosons** of the $SU(3)$-gauge theory of strong interactions; and sometimes, by extension, the gauge bosons of any **gauge theory** that is believed to behave similarly, namely a theory with **confinement** and no **Higgs mechanism**. Because of confinement gluons don't really exist as isolated particles but are important in describing the short distance structure. See **gauge boson**.

gravitational field

Synonym for a metric; in some contexts it is considered fixed, that is to say a **background field**, in others it is a **dynamical variable**, i.e., is allowed to vary and in classical theory is determined by the equations of motion, whereas in quantum theory (which would be a quantum theory of gravity) it is one of the variables of integration.

Hamiltonian, Hamiltonian framework

In a classical field theory, the Hamiltonian framework consists of spacetime which is split as $M \times \mathbb{R}$ with M being space and $\mathbb{R}$ being time. Associated to this decomposition, there is a distinguished real nonnegative function, the Hamiltonian, on the space of classical solutions, given by an integral of a local functional (energy density) over a spatial cycle. In a quantum theory, the Hamiltonian framework views the quantum field theory as an algebra of quantum observables derived from a theory on a spacetime manifold which is split as above. The Hamiltonian operator is a distinguished element of this algebra which is the quantum operator corresponding to the classical Hamiltonian. In every realization of the quantum theory (i.e. representation of this algebra) this element defines a selfadjoint operator in the Hilbert space of states which is also called the Hamiltonian. It is required that the Hamiltonian operator be nonnegative or at least bounded from below. In a theory defined by a **lagrangian**, the Hamiltonian is derived from the lagrangian by a certain general formula.

helicity

see **particle**

Higgs mechanism

In perturbative quantum field theory, the "spontaneous breaking" of gauge symmetry, in which a massless **gauge boson** and a massless scalar field combine to form a massive gauge boson.

higher derivative term

A term in a **lagrangian** which depends on higher than first derivatives of the **elementary fields** or is more than quadratic in first derivatives. Examples: For a real-valued scalar field ϕ the terms $|\nabla d\phi|^2$ or $|d\phi|^4$. In gauge theory with connection A and curvature F_A the terms $|\nabla F_A|^2$ or $|F_A|^4$.

infrared (IR) limit

Synonym for **low-energy limit.**

instanton

A solution of the equations of motion of a Euclidean-signature field theory which is localized in all directions of spacetime. Instanton contributions to functional integrals are often important because they break symmetries that are present in perturbation theory or otherwise contribute important quantum effects that are not seen perturbatively.

integrating out

The essence of passing from microscopic (i.e., valid at all energy scales) to **effective theories** (valid only at low energy scales) is integrating out (=averaging over) degrees of freedom which are irrelevant macroscopically. This is illustrated by the Wilson **renormalization** group picture, in which high frequency Fourier modes of fields are integrated out to obtain the effective theory at a lower scale. The expression "integrating out" is also used when one needs to refer to the Fubini's theorem for path integrals, i.e. to integrate first with respect to one set of fields and then with respect to another. If the first set of fields occurs in the **lagrangian** quadratically with no derivatives and with coefficient 1, it is called a "dummy" or "auxiliary" field and is integrated out just by setting it to be equal to its extremal value for the action. Otherwise, after the first integration, one usually obtains a nonlocal answer (a determinant if the first integration was Gaussian). See also **effective lagrangian.**

interactions

The effects which various particles in a theory have upon each other. Fundamentally, interactions are measured by means of S-matrices; in a **lagrangian** theory, each interaction typically has an associated term in the the lagrangian. The simplest such terms are *polynomial interactions*, which involve products of fields but not derivatives.

kinetic term

In classical field theory: A kinetic energy term in a **lagrangian.** For a classical particle $x(t)$ it is $\frac{1}{2}|dx(t)/dt|^2$. For a classical **bosonic** field it is half the norm squared of the time derivative of the field. Often "kinetic term" refers to the Lorentz invariant combination of first derivatives, so for a scalar field to $\frac{1}{2}|d\phi|^2$ rather than just $\frac{1}{2}|\partial_t\phi|^2$. Similarly, for a fermionic (**spinor** or **Rarita-Schwinger**) field the kinetic term is the Lorentz invariant term which is quadratic in the field and involves a single first derivative. In supersymmetric classical mechanics with odd path $\psi(t)$ the kinetic term is $\frac{1}{2}\langle\psi, d\psi(t)/dt\rangle$; for fields it involves the Dirac operator.

lagrangian, lagrangian density, lagrangian framework

In the lagrangian framework a field theory is described by giving a lagrangian density on **spacetime** written in terms of the **elementary fields** of a theory. The integral of this density is called the **action** (or sometimes the lagrangian)

of the theory. Most (but not all) quantum field theories are defined by giving lagrangian densities.

The *classical solutions* of the theory are the field configurations which satisfy the associated PDE's, which are **Euler-Lagrange equations**.

The lagrangian also describes a family of quantum field theories which arise by **renormalization**. In the lagrangian framework one wishes to compute **correlation functions** of the quantum theory, by doing **path integrals** over the space of all elementary field configurations. These correlation functions are the fundamental objects of the theory. They are usually computed perturbatively, using **Feynman diagrams**. Sometimes non-perturbative effects such as **instanton** corrections can also be computed.

It is possible to define theories and do computations in the lagrangian framework when the underlying spacetime manifold is not split as space $\times$ time. Examples of interest in both physics and mathematics exist for spacetime being a compact manifold with a metric of Euclidean signature. But unless the manifold of spacetime is split, there is no expectation of creating an algebra of operators with a Hamiltonian. Compare with **Hamiltonian framework**. See also **action, Euler-Lagrange equations, path integrals**.

large N limit

Some theories with $U(N)$, $SO(N)$, or $Sp(N)$ symmetry, and a number of fields proportional to N (or sometimes N^2) simplify in the large N limit. When it is possible to solve for the large N limit and construct a $1/N$ expansion around it, one sometimes gets important information that goes far beyond what one can see in perturbation theory.

lightcone, lightcone gauge

The locally closed subset of the vector space underlying Minkowski space consisting of all nonzero null vectors, i.e., vectors of length zero in the Lorentzian metric. Physically, these are velocity vectors of particles moving at the speed of light. Vectors inside the lightcone (i.e., vectors of positive length) are called *time-like*, those outside are called *space-like*, and those on the lightcone are called *light-like*.

A lightcone gauge is the choice of a frame of reference (coordinate system) in a relativistic theory adapted to a particle moving along the lightcone, i.e., moving at the speed of light. See also **Minkowski space**.

little group

The little group is the reductive part of the stabilizer subgroup inside the **Poincaré group** of a time-like or light-like nonzero vector in the dual space to the vector space underlying **Minkowski space**. In n dimensions the little group of a time-like dual vector is isomorphic to $\mathrm{Spin}(n-1)$; the little group of a light-like dual vector is isomorphic to $\mathrm{Spin}(n-2)$.

low energy (behavior, limit, effective Lagrangian, effective theory)

At low energies (i.e., in the **infrared limit**) a quantum field theory may be described approximately by a very different **effective theory** (often described by an **effective Lagrangian**) which ignores all high-energy effects.

magnetic charge

In classical field theory: In a $U(1)$-gauge theory on **Minkowski space** the integral of (a suitable multiple of) the curvature over a sphere at spatial infinity. When spacetime is a curved n-manifold an analogous quantity can be defined by taking the integral over any over any 2-cycle. More generally, in a gauge theory with gauge group G, there is a magnetic charge defined for each trace on the Lie algebra of G as the integral of (a suitable multiple of) the curvature.

In quantum field theory: The quantum operators which correspond to the classical observables described above.

Majorana spinor

See **spinor**.

mass

In classical mechanics mass is a basic characteristic of a particle appearing in Newton's equations of motion. In the classical theory of relativity, the mass of a particle characterizes the energy of this particle at rest. In QFT the mass of a **particle** characterizes the orbit in the **momentum** space which is the support of the restriction of the corresponding representation of the Poincaré group on the subgroup of translations. One says that a quantum field theory has a *mass gap* if there is a gap between the spectrum of the corresponding **Hamiltonian** on the orthogonal complement to the vacuum and 0. In the case when the classical approximation is applicable the masses of the theory are related to the eigenvalues of the Hessian of the classical potential at the minimum.

Maxwell theory

Synonym for a $U(1)$ gauge theory.

minimal coupling

See **gauging a symmetry**.

Minkowski space, super Minkowski space

Minkowski space is the basic underlying **spacetime** for relativistic quantum field theory. Minkowski space M^n of dimension n is an affine space with a translationally invariant Lorentzian metric of signature $(1, n-1)$. The underlying vector space V is n-dimensional with coordinates $(x_0, \ldots, x_{n-1})$ and the Lorentz metric on M^n is induced from the quadratic form

$$q(x_0, x_1, \ldots, x_{n-1}) = c^2 x_0^2 - \sum_{i=1}^{n-1} x_i^2$$

on V, where c is the speed of light. One usually chooses units so that $c = 1$.

Super Minkowski space is the basic spacetime for relativistic super quantum field theory. It is the supermanifold which is an odd vector bundle over ordinary Minkowski space. The vector bundle is some number of copies of the basic real spin bundle of Minkowski space. Sometimes the terminology "$N = k$ super Minkowski space of dimension n" refers to the superspace made from k copies of the basic spin bundle over M^n, though the usage is not uniform. (For example, in dimensions congruent to 2,6 modulo 8 there are two inequivalent minimal real spin representations.) See also **Poincaré group**.

moduli space of vacua (classical, quantum)

The classical moduli space of vacua is the set of vacuum states, i.e., solutions to the field equations (equations of motion); the quantum moduli space of vacua is the target space of a sigma model which describes an **effective, low-energy theory** which the original theory approaches in the **infrared limit**. It is typically a deformation of the classical moduli space, or more precisely a deformation of the zeros of a potential function on the classical moduli space generated by **quantum corrections**. See also **quantum corrections, vacuum.**

momentum

The vector space V^* dual to the vector space V underlying **Minkowski space** is called *momentum space.* The vector space V comes equipped with coordinates (spacetime coordinates). Momenta are the dual coordinates in momentum space. When one computes amplitudes of **Feynman diagrams** one usually works in the momentum space and computes the Fourier transform of **correlation functions** as an integral over the momentum space. In such computations one often has to introduce a momentum **scale** and to approximate the integral by integrating over the region where the momentum is bounded above and below.

monopole

A monopole is a particle carrying magnetic charge. The integral of the magnetic field (that is, of the curvature of the $U(1)$-gauge field of electromagnetism) over a sphere surrounding the monopole is nonzero (and proportional to the first Chern class of the $U(1)$ bundle restricted to this sphere). When a $U(1)$-gauge theory arises as the **low energy limit** of a nonabelian gauge theory with a semisimple gauge group, monopoles arise as **solitons**.

multiplet

See **superfield**.

Neveu-Schwarz sector

The subspace of the Hilbert space of a string theory (in the Neveu-Schwarz-Ramond formulation) for which the ground state is a boson.

Noether (current, charge)

Given a **lagrangian**, the space of classical solutions of the **Euler-Lagrange** equations has a natural symplectic structure. Thus, any infinitesimal symmetry α has an associated closed one-form $\eta_\alpha = df_\alpha$. The map $\alpha \to f_\alpha$ defines the moment map from the space of classical solutions to the dual space of a central extension of the Lie algebra of symmetries. This moment map is the Noether charge and f_α is the Noether charge of α.

From the lagrangian and its symmetry one can also construct a codimension-one form on the space of classical solutions called the Noether current with the property that the integral of this current over a space-like hypersurface of **spacetime** is the Noether charge.

In quantum field theory: The Noether current is the codimension-one form valued operator corresponding to the classical Noether current. Its integral over a space-like slice is the quantum operator which is the associated quantum Noether charge. See also **current**.

non-perturbative

A non-perturbative effect is an effect which cannot be seen from the **perturbation expansion** of a field theory. Such effects usually involve functions, like e^{-1/g^2}, of the **coupling** which are nonzero but have zero Taylor series. Examples are **chiral** symmetry breaking, dynamical mass generation, **instanton** effects, etc. A non-perturbative result or description in quantum field theory is a result or description which is valid beyond a **perturbation expansion**. Getting such results can involve doing perturbation expansions not only around constant field configurations but also around instantons.

observable

In classical theory, an observable is a function on the space of states. In quantum theory, an observable is an operator in the quantum space of states. A local observable is a local functional (operator).

on-shell, off-shell

On-shell is a term used in classical and quantum field theories in two quite different ways. In the classical framework (and in the cases when we apply the **semi-classical** approximation for the evaluation of the **path integral**) it signifies the restriction of our interest to field configurations which are solutions of the **Euler-Lagrange equations**. In quantum theories the term "on-shell" signifies the restriction of our attention to the subset of the **momentum** space which corresponds to actual particles. The term "off-shell" means that we are not restricting to particles and fields which are on-shell.

operator product expansion (OPE)

In quantum mechanics, local operators form an honest algebra. This is not the case in quantum field theory when **spacetime** is of dimension at least two; because of **ultraviolet** divergences the product of two local operators at the same point does not make sense. The only thing that makes sense is

to consider the product of two local operators located at nearby points x, y, and then decompose this product as a linear combination of local operators at x. The coefficients of this expansion are functions of $y - x$. This expansion is called the operator product expansion. Usually it makes sense only as an asymptotic expansion (for $y \to x$). However, in 2-dimensional conformal field theory it makes sense exactly, which gave rise to the theory of vertex operator algebras.

operator

A quantum theory is most generally defined as a kind of algebra, the algebra of observables, with a distinguished element called the **Hamiltonian.** A **realization** of the quantum theory is a representation of this algebra in a Hilbert space. Thus algebra elements are sent to operators in the Hilbert space (usually unbounded), which are called quantum operators (observables). A local operator is an operator obtained by quantizing a local functional in a classical theory. Mathematically a local operator is an operator-valued generalized function (or, more generally, tensor field) on the **spacetime**.

particle (quantum, relativistic)

A classical physical system of n particles is described by a system of **Euler-Lagrange equations** on the n-th power of the **spacetime** V. In this case we can think about the system as a law of motion for n particles in V. A quantum mechanical system is described by a **Hamiltonian** $\mathcal{H}$ acting in a space W of symmetric [in the **bosonic** case] or antisymmetric [in the **fermionic** case] functions of n points of space. Often the Hamiltonian H is a sum

$$H = \sum_{1 \leqq i \leqq n} \mathrm{Id}^{i-1} \otimes H_1 \otimes \mathrm{Id}^{n-i} + I$$

where H_1 is an operator on the space of functions on V and the interaction part I is given by an operator on W which goes to zero when the distance between points in V goes to infinity. In this case we say that our system is a system of n interacting quantum particles where each one is described by the operator H_1.

In a QFT a particle is an irreducible subrepresentation of the Poincaré group P on the space $\mathcal{H}$ of the theory. Any unitary irreducible representation ρ of P is described by: a) the support $\mathcal{O}$ in the **momentum** space V^* of the restriction of ρ on the subgroup of translations; and b) a unitary representation χ of $H \ltimes V$, where $H \subset L$ is the stabilizer in the Lorentz group of a point in $\mathcal{O}$. More precisely, the representation ρ is induced from H by χ. The Levi component O of H is a compact orthogonal group. In all interesting cases χ is finite-dimensional and therefore (since it is unitary) it is trivial on the unipotent radical of H. Therefore we can consider χ as a representation of O. The orbit $\mathcal{O}$ is a connected component of the subset of V defined by the equation $v^2 = m^2$, where the number $m \geq 0$ is called the **mass** of the particle. An irreducible subrepresentation of the complexification of χ of O is called the **helicity** of the particle. Since O is a compact orthogonal group there exists a natural equivalence class of "minimal" embeddings of $U(1) = SO(2)$ into O.

The restriction of χ on $U(1)$ is a direct sum of one-dimensional representations which are powers of the standard representation of $U(1)$. The highest power which appears in the decomposition is called the **spin** of a particle.

partition function

The partition function Z of a quantum theory is given, formally at least, by

$$Z = \int \mathcal{D}\phi e^{iS(\phi)},$$

where $\mathcal{D}$ is a measure on the space of all field configurations and S is the **action**. (In the case when the underlying manifold has Euclidean signature the integrand is $e^{-S(\phi)}$.) Thus, the partition function Z is the value of the **path integral** of the quantum theory with no insertions of local operators.

The name comes from statistical mechanics where the partition function is given by

$$Z = \sum_{\text{states}} e^{-E/kT}$$

where E is the energy of the state, T is the temperature and k is Boltzmann's constant.

To obtain a value for Z, even in quantum mechanics, requires **renormalization**, so in an isolated theory the partition function often makes no sense and is usually normalized to 1. However, in a family of theories it makes sense to talk of the partition function as a function of the family parameters, defined up to scaling, since one can define its logarithmic derivative. See also **expectation value, correlation function**.

path integral

The path integral (also called Feynman integral or functional integral) is a way to write solutions of the Schrödinger equation in the form of an integral (called the Feynman integral) over the space of configurations of classical **elementary fields**. The integration is over a complex-valued measure and so far there is no way to define this measure rigorously. If one performs the **Wick rotation** to the Euclidean picture one obtains a presentation of solutions of the heat equation in the form of a functional integral over a positive-valued measure. Such a presentation (known as the Feynman-Kac formula) is rigorous and is very useful in different areas of quantum mechanics. In the case of QFT there exists a presentation of **correlation functions** in terms of functional integrals. In the case of physically interesting theories one does not know how to rigorously define even the Euclidean form of the functional integral. In spite of this difficulty, the possibility of formal analysis of such a presentation makes the functional integral into one of the most powerful tools of contemporary QFT. See also **expectation value, correlation function, partition function**.

perturbation theory, expansion

Perturbation theory is an approach to quantum field theory which treats it as a perturbation of some **free theory**. This is one of the main methods of studying quantum theories. If a theory is given by a **lagrangian**, then perturbation

theory amounts to treating cubic and higher terms in the Lagrangian as small perturbations of the free theory—the theory given by the purely quadratic terms in the lagrangian—and expanding the **path integral** in an infinite asymptotic series of **Feynman diagrams**. This perturbation expansion is the stationary phase expansion of the functional integral near a critical point of the action.

photon

In physics the photon is the particle of light, i.e., the carrier of the electromagnetic field. It is massless and has no charge and spin one. More generally, the word is used to describe the gauge particle in a $U(1)$ gauge theory (or more generally, an abelian gauge theory); it has spin 1 and is invariant under $U(1)$ (i.e., has charge 0).

Poincaré algebra, super Poincaré algebra

The Poincaré algebra is the Lie algebra of the **Poincaré group**. A super Poincaré algebra is a real super Lie algebra whose underlying (bosonic) Lie algebra is the sum of the Poincaré algebra and an abelian Lie algebra and whose odd part is a real spinorial representation of the Lorentz group.

Poincaré group, super Poincaré group

The Poincaré group is the spin double covering of the connected component of the isometry group of **Minkowski space**. It is a semi-direct product of the spin double covering of the Lorentz group with normal subgroup consisting of translations. It acts transitively on **Minkowski space** with the stabilizer of a point being the Lorentz group.

A super Poincaré group is a super Lie group whose underlying ordinary Lie group is the Poincaré group and whose super Lie algebra is a super Poincaré algebra. It is the spin double covering of the connected component of the identity of the isometry group of **super Minkowski space**. It acts transitively on super Minkowski space respecting the odd distribution of left-invariant super vector fields. This supergroup has central extensions which are also sometimes called super Poincaré groups. See also **Minkowski space, lightcone, supersymmetry**.

potential

In classical field theory: For a classical particle $x(t)$ moving in a manifold X the potential is a function $V\colon X \to \mathbb{R}$ and the potential term in the **lagrangian** is $-V\big(x(t)\big)$. In a field theory, the scalar field takes values in a manifold X, and the potential is again a real-valued function on X which appears in the lagrangian in the same way. Spatial derivatives of fields also give potential energy terms in a lagrangian.

In quantum field theory: see **effective potential**.

primary field

Primary fields are fields in a two-dimensional conformal QFT which behave under the action of the Virasoro algebra as highest weight vectors. Since unitary representations of the Virasoro algebra are generated by highest weight vectors the study of any two-dimensional conformal QFT can be reduced to the study of primary fields.

propagator

A bilinear form (or linear operator) which is put on the edges of **Feynman diagrams** in **perturbation theory** (the inverse of the quadratic part of the **lagrangian**). It characterizes the motion of a free particle; namely, it is the 2-point function of a **free field**. The exact propagator of a quantum field is the 2-point function of this field. A cutoff propagator is a smeared (smoothed) propagator which is used to regularize divergences in amplitudes of Feynman diagrams.

quantum field

A quantum field is a local operator in a quantum field theory. This term is also used for the integration variable in the **path integral**.

quantum corrections

Modifications which must be made to the classical version of a theory in order to obtain the quantum theory. There can be quantum corrections to a **lagrangian**, to a **moduli space of vacua** and its metric, to **correlation functions**, etc.

quark

Elementary fermion fields that transform in the fundamental representation of the $SU(3)$ gauge group that governs the strong interactions are called quarks.

R-symmetry

Lie algebra, or Lie group, of symmetries normalizing the **super Poincaré group**, acting trivially on the Poincaré group, and acting faithfully on the odd part of the Lie algebra of the super Poincaré group.

Ramond sector

The subspace of the Hilbert space of a string theory (in the Neveu-Schwarz-Ramond formulation) for which the ground state is a **fermion**.

Rarita-Schwinger field

In classical *supergravity* theories in n dimensions: Let V be the standard n-dimensional representation of the spin group and S a spinor representation. Then a Rarita-Schwinger field takes values in the highest irreducible component of $S \otimes V$.

realization of a quantum field theory

A quantum field theory is most generally defined as a sort of algebra, called the algebra of observables, with a distinguished element called the **Hamiltonian**. A realization (=solution) of this quantum theory is a representation of this algebra in a Hilbert space in which the Hamiltonian goes to a nonnegative operator. The same initial theory (which is also referred to as the underlying, the fundamental, or the microscopic theory) can have several realizations, since the same algebra can have different representations.

In many cases, realizations of a quantum theory are in 1-1 correspondence with **classical vacuum states**, i.e., classical states of zero energy. In this case, for every classical vacuum state there is a quantum vacuum state localized near it. The space of vacuum states usually has a natural metric, and the **low energy effective theory** in such a situation is often the **σ-model** with target space being the space of classical vacuum states. Even in the case when the moduli spaces of quantum vacuum states and of classical vacuum states are diffeomorphic, there can be **quantum corrections** to the metric, so that the natural metrics are different. The low energy approximation is then a σ-model into the moduli space with the quantum metric.

There can be more serious differences between the classical and quantum moduli spaces generated by quantum corrections. For example, quantum mechanically an **effective potential** can be generated, so that the moduli space of quantum vacua is identified with the zeros of this effective potential on the classical moduli space (with the quantum corrections to the metric).

renormalization

Renormalization is a process of making sense of **ultraviolet** and **infrared** divergences arising in **Feynman diagram** integrals in quantum field theory. This is done in two steps: regularization and renormalization. Regularization consists in introducing a **cutoff** which makes the integrals converge but depend on the cutoff. There are many types of cutoff—momentum cutoff, dimensional regularization, etc. The integrals will usually tend to infinity as the cutoff goes to infinity. This is why one needs the second step—renormalization. This step consists in making finitely many parameters of the **lagrangian** depend on the regularization, so that they go to infinity as the regularization goes to infinity, but all the **correlation functions** remain finite. This is possible to do only for renormalizable theories, and not in a unique way. Various ways of doing it are called renormalization schemes.

renormalization group flow, equation

The Wilson renormalization group flow is the flow on the product of the space of all **lagrangians** on a fixed set of elementary fields (infinite dimensional) with the space of scales (essentially $\mathbb{R}^+$), whose flow-lines $\mathcal{L}(\Lambda)$ are defined by the condition that the **correlation functions** for $\mathcal{L}(\Lambda)$, computed with the propagator cut off at Λ, do not depend on Λ. The flow goes from high to low momenta. In a renormalizable case, the flow has a finite dimensional attracting manifold, which is the space of renormalizable theories.

A more frequently used version of this idea is the ordinary renormalization group equation (flow), which is a vector field (flow) on the subspace of renormalizable lagrangians times the space of scales. This flow is defined if one fixes a renormalization prescription which depends on a particular scale μ. Then the flow is a lifting of the dilation flow on the space of scales, and its flowlines $\mathcal{L}(\mu)$ are defined by the condition that the correlation functions of $\mathcal{L}(\mu)$ computed with the prescription at μ don't depend on μ.

S-matrix

See **Scattering**.

scale

A scale of a physical observable in a particular theory is a range of values which it normally takes. Usually this scale is characterized by a typical value of the observable. For instance, the scale of length in fluid mechanics is 1 cm and in string theory is 10^{-33} cm. In quantum field theory, one often says that a certain description of a system is valid at a certain range of scales. For example, quantum electrodynamics is a good theory at low energies, but at higher energies it should be treated as a part of an asymptotically free, nonabelian gauge theory. In the Wilsonian picture of the **renormalization group flow**, the space of theories is a fiber bundle over the space of scales, and the renormalization group flow is a lifting of the dilation flow from the base to the total space of this bundle.

scattering

Scattering in classical and quantum mechanics is the process when a group of particles, widely separated in the far past, interact for a short period of time and become widely separated in the far future. Since neither classical nor quantum mechanics allows creation of new particles the number of particles is preserved. The new feature of QFT is the necessary appearance of the process of the creation and the annihilation of particles. Therefore in QFT the number of particles is not preserved in the process of a collision. In the case of a QFT without zero-**mass** particles in at least four dimensions there exist two canonical isometric embeddings Φ^{in}, Φ^{out} from the space $\mathcal{H}_0$ of an appropriate **free field** theory into the space $\mathcal{H}$ of this QFT which associate with any state v of a free field theory the states $\Phi^{\text{in}}(v), \Phi^{\text{out}}(v) \in \mathcal{H}$ which behave as v in the far past, respectively in the far future. Moreover the images of Φ^{in} and Φ^{out} coincide. The induced unitary transformation $S = (\Phi^{\text{in}})^{-1}\Phi^{\text{out}} \in \text{Aut}(\mathcal{H}_0)$ is called **the S-matrix**. The S-matrix is equal to the residues of the **correlation functions** on the **mass-shell**.

Scattering amplitudes (and masses) are the only quantities that can actually be measured in a lab. Thus, these form the point of contact between quantum field theories and experiment.

semi-classical approximation

Any approximation in which one extracts as much information as one can from classical mechanics (sometimes from an effective classical system described by

an **effective potential**) about a quantum system is called a semi-classical approximation. In using this term, it is usually assumed that one knows how to quantize a quadratic **lagrangian**, and then a semi-classical approximation is made by expanding some classical system around a classical solution and making an approximation by quantizing only the quadratic terms of this expansion.

soliton

In classical field theory: A static (time-invariant) and stable solution, localized in space, to the classical field equations which is not a global minimum of energy but has energy exceeding the energy of the global minimum (the vacuum) by only a finite amount. Often the space of all static field configurations of finite energy is not connected, and a *topological soliton* is a minimum of energy in a component which contains no global minima. In **supersymmetric** theories some solitons are **BPS states**.

In quantum field theory: A quantum state which corresponds to a classical soliton.

In four dimensions, the most important solitons are **monopoles**.

solution of a theory

Same as **realization of a theory**.

spacetime

The underlying manifold on which the **elementary fields** and more general composite local fields exist and over which the **lagrangian density** is integrated; in physical examples it carries a Minkowski metric with a spatial slice being a codimension-one submanifold on which the metric is *negative* definite. Often in mathematical examples spacetime carries a Riemannian metric. See also **Minkowski space**.

spin-statistics theorem (in space time of dimension ≥ 3)**:**

The spin-statistics theorem (which holds under some reasonable hypotheses) says that in a quantum field theory the mod 2 grading of the super Hilbert space of the theory is given by the action of the 2-element center of the spin group. See also **boson, fermion** and compare with **ghosts**.

spinor (Weyl spinor, Majorana spinor, Dirac spinor)

In even dimensions, the space of Dirac spinors is an irreducible complex module over the Clifford algebra. The Lie algebra of the orthogonal group is contained in the even part of the Clifford algebra, and the space of Dirac spinors is the sum of two irreducible representations of the spin group: the spaces of Weyl spinors (or semi-spinors) of both **chiralities**, on which the chirality matrix "γ_5", a product in the Clifford algebra of the elements of an orthogonal basis, acts with opposite eigenvalues. **Chiral** refers to the fact that elements of determinant -1 of the orthogonal group transform γ_5 into $-\gamma_5$, and acting

by outer automorphisms on the spin group, permute the two kinds of Weyl spinors.

Majorana is an adjective, meaning "in some real form". Example: Majorana Weyl spinors exist if the signature is congruent to 0 modulo 8.

Γ-matrices are the expression in some basis of morphisms $S \otimes S \to V$ or $V \otimes S \to S$, for V the vector and S a spinor representation of the Lorentz group. In the physics literature, the existence of Majorana spinors or of some pairings is often expressed by the existence of a basis in which Γ-matrices have suitable reality/symmetry/positivity properties. When referring to morphisms $V \otimes S \to S$ giving S the structure of a module over the Clifford algebra, Γ^μ can be thought of as the element of the Clifford algebra corresponding to the basis vector e_μ, and $\Gamma^{\mu_1 \cdots \mu_p}$ is the antisymmetrized product of the Γ^{μ_i} in the Clifford algebra.

stress tensor

See **energy-momentum tensor**.

super Minkowski space

See **Minkowski space**.

superfield

See **supersymmetry**.

supergravity

Supergravity is an extension of general relativity to incorporate **supersymmetry**. Just as **Minkowski space** is a maximally symmetric solution of the equations of general relativity and has **Poincaré** symmetry, in a supergravity theory (if supersymmetry is not spontaneously broken) the maximally symmetric solution is invariant under a supergroup whose bosonic part is the Poincaré group. Supergravity theories can be described as ordinary theories containing fields of spin ≤ 2, the spin 2 field being the fluctuation in the metric tensor.

Some supergravity theories are constructed in a supermanifold formulation, in terms of differential geometry in superspace. Other supergravity theories are constructed "by hand," in terms of ordinary **component fields**. Generally, superspace constructions exist for the models in low dimensions and with a relatively small number of supersymmetries. The maximal models—dimension up to 11, and number of supersymmetries up to 32—are rather exceptional structures. Even when a superspace **lagrangian** does not exist, it is possible to describe solutions of the classical equations of motion of supergravity in terms of certain differential-geometric structures in superspace. The exceptional supergravity theories with many **supercharges** arise as **low energy limits** of string theory. It is believed that supergravity (like general relativity) cannot be quantized by itself and requires this embedding in string theory.

supersymmetry (SUSY)

Action of a **super Poincaré group** on a theory. One speaks of $N = i$ *supersymmetry* if the odd part of the super Poincaré group $\mathfrak{q}$ which is acting is the sum of i irreducible real representations of the Spin group. This convention is somewhat refined in dimension $d = 4k + 2$, where there are inequivalent irreducible real spin representations of $SO(1, d-1)$, so that one can refine the classification and talk of (i, j) supersymmetry.

In a classical field theory, supersymmetry (action of a **super Poincaré group** on the super-space of extremals) can sometimes be made *manifest* by expressing the fields as component fields of a superfield and the action as an integral (in the sense of Berezin) over **super Minkowski space**. **Superfields** are objects (functions, connections,...) over superspace, possibly subject to constraints. **Component fields** are objects over the reduced space (with no constraints). A collection of component fields carries the same information as a superfield. For functions: the restriction to the reduced space of the function and a suitable set of derivatives in the odd direction. For gauge fields: the restriction of the torsor (or bundle) with connection to the reduced space, and suitable curvature information in the odd direction to recover the torsor (or bundle) with connection on superspace up to gauge equivalence. *Auxiliary fields* are component fields which, using the **Euler Lagrange equations** for the **action** functional, can be expressed locally (algebraically) in terms of other component fields. Sometimes, by introducing auxiliary fields, one can transform an **on-shell supersymmetry** (action on the superspace of extremals) into an **off-shell supersymmetry** (action on the superspace of all fields). Without them, one only has at best an action of generators of the Lie algebra of the super Poincaré group on the superspace of all fields which *closes* **on shell**: the needed bracket relations hold only on the space of extremals.

Quantum theory: Let $\mathcal{H}$ be a Hilbert space on which the Poincaré group P acts, with positivity of energy. The space $\mathcal{H}$ is mod 2 graded, by the action of the 2-element center of P (see the **spin-statistics theorem**). If the action extends to an action of the super Poincaré group sP, the mass spectra of $\mathcal{H}^+$ and $\mathcal{H}^-$ agree. Conversely, such an agreement is often taken as evidence for existence of an action of sP. Irreducible representations of sP decompose into a sum of irreducible representations of P forming a *supermultiplet*. Such decompositions often mimic decompositions of superfields into component fields, in classical supersymmetric field theory.

symmetry breaking

In classical field theory: If a Lie group H acts as global symmetries of a classical field theory, then it can be *explicitly broken* by adding a term to the **lagrangian** which is not H-invariant. A solution to the classical field equations *spontaneously breaks* the symmetry group down to the stabilizer subgroup in H of the solution. If the theory has a **gauge field** with **gauge group** G, then G acts on the manifold X where the scalar field takes its values. A **vacuum** solution in which the scalar field takes constant value $x \in X$ is said to *spontaneously break* the gauge group to the subgroup $G' \subset G$ which stabilizes x. This means that the fluctuations around that vacuum include a

massless gauge field with gauge group G'; other components of the original gauge field become massive due to the **Higgs mechanism**.

In quantum field theory: A symmetry of the quantum theory is an automorphism of the algebra of quantum **operators**. A symmetry of a **realization** of a theory is a unitary transformation of the quantum Hilbert space which commutes with all of the quantum operators. A symmetry of a classical limit may not be a symmetry of the quantum theory if there is an **anomaly**. A symmetry group of the quantum theory can be *spontaneously broken* to a subgroup in a particular realization.

't Hooft loop

't Hooft loop is an operator in abelian gauge theory which maps to a **Wilson loop** under the abelian duality transformation.

tachyon

In classical physics, a tachyon is a particle that travels faster than light. Quantum mechanically, a tachyon would be a particle of negative mass squared. However, tachyons never arise as excitations of stable vacuum states. The reason for this is that, for example, if in expanding around a critical point one finds a bosonic field of negative mass squared, it means that the critical point in question is not a minimum of the energy. For fermions, instead, one finds that as long as the Lagrangian is real, the mass is real and hence the mass squared is positive.

An important example of a significant but not completely well-defined theory that has a tachyon in perturbation theory is the bosonic string theory. The bosonic string theory is not really well-defined because of the tachyon, but is a useful prototype in constructing string theories that are well-defined.

topological term

In classical field theory, a topological term is a term in the **lagrangian** which is locally a differential form representing a "secondary cohomology class." An example in gauge theory is a Chern-Simons term. An example for a σ-model is a **Wess-Zumino term**: for the field $\phi\colon M \to X$ it is locally described as $\phi^*\alpha$, where α is a local form on X whose differential $\beta = d\alpha$ represents a nontrivial cohomology class.

Sometimes "topological term" is used to describe a **θ-term**.

ultraviolet (UV)

The high-energy, short-wavelength end of the momentum spectrum; this is where difficulties of renormalization of quantum field theories typically arise.

vacuum

In classical field theory: A static (time-invariant) field configuration which is a global minimum of energy. In a **Poincaré**-invariant field theory on **Minkowski space** such a configuration is usually fixed under Poincaré transformations, and in any case is assumed to be so. Thus, in a vacuum field configuration a

scalar field is constant, a spinor field vanishes, and a connection (gauge field) is equivalent to the trivial connection. The **classical moduli space of vacua** is the manifold of constant values of the scalar fields. It is the manifold of minima of the **potential energy**, divided by the action of the **gauge group** in a gauge theory.

Quantum field theory: A vector (state) in the quantum Hilbert space of zero energy. Such a state is invariant under the Poincaré group. Sometimes the term is also used as a synonym for **realization of a quantum theory**. The **quantum moduli space** is the set of all possible realizations of a quantum theory.

(Perturbative) string theory: A conformal field theory. **Correlation functions** in the conformal field theory are then used to define string amplitudes.

Ward identities

Ward identities are identities in perturbative quantum gauge theories which ensure that the **renormalization** scheme does not destroy the gauge invariance.

wavelength

In relativistic quantum theories mass, energy, and distance are all related by the fundamental constants of the theory (the speed of light and Planck's constant). Thus, one often refers to energy **scales** by wavelength. Short wavelength (**UV** or **ultraviolet**) corresponds to high energy and long wavelength (**IR** or **infrared**) to low energy. See also **low energy** and **effective action**.

Wess-Zumino term

See **topological term**.

Weyl spinor

See **spinor**.

Wick rotation

Wick rotation is analytic continuation from real to imaginary time in quantum mechanics and quantum field theory. The existence of such a continuation is a consequence of the locality, Lorentz invariance of QFT, and positivity of energy. Wick rotation replaces the the problem of constructing a QFT by the problem of constructing a measure on a space of Euclidean field configurations. The latter is sometimes more tractable by rigorous methods. In perturbative QFT the Wick rotation replaces the finite dimensional integrals with pole singularities in **Feynman diagrams** by nonsingular Euclidean integrals.

Wilson line (loop)

The Wilson line (or loop) functional is a functional in gauge theory which assigns to a connection the trace of its holonomy along a fixed curve in spacetime. This functional can be quantized and the obtained quantum operator is called the Wilson line operator.

worldline

The curve traced out in spacetime by a (spatially localized) particle.

worldsheet

The surface swept out in spacetime by a (spatial) string.

Yang-Mills theory

Synonym for a nonabelian **gauge theory**.

Yukawa coupling

The coefficient of a cubic interaction term in a **lagrangian** of the form $\phi\psi\psi'$, where ϕ is an **elementary bosonic field** and ψ and ψ' are **elementary fermionic fields**.

Part 1

Classical Fields and Supersymmetry

Notes on Supersymmetry (following Joseph Bernstein)

Pierre Deligne and John W. Morgan

Introduction

These notes are based on the lectures given by Joseph Bernstein in the fall of 1996 at the Institute for Advanced Study. They contain both more and less than those lectures. Less: representations of super Poincaré groups, the Coleman-Mandula theorem and examples of super symmetric Lagrangians that were contained in Bernstein's lectures have been incorporated into other texts of this volume. More: more details have been given, and we have paid special attention to having precise and coherent sign conventions.

The purpose of Bernstein's lectures was to present, in a coherent and systematically mathematical way, material that exists in the physics literature. We have tried to remain faithful to that purpose in these notes. Our approach to super mathematics follows that of Leites [**1980**] and of the Chapters 3 and 4 of Manin [**1988**]. We have tried to avoid duplicating those works. We give complete proofs mainly when those references only sketch them. In our treatment of super algebra (§1), we have emphasized the methods which enable one to hide the signs. In our treatment of super calculus, we have emphasized the use of the functor of points. Our systematic treatment of signs has led us to conventions that at places differ from those used by the physicists. We have tried to point out as they occur those places where our conventions differ from theirs.

Another subtlety arises when one considers $\mathbb{Z}$-graded super objects. There are two fairly natural choices for the sign rules. Suppose x and y are elements with x of $\mathbb{Z}$-degree n and parity p and y of $\mathbb{Z}$-degree m and parity q. When permuting x and y, one may introduce either the sign $(-1)^{(n+p)(m+q)}$, or the sign $(-1)^{nm+pq}$. Both conventions have been used and have their advantages, as is explained in the Appendix to Section 1. Our point of view makes the second rule the natural one, and we adopt it.

We turn next in Section 2 to the notion of super manifolds. We give two basic approaches to these objects. The first uses the point of view of describing supermanifolds by their sheaf of "local functions". This allows us to give a description by coordinate charts and gluing functions. Our second, more geometric, approach uses the functor of points: a supermanifold M is completely determined by giving functorially in S the set of its S-points $M(S) := \mathrm{Hom}(S, M)$. Using this point of

view we define super Lie groups. We then discuss in Section 3 the basic objects of differential topology and geometry: vector bundles, the tangent bundle TM and cotangent bundle Ω^1_M. Finally, we discuss densities, i.e. things that can be integrated over a super manifold. We show that they are simply sections of the Berezinian $\mathrm{Ber}(\Omega^1_M)$ twisted by the orientation cover of M. This comes down to showing a change of variable formula for integration over open subsets $U \subset \mathbb{R}^{p|q}$. Namely, that under an isomorphism $\varphi\colon U \to V$ between such open sets we have

$$\int_V f du = \int_U \varphi^*(f) j(\varphi) du$$

where $j(\varphi) = \pm\mathrm{Ber}(d\varphi)$, the sign being the effect of φ on the orientation of the underlying reduced manifold. Then we introduce integral forms which can be integrated over codimension $(p|0)$ submanifolds.

In the fourth section, we discuss real structures and complexifications. When computing, physicists pass freely between real and complex variables. In the simplest cases, this reduces to using a moving frame which is a basis of the *complexified* tangent bundle. The justification can in other cases be more subtle, and one of the main purposes of this section is to give a precise framework in which the computations make sense.

We begin the section with the super analogues of Hilbert spaces and adjoints of operators. Such objects occur naturally in the quantization of symplectic super manifolds, either finite dimensional or, as in field theory, infinite dimensional. Our wish to have a formalism parallel to the classical one, with real functions giving rise to self-adjoint operators, and with the product of functions corresponding to the product of operators, up to terms of higher order in $\hbar$, leads us to a rather strange convention, for which the eigenvalues of an odd "self-adjoint" operator are in $i^{1/2}\mathbb{R}$. Our conventions differ from those used by the physicists.

Our next topic is cs-manifolds: supermanifolds which are complex in the odd direction, and integration on them. Integration on cs-manifolds occurs in manifestly supersymmetric descriptions of some Lagrangians (see e.g. [I-**Supersolutions**, §5.3]), and in some "path integrals".

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CHAPTER 1
Multilinear Algebra

§1.1. The sign rule

The ground rule of superalgebra is that all objects considered should be mod 2 graded, and that in all classical formulas, whenever the order in which two odd quantities appear is changed, a minus sign appears..

Here are some examples. For simplicity, we fix a ground field k of characteristic zero, which the reader may safely assume to be $\mathbb{R}$ or $\mathbb{C}$.

A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space:

$$V = V_0 \oplus V_1 \ .$$

An element v of V_0 (resp. V_1) is said to be even (resp. odd). Its parity in $\mathbb{Z}/2\mathbb{Z}$ is denoted $p(a)$. A *morphism* from a super vector space V to a super vector space W is a $\mathbb{Z}/2\mathbb{Z}$-degree preserving linear map from V to W. With this notion of morphisms, super vector spaces form an abelian category. The *parity reversing functor* $\prod$ is defined by

$$(\prod V)_0 := V_1, \ (\prod V)_1 := V_0 \ .$$

If V is finite dimensional, we define its *dimension* to be the pair of integers $(\dim V_0, \dim V_1)$, usually denoted $m_0|m_1$ for $m_i = \dim(V_i)$.

The *tensor product* of super vector spaces V and W is the tensor product of the underlying vector spaces, with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$(V \otimes W)_k = \underset{i+j=k}{\oplus} V_i \otimes W_j \ .$$

The tensor product functor is additive and exact in each variable. It has a unit object: if $\underline{1}$ is the vector space k in even degree, $\underline{1} \otimes V$ and $V \otimes \underline{1}$ are canonically isomorphic to V, by $1 \otimes v$, $v \otimes 1 \mapsto v$. It is associative: $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ is a canonical isomorphism from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$. The sign rule appears in the definition of the *commutativity isomorphism*

$$c_{V,W} \colon V \otimes W \to W \otimes V \tag{1.1.1}$$

which is defined by

$$v \otimes w \longmapsto (-1)^{p(v)p(w)} w \otimes v$$

for v and w homogeneous. From now on homogeneity of the relevant quantities will be tacitly assumed when writing formulas.

Let $(V_i)_{i\in I}$ be a finite family of super vector spaces. An ordering of I can be identified with a bijection $\sigma\colon [1,n] \to I$, where $n = |I|$. A tensor product of the V_i is obtained by choosing an ordering σ of I, and by parenthesizing the expression $V_{\sigma(1)} \otimes V_{\sigma(2)} \otimes \ldots \otimes V_{\sigma(n)}$. If $T' = V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}$ and $T'' = V_{\tau(1)} \otimes \ldots \otimes V_{\tau(n)}$ are two tensor products of the V_i, whichever way one composes associativity and commutativity isomorphisms to obtain an isomorphism from T' to T'', one always obtains the same isomorphism from T' to T''. For v_i homogeneous in V_i, it is given by

$$v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \longmapsto (-1)^N v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)}, \tag{1.1.2}$$

where N is the number of pairs of indices $i, j \in I$ such that v_i and v_j are odd and $\sigma^{-1}(i) < \sigma^{-1}(j)$ while $\tau^{-1}(i) > \tau^{-1}(j)$. This is the basic reason why the sign rule works.

Examples. $c_{WV} \circ c_{VW}$ is the identity of $V \otimes W$; the "hexagon" diagram

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{c_{U,V\otimes W}} & V \otimes W \otimes U \\ & {\scriptstyle c_{U,V}}\searrow \quad \nearrow{\scriptstyle c_{U,W}} & \\ & V \otimes U \otimes W & \end{array}$$

is commutative (if we had not suppressed the associativity isomorphism, this diagram would have had six sides).

A *super algebra* over k is a super vector space A, given with a morphism, called the product: $A \otimes A \to A$. By the definition of morphisms, the parity of the product of homogeneous elements of A is the sum of the parities of the factors.

The super algebra A is *associative* if $(xy)z = x(yz)$. A *unit* is an even element 1 (i.e. a morphism $\underline{1} \to A$) such that $1x = x1 = x$. By "super algebra", we will usually mean an associative super algebra with unit. For such a super algebra A, a left (resp. right) A-module is a super vector space M given with a morphism, also called product: $A \otimes M \to M$ (resp. $M \otimes A \to M$) obeying the usual identities, expressing that M is a module over A considered as an ordinary algebra. The sign rule has not entered so far. It changes the definition of commutativity: the super algebra A is *commutative* if the product of homogeneous elements obeys

$$xy = (-1)^{p(x)p(y)} yx \ .$$

As classically, if A is commutative, a left A-module is also a right A-module. The passage from left to right uses the sign rule:

$$m \cdot a := (-1)^{p(m)p(a)} a \cdot m. \ . \tag{1.1.3}$$

We will just say "*module*" for left (or right) module. The *tensor product* of A-modules

$$M \otimes_A N := (M \text{ as right module}) \otimes_A (N \text{ as left module})$$

is again an A-module, and the tensor product functor is associative, commutative and has a unit, the A-module A. The commutativity isomorphism is given by (1.1.1).

If A is an algebra, the *opposite* algebra A^0 is A, with the product

$$x \cdot_{\text{opp}} y := (-1)^{p(x)p(y)} y \cdot x \tag{1.1.4}$$

An element x of A is *central* if its homogeneous components obey $xy = (-1)^{p(x)p(y)}yx$ for all y in A. The *tensor product* of super algebras A, B is $A \otimes B$, with the product

$$(a \otimes b)(c \otimes d) := (-1)^{p(b)p(c)} ac \otimes bd \ . \tag{1.1.5}$$

1.1.6 Example. The algebra D with basis 1, η, with η odd and $\eta^2 = 1$, while classically commutative, is not a commutative super algebra. Its center is reduced to k and the opposite algebra has $\eta^2 = -1$. It is a *super skew field*: nonzero homogeneous elements are invertible.

Many constructions and theorems of algebra extend to the super case. Making the story interesting, there are also new phenomena. Keeping track of signs can be a nuisance. We will explain two methods to hide them, and to extend classical results to the super case without having to work, viz. in 1.2, the categorical method, and in 1.7, the "even rules" method.

§1.2. Categorical approach

Many constructions and theorems of multilinear algebra make sense and remain true in any abelian category $\mathcal{T}$ with a biadditive tensor product which is associative, commutative and with a unit $\underline{1}$ (a *tensor category*). Three examples of tensor categories are (i) vector spaces, (ii) super vector spaces and (iii) modules over a commutative super algebra A. The generalization to the categorical setting can require extra assumptions. It involves no sign. One can then specialize to the case of super vector spaces, or more generally to the case of modules over a commutative super algebra: the signs have been hidden in the commutativity isomorphism (1.1.1).

That $\mathcal{T}$ is a tensor category means that one can speak of the tensor product of a finite family of objects $(V_i)_{i \in I}$ with standard properties. No ordering of I has to be prescribed. For the empty family, one gets the unit object.

For super vector spaces, the tensor product $\underset{i \in I}{\otimes} V_i$ can be understood as follows. For each ordering σ: $[1, n] \xrightarrow{\sim} I$ of I, one has a tensor product

$$T_\sigma := V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)} \ ,$$

For variable σ, one has a transitive system of isomorphisms among the T_σ, and $\underset{i \in I}{\otimes} V_i$ is the "common value" of the T_σ. More pedantically, it is the projective limit of the T_σ: an element of $\underset{i \in I}{\otimes} V_i$ is the data for each σ of t_σ in T_σ, the t_σ corresponding to each other by (1.1.2). An inductive limit would have worked as well. Notice that in the case when all the V_i are the same super vector space, the symmetric group of I acts on the family $(V_i)_{i \in I}$, and hence (by transport of structure) on the tensor product $\otimes_{i \in I} V_i$.

Example. Let $(X_i)_{i\in I}$ be a finite family of nice topological spaces, say finite CW complexes. The Künneth formula in cohomology with coefficients in k says that

$$H^*\Big(\prod_{i\in I} X_i\Big) = \bigotimes_{i\in I} H^*(X_i)\ . \tag{1.2.1}$$

Here, $H^*(X_i)$ is viewed as a super vector space, the $\mathbb{Z}/2\mathbb{Z}$-grading being given by the cohomological degree modulo 2. In (1.2.1), no ordering of I is needed. However, to name decomposable elements of the tensor product, a convenient way is to choose an ordering $\sigma\colon [1,n] \to I$ of I and elements $x_i \in H^*(X_i)$. The element $x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$ of $\otimes H^*(X_i) = H^*(X_{\sigma(1)}) \otimes \ldots \otimes H^*(X_{\sigma(n)})$ depends on σ.

§1.3. Examples of the categorical approach

We will assume that $\mathcal{T}$ is a k-linear tensor category, i.e. that $k \subset \mathrm{End}(\underline{1})$. We now give examples of how classical constructions can be made categorical, then specialized to the case of super vector spaces.

1.3.1. A *Lie algebra* in $\mathcal{T}$ is an object $\mathcal{L}$ of $\mathcal{T}$, given with a morphism $[\ ,\ ]\colon \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$ obeying the usual identities. They are interpreted as follows:

Antisymmetry: vanishing of the sum of the bracket $[\ ,\ ]\colon \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$ and of the opposite bracket, by which we mean the composite $[\ ,\ ]\circ c_{\mathcal{L},\mathcal{L}}\colon \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$.

Jacobi identity: vanishing of the sum of three iterated brackets, that is to say the vanishing of the composite of $[\ ,[\ ,\ ]]\colon \mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$ and $1+\sigma+\sigma^2$, for σ the automorphism of $\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}$ associated to the cyclic permutation of the factors (see §1.2).

For super vector spaces, these conditions read

$$[x,y] + (-1)^{p(x)p(y)}[y,x] = 0$$
$$[x,[y,z]] + (-1)^{p(x)p(y)+p(x)p(z)}[y,[z,x]] + (-1)^{p(x)p(z)+p(y)p(z)}[z,[x,y]] = 0\ .$$

1.3.2. An *algebra* in $\mathcal{T}$ is an object A of $\mathcal{T}$, given with a product morphism $A\otimes A \to A$. Associativity, commutativity and unit are expressed by the commutativity of standard diagrams. For instance, a unit element is a morphism $\underline{1} \xrightarrow{u} A$ such that $A = \underline{1}\otimes A \xrightarrow{u} A\otimes A \to A$ and $A = A\otimes\underline{1} \xrightarrow{u} A\otimes A \to A$ are the identity. Commutativity means that the product $A\otimes A \to A$ is equal to the composite $A\otimes A \to A\otimes A \to A$ of the product with $c_{A,A}$.

By "algebra", we will usually mean associative algebra with a unit element. For such an algebra A, a left (resp. right) A-module in $\mathcal{T}$ is an object M of $\mathcal{T}$ given with a product morphism $A\otimes M \to M$ (resp. $M\otimes A \to M$) making standard diagrams commute. The tensor product over A of a right A-module M with a left A-module N is the cokernel of the difference of the two natural maps $M\otimes A\otimes N \to M\otimes N$.

For $\mathcal{T}$ the tensor category of super vector spaces, we recover the super algebras and the super modules of 1.1, and the tensor product over A.

1.3.3. If A is commutative, no distinction is to be made between left and right A-modules, and we will just say A-module. The tensor product of two A-modules

is again an A-module. The A-modules form a tensor category $\mathcal{T}_A$. Its unit is the A-module A. An algebra in $\mathcal{T}_A$ is called an A-algebra.

For $\mathcal{T}$ the tensor category of super vector spaces, we recover the rule (1.1.3) to transform a left module structure over a commutative super algebra into a right module structure.

If $A \to A'$ is a morphism of commutative algebras, the extension of scalars $M \mapsto M_{(A')} := A' \otimes_A M$: $\mathcal{T}_A \to \mathcal{T}_{A'}$ is compatible with tensor product. It hence transforms algebras into algebras, modules into modules, and Lie algebras into Lie algebras.

1.3.4. The symmetric power $\mathrm{Sym}^n(V)$ of an object V of $\mathcal{T}$ is a quotient of $V \otimes \ldots \otimes V$ (n factors): it is the coinvariants of the action of the symmetric group S_n described in §1.2. If infinite direct sums exist in $\mathcal{T}$, $\mathrm{Sym}^*(V) := \oplus \mathrm{Sym}^n(V)$ $(n \geq 0)$ is a commutative algebra. It is the commutative algebra freely generated by V: for any commutative algebra A,

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{Sym}^*(V), A) \xrightarrow{\sim} \mathrm{Hom}(V, A) \ .$$

For the case when V is a super vector space, one has

$$\mathrm{Sym}^*(V) = \mathrm{Sym}^*(V_0) \otimes \wedge^*(V_1)$$

where the operations on the right-hand-side are the usual ones in the category of ordinary vector spaces. The parity is the parity of the exterior degree. The degree * is the total degree.

1.3.5. Similarly, we define the exterior power $\wedge^n V$ as the coinvariants of the action of S_n on $V \otimes \ldots \otimes V$ (n factors) by

$$\sigma \longmapsto \varepsilon(\sigma) \cdot \text{action associated to the permutation of the factors,}$$

for $\varepsilon(\sigma)$: $S_n \to \{\pm 1\}$ the sign character.

1.3.6 Graded Objects. Define the tensor category $\mathcal{T}^{\mathrm{gr}}$ of graded objects of $\mathcal{T}$ to be the category of $\mathbb{Z}$-graded objects of $\mathcal{T}$, with the tensor product and associativity isomorphism of $\mathcal{T}$, and the following commutativity isomorphism: for V (resp. W) purely of degree n (resp. m),

$$c_{V,W} \text{ in } \mathcal{T}^{\mathrm{gr}} := (-1)^{nm} c_{V,W} \text{ in } \mathcal{T}\text{: } V \otimes W \to W \otimes V \ .$$

For V in $\mathcal{T}$, the exterior algebra $\wedge^* V$ is an object of $\mathcal{T}^{\mathrm{gr}}$, with $\wedge^n V$ in degree n. As object of $\mathcal{T}^{\mathrm{gr}}$, it can be described as the symmetric algebra of the object of $\mathcal{T}^{\mathrm{gr}}$ reduced to V in degree 1.

For $\mathcal{T}$ the category of super vector space, if V is a super vector space, $\wedge^* V$ is with this definition a commutative algebra in the category (super vector spaces)^{gr} of graded super vector spaces, with the commutativity isomorphism

$$(1.3.6.1) \qquad c_{V,W}\text{: } V \otimes W \to W \otimes V\text{: } v \otimes w \longmapsto (-1)^{\deg(v)\deg(w)+p(v)p(w)} w \otimes v \ .$$

For super vector spaces, another definition is sometimes used in the literature: $\wedge^n V := \mathrm{Sym}^n(\prod V)$. If n is odd, this definition differs by a parity change from the definition we use. For a description of the essential equivalence between the two points of view, we refer to the appendix.

1.3.7 Poincaré-Birkhoff-Witt Theorem. If A is an associative algebra in $\mathcal{T}$, the bracket $[x,y] := xy - yx$: $A \otimes A \to A$ is a Lie bracket. The defining formula for $[\ ,\ ]$ is to be interpreted as follows: it is the difference between the product $A \otimes A \to A$ and its composition with $c_{A,A}$: $A \otimes A \to A \otimes A$. For super algebras, this gives

$$[x,y] := xy - (-1)^{p(x)p(y)}yx\ .$$

Classically, the universal enveloping algebra $\mathcal{UL}$ of a Lie algebra $\mathcal{L}$ is defined by the universal property that for any associative algebra with unit A, one has

$$\mathrm{Hom}_{\mathrm{algebra}}(\mathcal{UL}, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Lie}}(\mathcal{L}, A)\ .$$

The algebra $\mathcal{UL}$ is a quotient of the tensor algebra $T\mathcal{L} := \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. One filters $\mathcal{UL}$ by the images of the $\bigoplus_{i \leq n} \mathcal{L}^{\otimes i}$. The Poincaré-Birkhoff-Witt theorem says that there is an isomorphism of graded algebras $\mathrm{Sym}^*(\mathcal{L}) \xrightarrow{\sim} \mathrm{Gr}\,\mathcal{UL}$. The map from $\mathrm{Sym}^*\mathcal{L}$ to $F^n\mathcal{UL}/F^{n-1}\mathcal{UL}$ lifts to $F^n\mathcal{UL}$, by

$$x_1 \cdots x_n \text{ in } \mathrm{Sym}^n\mathcal{L} \longmapsto \frac{1}{n!}\sum_{\sigma} x_{\sigma 1} \cdots x_{\sigma n} \text{ in } \mathcal{UL}\ . \tag{1.3.7.1}$$

The Poincaré-Birkhoff-Witt theorem is equivalent to the statement that the maps (1.3.7.1) define a vector space isomorphism from $\mathrm{Sym}^*(\mathcal{L})$ to $\mathcal{UL}$, with the product in $\mathcal{UL}$ agreeing with that of $\mathrm{Sym}^*\mathcal{L}$, modulo terms of lower degree. Transporting the product of $\mathcal{UL}$ by this isomorphism, one obtains a product $*$ on $\mathrm{Sym}^*\mathcal{L}$.

Let $\mathcal{L}$ be a Lie algebra in a tensor category $\mathcal{T}$. Our goal is to generalize Poincaré-Birkhoff-Witt in this categorical setting. To do this we will first define the analogue of the product $*$ (with induced multiplication on the associated graded being the usual product in $\mathrm{Sym}^*\mathcal{L}$), then prove that $(\mathrm{Sym}^*\mathcal{L}, *)$ has the universal property required of an universal enveloping algebra.

The components

$$*\colon \mathrm{Sym}^p\mathcal{L} \otimes \mathrm{Sym}^q\mathcal{L} \longrightarrow \bigoplus_{i \leq p+q} \mathrm{Sym}^i\mathcal{L}$$

of the product $*$ are defined by induction on $p+q$. In defining morphisms below we use the following conventions: (a) an iterated product $x_1 * \cdots * x_N$ is defined by induction on N to be $x_1 * (x_2 * \cdots * x_N)$; (b) formulas stand, in a hopefully self evident way, for morphisms from $\mathcal{L}^{\otimes N}$ to $\mathrm{Sym}^*\mathcal{L}$. For instance, $x_{\sigma 1} * \cdots * x_{\sigma N}$ stands for the composite of $\mathcal{L}^{\otimes N} \to \mathrm{Sym}^*\mathcal{L}$ considered in (a) and of the action (1.2) of σ^{-1} on $\mathcal{L}^{\otimes N}$. For $\mathcal{T}$ the tensor category of super vector spaces, before interpreting the given formula as giving images of elements, each term should be multiplied by the sign prescribed by the sign rule. (This enters, of course, in the action of the symmetric group S_N on $\mathcal{L}^{\otimes N}$.) We will also use the fact that defining a map with source $\mathrm{Sym}^p\mathcal{L} \otimes \mathrm{Sym}^q\mathcal{L}$ amounts to the same thing as defining a map with source $\mathcal{L}^{\otimes p} \otimes \mathcal{L}^{\otimes q}$, which is $S_p \times S_q$-invariant. The latter is what is really described.

For $p = 1$, and for $y_1 \cdots y_q$ denoting the projection $\mathcal{L}^{\otimes q} \to \mathrm{Sym}^q(\mathcal{L})$, we define $*$ by

$$\begin{aligned} x * (y_1 \cdots y_q) &:= (xy_1 \cdots y_q \text{ in } \mathrm{Sym}^{q+1}\mathcal{L}) \\ &\quad + \frac{1}{(q+1)!}\sum_{\sigma}\sum_{i}(q-i+1)y_{\sigma 1} * \cdots * [x, y_{\sigma i}] * \cdots * y_{\sigma q}\ . \end{aligned} \tag{1.3.7.2}$$

For $p = 0$, $*$ is defined so that $\underline{1} = \mathrm{Sym}^0\mathcal{L} \hookrightarrow \mathrm{Sym}^*\mathcal{L}$ is a left unit, and the case $p \geq 2$ is reduced to the case $p = 1$ by

$$(x_1 \cdots x_p) * Y = \frac{1}{p!} \sum_\sigma x_{\sigma 1} * \cdots * x_{\sigma p} * Y \tag{1.3.7.3}$$

for Y in $\mathrm{Sym}^*\mathcal{L}$. Note that for $q = 0$, (1.3.7.2) is a right unit property for $\underline{1} = \mathrm{Sym}^0\mathcal{L} \hookrightarrow \mathrm{Sym}^*\mathcal{L}$. By (1.3.7.3), that $\underline{1} = \mathrm{Sym}^0\mathcal{L} \hookrightarrow \mathrm{Sym}^*\mathcal{L}$ is a right unit amounts to having for all p

$$x_1 \cdots x_p \text{ in } \mathrm{Sym}^p\mathcal{L} = \frac{1}{p!} \sum_\sigma x_{\sigma 1} * \cdots * x_{\sigma p} \ . \tag{1.3.7.4}$$

This identity is proved by induction on p. By induction, the right side of (1.3.7.4) is

$$\frac{1}{p} \sum_i x_i * (x_1 \cdots \hat{x}_i \cdots x_p) \ .$$

If we expand each term by (1.3.7.2), the first summands add up to $x_1 \cdots x_p$ in $\mathrm{Sym}^p\mathcal{L}$, and the second summands cancel by the antisymmetry of the bracket.

Lemma 1.3.7.5. *The algebra* $(\mathrm{Sym}^*\mathcal{L}, *)$ *is associative, and the inclusion of* $\mathcal{L}$ *into* $\mathrm{Sym}^*\mathcal{L}$ *is a morphism of Lie algebras.*

The second statement results from the definition (the case $q = 1$ of (1.3.7.2))

$$x * y := xy + \frac{1}{2}[x, y] \ ,$$

Indeed, the product xy: $\mathcal{L} \otimes \mathcal{L} \to \mathrm{Sym}^2\mathcal{L}$ is symmetric while the second term $\frac{1}{2}[xy]$ is antisymmetric.

Associativity means that for all p, q and r, the two maps $(x*y)*z$ and $x*(y*z)$ from $\mathrm{Sym}^p\mathcal{L} \otimes \mathrm{Sym}^q\mathcal{L} \otimes \mathrm{Sym}^r\mathcal{L}$ to $\mathrm{Sym}^*\mathcal{L}$ are equal. We will prove this equality by induction on $p + q + r$. For p, q or $r = 0$, associativity follows from the unit property.

Assuming the induction hypothesis for $p+q+r \leq n$, we first show the vanishing of the map

$$\{x_1, \ldots, x_{n+1}\} := x_1 * x_2 * x_3 * \cdots * x_n - x_2 * x_1 * x_3 \cdots * x_n - [x_1, x_2] * x_3 * \cdots * x_n$$

from $\mathcal{L}^{\otimes(n+1)}$ to $\mathrm{Sym}^*\mathcal{L}$. The morphism $\{x_1, \ldots, x_{n+1}\}$ is antisymmetric in the first two variables . It is symmetric in the last $(n-1)$ variables. Indeed, for $3 \leq i \leq n$, the induction hypothesis gives

$$\{x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1}\} - \{x_1, \ldots, x_{i+1}, x_i, \ldots, x_{n+1}\} = \\ \{x_1, \ldots, [x_i, x_{i+1}], \ldots, x_{n+1}\} = 0 \ .$$

We next check that

$$\{x, y, z, \cdots\} + \{y, z, x, \cdots\} + \{z, x, y, \cdots\} = 0 \ .$$

This sum is indeed

$$\begin{aligned} x*[y,z]*\cdots \quad &+ y*[z,x]*\cdots \quad + z*[x,y]*\cdots \\ -[y,z]*x*\cdots \quad &- [z,x]*y*\cdots \quad - [x,y]*z*\cdots = \\ [x[y,z]]*\cdots \quad &+ [y,[z,x]]*\cdots \quad + [z[x,y]]*\cdots = 0 \end{aligned}$$

These properties ensure that for suitable morphisms A_i, one has

$$\{x_i, x_j, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}\} = A_i - A_j \ . \tag{1.3.7.6}$$

For any fixed i, one can for instance take $A_i = 0$ and define A_j for $j \neq i$ by the (i,j) instance of (1.3.7.6).

The definition (1.3.7.2) and (1.3.7.4) give

$$\begin{aligned} \frac{1}{n!} x* \sum_\sigma y_{\sigma 1} * \cdots * y_{\sigma n} = \ & \text{symmetrized product of } x, y_1, \ldots, y_n \\ & + \frac{1}{(n+1)!} \sum_\sigma \sum_i (n-i+1) y_{\sigma 1} * \cdots * [x, y_{\sigma i}] * \cdots * y_{\sigma n} \ . \end{aligned} \tag{1.3.7.7}$$

In the last sum, the terms with $i \neq 1$ can be expanded using $[x, y_{\sigma i}] = x*y_{\sigma i} - y_{\sigma i}*x$ and the induction hypothesis. The sum over i becomes

$$n[x, y_{\sigma 1}] * \cdots * y_{\sigma n} + n\, y_{\sigma 1} * x * \cdots * y_{\sigma n} - \sum_i y_{\sigma 1} * \cdots * y_{\sigma i} * x * \cdots * y_{\sigma n} \ ,$$

and (1.3.7.7) gives that

$$\sum_i \{x, y_i, y_1, \ldots, \hat{y}_i, \ldots, y_n\} = 0 \ .$$

In the notation of (1.3.7.6), this means that for any fixed i , $\sum_j (A_j - A_i) = 0$, so that $A_i = \frac{1}{n+1} \sum A_j$ is independent of i and the $\{x_1, \ldots, x_n\}$ vanish.

It remains to deduce associativity for $p, q, r > 0$ with $p+q+r = n+1$ from the induction hypothesis and the vanishing of $\{x_1, \ldots, x_{n+1}\}$. By 1.3.7.4, it suffices to check for Z in $\mathrm{Sym}^r(\mathcal{L})$ the equality of

$$(x_1 * \cdots * x_p) * ((y_1 * \cdots * y_q) * Z) = (x_1 * \cdots * x_p) * (y_1 * \cdots * y_q * Z)$$

with

$$((x_1 * \cdots * x_p) * (y_1 * \cdots * y_q)) * Z = (x_1 * \cdots * y_q) * Z \ .$$

This results from having for all p

$$(x_1 * \cdots * x_p) * Z = x_1 * \cdots * x_p * Z$$

(an equality of maps from $\mathcal{L}^{\otimes p} \otimes \mathrm{Sym}^{n+1-p}\mathcal{L}$ to $\mathrm{Sym}^*\mathcal{L}$). After symmetrization in $x_1, \ldots, x_p$, this holds true by the definition (1.3.7.3). It remains to check that

$$\begin{aligned} &(x_1 * \cdots x_i * x_{i+1} * \cdots * x_p) * Z - (x_1 * \cdots * x_{i+1} * x_i * \cdots * x_p) * Z \\ =&x_1 * \cdots * x_i * x_{i+1} * \cdots * x_p * Z - x_1 * \cdots * x_{i+1} * x_i * \cdots * x_p * Z \ . \end{aligned}$$

By the induction hypothesis, the left side is

$$(x_1 * \cdots * [x_i, x_{i+1}] * \cdots * x_p) * Z = x_1 * \cdots * [x_i, x_{i+1}] * \cdots * x_p * Z \ .$$

For the right side, the induction hypothesis (resp. the vanishing of $\{\cdots\}$) gives the same answer if $i \neq 1$ (resp. $i = 1$).

The algebra $(\mathrm{Sym}^*\mathcal{L}, *)$ deserves to be called the universal enveloping algebra of $\mathcal{L}$:

Lemma 1.3.7.8. *For A an associative algebra with unit in $\mathcal{T}$, one has*

$$\mathrm{Hom}_{\mathrm{algebra}}((\mathrm{Sym}^*\mathcal{L}, *), A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Lie}}(\mathcal{L}, A) \ .$$

Proof. Given $f\colon \mathcal{L} \to A$, it results from (1.3.7.4) that the only possible extension of f to an algebra morphism is given by

$$\mathrm{Sym}^n\mathcal{L} \to A\colon y_1 \cdots y_n \longmapsto \frac{1}{n!}\sum_\sigma f(y_{\sigma 1}) \cdots f(y_{\sigma n}) \ .$$

We leave it to the reader to check that this map is indeed a morphism of algebras.

Remark. For $\mathcal{T}$ the tensor category of super vector spaces, we get the Poincaré-Birkhoff-Witt theorem for super Lie algebras. Our arguments used characteristic zero in an essential way. For a proof valid for super Lie algebras $\mathcal{L}$ over a super commutative ring A, when $\mathcal{L}$ is free as an A-module and 2 and 3 are invertible in A, we refer to L. Corwin, Y. Ne'eman and S. Sternberg, Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry) Reviews of Modern Physics **47** (1975), p. 573–603.

§1.4. Free modules

A *free module* over a super algebra is a module which is free as an ungraded module, with a homogeneous basis.

Fix a commutative super algebra A. The standard free module $A^{p|q}$ is the module freely generated by even elements $e_1, \ldots, e_p$ and odd elements $e_{p+1}, \ldots, e_{p+q}$. A morphism $T\colon A^{p|q} \to A^{r|s}$ can be represented by a matrix of size $(r+s) \times (p+q)$, with blocks of even and of odd entries as follows:

$$\begin{array}{c} \\ r\{ \\ \\ s\{ \end{array} \overbrace{\phantom{\text{even}}}^{p} \ \overbrace{\phantom{\text{odd}}}^{q} \begin{pmatrix} \text{even} & \text{odd} \\ \\ \text{odd} & \text{even} \end{pmatrix}$$

We will represent an element x of $A^{p|q}$ by the column vector x^i such that $x = \sum e_i x^i$, and define the entries of the matrix of T by $T(e_j) = \sum e_i t^i_j$. With those conventions, $T(x)$ is given by the matrix product Tx and the composition of morphisms is given by a matrix product.

This description of endomorphisms works as well for right modules over a not necessarily commutative super algebra.

Warning. For M an A-module, the parity changed module $\prod M$ is not the super vector space $\prod M$, with the same right and left A-module structures as M: those A-module structures on $\prod M$ don't correspond to each other by (1.1.3). A correct definition of $\prod M$, for M a left or right module over a not necessarily commutative A is as follows: for right modules, the module structure remains the same; for left modules, the action of $a \in A$ is changed by the sign $(-1)^{p(a)}$. Let $\prod$ be the vector space k put in odd degree. If $\prod M$ is construed as being $\prod \otimes M$, by $1 \otimes m \xrightarrow{\sim} m$, this convention is an application of the sign rule.

With this definition of $\prod M$, one has

$$A^{p|q} \simeq A^p \times (\textstyle\prod A)^q \ .$$

§1.5. Free commutative algebras

The commutative A-algebra $A[t_1, \dots, t_p, \theta_1, \dots, \theta_q]$ freely generated by even quantities $t_1 \dots t_p$ and odd quantities $\theta_1, \dots, \theta_q$ is the symmetric algebra $\mathrm{Sym}^*(A^{p|q})$. It can be described as polynomial in the t_i, and exterior in the θ_j, in the usual sense of those words. More precisely, it is the tensor product over k of the following three commutative super algebras: A, $\mathrm{Sym}^*(k^p) \simeq k[t_1, \dots, t_p]$ (purely even), and $\wedge^* k^q$ (with $\wedge^*$ taken in its usual sense, and with the parity defined to be the parity of the exterior degree and the basis for k^q being $\theta_1, \dots, \theta_q$). One should, however, keep in mind that the tensor product of super algebras involves the sign (1.1.5).

§1.6. The trace

Let $T\colon A^{p|q} \to A^{p|q}$ be a morphism given by

$$\begin{array}{c} \overbrace{}^{p}\ \overbrace{}^{q} \\ \left(\begin{array}{cc} A & B \\ C & D \end{array}\right) \end{array}$$

The *trace*, also called, for emphasis, *super trace* of T, is defined by

$$\begin{aligned} \mathrm{Tr}(T) :&= \text{sum of diagonal entries of } A \\ &\quad - \text{sum of diagonal entries of } D\ . \end{aligned} \tag{1.6.1}$$

It is an element of A_0. Other notations: $\mathrm{str}(T)$, or, for endomorphisms of super vector spaces, $\mathrm{Tr}((-1)^F T)$. We now explain how the minus sign in (1.6.1) is forced on us by the categorical point of view. We will see at the same time that the trace is independent of the basis chosen.

The category of A-modules admits an inner Hom functor $\underline{\mathrm{Hom}}_A(M, N)$, characterized by the adjunction formula

$$\mathrm{Hom}_A(M, \underline{\mathrm{Hom}}_A(N, P)) = \mathrm{Hom}_A(M \otimes_A N, P)\ . \tag{1.6.2}$$

The even (resp. odd) part of $\underline{\mathrm{Hom}}_A(M, N)$ is the k-vector space of even (resp. odd) maps $f\colon M \to N$ for which

$$f(am) = (-1)^{p(f)p(a)} a f(m)\ , \quad \text{i.e. } f(ma) = f(m)a\ .$$

The A-module structure is given by $(af)(m) = af(m)$, and to $m \mapsto \varphi_m$, the identification in (1.6.2) associates $m \otimes n \mapsto \varphi_m(n)$. The even part of $\underline{\mathrm{Hom}}_A(M, N)$ is $\mathrm{Hom}_A(M, N)$. This is obvious by inspection, as well as a particular case of (1.6.2): if $\underline{1}$ is the A-module A, the unit for $\otimes$, the even part of an A-module H is $\mathrm{Hom}_A(\underline{1}, H)$, and one takes $M = \underline{1}$ in (1.6.2).

For $M = A^{p|q}$ and $N = A^{r|s}$, $\underline{\mathrm{Hom}}_A(M, N)$ can again be viewed as a space of $(r+s) \times (p+q)$ matrices. Entries can now have any parity and $\underline{\mathrm{Hom}}_A(M, N)_i$ consists of the matrices with the $r \times p$ and $s \times q$ blocks of parity i, the other two of

parity $i+1$. The map corresponding to a matrix is again given by a matrix product. In the matrix description, the A-module structure of $\underline{\mathrm{Hom}}_A(M,N)$ involves a sign:

$$a \cdot (t^i_j) = (\pm a t^i_j) \tag{1.6.3}$$

with $+$ (resp. $-$) for $1 \le i \le r$ (resp. $r+1 \le i \le r+s$).

In any tensor category with an inner Hom, the *dual* of M is

$$M^\vee := \underline{\mathrm{Hom}}(M, \underline{1}) \ .$$

Using just (1.6.2), one defines a morphism

$$M^\vee \otimes N \longrightarrow \underline{\mathrm{Hom}}(M,N) \ . \tag{1.6.4}$$

In the special case of A-modules, it is the map sending $\omega \otimes n$ to the map

$$m \longmapsto (-1)^{p(\omega)p(n)} n\omega(m)) \ ,$$

and it is an isomorphism if M is free of finite type.

One has an evaluation map

$$\mathrm{ev}\colon M^\vee \otimes M \to \underline{1}$$

and, if (1.6.4) is an isomorphism, a coevaluation map

$$\delta\colon \underline{1} \to M^\vee \otimes M$$

corresponding by (1.6.4) to the identity map of M. More generally, any morphism f in $\mathrm{Hom}(M,N) = \mathrm{Hom}(\underline{1}, \underline{\mathrm{Hom}}(M,N))$ gives, if (1.6.4) is an isomorphism,

$$\delta(f)\colon \underline{1} \to M^\vee \otimes N \ .$$

The *trace* of an endomorphism f of M is the endomorphism of $\underline{1}$

$$\mathrm{Tr}(f) := \mathrm{ev} \circ \delta(f) \ .$$

For A-modules, ev is $\omega \otimes m \mapsto \omega(m)$. For $M = A^{p|q}$, if e^i in the dual of M is defined by $e^i(e_j) = \delta^i_j$,

$$\delta(f) = \sum f(e_i) \otimes e^i \text{ in } N \otimes M^\vee = \sum (-1)^{p(e_i)} e^i \otimes f(e_i) \text{ in } M^\vee \otimes N \ ,$$

and for an endomorphism f of $A^{p|q}$

$$\mathrm{Tr}(f) = \mathrm{ev} \circ \delta(f) = \sum (-1)^{p(e_i)} e^i(f(e_i)) \ ,$$

which agrees with (1.6.1).

The categorical method can be used to prove that, in a setting where (1.6.4) is an isomorphism, the trace has the usual properties. It is defined not only on morphisms, but on the inner Hom: it is the composite

$$\mathrm{Tr}\colon \underline{\mathrm{Hom}}(M,M) \xleftarrow{\sim} M^\vee \otimes M \xrightarrow{\mathrm{ev}} 1 \ . \tag{1.6.5}$$

The inner Hom $\underline{\mathrm{Hom}}(M,M)$ is an associative algebra, hence also a Lie algebra, and (1.6.5) is a Lie algebra morphism, for the zero bracket on the unit object $\underline{1}$. For T, U in $\underline{\mathrm{Hom}}(M,M)$, this means that

$$\mathrm{Tr}(TU) = (-1)^{p(T)p(U)}\mathrm{Tr}(UT)\ . \tag{1.6.6}$$

For A-modules, and $M = A^{p|q}$, if $\underline{\mathrm{Hom}}(M,M)$ is viewed as a space of matrices, one computes that the trace (1.6.5) is given by

$$\begin{aligned}\mathrm{Tr}\colon T \longmapsto\ &\text{sum of diagonal entries of the } p\times p \text{ block}\\ &-(-1)^{p(T)}\ \text{sum of diagonal entries of the } q\times q \text{ block}\ ,\end{aligned} \tag{1.6.7}$$

generalizing (1.6.1).

From the categorical point of view, the facts that a morphism $u\colon M \to N$ has a transpose $u^t\colon N^\vee \to M^\vee$, that $\mathrm{Tr}(u) = \mathrm{Tr}(u^t)$, and that $u^{tt} = u$ don't pose any sign problems. The meaning of the formula $u^{tt} = u$ is that the natural biduality isomorphisms $M \xrightarrow{\sim} M^{\vee\vee}$ and $N \xrightarrow{\sim} N^{\vee\vee}$ make the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\ u\ } & N \\ \downarrow & & \downarrow \\ M^{\vee\vee} & \xrightarrow{\ u^{tt}\ } & N^{\vee\vee} \end{array}$$

Similar constructions and compatibilities hold for $\underline{\mathrm{Hom}}(M,N)$.

When one translates this into matrices, however, care is required: a basis e_i of a free module M defines a basis e_i' of the dual $M^\vee$, with $e_i'(e_j) = \delta_j^i$. Iterating defines a basis e_i'' of the bidual. In the natural isomorphism between M and its bidual, e_i maps not to e_i'', but to $(-1)^{p(e_i)}e_i''$. If $u \in \underline{\mathrm{Hom}}(M,N)$ has the matrix T in the bases $\{e_i\}$ for M and $\{f_j\}$ for N, the *transpose* of T is the matrix of u^t, in the bases $\{f_j'\}, \{e_i'\}$. The sign in the biduality isomorphism explains why, while $u^{tt} = u$ in the sense we explained, the transposition of matrices (for which our definition agrees with that of Manin [**1988**] Ch. 3, §3.1), is an operation of order 4.

§1.7. Even rules

It is often convenient to work with purely even objects. The "even rules" principle shows that if one is willing to extend scalars from k to algebras $k[\theta_1, \dots]$ generated by odd elements one can often reduce to the even context. The price one must pay is to show that things vary functorially under extensions of scalars.

By extension of scalars we have in mind the following. For V a super vector space and B a commutative super algebra, let $V(B)$ be the even part of $V_{(B)} := B \otimes V$ deduced from V by extension of scalars from k to B. It is a module over the even part B_0 of B, and a morphism $B' \to B''$ of super algebras induces a morphism of B_0'-modules from $V(B')$ to $V(B'')$.

Let us see how the "even rules" principle works in defining a morphism $f\colon \underset{i\in I}{\otimes} V_i \to V$. Such a morphism induces for each B a B_0-multilinear map $f(B)$ from the $V_i(B)$ to $V(B)$. It is functorial in B. Conversely, one has

Theorem 1.7.1. *Any system of B_0-multilinear maps from the $V_i(B)$ to $V(B)$, which is functorial in B, comes from a unique morphism $f\colon \otimes V_i \to V$.*

Proof. We may and shall assume that $I = [1, n]$ for some n. We first prove uniqueness. Fix $J \subset I$. If $v_i \in V$ is even for $i \notin J$, and odd for $i \in J$, we will recover $f(\otimes v_i)$ from $f(B)$ for $B = k[(\theta_j)_{j \in J}]$. Indeed, let $w_i = v_i$ for $i \notin j$ and $w_i = \theta_i v_i$ for $i \in J$; if J has N elements,

$$f(B)((w_i)) = (-1)^{N(N-1)/2} . \prod \theta_j . f(\otimes v_i) \ , \tag{1.7.2}$$

which determines $f(\otimes v_i)$. This proves uniqueness, and we now prove existence.

Let $F(B)$ be a functorial system of multilinear maps. If $(v_i)_{i \in I}$, J, B and the w_i are as in the proof of uniqueness, each w_i for i in J is annihilated by $B \to B/(\theta_j)$. So is $F(B)((w_i))$ and it follows that $F(B)((w_i))$ in $B \otimes V$ is a multiple of $\prod \theta_j$. Define $f(\otimes v_i)$ by 1.7.2. It remains to check that F is deduced from this f.

Let C be a commutative super algebra. It suffices to check, for any $J \subset [1, n]$, that if $u_i = c_i v_i$ with c_i in C and v_i in V both even for $i \notin J$, and odd for $i \in J$, one has

$$F(C)((u_i)) = f(C)((u_i)) \ .$$

By multilinearity, we reduce to the case where $c_i = 1$ for $i \notin J$. The u_i are then the images of the w_i by $B \to C$, $\theta_j \to c_j$ and the claimed equality follows from the definition 1.7.2 of f by functoriality.

Remarks. (i) The same result holds if B runs only over exterior algebras $k[\theta_1, \ldots, \theta_N]$, i.e. commutative super algebras freely generated by odd elements.

(ii) A similar result holds for modules over a fixed commutative superalgebra A, if B runs over commutative A-algebras and $V(B)$ is the B_0-module even part of $B \otimes_A V$.

Corollary 1.7.3. (i) *To give an algebra structure on a super vector space V, it suffices to give a functorial structure of B_0-algebra on the $V(B)$. The algebra will be Lie (resp. associative, commutative,...) if and only if the algebras $V(B)$ are.*

(ii) *For A an algebra, to give a structure of A-module on a super vector space V, it suffices to give a functorial structure of $A(B)$-module on $V(B)$.*

Proof. The algebra and module structures are defined by morphisms of the type considered in the theorem. The algebraic properties listed in the corollary are expressed by the commutativity of diagrams constructed from such maps, or, in the Lie case, by linear relations between composites of such maps. According to the theorem, to give the defining morphisms amounts to giving them functorially on the even part, after extension of scalars. The required commutativity of diagrams, or linear relations, follows from the same on the even part, after extension of scalars, because of the uniqueness statement.

§1.8. Examples of the "even rules" principle

1.8.1. Let A be a super algebra. For each commutative super algebra B, $A(B)$ is an algebra. It is also an algebra for the opposite multiplication: $x \cdot_{\text{opp}} y := y \cdot x$. According to Corollary 1.7.3, this defines a new algebra structure on A: the *opposite* super algebra. To compute $x \cdot_{\text{opp}} y$ for x and y odd, one extends scalars by two odd quantities θ, η, and expresses that

$$\theta x \cdot_{\text{opp}} \eta y = \eta y \cdot \theta x \ .$$

As x is odd, $\theta x \cdot_{\text{opp}} \eta y = -\theta\eta x \cdot_{\text{opp}} y$ while $\eta y \cdot \theta x = \theta\eta y \cdot x$, giving

$$x \cdot_{\text{opp}} y = -y \cdot x \quad \text{for} \quad x, y \quad \text{odd} \ .$$

Thus, we recover 1.1.4. The sign has been hidden in the compatibility between extension of scalars and tensor products:

$$(ax) \otimes (by) = (-1)^{p(x)p(b)} ab(x \otimes y) \ .$$

1.8.2. The signs in the definition of a super Lie algebra are similarly characterized by the requirement that, after any extension of scalars, the even part be an ordinary Lie algebra. For A an associative super algebra, the sign in the definition of the bracket is to get the usual bracket on the even part, after any extension of scalars. That this bracket is Lie follows from the same statement in the even case.

1.8.3. For M and N two A-modules, the formation of $\underline{\text{Hom}}_A(M, N)$ is compatible with extensions of scalars $k \mapsto B$, for B an exterior algebra $k[\theta_1, \ldots, \theta_N]$. The even part of $\underline{\text{Hom}}_A(M, N)$ is $\text{Hom}_A(M, N)$. If M is a free module of finite type, the restriction to the even part of the trace (1.6.7) on $\underline{\text{Hom}}_A(M, M)$ is the trace (1.6.1) on $\text{Hom}_A(M, M)$. That (1.6.7) is a morphism of Lie algebras results from the fact that for the trace (1.6.1), and endomorphisms f, g of M, one has

$$\text{Tr}(fg) = \text{Tr}(gf) \ .$$

The sign $(-1)^{p(T)}$ in the matrix description of the trace (1.6.7) is then a consequence of the sign in (1.6.3).

1.8.4. An (even) *derivation* of a super algebra A is a morphism of super vector spaces D: $A \to A$ such that

$$D(ab) = Da \cdot b + a \cdot Db \ . \tag{1.8.4.1}$$

If we extend scalars from k to $k[\varepsilon]/(\varepsilon^2)$, with ε even, the derivations of A are identified with the automorphisms of the $k[\varepsilon]$-algebra $A \otimes k[\varepsilon]$, which are the identity modulo ε, by

$$D \longmapsto \ (\text{automorphism } a \mapsto a + \varepsilon Da)$$

The derivations form the even part of the even or odd derivations, for which (1.8.4.1) is replaced by

$$D(ab) = Da \cdot b + (-1)^{p(D)p(a)} a \cdot Db \ . \tag{1.8.4.2}$$

The sign in (1.8.4.2) ensures that if D is odd, and that we extend scalars from k to $k[\theta]$ with θ odd, then θD is an even derivation.

If $\mathcal{D} \subset \underline{\text{Hom}}_k(A, A)$ is the super vector space of the even or odd derivations of A, and if we extend scalars to $B = k[\theta^1, \ldots, \theta^q]$, $\mathcal{D}_{(B)} := B \otimes \mathcal{D}$ is the B-module of even or odd B-linear derivations of $A_{(B)} = B \otimes A$. The even part of $\mathcal{D}_{(B)}$ are even derivations, classically known to form a Lie algebra. By Corollary 1.7.3, $\mathcal{D}$ is a super Lie algebra.

§1.9. Alternate description of super Lie algebras

Even if §1.7 gives a systematic way of considering only even elements, the use of odd elements can lead to more transparent formulations. I particularly like the following description (cf. Leites [**1980**] 1.9.4) of what a super Lie algebra $\mathfrak{L}$ over A is. It is the data of a quadratic map

$$x^2\colon \mathfrak{L}'_1 \to \mathfrak{L}'_0$$

of A'_0-modules, functorial in the extension of scalars A', obeying (1.9.1) below. The bilinear form $[x,y] = (x+y)^2 - x^2 - y^2$ associated to x^2 extends (by 1.7 applied to bilinear maps from $(\prod\mathfrak{L}, \prod\mathfrak{L})$ to $\mathfrak{L}$) to a bracket $\mathfrak{L}\otimes\mathfrak{L} \to \mathfrak{L}$. The axiom is

$$[x, x^2] = 0\ , \tag{1.9.1}$$

a form of the Jacobi identity reminiscent of the Bianchi identity.

Example. The Lie algebra freely generated by an odd element D has the basis $\{D, D^2\}$, with the only non-trivial bracket being $[D,D] = 2D^2$. The universal enveloping algebra is the free associative algebra generated by D. As required by the Poincaré-Birkhoff-Witt theorem, it admits the basis $\{(D^2)^n D^m | n \geq 0,\ m = 0 \text{ or } 1\}$.

§1.10. The Berezinian of an automorphism

Let L be a free module of finite type over a commutative super algebra A. We write $\mathrm{GL}(L)$ for the group of automorphisms of the A-module of L, and define $\mathrm{GL}_{p|q}(A) = \mathrm{GL}(A^{p|q})$. the *Berezinian* is a homomorphism

$$\mathrm{Ber}\colon \mathrm{GL}(L) \to \mathrm{GL}_{1|0} = A_0^*$$

which generalizes the determinant. We give three descriptions of it.

(A) The usual determinant $\det\colon \mathrm{GL}(n) \to \mathrm{GL}(1)$ induces the trace on the Lie algebras, and this characterizes it. Similarly, if ε is even of square zero and T an (even) endomorphism,

$$\mathrm{Ber}(1+\varepsilon T) = 1 + \varepsilon\,\mathrm{Tr}(T) \tag{1.10.1}$$

and, with enough preliminaries on super algebraic groups, (1.10.1) can be turned into a definition (cf. 2.10). For an automorphism $T = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix}$ of $A^{p|q}$ with no nonzero odd entries, the sign in the definition 1.6.1 of the trace gives

$$\mathrm{Ber}(T) = \det(T')\det(T'')^{-1}\ . \tag{1.10.2}$$

Generalizing 1.10.1, for T nilpotent, or in settings where the exponential series converges, one has

$$\mathrm{Ber}\,\exp(T) = \exp\,\mathrm{Tr}(T)\ . \tag{1.10.3}$$

(B) For ordinary free modules, the usual determinant of $T\colon L \to L$ is the scalar by which T acts on the top exterior power $\wedge^{\dim L} L$. This definition cannot be

generalized to the super case; for $q \neq 0$, $A^{p|q}$ has no top exterior power. Another, a priori, reason is that the Berezinian makes sense only for automorphisms, not for endomorphisms.

However, in the classical case, computing the Ext by a Koszul complex, one finds a canonical isomorphism

$$\mathrm{Ext}^{\dim(L)}_{\mathrm{Sym}^*(L^\vee)}(A, \mathrm{Sym}^*(L^\vee)) = \wedge^{\dim(L)} L \ . \tag{1.10.4}$$

In (1.10.4), A is viewed as a $\mathrm{Sym}^*(L^\vee)$-module by the augmentation map. For A a commutative super algebra, and L a free module of dimension $p|q$, the $\underline{\mathrm{Ext}}$ group $\underline{\mathrm{Ext}}^{\,n}_{\mathrm{Sym}^*(L^\vee)}(A, \mathrm{Sym}^*(L^\vee))$ vanishes for $n \neq p$. For $n = p$, it is free of rank $1|0$ if q is even, and of rank $0|1$ if q is odd. One defines

$$\mathrm{Ber}(T) = \text{ action of } T \text{ on } \underline{\mathrm{Ext}}^{\,p}_{\mathrm{Sym}^*(L^\vee)}(A, \mathrm{Sym}^*(L^\vee)) \ . \tag{1.10.5}$$

Example. For $L = A^{0|q}$, $\mathrm{Sym}^*(L^\vee)$ is isomorphic to $A[\theta_1, \dots, \theta_q]$. The Ext group $\underline{\mathrm{Ext}}^{\,i}_{A[\theta_1,\dots,\theta_q]}(A, A[\theta_1, \dots, \theta_q])$ vanishes for $i \neq 0$. For $A = k$, this results from $k[\theta_1, \dots, \theta_q]$ being an injective module over itself. In general, it results from the $A[\theta_1, \dots, \theta_q]$-module $A[\theta_1, \dots, \theta_q]$ being isomorphic, up to a parity change, to $\underline{\mathrm{Hom}}_A(A[\theta_1, \dots, \theta_q], A)$ (Frobenius property). The $\underline{\mathrm{Ext}}^{\,0}$ is

$$\underline{\mathrm{Hom}}_{A[\theta_1,\dots,\theta_q]}(A, A[\theta_1, \dots, \theta_q]) \ .$$

It admits the basis consisting of the one element $a \mapsto \theta_1 \dots \theta_q a$, hence is of dimension $1|0$ for q even, and $0|1$ for q odd. An automorphism T of L acts on $\mathrm{Sym}^*(L^\vee) \simeq A[\theta_1, \dots, \theta_q]$; it maps $\theta_1 \dots \theta_q$ to $\mathrm{Ber}(T)\theta_1 \dots \theta_q$.

(C) Let T be an automorphism of $A^{p|q}$, with matrix $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ as given in (1.4). Let B be the quotient of A by the ideal generated by the odd elements. It is the quotient of A_0 by a nilpotent ideal I. After extension of scalars to B, the matrix of T takes the form

$$\begin{pmatrix} K \bmod I & 0 \\ 0 & N \bmod I \end{pmatrix} \ .$$

It follows that the matrices with even entries K and N are invertible modulo the nilpotent ideal I. They are hence invertible, and one defines

$$\mathrm{Ber}(T) := \det(K - LN^{-1}M)\det(N)^{-1} \ , \tag{1.10.6}$$

a formula suggested by (1.10.2) and the decomposition

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} 1 & LN^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K - LN^{-1}M & 0 \\ M & N \end{pmatrix} \ .$$

The matrices K, $LN^{-1}M$ and N have entries in the commutative ring A_0, so that the determinants in (1.10.6) make sense. It is a slightly nontrivial exercise to prove that 1.10.6 is multiplicative. To compare with the point of view (A), one checks (1.10.1). To compare with (B), one checks that with both definitions one has the property (1.10.2), multiplicativity, and that for an automorphism of a short exact sequence $0 \to L' \to L \to L'' \to 0$,

$$\mathrm{Ber}(T) = \mathrm{Ber}(T').\mathrm{Ber}(T'') \ . \tag{1.10.7}$$

Those properties uniquely characterize the Berezinian.

§1.11. The Berezinian of a free module

Classically, one defines the determinant of a free module of rank n by $\det(L) := \wedge^n L$. Similarly, one could define the Berezinian of a free module of rank $p|q$ over a commutative super algebra A by

$$\mathrm{Ber}(L) := \underline{\mathrm{Ext}}^{\,p}_{\mathrm{Sym}^*(L^\vee)}(A, \mathrm{Sym}^*(L^\vee)) \tag{1.11.1}$$

(cf. 1.10 (B)). Another definition is obtained by turning around the definition 1.10 (B) of the Berezinian of an automorphism: a basis $(e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q})$ of L defines a one-element basis $[e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}]$ of $\mathrm{Ber}\, L$, and for any automorphism T,

$$[Te_1, \ldots, Te_{p+q}] = \mathrm{Ber}(T)[e_1 \ldots, e_{p+q}]\ . \tag{1.11.2}$$

To keep track of signs, we will consider $\mathrm{Ber}(L)$ as an object of the tensor category (1.3.6) of graded super modules. The grading will be called grading by cohomological degree, and $[e_1, \ldots, e_{p+q}]$ is of parity $q \bmod 2$ and cohomological degree p (as suggested by (1.11.1). For

$$0 \to L' \to L \to L'' \to 0 \tag{1.11.3}$$

a short exact sequence of free modules, if $e'_1, \ldots, e'_p, e'_{p+1}, \ldots, e'_{p+q}$ is a basis of L', and if $\tilde{e}''_1, \ldots, \tilde{e}''_r, \tilde{e}''_{r+1}, \ldots, \tilde{e}''_{r+s}$ lifts to L the basis $e''_1, \ldots, e''_r, e''_{r+1}, \ldots, e''_{r+s}$ of L'', we define an isomorphism

$$\mathrm{Ber}(L') \otimes \mathrm{Ber}(L'') \to \mathrm{Ber}(L) \tag{1.11.4}$$

by

$$\left[e'_1 \ldots e'_{p+q}\right] \otimes \left[e''_1 \ldots e''_{r+s}\right] \longmapsto \left[e'_1 \ldots e'_p \tilde{e}''_1 \ldots \tilde{e}''_r \tilde{e}'_{p+1} \ldots e'_{p+q} \tilde{e}''_{r+1} \ldots \tilde{e}''_{r+s}\right]\ .$$

With the sign rule (1.3.6.1), for any exact square

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L'_1 & \to & L' & \to & L'_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_1 & \to & L & \to & L_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L''_1 & \to & L'' & \to & L''_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the diagram

$$\begin{array}{ccc}
\otimes \mathrm{Ber}(L'^i_j) & \longrightarrow & \otimes \mathrm{Ber}(L'^i) \\
\big\downarrow & & \big\downarrow \\
\otimes \mathrm{Ber}(L_j) & \longrightarrow & \mathrm{Ber}(L)
\end{array}$$

is commutative.

Appendix to §1: Graded super vector spaces

Some classical constructions give objects which are graded and are subject to Koszul's sign rule. Examples: cohomology groups of all kinds (H^*, Tor_*, $\mathrm{Ext}^* \dots$), the de Rham complex and other standard complexes.

To handle their analogue in the super world, two points of view have been used. As we will see, they are basically equivalent. However, they lead to different sign conventions.

Point of View I

One considers such objects as graded objects in the super category, i.e., graded mod 2 graded objects. The grading which is the analogue of the grading in the classical case will be called the cohomological grading. The commutativity isomorphism for the tensor product of graded mod 2 graded vector spaces is defined as follows: for v of bidegree (p, n), viz. parity p and cohomological degree n, and w of bidegree (q, m),

$$c^{\mathbf{I}}_{V,W}\colon V \otimes W \to W \otimes V \quad \text{is} \quad v \otimes w \longmapsto (-1)^{pq+nm} w \otimes v.$$

This is the point of view adopted in §1.3.6.

Point of View II

One considers that the classical construction already lives in the super world, with parity being cohomological degree modulo 2. In the super world, one continues to obtain super objects; they also have a cohomological degree, with no influence on signs: the commutativity isomorphism of graded mod 2 graded vector spaces is defined as

$$c^{\mathbf{II}}_{V,W}\colon V \otimes W \to W \otimes V\colon v \otimes w \to (-1)^{pq} w \otimes v$$

for v of parity p and w of parity q. Of course, one could consider modules instead of vector spaces.

Comparison of I and II

In his lectures, Bernstein used the point of view **II**, which he prefers. In our rendition of it here, we have used the point of view **I**. Here are our reasons for doing so.

(A) In classical occurrences of super objects, the sign rule is violated by some of the standard conventions. For instance, for V and $V^\vee$ vector spaces in duality it is usual to define the duality between $\wedge^p V$ and $\wedge^p V^\vee$ by

$$\langle \omega_1 \wedge \dots \wedge \omega_p,\ v_1 \wedge \dots \wedge v_p \rangle = \det\left(\langle \omega_i, v_j \rangle\right) .$$

In the determinant, the diagonal term is $\langle \omega_1, v_1 \rangle \cdots \langle \omega_p, v_p \rangle$, with no sign, despite the fact that v_i passed over ω_j for $i < j$. Using the point of view **I** allows us to keep the inherited classical conventions, while consistently using the sign rule as far as parity is concerned.

(B) The point of view **I** is forced on us by the categorical method 1.2 to hide signs. Indeed, when mimicking in a tensor category $\mathcal{T}$ a classical construction which could

be construed as graded (or super), we are led to consider the category of graded (or just mod 2 graded) objects of $\mathcal{T}$, with the commutativity of tensor product being given by modifying the one induced from $\mathcal{T}$ by the Koszul sign rule. When $\mathcal{T}$ is the category of super vector spaces, this gives $c^{\mathbf{I}}$ of **I**.

(C) One does not have to decide early on whether an object should be seen as having a cohomological degree.

Example. Let A be a commutative super k-algebra. The point of view **I** suggests defining the module Ω^1_A of Kähler differentials as being an A-module (hence bimodule) Ω, provided with a morphism of super k-vector spaces $d\colon A \to \Omega$ such that

$$d(ab) = a \cdot db + da \cdot b, \tag{C.1}$$

which is universal. Later, when considering the de Rham complex, one may decide that Ω^1_A is of cohomological degree 1. In the point of view **II**, deciding that in the classical case (purely even A) Ω^1_A is odd requires d to be an odd map, and that (C.1) be replaced by

$$d(ab) = (-1)^{p(a)} a \cdot db + da \cdot b \tag{C.2}$$

(D) The point of view **I** minimizes the use of the parity change functor Π. It is replaced by the imposition of an odd cohomological degree. Using the functor Π leads to nightmares of signs, for the following reasons.

(i) Let Πk be k, viewed as an odd k-vector space. The functor Π is best viewed as being the tensor product with Πk. One has to decide whether it is $V \mapsto \Pi k \otimes V$ or $V \mapsto V \otimes \Pi k$. The two are canonically isomorphic, but lead to different sign conventions.

(ii) One has natural isomorphisms $(\Pi V) \otimes W \xrightarrow{\sim} \Pi(V \otimes W)$ and $(V \otimes \Pi W) \xrightarrow{\sim} \Pi(V \otimes W)$, exchanged by the commutativity of $\otimes$. The diagram

$$\begin{array}{ccc} \Pi V \otimes \Pi W & \longrightarrow & \Pi(V \otimes \Pi W) \\ \downarrow & & \downarrow \\ \Pi(\Pi V \otimes W) & \longrightarrow & \Pi\Pi(V \otimes W) \end{array}$$

is anticommutative, rather than commutative.

The point of view **II** has advantages too: one has only one parity to consider, for applying the sign rule, and some constructions are more natural. For example, if D^- is the standard odd line (coordinate ring $k[\theta]$, θ odd, $\theta^2 = 0$), the de Rham complex of a super manifold M is (up to a completion) the space of functions on the super manifold

$$\underline{\mathrm{Hom}}(D^-, M)\ ,$$

where the $\underline{\mathrm{Hom}}$ is defined by $\mathrm{Hom}(S, \underline{\mathrm{Hom}}(D^-, M)) = \mathrm{Hom}(S \times D^-, M)$ functorially in S. In the point of view **I**, one has to apply to the de Rham complex the functor "associated simply graded object" explained below.

The points of view **I** and **II** are equivalent, in the sense that the tensor categories $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{II}}$ of graded mod 2 graded vector spaces introduced in **I** and **II** are equivalent. That is to say, there is an equivalence of categories

$$\widetilde{\mathbf{s}}\colon \mathcal{T}_{\mathbf{I}} \longrightarrow \mathcal{T}_{\mathbf{II}}$$

and an isomorphism of functors

$$\alpha\colon \widetilde{\mathbf{s}}(V \otimes_{\mathbf{I}} W) \xrightarrow{\sim} \widetilde{\mathbf{s}}(V) \otimes_{\mathbf{II}} \widetilde{\mathbf{s}}(W)$$

compatible with the associativity and commutativity isomorphisms for $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$.

The functor $\widetilde{\mathbf{s}}$ is a regrading functor: $\widetilde{\mathbf{s}}(V)$ has the same underlying vector space as V, with v of parity p and cohomological degree n acquiring the parity $p+n$, and keeping its cohomological degree.

On the underlying vector space, $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$ are both the usual tensor product. For v in $V^{p,n}$ and w in $W^{q,m}$, α is defined as

$$\alpha\colon v \otimes w \longmapsto (-1)^{nq} v \otimes w.$$

The compatibility with the commutativity isomorphisms is the commutativity of the diagram

$$\begin{array}{ccc} V^{p.n} \otimes W^{q,m} & \xrightarrow{(-1)^{nq}} & V^{p,n} \otimes W^{q,m} \\ {\scriptstyle (-1)^{pq+nm}}\downarrow & & \downarrow{\scriptstyle (-1)^{(p+n)(q+m)}} \\ W^{q,m} \otimes V^{p,n} & \xrightarrow{(-1)^{mp}} & W^{q,m} \otimes V^{p,n} \end{array} .$$

The compatibility with the associativity isomorphisms comes from the identity

$$n_1(p_2 + p_3) + n_2 p_3 = n_1 p_2 + (n_1 + n_2) p_3 \ .$$

The functor "*associated simply* (mod 2) *graded*" is the composite of $\widetilde{\mathbf{s}}$ with the functor "forgetting the cohomological degree" to super vector spaces. We denote this composite by $\mathbf{s}$.

Example. Let A be a graded mod 2 graded algebra (point of view **I**). It is given by a multiplication

$$\cdot : A \otimes A \to A.$$

Applying $\mathbf{s}$, we obtain a super algebra, with a of parity p and cohomological degree n becoming of parity $p+n$. The new product

$$*\colon \mathbf{s}(A) \otimes \mathbf{s}(A) \xleftarrow{\sim} \mathbf{s}(A \otimes A) \xrightarrow{\mathbf{s}(\cdot)} \mathbf{s}(A)$$

is

$$x * y = (-1)^{nq} x \cdot y$$

for x in $A^{p,n}$ and y in $A^{q,m}$. If A is commutative, meaning that for x of bidegree (p,n) and y of bidegree (q,m) one has $xy = (-1)^{pq+nm} yx$, then the superalgebra $(\mathbf{s}(A), *)$ is a commutative super algebra.

CHAPTER 2
Super Manifolds: Definitions

§§2.1–2.7. Super manifolds as ringed spaces

2.1. Many kinds of spaces can be defined as topological spaces endowed with a sheaf of algebras. This is the case for C^∞ or C^r manifolds, complex manifolds, complex analytic spaces or, in algebraic geometry, schemes. Super manifolds can be defined in this way, too. Let $\mathcal{C}^\infty$ be the sheaf of C^∞-functions on $\mathbb{R}^p$. The space $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$, endowed with the sheaf $\mathcal{C}^\infty[\theta^1, \ldots, \theta^q]$ of commutative super $\mathbb{R}$-algebras, freely generated over $\mathcal{C}^\infty$ by odd quantities $\theta^1, \ldots, \theta^q$. The coordinate functions t^i of $\mathbb{R}^p$, and the θ^j are the *coordinates* of $\mathbb{R}^{p|q}$. A *super manifold* M of dimension $p|q$ is a topological space $|M|$ endowed with a sheaf of super $\mathbb{R}$-algebras, which is locally isomorphic to $\mathbb{R}^{p|q}$. Its *structural sheaf* is denoted $\mathcal{O}_M$, or simply $\mathcal{O}$. By abuse of language, the sections of $\mathcal{O}_M$ are sometimes called *functions* on M. The odd functions generate a nilpotent ideal J of $\mathcal{O}_M$, and $(|M|, \mathcal{O}_M/J)$ is a C^∞-manifold of dimension p, i.e. is locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty)$. By the definition of a super manifold, it suffices to check this for $\mathbb{R}^{p|q}$, in which case it is clear by inspection. One calls $(|M|, \mathcal{O}_M/J)$ the *reduced* manifold M_{red} of M.

In the case of C^∞ manifolds, the sheaf $\mathcal{O}$, even if initially given just as a sheaf of $\mathbb{R}$-algebras, can be viewed as a sheaf of actual functions: the *value* of $f \in \Gamma(U, \mathcal{O})$ at $x \in U$ is the unique real number λ such that $f - \lambda$ is not invertible in any neighborhood V of x. The association $f \mapsto$ function "value of f at variable x" embeds $\mathcal{O}$ into the sheaf of continuous functions.

For M a super manifold, a section f of $\mathcal{O}_M$ has an image f_{red} in $\mathcal{O}_{M_{\text{red}}}$, which can be viewed as a C^∞ function on $|M|$. In particular, one can still speak of the *value* of f at x in $|M|$; it is the only real number λ such that $f - \lambda$ is not invertible in any neighborhood of x. However, we will have to live with the fact that sections of $\mathcal{O}_M$, even if called "functions on M", are not determined by their values at each point. For instance, an odd function takes the value 0 at each point $x \in |M|$.

An *open submanifold* U of M is an open subset $|U|$ of $|M|$, endowed with the restriction to $|U|$ of $\mathcal{O}_M$.

Morphisms between super manifolds are defined to be morphisms of ringed spaces: a morphism $f\colon M \to N$ is a continuous map $|f|\colon |M| \to |N|$ given with a morphism of sheaves of super $\mathbb{R}$-algebras from $|f|^*\mathcal{O}_N$ to $\mathcal{O}_M$. If V is an open subset of $|N|$ and if v in $\mathcal{O}_N(V)$, its image in $\mathcal{O}_M(f^{-1}V)$ is called the pullback of v by f and denoted $v \circ f$, $v(f)$, or f^*v.

2.2 Remarks. (i) In the purely even case of ordinary C^∞-manifolds, this notion of morphism agrees with the usual one. The point is that for f a morphism from M to N, and v a section of $\mathcal{O}_N$, if $f(x) = y$, the value of $f^*(v)$ at x is necessarily $v(y)$. Indeed, for $\lambda \neq v(y)$, $f^*(v) - \lambda = f^*(v - \lambda)$ is invertible in a neighborhood of x, with inverse $f^*((v-\lambda)^{-1})$.

(ii) The structural sheaf $\mathcal{O}$ of a super manifold M is a sheaf of local rings: its stalk $\mathcal{O}_x$ at a point $x \in |M|$ is a local ring, with maximal ideal the kernel of the evaluation map $f \mapsto f(x)$. A morphism $f\colon M \to N$ induces a morphism from M_{red} to N_{red} and, by (i), is a morphism of locally ringed spaces: the induced homomorphism, from $\mathcal{O}_{f(x)}$ to $\mathcal{O}_x$, maps the maximal ideal to the maximal ideal. Our definition hence agrees with that of Manin [**1988**] Ch. 4, §1.2.

2.3 Examples. (i) The quotient map from $\mathcal{O}_M$ to $\mathcal{O}_M/J$ defines a morphism $M_{\mathrm{red}} \hookrightarrow M$.

(ii) The inclusion of an open submanifold U of M is a morphism $U \hookrightarrow M$.

(iii) Let M_0 be a C^∞ manifold, and let V be a locally free module over the sheaf $\mathcal{C}^\infty$ of C^∞ functions on M_0. The space M_0, endowed with the sheaf $\wedge^* V$ of $\mathbb{R}$-algebras graded by the exterior degree modulo 2, is a super manifold M, with $M_{\mathrm{red}} = M_0$. The inclusion of $\mathcal{C}^\infty$ into $\wedge^* V$ defines a morphism $M \to M_{\mathrm{red}}$, which is a retraction to the embedding of M_{red} in M. Any super manifold is isomorphic to one constructed in this way, but not canonically so: if M is of dimension $p|q$ with $p \geq 1$ and $q \geq 2$, no retraction $M \to M_{\mathrm{red}}$ is invariant under the automorphisms of M.

If $f\colon M \to \mathbb{R}^{p|q}$ is a morphism of super manifolds, its *coordinates* f_i $(1 \leq i \leq p)$ and φ_j $(1 \leq j \leq q)$ are the pullbacks of the coordinates of $\mathbb{R}^{p|q}$. The definitions of super manifolds and of morphisms are reasonable only because one has:

Proposition 2.4. *Let $U \subset \mathbb{R}^{p|q}$ be an open submanifold of $\mathbb{R}^{p|q}$. The map $f \mapsto$ coordinates (f_i, φ_j) of f is a bijection from*

(i) *the set of morphisms $f\colon M \to U$, to*

(ii) *the set of systems of p even functions f_i and q odd functions φ_j on M, such that the values $(f_1(x), \ldots, f_p(x))$ for x in $|M|$ are in $|U| \subset \mathbb{R}^p$.*

This is proved in Leites [**1980**] 2.1.7, or Manin [**1988**] Ch. 4, §1.8. The main point is that if $f_1, \ldots, f_p$ and $\varphi_1, \ldots, \varphi_q$ are respectively even and odd functions on M, and if F is a function on $\mathbb{R}^{p|q}$, one can give a natural meaning to $F(f_1, \ldots, f_p, \varphi_1, \ldots, \varphi_q)$.

Suppose first that there are no φ. This means that $q = 0$ and F is a C^∞-function on $\mathbb{R}^p$. Suppose also that M is an open submanifold of $\mathbb{R}^{n|m}$, with coordinates $x_1, \ldots, x_n, \chi_1, \ldots, \chi_m$. A function u on M can then be expressed in a unique way as $u = \sum u_J \chi^J$, where the u_J are C^∞ functions on $|M|$ and where, for $J \subset [1, m]$, χ^J is the product of the χ_j for j in J, taken in order of increasing j. In particular, each of the even functions f_i can be decomposed as $f_i = f_i^0 + r_i$, where f_i^0 is a C^∞-function on $|M|$ and where r_i is even and nilpotent. One defines $F(f_1, \ldots, f_p)$ by Taylor's formula:

$$F(f_1, \ldots, f_p) := \sum \partial^{\mathbf{k}} F(f_1^0, \ldots, f_p^0) r^{\mathbf{k}} / \mathbf{k}! \,. \tag{2.4.1}$$

In the sum, only finitely many terms don't vanish. A general super manifold M is locally isomorphic to an open submanifold of some $\mathbb{R}^{n|m}$, and one should check that (2.4.1) gives, on M, a result independent of the local isomorphism used. In general, a function F on $\mathbb{R}^{p|q}$ (coordinates t^i, θ^j) can be uniquely written as $F = \sum F_I \theta^I$, with each F_I a C^∞ function on $\mathbb{R}^p$, and

$$F(f_1, \dots, f_p, \varphi_1, \dots, \varphi_q) := \sum F_I(f_1, \dots, f_p)\varphi^I \ . \tag{2.4.2}$$

The morphism to $\mathbb{R}^{p|q}$ defined by $f_1, \dots, f_p, \varphi_1, \dots, \varphi_q$ is given by the continuous map $(f_{1\,\mathrm{red}}, \dots, f_{p\,\mathrm{red}})$ and the pullback of functions $F \mapsto F(f_1, \dots, f_p, \varphi_1, \dots, \varphi_q)$.

2.5. Thanks to Proposition 2.4, the definition 2.1 of super manifolds is equivalent to a definition in terms of an atlas. Local charts are open subsets of $\mathbb{R}^{p|q}$. Gluing maps are isomorphisms $f\colon U \to V$, for U and V open subsets of $\mathbb{R}^{p|q}$. They are given by even and odd functions $f_1, \dots, f_p, \varphi_1, \dots, \varphi_q$, where each of those functions, say, u, has a decomposition $\sum u_I \theta^I$, with u_I a C^∞ function on the open subset U of $\mathbb{R}^n$. To be an isomorphism, f must admit an inverse $V \to U$.

2.6 Remarks. (i) The formula (2.4.1) involves an arbitrarily large number of derivatives. For this reason, C^r super manifolds don't make sense.

(ii) In infinite dimension, one should not try to define super manifolds as being ringed spaces. This has nothing to do with "super." Already for infinite dimensional (complex) spaces modeled on a Banach space, Douady (le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, Ann. Inst. Fourier tXVI 1 (1966) p. 1-95) observed that the useful notion is defined in terms of local charts; the sheaf of functions is insufficient information.

(iii) The infinite dimensional super manifolds M we will meet will be spaces of "fields." We will not even try to define them as topological spaces endowed with additional structures. In the spirit of the functor of points, explained in 2.8, 2.9, we only care to know what is a morphism from a finite dimensional super manifold S to M, and only the functor $S \mapsto \mathrm{Hom}(S, M)$ will be defined.

(iv) It follows from Proposition 2.4 that finite products exist in the category of super manifolds. For U and V open submanifolds of $\mathbb{R}^{p|q}$ and $\mathbb{R}^{r|s}$, $U \times V$ is the open submanifold of $\mathbb{R}^{p+r|q+s}$ with $|U \times V| = |U| \times |V|$. By 2.4, the defining property $\mathrm{Hom}(M, U \times V) \xrightarrow{\sim} \mathrm{Hom}(M, U) \times \mathrm{Hom}(M, V)$ holds. The general case follows by gluing.

(v) More generally, if $f\colon M \to S$ is locally (on M) isomorphic to $\mathrm{pr}_2\colon \mathbb{R}^{p|q} \times S \to S$, then, for any morphism $u\colon S' \to S$, the fiber product $M \times_S S'$ exists. Its defining property is that for any T, $\mathrm{Hom}(T, M \times_S S') \xrightarrow{\sim} \mathrm{Hom}(T, M) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, S')$. The projection $f'\colon M' := M \times_S S' \to S'$ is said to be deduced from f by *base change* from S to S'.

(vi) Complex super manifolds can similarly be defined as topological spaces endowed with a sheaf of $\mathbb{C}$-algebras, locally isomorphic to some $(\mathbb{C}^p, \mathcal{O}[\theta^1, \dots, \theta^q])$, for $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}^p$. Same definition for "real analytic", with $\mathbb{C}^p$ replaced by $\mathbb{R}^p$ and $\mathcal{O}$ by the sheaf of real analytic functions.

(vii) In Manin [**1980**] Ch. 4, §1.6, analytic super spaces and super schemes are similarly defined as ringed spaces.

(viii) The reader may prefer to add to the definition of super manifold the condition that the underlying topological space be Hausdorff and σ-compact.

2.7. The difficulty that functions are not determined by their values is familiar in algebraic geometry. There, as here, it is due to the fact that the structural sheaf $\mathcal{O}$ can have nilpotent elements.

In complex algebraic geometry, the simplest example is afforded by the affine algebraic variety $\mathrm{Spec}(D)$ with algebra of functions the algebra of dual numbers $D := \mathbb{C}[\varepsilon]/(\varepsilon^2)$. A morphism from $\mathrm{Spec}(D)$ to the affine space A^n is simply a morphism of $\mathbb{C}$-algebras from $\mathbb{C}[t_1, \dots, t_n]$ to D. It is the same as the data of a point $z = (z_1, \dots, z_n)$ of A^n, and of a tangent vector v at z. The corresponding morphism of $\mathbb{C}$-algebras is $P \mapsto P(z) + \varepsilon \partial_v P(z)$.

Similarly, a morphism from $\mathbb{R}^{0|1}$, with ring of functions $\mathbb{R}[\theta]$, to $\mathbb{R}^{p|q}$ (coordinates $t_1, \dots, t_p, \theta_1, \dots, \theta_q$) is the data of a point $(x_1, \dots, x_p)$ of $\mathbb{R}^p$, and of an odd morphism ∂ from the local ring at x to $\mathbb{R}$, which obeys the Leibniz identity $\partial(fg) = \partial f.g(x) + (-1)^{p(f)} f(x).\partial g$. The corresponding morphism of rings is $f \mapsto f(x) + \theta \partial f$. (In §3.3 we shall see that such morphisms are odd tangent vectors to $\mathbb{R}^{p|q}$.)

In both cases, a space M can be thought of as a reduced space M_{red}, surrounded by a nilpotent fuzz. An automorphism $f\colon M \to M$ can be the identity on M_{red}, and shear the fuzz. Example: the automorphism $(t, \theta_1, \theta_2) \mapsto (t + \theta_1\theta_2, \theta_1, \theta_2)$ of $\mathbb{R}^{1|2}$, i.e. the automorphism with coordinates $(t + \theta_1\theta_2, \theta_1, \theta_2)$ (cf. 2.4) does not respect the projection $t\colon \mathbb{R}^{1|2} \to \mathbb{R}^{1|0}$.

In algebraic geometry, nonreduced space are defined to be singular. Not so here. Another, related, difference is that $\mathbb{R}^{0|1}$, and more generally $\mathbb{R}^{p|q}$, admits the group law

$$+\colon \mathbb{R}^{p|q} \times \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}\ , \tag{2.7.1}$$

with coordinates by $(t_1' + t_1'', \dots, t_p' + t_p'', \theta_1' + \theta_1'', \dots, \theta_q' + \theta_q'')$. Not so for $\mathrm{Spec}(D)$: there is no morphism from D to $D \otimes D = \mathbb{C}[\varepsilon', \varepsilon'']/(\varepsilon'^2, \varepsilon''^2)$ mapping ε to $\varepsilon' + \varepsilon''$. Indeed, $(\varepsilon' + \varepsilon'')^2 = \varepsilon'\varepsilon'' + \varepsilon''\varepsilon' = 2\varepsilon'\varepsilon'' \neq 0$. For ε' and ε'' odd, the mixed terms would cancel, and we obtain a group law.

§§2.8–2.9. The functor of points approach to super manifolds

2.8. The language of ringed spaces can be cumbersome. As in algebraic geometry, an approach closer to the geometric intuition is the language of the functor of points. This approach is also closer to the way physicists make computations.

Let S be a super manifold. An *S-point* of a super manifold M is a morphism $S \to M$. It should be thought of as a family of points of M parametrized by S. The set $M(S)$ of S-points of S is functorial in S: a morphism $u\colon T \to S$ induces the map $m \mapsto m \circ u$ from $M(S)$ to $M(T)$. An S-point of M can be identified with a section of the projection $M \times S \to S$. From this point of view, the functoriality in S is given by the base change by $T \to S$.

A morphism $f\colon M \to N$ induces for each S a map f_S from $M(S)$ to $N(S)$, and this map is functorial in S. This construction is a bijection from the set of morphisms $f\colon M \to N$ to the set of systems of maps $f_S\colon M(S) \to N(S)$, functorial in S. This fact is called Yoneda's lemma. Its proof is deceptively trivial. The inverse construction attaches to (f_S) the image by $f_M\colon \mathrm{Hom}(M, M) \to \mathrm{Hom}(M, N)$ of the identity map of M.

Examples. (i) For S reduced to a point, i.e. to $\mathbb{R}^{0|0}$, an S-point of M is simply a point of $|M|$.

(ii) By 2.4, an S-point of $\mathbb{R}^{p|q}$ can be identified with a system of p even functions and q odd functions on S.

2.9. To construct a super manifold M, it is often best to first construct the corresponding functor of points, and to prove that this functor is representable, i.e. corresponds to a super manifold. To construct a morphism $f: M \to N$ it is often best to construct the corresponding functorial map $M(S) \to N(S)$. This requires systematically working "over a base S". Instead of super manifolds, one should consider *families of super manifolds parametrized by* S, i.e. morphisms $M \to S$ which are isomorphic locally on M to a projection $U \times S \to S$. Together with $\varphi: M \to S$, one should systematically consider the families deduced from it by a base change $g: S' \to S$, i.e. the fiber products $M_{(S')} := M \times_S S'$, see (2.6(v), and one should use only constructions which are *geometric*, i.e. compatible with any base change. The construction $M \mapsto M_{\mathrm{red}}$ is not geometric, as in general $M_{\mathrm{red}} \times_S S' \neq (M \times_S S')_{\mathrm{red}}$.

We will need the relative version of Yoneda's lemma. Let $\varphi: M \to S$ be a family of super manifolds parametrized by S. For $g: T \to S$, a T-point of M/S is a section of $M_{(T)}$. If $\psi: N \to S$ is another family, giving an S-map $f: M \to N$, i.e. a map f such that $\psi f = \varphi$, is equivalent to giving functorially in T, a map f_T: (T-points of M/S) $\to$ (T-points of N/S)).

Functions can be defined by their values at each point too, if "point" is taken to mean "T-point": for M/S, giving a function f on M is equivalent to giving functorially in $T \to S$, for each T-point t of M/S, the value of f at t (the function t^*f on T). First proof: as in the proof of Yoneda's lemma, take $T = M$, t =identity. Second proof: reduce to Yoneda's lemma, identifying even (resp.odd) functions to maps to $\mathbb{R}^{1|0}$ (resp. $\mathbb{R}^{0|1}$).

§2.10. Super Lie groups

A *super Lie group* is a group object in the category of super manifolds: it is a super manifold G given with a product law $\mu: G \times G \to G$ such that there exists a unit $e: (\mathbb{R}^{0|0}) \to G$ and an inverse map: $G \to G$ making standard diagrams commute. By Yoneda's lemma, the data of a product law μ amounts to that of a composition law μ_S on the set of S-points $G(S)$, functorial in S, and the condition that G be a super Lie group is that each $G(S)$ be a group.

A convenient way to define a super Lie group is to first define the group-valued functor $G(S)$, and then to check that it is representable by a super manifold, which is then automatically a super Lie group. See the examples in 2.11 below. A useful representability criterion is the following. Suppose that a super Lie group H acts on a super manifold M, and that m is a point of M. For each S, $H(S)$ acts on $M(S)$, and m defines m_S in $M(S)$. Define $G(S)$ to be the subgroup of $H(S)$ which fixes m_S. Then $S \mapsto G(S)$ is representable by a submanifold of H, the stabilizer of m in G.

One has the usual relation between super Lie groups and super Lie algebras. For more details, see §3.3.5 and §3.8. The topological subtleties concern only the even part. For instance: If G is a super Lie group, with Lie algebra $\mathfrak{g}$, G_{red} is a Lie subgroup, with Lie algebra the even part $\mathfrak{g}_0$ of $\mathfrak{g}$. If G and H are connected super Lie groups, a morphism f of Lie algebras from $\mathfrak{g} = \mathrm{Lie}\, G$ to $\mathfrak{h} = \mathrm{Lie}\, H$ extends to a

(unique) morphism of groups if and only if f_0: $\mathfrak{g}_0 \to \mathfrak{h}_0$ extends to a morphism of ordinary Lie groups from G_{red} to H_{red}.

§2.11. Classical series of super Lie groups

Here are descriptions by their functors of points of the linear groups, the orthosymplectic groups, and the groups of the P and Q series (cf. Manin [**1988**], Ch. 4 §10).

$\mathrm{GL}(p|q)$. The group of S-points is the group of automorphisms of the sheaf of $\mathcal{O}_S$-modules $\mathcal{O}_S^{p|q}$. The description by matrices shows that it is an open submanifold of some affine superspace.

For each S, the Berezinian (1.10) provides a morphism from the group of S-points of $\mathrm{GL}(p,q)$ to the group of S-points of $\mathrm{GL}(1|0)$. Those morphisms define a morphism of super Lie groups

$$\text{(2.11.1)} \qquad \mathrm{Ber}\colon \mathrm{GL}(p|q) \to \mathrm{GL}(1|0)\ .$$

The kernel of the (2.11.1) is the special linear group $\mathrm{SL}(p|q)$.

$\mathrm{OSp}(n|2m)$: Fix an even supersymmetric nondegenerate bilinear form Φ on the super vector space $\mathbb{R}^{n|2m}$ with values in $\mathbb{R}^{1|0}$. In non-super terms, Φ is given by a nondegenerate symmetric bilinear form on $\mathbb{R}^n$, and a nondegenerate alternating form on $\mathbb{R}^{2m}$. The corresponding group $\mathrm{O}(\Phi)$ is the subgroup of $\mathrm{GL}(n|2m)$ preserving Φ: the group of S-points is the group of automorphisms of $\mathcal{O}^{n|2m} = \mathcal{O} \otimes_{\mathbb{R}} \mathbb{R}^{n|2m}$ preserving Φ.

$\pi\,\mathrm{Sp}(n|n)$: Same as above with Φ odd, and antisymmetric in the super sense. The groups of the P series are the kernels, in $\pi\,\mathrm{Sp}$, of the Berezinian.

Q series: Let D be the super division algebra $\mathbb{R}[\eta]$ with η odd and $\eta^2 = -1$ (cf. 1.1.6). The S-points of the Lie group $\mathrm{GL}(n,D)$ are the automorphisms of the sheaf of right $\mathcal{O} \otimes D$-modules $(\mathcal{O} \otimes D)^n$.

For $n = 1$, this is $(\mathcal{O} \otimes D)_0^*$: the invertible $a + \alpha\eta$, with a even and α odd, with the multiplication law

$$(a + \alpha\eta)(b + \beta\eta) = (ab + \alpha\beta) + (a\beta + \alpha b)\eta\ .$$

The Lie algebra is the free Lie algebra in one odd generator.

The map $a + \alpha\eta \mapsto a + \alpha$ identifies this group with the multiplicative group of $\mathcal{O} = \mathcal{O}_0 \oplus \mathcal{O}_1$. The map

$$a + \alpha\eta \longmapsto \alpha/a$$

is a morphism from $\mathrm{GL}(1,D)$ to $\mathbb{R}^{0|1}$. Indeed,

$$(a\beta + \alpha b)/(ab + \alpha b) = (\beta/b + \alpha/a)/(1 - (\alpha/a)\cdot(\beta/b)) = \beta/b + \alpha/a$$

as $\alpha^2 = \beta^2 = 0$. As in the story of Dieudonné's determinants (Dieudonné, les déterminants sur un corps non commutatif, Bull. S.M.F. 71 (1943), p. 27-45) for linear groups over skew fields, this morphism extends to a morphism of Lie groups, called the odd determinant

$$\mathrm{o\,det}\colon \mathrm{GL}(n,D) \to \mathbb{R}^{0|1}.$$

For $n \geq 2$, the kernels of the odd determinant, divided by the diagonal subgroup $\mathrm{GL}(1|0)$, are the Lie groups of the Q series.

CHAPTER 3
Differential Geometry of Super Manifolds

§3.1. Introduction

Many notions and results from ordinary differential geometry extend to super manifolds in a straightforward way. Vector bundles on M can be interpreted as sheaves of locally free super modules over $\mathcal{O}_M$. A super manifold M has a tangent bundle T_M. If M is of dimension $p|q$, the vector bundle T_M is a locally free super $\mathcal{O}_M$-module of dimension $p|q$. A morphism $f\colon M \to N$ of super manifolds induces a morphism $df\colon T_M \to f^*T_N$. Analogues of the inverse and of the implicit function theorem, as well as of the Frobenius theorem, hold. There is the usual relationship between actions of Lie algebras and local actions of Lie groups, and in particular between even vector fields and flows. From this follows a definition of the Lie derivative $\mathcal{L}_X$ of a section of any canonical vector bundle with respect to a vector field X.

The dual of the tangent bundle is the cotangent bundle Ω^1, and $d\colon \mathcal{O}_M \to \Omega^1_M$ is a universal derivation. For $\Omega^*_M = \wedge^*\Omega^1_M$, with $\wedge^*$ understood as in 1.3.5, 1.3.6, d extends uniquely to a square zero derivation, the *exterior derivative* $d\colon \Omega^*_M \to \Omega^{*+1}_M$. The Poincaré lemma holds. As a consequence, the cohomology of the topological space $|M|$ is $H^*(\Gamma(M, \Omega^*_M), d)$.

A connection on a vector bundle $\mathcal{V}$ is $\nabla\colon \mathcal{V} \to \Omega^1 \otimes \mathcal{V}$ obeying the Leibniz identity. It extends to $\nabla\colon \Omega^* \otimes \mathcal{V} \to \Omega^{*+1} \otimes \mathcal{V}$ and the curvature F in $\Omega^2 \otimes \underline{\mathrm{End}}\,(\mathcal{V})$ describes ∇^2. It is the obstruction to $\mathcal{V}$ being flat, that is to $\mathcal{V}$ admitting a local basis (e_i) with $\nabla e_i = 0$. As classically, one can consider more generally connections on G-torsors (a.k.a. principal G-bundles) with G being a super Lie group, the case of vector bundles corresponding to $G = \mathrm{GL}(r|s)$.

For M of dimension $p|q$ with $q \neq 0$, there are forms of all degrees $n \geq 0$: there are no top forms. An n-form can be integrated on a $n|0$-dimensional submanifold. What can be integrated on M are densities, which can be interpreted as sections of $\mathrm{Ber}(\Omega^1_M)$, tensored with the orientation sheaf of M. One can also define *integral forms*, which can be integrated on submanifolds of codimension $p|0$ and give rise to a super version of Stokes' theorem. Densities are the top integral forms.

§3.2. Vector bundles

In classical differential geometry, vector bundles over a manifold M can be viewed in two ways.

(i) As a fiber bundle V over M, with typical fiber $\mathbb{R}^n$ and structural group $\mathrm{GL}(n)$, for some n. It is a manifold V mapping to M, with an additional structure.

(ii) As a sheaf $\mathcal{V}$ of locally free modules of finite type over $\mathcal{C}^\infty_M$.

The same two points of view can be used in supergeometry. A vector bundle of rank $p|q$ over M is

(i′) a fiber bundle V over M, with typical fiber $\mathbb{R}^{p|q}$ and structural group $\mathrm{GL}(p,q)$;

(ii′) a sheaf of $\mathcal{O}_M$-super modules $\mathcal{V}$, locally free of dimension $p|q$.

The super manifold V corresponding to the super module $\mathcal{V}$ is most easily defined by its functor of points: after any base change, a section of V over M is an even section of $\mathcal{V}$. In other words, for any super manifold S, the set of S-points of V can be identified with the set of pairs (f,v), where $f\colon S \to M$ is an S-point of M and v is an even section of the pullback $f^*\mathcal{V}$ of $\mathcal{V}$ to S. To check that the functor $S \to \{$ set of pairs $(f,v)\}$ is representable can be reduced to the same question locally on M. Locally, $\mathcal{V}$ is free and we can choose a basis $e_1,\dots,e_p$, $f_1,\dots,f_q$ of $\mathcal{V}$. Proposition 2.4 then shows that the functor considered is represented by $\mathbb{R}^{p|q}\times M$. Another basis will define another isomorphism of V with $\mathbb{R}^{p|q}\times M$, deduced from the first one by $g\colon M \to \mathrm{GL}(p|q)$. This gives the fiber space structure of V.

Warnings. (i) It is dangerous to call "even" a vector bundle of dimension $(r|0)$: if M is of dimension $p|q$ with $q \neq 0$, the odd part $\mathcal{V}_1$ is not zero. Indeed, $\mathcal{V}$ is locally isomorphic to $\mathcal{O}^r_M$ and $\mathcal{O}_M$ has a non-trivial odd part.

(ii) Similarly, if M is not purely even, the mod 2 grading of a vector bundle $\mathcal{V}$ does not decompose it into the sum of an even and an odd bundle.

(iii) The parity changed $\mathcal{V}$, denoted $\prod\mathcal{V}$, has been defined in (1.4, Warning). The spaces V and $\prod V$ corresponding to $\mathcal{V}$ and $\prod\mathcal{V}$ don't have the same underlying topological space. If $\mathcal{V}_{\mathrm{red}}$ is the restriction of $\mathcal{V}$ to M_{red}, $|V|$ (resp. $|\prod V|$) is the vector bundle over $|M| = M_{\mathrm{red}}$ corresponding to $(\mathcal{V}_{\mathrm{red}})_0$ (resp. $(\mathcal{V}_{\mathrm{red}})_1$).

§3.3. The tangent bundle, the cotangent bundle and the de Rham complex

On $\mathbb{R}^{p|q}$ (coordinates t^i and θ^j), the even derivations $\partial/\partial t^i$ and the odd derivations $\partial/\partial\theta^j$ of the structural sheaf $\mathcal{O}$ are defined as follows. For $f = \sum f_I\theta^I$, with each f_I a C^∞ function on $\mathbb{R}^p$,

$$\partial/\partial t^i(f) := \sum \partial/\partial t^i(f_I)\theta^I \ .$$

For f written as $\sum_{j\notin I}(f_I\theta^I + f_{j,I}\theta^j\theta^I)$,

$$\partial/\partial\theta^j(f) := \sum f_{j,I}\theta^I \ .$$

As proved in Leites [**1980**] 2.2.3, one has

3.3.1 Proposition. *The $\mathcal{O}$-module of $\mathbb{R}$-linear derivations of $\mathcal{O}$ is free of dimension $p|q$, with basis the $\partial/\partial t^i$ and the $\partial/\partial\theta^j$.*

A super manifold M of dimension $p|q$ being locally isomorphic to $\mathbb{R}^{p|q}$, it follows that the $\mathcal{O}$-module of derivations of $\mathcal{O}_M$ is a vector bundle of dimension $p|q$. It is the *tangent bundle* T_M of M. The sections of T_M are called *vector fields*. These are the derivations on $\mathcal{O}_M$. The super vector space $\mathrm{Vect}(M)$ of all vector fields is a super Lie algebra, see (1.8.4).

The *cotangent bundle* of a super manifold M is the dual Ω^1_M of T_M. We will systematically write the duality pairing between the tangent and cotangent bundle as

$$\langle\ ,\ \rangle\colon T_M \otimes \Omega^1_M \to \mathcal{O}_M\ ,$$

with $\langle uD, v\omega\rangle = (-1)^{p(D)p(v)}uv\,\langle D,\omega\rangle$ for $u, v \in \mathcal{O}_M$. We will avoid the notation $\omega(D)$ which by the sign rule should be taken to mean $(-1)^{p(D)p(\omega)}\,\langle D,\omega\rangle$.

One defines $d\colon \mathcal{O} \to \Omega^1_M$ by

$$\langle D, df\rangle = Df\ .$$

This is the universal derivation of even parity with values in an $\mathcal{O}$-module. As classically, d extends uniquely to a square zero derivation d of the commutative graded super algebra $\Omega^*_M := \wedge^*\Omega^1_M$ (1.3.5):

$$d^2 = 0 \tag{3.3.2}$$

$$d(\alpha\beta) = d\alpha\cdot\beta + (-1)^p\alpha\cdot d\beta \quad \text{for} \quad \alpha \quad \text{in} \quad \Omega^p_M\ . \tag{3.3.3}$$

This defines the differential graded algebra of differential forms on a super manifold. If M is of dimension $p|q$ with $q \neq 0$, $\Omega^n_M \neq 0$ for all $n \geq 0$: there are no "top forms". A p-form can be restricted to a $p|0$ dimensional submanifold, and integrated if the submanifold is oriented (this is especially meaningful over a base S). Objects of a different kind, the integral forms explained in 3.12 below, can be integrated on submanifolds of codimension $p|0$.

3.3.4 Poincaré lemma. *The complex Ω^*_M is a resolution of the constant sheaf $\mathbb{R}$ on $|M|$.*

Proof. On $\mathbb{R}^{p|q}$, the de Rham complex Ω^*_M is the algebraic tensor product of the pullback of the de Rham complexes on the factors $\mathbb{R}^{p|0}$ and $\mathbb{R}^{0|q}$. This reduces the Poincaré lemma to the classical case of $\mathbb{R}^p$, and to the case of $\mathbb{R}^{0|q}$. The latter can further be reduced to the case of $\mathbb{R}^{0|1}$. For $\mathbb{R}^{0|1}$, Ω^n is $\mathbb{R}[\theta]\cdot(d\theta)^n$, with $d[(a+b\theta)(d\theta)^n] = b(d\theta)^{n+1}$. That Ω^* is a resolution of $\mathbb{R}$ follows by inspection.

The sheaves Ω^i_M are soft. It hence follows from the Poincaré lemma that the cohomology $H^*(|M|,\mathbb{R})$ of the topological space $|M|$ can be computed by the de Rham complex Ω^*_M:

$$H^*(|M|,\mathbb{R}) = H^*(\Gamma(M,\Omega^*_M))\ . \tag{3.3.5}$$

Remark. The algebra Ω^*_M is a commutative graded super algebra, in the sense of 1.3.6. Its relation with the parity changed tangent bundle is as follows. With the notation of the Appendix to §1, $\mathbf{s}(\Omega^*_M)$ is a commutative super algebra of functions on the parity changed tangent bundle, viewed as a fiber bundle over M. It consists of those functions which are fiberwise polynomial.

The parity changed tangent bundle ΠT represents the functor $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ of maps from $\mathbb{R}^{0|1}$ to M: for any S, its S-points are the S-morphisms from $\mathbb{R}^{0|1} \times S$ to $M \times S$. As $\mathbb{R}^{0|1}$ is a group, it acts on $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ by translations on $\mathbb{R}^{0|1}$. This defines an action of $\mathbb{R}^{0|1}$ on ΠT. The Lie algebra of $\mathbb{R}^{0|1}$ is of dimension $0|1$, with a basis reduced to one odd element D, with $[D, D] = 0$. It acts on ΠT. Possibly up to a sign, D acts on $\mathbf{s}(\Omega^*_M)$ as $\mathbf{s}(d)$: $\alpha \mapsto (-1)^q d\alpha$ for α of parity q.

3.3.6 The super Lie algebra of a super Lie group

A vector field X on a super Lie group G is said to be *left invariant* if for any S, and any S-point g of G, the corresponding vertical vector field on $G_S := G \times S \to S$ is invariant under the left translation by g. As usual, it is enough to consider the universal case $S = G$ and g the identity map: the vector field X_1 deduced from X, vertical relatively to pr_2: $G \times G \to G$, should be invariant by $(g, h) \to (hg, h)$.

By transport of structure, diffeomorphisms, for instance left translations, preserve brackets. It follows that the left invariant vector fields on G form a sub super Lie algebra of the algebra of all vector fields on G. By evaluation at the unit element e of G, the space of left invariant vector fields maps bijectively to the tangent space T_eG of G at e. Transporting to T_eG the bracket of left invariant vector fields, one obtains the *Lie algebra* of G.

Left invariant vector fields can be viewed as infinitesimal right translations: the left invariant vector field with value X at e is the corresponding derivative in h of gh: $G \times G \to G$.

Right invariant vector fields are defined similarly. The involution $g \mapsto g^{-1}$ of G exchanges left and right translations. It hence maps left (resp. right) invariant to right (resp. left) invariant vector fields. If by evaluation at e one identifies T_eG with the space of right invariant vector fields, the resulting bracket $[\ ,\ ]_r$ is the opposite of the Lie algebra bracket. Indeed, $g \mapsto g^{-1}$ induces $x \mapsto -x$ on T_eG, hence $[-x, -y] = -[x, y]_r$.

Example. Consider the super Lie group structure on $\mathbb{R}^{1|1}$ with product law

$$(t^1, \theta^1) \cdot (t^2, \theta^2) = (t^1 + t^2 + \theta^1\theta^2, \theta^1 + \theta^2)\ .$$

The left invariant vector fields are obtained by derivation in (t^2, θ^2), at $(t^2, \theta^2) = 0$, of the product law. At $(0, 0)$, one has

$$\begin{aligned} \partial_t(t^1 + t + \theta^1\theta) &= 1 & \partial_\theta(t^1 + t + \theta^1\theta) &= -\theta^1 \\ \partial_t(\theta^1 + \theta) &= 0 & \partial_\theta(\theta^1 + \theta) &= 1, \end{aligned}$$

and the left invariant vector fields are hence spanned by ∂_t and $-\theta\partial_t + \partial_\theta$. A similar computation shows that the right invariant vector fields are spanned by ∂_t and $\theta\partial_t + \partial_\theta$.

§3.4. The inverse and implicit function theorems

Results which classically are proved by using a sequence of successive approximations can often, in the super case, be proved by first using the classical result on the reduced variety, then finding "successive approximations" by working modulo higher and higher power of the ideal generated by the odd functions. As this ideal is nilpotent, the process terminates. An example is the inverse function theorem, as proved by Leites [**1980**] 2.3.1.

3.4.1. Theorem. *If, at one point x of M, the morphism $f: M \to N$ induces an isomorphism from the tangent space of M at x to the tangent space of N at $f(x)$, then f induces an isomorphism from a neighborhood of x to one of $f(x)$.*

In local coordinates, the condition on f is that its Jacobian matrix be invertible.

This theorem has the usual corollaries: implicit function theorem, Jacobian criterion for $f: M \to N$ to be a submersion (= family of manifolds parametrized by N, see 2.7) or an immersion, structure of morphisms of "constant rank". See Leites (1980) 2.3. One also has relative variants for families of manifolds.

§3.5. Distributions

Let M be a super manifold with tangent bundle T. A *distribution* on M is an $\mathcal{O}_M$-submodule $\mathcal{D}$ of T which is locally a direct factor. The bracket of vector fields, i.e. of derivations, induces an $\mathcal{O}$-linear map

$$\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D} \to T/\mathcal{D}$$

the *Frobenius bilinear form.* A distribution $\mathcal{D}$ is said to be *integrable* if its Frobenius bilinear form vanishes, i.e. if $\mathcal{D}$ is stable under bracket. As classically, one has the

3.5.1. Theorem. *If the distribution $\mathcal{D}$ is integrable, then the super manifold M admits locally a product structure $M = M' \times M''$, giving a decomposition $T = T' \oplus T''$ of the tangent bundle, with $\mathcal{D} = T'$.*

In other words, there exists locally a submersion $M \to M/\mathcal{D}$, for which $\mathcal{D}$ is the relative tangent bundle.

For $\mathcal{D}$ of dimension $1|0$, the theorem results from the

Lemma 3.5.2. *If an even vector field D does not vanish at the point x, then a neighborhood of x admits a coordinate system $(x^1, \ldots, \theta^q)$ for which $D = \partial/\partial x^1$.*

Proof. Let $\mathcal{J} \subset \mathcal{O}$ be the ideal generated by the odd functions. As D is even, it maps $\mathcal{J}$ to $\mathcal{J}$ and induces a vector field D_{red} on M_{red}. On the vector bundle $\mathcal{J}/\mathcal{J}^2$ on M_{red}, it also induces a connection in the direction of D_{red}, i.e. an operator $\partial: \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/\mathcal{J}^2$ such that $\partial(fx) = D_{\mathrm{red}}(f)x + f\partial x$. By classical results, one can find a local coordinate system on M_{red} and a local basis of $\mathcal{J}/\mathcal{J}^2$ for which D_{red} (resp. the connection in the direction of D_{red}) are given by $\partial/\partial x_1$.

By (3.4.1), if we lift the coordinates of M_{red} to even functions on M, and the basis of $\mathcal{J}/\mathcal{J}^2$ to sections of $\mathcal{J}$, we obtain a local coordinate system on M. In it, we will have

$$D = \partial/\partial x^1 \bmod \mathcal{J}^2 T \ .$$

This is the beginning of a sequence of successive approximations in which, starting from a local coordinate system in which $D = \partial/\partial x^1 \bmod \mathcal{J}^k T$ $(k \geq 2)$, one constructs a new one in which $D = \partial/\partial x^1 \bmod \mathcal{J}^{k+1} T$. To do this, one looks for a diffeomorphism

$$(x^i, \theta^j) \to (x^i + X^i, \theta^j + \chi^j) \tag{3.5.3}$$

with X^i, χ^j in $\mathcal{J}^k$, which transforms D into $\frac{\partial}{\partial x^1} \bmod \mathcal{J}^{k+1} T$. [Formula (3.5.3) describes, for any S, the coordinates of the image of the S-point with coordinates (x^i, θ^j), or equivalently the pullback of the coordinate functions.] Let X be the vector field $\sum X^i \partial/\partial x^i + \sum \chi^j \partial/\partial \theta^j$. One has

$$\text{transform of } D \equiv D - [X, D] \equiv D + [\partial/\partial x_1, X] \pmod{\mathcal{J}^{k+1} T}.$$

One observes that $\partial/\partial x_1$ is onto and hence the inductive step can be carried out. As explained in 3.4, this process terminates.

Proof of 3.5.1. Suppose $\mathcal{D}$ is of dimension $r|s$. As long as $r > 0$, one can choose in $\mathcal{D}$ an even vector field D, generating a direct factor. Applying 3.5.2, we may and shall assume that D is $\partial/\partial x^1$. Applying the assumption that $[\partial/\partial x^1, \mathcal{D}] \subset \mathcal{D}$, one checks that $\mathcal{D}$ comes from a distribution of dimension $(r-1, s)$ on the quotient space with coordinates $x^2, \ldots, \theta^s$. This distribution is also stable under bracket and, iterating that construction, one reduces to the case where $r = 0$.

We now suppose that $r = 0$. Let $D_1, \ldots, D_s$ be odd derivations forming a basis of $\mathcal{D}$. For $f_1, \ldots, f_s$ odd functions, $D := \sum f_i D_i$ is an even derivation. The exponential series $\exp(D) := \sum D^n/n!$ terminates and $\exp(D)$: $\mathcal{O} \to \mathcal{O}$ is an automorphism. There is a unique diffeomorphism of M, still noted $\exp(D)$, so that *pullback* of functions by the diffeomorphism is the map $\exp(D)$ on $\mathcal{O}$. If we perform this construction on $\mathbb{R}^{0|s} \times M$, taking the f_i to be the coordinate functions θ^i of $\mathbb{R}^{0|s}$, we obtain an automorphism of $\mathbb{R}^{0|s} \times M$, compatible with the projection to $\mathbb{R}^{0|s}$. Projecting to M, we obtain a morphism

$$\exp\left(\sum \theta^i D_i\right) : \mathbb{R}^{0|s} \times M \to M\ . \tag{3.5.4}$$

If $T \subset M$ is of codimension $0|s$, and transversal to $\mathcal{D}$, the induced morphism

$$\exp\left(\sum \theta^i D_i\right) : \mathbb{R}^{0|s} \times T \to M \tag{3.5.5}$$

is a diffeomorphism. We identify $\mathbb{R}^{0|s} \times T$ and M by this diffeomorphism. By construction the distribution $\mathcal{D}$ is tangent to $\mathbb{R}^{0|s}$-factors along $\{0\} \times T$. We need to show that the same is true everywhere, or in other words that (3.5.4) preserves the projection to T. The Campbell-Hausdorff formula for $\exp\left(\sum \alpha^i D_i\right) \exp\left(\sum \theta^i D_i\right)$ terminates. This shows that the diffeomorphisms $\exp\left(\sum \alpha^i D_i\right)$ respect the projection to T, which is the required submersion.

§3.6. Connections on vector bundles

The formalism of connections on vector bundles gives rise to no surprise, but some questions of signs and of notations have to be treated carefully. We concentrate on them.

A connection on $\mathcal{V}$ is a morphism in the category of sheaves of super vector spaces $\nabla\colon \mathcal{V} \to \Omega^1 \otimes \mathcal{V}$ obeying the Leibniz identity

$$\nabla(fv) = df \otimes v + f\nabla v. \tag{3.6.1}$$

For a vector field X, one then defines a section $\nabla_X v$ of $\mathcal{V}$ by

$$\nabla_X v = \langle X, \nabla v \rangle\,, \tag{3.6.2}$$

with $\langle X, \alpha \otimes v\rangle := \langle X, \alpha\rangle\, v$. From (3.6.1), it follows that

$$\nabla_X(fv) = Xf \cdot v + (-1)^{p(X)p(f)} f \nabla_X v\ . \tag{3.6.3}$$

One extends ∇ to an endomorphism of $\Omega^* \otimes \mathcal{V}$ by requiring that for α in Ω^p, one has

$$\nabla(\alpha x) = d\alpha \cdot x + (-1)^p \alpha \cdot \nabla x\ , \tag{3.6.4}$$

a formula which generalizes (3.3.3). With our convention of separating parity and cohomological degree, the sign in (3.6.4) does not depend on the parity of α. Only its cohomological degree matters. The operator ∇, of cohomological degree 1, is of parity 0.

The curvature, in $\Omega^2 \otimes \underline{\mathrm{End}}\,(\mathcal{V})$, is defined by

$$\nabla^2 x = F \cdot x \tag{3.6.5}$$

for x in $\Omega^* \otimes \mathcal{V}$. The curvature is of even parity. The product $F \cdot x$ has the following meaning: for T an even or odd endomorphism of $\mathcal{V}$, α in Ω^2, β in Ω^p and v in $\mathcal{V}$,

$$(\alpha \otimes T) \cdot (\beta \otimes v) = (-1)^{p(T)p(\beta)} (\alpha \wedge \beta) \otimes T(v)\ .$$

A vector field X defines a contraction operator ι_X on Ω^* of the same parity as X and of cohomological degree -1. It is characterized by the properties that $\iota_X f = 0$ for f in Ω^0, that $\iota_X \alpha = \langle X, \alpha\rangle$ for α in Ω^1 and that it is a derivation of the commutative graded super algebra Ω^*: for α in Ω^k,

$$\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^k (-1)^{p(X)p(\alpha)} \alpha \wedge \iota_X(\beta)\ .$$

Classically, for α a k-form and X_i vector fields, one defines $\alpha(X_1, \ldots, X_k)$ as follows: if $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_k$ for 1-forms α_i, then

$$\alpha(X_1, \ldots, X_k) = \det(\alpha_i(X_j))\ . \tag{3.6.6}$$

This definition does not follow the Koszul sign rule, but this is no obstruction to extending it to the super case, thanks to the separation of the roles of cohomological

degree and of parity. Using the "even rules" principle, one defines $\alpha(X_1, \ldots, X_k)$ by requiring (3.6.6) to hold when α and the X_i are of even parity.

For a k-form with values in a vector bundle $\mathcal{E}$, i.e. for α in $\Omega^k \otimes \mathcal{E}$, the sign rule requires in addition that $(\alpha \otimes e)(X_1, \ldots, X_k) = (-1)^{p(e)\sum p(X_i)} \alpha(X_1, \ldots, X_k)e$. One should also remember that $\alpha(X) = (-1)^{p(\alpha)p(X)} \langle X, \alpha \rangle$.

With these definitions, ∇_X acting on $\mathcal{V}$ gives rise to the usual formula

$$F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} .$$

Since F is even, one has

$$F(X, Y) = -\iota_X \iota_Y F .$$

A possible convention for defining the components of F in a moving frame D_α is

$$F_{ab} = -\iota_{D_a} \iota_{D_b} F .$$

With this convention,

$$F_{ab} = [\nabla_{D_a}, \nabla_{D_b}] - \nabla_{[D_a, D_b]}$$

and if ω^a is the coframe for which $\langle D_a, \omega^b \rangle = \delta_a^b$,

$$F = \frac{1}{2} \omega^a \wedge \omega^b F_{ab} \qquad \text{(sum on } a, b) .$$

§3.7. Actions of super Lie algebras; vector fields and flows; Lie derivative

Suppose a super Lie group G acts on a super manifold M. Differentiating at the identity e of G the map $G \times M \to M$, we obtain a map ρ from the tangent space $T_e G$ to vector fields on M. For X in $T_e G$, differentiating in g the identity $(gh)m = g(hm)$, one sees that $G \times M \to M$ maps the right invariant vector fields on G corresponding to X to $\rho(X)$. By 3.3.6, it follows that ρ is a morphism from the Lie aglebra of G to the opposite of the Lie algebra of vector fields on M. From Theorem 3.5.1, one deduces the converse relation, at least locally: Namely, given a morphism Φ from the super Lie algebra of G to the opposite of the super Lie algebra of vector fields on M there is a morphism from a neighborhood of $\{e\} \times M$ in $G \times M$ to M which is a local action whose differential induces Φ.

In particular, if $\mathcal{L}$ is the super Lie algebra of a super Lie group G, any super Lie algebra morphism from $\mathcal{L}$ to the tangent bundle T of M, with $\mathcal{O} \otimes \mathcal{L} \to T$ the embedding of a vector bundle direct factor, is locally isomorphic to a standard $\mathcal{L} \to T_{N \times G}$, with $\mathcal{L}$ mapping to the left invariant vector fields on G.

Examples. (i) The super Lie algebra $\mathbb{R}^{0|1}$ is generated by a single odd element D with $D^2 = \frac{1}{2}[D, D] = 0$. A non-vanishing odd vector field D on a supermanifold, with $D^2 = 0$, is, in a suitable local coordinate system, the vector field $\partial/\partial\theta^1$.

(ii) The super Lie algebra of the super Lie group $\mathbb{R}^{1|1}$ of 3.3.6 is generated by a single odd element D with $D^2 = 2[D, D] \neq 0$. If D is an odd vector field on a super manifold and D^2 is nonvanishing, then, in a suitable local coordinate system, D is $\partial/\partial\theta^1 + \theta^1 \partial/\partial x^1$.

(iii) An even vector field X on a supermanifold M defines a flow $\exp(tX)$: $\mathbb{R} \times M \to M$. For M noncompact, flow lines can escape to infinity in finite time and the flow

is defined only in some neighborhood of $\{0\} \times M$. For X an even vector field, and for $\mathcal{V}$ a bundle canonically associated to a supermanifold, the *Lie derivative* of a section of $\mathcal{V}$ is defined by

$$\mathcal{L}_X(s) := \lim_{t\to 0} \tfrac{1}{t}(\exp(tX)^* s - s) \ . \tag{3.7.1}$$

Example (iii) holds as well for a family $M \to S$ of supermanifolds and a vertical even vector field. The induced flow commutes with the projection to S. For odd vector fields, following the "even rules" principle of (1.7), $\mathcal{L}_X$ is defined so that it is compatible with changes of base and that for ε an odd function on the base, $\mathcal{L}_{\varepsilon X} = \varepsilon \mathcal{L}_X$. The Lie derivative $\mathcal{L}_X$ is $\mathcal{O}_S$-linear, of the same parity as X.

Proposition 3.7.2. *For any vector fields X, Y on a supermanifold M and any function f we have*

$$\mathcal{L}_X(f) = X(f) \quad \text{and} \quad \mathcal{L}_X(Y) = [X, Y].$$

Proof. The first equation is clear from the definition. To prove the second notice that the Lie deriviative satisfies the Leibniz rule in the sense that for any bundles $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ canonically associated to a supermanifold and for any canonical pairing $\mu\colon \mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{V}_3$ we have

$$\mathcal{L}_X(\mu(v_1 \otimes v_2)) = \mu(\mathcal{L}_X(v_1) \otimes v_2) + (-1)^{p(X)p(v_1)} \mu(v_1 \otimes \mathcal{L}_X(v_2)).$$

In particular,

$$\begin{aligned} X(Y(f)) = \mathcal{L}_X(Y(f)) &= \mathcal{L}_X(Y)(f) + (-1)^{p(X)p(Y)} Y(\mathcal{L}_X(f)) \\ &= \mathcal{L}_X(Y)(f) + (-1)^{p(X)p(Y)} Y(X(f)). \end{aligned}$$

We also have the super version of the Cartan formula.

Proposition 3.7.3: Cartan Formula. *For any vector field X let the contraction i_X be as defined in (3.6). Then we have the following equality of derivations on the de Rham complex Ω^*_M:*

$$\mathcal{L}_X = [d, i_X] = d i_X + i_X d.$$

Proof. The action of $\mathcal{L}_X$ on the de Rham complex Ω^*_M is determined by the following properties, all deduced from (3.7.1) and (3.7.2): it is a derivation of the commutative graded super algebra Ω^*_M (cf. §1, App.) of cohomological degree 0 and of parity that of X; it commutes with d; on functions, $\mathcal{L}_X f = Xf = \langle X, df \rangle$. The Cartan formula then results from the fact that the derivations on each side of the equality agree on $\mathcal{O}$ and Ω^1.

§3.8. Super Lie groups and Harish-Chandra pairs

The following construction reduces the relation between super Lie groups and Lie algebras to its classical analog. Define a *Harish-Chandra pair* to be a system consisting of a Lie group K, a Lie algebra $\mathfrak{g}$, acted on by K, and a K-equivariant embedding $\varepsilon\colon \mathrm{Lie}(K) \hookrightarrow \mathfrak{g}$, such that

$$\mathrm{ad}\,\varepsilon = d \ (\text{action of } K \text{ on } \mathfrak{g}) \ .$$

A super Lie group G defines a Harish-Chandra pair with $K = G_{\mathrm{red}}$, $\mathfrak{g} = \mathrm{Lie}\, G$ and ε the embedding of the even part $\mathfrak{g}_0$ in $\mathfrak{g}$. This construction is an equivalence of categories from super Lie groups to Harish-Chandra pairs with K an ordinary Lie group and $\mathrm{Lie}\, K$ the even part of $\mathfrak{g}$.

(3.8.1)

An action of a super Lie group G on a super manifold M is equivalent to an action of the underlying reduced group on M and a morphism from the super Lie algebra $\mathfrak{g}$ of G to the opposite to the super Lie algebra of vector fields on M which are compatible in the sense that the differential of the action of the reduced group at the identity agrees with the restriction to the even sub Lie algebra $\mathfrak{g}_0$.

§3.9. Densities

A *density* on an open submanifold U of $\mathbb{R}^{p|q}$ is an $\mathbb{R}$-linear $\mathbb{R}$-valued form on the space of functions with compact support on U, of the form

$$g = \sum g_I(\mathbf{t})\theta^I \longmapsto \sum_I \int d_I(\mathbf{t}) g_I(\mathbf{t}) dt^1 \dots dt^p \ , \tag{3.9.1}$$

for C^∞-functions d_I on $\mathbb{R}^p$. For μ a density, the linear form μ is usually written $g \mapsto \int \mu g$. For $V \subset U$, a density on U induces one on V, and densities form a sheaf on $\mathbb{R}^{p|q}$. It is a sheaf of super vector spaces, and it has a natural structure of $\mathcal{O}$-module, defined by

$$\int (\mu u) g = \int \mu(ug) \quad \text{for } u \text{ a local section of } \mathcal{O} \ . \tag{3.9.2}$$

Proposition 3.9.3. *The $\mathcal{O}$-module $\mathcal{D}$ of densities on $\mathbb{R}^{p|q}$ is free of rank $(1|0)$ if q is even, of rank $(0|1)$ if q is odd.*

Proof. We leave it to the reader to check that the following density $[dt^1 \cdots dt^p \, d\theta^1 \cdots d\theta^q]$ is a basis of $\mathcal{D}$: to $g = \sum g_I \theta^I$, it associates the integral $\int dt^1 \cdots dt^p h$ of the coefficient h of $\theta^q \cdots \theta^1$.

§3.10. Change of variables formula for densities

If $\varphi\colon U \to V$ is an isomorphism between open submanifolds of $\mathbb{R}^{p|q}$, φ transforms densities on U into densities on V. To check this, one may observe that the densities are the linear functionals which are continuous for the distribution topology on the components g_I of g, and that this topology is respected by the change of variables φ. Indeed, in components, $g \mapsto g \circ \varphi$ involves only change of variables, C^∞ derivations and multiplication by C^∞ functions.

This allows us to generalize 3.9 to a definition of a density on any supermanifold: it is a linear functional on the space of functions with compact support whose restriction to any local coordinate chart is of the form (3.9.1).

From 3.9.3, it follows that if $\varphi\colon U \to V$ is an isomorphism between open submanifolds of $\mathbb{R}^{p|q}$, then for some even function $j(\varphi)$, one has

$$\int_V [dt^1 \cdots dt^p \, d\theta^1 \cdots d\theta^q] g = \int_U [dt^1, \dots, dt^p \, , d\theta^1 \dots, d\theta^q] \, j(\varphi) g \circ \varphi \tag{3.10.1}$$

(change of variables formula).

Using the basis $\partial/\partial t^1, \dots, \partial/\partial t^p$, $\partial/\partial\theta^1, \dots, \partial/\partial\theta^q$ of the tangent bundle T of $\mathbb{R}^{p|q}$, one identifies it with $\mathcal{O}^{p|q}$, and $d\varphi\colon TU \to \varphi^* TV$ becomes an automorphism of $\mathcal{O}_U^{p|q}$. We denote by $\mathrm{Ber}(d\varphi)$ the Berezinian of this automorphism. We write $\mathrm{or}(\varphi)$ for the sign ± 1, equal to $+1$ where $\varphi\colon |U| \xrightarrow{\sim} |V|$ preserves the orientation of $\mathbb{R}^p$, and to -1 where it doesn't.

Proposition 3.10.2. *The change of variables factor $j(\varphi)$ in 3.10.1 is given by*

$$j(\varphi) = \mathrm{or}(\varphi)\mathrm{Ber}(d\varphi)\ . \tag{3.10.3}$$

A local coordinate system $t^1, \dots, \theta^q$ defines an orientation of $|M|$, a section $[dt^1, \dots, d\theta^q]$ of $\mathrm{Ber}(\Omega^1)$ and a density which, by anticipation, we denoted $[dt^1 \cdots d\theta^q]$ as well (see 3.9.3). For u with compact support, one defines

$$\int \big(\text{section } [dt^1, \dots, d\theta^q] \text{ of } \mathrm{Ber}(\Omega^1)\big)\, u := \int (\text{density } [dt^1 \cdots d\theta^q])u\ . \tag{3.10.4}$$

Proposition 3.10.5. *Let M be a super manifold. Let $\mathrm{or}_{|M|}$ denote the orientation sheaf of its reduced submanifold. Then there is a unique $\mathbb{R}$-linear map*

$$\int_M : \Gamma_0(\mathrm{Ber}(\Omega^1_M) \otimes \mathrm{or}_{|M|}) \to \mathbb{R}$$

from the super vector space of compactly supported sections of the Berezinian line bundle of M twisted by the orientation sheaf for $|M|$ with the following property. Let $U \subset M$ be an open subset with coordinates $t^1, \dots, \theta^q$. Let φ be a section of $\mathrm{Ber}(\Omega^1_M)$ with compact support contained in U which written in these coordinates is given by $[dt^1, \dots, d\theta^q]u$. Let τ be the section of $\mathrm{or}_{|U|}$ determined by these coordinates. Then $\int_M \varphi \otimes \tau$ agrees with

$$\int [dt^1 \cdots d\theta^q]u$$

as given in 3.10.4.

Proof. This is proved using a partition of unity to decompose any compactly supported section as a sum of sections supported in coordinate charts. The independence of the resulting sum on the chosen decomposition uses the change of variables formula (3.10.3).

Thus, what can be canonically integrated over a super manifold M is a compactly supported section of $\mathrm{Ber}(\Omega^1_M)$, twisted by the orientation bundle of $|M|$. When $|M|$ is oriented, we can integrate compactly supported sections of $\mathrm{Ber}(\Omega^1_M)$.

We will sketch the proof of a relative statement generalizing (3.10.2). For $M \to S$ a family of supermanifolds, a *relative density* on M is a suitably continuous $\mathcal{O}_S$-linear morphism $f_!\colon \mathcal{O}_M \to \mathcal{O}_S$, where $f_!$ is the direct image with compact supports. Notation: $\int_{M/S} \mu g$. In local coordinates, "suitably continuous" means that for $U \times V \to V$ with $U \subset \mathbb{R}^{p|q}$ and $V \subset \mathbb{R}^{r|s}$, the components of $\int_{U\times V/V} \mu g$ should be obtained from the components of g by integration along the fibers of $|U|\times|V| \to |V|$, relatively to some C^∞ relative densities on $|U| \times |V|$.

If M'/S' is deduced from M/S by a base change $u\colon S' \to S$:

$$\begin{array}{ccc} M' & \xrightarrow{u_M} & M \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{u} & S\,, \end{array}$$

a relative density μ on M has a pullback μ' to M', characterized by

$$\int \mu' \, u_M^*(g) f'^*(h) = u^* \left(\int \mu \, g \right) h$$

for g on M and h on S'. If M/S is of relative dimension $p|q$, the relative densities on M form a $\mathcal{O}_M$-module locally free of rank $1|0$ if q is even, $0|1$ if q is odd. In local coordinates, for M an open subspace of $\mathbb{R}^{p|q} \times S$, the relative density $[dt^1 \cdots d\theta^q]$ (the pullback of the density $[dt^1 \cdots d\theta^q]$ of $\mathbb{R}^{p|q}$) is a basis.

Let φ be a S-isomorphism between open subsets U and V of $\mathbb{R}^{p|q} \times S$. The module $\mathcal{D}$ of relative densities being of rank $1|0$ or $0|1$ (according to the parity of q), we know a priori that for some even function $j(\varphi)$, one has

$$\varphi^*[dt^1, \ldots, d\theta^q] = [dt^1, \ldots, d\theta^q] j(\varphi)$$

on U, i.e. that for g on V,

$$\int_{V/S} [dt^1, \ldots, d\theta^q] g = \int_{U/S} [dt^1, \ldots, d\theta^q] j(\varphi) g \circ \varphi \ . \tag{3.10.6}$$

We identify the relative tangent bundle T of $\mathbb{R}^{p|q} \times S$ with the pullback of the tangent bundle $\mathcal{O}^{p|q}$ of $\mathbb{R}^{p|q}$, and write $\mathrm{Ber}(d\varphi)$ for the Berezinian of $d\varphi$: $TU \to \varphi^* TV$, identified with an automorphism of $\mathcal{O}_U^{p|q}$. We write $\mathrm{or}(\varphi)$ for the sign ± 1, equal to $+1$ where φ: $|U| \to |V|$, restricted to each fiber over $|S|$, preserves the orientation of $\mathbb{R}^p$, and to -1 where it doesn't. The generalization of (3.10.2) to the relative case is:

Proposition 3.10.7. *With notation as above, one has again*

$$j(\varphi) = \mathrm{or}(\varphi) \mathrm{Ber}(d\varphi) \ .$$

Both sides of (3.10.7) are compatible with a base change $S' \to S$, and at the cost of considering not just φ but also any morphism deduced from φ by base change (this changes nothing in the setting), it suffices to check (3.10.7) at each point, in the sense of 2.9: for u a section of U/S, one should check (3.10.7) along u (i.e. on S, after pullback by u). By translation invariance, we may and shall assume that u and $\varphi(u)$ are the section 0 of $\mathbb{R}^{p|q} \times S/S$.

Lemma 3.10.8. *Proposition 3.10.7 holds for $U = V = \mathbb{R}^{p|q} \times S$, and for φ a family of linear maps from $\mathbb{R}^{p|q}$ to $\mathbb{R}^{p|q}$.*

Proof. By translation invariance, if φ is linear, $j(\varphi)$ or (φ) as well as $\mathrm{Ber}(d\varphi)$ are pullbacks of functions on S. These functions are multiplicative: they define morphisms

$$j \, , \mathrm{or} \text{ and } \mathrm{Ber}\colon \ \mathrm{GL}(p|q) \to \mathrm{GL}(1|0) \ .$$

Matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ being commutators, the morphisms j, or and Ber are determined by their restriction to $\mathrm{GL}(p) \times \mathrm{GL}(q) \subset \mathrm{GL}(p|q)$. This group being an ordinary manifold, it suffices to prove (3.10.8) for φ any point of $\mathrm{GL}(p) \times \mathrm{GL}(q)$, i.e. for S a point and for φ: $\mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ linear, of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

By a compatibility with products, one can even consider just the cases of $\mathbb{R}^{p|0}$ and $\mathbb{R}^{0|q}$. In the first case, one has just the classical change of variables formula on $\mathbb{R}^p$, for a linear change of variable, leading to a factor $|\det(\varphi)| = \mathrm{or}(\varphi)\det(d\varphi)$. In the case of $\mathbb{R}^{0|q}$, one observes that for a linear map φ: $\mathbb{R}^{0|q} \to \mathbb{R}^{0|q}$, the degree in θ is preserved by φ^* and that

$$\varphi^*(\theta^1 \cdots \theta^q) = \det(\text{matrix of } \varphi) \cdot \theta^1 \cdots \theta^q \ .$$

This completes the proof of Lemma 3.10.8.

Proof of 3.10.7. Suppose that φ: $U \to V$ maps 0 to 0. We deform φ to its linearization by the standard operation of zooming in at 0. For any λ, let $R(\lambda)$ be the homothety of ratio λ. Define

$$\varphi_\lambda := R(\lambda^{-1})\varphi R(\lambda) \ .$$

By multiplicativity of $\varphi \mapsto j(\varphi)$ for a composite map, $j(\varphi_\lambda)$ at 0 is independent of λ. The family φ_λ extends to $\lambda = 0$, with $\varphi_0 =$ linear part $d\varphi$ of φ. By constancy, it suffices to prove 3.10.7 for φ_0, to which 3.10.8 applies.

§3.11. The Lie derivative of sections of $\mathrm{Ber}(\Omega^1_M)$

The Lie derivative has been defined in 3.7. One has:

Proposition. *For v a section of $\mathrm{Ber}(\Omega^1_M)$ one has*

$$\mathcal{L}_{fX}(v) = (-1)^{p(f)p(X)}\mathcal{L}_X(fv) \ . \tag{3.11.1}$$

By the sign rule for modules over commutative super algebras, $vf := (-1)^{p(f)p(v)}fv$. If we define $vX := (-1)^{p(v)p(X)}\mathcal{L}_X v$, (3.11.1) can be rewritten

$$(vf)X = v(fX) \ , \tag{3.11.2}$$

an identity which allows us to define

$$d\colon \mathrm{Ber}(\Omega^1_M) \otimes_{\mathcal{O}_M} T \to \mathrm{Ber}(\Omega^1_M) \tag{3.11.3}$$

by $d(v \otimes X) = vX$.

Proof. We will prove (3.11.1) in a relative setting as well. The question being local and compatible with changes of base, applying the "even rules" principle, we may and shall suppose X and f even. We may and shall assume in addition that the reduced fibers of M/S are oriented, and that v has compact support.

By transport of structures, integration of sections with compact support of $\mathrm{Ber}(\Omega^1_{M/S})$ is invariant by orientation preserving diffeomorphisms, and in particular

$$\int_{M/S} \exp(tX)^* v = \int_{M/S} v \ ,$$

hence by passage to the limit for $t \to 0$

$$\int_{M/S} \mathcal{L}_X v = 0 \ . \tag{3.11.4}$$

For any function u, $\mathcal{L}_X(uv) = \mathcal{L}_X u \cdot v + u \cdot \mathcal{L}_X v = Xu \cdot v + u \cdot \mathcal{L}_X v$. Applying (3.11.4) to uv, we obtain

$$\int u \mathcal{L}_X v = - \int Xu \cdot v$$

and, for f even,

$$\int u \mathcal{L}_{fX} v = - \int fXu \cdot v = - \int Xu \cdot (fv) = \int u \mathcal{L}_X (fv) \ .$$

This holding for any u, (3.11.1) follows.

§3.12. Integral forms

For an ordinary manifold M of dimension n, one has on Ω^*_M the operations $\alpha\wedge$ for α in Ω^1_M and i_X for X in the tangent bundle T. They turn Ω^*_X into a module over the Clifford algebra generated by $\Omega^1_M \oplus T$, meaning that $(\alpha\wedge)^2 = (i_X)^2 = 0$ and that $i_X \circ \alpha\wedge + \alpha\wedge \circ i_X = \langle X, \alpha \rangle$. One can define Ω^* as being the Clifford module freely generated by $\mathcal{O}$, with the relations $i_X f = 0$ for f in $\mathcal{O}$. One could also define it as the Clifford module freely generated by $\omega := \Omega^n$, with the relations $\alpha\wedge v = 0$ for v in ω.

From the second point of view, the exterior derivative d can be characterized by the properties that $d = 0$ on ω and that $di_X + i_X d = \mathcal{L}_X$. Indeed, the formula

$$d(i_{X_1} \cdots i_{X_p} v) = \mathcal{L}_{X_1}(i_{X_2} \cdots i_{X_p} v) - i_{X_1} d(i_{X_2} \cdots i_{X_p} v) \tag{3.12.1}$$

gives inductively a way to compute d. As $d = 0$ on ω, Cartan's formula reduces for v in ω to

$$di_X v = \mathcal{L}_X v \ ,$$

which shows that $\mathcal{L}_{fX} v = \mathcal{L}_X fv$. One can check using just this property that (3.12.1), used as a definition of d, will not lead to contradiction.

In the case of a supermanifold M of dimension $p|q$, one can use the second construction to construct a complex I^*_M, with top component $\omega := \mathrm{Ber}(\Omega^1_M)$. If $q \neq 0$, this complex is not the de Rham complex. The de Rham complex is bounded from below, while I^*_M, the complex of *integral forms*, is bounded from above.

More precisely, I^*_M is defined as follows.

(a) As a graded $\mathcal{O}$-module, it is freely generated from $\omega := \mathrm{Ber}(\Omega^1)$, put in cohomological degree p, by $\mathcal{O}$-linear operations i_X of cohomological degree -1 obeying $[i_X, i_Y] = 0$:

$$i_X i_Y + (-1)^{p(X)p(Y)} i_Y i_X = 0 \ ,$$

and $i_{fX} = f i_X$. Iterated "contractions" i_X induce isomorphisms

$$\wedge^i T \otimes \omega \xrightarrow{\sim} I^{p-i} \ . \tag{3.12.2}$$

(b) Operations $\alpha\wedge$ of comological degree 1, for α in Ω^1, are defined by $\alpha\wedge v = 0$ for v in ω and $[i_X, \alpha\wedge] = \langle X, \alpha\rangle$, i.e. $i_X(\alpha\wedge\eta) - (-1)^{p(X)p(\alpha)}\alpha\wedge i_X\eta = \langle X, \alpha\rangle\,\eta$. The operations $\alpha\wedge$ give I^* the structure of a module over Ω^*.

(c) An *exterior derivative* d is defined. It is characterized as being of cohomological degree 1 (implying that $d = 0$ on ω) and $[d, i_X] = \mathcal{L}_X$.

Exercise. Check that this is a valid definition, and that for α in Ω^i and $\eta \in I^*$,

$$d(\alpha\wedge\eta) = d\alpha\wedge\eta + (-1)^i\alpha\wedge d\eta\ .$$

Proposition 3.12.3. *Let M be a supermanifold of dimension $p|q$ with $|M|$ oriented. Let η be a compactly supported section of I^{p-1}. Then*

$$\int_M d\eta = 0.$$

Proof. By (3.12.2) every $\eta \in I^{p-1}$ is a linear combination of elements of the form $i_{X_a}(\alpha_a)$ for X_a a vector field and $\alpha_a \in \omega$. If η has compact support, we can take the α_a to have their support in a slightly larger compact set. Thus, it suffices to prove this result for $\eta = i_X(\alpha) \in I^{p-1}$. But, by (3.10.5) and (3.11.4) we have

$$0 = \int_M \mathcal{L}_X(\alpha) = \int_M di_X(\alpha) + i_X d\alpha = \int_M di_X(\alpha).$$

§3.13. A second definition of integral forms

Another way to define the complex I^*_M is by duality, repeating the definition we gave for densities: once an orientation of $M_{\rm red}$ has been chosen, and for a suitable topology on Ω^i, the space of sections with compact support of I^{p-i} is the dual of the space of sections of Ω^i. In the approach of 3.12, the duality is

$$\alpha, \eta \longmapsto \int_M \alpha\wedge\eta\ .$$

Contractions, products with forms and d are then defined as transposes of the same operations for Ω^*:

$$\int \alpha\wedge(\beta\wedge\eta) = \int (\alpha\wedge\beta)\wedge\eta\ ,$$
$$\int \alpha\wedge i_X\eta = -(-1)^{p(\alpha)p(X)}\int i_X(\alpha)\wedge\eta\ ,$$
$$\int \alpha\wedge d\eta = -(-1)^{\deg(\alpha)}\int d\alpha\wedge\eta\ .$$

From the point of view of 3.12, the last identity results from $\int d\eta = 0$, as established in (3.12.3).

The virtue of integral forms, explaining their name, and the reason Bernstein and Leites [**1977**] introduced them, is that an integral form of dimension $p - i$ can be integrated on submanifolds of codimension $i|0$. We will give two explanations of what happens, leaving proofs as exercises for the reader.

§3.14. Generalized functions

Generalized functions on a super manifold M can be defined as the elements either of a completion of the space $\Gamma(M, \mathcal{O})$ of functions, or as the dual of the space of compactly supported densities with the C^∞ topology. In a local coordinate system $x^1, \dots, x^p$, $\theta^1, \dots, \theta^q$, they are written $\sum T_I \theta^I$, with the T_I generalized functions, a.k.a. distributions, on $\mathbb{R}^p$. Pullbacks by submersive maps are defined. The generalized functions form an $\mathcal{O}$-module $\mathcal{O}\hat{}$, and for any vector bundle $\mathcal{V}$, the *generalized sections* of $\mathcal{V}$ are the sections of $\mathcal{O}\hat{} \otimes_{\mathcal{O}} \mathcal{V}$.

Example. On $\mathbb{R}$, let Y be the generalized (L^1) function defined by $Y(x) = 0$ for $x < 0$, $Y(x) = 1$ for $x > 0$. One has

$$dY = \delta(x)dx \tag{3.14.1}$$

For $f\colon M \to \mathbb{R}^{1|0}$ an even function on a super manifold M, submersive where it vanishes, $\int_{f \geq 0} \eta$ is defined to be $\int Y(f)\eta$. This integral depends not only on the region $f \geq 0$ of M_{red}, but also on the chosen "boundary", the submanifold of codimension $1|0$ given by the equation $f = 0$. For instance, on $\mathbb{R}^{1|2}$,

$$Y(x + \theta^2\theta^1) = Y(x) + \delta(x)\theta^2\theta^1$$

and for the standard orientation of $\mathbb{R}$

$$\int_{x+\theta^2\theta^1 \geq 0} [dx\, d\theta^1 d\theta^2] u(x) = u(0) + \int_{x \geq 0} [dx\, d\theta^1 d\theta^2] u(x)\ .$$

If $N \subset M$ is a submanifold of codimenson $i|0$, a normal orientation of N defines a generalized section δ_N of Ω^i_M: if $f_1, \dots, f_i$ is a system of equations, compatible with the normal orientation,

$$\delta_N := \delta(f_1) \cdots \delta(f_i) df_1 \wedge \cdots \wedge df_i$$

does not depend on the choice of equations. If η is an integral form in I_M^{p-i}, its restriction to I_N^{p-i} is defined by

$$\int_N f(\eta|N) = \int_M f\delta_N \wedge \eta$$

for f with compact support on M.

A Stokes formula (Bernstein and Leites (1977)) for integrals $\int_{f \geq 0}$ over domains with boundary then results from the formula $\int d\eta = 0$.

§3.15. Integral forms as functions of infinitesimal submanifold elements

Let V be a vector space of dimension N. Fix $p \leq N$ and let G be the Grassmannian variety of subspaces F of V of dimension p. A p-form $\alpha \in \wedge^p V^*$ induces on each $F \in G$ an element of $\det(F^*) = \wedge^p F^*$. If $\det(F^*)$ is the line bundle on G with

fiber $\det(F^*)$ at F, one has in algebraic geometry, i.e. for "polynomial" sections of $\det(F^*)$,

$$\wedge^p V^* \xrightarrow{\sim} H^0(G, \det(F^*)) \ .$$

It follows that to give a p-form on a manifold M is the most general way to give a form α_N of maximal degree on all submanifolds N of M of dimension p, if one insists that (a) at $n \in N$, the value of α_N at n depends only on the tangent space $T_{N,n} \subset T_{M,n}$ of N at n; (b) at fixed $n \in M$, α_N on $T \subset T_{M,n}$ is "polynomial" in T.

In the case of super manifolds, the appearance of the integral forms is similarly made natural by the following

Lemma 3.15.1. *Let V be a super vector space of dimension $N|M$. Let G be the super scheme of subspaces F of dimension $(N-p)|M$. On G, let ω be the line bundle* $\mathrm{Ber}(F^\vee)$. *Then,*

$$H^0(G, \omega) = \wedge^p V \otimes \mathrm{Ber}(V^*) \ .$$

For submanifolds of dimension $i|j$ of a super manifold of dimension $N|M$, such a game cannnot be played if $j \neq 0$ or M:

Lemma 3.15.2. *Let V be a super vector space of dimension $N|M$ and G be the super scheme of subspaces of codimension $i|j$, with $j \neq 0$, M. On G, let ω be the line bundle* $\mathrm{Ber}(F^\vee)$. *Then $H^0(G, \omega) = 0$.*

Sketch of proof of 3.15.2. We first consider the case when $i = 0$. The reduced scheme G_{red} is then the ordinary grassmannian of j-dimensonal subspaces F of the M-dimensional space V_1. On G_{red}, the restriction of ω is the line bundle $\det(F)$. The normal bundle of G_{red} in G is the odd bundle $\mathrm{Hom}(F, V_0)$. Working modulo successive powers of the ideal of G_{red} in G, this reduces us to checking, in ordinary algeraic geometry, the vanishing of the

$$H^0(G_{\mathrm{red}}, \det(F) \otimes \wedge^a(\mathrm{Hom}(F, V_0)^\vee)) \ .$$

Now we consider the case $i > 0$. Let $\widetilde{G}$ be the super scheme of partial flags of type $i|0 \subset i|j$, and H be the super scheme of subspaces of dimension $i|0$. One has forgetful maps

$$G \xleftarrow{p} \widetilde{G} \xrightarrow{q} H \ .$$

As p is smooth, it suffices to check that $H^0(\widetilde{G}, p^*\omega) = 0$. It then suffices to check such a vanishing for the fibers of q, which are super schemes of subspaces of dimension $0|j$ in a space of dimension $(N-i)|M$, and these have vanishing cohomology.

CHAPTER 4
Real Structures

§§4.1–4.3. Real structures and $*$-operations

4.1. A *real structure* on a complex super algebra A is the data of a real super algebra $A_{\mathbb{R}}$ and of an isomorphism of complex super algebras from$A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ to A. This amounts to giving an $\mathbb{C}$-antilinear involutive automorphism σ: $A \to A$: take for $A_{\mathbb{R}}$ the fixed points of σ.

Another kind of real structure is the data of an involutive $\mathbb{C}$-antilinear isomorphism from A to the opposite algebra: a $*$-operation, obeying

$$(xy)^* = (-1)^{p(x)p(y)} y^* x^* \tag{4.1.1}$$

For a commutative super algebra, such a $*$-operation is the same thing as a real structure.

In the classical (non super) setting, algebras with a $*$-operation are a natural "noncommutative" analogue of topological spaces, when topological spaces are viewed as encoded by the commutative algebra of continuous functions on them.

The formula (4.1.1) is the natural extension of the formula $(xy)^* = y^* x^*$ to the case of super algebras, and super algebras with a $*$-operation do occur as quantizations of super symplectic manifolds.

4.2. The notions 4.1 of real structures depend, from the categorical point of view, on the fact that the complex tensor category $\mathcal{T}_{\mathbb{C}}$ of complex super vector spaces is deduced by extension of scalars from $\mathbb{R}$ to $\mathbb{C}$ from the real tensor category $\mathcal{T}_{\mathbb{R}}$ of real super vector spaces. The complex tensor category $\mathcal{T}_{\mathbb{C}}$ is also deduced by extension of scalars from the following "quaternionic" real tensor category $\mathcal{T}'_{\mathbb{R}}$. The objects of $\mathcal{T}'_{\mathbb{R}}$ are the complex super vector spaces V, given with a $\mathbb{C}$-antilinear automorphism σ: $V \to V$ for which

$$\sigma^2 x = (-1)^{p(x)} x \ . \tag{4.2.1}$$

The tensor product is that of $\mathcal{T}_{\mathbb{C}}$, with $\sigma(x \otimes y) = \sigma(x) \otimes \sigma(y)$. The extension of scalars from $\mathcal{T}'_{\mathbb{R}}$ to $\mathcal{T}_{\mathbb{C}}$ is "forgetting σ".

To the "real form" $\mathcal{T}'_{\mathbb{R}}$ of $\mathcal{T}_{\mathbb{C}}$ correspond new notions of real structures.

Examples. (i) A purely odd object of $\mathcal{T}'_{\mathbb{R}}$ can be identified with a vector space over the field $\mathbb{H}$ of quaternions, with σ being the multiplication by j.

(ii) In $\mathcal{T}'_{\mathbb{R}}$, one has a notion of "positive definite" symmetric bilinear form on an object V. Concretely, a symmetric bilinear form ψ is the data of a symmetric form ψ_0 on V_0 and of an alternating form ψ_1 on V_1, with $\psi_i(\sigma x, \sigma y) = \overline{\psi_i(x,y)}$.

From this reality condition it follows that $\psi_i(x, \sigma x)$ is real:

$$\overline{\psi_i(x,\sigma x)} = \psi_i(\sigma x, \sigma^2 x) = (-1)^i \psi_i(\sigma x, x) = \psi_i(x, \sigma x) \ ,$$

and the positivity condition is

$$\psi_i(x, \sigma x) > 0 \qquad \text{for} \qquad x \neq 0 \ .$$

One should beware that the tensor product of two forms which are positive in this sense is not positive, except when one of the underlying super vector spaces is purely even.

(iii) A $\mathcal{T}'_{\mathbb{R}}$-real structure σ (resp. $*$-operation $*$) on a complex super algebra A is σ (resp. $*$): $A \to A$, $\mathbb{C}$-antilinear, obeying (4.2.1) and $\sigma(xy) = \sigma x \cdot \sigma y$ (resp. $(xy)^* = (-1)^{p(x)p(y)} y^* x^*$).

4.3. Manin (1988), Ch. 3, §6 considers additional notions of reality, with no categorical underpinning. A typical one is $*$-operations ρ: $A \to A$, antilinear, involutive, with

$$(ab)^\rho = b^\rho a^\rho \ . \tag{4.3.1}$$

Our understanding is that such objects arose only because physicists did not consistently apply the sign rule when considering super Hilbert spaces. They can be reduced to $*$-operations. Indeed, if ρ: $A \to A$ is antilinear, define

$$a^* = \begin{cases} a^\rho & \text{for } a \text{ even;} \\ ia^\rho & \text{for } a \text{ odd} \ . \end{cases} \tag{4.3.2}$$

Then, ρ is involutive if and only if $*$ is, and (4.3.1) holds for ρ if and only if (4.4.1) holds for $*$. In particular, for A commutative, (4.3.1) gives $(ab)^* = a^* b^*$.

§4.4. Super Hilbert spaces

From now on, our setting for reality will be 4.1. According to the sign rule, a *Hermitian form* on a complex super vector space H is a morphism ψ: $H \otimes_{\mathbb{R}} H \to \mathbb{C}$ of $\mathbb{R}$-super vector spaces, $\mathbb{C}$-antilinear in the first variable and $\mathbb{C}$-linear in the second, with

$$\overline{\psi(x,y)} = (-1)^{p(x)p(y)} \psi(y,x) \ . \tag{4.4.1}$$

It follows from (4.4.1) that $\psi(x,x)$ is real for x even, and purely imaginary for x odd. As ψ is assumed even, $\psi(x,y) = 0$ for x and y of distinct parities.

The tensor product of Hermitian forms ψ and ψ' on H and H' is defined using the sign rule:

$$\psi \otimes \psi'(x \otimes x', y \otimes y') = (-1)^{p(x')p(y)} \psi(x,y) \psi'(x', y') \ .$$

In finite dimensions, a *super Hilbert space* is an Hermitian super space $(H, \langle\ \ \rangle)$ for which

$$\begin{aligned} \langle x, x\rangle &> 0 \qquad \text{for } x \neq 0 \text{ even;} \\ i^{-1}\langle x, x\rangle &> 0 \qquad \text{for } x \neq 0 \text{ odd .} \end{aligned} \tag{4.4.2}$$

The *adjoint* T^* of T: $H \to H$ (homogeneous but not necessarily even) is defined by

$$\langle x, Ty\rangle = (-1)^{p(x)p(T)}\langle T^*x, y\rangle \tag{4.4.3}$$

Categorical reasoning, or a direct argument, shows that $T \mapsto T^*$ is a $*$-operation: it is $\mathbb{C}$-antilinear, $T^{**} = T$ and

$$(TU)^* = (-1)^{p(T)p(U)}U^*T^* \ . \tag{4.4.4}$$

Let H be a super Hilbert space. On the ordinary vector space $H_0 \oplus H_1$, we define an ordinary Hilbert space structure $(\ ,\ )$ by

$$\begin{aligned} &(x, y) = 0 && \text{if } x \text{ and } y \text{ are not of the same parity} \\ &(x, y) = \langle x, y\rangle && \text{if } x \text{ and } y \text{ are even} \\ &i(x, y) = \langle x, y\rangle && \text{if } x \text{ and } y \text{ are odd .} \end{aligned} \tag{4.4.5}$$

The adjoint † for $(\ ,\ )$, and the adjoint (4.4.3) are related by

$$T^* = \begin{cases} T^\dagger & \text{for } T \text{ even;} \\ iT^\dagger & \text{for } T \text{ odd .} \end{cases} \tag{4.4.6}$$

(cf. (4.3.2)).

The positivity condition (4.4.2) is stable under tensor product: as $H \otimes H'$ is the orthogonal direct sum of the $H_i \otimes H'_j$, it suffices to check (4.4.2) for x in each of those four summands. The least obvious case is that of the component $H_1 \otimes H'_1$, for which we have

$$\begin{aligned} \langle x \otimes x', y \otimes y'\rangle &= -\langle x, y\rangle\langle x', y'\rangle = -i^2(x, y)(x', y') \\ &= (x, y)(x', y') \ . \end{aligned}$$

This formula shows that the Hermitian form $\langle\ ,\ \rangle$ on $H_1 \otimes H'_1$ is the ordinary tensor product of the positive Hermitian forms $(\ ,\ )$ on the vector spaces H_1 and H'_1. It is hence positive definite. More generally, the ordinary Hilbert space structure $(\ ,\ )$ of $H \otimes H'$ is the ordinary tensor product of the ordinary Hilbert space structures $(\ ,\ )$ of H and H'.

Passing to infinite dimensions creates the usual analytic difficulties, unchanged in the super setting.

§4.5. SUSY quantum mechanics

In relativistic SUSY quantum mechanics, the Hilbert space H of states of a physical system $\mathcal{S}$ is a super Hilbert space. It carries an action of the *Poincaré* group and the

parity is given by the action of its order 2 central element (spin-statistics relation). The Hilbert space corresponding to n indistinguishable copies of $\mathcal{S}$ is $\mathrm{Sym}^n(H)$, where the symmetric power is taken in the super sense. In the non-relativistic setting, one similarly has a projective action of the galilean group. Its restriction to the special orthogonal group defines a linear action of the spin group, and the action of $\mathrm{Ker}(\mathrm{Spin} \to \mathrm{SO})$ defines parity.

The definitions (4.4.2) of super Hilbert space and (4.4.3) of super adjoint disagree with those of the physicists and have the unsettling consequence that if T is odd and self-adjoint, its eigenvalues are in $i^{1/2}\mathbb{R}$. However, the corresponding eigenvectors don't correspond to physical states, being neither even nor odd. This saves our conventions from being physically absurd.

It would have been as consistent as (4.4.2) to impose as positivity condition $i\langle x, x\rangle > 0$ for x odd. For a discussion of related sign conventions, we refer to [I-**Signs**].

A representation ρ of a real super Lie algebra $\mathfrak{g}$ on H is *unitary* if the $\rho(X)$ for X in $\mathfrak{g}$ are anti-Hermitian: $\rho(X) = -\rho(X)^*$. For X odd, this requires $\rho([X, X])$ to have a spectrum in $-i\mathbb{R}^+$. Indeed, the spectrum of $\rho(X)$ is in $i \cdot i^{1/2}\mathbb{R} = i^{3/2}\mathbb{R}$, and $\rho([X, X]) = 2\rho(X)^2$.

§4.6. Real and complex super manifolds

In the classical (non super) setting, two points of view are useful in relating real and complex analytic manifolds. In the first, starting from a complex analytic manifold M, one defines a *real structure* on M as being an involution ρ of M, antilinear in the sense that for f a holomorphic function on $U \subset M$, $\rho^*(f)$ is antiholomorphic, that is $\rho^*(f)^-$ is again holomorphic. The fixed locus M^ρ of ρ, together with the sheaf of sections of $\mathcal{O}|M^\rho$ fixed by $f \mapsto \overline{\rho^*(f)}$, is then a real analytic manifold. One says that M is a *complexification* of M^ρ. Any real analytic manifold N can be obtained in this way, and the germ of M along $N = M^\rho$ is uniquely determined (up to unique isomorphism) by N.

In the second point of view, a complex analytic structure on M is an additional structure on the underlying real analytic manifold $M_\mathbb{R}$. The additional structure is that of an integrable complex structure on the tangent bundle.

The conjugate $\bar{M}$ of the complex analytic manifold M is the space M, with the sheaf $\mathcal{O}_{\bar{M}}$ of antiholomorphic functions on M. The morphism from $\mathcal{O}_M$ to $\mathcal{O}_{\bar{M}}$: $f \mapsto \bar{f}$, is a $\mathbb{C}$-antilinear isomorphism of sheaf of rings. It is convenient to write $m \mapsto \bar{m}$ for the identity map $M \to \bar{M}$: if f is holomorphic on M, then so is $\overline{f(\bar{m})}$ on $\bar{M}$.

From the first point of view, the real analytic variety $M_\mathbb{R}$ *underlying* the complex analytic variety M is to be described as the fixed point set of a real structure on a complexification. One can take as complexification $M \times \bar{M}$, the real structure being $(x, y) \mapsto (\bar{y}, \bar{x})$. The underlying real analytic variety $M_\mathbb{R}$ is embedded in $M \times M^-$ diagonally. In local coordinates: complex-valued real analytic functions on $U \subset \mathbb{C}^N$ can be identified with complex analytic functions of the z_i and $\bar{z}_i$, defined in a neighborhood of $\{(z, \bar{z}) | z \in U\}$. The physicists often use this identification and, with less justification, write $f(z, \bar{z})$ for a C^∞-function on U.

The same two points of view can be used to relate real and complex super manifolds. A *complex conjugate* of a complex analytic super manifold $(M, \mathcal{O}_M)$ is

a complex analytic super manifold $(\bar{M}, \mathcal{O}_{\bar{M}})$, provided with a $\mathbb{C}$-antilinear isomorphism $m \to \bar{m}$ of ringed spaces. For instance; $|\bar{M}|$ is $|M|$, and $\mathcal{O}_{\bar{M}}$ is $\mathcal{O}_M$, with the complex conjugate $\mathbb{C}$-algebra structure, and the structural isomorphism is the identity. In an obvious sense, M is the complex conjugate of $\bar{M}$. It is convenient to write $f \mapsto \bar{f}$ for the isomorphism of $\mathcal{O}_M$ with $\mathcal{O}_{\bar{M}}$: for f holomorphic on U, $\bar{f}$ is holomorphic on $\bar{U}$, and $\overline{(\lambda f)} = \bar{\lambda}\bar{f}$.

A *real structure* on a complex super manifold M is an involutive isomorphism $\rho: M \to \bar{M}$. In other words, it is an involutive automorphism ρ of the topological space $|M|$, given with an $\mathbb{C}$-antilinear isomorphism $\rho^*\mathcal{O}_M \to \mathcal{O}_M$, usually written $f \mapsto \bar{f}$. The fixed point set M^ρ of M, provided with the sheaf $\mathcal{O}^\rho$ of complex conjugation invariant sections of $\mathcal{O}|M^\rho$, is then a real analytic super manifold. One again says that M is a complexification of M^ρ, and a real analytic manifold N determines the germ (along N) of its complexifications.

The underlying real analytic super manifold $M_{\mathbb{R}}$ of a complex analytic super manifold M is the real locus in $M \times M^-$ of the real structure $(x, y) \to (\bar{y}, \bar{x})$. Example: for $\mathbb{C}^{0|1}$, with coordinate θ, the ring of functions on $(\mathbb{C}^{0|1})_{\mathbb{R}}$ is the fixed points of complex conjugation in $\mathbb{C}[\theta, \bar{\theta}]$. It is $\mathbb{R}[\theta + \bar{\theta}, (\theta - \bar{\theta})/i]$.

Let $z^1, \ldots, z^p, \theta^1, \ldots, \theta^q$ be a local complex coordinate system on M. By abuse of notation, physicists will often write $f(z^1, \ldots, \theta^q, \bar{z}^1, \ldots, \bar{\theta}^q)$ for a C^∞-function on $M_{\mathbb{R}}$.

§4.7. Complexifications, in infinite dimensions

When dealing with function spaces it can also be convenient to work in complexified spaces, with reality conditions coming in as an afterthought, if needed. If X and Y have complexifications $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, the complexification of $\underline{\mathrm{Hom}}(X, Y)$ is the space $\underline{\mathrm{Hom}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ of holomorphic maps f from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$. One requires f to be defined only in a suitable neighborhood of $X \subset X_{\mathbb{C}}$, and that $f(X)$ be close to $Y \subset Y_{\mathbb{C}}$.

Example. (i) If v is a section of the complexified tangent bundle of X, with X compact and X and v real analytic, $\exp(tv)$ (t small) belongs to $\mathrm{Diff}(X)_{\mathbb{C}}$.

(ii) (After G. Segal), Let S^1 be the unit circle in $\mathbb{C}$. A point of $\mathrm{Diff}(S^1)_{\mathbb{C}}$ is a holomorphic map from an annulus $1-\eta < r < 1+\eta$ to an annulus $1-\varepsilon < r < 1+\varepsilon$. It should have a restriction to S^1 close to a diffeomorphism of S^1. Composition is only partially defined. However, if f maps S^1 to the interior of the unit disc, one can associate to f the annulus $[f]$ between $f(S^1)$ and S^1 in $\mathbb{C}$. Both boundary components of $[f]$ are parametrized by S^1, using the identity map, resp. f. To the composition of maps corresponds the (always defined) gluing of annuli. Because of this, a substitute for $\mathrm{Diff}(S^1)_{\mathbb{C}}$ is the monoid of isomorphism classes of annuli with parametrized boundaries.

In conformal field theory, one meets complex (projective) representations of the real semigroup underlying the complex semigroup $\mathrm{Diff}(S^1)_{\mathbb{C}}$. At the Lie algebra level, if W is the Lie algebra of vector fields on S^1, with complexification $W_{\mathbb{C}}$, a complex (projective) representation of the real Lie algebra $r(W_{\mathbb{C}})$ underlying $W_{\mathbb{C}}$ is the same as a complex (projective) representation of $r(W_{\mathbb{C}}) \otimes \mathbb{C} = W_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C} = W_{\mathbb{C}} \times W_{\mathbb{C}}$.

§4.8. cs manifolds

Ordinary C^∞-manifolds are ringed spaces locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty)$. It would make no difference to define them as ringed spaces locally isomorphic to ($\mathbb{R}^p$, sheaf of $\mathbb{C}$-algebras $\mathcal{C}^\infty \otimes \mathbb{C}$), i.e. to take as structural sheaf the sheaf of complex-valued C^∞-functions. From the sheaf of $\mathbb{C}$-algebras of complex-valued C^∞-functions one indeed recovers the real-valued functions as those which take real values at every point. In the context of super manifolds, there is no such equivalence, and the following notion is of interest.

Definition 4.8.1. A cs *manifold* (c for "complex", s for "super") of dimension $p|q$ is a topological space X with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}$, such that locally $(X, \mathcal{O})$ is isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty[\theta^1, \dots, \theta^q] \otimes \mathbb{C})$.

As we explained, a cs manifold of dimension $p|0$ can be identified with an ordinary C^∞-manifold of dimension p. A cs manifold of dimension $0|q$ is the same thing as a complex super manifold of dimension $0|q$. If M is a super manifold, the sheaf $\mathcal{O}_{\mathbb{C}} = \mathcal{O} \otimes \mathbb{C}$ of complex functions on M is the structural sheaf of a cs manifold M^{cs}. In general, there are more morphisms from M^{cs} to N^{cs} than from M to N.

Variant. Replacing in 4.7 the sheaf of C^∞-functions by the sheaf of real analytic functions, one obtains the *analytic* cs *manifolds*. By a completion for a suitable topology of the structural sheaf, an analytic cs manifold defines a cs manifold.

Example. If M is a complex analytic super manifold, and if ρ is a real structure on M_{red}, the space $|M|^\rho$, with the restriction to $|M|^\rho$ of the structural sheaf of M, is an analytic cs manifold (of which M is said to be a *complexification*).

§4.9. Integration on cs manifolds; examples

In the classical situation, the real locus M^ρ of a real structure ρ on a complex manifold M can serve as a cycle of integration: once it is oriented, and provided one has suitable control at infinity, one can integrate on M^ρ a holomorphic differential form of maximal degree. The integral depends on M^ρ only up to suitable homologies.

Example 4.9.1. To compute $\int e^{-(x^2/2)+iax} dx$, it is better to go over to the holomorphic picture, where the integration cycle $\mathbb{R} \subset \mathbb{C}$ is needed only up to homologies given by 2-chains on which the integrand decays fast enough. In this setting, the completing the square change of variables $y = x - ia$ makes good sense, giving the answer $\sqrt{2\pi}\, e^{-a^2/2}$.

At the other extreme of $0|q$ manifolds, integration is a purely algebraic operation. For $M^{0|q}$ complex, integration of a section of the Berezinian line bundle makes sense. No real structure, and no orientation, are needed.

For a general complex super manifold M, integration of a holomorphic section of the Berezinian line bundle $\mathrm{Ber}(\Omega^1_M)$ requires only an integration cycle of half the real dimension on M_{red}, and suitable control at infinity. A variant of this principle is that for cs manifolds, with M_{red} oriented, integration of sections with compact support of $\mathrm{Ber}(\Omega^1_M)$ makes sense. Similar statements hold in a relative setting.

Example 4.9.2. Let M be a super manifold of dimension $p|q$. In the complexified tangent bundle $T_{\mathbb{C}}$, fix a locally direct factor τ of dimension $0|s$ for some s. If τ is integrable, i.e. stable under bracket, $|M|$ endowed with the sheaf $\mathcal{O}^\tau_{\mathbb{C}}$ of complex

functions on M, annihilated by τ, is a cs manifold of dimension $p|(q-s)$. We denote it M/τ. Some supersymmetric Lagrangians involve integration on such cs spaces ("integration of chiral functions"); see [I-**Supersolutions**, §5.3].

Example 4.9.3. If one tries by a "Wick rotation" $t \mapsto it$ to convert a super Minkowski space (see [I-**Supersolutions**, §1.1]) into a Euclidean analogue, one often runs into trouble, in that the real spin representation of $\mathrm{Spin}(1,n)$ used to define the super Minkowski space is not a real representation of $\mathrm{Spin}(n+1)$. In such cases, the Euclidean analogue of super Minkowski space is only a cs manifold whose underlying reduced space is Euclidean space $\mathbb{R}^{n+1}$. Corresponding Euclidean field theories involve integration on this cs manifold.

Consider for instance the case of the super Minkowski space of dimension $4|4$. Its construction ([I-**Supersolutions**, §2.4]) starts with the unique irreducible spinorial real representation S of the double covering $\mathrm{SL}(2,\mathbb{C})$ of the Lorentz group $\mathrm{SO}(1,3)$. As a representation of $\mathrm{SL}(2,\mathbb{C})$ (which is viewed as a real Lie group), S is the 4-dimensional real vector space underlying the complex 2-dimensional defining representation of $\mathrm{SL}(2,\mathbb{C})$. The complexification of $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$, with the real locus $\mathrm{SL}(2,\mathbb{C})$ embedded by $g \mapsto (g,\bar{g})$; the complexification $S_{\mathbb{C}}$ of S is the sum of the defining representations of the two factors $\mathrm{SL}(2,\mathbb{C})$. The real form of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ corresponding to the real form $\mathrm{SO}(4)$ of $\mathrm{SO}(1,3)_{\mathbb{C}}$ is $\mathrm{SU}(2)\times\mathrm{SU}(2)$. There is no real form of $S_{\mathbb{C}}$ stable under $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Indeed, the defining complex representation W of $\mathrm{SU}(2)$ is not real, but rather quaternionic: it admits an intertwining operator $j\colon W \to W$, $\mathbb{C}$-antilinear of square -1.

Example 4.9.4. There is a far-reaching analogy between Riemann surfaces, i.e. complex curves, and complex super manifolds of dimension $1|1$, provided with a $0|1$ distribution τ "as non-integrable as possible": for T the tangent bundle, the Frobenius pairing $\tau\otimes\tau \to T/\tau$ should be an isomorphism. Such objects are called SUSY curves. Let $\mathcal{M}$ be the moduli space of SUSY curves of genus g. Let us ignore the point that because SUSY curves can have nontrivial automorphism, $\mathcal{M}$ is a stack, rather than a complex super manifold.

The moduli space $\mathcal{M}$ has a natural real structure. This expresses the fact that the complex conjugate of a SUSY curve is again a SUSY curve, and more generally that if $p : X \to S$ is a complex analytic family of SUSY curves parametrized by S, then $\bar{p}\colon \bar{X} \to \bar{S}$ is again a family of SUSY curves.

The complexification $\mathcal{M}\times\bar{\mathcal{M}}$ of $\mathcal{M}_{\mathbb{R}}$, the real analytic space underlying $\mathcal{M}$, can hence be rewritten as being $\mathcal{M}\times\mathcal{M}$, with real structure $(x,y)\mapsto(\bar{y},\bar{x})$. The reduced space $(\mathcal{M}\times\mathcal{M})_{\mathrm{red}}$ is the space of pairs of Riemann surfaces Σ_1, Σ_2, each provided with a square root of the canonical bundle. In it, the locus where Σ_1 and Σ_2 are complex conjugate defines a cs manifold. It is to integration on this cs manifold that one refers when one speaks of "summing independently over spin structures for left and right movers" ([II-**Strings**, §9.2]).

Example 4.9.5. If X is a super manifold of dimension $0|q$, for any super manifold Y, the mapping space $\underline{\mathrm{Hom}}(X,Y)$ is again a finite dimensional super manifold.

For simplicity, let us assume that X is connected, hence isomorphic to $\mathbb{R}^{0|q}$. Let $X_{\mathbb{C}}$ be the complexification of X. If Y is purely even and $q=1$, the cs manifold $\underline{\mathrm{Hom}}(X,Y)^{\mathrm{cs}}$ depends only on $X_{\mathbb{C}}$ and Y. If Y is real analytic, this can be thought of as follows. Let $Y_{\mathbb{C}}$ be a complexification of Y. Then $\underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})$, computed in the holomorphic world, is a complexification of $\underline{\mathrm{Hom}}(X,Y)$. The corresponding reduced

space is $Y_{\mathbb{C}}$, and $\underline{\mathrm{Hom}}(X,Y)^{\mathrm{cs}}$ is the cs space with complexification $\underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})$, defined by the real structure Y of $Y_{\mathbb{C}} = \underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})_{\mathrm{red}}$.

The same does not hold for $q > 1$. For simplicity, take $q = 2$. Fix coordinates θ_1, θ_2 on $X_{\mathbb{C}}$. Take $Y = \mathbb{R}$, of dimension $1|0$, and $Y_{\mathbb{C}} = \mathbb{C}$. A holomorphic map from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$ (rather, a family of such maps, parametrized by some S) can be written

$$\theta_1, \theta_2 \longmapsto a + \alpha_1\theta_1 + \alpha_2\theta_2 + b\theta_1\theta_2 \ .$$

The space $\underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})$ is hence $\mathbb{C}^{2|2}$, with coordinates $(a, b, \alpha_1\alpha_2)$. The reduced space is $\mathbb{C}^2$, with coordinates (a, b). The real structure of Y tells us what it means for a to be real. For b, we need a real structure on X as well. To get a cs space, we would need to divide out by translations in b.

Similar troubles occur in infinite dimensional map spaces. To obtain a cs structure, one sometimes needs to "eliminate auxiliary fields".

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Notes on Spinors

Pierre Deligne

Introduction

In these notes, we collect the properties of spinors in various dimensions and, over $\mathbb{R}$, for spaces of various signatures. Such information is needed to discuss the possible supersymmetries in various dimensions (super Poincaré groups), and the possible Lorentz invariant kinetic and mass terms for fermions in lagrangians.

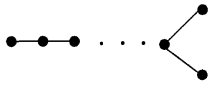
The exposition owes a lot to Bourbaki's treatment in Alg. Ch. 9, and through Bourbaki to C. Chevalley's "The algebraic theory of spinors". The super Brauer group of §3 was first considered by C. T. C. Wall (1963), under the name "graded Brauer group" and with a different, but equivalent, definition. I have learned of the analogy between spinorial and oscillator representations (2.3, 2.5) from a lecture R. Howe gave in 1978. The uniform treatment in 6.1 of the Minkowski signature cases was inspired by a conversation with Witten.

§1. Overview
§2. Clifford Modules
§3. Reality of Spinorial Representations and Signature Modulo 8
§4. Pairings and Dimension Modulo 8, Over $\mathbb{C}$
§5. Passage to Quadratic Subspaces
§6. The Minkowski Case

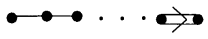
CHAPTER 1
Overview

1.1. A *quadratic vector space* is a vector space given with a nondegenerate quadratic form Q. Let V be a complex quadratic vector space. If $\dim(V) \geq 3$, the complex spin group $\mathrm{Spin}(V)$ is the universal covering of the special orthogonal group $\mathrm{SO}(V)$. For V of any dimension ≥ 1, it is a double covering of $\mathrm{SO}(V)$. For $\dim(V) = 1$ or 2, it is described in the table below.

For $n \geq 3$, the Dynkin diagram of the group $\mathrm{Spin}(n)$ is:



$n = 2r$



$n = 2r + 1$

where r is the rank. The spin or semi-spin representations are the fundamental representations corresponding to the vertex, or vertices, at the right of the diagram. They are either self-dual, or permuted by duality. If one restricts the spin representation of $\mathrm{Spin}(2r+1)$ to $\mathrm{Spin}(2r)$, one obtains the sum of the two semi-spin representations of $\mathrm{Spin}(2r)$. Each of the two semi-spin representations of $\mathrm{Spin}(2r)$ restricts as the spin representation of $\mathrm{Spin}(2r-1)$. For a proof, see 5.1.

The Dynkin diagram makes clear the low rank exceptional isomorphisms. We list in the next table $n = 1, 2, 3, 4, 5$ or 6, the Dynkin diagram, a group G to which $\mathrm{Spin}(n)$ is isomorphic, the description of spin or semi-spin representations as representations of G, and the description of the defining representation of $\mathrm{SO}(n) = \mathrm{Spin}(n)/(\mathbb{Z}/(2))$ as a representation of G.

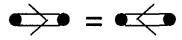
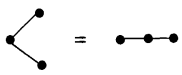
n	diagram	G	(semi-)spin	defining orthogonal
1	none	$\mathbb{Z}/2$	nontrivial character	trivial character
2	none	$\mathbb{G}_m$	characters z, z^{-1}	$z^2 \oplus z^{-2}$
3	•	$\mathrm{SL}(2)$	defining V	adjoint $= \mathrm{Sym}^2(V)$
4	:	$\mathrm{SL}(2) \times \mathrm{SL}(2)$	defining V_1, V_2	$V_1 \otimes V_2$
5		$\mathrm{Sp}(4)$	defining V	$\wedge^2 V$/fixed line
6		$\mathrm{SL}(4)$	defining V, and V^*	$\wedge^2 V$

Table 1.1.1

The symmetric bilinear form on the defining orthogonal representation is given by

$n = 3$: Killing form $\mathrm{Tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$, or
$\langle v^2, w^2 \rangle = \psi(v,w)^2$, for ψ the symplectic (= volume) form of V.

$n = 4$: $\langle v_1 \otimes v_2,\ w_1 \otimes w_2 \rangle = \psi(v_1, w_1)\psi_2(v_2, w_2)$.

$n = 5$: $\langle v_1 \wedge v_2,\ w_1 \wedge w_2 \rangle = \psi(v_1, w_1)\psi(v_2, w_2) - \psi(v_1, w_2)\psi(v_2, w_1)$.

$n = 6$: $\langle v_1 \wedge v_2,\ w_1 \wedge w_2 \rangle = v_1 \wedge v_2 \wedge w_1 \wedge w_2 \in \wedge^4 V = \mathbb{C}$.

For $n = 8$, the Dynkin diagram makes clear that the vector and the two semi-spinor representations play symmetric roles (triality). The octonionic model for them, given in 6.5, makes this symmetry explicit.

1.2. Let V be a real quadratic vector space, with complexification $V_{\mathbb{C}}$. The real form V of $V_{\mathbb{C}}$ defines real forms $\mathrm{SO}(V)$ and $\mathrm{Spin}(V)$ of the complex algebraic groups $\mathrm{SO}(V_{\mathbb{C}})$ and $\mathrm{Spin}(V_{\mathbb{C}})$. For $\dim V \geq 3$, the group of real points of $\mathrm{Spin}(V)$ is connected, and is the unique nontrivial double covering of the connected component of the group of real points of $\mathrm{SO}(V)$.

The *spinorial representations* of $\mathrm{Spin}(V)$ are the real representations which, after extensions of scalars to $\mathbb{C}$, become sums of spin or semi-spin representations. The dual of a spinorial representation is again spinorial.

We will be interested in the following kind of morphisms of representations.

(1.2.1) symmetric morphisms $S \otimes S \to V$, for S spinorial. Such morphisms enter into the construction of super Minkowski spaces and super Poincaré groups.

(1.2.2) $V \otimes S_1 \to S_2$, for S_1 and S_2 spinorial. Such morphisms enter into the construction of Dirac operators $\not{D}$ between spin bundles.

(1.2.3) $S \otimes S \otimes V \to$ trivial. Such morphisms enter into kinetic terms $\psi \not{D} \psi$ in Lagrangians.

(1.2.4) $S \otimes S \to$ trivial. Such morphisms enter into mass terms $\psi M \psi$ in Lagrangians.

1.3. The nature of the spinorial representations and of the morphisms (1.2.1), (1.2.2), (1.2.3), (1.2.4) is controlled by the signature and the dimension modulo 8. We explain those two modulo 8 periodicities in 1.4 and 1.5 and, with more details, in §3 and §4.

1.4. The signature modulo 8 determines how many nonisomorphic irreducible spinorial representations there are, and whether they are real, complex or quaternionic. As (V, Q) and $(V, -Q)$ define the same spin group, only the signature taken up to sign matters.

Let p and q be the number of $+$ and $-$ signs of Q, in an orthogonal basis. The signature $p - q$ and the dimension $p + q$ have the same parity. If they are odd, there is over the complex numbers a unique irreducible spinorial representation. If they are even, there are two. It follows that over the reals there is in the odd case a unique irreducible spinorial representation. It is either real or quaternionic. In the

even case, either there are two, real or quaternionic, or there is only one, and it is complex. Here is a table showing for which signatures each case occurs:

signature $p-q$ mod 8	real, complex or quaternionic
0	$\mathbb{R}$, $\mathbb{R}$
1 or 7	$\mathbb{R}$
2 or 6	$\mathbb{C}$
3 or 5	$\mathbb{H}$
4	$\mathbb{H}$, $\mathbb{H}$

Table 1.4.1

1.5. Over the complex numbers, the dimension modulo 8 controls the existence and symmetry properties of the morphisms (1.2.1), (1.2.2), (1.2.3) and (1.2.4).

For n odd, the spin representation S of $\mathrm{Spin}(n)$ is orthogonal or symplectic. There is a morphism of representations from $S \otimes S$ to the defining representation V; it is unique, hence it is symmetric or antisymmetric. Here and below, "unique" is understood projectively: uniqueness up to a scalar factor of a nonzero morphism.

For n even, the semi-spin representations S^+ and S^- are orthogonal, symplectic, or duals of each other. There are unique morphisms of representations $V \otimes S^+ \to S^-$ and $V \otimes S^- \to S^+$, and no morphism $V \otimes S^+ \to S^+$ or $V \otimes S^- \to S^-$. If S^+ and S^- are duals of each other, the morphism $V \otimes S^+ \to S^-$ (resp. $V \otimes S^- \to S^+$) corresponds to a morphism $S^+ \otimes S^+ \to V$ (resp. $S^- \otimes S^- \to V$), which is either symmetric or antisymmetric. If S^+ and S^- are self-dual, they correspond to a morphism $S^+ \otimes S^- \to V$.

Here is a table showing in which dimension each case occurs:

n mod 8	forms on spinors	symmetry of spinors, spinors $\to V$
0	S^+ and S^- orthogonal	$S^+ \otimes S^- \to V$
1	orthogonal	symmetric
2	S^+ dual to S^-	symmetric (on S^+, and on S^-)
3	symplectic	symmetric
4	S^+ and S^- sympletic	$S^+ \otimes S^- \to V$
5	symplectic	antisymmetric
6	S^+ dual to S^-	antisymmetric (on S^+, and on S^-)
7	orthogonal	antisymmetric

Table 1.5.1

1.6. To go from 1.5 to information over $\mathbb{R}$, one proceeds as follows.

(A) When the irreducible complex spinorial representations are real, i.e. admit a real form, the real story is the same as the complex story.

(B) Suppose now that they are quaternionic. After extension of scalars to $\mathbb{C}$, the quaternions become a 2×2 matrix algebra: $\mathbb{H}_{\mathbb{C}} = \mathrm{End}(W)$, the real form $\mathbb{H}$ of $\mathrm{End}(W)$ being induced by an antilinear automorphism σ of W, with $\sigma^2 = -1$, and the conjugation $*$ of $\mathbb{H}$ being the transposition relative to any symplectic form ψ_W on W. The form ψ_W can and will be chosen to be real, in the sense that

$\psi_W(\sigma x, \sigma y) = \overline{\psi_W(x,y)}$. The $\psi_W(x, \sigma x)$ are then real, and ψ_W can and will be normalized so that $\psi_W(x, \sigma x) > 0$ for $x \neq 0$.

If S is an irreducible real representation, with field of endomorphisms $\mathbb{H}$, the complexification $S_{\mathbb{C}}$ of S is acted upon by $\mathrm{End}(W)$ and hence can be written as

$$S_{\mathbb{C}} = S_0 \otimes W \ ,$$

for $S_0 = \mathrm{Hom}_{\mathrm{End}(W)}(W, S_{\mathbb{C}})$. The complex conjugation σ_S of $S_{\mathbb{C}}$ can be written $\sigma_S = \sigma_0 \otimes \sigma$, with σ: $S_0 \to S_0$ $\mathbb{C}$-antilinear and $\sigma_0^2 = -1$.

The complex representation S_0 is irreducible, and 1.5 applies to it. If S_0 is orthogonal, $\psi(X,Y) \mapsto \phi^\sigma(X,Y) := \overline{\phi(\sigma x, \sigma y)}$ is a $\mathbb{C}$-antilinear involution on the one-dimensional space of symmetric bilinear forms on S_0. If ϕ is real, i.e. if $\phi = \phi^\sigma$, $\phi \otimes \psi_W$ on $S_{\mathbb{C}}$ is the complexification of a symplectic form on S, relative to which the conjugation $*$ of $\mathbb{H}$ is transposition.

Similarly, if S_0 is orthogonal, S is symplectic. If two irreducible spinorial representations S^+ and S^- are quaternionic, if S_0^+ and S_0^- are in duality, so are S^+ and S^-. The same applies to pairings with values in V.

(C) If the irreducible spinorial representation S is complex, i.e. if its commutant is $\mathbb{C}$, by complexification it becomes the sum of the two semi-spinorial representations. If they are duals of each other (resp. orthogonal, resp. symplectic), S admits an invariant Hermitian (resp. complex orthogonal, complex symplectic) form.

1.7. In *Minkowski signature*, $(+,-,-,\dots)$, and with a positive light cone chosen on V, if S is an irreducible spinorial representation, there is up to a real factor a unique symmetric bilinear maps B: $S \otimes S \to V$. It can be normalized so that $Q(s) := \frac{1}{2} B(s,s)$ takes values in the closed positive cone. Such a Q is now unique up to a positive real factor.

If S is complex, B is the real part of a Hermitian bilinear form with values in $V_{\mathbb{C}}$: if J is the complex structure, Q is invariant under the circle group of the $\exp(\theta J)$ ($\theta \in \mathbb{R}$). After extension of scalars to $\mathbb{C}$, S becomes the sum of the two semi-spinorial representations, and B corresponds to a pairing $S^+ \otimes S^- \to V$.

If S is quaternionic, the quaternions with absolute value one preserve Q. After extension of scalars to $\mathbb{C}$, S becomes the sum $S_0 \otimes W$ of two copies of an irreducible spinorial representation S_0, and B becomes the tensor product of an antisymmetric pairing $S_0 \otimes S_0 \to V_{\mathbb{C}}$ with a scalar alternating form $W \otimes W \to \mathbb{C}$. The quaternions become $\mathrm{End}(W)$.

The next table gives the nature of irreducible spinorial representations, as a function of n mod 8, for $\mathrm{Spin}(1, n-1)$. If $\mathrm{Spin}(1, n-1)$ has two irreducible spinorial representations, they have isomorphic restrictions to $\mathrm{Spin}(1, n-2)$. The last column of the table describes the restriction of an irreducible representation of $\mathrm{Spin}(1, n-1)$ to $\mathrm{Spin}(1, n-2)$. It is either an irreducible spinorial representation, or twice it, or

the sum of two distinct spinorial representations. Notation: S, $2S$, $S^+ + S^-$.

n mod 8	nature	restriction to $\mathrm{SO}(1, n-2)$
1	$\mathbb{R}$	S
2	$\mathbb{R}$, $\mathbb{R}$ (S^+ and S^-, in duality)	S
3	$\mathbb{R}$	$S^+ + S^-$
4	$\mathbb{C}$	$2S$
5	$\mathbb{H}$	$2S$
6	$\mathbb{H}$, $\mathbb{H}$ (S^+ and S^-, in duality)	S
7	$\mathbb{H}$	$S^+ + S^-$
8	$\mathbb{C}$	S

The cases $n = 3$, 4, 6 and 10 are particularly interesting. The spin groups acting on an irreducible spinorial representation S are respectively $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$, $\mathrm{SL}(2, \mathbb{H})$ and deserving to be called $\mathrm{SL}(2, \mathbb{O})$. In those dimensions, the spin group acts transitively on non-zero spinors and by the symmetric pairing $S \otimes S \to V$, $s \otimes s$ is mapped to an isotropic vector.

CHAPTER 2
Clifford Modules

2.1. Let k, V and Q be a commutative ring, a k-module and a quadratic form on V. The *Clifford algebra* $C(V,Q)$ is the associative k-algebra with unit generated by the k-module V with the relations

$$x^2 = Q(x) \cdot 1 \tag{2.1.1}$$

for $x \in V$. This is Bourbaki's definition (Alg. Ch. 9 §9, n° 1). Some authors prefer to use as defining relations $x^2 = -Q(x) \cdot 1$. We will often write $C(V)$, or $C(Q)$, instead of $C(V,Q)$.

It results immediately from this definition that

(A) $C(Q)$ is mod 2 graded, the image of V being odd. In other words: $C(Q)$ is a super algebra. Indeed, the defining relations are in the even part of the tensor algebra on V. We will write $p(x)$ for the parity of a homogeneous element x of $C(Q)$.

(B) The algebra $C(Q)$ admits a unique anti-involution β which is the identity on the image of V. Indeed, the opposite algebra $C(Q)^0$ is a solution of the same universal problem as $C(Q)$ is. "Opposite" is taken in the ungraded sense, not in the super sense. Bourbaki's terminology: β is the principal antiautomorphism of $C(Q)$. By definition, $\beta(xy) = \beta(y)\beta(x)$.

If we apply (2.1.1) to $x+y$, x and y and take the difference, we obtain the polarized form of (2.1.1):

$$xy + yx = \Phi(x,y) \cdot 1 \tag{2.1.2}$$

for Φ the bilinear form $Q(x+y) - Q(x) - Q(y)$ associated to Q.

When 2 is invertible in k, (2.1.1) is equivalent to (2.1.2): take $x = y$ in (2.1.2). In general, if $X \subset V$ generates V, (2.1.1) is implied by (2.1.1) for $x \in X$, and (2.1.2) for $x \neq y$ in X. This makes it clear that

(C) The formation of $C(Q)$ is compatible with extensions of scalars.

(D) If (V,Q) is the orthogonal direct sum of (V',Q') and (V'',Q''), the Clifford algebra $C(Q)$ is the tensor product, in the sense of super algebras (Supersymmetry 1.1 (1.1.5)), of $C(Q')$ and $C(Q'')$. If β' and β'' are the principal antiautomorphisms of $C(Q')$ and $C(Q'')$, the principal antiautomorphism β of $C(Q)$ is given by

$$\beta(x \otimes y) = (-1)^{p(x)p(y)} \beta(x) \otimes \beta(y) \ .$$

Indeed, $C(Q)$ is generated by the k-modules V' and V'', with the relations (2.1.1) for $x \in V'$ or $x \in V''$ and the relations (2.1.2) for $x \in V'$ and $y \in V''$. For $x \in V'$ and $y \in V''$, (2.1.2) means that x and y supercommute, and the first assertion of (D) follows. For the second, one observes that for x in $C(V')$ and y in $C(V'')$, one has

$$\begin{aligned}\beta(x \otimes y) &= \beta((x \otimes 1)(1 \otimes y)) = \beta(1 \otimes y)\beta(x \otimes 1) \\ &= (1 \otimes \beta(y))(\beta(x) \otimes 1) = (-1)^{p(x)p(y)}\beta(x) \otimes \beta(y) \ .\end{aligned}$$

(E) The identity of V extends to an isomorphism from the opposite of the super algebra $C(Q)$ to the super algebra $C(-Q)$.

Indeed, each defining relation $x \cdot x = Q(x) \cdot 1$ is replaced by its opposite $x \cdot x = -Q(x) \cdot 1$.

We now take $k = \mathbb{C}$. From now on, we assume V non reduced to 0.

Proposition 2.2 (Bourbaki Alg. Ch. 9 §9, no. 4). *Let V be a complex quadratic vector space.*

(i) *If V is of dimension $2n > 0$ the super algebra $C(V)$ is isomorphic to $\underline{\mathrm{End}}\,(S)$, with S of dimension $2^{n-1}|2^{n-1}$.*

(ii) *Let D be the super algebra $\mathbb{C}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$. If V is of dimension $2n+1$, $C(V)$ is isomorphic to $\underline{\mathrm{End}}_D(D^N) \sim D \otimes M_N(\mathbb{C})$ for $N = 2^n$.*

Proof of (i). The quadratic space V is isomorphic to $L \oplus L^\vee$ with $\dim L = n$ and $Q(\ell + \alpha) = \langle \ell, \alpha \rangle$. Take $S = \wedge^* L^\vee$. One defines

$$C(L \oplus L^\vee) \to \underline{\mathrm{End}}\,(S) \tag{2.2.1}$$

by mapping α in $L^\vee$ to the odd endomorphism $\alpha\wedge$, and ℓ in L to ι_ℓ, the odd derivation of the exterior algebra $\wedge^* L$ (viewed as a commutative super algebra) for which $\iota_\ell(\alpha) = \langle \ell, \alpha \rangle$. A decomposition of L as a direct sum of lines L_i induces a dual decomposition of $L^\vee$ as the sum of the dual lines ${L^\vee}_i$, and a decomposition of V as the orthogonal direct sum of the hyperbolic planes $V_i := L_i \oplus {L^\vee}_i$. The super algebra $C(V)$ is the tensor product of the super algebra $C(V_i)$, the graded vector space $S = \wedge^* L^\vee$ is the tensor product of the graded vector spaces $S_i := \wedge^* {L^\vee}_i$, and (2.2.1) is the tensor product of the similarly defined morphisms $C(V_i) \to \underline{\mathrm{End}}\,(S_i)$. To prove that (2.2.1) is an isomorphism, it hence suffices to check it when $\dim(L) = 1$, a case left to the reader.

Proof of (ii). For $\dim(V) = 1$, $C(V)$ is isomorphic to D. For $\dim(V) \geq 3$, we write V as the orthogonal direct sum of a line V_1 and of V_{2n}, of dimension $2n$. For S of dimension $2^{n-1}|2^{n-1}$, $C(V)$ is isomorphic to $\underline{\mathrm{End}}\,(S) \otimes D \sim \underline{\mathrm{End}}_D(S \otimes D) \sim \underline{\mathrm{End}}_D(D^N)$, the D-module ΠD being isomorphic to the D-module D.

2.3. The construction used in the proof of 2.2. (i) is an odd analogue of that of the Schrödinger representation of Heisenberg groups, or Lie algebras. The analogy goes as follows. Let V be symplectic, rather than orthogonal, over a field k. Provide $k \oplus V$ with the Lie algebra structure for which k is central and for which the bracket of elements of V is given by $[v_1, v_2] = \psi(v_1, v_2)$. It is the Heisenberg Lie algebra

$\mathcal{H}$ associated to V. Write V as a direct sum $L \oplus L^\vee$, with $\psi(\ell' + \alpha', \ell'' + \alpha'') = \alpha''(\ell') - \alpha'(\ell'')$. The Lie algebra $\mathcal{H}$ acts on the vector space $\mathrm{Sym}^*(L^\vee)$, the algebra of polynomial functions on L, with $1 \in k$ acting as the identity, α in $L^\vee$ acting by multiplication by α, and ℓ in L acting by ∂_ℓ. This turns $\mathrm{Sym}^*(L^\vee)$ into a module over the universal envelopping algebra $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$. Let 1 be the unit element of $\mathcal{U}(\mathcal{H})$, and $1_{\mathcal{H}}$ be the central element $1 \in k$ of $\mathcal{H}$. Both act as the identity on $\mathrm{Sym}^*(L^\vee)$, which is hence a module over the quotient $\mathcal{U}_1(\mathcal{H})$ of $\mathcal{U}(\mathcal{H})$ by the relation $1 = 1_{\mathcal{H}}$. If we repeat this in the super case, for a purely odd super vector space V, the symplectic structure is, in concrete terms, a symmetric bilinear form on the underlying vector space. The sum $k \oplus V$ is a super Lie algebra $\mathcal{H}$, and $\mathcal{U}_1(\mathcal{H})$ is the Clifford algebra $C(V)$.

In the even case, $\mathcal{U}_1(\mathcal{H})$ maps isomorphically to the algebra of all polynomial differential operators acting on $\mathrm{Sym}^*(L^\vee)$. In the odd case, the analogue is all endomorphisms of $\wedge^* L^\vee$. Physicists use the same terminology in both cases: 1 in $\mathrm{Sym}^*(L^\vee)$ (resp. $\wedge^* L^\vee$) is the vacuum, $L^\vee$ acts by creation operators, and L by annihilation operators.

The Poincaré-Birkhoff-Witt theorem implies, in the even case, that $\mathrm{Sym}^*(V) \xrightarrow{\sim} \mathrm{Gr}\, \mathcal{U}_1(\mathcal{H})$, where $\mathcal{U}_1(\mathcal{H})$ is filtered by the images of the $\bigoplus_{i \leq n} V^{\otimes i}$. In the odd case, it implies that $\wedge^* V \xrightarrow{\sim} \mathrm{Gr}\, C(V)$.

2.4. We return to the case of a complex quadratic vector space V. By 2.2 (i), if the dimension of V is even, the Clifford algebra $C(V)$ is isomorphic to a matrix algebra, hence has up to isomorphism a unique simple module S. The group of automorphisms of S is the multiplicative group $\mathbb{C}^*$. The module S admits a mod 2 grading $S = S^+ \oplus S^-$ compatible with the grading of $C(V)$, unique up to parity change. From the super point of view, $C(V)$ is a super matrix algebra, hence has exactly two isomorphism classes of simple super modules, exchanged by parity change. The even part $C^+(V)$ of $C(V)$ is $\mathrm{End}(S^+) \times \mathrm{End}(S^-)$. As $S^+, S^- \neq \{0\}$, the center Z of $C^+(V)$ is $\mathbb{C} \times \mathbb{C}$, and the two isomorphism classes of simple super modules are distinguished by the character through which Z acts on S^+.

It follows that any automorphism g of $C(V)$ can be extended to an automorphism of $(C(V)$, module $S)$. The extension is unique up to an automorphism $\lambda \in \mathbb{C}^*$ of S. If g respects the grading of $C(V)$, the extension respects or permutes the homogeneous components $S^\pm$ of S, depending on the action of g on the center of $C^+(V)$.

In particular, $\mathrm{O}(V)$ acts projectively on S. The parity respecting subgroup is $\mathrm{SO}(V)$.

The projective action of the group $\mathrm{SO}(V)$ on S induces a projective action of its Lie algebra $\mathfrak{so}(V)$. In algebraic language, this can be expressed as follows. The construction of the projective action can be repeated after extension of scalars from $\mathbb{C}$ to the ring of dual numbers $B := \mathbb{C}[\varepsilon]/(\varepsilon^2)$, and interpreting $\mathfrak{so}(V)$ as $\mathrm{Ker}(\mathrm{SO}(V)(B) \to \mathrm{SO}(V)(\mathbb{C}))$, one obtains a projective action ρ: $\mathfrak{so}(V) \to \mathrm{End}(S)/\mathbb{C}$ of $\mathfrak{so}(V)$ on S. Equivalent description: as $C(V)$ is a matrix algebra, any derivation of $C(V)$ is inner. In particular, the derivation defined by x in $\mathfrak{so}(V)$ can be written as $a \mapsto [f(x), a]$. As this derivation is even, $f(x)$ is in $C^+(V)$. It is well defined up an additive constant, and $\rho(x)$ is multiplication by $f(x)$. As $C(Q)$ is, as an ordinary Lie algebra, the Lie algebra product of $\mathbb{C}$ and of $[C(V), C(V)]$, f can be normalized to be a Lie algebra morphism from $\mathfrak{so}(V)$ to $[C(V), C(V)]$ (usual bracket). Here is a direct construction of f, so normalized.

Identifying $\mathfrak{so}(V)$ with $\wedge^2(V)$ by

$$x \wedge y \longmapsto \text{endomorphism } (v \mapsto \Phi(y,v)x - \Phi(x,v)y) \text{ of } V\ , \tag{2.4.1}$$

one defines f to be

$$f\colon\ x \wedge y \longmapsto \frac{1}{2}(xy - yx)\ . \tag{2.4.2}$$

We have to check that for v in V, $[f(x\wedge y), v]$ is given by (2.4.1). As $xy + yx$ is a scalar, one has indeed, for $\{\ ,\ \}$ the super bracket in $C(V)$

$$\begin{aligned}[f(x \wedge y), v] = [xy, v] = \{xy, v\} &= x\{y, v\} - \{x, v\}y \\ &= \Phi(y,v)x - \Phi(x,v)y\ .\end{aligned}$$

If the dimension of V is odd, $C^+(V)$ is a matrix algebra and the same argument shows that $\mathfrak{so}(V)$ acts projectively on a simple $C^+(V)$-module S^+ (unique up to isomorphism). This action lifts uniquely to a Lie algebra morphism from $\mathfrak{so}(V)$ to $[C^+(V), C^+(V)] = C^+(V) \cap [C(V), C(V)]$. This lifting is again given by (2.4.1), (2.4.2).

Variant: if the dimension of V is odd, the super algebra $C(V)$ has up to isomorphism a unique simple super module S. The group of (even) automorphisms is $\mathbb{C}^*$. The group $\mathrm{O}(V)$ acts projectively on S (respecting the mod 2 grading), and so does the Lie algebra $\mathfrak{so}(V)$. The action is by multiplication by $f(D)$, with f as in (2.4.2).

2.5. In the spirit of 2.3, the construction used in 2.4 is analogous to the one by which, using the uniqueness of an irreducible representation of the Heisenberg commutation relations, one gets on the representation space a projective action of a real symplectic group — or an actual representation of its double covering, the metaplectic group. This is, however, an analytic story, involving infinite dimensional Hilbert spaces, as evidenced by the fact that the metaplectic double covering of the real symplectic group is not algebraic. The Lie algebra story, however, has a purely algebraic analogue. With the notations of 2.3, the symmetrized product gives a vector space isomorphism

$$\mathrm{Sym}^*(V) \xrightarrow{\sim} \mathcal{U}_1(k \oplus V) \tag{2.5.1}$$

The image of $\mathrm{Sym}^2(V)$ by this map is a Lie algebra normalizing V, and it is identified by its action on V with the Lie algebra of the symplectic group.

Assume V is the dual of E, so that $\mathrm{Sym}^*(V)$ is the polynomial functions on E. If we replace ψ by $t\psi$, and transport the product on the corresponding $\mathcal{U}_1(k\oplus V)$ to $\mathrm{Sym}^*(V)$ by (2.5.1), we obtain on $\mathrm{Sym}^*(V)$ a product $*_t$ depending on t. It comes from a $k[t]$-algebra structure on $\mathrm{Sym}^*(V) \otimes k[t]$.

One has

$$f *_t g = fg + \tfrac{1}{2}\{f, g\}t + O(t^2),$$

for $\{\ ,\ \}$ the Poisson bracket on the symplectic manifold E. A weakened version of the fact that $\mathrm{Sym}^*(V) \subset \mathcal{U}_1(k \oplus V)$ acts on V as Lie $\mathrm{Sp}(V)$ is the fact that the Hamiltonian vector fields on E given by quadratic functions are the infinitesimal symplectic transformations.

2.6. We return to the case of a complex quadratic vector space V. The Lie algebra morphism (2.4.1), (2.4.2) from $\mathfrak{so}(V)$ to $C^+(V)$ induces an algebra morphism

$$\mathcal{U}\mathfrak{so}(V) \to C^+(V)\ . \tag{2.6.1}$$

For x, y in V, $\frac{1}{2}(xy - yx)$ is in the image of $\mathfrak{so}(V)$; it differs from xy by the constant $\frac{1}{2}(xy + yx) = \frac{1}{2}\phi(x, y)$, and it follows that (2.6.1) is onto. The *spinorial* representations S of $\mathfrak{so}(V)$ are those for which $\mathcal{U}\mathfrak{so}(V) \to \mathrm{End}(S)$ factors through $C^+(V)$. By (2.6.1), they are identified with $C^+(V)$-modules. By 2.2, if S is a simple super $C(V)$-module, the simple spinorial representations are isomorphic to

for n even: S^+ or S^-;

for n odd: S^+ (isomorphic to S^-).

For real quadratic vector spaces, *spinorial* representations are those which become spinorial after extension of scalars to $\mathbb{C}$. They are identified by (2.6.1) to $C^+(V)$-modules.

The *spinorial representations* of the spin group $\mathrm{Spin}(V)$ are those representations obtained by integrating the spinorial representations of the Lie algebra $\mathfrak{so}(V)$.

Remark. If S is a super module over $C(V)$, the induced $C^+(V)$-module structure turns S^+ and S^- into spinorial representations of $\mathfrak{so}(V)$. The maps $V \otimes S^\pm \to S^\mp$ induced by the module structure are morphisms of representations. Indeed, for $a \in \mathfrak{so}(V) \subset C^+(V)$, $a(vs) = [a, v]s + v(as)$.

If V is odd-dimensional, let Z be the commutant of $C^*(V)$ in $C(V)$ and let Z^- be its odd part. It is one-dimensional. The multiplication by a generator z of Z^- is an isomorphism of representations $S^\pm \to S^\mp$.

2.7. In the same way that, in 2.4, the Lie algebra $\mathfrak{so}(V)$ is identified with a sub-Lie algebra of $C^+(V)$, the spin group $\mathrm{Spin}(V)$ can be realized as a subgroup of the multiplicative group of $C^+(V)$. Let G be the group of invertible elements in $C^+(V)$ or $C^-(V)$ which normalize V. We let it act on V by

$$\rho(g):\ v \longmapsto (-1)^{p(g)} gvg^{-1}.$$

This action respects the quadratic form $Q(v) = v^2$ on V. If g fixes V, g is in the center (in the super sense) of $C(V)$, hence is a scalar (3.4 and 3.3.1): we have an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G \longrightarrow \mathrm{O}(V). \tag{2.7.1}$$

Elements w of V with $Q(w) \neq 0$ are in $G^- := G \cap C^-(Q)$, and $\rho(w)$ is the reflection relative to the hyperplane orthogonal to w. As each element of $\mathrm{O}(V)$ is a product of reflections, the sequence (2.7.1) is exact on the right. As a product of k reflections has determinant $(-1)^k$, it follows from (2.7.1) that $G^+ := G \cap C^+(Q)$ maps onto $\mathrm{SO}(V)$, while G^- maps onto $\mathrm{O}(V) - \mathrm{SO}(V)$. We have

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G^+ \longrightarrow \mathrm{SO}(V) \longrightarrow 1. \tag{2.7.2}$$

If $g \in G$, applying the principal antiautomorphism β to the defining relation

$$gv = (-1)^{p(g)} vg,$$

we see that $\beta(g) \in G$ with $\rho(\beta(g)) = \rho(g)^{-1}$. It follows that $g\beta(g)$ is in $\mathbb{C}^*$, and $g \mapsto g\beta(g)$ is a homomorphism from G to $\mathbb{C}^*$: $gh\beta(gh) = g(h\beta(h))\beta(g) = g\beta(g)h\beta(h)$. The *Spin group* is the kernel of

$$g\beta(g) : G^+ \longrightarrow \mathbb{C}^*. \tag{2.7.3}$$

On $\mathbb{C}^* \subset G^+$, $g\beta(g)$ is the squaring map. It follows that the group $\mathrm{Spin}(V)$ is a double covering of $\mathrm{SO}(V)$: we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mu_2 & \to & \mathrm{Spin} & \to & \mathrm{SO} & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \mathbb{C}^* & \to & G^+ & \to & \mathrm{SO} & \to & 1 \end{array} \tag{2.7.4}$$

$$\lambda^2 \searrow \quad \downarrow g\beta(g)$$

$$\mathbb{C}^*$$

For real quadratic vector spaces, or over any field k, the groups of k-points of the algebraic group G^+ and $\mathrm{Spin}(V)$ maintain the same description, but in (2.7.4) $\mathrm{Spin}(k) \to \mathrm{SO}(k)$ is in general not surjective, as $\lambda \mapsto \lambda^2$: $k^* \to k^*$ isn't.

CHAPTER 3
Reality of Spinorial Representations and Signature Modulo 8

3.1. Let V be a real quadratic vector space. By 2.6, the reality properties of the irreducible spinorial representations of $\mathfrak{so}(V)$ depend only on the structure of the $\mathbb{R}$-algebra $C^+(V)$. As what we care about is the category of $C^+(V)$-modules, and more precisely the number of isomorphism classes of simple modules, and their fields of endomorphisms, $C^+(V)$ matters only up to Morita equivalence. It will be convenient to consider the full super algebra $C(V)$, and to encode its properties in terms of the super Brauer group (C.T.C. Wall (1963)) whose properties we review in 3.2 to 3.5.

3.2. Fix a ground field k of characteristic $\neq 2$. A super k-algebra A is a (super) *division algebra* if $A \neq 0$ and if every nonzero homogeneous element a of A is invertible. Every super module M over a super division algebra is free: if A is purely even, it is the sum of A-vector spaces in even and odd degree; if A is not purely even, a basis of M^+ over the even part A^+ of A (which is an ordinary division algebra) is a basis of M over A.

From now on, we will consider only finite dimensional super algebras over k.

Example. If k is algebraically closed, the only finite dimensional super division algebras A over k are, up to isomorphism, k itself and $D := k[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$. The super algebra D is not super commutative. Its super center is reduced to k. It is isomorphic to the opposite super algebra D^0, which is $k[\varepsilon^0]$ with ε^0 odd and $(\varepsilon^0)^2 = -1$. Left and right multiplications turn the super vector space D into a $D \otimes D^0$-module, where $\otimes$ is the super tensor product of super algebras (Supersymmetry (1.1.5)), and

$$D \otimes D^0 \xrightarrow{\sim} \underline{\mathrm{End}}_k(D) \ , \tag{3.2.1}$$

a $1|1$ matrix algebra. Indeed, in the basis $1, \varepsilon$ of D, the matrix of multiplication by $\varepsilon \otimes 1$ (resp. $1 \otimes \varepsilon^0$) is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$). Those matrices generate $M_{1|1}$ and both sides of (3.2.1) have the same dimension. Over an algebraically closed field, D hence appears as a square root of the matrix algebra $M_{1|1}$.

3.3. A (finite dimensional) super algebra A over k is *central simple* if $A \neq 0$ and, after extension of scalars to an algebraic closure $\bar{k}$ of k, it becomes isomorphic to

a matrix algebra $M_{r|s}$, or to $M_{r|s} \otimes D$. The latter is isomorphic to $M_{r+s} \otimes D$, the super D-module $k^{r|s} \otimes D$ being isomorphic to D^{r+s}. The following statements about an algebra are true as soon as they are true after an extension of scalars. To check them for central simple algebras, it hence suffices to check them after extension of scalars to an algebraic closure $\bar{k}$ of k.

(3.3.1) The super center is reduced to k.

(3.3.2) As in (3.2.1), $A \otimes A^0 \xrightarrow{\sim} \underline{\mathrm{End}}_k(A)$.

(3.3.3) For any A-module $L \neq 0$, with commutant $B := \underline{\mathrm{End}}_A(L)$, B is central simple and one has

$$A \xrightarrow{\sim} \underline{\mathrm{End}}_B(L) \ .$$

The super center of a tensor product is the tensor product of the super centers, and this reduces (3.3.1) to the cases of $M_{r|s}$ and D. The map (3.3.2) for a tensor product is the tensor product of the maps (3.3.2) for the factors, and this reduces (3.3.2) to (3.2.1) and to (3.3.2) for $M_{r|s}$, i.e. for $\underline{\mathrm{End}}_k(V)$ for $V = k^{r|s}$. This case is checked by identifying the $\underline{\mathrm{End}}\,(V) \otimes \mathrm{End}(V)^0$-module $\underline{\mathrm{End}}\,(V)$ to the $\underline{\mathrm{End}}\,(V) \otimes \mathrm{End}(V^\vee)^0$-module $V \otimes V^\vee$. If $A = \underline{\mathrm{End}}\,(V)$, any A-module is of the form $V \otimes W$, and (3.3.3) results from the commutant being $\underline{\mathrm{End}}\,(W)$. If $A = D \otimes \mathrm{End}_k(V)$, for V a purely even k-vector space, any A-module is of the form $D \otimes V \otimes W$ for W a purely even k-vector space, and (3.3.3) results from the commutant being $D^0 \otimes \mathrm{End}_k(W)$.

It also results from (3.2.1) that the tensor product (in the super sense) of central simple algebras is central simple.

3.4. Let A be a central simple algebra. If S is a simple super module, and f a homogeneous endomorphism, the kernel and images of f are submodules. If $f \neq 0$, f is hence invertible: the commutant $B = \underline{\mathrm{End}}_A(S)$ is a super division algebra (super Schur's lemma). By (3.3.3), A is isomorphic to $M_{r|s} \otimes B^0$ for some r and s. If S' is another simple module, $\underline{\mathrm{Hom}}_A(S, S')$ is compatible with extension of scalars, hence, as one sees over $\bar{k}$, nonzero. Picking a homogeneous element in $\mathrm{Hom}_A(S, S')$, one sees that S and S' are isomorphic, possibly up to parity change. It follows that their commutants are isomorphic.

Two central simple algebras are said to be *similar* if the commutants of their simple modules are isomorphic, i.e. if they are both of the form $M_{r|s} \otimes B$ for the same central simple division algebra B.

The *super Brauer group* $\mathrm{s\,Br}(k)$ is the set of similarity classes of central simple algebras over k, with product the tensor product. That it is a (commutative) group follows from (3.3.2).

Let B be a central simple division algebra, and $A = M_{r|s} \otimes B$. If A is not purely even, then

(a) if B is purely even, $A^+ \simeq M_r(B) \times M_s(B)$,

(b) if B is not purely even, $A \simeq M_n \otimes B$ with $n = r + s$ and

$$A^+ \simeq M_n \otimes B^+ \ .$$

In both cases, the Morita equivalence class of A^+ depends only on the class of A in $\mathrm{s\,Br}(k)$.

If V is a quadratic vector space over k, $C(V)$ is central simple. Indeed, after extension of scalars to $\bar{k}$, $C(V)$ becomes isomorphic to the tensor product of $\dim(V)$ copies of D. If V is hyperbolic, $C(V)$ is a matrix algebra. This can be checked as in the proof of 2.2 (i), or writing $(V,Q) = (V_1,Q_1) \oplus (V_1,-Q_1)$, it can be deduced from (3.3.2) and 2.1 (D)(E). It then follows from 2.1 (D) that $V \mapsto C(V)$ induces a homomorphism

$$W(k) \to \mathrm{s\,Br}(k) \tag{3.4.1}$$

from the Witt group of k to the super Brauer group.

Remark 3.5. Using arguments parallel to those of Bourbaki Alg. Ch. 8, one can show that for A a nonzero finite dimensional super algebra over k, the following conditions are equivalent.

(i) A is central simple;

(ii) $A \otimes A^0 \xrightarrow{\sim} \mathrm{End}_k(A)$;

(iii) The super center of A is reduced to k, and the super A-module A is semi-simple (equivalently: as an ungraded algebra, A is semi-simple).

(iv) A is isomorphic to $M_{r|s} \otimes L$ for some r and s and some super division algebra L over k with super center reduced to k.

Proposition 3.6 (C. T. C. Wall (1963) p. 195). *The super Brauer group of* $\mathbb{R}$ *is cyclic of order* 8, *generated by the class of* $\mathbb{R}[\varepsilon]$ *with* ε *odd and* $\varepsilon^2 = 1$. *For* V *a real quadratic vector space,* $C(V)$ *is similar to* $\mathbb{R}[\varepsilon]^{\otimes \mathrm{sgn}(V)}$. *The similarity class of* $C(V)$ *and the Morita equivalence class of* $C^+(V)$ *(for* $V \neq 0$*) hence depend only on the signature of* V *modulo* 8.

That $\mathrm{s\,Br}(\mathbb{R})$ is of order 8 results from the following lemma, valid for any field k of characteristic $\neq 2$.

Lemma 3.7 (C. T. C. Wall (1963) Th. 3). *The group* $\mathrm{s\,Br}(k)$ *is an iterated extension of* $\mathbb{Z}/2$ *by* k^*/k^{*2} *by the ordinary Brauer group of* k.

Proof. The morphism $\mathrm{s\,Br}(k) \to \mathbb{Z}/2$ is given by extension of scalars to $\bar{k}$, for $\bar{k}$ an algebraic closure of k. By 3.2 (example), one has indeed $\mathrm{s\,Br}(\bar{k}) = \mathbb{Z}/2$.

The kernel $\mathrm{s\,Br}(k)'$ corresponds to central simple super algebras A which by extension of scalars to $\bar{k}$ become isomorphic to a matrix algebra $M_{r|s}$. The algebra $A_{\bar{k}}$, isomorphic to $M_{r|s}$, has two isomorphism classes of simple modules, exchanged by parity change. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on the set $I(A_{\bar{k}})$ of those isomorphism classes, defining

$$\alpha_A \colon \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}/2 \ .$$

Using that the simple $(A' \otimes A'')_{\bar{k}}$-modules are the tensor product of simple $A'_{\bar{k}}$ and $A''_{\bar{k}}$ modules, one checks that $\alpha_{A \otimes A'} = \alpha_A + \alpha_{A'}$ and $A \mapsto \alpha_A$ induces a homomorphism

$$\mathrm{s\,Br}(k)' \to \mathrm{Hom}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) = k^*/k^{*2} \ , \tag{3.7.1}$$

where to u in k^* corresponds the homomorphism giving the action of Galois on $\pm\sqrt{u}$.

If A is purely even, $A_{\bar{k}}$ is isomorphic to $M_n(\bar{k})$, one element of $I(A)$ is purely even, the other purely odd, and α_A is trivial. If A is not purely even, $A_{\bar{k}}$ is isomorphic to $M_{r|s}(\bar{k})$ with $r, s > 0$, the center Z^+ of A^+ is of dimension 2, becoming isomorphic to $\bar{k} \times \bar{k}$ over $\bar{k}$, and $I(A)$ is in bijection with the two characters $Z^+ \to \bar{k}$ of Z^+, a simple module S corresponding to the character by which Z^+ acts on S^+. For A the Clifford algebra $C(k^2, x^2 - ay^2)$, one has $Z^+ = A^+ = \langle 1, e_1e_2 \rangle$ and $(e_1e_2)^2 = -e_1^2e_2^2 = a$. This shows that (3.7.1) is onto.

The kernel $\mathrm{s\,Br}(k)''$ of (3.7.1) corresponds to the central simple algebras, for which $A_{\bar{k}}$ is isomorphic to $M_{r|s}$ and for which either A is purely even or A^+ has a center isomorphic to $k \times k$. If A is a super division algebra, A must be an ordinary, purely even, division algebra, and $\mathrm{s\,Br}(k)'' = \mathrm{Br}(k)$.

3.8 Proof of 3.6. For A a central simple algebra over $\mathbb{R}$, let t be the trace for the underlying ungraded algebra, let sgn^+ (resp. sgn^-) be the signatures of the quadratic forms $t(xy)$ on A^+ (resp. A^-), and define

$$s(A) := \mathrm{sgn}^+ + \mathrm{sgn}^- i \in \mathbb{Z}[i] \subset \mathbb{C} .$$

For a tensor product $A_1 \otimes A_2$, t vanishes on the $A_1^{\pm} \otimes A_2^{\pm}$ other than $A_1^+ \otimes A_2^+$, where it is the tensor product of t for A_1 and A_2. On $A_1^{\pm} \otimes A_2^{\pm}$, $t(xy)$ is related to the tensor product of the bilinear forms $t_1(x_1y_1)$ and $t_2(x_2y_2)$ by

$$\begin{aligned} &A_1^+ \otimes A_2^+, A_1^+ \otimes A_2^- \text{ or } A_1^- \otimes A_2^+ : && t(xy) = t_1(x_1y_1) \otimes t_2(x_2y_2) \\ &A_1^- \otimes A_2^- : && t(xy) = -t_1(x_1y_1) \otimes t_2(x_2y_2), \end{aligned}$$

the latter because $(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = -(x_1x_2) \otimes (y_1y_2)$.

The signature being multiplicative, and changing sign when the bilinear form is replaced by its opposite, we get

$$s(A_1 \otimes A_2) = s(A_1)s(A_2).$$

For a matrix algebra $M_{r,s}$, off-diagonals contribute an isotropic form, and

$$s(M_{r,s}) = r + s$$

For $\mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$, one has $s = 1 + i$, of order 8 in $\mathbb{C}^*/(\mathbb{R}^+)^*$. As $\mathrm{s\,Br}(\mathbb{R})$ is of order 8, it follows that $\mathrm{s\,Br}(\mathbb{R})$ is cyclic of order 8 with $\mathbb{R}[\varepsilon]$ as generator. The remaining statements in 3.6 result from 3.4.

3.9. Let us identify $\mathrm{s\,Br}(\mathbb{R})$ with the set of isomorphism classes of super division algebras over $\mathbb{R}$. Write ε (resp. ε^0) for an odd quantity with $\varepsilon^2 = 1$ (resp. $(\varepsilon^0)^2 = -1$), and write $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ in purely even degree. The bijection 3.6 between $\mathrm{s\,Br}(\mathbb{R})$ and $\mathbb{Z}/8$ is as follows.

$$\begin{aligned} &0 \longmapsto \mathbb{R} && 4 \longmapsto \mathbb{H} \\ &1 \longmapsto \mathbb{R}[\varepsilon] && 5 \longmapsto \mathbb{H} \otimes \mathbb{R}[\varepsilon] \\ &2 \longmapsto \mathbb{C}[\varepsilon], \varepsilon z = \bar{z}\varepsilon, && 6 \longmapsto \mathbb{C}[\varepsilon^0], \varepsilon z = \bar{z}\varepsilon^0, \\ &3 \longmapsto \mathbb{H} \otimes \mathbb{R}[\varepsilon^0] && 7 \longmapsto \mathbb{R}[\varepsilon^0] \end{aligned}$$

The image of 2 is the class of $\mathbb{R}[\varepsilon_1, \varepsilon_2]$ with $\varepsilon_i^2 = 1$, $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$. One has $(\varepsilon_1\varepsilon_2)^2 = -1$ and the claim for 2 follows. Passing to the opposite algebra, we get the image of 6. As the class of $\mathbb{H}$ is of order 2, it must be the image of 4. One concludes by using that $5 = 4 + 1$ and that n and $-n$ correspond to opposite algebras.

By 3.4, to justify 1.4, it suffices to read the even part of the super algebras in this table.

CHAPTER 4
Pairings and Dimension Modulo 8, over $\mathbb{C}$

Let V be a complex quadratic vector space, and S a spinorial representation of $\mathrm{Spin}(V)$ or, what amounts to the same, a $C^+(V)$-module (2.6). Let β be the principal antiautomorphism of $C(V)$ (2.1 (B)).

Proposition 4.1. *A bilinear form* $(\ ,\ )$ *on* S *is invariant by* $\mathrm{Spin}(V)$ *if and only if for* a *in* $C^+(V)$,

$$(as,t) = (s,\beta(a)t)\ . \tag{4.1.1}$$

If (4.1.1) holds, for $g \in G^+ \subset C^+(V)$ (2.7), one has

$$(gs,gt) = (s,\beta(g)gt) = \beta(g)g(s,t)\ . \tag{4.1.2}$$

In particular if g is in the spin group, i.e. if $\beta(g)g = 1$, g leaves $(\ ,\ )$ invariant.

Conversely, if G leaves $(\ ,\ )$ invariant, its Lie algebra $\mathfrak{so}(V) \subset C^+(V)$ leaves it invariant too: for x in $\mathfrak{so}(V)$,

$$(xs,t) + (s,xt) = 0\ .$$

As $\beta(x) = -x$ (a consequence of (2.4.1) (2.4.2)), this gives (4.1.1) for $a = x$ and one concludes by using that $\mathfrak{so}(V)$ generates $C^+(V)$ (2.6).

4.2 Variants. (i) If S is a $C(V)$-module, and if a bilinear form $(\ ,\ )$ on S is such that $(as,t) = (s,\beta(a)t)$, for a in $C(V)$, the same proof shows that for g in the subgroup G of $C(V)^*$ (2.6), (4.1.2) holds. In particular, $(\ ,\ )$ is invariant by the kernel of $\beta(g)g$: $G \to \mathbb{C}^*$. The converse holds. The condition $(as,t) = (s,\beta(a)t)$ for a in $C(V)$ is equivalent to

$$(vs,t) = (s,vt) \qquad \text{for } v \text{ in } V\ . \tag{4.2.1}$$

(ii) Let V be a complex quadratic vector space. We have $Q(v) = \langle v,v\rangle$ and we use the symmetric bilinear form $\langle\ ,\ \rangle$ to identify V with its dual. Let S and $S^\vee$ be vector spaces in duality, provided with Γ: $S \otimes S \to V$ and Γ': $S^\vee \otimes S^\vee \to V$, both symmetric. The self-duality of V allows us to identify Γ and Γ' with γ: $V \otimes S \to S^\vee$ and γ': $V \otimes S^\vee \to S$. Assume that γ and γ' turn $S \oplus S^\vee$ into a $C(V)$-module. The vector spaces S and $S^\vee$ become $C^+(V)$-modules, hence representations of $\mathrm{Spin}(V)$.

Claim: the duality pairing $S \otimes S^\vee \to \mathbb{C}$, Γ and Γ' are morphisms of representations.

By the remark in 2.6, γ and γ' are morphisms of representations of $\mathrm{Spin}(V)$, so that it suffices to check that the duality pairing is a morphism too. Define, on $S \oplus S^\vee$, $\varepsilon(s+s', t+t') = s'(t) + t'(s)$. The symmetry of Γ and Γ' translates as

$$\varepsilon(vs, t) = \langle \Gamma(s,t), v \rangle = \varepsilon(s, vt)$$
$$\varepsilon(vs', t') = \langle \Gamma'(s',t')v \rangle = \varepsilon(s', vt) \ ,$$

and (i) gives the G-invariance of ε, a fortiori the $\mathrm{Spin}(V)$-invariance of the duality pairing of S with S'.

4.3. It follows that the pairings between irreducible spinorial representations of $\mathrm{Spin}(V)$ are encoded in $(C^+(V), \beta)$. As in §3, it is convenient to consider the full Clifford algebra.

This leads to considering central simple super algebras A, endowed with an even involution $\beta\colon A \to A$ which is an antiautomorphism of the underlying ungraded algebra:

$$\beta(xy) = \beta(y)\beta(x). \tag{4.3.1}$$

The tensor product is defined as follows: take the tensor product of super algebras, and define β as extending the involutions of the factors:

$$\beta(a \otimes b) = (-1)^{p(a)p(b)} \beta(a) \otimes \beta(b). \tag{4.3.2}$$

Tensor product is associative and commutative. For Clifford algebras, endowed with their principal antiautomorphisms, it corresponds to the orthogonal direct sum of quadratic vector spaces.

4.4 Example. Let S be a super vector space, and $A := \underline{\mathrm{End}}(S)$. Antiautomorphisms β of the underlying ungraded algebra correspond to nondegenerate bilinear forms on the ordinary vector space underlying S, taken up to a factor, by

$$(ax, y) = (x, \beta(a)y).$$

Involutive β correspond to symmetric or antisymmetric forms. Even β correspond to even or odd forms.

Suppose given two super vector spaces S_i $(i = 1, 2)$ provided with even or odd, symmetric or antisymmetric bilinear forms. Let (A, β) be the tensor product of the corresponding (A_i, β). One has

$$A = A_1 \otimes A_2 = \underline{\mathrm{End}}(S_1) \otimes \underline{\mathrm{End}}(S_2) \xrightarrow{\sim} \underline{\mathrm{End}}(S_1 \otimes S_2).$$

We leave it to the reader to check that the involution β of A corresponds to the following bilinear form on $S_1 \otimes S_2$:

$$(s_1 \otimes s_2, t_1 \otimes t_2) + (-1)^{(p(s_1)+p(t_1))p(s_2)} (s_1, s_2)(t_1, t_2). \tag{4.4.1}$$

If the form $(\ ,\ )_i$ on S_i is of parity p_i and of sign s_i (+ for symmetric, − for antisymmetric), the parity and sign of the form (4.4.1) on $S_1 \otimes S_2$ are given by

$$p = p_1 + p_2, \quad s = s_1 s_2 (-1)^{p_1 p_2} \ . \tag{4.4.2}$$

4.5 Remark. (4.3.1) flouts the sign rule. This could be repaired by considering rather γ defined by

$$\begin{aligned}\gamma(a) &:= \beta(a) \text{ for } a \text{ even},\\ &:= i\beta(a) \text{for } a \text{ odd}\ .\end{aligned}$$

The condition (4.3.1) translates into $\gamma(ab) = (-1)^{p(a)p(b)}\gamma(b)\gamma(a)$, involutivity of β translates as the square of γ being the parity automorphism "multiplication by $(-1)^{p(a)}$", and (4.3.2) becomes $\gamma = \gamma' \otimes \gamma''$.

4.6. Let us call (A, β) neutral if A is of the form $\underline{\mathrm{End}}\,(S)$ and if β is given as in 4.4 by an even symmetric form $(\ ,\ )$ on S: the sum of symmetric forms on S^+ and S^-.

Define (A, β) and (B, β) to be *similar* if for suitable neutrals (M, β) and (N, β), one has

$$(A, \beta) \otimes (M, \beta) \simeq (B, \beta) \otimes (N, \beta).$$

Similarity is an equivalence relation, and is stable under tensor product. Let $B(\mathbb{C})$ be the set of similarity classes. Tensor product induces on $B(\mathbb{C})$ a composition law associative, commutative and with unit. The following proposition shows that it is a group.

Proposition 4.7. *Let (A, β) be as in 4.3, and let* Tr *be the trace of the underlying ungraded algebra. Let α be the parity automorphism $a \mapsto (-1)^{p(a)}a$. Then $(A, \beta) \otimes (A^0, \alpha\beta)$ is neutral. More precisely, by the isomorphism*

$$A \otimes A^0 \xrightarrow{\sim} \underline{\mathrm{End}}(A)$$

and its involution corresponds to the even symmetric form $\mathrm{Tr}(x\beta(y))$ *on A.*

Proof. We have

$$\mathrm{Tr}((a \otimes b \cdot x)\beta(y)) = (-1)^{p(b)p(x)}\mathrm{Tr}(axb\beta(y)) = (-1)^{p(b)p(x)}\mathrm{Tr}(xb\beta(y)a)\ .$$

As $b\beta(y)a = \beta(\beta(a)y\beta(b)) = (-1)^{p(b)p(y)}\beta(\beta(a) \otimes \beta(b) \cdot y)$, this equals

$$= (-1)^{p(b)(p(x)+p(y))}\mathrm{Tr}(x \cdot \beta(\beta(a) \otimes \beta(b) \cdot y)\ .$$

Both sides are zero if $p(a) + p(b) + p(x) + p(y)$ is odd. We hence may replace $p(b)(p(x) + p(y))$ by $p(b)(p(a) + p(b))$, giving

$$= \mathrm{Tr}(x.\beta((-1)^{p(a)p(b)}\beta(a) \otimes \alpha\beta(b) \cdot y),$$

and $a \otimes b \mapsto (-1)^{p(a)\,p(b)}\beta(a) \otimes \alpha\beta(b)$ is the tensor product of β and $\alpha\beta$.

Proposition 4.8. *The group $B(\mathbb{C})$ is cyclic of order 8, generated by the class of the Clifford algebra $C(V)$ for* $\dim(V) = 1$.

Proof. We first consider the kernel $B'(\mathbb{C})$ of the morphism "forgetting β" from $B(\mathbb{C})$ to $\mathrm{s\,Br}(\mathbb{C}) = \mathbb{Z}/2$, i.e. we consider the (A, β) for which A is a matrix algebra $M_{r|s}$. By 4.4, there are four similarity classes of them, characterized by a parity p and a sign s: the parity, and the symmetry, of a bilinear form on a simple A-module defining β. The product is given by (4.4.2), showing that $B'(\mathbb{C})$ is cyclic of order 4, with generators the elements for which the parity p is odd.

The Clifford algebra $C(V)$ with $\dim(V) = 1$ maps to a generator of $\mathrm{s\,Br}(\mathbb{C})$. It remains to check that its square, the Clifford algebra $C(W)$ with $\dim(W) = 2$, is a generator of $B'(\mathbb{C})$. The algebra $C(W)$ is a matrix algebra $M_{1|1} = \mathrm{End}(\mathbb{C}^{1,1})$ with W as its odd part. The transposition relative to the odd symmetric form xy on $\mathbb{C}^{1,1}$ fixes W, hence is the principal antiautomorphism, giving for $C(W)$ the sign $+$ and the parity 1: the parity is odd as required. Applying (4.4.2), we also see that for $\dim(V)$ even, the parity p and the sign s of $C(V)$ are given in terms of $\dim(V)$ mod 8 by the table

(4.8.1)

dimension mod 8	s	p
0	$+$	0
2	$+$	1
4	$-$	0
6	$-$	1

4.9. Suppose V odd-dimensional, and let S be the even part of a simple super $C(V)$-module. Let Z be the center of the ungraded algebra $C(V)$, and z be a generator of Z^-. Then, $v, s \mapsto zvs$ is a morphism of representations of the Spin group: $V \otimes S \to S$.

If V is even-dimensional and S a simple super $C(V)$-module, the Clifford multiplication induces morphisms of representations of the Spin group: $V \otimes S^+ \to S^-$ and $V \otimes S^- \to S^+$.

We now check the uniqueness claim in 1.5 that, up to a scalar factor, these morphisms are the only morphisms $V \otimes S_1 \to S_2$, for S_1 and S_2 irreducible spinorial representations. If V is of dimension 1 or 2, this is clear by inspection, using the description 1.1. If $\dim V \neq 2$, the representation V is irreducible. As $\mathrm{Hom}(V \otimes S_1, S_2) = \mathrm{Hom}(V, \underline{\mathrm{Hom}}\,(S_1, S_2))$, it suffices to check the

Lemma 4.9.1. (i) *For $d := \dim(V)$ odd, the irreducible representation V occurs exactly once in* $\underline{\mathrm{End}}\,(S)$.

(ii) *For d even, V occurs exactly twice in* $\underline{\mathrm{End}}\,(S)$.

Proof. By the odd analogue of the Poincaré-Birkhoff-Witt Theorem, the Clifford algebra $C(V)$ is, as a representation of $\mathrm{O}(V)$, isomorphic to $\wedge^* V$, with $C^+(V)$ isomorphic to $\wedge^{2*} V$.

If d is odd, $\underline{\mathrm{End}}\,(S) \simeq C^+(V) \simeq \wedge^{2*} V$. As $\wedge^i V \simeq \wedge^{d-i} V$ as a representation of $\mathrm{SO}(V)$, this can be rewritten

$$S \otimes S = \bigoplus \wedge^i V \quad (\text{sum for } 0 \leq i \leq (d-1)/2)\ .$$

The $\wedge^i V$ for $i \leq (d-1)/2$ are irreducible, nonisomorphic representations: they are the trivial representation, and all fundamental representations except the spinorial one. The representation V occurs only once, for $i = 1$.

If d is even, $\underline{\mathrm{End}}\,(S) \simeq C(V) \simeq \wedge^* V$, which decomposes as follows into irreducible representations: twice $\bigoplus_{i<d/2} \wedge^i V$ and the two constituents of $\wedge^{d/2} V$ (the eigenspace of $*$). The representation V occurs twice.

We now check the table in 1.5 for $d := \dim(V)$ even. Let $(\ ,\ )$ be the nondegenerate bilinear form on S for which

$$(vs, t) = (s, vt) \tag{4.9.2}$$

for v in V. It is unique up to a scalar factor. Its parity and sign (4.4) are given by (4.8.1). If of odd parity, it makes of S^+ the dual of S^-. If even, it gives a nondegenerate form on S^+ and S^-, of sign given by (4.8.1). This confirms the column "forms on spinors" in 1.5.

The morphism of representations (2.6, Remark)

$$V \otimes S \to S \tag{4.9.3}$$

induced by the Clifford module structure of S can, as V is self dual, be reinterpreted as a morphism from $S \otimes S^\vee$ to V, or, using $(\ ,\ )$ to identify S and $S^\vee$, as a morphism of representations

$$\Gamma\colon S \otimes S \to V\ . \tag{4.9.4}$$

Its defining characteristic property is that

$$\langle \Gamma(s,t), v\rangle = (vs, t)\ . \tag{4.9.5}$$

As (4.9.3) is odd, the parity of (4.9.4) is opposite to that of $(\ ,\ )$. By (4.9.2), its sign (symmetric, or antisymmetric) is the same as that of $(\ ,\ )$ and (4.8.1) confirms the column "symmetry of spinors, spinors $\to V$" of 1.5.

Corollary 4.9.6. *Let L^+ and L^- be spinorial representations of* $\mathrm{Spin}(V)$, *and $\bullet\colon V \otimes L^\pm \to L^\mp$ be morphisms of representations which turn $L := L^+ \oplus L^-$ into a $C(V)$-module:*

$$v \bullet v \bullet \ell = Q(v)\ell\ .$$

Then the resulting $C^+(V)$-module structure of $L^\pm$ induces its structure of representation of $\mathrm{Spin}(V)$.

Proof. We first assume that L^+ and L^- are isotypic. Let S be a simple super $C(V)$-module, with its resulting structure of representation of $\mathrm{Spin}(V)$. Replacing S by ΠS if needed, we may assume that $L^\pm$ is a multiple of $S^\pm$: $L^\pm = S^\pm \otimes W^\pm$. As the morphism $V \otimes S^\pm \to S^\mp$ induced by the module structure is the unique morphism of representations from $V \otimes S^\pm \to S^\mp$, the morphisms $V \otimes L^\pm \to L^\mp$ are induced by $\alpha^\pm\colon W^\pm \to W^\mp$. That they turn L into a $C(V)$-module implies that α^+ and α^- are inverses of each other: L, as a representation of $\mathrm{Spin}(V)$ and as a $C(V)$-module, is a multiple of S.

If L^+ or L^- is not isotypic, the dimension of V is even, they are two irreducible spinorial representations S_1 and S_2 and if $L_i^\pm$ is the S_i-isotypic part of $L^\pm$, as there is no morphism $V \otimes S_i \to S_i$ the data $V \otimes L^\pm \to L^\mp$ breaks into the direct sum of $(V \otimes L_1^+ \to L_2^-,\ V \otimes L_2^- \to L_1^+)$ and of $(V \otimes L_2^+ \to L_1^-,\ V \otimes L_1^- \to L_2^+)$. This reduces us to the already considered isotypic case.

4.10 Lemma. *Fix (A,β) as in 4.3, with A nontrivial in $\mathrm{sBr}(\mathbb{C})$, i.e. of the form $A^+ \otimes Z$, where A^+ is a purely even matrix algebra and where the center (not super center) Z of A is of dimension $1|1$. Whether β acts by $+1$ or by -1 on Z^- depends only on the similarity class of (A,β).*

Proof. Take (B,β) with $B = \underline{\mathrm{End}}\,(S)$ and β defined, as in 4.4, by a nondegenerate bilinear form on S of parity p and sign s. The involution β of B^+ acts trivially on the center $Z(B^+)$ if $p = 0$, nontrivially if $p = 1$. The center $Z(B^+)$ is $\mathbb{C}$ if B is purely even, $\mathbb{C} \times \mathbb{C}$ otherwise. Let ν be 1 in the first case, and a nonzero element $(a,-a)$ in the second. One has $\beta(\nu) = (-1)^p\nu$. For z in Z^-, $\nu \otimes z$ commutes with $(B \otimes A)^+$. Indeed, both ν and z commute with $B^+ \otimes A^+$, and anticommute with $B^- \otimes A^-$. The negative part $Z^-(B \otimes A)$ of the center of $B \otimes A$ is hence νZ^-, and

$$\beta\Big|Z^-(B \otimes A) = (-1)^p\beta\Big|Z^-(A)\ . \tag{4.10.1}$$

Indeed, $\beta(\nu z) = \beta(z)\beta(\nu) = \beta(\nu)\beta(z) = (-1)^p\nu\beta(z)$. If (B,β) is neutral, $p = 0$ and 4.10 follows.

4.11 Corollary. *The similarity class of (A^+,β) depends only on that of (A,β).*

Proof. It is the difference of (A,β) and (Z,β) in $B(\mathbb{C})$.

From 4.10 and 4.11, we get two similarity invariants of (A,β) as in 4.10, i.e. of (A,β) whose class in $B(\mathbb{C}) = \mathbb{Z}/8$ is odd: the sign $s(\beta)$ by which β acts on Z^-, and the sign $s(A^+)$ of the form corresponding to (A^+,β) by 4.4. The class 1 in $B(\mathbb{C}) = \mathbb{Z}/8$ is represented by $(\mathbb{C}[\varepsilon],\beta)$ with ε odd, $\varepsilon^2 = 1$ and β the identity. By 4.7, the class -1 is represented by $\mathbb{C}[\varepsilon]$ with $\beta\varepsilon = -\varepsilon$. The classes 0 and 4 can be represented by purely even algebras, and writing any odd element of $\mathbb{Z}/8$ as (0 or 4) $\pm$ 1, one deduces from (4.8.1) the following table:

n	$s(A^+)$	$s(\beta)$
1	+	+
3	−	−
5	−	+
7	+	−

(4.11.1)

4.12. We now check the table in 1.5 for V odd-dimensional. The Clifford algebra is as in 4.10: $C(V) = C^+(V) \otimes Z$. Let S be a simple super $C(V)$-module. The $C^+(V)$-module S^+ is then the unique irreducible spinorial representation of $\mathrm{Spin}(V)$. On S^+, there is a nondegenerate bilinear form $(\ ,\ )$ for which

$$(as,t) = (s,\beta(a)t) \tag{4.12.1}$$

for a in $C^+(V)$. It is invariant under $\mathrm{Spin}(V)$. Its sign is the sign $s(C^+(V))$ attached to $(C(V),\beta)$ in 4.4, and (4.11.1) confirms the column "forms on spinors" in 1.5.

Multiplication by zv gives a morphism of representations

$$V \otimes S^+ \to S^+ \tag{4.12.2}$$

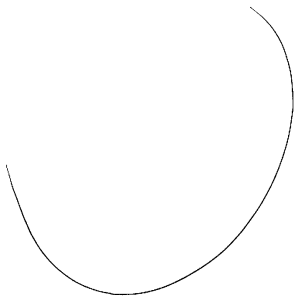
which, as in 4.9, defines

$$S^+ \otimes S^+ \to V \tag{4.12.3}$$

characterized by

$$\langle f(s \otimes t), v \rangle = (zvs, t) \ . \tag{4.12.4}$$

One has $\beta(zv) = \beta(v)\beta(z) = v\beta(z) = \beta(z)v$ and, by (4.12.1), the sign of (4.12.3) is that of $(\ ,\ )$ if $s(\beta) = +$, and is the opposite if $s(\beta) = -$. The table (4.11.1) confirms the column "symmetry of spinors, spinors $\to V$" of 1.5.



CHAPTER 5
Passage to Quadratic Subspaces

In this Chapter we work over $\mathbb{C}$. As promised in §1.1, we first check the

Proposition 5.1. *For $r > 0$, if one restricts the spin representation of* $\mathrm{Spin}(2r+1)$ *to* $\mathrm{Spin}(2r)$, *one obtains the sum of the two semi-spin representations of* $\mathrm{Spin}(2r)$. *Each of the two semi-spin representations of* $\mathrm{Spin}(2r)$ *restricts as the spin representation of* $\mathrm{Spin}(2r-1)$.

Proof. Let V be the orthogonal direct sum of W and of a line L, and put $D := C(L)$. If $\dim(W)$ is odd, one has a tensor decomposition $C(W) = C^+(W) \otimes Z$. The commutant Z of $C^+(W)$ is of dimension $1|1$, and $C(V) = C^+(W) \otimes (Z \otimes D) \simeq C^+(W) \otimes M_{1|1}$. If S is a simple $C^+(W)$-module, $S \otimes \mathbb{C}^{1,1}$ is a simple $C(V)$-module. As a $C^+(W)$-module, this is $S \oplus \Pi S$, and the second statement follows.

Suppose now that $\dim(W)$ is even. For some r, $C(W) \simeq M_{r|r}$. If S is a simple $C(W)$-super module, $S \otimes D$ is a simple $C(W) \otimes D$-module. Its even part (the spin representation of $\mathrm{Spin}(V)$) is, as a $C^+(W)$-module, isomorphic to $S^+ \oplus S^-$ and the first statement follows.

5.2. Let $V \neq 0$ be an even-dimensional complex quadratic space, and let S be a simple super $C(V)$-module. Let $(\ ,\ )$ be a nondegenerate bilinear form on S for which

$$(vs, t) = (s, vt)\ . \tag{5.2.1}$$

As we saw in 4.4, this form is unique up to a factor.

Let $v \neq 0$ be an isotropic vector in V. Define $V_1 = v^\perp / \mathbb{C}v$. The square zero endomorphism v of S super commutes with $v^\perp$. Its kernel $\ker(v)$ is hence stable under $v^\perp$. The action of $v^\perp$ on $\mathrm{Ker}(v)$ factors through V_1, and turns $\mathrm{Ker}(v)$ into a $C(V_1)$-module.

Proposition 5.3. *With the notations of 5.2*

(i) $\mathrm{Ker}(v) = \mathrm{Im}(v)$;

(ii) $\mathrm{Ker}(v)$ *is an irreducible representation of* $C(V_1)$;

(iii) *The form* $(\ ,\ )$ *vanishes on* $\mathrm{Ker}(v)$;

(iv) *Define the form* $(\ ,\ )_1$ *on* $\mathrm{Ker}(v) = \mathrm{Im}(v)$ *by* $(s, vt)_1 = (-1)^{p(s)}(s,t)$. *It is nondegenerate. The* $C(V_1)$*-module* $\mathrm{Ker}(v)$, *and* $(\ ,\ )_1$, *obey (5.2.1).*

Proof (i), (ii). Lift V_1 in $v^\perp$ and let H be the orthogonal complement of V_1 in V. The quadratic space V becomes the orthogonal direct sum of V_1 and H, with $\dim(H) = 2$ and $v \in H$. Let S_1 be an irreducible super representation of $C(V_1)$, and let S_H be one of $C(H)$. The super vector space S_H is of dimension $1|1$ and, possibly after parity change and in a suitable basis, the matrix of multiplication by v is

$$v: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{5.3.1}$$

One has $C(V) = C(V_1) \otimes C(H)$ and, possibly after a parity change for S_1, the $C(V)$-module S is $S_1 \otimes S_H$. This reduces (i) to the case of H, clear on (5.3.1), which also shows that the $C(V_1)$-module $\mathrm{Ker}(v)$ is isomorphic to S_1, proving (ii).

(iii) As $(vs, vt) = (s, vvt)$ and $v^2 = Q(v) = 0$, (iii) results from (i).

(iv) As $(\ ,\ )$ is nondegenerate, for $s \neq 0$ in $\mathrm{Ker}(v)$ there exists t such that $(s,t) \neq 0$. By (iii), $t \notin \mathrm{Ker}(v)$ and $(s, vt)_1 \neq 0$: the form $(\ ,\ )_1$ is nondegenerate and it remains to check that it obeys (5.2.1). Indeed, if $t_1 = vt$ and if w is in $v^\perp$, one has

$$\begin{aligned}(ws, t_1) &= (-1)^{p(ws)}(ws, t) = (-1)^{p(ws)}(s, wt) \\ &= (-1)^{p(ws)+p(s)}(s, vwt)_1 = -(s, -wvt) = (s, wt)_1 .\end{aligned}$$

Proposition 5.4. *Under the same assumptions, let* Γ: $S \otimes S \to V$ *be defined by (4.9.5). For* s, t *in* $\mathrm{Ker}(v)$ *one has in* V

$$\Gamma(s,t) = (-1)^{p(s)}(s,t)_1 v . \tag{5.4.1}$$

Proof. We check that both sides have the same inner product with any $w \in V$. If $t = vx$, the left side indeed gives

$$\begin{aligned}(w, \Gamma(s,t)) &= (ws, vx) = (vws, x) = ((vw + wv)s, x) \\ &= (v, w)(s, x) = (-1)^{p(s)}(v, w)(s,t)_1 .\end{aligned}$$

CHAPTER 6
The Minkowski Case

6.1 Theorem. *Let (V, Q) over $\mathbb{R}$ be of signature $(+, -, \ldots, -)$. Let $S_{\mathbb{R}}$ be an irreducible real spinorial representation of* $\mathrm{Spin}(V, Q)$. *The commutant Z of $S_{\mathbb{R}}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $^-$ be the standard anti-involution of Z.*

(i) *Up to a real factor, there exists a unique symmetric morphism $\Gamma\colon S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$. It is invariant under the group Z^1 of elements of norm 1 of Z.*

(ii) *For $v \in V$, if $Q(v) > 0$, the form $\langle v, \Gamma(s,t)\rangle$ on $S_{\mathbb{R}}$ is positive or negative definite.*

The set of v for which $Q(v) > 0$ has two connected components. It follows from (ii) that for v in one of them, call it C, the form $\langle v, \Gamma(s,t)\rangle$ is positive definite and that for v in the other, i.e. for $-v$ in C, it is negative definite.

Let d be the dimension of V. The signature is $2 - d$ and, by the table 1.4.1, proved in 3.1, 3.9, Z and the complexification of $S_{\mathbb{R}}$ are given as follows as a function of d modulo 8. For d even, S^+ and S^- denote the semi-spinorial representations. For d odd, S denotes the spin representation.

$$\begin{array}{lll} 2 & : & \mathbb{R}\,, S^+ \text{ or } S^- \\ 3 \text{ or } 1 & : & \mathbb{R}\,, S \\ 4 \text{ or } 0 & : & \mathbb{C}\,, S^+ + S^- \\ 5 \text{ or } 7 & : & \mathbb{H}\,, 2S \\ 6 & : & \mathbb{H}\,, 2S^+ \text{ or } 2S^- \end{array} \tag{6.1.1}$$

Proof. We first show that if d is congruent to 2 modulo 8, a morphism Γ with the listed properties exists. In this case, the signature is 0 modulo 8, so that $C(V)$ is a matrix algebra. Let $S = S^+ \oplus S^-$ be a simple $C(V)$-module. The possible $S_{\mathbb{R}}$ are S^+ and S^-. The pairing $(\ ,\ )$ on S, relative to which the principal antiautomorphism β is transposition, is odd and the corresponding $\Gamma\colon S \otimes S \to V$ is even and symmetric ((1.5.1) proved in 4.8, 4.9).

Fix v with $Q(v) > 0$ and let V_1 be its orthogonal complement. By 5.1, the graded module S remains simple as a graded $C(V_1)$-module, and S^+ remains absolutely irreducible as a representation of the group $\mathrm{Spin}(V_1)$. The symmetric bilinear form $\langle v, \Gamma(s,t)\rangle$ on S^+ is not identically zero: if it were zero for one v, it would

be zero for all v by $\mathrm{Spin}(V)$-invariance, and Γ would be zero. This form is invariant under the compact group $\mathrm{Spin}(V_1)$. As S^+ is an irreducible representation of $\mathrm{Spin}(V_1)$, it must be definite. The same applies to S^-.

We now show, in all dimensions, the existence of a morphism $S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$ with the properties listed in 6.1 (i) (ii). Embed V in a Minkowski space V_0 of dimension congruent to 2 modulo 8: V_0 is the orthogonal direct sum of V and of a negative definite quadratic space of a suitable dimension. Let pr be the orthogonal projection of V_0 onto V.

Let $S_0 = S_0^+ \oplus S_0^-$ be a simple $C(V_0)$-module. As V_0 is of dimension congruent to 2 modulo 8, there exists a morphism of representations of $\mathrm{Spin}(V_0)$: Γ_0: $S_0^\pm \otimes S_0^\pm \to V_0$, with the properties listed in 6.1 (i)(ii). As $\mathrm{Spin}(V) \subset \mathrm{Spin}(V_0)$, we can by restriction consider S_0 as a $\mathrm{Spin}(V)$-representation. The action of $\mathrm{Spin}(V)$ comes from the structure of $C(V)$-module of S_0, deduced from its structure of $C(V_0)$-module by the inclusion of $C(V)$ in $C(V_0)$. It follows that S_0 is a spinorial representation of $\mathrm{Spin}(V)$. Further, S_0 being a faithful $C(V_0)$-module, and hence a faithful $C(V)$-module, any irreducible real spinorial representation of $\mathrm{Spin}(V)$ occurs as a direct factor of S_0: there exists a morphism of $\mathrm{Spin}(V)$-representations $S_{\mathbb{R}} \hookrightarrow S_0^+$ or $S_{\mathbb{R}} \hookrightarrow S_0^-$. On $S_{\mathbb{R}}$, we now define Γ_1 as the orthogonal projection

$$\Gamma_1(s,t) = \mathrm{pr}\,\Gamma_0(s,t)$$

of Γ_0 on V. From the same properties of Γ_0, it follows that the form Γ_1 is symmetric and that if $Q(v) > 0$ with v in V, the form $\langle v, \Gamma_1(s,t)\rangle = \langle v, \Gamma_0(s,t)\rangle$ on $S_{\mathbb{R}}$ is definite. It remains to average Γ_0 over the compact group K of elements of norm 1 in Z to obtain the required

$$\Gamma(s,t) := \int_K \Gamma_1(ks, kt)dk\ .$$

We now prove the uniqueness claim in (i). Let $S_{\mathbb{C}} := S_{\mathbb{R}} \otimes \mathbb{C}$ be the complexification of $S_{\mathbb{R}}$. It suffices to show that symmetric morphisms of representations $S_{\mathbb{C}} \otimes S_{\mathbb{C}} \to V_{\mathbb{C}}$ are unique up to a (complex) factor.

Case 1. $Z = \mathbb{R}$, that is $S_{\mathbb{C}}$ is an irreducible representation. It is spinorial for d odd, semi-spinorial for d even. If we exclude the case $d = 2$, V is an absolutely irreducible representation, and over $\mathbb{C}$, $V_{\mathbb{C}}$ occurs at most once in $S_{\mathbb{C}} \otimes S_{\mathbb{C}}$ (4.9.1). Uniqueness follows.

The case $d = 2$ is easy to treat directly: V decomposes as $D_1 \oplus D_2$, for D_1 and D_2 the two isotropic lines, and the $S^\pm$ are of dimension one with tensor square isomorphic, respectively, to D_1 and D_2.

Case 2. $Z = \mathbb{H}$, i.e. $S_{\mathbb{C}}$ is twice an irreducible representation S_0: $S_{\mathbb{C}} = S_0 \otimes_{\mathbb{C}} W$ with $\dim(W) = 2$. We know the existence of a symmetric Z^1-invariant Γ. After complexification, the Z^1-invariance amounts to $\mathrm{SL}(W)$-invariance, and means that Γ is the tensor product of Γ_0: $S_0 \otimes S_0 \to V_{\mathbb{C}}$ and of the unique (up to a factor) antisymmetric ψ: $W \otimes W \to \mathbb{C}$. Symmetry of Γ implies antisymmetry of Γ_0: there is a morphism Γ_0: $S_0 \otimes S_0 \to V_{\mathbb{C}}$ (unique up to a factor by 4.9), and it is antisymmetric. Any symmetric Γ: $S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$ is, after complexification, a multiple of $\Gamma_0 \otimes \psi$; uniqueness follows.

Case 3. $Z = \mathbb{C}$, i.e. $S_{\mathbb{C}}$ is the sum of two inequivalent (complex conjugate) representations. This can happen only for d even, with $S_{\mathbb{C}} = S^+ \oplus S^-$ the sum of the two semi-spinorial representations. We know the existence of a symmetric Z^1-invariant Γ. After complexification, the Z^1-invariance amounts to saying that Γ is a linear combination of a morphism $S^+ \otimes S^- \to V_{\mathbb{C}}$ and of its symmetric. It follows that there is a morphism of representations Γ_0: $S^+ \otimes S^- \to V_{\mathbb{C}}$. By 4.9, any symmetric morphism Γ: $S \otimes S \to V$ is a multiple of the sum of Γ_0 and of its symmetric. Uniqueness follows.

Corollary 6.2. *Let S be a real spinorial representation, not necessarily irreducible, and Γ: $S \otimes S \to V$ be symmetric and such that for v in the open positive cone C of V, $\langle v, \Gamma(s,t)\rangle$ be positive definite.*

(i) *S is the direct sum of irreducible subrepresentations $S^{(\alpha)}$, with $\Gamma = \sum \Gamma^{(\alpha)}$.*

(ii) *There is a unique $\check{\Gamma}$: $S^\vee \otimes S^\vee \to V$ such that, if Γ and $\check{\Gamma}$ are reinterpreted as morphisms γ: $V \to \underline{\mathrm{Hom}}\,(S, S^\vee)$ and $\check{\gamma}$: $V \to \underline{\mathrm{Hom}}\,(S^\vee, S)$, the Clifford relations*

$$\check{\gamma}(v)\gamma(v), \gamma(v)\check{\gamma}(v) = Q(v) \tag{6.2.1}$$

hold. For v in C one has again $\langle v, \check{\Gamma}(s,t)\rangle$ symmetric and positive definite.

In (ii), there is a factor of 2 ambiguity. Our choice here is to write $Q(v) = (v,v)$ and to define γ by

$$\langle \gamma(v)(s), t\rangle = (\Gamma(s,t), v)$$

It results from 4.9.6 that the structure of $\mathrm{Spin}(V)$-representation of S is induced by the $C(V)$-module structure of $S \oplus S^\vee$ defined by (6.2.1).

Proof. If there are two nonisomorphic irreducible real spinorial representations S_1 and S_2, after extension of scalars to $\mathbb{C}$, they must be multiples of irreducible complex spinorial representations S_1' and S_2'. This results from there being at most two such. The existence of nontrivial morphisms $S_i \otimes S_i \to V$ $(i = 1, 2)$ implies over $\mathbb{C}$ that of nontrivial morphisms $S_i' \otimes S_i' \to V_{\mathbb{C}}$. By 4.9.1 (ii), it in turn implies the nonexistence of morphisms $S_1' \otimes S_2' \to V_{\mathbb{C}}$, hence the nonexistence of morphisms $S_1 \otimes S_2 \to V$. The representation S is the direct sum of a multiple $S^{(1)}$ of S_1 and of a multiple $S^{(2)}$ of S_2, and Γ, being a morphism of representations, is the sum of morphisms $\Gamma^{(i)}$: $S^{(i)} \otimes S^{(i)} \to V$. This reduces the proof of (i), and the existence statement in (ii), to the case where S is isotypic: a multiple of an irreducible real representation S_0. Let Z be the field of endomorphisms of S_0. Define $W := \mathrm{Hom}(S_0, S)$. It is a right Z-vector space, and

$$W \otimes_Z S_0 \xrightarrow{\sim} S\ .$$

Fix a symmetric morphism Γ_0: $S_0 \otimes S_0 \to V$ with $(\Gamma(s,t), v\rangle$ positive definite for v in C. If Γ: $S \otimes S \to V$ is symmetric, for each w in W, $\Gamma(w \otimes s_0, w \otimes t_0)$ is symmetric too, hence a multiple of Γ_0 (6.1 (i)):

$$\Gamma(w \otimes s_0, w \otimes t_0) = F(w)\Gamma_0(s_0, t_0)\ .$$

Fixing s_0 and t_0, one sees that the function F is quadratic, and the Z^1-invariance of Γ_0 (6.1 (i)) implies that it is invariant by $Z^1 \subset Z^*$. It is hence of the form $\Phi(w,w)$, for some Z-Hermitian form Φ on W. The quadratic function F, or equivalently the Hermitian form Φ, uniquely determines Γ, and any Hermitian Φ is obtained, as one

sees by writing (W, Φ) as an orthogonal sum of subspaces of dimension 1 over Z. The Γ for which $\langle \Gamma(s,t), v\rangle$ is positive definite for v in C correspond to the positive definite forms Φ. Decomposing (W, Φ) as an orthogonal direct sum of subspaces of dimension 1 over Z, we obtain (i).

For v in V, the relation (6.2.1) between Γ and $\check{\Gamma}$ means that, if $Q(v) \neq 0$, $\langle \check{\Gamma}(s',t'), v\rangle$ is $Q(v)^{-1}$ times the inverse of the bilinear form $\langle \Gamma(s,t), v\rangle$. This makes the uniqueness of $\check{\Gamma}$ clear, as well as its symmetry and positivity property. In coordinates, for a basis e_μ of V and e_α of S, (6.2.1) reads

$$\frac{1}{2}\left(\Gamma^{\mu}_{\alpha\beta}\check{\Gamma}^{\nu\beta\gamma} + \text{ same with } \mu,\ \nu \text{ permuted}\right) = g^{\mu v}\delta^{\gamma}_{\alpha}\ . \tag{6.2.2}$$

To prove the existence of $\check{\Gamma}$, we begin, as in 6.1, by treating the case where V is of dimension congruent to 2 modulo 8. In this case, if S is irreducible, there are up to scalar factors unique morphisms $\Gamma\colon S \otimes S \to V$ and $\widetilde{\Gamma}\colon S^\vee \otimes S^\vee \to V$; they are deduced from the Clifford module structure of $S \oplus S^\vee$, and (6.2.1) follows, for $\widetilde{\Gamma}$ replaced by a suitable scalar multiple.

Going from V to a subspace as in 6.1, and projecting Γ and $\widetilde{\Gamma}$ orthogonally onto that subspace, we obtain in any dimension systems $(S, \Gamma, \Gamma^\vee)$ as in (ii), where S can be assumed to contain any preassigned irreducible spinorial representation S_1. A decomposition of S as in (i) induces a similar decomposition of $(S^\vee, \Gamma^\vee)$ and we obtain a system $(S_1, \Gamma, \Gamma^\vee)$ as in (ii). By (i), taking direct sums of such systems, we prove existence in (ii).

6.3 Remark. (i) The proof of 6.2 (i) gives us a classification of the (S, Γ) for S a multiple of S_0, with commutant Z. Once Γ_0 for S_0 has been chosen, Γ corresponds to a positive definite Z-Hermitian form on $W := \mathrm{Hom}(S_0, S)$.

(ii) Let us compute the group of automorphisms of a structure of the following kind: V is a vector space with Minkowski metric $(\ ,\ )$ and positive cone C, S and $S^\vee$ are vector spaces in duality, $\Gamma\colon S \otimes S \to V$ is symmetric and positive: $\langle \Gamma(s,t), v\rangle$ is positive definite for v in C^0, and for some (unique) $\check{\Gamma}\colon S^\vee \otimes S^\vee \to V$, (6.2.1) holds. In fact, we will compute the group of automorphisms acting on V with determinant 1.

By (6.2.1), $S \oplus S^\vee$ is a $C(V)$-module. This turns S and $S^\vee$ into representations of $\mathrm{Spin}(V)$. By 4.2 (ii), the representations S and $S^\vee$ are contragredient, and Γ, $\check{\Gamma}$ are morphisms of representations: $\mathrm{Spin}(V)$ acts by automorphisms. If g acts on V with determinant 1, and respects the positive cone C, its image in $\mathrm{O}(V)$ is in the connected component of $\mathrm{O}(V)$, the image of $\mathrm{Spin}(V)$: we have $g = g_1 g_2$, with g_1 in $\mathrm{Spin}(V)$ and g_2 acting trivially on V, hence on $\mathrm{Spin}(V)$. By (i), if S is isotypic: $S = S_0 \otimes_Z W$, g_2 is in the unitary group of the Z-Hermitian space W. In the nonisotypic case: S the sum of the $S_i \otimes_{Z_i} W_i$ $(i = 1, 2)$, it is in the product of two such unitary groups. The group $\mathrm{U}(W)$ (resp. $\mathrm{U}(W_1) \times \mathrm{U}(W_2)$) is called the R-group.

6.4. The cases where V is of dimension $d = 3, 4, 6$ or 10 (i.e. $d-2 = 2^t$, $t = 0, 1, 2, 3$) are particularly interesting. In those dimensions, an irreducible real spinorial representation S is of dimension 2^{t+1}. For $d = 3, 4, 6$, the commutant is respectively $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, S is of dimension 2 over Z and

$$\mathrm{Spin}(1, d-1) \xrightarrow{\sim} \mathrm{SL}(2, Z)\ .$$

6.5. We now explain how the case $d = 10$ is similarly related to the octonion algebra $\mathbb{O}$. To avoid unnecessary signs, we will work in signature $(9,1)$ rather than $(1,9)$. This does not affect the spin group. We write the Minkowski space V as the orthogonal direct sum of W positive definite of dimension 8 and of H hyperbolic of dimension 2. We choose an isotropic basis $\{e, f\}$ of H with $Q(e+f) = 1$.

Here is a model for W, its quadratic form, the irreducible $C(W)$-module $S = S_W^+ \oplus S_W^-$, the Clifford multiplication ρ, quadratic forms on S_W^+ and S_W^- for which $(\rho(w)s, t) = (s, \rho(w)t)$ and the resulting pairing (4.9.4) Γ: $S_W^+ \otimes S_W^- \to W$:

W, S_W^+ and S_W^-: the space $\mathbb{O}$ of octonions;

quadratic forms on W, S_W^+ and S_W^-: the octonion reduced norm $N(x) = x\bar{x}$;

morphisms $W \otimes S_W^+ \xrightarrow{\rho} S_W^-$, $S_W^+ \otimes S_W^- \xrightarrow{\Gamma} V$ and $S_W^- \otimes W = W \otimes S_W^- \xrightarrow{\rho} S_W^+$: $x \otimes y \mapsto \bar{x}\bar{y}$.

One has to check that (a) ρ turns $S_W^+ \oplus S_W^-$ into an irreducible $C(W)$-module, (b) $(\rho(w)s, t) = (s, \rho(w)t)$ and (c) $(w, \Gamma(s,t)) = (\rho(w)s, t)$. The bilinear form $(\ ,\ )$ associated to the octonion norm is $\mathrm{Tr}(x\bar{y})$, where Tr is the octonion reduced trace $x + \bar{x}$. The proof will use that in $\mathbb{O}$, the subalgebra generated by two elements x and y (to which one can add $\bar{x} = \mathrm{Tr}(x) \cdot 1 - x$ and $\bar{y} = \mathrm{Tr}(\bar{y}) \cdot 1 - y$) is associative, and that $\mathrm{Tr}(x(yz)) = \mathrm{Tr}((xy)z)$, $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ and $\mathrm{Tr}(\bar{x}) = \mathrm{Tr}(x)$.

Proof of (a). That $\rho(w)^2 = N(w)$ is seen as follows: for w in W and s in S_W^+, $\rho(w)^2 s = (\bar{w}\bar{s})^- \bar{w} = (sw)\bar{w} = s(w\bar{w}) = N(w)s$. For s in S_W^-, one repeats this argument in the opposite algebra $\mathbb{O}^0$. Irreducibility is clear.

Proof of (b). For s in S_W^+ and t in S_W^-, $(\rho(w)s, t) = \mathrm{Tr}(\bar{w}\bar{s}\bar{t})$ while $(s, \rho(w)t) = \mathrm{Tr}(\bar{s}\bar{t}\bar{w})$. For s in S_W^- and t in S_W^+, one repeats this argument in $\mathbb{O}^0$.

Proof of (c). For s in S_W^+ and t in S_W^-, $(w, \Gamma(s,t)) = \mathrm{Tr}(\bar{w}(\bar{s}\bar{t}))$ while $(\rho(w)s, t) = \mathrm{Tr}((\bar{w}\bar{s})\bar{t})$.

In this octonionic description of W, S_W^+ and S_W^-, triality is in evidence.

An irreducible $C(H)$-super module is of dimension $1|1$. Up to parity change, it has a basis α (even), β (odd) in which the matrices of e and f are

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{6.5.1}$$

In those models, the even part of the irreducible super $C(V)$-module $S = (S_W^+ \oplus S_W^-) \otimes \langle \alpha, \beta \rangle$ is

$$S^+ = S_W^+ \alpha \oplus S_W^- \beta = \mathbb{O}\alpha \oplus \mathbb{O}\beta . \tag{6.5.2}$$

An element $v = a + \lambda e + \mu f$ of V is isotropic if $a\bar{a} + \lambda\mu = 0$. An element $\bar{x}\alpha + y\beta$ of S^+ is in the kernel of the multiplication by v if

$$\begin{aligned} \bar{a}x - \lambda y &= 0 \\ ay + \mu x &= 0 . \end{aligned} \tag{6.5.3}$$

Suppose v nonzero and isotropic. If $a \neq 0$, λ and μ are then nonzero and by the isotropy equation $a\bar{a} = -\lambda\mu$, the first equation (6.3.3) implies the second. If $a = 0$, one of λ and μ is 0 and the equations (6.5.3) reduce either to $x = 0$ or to $y = 0$.

In the *octonion plane* $\mathbb{O}^2$, the *octonion lines* are the subspaces $y = ax$ (for a in $\mathbb{O}$), as well as the subspace $x = 0$. We have obtained a model of S^+ as the octonion plane. In this model, the map $\mathbb{R}v \mapsto \text{Ker}$ (multiplication by v) is a bijection from the set of isotropic lines in V to the set of octonion lines. The octonion line $y = ax$ corresponds to the line in V spanned by $(\bar{a}, 1, -N(a))$ and the line $x = 0$ to the line spanned by $(0, 0, 1)$.

It is not unreasonable to define $\mathrm{GL}(2, \mathbb{O})$ as the group of $\mathbb{R}$-linear transformations of the octonion plane, which transform octonion lines into octonion lines. The Spin group, as it respects the Clifford multiplication, is contained in $\mathrm{GL}(2, \mathbb{O})$.

Proposition 6.6. *After extension of scalars to* $\mathbb{C}$*, the algebraic group* $\mathrm{GL}(2, \mathbb{O})$ *becomes the group generated by* $\mathrm{Spin}(V)$ *and the multiplicative group of homotheties.*

Sketch of proof. The map $v \mapsto \mathrm{Ker}(v)$ maps the quadric of isotropic lines to the Grassmannian of 8-dimensional subspaces of $\mathbb{O}^2$. It follows that the Zariski closure in the complex Grassmannian of the space of real octonion lines is a homogeneous space of $\mathrm{Spin}(V)$, and a quadric. The automorphism group of a quadric (viewed as an algebraic variety) is the projective orthogonal group. This gives $\mathrm{GL}(2, \mathbb{O}) \to \mathrm{PO}$. Using the nonassociativity of $\mathbb{O}$, one checks that the only elements of $\mathrm{GL}(2, \mathbb{O})$ respecting each octonion line are the homotheties. It remains to check that $\mathrm{GL}(2, \mathbb{O})$ maps in fact to PSO. There are, over the complex numbers, two kinds of maximal isotropic subspaces of $V_{\mathbb{C}}$, and SO is the subgroup of O which does not permute the two kinds. One of the two kinds is singled out by the property that for L of that kind, the intersection of the octonion lines relative to v in L is not reduced to zero.

The intersection of the group of homotheties and of $\mathrm{Spin}(V) \subset \mathrm{GL}(2, \mathbb{O})$ is the center of $\mathrm{Spin}(V)$. It is the group of 4th roots of 1. There is hence a "determinant"

$$\det\colon \mathrm{GL}(2, \mathbb{O}) \to \text{ multiplicative group },$$

which on a scalar λ takes the value $\det(\lambda) = \lambda^4$ and, if $\mathrm{SL}(2, \mathbb{O})$ is defined to be the kernel of det, 6.6 gives

$$\mathrm{Spin}(V) = \mathrm{SL}(2, \mathbb{O}) \ . \tag{6.6.1}$$

6.7 Remark. Over $\mathbb{R}$, the group $\mathrm{Spin}(V)$ acts transitively on $S^+ - \{0\}$. Indeed, it permutes transitively the octonion lines $\mathrm{Ker}(v)$ (v isotropic) and the stabilizer of an octonion line L acts on L as a group of orthogonal similitudes.

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Classical Field Theory

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Introduction

A physical system is usually described in terms of states and observables. (See the discussion in [I-**Faddeev**, §1.1].) In the Hamiltonian framework of classical mechanics, the states form a symplectic manifold $(\mathcal{M}, \Omega)$ and the observables are functions on $\mathcal{M}$. The dynamics of a (time-invariant) system is described by a one-parameter group of symplectic diffeomorphisms; the generating function is the *energy* or *hamiltonian*. The system is said to be *free* if $(\mathcal{M}, \Omega)$ is an affine symplectic space and the motion is by a one-parameter group of affine symplectic transformations. This general description applies to systems which include classical particles, strings, fields, and other types of objects. Often the dynamics of the theory is embedded in a larger symmetry group. For example, in relativistic field theories one assumes that $(\mathcal{M}, \Omega)$ carries a representation of the Poincaré group.

Many classical systems admit a *lagrangian* description in which $(\mathcal{M}, \Omega)$ is derived from a relatively simple expression, called the lagrangian density. One of the main features of a lagrangian description is that the conserved quantity—called the *Noether charge*—corresponding to a symmetry is computed directly from the lagrangian. Furthermore, in field theories there is a local *Noether current* which integrates to the global charge and which gives rise to local conservation laws. This reflects the physical fact that, for example, we can measure energy in any region of space, not just the total energy over all of space. In this text we develop the basic ideas of classical lagrangian field theory. The examples we have in mind are the ones which arise in relativistic quantum field theory as treated in other parts of the book. One should be careful in trying to apply the formalism developed here to constrained systems (e.g., nonholonomic constraints in classical mechanical systems, or constraints imposed on superfields in the superspace descriptions of supersymmetric fields theories). In most of the exposition, we will assume that fields are arbitrary sections of some fixed fiber bundle E over spacetime. Also, we develop the formalism in a purely local way from the lagrangian density, hardly mentioning its integral, the *action*. If one is interested in calculus of variations problems, then the emphasis is different: the action is of primary interest and boundary conditions play a crucial role. A last warning is that not all classical field theories admit a lagrangian description. There is a free example which is important in two-dimensional conformal field theory: a free chiral scalar field in two dimensions.

In Chapter 1 we review some classical mechanics. In a few standard examples we describe the classical equations of motion and the construction of a symplectic structure on the space of classical solutions. Noether's theorem—the construction of a conserved quantity from a one-parameter group of symmetries—is also discussed. We treat both nonrelativistic and relativistic examples, and in §1.4 we show how to obtain nonrelativistic Galilean spacetime as a limit of relativistic Minkowski spacetime.

The general theory of classical fields is laid out in Chapter 2. A classical lagrangian field theory consists of a spacetime M, a space of fields $\mathcal{F}$, and a lagrangian density L. We are mostly interested in Minkowski spacetime, but the theory is quite general and applies to spacetimes which are curved Lorentzian or Riemannian manifolds as well. The fields are some sort of functions on M, more precisely sections of a fiber bundle E over M. The lagrangian density is a density on M for each point of $\mathcal{F}$. It is assumed[1] to be of a local nature on M. More precisely, for some k the value of $L(\phi)$ at a point m of M should depend only on the k-jet of ϕ at m. Usually $k = 1$. The lagrangian density L determines Euler-Lagrange equations $\underline{D}L = 0$, also called equations of motion, which cut out the space of extremals $\mathcal{M} \subset \mathcal{F}$. If ϕ is a field, the tangent space of $\mathcal{F}$ at ϕ is the space of sections of the vector bundle $\phi^*T(E/M)$ over M. The Euler-Lagrange equation $\underline{D}L$ is a morphism of vector bundles from $\phi^*T(E/M)$ to the bundle of densities on M. Its characteristic property is that for a deformation with compact support $\phi[u]$ of ϕ, one has

$$\int \frac{d}{du} L(\phi)[u]\Big|_{u=0} = \int \underline{D}L \cdot \frac{d}{du}\phi[u]\Big|_{u=0} .$$

The formalism naturally takes place in the double complex of differential forms on $\mathcal{F} \times M$. This is qualified in two ways: (i) as we want $L(\phi)$ to be a density, rather than a differential form of maximal degree, this double complex should be twisted by the orientation bundle of M; and (ii) we want to consider only (p, q)-forms α which are *local* on M: to a field ϕ and tangent vectors $\xi_1, \ldots, \xi_p \in \phi^*T(E/M)$, the form α attaches a q-form $\alpha(\phi; \xi_1, \ldots, \xi_p)$ on M, and for some k the value of $\alpha(\phi; \xi_1, \ldots \xi_p)$ at a point m of M should depend only on the k-jet at m of ϕ and the ξ_i. The cohomology of the double complex of local forms has been investigated by F. Takens[2]. Write the exterior differential as $D = \delta + d$, with δ of degree $(1, 0)$ and d of degree $(0, 1)$. Takens' main result is Theorem 2.15: For $p > 0$ the complex $(\Omega^{p,\bullet}_{\text{loc}}, d)$ of local differential forms is exact except in top degree. One can view the $\Omega^{p,q}_{\text{loc}}$ as an inductive limit, in k, of spaces of global sections of soft sheaves on E, and this makes the exactness of $(\Omega^{p,\bullet}_{\text{loc}}, d)$ ($p > 0$, $\bullet \neq$ top) a local question on E. Let $J^k(E)$ be the bundle over M of k-jets of sections of E. Takens also observes that the associated simple complex $\Omega^\bullet_{\text{loc}}$ is the inductive limit over k of the de Rham complexes of the $J^k(E)$. As the projections $J^k(E) \to E$ are fibrations, with fibers affine spaces, they induce isomorphisms in cohomology. It follows that the cohomology of E maps isomorphically to that of $\Omega^\bullet_{\text{loc}}$. We include a proof of Takens' results in an appendix to Chapter 2.

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[1] Although we restrict to lagrangians which depend locally on fields, in quantum field theory one meets *effective lagrangians* which are not local. Some of the formal aspects carry over to nonlocal lagrangians, but we have not taken the trouble to distinguish them.

[2] See References at the end of the manuscript.

If the lagrangian density $L(\phi)$ depends only on the first jet of ϕ, it defines a $(1, n-1)$-form γ (where $n = \dim M$), which we call the *variational 1-form*, characterized by the following properties: (i) the value of the $(n-1)$-form $\gamma(\phi; \xi)$ at $m \in M$ depends only on the 1-jet of ϕ at m, and on the value of ξ at m; and (ii)

$$\underline{D}L = \delta L + d\gamma.$$

The variational $(1, n-1)$-form γ encodes the usual integration by parts argument which occurs in computing Euler-Lagrange equations. For more general lagrangians, the choice of a local $(1, n-1)$-form γ with $\underline{D}L = \delta L + d\gamma$ should be considered part of the definition of the theory. Such a form γ always exists and, by Takens' theorem, it is unique up to the addition of $d\beta$, for β a local $(1, n-2)$-form.

Classical mechanics corresponds to $M = \mathbb{R}$ (time). In classical mechanics, one is used to the following package: (i) the space of extremals $\mathcal{M}$ is a symplectic manifold; (ii) symmetries give symplectic automorphisms of $(\mathcal{M}, \omega)$, and infinitesimal symmetries ξ are given by generating functions Q, with $dQ = -\iota(\xi)\omega$. A particular case is the one-parameter group of time translations, whose generating function is minus the hamiltonian. In (ii), Q is ambiguous up to an additive constant (and its existence can be obstructed by $H^1(\mathcal{M}, \mathbb{R})$). This ambiguity can often be removed by refining (i) to: (i′) the space of extremals carries a canonical principal $\mathbb{R}$-bundle T with connection ∇ whose curvature is ω. Infinitesimal automorphisms of $(\mathcal{M}, T, \nabla)$ can then be identified with pairs (ξ, Q) consisting of a vector field ξ on $\mathcal{M}$ and a function Q such that $dQ = -\iota(\xi)\omega$.

In field theory one has a similar Hamiltonian interpretation if M is given as time $\times$ space, or if one has a suitable notion of space-like hypersurface. A new problem is to obtain counterparts local on M of those constructions. These local counterparts do not require a notion of space-like hypersurface, or a splitting into time $\times$ space, but rather exist for any spacetime M. In particular, they make sense for Euclidean analogs, obtained by a Wick rotation $t \mapsto it$. For the 2-form on $\mathcal{M}$, which we now denote 'Ω', this is done in a paper of G. Zuckerman:[3] if ω is the local $(2, n-1)$-form $\delta\gamma$ on $\mathcal{F} \times M$, then the 2-form Ω on $\mathcal{M}$ is deduced from ω by integration on any space-like hypersurface. Similarly, one can express the generating functions of infinitesimal symmetries as integrals over space-like hypersurfaces of conserved currents, called *Noether currents*. Let ξ be a vector field on $\mathcal{F}$ which is local: ξ_ϕ at $m \in M$ depends only on some jet of ϕ at m. Suppose it is a generalized symmetry of L, in the sense that for some given $(0, n-1)$-local form α_ξ, one has

$$\mathrm{Lie}(\xi)L = d\alpha_\xi.$$

Then the Noether current $j_\xi := \iota(\xi)\gamma - \alpha_\xi$ is *conserved*: $dj_\xi(\phi) = 0$ for all ϕ in $\mathcal{M}$,. The vector field ξ is tangent to $\mathcal{M}$, and the integral Q_ξ of j_ξ on a space-like hypersurface is a corresponding generating function.

It can be more natural to consider symmetries which act on $\mathcal{F}$ and M simultaneously. We will say that a symmetry is *manifest* if it preserves L and γ exactly. For example, time translation is usually a manifest symmetry when it acts on both $\mathcal{F}$ and M. If we let it act only on fields, not on spacetime, it is only a generalized symmetry (with $\alpha = -\iota(\partial_t)L$).

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[3] See References.

Theories whose field content includes a connection or metric possess an infinite dimensional group of local "gauge" symmetries. (For metrics the gauge symmetry group is the group of diffeomorphisms.) We can freeze the metric or connection at fixed values g_0, A_0; then local symmetries which fix g_0 or A_0 act as global symmetries in the theory of the remaining fields. The Noether current for these global symmetries may be computed by differentiating the total lagrangian with respect to the metric or connection, as explained in §2.8 and §2.9. The derivative with respect to the metric is called the *energy-momentum* tensor.

In §2.10 we discuss time-invariant field configurations of finite energy on spacetimes which are time × space. Among these we find *classical vacua* and *solitons*. We also explain briefly perturbations around a classical vacuum and the Higgs mechanism.

Chapter 3 summarizes the basic free lagrangian field theories on Minkowski spacetime. We treat scalar fields, spinor fields, and abelian gauge fields (connections). Our goal is to illustrate the general theory in the simplest case and to record useful formulas.

In classical physics one of the main applications of field theory is to electromagnetism. Chapter 4, which is a discussion of gauge theory in general, begins with a brief treatment of Maxwell's equations in the lagrangian framework. Some familiarity with this material is necessary to understand the intuition behind more complicated models with gauge fields, which are encountered in many lectures in these volumes. In §4.2 we review the basic geometry of connections in principal bundles, paying special attention to the universal connection (in terms of which we write gauge theory lagrangians). Then in §4.3 we write the lagrangian for Yang-Mills theory and describe some additional "θ-terms" which may appear in low dimensions. Finally, we define electric and magnetic charge in §4.4 and discuss the relationship to global gauge transformations.

The general bosonic lagrangians without gravity usually include only scalar fields and gauge fields. We discuss a general lagrangian for these fields in Chapter 5; it includes many important bosonic theories as special cases.

There are special *topological terms* in lagrangians which are invariant under (orientation-preserving) diffeomorphisms. Some, like the θ-terms mentioned above, are related to primary topological invariants. More subtle are the ones associated to secondary invariants, like the Wess-Zumino-Witten term in a σ-model or the Chern-Simons term in three-dimensional gauge theory. In Chapter 6 we briefly introduce the main examples and explain how the action acquires a more subtle geometric meaning. The geometric home for the lagrangian is a "Γ-calculus" which extends the usual calculus of differential forms, as we indicate in §6.3.

Finally, in Chapter 7 we discuss the "Wick rotation" of a lagrangian from Minkowski spacetime to Euclidean spacetime. For reference we collect the signs and factors of $\sqrt{-1}$ which occur in this analytic continuation.

To a large extent this text presents a preliminary version of this material; we are not satisfied at all with our understanding in many places. Still, we felt it important to include some mathematical framework for the computations in lagrangian field theory. Our treatment is guided by what is needed to follow the lectures and problems recorded in these volumes, and we hope at least to have provided sufficient background for that. Other mathematical accounts appear in the references.

The definition of a classical lagrangian that we use (Definition 2.39) is adopted from a lecture of Joseph Bernstein. The basic formalism appears in the aforementioned paper of Gregg Zuckerman. In preparing this text we also benefited from many discussions with David Kazhdan, John Morgan, Nati Seiberg, and Ed Witten, among others.

CHAPTER 1
Classical Mechanics

The equation of motion of classical mechanics is the Euler-Lagrange equation for extremizing the action integral. From this extremal description we construct a canonical closed 2-form ω on the space $\mathcal{M}$ of classical evolutions. It turns out that ω is nondegenerate, and so $\mathcal{M}$ is a symplectic manifold. More precisely, we construct on $\mathcal{M}$ a canonical $\mathbb{R}$-torsor[4] with connection whose curvature is ω. As usual in symplectic geometry, to an infinitesimal symmetry we associate a function which is called the *Noether charge*. (It is often called the *momentum* or *momentum map*.)

In quantum theory we apply $\exp(\frac{i}{\hbar}\cdot)$ to turn this $\mathbb{R}$-torsor into a unitary line bundle with connection whose curvature is $\frac{i}{\hbar}\omega$.

§1.1. The nonrelativistic particle

We treat three cases: the free particle, a system of particles with potential, and the electromagnetic field.

Free Particle

Let $X = \mathbb{R}^n$ be Euclidean space with its standard inner product $\langle \cdot, \cdot \rangle$. The evolution of a classical *free* particle of mass m is described by a map x from $\mathbb{R}$ (time) to X. The *lagrangian density*[5] is the density on $\mathbb{R}$

$$L = \frac{m}{2}|\dot{x}|^2\, dt, \tag{1.1}$$

where $|\dot{x}|^2$ is the inner product $\langle dx/dt, dx/dt \rangle$. The integral of the lagrangian density

$$S = \int_{t_0}^{t_1} L \tag{1.2}$$

from time t_0 to time t_1 is the *action integral*, or simply *action*. If we deform x we have

$$\begin{aligned} \delta L &= m\langle \dot{x}, \delta\dot{x} \rangle\, dt \\ &= -m\langle \ddot{x}, \delta x \rangle\, dt - d\Big\{ m\langle \dot{x}, \delta x \rangle \Big\}. \end{aligned} \tag{1.3}$$

[4]In this context '$\mathbb{R}$-torsor' means 'principal $\mathbb{R}$-bundle'.

[5]We should use the *density* $|dt|$ in place of the *form* dt in (1.1), but in this section we simply orient $\mathbb{R}$ and so identify 1-forms and densities.

Here δ is the differential on the space $\mathcal{F}$ of trajectories x of the particle, d is the differential on $\mathbb{R}$, and the second minus sign[6] in the second line of (1.3) arises since δ and d anticommute on $\mathcal{F} \times \mathbb{R}$. Integrating we find

$$\delta S = \delta \int_{t_0}^{t_1} L = -\int_{t_0}^{t_1} dt\, m\langle \ddot{x}(t), \delta x(t)\rangle + \Big[m\langle \dot{x}(t), \delta x(t)\rangle\Big]_{t_0}^{t_1}. \tag{1.6}$$

The extremality condition is that the 1-form δS on $\mathcal{F}$ vanish on deformations of x with compact support in (t_0, t_1). This leads to the classical equation of motion

$$\ddot{x} = 0 \tag{1.7}$$

whose solutions are uniform motion. The boundary term leads one to consider, for each time t, the 1-form

$$\gamma(t) = m\langle \dot{x}(t), \delta x(t)\rangle \tag{1.8}$$

on the space $\mathcal{F}$ of all paths x. On the subspace $\mathcal{M}$ of extremals (solutions to (1.7)) the action S is a function whose differential is

$$\delta S = \gamma(t_1) - \gamma(t_0) \qquad \text{on } \mathcal{M}. \tag{1.9}$$

It follows that the 2-form on $\mathcal{M}$ defined by

$$\begin{aligned} \omega(t) :&= \delta\gamma(t) \\ &= m\langle \delta\dot{x}(t) \wedge \delta x(t)\rangle \end{aligned} \tag{1.10}$$

is independent of t. It is even independent of t for a specific reason, namely (1.9), with a compatibility among the reasons if three times $t_0 < t_1 < t_2$ are considered:

$$\int_{t_0}^{t_1} L + \int_{t_1}^{t_2} L = \int_{t_0}^{t_2} L. \tag{1.11}$$

This can be rephrased as defining an $\mathbb{R}$-torsor with connection (T, ∇) on $\mathcal{M}$ whose curvature is ω: for each fixed t_0 it is the trivial $\mathbb{R}$-torsor $T(t_0)$ with the connection $\nabla(t_0)$ given by $\gamma(t_0)$. By (1.9) addition of $-\int_{t_0}^{t_1} L$ gives an isomorphism from $\big(T(t_0), \nabla(t_0)\big)$ to $\big(T(t_1), \nabla(t_1)\big)$, and by (1.11) these isomorphisms form a compatible system of isomorphisms. The desired (T, ∇) is the "common value" (projective limit) of the $\big(T(t), \nabla(t)\big)$.

If we fix t_0, the map $x \mapsto \big(x(t_0), \dot{x}(t_0)\big)$ from the space $\mathcal{M}$ of extremals to the tangent bundle TX of X is an isomorphism. If we use (1.8) to map the tangent bundle to the cotangent bundle, then γ (resp. ω) is the pullback of the canonical 1-form (resp. canonical 2-form) on the cotangent bundle.

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[6]It is more usual to let δx denote a tangent vector to $\mathcal{F}$, in which case there is no minus sign. We emphasize that our computations take place on $\mathcal{F} \times \mathbb{R}$. For example, '$x$' in (1.1) is the evaluation map $\mathcal{F} \times \mathbb{R} \to \mathbb{R}$ and '$\dot{x}$' is its time derivative. To convert (1.3) into the more usual formula, let $\hat{\xi}$ be a vector (field) on $\mathcal{F}$, and set

$$\hat{\xi}x = \iota(\hat{\xi})\delta x = X. \tag{1.4}$$

Apply the contraction $\iota(\hat{\xi})$ to (1.3). Note that in commuting $\iota(\hat{\xi})$ past d in the last term we pick up a minus sign. Thus we find

$$\iota(\hat{\xi})\delta L = \hat{\xi}L = -m\langle \ddot{x}, X\rangle\, dt + d\Big\{m\langle \dot{x}, X\rangle\Big\}. \tag{1.5}$$

System of Classical Particles with Potential

More generally, if we consider a system of classical particles with rigid constraints, the configuration space is a Riemannian manifold X with Riemannian structure given by twice the kinetic energy. Evolution is described by a map from $\mathbb{R}$ (time) to X. The lagrangian density is

$$L = \frac{1}{2}|\dot{x}|^2\, dt; \tag{1.12}$$

here the masses are included in the metric. If in addition we have an external field of forces depending on a potential, or interaction between the particles described by potentials, then the potentials are encoded by a real-valued function V on X, and the lagrangian density is

$$L = \left\{\frac{1}{2}|\dot{x}|^2 - V(x)\right\} dt. \tag{1.13}$$

The free story can be repeated with the following changes. The Euler-Lagrange equation (1.7) is now Newton's law

$$\nabla_t \dot{x} + \operatorname{grad} V = 0. \tag{1.14}$$

The 1-form on the space of x is

$$\gamma(t) = \langle \dot{x}(t), \delta x(t) \rangle \tag{1.15}$$

and the symplectic 2-form is

$$\omega(t) = \langle \delta\dot{x}(t) \wedge \delta x(t) \rangle. \tag{1.16}$$

The construction of the $\mathbb{R}$-torsor is as before. The identification of the space of solutions with the initial data at a fixed time depends on suitable completeness assumptions.

Electromagnetic Field

An *electromagnetic field* can be described as an $\mathbb{R}$-torsor with connection (P, ∇) on spacetime $\mathbb{R} \times X$. For now choose a trivialization of P (a "gauge"), and so write ∇ as a 1-form α. We separate the time and space components by

$$\alpha = V\, dt + A. \tag{1.17}$$

Then V is the *scalar potential* and A is the *vector potential*. The evolution of a single charged particle is described as before by a map $x\colon \mathbb{R} \to X$, and for a particle of mass m and charge q the lagrangian density is

$$L = \frac{m}{2}|\dot{x}|^2\, dt - qx^*(\alpha). \tag{1.18}$$

The action $S = \int_{t_0}^{t_1} L$ is the sum of a kinetic energy term and of q times the parallel transport along x from $x(t_0)$ to $x(t_1)$, computed in the chosen gauge. In δL the boundary term is now

$$\gamma(t) = m\langle \dot{x}(t), \delta x(t) \rangle - qA\big(\delta x(t)\big). \tag{1.19}$$

Fix t_0; then under suitable completeness assumptions the map $x \mapsto \big(x(t_0), \dot{x}(t_0)\big)$ identifies the space $\mathcal{M}$ of extremals with the tangent bundle TX. Notice that the map from the tangent bundle to the cotangent bundle given by (1.18) is the previous map shifted by qA.

If we change the gauge, the space of extremals does not change. This becomes clear if we do not interpret $\int_{t_0}^{t_1} L$ as a number, but rather as an isomorphism from $qP_{(t_0,x(t_0))}$ to $qP_{(t_1,x(t_1))}$, the sum of a kinetic energy term and of parallel transport. This isomorphism is manifestly gauge independent. On the space of extremals $\mathcal{M}$ we continue to have an $\mathbb{R}$-torsor with connection whose curvature is ω, independent of the gauge. For t_0 fixed it is naturally $\big(t_0, x(t_0)\big)^*(qP)$ with connection given by $\big(t_0, x(t_0)\big)^*(q\nabla) + m\langle \dot{x}(t_0), \delta x(t_0)\rangle$. Isomorphisms between this $\mathbb{R}$-torsor with connection for different choices of t_0 are given by the action—parallel transport plus a kinetic energy term.

In this example we see that the action need not be a number, but rather can be an element of an $\mathbb{R}$-torsor. We discuss such *topological terms* further in Chapter 6.

§1.2. The relativistic particle

Let X be n-dimensional Minkowski spacetime. This is standard n-dimensional affine space with a Lorentz metric. Fix affine coordinates $t, x^1, x^2, \dots, x^{n-1}$ so that the metric takes the form

$$c^2(dt)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.20}$$

The corresponding basis of the underlying vector space of translations is called an *inertial frame*. Here c is the speed of light. It is often convenient to set $x^0 = ct$.

The worldline of a relativistic particle is represented by a map

$$\begin{aligned} x\colon \mathbb{R} &\longrightarrow X \\ x(\tau) &= \big(t(\tau), x^1(\tau), \dots, x^{n-1}(\tau)\big) \end{aligned} \tag{1.21}$$

with $\langle dx/d\tau, dx/d\tau\rangle \geq 0$ and $dt/d\tau > 0$. The lagrangian density of a free particle of *rest mass* m_0 is

$$L = -m_0 c \left\langle \frac{dx}{d\tau}, \frac{dx}{d\tau} \right\rangle^{1/2} d\tau. \tag{1.22}$$

For a physical particle $dx/d\tau$ lies in the positive light cone; in particular, we have $\langle \frac{dx}{d\tau}, \frac{dx}{d\tau}\rangle > 0$. This lagrangian is invariant by the Poincaré group of symmetries of X, as well as by the group $\mathrm{Diff}^+(\mathbb{R})$ of reparametrizations of x. An action integral is attached to a region R of spacetime bounded by two space-like hypersurfaces H_0 and H_1. One can, for instance, take for R the region $t_0 \leq t \leq t_1$. For a free particle of rest mass m_0, the action integral is $-m_0 c$ times the arc length of the portion of the path x contained in R.

The extremals for this action are straight lines. If S is the action integral for the region R bounded by hypersurfaces H_0, H_1, and if we deform x, then δS is the sum of two terms: (i) an integral $\int_{\tau_0}^{\tau_1}$, where $x(\tau_i) \in H_i$; and (ii) boundary terms. The boundary term for H_1 is

$$\gamma[H_1] = -m_0 c\, \frac{\langle \dot{x}(\tau_1), \delta x(\tau_1)\rangle}{|\dot{x}(\tau_1)|}. \tag{1.23}$$

Note that this expression is Diff($\mathbb{R}$)-invariant. As previously, $\omega := \delta\gamma[H_1]$ is independent of H_1 on the space of extremals and turns it into a symplectic manifold. More precisely, the space of extremals carries an $\mathbb{R}$-torsor T with connection whose curvature is ω. For each choice of space-like hypersurface H it can be identified with the trivial $\mathbb{R}$-torsor, with connection given by $\gamma[H]$. Action integrals give a transitive system of isomorphisms between the descriptions of (T, ∇) given by different H.

To go to the nonrelativistic limit we proceed as follows. We work in our chosen coordinate system and consider velocities which are small relative to the speed of light c. If we take for coordinate τ the time t, then the path is

$$x(t) = \big(t, x^1(t), \dots, x^{n-1}(t)\big). \tag{1.24}$$

The lagrangian is then

$$\begin{aligned} L &= -m_0c^2\sqrt{1 - v^2/c^2}\, dt \\ &= \left\{-m_0c^2 + \frac{1}{2}m_0v^2 + O(\frac{v^4}{c^2})\right\} dt, \end{aligned} \tag{1.25}$$

where

$$v^2 = \sum_i \left(\frac{dx^i}{dt}\right)^2 \tag{1.26}$$

is the velocity squared. The second line of (1.25) shows that the nonrelativistic limit $v/c \to 0$ of L is the kinetic energy $\frac{1}{2}m_0v^2$ minus the potential energy m_0c^2 of the particle at rest. The reader can also check that the nonrelativistic limit of (1.23) for the hypersurface $H_1 : \{t = \text{constant}\}$ is (1.8) (for $m = m_0$).

In the relativistic setting it does not make sense to introduce rigid constraints on a system of particles, nor to introduce potentials. On the other hand, a background electromagnetic field can be introduced on Minkowski spacetime analogously to the nonrelativistic case.

§1.3. Noether's theorem

We have seen a number of examples of how, because equations of motion are Euler-Lagrange equations, the space of classical evolutions $\mathcal{M}$ carries a symplectic structure ω. More precisely, it carries a canonical $\mathbb{R}$-torsor T with connection ∇ whose curvature is ω. Any automorphism of the data used to define the extremality condition induces an automorphism of $(\mathcal{M}, T, \nabla)$. The same holds for infinitesimal automorphisms. Infinitesimal automorphisms of $(\mathcal{M}, T, \nabla)$ can be identified with functions Q on $\mathcal{M}$. Namely, if a vector field $\tilde{\xi}$ on T is such an infinitesimal automorphism, let ξ denote the projection to $\mathcal{M}$ and $\hat{\xi}$ the horizontal lift of ξ to T. Then

$$\tilde{\xi} = \hat{\xi} + Q\zeta, \tag{1.27}$$

where ζ is the infinitesimal action of $\mathbb{R}$ which defines the torsor structure. The infinitesimal automorphism $\tilde{\xi}$ of T respects the connection if and only if

$$dQ = -\iota(\xi)\omega. \tag{1.28}$$

If the curvature ω is nondegenerate, i.e., turns $\mathcal{M}$ into a symplectic manifold, then (1.28) shows that vector field ξ is determined by the generating function Q. For functions Q_1, Q_2 corresponding to vector fields ξ_{Q_1}, ξ_{Q_2}, we compute the vector field which corresponds to the Poisson bracket $\{Q_1, Q_2\}$ to be

$$\xi_{\{Q_1,Q_2\}} = [\xi_{Q_1}, \xi_{Q_2}]. \tag{1.29}$$

(Note: Brackets of infinitesimal automorphisms are given by the *opposite* of Poisson brackets, in the same way that the Lie algebra of diffeomorphisms is the set of vector fields with bracket *opposite* to the usual brackets of vector fields.)

If we have a trivialization of (T, ∇) which is preserved by an infinitesimal automorphism $\tilde{\xi}$, then

$$Q = \iota(\xi)\gamma, \tag{1.30}$$

where γ is the connection form on $\mathcal{M}$ induced from the trivialization. If the trivialization changes, then there is an additional term in (1.30). In §2 we develop a more systematic formalism in which to make computations, so in this section when the trivialization changes we simply report the result.

We apply these ideas to the free particle in both the nonrelativistic and relativistic settings. Consider first a free nonrelativistic particle of mass m_0 moving in $\mathbb{R}^n$. Let $x^1, \dots, x^n$ be standard coordinates on $\mathbb{R}^n$ and δ_{ij} the standard metric. The isometries of $\mathbb{R}^n$ induce automorphisms of $(\mathcal{M}, T, \nabla)$ which fix the trivialization at a fixed time t_0. Thus we can use (1.30) in conjunction with (1.8) to compute the charges. The charge corresponding to the infinitesimal translation $\xi = \partial/\partial x^i$ is the *linear momentum*

$$p_i = m_0 \delta_{ij} \dot{x}^j. \tag{1.31}$$

The charge corresponding to an infinitesimal rotation $\xi = x^i \partial/\partial x^j - x^j \partial/\partial x^i$ is the *angular momentum*

$$M^i_j = m_0 \delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i). \tag{1.32}$$

Now the *hamiltonian* or *energy* E is the charge associated to *minus* time translation in the domain $\mathbb{R}$, and this does not preserve the trivialization of (T, ∇) induced at a fixed time. Rather, the derivative of the trivialization at t_0 is $L_0(t_0)$, where we write the lagrangian density as $L_0\, dt$. (The change in trivialization comes because infinitesimal time translation only preserves the lagrangian up to a total derivative.) Accounting for this we compute the nonrelativistic energy (see Example 2.105).

$$E = \frac{1}{2} m_0 |\dot{x}|^2. \tag{1.33}$$

There is an additional symmetry called a *Galilean boost*. We write the infinitesimal version as a time-varying vector field $\xi = t\partial/\partial x^i$. Again there is an additional term in the formula for the charge N_i since the lagrangian is only preserved up to a total derivative:

$$N_i = m_0 \delta_{ij}(t\dot{x}^j - x^j). \tag{1.34}$$

This is simply minus the initial position times the mass.

Consider now a relativistic particle. The lagrangian (1.22) is invariant under the isometries of X, the Poincaré group, and we would like to compute the corresponding charges. Fix a splitting of Minkowski spacetime into time $\times$ space and use the hypersurface $t = t_0$ to trivialize (T, ∇). We parametrize the worldline as in (1.24). The subgroup of Poincaré which fixes this hypersurface also fixes this trivialization and we use (1.30) and (1.23) to compute the associated charges. Thus the momentum p_i is the charge corresponding to $\xi = \partial/\partial x^i$, which we compute to be

$$p_i = \frac{m_0 \delta_{ij} \dot{x}^j}{\sqrt{1 - v^2/c^2}} = m(v) \delta_{ij} \dot{x}^j. \tag{1.35}$$

Here we have introduced the *relativistic mass*

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \tag{1.36}$$

Note that the nonrelativistic limit $v/c \to 0$ of the relativistic mass $m(v)$ is the rest mass m_0. Similarly, an infinitesimal spatial rotation $\xi = x^i \partial/\partial x^j - x^j \partial/\partial x^i$ has charge the *angular momentum*

$$M^i_j = \frac{m_0 \delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i)}{\sqrt{1 - v^2/c^2}} = m(v) \delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i). \tag{1.37}$$

Now the charge corresponding to $\xi = -\partial/\partial t$ is the *relativistic energy* E. As in the nonrelativistic case the trivialization changes according to the lagrangian (1.25) (divided by the standard density dt), and so we compute Einstein's famous formula

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = m(v) c^2. \tag{1.38}$$

Note that $(E, -p_i)$ transforms in the dual to the standard representation of the Lorentz group. In $n = 4$ dimensions this quantity is called the *4-momentum*. Finally, the infinitesimal Lorentz boost

$$\xi = \frac{x^i}{c^2} \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^i} \tag{1.39}$$

has corresponding charge

$$N_i = \frac{m_0 \delta_{ij}(t \dot{x}^j - x^j)}{\sqrt{1 - v^2/c^2}} = m(v) \delta_{ij}(t \dot{x}^j - x^j). \tag{1.40}$$

Here again the symmetry does not preserve the trivialization, so there is an extra term to compute. The charge N_i is simply minus the initial position of the particle times the relativistic mass.

The nonrelativistic limits of (1.35), (1.37), (1.38), (1.40) give (1.31), (1.32), (1.33), (1.34). (From E in (1.38) we must subtract the rest energy $m_0 c^2$ to obtain the nonrelativistic energy (1.33).)

§1.4. Synthesis

Let M_c denote n-dimensional Minkowski spacetime with metric g_c given in (1.20). Let M_∞ denote the limiting space as $c \to \infty$. Of course, the affine spaces which underlie M_c and M_∞ are identical. Let W be the vector space of translations of M_∞. The limit $\lim_{c\to\infty} g_c^{-1} = g_\infty^{-1}$ of the inverse metric is a degenerate quadratic form on W^*. It has a one-dimensional kernel $S^\circ \subset W^*$ whose annihilator $S \subset W$ determines a codimension one foliation of M_∞ by affine subspaces. Then g_∞^{-1} on W^*/S° is inverse to a negative definite metric on S. We think of this foliation (with the metric on the leaves) as defining the simultaneous spatial events in M_∞. We also fix a scale $dt \in S^\circ$ for time; then there is a one-dimensional space of distinguished affine time functions which differ by a constant. The group G_∞ of affine transformations of M_∞ which preserve g_∞^{-1} and dt is the *Galilean group*. Let G_c denote the group of affine transformations of M_c which preserve g_c; the double cover of its identity component is the Poincaré group. Note that G_c, G_∞ act on the same affine space, so there is a well-defined sense in which $G_c \to G_\infty$ as $c \to \infty$.

The translations sit inside G_c and G_∞. In G_∞ the spatial translations lie in a subgroup H of dimension $2(n-1)$ generated by vector fields $t\partial + \partial'$, where t is any affine time function and ∂, ∂' are spatial translations. Vector fields $t\partial$ generate affine transformations called *Galilean boosts*; they are the $c \to \infty$ limit of Lorentz boosts (1.39).

There is a nontrivial central extension of G_∞ which restricts nontrivially on H. For a geometric picture we begin with a fixed $(n+1)$-dimensional Minkowski spacetime N with vector space of translations V. For each spacelike vector $v \in V$ we consider the subgroup G_v of the Poincaré group which fixes v. It acts on the quotient affine space $M_v = N/\mathbb{R}\cdot v$. This quotient space M_v inherits a Minkowski metric we describe in two equivalent ways: it is the metric induced from the orthogonal space $(\mathbb{R}\cdot v)^\perp \subset V$, or equivalently its inverse is the subspace metric on the annihilator $(V/\mathbb{R}\cdot v)^* \cong (\mathbb{R}\cdot v)^\circ \subset V^*$. The subgroup G_v is the trivial central extension of the Poincaré group of M_v by translations along v. Now consider $\ell \in V$ a lightlike vector. Set $M_\ell = N/\mathbb{R}\cdot\ell$, and let G_ℓ be the subgroup of the Poincaré group of N which fixed ℓ. Now the inverse metric on V^* restricts to a degenerate form on $(V/\mathbb{R}\cdot\ell)^* \cong (\mathbb{R}\cdot\ell)^\circ \subset V^*$. Also, we take the functional $\langle \ell, \cdot \rangle$ to be the dt above. The group of affine transformations of M_ℓ which preserves the degenerate form and ℓ is the Galilean group. Thus G_ℓ is the central extension of this group by translations along v. Note that the subspace of $(V/\mathbb{R}\cdot\ell)^* \cong (\mathbb{R}\cdot\ell)^\circ$ previously called S° is here $(\ell^\perp)^\circ$, the annihilator of the orthogonal subspace to ℓ. (Since ℓ is lightlike, we have $\mathbb{R}\cdot\ell \subset \ell^\perp$, which is why it makes sense to define $dt = \langle \ell, \cdot \rangle$.) Also, S is simply $\ell^\perp/\mathbb{R}\cdot\ell$.

For an explicit description introduce coordinates $t^+, t^-, x^1, \dots x^{n-1}$ on N so that the metric is[7]

$$dt^+ dt^- - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.41}$$

[7]Here the speed of light is inessential, so we set it to one.

Take $\ell = \partial_+ = \partial/\partial t^+$. Then $(\ell^\perp)^\circ$ is the span of dt^-, and the possible time functions are $\{at^- + b : a, b \in \mathbb{R}\}$. The Lie algebra of G_ℓ is spanned by

$$\begin{aligned} & \partial_+ && \text{(central element)} \\ \partial_- = \frac{\partial}{\partial t^-}, \quad & \partial_i = \frac{\partial}{\partial x^i} && \text{(translations)} \\ & x^i\partial_j - x^j\partial_i && \text{(spatial rotations)} \\ & B_i = t^-\partial_i + x^i\partial_+ && \text{(Galilean boosts)} \end{aligned} \tag{1.42}$$

The nontrivial bracket

$$[\partial_i, B_j] = \delta_{ij}\partial_+ \tag{1.43}$$

reflects the nontrivial central extension. Note from (1.31) and (1.34) that for the free nonrelativistic particle the Poisson bracket of the Noether charges corresponding to ∂_i and B_j is

$$\{p_i, N_j\} = -\delta_{ij}m_0. \tag{1.44}$$

Comparing (1.43) and (1.44) we see that the conserved quantity corresponding to ∂_+ in a theory should be identified with the mass. Thus in the Galilean theory there is an additional conserved quantity over the relativistic theory—the mass. (The bracket of elements in the centrally extended Galilean algebra and the Poisson bracket of the corresponding Noether charges are opposite. This is the usual situation for *left* group actions on symplectic manifolds; see §5 of [I-**Signs**].)

Quite generally, suppose we have a classical theory whose state space $(\mathcal{M}, \omega)$ carries a symplectic action of either the Galilean group or the Poincaré group. Infinitesimally we have an antihomomorphism from the Lie algebra $\mathfrak{g}$ of that group into the Lie algebra of vector fields on $\mathcal{M}$. We assume that each vector field ξ in the image satisfies $-\iota(\xi)\omega$ is exact, i.e., is the symplectic gradient of some function. For a general group G, the existence and uniqueness of a lift $\mathfrak{g} \to C^\infty(\mathcal{M})$ which is an antihomomorphism of Lie algebras is measured by $H^1(\mathfrak{g})$ and $H^2(\mathfrak{g})$. For the Poincaré group there is a unique lift; for the Galilean group there is a lift of the central extension we constructed above, unique up to a shift of the total energy. (If spacetime has dimension 3, then we can also shift angular momentum.) The central element then maps to a locally constant function which is the total mass of the system. In the Poincaré case the vector space V of translations maps to a vector space of functions which—after choosing an inertial frame—is $(-E, p_1, \ldots, p_{n-1})$, where E is energy and p_i are linear momenta. Under Lorentz transformations this transforms as the coefficients of an element in[8] V^*, and out of it we construct an invariant using the metric:

$$E^2/c^2 - |p|^2 = m_0^2c^2. \tag{1.45}$$

This defines the "rest mass" m_0 of the system. It is a Poincaré invariant, though in general not locally constant, function on $\mathcal{M}$. In the Galilean case there is a Galilean-invariant codimension one subspace $S \subset V$ of translations in spatial directions, and

[8]Note that the vector dual to $(E, -p)$ is $(E/c^2, p)$.

S carries a metric. If the Galilean algebra, and not a central extension, were to lift to functions, then we would conclude that the norm square $|p|^2$ of the total momentum is Galilean-invariant. But clearly $|p|^2$ typically changes under boosts, and this shows why boosts and spatial translations have a nonzero commutator in the central extension.

We can formulate the theory of the free nonrelativistic particle by considering its worldline in the space M_∞. The difference from the relativistic situation is that the time functions provide distinguished parametrizations of the worldline up to translation. This allows us to "couple" the free nonrelativistic particle to an arbitrary potential function V, as in (1.13); the integrated lagrangian (action) does not change if we translate time and so is Galilean invariant. By contrast, as stated earlier we cannot introduce a Poincaré invariant coupling of the relativistic particle to a potential function. Rather, we can couple it to *fields*, specifically to an electromagnetic field (abelian connection) and to a gravitational field (variable Lorentz metric). In the next chapter we take up the general theory of fields.

CHAPTER 2
Lagrangian Theory of Classical Fields

§2.1. Dimensional analysis

It is often useful to follow the advice we give to beginning students: Check units!

Every physical quantity has units attached to it. The basic units are units of mass, length, and time. The number measuring a physical quantity Q depends on the choice of units, and Q is said to be of dimension $M^aL^bT^c$ if the number q measuring Q is multiplied by $\lambda^{-a}\mu^{-b}\nu^{-c}$ when the units are multiplied by λ, μ, ν. Notation: $[Q] = M^aL^bT^c$. For example, the action integral S of (1.2), where the integral is taken between prescribed instants, has the dimension of an action:

$$[S] = ML^2T^{-1}. \tag{2.1}$$

The dimension of a p-form is defined to be the dimension of its value on a fixed p-vector. The lagrangian density (1.1) (a 1-form on the time line), the variational 1-form γ of (1.8) (a 1-form on the space of trajectories), and the closed 2-form ω all have the dimension of an action.

The dimension of the conserved quantity Q corresponding to an infinitesimal Galilean transformation η is given by

$$[Q] = [\text{action}][\eta]. \tag{2.2}$$

In the relativistic setting we will usually impose $c = 1$, so that a unit of length gives one of time. Then [action] becomes ML, and the conserved quantities corresponding to infinitesimal generators of the Poincaré group once again have

$$[Q] = [\text{action}][\eta]. \tag{2.3}$$

We can consider electric charge to have units C and take the constant q in (1.18) to have $[q] = C$; then the 1-form α in (1.17) which represents the electromagnetic field has units of action divided by charge:

$$[\alpha] = ML^2T^{-1}C^{-1}. \tag{2.4}$$

Alternatively, we can work in a system of units in which charge is expressed in terms of mass, length, and time by

$$C^2 = ML^3T^{-2}. \tag{2.5}$$

This comes from declaring the constant k in Coulomb's law "$F = kq_1q_2/r^2$" to be dimensionless.

Universal physical constants allow us to convert units. In relativistic theories the speed of light

$$[c] = LT^{-1} \tag{2.6}$$

identifies time with length. In quantum theories Planck's constant

$$[\hbar] = ML^2T^{-1} \tag{2.7}$$

has units of action, which eliminates one of M, L, T. In theories of gravity the Newton constant G in the Newton force law "$F = Gm_1m_2/r^2$" has units

$$[G] = L^3M^{-1}T^{-2}, \tag{2.8}$$

which again allows us to eliminate one of M, L, T. In a relativistic quantum field theory we can use $c, \hbar$ to express all units in terms of mass, and so each physical quantity has a *mass dimension*.[9] If the theory includes gravity (e.g. string theory), then we can use $c, \hbar, G$ to express everything in terms of dimensionless quantities.

§2.2. Densities and twisted differential forms

In this section M is an ordinary (not super) manifold.

If M is oriented of dimension n, then n-forms with compact support can be integrated on M. Changing the orientation multiplies the integral by -1. This leads to the consideration of *densities*, defined to be sections of

$$\operatorname{Dens} M := \textstyle\bigwedge^n T^*M \otimes \mathfrak{o}_M, \tag{2.9}$$

where $\mathfrak{o}_M$ is the orientation line bundle. On any M, oriented (and orientable) or not, the integral of a density ω with compact support is unambiguously defined: one writes ω as a sum of densities ω_i with support in orientable local charts U_i, one orients each U_i, and doing so identifies ω_i with an n-form on U_i, and one defines $\int_M \omega := \sum \int_{U_i} \omega_i$.

More generally, we will have to consider the complex of *twisted forms*, the tensor product of $\Omega^\bullet_M$ with the local system $\mathfrak{o}_M$. We treat $\mathfrak{o}_M$ as being in cohomological degree $-n$, and define

$$\Omega_M^{|-p|} := \Omega_M^{n-p} \otimes \mathfrak{o}_M. \tag{2.10}$$

A compactly supported element of $\Omega_M^{|-p|}$ can be integrated on a normally oriented submanifold of M of codimension p. Elements of $\Omega_M^{|-p|}$ can be viewed as sections of

$$\textstyle\bigwedge^p TM \otimes \operatorname{Dens} M; \tag{2.11}$$

the isomorphism with the description (2.10) comes by contracting a p-vector field with a density.

In the sequel we often say simply 'form' for 'twisted form'.

In the case of supermanifolds, densities are related not to an exterior power of the cotangent bundle, but to its Berezinian, and (2.11) defines the components of the complex of integral densities. See [I-**Supersymmetry**, §§3.9–12] for details.

On a Riemannian manifold M the Hodge $*$-operator is an isomorphism

$$*\colon \Omega_M^q \longrightarrow \Omega_M^{|-q|}. \tag{2.12}$$

[9]Then for a physical quantity Q instead of writing $[Q] = M^n$ we usually write $[Q] = n$.

§2.3. Fields and lagrangians

Let M be a smooth manifold of dimension n. We formulate field theory on M. In the standard physical setup $M = M^n$ is affine Minkowski spacetime. For $n = 1$ this is M^1, which is (affine) time; it is the appropriate "spacetime" M for classical mechanics. In usual examples of field theory we can analytically continue to imaginary time and so obtain a field theory on Euclidean space. Often field theories can also be formulated for curved metrics—of Lorentz or Euclidean signature. We also allow the spacetime M to be a supermanifold. As we cautioned in the introduction, our framework is not adequate for many types of constrained mechanical systems.

Fields on M are (smooth) sections of a given fiber bundle $E \to M$. Let $\mathcal{F}$ denote the space of all sections. For example, the basic field in a *σ-model* is a map $\phi\colon M \to X$ for some auxiliary manifold X. In this case we simply have $E = M \times X$. We can also study a twisted version in which E is not a product. An important case is when E is a vector bundle. The basic field in a *gauge theory* is a connection A with gauge group some specified Lie group G. If we fix a principal G-bundle $P \to M$, then A is a section of a certain associated bundle of affine spaces. For many purposes it is best not to fix P and rather to view the collection of all connections as a category. However, in this section we simply view P as fixed. There is an evaluation map

$$e\colon \mathcal{F} \times M \longrightarrow E. \tag{2.13}$$

Again we allow E—and so also $\mathcal{F}$—to be a supermanifold, even if M is an ordinary manifold.

Of course, a field theory typically contains several fields ϕ_i and correspondingly $E = \times E_i$ is a fiber product. Roughly speaking, each E_i decomposes into an intrinsic part times an extrinsic part. The intrinsic part is associated to the principal frame bundle of M via a representation of GL (or Spin), and the representation determines the type of field. Thus a *scalar field* is associated to the trivial representation. The basic field $\phi\colon M \to X$ in a σ-model is a typical example. A scalar field may take values in a nonlinear space, but the extrinsic values of other types of fields are linear. A *p-form field* is a p-form on M; it may take values in some vector bundle over M. Physicists often use the word 'vector field' to refer to a 1-form field. They also use 'vector field' or 'gauge field' to refer to a *connection*, which is a type of field which was discussed above. There are also "connectionlike" versions of p-forms (see §6.3), but only for abelian structure groups. If M is a spin manifold then we can also consider *spinor fields*, which are sections of a spin bundle possibly tensored with another extrinsic vector bundle. The precise choice of spinor bundle varies with the example. Theories of gravity also include a *metric* on X, sometimes called a 'gravitational field'. In theories of supergravity there is also a *Rarita-Schwinger field*, which is a section of an irreducible subbundle of $S \otimes T^*M$, where S is a spin bundle.

The *spin* of a field depends on the representation of Spin (or GL) which defines it. Namely, we decompose the complexified representation under the Spin(2) subgroup which double covers the group of rotations in some 2-plane. We obtain a sum of one-dimensional representations which we label by half-integers $0, \pm 1/2, \pm 1, \dots$. Replace these numbers by their absolute values; then the largest number which occurs is the spin of the field. Thus a scalar field has spin 0, a spinor field spin $1/2$, a p-form field or connection spin 1, a Rarita-Schwinger field spin $3/2$, and a metric spin 2.

There are physical reasons to restrict to these values of the spin in quantum field theories (see [II-**Dynamics of QFT**, §2.4]).

In unitary quantum field theories there is a connection between spin and *statistics*, the statistics being whether the field is even or odd (in the sense of supergeometry): fields of integral spin are even and fields of half-integral spin are odd. The usual physical terminology is that even fields are *bosons* and odd fields are *fermions*. This spin-statistics connection is violated in some nonunitary topological field theories.

In some theories there are local gauge symmetries (see Definition 2.93) which act on the fields. This occurs for fields which are p-forms ($p \geq 1$) and for connections in gauge theory. (A p-form α is gauge equivalent to $\alpha + d\beta$ for β any $(p-1)$-form.) The gauge symmetries of a connection are as usual. Local symmetries are sections of a bundle of groups over M. Let $\overline{\mathcal{F}}$ denote the quotient of the space of fields $\mathcal{F}$ by the action of local symmetries. In our notation we treat $\overline{\mathcal{F}}$ as a manifold, though this may not be true and we may need to work equivariantly on some space which projects onto $\overline{\mathcal{F}}$. (For example, in gauge theory we often fix a basepoint on each component of M. Then the group of gauge transformations which equal the identity at the basepoints acts freely on the space of connections, and the finite dimensional group of automorphisms at the basepoints acts on the quotient.)

Our formulation is not meant to include theories of gravity, where one of the fields is a metric and the group of diffeomorphisms of M acts as a local symmetry. We will, however, use the metric as a *background field* when we discuss the energy-momentum tensor. Then in some cases (e.g., with spinor fields) the fiber bundle $E \to M$ varies with the metric. The appropriate modification of this setup is described later when we discuss the energy-momentum tensor.

We will use a complex which is, basically, the de Rham complex of $\mathcal{F} \times M$. Reflecting the product structure of $\mathcal{F} \times M$, it is a double complex. Suppose first that M is oriented, of dimension n. The complex $\Omega^{\bullet,|\bullet|}$ we will use is then the de Rham complex shifted by n: in other words, $\Omega^{p,|-q|}(\mathcal{F} \times M)$ is the space of p-forms on $\mathcal{F}$ with values in the space of $(n-q)$-forms on M. We let δ be the exterior derivative of $\mathcal{F}$, d the exterior derivative[10] of forms on M, and $D = \delta + d$ the total exterior derivative. For general M we use twisted $(n-q)$-forms: instead of the de Rham complex of M, we use $\Omega_M^{|\bullet|} = \Omega_M^\bullet \otimes \mathfrak{o}_M$.

More relevant is the subcomplex $\Omega_{\mathrm{loc}}^{\bullet,|\bullet|}$ of *local* elements of $\Omega^{\bullet,|\bullet|}$, where $\alpha \in \Omega^{p,|-q|}$ is said to be local if for some k, at any $\phi \in \mathcal{F}$ the value of the twisted $(n-q)$-form $\alpha(\phi; \hat{\xi}_1, \dots, \hat{\xi}_p)$ at $m \in M$ depends only on the k-jet at m of $\phi, \hat{\xi}_1, \dots, \hat{\xi}_p$. Here $\hat{\xi}_i$ are tangents to $\mathcal{F}$ at ϕ. If $\mathcal{F}$ is the space of sections of a bundle E over M, and if $\pi^{(k)}\colon J^kE \to M$ is the bundle of k-jets of sections of $\pi\colon E \to M$, such an α is a global section over J^kE of $\Omega^p_{J^kE/M} \otimes \pi^{(k)*}\Omega_M^{|-q|}$. Note that k cannot be kept fixed, as d maps $\Omega^p_{J^kE/M} \otimes \pi^{(k)*}\Omega_M^{|-q|}$ to $\Omega^p_{J^{k+1}E/M} \otimes \pi^{(k+1)*}\Omega_M^{|-q+1|}$.

As mentioned in the Introduction, the double complex $\Omega_{\mathrm{loc}}^{\bullet,|\bullet|}$ has been investigated by Takens, who proved the following.

---

[10] Our sign convention is that for $\alpha \in \Omega^p(\mathcal{F})$ and $\beta \in \Omega^{|-q|}(M)$,

$$d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d\beta. \tag{2.14}$$

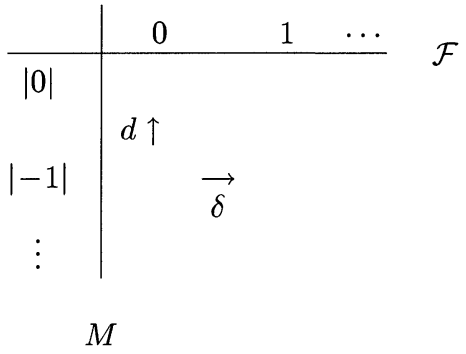
Theorem 2.15 (Takens). *For $p > 0$ the complex $\left(\Omega^{p,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M), d\right)$ of local differential forms is exact except in the top degree $|\bullet| = 0$.*

This theorem plays a crucial role in the discussion of generalized symmetries (§2.6), and also plays a part in the general discussion of this section. We also make use of the following generalization, proved by the same method.

Theorem 2.16 (Takens). *Let $V_i \to E$, $i = 1, \ldots, p$, be vector bundles, where we require $p \geq 1$. Let $V = \times_E V_i$ be the fiber product. For $\phi \in \mathcal{F}$ a section of E, let $\mathcal{V}_\phi$ be the space of sections of $\phi^* V \to M$. Consider in $\Omega^{0,|\bullet|}_{\mathrm{loc}}(\mathcal{V}_\phi \times M)$ the subcomplex of forms $\alpha(\phi; \zeta_1, \ldots, \zeta_p)$ which are $\mathbb{R}$-multilinear in ζ_i. Then this subcomplex is exact except in the top degree $|\bullet| = 0$.*

In the appendix to Chapter 2 we give a proof of these theorems.

We will use the following picture to depict elements in the double complex:



We remark that certain topological (terms in) lagrangians do not fit into this formalism. We discuss the necessary modifications in Chapter 6.

The basic ingredient in a classical field theory is a *lagrangian (density)*

$$L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M); \tag{2.17}$$

it has units of action:

$$[L] = ML^2T^{-1}. \tag{2.18}$$

The notation in (2.15) indicates that for each field ϕ in $\mathcal{F}$ we have a density $L(\phi)$ on M, while (2.16) indicates that the integral of $L(\phi)$ on a fixed compact region of M has the dimension of an action. Typically, $L(\phi)$ is a local functional of ϕ and, for most fundamental lagrangians, the value of $L(\phi)$ at a point m depends only on the 1-jet of ϕ at m (gravity is an exception).

Typically, the integral $\int_M L(\phi)$ is divergent. Even if, as often, one can restrict one's attention to fields decaying rapidly at spatial infinity, so that the integral between two space-like hypersurfaces converges, the integration over time will diverge. Computations given below show that otherwise the expected symplectic structure on the space of extremals would vanish. However, for a deformation with compact support of ϕ, and if we assume L local, then $\delta L(\phi)$ is a density with compact support, which can be integrated. One says that ϕ_0 is *extremal* if $\int \delta L = 0$ for any such deformation of ϕ_0. We now make this more precise. One considers families of fields $\phi[u]$, with $\phi[0] = \phi_0$, and with $\phi[u]$ independent of u in the complement

of a compact region of M. The density $\frac{d}{du}L(\phi[u])\big|_{u=0}$ then has compact support. The extremality condition is that its integral should vanish, for any $\phi[u]$. It is suggestive, if abusive, to write the condition $\frac{d}{du}\int_M L(\phi[u]) = 0$.

Let us suppose that the fields ϕ are sections of a bundle E over M. For a deformation $\phi[u]$ of ϕ_0, the density $\frac{d}{du}L(\phi[u])$ at $u = 0$ will depend only on the field of vertical vectors $\frac{d}{du}\phi$ on E, along ϕ_0, which is a section $\hat{\xi}$ of $\phi_0^*T(E/M)$. The $\mathbb{R}$-linear form

$$\hat{\xi} \longmapsto \int_M \frac{d}{du}L(\phi) \tag{2.19}$$

on the sections with compact support of $\phi_0^*T(E/M)$ can be uniquely written as

$$\int_M \underline{D}L(\hat{\xi}) \tag{2.20}$$

for $\underline{D}L$ a $(\phi_0^*T(E/M))^*$-valued density. The unicity of $\underline{D}L$ is clear, and its existence is a local question. It is proved by the usual integration by parts argument leading to Euler-Lagrange equations. Extremality of ϕ_0 means that $\underline{D}L = 0$ at ϕ_0 (Euler-Lagrange equations). We note that $\underline{D}L$ is in $\Omega^{1,|0|}$, with a value at $m \in M$ depending only on some jet of ϕ at m, and on the value at m of $\hat{\xi}$.

This formalism does not hold if, as is typical of superspace formulations of supersymmetric theories, the fields ϕ are sections of a bundle E subjected to constraints. The problem is that $\hat{\xi}$ is no longer arbitrary in $\phi_0^*T(E/M)$. Let $\mathcal{M}$ be the space of extremals. We now explain that if spacetime M is time $\times$ space, or at least if there is a suitable notion of space-like hypersurface, then the construction on $\mathcal{M}$ of a closed 2-form—on the model of Chapter 1—uses only conditions on L which hold for such constrained superspace formulations. The crucial condition is that the constraints allow ϕ to be deformed independently in disjoint regions of M. (Formally: the sheaf of fields, a subsheaf of the sheaf of C^∞-sections of E/M, is soft). In practice, for superspace formulations, this means that superfields can be described in components.

Let H_1 and H_2 be space-like hypersurfaces with, to simplify the picture, H_1 before H_2. In superspace formulations, H_1 and H_2 should be codimension $1|0$ submanifolds of M. We consider

$$\int_{H_1}^{H_2} L(\phi)\ , \tag{2.21}$$

which we at first assume to converge. If ϕ_0 is extremal, and if $\phi[u]$ deforms ϕ_0 ($\phi[0] = \phi_0$), then

$$\frac{d}{du}\int_{H_1}^{H_2} L(\phi[u]) \tag{2.22}$$

will in general not vanish at $u = 0$, except if the support of the deformation is between H_1 and H_2. For deformations with compact support, it will be the difference of boundary terms attached to H_1 and H_2:

$$\frac{d}{du}\int_{H_1}^{H_2} L(\phi[u]) = \Gamma_2(\hat{\xi}) - \Gamma_1(\hat{\xi})\ . \tag{2.23}$$

In this formula, Γ_i is a 1-form on $\mathcal{F}$, defined only on $\mathcal{M}$, whose value at a tangent vector $\hat{\xi}$ at an extremal ϕ_0 depends only on some jet of $\hat{\xi}$ along H_i. We now restrict those 1-forms to (the tangent bundle of) $\mathcal{M}$. Equation (2.23) means that, with δ the de Rham differential on $\mathcal{M}$,

$$\delta \int_{H_1}^{H_2} L = \Gamma_2 - \Gamma_1 \ , \tag{2.24}$$

so that the exterior derivative $\delta\Gamma_i$ is independent of i. One defines the canonical closed 2-form on $\mathcal{M}$ by

$$\Omega := \delta\Gamma_i \ . \tag{2.25}$$

Remark 2.26. If the integral of $L(\phi)$ on the whole of $\mathcal{M}$ did make sense, this construction would collapse: $\Gamma_2(\phi)$ would be δ of the integration of $L(\phi)$ from the infinite past to H_2, giving $\Omega = \delta\Gamma_2 = 0$.

Remark 2.27. Instead of assuming $\int_{H_1}^{H_2}$ to exist, one may consider only fields which coincide with a fixed extremal ϕ_* at spatial infinity. The given construction then defines Γ_i for tangent vectors to $\mathcal{M}$ whose support intersects the region of spacetime in between any two space-like hypersurfaces in a compact set.

Remark 2.28. As in Chapter 1, the construction gives an $\mathbb{R}$-torsor T with connection ∇ on $\mathcal{M}$, whose curvature is Ω. The choice of a space-like hypersurface H trivializes T, and the trivializations given by H_1 and H_2 differ by $\int_{H_1}^{H_2} L(\phi)$.

Remark 2.29. Adding to the lagrangian density an exact term $d\alpha$, for α in $\Omega^{0,|-1|}_{\mathrm{loc}}$, does not change the space of extremals. For H a space-like hypersurface, it changes the corresponding 1-form Γ on $\mathcal{M}$ by

$$\hat{\xi} \longmapsto \int_H \frac{d}{du}\alpha(\phi[u]) \quad \text{at} \quad u = 0 \ . \tag{2.30}$$

The torsor with connection (T, ∇) for L, and the one (T', ∇') for $L + d\alpha$, can be identified, by sending the trivialization 0_H of T corresponding to a space-like hypersurface H to the trivialization $0'_H - \int_H \alpha$ of T'. In Chapter 1 (particle in an electromagnetic field), we saw one example where a change of gauge changes $L(\phi)$ by an exact term, and where the torsor T of Remark 2.28 is defined, but where a trivialization of T is given by the choice of H and of a gauge along H.

Remark 2.31. We now give a variant of the definition of Ω, to make clearer on what it depends. Let $\phi[u, v]$ be a 2-parameter family of extremals. At $\phi_0 = \phi[0, 0]$ in $\mathcal{M}$, we have the two tangent vectors $\hat{\xi}_1 = \frac{\partial}{\partial u}\phi[u, v]$ at $(0, 0)$ and $\hat{\xi}_2 = \frac{\partial}{\partial u}\phi[u, v]$ at $(0, 0)$. What is $\Omega(\hat{\xi}_1, \hat{\xi}_2)$? Let H be a space-like hypersurface. It has a past and a future side, and we identify a neighborhood of H with $H \times (-1, 1)$, with $h \in H$ corresponding to $(h, 0)$ and the future side of H corresponding to $H \times [0, 1)$. Let $\phi_1[u, v]$ be a family of fields, deforming $\phi[u, 0]$, agreeing with $\phi[u, v]$ in the future, and with $\phi[u, 0]$ in the past: for some a, b with $-1 < a < b < 1$,

$$\begin{aligned} &\phi_1[u, 0] = \phi[u, 0]; \\ &\phi_1[u, v] = \phi[u, v], \qquad \text{at } (h, t) \text{ for } t > b; \\ &\phi_1[u, v] = \phi[u, 0], \qquad \text{at } (h, t) \text{ for } t < a. \end{aligned} \tag{2.32}$$

Let $\phi_2[u,v]$ be a similar family, with the roles of u and v interchanged. Then

$$\Omega(\hat{\xi}_1,\hat{\xi}_2)=\int \partial_u\partial_v L(\phi_1)-\partial_u\partial_v L(\phi_2) \quad \text{at} \quad (u,v)=(0,0)\ . \tag{2.33}$$

The integrand vanishes for $t>b$ (where $L(\phi_1)=L(\phi_2)$) and for $t<a$ (where $\partial_v L(\phi_1)=\partial_u L(\phi_2)=0$). If $\phi[u,v]=\phi_0$ at (h,t) for h outside a compact of H, one can take $\phi=\phi_1=\phi_2$ outside of such a compact, and the integrand has then compact support.

We now relate the definition of Ω to the previous one. We first consider a one-parameter family $\phi[u]$ of extremals, with $\phi[0]=\phi_0$. Let $\phi_1[u]$ agree with $\phi[u]$ for $t>b$, and with ϕ_0 for $t<a$ $(-1<a<b<0)$. We have

$$\Gamma_H(\hat{\xi})=\int_{t=-1}^{t=0}\frac{d}{du}L(\phi_1[u]) \quad \text{at} \quad u=0, \tag{2.34}$$

with Γ_H the boundary term attached to H. For commuting vector fields, we have $d\alpha(X,Y)=X\alpha(Y)-Y\alpha(X)$. Applying this to the (u,v) plane, and to ∂_u, ∂_v to compute $d\Gamma_H$, we get the formula given for $\Omega(\hat{\xi}_1,\hat{\xi}_2)$.

The construction of the closed 2-form Ω on the space $\mathcal{M}$ of extremals is reassuring, but not particularly useful. What is more interesting are analogs local on M of Ω and for Ω nondegenerate of functions on $\mathcal{M}$ corresponding to infinitesimal symmetries. Such local analogs will continue to make sense after a Wick rotation, in Euclidean field theory.

We will see in §2.4 that if the space of fields is the space of sections of a bundle E over M, and if the value of $L(\phi)$ at $m\in M$ depends only on the first jet of ϕ at m, there is a unique

$$\gamma\in\Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F}\times M) \tag{2.35}$$

such that

$$\underline{D}L=\delta L+d\gamma \tag{2.36}$$

and that γ is "linear over functions": at ϕ, for $\hat{\xi}$ a tangent vector to $\mathcal{F}$ at ϕ, identified with a section of $\phi^*T(E/M)$, the value of the form $\gamma(\hat{\xi})$ at $m\in M$ should depend only on $\hat{\xi}$ at m. More generally, we have the following

Definition 2.37. A form $\beta\in\Omega^{1,|\bullet|}_{\mathrm{loc}}(\mathcal{F}\times M)$ is *linear over functions* at (ϕ,m) if for every $\hat{\xi}\in T_\phi\mathcal{F}$ and every function f on M,

$$\beta_{(\phi,m)}(f\hat{\xi})=f(m)\beta_{(\phi,m)}(\hat{\xi}). \tag{2.38}$$

This γ is a local counterpart to the 1-form Γ_H on $\mathcal{M}$ attached to a space-like hypersurface H. Indeed, Γ_H is the integral of γ on H. Taking an exterior derivative in the $\mathcal{F}$ direction, we obtain a local analog of the 2-form Ω: the 2-form Ω is the integral on H of $\omega:=\delta\gamma$ in $\Omega^{2,|-1|}_{\mathrm{loc}}(\mathcal{F}\times M)$. This integral is a 2-form on the whole of $\mathcal{F}$, but it is independent of H only as a 2-form on $\mathcal{M}$.

In the case of gauge theories, even if L and γ are invariant by gauge transformations, one should not expect γ to vanish in the direction of the gauge orbits.

For more general Lagrangians, to localize the construction of Ω, one should *choose* γ in $\Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F}\times M)$ such that $\underline{D}L=\delta L+d\gamma$. We call γ the *variational 1-form*. The pair (L,γ) defines a field theory.

Definition 2.39. $\mathcal{L} = L + \gamma$ defines a *classical (lagrangian) field theory* if

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma \tag{2.40}$$

is linear over functions.

Equation (2.40) is a simple rewriting of (2.36).

Since $\underline{D}L$ is uniquely determined by L, and is local, it follows from Theorem 2.15 that the difference between any two choices for the variational 1-form γ is d-exact. A d-exact change in γ leads to a d-exact change in the local symplectic form ω, defined below in (2.44). In the Hamiltonian situation, where spacetime is time $\times$ space, the symplectic form Ω on the space of classical solutions and the $\mathbb{R}$-torsor with connection whose curvature is Ω do not depend on the choice of γ.

For $K \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ we can form a new lagrangian $L + dK$. Then $\gamma + \delta K$ is a valid choice of variational 1-form, and the new total lagrangian is $\mathcal{L} + DK$. In this case the local symplectic form ω is unchanged, as are the equations of motion $(D\mathcal{L})^{1,|0|}$. In the Hamiltonian situation the global symplectic form Ω is unchanged, and there is an isomorphism of the $\mathbb{R}$-torsors for L and $L + dK$ constructed from K.

If γ is linear over functions, in the sense of Definition 2.37, we write

$$\gamma = \delta\phi \wedge \pi. \tag{2.41}$$

In this formula ϕ is a section of E over $\mathcal{F} \times M$, $\delta\phi$ is the corresponding local $(1,0)$-form with values in $\phi^*T(E/M)$, and π is a local $(0,|-1|)$-form with values in $\phi^*T^*(E/M)$. If ϕ is a real scalar field $\phi\colon M \to \mathbb{R}$, then $\pi \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ is called the *conjugate momentum* (density) to ϕ. Roughly speaking, at least locally we can choose a coordinate system (on the fibers of E) to write any set of fields as a collection of real scalar fields and so obtain conjugate momenta.

As explained above, we have the

Definition 2.42. The space $\mathcal{M} \subset \mathcal{F}$ of *classical solutions* is the space of ϕ such that the restriction of $(D\mathcal{L})^{1,|0|}$ to $\{\phi\} \times M$ vanishes:

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma = \underline{D}L = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.43}$$

Physicists refer to the submanifold $\mathcal{M}$ (or $\mathcal{M} \times M$) as *on-shell*; its complement in $\mathcal{F}$ (or $\mathcal{F} \times M$) is *off-shell*, though that term is usually used to describe all of $\mathcal{F}$ (or $\mathcal{F} \times M$). We let $\overline{\mathcal{M}}$ denote the quotient of $\mathcal{M}$ by gauge symmetries; as with $\overline{\mathcal{F}}$ we treat $\overline{\mathcal{M}}$ as a manifold. In some contexts $\overline{\mathcal{M}}$ is called the *moduli space*.

Define

$$\omega := \delta\gamma, \qquad \text{in } \Omega^{2,|-1|}_{\text{loc}}(\mathcal{F} \times M). \tag{2.44}$$

Restricted to the space of classical solutions we have

$$\omega = D\mathcal{L} \qquad \text{on } \mathcal{M} \times M, \tag{2.45}$$

and so

$$D\omega = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.46}$$

We call ω the *local symplectic form*, since in the Hamiltonian situation its integral over a spacelike hypersurface is the global symplectic form on the space of classical solutions. ω also has units of action:

$$[\omega] = ML^2T^{-1}. \tag{2.47}$$

The picture of our on-shell data is:

$$\begin{array}{c|ccccc} & 0 & & 1 & & 2 \\ \hline |0| & L & \to & 0 & & \\ & & & \uparrow & & \\ |-1| & & & \gamma & \to & \omega \end{array} \qquad \mathcal{M}$$

$$M$$

Elements of $\Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ are called *currents*. A current j is *conserved* if

$$dj = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.48}$$

Below we discuss symmetries and show how they lead to conserved (Noether) currents. Such currents are local counterparts of generating functions, on the symplectic manifold $\mathcal{M}$ of extremal fields, of infinitesimal symplectic transformations. More precisely, their integral on a spacelike hypersurface, to the extent that it makes sense, is such a generating function. The current being conserved, its integral is invariant under suitable deformations of the spacelike hypersurface. Another source of conserved currents is topology. Namely, in some theories the space of fields has nontrivial topology and topological invariants are constructed by integrating topological currents. Note that the units of a current are not fixed; see (2.3).

§2.4. First order lagrangians

Let $\pi\colon E \to M$ be a fiber bundle. For s_1 and s_2 two local sections of E/M defined in a neighborhood of $m \in M$, the relation "in local coordinates, the derivatives of order $\le k$ at m of s_1 and s_2 coincide" is an equivalence relation. The equivalence classes are called the k-jets of sections of E/M at m. They form the fiber at m of a new fiber bundle $J^k(E) \to M$. Local coordinate systems on $J^k(E)$ can be obtained as follows. One fixes near $m \in M$ a local coordinate system $\{x^\mu\}$ on M, a trivialization $E = M \times F$ of E, and local coordinates $\{y^a\}$ on F. Sections of $E \to M$ near m are given by functions $\phi^a(x^\mu)$ from M to F. The x^μ and $\partial^n\phi^a$ with $|n| \le k$ form a local coordinate system.

A local section ϕ of $E \to M$ defines a section $j^k(\phi)$ of $J^k(E) \to M$, with value at $m \in M$ the k-jet of ϕ at m.

Definition 2.49. A lagrangian $L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M)$ *depends only on the k-jet of the fields* if there exists a morphism $\ell\colon J^k(E) \to \mathrm{Dens}(M)$ of bundles over M such that

$$L(\phi) = \ell\big(j^k(\phi)\big). \tag{2.50}$$

Example 2.51. Most "fundamental" lagrangians L depend only on the 1-jet of the fields. (A notable exception occurs in theories of gravity, since the curvature of a metric depends on the 2-jet of the metric.) A simple example to keep in mind is a theory of a scalar field $\phi\colon M \to X$ with values in a Riemannian manifold X. Then $E = M \times X$ and the fiber of $J^1E \to E$ at (m, x) is $\mathrm{Hom}(T_mM, T_xX)$. Let $V\colon X \to \mathbb{R}$ be a (potential energy) function. The theory of the scalar field in the potential V has

$$\ell(m, x, T) = \frac{1}{2}|T|^2 - V(x), \tag{2.52}$$

where $T \in \mathrm{Hom}(T_mM, T_xX)$.

We now assume that L is a lagrangian density depending only on the first jet of the field ϕ. The condition $\underline{D}L = \delta L + d\gamma$ means, in integrated form, that for U a compact integration domain with smooth boundary, one has, for $\xi \in \Gamma\big(M, \phi^*T(E/M)\big)$ a tangent vector of $\mathcal{F}$ at ϕ,

$$\xi \int_U L(\phi) = \int_U \iota(\xi)\underline{D}L + \int_{\partial U} \iota(\xi)\gamma. \tag{2.53}$$

The last sign comes from the fact that $\iota(\xi)d\gamma = -d\iota(\xi)\gamma$. If γ is assumed to satisfy $\iota(f\xi)\gamma = f\iota(\xi)\gamma$ for f a function on M, i.e., if $\gamma(\phi, \xi)$ at m depends only on the value of ξ at m, this formula makes it clear that γ is unique. Indeed, the formula determines $\int_{\partial U} f\iota(\xi)\gamma$ for any U and f, hence $\iota(\xi)\gamma$ for any ξ. As any ξ can be decomposed into sections of $\Gamma\big(M, \phi^*T(E/M)\big)$ with small support, existence is a local question on E and M. Locally on M, one can trivialize E as $M \times F$. Let us choose local coordinate systems $\{x^\mu\}$ on M and $\{y^a\}$ on F. This gives a local coordinate system $\{x^\mu, y^a, y^a_\mu\}$ on $J^1(E)$. By assumption, $L = \ell(j^1(\phi))$ with

$$\ell = \ell(x^\mu, y^a, y^a_\mu)|d^nx|. \tag{2.54}$$

The $|-1|$-form $\iota(\xi)\gamma$ on M is then given by the standard integration by parts:

$$\iota(\xi)\gamma = \sum \frac{\partial \ell}{\partial y^a_\mu}\xi^a\iota(\partial_\mu)|d^nx|. \tag{2.55}$$

§2.5. Hamiltonian theory

Suppose that spacetime is $M = M^1 \times N$ for some manifold N, where M^1 is affine one-dimensional Minkowski space, i.e., the affine real line with its standard metric. We view M^1 as time and N as space. We also assume that the fiber bundle $E \to M^1 \times N$ is pulled back from a fixed bundle on N, or equivalently that time translation has been lifted to E. In this case we integrate the local symplectic form ω to obtain a closed 2-form Ω on the space of classical solutions $\mathcal{M}$:

$$\Omega = \int_{\{t\}\times N} \omega \ \in \Omega^2(\mathcal{M}). \tag{2.56}$$

Typically N is noncompact and so to ensure convergence we only evaluate Ω on tangent vectors to $\mathcal{M}$ with compact support in spatial directions, or at least with

sufficient decay at spatial infinity. The hyperbolicity of the classical equations of motion implies finite propagation speed of the classical solutions, and so the decay conditions are uniform in time. By (2.46) the right hand side of (2.56) is independent of $t \in M^1$, and also Ω is a closed 2-form on $\mathcal{M}$.

A generalized infinitesimal local symmetry is a construction as follows (see Definition 2.93). One gives: (i) a vector bundle V over M; (ii) for each section ζ of V, a vector field X_ζ on the space $\mathcal{F}$ of fields and a $(0, |-1|)$-form α_ζ such that $\mathrm{Lie}(X_\zeta)L = d\alpha_\zeta$. One requires that X and α be local: the value at $m \in M$ of $X_\zeta(\phi) \in \Gamma(M, \phi^*T(E/M))$ should depend only on the k-jet of ϕ and ζ at m, and similarly for α. One also requires X and α to be additive in ζ. The basic example is the algebra of infinitesimal gauge symmetries, for which V is the adjoint bundle of a principal bundle.

In the presence of such symmetries, one cannot hope for the closed 2-form Ω on $\mathcal{M}$ to be nondegenerate. Indeed, the X_ζ are tangent to $\mathcal{M}$, because X_ζ is a generalized symmetry (see Definition 2.71). We claim that the X_ζ preserve Ω and are in the kernel of Ω, at least for ζ with compact support so that everything is well-defined. Indeed, decomposing ζ one can assume it has a small support. Let us choose t such that $\{t\} \times N$ does not meet the support of ζ. If we compute Ω using $\{t\} \times N$, then Ω does not see ζ and the claim follows. Therefore, Ω is the pullback of a 2-form $\overline{\Omega}$ on the quotient $\overline{\mathcal{M}}$ of $\mathcal{M}$ by the local symmetries (or on the quotient by any subgroup). The best one can hope for, then, is that $\overline{\Omega}$ is a symplectic structure.

If $j \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a conserved current then the associated *charge* Q_j is

$$Q_j = \int_{\{t\}\times N} j. \tag{2.57}$$

If ϕ is in $\mathcal{M}$, then since $dj = 0$ the right hand side is independent of t. This is a global conservation law. Local conservation laws are obtained by considering a domain $U \subset N$. For simplicity assume the closure of U is compact with smooth boundary ∂U. Let

$$q_t = \int_{\{t\}\times U} j \tag{2.58}$$

be the total charge contained in U at time t. Write

$$j = dt \wedge j_1 + j_2, \tag{2.59}$$

where j_1, j_2 do not involve dt. Stokes' theorem applied to integration over the fibers of the projection $M^1 \times U \to M^1$ implies

$$\frac{dq_t}{dt} + \int_{\{t\}\times \partial U} j_1 = 0. \tag{2.60}$$

This says that the rate of change of the total charge in U is minus the flux through the boundary. The units of a charge are the same as that of the current.

In theories with local symmetry the global charge is gauge invariant since we require the local current to be gauge invariant up to an exact form.

A field ϕ on $M = M^1 \times N$ is *static* if its time derivative $\mathrm{Lie}(\partial/\partial t)\phi$ vanishes. (Recall that ϕ is a section of $E \to M^1 \times N$ and we have fixed a lift of $\partial/\partial t$ to E.) When there are gauge symmetries we say that ϕ is static if locally on N there is a gauge in which $\mathrm{Lie}(\partial/\partial t)\phi = 0$. For a p-form field α such a local gauge exists if and only if the gauge-invariant field $d\alpha$ is invariant under t. The same assertion holds for abelian connections α, where $d\alpha$ is replaced by the curvature. The proof of this assertion is straightforward.[11] The local gauge in which α is static is unique up to static local gauge transformations. There does not seem to be an analogous criterion for static nonabelian connections in terms of the curvature.

Let $\mathcal{F}_N$ denote the space of static fields and $\overline{\mathcal{F}}_N$ the quotient by static gauge symmetries. In §2.10 we define the energy of a static field and so a subspace $\mathcal{FE}_N$ of static fields of finite energy.

§2.6. Symmetries and Noether's theorem

We assume that the space of fields $\mathcal{F}$ is the space of sections of a fiber bundle E over M, and that L in $\Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a lagrangian density. An automorphism $g\colon \mathcal{F} \to \mathcal{F}$ of $\mathcal{F}$ is *local* if for some k the value of $g(\phi)$ at $m \in M$ depends only on the k-jet of ϕ at m, and if the same condition holds for the inverse g^{-1} of g. A *generalized symmetry* of L is a local automorphism g of $\mathcal{F}$, given with α in $\Omega^{0,|-1|}_{\mathrm{loc}}$ such that

$$L(g(\phi)) - L(\phi) = d\alpha(\phi)\ . \tag{2.66}$$

By the locality assumption, deformations with compact support of ϕ and $g\phi$ correspond to each other, and by (2.66), the integral of the variation of $L(\phi)$ is preserved. It follows that g preserves the space $\mathcal{M}$ of extremals. As in §2.3, the lagrangian density L gives rise to an $\mathbb{R}$-torsor with connection (T, ∇) on $\mathcal{M}$. A generalized symmetry acts on (T, ∇) as follows. Let H be a space-like hypersurface. It defines a trivialization 0_H of T. We map 0_H at ϕ to

$$g(0_H \text{ at } \phi) = \left(0_H - \int_H \alpha(\phi)\right) \text{ at } g(\phi)\ . \tag{2.67}$$

---

[11] Write

$$\begin{aligned} &\alpha = dt \wedge A(t) + B(t), \qquad A(t) \in \Omega^{p-1}_N, \quad B(t) \in \Omega^p_N, \\ &\beta = dt \wedge P(t) + Q(t), \qquad P(t) \in \Omega^{p-2}_U, \quad Q(t) \in \Omega^{p-1}_U \end{aligned} \tag{2.61}$$

on some small open set $U \subset N$. Then $\alpha + d\beta$ is time independent if and only if

$$\dot{A} - d\dot{P} + \ddot{Q} = 0 \tag{2.62}$$

$$\dot{B} + d\dot{Q} = 0, \tag{2.63}$$

whereas $d\alpha$ is time independent if and only if

$$\ddot{B} - d\dot{A} = 0 \tag{2.64}$$

$$d\dot{B} = 0. \tag{2.65}$$

The integrability condition for (2.63) is (2.65), so we can solve for $\dot{Q}$. Then we solve (2.62) for $\dot{P}$; the integrability condition is (2.64). (A slightly different argument is required for $p = 1$ since in that case $P \equiv 0$.)

The resulting map $g\colon T \to T$ is independent of the choice of H. Indeed, if H_1 and H_2 are two space-like hypersurfaces, with H_1 before H_2, we have at ϕ

$$0_{H_2} - 0_{H_1} = \int_{H_1}^{H_2} L(\phi)\ . \tag{2.68}$$

As $\int_{H_1}^{H_2} L(g\phi) = \int_{H_1}^{H_2} L(\phi) + [\int \alpha(\phi)]_{H_1}^{H_2}$, this is compatible with 0_{H_1} and 0_{H_2} at ϕ to be mapped, respectively, to

$$0_{H_1} - \int_{H_1} \alpha(\phi) \quad \text{and} \quad 0_{H_2} - \int_{H_2} \alpha(\phi) \tag{2.69}$$

at $g(\phi)$. Indeed, at $g(\phi)$,

(2.70)
$$\left(0_{H_2} - \int_{H_2} \alpha(\phi)\right) - \left(0_{H_1} - \int_{H_1} \alpha(\phi)\right) = \int_{H_1}^{H_2} L(g(\phi)) - \left[\int \alpha(\phi)\right]_{H_1}^{H_2}$$
$$= 0_{H_2} - 0_{H_1} \text{ at } \phi.$$

We now consider the infinitesimal analog of this construction. A vector field $\hat{\xi}$ on $\mathcal{F}$ is *local* if for some k the value of $\hat{\xi}_\phi \in \Gamma\big(M, \phi^* T(E/M)\big)$ at $m \in M$ depends only on the k-jet of ϕ at m. In other words, if q is the projection from $J^k(E)$ to E, then $\hat{\xi}$ is given by a section of $q^*T(E/M)$ on $J^k(E)$.

Definition 2.71. A *generalized infinitesimal symmetry* of L is a local vector field $\hat{\xi}$ on $\mathcal{F}$, given with $\alpha_{\hat{\xi}}$ in $\Omega^{0,|-1|}_{\mathrm{loc}}$ such that

$$\mathrm{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M. \tag{2.72}$$

Here, $\mathrm{Lie}(\hat{\xi})L$ is simply $\iota(\hat{\xi})\delta L$. Such a symmetry should not be expected to integrate to a generalized (local) symmetry, as defined above. For instance, the vector field on $\mathcal{F}$ corresponding to an infinitesimal time translation is local. A finite time translation on $\mathcal{F}$ is not. To make it local, one would need to act both on $\mathcal{F}$ and M, as we will do later.

One can, however, repeat the previous construction, if one views a generalized infinitesimal symmetry as a generalized symmetry (g, α) depending on an infinitesimal parameter ε, with $(g(0), \alpha(0)) = (\text{Identity}, 0)$. Formally, this means working over $\mathrm{Spec}(\mathbb{R}[\varepsilon]/(\varepsilon^2))$. Doing so, we get a lifting of $g(\varepsilon)$ to T, i.e., a lifting of the vector field $\hat{\xi}$ on $\mathcal{M}$ to T, respecting the connection. Such a lifting corresponds to a function $Q_{\hat{\xi}}$ on $\mathcal{M}$, such that

$$dQ_{\hat{\xi}} = -\iota(\hat{\xi})\Omega\ , \tag{2.73}$$

for Ω the curvature form of T. The function $Q_{\hat{\xi}}$ is the difference between the lifting of $\hat{\xi}$ and its horizontal lifting. For any space-like hypersurface H

$$Q_{\hat{\xi}} = \Gamma_H(\hat{\xi}) - \int_H \alpha_{\hat{\xi}} = \int_H \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}}\ . \tag{2.74}$$

The *Noether current* of $(\hat{\xi}, \alpha_{\hat{\xi}})$ is the local $(0, |-1|)$-form

$$j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} \ . \tag{2.75}$$

By construction, its integral on a space-like hypersurface H is the charge $Q_{\hat{\xi}}$, giving the lifting of the vector field $\hat{\xi}$ on $\mathcal{M}$ to (T, ∇). For ϕ in $\mathcal{M}$, $\int_H j_{\hat{\xi}}$ is hence independent of H. It follows that $j_{\hat{\xi}}$ is a conserved current.

To prove that the Noether current $j_{\hat{\xi}}$ is conserved, i.e. that $j_{\hat{\xi}}(\phi)$ is closed when ϕ is extremal, we have used a global argument, relying on a notion of "space-like hypersurface" H. This leads to extraneous convergence problems. They could be avoided by using instead hypersurfaces homologous to zero, in a small neighborhood of a point. The corresponding line bundle on $\mathcal{M}$ has trivial curvature, but this does not spoil the argument. Here is another way out, which gives as well a local counterpart of (2.73).

Proposition 2.76. *Suppose $(\hat{\xi}, \alpha_{\hat{\xi}})$ is a generalized infinitesimal symmetry of $\mathcal{L}$. Then, for some $\beta_{\hat{\xi}}$ in $\Omega^{1,|-2|}_{\mathrm{loc}}(\mathcal{F} \times M)$, the identity*

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M \tag{2.77}$$

holds on shell. The Noether current $j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}}$ is conserved,

$$dj_{\hat{\xi}} = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.78}$$

and

$$\delta j_{\hat{\xi}} = -\iota(\hat{\xi})\omega + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M. \tag{2.79}$$

Remark 2.80. Both (2.77) and (2.79) hold as equalities, on $\mathcal{M} \times M$, of $(1, |-1|)$-forms on $\mathcal{F} \times M$: at ϕ in $\mathcal{M}$, they give rise to an equality of $|-1|$-forms on M when evaluated against any tangent vector η of $\mathcal{F}$ at ϕ, whether or not it is tangent to $\mathcal{M}$.

The formula (2.79) is the promised local counterpart of (2.73). We summarize (2.72) and (2.77) in the diagrams:

	0
$\|0\|$	$\mathrm{Lie}(\hat{\xi})L$
	$\uparrow$
$\|-1\|$	$\alpha_{\hat{\xi}}$

$\mathcal{F}$

M

	0	1
$\|0\|$	$\mathrm{Lie}(\hat{\xi})L$	
	$\uparrow$	
$\|-1\|$	$\alpha_{\hat{\xi}}$	$\to \ \mathrm{Lie}(\hat{\xi})\gamma$
		$\uparrow$
$\|-2\|$		$\beta_{\hat{\xi}}$

$\mathcal{M}$

M

Proof. The vector field $\hat{\xi}$ is tangent to $\mathcal{M}$. Since on $\mathcal{M} \times M$ we have $\underline{D}\, L = \delta L + d\gamma = 0$, we also have on $\mathcal{M} \times M$

$$\mathrm{Lie}(\hat{\xi})\underline{D}\, L = \delta\, \mathrm{Lie}(\hat{\xi})L + d\, \mathrm{Lie}(\hat{\xi})\gamma = 0, \tag{2.81}$$

in other words,

$$d(-\delta\alpha_{\hat{\xi}} + \mathrm{Lie}(\hat{\xi})\gamma) = 0. \tag{2.82}$$

If ϕ is in $\mathcal{M}$, then this identity holds when evaluated on any tangent vector $\eta \in \Gamma(M, \phi^* T(E/M))$ of $\mathcal{F}$ at ϕ. As $\iota(\eta)(-\alpha_{\hat{\xi}} + \mathrm{Lie}(\hat{\xi})\gamma)$ is local in η, it follows from Takens that for some $\beta_{\hat{\xi}}(\phi; \eta)$, local in η, one has

$$-\delta\alpha_{\hat{\xi}}(\phi) + \mathrm{Lie}(\hat{\xi})\gamma(\phi) = d\beta_{\hat{\xi}}\ . \tag{2.83}$$

Suppose we have chosen locally on M coordinate systems $U_i \hookrightarrow \mathbb{R}^n$, trivializations of E as $F \times U_i$ and local coordinate systems on F. In terms of those, and of a partition of unity attached to the resulting covering of E, $\beta_{\hat{\xi}}$ is given by Takens by explicit local formulas. Those formulas define $\beta_{\hat{\xi}}$ on $\mathcal{F} \times M$, with (2.77) holding on $\mathcal{M} \times M$. *A fortiori*, (2.77) holds as an identity of $(1, |-1|)$-forms on $\mathcal{M} \times M$, i.e., when applied to vectors tangent to $\mathcal{M} \times M$ only. Applying δ to it, we obtain that $\hat{\xi}$ preserves the $(2, |-1|)$-form ω on $\mathcal{M} \times M$ up to an exact derivative:

$$\mathrm{Lie}(\hat{\xi})\omega = d(-\delta\beta_{\hat{\xi}})\ . \tag{2.84}$$

As $\hat{\xi}$ is a vector field on $\mathcal{F}$, Cartan's formula takes the form

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\iota(\hat{\xi})\gamma + \iota(\hat{\xi})\delta\gamma = \delta\iota(\hat{\xi})\gamma + \iota(\hat{\xi})\omega\ . \tag{2.85}$$

Plugging this in (2.77), we obtain (2.79) on $\mathcal{M} \times M$.

A vector field ξ on $\mathcal{F} \times M$ is said to be *decomposable and local* if it is the sum of a local vector field $\hat{\xi}$ on $\mathcal{F}$ and of a vector field η on M. For such a vector field, the Lie derivative $\mathrm{Lie}(\xi)$ preserves the bigrading of $\Omega^{\bullet,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M)$. We say that (ξ, α_ξ) is a *generalized infinitesimal symmetry* if α_ξ in $\Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is such that

$$\mathrm{Lie}(\xi)L = d\alpha_\xi\ . \tag{2.86}$$

The Lie derivative $\mathrm{Lie}(\xi)$ is the sum of $\mathrm{Lie}(\hat{\xi})$ and of $\mathrm{Lie}(\eta)$. As $\mathrm{Lie}(\eta)L = d\iota(\eta)L$, it hence follows from (2.86) that

$$\mathrm{Lie}(\hat{\xi})L = d(\alpha_\xi - \iota(\eta)L)\ : \tag{2.87}$$

the vector field $\hat{\xi}$ on $\mathcal{F}$, and $\alpha_{\hat{\xi}} := \alpha_\xi - \iota(\eta)L$, form a generalized infinitesimal symmetry. The corresponding Noether current is

$$j_\xi = \iota(\hat{\xi})\gamma + \iota(\eta)L - \alpha_\xi = \left[\iota(\xi)\mathcal{L}\right]^{0,|-1|} - \alpha_\xi\ . \tag{2.88}$$

In this formula, as before, $\mathcal{L} = L + \gamma$. We say that ξ is a *manifest* symmetry of L if $\mathrm{Lie}(\xi)L = 0$. For a manifest symmetry, one can take $\alpha_\xi = 0$, hence $\alpha_{\hat{\xi}} = -\iota(\eta)L$, and the formula (2.88) for the Noether current then simplifies to

$$j_\xi = \left[\iota(\xi)\mathcal{L}\right]^{0,|-1|} \qquad \text{(for a manifest symmetry of } L\text{)}. \tag{2.89}$$

We say that ξ is a *manifest* symmetry of $\mathcal{L} = L + \gamma$ if $\mathrm{Lie}(\xi)\mathcal{L} = 0$, i.e. if ξ preserves L and γ. As $\mathrm{Lie}(\xi) = \mathrm{Lie}(\hat{\xi}) + \mathrm{Lie}(\eta)$, this gives

$$\mathrm{Lie}(\hat{\xi})\gamma = -(d\iota(\eta)\gamma + \iota(\eta)d\gamma)\ . \tag{2.90}$$

On shell, $\iota(\eta)d\gamma = -\iota(\eta)\delta L = \delta\iota(\eta)L = -\delta\alpha_{\hat{\xi}}$, hence $\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d(-\iota(\eta)\gamma)$: the formula (2.77) holds for the generalized symmetry $(\hat{\xi}, \alpha_{\hat{\xi}})$ and $\beta_{\hat{\xi}} = -\iota(\eta)\gamma$. For this choice of $\beta_{\hat{\xi}}$, (2.79) becomes

$$\delta j_\xi = -\iota(\hat{\xi})\omega - d\iota(\eta)\gamma\ . \tag{2.91}$$

Proposition 2.92. *Suppose that L depends only on the first jet of the fields, that γ is the canonical variational form, that $\xi = \hat{\xi} + \eta$ is a manifest symmetry of L, and that it is induced by an infinitesimal automorphism of the bundle $E \to M$, that is, by a vector field X on E projecting to η on M. Then, ξ is a manifest symmetry of $\mathcal{L}$.*

Integrating ξ, this reduces to the statement that if an automorphism of the bundle $E \to M$ preserves L, it also preserves γ. As γ is canonically deduced from L, this is an application of transport of structures.

This argument seems to assume that X can be integrated, i.e. that $\exp(tX)e$ does not go to infinity in finite time. However, as the question is local, integrability is immaterial. One only needs to know that for $t \leq t_0$ and for suitable $E' \to M'$ with E' open in E and M' in M, $\exp(tX)$ and $\exp(t\eta)$ map E' and M' isomorphically to $E'_t \subset E$ and to $M'_t \subset M$.

Following Zuckerman, we give the formal definition of a local, or gauge, symmetry.

Definition 2.93. A *generalized infinitesimal local symmetry* of a theory $\mathcal{L} = L + \gamma$ is specified by a vector bundle $V \to M$ and linear maps

$$\begin{aligned} \zeta &\longmapsto X_\zeta \\ \zeta &\longmapsto \alpha_\zeta, \end{aligned} \tag{2.94}$$

where ζ is a section of V and X_ζ is a generalized infinitesimal symmetry of $\mathcal{L}$:

$$\mathrm{Lie}(X_\zeta)L = d\alpha_\zeta \qquad \text{on } \mathcal{F} \times M. \tag{2.95}$$

We require that the dependence of X_ζ and α_ζ on ζ be local.

Two examples: In gauge theory V is the adjoint bundle of a principal bundle and X_ζ acts by infinitesimal gauge transformations. In theories of gravity $V = TM$ and X_ζ acts by the Lie derivative of ζ.

Proposition 2.96. *The Noether current attached to a generalized infinitesimal local symmetry is d-exact on* $\mathcal{M} \times M$.

Proof. This follows directly from Theorem 2.16.

Assuming suitable decay at infinity, we see that the Noether charge associated to a local infinitesimal symmetry vanishes.

Finally, we introduce a bracket operation on Noether currents. There is a standard Poisson bracket of functions on $\overline{\mathcal{M}}$ in the Hamiltonian situation. If $j = j^\mu(x)\,\partial_\mu \otimes |d^n x|$ is a current, then we can view $j^\mu(x)$ for x fixed as defining a (singular) function on $\overline{\mathcal{M}}$ and so compute Poisson brackets of components of currents. This is a standard procedure in physics. Such computations go under the name *current algebra*. The bracket we introduce on *Noether* currents is defined in arbitrary field theories: we need not be in a Hamiltonian situation and there is no nondegeneracy assumption. The bracket (2.101) below is often simpler to compute in practice than the standard Poisson bracket of distributions on $\overline{\mathcal{M}}$.

Consider a theory $\mathcal{L} = L + \gamma$ on $\mathcal{F} \times M$ for some manifold M.

Definition 2.97. We call (j, ξ) a *Noether pair* if $j \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a current, ξ is a generalized infinitesimal symmetry of $\mathcal{L}$, and if

$$\mathrm{Lie}(\xi)L = d\alpha \qquad \text{on } \mathcal{F} \times M, \tag{2.98}$$

where

$$\alpha = \left[\iota(\xi)\mathcal{L}\right]^{0,|-1|} - j. \tag{2.99}$$

Let (j_i, ξ_i), $i = 1, 2$ be Noether pairs. We define the bracket

$$\left\{(j_1, \xi_1)\,,\,(j_2, \xi_2)\right\} = \left(\mathrm{Lie}(\xi_1)j_2 + C\,,\,[\xi_1, \xi_2]\right), \tag{2.100}$$

where

$$C = C(\xi_1, j_1; \xi_2, j_2) = \left[-\iota(\xi_2)\,\mathrm{Lie}(\xi_1)\mathcal{L} + \mathrm{Lie}(\xi_2)\big(\iota(\xi_1)\mathcal{L} - j_1\big)\right]^{0,|-1|}. \tag{2.101}$$

A straightforward computation shows that the right hand side of (2.100) is a Noether pair.

Remark 2.102. The formula for C may be changed by an exact term, though this particular choice is nice: for this choice the quantity α defined in (2.99) for the bracket is

$$\mathrm{Lie}(\xi_1)\alpha_2 - \mathrm{Lie}(\xi_2)\alpha_1, \tag{2.103}$$

where α_i are the corresponding quantities for ξ_i. The precise formula for C is not valuable; the important term in the bracket of currents is $\mathrm{Lie}(\xi_1)j_2$.

For manifest symmetries $C = 0$, and in all cases C is exact on-shell (after imposing the equations of motion). The bracket makes the space of Noether pairs into a Lie algebra (off-shell). For nonmanifest symmetries, this Lie algebra is typically infinite dimensional and becomes finite dimensional only after we impose the equations

of motion. Then it is nice to work modulo exact forms. Hence define an e-Noether pair $[j,\xi]$ to be an equivalence class of Noether pairs (j,ξ) where $(j_1,\xi) \sim (j_2,\xi)$ if $j_2 - j_1$ is d-exact. Thus j in $[j,\xi]$ is determined on-shell by ξ up to an element of the local cohomology group $H^{0,|-1|}_{\mathrm{loc}}(\mathcal{M} \times M)$. Under the bracket (2.100) the set of e-Noether pairs is a Lie algebra on-shell. If we restrict the symmetries ξ to lie in some Lie algebra $\mathfrak{h}$, then the corresponding set of e-Noether pairs is a central extension of $\mathfrak{h}$ by $H^{0,|-1|}_{\mathrm{loc}}(\mathcal{M} \times M)$.

Example 2.110 in §2.7 illustrates the difference between on-shell and off-shell symmetries. Note that the algebra of symmetries generated off-shell is infinite dimensional.

Suppose now that we are in the Hamiltonian situation $M = M^1 \times N$ and that $(\overline{\mathcal{M}}, \overline{\Omega})$ is symplectic. We work with Noether pairs (j,ξ) such that ξ projects to a vector field $\bar{\xi}$ on $\overline{\mathcal{M}}$ and j is gauge invariant. Then by (2.73) $\bar{\xi}$ is the symplectic gradient of the global charge $\overline{Q} \in \Omega^0(\overline{\mathcal{M}})$. The charge $\overline{Q}$ only depends on the equivalence class of (j,ξ) in the set of e-Noether pairs. Let $[j_i,\xi_i]$ be e-Noether pairs with global charges $\overline{Q}_i$. Set $[j,\xi] = \{[j_1,\xi_1], [j_2,\xi_2]\}$. Then the global charge $\overline{Q}$ associated to j is the Poisson bracket

$$\overline{Q} = \{\overline{Q}_1, \overline{Q}_2\}_{\overline{\mathcal{M}}}. \tag{2.104}$$

Equation (2.104) relates the bracket on Noether pairs to the Poisson bracket on $\overline{\mathcal{M}}$; it follows directly from (2.100), the definition of Poisson bracket, and Stokes' theorem.

Recall that if $\mathfrak{h}$ is a Lie algebra of Hamiltonian vector fields on a symplectic manifold, then the sub-Poisson algebra of functions whose symplectic gradients lie in $\mathfrak{h}$ is a central extension of $\mathfrak{h}$ by the space of locally constant functions. For a Lie algebra $\mathfrak{h}$ of symmetries of $\mathcal{L}$ as above, the set of e-Noether pairs $[j,\xi]$ with $\xi \in \mathfrak{h}$ is a local version of the global central extension defined by the charges.

§2.7. More on symmetries

This section is not strictly speaking needed for the theoretical development. Rather, we present examples of symmetries and Noether currents. These examples also illustrate the calculus we use to compute in function spaces.

Example 2.105. For a system of nonrelativistic point particles (1.13) time translation is a symmetry. As in the discussion of Example 2.110 it is manifest[12] for the vector field ξ in (2.120). Explicitly,[13]

$$\begin{aligned} \iota(\xi)\delta x &= -\dot{x}, \\ \iota(\xi)dt &= 1. \end{aligned} \tag{2.107}$$

[12] In Example 2.131 below we discuss time translation as a nonmanifest symmetry.

[13] Beware that usually the vector field $\hat{\xi}$ is omitted and one simply writes

$$\delta x = -\dot{x} \tag{2.106}$$

instead of the first equation in (2.107). In (2.106) 'δ' does not denote the differential on $\mathcal{F}$, but rather the vector field $\hat{\xi}$. Then signs are different when commuting d and δ.

Then we compute directly from (1.13) and (1.15) that $\mathrm{Lie}(\xi)\mathcal{L} = 0$. The associated Noether current

$$\begin{aligned} j_\xi &= \iota(\xi)\gamma + \iota(\xi)L \\ &= -|\dot{x}|^2 + \left(\frac{1}{2}|\dot{x}|^2 - V(x)\right) \\ &= -\left(\frac{1}{2}|\dot{x}|^2 + V(x)\right) \end{aligned} \tag{2.108}$$

is *minus* the *hamiltonian*. The Hamiltonian formulation of classical mechanics involves the hamiltonian function (2.108) and the symplectic form (1.16) on the symplectic manifold $\mathcal{M}$. Noether currents coincide with charges.

In this example the field is a map $M^1 \to X$, where M^1 is affine time and X is a Riemannian manifold. One should view 'x' in these formulas as the evaluation map (2.13)

$$x\colon \mathcal{F} \times M^1 \longrightarrow X, \tag{2.109}$$

though we often also denote a specific field $x\colon M^1 \to X$ with the same letter. The coordinate t on M^1 is a real-valued function on $\mathcal{F} \times M^1$ which is constant in the $\mathcal{F}$ direction.

Example 2.110. Let X be a real inner product space of dimension r with inner product $\langle \cdot, \cdot \rangle$, and for $x\colon M^1 \to X$ let

$$L = \left\{\frac{1}{2}|\dot{x}|^2 - \frac{1}{2}|x|^2\right\}|dt|. \tag{2.111}$$

This lagrangian describes a system of r identical harmonic oscillators. The equation of motion (1.14) is

$$\ddot{x} = -x. \tag{2.112}$$

(We omit the argument 't' from equations for readability.) Consider the vector field $\hat{\xi}$ on $\mathcal{F} = \mathrm{Map}(M^1, X)$ given by

$$\iota(\hat{\xi})\delta x = Ax + B\dot{x} \tag{2.113}$$

for some $A, B \in \mathrm{End}(X)$. Then

$$\iota(\hat{\xi})\delta\dot{x} = A\dot{x} + B\ddot{x}. \tag{2.114}$$

Assume that A is skew-symmetric and B is symmetric. We compute

$$\mathrm{Lie}(\hat{\xi})L = \iota(\hat{\xi})\delta L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M^1, \tag{2.115}$$

for

$$\alpha_{\hat{\xi}} = \frac{1}{2}\langle B\dot{x}, \dot{x}\rangle - \frac{1}{2}\langle Bx, x\rangle. \tag{2.116}$$

Thus $\hat{\xi}$ is a generalized infinitesimal symmetry for A skew-symmetric and B symmetric. Also,

$$\text{Lie}(\hat{\xi})\gamma = \iota(\hat{\xi})\delta\gamma + \delta\iota(\hat{\xi})\gamma = \langle B\dot{x}, \delta\dot{x}\rangle + \langle B\ddot{x}, \delta x\rangle. \tag{2.117}$$

From (2.116) we find

$$\delta\alpha_{\hat{\xi}} = \langle B\dot{x}, \delta\dot{x}\rangle - \langle Bx, \delta x\rangle, \tag{2.118}$$

so that on-shell—that is, after imposing the equation of motion (2.112)—we have

$$\text{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M^1, . \tag{2.119}$$

as implied by Proposition 2.76.

Only when $B = 0$ is the symmetry manifest. In that case it is the internal symmetry generated by an infinitesimal rotation in X. For $A = 0$ and $B = \text{id}$ we see that $\hat{\xi}$ is the vector field on $\mathcal{F}$ induced from the vector field $\partial/\partial t$ on M^1. We make this symmetry manifest by forming the vector field

$$\xi = -\hat{\xi} + \partial/\partial t; \tag{2.120}$$

then $\text{Lie}(\xi)\mathcal{L} = 0$. In other cases the symmetry is nonmanifest and is not induced by any symmetry of M^1. Also, note that on-shell the group of symmetries generated by (2.113), (2.114) is isomorphic to the unitary group $U(r)$, whereas off-shell it is infinite dimensional. We meet this same phenomenon in the more complicated situation of supersymmetric theories with no off-shell formulation, i.e., no auxiliary fields. Then the supersymmetry algebra only closes on-shell and the supersymmetry is not manifest.

The conserved quantity associated to a diagonal element of the Lie algebra $\mathfrak{u}(r)$ with one nonzero entry is the energy of one of the r oscillators.

Next, we give an example in field theory where β_ξ in Proposition 2.76 is nonzero.

Example 2.121. Let $M = M^2$ denote two-dimensional Minkowski spacetime with coordinates x^0, x^1. Here x^0 is the speed of light times a standard time coordinate. For convenience fix the orientation $\{x^0, x^1\}$ and use it to identify twisted forms with forms. Note that $*dx^0 = dx^1$ and $*dx^1 = dx^0$. Let $\mathcal{F} = \{\phi\colon M^{1,1} \to \mathbb{R}\}$ be the set of real scalar fields. The free (massless) lagrangian is

$$\begin{aligned} L &= \frac{1}{2} d\phi \wedge *d\phi \\ &= \frac{1}{2} |d\phi|^2 \, dx^0 \wedge dx^1 \\ &= \frac{1}{2} \{(\partial_0\phi)^2 - (\partial_1\phi)^2\} \, dx^0 \wedge dx^1. \end{aligned} \tag{2.122}$$

From this we derive

$$\gamma = \partial_0\phi \, \delta\phi \wedge dx^1 + \partial_1\phi \, \delta\phi \wedge dx^0 \tag{2.123}$$

$$\omega = \partial_0\delta\phi \wedge \delta\phi \wedge dx^1 + \partial_1\delta\phi \wedge \delta\phi \wedge dx^0 \tag{2.124}$$

and the equation of motion

$$\partial_0^2\phi - \partial_1^2\phi = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.125}$$

Consider infinitesimal translation in the x^0 (time) direction. It defines a vector field ξ on $\mathcal{F} \times M$ by

$$\begin{aligned} \iota(\xi)\delta\phi &= -\partial_0\phi \\ \iota(\xi)dx^0 &= 1 \\ \iota(\xi)dx^1 &= 0. \end{aligned} \tag{2.126}$$

Then a routine computation shows that ξ is a *manifest* symmetry: $\mathrm{Lie}(\xi)L = \mathrm{Lie}(\xi)\gamma = 0$ on $\mathcal{F} \times M$. (By Proposition 2.92 we need only check $\mathrm{Lie}(\xi)L = 0$.) But we can also regard infinitesimal time translation as a *nonmanifest* symmetry $\hat{\xi}$ by letting it operate only along $\mathcal{F}$:

$$\begin{aligned} \iota(\hat{\xi})\delta\phi &= -\partial_0\phi \\ \iota(\hat{\xi})dx^0 &= 0 \\ \iota(\hat{\xi})dx^1 &= 0. \end{aligned} \tag{2.127}$$

Then we compute

$$\mathrm{Lie}(\hat{\xi})L = d\alpha \qquad \text{on } \mathcal{F} \times M, \tag{2.128}$$

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha + d\beta \qquad \text{on } \mathcal{M} \times M, \tag{2.129}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\left[(\partial_0\phi)^2 - (\partial_1\phi)^2\right] dx^1 \in \Omega^{0,|-1|}_{\mathrm{loc}}, \\ \beta &= \partial_1\phi\,\delta\phi \qquad\qquad\qquad\qquad \in \Omega^{1,|-2|}_{\mathrm{loc}}. \end{aligned} \tag{2.130}$$

A simple example from mechanics illustrates the importance of locality in Definition 2.71.

Example 2.131. Consider a free particle $x\colon M^1 \to \mathbb{R}$ with lagrangian $L = \frac{m}{2}\dot{x}^2\,dt$ and the canonical $\gamma = m\dot{x}\,\delta x$. Let $\hat{\xi}$ be infinitesimal time translation, considered[14] as a motion only along $\mathcal{F}$:

$$\begin{aligned} \iota(\hat{\xi})\delta x &= -\dot{x}, \\ \iota(\hat{\xi})dt &= 0. \end{aligned} \tag{2.132}$$

Then

$$\mathrm{Lie}(\hat{\xi})L = -m\dot{x}\ddot{x}\,dt = da_{\hat{\xi}} \tag{2.133}$$

[14]In Example 2.105 we discuss time translation as a manifest symmetry.

for

$$a_{\hat{\xi}} = -\frac{1}{2}m\dot{x}^2, \tag{2.134}$$

so that $\hat{\xi}$ is a generalized infinitesimal symmetry. Imposing the equation of motion $\ddot{x} = 0$ we compute

$$\mathrm{Lie}(\hat{\xi})\gamma - \delta a_{\hat{\xi}} = 0 \qquad \text{on } \mathcal{M} \times M^1, \tag{2.135}$$

as it must by Proposition 2.76. Now fix a time t_0 and a function $F\colon M^1 \to \mathbb{R}$ and consider instead

$$\tilde{a}_{\hat{\xi}} = -\frac{1}{2}m\dot{x}^2 + F\big(x(t_0)\big). \tag{2.136}$$

(We could put any function on $\mathcal{F}$ here; we take one of this explicit form for ease of writing.) Since $F\big(x(t_0)\big)$ is independent of t we have $\mathrm{Lie}(\hat{\xi})L = d\tilde{a}_{\hat{\xi}}$, but now

$$\mathrm{Lie}(\hat{\xi})\gamma - \delta\tilde{a}_{\hat{\xi}} = -F'\big(x(t_0)\big)\,\delta x(t_0) \qquad \text{on } \mathcal{M} \times M^1 \tag{2.137}$$

is nonzero unless F is constant.

Note that $F\big(x(t_0)\big)$ is a *nonlocal* function of x—its value at $t \in M^1$ does not just depend on a finite jet of x at t. This example shows that without locality in Definition 2.71, the energy would be completely ill-defined.

§2.8. Computing Noether's current by gauging symmetries

Let us make the following assumptions: (a) the space $\mathcal{F}$ of fields is the space of maps from M to a manifold X, that is, E is a product $M \times X$; (b) the lagrangian density L depends only on the first order jet of fields; (c) γ is the canonical variational form; (d) a Lie group G acts on X; it preserves L, hence γ.

Let P be a principal G-bundle on M, and let E^P be the twisted form of E defined by P: a section p of P over an open subset U of M defines an isomorphism $\sigma(p)\colon E_U \to E^P_U$, and for g a map from U to G, $\sigma(pg) = \sigma(p)g$. Let $\mathcal{F}_1$ be the space of pairs (A, ϕ), where A is a connection on P and where ϕ is a section of E^P. On $\mathcal{F}_1 \times M$, let $L_1(A, \phi)$ be the lagrangian density whose value at m in M is computed as follows. Choose a local trivialization p of P which is horizontal (for A) at m. The section $\sigma(p)^{-1}\phi$ of E can be identified with a map from M to X, and

$$L_1(A, \phi) \text{ at } m := L(\sigma(p)^{-1}\phi) \text{ at } m\ . \tag{2.138}$$

The assumptions (b) and (d) ensure that the second member does not depend on the choice of p. The value of $L_1(A, \phi)$ at m depends only on the value of A at m and on the first jet of ϕ at m. We let γ_1 be the canonical variational 1-form.

In local coordinates, L is computed as follows. Over a local coordinate system $x\colon U \hookrightarrow \mathbb{R}^n$ of M, one chooses a trivialization of P. This identifies E^P with $M \times X$. The connection A is a 1-form on M with values in $\mathrm{Lie}(G)$, which the action of G on X turns into a 1-form on M with values in vector fields on X. The original lagrangian L is of the form

$$L(\phi) = \ell(\phi; \phi_\imath)dx \tag{2.139}$$

with $\phi_i = \partial\phi/\partial x_i$. The new one is

$$L_1(A,\phi) = \ell(\phi; \phi_i + A_i(\phi))dx \ . \tag{2.140}$$

It follows that if the variational 1-form $\gamma(\phi)$ is $\sum c_I(\phi;\phi_i)dx^I$, then

$$\gamma_1(A,\phi) = \sum c_I(\phi, \phi_i + A_i(\phi))dx^I \ . \tag{2.141}$$

The gauge group $\mathcal{G}_P$ of automorphisms of P acts on $\mathcal{F}_1$ by transport of structures. As L_1 and γ_1 have been defined (in terms of structures preserved by $\mathcal{G}_P$), they are preserved. If P is the trivial principal bundle G, so that $E^P = E$, and if A is the trivial connection A_0, the group G acting on P (on the left) by constant gauge transformations respects A_0; its action on ϕ is the action of G on $\mathcal{F}$, and

$$L(A_0,\phi) = L(\phi) \ . \tag{2.142}$$

The construction of $\mathcal{F}_1$, L_1 from $\mathcal{F}$, L and the action of G is called "gauging the symmetry". Our aim in this section is to explain how the Noether current attached to the manifest infinitesimal symmetry of L given by the action of Lie(G) can be computed by differentiating $L_1(A,\phi)$ in A. We will work in a more general framework than the one above, which covers as well the case where the fields are a map ϕ: $M \to X$ and a ϕ^*TX-valued spinor.

Let P be a fixed principal G-bundle on M, and let con(P) be the bundle over M whose sections are the connections on P. Let us first assume only that E is a fiber space over con(P), so that the space of fields $\mathcal{F}$ fibers over the space $\mathcal{A}$ of connections, that the action of the gauge group $\mathcal{G}_P$ of automorphisms of P on $\mathcal{A}$ is lifted to an action of $\mathcal{F}$, that this lifted action is local—$g\phi$ at $m \in M$ depends only on some jet of g and ϕ at m—and that it preserves a lagrangian density L. For A in $\mathcal{A}$, let $\mathcal{F}_A$ be the fiber of $\mathcal{F}$ over $\mathcal{A}$ at A. It is the space of sections of E_A, the inverse image in E of the section A of con(P). The lagrangian density L induces a lagrangian density L_A on $\mathcal{F}_A \times M$. Let ϕ in $\mathcal{F}_A$ be an extremal for L_A. This means that $\underline{D}L$ at ϕ, a morphism of vector bundles from $\phi^*T(E/M)$ to densities on M, factors through $A^*T(\mathrm{con}(M)/M)$. As con(M)/M is an affine space bundle over M, the latter is independent of A. It is the vector bundle of 1-forms with values in the adjoint bundle $\mathfrak{g}^P$.

The bundle $\underline{\mathrm{Hom}}(\Omega^1(\mathfrak{g}^P), \Omega^{|0|})$ is identified with $\Omega^{|-1|}(g^{P\vee})$: to an element $\beta\otimes\xi'$ in $\Omega^{|-1|}(g^{P\vee})$ corresponds the morphism $\alpha\otimes\xi \mapsto \alpha\wedge\beta.\langle\xi,\xi'\rangle$. To ϕ in $\mathcal{F}_A$, extremal for L_A, we[15] have attached a section J_1 of $\Omega^{|-1|} \otimes \mathfrak{g}^{P\vee}$. The gauge invariance of L gives:

Proposition 2.144. *One has*

$$d_A J_1 = 0.$$

In particular, if the infinitesimal gauge transformation ξ is horizontal for A, then $J_1(\xi) = \langle\xi, J_1\rangle$ is a conserved current.

[15]Physicists typically summarize the definition of J_1 by the equation

$$\delta S(\phi) = \int_M J_1^\mu(\phi)\,\delta A_\mu, \tag{2.143}$$

where S is the action.

Proof. Let ξ be an infinitesimal gauge transformation, a section of g^P. Then, ξ induces a vector field on $\mathcal{A}$ whose value at A is $-d_A\xi$. If ξ has compact support, one has

$$\int \langle -d_A\xi \wedge J_1 \rangle = 0 \ . \tag{2.145}$$

Suppose indeed that $g(u)$ is a family of sections with compact support of the gauge group, with $\frac{d}{du}g(u) = \xi$ at $u = 0$. By gauge invariance of L and extremality of ϕ, we have

$$\int \langle -d_A\xi \wedge J_1 \rangle = \int \left\langle \underline{D}L, \frac{d}{du}g(u)\phi \right\rangle = \int \frac{d}{du}L(g(u)\phi) = 0 \ . \tag{2.146}$$

By integration by parts, i.e., using that

$$d\,\langle \xi, J_1 \rangle = \langle d_A\xi \wedge J_1 \rangle + \langle \xi \wedge d_A J_1 \rangle \,, \tag{2.147}$$

equation (2.145) gives that for any ξ with compact support,

$$\langle \int \xi \wedge d_A J_1 \rangle = 0 \ , \tag{2.148}$$

so that $d_A J_1 = 0$. The second statement follows from (2.147).

We now give conditions under which the current $J_1(\xi)$ can be interpreted as a Noether current. We suppose that:

(a) the fiber bundle E over M is a fiber product $\mathrm{con}(P) \times E'$, so that the space $\mathcal{F}$ of fields is a product $\mathcal{A} \times \mathcal{F}'$;

(b) the action of $\mathcal{G}_P$ on E is deduced from its action on $\mathrm{con}(P)$, and from an action of G^P on E': the value at m of $g\phi'$, for g in $\mathcal{G}_P$ and ϕ' a section of E', depends only on the values of g and ϕ' at m;

(c) the value of $L(A, \phi')$ at m depends only on the value of A at m, and of the 1-jet of ϕ at m.

The assumptions (a) (b) (c) hold for a lagrangian obtained by gauging a symmetry. The assumption (c) ensures that the canonical variational 1-form γ depends only on $\delta\phi'$, not on δA. It agrees with the γ_A for the L_A on $\mathcal{F}' = \mathcal{F}_A$.

As $\mathcal{G}_P$ acts on $\mathcal{F}'$, an infinitesimal gauge transformation ξ induces a vector field $\xi^\wedge$ on $\mathcal{F}'$. If ξ is horizontal for A, i.e. fixes A, then the vector field $\xi^\wedge$ on $\mathcal{F}'$ is a manifest symmetry of L_A.

Proposition 2.149. *If the lagrangian density L is invariant by gauge transformations, if the assumptions (a) (b) (c) above hold, and if ξ is an infinitesimal gauge transformation which fixes the connection A on P, i.e. if $d_A\xi = 0$, then the Noether current for the manifest symmetry $\xi^\wedge$ of the lagrangian L_A is the conserved current $J_1(\xi)$ of Proposition 2.144.*

Proof. Let ξ be an infinitesimal gauge transformation. By gauge invariance of L,

$$\text{Lie}(\xi)L = \iota(\xi)\delta L = \iota(-d_A\xi)\delta L + \iota(\xi^\wedge)\delta L_A = 0 \ . \tag{2.150}$$

As L depends only on the 0th-jet of A, we have

$$\iota(-d_A\xi)\delta L = \iota(-d_A\xi)\underline{D}L = -d_A\xi \wedge J_1 \ . \tag{2.151}$$

We also have

$$\iota(\xi^\wedge)\delta L_A = \iota(\xi^\wedge)\underline{D}L_A - \iota(\xi^\wedge)d\gamma_A = \underline{D}L_A(\xi^\wedge) + d\iota(\xi^\wedge)\gamma_A \ , \tag{2.152}$$

and so

$$d\iota(\xi^\wedge)\gamma_A = d_A\xi \wedge J_1 - \underline{D}L_A(\xi^\wedge) \ . \tag{2.153}$$

For any function f on M, we have $(f\xi)^\wedge = f\xi^\wedge$. Taking (2.153) for $f\xi$, and subtracting f times (2.153) for ξ, we get for any f

$$df \wedge \iota(\xi^\wedge)\gamma_A = df \wedge J_1(\xi) \ , \tag{2.154}$$

hence

$$\iota(\xi^\wedge)\gamma_A = J_1(\xi) \ . \tag{2.155}$$

For $d_A\xi = 0$, the left side is the Noether current corresponding to the manifest symmetry $\xi^\wedge$.

Elements in the center of $\mathfrak{g}$ determine infinitesimal "global" gauge transformations which act trivially on all of $\mathcal{A}$, and so by Proposition 2.144 lead to currents defined on the entire space of classical solutions of a theory $\mathcal{L}$. In §4.4 we relate these currents to *electric charge*.

§2.9. The energy-momentum tensor

First approach

Let M denote (affine) n-dimensional Minkowski spacetime. Fix a coordinate system $x^0, x^1, \dots, x^{n-1}$ with respect to which the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2, \tag{2.156}$$

and the natural density is

$$|d^n x| = |\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \,|. \tag{2.157}$$

Here $x^0 = ct$ is the speed of light times the standard time coordinate. Of course, the metric and density have their geometric units, a power of length:

$$[g] = L^2, \qquad [\,|d^n x|\,] = L^n. \tag{2.158}$$

Suppose $\mathcal{L} = L + \gamma$ describes a field theory on M. We assume that the Poincaré group is a manifest symmetry group of the theory. Now the constant vector field

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{2.159}$$

on M induces a vector field on $\mathcal{F} \times M$, which we are assuming is a symmetry, and we denote *minus* the $\big(0, |-1|\big)$-component of the associated Noether current by

$$\Theta_{\mu\nu} * dx^\nu = \Theta_{\mu\nu} g^{\nu\nu'} \, \iota(\partial_{\nu'}) |d^n x| \tag{2.160}$$

for some functions

$$\Theta_{\mu\nu} \colon \mathcal{F} \times M \longrightarrow \mathbb{R}. \tag{2.161}$$

The tensor $\Theta = (\Theta_{\mu\nu})$ is called the energy-momentum tensor. The conservation law (2.78) is

$$\sum_\nu \partial_\nu \Theta_{\mu\nu} = 0. \tag{2.162}$$

The units of the functions $\Theta_{\mu\nu}$ are those of a spatial momentum density:

$$[\Theta_{\mu\nu}] = \frac{M}{L^{n-2}T} = \frac{ML}{T} \frac{1}{L^{n-1}}. \tag{2.163}$$

Physically correct units for the components involving $x^0 = ct$ are obtained by re-expressing Θ in the coordinate system $\{t, x^1, \ldots, x^n\}$.

A vector field $\eta = \eta^\mu \partial_\mu$ on M defines a current

$$\eta \cdot \Theta = \Theta_{\mu\nu} \eta^\mu * dx^\nu. \tag{2.164}$$

Lemma 2.165. *Θ is symmetric if and only if $\eta \cdot \Theta$ is conserved for every infinitesimal Lorentz transformation η.*

Proof. Let $g_{\mu\nu}$ denote the Minkowski metric. Then a skew form $B = (B^{\nu\lambda})$ corresponds to an infinitesimal Lorentz transformation

$$\eta = B^{\mu\lambda} g_{\lambda\sigma} x^\sigma \partial_\mu. \tag{2.166}$$

Using (2.162) we find

$$\begin{aligned} d(\eta \cdot \Theta) &= d(\Theta_{\mu\nu} B^{\mu\lambda} g_{\lambda\sigma} x^\sigma * dx^\nu) \\ &= \Theta_{\mu\nu} B^{\mu\lambda} g_{\lambda\sigma} g^{\sigma\nu} \, |d^n x| \\ &= \Theta_{\mu\nu} B^{\mu\nu} \, |d^n x|. \end{aligned} \tag{2.167}$$

The conclusion follows.

In practice Θ is symmetric only for theories of scalar fields. For other types of fields there is an "improved" symmetric energy-momentum tensor, as we discuss below.

The various components of the energy-momentum tensor have a physical interpretation, as the name suggests. (See the general discussion at the end of §1.4.) In this paragraph we work on-shell. We also insert factors of c to obtain correct physical units. Let

$$|d^{n-1}x| = |\, dx^1 \wedge \cdots \wedge dx^{n-1} \,| \tag{2.168}$$

be the canonical density on a time slice $\{x^0 = \text{constant}\}$. Then $c\Theta_{00}\,|d^{n-1}x|$ is the *energy density* and $-\Theta_{i0}\,|d^{n-1}x|$ $(i \geq 1)$ the *momentum density*. Assuming suitable decay at infinity, we integrate these densities over space to obtain minus the charges associated to the infinitesimal translations ∂_μ:

$$\begin{aligned} E &= \int_{\{x^0=\text{constant}\}} c\Theta_{00}\,|d^{n-1}x|, \\ P_i &= \int_{\{x^0=\text{constant}\}} -\Theta_{i0}\,|d^{n-1}x| \end{aligned} \tag{2.169}$$

(The minus sign comes since $\Theta_{i\nu}$ is *minus* the Noether current associated to ∂_i, whereas the momentum P_i is the Noether charge associated to ∂_i. Recall that $-E$ is the Noether charge associated to $c\partial_0$.) The conservation law asserts that on-shell E and P_i are independent of time. E is the *total energy* or *hamiltonian* and P_i the *momentum* in the ith spatial direction. The square of the *(rest) mass* of a field configuration is

$$M^2 = E^2/c^4 - \sum_i P_i^2/c^2, \tag{2.170}$$

as in (1.45). The remaining components of Θ comprise the *stress tensor* $(\Theta_{ij})_{i,j\geq 1}$.

Second approach

For field theories which can be formulated in an arbitrary background metric, another approach to energy-momentum tensors is available. The relation with Noether currents is to be obtained by arguments parallel to those of §2.8, but with the group of diffeomorphisms replacing the group of gauge transformations.

Let $\text{met}(M) \to M$ denote the fiber bundle of Lorentzian metrics on the tangent bundle. Taking the inverse metric, we will view it as an open subbundle of the bundle of contravariant symmetric 2-tensors $g^{\mu\nu}$, and identify its relative tangent bundle $T\big(\text{met}(M)/M\big)$ with the pull-back from M of the bundle $\text{Sym}^2(T)$ of contravariant symmetric 2-tensors. In a Euclidean context, one would rather take for $\text{met}(M)$ the fiber bundle whose sections are the Riemannian structures on M.

In §2.8, we assumed the fiber bundle E over M to be a product $\text{con}(P) \times_M E'$. To similarly assume here that $E = \text{met}(M) \times_M E'$ is possible only at the cost of excluding spinor (or Rarita-Schwinger) fields. Indeed, for V a vector space given with a symmetric bilinear form g, the corresponding space of spinors depends on g (and on a lifting of the structural group from an orthogonal to a spin group). When g varies, the spaces of spinors form a vector bundle on the space of g's. Because of this, we will only assume that E is a fiber bundle $E \to \text{met}(M)$ over $\text{met}(M)$.

In §2.8, we assumed that the gauge group acts on $\mathcal{F}$. Here, if among the fields there is a connection over some principal G-bundle P, it is not convenient

to assume an action on $\mathcal{F}$ of the group of diffeomorphisms. What will act is an extension of the group of diffeomorphisms by the gauge group: the group of pairs (f, φ) of a diffeomorphism f and of a lifting φ: $P \xrightarrow{\sim} f^*P$ of f to P. If P is trivial, a trivialization of P defines a splitting of this extension. Often, it is more natural to use that a connection on P provides a lifting to P of infinitesimal diffeomorphisms (= vector fields on M), the horizontal lifting. One should however keep in mind that this horizontal lifting is not compatible with brackets of vector fields.

Because of this, we will at first only assume given a lagrangian density L on $\mathcal{F} \times M$ which obeys the weak version of diffeomorphism invariance explained below. A diffeomorphism of M acts on $\mathrm{Met}(M)$. An infinitesimal diffeomorphism of M, that is, a vector field ξ, induces on $\mathrm{Met}(M)$ the vector field whose value at a metric g is $-\mathrm{Lie}(\xi)(g^\vee)$, for g the inverse metric. One has

$$-\mathrm{Lie}(\xi)(g^\vee)^{\mu\nu} = (\nabla^\mu \xi)^\nu + (\nabla^\nu \xi)^\mu, \tag{2.171}$$

where the tensor indices are moved up and down by the metric tensor and its inverse. At m, this is easily checked in a coordinate system in which the metric g is constant $+O(x^2)$, so that the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ vanish at m. The diffeomorphism invariance assumption is the following: for any vector field ξ on M, there is a local vector field $\tilde{\xi}$ on $\mathcal{F}$, projecting to $-\mathrm{Lie}(\xi)(g^\vee)$ on $\mathrm{Met}(M)$, which is a generalized symmetry of the lagrangian density L.

For g in $\mathrm{Met}(M)$, let $\mathcal{F}_g$ be the fiber of $\mathcal{F} \to \mathrm{Met}(M)$ at g. If E_g is the inverse image of g in E, then $\mathcal{F}_g$ is the space of sections of the fiber bundle E_g over M. The lagrangian density L induces a lagrangian density L_g on $\mathcal{F}_g \times M$.

Suppose that ϕ in $\mathcal{F}_g$ is an extremal for L_g. The Euler-Lagrange equation

$$\underline{D}L\colon \phi^*T(E/M) \to \text{ densities on } M \tag{2.172}$$

then vanishes on $\phi^*T\big(E/\mathrm{met}(M)\big)$, hence factors through

$$\underline{D}_g L\colon g^*T\big(\mathrm{met}(M)/M\big) \to \text{ densities on } M\ . \tag{2.173}$$

This expresses that if $\phi[u]$ is a deformation with compact supports of ϕ, inducing a deformation $g[u]$ of the metric, then

$$\int \frac{d}{du} L(\phi[u]) \qquad \text{at } u = 0 \tag{2.174}$$

depends only on $\frac{d}{du}(g[u])$ at $u = 0$, and not on the concomitant variation of the other fields. This allows to unambiguously attach to ϕ (assumed to be extremal for L_g) a symmetric 2-tensor $T_{\mu\nu}$ with values in densities on M such that (2.173) is[16]

$$a^{\mu\nu} \longmapsto \frac{1}{2}\, a^{\mu\nu}\, T_{\mu\nu} \tag{2.176}$$

for $a^{\mu\nu}$ in $g^*T(E/M)$ a symmetric contravariant 2-tensor. In local coordinates $\{x^\mu\}$ on M we write the energy-momentum tensor as

$$T_g = T_{\mu\nu}(g, \phi)\, dx^\mu \otimes dx^\nu\, \mu_g(x), \tag{2.177}$$

where μ_g is the canonical density given by the Riemannian metric g.

[16] Physicists write (2.176) as

$$\delta S(\phi) = \frac{1}{2} \int_M T_{\mu\nu}(\phi)\, \delta g^{\mu\nu}, \tag{2.175}$$

where S is the action. In our definition we restrict to ϕ which are extremal for L_g.

Remark 2.178. We have defined $T_{\mu\nu}$ only on-shell, that is, for ϕ an extremal of L_g. This does not provide a unique differential expression in ϕ giving $T_{\mu\nu}$. Suppose, however that we obtained a differential expression θ in ϕ which (even locally on M) gives $T_{\mu\nu}$ for any extremal. Suppose also that θ depends only on the k-jet of the section ϕ of E_g: θ is a section on $J^k(E_g/M)$ of the pull-back of $\mathrm{Sym}^2(T)\otimes$ densities on M. Then, if any k-jet is the k-jet of an extremal (the validity of this depends on which Cauchy data the Euler Lagrange equations for L_g require), then θ is unique and defines $T_{\mu\nu}$ off-shell.

Proposition 2.179. *Under the assumptions above, if ϕ in $\mathcal{F}_g$ is extremal for L_g, the corresponding $T_{\mu\nu}$ obeys the following conservation law: its covariant derivative $T_{\mu\nu;\rho}$ is such that its contraction $T_{\mu\nu}{}^{\nu}$ vanishes.*

Proof. Let ξ be a vector field with compact support on M. It induces the vector field $-\mathrm{Lie}(\xi)(g^\vee)$ on $\mathrm{Met}(M)$ and, by assumption, this vector field can be lifted to $\hat{\xi}$ on $\mathcal{F}$ in such a way that $\mathrm{Lie}(\hat{\xi})L$ is an exact differential. It follows that

$$\int_M \langle T, -\mathrm{Lie}(\xi)(g^\vee)\rangle = 0\ . \tag{2.180}$$

Our computations will be local. Let us choose an orientation of M to identify densities with forms of maximal degree n. Instead of viewing T as a section of $\Omega^1\otimes\Omega^1\otimes\Omega^n$, one can use g to identify Ω^1 with T^1, and view T as a section of $\Omega^1\otimes T^1\otimes\Omega^n$, mapped isomorphically by contraction to $\Omega^1\otimes\Omega^{n-1}$. Raising and lowering indices by g, $-\mathrm{Lie}(\xi)(g^\vee)$ similarly corresponds to a symmetrization of $\nabla\xi$ in $\Omega^1\otimes T^1$, and in (2.180), the symbol $\langle\ \ \rangle$ denotes contraction of Ω^1 with T^1 while simultaneously wedging Ω^{n-1} and Ω^1. One has, for $\langle\ \ \rangle$ denoting only contraction of Ω^1 and T^1,

$$\nabla\langle T,\xi\rangle = \langle\nabla T,\xi\rangle + \langle T,\nabla\xi\rangle\ . \tag{2.181}$$

After wedging, $\nabla\langle T,\xi\rangle$ becomes $d\langle T,\xi\rangle$ and (2.181) gives

$$\int_M \langle \nabla T \text{ wedged }, \xi\rangle = 0\ . \tag{2.182}$$

As this holds for all ξ, it follows that ∇T in $\Omega^1\otimes\Omega^1\otimes\Omega^{n-1}$, projected to $\Omega^1\otimes\Omega^n$ by $\alpha\otimes\beta\otimes\gamma\mapsto\beta\otimes(\alpha\wedge\gamma)$, vanishes. This is equivalent to $T_{\mu\nu}{}^{\nu}=0$.

Corollary 2.183. *Let us view T as a 1-form with values in $(n-1)$-forms (or better, with values in $\Omega^{|-1|}$). If ϕ in $\mathcal{F}_g$ is extremal for L_g, and if ζ is a Killing vector field, i.e., an infinitesimal isometry, then $T(\zeta)$ is closed.*

Proof. As $\nabla\zeta=0$, we have

$$(\nabla T)(\zeta) = \nabla(T(\zeta))\ , \tag{2.184}$$

which, applying $\wedge$: $\Omega^1\otimes\Omega^{n-1}\to\Omega^n$, gives

$$0 = dT(\zeta)\ . \tag{2.185}$$

We managed to relate the conserved currents in Corollary 2.183 to Noether's currents only under the following restrictive assumptions: (a) E is a fiber product $\mathrm{met}(M) \times_M E'$, making the space $\mathcal{F}$ of fields a product $\mathrm{Met}(M) \times \mathcal{F}'$; (b) the Lagrangian density $L(g, \phi')$ at $m \in M$ depends only on the value of g at m and on the 1-jet of the section ϕ' of E' at m; (c) we take for γ the canonical choice; (d) to each vector field ζ on M is attached, by a local rule, a vector field ξ_ζ on $\mathcal{F}'$, so that $(\zeta, -\mathrm{Lie}(\zeta)(g^\vee), \xi_\zeta)$ is a manifest symmetry of L; (e) the map $\zeta \mapsto \xi_\zeta$ is linear over functions.

Remarks 2.186. (i) (a) is needed to make sense of (b); it fails for spinor fields.
(ii) Condition (d) fails for tensor fields, if $\zeta \mapsto \xi_\zeta$ is given by a Lie derivative. However, it holds for connections on a fixed principal G-bundle P, if ξ_ζ is defined as follows. Given a connection ∇_A, use it to lift ζ to P. The flow $\exp(t\zeta)$ generated by ζ is then lifted to P, and one takes

$$\xi_\zeta = \frac{d}{dt}(\exp(-t\zeta)^*(\nabla_A)) \quad \text{at} \quad t = 0 \,. \tag{2.187}$$

The vector field ξ_ζ is obtained by contracting the curvature 2-form F_A with ζ, and is hence linear over the functions.
(iii) Condition (a) allows us to define T off-shell as the restriction to the tangent space to $\mathrm{Met}(M)$ of $\underline{D}L$ (or δL; this amounts to the same by (b)).

Assuming (a) to (e), we now repeat the arguments of §2.8. The manifest symmetry of L means the vanishing of a Lie derivative of L: at (g, ϕ')

$$d\iota(\zeta)L + \underline{D}L(-\mathrm{Lie}(\zeta)g^\vee) + \iota(\xi_\zeta)\delta L = 0 \,. \tag{2.188}$$

Writing $I(\zeta)$ for this identity, we now express that $I(f\zeta) - fI(\zeta)$ vanishes. The first term contributes $df \wedge \iota(\zeta)L$. As $-\mathrm{Lie}(\zeta)(g^\vee)$ is $\nabla\zeta$, changed to a twice contravariant tensor and symmetrized and that $\nabla(f\zeta) - f\nabla\zeta = df \otimes \zeta$, the second term contributes $\langle T, \mathrm{grad}\, f \otimes \zeta \rangle$. As $\delta L = \underline{D}L - d\gamma$, with $\underline{D}L$ linear over functions, and that $\iota(\xi)d\gamma - d\iota(\xi)\gamma$, the third term contributes $df \wedge \iota(\xi_\zeta)\gamma$. With the notations of Corollary 2.183, the contribution of the second term can be written $df \wedge T(\zeta)$, giving

$$df \wedge (\iota(\zeta)L + T(\zeta) + \iota(\xi_\zeta)\gamma) = 0 \,. \tag{2.189}$$

As this holds for all f, we can suppress the $df\wedge$. If ζ is an infinitesimal isometry of g, then (ζ, ξ_ζ) is a manifest infinitesimal symmetry of L_g, and we get

Proposition 2.190. *Under the assumptions made, if ζ is a Killing vector field, the conserved current $T(\zeta)$ of (3.5) is the opposite of the Noether current of the corresponding symmetry of L_g.*

§2.10. Finite energy configurations, classical vacua, and solitons

In this section we work in the Hamiltonian framework $M = M^1 \times N$. Let t be an affine coordinate on M^1 and ∂_t the vector field which generates unit time translation. Recall that a field ϕ is *static* if

$$\frac{\partial \phi}{\partial t} = \iota(\hat{\xi}_t)\delta\phi = 0, \tag{2.191}$$

where $\hat{\xi}_t$ is the vector field on $\mathcal{F}$ induced by the action of ∂_t. Let $\mathcal{F}_N$ denote the space of static fields and $\overline{\mathcal{F}}_N$ the quotient by gauge symmetries. Equation (2.191) asserts that the vector field $\hat{\xi}_t$ vanishes on $\mathcal{F}_N$.

Consider a theory $\mathcal{L} = L + \gamma$ which we assume is manifestly invariant under time translation.[17] The *energy density* Θ is *minus* the canonical Noether current for ∂_t:

$$\Theta = -\big(\iota(\xi_t)\mathcal{L}\big)^{0,|-1|}, \tag{2.192}$$

where $\xi_t = \partial_t - \hat{\xi}_t$. The energy at time t of a field ϕ is (see (2.169))

$$E_\phi(t) = \int_{\{t\}\times N} \Theta(\phi). \tag{2.193}$$

For a static field the energy is constant in time. Let $\mathcal{FE}_N$ denote the space of finite energy static fields and $\overline{\mathcal{FE}}_N$ the quotient by gauge symmetries. These are subspaces of $\mathcal{F}_N$ and $\overline{\mathcal{F}}_N$, respectively. Define $\mathcal{M}_N \subset \mathcal{FE}_N$ (and $\overline{\mathcal{M}}_N \subset \overline{\mathcal{FE}}_N$) to be the space of static classical solutions of finite energy.

The energy density of a static field is simply related to the lagrangian density.

Proposition 2.194. *(i) We have*

$$\Theta = -\iota(\partial_t)L \qquad \text{on } \mathcal{F}_N; \tag{2.195}$$

(ii) If ϕ is a critical point of energy on $\mathcal{FE}_N$, then ϕ is a solution to the classical equations, i.e., $\phi \in \mathcal{M}_N$. Conversely, every element of $\mathcal{M}_N$ is a critical point of energy.

Proof. By (2.191) we have $\iota(\hat{\xi}_t)\gamma = 0$ on $\mathcal{F}_N$. Hence on $\mathcal{F}_N$

$$\Theta = -\iota(\xi_t)\mathcal{L} = -\iota(\partial_t)L + \iota(\hat{\xi}_t)\gamma = -\iota(\partial_t)L, \tag{2.196}$$

which is (i). For (ii) we first note that on $\mathcal{F}_N$ we have by (2.191) that

$$\begin{aligned} 0 = \mathrm{Lie}(\xi_t)\gamma &= d\iota(\partial_t)\gamma + \iota(\partial_t)d\gamma - \delta\iota(\hat{\xi}_t)\gamma - \iota(\hat{\xi}_t)\delta\gamma \\ &= d\iota(\partial_t)\gamma + \iota(\partial_t)d\gamma. \end{aligned} \tag{2.197}$$

Using (2.195) we compute that on $\mathcal{F}_N$

$$\begin{aligned} \delta\Theta &= \iota(\partial_t)\delta L \\ &= \iota(\partial_t)(D\mathcal{L})^{1,|0|} - \iota(\partial_t)d\gamma \\ &= \iota(\partial_t)(D\mathcal{L})^{1,|0|} + d\iota(\partial_t)\gamma. \end{aligned} \tag{2.198}$$

This is the integration by parts equation for the functional E on $\mathcal{F}_N$—analogous to (2.40) for the functional L—and so the Euler-Lagrange equation for E is

$$\iota(\partial_t)(D\mathcal{L})^{1,|0|} = 0, \tag{2.199}$$

[17] This excludes, for example, the theory of a nonrelativistic particle moving in a time-dependent potential.

which is equivalent to the equation of motion $(D\mathcal{L})^{1,|0|} = 0$.

The global minima of energy are called *vacuum solutions*. For field theories on Minkowski spacetime the set of vacuum solutions is referred to as the *classical moduli space of vacua*. We denote it $\mathcal{M}_{\text{vac}}$. A vacuum solution in Minkowski spacetime is usually assumed to be Poincaré invariant (if the theory is Poincaré invariant). This means that scalar fields are constant, gauge fields are gauge equivalent to a trivial connection, spinor fields vanish, and p-form fields ($p \geq 1$) are exact. The classical moduli space is then the space of constant values of the scalar fields. Now the energy density $\Theta(\phi_0)$ of a constant scalar field $\phi_0 \colon M \to X$ satisfies

$$dt \wedge \Theta(\phi_0) = V(\phi_0)\,|dt|\,|d^{n-1}x| \tag{2.200}$$

for a *potential energy* function $V \colon X \to \mathbb{R}$. Assume[18] that $V \geq 0$. The only finite energy constant scalar fields have $V(\phi_0) = 0$, and so

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \tag{2.202}$$

is the classical moduli space. If there is also a connection field in the theory with gauge group G, then G acts on X and V is an invariant function. In that case the effect of dividing by global gauge transformations is that the classical moduli space is the quotient

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{2.203}$$

Returning for a moment to a general field theory $\mathcal{L} = L + \gamma$ on a manifold M, fix a field configuration $\phi_0 \in \mathcal{F}$. Then there is a *perturbation theory* for the fluctuations around ϕ_0 in which the space of fields is $T_{\phi_0}\mathcal{F}$ and the Nth order perturbative lagrangian is the Nth order jet of L at ϕ_0.

On Minkowski spacetime M we often perturb around a vacuum solution.[19] Recall that at a vacuum all gauge fields A_0 are trivial, and we can use them to trivialize all bundles. Then any scalar ϕ_0 is a constant in a manifold X. In the perturbation theory the fluctuations of a trivial connection A_0 on a principal G-bundle P lie in $\Omega^1_M(\operatorname{ad} P) \cong \Omega^1_M(\mathfrak{g})$, where we use the trivialization. The fluctuations $\tilde{\phi}$ of the constant scalar ϕ_0 lie in $T_{\phi_0}X$. Spinor fields, Rarita-Schwinger fields, and p-form fields all vanish at a vacuum configuration and we consider the fields in the original lagrangian to be fluctuations about zero. Now since we are at a vacuum solution the perturbative lagrangian starts out with quadratic terms. In the quadratic approximation $\tilde{\phi}$ is typically a free scalar field in the Euclidean space $T_{\phi_0}X$ with mass[20] $\operatorname{Hess}_{\phi_0} V$; there may be higher derivative terms as well. There is a mass matrix for the spinor fields which also depends on ϕ_0. The *massless* fluctuations of the gauge field lie in $\Omega^1_M(\mathfrak{g}_{\phi_0})$, where $\mathfrak{g}_{\phi_0}$ is the Lie algebra of the stabilizer subgroup G_{ϕ_0} at ϕ_0 of the G action on X. Often G_{ϕ_0} is called the *unbroken* gauge group, though this terminology is confusing. Other components of the gauge fluctuations are massive due to the *Higgs mechanism*, which we illustrate in the following example.

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[18] In many problems the energy density is nonnegative for all fields: $\Theta_t(\phi) \geq 0$ for all ϕ, where

$$dt \wedge \Theta(\phi) = \Theta_t(\phi)\,|dt|\,|d^{n-1}x|. \tag{2.201}$$

[19] This is the first step in a perturbative construction of a quantum theory around the chosen vacuum.

[20] Free fields and this use of 'mass' are discussed in §3.

Example 2.204. We work on n-dimensional Minkowski spacetime M^n. The fields are a complex scalar field ϕ, a real spinor field[21] ψ, and a $\mathbb{T} \times \mathbb{T}$ connection A. As in any gauge theory, we work on the quotient of the space of fields by gauge transformations. Let $\mathbb{T} \times \mathbb{T}$ act on $\mathbb{C}$ by $(\lambda_1, \lambda_2) \cdot z = \lambda_1 z \lambda_2^{-1}$, and suppose that ϕ is a section of the associated hermitian line bundle. The spinor field ψ is not coupled to A. Let

$$L = \left(\| d_A \phi \|^2 - \frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \psi \not{D} \psi \rangle - \|\phi\|^2 (1 - \|\phi\|^2)^2 - \phi\psi\psi \right) |d^n x|. \tag{2.205}$$

The potential energy is

$$V(\phi_0) = \|\phi_0\|^2 (1 - \|\phi_0\|^2)^2 \tag{2.206}$$

and $V^{-1}(0)$ consists of the origin and the unit circle. The classical moduli space is the quotient of $V^{-1}(0)$ by $\mathbb{T} \times \mathbb{T}$ and so consists of two isolated points. They are the two classical vacua. At the origin the perturbative lagrangian is the same as (2.205), except that A should now be viewed as a perturbation $A = A_0 + \alpha$ of a trivial connection; α is an $(i\mathbb{R} \oplus i\mathbb{R})$-valued 1-form on M. We use A_0 to construct a trivializing section of the hermitian line bundle, and so view ϕ as a map $M \to \mathbb{C}$. Then the perturbative lagrangian is

$$L_0 = \left(\| d\phi + \alpha \cdot \phi \|^2 - \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\langle \psi \not{D} \psi \rangle - \|\phi\|^2 (1 - \|\phi\|^2)^2 - \phi\psi\psi \right) |d^n x|. \tag{2.207}$$

From the quadratic part of L_0 we read off that ϕ is massive (with mass 1), ψ is massless, and α is massless. The entire gauge group $\mathbb{T} \times \mathbb{T}$ is unbroken.

Now we expand around $\phi_0 = 1$, which is most easily accomplished by substituting $\phi = 1 + \tilde{\phi}$ into L_0:

$$\begin{aligned} L_1 &= \Big(\| d\tilde{\phi} + \alpha \cdot (1 + \tilde{\phi}) \|^2 - \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\langle \psi \not{D} \psi \rangle \\ &\qquad - \|1 + \tilde{\phi}\|^2 (1 - \|1 + \tilde{\phi}\|^2)^2 - (1 + \tilde{\phi})\psi\psi \Big) |d^n x| \\ &= \Big(\| d\tilde{\phi} \|^2 - \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\langle \psi \not{D} \psi \rangle - 4|(\operatorname{Re} \tilde{\phi})|^2 \\ &\qquad - \psi\psi + \|\alpha_1 - \alpha_2\|^2 \Big) |d^n x| + \text{higher order terms.} \end{aligned} \tag{2.208}$$

In the last line we wrote only the quadratic part; it contains the information about masses. Also, we wrote $\alpha = (\alpha_1, \alpha_2)$; the last term is the norm square of $\alpha \cdot 1 = \alpha_1 - \alpha_2$. The imaginary part of the complex scalar field is gauged away by constant gauge transformations (which preserve the trivial connection A_0). Thus the only scalar field is the *real* scalar field $\operatorname{Re} \tilde{\phi}$ with mass square 4. The spinor ψ is now massive with mass 2 (and so mass square 4). Set[22] $\sqrt{2}\beta = \alpha_1 - \alpha_2$ and $\sqrt{2}\gamma = \alpha_1 + \alpha_2$. Then the quadratic part of the lagrangian involving α is

$$-\frac{1}{2}|d\beta|^2 - \frac{1}{2}|d\gamma|^2 + 2\|\beta\|^2. \tag{2.209}$$

[21] The precise spinor representation S of $\mathrm{Spin}(1, n-1)$ is not crucial, but we do assume that there is an invariant skew form $\bigwedge^2 S \to \mathbb{R}$ so that we can write a mass term.

[22] Here the massless field γ is canonical, but the massive field β is somewhat arbitrary.

So we see that β has mass square 4 and γ is massless. Note that γ lies in the Lie algebra of the stabilizer of $1 \in \mathbb{C}$, which is the diagonal $\mathbb{T} \subset \mathbb{T} \times \mathbb{T}$. This is the unbroken gauge group. The appearance of a mass for β is an example of the Higgs mechanism.

In general, if H is a group of global symmetries of $\mathcal{L}$, then H acts on the space of static classical solutions $\mathcal{M}_N$. For $\phi \in \mathcal{M}_N$ we have the subgroup $\operatorname{Stab}\phi \subset H$ which fixes ϕ. We say that ϕ *spontaneously breaks* H down to $\operatorname{Stab}\phi$, and that $\operatorname{Stab}\phi$ is the group of *unbroken* symmetries. (For example, we asserted above that in Minkowski spacetime Poincaré symmetry is unbroken at a vacuum solution.) If ϕ_0 is a vacuum solution, the homogeneous space $H/\operatorname{Stab}\phi_0$ is embedded in $V^{-1}(0)$ by the H action. Then the perturbative scalar fields with values in $\mathfrak{h}/\mathfrak{stab}\,\phi_0 \subset T_{\phi_0}X$ are massless. These fields are called (classical) *Goldstone bosons*. They are massless scalar fields guaranteed by the symmetry. Of course, there may be other massless scalar fields which are not related to symmetry.

Example 2.210. Consider a theory of a complex scalar $\phi\colon M \to \mathbb{C}$ with potential (2.206). Now there is a global $\mathbb{T}$ symmetry and the moduli space consists of the origin and the unit circle. We do *not* divide by the global symmetry; rather, at a classical vacuum on the unit circle there is a single (real) Goldstone boson field due to the symmetry. At the origin $\mathbb{T}$ acts trivially and the entire complex scalar is massive.

A *soliton* is a static classical solution whose energy is not a global minimum. In many examples the space $\overline{\mathcal{FE}}_N$ of finite energy static fields is not connected, and a soliton is a minimum energy configuration in a component where the global minimum is not achieved. For example, consider a scalar field $\phi\colon M^2 \to \mathbb{R}$ on two-dimensional Minkowski spacetime with lagrangian density

$$L = \left(\frac{1}{2}|d\phi|^2 - V(\phi)\right)|d^2x|. \tag{2.211}$$

Suppose $V \geq 0$ and $V^{-1}(0) = \{a, b\}$. Then the space of finite energy static fields has 4 components: A static field $\phi(t, x) = \phi(x)$ depends only on the spatial variable, and the finite energy condition means $\lim_{x\to\infty}\phi(x)$ and $\lim_{x\to-\infty}\phi(x)$ lie in $V^{-1}(0) = \{a, b\}$. There are 2 vacuum solutions $\phi(x) \equiv a$ and $\phi(x) \equiv b$. There are solitons with $\lim_{x\to-\infty}\phi(x) = a$, $\lim_{x\to\infty}\phi(x) = b$ and also (anti-)solitons with $\lim_{x\to-\infty}\phi(x) = b$, $\lim_{x\to\infty}\phi(x) = a$.

§2.11. Dimensional reduction

Suppose $\mathcal{L} = L + \gamma$ is a Poincaré invariant field theory on n-dimensional Minkowski spacetime M^n. We obtain a theory on M^{n-1} as follows. As usual let $x^0, \ldots, x^{n-1}$ denote coordinates on M^n, and $x^0, \ldots, x^{n-2}$ coordinates on M^{n-1}, viewed as the quotient of M^n by translations in the x^{n-1} direction. Let $\hat{\xi}_{n-1}$ denote the vector field on $\mathcal{F}$ induced by the action of ∂_{n-1}, and define

$$\mathcal{F}_{n-1} = \{\phi \in \mathcal{F} : \operatorname{Lie}(\hat{\xi}_{n-1})\phi = 0\}. \tag{2.212}$$

These are the fields which are constant in the x^{n-1} direction. We identify $\mathcal{F}_{n-1}$ with a space of fields on M^{n-1}. Then the dimensionally reduced theory is

$$\mathcal{L}_{n-1} = \mathcal{L} \otimes \partial_{n-1} \in \Omega^0_{\mathrm{loc}}(\mathcal{F}_{n-1} \times M^{n-1}). \tag{2.213}$$

A scalar field on M^n reduces to a scalar field on M^{n-1}, but for fields of higher spin the identification of $\mathcal{F}_{n-1}$ with fields on M^{n-1} is more complicated. For example, a 1-form on M^n reduces to a 1-form plus a scalar field on M^{n-1}.

More generally, if $M \to M'$ is a fiber bundle with a section of $\operatorname{Det} T(M/M')$, then we can reduce a field theory on M to a field theory on M'. Instead of restricting to fields constant along the fibers (dimensional reduction) we can also include fields which fluctuate along the fibers (compactification); from the point of view of M' there is an infinite number of such fields.

Appendix: Takens' acyclicity theorem

This appendix gives the proof of Theorem 2.16. Our setting is slightly different from that of Takens, who works on the infinite jet bundle $J^\infty_{E/M}$ of E/M. To go from local to global results, he uses that if $0 \to \mathcal{F}_0 \to \cdots \to \mathcal{F}_n \to \mathcal{F}_{n+1}$ is an exact sequence of sheaves, with $\mathcal{F}_i$ soft for $i \leq n$, the sequence of groups of global sections is again exact. In order that the space of "local" lagrangian densities, or forms, be a space of global sections of some sheaf, he is led to define "local" to mean "depending only on k-jets of fields and their variations, where k is bounded *locally* on $J^\infty_{E/M}$". For us, k is globally bounded, and the sheaf theoretic argument has to be unraveled, replaced by a direct use of partitions of unity.

We fix a submersion $E \to M$. For fixed $p > 0$, we are to prove that the corresponding complex $(\Omega^{p,*}_{\mathrm{loc}}, d)$ is acyclic, except in top degree. It will be more convenient to prove a more general statement. Fix vector bundles $V_1, \dots, V_p$ on E. Let V be their product. The complex $(\Omega^{0,*}_{\mathrm{loc}}, d)$ for $V \to M$ is the complex of forms α on M depending (locally) on a section ϕ of E/M and on sections $\xi_1, \dots, \xi_p$ of $\phi^* V_i$. For some k, the value of $\alpha(\phi, \xi_1, \dots, \xi_p)$ at $m \in M$ should depend only on the k-jet of ϕ and the k-jet of $\xi_1, \dots, \xi_p$ at m. We denote $\Omega_{\mathrm{loc,multi}}$ the subcomplex consisting of the α depending $\mathbb{R}$-linearly on each of $\xi_1, \dots, \xi_p$. The following is a restatement of Theorem 2.16.

Theorem 2.214. *If $p > 0$, the complex $\Omega_{\mathrm{loc,multi}}$ is acyclic except in top degree.*

If we take all V_i to be the relative tangent bundle $T_{E/M}$, then $(\Omega^{p,*}_{\mathrm{loc}}, d)$ is a direct factor of $\Omega_{\mathrm{loc,multi}}$: it is the antisymmetric part for the action of the symmetric group S_p. The acyclicity of $(\Omega^{p,*}_{\mathrm{loc}}, d)$ for $p > 0$ (except in top degree) hence follows from the theorem.

Proof. For e_0 in E, with image m_0 in M, one can choose (a) a neighborhood U of m_0, and a local coordinate system $x\colon U \hookrightarrow \mathbb{R}^d$, (b) a neighborhood W of e_0, a decomposition $W = F \times U$ and a local coordinate system $F \hookrightarrow \mathbb{R}^f$, and (c) trivializations of the V_i on W. In such a local coordinate system, α in $\Omega_{\mathrm{loc,multi}}$ can uniquely be written as a finite sum

$$\alpha(\phi; \xi_1, \dots, \xi_p) = \sum \alpha_{n_1,\dots,n_p}(\phi) \partial^{n_1}\xi_1 \dots \partial^{n_p}\xi_p \ . \tag{2.215}$$

In the formula, each n_i is a multi-index $(n_{i1}, \dots, n_{id})$ and $\partial^{n_i} := \partial_1^{n_{i1}} \dots \partial_d^{n_{id}}$ is the corresponding iterated derivative in the coordinate system $(x_1, \dots, x_d)$. We define as usual $|n_i| = \sum n_{ij}$. For some k each $\alpha_{n_1,\dots,n_p}$ depends only on the k-jet of ϕ and takes values in $\otimes V_i^\vee \otimes \Omega^*_M$: it is a section on $J^k(E/M)$ of the tensor product of the inverse images of the $V_i^\vee$ and of Ω^*_M. We write F_* for the increasing filtration by

$\sum |n_i|$. It does not depend on the local coordinate systems used. The differential d maps F_N to F_{N+1}.

Replacing in (2.215) each ∂^{n_i} by its symbol in $\mathrm{Sym}^{|n_i|}(T_M)$, we obtain the following description of $\mathrm{Gr}^F(\Omega^q_{\mathrm{loc,multi}})$. It is the inductive limit in k of the space of sections over $J^k_{E/M}$ of the vector bundle

$$\otimes_{*\leq k} \mathrm{Sym}^*(T_M) \otimes V^{i\vee} \otimes \Omega^q_M \tag{2.216}$$

with each factor to be replaced by its pull-back to $J^k_{E/M}$. The degree N is the sum of the symmetric algebra degrees.

The differential d induces $\mathrm{Gr}^F_N(d)$: $\mathrm{Gr}^F_N(\Omega^q_{\mathrm{loc,multi}}) \to \mathrm{Gr}^F_{N+1}(\Omega^{q+1}_{\mathrm{loc,multi}})$, which is linear over the functions. It is deduced from a morphism of vector bundles on M

$$\left(\bigotimes_1^p \mathrm{Sym}^*(T_M)\right) \otimes \Omega^q_M \to \left(\bigotimes_1^p \mathrm{Sym}^*(T_M)\right) \otimes \Omega^{q+1}_M \tag{2.217}$$

by pull-back and tensorization by $\otimes V^{i\vee}$. Locally on M, for $\{e_j\}$ a basis of T_M, (2.217) is

$$s \otimes \alpha \longmapsto \sum_j \left(\left(\sum_{i=1}^p 1 \otimes \cdots \otimes (e_j \text{ at } i^{\text{th}} \text{ place}) \otimes \cdots \otimes 1 \right) \cdot s \right) \otimes e^j \wedge \alpha \ . \tag{2.218}$$

The crux of the matter is now the following

Lemma 2.219. *On M, for each integer N, the complex of vector bundles with components*

$$(\textit{part of degree } (N+q) \textit{ of } \bigotimes_1^p \mathrm{Sym}^*(T_M)) \otimes \Omega^q_M$$

and differential the morphisms (2.218) *of vector bundles is acyclic, except in top degree.*

Proof. This is to be checked point by point. One then recognizes in the complex a variant of the Koszul complex, and one can write an explicit homotopy. In more detail: for T the tangent space at a point of M, one has to consider a complex with components

$$\left(\bigotimes_1^p \mathrm{Sym}^*(T)\right) \otimes \wedge^q(T^\vee) \ . \tag{2.220}$$

One has $\bigotimes_1^p \mathrm{Sym}^*(T) = \mathrm{Sym}^*\left(\bigoplus_1^p T\right)$. If we identify T with its image $\Delta T \subset \bigoplus_1^p T$ by the diagonal embedding, and if S is any supplement, we have

$$\mathrm{Sym}^*\left(\bigoplus_1^p T\right) = \mathrm{Sym}^*(S) \otimes \mathrm{Sym}^*(T) \ , \tag{2.221}$$

and the complex (2.220) becomes the tensor product of $\mathrm{Sym}^*(S)$ (of cohomological degree zero) by the complex whose component of degree q is

$$\mathrm{Sym}^*(T) \otimes \wedge^q T^\vee \ , \tag{2.222}$$

and whose differential is

$$d(s \otimes \alpha) = \sum_j (e_j s) \otimes (e^j \wedge \alpha) \tag{2.223}$$

for e_j a basis of T and e^j the dual basis of $T^\vee$.

The complex (2.222) is a version of the Koszul complex. It is multiplicative in T: for $T = T' \oplus T''$, it is the tensor product of the similar complexes of T' and T''. To check its acyclicity except in top degree, it hence suffices to check it for $\dim(T) = 1$, in which case the complex reduces to

$$\mathbb{R}[x] \xrightarrow{x} \mathbb{R}[x] \tag{2.224}$$

in degree 0 and 1. One can also write an explicit homotopy H: for P and α homogeneous, define $H(P\alpha)$ to be 0 if P is of degree 0 and α of maximal degree $\dim(T)$. Otherwise

$$H(P\alpha) := \left(\sum \partial_{e^j}(P) \otimes \iota(e_j)\alpha\right) / (\deg P + \dim T - \deg \alpha) \tag{2.225}$$

One checks that $dH + Hd$ is the identity minus the projection to the part of bidegree 0 in P and $\dim(T)$ in α.

Lemma 2.219 implies Theorem 2.214, as the acyclicity (except at the top) of an associated graded to $\Omega_{\text{loc,multi}}$ implies the same acyclicity for $\Omega_{\text{loc,multi}}$ itself. From the proof of Corollary 2.183 one can in addition deduce a local formula to attach to α such that $d\alpha = 0$ a β for which $\alpha = d\beta$. Indeed, let us cover E by open sets W_ℓ for which one has local coordinates as in (a), (b), (c) before (2.215). On each W_ℓ, a choice of local coordinates splits the filtration F, and the proof of Lemma 2.219 gives a homotopy operator H_ℓ of filtration -1 such that outside of top degree $\mathrm{Id} - (dH_\ell + H_\ell d)$ is of filtration -1, i.e., maps F_N to F_{N-1}. Let χ_ℓ be a partition of unity subordinate to the covering W_ℓ and define $H(\alpha) = \sum H_\ell(\chi_\ell \alpha)$. We again have $\mathrm{Id} - (dH + Hd)$ of filtration -1. If α is in F_N, and closed,

$$\alpha' := \alpha - (dH + Hd)\alpha = \alpha - d(H\alpha) \tag{2.226}$$

is in F_{N-1}. One has $\alpha = dH\alpha + \alpha'$, α' is closed, and repeating the argument for α', one eventually obtains β with $\alpha = d\beta$.

CHAPTER 3
Free Field Theories

§3.1. Coordinates on Minkowski spacetime

Let M denote affine n-dimensional Minkowski spacetime. Fix a coordinate system $x^0, x^1, \ldots, x^{n-1}$ with respect to which the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{3.1}$$

Note that $x^0 = ct$ for t the standard time coordinate. The wave operator on functions is

$$\begin{aligned} \Box &= (-1)^{n-1} * d * d \\ &= -d^* d \\ &= \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 \\ &= g^{\mu\nu}\partial_\mu\partial_\nu, \end{aligned} \tag{3.2}$$

where we use the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{3.3}$$

The canonical density is

$$|d^n x| = |\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \,|. \tag{3.4}$$

For a function f we have

$$d * df = \Box f \; |d^n x|. \tag{3.5}$$

The symbol 'g' usually denotes the Minkowski metric, though occasionally it denotes a general metric as is clear from the context.

Lagrangians in Minkowski spacetime are real (see Chapter 7). Thus equations involving complex quantities have complex conjugate equations which also hold, and which we usually omit.

We define global charges by integrating over the spacelike submanifold $\{x^0 = 0\}$ with canonical density

$$|d^{n-1} x| = |\, dx^1 \wedge \cdots \wedge dx^{n-1} \,|. \tag{3.6}$$

Indices $\mu, \nu, \ldots$ run from 0 to $n-1$; indices $i, j, \ldots$ run from 1 to $n-1$.

§3.2. Real scalar fields

A real scalar field is a map $\phi\colon M \to \mathbb{R}$. The free lagrangian for a field of mass $m \geq 0$ is

$$\begin{aligned} L &= \Big\{\frac{1}{2}\,|d\phi|^2 - \frac{m^2}{2}\,\phi^2\Big\}\,|d^n x| \\ &= \frac{1}{2}\,d\phi \wedge *d\phi - \frac{m^2}{2}\,\phi^2\,|d^n x| \\ &= \Big\{\frac{1}{2} g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \frac{m^2}{2}\phi^2\Big\}\,|d^n x|. \end{aligned} \tag{3.7}$$

It is instructive to check the units in the lagrangian. Comparing the two terms we see that m must have units L^{-1} (to match the units of d). In a relativistic quantum theory we can replace m by $mc/\hbar$; then m has units of mass. Of course, in that context we usually work in a natural system of units with $c = \hbar = 1$. In that case ϕ has mass dimension $(n-2)/2$. The differential of L along the space of fields ϕ is

$$\begin{aligned} \delta L &= -d\delta\phi \wedge *d\phi - m^2\phi\,\delta\phi\,|d^n x| \\ &= -\delta\phi \wedge \big\{d * d\phi + m^2\phi\,|d^n x|\big\} - d\big\{\delta\phi \wedge *d\phi\big\}. \end{aligned} \tag{3.8}$$

So from (3.5) the classical field equation is

$$(\Box + m^2)\phi = 0 \tag{3.9}$$

and the variational 1-form is

$$\gamma = \delta\phi \wedge *d\phi. \tag{3.10}$$

The local symplectic form $\omega = \delta\gamma$ is

$$\omega = *d\delta\phi \wedge \delta\phi. \tag{3.11}$$

Equation (3.9) is most easily analyzed through the Fourier transform

$$\begin{aligned} \hat\phi(k) &= \frac{1}{(2\pi)^{n/2}} \int_V e^{-\sqrt{-1}\langle k,x\rangle}\phi(x)\,|d^n x|, \\ \phi(x) &= \frac{1}{(2\pi)^{n/2}} \int_{V^*} e^{+\sqrt{-1}\langle k,x\rangle}\hat\phi(k)\,|d^n k|. \end{aligned} \tag{3.12}$$

Then equation (3.9) transforms to

$$\big(-|k|^2 + m^2\big)\hat\phi(k) = 0. \tag{3.13}$$

Thus the Fourier transform $\hat\phi$ is supported on the *mass shell*

$$\mathcal{O}_m = \{k \in M : |k|^2 = m^2\}. \tag{3.14}$$

Note that since ϕ is real we have

$$\hat\phi(-k) = \overline{\hat\phi(k)}. \tag{3.15}$$

Write $k = (k_0, \ldots, k_{n-1})$; then k_i has units L^{-1} and is called the *wave number*. The *frequency* $\omega = k_0 c$ has units T^{-1}.

We can formulate the theory for an arbitrary metric g on M as

$$L_g = \left(\frac{1}{2} g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - \frac{m^2}{2}\,\phi^2\right) \mu_g(x), \tag{3.16}$$

where μ_g is the canonical density associated to the metric g. Now a computation shows

$$\delta\mu_g = \frac{1}{2} g^{-1} \cdot \delta g\ \mu_g = -\frac{1}{2}\delta g^{-1} \cdot g\ \mu_g, \tag{3.17}$$

where

$$\delta g^{-1} \cdot g = \delta g^{\mu\nu} g_{\mu\nu}. \tag{3.18}$$

So the energy-momentum tensor, as defined in (2.176), is

$$T_g = d\phi \cdot d\phi\ \mu_g - g\, L_g. \tag{3.19}$$

In local coordinates we write (see (2.177))

$$T_{\mu\nu} = \partial_\mu\phi\, \partial_\nu\phi + \left(-\frac{1}{2}|d\phi|^2 + \frac{m^2}{2}\phi^2\right) g_{\mu\nu}. \tag{3.20}$$

Specialize to Minkowski spacetime. The energy density $T_{00}\,|d^{n-1}x|$ is given by

$$T_{00} = \left(\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\sum_{i=1}^{n-1}(\partial_i\phi)^2 + \frac{m^2}{2}\phi^2\right). \tag{3.21}$$

Note that this is nonnegative and only vanishes for $\phi \equiv 0$. Minus the momentum density $T_{i0}\,|d^{n-1}x|$ is given by

$$T_{i0} = \partial_i\phi\partial_0\phi, \qquad i = 1, 2, \ldots, n-1. \tag{3.22}$$

By Proposition 2.190 this agrees with the energy-momentum tensor (2.160) given by the canonical Noether currents of translations, as we now verify directly. The vector field ξ_μ induced by ∂_μ is

$$\begin{aligned} \iota(\xi_\mu)dx^\nu &= \delta_\mu^\nu \\ \iota(\xi_\mu)\delta\phi &= -\partial_\mu\phi. \end{aligned} \tag{3.23}$$

Then *minus* the associated Noether current is

$$\begin{aligned} -\iota(\xi_\mu)(L+\gamma) &= -L * dx^\mu + \partial_\mu\phi * d\phi \\ &= (-g_{\mu\nu}L + \partial_\mu\phi\partial_\nu\phi) * dx^\nu, \end{aligned} \tag{3.24}$$

where $g_{\mu\nu}$ is the Lorentz metric.

The "trace" of T is

$$g^{\mu\nu}T_{\mu\nu} = (1 - \frac{n}{2}) g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi + \frac{m^2 n}{2}\phi^2, \tag{3.25}$$

which vanishes if $n = 2$ and $m = 0$. Thus the theory of a massless scalar field is conformally invariant in 2 dimensions.

§3.3. Complex scalar fields

Our convention is that if '$\langle\cdot,\cdot\rangle$' denotes a real bilinear form, then it also denotes the extension to a *bilinear* form over the complex numbers. If $\langle\cdot,\cdot\rangle$ is a real inner product on a real vector space W, then the associated hermitian norm on the complexification $W_{\mathbb{C}}$ is

$$w \longmapsto \langle \bar{w}, w\rangle. \tag{3.26}$$

Over the reals or complexes we always use the notation

$$|w|^2 = \langle w, w\rangle. \tag{3.27}$$

A complex scalar field is a map $\Phi\colon M \to \mathbb{C}$. The free lagrangian for a field of mass $m \geq 0$ is

$$L = \Big\{ \langle d\overline{\Phi}, d\Phi\rangle - m^2\langle\overline{\Phi}, \Phi\rangle \Big\} |d^n x|. \tag{3.28}$$

This theory is equivalent to a theory of two uncoupled free real scalars ϕ_1, ϕ_2; simply set

$$\Phi = \frac{\phi_1 + \sqrt{-1}\,\phi_2}{\sqrt{2}}. \tag{3.29}$$

Then (3.28) reduces to the sum of two copies of (3.7). The variation of (3.28) is

$$\begin{aligned} \delta L &= \big\{ -\langle d\delta\overline{\Phi} \wedge d\Phi\rangle + \langle d\overline{\Phi} \wedge d\delta\Phi\rangle - m^2\delta\overline{\Phi}\,\Phi - m^2\overline{\Phi}\,\delta\Phi \big\}\, |d^n x| \\ &= -\delta\overline{\Phi}\big\{ d * d\Phi + m^2\Phi\,|d^n x| \big\} - \delta\Phi\big\{ d * d\overline{\Phi} + m^2\overline{\Phi}\,|d^n x| \big\} - d\gamma \end{aligned} \tag{3.30}$$

for the variational 1-form

$$\gamma = \delta\overline{\Phi} \wedge *d\Phi + \delta\Phi \wedge *d\overline{\Phi}. \tag{3.31}$$

So the equation of motion is

$$(\Box + m^2)\Phi = 0. \tag{3.32}$$

The analysis of (3.32) proceeds as in the real case. Note the significant difference that there is no reality condition (3.15) in the Fourier transform.

The circle group $\mathbb{T} \subset \mathbb{C}$ operates on Φ by scalar multiplication, and this symmetry manifestly preserves the lagrangian. (In the language of §2 it is an "internal" symmetry.) The corresponding infinitesimal symmetry ξ is

$$\iota(\xi)\delta\Phi = \sqrt{-1}\Phi. \tag{3.33}$$

(Of course, we also have the conjugate equation $\iota(\xi)\delta\overline{\Phi} = -\sqrt{-1}\ \overline{\Phi}$.) So the associated Noether current (see (2.88)) is

$$\begin{aligned} j_\xi &= \iota(\xi)\big(L + \gamma\big) \\ &= \sqrt{-1}\big(\Phi * d\overline{\Phi} - \overline{\Phi} * d\Phi\big). \end{aligned} \tag{3.34}$$

The corresponding global charge is

$$Q_\xi = \sqrt{-1}\int_{\{x^0=0\}} (\Phi\partial_0\overline{\Phi} - \overline{\Phi}\partial_0\Phi)\,|d^{n-1}x|. \tag{3.35}$$

The energy-momentum tensor is

$$T_g = d\overline{\Phi}\cdot d\Phi\;\mu_g - g\,L_g. \tag{3.36}$$

or (see (2.177))

$$T_{\mu\nu} = \partial_\mu\overline{\Phi}\,\partial_\nu\Phi + \Big(-\frac{1}{2}\langle d\overline{\Phi}, d\Phi\rangle + \frac{m^2}{2}\langle\overline{\Phi},\Phi\rangle\Big)\,g_{\mu\nu}. \tag{3.37}$$

We evaluate the energy-momentum tensor for a plane wave

$$\Phi(x) = ae^{\imath k(x)}, \tag{3.38}$$

where $k = (k_0, \ldots, k_{n-1})$ is in the dual of V, the vector space underlying M. We restore the constants necessary to treat m as a mass, so replace m by $mc/\hbar$. Components of the *wave number* $(k_1, \ldots, k_{n-1})$ have units L^{-1} and the *frequency* $\omega = k_0 c$ has units T^{-1}. The complex constant a has units $\sqrt{M/TL^{n-4}}$. By the equation of motion (3.32), we have

$$|k|^2 = m^2c^2/\hbar^2. \tag{3.39}$$

Then (3.37) reduces to

$$T_{\mu\nu} = |a|^2 k_\mu k_\nu. \tag{3.40}$$

§3.4. Spinor fields

Let V be the vector space underlying Minkowski spacetime M, and let $\mathrm{Spin}(V)$ denote the Lorentz group. Suppose S is a *real* spin representation of $\mathrm{Spin}(V)$. This means that there are symmetric pairings

$$\begin{aligned} \Gamma\colon S^*\otimes S^* &\longrightarrow V \\ \tilde{\Gamma}\colon S\otimes S &\longrightarrow V \end{aligned} \tag{3.41}$$

which satisfy a Clifford relation. Let $\{e_\mu\}$ be a basis of V and $\{f^a\}$ a basis of S. We use the dual bases $\{e^\mu\}, \{f_a\}$ for V^*, S^*. Write

$$\begin{aligned} \Gamma(f_a, f_b) &= \Gamma^\mu_{ab} e_\mu, \\ \tilde{\Gamma}(f^a, f^b) &= \tilde{\Gamma}^{\mu ab} e_\mu; \end{aligned} \tag{3.42}$$

then the Clifford relation is

$$\tilde{\Gamma}^{\mu ab}\Gamma^\nu_{bc} + \tilde{\Gamma}^{\nu ab}\Gamma^\mu_{bc} = 2g^{\mu\nu}\delta^a_c. \tag{3.43}$$

Let ΠS denote the odd vector space which is S with odd parity. A *spinor field* (spin 1/2 fermion) is a map $\psi\colon M \to \Pi S$. With respect to the basis we write

$$\psi(x) = \psi_a(x) f^a. \tag{3.44}$$

A *dual spinor field* is a map $\lambda\colon M \to \Pi S^*$. The analysis of spinor fields below adapts easily to dual spinor fields.

A mass pairing for spinor fields is a skew-symmetric pairing

$$M\colon \textstyle\bigwedge^2 S \longrightarrow \mathbb{R}. \tag{3.45}$$

Nonzero pairings need not exist.[23] If there is a nonzero mass pairing M, then there exists a *normalized* skew-symmetric pairing

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R}; \tag{3.46}$$

the normalization condition is

$$\tilde{\Gamma}^{\mu ab} = \Gamma^{\mu}_{a'b'} \epsilon^{aa'} \epsilon^{bb'}. \tag{3.47}$$

For simplicity assume that ϵ is irreducible. Then we can write

$$M = m\epsilon, \qquad m \in \mathbb{R}. \tag{3.48}$$

We write

$$\psi M \psi = M^{ab} \psi_a \psi_b, \tag{3.49}$$

where the name of a bilinear form is written between the arguments. The kinetic term in the free lagrangian is the *Dirac form*

$$\psi \not{D} \psi = \tilde{\Gamma}^{\mu}(\psi, \partial_\mu \psi) = \tilde{\Gamma}^{\mu ab} \psi_a \, \partial_\mu \psi_b. \tag{3.50}$$

Equation (3.50) defines a symmetric (in the graded sense) bilinear form up to an exact term, so it is exactly symmetric after integrating over M, assuming no contribution at infinity. The free lagrangian is

$$L = \left\{ \frac{1}{2} \psi \not{D} \psi - \frac{1}{2} \psi M \psi \right\} |d^n x|. \tag{3.51}$$

The variation of L is

$$\delta L = \left\{ \frac{1}{2} \delta\psi \not{D} \psi + \frac{1}{2} \psi \not{D} \delta\psi - \delta\psi \, M \, \psi \right\} |d^n x|. \tag{3.52}$$

Now

$$\begin{aligned} \psi \not{D} \delta\psi &= \tilde{\Gamma}^{\mu ab} \psi_a \, \partial_\mu \delta\psi_b \\ &= -\tilde{\Gamma}^{\mu ab} \partial_\mu \psi_a \, \delta\psi_b + \tilde{\Gamma}^{\mu ab} \partial_\mu (\psi_a \, \delta\psi_b), \end{aligned} \tag{3.53}$$

[23] For example, for $n = 2$ take S to be a half-spinor representation. It is one-dimensional, so $\bigwedge^2 S = 0$.

and so

$$\delta L = \delta\psi\Big(\not{D}\psi - M\,\psi\Big)\,|d^n x| - d\gamma \tag{3.54}$$

for the variational 1-form

$$\begin{aligned}\gamma &= \frac{1}{2}\tilde{\Gamma}^{\mu}(\psi, \delta\psi)\,\iota(\partial_\mu)|d^n x| \\ &= \frac{1}{2}(\tilde{\Gamma}^{\mu ab}\psi_a\,\delta\psi_b)\,\iota(\partial_\mu)|d^n x|.\end{aligned} \tag{3.55}$$

Now M determines a map $S \to S^*$ and $\tilde{\Gamma}$ a map $S \to V \otimes S^*$, so that both M and $\not{D}$ define operators from spinor fields to dual spinor fields. We use the same letters $M, \not{D}$ to denote these operators. Then the equation of motion is

$$\not{D}\psi = M\psi. \tag{3.56}$$

We analyze (3.56) via the Fourier transform (3.12):

$$\sqrt{-1}\,\tilde{\Gamma}^{\mu ab}k_\mu\hat{\psi}_b(k) = M^{ab}\hat{\psi}_b(k) \qquad \text{for all } a. \tag{3.57}$$

Multiply both sides of (3.57) by $\sqrt{-1}\,\tilde{\Gamma}^{\nu}_{ca}k_\nu$. After some simplification using (3.43), (3.47), and (3.57) we find

$$|k|^2\hat{\psi}_c(k) = m^2\hat{\psi}_c(k) \qquad \text{for all } c. \tag{3.58}$$

Thus the Fourier transform of a solution is supported on the mass shell $\mathcal{O}_m$. Furthermore, equation (3.57) defines a subbundle $\Pi S'$ of the trivial bundle $\mathcal{O}_m \times \Pi S$, and $\hat{\psi}$ is a section of $\Pi S'$. The rank of S' is[24] $\dim S/2$. Again we have a reality condition (3.15). For $m > 0$ the subgroup of $\mathrm{Spin}(V)$ which stabilizes $k \in \mathcal{O}_m$ is isomorphic to $\mathrm{Spin}(n-1)$; the fiber of S' at k is a spin representation of $\mathrm{Spin}(n-1)$. For $m = 0$ the stabilizer has reductive part isomorphic to $\mathrm{Spin}(n-2)$, and the fiber of S' at k is a spin representation of $\mathrm{Spin}(n-2)$.

In some cases the minimal real spin representation S admits an action of $\mathbb{C}$ or $\mathbb{H}$ commuting with $\mathrm{Spin}(V)$. Then if $\tilde{\Gamma}$ is sesquilinear we have an action of $\mathbb{T}$ or Sp_1 which preserves the kinetic term $\psi\not{D}\psi$. (The mass pairing (3.45) may break this symmetry.) If there are multiple copies of the spin representation, then there is a larger compact group of manifest symmetries which rotate the various copies. Each such symmetry has a Noether current which may be computed from (3.55).

As for the energy-momentum tensor we first compute minus the canonical Noether current associated to translations, as in (2.160). Translation ∂_μ induces a vector field ξ_μ with

$$\begin{aligned}\iota(\xi_\mu)dx^\nu &= \delta^\nu_\mu \\ \iota(\xi_\mu)\delta\psi &= -\partial_\mu\psi.\end{aligned} \tag{3.59}$$

Using (3.51) and (3.55) we compute $-\iota(\xi_\mu)(L+\gamma) = \Theta_{\mu\nu} * dx^\nu$, where

$$\Theta_{\mu\nu} = \frac{1}{2}\tilde{\Gamma}^{\sigma ab}\psi_a\partial_\mu\psi_b g_{\nu\sigma} - (\frac{1}{2}\psi\not{D}\psi - \frac{1}{2}\psi M\psi)g_{\mu\nu}. \tag{3.60}$$

This is *not* symmetric in μ, ν (as expected).

[24]There is an exceptional case: For a chiral spinor field in 2 dimensions equation (3.57) says that the Fourier transform is supported on half of the forward lightcone. In other words, a solution to the Dirac equation is either left- or right-moving, depending on the chirality. See [**I-Supersolutions**, §2.6] for the precise formulas.

§3.5. Abelian gauge fields

We restrict ourselves to the massless case. In 3 dimensions the Chern-Simons functional serves as a mass term (see Problem FP4 of [I-**Homework**]). In any dimension the Higgs mechanism can be used to introduce a mass. For another discussion of this material, see the solution to Problem FP3 of [I-**Homework**].

The standard physical interpretation (Maxwell's equations) is discussed in §4.1. The basics of principal bundles and connections are reviewed in §4.2.

We choose as gauge group the circle group $\mathbb{T}$ of unit norm complex numbers. On the Lie algebra $\mathrm{Lie}(\mathbb{T}) \cong \sqrt{-1}\,\mathbb{R}$ we have the positive definite inner product

$$\langle a, b\rangle = -ab, \qquad a, b \in \sqrt{-1}\,\mathbb{R}. \tag{3.61}$$

The field in a gauge theory is a connection A on M with structure group $\mathbb{T}$, and we work with fields up to gauge equivalence. The space of equivalence classes is a real affine space. The lagrangian is

$$\begin{aligned} L &= -\frac{1}{2}\langle F_A \wedge *F_A\rangle \\ &= -\frac{1}{2}|F_A|^2\,|d^n x| \\ &= -\frac{1}{4}\langle F_{\mu\nu}, F_{\mu'\nu'}\rangle\, g^{\mu\mu'} g^{\nu\nu'}\,|d^n x|, \end{aligned} \tag{3.62}$$

where the curvature is

$$F_A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{3.63}$$

Note that since

$$*(dx^\mu \wedge dx^\nu) = g^{\mu\mu'} g^{\nu\nu'} \iota(\partial_{\nu'})\iota(\partial_{\mu'})|d^n x|, \tag{3.64}$$

we have

$$*F_A = \frac{1}{2} F_{\mu\nu}\, g^{\mu\mu'} g^{\nu\nu'} \iota(\partial_{\nu'})\iota(\partial_{\mu'})|d^n x|. \tag{3.65}$$

Now δA is an imaginary 1-form on $\mathcal{F}_A \times M$, and

$$\delta F_A = -d\delta A; \tag{3.66}$$

the sign comes from commuting δ past d. Thus the variation of the lagrangian is

$$\begin{aligned} \delta L &= \langle d\delta A \wedge *F_A\rangle \\ &= d\langle \delta A \wedge *F_A\rangle - \langle \delta A \wedge d * F_A\rangle. \end{aligned} \tag{3.67}$$

Hence the canonical 1-form is

$$\begin{aligned} \gamma &= -\langle \delta A \wedge *F_A\rangle \\ &= -\langle \delta A_\mu, F_{\nu\sigma}\rangle g^{\mu\sigma} g^{\nu\rho}\, \iota(\partial_\rho)|d^n x| \end{aligned} \tag{3.68}$$

and the classical equation of motion is

$$d * F_A = 0, \tag{3.69}$$

or

$$g^{\rho\mu}\,\partial_\rho F_{\mu\nu} = 0 \qquad \text{for all } \nu. \tag{3.70}$$

This is an affine equation for A, and since gauge transformations act by an affine action, the moduli space $\overline{\mathcal{M}}$ of classical solutions modulo gauge equivalence is an affine subspace of the space of fields modulo equivalence. The local symplectic form is

$$\begin{aligned} \omega &= \langle \delta A \wedge *d\delta A \rangle \\ &= \langle (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \wedge \delta A_\sigma \rangle\, g^{\nu\sigma} g^{\mu\rho}\, \iota(\partial_\rho) |d^n x|. \end{aligned} \tag{3.71}$$

Of course, the detailed analysis involves choosing an origin, so we may as well from the beginning work with the vector space of translations associated to connections. Hence in this paragraph only we take the field to be a *real* 1-form α up to addition of an exact real function. The lagrangian is

$$L = -\frac{1}{2} d\alpha \wedge *d\alpha = -\frac{1}{2} |d\alpha|^2\, |d^n x|, \tag{3.72}$$

and the equation of motion is

$$d * d\alpha = 0. \tag{3.73}$$

As usual, we consider the Fourier transform $\hat{\alpha} = \hat{\alpha}_\mu dx^\mu$ on V^*. The vector space of classical solutions is the first cohomology space of the complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{*d*d} \Omega^1(M). \tag{3.74}$$

On the Fourier transforms we have

$$\begin{aligned} d\hat{f}(k) &= \sqrt{-1}\, \hat{f}(k) k \\ *\, d * d\hat{\alpha}(k) &= |k|^2 \hat{\alpha}(k) - \langle \hat{\alpha}(k), k \rangle k. \end{aligned} \tag{3.75}$$

An easy argument shows that the first cohomology is isomorphic to the collection of Real functions $\{\hat{\alpha}\colon \text{lightcone} \to V^* \otimes \mathbb{C}\}$ on the lightcone which satisfy $\langle \hat{\alpha}(k), k \rangle = 0$ modulo the set of functions $\{\hat{\alpha}(k) = \hat{f}(k)k\}$, where $\hat{f}$ ranges over the complex functions on the lightcone. (The Reality condition is that of the Fourier transform.) The subgroup of $\mathrm{Spin}(V)$ which stabilizes k has reductive part isomorphic to $\mathrm{Spin}(n-2)$, and $\hat{\alpha}$ is a section of a bundle whose fiber at k is the vector representation of $\mathrm{Spin}(n-2)$.

Return now to the abelian gauge field and lagrangian (3.62) To compute the energy-momentum tensor, we first compute the Noether current j_μ of *minus* translation by ∂_μ (see (2.160)). Note that ∂_μ induces the vector field $\hat{\xi}_\mu$ on the space $\mathcal{F}_A$ of fields defined by (see (4.40))

$$\hat{\xi}_\mu A = -\iota(\partial_\mu) F_A. \tag{3.76}$$

Let $\xi_\mu = \partial_\mu + \hat{\xi}_\mu$. Then

$$\begin{aligned} j_\mu &= -\iota(\partial_\mu + \hat{\xi}_\mu)(L + \gamma) \\ &= -\frac{1}{2}\big[\iota(\partial_\mu)F_A \wedge *F_A - F_A \wedge \iota(\partial_\mu) * F_A\big]. \end{aligned} \tag{3.77}$$

We find after some computation that

$$\Theta_{\mu\nu} = -\langle F_{\mu\rho}, F_{\nu\sigma}\rangle\, g^{\rho\sigma} + \frac{1}{2}|F_A|^2 g_{\mu\nu}, \tag{3.78}$$

where $\Theta_{\mu\nu}$ is defined in (2.160). Notice that it is symmetric.

We can compute the energy-momentum tensor more directly using the alternative definition (2.176). Couple the lagrangian (3.62) to an arbitrary metric g:

$$L_g = -\frac{1}{2}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle\, g^{\mu\rho} g^{\nu\sigma} \mu_g(x). \tag{3.79}$$

Using (3.17) and differentiating with respect to g^{-1} we find the energy-momentum tensor to be (see (2.177))

$$T_{\mu\nu} = -\langle F_{\mu\rho}, F_{\nu\sigma}\rangle g^{\rho\sigma} + \frac{1}{2}|F_A|^2 g_{\mu\nu}, \tag{3.80}$$

which agrees with (3.78), as it must by Proposition 2.190. In (4.8) we rewrite (3.80) in terms of the electric and magnetic components.

For a 1-form field α the energy-momentum tensor computed by differentiating the metric is again (3.80) (where we interpret $F_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu$). However, the canonical Noether current j_μ differs from (3.77) since the induced vector field $\hat{\xi}_\mu$ acts with an additional term over (3.76):

$$\hat{\xi}_\mu \alpha = -\operatorname{Lie}(\partial_\mu)\alpha = -\iota(\partial_\mu) d\alpha - d\iota(\partial_\mu)\alpha. \tag{3.81}$$

Then j_μ picks up an extra term

$$-d\iota(\partial_\mu)\alpha \wedge *d\alpha = -d\big(\iota(\partial_\mu)\alpha \cdot *d\alpha\big) + \iota(\partial_\mu)\alpha \cdot d * d\alpha. \tag{3.82}$$

This is written as the sum of an exact term and a term which vanishes on-shell.

The "trace" of T,

$$g^{\mu\nu} T_{\mu\nu} = (\frac{n}{2} - 2)|F_A|^2, \tag{3.83}$$

vanishes when $n = 4$. This says that the lagrangian (3.62) is conformally invariant in $n = 4$ dimensions.

CHAPTER 4
Gauge Theory

§4.1. Classical electromagnetism

As in §3.5 we consider an abelian gauge field on Minkowski space M. To make contact with the usual formulas of electromagnetism we take the gauge group to be the multiplicative group $\mathbb{R}^{>0}$ rather than the circle group $\mathbb{T}$. (In classical physics the electric charge is not quantized, so this makes more physical sense.) Thus the gauge field A is a connection on a principal $\mathbb{R}^{>0}$-bundle. The curvature F_A is a *real* 2-form. The formulas of §3.5 hold, only now (3.61) is replaced by

$$\langle a, b\rangle = ab, \qquad a, b \in \mathbb{R}. \tag{4.1}$$

We work in any dimension n, though of course the electromagnetism of our world is $n = 4$. Choose a splitting $M = M^1 \times N$ of Minkowski spacetime into time $\times$ space. For now N can be any Riemannian manifold. Let $x^0 = ct$ be the standard coordinate on M^1. Define the *electric* and *magnetic* fields

$$\begin{aligned} E_A &\in \Omega^1(N) \\ B_A &\in \Omega^2(N) \end{aligned} \tag{4.2}$$

by

$$F_A = B_A - dt \wedge E_A. \tag{4.3}$$

E_A and B_A depend on time t, but neither contains dt:

$$\iota(\partial_0)E_A = \iota(\partial_0)B_A = 0. \tag{4.4}$$

Let $*_N$ be the Hodge star operator for the *positive* definite metric on N. Then if $*$ is the Hodge $*$ operator on $M^1 \times N$ (with its usual metric of signature $+---\cdots$), we have

$$*F_A = \frac{1}{c} *_N E_A + c\, dt \wedge *_N B_A. \tag{4.5}$$

So the lagrangian (3.62) is

$$L = \frac{1}{2}\Big(\frac{|E_A|^2}{c^2} - |B_A|^2\Big)\, |d^n x|, \tag{4.6}$$

where the norms of the forms E_A, B_A are computed in the positive definite metric on N. A multiplicative constant is usually inserted in L, depending on the units used; we omit it.[25] The equation of motion (3.69) together with the Bianchi identity $dF_A = 0$ comprise Maxwell's law in empty space:

$$\begin{aligned} dB_A &= 0 & dE_A &= -\frac{\partial B_A}{\partial t} \\ d{*_N}E_A &= 0 & c^2\, d{*_N}B_A &= {*_N}\frac{\partial E_A}{\partial t} \end{aligned} \tag{4.7}$$

The initial components of the energy-momentum tensor (3.80) reduce to

$$\begin{aligned} T_{00} &= \frac{1}{2}\Big(\frac{|E_A|^2}{c^2} + |B_A|^2\Big) \\ -T_{i0} &= \sum_{j=1}^{n-1}(B_A)_{ij}(E_A)_j, \end{aligned} \tag{4.8}$$

the classical expressions for the energy density and the Poynting vector, while the space components T_{ij} comprise the Maxwell stress tensor. (Recall that $-T_{i0}$ integrates to the field momentum.)

We can couple the electromagnetic field to a current $J \in \Omega^{|-1|}(M)$. The lagrangian density is

$$L = -\frac{1}{2}F_A \wedge {*F_A} + J \wedge A. \tag{4.9}$$

The second term is not well-defined in general, since A is a connection, not a 1-form. In fact, if J is a *conserved* current

$$\cdot \quad dJ = 0, \tag{4.10}$$

then we can write $J = dK$ and so by Stokes' theorem

$$\int_M J \wedge A = -\int_M K \wedge F_A \tag{4.11}$$

is well-defined, assuming suitable decay at infinity. (We discuss such "topological terms" in Chapter 6.) A typical conserved current is the Poincaré dual of a *closed* curve in M, in which case (4.11) is the holonomy of A around the curve. In coordinates we write

$$J = J^\mu\, \iota(\partial_\mu)|d^n x|, \tag{4.12}$$

where $J^0 = c\rho$ is c times the *charge density* ρ and minus the spatial components $-J^i$ comprise the *current density* j:

$$J = c\rho\, |d^{n-1}x| - c\, dt \wedge j. \tag{4.13}$$

[25] In the "mks" system of units, the constant is written $\epsilon_0 c^2 = 10^7/4\pi$.

We can treat J as an external field, which is fixed, or as a dynamical field which varies. For example, we can take J to be Poincaré dual to the path of a dynamical moving point charge of rest mass m_0 and charge q. In that case we add the kinetic term (1.22) for the point particle to the lagrangian. Then the fields are the particle $x\colon \mathbb{R}^1 \to M$ and the connection A on M. The classical action is

$$S = \int_{T_1}^{T_2} -m_0 c \left|\frac{dx}{d\tau}\right| |d\tau| - q\int_{T_1}^{T_2} x^*A - \int_M \frac{1}{2} F_A \wedge *F_A. \tag{4.14}$$

In these formulas we compute the action for the proper time interval $[T_1, T_2] \subset \mathbb{R}$. The second term should be interpreted as a multiple of the logarithm of the parallel transport of A along a piece of the path x; see §6.1. It is not a real number, but rather lives in a real torsor (depending on the endpoints). Note that the path is oriented.

The equation of motion for the lagrangian (4.9) is

$$d * F_A = J. \tag{4.15}$$

Rewriting in terms of E_A, B_A, ρ, j we have

$$\begin{aligned} d{*_N} E_A &= c^2 \rho \, |d^{n-1}x| \\ c^2 d{*_N} B_A &= {*_N}\frac{\partial E_A}{\partial t} + c^2 j, \end{aligned} \tag{4.16}$$

which are two of Maxwell's equations. (The top two equations in (4.7) remain unchanged.)

The action (4.14) is invariant under reparametrizations of the path x. We compute the equation of motion ignoring boundary terms. The equation for A is a special case of (4.16) where the current J is Poincaré dual to q times the path x. The variation of the first term with respect to the path x—after integration by parts—leads to a contribution to the equation of motion of (see (1.25))

$$-\frac{\partial}{\partial t}\Big(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\Big), \tag{4.17}$$

where v is the velocity measured in our fixed system of coordinates. To vary the second term with respect to x, we use the formula that the variation of the holonomy is the integral of the curvature evaluated on the variation of the path. Thus the contribution of this term to the equation of motion is the vector associated via the metric to the 1-form

$$q\iota\big(\partial/\partial t + v\big)F_A = qE_A - q\iota(v)B_A. \tag{4.18}$$

(We omit the dt component of the 1-form.) So the equation for x is

$$\frac{\partial}{\partial t}\Big(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\Big) = q(E_A - \iota(v)B_A)^*, \tag{4.19}$$

'$*$' denoting the dual vector. In four dimensions this is the Lorentz force law.

§4.2. Principal bundles and connections

In this section we establish some notation and review standard notions. Let M be an arbitrary (super)manifold. All vector fields are even.

Let $P \to M$ be a principal bundle with structure group a Lie group G, often called the *gauge group*. If M is a supermanifold, then so is P. We take G to act on the *right*. Elements $\zeta \in \mathfrak{g}$ in the Lie algebra of G induce vertical vector fields $\hat{\zeta}$ on P, and $\widehat{[\zeta_1, \zeta_2]} = [\hat{\zeta}_1, \hat{\zeta}_2]$.

Suppose V is a space with a *left* G action. In many interesting cases V is a linear space and the action is linear, but it need not be. There is an associated bundle $V^P \to M$ whose sections are equivariant maps $f\colon P \to V$. So f satisfies

$$f(pg) = g^{-1} \cdot f(p), \qquad p \in P, \quad g \in G, \tag{4.20}$$

where '$\cdot$' denotes the action of G on V. Write $g = \exp(t\zeta)$ and differentiate with respect to t to find

$$\hat{\zeta} f = -\zeta \cdot f, \qquad \zeta \in \mathfrak{g}, \tag{4.21}$$

where now '$\cdot$' denotes the infinitesimal action of $\mathfrak{g}$ on V.

The bundle associated to the adjoint action of G on $\mathfrak{g}$ is the adjoint bundle $\mathfrak{g}^P = \operatorname{ad} P = P \times_G \mathfrak{g}$. A section of the adjoint bundle is then a map $\epsilon\colon P \to \mathfrak{g}$ with $\epsilon(pg) = \operatorname{Ad}_{g^{-1}} \epsilon(p)$. Using the infinitesimal $\mathfrak{g}$ action on P we can identify ϵ with a G-invariant vertical vector field $\hat{\epsilon}$ on P. Namely, define $\hat{\epsilon}_p$ to be the vertical vector corresponding to the infinitesimal action by $\epsilon(p)$. Then by (4.21)

$$\hat{\epsilon}_1(\epsilon_2) = -[\epsilon_1, \epsilon_2] \tag{4.22}$$

and the corresponding vertical vector field is $[\hat{\epsilon}_1, \hat{\epsilon}_2]$. The fiber of $\operatorname{ad} P$ at $m \in M$ is the Lie algebra of infinitesimal automorphisms of the fiber of P at m. It acts on the *left*, which explains the minus in (4.22). Equation (4.22) is a special case of the following: If $f\colon P \to V$ is a section of an associated bundle, then

$$\hat{\epsilon} f = -\epsilon \cdot f. \tag{4.23}$$

(Compare (4.21).)

There is also an adjoint bundle of groups $P \times_G G \to M$. Sections of this bundle act as automorphisms of P, often called *gauge transformations*.

A *connection* on P is a G-invariant distribution on P which projects isomorphically onto TM. Thus a vector field η on M has a horizontal lift $\tilde{\eta}$ which is a G-invariant vector field on P. Equivalently, a connection is encoded in a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$ which satisfies

$$\begin{aligned} \iota(\hat{\zeta})A &= \zeta, & \zeta &\in \mathfrak{g}; \\ R_g^* A &= \operatorname{Ad}_{g^{-1}}(A), & g &\in G. \end{aligned} \tag{4.24}$$

Here $R_g\colon P \to P$ is the (right) action of g on P. The *curvature* $F_A \in \Omega^2_P(\mathfrak{g})$ is

$$F_A = dA + \frac{1}{2}[A \wedge A]. \tag{4.25}$$

It follows from (4.24) that $F_A \in \Omega^2_M(\operatorname{ad} P)$ is a 2-form on the base M with values in the adjoint bundle. If η_1, η_2 are vector fields on M, then

$$-\Big([\tilde{\eta}_1, \tilde{\eta}_2] - \widetilde{[\eta_1, \eta_2]}\Big) \tag{4.26}$$

is a G-invariant vertical vector field on P, and by (4.25) it corresponds to the section

$$\iota(\eta_2)\iota(\eta_1)F_A = -\iota([\tilde{\eta}_1, \tilde{\eta}_2])A \tag{4.27}$$

of the adjoint bundle.

Let $\epsilon\colon P \to \mathfrak{g}$ be a section of the adjoint bundle and $\hat{\epsilon}$ the associated G-invariant vertical vector field. Then for any vector field η on M,

$$[\tilde{\eta}, \hat{\epsilon}] = \widehat{\tilde{\eta}\epsilon}. \tag{4.28}$$

A connection A on P induces a covariant derivative ∇ on any associated bundle $V^P = P \times_G V$. For a vector field η on M, and a section $f\colon P \to V$ of V^P,

$$\nabla_\eta f = \tilde{\eta} f. \tag{4.29}$$

The covariant derivative may be viewed as an operator $d_A\colon \Omega^0(V^P) \to \Omega^1(V^P)$, defined by

$$d_A f = (d + A)f. \tag{4.30}$$

If G is a linear algebraic group, then we have the Tannakian statement: A connection on P is equivalent to a system of connections on all associated vector bundles which is compatible with the tensor product of representations.

Automorphisms of P—gauge transformations—act on connections by pullback. An automorphism $\varphi\colon P \to P$ may be represented by an equivariant map $g\colon P \to G$, defined by

$$\varphi(p) = p \cdot g(p). \tag{4.31}$$

Then if $A \in \Omega^1_P(\mathfrak{g})$ is a connection, we have

$$\varphi^* A = \operatorname{Ad}_{g^{-1}} A + g^{-1} dg, \tag{4.32}$$

where $g^{-1}dg$ is the pullback by g of the Maurer-Cartan form on G.

The *Bianchi identity* is simply the Jacobi identity for horizontal vector fields on P:

$$[[\nabla_{\eta_1}, \nabla_{\eta_2}], \nabla_{\eta_3}] + [[\nabla_{\eta_2}, \nabla_{\eta_3}], \nabla_{\eta_1}] + [[\nabla_{\eta_3}, \nabla_{\eta_1}], \nabla_{\eta_2}] = 0. \tag{4.33}$$

Equivalently,

$$d_A F_A = 0, \tag{4.34}$$

where $d_A\colon \Omega^2_M(\operatorname{ad} P) \to \Omega^3_M(\operatorname{ad} P)$ is the extension of the differential d using the connection A.

The set of all connections on P is an affine space $\mathcal{A}_P$. One sees this easily from (4.24), which are affine equations. The associated vector space of translations is $\Omega^1_M(\operatorname{ad} P)$. The group $\mathcal{G}_P$ of gauge transformations acts on $\mathcal{A}_P$ on the left using pullback by the inverse (or pushforward of the associated horizontal distribution).

Let $\operatorname{Aut} P$ denote the group of all diffeomorphisms $\varphi\colon P \to P$ which commute with the right G action. Such a φ induces a diffeomorphism of M, and there is an exact sequence

$$1 \longrightarrow \mathcal{G}_P \longrightarrow \operatorname{Aut} P \longrightarrow \operatorname{Diff} M. \tag{4.35}$$

The last map may not be onto; the image is the subgroup of Diff M which preserves the topological type of $P \to M$. Also, the sequence (4.35) is not usually split. Infinitesimally—at the level of Lie algebras—we do have a short exact sequence

$$0 \longrightarrow \operatorname{Lie}(\mathcal{G}_P) \longrightarrow \operatorname{Lie}(\operatorname{Aut} P) \longrightarrow \mathcal{X}(M) \longrightarrow 0. \tag{4.36}$$

Furthermore, a connection A gives a splitting as vector spaces (but not as Lie algebras): to a vector field $\eta \in \mathcal{X}(M)$ we attach its horizontal lift $\tilde{\eta}$.

It is often more geometric (and more physical) not to fix a particular bundle P. Then instead of a single affine space of connections, we study the *category* $\mathcal{C}_M(G)$ of all connections on all principal G-bundles over M. A morphism in $\mathcal{C}_M(G)$ is an isomorphism of principal bundles which preserves the given connections. The set of equivalence classes $\overline{\mathcal{C}_M(G)}$ may be identified as a disjoint union of spaces $\mathcal{A}_P/\mathcal{G}_P$, where P runs over a set of representatives of topological types of G-bundles on M. In a lagrangian field theory including connections, the category $\mathcal{C}_M(G)$ is part of the "space" of fields.

Fix a bundle P. On the product bundle

$$\mathbf{P} = \mathcal{A}_P \times P \longrightarrow \mathcal{A}_P \times M \tag{4.37}$$

there is a universal connection $\mathbf{A}$. Its restriction to $\{A\} \times P$ is A, and its restriction to $\mathcal{A}_P \times \{p\}$ is zero. It is straightforward to compute the curvature of $\mathbf{A}$ at (A, m): evaluated on $\eta_1, \eta_2 \in T_m M$ and $\alpha_1, \alpha_2 \in \Omega^1_M(\operatorname{ad} P)$ we obtain

$$\begin{aligned} \iota(\eta_2)\iota(\eta_1)F_{\mathbf{A}} &= \iota(\eta_2)\iota(\eta_1)F_A \\ \iota(\eta)\iota(\alpha)F_{\mathbf{A}} &= \iota(\eta)\alpha \\ \iota(\alpha_2)\iota(\alpha_1)F_{\mathbf{A}} &= 0. \end{aligned} \tag{4.38}$$

These are equations for elements of the fiber $(\operatorname{ad} P)_m$.

The group $\operatorname{Aut} P$ acts on $\mathbf{P}$ (on the left) by the product of its actions on $\mathcal{A}_P$ and P. The universal connection $\mathbf{A}$ is invariant under this action. Lagrangians in gauge theory are gauge invariant functions (partial densities) on $\mathcal{A}_P \times M$ which are computed from $\mathbf{A}$, and the $(\operatorname{Aut} P)$-invariance of $\mathbf{A}$ implies that such lagrangians are invariant under subgroups of Diff M.

From (4.35) we see that Diff M acts on the space $\overline{\mathcal{C}_M(G)}$ of equivalence classes of connections. Also, the discussion after (4.36) shows that there is a canonical lift of $\mathcal{X}(M)$ to vector fields on the universal bundle $\mathbf{P}$. First, if $A \in \mathcal{A}_P$ is a connection and η a vector field on M, then η acts on A using the horizontal lift $\tilde{\eta}$:

$$\operatorname{Lie}(\tilde{\eta})A = \iota(\tilde{\eta})dA = \iota(\eta)F_A. \tag{4.39}$$

Since diffeomorphisms of M act on fields by pullback, the induced vector field $\hat{\xi}_\eta$ on $\mathcal{A}_P$ is

$$\hat{\xi}_\eta = -\iota(\eta)F_A \qquad \text{at } A \in \mathcal{A}_P. \tag{4.40}$$

Thus

$$\xi_\eta = -\iota(\tilde{\eta})F_A + \eta \tag{4.41}$$

is the vector field on $\mathcal{A}_P \times M$ induced by η. Its horizontal lift $\tilde{\xi}_\eta$ to $\mathbf{P}$ (using the universal connection $\mathbf{A}$) is the desired induced vector field on $\mathbf{P}$. For vector fields η_1, η_2 on M, we compute

$$[\hat{\xi}_{\eta_1}, \hat{\xi}_{\eta_2}] = \hat{\xi}_{[\eta_1,\eta_2]} - d_A\big(F_A(\eta_1, \eta_2)\big) \tag{4.42}$$

$$[\tilde{\eta}_1, \tilde{\xi}_{\eta_2}] = \widetilde{[\eta_1, \eta_2]} - F_A(\eta_1, \eta_2) \tag{4.43}$$

Equation (4.42) asserts that the map $\eta \mapsto \hat{\xi}_\eta$ is a homomorphism[26] up to infinitesimal gauge transformations, as is evident from the sequence (4.36). Equation (4.43) implies that if η_1 and η_2 commute, then the symmetry ξ_{η_2} commutes with η_1 up to a gauge transformation. This is used in superspace formulations of supersymmetric gauge theories, where the symmetry generated by the vector fields Q_a on superspace commute up to a gauge transformation with the action of vector fields D_a used to build lagrangians.

§4.3. Pure Yang-Mills theory

The data which define a pure gauge theory are:

$$\begin{array}{ll} G & \text{Lie group with Lie algebra } \mathfrak{g} \\ \langle \cdot, \cdot \rangle & \text{bi-invariant inner product on } \mathfrak{g} \end{array} \tag{4.44}$$

We work on Minkowski n-space M. The field in the theory is a connection A on a principal G-bundle over M. As explained in §4.2, the collection of fields is best regarded as a category. We write lagrangians which are invariant under gauge transformations.

Recall (4.25) that for a nonabelian group the curvature F_A is a nonlinear function of the connection A. So nonabelian gauge theories are not free—the lagrangian is not quadratic in the field A. For perturbation theory one introduces a coupling constant (or several if G is a product of simpler groups). We simply absorb these constants in the inner product $\langle \cdot, \cdot \rangle$.

The pure Yang-Mills lagrangian is given by the same formulas as (3.62):

$$L = -\frac{1}{2}\langle F_A \wedge *F_A \rangle. \tag{4.45}$$

[26]When a Lie algebra acts on a manifold on the left we expect an *anti*homomorphism. But if we view the space of vector fields as the Lie algebra of the group of diffeomorphisms, then the induced bracket is *minus* the usual Lie bracket of vector fields.

The manipulations following (3.62) are valid for arbitrary gauge groups, except that we must replace d by d_A in the nonabelian case. Thus the equation of motion is

$$d_A * F_A = 0, \tag{4.46}$$

and the variational 1-form and local symplectic form are

$$\begin{aligned} \gamma &= -\langle \delta A \wedge *F_A \rangle, \\ \omega &= \langle \delta A \wedge *d_A \delta A \rangle. \end{aligned} \tag{4.47}$$

Equation (4.46) is called the *Yang-Mills equation*. Of course, the connection A always satisfies the Bianchi identity (4.34).

The discussion of the energy-momentum tensor carries over without change, and formula (3.80) holds:

$$T_{\mu\nu} = -\langle F_{\mu\rho}, F_{\nu\sigma} \rangle g^{\rho\sigma} + \frac{1}{2}|F|^2 g_{\mu\nu}. \tag{4.48}$$

Pure Yang-Mills theory is conformally invariant in 4 dimensions (see (3.83)).

The energy density is

$$T_{00} = \frac{1}{2}\sum_i |F_{0i}|^2 + \frac{1}{2}\sum_{i<j} |F_{ij}|^2, \tag{4.49}$$

where $i, j = 1, \dots, n-1$ run over spatial indices. Therefore, the field configurations of minimal energy are flat, and the moduli space of vacua on Minkowski space is a point—the equivalence class of the trivial connection.

Fundamental lagrangians in physical theories are constrained by renormalizability. With that criterion there are a few terms one can add to the Yang-Mills lagrangian (4.45) in dimensions 2, 3, and 4. These terms are topological in nature. We often refer to them as "θ-terms".

In $n = 2$ dimensions suppose

$$\langle\!\langle \cdot \rangle\!\rangle : \mathfrak{g} \longrightarrow \mathbb{R} \tag{4.50}$$

is a trace on $\mathfrak{g}$, i.e., a linear map for which $\langle\!\langle\, [a, b] \,\rangle\!\rangle = 0$ for all $a, b \in \mathfrak{g}$. Then an additional possible term in the lagrangian is

$$L_2 = \frac{\theta}{2\pi}\langle\!\langle F_A \rangle\!\rangle. \tag{4.51}$$

We include the constant $\theta \in \mathbb{R}$ since this is the form in which this θ-term is usually written. There is a constant θ for each independent trace on $\mathfrak{g}$. On a compact 2-manifold M (or on the space of fields with finite action on Minkowski space) the integral of (4.51) is locally constant on the space of fields. Thus L_2 does not affect the equations of motion. It does, however, enter into the formula for the variational 1-form. Namely, the variational 1-form for pure Yang-Mills plus L_2 is

$$\gamma = -\langle \delta A \wedge *F_A \rangle + \frac{\theta}{2\pi}\langle\!\langle \delta A \rangle\!\rangle. \tag{4.52}$$

The local symplectic form and energy-momentum tensor are unchanged.

In $n = 3$ dimensions there is a Chern-Simons term; we discuss it in §6.2. It is different than the θ-terms (4.51) and (4.54): the Chern-Simons term is not locally constant on the space of fields, and so does have a dynamical effect.

In $n = 4$ dimensions there is a θ-term associated to an invariant bilinear form

$$\langle\langle \cdot \rangle\rangle : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}. \tag{4.53}$$

(The form need not be an inner product; in particular, it need not be identical to (4.44).) The θ-term is

$$L_4 = \frac{\theta}{16\pi^2} \langle\langle F_A \wedge F_A \rangle\rangle. \tag{4.54}$$

As with L_2, the integral of L_4 is locally constant on the space of fields. The modified variational 1-form is

$$\gamma = -\langle \delta A \wedge *F_A \rangle + \frac{\theta}{8\pi^2} \langle\langle \delta A \wedge F_A \rangle\rangle. \tag{4.55}$$

§4.4. Electric and magnetic charge

To begin, let spacetime be a product $M = M^1 \times N$. For a classical electromagnetic field A, as in §4.1, the electric charge "enclosed" by a closed codimension 1 submanifold $\Sigma \subset N$ is

$$\mathcal{E}_A(\Sigma) = \int_\Sigma *F_A = \frac{1}{c} \int_\Sigma *_N E_A, \tag{4.56}$$

and the magnetic charge enclosed by a closed 2-dimensional submanifold $\Sigma \subset N$ is

$$\mathcal{B}_A(\Sigma) = \int_\Sigma F_A = \int_\Sigma B_A. \tag{4.57}$$

In each case Σ need not be a boundary, hence the quotes around 'enclosed'. We use the same terminology for a general spacetime M, which may also be Euclidean—then only the first equations in (4.56) and (4.57) make sense. For a gauge field with general gauge group G, electric and magnetic charge are associated to a trace

$$\tau : \mathfrak{g} \longrightarrow \mathbb{R} \tag{4.58}$$

on the Lie algebra $\mathfrak{g}$:

$$\begin{aligned} \mathcal{E}^\tau_A(\Sigma) &= \int_\Sigma *\tau(F_A), \\ \mathcal{B}^\tau_A(\Sigma) &= \int_\Sigma \tau(F_A). \end{aligned} \tag{4.59}$$

Magnetic charge is an example of a *topological* charge; $\mathcal{B}^\tau_A(\Sigma)$ is a topological invariant which depends only on the topological type of the principal bundle underlying A. Hence it is a locally constant function on the space of fields—in particular,

the space of classical solutions. It is *central* in the sense that it Poisson commutes with any other charge.

Electric charge is not generally conserved in a finite region. For example, in classical electromagnetism it follows from (4.16) that the time derivative of the electric charge enclosed by Σ is proportional to the integral of the spatial current j through Σ. Also, if $\Sigma = \partial\Omega$ then the electric charge enclosed by Σ is proportional to the integral of the charge density ρ over Ω. This is the usual statement of Gauss' law. We generalize to field theory, where the current J is computed from matter fields, as follows. We work with a gauge field A for a general gauge group G. Elements ϵ in the center of $\mathfrak{g}$ give infinitesimal global gauge transformations ϵ which act trivially on the space of connections: $d_A\epsilon = 0$ for all A. From §2.8 it follows that if L'_A is a gauge invariant "matter lagrangian" in which A is a fixed background field and which only depends on the 1-jet of the matter fields ϕ, then the Noether current j_ϵ for ϵ in the theory $\mathcal{L}'_A$ is computed by differentiating L'_A with respect to A (see Proposition 2.149). We now consider A as a dynamical variable by adding the Yang-Mills lagrangian to L'_A. We write the signs for a Lorentz manifold.

Proposition 4.60. *Let $L'_A(\phi)$ be the matter lagrangian described above, and set*

$$L(A,\phi) = -\frac{1}{2}\langle F_A \wedge *F_A\rangle + L'_A(\phi). \tag{4.61}$$

Suppose Ω is a spacelike $(n-1)$-dimensional region with smooth boundary Σ. Then if ϵ is the global gauge transformation corresponding to a central element $\epsilon \in \mathfrak{g}$, on-shell we have

$$\mathcal{E}_A^{\tau(\epsilon)}(\Sigma) = \int_\Omega j_\epsilon, \tag{4.62}$$

where $\tau(\epsilon) \in \mathfrak{g}^*$ is the trace $\zeta \mapsto \langle \epsilon, \zeta\rangle$ on $\mathfrak{g}$ and j_ϵ is the canonical Noether current for ϵ in the theory $\mathcal{L}'_A$.

Remark 4.63. In Minkowski space we often take Ω to be a spacelike hyperplane—the limit of large balls—in which case Σ is the sphere at "spatial infinity"—the limit of large spheres. Then the limit of $\mathcal{E}_A^{\tau(\epsilon)}(\Sigma)$ is the total electric charge in the system.

Proof. There is a canonical variational 1-form γ since the lagrangian only depends on the 1-jet of the fields. Let $\mathcal{L} = L + \gamma = \mathcal{L}_{YM} + \mathcal{L}'_A$ be the total lagrangian. The equation of motion is

$$(D\mathcal{L}_{YM})^{1,|0|} + (D\mathcal{L}'_A)^{1,|0|} = 0. \tag{4.64}$$

Evaluate this on $\alpha\epsilon$, where $\alpha \in \Omega^1_M$ is an arbitrary 1-form and $\alpha\epsilon$ is viewed as a tangent to the space of connections. By manipulations as in §3.5 we have

$$\iota(\alpha\epsilon)(D\mathcal{L}_{YM})^{1,|0|} = -\alpha \wedge \langle \epsilon, d_A * F_A\rangle. \tag{4.65}$$

By Proposition 2.149 the second term is $j_\epsilon(\alpha)$. Since α is arbitrary,

$$j_\epsilon = \langle \epsilon, d_A * F_A\rangle = d\langle \epsilon, *F_A\rangle. \tag{4.66}$$

Equation (4.62) is the integral version of (4.66).

Formula (4.62) is not modified if we add a θ-term (4.51) or (4.54) to (4.61). (This has the effect of adding a multiple of F_A to $*F_A$ in (4.65), but that extra piece vanishes by Bianchi.) However, formula (4.62) is modified in the theory of monopoles; see [II-**Dynamics of QFT**, §9.6].

CHAPTER 5
σ-Models and Coupled Gauge Theories

§5.1. Nonlinear σ-models

We begin with some preliminary remarks. First, if M is a manifold and $E \to M$ a real or complex vector bundle with connection ∇, then we form the twisted de Rham complex

$$\Omega^0_M(E) \xrightarrow{d_\nabla = \nabla} \Omega^1_M(E) \xrightarrow{d_\nabla} \Omega^2_M(E) \xrightarrow{d_\nabla} \cdots . \tag{5.1}$$

It is not a complex in general, but rather

$$d^2_\nabla = R_\nabla, \tag{5.2}$$

where R_∇ is the curvature of ∇. Next, if $\phi\colon M \to X$ is a map between manifolds, then $d\phi \in \Omega^1_M(\phi^* TX)$. If now TX has a connection ∇, then

$$d_\nabla d\phi = \phi^* T_\nabla, \tag{5.3}$$

where T_∇ is the torsion of the connection ∇. We will apply this to the Levi-Civita connection of a Riemannian manifold; its torsion vanishes.

The data which define a nonlinear σ-model are:

$$\begin{array}{ll} X & \text{Riemannian manifold} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential energy function} \end{array} \tag{5.4}$$

The field in the theory is a map $\phi\colon M \to X$; the space of fields is $\mathcal{F} = \mathrm{Map}(M, X)$. Diffeomorphisms of X act on $\mathcal{F}$ by composition, and diffeomorphisms of M act on $\mathcal{F}$ using composition by the *inverse*. In each case the subgroup of isometries preserves the lagrangian (5.6) below. There is a canonical evaluation map

$$e\colon \mathcal{F} \times M \longrightarrow X \tag{5.5}$$

which is invariant under the product action of $\mathrm{Diff}\, M$ on $\mathcal{F} \times M$. Infinitesimal symmetries act on derivatives of the field ϕ via the Levi-Civita covariant derivative ∇. This introduces curvature terms into the bracket of infinitesimal symmetries in general. Since e is invariant under $\mathrm{Diff}\, M$, and since the lagrangian (5.6) below

can be written in terms of e, it is easy to check that isometries of M are manifest symmetries (as are isometries of X).

The σ-model lagrangian with potential is

$$\begin{aligned} L &= \left\{ \frac{1}{2}|d\phi|^2 - \phi^* V \right\} |d^n x| \\ &= \frac{1}{2}\langle d\phi \wedge *d\phi \rangle - \phi^* V \, |d^n x|. \end{aligned} \tag{5.6}$$

Note that $|d\phi|$ is computed using both the metric on M and the metric on X. We carry out the familiar analysis:

$$\begin{aligned} \delta L &= \langle \delta_\nabla d\phi \wedge *d\phi \rangle - \langle \delta\phi, \phi^* \operatorname{grad} V \rangle \, |d^n x| \\ &= -d\{ \langle \delta\phi \wedge *d\phi \rangle \} - \langle \delta\phi \wedge d_\nabla * d\phi \rangle - \langle \delta\phi, \phi^* \operatorname{grad} V \rangle \, |d^n x|. \end{aligned} \tag{5.7}$$

Hence the equation of motion is

$$\Box_\nabla \phi = -\phi^* \operatorname{grad} V, \tag{5.8}$$

where $\Box_\nabla$ is the covariant extension of the wave operator (3.2). Despite this notation one shouldn't lose sight of Newton's law, which is the special case $n = 1$ of equation (5.8). The variational 1-form and local symplectic form are

$$\begin{aligned} \gamma &= \langle \delta\phi \wedge *d\phi \rangle, \\ \omega &= \langle *d_\nabla \delta\phi \wedge \delta\phi \rangle. \end{aligned} \tag{5.9}$$

It is easy to couple this theory to an arbitrary metric g on M:

$$L_g = \left(\frac{1}{2} g^{\mu\nu} \, \partial_\mu \phi \, \partial_\nu \phi - \phi^* V \right) \mu_g(x). \tag{5.10}$$

The computation leading to (3.19) is essentially unchanged, and so the energy-momentum tensor is

$$T_g = \langle d\phi \cdot d\phi \rangle \mu_g - g\, L, \tag{5.11}$$

or more explicitly (see (2.177))

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi + \left(-\frac{1}{2}|d\phi|^2 + \phi^* V \right) g_{\mu\nu}. \tag{5.12}$$

On Minkowski space the moduli space of vacua—field configurations which minimize the energy density T_{00}—is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \tag{5.13}$$

assuming that 0 is the minimum value of the potential energy V.

If ξ is a Killing vector field (infinitesimal isometry) on X, then the associated Noether current is

$$j = \langle \xi, *d\phi \rangle. \tag{5.14}$$

There are also topological terms one can add to a pure σ-model, usually called *Wess-Zumino terms*. We describe them in Chapter 6.

§5.2. Gauge theory with bosonic matter

The theory in this section is the most general bosonic theory without gravity[27] (though we do not include all possible terms—e.g. topological terms—in the lagrangian). We can also describe it as a gauged nonlinear σ-model, or as a gauge theory with bosonic matter. The data which defines the theory are:

(5.15)

G	Lie group with Lie algebra $\mathfrak{g}$
$\langle \cdot, \cdot \rangle$	bi-invariant scalar product on $\mathfrak{g}$
X	Riemannian manifold on which G acts by isometries
$V\colon X \longrightarrow \mathbb{R}$	potential function invariant under G

The fields are

(5.16)

A	connection on some principal G-bundle $P \longrightarrow M$
ϕ	section of the associated bundle $P \times_G X \longrightarrow M$

It is often convenient to view ϕ as an equivariant map $\phi\colon P \to X$. The lagrangian combines (4.45) and (5.6):

$$L = \left(-\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^* V\right)|d^n x|. \tag{5.17}$$

There is a new term

$$\langle \delta A \cdot \phi, *d_A\phi \rangle \tag{5.18}$$

in δL from the coupling of A and ϕ, so a new term in the equations of motion. The variational 1-form is

$$\gamma = \langle \delta\phi \wedge *d_A\phi \rangle - \langle \delta A \wedge *F_A \rangle. \tag{5.19}$$

The energy-momentum tensor is

$$T_{\mu\nu} = (\partial_A)_\mu\phi\,(\partial_A)_\nu\phi - \langle F_{\mu\rho}, F_{\nu\sigma}\rangle\, g^{\rho\sigma} + \left(\frac{1}{2}|F_A|^2 - \frac{1}{2}|d_A\phi|^2 + \phi^* V\right) g_{\mu\nu} \tag{5.20}$$

The moduli space of vacua is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \,/\, G \tag{5.21}$$

assuming that 0 is the minimum value of the potential energy V.

In §2.8 we described a simple example of "gauging a symmetry".

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[27] By this we mean a theory of scalar and gauge fields only. There are also models with p-form fields for $p \geq 2$, for example. The lagrangian here covers most *fundamental* (vs. *effective*) lagrangians without gravity.

CHAPTER 6
Topological Terms

We have already introduced "θ-terms" in gauge theory in §4.3. These are a sort of topological term related to *primary* topological invariants, in this case characteristic classes of principal bundles. Corresponding terms occur in a σ-model as well. Namely, if $\phi\colon M \to X$ is a field in a σ-model defined on an oriented spacetime M^n, and $\omega \in \Omega^n(X)$ is a closed differential form, then we can insert a term

$$-c\,\phi^*\omega \tag{6.1}$$

into a lagrangian. (The minus sign indicates that (6.1) is a contribution to the potential energy.) Here $c \in \mathbb{R}$ is a constant; if ω has periods which are $2\pi\hbar$ times integers, then only $c \pmod{\mathbb{Z}}$ enters into the quantum theory. In this chapter we consider a different type of topological term which is related to *secondary* topological invariants. For example, in a gauge theory the holonomy of a connection is a secondary invariant associated to a first Chern class; in three dimensions the Chern-Simons invariant is associated to a four-dimensional characteristic class. In σ-models we meet *Wess-Zumino-Witten terms* which are secondary invariants associated to cohomology classes in the target space X.

To fit such terms into the general theory of Chapter 2, we need local differential-geometric objects which integrate to these secondary invariants. We indicate an extension of ordinary calculus to include these objects briefly, though a systematic development of foundations for this extension is lacking. In the first few sections we focus instead on examples.

For both θ-terms and topological terms we need to work over an oriented spacetime. The orientation allows us to pass from differential forms to densities. Note that neither θ-terms nor topological terms depend on a metric on spacetime.

§6.1. Gauge theory

The simplest example of a topological term was already discussed at the end of §1.1 and in §4.1. Namely, consider a theory on a spacetime M^n which includes a connection A for the group $\mathbb{R}^{>0}$. Suppose $x\colon \mathbb{R} \to M$ is a parametrized path which may be fixed or variable. Then we can introduce a term

$$-q\,x^*A \tag{6.2}$$

into the lagrangian of the theory, where $q \in \mathbb{R}$ is a constant. Note that (6.2) is meant to be integrated over $\mathbb{R}$, not over spacetime M. It is unchanged by orientation-preserving diffeomorphisms of $\mathbb{R}$; in other words, it only depends on the image of x viewed as an oriented submanifold of M.

Now $-q\,x^*A$ is different from other terms in lagrangians we have seen so far. A connection A on an $\mathbb{R}^{>0}$-bundle $P \to M$ is an element of $\Omega^1(P)$, and x^*A a 1-form on the pullback x^*P. It is *not* a 1-form on the base $\mathbb{R}$. To compute the action we can fix a trivialization of x^*P. Taking into account the dependence on this trivialization, the action over an interval $[T_1, T_2] \subset \mathbb{R}$ is a homomorphism of $\mathbb{R}$-torsors, the $\mathbb{R}$-torsors being q times the fibers of x^*P at T_1 and T_2. In the lagrangian approach to the quantum theory it is not the action S, but rather the exponential $\exp(\sqrt{-1}\,S/\hbar)$ which enters. Then this term in the action is parallel transport in the circle bundle associated to P via the homomorphism

$$\begin{aligned} \mathbb{R}^{>0} &\longrightarrow \mathbb{T} \\ x &\longmapsto (\sqrt{-1})^{qx/\hbar} \end{aligned} \tag{6.3}$$

For usual kinetic and potential terms, the exponentiated action is an element of $\mathbb{T}$. This parallel transport is instead a homomorphism of circle torsors.

The equation of motion (Lorentz force law) arising from (6.2) in classical electromagnetism is discussed at the end of §4.1.

In the quantum theory of electromagnetism the electric charge q is quantized in suitable units. It makes sense from the beginning to regard the gauge field A as a connection in a $\mathbb{T}$-bundle (as opposed to an $\mathbb{R}^{>0}$-bundle). More generally, in quantum Yang-Mills theories the gauge field is a connection for a *compact* gauge group G. Then it makes sense to write a term $x^*\tau(A)$ in the lagrangian, where the trace $\tau\colon \mathfrak{g} \to \mathbb{R}$ is $1/\sqrt{-1}$ times the differential of an abelian character $G \to \mathbb{T}$. This term in the exponentiated action is again interpreted as a homomorphism of circle torsors.

In three-dimensional gauge theory there is a topological term due to Chern and Simons. Let M^3 be a 3-dimensional oriented spacetime and consider a theory with gauge field A. If G is the gauge group, and $\langle \cdot, \cdot \rangle$ an invariant inner product on the Lie algebra $\mathfrak{g}$, then the *Chern-Simons term* is

$$L_{CS} = \langle A \wedge F_A \rangle - \frac{1}{6}\langle A \wedge [A \wedge A] \rangle. \tag{6.4}$$

If A is a connection on P, then this term is a 3-form on the total space of P. Assume for convenience that $M^3 = M^1 \times \Sigma^2$ is the product of time M^1 and a space Σ^2 which is a closed oriented surface. Then once again the exponentiated action on $[T_1, T_2] \times \Sigma$ is best interpreted as a homomorphism between $\mathbb{T}$-torsors $\mathcal{T}_1$ and $\mathcal{T}_2$ defined from (6.4) at $\{T_1\} \times \Sigma$ and $\{T_2\} \times \Sigma$. (For this we need $\langle \cdot, \cdot \rangle$ to lie in a distinguished lattice of inner products.) For trivializable bundles P the torsors are trivialized by a choice of trivialization of P, and may be constructed by such trivializations. We omit the details. (For nontrivializable bundles we need a refinement of the inner product $\langle \cdot, \cdot \rangle$ to an element of $H^4(BG; \mathbb{Z})$ in order to define the torsors and the exponentiated action.)

Locally we can break gauge invariance and fix a trivialization of the bundle which carries the connection A. Then (6.4) pulls down to a local 3-form on M. We compute

$$\delta L_{CS} + d\gamma = \langle \delta A \wedge 2F_A \rangle, \tag{6.5}$$

where

$$\gamma = \langle A \wedge \delta A \rangle. \tag{6.6}$$

So for pure Chern-Simons theory the equation of motion is

$$F_A = 0. \tag{6.7}$$

For the particular spacetime $M^3 = M^1 \times \Sigma^2$, the space of solutions $\overline{\mathcal{M}}$ up to gauge equivalence is the moduli space of flat connections on Σ. The local symplectic form

$$\omega = \langle \delta A \wedge \delta A \rangle \tag{6.8}$$

integrates on Σ to (twice) the usual symplectic form on $\overline{\mathcal{M}}$.

The equations of motion (6.7) and the local symplectic form (6.8) are gauge invariant, whereas the variational 1-form (6.6) is not. This suggests that a Noether charge is no longer a well-defined real number. Indeed, if ϵ is a central element of $\mathfrak{g}$, interpreted as a constant gauge transformation, then the associated Noether current from (6.6) is formally

$$d\langle \epsilon \wedge A \rangle. \tag{6.9}$$

The integral over a region $\Omega \in \Sigma$ is minus the logarithm of the holonomy around $\partial\Sigma$ of the bundle associated to the exponential of the trace $\langle \epsilon, \cdot \rangle$ on $\mathfrak{g}$. This logarithm is well-defined only up to integer shifts. (If the Chern-Simons term is added to the lagrangian of a standard 3-dimensional theory, as in (4.61), then this term contributes to the electric charge (4.62).)

§6.2. Wess-Zumino-Witten terms

Consider a σ-model with field $\phi\colon M \to X$, where the spacetime M is an oriented n-manifold. Under the simplifying hypothesis $H_{n-1}(X) = H_n(X) = 0$, we construct a term in an action associated to a closed form $\Omega \in \Omega^{n+1}(X)$ with periods in $2\pi\hbar\mathbb{Z}$. To remove the simplifying topological assumption we need a more precise form of this data, as we explain in §6.3. For simplicity, consider a product spacetime $M = M^1 \times N^{n-1}$, where we assume space N to be a compact oriented n-manifold.

We first construct a $\mathbb{T}$-torsor $\mathcal{T}_f = \mathcal{T}_f(\Omega)$ for each map $f\colon N \to X$ as follows. Since $H_{n-1}(X) = 0$ there exist extensions $F\colon Y^n \to X$ to oriented n-chains Y^n with $\partial F = f$. Each such F trivializes $\mathcal{T}_f$. If F_1, F_2 are two such extensions, then since $H_n(X) = 0$ we can find an oriented $(n+1)$-chain $H\colon Z^{n+1} \to X$ with $\partial H = F_2 - F_1$. Then the isomorphism from the F_1-trivialization to the F_2-trivialization is

$$\exp\Big[\frac{\sqrt{-1}}{\hbar}\int_Z H^*\Omega\Big]. \tag{6.10}$$

Because of the assumption on the periods of Ω, this is independent of the choice of H.

We can allow $H_{n-1}(X)$ to be torsion in this construction if we specify a cohomology class $c \in H^n(X; 2\pi\hbar\mathbb{Z})$ such that the de Rham class of Ω is the image $c_{\mathbb{R}}$

of c in real cohomology. Then a multiple of f bounds, and we essentially use the choice of c to divide by that multiple in the abelian group $\mathbb{T}$.

For another perspective, consider the usual diagram

$$\begin{array}{ccc} \mathrm{Map}(N,X)\times N & \xrightarrow{\ e\ } & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ \mathrm{Map}(N,X) & & \end{array} \tag{6.11}$$

Then $\pi_* e^*\Omega$ is a 2-form on $\mathrm{Map}(N,X)$ with periods in $2\pi\hbar\mathbb{Z}$, and $\pi_* e^* c$ is a refinement in $H^2\big(\mathrm{Map}(N,X); 2\pi\hbar\mathbb{Z}\big)$. So there exist principal $\mathbb{T}$-bundles with connection over $\mathrm{Map}(N,X)$ whose curvature is $-(\sqrt{-1}/\hbar)\pi_* e^*\Omega$ and whose Chern class is $\pi_* e^* c/2\pi\hbar$. The construction above gives a particular such $\mathbb{T}$-bundle; it can be extended to produce a connection as well.

Now if $\phi\colon [T_1,T_2]\times N \to X$ is a field, the exponentiated action $\exp(\sqrt{-1}\,S/\hbar)$ of ϕ is naturally an element of the circle torsor

$$\mathcal{T}_{\partial\phi} = \mathcal{T}_{\phi|\{T_2\}\times N}\cdot \mathcal{T}^{-1}_{\phi|\{T_1\}\times N}. \tag{6.12}$$

Namely, if F_i is a trivialization of $\mathcal{T}_{\phi|\{T_i\}\times N}$, then $F_2+\phi-F_1$ bounds an $(n+1)$-chain $H\colon Z^{n+1}\to X$, and we use formula (6.10) to compute the action in this trivialization.

We remark that the gauge theory constructions of §6.1 may be understood—at least heuristically—as special cases in which $X = BG$ is the classifying space of the gauge group G.

§6.3. Smooth Deligne cohomology

Fix a smooth manifold X and an integer[28] n. Let us consider on X the constant sheaf $\mathbb{Z}$, the constant sheaf $\mathbb{R}$, and the following complex of sheaves, denoted $F^n(\Omega)$ (or simply $F(\Omega)$): the subcomplex of the de Rham complex given by

$$F(\Omega)^p = \begin{cases} 0, & \text{if } p<n; \\ \Omega^p, & \text{if } p\geq n. \end{cases} \tag{6.13}$$

Both the cohomology of X with coefficients in $\mathbb{Z}$ and the hypercohomology of X with coefficients in $F(\Omega)$ map to the cohomology of X with coefficients in $\mathbb{R}$. The first because $\mathbb{Z}$ maps to $\mathbb{R}$. The second is just the cohomology of the complex $\Omega^n(X)\to\Omega^{n+1}(X)\to\cdots$ starting in degree n, which is a subcomplex of the de Rham complex computing the real cohomology. Suppose we have somehow lifted those maps at the cochain level. In other words, suppose we have found natural complexes $(C^*_{\mathbb{Z}}, d_{\mathbb{Z}})$, $(C^*_{F^n}, d_{F^n})$ (or simply (C^*_F, d_F)), and $(C^*_{\mathbb{R}}, d_{\mathbb{R}})$ with $H^*(C^*_{\mathbb{Z}}) = H^*(X,\mathbb{Z})$, $H^*(C^*_F) = \mathbb{H}^*\big(X, F(\Omega)\big)$, and $H^*(C^*_{\mathbb{R}}) = H^*(X,\mathbb{R})$, and found morphisms of complexes

$$\varphi_{\mathbb{Z}}, \varphi_F\colon C^*_{\mathbb{Z}}, C^*_{F^n} \longrightarrow C^*_{\mathbb{R}} \tag{6.14}$$

[28] Our choice of indexing makes the description of products below more natural. However, there is a shift in the application to lagrangian field theory: it is cocycles for the cohomology group D^{n+1} which enter lagrangians for n-dimensional spacetimes.

inducing the maps we described in cohomology. This can be done in many ways—one way will be described later—but the philosophy of cohomological algebra tells that they are essentially equivalent.

One can then form a mapping cone (K, d) with

$$\begin{aligned} K^p &:= C_{\mathbb{Z}}^p \oplus C_F^p \oplus C_{\mathbb{R}}^{p-1} \\ d &:= d_{\mathbb{Z}} + d_F - d_{\mathbb{R}} + \varphi_{\mathbb{Z}} - \varphi_F. \end{aligned} \tag{6.15}$$

A p-cycle $c = (c_{\mathbb{Z}}, c_F, c_{\mathbb{R}})$ of K is the data of p-cycles $c_{\mathbb{Z}}$ and c_F of $C_{\mathbb{Z}}$ and C_F, and of a homology between their images in $C_{\mathbb{R}}$. The mapping cone K behaves as if one had a short exact sequence of complexes $0 \to K \to C_{\mathbb{Z}} \oplus C_F \to C_{\mathbb{R}} \to 0$. For instance, one has a long exact sequence of cohomology groups. The *smooth Deligne cohomology* group $D^{p,n}$ is $H^p(K)$. For $p < n$ it is $H^{p-1}(X, \mathbb{R}/\mathbb{Z})$. For $p > n$ it is $H^p(X, \mathbb{Z})$. We will be mainly interested in $D^n := D^{n,n}$. It sits in an exact sequence

$$H^{n-1}(X,\mathbb{Z}) \to H^{n-1}(X,\mathbb{R}) \to D^n \to H^n(X,\mathbb{Z}) \oplus (\text{closed } n\text{-forms}) \to H^n(X,\mathbb{R}) \tag{6.16}$$

and it is an extension of the group of closed n-forms with integral periods by $H^{n-1}(X, \mathbb{R}/\mathbb{Z})$.

This description of D^n is not the most economical, but it suggests the functorial properties to be expected. *Products*: One has product maps on $\mathbb{Z}$ and $\mathbb{R}$, as well as product maps $F^n \otimes F^m \to F^{n+m}$. If the corresponding cup-product in cohomology is expressed at the cochain level, giving products in $C_{\mathbb{Z}}$, $C_{\mathbb{R}}$, and C_F compatible with $\varphi_{\mathbb{Z}}$ and $\varphi_{\mathbb{R}}$, one obtains products $D^{p,n} \otimes D^{q,m} \to D^{p+q,n+m}$. Indeed, homologies c and c' between the images of $c_{\mathbb{Z}}$ and c_F (resp. $c'_{\mathbb{Z}}$ and c'_F) give a homology between the images of $c_{\mathbb{Z}} c'_{\mathbb{Z}}$ and $c_F c'_F$. In fact, there are two naturally cohomologous homologies: $\varphi(c_{\mathbb{Z}})c' + c\varphi(c'_F)$ and $c\varphi(c'_{\mathbb{Z}}) + \varphi(c_F)c'$. *Integration*: If a proper submersion $f\colon X \to Y$ of relative dimension d has oriented fibers, integration along the fibers f has meaning in integral and real cohomology, as well as from F^n on X to F^{n-d} on Y. Expressed compatibly at the cochain level, it should provide $\int_{X/Y}\colon D^{p,n}(X) \to D^{p-d,n-d}(Y)$, and in particular $D^n(X) \to D^{n-d}(Y)$.

One way to find complexes $C_{\mathbb{Z}}^*$, C_F^*, and $C_{\mathbb{R}}^*$ is to use the Čech method for computing cohomology. If $\{U_i\}_{i\in I}$ is an open covering of X such that the nonempty intersections $U_{i_0 \dots i_p} := U_{i_0} \cap \dots \cap U_{i_p}$ are contractible, one can use for $C_{\mathbb{Z}}^*$ the Čech complex

$$C_{\mathbb{Z}}^p = \prod \Gamma(U_{i_0 \dots i_p}, \mathbb{Z}), \tag{6.17}$$

and for C_F^* and $C_{\mathbb{R}}^*$ the simple complex associated to the double complex

$$\begin{aligned} C_F^{p,q} &= \prod \Gamma\big(U_{i_0 \dots i_p}, F(\Omega)^q\big) \\ C_{\mathbb{R}}^{p,q} &= \prod \Gamma\big(U_{i_0 \dots i_p}, \Omega^q\big). \end{aligned} \tag{6.18}$$

For $n > 0$ the map $\mathbb{Z} \oplus F \to \Omega^*$ of complexes of sheaves on X is injective. If so is $\varphi_{\mathbb{Z}} - \varphi_F\colon C_{\mathbb{Z}}^* \oplus C_F^* \to C_{\mathbb{R}}^*$, the mapping cone K has the same cohomology, up

to a shift of index by 1, as the cokernel of $\varphi_{\mathbb{Z}} - \varphi_F$, a complex whose cohomology is the hypercohomology of the complex of sheaves

$$C^\infty(\text{values in } \mathbb{R}/\mathbb{Z}) \longrightarrow \Omega^1 \longrightarrow \ldots \longrightarrow \Omega^{n-1} \longrightarrow 0. \tag{6.19}$$

If we denote by $\tilde{\Omega}^*$ this complex, we have in the Čechist computation

$$\operatorname{coker}(\varphi_{\mathbb{Z}} - \varphi_F) = \prod \Gamma(U_{i_0 \ldots i_p}, \tilde{\Omega}^p), \tag{6.20}$$

and $D^{p,n}$ is the cohomology group H^{p-1} of the complex.

For $n = 1$ we find that D^1 is the group of C^∞ maps from X to the circle $\mathbb{R}/\mathbb{Z}$. For $n = 2$ elements of D^2 are represented by cocycles

$$\begin{array}{ccc} \alpha_i & \longrightarrow & 0 \\ & & \uparrow \\ & & g_{ij} \longrightarrow 0 \end{array} \tag{6.21}$$

where the diagram indicates the equations

$$\begin{aligned} g_{ij} + g_{jk} &= g_{ik} \\ \alpha_i - \alpha_j &= dg_{ij}. \end{aligned} \tag{6.22}$$

The $\exp(2\pi\sqrt{-1}\, g_{ij})$ are the transition functions for a $\mathbb{T}$-bundle, trivialized on the U_i, and the $2\pi\sqrt{-1}\,\alpha_i$, viewed as local connection 1-forms, provide this $\mathbb{T}$-bundle with a connection. The group D^2 is the group of isomorphism classes of $\mathbb{T}$-bundles with connection. More precisely, a cocycle defines a bundle with connection and a homology defines an isomorphism between bundles.

In §6.2, what we needed to avoid assumptions on the homology of X is not just a closed $(n+1)$-form with integral periods, but rather a class in D^{n+1}, or rather a cocycle giving such a class. If $H^n(X, \mathbb{R}/\mathbb{Z}) = 0$ (equivalently: $H_n(X, \mathbb{Z}) = 0$), one has

$$D^{n+1} \xrightarrow{\sim} (\text{closed } (n+1)\text{-forms with integral periods}). \tag{6.23}$$

If $H^{n-1}(X, \mathbb{R}/\mathbb{Z}) = 0$ (equivalently: $H_{n-1}(X, \mathbb{Z}) = 0$), the ambiguity in the choice of a cocycle becomes irrelevant: if c_1 and c_2 are two cocycles representing the same class in D^{n+1}, not only are they homologous, $c_1 - c_2 = dc$, but any two choices c' and c'' for c are homologous: $c' - c'' = d\tilde{c}$.

If a cocycle c is chosen and if ϕ is a field, i.e., a map from M to X, then $\phi^* c$ is a cocycle giving a class in $D^{n+1}(M)$. For N in M a compact subvariety of dimension $n-1$, integration of $\phi^* c$ on N should produce a $\mathbb{T}$-principal homogeneous space—a one-dimensional complex vector space $\mathcal{L}(N, \phi^* c)$ with metric attached to N and $\phi^* c$. Indeed, integration on N maps $D^{n+1}(M)$ to D^2 of a point. If N is the boundary of a singular chain S, integration of $\phi^* c$ on S should produce a unit vector in $\mathcal{L}(N, \phi^* c)$. For instance, in a Hamiltonian picture, a space-like hypersurface N, supposed here compact, would give $\mathcal{L}(N, \phi^* c)$ and the slice S between two such hypersurfaces N_1 and N_2 would provide an isomorphism (the action integral) from $\mathcal{L}(N_1, \phi^* c)$ to $\mathcal{L}(N_2, \phi^* c)$.

A systematic treatment of those expectations has yet to be given.

CHAPTER 7
Wick Rotation: From Minkowski Space to Euclidean Space

A basic constraint on a Minkowski space action is that it be real. An action S_M is the integral of a lagrangian density L_M over Minkowski space M:

$$S_M = \int_M L_M. \tag{7.1}$$

Choose a time t on M. Then we (Wick) rotate to Euclidean space E by introducing imaginary time

$$\tau = \sqrt{-1}\, t. \tag{7.2}$$

By convention the Euclidean action is $1/\sqrt{-1}$ times the rotated Minkowski action:

$$\frac{1}{\sqrt{-1}} S_M = S_E = \int_E L_E. \tag{7.3}$$

Note that $e^{\sqrt{-1}\, S_M} = e^{-S_E}$. Also, S_E is not real in general.

We describe the continuation to Euclidean space more precisely for a σ-model. The field is a map $\phi : M \longrightarrow X$ into some Riemannian manifold. The complexification of the space of maps $M \longrightarrow X$ is the space of *holomorphic* maps $M_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ between the complexified spaces. The lagrangian extends to a holomorphic function on this space, and the Euclidean action is the restriction of this continuation to maps $E \longrightarrow X$. (Note that $E_{\mathbb{C}} = M_{\mathbb{C}}$ so $E \subset M_{\mathbb{C}}$.) There is a similar picture for other types of fields.

We consider four types of terms which typically occur in an action: kinetic terms for bosons, potential terms, topological terms (also θ-terms), and kinetic terms for fermions.

In this chapter set the speed of light $c = 1$. We use the conventions in §3.1 for Minkowski space M. So the metric is

$$g_M = dt^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{7.4}$$

On Euclidean space E we use the positive definite metric

$$g_E = d\tau^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2. \tag{7.5}$$

So as to avoid confusion, we fix the standard orientations $t, x^1, \ldots, x^{n-1}$ on M and $\tau, x^1, \ldots, x^{n-1}$ on E, and we write lagrangians as forms rather than densities, though we often omit the '$\wedge$' sign. Let

$$d^{n-1}x = dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{7.6}$$

§7.1. Kinetic terms for bosons

Consider a particle of mass m moving in some Riemannian manifold X. It is described by a map $x : M^1 \to X$. Then the kinetic energy density is (see (1.13))

$$L_M = \frac{m}{2} \left| \frac{dx}{dt} \right|^2 dt. \tag{7.7}$$

The continuation to imaginary time—after dividing by $\sqrt{-1}$—is

$$L_E = \frac{m}{2} \left| \frac{dx}{d\tau} \right|^2 d\tau. \tag{7.8}$$

In higher dimensions, we consider a real scalar field on Minkowski space, which is a real function $\phi : M \to \mathbb{R}$. The kinetic lagrangian is

$$L_M = \frac{1}{2} |d\phi|^2_M \, dt \, d^{n-1}x, \tag{7.9}$$

where $|\cdot|_M$ is the norm (7.4) on M. The continuation to E is

$$L_E = \frac{1}{2} |d\phi|^2_E \, d\tau \, d^{n-1}x, \tag{7.10}$$

where $|\cdot|_E$ is the Euclidean norm (7.5).

For a gauge field[29] A the kinetic term is the Yang-Mills lagrangian

$$L_M = -\frac{1}{2} |F_A|^2_M \, dt \, d^{n-1}x. \tag{7.11}$$

The continuation to Euclidean space is

$$L_E = \frac{1}{2} |F_A|^2_E \, d\tau \, d^{n-1}x. \tag{7.12}$$

§7.2. Potential terms

For the particle $x : \mathbb{R} \to X$, the potential energy is described by a function $V : X \to \mathbb{R}$. The corresponding term in the lagrangian is

$$L_M = -V(x(t))\, dt. \tag{7.13}$$

The continuation to imaginary time is

$$L_E = V(x(\tau))\, d\tau. \tag{7.14}$$

The extension to higher dimensions is the same: *Potential terms appear with a $-$ sign in Minkowski actions and with a $+$ sign in Euclidean actions.*

[29]For a concrete example of rotating a gauge field to Euclidean space, see the solution to Problem 11 of [II-**Dynamics of QFT, Exercises**].

§7.3. Topological terms and θ-terms

Let α be a *real* 1-form on X and consider the lagrangian (for $x\colon \mathbb{R} \to X$)

$$L_M = -x^*\alpha. \tag{7.15}$$

The continuation to imaginary time is innocuous except for the conventional division by $\sqrt{-1}$:

$$L_E = \sqrt{-1}\, x^*\alpha. \tag{7.16}$$

Hence in the Euclidean (imaginary time) lagrangian the topological term is imaginary. A similar factor is introduced into the continuation of any topological term (in the sense of Chapter 6).

The θ-terms in gauge theory (§4.3) behave similarly. For example, the continuation of

$$L_M = \frac{\theta}{2\pi}\langle\langle F_A \rangle\rangle \tag{7.17}$$

to Euclidean space is

$$L_E = -\sqrt{-1}\,\frac{\theta}{2\pi}\langle\langle F_A \rangle\rangle. \tag{7.18}$$

§7.4. Kinetic terms for fermions

We begin with the particle $x : \mathbb{R} \to X$ and add an odd field ψ which is a section of $x^*\Pi TX$, the parity-reversed pullback of the tangent bundle. Then ψ is real and in real time its kinetic term in the lagrangian is

$$L_M = \frac{m}{2}\,\langle \psi,\, \frac{d\psi}{dt} \rangle\, dt. \tag{7.19}$$

Rotating to imaginary time and dividing by $\sqrt{-1}$, we obtain

$$L_E = -\sqrt{-1}\,\frac{m}{2}\,\langle \psi,\, \frac{d\psi}{d\tau} \rangle\, d\tau. \tag{7.20}$$

Consider now a real spinor field ψ in n-dimensional Minkowski spacetime. We retain the notation of §3.4, but we set the mass $M = 0$. (The mass term is a potential energy term, so is covered by the discussion of §7.2.) The lagrangian in Minkowski space is

$$\begin{aligned} L_M &= \frac{1}{2}\psi \not{D}_M \psi \, dt\, d^{n-1}x \\ &= \frac{1}{2}(\tilde{\Gamma}_M)^{\mu a b}\psi_a \partial_\mu \psi_b \, dt\, d^{n-1}x. \end{aligned} \tag{7.21}$$

The first observation when rotating to Euclidean space is that a real spin representation S of $\mathrm{Spin}(1, n-1)$ extends to a *complex* representation $S_{\mathbb{C}}$ of the complex spin group $\mathrm{Spin}_{\mathbb{C}}(n)$, but the restriction to $\mathrm{Spin}(n) \subset \mathrm{Spin}_{\mathbb{C}}(n)$ is not necessarily real. Thus in the Euclidean theory we take the field

$$\psi\colon E \longrightarrow \Pi S_{\mathbb{C}} \tag{7.22}$$

to be a *complex* spinor field. Again: *There are no reality conditions on spinor fields in Euclidean space.* Define pairings

$$\begin{aligned} \Gamma_E &\colon S^*_{\mathbb{C}} \otimes S^*_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ \tilde{\Gamma}_E &\colon S_{\mathbb{C}} \otimes S_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \end{aligned} \tag{7.23}$$

by

$$\begin{aligned} (\Gamma_E)^0_{ab} &= -\sqrt{-1}\,(\Gamma_M)^0_{ab}, & (\tilde{\Gamma}_E)^{0ab} &= -\sqrt{-1}\,(\tilde{\Gamma}_M)^{0ab}, \\ (\Gamma_E)^i_{ab} &= -(\Gamma_M)^i_{ab}, & (\tilde{\Gamma}_E)^{iab} &= -(\tilde{\Gamma}_M)^{iab}, \end{aligned} \tag{7.24}$$

where $i = 1, \dots, n-1$ runs over the spatial indices. Then $\Gamma_E, \tilde{\Gamma}_E$ satisfy a Clifford relation

$$(\tilde{\Gamma}_E)^{\mu ab}(\Gamma_E)^\nu_{bc} + (\tilde{\Gamma}_E)^{\nu ab}(\Gamma_E)^\mu_{bc} = -2g_E^{\mu\nu}\delta^a_c. \tag{7.25}$$

The Euclidean Dirac form is

$$\psi \not{D}_E \psi = (\tilde{\Gamma}_E)^{\mu ab}\psi_a \partial_\mu \psi_b, \tag{7.26}$$

and the factors in (7.24) are chosen so that the lagrangian (7.21) rotates to

$$L_E = \frac{1}{2}\psi \not{D}_E \psi \, d\tau \, d^{n-1}x \tag{7.27}$$

in Euclidean space.

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Supersolutions

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Introduction

This text develops the basic classical supersymmetric field theories in all dimensions, but without supergravity. We emphasize at the start that although we hope to provide a bit of a fresh viewpoint in some places, this is standard material for the working quantum field theorist (string theorist, M-theorist, ...)—it was largely developed in the physics literature of the 1970s and 1980s. This account began as the solutions to the Superhomework problems in [I-**Homework**], but then developed into a larger project. Unfortunately, we often sink into an unpalatable morass of indices which may obscure the underlying geometry.[1] So we hope in this introduction to at least provide a roadmap to the more conceptual parts of the text. Certainly some readers may find the plethora of formulas useful. Indeed, we have gone to great lengths[2] to ensure their accuracy, though undoubtedly errors have crept in. Since there are so many formulas, we have boxed the ones we thought most crucial or most likely to be referenced. One contribution here is a consistent choice of conventions in all dimensions and with varying amounts of supersymmetry. Our sign conventions are summarized in [I-**Signs**]. Throughout we freely use notions of supergeometry as developed in [I-**Supersymmetry**] and notions of classical field theory as developed in [I-**Classical Fields**]. Knowledge of spinors [I-**Spinors**] is also indispensable.

The basic Poincaré-invariant classical field theories of a scalar field (spin 0), spinor field (spin 1/2), and gauge field (spin 1) exist in any dimension n. Here we are interested in field theories which are invariant under a supergroup which extends the Poincaré group. Such a *super Poincaré group* is determined by a real spin representation S^* of the Lorentzian spin group $\mathrm{Spin}(1, n-1)$ together with a symmetric pairing $S^* \otimes S^* \to V$ into the vectors, as we explain in §1.1. A theory has *minimal supersymmetry* if S^* is a minimal (that is, irreducible) spin representation; otherwise, it is said to have *extended supersymmetry*. We construct theories on Minkowski space which obey the spin-statistics connection, so that odd symmetries map fields of integral spin to fields of half-integral spin. It is convenient to measure the amount of supersymmetry in terms of $s = \dim S^*$. The larger s is, the greater the mixing of fields of different spin. As we consider theories only[3]

[1] As Mel Brooks famously said, "We mock the thing we are to be."

[2] in units where $c = 1$

[3] For a physical explanation of this restriction, see [II-**Dynamics of QFT**, §2.4]

with fields of spins at most one, we are restricted to small values of s. Specifically, supersymmetric σ-models—which only have fields of spins at most one-half—must have $s \leq 8$ and supersymmetric gauge theories must have $s \leq 16$. Because of this restriction the theory has a small set of general examples and the study of the theory is centered on the examples. We mostly consider theories with $s = 2, 4, 8, 16$ supersymmetries and these occur in the maximal dimensions $n = 3, 4, 6, 10$. "Maximal" means, for example, that only in dimensions ≤ 6 are there real spin representations of dimension 8. (We also say a bit in §1.3 about theories with a single supersymmetry and there is more in Problem FP2 of [I-**Homework**], which we recommend as a starting point before tackling the more complicated theories with more supersymmetry. Other problems in [I-**Homework**] also deal with formal properties of supersymmetric field theories.) By the exceptional isomorphisms among low dimensional Lie groups, the Lorentzian spin group in dimensions $n = 3, 4, 6$ is $SL(2, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. (There is also a sense in which the spin group in 10 dimensions is "$SL(2, \mathbb{O})$" for the octonions $\mathbb{O}$: see [I-**Spinors**, §6.5–§6.7].) Our treatment emphasizes this exceptional isomorphism, both in its theoretical and computational aspects. Throughout we focus on theories in the maximal dimension, though in some cases (see Chapter 9, for example) we work out some lower dimensional theories obtained by dimensional reduction. (Not everything in lower dimensions comes by dimensional reduction from the maximal dimension, but we barely mention the exceptions.)

A supersymmetric field theory is first of all an ordinary (lagrangian) field theory on Minkowski space, described by a set of fields and a lagrangian. As usual the Poincaré symmetry is manifest. In addition, there are *nonmanifest* odd infinitesimal symmetries corresponding to the odd generators of the super Poincaré algebra. In this approach one checks by direct computation that the theory is supersymmetric, i.e., that the odd transformation of the fields is a symmetry of the lagrangian. This is the approach we follow in Chapter 3 for σ-models and in Chapter 6 for pure Yang-Mills theories. Although there are heuristic arguments to motivate the form of the odd symmetries, they are not geometric. Moreover, in many examples the bracketing relations of the super Poincaré algebra are only obeyed on-shell. It is more satisfying to use a *superspace formulation* in which the entire super Poincaré symmetry is manifest. In this approach the spacetime M of the classical field theory is *super Minkowski space*, a supermanifold whose underlying ordinary manifold is ordinary Minkowski space. The super Poincaré group acts as symmetry transformations of super Minkowski space. The fields in the theory, usually called *superfields*, are "functions" on super Minkowski space. Then super Poincaré acts on superfields by pullback, and we can write manifestly supersymmetric lagrangians. By restricting superfields and their derivatives to the underlying ordinary Minkowski space, we recover the *component fields* of the ordinary field theory on Minkowski space, and by partial integration we recover the *component lagrangian*. These general ideas are explained in Chapter 1.

Although superspace formulations have the overriding virtue of exhibiting the supersymmetry in a manifest way, there are several drawbacks. First and foremost, there is not always a superspace formulation! Unfortunately, that is the case for theories with the maximal allowed supersymmetry: σ-models with 8 supersymmetries and Yang-Mills theories with 16 supersymmetries. There are good superspace formulations for all theories with at most 4 supersymmetries and also for pure gauge theories with at most 8 supersymmetries. Of course, one can use the superspace

formulation with 4 supersymmetries even for theories with more supersymmetry; then some of the supersymmetry is manifest and some is nonmanifest. Another disadvantage of superspace formulations is that in many cases the superfields are constrained. This complicates the classical field theory computations. Also, the constraints vary from case to case and add further to the specificity of the subject. To a physicist the most damning criticism of the superspace formulation is that it obscures the physical intuition, which is based on experience with ordinary fields on Minkowski space. These drawbacks are balanced by the manifest geometric realization of odd symmetries, particularly when applied to the quantum theory. (For example, the possible form of quantum corrections to the classical action is clearer in the superspace formulation.) So we persist in presenting the superspace point of view alongside the component approach.

Chapters 1 and 2 are preliminary. Chapter 1 describes features of classical supersymmetric field theories which go beyond the general ideas of classical field theory. In Chapter 2 we describe the coordinates and bases we use to compute in 3, 4, and 6 dimensions. As mentioned before, they are based on the special form of the spin group in those cases. We list many formulas for easy future reference.

Our real work begins in Chapter 3, where we study supersymmetric σ-models in components. The quantum theories are often discussed in 2 dimensions, but we treat the classical theories in the maximal dimensions $n = 3, 4, 6$ (where they have minimal supersymmetry). In the linear case we give a uniform description of the theory and a uniform proof of supersymmetry. In the nonlinear case the scalar field is a map from ordinary Minkowski space to a curved Riemannian manifold X. Supersymmetry constrains X: in the $n = 4$ theory X must be Kähler and in the $n = 6$ theory X must by hyperkähler. In §3.3 we give a uniform motivation for these constraints. In the nonlinear case there are curvature terms in the lagrangian, and as they vary in the three cases we check supersymmetry on a case-by-case basis. In addition to the kinetic terms, there are supersymmetric potential terms one can add in the $n = 3, 4$ theories; they are summarized in §3.4.

Chapters 4 and 5 present the superspace formulation of supersymmetric σ-models in $n = 3, 4$ dimensions, including potential terms. In Chapter 4 we not only derive the component lagrangian and supersymmetry transformations, but we also discuss the classical field theory computations directly in superspace, including the computation of the Noether current for the supersymmetries. This is the only example we work out in such detail in superspace. The superspace lagrangian of the $n = 4$ theory is usually given in terms of a local Kähler potential on the target manifold X, and so does not make sense globally. In §5.2 we point out a global lagrangian in case the target manifold is Hodge.

In Chapter 6 we add gauge fields to the mix. We begin in §6.1 with pure supersymmetric Yang-Mills theories in components. The usual argument which verifies supersymmetry relies on a "Fierz identity," something we do not encounter in our treatment. Instead we rely on a geometric property of spinors which only holds in dimensions $n = 3, 4, 6, 10$ with the minimal real spinor representation: the quadratic pairing on spinors takes values on the light cone. (In other cases the values are positive timelike or lightlike, not just lightlike.) That same property enters again in §6.3, where we describe some common features of the superspace formulation of the theory in $n = 3, 4, 6$ dimensions. In each case the basic superfield is a connection on super Minkowski space constrained to have vanishing curvature along a canonical odd distribution. The aforementioned special property of spinors

is used to see that this constraint leads to the correct physical component fields. On the other hand, the auxiliary fields[4] differ in each case; they are treated in later chapters.

The most general theory without gravity is a gauge theory with matter—a coupled system consisting of scalar fields, spinor fields, and gauge fields. The general supersymmetric theories are summarized in Theorem 6.33. Of interest in applications are the nontrivial potential energy functions which appear; they lead to nontrivial classical moduli spaces. We describe the coupled systems in dimensions $n = 3, 4, 6$ with minimal supersymmetry. (Recall that there is no possible σ-model with $s = 16$ supersymmetries, only the pure Yang-Mills theory described previously.) Systems with $s = 2, 4, 8$ supersymmetries in lower dimensions are often, but not always, obtained by dimensional reduction from the theories described here. For $n = 3, 4$ there is a good superspace description from which the component description may be derived. This is the subject of Chapters 7 and 8. In the $n = 4$ case the appearance of the moment map in the component lagrangian has a geometric explanation, at least if the target Kähler manifold is Hodge (see §8.3). In Chapters 7 through 11 we only describe the Yang-Mills terms in the lagrangian, having treated the σ-model terms in Chapters 4 and 5. Chapter 10 gives the superspace description of super Yang-Mills theory in $n = 6$ dimensions. (Recall here that there is no superspace description of the σ-model, hence none of the coupled system.) For each of the cases $n = 3, 4, 6$ we prove that the category[5] of constrained connections is equivalent to the category of component fields. For the $n = 6$ case the proof is presented in Chapter 11.

A connection is locally specified by n Lie algebra-valued functions in n dimensions, and the dimensional reduction to $n - k$ dimensions still has n functions: $n - k$ of them transform as a connection and k of them as scalar fields (see §9.1). There are two cases of dimensionally reduced super Yang-Mills theory which are of particular interest—the reduction of the $n = 4$ theory to 2 dimensions and the reduction of the $n = 6$ theory to 4 dimensions. The former is the subject of Chapter 9 and the latter the subject of §10.2.

Finally, in §5.5 and §10.3 we describe theories in $n = 4$ dimensions with extended supersymmetry on the minimal super Minkowski space. Then four of the supersymmetries are manifest and the extended supersymmetry is nonmanifest.

In several places we refer to the $n = 3, 4, 6$ cases with minimal supersymmetry as the $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ cases.

Writing this has been an absolute *cauchemar de signes*! Standard differential geometry has some bad signs, classical field theory has tricky sign conventions, and odd variables add a whole new level of complication. Our sign conventions are explained in the previous texts in Part 1 and are summarized in [I-**Signs**].

During the preparation of this manuscript we received advice and help from many people. Certainly Edward Witten's Superhomework [I-**Homework**] was our starting point for organizing the material, and Ed was invaluable at many other times. Input from our classmates, notably David Kazhdan and John Morgan, is also reflected here, as is the advice of Joseph Bernstein, Nati Seiberg, and many others. However, they should not be held accountable for any errors.

[4] An *auxiliary field* has an algebraic equation of motion and does not correspond to a physical degree of freedom.

[5] Because of the local gauge transformations, the collection of connections (gauge fields) forms a category, not a set.

CHAPTER 1
Preliminary Topics

A supersymmetric field theory has as symmetry group a *super Poincaré group*. In §1.1 we define these supergroups as symmetry groups of *super Minkowski space*. In many cases supersymmetric theories have a superspace formulation in terms of fields defined on super Minkowski space. Passage from the superspace formulation to the formulation in "components" on ordinary Minkowski space is the subject of §1.2. The simplest example is in one dimension (time) with one odd variable; this is described in §1.3.

§1.1. Super Minkowski spaces and super Poincaré groups

A super Minkowski space M is constructed as follows. The starting data are a Minkowski space $\check{M}$ with vector space of translations V, a positive cone C of time-like vectors in V, a real spinorial representation S of $\mathrm{Spin}(V)$, and a symmetric morphism Γ of representations of $\mathrm{Spin}(V)$:

$$\Gamma\colon S^* \otimes S^* \longrightarrow V. \tag{1.1}$$

which is positive definite in the sense that $\Gamma(s^*, s^*) \in \overline{C}$ for $s^* \in S^*$, with $\Gamma(s^*, s^*) = 0$ only for $s^* = 0$.

Minkowski space is an affine space: it has no origin. By the classification in [I-**Spinors**, §6.3(i)], once the representation S is given, positive definite Γ as in (1.1) exist and are unique up to automorphisms of the representation S. If S is an irreducible real spinorial representation, nonzero symmetric morphisms $S^* \otimes S^* \to V$ are unique up to a factor, are positive or negative definite, and are respected by elements of norm 1 in the field of endomorphisms of S (isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) [I-**Spinors**, §6.1].

Note that so far we have not used the metric on V, only its conformal structure and the choice of a positive cone.

By [I-**Spinors**, §6.2] there is a unique symmetric morphism

$$\tilde{\Gamma}\colon S \otimes S \to V \tag{1.2}$$

related to Γ by the following formula. With respect to a basis $\{e_\mu\}$ of V and $\{f^a\}$ of S, write[6] $\Gamma(f_a, f_b) = \Gamma^\mu_{ab} e_\mu$ and $\tilde{\Gamma}(f^a, f^b) = \tilde{\Gamma}^{\mu ab} e_\mu$. Then

$$\boxed{\Gamma^\mu_{ab}\tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu}\delta^c_a\ .} \tag{1.3}$$

[6]We use the usual summation convention throughout: Repeated indices are summed if one index is "upstairs" and one is "downstairs." Also, the Kronecker δ-symbol has its usual meaning.

If one uses the Minkowski bilinear form $g(\cdot,\cdot) = (\cdot,\cdot)$ on V to convert (1.1) and (1.2) into morphisms

$$\gamma\colon V \to \mathrm{Hom}(S^*, S), \qquad \tilde{\gamma}\colon V \to \mathrm{Hom}(S, S^*) \tag{1.4}$$

i.e., if one defines γ and $\tilde{\gamma}$ by

$$\begin{aligned} t^*\big(\gamma(v)(s^*)\big) &= (\Gamma(s^*, t^*), v) \quad \text{and} \\ \tilde{\gamma}(v)(s)(t) &= (\tilde{\Gamma}(s,t), v), \end{aligned} \tag{1.5}$$

then (1.3) means that for v in V,

$$\begin{aligned} &\tilde{\gamma}(v)\gamma(v)\colon S^* \to S^* \quad \text{and} \\ &\gamma(v)\tilde{\gamma}(v)\colon S \to S \end{aligned}$$

are multiplication by (v,v). In other words, γ and $\tilde{\gamma}$ turn $S \oplus S^*$ into a module over the Clifford algebra $C(V)$. By [I-**Spinors**, §4.9.6] the action of $\mathrm{Spin}(V)$ on S and S^* is induced by this $C(V)$-module structure. Taking the trace in $\tilde{\gamma}(v)\gamma(v) = (v,v)$, i.e., contracting the indices a and c in (1.3), one obtains

$$\Gamma^{\mu}_{ab}\,\tilde{\Gamma}^{\nu ab} + \Gamma^{\nu}_{ab}\,\tilde{\Gamma}^{\mu ab} = 2\,\dim(S) g^{\mu\nu}\ . \tag{1.6}$$

The identity (1.3) also means that if $(v,v) \neq 0$, the inverse of the symmetric bilinear form $(\Gamma(s^*,t^*),v)$ on S^* is the bilinear form $(v,v)^{-1}(\tilde{\Gamma}(s,t),v)$ on S. For v in C, the bilinear form $(\tilde{\Gamma}(s,t),v)$ is hence positive definite, and $\tilde{\Gamma}\colon S \otimes S \to V$ is positive definite.

By the unicity of Γ up to automorphisms of S, it follows that if the representation S of $\mathrm{Spin}(V)$ is isomorphic to its dual, there exists an isomorphism of representations $\alpha\colon S \to S^*$ such that

$$\tilde{\Gamma}(s,t) = \Gamma(\alpha(s), \alpha(t))\ . \tag{1.7}$$

This isomorphism corresponds to a nondegenerate $\mathrm{Spin}(V)$-invariant bilinear form ϵ on S:

$$\epsilon(s,t) = \alpha(s)(t)\ . \tag{1.8}$$

The form $\epsilon(t,s)$ corresponds to the transpose ${}^t\alpha$ of α. One has also $\tilde{\Gamma}(s,t) = \Gamma({}^t\alpha(s), {}^t\alpha(t))$. Indeed, expressing for $(v,v) \neq 0$ that the nondegenerate forms $(v, \tilde{\Gamma}(s,t))$ and $(v, \Gamma(\alpha(s), \alpha(t)))$ have the same inverse, one finds that $\Gamma(s',t') = \tilde{\Gamma}(\alpha^*(s'), \alpha^*(t'))$, with $\alpha^* = {}^t\alpha^{-1}$.

In coordinates, (1.7) reads

$$\boxed{\tilde{\Gamma}^{\mu ab} = \Gamma^{\mu}_{cd}\,\epsilon^{ac}\epsilon^{bd}\ .} \tag{1.9}$$

We now assume that S is irreducible, and is isomorphic to its dual. This excludes the cases where the dimension n of V is congruent to 2 or 6 modulo 8. We treat in turn the cases where the field of endomorphisms Z of S is isomorphic to $\mathbb{R}$ ($n \equiv 1$ or $3 \bmod 8$), $\mathbb{H}$ ($n \equiv 5$ or $7 \bmod 8$) or $\mathbb{C}$ ($n \equiv 0$ or $4 \bmod 8$). Note that (1.7) holds up to a factor for any invariant non-zero bilinear form ϵ.

Case $Z = \mathbb{R}$. Up to a factor, S admits a unique invariant bilinear form, symmetric for $n \equiv 1(\mathrm{mod}8)$ and alternating for $n \equiv 3(\mathrm{mod}8)$. The form ϵ for which (1.7) holds is unique up to sign.

Case $Z = \mathbb{H}$. Up to a factor, S admits a unique invariant bilinear form such that

$$\epsilon(sh, t) = \epsilon(s, t\bar{h}), \qquad h \in H, \quad s, t \in S. \tag{1.10}$$

(We take S to be a *right* $\mathbb{H}$-module.) After extension of scalars from $\mathbb{R}$ to $\mathbb{C}$, S becomes $S_0 \otimes W$, with S_0 irreducible and W of dimension 2, $\mathbb{H}$ becomes $\mathrm{End}(W)$ and (1.10) means that $\epsilon = \epsilon_0 \otimes \epsilon_W$, for ϵ_0 invariant on S_0 and ϵ_W the alternating form on W. The form ϵ for which (1.10) holds is symmetric for $n \equiv 5(\mathrm{mod}8)$ and alternating for $n \equiv 7(\mathrm{mod}8)$, as ϵ_0 is respectively alternating and symmetric [I-**Spinors**, §1.5.1]. The form ϵ for which (1.10) and (1.7) holds is unique up to sign.

Case $Z = \mathbb{C}$. After extension of scalars to $\mathbb{C}$, S becomes the sum of two irreducible representations, orthogonal for $n \equiv 0(\mathrm{mod}8)$ and symplectic for $n \equiv 4(\mathrm{mod}8)$. Over $\mathbb{R}$, the invariant bilinear form will correspondingly be orthogonal or symplectic, with $\epsilon(zs, t) = \epsilon(s, zt)$. The ones for which (1.7) holds form a U_1-orbit under $\epsilon(s, t) \mapsto \epsilon(s, ut)$.

In all cases, ϵ is symmetric or antisymmetric, so that the resulting isomorphism $\alpha\colon S \to S^*$ is unambiguous, at least up to sign. Transporting ϵ by α, we obtain the dual form $\tilde{\epsilon}$ on S^*, which is completely unambiguous. One has[7]

$$\boxed{\epsilon^{ab}\,\tilde{\epsilon}_{cb} = \delta^a_c\ .} \tag{1.11}$$

If the dimension is congruent to 2 or 6 (mod8)—i.e., for the range of dimensions of physical interest, in dimensions 2, 6 and 10—there are super Minkowski spaces based on unequal number of copies of the half spinor representations S^+ and S^-. The invariant pairing in those dimensions is between S^+ and S^-, so that Γ and $\tilde{\Gamma}$ encode information not related by a self-duality pairing.

Let $\mathcal{L}$ be the following super Lie algebra: $\mathcal{L}^0 = V$, $\mathcal{L}^1 = S^*$, the subalgebra $\mathcal{L}^0 = V$ is central, and the Lie bracket on S^* is -2Γ. The corresponding super Lie group $\exp\mathcal{L}$ is defined to be a simply connected super Lie group, given with an isomorphism of $\mathcal{L}$ with its Lie algebra of left-invariant vector fields. It can be constructed as follows. As a space it is

$$\exp\mathcal{L} = V \times \Pi S^*. \tag{1.12}$$

In other words, for any supercommutative ring R the set of its R-points is the even part of $\mathcal{L}_R = \mathcal{L} \otimes R \cong R \otimes \mathcal{L}$. The group law is derived from the Hausdorff formula, which terminates after the quadratic term since $[x, [y, z]] = 0$ for all $x, y, z \in \mathcal{L}$:

(1.13)

$$(v_1, s_1^*)(v_2, s_2^*) = (v_1 + v_2 + \frac{1}{2}[s_1^*, s_2^*], s_1^* + s_2^*), \qquad v_1, v_2 \in R^0 \otimes V, \quad s_1^*, s_2^* \in R^1 \otimes S^*.$$

[7]We do not set a convention for raising and lowering a single index with ϵ in the antisymmetric case, since, as was just explained, this involves an arbitrary sign choice we prefer not to make. However, there is no ambiguity in raising and lowering an even number of indices, as for example in (1.10). Thus $\epsilon_{12} = \epsilon^{ab}\epsilon_{1a}\epsilon_{2b} = \epsilon^{21}\epsilon_{12}\epsilon_{21} = -\epsilon^{21} = \epsilon^{12}$. Beware: Many standard physics texts use a different convention here.

The sign rule applied to $[s_1^*, s_2^*] = -2\Gamma(s_1^*, s_2^*)$ gives

$$\frac{1}{2}[b_1 \otimes t_1^*, b_2 \otimes t_2^*] = b_1 b_2 \Gamma(t_1^*, t_2^*), \qquad b_1, b_2 \in R^1, \quad t_1^*, t_2^* \in S^*. \tag{1.14}$$

Fix bases $\{e_\mu\}$ of V and $\{f^a\}$ of $S = (S^*)^*$. As before, write $\Gamma(f_a, f_b) = \Gamma_{ab}^\mu e_\mu$ and $\tilde{\Gamma}(f^a, f^b) = \tilde{\Gamma}^{\mu ab} e_\mu$. In the super Lie algebra $\mathcal{L}$, with the basis $\{e_\mu; f_a\}$, the nontrivial brackets are

$$[f_a, f_b] = -2\Gamma_{ab}^\mu e_\mu. \tag{1.15}$$

In the physics literature it is customary to denote e_μ, f_a by P_μ, Q_a, and we sometimes follow that convention. Thus we write

$$\boxed{[Q_a, Q_b] = -2\Gamma_{ab}^\mu P_\mu.} \tag{1.16}$$

The bases of V^*, S induce a coordinate system (x^μ, θ^a) on $\exp(\mathcal{L}) = V \times \Pi S^*$: the coordinates of $P \in (\mathcal{L} \otimes R)^0$ are $x^\mu(P) = \langle e^\mu, P \rangle$ and $\theta^a(P) = \langle f^a, P \rangle$. If $\{e_\mu\}$ and $\{f_a\}$ are the dual bases of V and S^*, then $P = e_\mu x^\mu + f_a \theta^a$. Let ∂_μ and ∂_a be the corresponding vector fields. In other words,

$$\partial_\mu x^\nu = \delta_\mu^\nu, \qquad \partial_a \theta^b = \delta_a^b, \qquad \partial_\mu \theta^a = \partial_a x^\mu = 0. \tag{1.17}$$

The Lie algebra $\mathcal{L}$ is identified with the tangent space to $\exp \mathcal{L}$ at the origin via $e_\mu \mapsto \partial_\mu \big|_0$ and $f_a \mapsto \partial_a \big|_0$. We shall write D_a for the left-invariant vector field which is ∂_a at 0, and we write τ_{Q_a} for the right-invariant vector field which is ∂_a at 0. Then

$$\begin{aligned} D_a &= \partial_a - \theta^b \Gamma_{ab}^\mu \partial_\mu, \\ \tau_{Q_a} &= \partial_a + \theta^b \Gamma_{ab}^\mu \partial_\mu \end{aligned} \tag{1.18}$$

at a point with coordinates (x^μ, θ^b). The vector fields ∂_μ are both left- and right-invariant. The bracketing relations are

$$\boxed{\begin{aligned} [D_a, D_b] &= -2\Gamma_{ab}^\mu \partial_\mu, \\ [\tau_{Q_a}, \tau_{Q_b}] &= 2\Gamma_{ab}^\mu \partial_\mu, \\ [D_a, \tau_{Q_b}] &= 0. \end{aligned}} \tag{1.19}$$

The D_a, being left-invariant, have the same brackets as the abstract Lie algebra $\mathcal{L}$. (See (1.15).) The vector fields D_a and τ_{Q_b} commute, since right and left translations commute.

Super Minkowski space M is the underlying supermanifold of the super Lie group $\exp(\mathcal{L})$. We let $\exp(\mathcal{L})$ act on the left on M by left translations. Corresponding to (1.12) we have a splitting

$$M = \check{M} \times \Pi S^*. \tag{1.20}$$

It is well-defined since the ambiguity in identifying M with $\exp(\mathcal{L})$ is translation by an element of V, and such translations preserve the splitting (1.12). The *super*

Poincaré group is the semi-direct product $\mathrm{Spin}(V) \ltimes \exp(\mathcal{L})$. It acts on M, and the structures we will consider on M are invariant under the super Poincaré group, with one notable exception—the splitting (1.20). For example, the left translates of $S^* = \mathcal{L}^1 \subset \mathcal{L}$ form a left-invariant distribution τ on M which is preserved by the action of the super Poincaré group.

As we saw, bases $\{e^\mu\}$ of V^* and $\{f^a\}$ of S give a coordinate system (x^μ, θ^a) on $M = \exp(\mathcal{L})$. It will often be more convenient to make computations not in this coordinate system, but rather using the left-invariant moving frame consisting of the vector fields D_a, ∂_μ and/or the dual moving coframe consisting of the 1-forms ρ^a, ω^μ. We fix the sign in the duality by putting vectors to the left of forms:

$$\langle D_a, \rho^b \rangle = \delta_a^b. \tag{1.21}$$

Also, we will sometimes use *complex* bases and so complex vector fields and differential forms. The complex conjugate of a product is the product of the complex conjugates in the *same* order, even for odd elements.[8]

The infinitesimal generators of the action of $\exp(\mathcal{L})$ on M are the right-invariant vector fields. The τ_{Q_a} defined above are a basis of the odd right-invariant vector fields.

A spinor field $\psi = \psi_a f^a$ on $\check{M}$ is a map $\psi\colon \check{M} \to \Pi S$. The form $\tilde{\Gamma}$ then determines a symmetric (in the super sense) bilinear form on spinor fields, which we denote with the usual physicists' notation:

$$\boxed{\psi \not{D} \psi = \tilde{\Gamma}^\mu(\psi, \partial_\mu \psi) = \tilde{\Gamma}^{\mu a b} \psi_a \partial_\mu \psi_b.} \tag{1.22}$$

We call (1.22) the *Dirac form*. One should understand '$\not{D}$' in this formula as the name of the bilinear form, which has been inserted between the arguments. The Dirac form is symmetric up to an exact term, so is exactly symmetric when integrated over $\check{M}$ (assuming no contributions at infinity). A dual spinor field $\lambda = \lambda^a f_a$ is a map $\lambda\colon \check{M} \to \Pi S^*$, and we have the (dual) Dirac form

$$\boxed{\lambda \not{D} \lambda = \Gamma^\mu(\lambda, \partial_\mu \lambda) = \Gamma^\mu_{ab} \lambda^a \partial_\mu \lambda^b.} \tag{1.23}$$

If there is a duality pairing (1.8), then we can use it to identify dual spinors with spinors, in which case (1.23) is identified with (1.22).

We complexify V and S to construct complexified super Minkowski space $M_{\mathbb{C}}$. Inside $\check{M}_{\mathbb{C}}$ we may consider a real affine subspace E of Euclidean signature.[9] However, the complex spinors $S_{\mathbb{C}}$ generally will not carry a real structure under the Euclidean spin group $\mathrm{Spin}(E)$, and so the corresponding superspace $\check{M} \times \Pi S_{\mathbb{C}}$ is complex in odd directions. It is a cs supermanifold [I-**Supersymmetry**, §4.8]. The lagrangians we construct can be rotated to Euclidean space.[10]

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[8] See §5 of [I-**Signs**].

[9] With our sign conventions the metric will be *negative* definite; simply change the sign of the metric to obtain a positive definite metric.

[10] For more details on the transition from Minkowski space to Euclidean space, see Chapter 7 of [I-**Classical Fields**].

Remark 1.24. The super Minkowski space is usually called "$N = k$ superspace" if S is the sum of k irreducible real representations of $\mathrm{Spin}(V)$. However, the usage fluctuates somewhat for $n = \dim(\check{M})$ small. For example, if $\dim(\check{M}) = 2$, then $N = 1$ usually means that S is the sum of the two inequivalent real semi-spinorial representations S^+ and S^-. It is more informative to use the notation '$M^{n|s}$' to indicate s supersymmetries in n dimensions, and in dimensions $n \equiv 2(\mathrm{mod}4)$ to use a special notation to indicate the precise spinorial representation: '$M^{n|(s^+,s^-)}$' denotes a super Minkowski space of dimension $n|s^+ + s^-$ built from a spinorial representation which is a sum of copies of S^+, of dimension s^+, and a sum of copies of S^-, of dimension s^-.

The connected group of outer automorphisms of the super Poincaré group which fix the Poincaré subgroup is called the *R-symmetry group*. Dimensional reduction leads to an R-symmetry group (double covering rotations in the extra spatial dimensions), but it is not the only source of R-symmetries.

§1.2. Superfields, component fields, and lagrangians[11]

For many theories we work directly on super Minkowski space M. In such situations we say the theory has a "superspace formulation." Fields on M are called *superfields*. Just as in ordinary nonsupersymmetric field theory, typical examples are: (i) maps $\Phi\colon M \to X$ into an ordinary manifold X; and (ii) a connection A on a principal bundle $\mathcal{P} \to M$. Very often superfields are constrained. For example, in the four-dimensional supersymmetric σ-model we constrain certain derivatives of Φ to vanish, and in all super gauge theories we constrain certain components of the curvature to vanish. The usual elementary facts about functions apply to superfields. Thus elementary operations such as composition, sum, and product produce new superfields. We can also produce new superfields by differentiation, for example by the vector fields D_a, τ_{Q_a} defined in (1.18).

Component fields are ordinary (even and odd) fields on Minkowski space $\check{M}$. We view these as maps from $\check{M}$ into a supermanifold. Given a superfield Φ we define its component fields to be the restriction to $\check{M}$ of certain derivatives of Φ. Let

$$i\colon \check{M} \longrightarrow M \tag{1.25}$$

denote the inclusion of Minkowski space into super Minkowski space. Then for a map $\Phi\colon M \to X$ we typically define a *multiplet* of component fields (ϕ, ψ, F) with formulas like

$$\begin{aligned} \phi &= i^*\Phi, \\ \psi &= i^*D\Phi, \\ F &= i^*DD\Phi. \end{aligned} \tag{1.26}$$

For specific instances, see (4.2) or (5.3). The definition of the multiplet is chosen so that the supermanifold of superfields is isomorphic to the supermanifold of multiplets. (In gauge theory this is an equivalence of categories rather than a bijection.)

[11] Our formulas in this subsection are meant to be suggestive only. Precise versions depend on the particular theory; they appear in subsequent sections. For example, versions of (1.26) for the supersymmetric σ-model appear in (4.7) and (5.7). Also, versions of (1.29) are used in the supersymmetric σ-model to compute (4.14) and (5.10).

In (1.26) ϕ is a map $\phi\colon \check{M} \to X$ and ψ is a spinor field on $\check{M}$ with values in ϕ^*TX. Note that ϕ, F are even, whereas ψ is odd. The field F is *auxiliary* in the sense that for fundamental lagrangians no derivatives of F occur in the lagrangian. The classical field equation for F (keeping the other fields fixed) expresses F in terms of the other fields. In some cases there are no auxiliary fields. If X is flat space, then the formulas (1.26) are well-defined as written. If X is curved, however, then the outermost D in the definition of F acts on a section of the nontrivial bundle Φ^*TX, and we use a covariant derivative to define the action of D. In the simplest supersymmetric σ-models X is Riemannian and we use the pullback of the Levi-Civita connection. In other models different connections on TX (e.g. with torsion) might be more convenient. In gauge theories we meet superfields Λ which are sections of nontrivialized bundles associated to the principal bundle[12] $\mathcal{P}$. A typical example is the adjoint bundle. Since one of the fields in the theory is a connection A on $\mathcal{P}$, we use it to define derivatives of Λ. Our covariant definitions of component fields are global—we never choose coordinate systems or local trivializations.

Component fields are defined in terms of the left-invariant vector fields D. *Lagrangian densities* on superspace are also written in terms of D. Recall that a lagrangian density L is a section of $\operatorname{Dens} M \to \mathcal{F} \times M$, where $\mathcal{F}$ is the supermanifold of superfields and $\operatorname{Dens} M$ the density bundle of M, pulled back over the product. Let η^a be odd parameters. Then the element $\eta^a Q_a$ induces an even vector field ξ on $\mathcal{F} \times M$. The corresponding one-parameter group combines the motion on M with the *inverse*[13] pullback on fields. We now give a general argument that the lagrangians we construct are invariant under ξ. First, note that in our two examples—a σ-model with field $\Phi\colon M \to X$ and a gauge theory with connection A on a principal bundle $\mathcal{P} \to M$ over super Minkowski space—the connection $\mathcal{A}$ we use to define the action of D on superfields depends on the field, so is best thought of as living on $\mathcal{F} \times M$. For example, in the σ-model $\mathcal{A}$ is the pullback of the Levi-Civita connection on TX by the evaluation map $\mathcal{F} \times M \to X$. In the gauge theory $\mathcal{A}$ is the natural "universal connection" on $\mathcal{P} \to \mathcal{F} \times M$ which is flat in the $\mathcal{F}$ direction. Now for the σ-model $\mathcal{A}$ is canonically trivial along the orbits of ξ, since ξ is tangent to the fibers of the evaluation map. Hence the actions of ξ and D on superfields via the connection $\mathcal{A}$ do (super)commute. In the gauge theory case the same statement is true up to gauge transformations, as the reader may verify with some computation (cf. the last remark in [I-**Classical Fields**, §4.2]). Therefore, if L is a gauge-invariant lagrangian density constructed from derivatives of superfields by the D's, then L is invariant under ξ. This is the statement that "L is supersymmetric".

Next, we derive the general shape of the formula for the action of $\eta^a Q_a$ on component fields. We denote a typical component field as

$$f = i^* D^r \Phi. \tag{1.27}$$

Implicit in the notation is that D acts by covariant derivative where necessary. Now $\eta^a Q_a$ generates a diffeomorphism $\exp(\eta^a \tau_{Q_a})$ of M, and we define[14] $\hat{\xi} f$ to be the

[12] In other places we call a 'principal G bundle' a 'G-torsor'.

[13] We use pullback by the *inverse* to achieve a *left* action on functions. This introduces a minus sign into the infinitesimal action of vector fields on functions. For example, ∂_μ acts on functions by *minus* the indicated derivative.

[14] Usually, $\hat{\xi}$ is denoted as 'δ', but since we use 'δ' as the differential along $\mathcal{F}$, we avoid this notation.

change in the component f under this diffeomorphism acting on Φ by the *inverse* pullback:

$$\hat{\xi} f := \frac{D}{Dt}\Big|_{t=0} i^* D^r \big(\exp(-t\eta^a \tau_{Q_a})^* \Phi\big). \tag{1.28}$$

We write 'D/Dt' to indicate that we may have to use a covariant derivative to differentiate in t. To compute (1.28) we use the fact that $\exp(-t\eta^a \tau_{Q_a})$ preserves D and so

$$\begin{aligned} \hat{\xi} f &= \frac{D}{Dt}\Big|_{t=0} i^* \exp(-t\eta^a \tau_{Q_a})^* (D^r \Phi) \\ &= -\eta^a i^* \tau_{Q_a} D^r \Phi \\ &= -\eta^a i^* D_a D^r \Phi. \end{aligned} \tag{1.29}$$

At the last stage we use the fact that τ_{Q_a} and D_a agree when restricted to $\check{M}$. In practice we use (1.29) and the commutation relations among the Ds to express $\hat{\xi} f$ in terms of other component fields.

With our sign conventions, the infinitesimal symmetry P_μ acts on a component field f by $-\partial_\mu f$.

The supersymmetry transformations (1.29) on components obey the commutation relations derived from those of the vector fields Q if there are no covariant derivatives. Explicitly, if η_1, η_2 are odd parameters and $\hat{\xi}_i$ the vector field which corresponds to $\eta_i^a Q_a$, then[15]

$$\boxed{[\hat{\xi}_1, \hat{\xi}_2] f = 2\eta_1^a \eta_2^b \Gamma_{ab}^\mu \partial_\mu f.} \tag{1.30}$$

The same is true for nonlinear σ-models, but in gauge theory these commutation relations may be altered by curvature terms which act as infinitesimal gauge transformations.

The most straightforward way to define a *component lagrangian* for the component fields on $\check{M}$, starting from a lagrangian L in superspace, is to integrate L over the odd variables using the splitting (1.20). In other words, the splitting (1.20) determines a projection

$$\pi \colon M \longrightarrow \check{M} \tag{1.31}$$

and so an integration

$$\pi_* \colon \mathrm{Dens}(M) \longrightarrow \mathrm{Dens}(\check{M}) \tag{1.32}$$

on densities. However, we find better formulas in certain cases by a procedure which differs from this by an exact term. First, the metric on the vector space V determines a canonical positive density $|d^n x|$ on Minkowski space $\check{M}$; it is invariant

[15] In the abstract supersymmetry algebra we have $[\eta_1^a Q_a, \eta_2^b Q_b] = -\eta_1^a \eta_2^b [Q_a, Q_b] = 2\eta_1^a \eta_2^b \Gamma_{ab}^\mu P_\mu$. There are two minus signs which cancel in passing to (1.30): a minus sign for brackets since we use left actions, and the minus sign in the action of P_μ. (See §5 of [I-**Signs**].)

under the Poincaré group. Similarly, if we fix a volume form[16] on S^*, then together with the metric on V this determines a canonical density $|d^n x|\, d^s\theta$ on super Minkowski space M; it is invariant under the super Poincaré group. We define a lagrangian *function* ℓ on M by

$$L = |d^n x|\, d^s\theta\, \ell. \tag{1.33}$$

Then we define the component lagrangian density $|d^n x|\, \check{L}$ on Minkowski space $\check{M}$ by applying some definite combination of D operators, which we symbolically denote 'D^s', to ℓ and then restricting to $\check{M}$:

$$|d^n x|\, \check{L} = |d^n x|\, (i^* D^s \ell). \tag{1.34}$$

Notice that the component lagrangian *function* $\check{L}$ is naturally expressed in terms of component fields. Often i^*D^s is chosen to agree with integration over the odd variables in the splitting (1.20), but in some cases not. For an example of the former (on $M^{3|2}$) see (2.40); for an example of the latter (on $M^{4|4}$) see the discussion following (2.72). In any case we suppose there is a Poincaré invariant differential operator Δ on $\check{M}$ such that

$$\begin{aligned} \pi_* L &= |d^n x|\, (i^* D^s \ell + \Delta i^* \ell) \\ &= |d^n x|\, (\check{L} + \Delta i^* \ell). \end{aligned} \tag{1.35}$$

For integration on $M^{4|4}$ this is formula (2.71).

Theorem 1.36. *The component lagrangian* (1.34) *is supersymmetric.*

Here the supersymmetry is *nonmanifest* so that $\mathrm{Lie}(\hat{\xi})(|d^n x|\, \check{L})$ is a nonzero exact form, where $\hat{\xi}$ is defined in (1.29).

Proof. The verification is easy if the component lagrangian is defined by ordinary integration (1.32), i.e., if $\Delta = 0$ in (1.35):

$$\begin{aligned} \mathrm{Lie}(\hat{\xi})\big(|d^n x|\, \check{L}\big) &= \mathrm{Lie}(\hat{\xi})(\pi_* L) \\ &= \pi_*\big(\mathrm{Lie}(-\eta^a \tau_{Q_a}) L\big) \\ &= \pi_*\big(d\iota(-\eta^a \tau_{Q_a}) L\big) \\ &= d\Big\{\pi_*\big(-\iota(\eta^a \tau_{Q_a}) L\big)\Big\}, \qquad \text{if } \Delta = 0. \end{aligned} \tag{1.37}$$

In this computation 'Lie' denotes the Lie derivative, and we use the Cartan formula $\mathrm{Lie}(\zeta) = d\iota(\zeta) + \iota(\zeta)d$ to define the *integral density*[17] $\iota(\eta^a \tau_{Q_a}) L$. Even if $\Delta \neq 0$ we use (1.35) to rewrite this last expression. First, use (1.31) to lift $|d^n x|$ to M, where

[16] Neither the superspace lagrangian L nor the component lagrangian function $\check{L}$ depends on this volume form; only the superspace lagrangian function ℓ in (1.33) and the choice of D^s in (1.34) depend on it.

[17] See [I-**Supersymmetry**, §§3.9–3.12] for an explanation of integral forms and integral densities.

we view it as a twisted differential form (twisted by the orientation bundle). Then

(1.38)
$$\begin{aligned} d\Big\{\pi_*\big(-\iota(\eta^a Q_a)L\big)\Big\} &= d\Big\{\pi_*\big(-\iota(\eta^a\partial_a + \eta^a\Gamma^\mu_{ab}\theta^b\partial_\mu)\,|d^nx|\,d^s\theta\,\ell\big)\Big\} \\ &= d\Big\{\pi_*\big(-\iota(\eta^a\Gamma^\mu_{ab}\theta^b\partial_\mu)\,|d^nx|\,d^s\theta\,\ell\big)\Big\} \\ &= d\Big\{i^*D^s\big(-\iota(\eta^a\Gamma^\mu_{ab}\theta^b\partial_\mu)\,|d^nx|\,\ell\big) \\ &\qquad\qquad + \Delta i^*\big(-\iota(\eta^a\Gamma^\mu_{ab}\theta^b\partial_\mu)\,|d^nx|\,\ell\big)\Big\} \\ &= d\Big\{i^*D^s\big(-\iota(\eta^a Q_a)\,|d^nx|\,\ell\big)\Big\}. \end{aligned}$$

In the penultimate step we use the fact that $i^*\theta^b = 0$. Now if $\Delta \neq 0$ we write

$$\Delta = \Delta^\mu\partial_\mu \tag{1.39}$$

for some constant differential operators Δ^μ. The expression (1.39) is not in general unique. Now

$$\begin{aligned} \mathrm{Lie}(\hat\xi)\big(|d^nx|\,\Delta i^*\ell\big) &= |d^nx|\,\Delta i^*(-\eta^a\tau_{Q_a}\ell) \\ &= |d^nx|\,\Delta i^*(-\eta^a D_a\ell) \\ &= d\Big\{-\iota(\partial_\mu)\,|d^nx|\,\Delta^\mu i^*(\eta^a D_a\ell)\Big\}. \end{aligned} \tag{1.40}$$

Putting together (1.35), (1.37), (1.38), and (1.40) we have the desired result

(1.41)
$$\mathrm{Lie}(\hat\xi)\big(|d^nx|\,\check L\big) = d\Big\{i^*D^s\big(-\iota(\eta^a Q_a)\,|d^nx|\,\ell\big) - \iota(\partial_\mu)|d^nx|\,\Delta^\mu i^*(\eta^a D_a\ell)\Big\}$$

(1.42)
$$= d\Big\{-\iota(\partial_\mu)|d^nx|\big(\Gamma^\mu_{ab}i^*D^s(\eta^a\theta^b\ell) + \Delta^\mu i^*(\eta^a D_a\ell)\big)\Big\}.$$

Equation (1.41) gives one contribution to the *supercurrent*, which is minus the Noether current for the supersymmetry transformation $\hat\xi$. If $\Delta \neq 0$, then the supercurrent we derive from these formulas depends on the choice we make in (1.39).

We sometimes define (a term in) a lagrangian on $\check M$ simply by restricting a scalar superfield S on M to Minkowski space $\check M$:

$$\check L = i^*S. \tag{1.43}$$

Of course, (1.34) is of this form with $S = D^s\ell$. In general (1.43) is *not* supersymmetric. Put differently, only in special cases does the assumption that $\hat\xi S = 0$ on M imply that $\mathrm{Lie}(\hat\xi)(|d^nx|\,\check L)$ is exact.

For some theories there is no (known) superspace formulation and we work directly in components. Then the supersymmetry transformation is defined directly on the component fields and a direct computation is needed to verify that a (component) lagrangian is supersymmetric.

§1.3. A simple example[18]

Here we consider a supersymmetric version of a particle moving on a line. Consider $M^{1|1}$ with coordinates t, θ. The basic odd vector fields are

$$\begin{aligned} D &= \partial_\theta - \theta\partial_t \\ \tau_Q &= \partial_\theta + \theta\partial_t, \end{aligned} \tag{1.44}$$

with $[D, D] = -2\partial_t$ and $[\tau_Q, \tau_Q] = 2\partial_t$. A superfield $\Phi\colon M^{1|1} \to \mathbb{R}$ has components

$$\begin{aligned} \phi &= i^*\Phi, \\ \psi &= i^*D\Phi. \end{aligned} \tag{1.45}$$

Then for $\hat{\xi}$ corresponding to $\eta^a Q_a$ we have, according to (1.29),

$$\begin{aligned} \hat{\xi}\phi &= -\eta i^* D\Phi = -\eta\psi, \\ \hat{\xi}\psi &= -\dot{\eta} i^* D^2\Phi = \eta\dot{\phi}, \end{aligned} \tag{1.46}$$

where $\dot{\phi} = \partial_t\phi$. We check the bracketing in the simplest case:

$$\begin{aligned} (\hat{\xi}_1\hat{\xi}_2 - \hat{\xi}_2\hat{\xi}_1)\phi &= \hat{\xi}_1(-\eta_2\psi) - \hat{\xi}_2(-\eta_1\psi) \\ &= -\eta_2\hat{\xi}_1\psi + \eta_1\hat{\xi}_2\psi \\ &= -\eta_2\eta_1\dot{\phi} + \eta_1\eta_2\dot{\phi} \\ &= 2\eta_1\eta_2\dot{\phi}. \end{aligned} \tag{1.47}$$

This is consistent with (1.30).[19]

A fundamental lagrangian for the field Φ is

$$L = |dt|\, d\theta \left\{ -\frac{1}{2} D\Phi\partial_t\Phi \right\}. \tag{1.52}$$

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[18] Further analysis of this example appears in Problem FP2 of [I-**Homework**].

[19] It is important to use the odd parameter η in these computations. If we write informally

$$\Phi(t, \theta) = \phi(t) + \theta\psi(t), \tag{1.48}$$

then the action of $-\tau_Q$ is

$$-\tau_Q\Phi(t, \theta) = -\psi(t) - \theta\dot{\phi}(t), \tag{1.49}$$

from which we might *erroneously* conclude

$$\begin{aligned} (-\tau_Q)\phi &= -\psi, \\ (-\tau_Q)\psi &= -\dot{\phi}. \end{aligned} \tag{1.50}$$

Multiplying by η we do *not* obtain (1.46). Rather, instead of (1.49) we should write

$$(-\eta\tau_Q)\Phi(t, \theta) = -\eta\psi(t) + \theta\eta\dot{\phi}(t) \tag{1.51}$$

and correctly deduce (1.46). The process of taking components involves odd derivatives; the odd parameter η protects us from making a sign mistake.

The component lagrangian (1.34) is easy to compute:

$$\check{L} = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}\psi\dot{\psi}. \tag{1.53}$$

We evaluate the expression in braces on the right hand side of (1.41), noting that $\Delta^z = 0$ in this case:

$$\begin{aligned} i^*D\big(-\iota(\eta Q)\,|dt|\,\frac{-1}{2}D\Phi\partial_t\Phi\big) &= \frac{1}{2}i^*D(\eta\theta D\Phi\partial_t\Phi) \\ &= -\frac{1}{2}\eta\psi\dot{\phi}. \end{aligned} \tag{1.54}$$

We check (1.41) by computing the left hand side directly from (1.53) using (1.46):

$$\begin{aligned} \hat{\xi}\check{L} &= -\big(\partial_t(\eta\psi)\dot{\phi}\big) + \frac{1}{2}\eta\dot{\phi}\dot{\psi} + \frac{1}{2}\psi\partial_t(\eta\dot{\phi}) \\ &= \partial_t(-\frac{1}{2}\eta\psi\dot{\phi}). \end{aligned} \tag{1.55}$$

To compute the Noether current j corresponding to the symmetry $\hat{\xi}$, we first record the variational 1-form computed from (1.53):

$$\gamma = \dot{\phi}\,\delta\phi + \frac{1}{2}\psi\,\delta\psi. \tag{1.56}$$

Then the current is

$$\begin{aligned} j &= \iota(\hat{\xi})\gamma - (-\frac{1}{2}\eta\psi\dot{\phi}) \\ &= (-\eta\dot{\phi}\psi - \frac{1}{2}\eta\psi\dot{\phi}) - (-\frac{1}{2}\eta\psi\dot{\phi}) \\ &= -\eta\dot{\phi}\psi. \end{aligned} \tag{1.57}$$

The *supercharge* is defined to be *minus* the Noether current of Q, which is $\dot{\phi}\psi$.

CHAPTER 2
Coordinates on Superspace

Our task in this chapter is to develop the formalism we need to compute in our main examples: the super Minkowski spaces $M^{3|2}$, $M^{4|4}$, and $M^{6|(8,0)}$. Most of the formulas we derive also apply to differential geometric computations (including spinors) in 3-, 4-, and 6-dimensional ordinary Minkowski space. The entire treatment is based on the exceptional isomorphism of Lie groups in low dimensions, as explained in §2.1. After a brief interlude (§2.2) on dimensional reduction, we flush out the definitions into concrete formulas (§§2.3–2.5). The final section §2.6 records some specializations to dimension 2.

§2.1. $M^{3|2}$, $M^{4|4}$, $M^{6|(8,0)}$ and their complexifications

We consider the cases $n = \dim(V) = 3, 4, 6$ with minimal supersymmetry. The irreducible real spinorial representation S of $\mathrm{Spin}(V)$ is respectively of dimension 2, 4, and 8, with algebra of endomorphisms D respectively reduced to $\mathbb{R}$ ($n = 3$) or isomorphic to $\mathbb{C}$ ($n = 4$) or $\mathbb{H}$ ($n = 6$). Once Γ is chosen, V can be identified with the space of D-hermitian symmetric sesquilinear forms on $\mathrm{Hom}_D(S, D)$, with the positive cone C corresponding to positive forms. Namely, as will be clear from the complexified description, there is up to a real factor a unique symmetric morphism of representations $\Gamma\colon S^* \otimes S^* \to V$, and $\Gamma(\bar{a}s, t) = \Gamma(s, at)$ for $a \in D$. It induces an isomorphism of the symmetric quotient of $S^* \otimes_D S^*$ with V. This symmetric quotient is identified with the space of hermitian symmetric sesquilinear forms on $\mathrm{Hom}_D(S^*, D)$ by letting $s \otimes s$ correspond to the form for which $\langle \alpha, \alpha \rangle = \overline{\alpha(s)}\alpha(s)$.

For V of dimension 4 (resp. 6) we choose an isomorphism of D with $\mathbb{C}$ (resp. $\mathbb{H}$). On the super Minkowski space M, the odd distribution τ then carries a $\mathbb{C}$- (resp left $\mathbb{H}$-) module structure, and this structure is respected by the super Poincaré group.

Over $\mathbb{C}$ one has the exceptional isomorphisms $\mathrm{Spin}(3) \cong SL(2)$, $\mathrm{Spin}(4) \cong SL(2) \times SL(2)$, and $\mathrm{Spin}(6) \cong SL(4)$. They make it convenient to describe the complexification $V_{\mathbb{C}}$ of V starting from that of S. The real form V corresponds to a complex conjugation (antilinear automorphism) of $V_{\mathbb{C}}$.

After complexification $\mathbb{H}$ becomes isomorphic to the 2×2 matrix algebra. This enables us to fix a 2-dimensional complex vector space W, a pseudoreal (or quaternionic) structure on W (that is, an antilinear automorphism j of square -1) and an isomorphism of $\mathbb{H}$ with the ring of real endomorphisms of W, where 'real' means commuting with j. We take these endomorphisms to act on the right. The model

is $W = \mathbb{H}$, the complex structure given by left multiplication by $I \in \mathbb{H}$, the pseudoreal structure given by left multiplication by $J \in \mathbb{H}$. The real endomorphisms are the right multiplications by elements of $\mathbb{H}$. We fix on W a symplectic form ϵ_W which is real, i.e., $\epsilon_W(jx, jy) = \overline{\epsilon_W(x, y)}$, and positive definite, i.e., $\epsilon_W(x, jx) > 0$ for $x \neq 0$. Note that $\epsilon_W(x, jx)$ is real since ϵ_W is. We also fix a basis $\{e', e'' = j(e')\}$ of W for which $\epsilon_W(e', e'') = 1$ and for which the $\mathbb{H}$-module multiplication by $z \in \mathbb{C} \subset \mathbb{H}$ is the multiplication by z on e' and by $\bar{z}$ on e''. In the model $W = \mathbb{H}$ introduced above, one may take $e' = 1$, $e'' = J$.

We now describe, case by case, the complexifications $S_{\mathbb{C}}$ and $V_{\mathbb{C}}$ of S and V, the symmetric morphism $\Gamma\colon S^*_{\mathbb{C}} \otimes S^*_{\mathbb{C}} \to V_{\mathbb{C}}$, the D-module structure of $S_{\mathbb{C}}$, and $\mathrm{Spin}(V)$. For $M^{3|2}$ and $M^{4|4}$ we give a duality pairing $\epsilon\colon S_{\mathbb{C}} \otimes S_{\mathbb{C}} \to \mathbb{C}$. In these cases the form $\tilde{\Gamma}\colon S_{\mathbb{C}} \otimes S_{\mathbb{C}} \to V_{\mathbb{C}}$ is constructed using (1.9). (We write the formulas in coordinates below.) For $M^{6|(8,0)}$ there is no duality pairing and we specify $\tilde{\Gamma}$ explicitly.

3|2 *case.* Here there is no need to complexify to have a nice description, so we directly describe the real superspace. Let S be a 2-dimensional real vector space with volume form ϵ, viewed as a symplectic form on S. Define

$$\begin{aligned} V &= \mathrm{Sym}^2(S^*) \\ \Gamma\colon (s, t) &\longmapsto st \end{aligned} \tag{2.1}$$

There is an isomorphism $SL(S) \xrightarrow{\sim} \mathrm{Spin}(V)$.

4|4 *case.* Let S' and S'' be 2-dimensional complex vector spaces with complex volume forms ϵ', ϵ'', viewed as symplectic forms on S' and S''. Define

$$\begin{aligned} S_{\mathbb{C}} &= S' \oplus S'' \\ V_{\mathbb{C}} &= S'^* \otimes S''^* \\ \Gamma\colon (s'_1 + s''_1,\ s'_2 + s''_2) &\longmapsto (s'_1 \otimes s''_2) + (s'_2 \otimes s''_1) \\ \epsilon &= \epsilon' \oplus \epsilon'' \end{aligned} \tag{2.2}$$

The $\mathbb{C}$-module structure of $S_{\mathbb{C}}$ is multiplication by z on S' and multiplication by $\bar{z}$ on S''. There is an isomorphism $SL(S') \times SL(S'') \xrightarrow{\sim} \mathrm{Spin}(V_{\mathbb{C}})$. The bigger group

$$\{\langle g, g' \rangle\colon \det g' \det g'' = 1\} \subset GL(S') \times GL(S'') \tag{2.3}$$

acts on S' and S'' preserving the bilinear form on $V_{\mathbb{C}}$. The new symmetry in this larger group is called *R-symmetry*; it is the action of unit norm scalars in $\mathbb{C}$ on $S_{\mathbb{C}}$.

6|(8, 0) *case.* Let S_0 be a 4-dimensional complex vector space with a volume form ν. Define

$$\begin{aligned} S_{\mathbb{C}} &= S_0 \otimes W \\ S^*_{\mathbb{C}} &= S_0^* \otimes W^* \\ V_{\mathbb{C}} &= \textstyle\bigwedge^2 S_0^* \\ \Gamma\colon (w' \otimes s'_0, w'' \otimes s''_0) &\longmapsto \epsilon_W(w', w'')\ (s'_0 \wedge s''_0) \end{aligned} \tag{2.4}$$

Then $SL(S_0) \xrightarrow{\sim} \mathrm{Spin}(V_{\mathbb{C}})$. The $\mathbb{H}$-module structure of $S_{\mathbb{C}}$ is induced from that of W. The volume form ν defines a symmetric pairing

$$\nu\colon \textstyle\bigwedge^2 S_0 \otimes \bigwedge^2 S_0 \longrightarrow \mathbb{C} \tag{2.5}$$

via which we identify $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^*$. This pairing is, up to a factor of 2, the inverse metric. Then $\tilde{\Gamma}$ is the composition

$$\tilde{\Gamma}\colon S_{\mathbb{C}} \otimes S_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}^* \longrightarrow V_{\mathbb{C}}, \tag{2.6}$$

where the first arrow is given by a similar formula to that above for Γ and the second arrow is constructed from the duality (2.5). To make (1.3) valid, we define the metric g on $V_{\mathbb{C}}$ to be such that the inverse metric g^{-1} on $\bigwedge^2 S_0$ is $\nu/2$.

In the $6|(8,0)$ and $4|4$ cases we now describe the real spaces V and S as fixed points of an antilinear involution of $V_{\mathbb{C}}$ and $S_{\mathbb{C}}$. The real structures must be compatible with Γ as well as with the symmetric bilinear form g on $V_{\mathbb{C}}$. The induced form on V is to have signature $(1, n-1)$. The real description above of the $3|2$ case does give the required signature $(1,2)$.

$4|4$ *case.* Fix an antilinear involution on $S_{\mathbb{C}} = S' \oplus S''$ which exchanges S' and S'' as well as ϵ' and ϵ''. From this we get real structures on $S_{\mathbb{C}}$ and $V_{\mathbb{C}} = S'^* \otimes S''^*$. The symmetric bilinear form $\epsilon' \otimes \epsilon''$ is real of signature $(1,3)$.

$6|(8,0)$ *case.* A pseudoreal structure j on S_0 induces real structures on $S_{\mathbb{C}} = S_0 \otimes W$ as well as on $V = \bigwedge^2 S_0^*$. The vector $s \wedge j(s) \in V$ is real for any $s \in S_0$, since

$$j\big(s \wedge j(s)\big) = j(s) \wedge (-s) = s \wedge j(s). \tag{2.7}$$

If the volume form ν on S_0 is real, the corresponding quadratic form on the real form of V has signature $(1,5)$ or $(5,1)$. We want (ν, j) to give signature $(1,5)$. This is the case if, for a basis of S_0 of the form $\{s, j(s), t, j(t)\}$, one has $\nu\big(s \wedge j(s) \wedge t \wedge j(t)\big)$ real and positive.

§2.2. Dimensional reduction

The super Lie algebras $V^3 \oplus S^2$ and $V^4 \oplus S^4$ underlying $M^{3|2}$ and $M^{4|4}$ are related as follows. Any (real) isomorphism $\alpha\colon \mathbb{C} \otimes_{\mathbb{R}} S^2 \xrightarrow{\sim} S^4$ extends to $\alpha\colon V^3 \hookrightarrow V^4$. The image $\alpha(V^3)$ is the orthogonal complement to a one-dimensional spacelike subspace $T \subset V^4$. This defines embeddings

$$M^{3|2} \hookrightarrow M^{4|4} \tag{2.8}$$

$$M^{3|2} \hookrightarrow M^{3|4} := M^{4|4}/T. \tag{2.9}$$

Notice that the left translates of S'' form a two-dimensional integrable odd distribution Λ on $M^{3|4}$, and the composition of (2.9) with the quotient map

$$M^{3|2} \hookrightarrow M^{3|4} \longrightarrow M^{3|4}/\Lambda \tag{2.10}$$

is an isomorphism.

Similarly, for the super Lie algebras $V^4 \oplus S^4$ and $V^6 \oplus S^8$ underlying $M^{4|4}$ and $M^{6|(8,0)}$, any isomorphism $\alpha\colon \mathbb{H} \otimes_{\mathbb{C}} S^4 \xrightarrow{\sim} S^8$ extends to $\alpha\colon V^4 \hookrightarrow V^6$. The image $\alpha(V^4) \subset V^6$ is the orthogonal complement to a two-dimensional spacelike subspace T. This defines embeddings

$$\begin{aligned} &M^{4|4} \hookrightarrow M^{6|(8,0)} \\ &M^{4|4} \hookrightarrow M^{4|(8,0)} := M^{6|(8,0)}/T. \end{aligned} \tag{2.11}$$

After complexification we obtain the following picture.

$M^{3|2} \hookrightarrow M^{4|4}$. The map α complexifies to isomorphisms $\alpha' \colon S^2_{\mathbb{C}} \to S'$ and $\alpha'' \colon S^2_{\mathbb{C}} \to S''$ which are complex conjugate. The extension to $V^3_{\mathbb{C}} = \mathrm{Sym}^2(S^2_{\mathbb{C}})^*$ is given by $st \mapsto \alpha'(s)\alpha''(t)+\alpha'(t)\alpha''(s)$, with values in $S^4_{\mathbb{C}} = S' \otimes S''$. Identify $S' \otimes S'' \cong S^2_{\mathbb{C}} \otimes S^2_{\mathbb{C}}$ via (α', α''); then the subspace $T \subset S'^* \otimes S''^*$ is identified with the orthogonal complement $\bigwedge^2(S^2_{\mathbb{C}})^*$ to $\mathrm{Sym}^2(S^2_{\mathbb{C}})^*$. The extension $\alpha \colon V^3_{\mathbb{C}} \to V^4_{\mathbb{C}}$ is compatible with the symmetric bilinear forms if, for some λ, we have

$$\alpha'(\epsilon) = \frac{\lambda}{\sqrt{2}}\,\epsilon', \qquad \alpha''(\epsilon) = \frac{\lambda^{-1}}{\sqrt{2}}\,\epsilon''. \tag{2.12}$$

$M^{4|4} \hookrightarrow M^{6|(8,0)}$. The complexified map $\alpha \colon S^4_{\mathbb{C}} = S' \oplus S'' \to S^8_{\mathbb{C}} = S_0 \otimes W$ is of the form $\alpha' e' + \alpha'' e''$ for $(\alpha', \alpha'') \colon S' \oplus S'' \to S_0$ an isomorphism. (Recall that $\{e', e''\}$ is a fixed basis of W.) The reality of our starting α means that the pseudoreal structure of $\mathrm{Hom}(S' \oplus S'', S_0)$ transforms α' into α'' and α'' into $-\alpha'$. The extension to $V^4_{\mathbb{C}} = S'^* \otimes S''^*$ is given by $s' \otimes s'' \mapsto \alpha'(s') \wedge \alpha''(s'')$, with values in $\bigwedge^2 S_0^*$. The subspace T of $\bigwedge^2 S_0^*$ is $\bigwedge^2 \alpha'(S') \oplus \bigwedge^2 \alpha''(S'')$. The extension $\alpha \colon V^4_{\mathbb{C}} \to V^6_{\mathbb{C}}$ will be compatible with the symmetric bilinear forms if

$$\alpha'(\epsilon') \wedge \alpha''(\epsilon'') = -\nu_0. \tag{2.13}$$

§2.3. Coordinates on $M^{3|2}$

A basis $\{f_a\}_{a=1,2}$ of S^* gives a basis $\{e_{ab} = f_a f_b\}$ of $V = \mathrm{Sym}^2(S^*)$. Notice that $e_{ab} = e_{ba}$. We use on V^* not the dual basis, but rather the basis $\{f^a f^b\}$ of $\mathrm{Sym}^2(S)$, in duality with $\mathrm{Sym}^2(S^*)$ by

$$\langle s^*t^*, st\rangle = \frac{1}{2}\big(\langle s^*, s\rangle\langle t^*, t\rangle + \langle s^*, t\rangle\langle t^*, s\rangle\big), \qquad s^*, t^* \in S^*, \quad s, t \in S. \tag{2.14}$$

The chosen bases of V and S^* give us vector fields ∂_a and ∂_{ab}, while the bases of V^* and S give us coordinates y^{ab} and θ^a. We have

$$\begin{aligned} \partial_a \theta^b &= \delta_a^b \\ \partial_{ab} y^{a'b'} &= \frac{1}{2}(\delta_a^{a'}\delta_b^{b'} + \delta_a^{b'}\delta_b^{a'}) \\ \partial_a y^{bc} &= \partial_{ab}\theta^c = 0. \end{aligned} \tag{2.15}$$

The left-invariant vector field D_a corresponding to f_a in S^* is

$$D_a = \partial_a - \theta^b \partial_{ab}, \tag{2.16}$$

and corresponding to the relation $[f_a, f_b] = -2f_{ab}$ in the Lie algebra we have

$$[D_a, D_b] = -2\partial_{ab}. \tag{2.17}$$

The right-invariant vector field τ_{Q_a} corresponding to f_a in S^* is

$$\tau_{Q_a} = \partial_a + \theta^b \partial_{ab} \tag{2.18}$$

with

$$[\tau_{Q_a}, \tau_{Q_b}] = 2\partial_{ab}. \tag{2.19}$$

We use $\{D_a, \partial_{ab}\}$ as a left-invariant moving frame on $M^{3|2}$. Dually, we use the left-invariant moving coframe $\{\rho^a, \omega^{ab}\}$ which has the following duality pairings:[20]

$$\begin{aligned} \langle D_a, \rho^b \rangle &= \delta_a^b \\ \langle \partial_{ab}, \omega^{a'b'} \rangle &= \frac{1}{2}(\delta_a^{a'}\delta_b^{b'} + \delta_a^{b'}\delta_b^{a'}) \\ \langle D_a, \omega^{bc} \rangle &= \langle \partial_{ab}, \rho^c \rangle = 0. \end{aligned} \tag{2.20}$$

This gives

$$\begin{aligned} \rho^a &= d\theta^a \\ \omega^{ab} &= dy^{ab} - (\theta^a d\theta^b + \theta^b d\theta^a) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} d\rho^a &= 0 \\ d\omega^{ab} &= -2\rho^a \wedge \rho^b. \end{aligned} \tag{2.22}$$

The symmetric pairing Γ is

$$\Gamma(f_a, f_b) = e_{ab} = \Gamma_{ab}^{(a'b')} e_{a'b'} = \frac{1}{2}(\delta_a^{a'}\delta_b^{b'} + \delta_a^{b'}\delta_b^{a'})e_{a'b'}. \tag{2.23}$$

Similarly,

$$\tilde{\Gamma}(f^a, f^b) = \tilde{\Gamma}^{(a'b')ab} e_{a'b'} = \frac{1}{2}(\epsilon^{a'a}\epsilon^{b'b} + \epsilon^{a'b}\epsilon^{b'a})e_{a'b'}, \tag{2.24}$$

where ϵ is the volume form on S. The metric on Minkowski space $\check{M}^3$ is the restriction to $\check{M}^3$ of

$$g = 2\epsilon_{aa'}\epsilon_{bb'}\, \omega^{ab} \otimes \omega^{a'b'}, \tag{2.25}$$

and the inverse metric is the restriction to $\check{M}^3$ of

$$g^{-1} = \frac{1}{2}\epsilon^{aa'}\epsilon^{bb'}\, \partial_{ab} \otimes \partial_{a'b'}. \tag{2.26}$$

The reader should be mindful of (2.20), which implies

$$g^{-1}(\omega^{ab}, \omega^{cd}) = \frac{1}{4}(\epsilon^{ac}\epsilon^{bd} + \epsilon^{ad}\epsilon^{bc}). \tag{2.27}$$

The Dirac form (1.22) on a spinor field $\psi = \psi_a f^a$ is

$$\psi \not{D} \psi = -\epsilon^{ab}\epsilon^{cd}\psi_a \partial_{bc} \psi_d. \tag{2.28}$$

[20]Recall the convention (1.21) that we write vector fields to the left of differential forms: $\langle D, df \rangle = \iota(D)df = Df$, where $\iota(D)$ is contraction by the vector field D.

The Dirac form (1.22) on a dual spinor field $\lambda = \lambda^a f_a$ is

$$\lambda \not{D} \lambda = \lambda^a \partial_{ab} \lambda^b. \tag{2.29}$$

We also introduce the Dirac *operator*

$$(\not{D}\psi)_b = -\epsilon^{cd} \partial_{bc} \psi_d, \tag{2.30}$$

which maps sections of S to sections of S. Then (2.28) and (2.30) are compatible in the sense that

$$\epsilon^{ab} \chi_a (\not{D}\psi)_b = \chi \not{D} \psi \tag{2.31}$$

for all spinor fields χ, ψ. The wave operator is

$$\begin{aligned} \Box &= \frac{1}{2} \epsilon^{aa'} \epsilon^{bb'} \partial_{ab} \partial_{a'b'} \\ &= \partial_{11} \partial_{22} - \partial_{12}^2; \end{aligned} \tag{2.32}$$

the sign is chosen so that the continuation to Euclidean space is the nonnegative laplace operator. The reader may check that $\not{D}^2 = -\Box$.

Preferred bases are those for which $\epsilon^{12} = -\epsilon^{21} = 1$ and hence[21] $\epsilon_{12} = -\epsilon_{21} = 1$. Put differently, $\epsilon(f_1, f_2) = 1$. In a preferred basis we introduce standard coordinates $\{x^\mu\}$ on Minkowski space $\check{M}^3$ by

$$\begin{aligned} y^{11} &= \frac{x^0 + x^1}{2}, \\ y^{22} &= \frac{x^0 - x^1}{2}, \\ y^{12} &= \frac{x^2}{2}. \end{aligned} \tag{2.33}$$

The dual vector fields ∂_{ab} and $\partial/\partial x^\mu$ are related by

$$\begin{aligned} \partial_{11} &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \\ \partial_{22} &= \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}, \\ \partial_{12} &= \frac{\partial}{\partial x^2}. \end{aligned} \tag{2.34}$$

The metric (2.25) is then the standard metric

$$g = (dx^0)^2 - (dx^1)^2 - (dx^2)^2. \tag{2.35}$$

The density $|d^3x|$ in (1.33) is the standard positive density

$$|d^3x| = |dx^0 dx^1 dx^2|. \tag{2.36}$$

[21] See (1.11).

It is convenient in our gauge theory computations to work with the Hodge $*$ operator. For this we stipulate that x^0, x^1, x^2 is a positively oriented coordinate system.[22] Then the volume form is the restriction to $\check{M}^3$ of

$$\text{vol} = 4\omega^{11} \wedge \omega^{12} \wedge \omega^{22} \tag{2.37}$$

and the $*$ operator acts by

$$\begin{aligned} *\omega^{ab} &= \epsilon_{a'b'}\, \omega^{aa'} \wedge \omega^{bb'} \\ *(\omega^{aa'} \wedge \omega^{bb'}) &= \frac{1}{4}\,(\epsilon^{ab}\omega^{a'b'} + \epsilon^{ab'}\omega^{a'b} + \epsilon^{a'b}\omega^{ab'} + \epsilon^{a'b'}\omega^{ab}). \end{aligned} \tag{2.38}$$

For integration we introduce the notation

$$\int d^2\theta = -\frac{1}{2}\int d\theta^a d\theta^b \epsilon_{ab} = \int d\theta^2 d\theta^1. \tag{2.39}$$

Explicit in (2.39) is our choice of volume form on spinors. Using the identification between integration and differentiation, we write

$$\int d^2\theta = i^*\partial_2\partial_1 = -\frac{1}{2} i^*\epsilon^{ab} D_a D_b = -i^* D^2 = \frac{1}{2} i^*(D_2 D_1 - D_1 D_2), \tag{2.40}$$

where we have introduced the notation

$$D^2 = \frac{1}{2}\epsilon^{ab} D_a D_b. \tag{2.41}$$

The following relations are easy to verify:

$$D_a D_b = -(\partial_{ab} - \epsilon_{ab} D^2) \tag{2.42}$$

$$D_a D^2 = -D^2 D_a = -\epsilon^{bc}\partial_{ab} D_c \tag{2.43}$$

$$D^2 D^2 = -\Box \tag{2.44}$$

§2.4. Coordinates on $M^{4|4}$

We fix a basis $\{f_a\}_{a=1,2}$ of S'^* and let $\{\bar{f}_{\dot{a}}\}$ be the complex conjugate basis of S''^*. As a basis for $V_{\mathbb{C}} = S'^* \otimes S''^*$ we use $\{e_{a\dot{b}} = f_a \bar{f}_{\dot{b}}\}$. Those bases and their duals give us a complex coordinate system $\{y^{a\dot{b}}, \theta^a, \bar{\theta}^{\dot{a}}\}$, with θ^a and $\bar{\theta}^{\dot{a}}$ complex conjugate and $y^{a\dot{b}}$ the complex conjugate to $y^{b\dot{a}}$. We write ∂_a, $\bar{\partial}_{\dot{a}}$, and $\partial_{a\dot{b}}$ for the corresponding vector fields. Note our convention about indices: If both 'a' and '$\dot{a}$' appear in the same equation, then $a = \dot{a}$. This allows us to write reality conditions and equations such as (2.78) and (2.79).

Let D_a and $\overline{D}_{\dot{a}}$ be the left-invariant complex vector fields corresponding to f_a and $\bar{f}_{\dot{a}}$. One has

$$\begin{aligned} D_a &= \partial_a - \bar{\theta}^{\dot{b}}\partial_{a\dot{b}} \\ \overline{D}_{\dot{a}} &= \bar{\partial}_{\dot{a}} - \theta^b \partial_{b\dot{a}}. \end{aligned} \tag{2.45}$$

[22]The orientation makes it easier to write intermediate steps in some of our computations, but the final formulas do not depend on it.

Corresponding to the relations $[f_a, f_b] = [\bar{f}_{\dot{a}}, \bar{f}_{\dot{b}}] = 0$ and $[f_a, \bar{f}_{\dot{b}}] = -2e_{a\dot{b}}$ in the Lie algebra, one has

$$\begin{aligned} [D_a, D_b] &= [\overline{D}_{\dot{a}}, \overline{D}_{\dot{b}}] = 0 \\ [D_a, \overline{D}_{\dot{b}}] &= -2\partial_{a\dot{b}}. \end{aligned} \tag{2.46}$$

The similar right-invariant complex vector fields are

$$\begin{aligned} \tau_{Q_a} &= \partial_a + \bar{\theta}^{\dot{b}}\partial_{a\dot{b}} \\ \tau_{\overline{Q}_{\dot{a}}} &= \bar{\partial}_{\dot{a}} + \theta^b \partial_{b\dot{a}} \end{aligned} \tag{2.47}$$

with

$$\begin{aligned} [\tau_{Q_a}, \tau_{Q_b}] &= [\tau_{\overline{Q}_{\dot{a}}}, \tau_{\overline{Q}_b}] = 0 \\ [\tau_{Q_a}, \tau_{\overline{Q}_{\dot{b}}}] &= 2\partial_{a\dot{b}}. \end{aligned} \tag{2.48}$$

We use $\{D_a, \overline{D}_{\dot{a}}, \partial_{a\dot{b}}\}$ as a left-invariant moving frame and also use the dual left-invariant moving coframe $\{\rho^a, \overline{\rho}^{\dot{a}}, \omega^{a\dot{b}}\}$. The nontrivial duality pairings are

$$\begin{aligned} \langle D_a, \rho^b \rangle &= \delta_a^b \\ \langle \overline{D}_{\dot{a}}, \overline{\rho}^{\dot{b}} \rangle &= \delta_{\dot{a}}^{\dot{b}} \\ \langle \partial_{a\dot{b}}, \rho^{a'\dot{b}'} \rangle &= \delta_a^{a'} \delta_{\dot{b}}^{\dot{b}'}. \end{aligned} \tag{2.49}$$

One has

$$\begin{aligned} \rho^a &= d\theta^a \\ \overline{\rho}^{\dot{a}} &= d\bar{\theta}^{\dot{a}} \\ \omega^{a\dot{b}} &= dy^{a\dot{b}} - (\bar{\theta}^{\dot{b}} d\theta^a + \theta^a d\bar{\theta}^{\dot{b}}) \end{aligned} \tag{2.50}$$

and, dual to the bracket relations,

$$\begin{aligned} d\rho^a &= d\overline{\rho}^{\dot{a}} = 0 \\ d\omega^{a\dot{b}} &= -2\rho^a \wedge \overline{\rho}^{\dot{b}}. \end{aligned} \tag{2.51}$$

The symmetric pairing Γ is

$$\Gamma(f_a, \bar{f}_{\dot{b}}) = e_{a\dot{b}} = \Gamma_{a\dot{b}}^{(a'\dot{b}')} e_{a'\dot{b}'} = (\delta_a^{a'} \delta_{\dot{b}}^{\dot{b}'}) e_{a'\dot{b}'}. \tag{2.52}$$

Similarly, the symmetric pairing $\tilde{\Gamma}$ is

$$\tilde{\Gamma}(f^a, \bar{f}^{\dot{b}}) = \tilde{\Gamma}^{(a'\dot{b}')a\dot{b}} e_{a'\dot{b}'} = (\epsilon^{a'a} \epsilon^{\dot{b}'\dot{b}}) e_{a'\dot{b}'}, \tag{2.53}$$

where ϵ is the volume form on S', S''. The metric on Minkowski space $\check{M}^4$ is the restriction to $\check{M}^4$ of

$$g = 2\epsilon_{aa'} \epsilon_{\dot{b}\dot{b}'}\, \omega^{a\dot{b}} \otimes \omega^{a'\dot{b}'}, \tag{2.54}$$

and the inverse metric is the restriction to $\check{M}^4$ of

$$g^{-1} = \frac{1}{2}\epsilon^{aa'}\epsilon^{\dot{b}\dot{b}'}\,\partial_{a\dot{b}} \otimes \partial_{a'\dot{b}'}. \tag{2.55}$$

The Dirac form (1.22) pairs a positive chirality spinor field $\psi = \psi_a f^a$ with a negative chirality spinor field $\bar{\psi} = \bar{\psi}_{\dot{a}}\bar{e}^{\dot{a}}$:

$$\bar{\psi}\not{D}\psi = \epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\bar{\psi}_{\dot{d}}\partial_{a\dot{c}}\psi_b. \tag{2.56}$$

The Dirac form (1.22) on dual spinor fields is

$$\bar{\lambda}\not{D}\lambda = \bar{\lambda}^{\dot{b}}\partial_{a\dot{b}}\lambda^a. \tag{2.57}$$

The Dirac operator on spinor fields exchanges sections of S' and S'':

$$\begin{aligned}(\not{D}\psi)_{\dot{c}} &= -\epsilon^{ab}\partial_{a\dot{c}}\psi_b \\ (\not{D}\bar{\psi})_c &= -\epsilon^{\dot{a}\dot{b}}\partial_{c\dot{a}}\bar{\psi}_{\dot{b}}\end{aligned} \tag{2.58}$$

The wave operator is

$$\begin{aligned}\Box &= \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\partial_{a\dot{c}}\partial_{b\dot{d}} \\ &= \partial_{1\dot{1}}\partial_{2\dot{2}} - \partial_{1\dot{2}}\partial_{2\dot{1}},\end{aligned} \tag{2.59}$$

and $\not{D}^2 = -\Box$.

Preferred bases are those for which $\epsilon'(f_1, f_2) = \epsilon''(\bar{f}_{\dot{1}}, \bar{f}_{\dot{2}}) = 1$. In a preferred basis we introduce standard coordinates $\{x^\mu\}$ on Minkowski space V by

$$\begin{aligned}y^{1\dot{1}} &= \frac{x^0 + x^1}{2}, \\ y^{2\dot{2}} &= \frac{x^0 - x^1}{2}, \\ y^{1\dot{2}} &= \frac{x^2 + ix^3}{2}, \\ y^{2\dot{1}} &= \frac{x^2 - ix^3}{2}.\end{aligned} \tag{2.60}$$

The dual vector fields $\partial_{a\dot{b}}$ and $\partial/\partial x^\mu$ are related by

$$\begin{aligned}\partial_{1\dot{1}} &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \\ \partial_{2\dot{2}} &= \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}, \\ \partial_{1\dot{2}} &= \frac{\partial}{\partial x^2} - i\frac{\partial}{\partial x^3}, \\ \partial_{2\dot{1}} &= \frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}.\end{aligned} \tag{2.61}$$

The metric (2.54) is then the standard metric

$$g = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{2.62}$$

The density $|d^4x|$ in (1.33) is the standard positive density

$$|d^4x| = |dx^0 dx^1 dx^2 dx^3|. \tag{2.63}$$

To define self-duality we need to orient $\check{M}^4$, and so we stipulate that x^0, x^1, x^2, x^3 is an oriented coordinate system.[23] Then for a symmetric tensor F_{ab} the 2-form

$$\frac{1}{2} F_{ab} \epsilon_{\dot{c}\dot{d}}\, \omega^{a\dot{c}} \wedge \omega^{b\dot{d}} \tag{2.64}$$

is self-dual, and for a symmetric tensor $\overline{F}_{\dot{c}\dot{d}}$ the 2-form

$$\frac{1}{2} \epsilon_{ab} \overline{F}_{\dot{c}\dot{d}}\, \omega^{a\dot{c}} \wedge \omega^{b\dot{d}} \tag{2.65}$$

is anti-self-dual. Our convention is that a self-dual 2-form σ satisfies $*\sigma = -\sqrt{-1}\,\sigma$. Then the analytic continuation to Euclidean space gives the usual notion of self-duality. The complex conjugate of a self-dual form (in Minkowski space) is anti-self-dual.

For the 2-form

$$F = \frac{1}{2}(F_{ab}\epsilon_{\dot{c}\dot{d}} + \epsilon_{ab}\overline{F}_{\dot{c}\dot{d}})\, \omega^{a\dot{c}} \wedge \omega^{b\dot{d}} \tag{2.66}$$

we have the formulas

$$\begin{aligned} |F|^2 &= \frac{1}{4}(F_{ab}F_{cd}\epsilon^{ac}\epsilon^{bd} + \overline{F}_{\dot{a}\dot{b}}\overline{F}_{\dot{c}\dot{d}}\epsilon^{\dot{a}\dot{c}}\epsilon^{\dot{b}\dot{d}}), \\ F \wedge F &= \frac{i}{4}(F_{ab}F_{cd}\epsilon^{ac}\epsilon^{bd} - \overline{F}_{\dot{a}\dot{b}}\overline{F}_{\dot{c}\dot{d}}\epsilon^{\dot{a}\dot{c}}\epsilon^{\dot{b}\dot{d}})\, d^4x, \end{aligned} \tag{2.67}$$

where d^4x is the standard volume form

$$d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{2.68}$$

In general the 2-form F is complex; then '$|F|^2$' in (2.67) is not a norm, but rather is a quadratic form. The 2-form F is real if $\overline{F}_{\dot{a}\dot{b}} = \overline{F_{ab}}$.

We mention the fact that the analytic continuation of the $*$ operator on 2-forms in Minkowski space of *any* dimension is $-\sqrt{-1}$ times the $*$ operator on 2-forms in Euclidean space.

A complex function $\Phi\colon M^{4|4} \to \mathbb{C}$ is *chiral* if $\overline{D}_{\dot{a}}\Phi = 0$ and *antichiral* if $D_a\Phi = 0$.

Following (2.39) we introduce the notations

$$\begin{aligned} \int d^2\bar{\theta} &= -\frac{1}{2}\int d\bar{\theta}^{\dot{a}} d\bar{\theta}^{\dot{b}} \epsilon_{\dot{a}\dot{b}} = \int d\bar{\theta}^{\dot{2}} d\bar{\theta}^{\dot{1}} \\ \int d^4\theta &= \int d^2\theta d^2\bar{\theta}. \end{aligned} \tag{2.69}$$

[23] The orientation only serves to choose which of (2.64) and (2.65) below is self-dual and which is anti-self-dual. It does not enter in a fundamental way in our formulae.

We discuss the meaning of $\int d^2\theta$ and $\int d^2\bar{\theta}$ in §5.3; in any case we compute them using (2.40) and the analogous formula with the conjugate vector fields. Note that we will use the notation

$$\overline{D}^2 = \frac{1}{2}\epsilon^{\dot{a}\dot{b}}\overline{D}_{\dot{a}}\overline{D}_{\dot{b}}, \tag{2.70}$$

analogous to (2.41). For integration over all odd coordinates we find the formulas

$$\begin{aligned}\int d^4\theta &= i^*\partial_2\partial_1\bar{\partial}_{\dot{2}}\bar{\partial}_{\dot{1}} \\ &= \frac{1}{2}\, i^*\left\{D_1\overline{D}_{\dot{1}}\overline{D}_{\dot{2}}D_2 + D_2\overline{D}_{\dot{2}}\overline{D}_{\dot{1}}D_1\right\} - \Box i^* \qquad (2.71)\\ &= \frac{1}{2}\, i^*\left\{D^2\overline{D}^2 + \overline{D}^2D^2\right\} + \Box i^*. \qquad (2.72)\end{aligned}$$

In each of the last two lines the term in braces disagrees with the integral by a multiple of the wave operator. In a lagrangian this wave operator gives an exact term which is usually omitted anyway. The particular combination of derivatives in the first term of (2.71) has the virtue that it exactly annihilates *chiral* and *antichiral* superfields. This is an advantage over the usual integral and leads to nicer formulas. Hence we compute component lagrangians (1.34) using this improved expression, which is the usual integral plus the wave operator composed with restriction to V. Formula (2.72) is useful in some computations. Both of these integration formulas follow from direct computation.

The following relations are easy to verify.

$$D_aD^2 = D^2D_a = \overline{D}_{\dot{a}}\overline{D}^2 = \overline{D}^2\overline{D}_{\dot{a}} = 0 \tag{2.73}$$

$$D_a\overline{D}^2 = \overline{D}^2D_a - 2\epsilon^{\dot{b}\dot{c}}\partial_{a\dot{b}}\overline{D}_{\dot{c}} \tag{2.74}$$

$$\overline{D}_{\dot{a}}D^2 = D^2\overline{D}_{\dot{a}} - 2\epsilon^{bc}\partial_{b\dot{a}}D_c \tag{2.75}$$

$$\overline{D}^2D^2 = -4\Box \qquad \text{on chiral superfields} \tag{2.76}$$

$$D^2\overline{D}^2 = -4\Box \qquad \text{on antichiral superfields} \tag{2.77}$$

Next, we describe the dimensional reduction map $M^{3|2} \hookrightarrow M^{4|4}$. Decorate the bases (of spinors and vectors) and coordinates in $M^{3|2}$ with a '$\hat{}$'. We take the inclusion to be compatible with the ϵ and Γ tensors:

$$\begin{aligned}\hat{f}_a &\longmapsto \frac{f_a + \bar{f}_{\dot{a}}}{\sqrt{2}} \\ \hat{e}_{ab} &\longmapsto \frac{e_{a\dot{b}} + e_{b\dot{a}}}{2}.\end{aligned} \tag{2.78}$$

The image of the inclusion on spinors is the subspace where $\theta^a = \bar{\theta}^{\dot{a}}$ and both are real, described by the equations

$$\theta^a = \bar{\theta}^{\dot{a}} = \frac{\hat{\theta}^a}{\sqrt{2}}. \tag{2.79}$$

The image of the inclusion on vectors is the subspace where $y^{1\dot{2}} = y^{2\dot{1}}$, or equivalently where $x^3 = 0$, described by the equations

$$\begin{aligned} y^{1\dot{1}} &= \hat{y}^{11} \\ y^{2\dot{2}} &= \hat{y}^{22} \\ y^{1\dot{2}} = y^{2\dot{1}} &= \hat{y}^{12}. \end{aligned} \tag{2.80}$$

As an exercise the reader may check that (2.74)–(2.75) reduce to (2.42)–(2.44) under dimensional reduction.

The R-symmetry is generated by an even vector field R with brackets

$$\begin{aligned} [R, D_a] &= D_a \\ [R, \overline{D}_{\dot{a}}] &= -\overline{D}_{\dot{a}} \\ [R, \partial_{a\dot{b}}] &= 0. \end{aligned} \tag{2.81}$$

Note that R is purely imaginary: $\overline{R} = -R$.

§2.5. Coordinates on $M^{6|(8,0)}$

A basis $\{f_a\}_{a=1,2,3,4}$ of S_0^* gives us a basis $e'^* \otimes f_a, e''^* \otimes f_a$ of $S_{\mathbb{C}}^*$ and a basis $\{e_{ab} = f_a \wedge f_b\}_{a<b}$ of $V_{\mathbb{C}}$. For the basis elements of $S_{\mathbb{C}}^*$ we use the notation 'f_{ai}', where $f_{a1} = e'^* \otimes f_a$ and $f_{a2} = e''^* \otimes f_a$. Let y^{ab}, θ^{ai} be the corresponding coordinate system. For $b < a$ define $e_{ba} = -e_{ab}$ and $y^{ba} = -y^{ab}$. Write ∂_{ai} and ∂_{ab} for the corresponding complex vector fields on $M^{6|(8,0)}$.

Let ϵ_{ij} denote the symplectic form on the vector space W^*. We use our standard choice $\epsilon_{12} = 1$.

The left-invariant complex vector field D_{ai} corresponding to e_{ai} is

$$D_{ai} = \partial_{ai} - \epsilon_{ij}\theta^{bj}\partial_{ab} \tag{2.82}$$

with brackets

$$[D_{ai}, D_{bj}] = -2\epsilon_{ij}\partial_{ab}. \tag{2.83}$$

The similar right-invariant vector fields are

$$\tau_{Q_{ai}} = \partial_{ai} + \epsilon_{ij}\theta^{bj}\partial_{ab} \tag{2.84}$$

with

$$[\tau_{Q_{ai}}, \tau_{Q_{bj}}] = 2\epsilon_{ij}\partial_{ab}. \tag{2.85}$$

We will use $\{D_{ai}, \partial_{ab}\}$ as a left-invariant moving frame with dual coframe $\{\rho^{ai}, \omega^{ab}\}$. The nonzero duality pairings are

$$\begin{aligned} \langle D_{ai}, \rho^{bj} \rangle &= \delta_a^b \delta_i^j \\ \langle \partial_{ab}, \omega^{cd} \rangle &= \delta_a^c \delta_b^d - \delta_a^d \delta_b^c. \end{aligned} \tag{2.86}$$

One has

$$\begin{aligned} \rho^{ai} &= d\theta^{ai} \\ \omega^{ab} &= dy^{ab} + \epsilon_{ij}\theta^{ai}d\theta^{bj} - \epsilon_{ij}\theta^{bi}d\theta^{aj} \end{aligned} \tag{2.87}$$

with differentials

$$\begin{aligned} d\rho^{ai} &= 0 \\ d\omega^{ab} &= 2\epsilon_{ij}\,\rho^{ai}\wedge\rho^{bj}. \end{aligned} \tag{2.88}$$

The symmetric pairing Γ is

$$\Gamma(f_{ai}, f_{bj}) = \epsilon_{ij}e_{ab} = \frac{1}{2}\Gamma^{(pq)}_{(ai)(bj)}e_{pq} = \frac{1}{2}\big[\epsilon_{ij}(\delta^p_a\delta^q_b - \delta^q_a\delta^p_b)\big]e_{pq}. \tag{2.89}$$

The symmetric pairing $\tilde{\Gamma}$ is

$$\tilde{\Gamma}(f^{ai}, f^{bj}) = \frac{1}{2}\tilde{\Gamma}^{(pq)(ai)(bj)}e_{pq} = \frac{1}{2}(\epsilon^{ij}\nu^{pqab})e_{pq}, \tag{2.90}$$

where ν is the volume form. The metric on Minkowski space $\check{M}^6$ is the restriction to $\check{M}^6$ of

$$g = \frac{1}{2}\nu_{pqrs}dy^{pq}\otimes dy^{rs} = \sum_{\substack{p<q\\ r<s}} 2\nu_{pqrs}\omega^{pq}\otimes\omega^{rs}, \tag{2.91}$$

and the inverse metric is the restriction to $\check{M}^6$ of

$$g^{-1} = \frac{1}{8}\nu^{pqrs}\,\partial_{pq}\otimes\partial_{rs} = \sum_{\substack{p<q\\ r<s}} \frac{1}{2}\nu^{pqrs}\,\partial_{pq}\otimes\partial_{rs}. \tag{2.92}$$

The volume form and its inverse are related by $\nu^{1234}\nu_{1234} = 1$. The Dirac form (1.22) on spinor fields $\psi = \psi_{ai}f^{ai}$ is

$$\psi\not{D}\psi = \frac{1}{2}\nu^{abcd}\epsilon^{ij}\psi_{ai}\partial_{bc}\psi_{dj} \tag{2.93}$$

On dual spinor fields $\lambda = \lambda^{ai}f_{ai}$ we have the Dirac form (1.23)

$$\lambda\not{D}\lambda = \epsilon_{ij}\lambda^{ai}\partial_{ab}\lambda^{bj}. \tag{2.94}$$

Preferred bases are those for which $\nu^{1234} = -1$. We also use the dual form with $\nu_{1234} = -1$. A real structure is given by the pseudoreal structure on S_0 for which $f_3 = j(f_1)$ and $f_4 = j(f_2)$. Then

$$\begin{aligned} \overline{D_{11}} &= D_{32} \\ \overline{D_{21}} &= D_{42} \\ \overline{D_{31}} &= -D_{12} \\ \overline{D_{41}} &= -D_{22}. \end{aligned} \tag{2.95}$$

In a preferred basis we introduce standard coordinates $\{x^\mu\}$ on Minkowski space $\check{M}^6$ by

$$\begin{aligned} y^{12} &= \frac{x^4 + ix^5}{2}, \\ y^{13} &= \frac{x^0 + x^1}{2}, \\ y^{14} &= \frac{x^2 + ix^3}{2}, \\ y^{23} &= \frac{x^2 - ix^3}{2}, \\ y^{24} &= \frac{x^0 - x^1}{2}, \\ y^{34} &= \frac{x^4 - ix^5}{2}. \end{aligned} \tag{2.96}$$

The dual vector fields ∂_{ab} and $\partial/\partial x^\mu$ are related by

$$\begin{aligned} \partial_{12} &= \frac{\partial}{\partial x^4} - i\frac{\partial}{\partial x^5}, \\ \partial_{13} &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \\ \partial_{14} &= \frac{\partial}{\partial x^2} - i\frac{\partial}{\partial x^3}, \\ \partial_{23} &= \frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}, \\ \partial_{24} &= \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}, \\ \partial_{34} &= \frac{\partial}{\partial x^4} + i\frac{\partial}{\partial x^5}. \end{aligned} \tag{2.97}$$

The metric (2.91) is then the standard metric

$$g = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2 - (dx^5)^2. \tag{2.98}$$

Next, we describe the dimensional reduction map $M^{4|4} \hookrightarrow M^{6|(8,0)}$. Denote the bases (of spinors and vectors) and coordinates in $M^{6|(8,0)}$ with a '~'. We take the inclusion maps to be

$$\begin{aligned} f_1 &\longmapsto \tilde{f}_{1(1)} & e_{1\dot{1}} &\longmapsto \tilde{e}_{13} \\ f_2 &\longmapsto \tilde{f}_{2(1)} & e_{2\dot{2}} &\longmapsto \tilde{e}_{24} \\ \bar{f}_{\dot{1}} &\longmapsto \tilde{f}_{3(2)} & e_{1\dot{2}} &\longmapsto \tilde{e}_{14} \\ \bar{f}_{\dot{2}} &\longmapsto \tilde{f}_{4(2)} & e_{2\dot{1}} &\longmapsto \tilde{e}_{23} \end{aligned} \tag{2.99}$$

The image of the inclusion on spinors is described by the equations

$$\begin{aligned} \theta^{1(1)} &= \theta^1 & \theta^{1(2)} &= 0 \\ \theta^{2(1)} &= \theta^2 & \theta^{2(2)} &= 0 \\ \theta^{3(1)} &= 0 & \theta^{3(2)} &= \bar{\theta}^{\dot{1}} \\ \theta^{4(1)} &= 0 & \theta^{4(2)} &= \bar{\theta}^{\dot{2}}. \end{aligned} \tag{2.100}$$

The image of the inclusion on vectors is described by the equations

$$\begin{aligned} \tilde{y}^{12} &= 0 & \tilde{y}^{23} &= y^{2\dot{1}} \\ \tilde{y}^{13} &= y^{1\dot{1}} & \tilde{y}^{24} &= y^{2\dot{2}} \\ \tilde{y}^{14} &= y^{1\dot{2}} & \tilde{y}^{34} &= 0 \end{aligned} \tag{2.101}$$

There is an R-symmetry group $SU(2)_R$ which acts on $M^{6|(8,0)}$. The action of a matrix $S = (S_i^j)$ is

$$D_{ai} \longrightarrow D_{ai'} S_i^{i'}. \tag{2.102}$$

Demanding that this preserve (2.83) requires $S \in SL(2;\mathbb{C})$; asking that the real structure also be preserved reduces us to $SU(2)$. More conceptually, this $SU(2)_R = Sp(1)_R$ symmetry group is the group of real endomorphisms of the complex symplectic vector space W.

There is also a dimensional reduction map $M^{4|8} \hookrightarrow M^{6|(8,0)}$, described simply by equations (2.101). Usually $M^{4|8}$ is described with vector fields $\{D_a^{(i)}, \overline{D}_{\dot{a}(j)}\}_{\substack{a=1,2 \\ i,j=1,2}}$ with bracketing (2.46) supplemented by a δ-function:

$$[D_a^{(i)}, \overline{D}_{\dot{a}(j)}] = -2\delta_j^i \partial_{a\dot{a}}. \tag{2.103}$$

One possible correspondence between these vector fields and the $\{\tilde{D}_{ai}\}_{\substack{a=1,2,3,4 \\ i=1,2}}$ is:

$$\begin{aligned} \tilde{D}_{11} &= D_1^{(1)} & \tilde{D}_{12} &= D_1^{(2)} \\ \tilde{D}_{21} &= D_2^{(1)} & \tilde{D}_{22} &= D_2^{(2)} \\ \tilde{D}_{31} &= -\overline{D}_{\dot{1}(2)} & \tilde{D}_{32} &= \overline{D}_{\dot{1}(1)} \\ \tilde{D}_{41} &= -\overline{D}_{\dot{2}(2)} & \tilde{D}_{42} &= \overline{D}_{\dot{2}(1)} \end{aligned} \tag{2.104}$$

A new feature of $M^{4|8}$ is an enlarged R-symmetry group $U(2)_R$. It acts trivially on the even variables. The representation of $U(2)_R$ on each of the pairs $\{D_a^{(1)}, D_a^{(2)}\}$ of odd vector fields is the standard one, whereas on each of the pairs $\{\overline{D}_{\dot{a}(1)}, \overline{D}_{\dot{a}(2)}\}$ of odd vector fields $U(2)_R$ acts by the conjugate to the standard action. Under the dimensional reduction $M^{4|4} \hookrightarrow M^{4|(8,0)}$, specified by (2.100), only the subgroup $U(1)_R \subset U(2)_R$ given by (2.81) survives. Notice that this subgroup is the induced action by the double cover in the spin group of rotations in the x^4-x^5 plane.

§2.6. Low dimensions

The basic superspace $M^{1|1}$ in dimension one is built starting from a one-dimensional odd space S and letting $V = S^{\otimes(-2)}$. In terms of the obvious left-invariant vector fields D and ∂ we have

$$[D, D] = -2\partial. \tag{2.105}$$

Analogs of other formulas we have developed above are easy to come by. Dimensional reduction from the higher dimensional spaces sets all spacelike coordinates

of Minkowski space to zero leaving just the time coordinate x^0. In this way we also obtain spaces $M^{1|2}$, $M^{1|4}$, and $M^{1|8}$ with more odd variables. As these spaces are reduced from $M^{3|2}$, $M^{4|4}$, and $M^{6|(8,0)}$, we obtain R-symmetry groups Spin(2), Spin(3), and Spin(5) from rotations in the additional spatial directions.

The basic[24] superspace $M^{2|(1,1)}$ in dimension two is simply a product

$$M^{2|(1,1)} = M^{1|1} \times M^{1|1}. \tag{2.106}$$

(Note that the metric (2.109) is not a product, however.) This corresponds to the fact that there are real semi-spinors in 2 dimensions (termed *Majorana-Weyl spinors* in the physics literature). For S^+, S^- real one-dimensional vector spaces, we set $S = S^+ \oplus S^-$ and $V = (S^+)^{\otimes(-2)} \oplus (S^-)^{\otimes(-2)}$ with the obvious bilinear pairing. We can see $M^{2|(1,1)}$ as coming from $M^{3|2}$ by dimensional reduction: in $M^{3|2}$ set $y^{12} = 0$. It is customary to use '+' for the odd index '1' and '−' for the odd index '2'. (These indices appear beginning with formula (2.15).) Also, we set $y^+ = y^{11}$ and $y^- = y^{22}$ with similar notation $\partial_+ = \partial_{11}$ and $\partial_- = \partial_{22}$.

$N = 2$ superspace in 2 dimensions is $M^{2|(2,2)}$, and we view it as embedded in $M^{4|4}$ (dimensional reduction) by setting $y^{1\dot{2}} = y^{2\dot{1}} = 0$, or equivalently $x^2 = x^3 = 0$. The real description of $M^{2|(1,1)}$ makes clear that there is also a real description of $M^{2|(2,2)}$, but we use the complex vector fields and forms induced from $M^{4|4}$ instead. Again we use '+' for the odd index '1' and '−' for the odd index '2' (which appear beginning with formula (2.45).) The '+' and'−' on the bosonic coordinates are the usual lightcone coordinates in two dimensions. Thus the left-invariant odd vector fields are $D_+, \overline{D}_+, D_-, \overline{D}_-$, and we denote the left-invariant even vector fields by ∂_+, ∂_-. The bracketing relations are

$$\begin{gathered} [D_+, D_+] = [D_+, D_-] = [D_+, \overline{D}_-] = [\overline{D}_+, \overline{D}_+] = 0 \\ [D_-, D_-] = [\overline{D}_+, \overline{D}_-] = [D_-, \overline{D}_+] = [\overline{D}_-, \overline{D}_-] = 0 \\ [D_+, \overline{D}_+] = -2\partial_+ \\ [D_-, \overline{D}_-] = -2\partial_-. \end{gathered} \tag{2.107}$$

For convenience we record the nonzero components of $\Gamma, \tilde{\Gamma}$:

$$\begin{aligned} \Gamma^+_{++} &= \Gamma^-_{--} = 1 \\ \tilde{\Gamma}^{+--} &= \tilde{\Gamma}^{-++} = 1. \end{aligned} \tag{2.108}$$

The formulas in $M^{4|4}$ all apply, though many simplify. For example, the metric (2.54), the inverse metric (2.55), the Dirac form (2.56), the Dirac operator (2.58), and the wave operator (2.59) reduce to

$$g = 2(dy^+ \otimes dy^- + dy^- \otimes dy^+) \tag{2.109}$$

$$g^{-1} = \frac{1}{2}(\partial_+ \otimes \partial_- + \partial_- \otimes \partial_+) \tag{2.110}$$

$$\bar{\psi} \not{D} \psi = \bar{\psi}_+ \partial_- \psi_+ + \bar{\psi}_- \partial_+ \psi_- \tag{2.111}$$

$$(\not{D}\psi)_+ = -\partial_+ \psi_- \qquad (\not{D}\psi)_- = \partial_- \psi_+ \tag{2.112}$$

$$(\not{D}\bar{\psi})_+ = -\partial_+ \bar{\psi}_- \qquad (\not{D}\bar{\psi})_- = \partial_- \bar{\psi}_+ \tag{2.113}$$

$$\Box = \partial_+ \partial_-. \tag{2.114}$$

[24] $M^{2|(1,1)}$ is usually termed $N = 1$ superspace, though it is not the minimal possibility—there is a superspace $M^{2|(s^+,s^-)}$ for any nonnegative integers s^+, s^-.

The relationship to standard coordinates x^0, x^1 is

$$\begin{aligned} y^+ &= \frac{x^0 + x^1}{2}, \\ y^- &= \frac{x^0 - x^1}{2}, \end{aligned} \tag{2.115}$$

and

$$\begin{aligned} \partial_+ &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \\ \partial_- &= \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}. \end{aligned} \tag{2.116}$$

There is a 2-dimensional space of R-symmetries with basis J_+, J_- described by the brackets

$$\begin{aligned} &[J_+, D_+] = \ D_+ && [J_-, D_-] = \ D_- \\ &[J_+, \overline{D}_+] = -\overline{D}_+ && [J_-, \overline{D}_-] = -\overline{D}_- \\ &[J_+, D_-] = [J_+, \overline{D}_-] = 0 && [J_-, D_+] = [J_-, \overline{D}_+] = 0. \end{aligned} \tag{2.117}$$

The R-symmetry (2.81) in $M^{4|4}$ induces $J_+ + J_-$ on $M^{2|(2,2)}$. The double cover in the spin group of rotation in the x^2-x^3 plane in $M^{4|4}$ induces the R-symmetry $J_+ - J_-$ of $M^{2|(2,2)}$.

As in $M^{4|4}$, a complex function $\Phi\colon M^{2|(2,2)} \to \mathbb{C}$ is *chiral* if

$$\overline{D}_+\Phi = \overline{D}_-\Phi = 0; \tag{2.118}$$

then the complex conjugate $\overline{\Phi}$ is *antichiral*. There is a new possibility as well. Namely, a complex function $\Sigma\colon M^{2|(2,2)} \to \mathbb{C}$ is *twisted chiral* if

$$\overline{D}_+\Sigma = D_-\Sigma = 0; \tag{2.119}$$

then the complex conjugate $\overline{\Sigma}$ is *twisted antichiral*.

Formulas (2.71) and (2.72) for computing $\int d^4\theta$ are, of course, valid on $M^{2|(2,2)}$. Again we use $\int d^4\theta + \Box i^*$ to compute component lagrangians when chiral and antichiral fields are involved. When twisted chiral and twisted antichiral fields are involved, it is more convenient to use $\int d^4\theta - \Box i^*$, which equals the first term in (2.72). This has the advantage that the expression $D^2\overline{D}^2 + \overline{D}^2D^2$ annihilates both twisted chiral and twisted antichiral fields. In a more complicated (term in a) lagrangian involving both chiral and twisted chiral fields one would have to simply choose a procedure and then deal with the exact terms which would most likely appear.

CHAPTER 3
Supersymmetric σ-Models

A scalar field is a map $\phi\colon \check{M}^n \to X$ from ordinary n-dimensional Minkowski space to a manifold X. If X is a linear space, then a theory with the field ϕ is called a linear σ-model; if X is curved, then it is a nonlinear σ-model. In supersymmetric σ-models there is in addition a spinor field. After some linear algebra preliminaries in §3.1, we begin in §3.2 with the linear case, where we show that a supersymmetric extension of the ordinary free σ-model exists in dimensions $n = 3, 4, 6$. We label these three cases according to the ring $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ which underlies the spin group $SL(2, \mathbb{F})$. In the nonlinear case (X a general Riemannian manifold) supersymmetry imposes constraints on the target manifold X: For $\mathbb{F} = \mathbb{C}$ it must be Kähler and for $\mathbb{F} = \mathbb{H}$ it must be hyperkähler. The sufficiency of these constraints is explained at the beginning of §3.3, after which we derive the exact form of the component lagrangian. In the bosonic case any function $V\colon X \to \mathbb{R}$ may serve as a potential energy function, but supersymmetry constrains the form of V. The constraints are described in §3.4, though the proofs are deferred to future chapters. We make some brief remarks about the superspace formulation in §3.5 and about the relationship among the different theories in §3.6.

§3.1. Preliminary remarks on linear algebra

We discuss simultaneously the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Since $\mathbb{H}$ is noncommutative, we must be careful about the order of multiplication. Let $^-$ denote the standard conjugation on $\mathbb{F}$, which is trivial for $\mathbb{F} = \mathbb{R}$. For $x, y \in \mathbb{F}$ we have

$$\overline{xy} = \bar{y}\bar{x}. \tag{3.1}$$

The extension of (3.1) to superalgebra has a sign, in accordance with the sign rule. It is useful to note that:

$$\mathrm{Re}(\lambda\mu) = \mathrm{Re}(\mu\lambda) = \mathrm{Re}(\bar{\lambda}\bar{\mu}) = \mathrm{Re}(\bar{\mu}\bar{\lambda}), \qquad \lambda, \mu \in \mathbb{F}. \tag{3.2}$$

Let V be the vector space of translations of $\check{M}^n$ ($n = 3, 4, 6$). If S is an irreducible real spinorial representation of $\mathrm{Spin}(V)$, the field $\mathbb{F}$ of endomorphisms of the representation S is respectively isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$; the vector space S is of dimension 2 over $\mathbb{F}$; and $\mathrm{Spin}(V) \cong SL_{\mathbb{F}}(S)$. We choose S and a *right* vector space structure over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ commuting with the action of $\mathrm{Spin}(V)$. The dual $S^* = \mathrm{Hom}_{\mathbb{F}}(S, \mathbb{F})$ is then a *left* $\mathbb{F}$-vector space, and $\mathrm{Spin}(V) \cong SL_{\mathbb{F}}(S^*)$ as

well. We write $\overline{S}$ for S with the left vector space structure $\lambda s := s\bar{\lambda}$ and $\overline{S^*}$ for S^* with the right vector space structure $s^*\lambda := \bar{\lambda}s^*$. The $\mathbb{F}$-dual S^* can be identified with the $\mathbb{R}$-dual $\mathrm{Hom}_{\mathbb{R}}(S,\mathbb{R})$ by $\alpha \mapsto$ real part of the $\mathbb{F}$-linear form α. If $\{f^a\}$ is a basis of S over $\mathbb{F}$, then there are induced bases $\{\bar{f}^{\dot{a}}\}$, $\{f_a\}$, $\{\bar{f}_{\dot{a}}\}$ of $\overline{S}$, S^*, $\overline{S^*}$.

As explained at the beginning of §2.1 there are invariant symmetric $\mathbb{R}$-bilinear forms $\Gamma\colon S^* \otimes_{\mathbb{R}} S^* \to V$ and $\tilde{\Gamma}\colon S \otimes_{\mathbb{R}} S \to V$. They factor through $\overline{S^*} \otimes_{\mathbb{F}} S^*$ and $S \otimes_{\mathbb{F}} \overline{S}$, respectively. In general, if A, B, C are respectively $\mathbb{F}$ right, $\mathbb{F}$ left, $\mathbb{R}$ vector spaces, then an $\mathbb{R}$-linear map $T\colon A \otimes_{\mathbb{F}} B \to C$ gives rise to an $(\mathbb{F},\mathbb{F})$-bilinear form $T_1\colon A \otimes_{\mathbb{R}} B \to C \otimes_{\mathbb{R}} \mathbb{F}$, characterized by

$$T(a,b) = \mathrm{Re}\big(T_1(a,b)\big). \tag{3.3}$$

Conversely, the real part of any $(\mathbb{F},\mathbb{F})$-bilinear T_1 obeys

$$\mathrm{Re}\, T_1(a\lambda, b) = \mathrm{Re}\, T_1(a,b)\lambda = \mathrm{Re}\, \lambda T_1(a,b) = \mathrm{Re}\, T_1(a,\lambda b), \tag{3.4}$$

so factors through $A \otimes_{\mathbb{F}} B$. We apply this construction to Γ and $\tilde{\Gamma}$, and denote by $\Gamma^{\mu}_{a\dot{b}}$ and $\tilde{\Gamma}^{\mu\dot{a}b}$ the coordinates of Γ_1 and $\tilde{\Gamma}_1$ in bases $\{e_\mu\}$ of V and $\{f^a\}$ of S, up to a factor:

$$\begin{aligned} \Gamma(x^a f_a, y^b f_b) &= \frac{1}{\kappa^2}\,\mathrm{Re}(y^b \Gamma^{\mu}_{b\dot{a}} \overline{x^a})\, e_\mu, \\ \tilde{\Gamma}(f^a x_a, f^b y_b) &= \quad \mathrm{Re}(\overline{y_b}\tilde{\Gamma}^{\mu\dot{b}a} x_a)\, e_\mu. \end{aligned} \tag{3.5}$$

Here κ is the factor

$$\kappa = \begin{cases} 1, & \mathbb{F} = \mathbb{R}; \\ \sqrt{2}, & \mathbb{F} = \mathbb{C}, \mathbb{H}. \end{cases} \tag{3.6}$$

On the left hand side of these equations, $\Gamma, \tilde{\Gamma}$ denote the real forms acting on elements of the underlying real vector space. Thus, for example, we might write $\Gamma\big(\mathrm{Re}(x^a f_a), \mathrm{Re}(y^b f_b)\big)$ to distinguish it from (2.52). The factor of κ^2 is introduced to make these formulas compatible with (2.52) and similar formulas of Chapter 2. The symmetry of Γ and $\tilde{\Gamma}$ becomes the hermiticity property

$$\begin{aligned} \overline{\Gamma^{\mu}_{a\dot{b}}} &= \Gamma^{\mu}_{b\dot{a}}, \\ \overline{\tilde{\Gamma}^{\mu\dot{a}b}} &= \tilde{\Gamma}^{\mu\dot{b}a}. \end{aligned} \tag{3.7}$$

The Clifford relation (1.3) with this normalization is

$$\Gamma^{\mu}_{a\dot{b}}\tilde{\Gamma}^{\nu\dot{b}c} + \Gamma^{\nu}_{a\dot{b}}\tilde{\Gamma}^{\mu\dot{b}c} = 2g^{\mu\nu}\delta^c_a \tag{3.8}$$

and its conjugate. We emphasize that $\Gamma^{\mu}_{a\dot{b}}, \tilde{\Gamma}^{\mu\dot{a}b}$ lie in $\mathbb{F}$, whereas $g_{\mu\nu}, g^{\mu\nu}$ are real.

For the quaternion case $\mathbb{F} = \mathbb{H}$ we can choose bases so that the pairing Γ is given by

$$\begin{aligned} &\Gamma^0_{1\dot{1}} = 1, \quad \Gamma^1_{1\dot{1}} = 1, \\ &\Gamma^0_{2\dot{2}} = 1, \quad \Gamma^1_{2\dot{2}} = -1, \\ &\Gamma^2_{1\dot{2}} = 1, \quad \Gamma^3_{1\dot{2}} = -i, \quad \Gamma^4_{1\dot{2}} = j, \quad \Gamma^5_{1\dot{2}} = k, \end{aligned} \tag{3.9}$$

where i, j, k are the usual quaternions. Then the pairing $\tilde{\Gamma}$ is given by

$$\begin{aligned} &\Gamma^{0i1} = 1, \quad \Gamma^{1i1} = -1, \\ &\Gamma^{0\dot{2}2} = 1, \quad \Gamma^{1\dot{2}2} = 1, \\ &\Gamma^{2\dot{2}1} = -1, \quad \Gamma^{3\dot{2}1} = -i, \quad \Gamma^{4\dot{2}1} = j, \quad \Gamma^{5\dot{2}1} = k. \end{aligned} \tag{3.10}$$

In this basis the metric has the standard form (2.98). It is useful to note that the vector $\Gamma(f_a, f_a)$ is lightlike (has vanishing norm square) for all a; this property will be important in Chapter 6. The pairings for the complex case $\mathbb{F} = \mathbb{C}$ are obtained by omitting the entries with j, k; those for the real case $\mathbb{F} = \mathbb{R}$ by taking the real part of all entries. Then the chosen bases agree with those in §2.3, §2.4. Recall from (1.9) that for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ we have a skew tensor ϵ, and the pairings $\tilde{\Gamma}$ and Γ are related by

$$\tilde{\Gamma}^{\mu\dot{b}a} = \Gamma^{\mu}_{c\dot{d}} \epsilon^{ac} \epsilon^{\dot{b}\dot{d}}, \qquad \mathbb{F} = \mathbb{R}, \mathbb{C}. \tag{3.11}$$

In §3.3 we have use for the following. Suppose V_R is a right $\mathbb{H}$ vector space and V_L a left $\mathbb{H}$ vector space. Suppose $\tilde{\Gamma}$ is a symmetric real-valued bilinear form on V_R which factors through $V_R \otimes_{\mathbb{H}} \overline{V_R}$ and h a symmetric real-valued bilinear form on V_L which factors through $\overline{V_L} \otimes_{\mathbb{H}} V_L$. Then we claim that there is a well-defined symmetric real bilinear form B on the real vector space $V = V_R \otimes_{\mathbb{H}} V_L$ constructed from $\tilde{\Gamma}, h$. In fact, if $\tilde{\Gamma}_1, h_1$ are the $\mathbb{H}$-valued pairings constructed as in (3.3), then

$$B(f \otimes v, f' \otimes v') = \mathrm{Re}\Big\{\tilde{\Gamma}_1(f, \bar{f}')\, h_1(\bar{v}, v)\Big\}, \qquad f, f' \in V_R, \quad v, v' \in V_L. \tag{3.12}$$

We easily check that this factors through the quaternionic tensor product using (3.2). Relative to bases $\{f^a\}$ of V_R and v_i of V_L, any vector in V can be written

$$f^a \psi^{\imath}_a \otimes v_i = f^a \otimes \psi^i_a v_{\imath}, \qquad \psi^i_a \in \mathbb{H}. \tag{3.13}$$

Then the formula for B is

$$B(f^b \psi^j_b v_j, f^a \lambda^{\imath}_a v^a) = \mathrm{Re}(\overline{\psi^j_b} \tilde{\Gamma}^{ab} \lambda^{\imath}_a h_{ij}). \tag{3.14}$$

§3.2. The free supersymmetric σ-model

The bosonic σ-model lagrangian in n-dimensional Minkowski space $\check{M}^n$ for a scalar field $\phi\colon \check{M}^n \to X$ with values in a Riemannian manifold X is

$$L = \frac{1}{2} |d\phi|^2 \, |d^n x|. \tag{3.15}$$

We look for supersymmetric extensions with various amounts of supersymmetry. With a single supersymmetry the supersymmetric extension for $n = 1$ is given in (1.53). (That lagrangian is in $\check{M}^1$, but in fact it is the dimensional reduction of a lagrangian in $\check{M}^2$; see Problem FP6 of [I-**Homework**].) To construct models with more supersymmetry we first consider the linear case $X = \mathbb{R}^k$ for some k; in the next subsection we treat nonlinear σ-models. The linear model is free, and

to predict which supersymmetric extensions exist we invoke an argument from the quantum theory. Namely, upon quantization we obtain a representation of the supersymmetry algebra, and the representation theory shows that models are possible with $s = 2, 4, 8$ supersymmetries for $k = 1, 2, 4$. That is, the smallest model has a scalar field in $\mathbb{R}^k$ for $k = 1, 2, 4$ and a single spinor field. The maximal dimension in which such models can occur is $n = 3, 4, 6$, and the models in these dimensions have minimal supersymmetry.

We refer to the theories in $n = 3, 4, 6$ according to the field $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

To treat the various cases simultaneously, we introduce constants κ in (3.6) and

$$A = \begin{cases} \frac{1}{2}, & \mathbb{F} = \mathbb{R}; \\ 1, & \mathbb{F} = \mathbb{C}, \mathbb{H}. \end{cases} \tag{3.16}$$

(The constant A enters the lagrangian (3.19) and the constant κ enters the supersymmetry transformation laws (3.23), as well as (3.5) above.)

The fields in the theory are

$$\begin{aligned} \phi &\colon \check{M}^n \longrightarrow \mathbb{F} \\ \psi &\colon \check{M}^n \longrightarrow \Pi S, \end{aligned} \tag{3.17}$$

where 'Π' denotes parity reversal. (The spinor field ψ is odd.)

Theorem 3.18. *The σ-model lagrangian*

$$\boxed{L = A\Big\{\langle d\phi, d\bar{\phi}\rangle + \mathrm{Re}(\psi \not{D} \bar{\psi})\Big\}\, |d^n x|} \tag{3.19}$$

is supersymmetric in the three cases $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Proof. The lagrangian (3.19) in more detail is[25]

$$\boxed{L = A\Big\{g^{\mu\nu}\partial_\mu\phi \cdot \partial_\nu\bar{\phi} + \frac{1}{2}\psi_a \cdot \tilde{\Gamma}^{\mu\dot{b}a} \cdot \partial_\mu\bar{\psi}_{\dot{b}} - \frac{1}{2}\partial_\mu\psi_a \cdot \tilde{\Gamma}^{\mu\dot{b}a} \cdot \bar{\psi}_{\dot{b}}\Big\}\, |d^n x|,} \tag{3.21}$$

where '$\cdot$' denotes multiplication in $\mathbb{F}$. We usually omit '$\cdot$'. In case $\mathbb{F} = \mathbb{R}, \mathbb{C}$ we can combine the two fermion terms at the cost of an exact term. For odd parameters $\eta^a \in \mathbb{F}$ we let $\hat{\xi}$ denote the *real* vector field on the space of fields which corresponds to the abstract supersymmetry transformation

$$\frac{\kappa^2}{2}(\eta^a Q_a - \overline{Q}_{\dot{a}}\bar{\eta}^{\dot{a}}), \tag{3.22}$$

where $\bar{\eta}^{\dot{a}} = \overline{\eta^a}$. (Recall the abstract supersymmetry algebra (1.16).) We postulate

$$\boxed{\begin{aligned} \hat{\xi}\phi &= \kappa\psi_a\,\eta^a \\ \hat{\xi}\psi_a &= \kappa\partial_\mu\phi\,\bar{\eta}^{\dot{b}}\,\Gamma^\mu_{a\dot{b}}, \end{aligned}} \tag{3.23}$$

[25]The fermion kinetic term is A times

$$\tilde{\Gamma}(f^a\psi_a, f^b\partial_\mu\psi_b) = \tilde{\Gamma}(f^b\partial_\mu\psi_b, f^a\psi_a) = \mathrm{Re}(\bar{\psi}_{\dot{a}}\tilde{\Gamma}^{\mu\dot{a}b}\partial_\mu\psi_b) = \mathrm{Re}(\psi_a\tilde{\Gamma}^{\mu\dot{b}a}\partial_\mu\bar{\psi}_{\dot{b}}). \tag{3.20}$$

and of course we have the conjugate equations

$$\begin{aligned} \hat{\xi}\bar{\phi} &= -\kappa\bar{\eta}^{\dot{a}}\,\bar{\psi}_{\dot{a}} \\ \hat{\xi}\bar{\psi}_{\dot{a}} &= \quad \kappa\Gamma^{\mu}_{b\dot{a}}\,\eta^{b}\,\partial_{\mu}\bar{\phi}\,. \end{aligned} \tag{3.24}$$

There is a minus sign in passing from $\hat{\xi}\phi$ to $\hat{\xi}\bar{\phi}$ since we commute the odd quantities η^a and ψ_a. These formulas are motivated by Lorentz invariance, parity, and power counting.

We must check that $\mathrm{Lie}(\hat{\xi})L$ is exact and that the transformations (3.23) generate the supersymmetry algebra on-shell.[26] The variation of the bosonic term in L is

$$\hat{\xi}(Ag^{\mu\nu}\,\partial_{\mu}\phi\cdot\partial_{\nu}\bar{\phi}) = \kappa A g^{\mu\nu}\{\partial_{\mu}\psi_{a}\,\eta^{a}\,\partial_{\nu}\bar{\phi} - \partial_{\mu}\phi\,\bar{\eta}^{\dot{a}}\,\partial_{\nu}\bar{\psi}_{\dot{a}}\}. \tag{3.25}$$

The second term in (3.21) has variation

(3.26)

$$\begin{aligned} \hat{\xi}(\frac{A}{2}\,\psi_{a}\cdot\tilde{\Gamma}^{\mu\dot{b}a}\cdot\partial_{\mu}\bar{\psi}_{\dot{b}}) &= \frac{\kappa A}{2}\{\partial_{\nu}\phi\,\bar{\eta}^{\dot{c}}\,\Gamma^{\nu}_{a\dot{c}}\tilde{\Gamma}^{\mu\dot{b}a}\,\partial_{\mu}\bar{\psi}_{\dot{b}} + \psi_{a}\tilde{\Gamma}^{\mu\dot{b}a}\Gamma^{\nu}_{c\dot{b}}\eta^{c}\partial_{\mu}\partial_{\nu}\bar{\phi}\} \\ &= \frac{\kappa A}{2}\{\partial_{\nu}\phi\,\bar{\eta}^{\dot{c}}\,\Gamma^{\nu}_{a\dot{c}}\tilde{\Gamma}^{\mu\dot{b}a}\,\partial_{\mu}\bar{\psi}_{\dot{b}} - \partial_{\nu}\psi_{a}\,\tilde{\Gamma}^{\mu\dot{b}a}\Gamma^{\nu}_{c\dot{b}}\,\eta^{c}\,\partial_{\mu}\bar{\phi}\} \\ &\qquad + \partial_{\nu}\{\frac{\kappa A}{2}\,\psi_{a}\,\tilde{\Gamma}^{\mu\dot{b}a}\Gamma^{\nu}_{c\dot{b}}\,\eta^{c}\,\partial_{\mu}\bar{\phi}\}. \end{aligned}$$

The variation of the third term in (3.21) is the conjugate

(3.27)

$$\begin{aligned} \hat{\xi}(-\frac{A}{2}\,\partial_{\mu}\psi_{a}\cdot\tilde{\Gamma}^{\mu\dot{b}a}\cdot\bar{\psi}_{\dot{b}}) = \frac{\kappa A}{2}\{&-\partial_{\mu}\psi_{a}\,\tilde{\Gamma}^{\mu\dot{b}a}\Gamma^{\nu}_{c\dot{b}}\,\eta^{c}\,\partial_{\nu}\bar{\phi} + \partial_{\mu}\phi\,\bar{\eta}^{\dot{c}}\,\Gamma^{\nu}_{a\dot{c}}\tilde{\Gamma}^{\mu\dot{b}a}\,\partial_{\nu}\bar{\psi}_{\dot{b}}\} \\ &- \partial_{\nu}\{\frac{\kappa A}{2}\partial_{\mu}\phi\,\bar{\eta}^{\dot{c}}\,\Gamma^{\nu}_{a\dot{c}}\tilde{\Gamma}^{\mu\dot{b}a}\bar{\psi}_{\dot{b}}\}. \end{aligned}$$

Adding (3.26) and (3.27) and using the Clifford relation (3.8) we see

$$\hat{\xi}(\text{fermion term}) = \frac{\kappa A}{2}g^{\mu\nu}\{\partial_{\mu}\phi\,\bar{\eta}^{\dot{a}}\,\partial_{\nu}\bar{\psi}_{\dot{a}} - \partial_{\mu}\psi_{a}\,\eta^{a}\,\partial_{\nu}\bar{\phi}\} + EXACT, \tag{3.28}$$

where

$$EXACT = \partial_{\nu}\{\frac{\kappa A}{2}\psi_{a}\,\tilde{\Gamma}^{\mu\dot{b}a}\Gamma^{\nu}_{c\dot{b}}\,\eta^{c}\,\partial_{\mu}\bar{\phi} \;-\; \frac{\kappa A}{2}\partial_{\mu}\phi\,\bar{\eta}^{\dot{c}}\,\Gamma^{\nu}_{a\dot{c}}\tilde{\Gamma}^{\mu\dot{b}a}\,\bar{\psi}_{\dot{b}}\}. \tag{3.29}$$

The nonexact term in (3.28) cancels against (3.25), which proves that $\hat{\xi}$ is a nonmanifest symmetry of L.

We now verify that the supersymmetry algebra holds on-shell. The vector field (3.22) is $\kappa^2\,\mathrm{Re}(\eta^a Q_a)$, and in the abstract supersymmetry algebra (1.16) we have, according to (3.5),

$$\begin{aligned} \big[\mathrm{Re}(\eta_1^a Q_a), \mathrm{Re}(\eta_2^b Q_b)\big] &= -\big[\mathrm{Re}(\eta_2^a Q_a), \mathrm{Re}(\eta_1^b Q_b)\big] \\ &= 2\,\Gamma\big(\mathrm{Re}(\eta_2^a Q_a), \mathrm{Re}(\eta_1^b Q_b)\big) \\ &= \frac{2}{\kappa^2}\,\mathrm{Re}(\eta_1^b\Gamma^{\mu}_{b\dot{a}}\bar{\eta}_2^{\dot{a}})P_{\mu} \\ &= \frac{2}{\kappa^2}\,\mathrm{Re}(\bar{\eta}_1^{\dot{b}}\Gamma^{\mu}_{a\dot{b}}\eta_2^{a})P_{\mu}. \end{aligned} \tag{3.30}$$

[26] By contrast, in one dimension (1.47) holds everywhere, that is, off-shell. We can achieve off-shell supersymmetry for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ by introducing an auxiliary field; see §4.1 and §5.1.

So on a field f we must verify (see (1.30))

$$[\hat{\xi}_1, \hat{\xi}_2]f = 2\kappa^2 \operatorname{Re}(\bar{\eta}_1^{\dot{b}} \Gamma^{\mu}_{a\dot{b}} \eta_2^a)\, \partial_\mu f. \tag{3.31}$$

The check for ϕ is straightforward:

$$\hat{\xi}_1 \hat{\xi}_2 \phi = \hat{\xi}_1(\kappa \psi_a \eta_2^a) = \kappa^2 \partial_\mu \phi\, \bar{\eta}_1^{\dot{b}} \Gamma^{\mu}_{a\dot{b}} \eta_2^a, \tag{3.32}$$

and so

$$\begin{aligned} [\hat{\xi}_1, \hat{\xi}_2]\phi &= \kappa^2 \partial_\mu \phi (\bar{\eta}_1^{\dot{b}} \Gamma^{\mu}_{a\dot{b}} \eta_2^a - \bar{\eta}_2^{\dot{b}} \Gamma^{\mu}_{a\dot{b}} \eta_1^a) \\ &= 2\kappa^2 \operatorname{Re}(\bar{\eta}_1^{\dot{b}} \Gamma^{\mu}_{a\dot{b}} \eta_2^a)\, \partial_\mu \phi, \end{aligned} \tag{3.33}$$

as desired. For ψ we must use the equation of motion (3.39), which appears below. Using (3.10) we write this equation as

$$\begin{aligned} \partial_0 \psi_1 - \partial_1 \psi_1 &= \partial_2 \psi_2 - \partial_3 \psi_2 i + \partial_4 \psi_2 j + \partial_5 \psi_2 k, \\ \partial_0 \psi_2 + \partial_1 \psi_2 &= \partial_2 \psi_1 + \partial_3 \psi_1 i - \partial_4 \psi_1 j - \partial_5 \psi_1 k. \end{aligned} \tag{3.34}$$

The action of the supersymmetry transformation on ψ is

$$\hat{\xi}_1 \hat{\xi}_2 \psi_a = \hat{\xi}_1(\kappa \partial_\mu \phi\, \bar{\eta}_2^{\dot{b}} \Gamma^{\mu}_{a\dot{b}}) = \kappa^2 \partial_\mu \psi_c\, \eta_1^c \bar{\eta}_2^{\dot{b}} \Gamma^{\mu}_{a\dot{b}}. \tag{3.35}$$

So

$$[\hat{\xi}_1, \hat{\xi}_2]\psi_a = \kappa^2 \partial_\mu \psi_c (\eta_1^c \bar{\eta}_2^{\dot{b}} - \eta_2^c \bar{\eta}_1^{\dot{b}}) \Gamma^{\mu}_{a\dot{b}}. \tag{3.36}$$

Checking explicitly for $a = 1, 2$ using (3.9) and (3.34) we find the desired result

$$[\hat{\xi}_1, \hat{\xi}_2]\psi_a = 2\kappa^2 \operatorname{Re}(\bar{\eta}_1^{\dot{b}} \Gamma^{\mu}_{c\dot{b}} \eta_2^c)\, \partial_\mu \psi_a. \tag{3.37}$$

This completes the proof.

We compute the equations of motion and the variational 1-form. Write the lagrangian (3.21) as $L = \check{L}\, |d^n x|$. Then

$$\begin{aligned} \frac{1}{A}\delta \check{L} = {} & \delta\phi(-g^{\mu\nu}\partial_\mu \partial_\nu \bar{\phi}) + (-g^{\mu\nu}\partial_\mu \partial_\nu \phi)\, \delta\bar{\phi} + \delta\psi_a(-\tilde{\Gamma}^{\mu\dot{b}a} \partial_\mu \bar{\psi}_{\dot{b}}) + (\partial_\mu \psi_a\, \tilde{\Gamma}^{\mu\dot{b}a})\, \delta\bar{\psi}_{\dot{b}} \\ & + \partial_\nu \{ g^{\mu\nu} \delta\phi\, \partial_\mu \bar{\phi} + g^{\mu\nu} \partial_\mu \phi\, \delta\bar{\phi} + \frac{1}{2} \delta\psi_a\, \tilde{\Gamma}^{\nu\dot{b}a}\, \bar{\psi}_{\dot{b}} - \frac{1}{2} \psi_a\, \tilde{\Gamma}^{\nu\dot{b}a}\, \delta\bar{\psi}_{\dot{b}} \}. \end{aligned} \tag{3.38}$$

This leads to the equations of motion

$$\boxed{\begin{aligned} g^{\mu\nu}\partial_\mu \partial_\nu \phi &= 0 \\ \partial_\mu \psi_a \tilde{\Gamma}^{\mu\dot{b}a} &= 0 \end{aligned}} \tag{3.39}$$

and the variational 1-form

(3.40)
$$\boxed{\gamma = A(g^{\mu\nu}\delta\phi\,\partial_\mu\bar\phi + g^{\mu\nu}\partial_\mu\phi\,\delta\bar\phi + \frac{1}{2}\delta\psi_a\,\tilde\Gamma^{\nu\dot b a}\,\bar\psi_{\dot b} - \frac{1}{2}\psi_a\,\tilde\Gamma^{\nu\dot b a}\,\delta\bar\psi_{\dot b})\,\frac{\partial}{\partial x^\nu}\otimes|d^n x|.}$$

In coordinate-free notation the equations of motion are

(3.41)
$$\boxed{\begin{aligned}\Delta\phi &= 0\\ \not{D}\psi &= 0.\end{aligned}}$$

The Noether current associated to $\hat\xi$ is computed from (3.40) and (3.29) to be

(3.42)
$$\begin{aligned}\iota(\hat\xi)\gamma - (\frac{\kappa A}{2}\psi_a\,\tilde\Gamma^{\mu\dot b a}\Gamma^\nu_{c\dot b}\,\eta^c\,\partial_\mu\bar\phi - \frac{\kappa A}{2}\partial_\mu\phi\,\bar\eta^{\dot c}\,\Gamma^\nu_{a\dot c}\tilde\Gamma^{\mu\dot b a}\,\bar\psi_{\dot b}\,)\,\frac{\partial}{\partial x^\nu}\otimes|d^n x|\\ = \kappa A(\psi_a\,\tilde\Gamma^{\nu\dot b a}\Gamma^\mu_{c\dot b}\,\eta^c\,\partial_\mu\bar\phi - \partial_\mu\phi\,\bar\eta^{\dot c}\,\Gamma^\mu_{a\dot c}\tilde\Gamma^{\nu\dot b a}\bar\psi_{\dot b})\,\frac{\partial}{\partial x^\nu}\otimes|d^n x|.\end{aligned}$$

(We use the Clifford identity (3.8) in this computation.) Thus for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ the *supercurrent*, which is *minus* the Noether current for Q_a, is

(3.43)
$$\boxed{j_a = \frac{2A}{\kappa}\psi_b\,\tilde\Gamma^{\nu\dot c b}\Gamma^\mu_{a\dot c}\,\partial_\mu\bar\phi\,\frac{\partial}{\partial x^\nu}\otimes|d^n x|, \qquad \mathbb{F} = \mathbb{R}, \mathbb{C}.}$$

(Recall the factor $\kappa^2/2$ in (3.22).) As a check, in the case $\mathbb{F} = \mathbb{R}$ this reduces in one dimension with one supersymmetry to (1.57). Other formulas from that section also check against those here.

§3.3. Nonlinear supersymmetric σ-model

Now consider the supersymmetric σ-model of the previous subsection where we replace the target X by a curved Riemannian manifold. Then ψ transforms as an element of the pullback tangent bundle, and it is easy to see that $\dim X$ is divisible by $\dim \mathbb{F}$ if the linearization is to reduce to the linear σ-model. But more is true: To write formulas (3.23) we need to be able to multiply tangent vectors in X by elements of $\mathbb{F}$. Thus we assume that TX is endowed with a *left* $\mathbb{F}$-structure. Of course, this is no extra condition for $\mathbb{F} = \mathbb{R}$. For $\mathbb{F} = \mathbb{C}, \mathbb{H}$ the $\mathbb{F}$ multiplication is a tensor field on X (which is called an almost complex structure for $\mathbb{F} = \mathbb{C}$). Let T denote the covariant derivative of this tensor. Then X is *Kähler* ($\mathbb{F} = \mathbb{C}$) or *hyperkähler* ($\mathbb{F} = \mathbb{H}$) if and only if T vanishes. The fields in the theory are now

(3.44)
$$\begin{aligned}\phi &\in C^\infty(\check M^n, X)\\ \psi &\in C^\infty(\check M^n, \Pi S \otimes_{\mathbb{F}} \phi^* TX).\end{aligned}$$

The lagrangian (3.19) continues to make sense if we use the Riemannian metric h on TX as well as the Levi-Civita covariant derivative ∇. But for ∇ to be well-defined on TX as an $\mathbb{F}$-bundle, we need T to vanish. Therefore, we now assume that X is Kähler for $\mathbb{F} = \mathbb{C}$ and that X is hyperkähler for $\mathbb{F} = \mathbb{H}$. For $\mathbb{F} = \mathbb{H}$ we explained above that if $\{v_i\}$ is a basis of TX at some point, then the Riemannian

metric is represented by a quaternionic matrix (h_{ij}). Using (3.14) we write the lagrangian (3.21) as

$$L = \mathrm{Re}\Big\{ g^{\mu\nu}\partial_\mu\phi^j \cdot h_{ij} \cdot \partial_\nu\bar{\phi}^i + \bar{\psi}^j_{\dot{b}} \cdot \tilde{\Gamma}^{\mu\dot{b}a} \cdot \nabla_\mu\psi^i_a \cdot h_{ij} \Big\}\, |d^6x|. \tag{3.45}$$

In (3.57) below we abbreviate the second term as '$\langle\bar{\psi}_{\dot{b}}\tilde{\Gamma}^{\mu\dot{b}a}\nabla_\mu\psi_a\rangle$'. In all cases the lagrangian (3.19) ceases to be supersymmetric if X is not flat. Rather, we have the following result. Denote the metric h on X as '$\langle\cdot,\cdot\rangle$' and let R be the Riemann curvature tensor.

Theorem 3.46. (*i*) ($\mathbb{F} = \mathbb{R}$) *The nonlinear σ-model lagrangian*

$$\boxed{L = \Big\{ \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi\not{D}_\phi\psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, \phi^*R(\psi_b,\psi_c)\psi_d\rangle \Big\}\, |d^3x|} \tag{3.47}$$

is supersymmetric for any Riemannian manifold X.

(*ii*) ($\mathbb{F} = \mathbb{C}$) *The nonlinear σ-model lagrangian*

$$\boxed{L = \Big\{ \langle\overline{d\phi}, d\phi\rangle + \mathrm{Re}\langle\bar{\psi}\not{D}_\phi\psi\rangle - \frac{1}{4}\epsilon^{ac}\epsilon^{\dot{b}\dot{d}}\langle\psi_a, \phi^*R(\psi_c,\bar{\psi}_{\dot{d}})\bar{\psi}_{\dot{b}}\rangle \Big\}\, |d^4x|} \tag{3.48}$$

is supersymmetric if and only if X is Kähler.

(*iii*) ($\mathbb{F} = \mathbb{H}$) *The nonlinear σ-model lagrangian*

$$\boxed{\begin{aligned} L = \Big\{ &\frac{1}{16}\tilde{h}_{\alpha\beta}\epsilon_{ij}\nu^{abcd}(\partial_{ab}\phi)^{\alpha i}(\partial_{cd}\phi)^{\beta j} + \frac{1}{4}\tilde{h}_{\alpha\beta}\nu^{abcd}\psi^\alpha_a(\phi^*\nabla_{cd})\psi^\beta_b \\ &- \frac{1}{24}\nu^{abcd}\phi^*\Omega_{\alpha\beta\gamma\delta}\psi^\alpha_a\psi^\beta_b\psi^\gamma_c\psi^\delta_d \Big\}\, |d^6x|. \end{aligned}} \tag{3.49}$$

is supersymmetric if and only if X is hyperkähler.

In all cases the Dirac form involves the covariant derivative ∇, and so it also depends on ϕ. For example, in the case $\mathbb{F} = \mathbb{R}$ we have

$$\frac{1}{2}\langle\psi\not{D}_\phi\psi\rangle = \frac{1}{2}\tilde{\Gamma}^{\mu ab}\langle\psi_a, (\phi^*\nabla_\mu)\psi_b\rangle. \tag{3.50}$$

Note the use of angle brackets on the left hand side to denote the inner product on TX. The supersymmetry transformation laws are the same as for the linear case.[27] We often omit 'ϕ' and 'ϕ^*' from the notation.

We explain (3.49), where we have switched from the quaternionic notation used previously to a complex notation. The complexification of the tangent bundle of a hyperkähler manifold X of real dimension $4n$ can be written

$$T_{\mathbb{C}}X \cong V \otimes W^*, \tag{3.51}$$

where V is a complex vector bundle of complex rank $2n$ and W^* is the dual of the fixed complex symplectic vector space described at the beginning of §2.1. Furthermore, V carries a pseudoreal (quaternionic) structure J and a skew-symmetric

[27] Physicists often use a spinor field ψ which is not intrinsic, so have different formulas.

bilinear form $\tilde{h}$ which encodes the metric. The complexified Riemann curvature tensor, viewed as a functional on $(T_{\mathbb{C}}X)^{\otimes 4} \cong V^{\otimes 4}$, is a totally symmetric tensor Ω. Let S be the real spin representation of $\mathrm{Spin}(1,5)$. As described in §2.1, its complexification is $S_0 \otimes W$ for a 4-dimensional complex vector space S_0. The spinor field ψ is a real odd section of $\phi^*V \otimes_{\mathbb{C}} S_0$. Note there is a trace map

$$\phi^*(T_{\mathbb{C}}X) \otimes S_{\mathbb{C}} \cong (\phi^*V \otimes W^*) \otimes (S_0 \otimes W) \longrightarrow \phi^*V \otimes S_0 \tag{3.52}$$

which expresses the quaternionic tensor product in (3.44). (All tensor products in (3.52) are over $\mathbb{C}$.) The pseudoreal structures on S_0 and ϕ^*V give a real structure on $\phi^*V \otimes S_0$. Recall that S_0 carries a volume form ν. Now fix a local framing of V (indices $\alpha, \beta, \cdots = 1, \dots, 2n$), a basis of W (indices $i, j, \cdots = 1, 2$), and a basis of S_0 (indices $a, b, \cdots = 1, 2, 3, 4$). The inner product of two tangent vectors ξ_1, ξ_2 is

$$\langle \xi_1, \xi_2 \rangle = \tilde{h}_{\alpha\beta}\epsilon_{ij}\xi_1^{\alpha i}\xi_2^{\beta j}. \tag{3.53}$$

We will find it useful to lift a spinor field ψ to an odd section of $\phi^*(T_{\mathbb{C}}X) \otimes S_{\mathbb{C}}$ via (3.52):

$$\psi_{aj}^{\alpha i} = \psi_a^{\alpha}\delta_j^i. \tag{3.54}$$

Our normalization of the tensor Ω in terms of the Riemann curvature R is

$$R_{(\alpha i)(\beta j)(\gamma k)(\delta \ell)} = \epsilon_{ij}\epsilon_{k\ell}\Omega_{\alpha\beta\gamma\delta}. \tag{3.55}$$

In this notation the supersymmetry variation of the fields (3.23) is

$$\boxed{\begin{aligned} (\hat{\xi}\phi)^{\alpha i} &= -\sqrt{2}\eta^{ai}\psi_a^{\alpha} \\ \hat{\xi}\psi_a^{\alpha} &= \sqrt{2}\,\epsilon_{ij}\eta^{bj}(\partial_{ab}\phi)^{\alpha i} \end{aligned}} \tag{3.56}$$

for $\eta^{ai} \in \mathbb{C}$.

Proof of Theorem 3.46. We review the proof of Theorem 3.18 to determine what changes when X is curved. (In this paragraph only we use quaternionic notation for the $\mathbb{F} = \mathbb{H}$ case.) The variation of the lagrangian L has a new contribution from varying the covariant derivative in the fermion kinetic term, and this cancels the variation of the curvature term. We do this computation presently, but first we examine changes in the previous computations. For $\mathbb{F} = \mathbb{R}$ they are unaltered except for the substitution of covariant derivatives. But for $\mathbb{F} = \mathbb{C}, \mathbb{H}$ our assumption that X is Kähler, hyperkähler is crucial. For in the "integration by parts" formula (3.26) there is an extra term from differentiating the $\mathbb{F}$-structure:

$$\begin{aligned} \hat{\xi}_\psi\Big(\frac{A}{2}\langle \psi_a \cdot \tilde{\Gamma}^{\mu \dot{b} a} \cdot \nabla_\mu \bar{\psi}_{\dot{b}} \rangle\Big) = \frac{\kappa A}{2}\big\{ &\langle \partial_\nu \phi\, \bar{\eta}^{\dot{c}}\, \Gamma^\nu_{a\dot{c}} \tilde{\Gamma}^{\mu \dot{b} a}\, \nabla_\mu \bar{\psi}_{\dot{b}} \rangle - \langle \partial_\nu \psi_a\, \tilde{\Gamma}^{\mu \dot{b} a} \Gamma^\nu_{c\dot{b}}\, \eta^c\, \partial_\mu \bar{\phi} \rangle \big\} \\ &+ \partial_\nu \Big\{ \frac{\kappa A}{2} \langle \psi_a\, \tilde{\Gamma}^{\mu \dot{b} a} \Gamma^\nu_{c\dot{b}}\, \eta^c\, \partial_\mu \bar{\phi} \rangle \Big\} \\ &- \frac{\kappa A}{2}\big\{ \langle \psi_a\, T_\nu\, \tilde{\Gamma}^{\mu \dot{b} a} \Gamma^\nu_{c\dot{b}}\, \eta^c\, \partial_\mu \bar{\phi} \rangle + \langle \psi_a\, \tilde{\Gamma}^{\mu \dot{b} a} \Gamma^\nu_{c\dot{b}}\, \eta^c\, T_\nu\, \partial_\mu \bar{\phi} \rangle \big\}. \end{aligned} \tag{3.57}$$

The subscript on $\hat{\xi}$ indicates that we only vary the fermion; the variation of ϕ is computed below in (3.60). We also pick up the conjugate terms in (3.27). Our assumption on X is that $T = 0$, so that this extra term does not appear. With this assumption the verification of the supersymmetry algebra also goes through as before.

We turn to the variation of the covariant derivative and the curvature term. Here it is easier to proceed on a case-by-case basis. First, consider $\mathbb{F} = \mathbb{R}$ so that X is any Riemannian manifold. Let R denote the Riemann curvature tensor. Fix a basis $\{v_i\}$ of TX at a point. Then as usual we write

$$R_{ijk\ell} = \langle v_i, R(v_k, v_\ell)v_j\rangle. \tag{3.58}$$

The curvature tensor obeys certain symmetries and satisfies the Bianchi identity:

$$\begin{aligned} &R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \\ &R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0. \end{aligned} \tag{3.59}$$

Now the variation in ϕ of (3.50) is

$$\begin{aligned} \hat{\xi}_\phi\big(\frac{1}{2}\tilde{\Gamma}^{\mu ab}\langle\psi_a, (\phi^*\nabla_\mu)\psi_b\rangle\big) &= \frac{1}{2}\tilde{\Gamma}^{\mu ab}\langle\psi_a, R(\hat{\xi}\phi, \partial_\mu\phi)\psi_b\rangle \\ &= \frac{1}{2}\tilde{\Gamma}^{\mu ab}\eta^c\langle\psi_a, R(\psi_c, \partial_\mu\phi)\psi_b\rangle \\ &= -\frac{1}{2}\tilde{\Gamma}^{\mu ab}\eta^c R_{ijk\ell}\psi_a^i\psi_b^j\psi_c^k(\partial_\mu\phi)^\ell. \end{aligned} \tag{3.60}$$

The last minus sign comes from commuting ψ_b and ψ_c. We claim that this cancels against the variation in ψ of the curvature term in the lagrangian (3.47). That variation is

(3.61)

$$\begin{aligned} &\hat{\xi}_\psi\big(\frac{1}{12}\epsilon^{ab}\epsilon^{cd}R_{ijk\ell}\psi_a^i\psi_d^j\psi_b^k\psi_c^\ell\big) \\ &\quad= \frac{1}{12}\epsilon^{ab}\epsilon^{cd}R_{\imath jk\ell}\eta^e\Big[\Gamma_{ae}^\mu(\partial_\mu\phi)^\imath\psi_d^j\psi_b^k\psi_c^\ell - \Gamma_{de}^\mu\psi_a^\imath(\partial_\mu\phi)^j\psi_b^k\psi_c^\ell \\ &\qquad\qquad + \Gamma_{be}^\mu\psi_a^i\psi_d^j(\partial_\mu\phi)^k\psi_c^\ell - \Gamma_{ce}^\mu\psi_a^\imath\psi_d^j\psi_b^k(\partial_\mu\phi)^\ell\Big] \\ &\quad= \frac{1}{12}\eta^e\Gamma_{ae}^\mu\Big[\epsilon^{ab}\epsilon^{cd}R_{ijk\ell} - \epsilon^{db}\epsilon^{ca}R_{jik\ell} + \epsilon^{da}\epsilon^{cb}R_{jk\imath\ell} - \epsilon^{dc}\epsilon^{ab}R_{jk\ell i}\Big](\partial_\mu\phi)^i\psi_d^j\psi_b^k\psi \\ &\quad= \frac{1}{12}\eta^e\Gamma_{ae}^\mu\epsilon^{ab}\epsilon^{cd}\Big[R_{ijk\ell} + R_{ijk\ell} + R_{kji\ell} + R_{kji\ell}\Big](\partial_\mu\phi)^i\psi_d^j\psi_b^k\psi_c^\ell \\ &\quad= \frac{1}{6}\eta^e\Gamma_{ae}^\mu\epsilon^{ab}\epsilon^{cd}(R_{\imath jk\ell} + R_{kj\imath\ell})(\partial_\mu\phi)^\imath\psi_d^j\psi_b^k\psi_c^\ell \\ &\quad= \frac{1}{6}\Gamma_{ae}^\mu\epsilon^{ab}(-\eta^d\epsilon^{ec} - \eta^c\epsilon^{de})(R_{\imath jk\ell} + R_{kj\imath\ell})(\partial_\mu\phi)^\imath\psi_d^j\psi_b^k\psi_c^\ell \\ &\quad= \frac{1}{6}(-\tilde{\Gamma}^{\mu bc}\eta^d + \tilde{\Gamma}^{\mu bd}\eta^c)(R_{ijk\ell} + R_{kji\ell})(\partial_\mu\phi)^\imath\psi_d^j\psi_b^k\psi_c^\ell \\ &\quad= \frac{1}{3}\tilde{\Gamma}^{\mu bd}\eta^c(R_{ijk\ell} + R_{kji\ell})(\partial_\mu\phi)^i\psi_d^j\psi_b^k\psi_c^\ell. \end{aligned}$$

In the third to last step we use the fact that $\mathbb{F}^2$ is two-dimensional, so that the cyclic sum $\eta^e\epsilon^{cd} + \eta^d\epsilon^{ec} + \eta^c\epsilon^{de}$ vanishes. To compare (3.61) to (3.60) we need to move around indices:

$$\hat{\xi}_\psi\big(\frac{1}{12}\epsilon^{ab}\epsilon^{cd}R_{ijk\ell}\psi_a^\imath\psi_d^j\psi_b^k\psi_c^\ell\big) = \frac{1}{3}\tilde{\Gamma}^{\mu ab}\eta^c(R_{\ell ijk} + R_{\imath jk\ell})\psi_a^\imath\psi_b^\jmath\psi_c^k(\partial_\mu\phi)^\ell. \tag{3.62}$$

Using the Bianchi identity (3.59) and other symmetries we find

$$\begin{aligned}\tilde{\Gamma}^{\mu ab}\eta^c R_{\ell ijk}\psi_a^\imath\psi_b^\jmath\psi_c^k(\partial_\mu\phi)^\ell &= -\tilde{\Gamma}^{\mu ab}\eta^c(R_{ij\ell k} + R_{\jmath\ell\imath k})\psi_a^i\psi_b^\jmath\psi_c^k(\partial_\mu\phi)^\ell \\ &= \tilde{\Gamma}^{\mu ab}\eta^c(R_{ijk\ell} + R_{\imath\ell\jmath k})\psi_a^i\psi_b^\jmath\psi_c^k(\partial_\mu\phi)^\ell\end{aligned} \tag{3.63}$$

from which

$$\tilde{\Gamma}^{\mu ab}\eta^c R_{\ell ijk}\psi_a^\imath\psi_b^j\psi_c^k(\partial_\mu\phi)^\ell = \frac{1}{2}\tilde{\Gamma}^{\mu ab}\eta^c R_{ijk\ell}\psi_a^\imath\psi_b^\jmath\psi_c^k(\partial_\mu\phi)^\ell. \tag{3.64}$$

Substituting into (3.62) we find

$$\hat{\xi}_\psi\big(\frac{1}{12}\epsilon^{ab}\epsilon^{cd}R_{ijk\ell}\psi_a^i\psi_d^j\psi_b^k\psi_c^\ell\big) = \frac{1}{2}\tilde{\Gamma}^{\mu ab}\eta^c R_{ijk\ell}\psi_a^i\psi_b^\jmath\psi_c^k(\partial_\mu\phi)^\ell \tag{3.65}$$

which cancels (3.60).

To complete the verification that (3.47) is supersymmetric, we check that the variation in ϕ of the curvature term vanishes:

$$\hat{\xi}_\phi\big(\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle\big) = \epsilon^{ab}\epsilon^{cd}\eta^e\langle\psi_a, (\nabla_{\psi_e}R)(\psi_b,\psi_c)\psi_d\rangle. \tag{3.66}$$

Now the symmetries of R and the (second) Bianchi identity imply that the expression $\langle\psi_a, (\nabla_{\psi_e}R)(\psi_b,\psi_c)\psi_d\rangle$ is symmetric in a,d and in b,c,e. In addition, it is symmetric under the interchange $a,d \leftrightarrow b,c$. Altogether this implies that it is totally symmetric in a,b,c,d,e, whence (3.66) vanishes. This completes the proof for $\mathbb{F} = \mathbb{R}$.

Next, consider $\mathbb{F} = \mathbb{C}$. Thus X is Kähler and we work with the complexified tangent bundle as usual. We consider ψ_a to be a complex vector of type $(1,0)$. The Riemann curvature tensor is of type $(1,1)$. If $\{v_i\}$ is a basis of type $(1,0)$ vectors at a point of X, we write

$$R_{i\bar{\jmath}k\bar{\ell}} = \langle v_i, R(v_k, \overline{v_\ell})\overline{v_\jmath}\rangle. \tag{3.67}$$

Then the symmetries of the curvature tensor are

$$R_{i\bar{\jmath}k\bar{\ell}} = R_{k\bar{\jmath}i\bar{\ell}} = R_{i\bar{\ell}k\bar{\jmath}} = -R_{\bar{\jmath}ik\bar{\ell}} = -R_{i\bar{\jmath}\bar{\ell}k}. \tag{3.68}$$

Now the variation in ϕ of the fermion kinetic term in (3.48) is

$$\begin{aligned}&\hat{\xi}_\phi\big(\frac{1}{2}\langle\psi\not{D}\psi\rangle - \frac{1}{2}\langle(\not{D}\bar{\psi})\psi\rangle\big) \\ &\quad= \frac{1}{2}\tilde{\Gamma}^{\mu a\dot{b}}\langle\bar{\psi}_{\dot{b}}, R(\hat{\xi}\phi,\partial_\mu\phi)\psi_a\rangle - \frac{1}{2}\tilde{\Gamma}^{\mu a\dot{b}}\langle R(\hat{\xi}\phi,\partial_\mu\phi)\bar{\psi}_{\dot{b}}, \psi_a\rangle \\ &\quad= \tilde{\Gamma}^{\mu a\dot{b}}\langle\psi_a, R(\hat{\xi}\phi,\partial_\mu\phi)\bar{\psi}_{\dot{b}}\rangle \\ &\quad= -\kappa\tilde{\Gamma}^{\mu a\dot{b}}R_{i\bar{\jmath}k\bar{\ell}}\psi_a^i\bar{\psi}_{\dot{b}}^{\bar{\jmath}}\big[\eta^e\psi_e^k(\partial_\mu\bar{\phi})^{\bar{\ell}} - \bar{\eta}^{\dot{e}}(\partial_\mu\phi)^k\bar{\psi}_{\dot{e}}^{\bar{\ell}}\big],\end{aligned} \tag{3.69}$$

where $\kappa = \sqrt{2}$. We claim this cancels against the variation in ψ of the curvature term in the lagrangian (3.48). That variation is

(3.70)

$$\begin{aligned}
&\hat{\xi}_\psi\big(-\frac{1}{4}\epsilon^{ac}\epsilon^{\dot b\dot d}R_{i\bar j k\bar\ell}\psi_a^i\bar\psi_{\dot b}^{\bar j}\psi_c^k\bar\psi_{\dot d}^{\bar\ell}\big)\\
&\quad= -\frac{1}{4}\kappa\epsilon^{ac}\epsilon^{\dot b\dot d}R_{i\bar j k\bar\ell}\bar\eta^{\dot e}[\Gamma^\mu_{a\dot e}(\partial_\mu\phi)^i\bar\psi_{\dot b}^{\bar j}\psi_c^k\bar\psi_{\dot d}^{\bar\ell}+\Gamma^\mu_{c\dot e}\psi_a^i\bar\psi_{\dot b}^{\bar j}(\partial_\mu\phi)^k\bar\psi_{\dot d}^{\bar\ell}]\\
&\qquad\quad+\frac{1}{4}\kappa\epsilon^{ac}\epsilon^{\dot b\dot d}R_{i\bar j k\bar\ell}\eta^e[\Gamma^\mu_{e\dot b}\psi_a^i(\partial_\mu\bar\phi)^{\bar j}\psi_c^k\bar\psi_{\dot d}^{\bar\ell}+\Gamma^\mu_{e\dot d}\psi_a^i\bar\psi_{\dot b}^{\bar j}\psi_c^k(\partial_\mu\bar\phi)^{\bar\ell}]\\
&\quad= -\frac{1}{2}\kappa\Gamma^\mu_{a\dot e}\epsilon^{ac}\epsilon^{\dot b\dot d}\bar\eta^{\dot e}R_{i\bar j k\bar\ell}(\partial_\mu\phi)^i\bar\psi_{\dot b}^{\bar j}\psi_c^k\bar\psi_{\dot d}^{\bar\ell}+\frac{1}{2}\kappa\Gamma^\mu_{e\dot b}\epsilon^{ac}\epsilon^{\dot b\dot d}\bar\eta^{\dot e}R_{i\bar j k\bar\ell}\psi_a^i(\partial_\mu\bar\phi)^{\bar j}\psi_c^k\bar\psi_{\dot d}^{\bar\ell}\\
&\quad= -\frac{1}{2}\kappa R_{i\bar j k\bar\ell}\Big[-\Gamma^\mu_{a\dot e}\epsilon^{ac}(-\epsilon^{\dot d\dot e}\bar\eta^{\dot b}-\epsilon^{\dot e\dot b}\bar\eta^{\dot d})(\partial_\mu\phi)^i\bar\psi_{\dot b}^{\bar j}\\
&\qquad\qquad+\Gamma^\mu_{e\dot b}(-\epsilon^{cd}\eta^a-\epsilon^{ea}\eta^c)\epsilon^{\dot b\dot d}\psi_a^i(\partial_\mu\bar\phi)^{\bar j}\Big]\psi_c^k\bar\psi_{\dot d}^{\bar\ell}\\
&\quad= -\frac{1}{2}\kappa R_{i\bar j k\bar\ell}\big[(-\tilde\Gamma^{\mu c\dot d}\bar\eta^{\dot b}+\tilde\Gamma^{\mu c\dot b}\bar\eta^{\dot d})(\partial_\mu\phi)^i\bar\psi_{\dot b}^{\bar j}\\
&\qquad\qquad+(\tilde\Gamma^{\mu c\dot d}\eta^a-\tilde\Gamma^{\mu a\dot d}\eta^c)\psi_a^i(\partial_\mu\bar\phi)^{\bar j}\big]\psi_c^k\bar\psi_{\dot d}^{\bar\ell}\\
&\quad= -\kappa\tilde\Gamma^{\mu c\dot d}R_{i\bar j k\bar\ell}\big[\eta^e\psi_e^i(\partial_\mu\bar\phi)^{\bar j}-\bar\eta^{\dot e}(\partial_\mu\phi)\bar\psi_{\dot e}^{\bar j}\big]\psi_c^k\bar\psi_{\dot d}^{\bar\ell}\\
&\quad= \kappa\tilde\Gamma^{\mu a\dot b}R_{i\bar j k\bar\ell}\psi_a^i\bar\psi_{\dot b}^{\bar j}\big[\eta^e\psi_e^k(\partial_\mu\bar\phi)^{\bar\ell}-\bar\eta^{\dot e}(\partial_\mu\phi)^k\bar\psi_{\dot e}^{\bar\ell}\big].
\end{aligned}$$

This cancels (3.69) as claimed.

To complete the proof that (3.48) is supersymmetric, we verify that the variation in ϕ of the curvature term vanishes:

$$\begin{aligned}
(3.71)\quad \hat{\xi}_\phi\big(\epsilon^{ac}\epsilon^{\dot b\dot d}\langle\psi_a,R(\psi_c,\bar\psi_{\dot d})\bar\psi_{\dot b}\rangle\big)&=\kappa\epsilon^{ac}\epsilon^{\dot b\dot d}\eta^e\langle\psi_a,(\nabla_{\psi_e}R)(\psi_c,\bar\psi_{\dot d})\bar\psi_{\dot b}\rangle\\
&\quad-\kappa\epsilon^{ac}\epsilon^{\dot b\dot d}\bar\eta^{edd}\langle\psi_a,(\nabla_{\bar\psi_{\dot e}}R)(\psi_c,\bar\psi_{\dot d})\bar\psi_{\dot b}\rangle.
\end{aligned}$$

The Bianchi identity implies that the factor $\langle\psi_a,(\nabla_{\psi_e}R)(\psi_c,\bar\psi_{\dot d})\bar\psi_{\dot b}\rangle$ in the first term is symmetric in e,c. Combining with (3.68) we see that it is symmetric in a,e,c. Hence the first term vanishes. The argument for the second term is similar.

Finally, consider $\mathbb{F}=\mathbb{H}$. The variation in ϕ of the fermion kinetic term in (3.49) is

(3.72)

$$\begin{aligned}
\hat{\xi}_\phi\big(\frac{1}{4}\tilde h_{\alpha\beta}\nu^{abcd}\psi_a^\alpha\nabla_{cd}\psi_b^\beta\big)&=\hat{\xi}_\phi\big(\frac{1}{8}\nu^{abcd}\epsilon^{ij}\langle\psi_{ai},\nabla_{cd}\psi_{bj}\rangle\big)\\
&=\frac{1}{8}\nu^{abcd}\epsilon^{ij}\langle\psi_{ai},R(\hat\xi\phi,\partial_{cd}\phi)\psi_{bj}\rangle\\
&=\frac{1}{8}\nu^{abcd}\epsilon^{ij}R_{(\alpha i')(\beta j')(\gamma k)(\delta\ell)}\psi_{ai}^{\alpha i'}\psi_{bj}^{\beta j'}(\hat\xi\phi)^{\gamma k}(\partial_{cd}\phi)^{\delta\ell}\\
&=\frac{\kappa}{4}\nu^{abcd}\epsilon_{k\ell}\Omega_{\alpha\beta\gamma\delta}\eta^{ek}\psi_a^\alpha\psi_b^\beta\psi_e^\gamma(\partial_{cd}\phi)^{\delta\ell},
\end{aligned}$$

where $\kappa = \sqrt{2}$. This cancels against the variation in ψ of the curvature term in (3.49), which is

(3.73)

$$\begin{aligned}\hat{\xi}_\psi\big(-\frac{1}{24}\nu^{abcd}\Omega_{\alpha\beta\gamma\delta}\psi_a^\alpha\psi_b^\beta\psi_c^\gamma\psi_d^\delta\big) &= \frac{\kappa}{24}\nu^{abcd}\Omega_{\alpha\beta\gamma\delta}\epsilon_{ij}\eta^{ej}\Big[(\partial_{ae}\phi)^{\alpha i}\psi_b^\beta\psi_c^\gamma\psi_d^\delta - \psi_a^\alpha(\partial_{be}\phi)^{\beta i}\psi_c^\gamma\psi_d^\delta \\ &\qquad + \psi_a^\alpha\psi_b^\beta(\partial_{ce}\phi)^{\gamma i}\psi_d^\delta - \psi_a^\alpha\psi_b^\beta\psi_c^\gamma(\partial_{de}\phi)^{\delta i}\Big] \\ &= -\frac{\kappa}{6}\epsilon_{ij}\Omega_{\alpha\beta\gamma\delta}\nu^{abcd}\eta^{ej}\psi_b^\beta\psi_c^\gamma\psi_d^\delta(\partial_{ae}\phi)^{\alpha i}.\end{aligned}$$

Now we use the fact that $\dim S_0 = 4$, so that the cyclic sum in a, b, c, d, e of $\nu^{abcd}\eta^{ej}$ vanishes. After relabeling some indices we conclude that (3.73) simplifies to

$$\frac{\kappa}{4}\nu^{abcd}\epsilon_{k\ell}\Omega_{\alpha\beta\gamma\delta}\eta^{ek}\psi_a^\alpha\psi_b^\beta\psi_e^\gamma(\partial_{cd}\phi)^{\delta\ell}, \tag{3.74}$$

which cancels (3.72).

As in the previous cases, we complete the proof by showing that the variation in ϕ of the curvature term vanishes. That variation is a multiple of

$$\hat{\xi}_\phi\big(\nu^{abcd}\Omega(\psi_a,\psi_b,\psi_c,\psi_d)\big) = -\nu^{abcd}\eta^{ei}(\nabla_{\psi_{ei}}\Omega)(\psi_a,\psi_b,\psi_c,\psi_d). \tag{3.75}$$

Now the Bianchi identity and the symmetry of Ω imply that $(\nabla_{\psi_{ei}}\Omega)(\psi_a,\psi_b,\psi_c,\psi_d)$ is totally skew in a, b, c, d, e. Since the skew-symmetrization of $\nu^{abcd}\eta^{ei}$ vanishes ($\dim S_0 = 4$), the entire expression (3.75) vanishes.

In the nonlinear model the formulas (3.40) and (3.43) for the variational 1-form and the supercurrent are replaced by the covariant versions; that is, we use covariant derivatives in place of ordinary derivatives. We do not bother recording them here; see Theorem 6.33. The equations of motion (3.41) are affected in a more drastic way—the right hand sides are nonzero expressions in the curvature and its covariant derivative. They vanish if we set $\psi = 0$. (In fact, the terms are quadratic, cubic, and quartic in ψ.) The interested reader can work out the precise formulas, some of which appear in §4 and §5.

§3.4. Supersymmetric potential terms

To the bosonic σ-model lagrangian (3.15) with target X we can add a potential term. Let

$$V\colon X \longrightarrow \mathbb{R} \tag{3.76}$$

be a real-valued function, called the *potential energy*. We assume that V is bounded below. Then the bosonic lagrangian with potential is

$$L = \Big\{\frac{1}{2}|d\phi|^2 - \phi^*V\Big\}\,|d^nx|. \tag{3.77}$$

The moduli space of classical vacua for this theory—that is, the space of static field configurations of minimal energy—is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \tag{3.78}$$

if we assume that the minimum value of V occurs at 0.

Now we consider supersymmetric extensions of (3.77). One basic principle, which follows from the supersymmetry algebra, is that in a supersymmetric theory the potential energy is nonnegative. That is manifest in our formulas below. Our starting point is the nonlinear σ-models developed in the previous section.

We will not carry out a detailed analysis in components, but rather content ourselves with pointing out a few elementary features. First, note that the variation of the potential term is, using (3.23),

$$\hat{\xi}(\phi^* V) = -\kappa \eta^a \phi^* \big(dV(\psi_a)\big). \tag{3.79}$$

Thus there must be an additional term in the lagrangian to cancel this. Power counting—that is, weighting the boson 0, the fermion 1/2, and a spacetime derivative 1—shows that in addition we must change the supersymmetry transformation law (3.23) for the fermion. The form of the new transformation is

$$\hat{\xi}\psi_a = \kappa \Gamma^{\mu}_{a\dot{b}}\, \bar{\eta}^{\dot{b}}\, \partial_\mu \phi + \kappa \epsilon_{ab} \eta^b \phi^* F_a \tag{3.80}$$

for some vector fields F_a on X, and the additional term in the lagrangian takes the form

$$\phi^* A^{ab}(\psi_a, \psi_b) \tag{3.81}$$

for some bilinear forms A^{ab} on X. Imposing the condition that the lagrangian be supersymmetric gives relations among V, A^{ab}, F_a. In fact, one can analyze those conditions in the dimensional reduction of the models to one spacetime dimension. Such an analysis leads to the form of the lagrangians in the following result.

Theorem 3.82. (*i*) ($\mathbb{F} = \mathbb{R}$) *Let X be a Riemannian manifold and $h\colon X \to \mathbb{R}$ a real function. Then the lagrangian*

$$\boxed{\begin{aligned} L = \Big\{ \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle \psi \not{D}_\phi \psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle \psi_a, \phi^* R(\psi_b, \psi_c)\psi_d\rangle \\ - \frac{1}{2}\phi^* |\operatorname{grad} h|^2 - \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b)\Big\}\, |d^3x|. \end{aligned}} \tag{3.83}$$

is supersymmetric. The supersymmetry transformation law is

$$\boxed{\begin{aligned} \hat{\xi}\phi &= -\eta^a \psi_a \\ \hat{\xi}\psi_a &= \Gamma^{\mu}_{ab}\, \eta^b\, \partial_\mu \phi + \epsilon_{ab}\eta^b \phi^* \operatorname{grad} h \end{aligned}} \tag{3.84}$$

(*ii*) ($\mathbb{F} = \mathbb{C}$) *Let X be a Kähler manifold and $W\colon X \to \mathbb{R}$ a holomorphic function. Then the lagrangian*

(3.85)

$$\boxed{\begin{aligned} L = \Big\{ \langle d\bar{\phi}, d\phi\rangle + \frac{1}{2}\langle \bar{\psi} \not{D}_\phi \psi\rangle - \frac{1}{2}\langle (\not{D}_\phi \bar{\psi})\psi\rangle - \frac{1}{4}\epsilon^{ac}\epsilon^{\dot{b}\dot{d}}\langle \psi_a, \phi^* R(\psi_c, \bar{\psi}_{\dot{d}})\bar{\psi}_{\dot{b}}\rangle \\ - \phi^* \|\operatorname{grad} W\|^2 - \operatorname{Re}\big[\epsilon^{ab}\phi^*(\operatorname{Hess} W)(\psi_a, \psi_b)\big]\Big\}\, |d^4x|. \end{aligned}}$$

is supersymmetric. The supersymmetry transformation law is

$$\boxed{\begin{aligned}\hat{\xi}\phi &= -\sqrt{2}\,\eta^a\,\psi_a \\ \hat{\xi}\psi_a &= \sqrt{2}\,\Gamma^{\mu}_{a\dot{b}}\,\bar{\eta}^{\dot{b}}\,\partial_\mu\phi + \sqrt{2}\,\epsilon_{ab}\eta^b\phi^*\,\mathrm{grad}\,W\end{aligned}} \tag{3.86}$$

Here $\mathrm{Hess}\,h$ is the covariant hessian

$$\mathrm{Hess}\,h = \nabla dh, \tag{3.87}$$

which is a symmetric bilinear form. Similarly,

$$\mathrm{Hess}\,W = \nabla\partial W. \tag{3.88}$$

The double bars denote the hermitian norm:

$$\|\,\mathrm{grad}\,W\|^2 = \langle\overline{\mathrm{grad}\,W}, \mathrm{grad}\,W\rangle. \tag{3.89}$$

For $\mathbb{F} = \mathbb{R}$ the potential energy is

$$V_{\mathbb{R}} = \frac{1}{2}|\,\mathrm{grad}\,h|^2; \tag{3.90}$$

for $\mathbb{F} = \mathbb{C}$ it is

$$V_{\mathbb{C}} = \|\,\mathrm{grad}\,W\|^2. \tag{3.91}$$

In both cases it is nonnegative. The function W is called the *superpotential*. The moduli space of classical vacua is

$$\mathcal{M}^{\mathbb{R}}_{\mathrm{vac}} = \mathrm{Crit}(h) \tag{3.92}$$

for $\mathbb{F} = \mathbb{R}$ and

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{vac}} = \mathrm{Crit}(W) \tag{3.93}$$

for $\mathbb{F} = \mathbb{C}$. Here 'Crit' denotes the set of critical points.

We defer the proof of Theorem 3.82 to §4 and §5, where we give a manifestly supersymmetric version in superspace. There we also derive the supercurrent for the model with potential.

For the $\mathbb{F} = \mathbb{H}$ case there is no way to add a potential energy term and preserve supersymmetry. Presumably this follows in a straightforward way from the component analysis indicated above. On the other hand, the dimensional reduction of this 6-dimensional theory to 4 dimensions (and below) permits a supersymmetric potential energy term—the bosonic potential energy V is the norm square of a vector field on X which preserves the hyperkähler structure.

§3.5. Superspace construction

Consider a superspace $M^{n|s}$ in any dimension built out of any real spin representation, and let X be any Riemannian manifold. Introduce a scalar superfield

$$\Phi\colon M^{n|s} \longrightarrow X. \tag{3.94}$$

Let $i\colon \check{M}^n \hookrightarrow M^{n|s}$ be the inclusion. There are component fields

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= i^* D_a \Phi \end{aligned} \tag{3.95}$$

of the type we need, but in general higher derivatives of Φ lead to more components.

It is tempting to write a lagrangian

$$L = \check{L}\,|d^n x| \tag{3.96}$$

of the form $\check{L} = i^*S$ for some function S on superspace. We need 4 fermionic derivatives in superspace to have 2 spacetime derivatives on the scalar ϕ, and the natural invariant expression is

$$\check{L} = i^*\Big\{ k g_{\mu\nu} \tilde{\Gamma}^{\mu ac} \tilde{\Gamma}^{\nu bd} D_a D_b \langle D_c\Phi, D_d\Phi\rangle \Big\}, \tag{3.97}$$

where k is a constant. In fact, the term

$$\check{L} = i^*\Big\{ -k g_{\mu\nu} \tilde{\Gamma}^{\mu ac} \tilde{\Gamma}^{\nu bd} \langle D_a D_c\Phi, D_b D_d\Phi\rangle \Big\} \tag{3.98}$$

contains a multiple of the bosonic lagrangian (3.15), and this fixes the constant k. It is not hard to see that we also obtain the kinetic term for the fermion ψ. In addition—if there are at least 2 supersymmetries—we find terms with no derivatives built out of *auxiliary fields*, which are restrictions to $\check{M}^n$ of second derivatives of Φ. But in general this construction fails to give a supersymmetric lagrangian: there are too many component fields. We can eliminate some of them from the supersymmetry transformation laws derived from (1.29), but there is no guarantee that (3.97) is supersymmetric, nor that the supersymmetry algebra closes on-shell. In fact, this construction does work for $s = 1$ supersymmetry (1.52) and $s = 2$ supersymmetries (§4). For $s = 1$ there are no auxiliary fields; for $s = 2$ there is an auxiliary real scalar field. For $s = 4$ supersymmetries (§5) the superspace model presented here works if we impose a *constraint* on Φ which eliminates many of the component fields. There is then a single auxiliary complex scalar field. In all of these cases we express (3.97) in a form which is manifestly supersymmetric.

§3.6. Dimensional reduction

In a standard way we can reduce a theory on $\check{M}^n$ to a theory on $\check{M}^m$ for $m < n$ by considering fields on $\check{M}^n$ invariant under an $(n-m)$-dimensional space of spatial translations. The scalar field ϕ reduces to a scalar field on $\check{M}^m$, and the spinor field ψ reduces to (nonchiral) spinor fields on $\check{M}^m$. (A single spinor field on $\check{M}^n$ gives possibly many spinor fields on $\check{M}^m$. For example, the 8-component chiral spinor field in the 6-dimensional $\mathbb{F} = \mathbb{H}$ model reduces to 4 right handed and 4 left handed

spinor fields in $\check{M}^2$.) The reduced model has the same number of supersymmetries as the original.

In particular, we can relate the three supersymmetric σ-models considered here. For example, the dimensional reduction of the linear $\mathbb{F} = \mathbb{C}$ model to 3 dimensions gives two copies of the $\mathbb{F} = \mathbb{R}$ model. Similarly, the dimensional reduction of the nonlinear $\mathbb{F} = \mathbb{C}$ model with target a Kähler manifold X gives the nonlinear $\mathbb{F} = \mathbb{R}$ model with target the underlying real manifold. In this case the fermion $\hat{\psi}$ of the 3-dimensional model is defined in terms of the fermion ψ of the 4-dimensional model by

$$\boxed{\hat{\psi} = \frac{\psi + \bar{\psi}}{\sqrt{2}}.} \tag{3.99}$$

Then the lagrangian (3.48) reduces to twice the lagrangian (3.47). For example, if we plug (3.99) into the curvature term in (3.47) we have 6 nonzero terms with a factor of $1/48$ in front, and twice $6 \cdot 1/48$ is $1/4$. The sign also works. The dimensional reduction of the potential term (3.85) in the $\mathbb{F} = \mathbb{C}$ model gives the potential term (3.83) with

$$\boxed{h = \operatorname{Re} W.} \tag{3.100}$$

To dimensionally reduce the $\mathbb{F} = \mathbb{H}$ model to 4 dimensions, we must fix a Kähler structure on the hyperkähler target manifold X. In fact, there is a 2-sphere of such Kähler structures parametrized by the unit imaginary quaternions acting on the vector space W^* (cf. (3.51)). We choose the basis $\{e_1, e_2\}$ of W so that $V \otimes \mathbb{C}e^1$ is the bundle of type $(1,0)$ tangent vectors to X and $V \otimes \mathbb{C}e^2$ the bundle of type $(0,1)$ tangent vectors to X. As for the spinor fields, recall from §2.2 that under dimensional reduction we identify $S_0 \cong S' \oplus S''$, where $S' \oplus S''$ is the complexified spinor representation of $\mathrm{Spin}(1,3)$. Thus the spinor fields of the dimensionally reduced theory transform in $(S' \oplus S'') \otimes \phi^* V$. In indices, if $\hat{\xi}^{\alpha i}$ is a complex tangent vector to X, then we define

$$\boxed{\xi^\alpha = \hat{\xi}^{\alpha 1}, \qquad \bar{\xi}^{\bar{\alpha}} = \hat{\xi}^{\alpha 2}.} \tag{3.101}$$

Note ξ is real if $\bar{\xi}^{\bar{\alpha}}$ is the complex conjugate of ξ^α. If $\hat{\psi}_a^\alpha$ is a spinor field on $\check{M}^6$, we define a spinor field ψ on $\check{M}^4$ by

$$\boxed{\begin{aligned} \psi_1^\alpha &= \hat{\psi}_1^\alpha, & \bar{\psi}_{\dot{1}}^{\bar{\alpha}} &= J_\beta^{\bar{\alpha}} \hat{\psi}_3^\beta, \\ \psi_2^\alpha &= \hat{\psi}_2^\alpha, & \bar{\psi}_{\dot{2}}^{\bar{\alpha}} &= J_\beta^{\bar{\alpha}} \hat{\psi}_4^\beta, \end{aligned}} \tag{3.102}$$

where J is the pseudoreal structure on V. The reality condition on ψ is that $\bar{\psi}_{\dot{a}}^{\bar{\alpha}}$ is the complex conjugate of ψ_a^α. Armed with these formulas, together with (2.99) and (3.55), we dimensionally reduce (3.49) to $\check{M}^4$, i.e., we evaluate (3.49) on fields ϕ, ψ which satisfy $\partial_{12}\phi = \partial_{12}\psi = 0$. A bit of computation shows that we recover (3.48).

CHAPTER 4
The Supersymmetric σ-Model in Dimension 3

In this section we discuss the σ-model with 2 supersymmetries. We already described it in components in Chapter 3. Here we give a manifestly supersymmetric treatment in superspace. We begin in §4.1 by deriving the component fields and supersymmetry transformation laws, following the general principles laid out in §1.2. Then in §4.2 we state the superspace lagrangian and derive the component lagrangian. We recover the lagrangian of Theorem 3.46(i) except with the addition of an auxiliary field. One virtue of the auxiliary field is that now the bracketing relations among the supersymmetry transformations (4.14) are precisely those of the super Poincaré algebra; without the auxiliary field we need to impose the equations of motion to get the correct algebra. In §4.3 we prove Theorem 3.82(i), which describes the supersymmetric potential term. We carry out the analysis of the classical theory in §4.4—we compute the variational 1-form, equations of motion, symplectic structure, supercurrent—directly in superspace and from that rederive the component expressions obtained in Chapter 3. We conclude in §4.5 by briefly considering the dimensional reduction to 2 dimensions, where the Poisson brackets of the supercharges leads to a central extension of the super Poincaré algebra.

§4.1. Fields and supersymmetry transformations on $M^{3|2}$

Linear Case. We begin with a real *scalar superfield*

$$\Phi\colon M^{3|2} \longrightarrow \mathbb{R}. \tag{4.1}$$

Define the component fields as

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= i^* D_a \Phi \\ F &= i^*(-D^2)\Phi. \end{aligned} \tag{4.2}$$

So ϕ is a real function on Minkowski space $\check{M}^3$, the field $\psi = \psi_a f^a$ is a spinor field on $\check{M}^3$, and F is again a real function on $\check{M}^3$. Note that ϕ and F are *even* whereas ψ is *odd*. We will see that the field F only enters algebraically, and its equations of motion are algebraic. Such a field is termed *auxiliary* since we can solve for it algebraically in terms of the other fields. The collection of component fields is termed a *multiplet*, so here (ϕ, ψ, F) is a *real scalar multiplet*.

Let $\hat{\xi}$ be the vector field on the space $\mathcal{F}_{(\phi,\psi,F)}$ of component fields induced by the supersymmetry transformation $\eta^a Q_a$, where η^a are odd parameters. We compute the action of $\hat{\xi}$ on component fields using (1.29). For the lowest component ϕ this is straightforward:

$$\hat{\xi}\phi = -\eta^a i^* D_a \Phi = -\eta^a \psi_a. \tag{4.3}$$

For the action on ψ_a we use (2.42):

$$\begin{aligned} \hat{\xi}\psi_a &= -\eta^b i^* D_b D_a \Phi \\ &= \eta^b i^* (\partial_{ab} - \epsilon_{ba} D^2)\Phi \\ &= \eta^b \partial_{ab}\phi - \epsilon_{ab}\eta^b F. \end{aligned} \tag{4.4}$$

To compute the action on F we use (2.43):

$$\begin{aligned} \hat{\xi}F &= -\eta^a i^* D_a(-D^2)\Phi \\ &= -\eta^a \epsilon^{bc} i^* \partial_{ab} D_c \Phi \\ &= -\eta^a \epsilon^{bc} \partial_{ab}\psi_c \\ &= \eta^a (\not{D}\psi)_a. \end{aligned} \tag{4.5}$$

The reader should check the supersymmetry algebra (1.30) and should also compare (4.3) and (4.4) to (3.23).

Nonlinear Case. For the *nonlinear* σ-model into a Riemannian manifold X the superfield is a map

$$\Phi\colon M^{3|2} \longrightarrow X. \tag{4.6}$$

The components are defined by the (covariant) extensions of (4.2) using the Levi-Civita covariant derivative:

$$\boxed{\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= i^* D_a \Phi \\ F &= i^*(-\mathcal{D}^2)\Phi. \end{aligned}} \tag{4.7}$$

Now $\phi\colon \check{M}^3 \to X$ is a map into the target manifold, i.e., the field of the underlying bosonic σ-model. Then $\psi = \psi_a f^a$ is a spinor field on $\check{M}^3$ with values in ϕ^*TX, i.e., $\psi\colon \check{M}^3 \to \Pi S \otimes \phi^*TX$. (Recall that '$\Pi$' denotes parity reversal.) For pedagogical purposes we use the symbol '$\mathcal{D}_a$' to denote the covariant derivative of a section of ϕ^*TX in the direction D_a and '∇_{ab}' for the covariant derivative in the direction ∂_{ab}. Then $\mathcal{D}^2 = \frac{1}{2}\epsilon^{ab}\mathcal{D}_a\mathcal{D}_b$, and F is a scalar with values in ϕ^*TX, i.e., a section of ϕ^*TX on $\check{M}^3$.

To compute with the nonlinear superfield Φ it is useful to recall that

$$\mathcal{D}_a D_b \Phi = \mathcal{D}_b D_a \Phi, \tag{4.8}$$

since the Levi-Civita connection is torsionfree. Also,

$$R(D_a, D_b) = \mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a + 2\nabla_{ab} \qquad \text{on } \Phi^*TX \tag{4.9}$$

by the definition of the curvature R of Φ^*TX and the sign rule. One must also be careful with the sign rule when applying the Bianchi identity.

Now the analogs of equations (2.43) for covariant derivatives are easily derived:

$$\mathcal{D}_a\mathcal{D}^2\Phi = -\epsilon^{bc}\big(\nabla_{ab} - \frac{1}{3}R(D_a\Phi, D_b\Phi)\big)D_c\Phi, \tag{4.10}$$

$$\mathcal{D}^2 D_a\Phi = \epsilon^{bc}\big(\nabla_{ab} - \frac{1}{6}R(D_a\Phi, D_b\Phi)\big)D_c\Phi. \tag{4.11}$$

To illustrate the computations, we derive (4.11) for $a = 1$:

(4.12)

$$\begin{aligned} 2\mathcal{D}^2 D_1\Phi &= \mathcal{D}_1\mathcal{D}_2 D_1\Phi - \mathcal{D}_2\mathcal{D}_1 D_1\Phi \\ &= \big[R(D_1\Phi, D_2\Phi)D_1\Phi - \mathcal{D}_2\mathcal{D}_1 D_1\Phi - 2\nabla_{12}D_1\Phi\big] + \mathcal{D}_2\partial_{11}\Phi \\ &= 2D_2\partial_{11}\Phi - 2\nabla_{12}D_1\Phi + R(D_1\Phi, D_2\Phi)D_1\Phi \\ &= 2\nabla_{11}D_2\Phi - 2\nabla_{12}D_1\Phi + R(D_1\Phi, D_2\Phi)D_1\Phi. \end{aligned}$$

At the last stage, and also when we compare (4.12) with (4.11), we use the Bianchi identity $R(D_1\Phi, D_1\Phi)D_2\Phi = -2R(D_1\Phi, D_2\Phi)D_1\Phi$.

Using (4.10) and (4.11) it is a routine matter to derive the transformation laws for the fields in the nonlinear case. In fact, the only change comes in (4.5):

$$\begin{aligned} \hat{\xi}F &= -\eta^a i^*\mathcal{D}_a(-\mathcal{D}^2)\Phi \\ &= -\eta^a\epsilon^{bc}i^*\big(\nabla_{ab} - \frac{1}{3}R(D_a\Phi, D_b\Phi)\big)D_c\Phi \\ &= -\eta^a\epsilon^{bc}\big(\nabla_{ab} - \frac{1}{3}R(\psi_a, \psi_b)\big)\psi_c \\ &= \eta^a\big[(\not{D}\psi)_a + \frac{1}{3}\epsilon^{bc}R(\psi_a, \psi_b)\psi_c\big]. \end{aligned} \tag{4.13}$$

For convenience we collect the transformation formulas for the component fields:

$$\boxed{\begin{aligned} \hat{\xi}\phi &= -\eta^a\psi_a \\ \hat{\xi}\psi_a &= \eta^b(\partial_{ab}\phi - \epsilon_{ab}F) \\ \hat{\xi}F &= \eta^a\big[(\not{D}\psi)_a + \frac{1}{3}\epsilon^{bc}R(\psi_a, \psi_b)\psi_c\big]. \end{aligned}} \tag{4.14}$$

These formulas have easy reductions to $M^{2|2}$. As mentioned after (2.106) we simply set $\partial_{12} = \nabla_{12} = 0$ and relabel indices as 1=+ and 2=−:

$$\boxed{\begin{aligned} \hat{\xi}\phi &= -\eta^+\psi_+ - \eta^-\psi_- \\ \hat{\xi}\psi_+ &= \eta^+\partial_+\phi - \eta^- F \\ \hat{\xi}\psi_- &= \eta^-\partial_-\phi + \eta^+ F \\ \hat{\xi}F &= -\eta^+\nabla_+\psi_- + \eta^-\nabla_-\psi_+ - \eta^+R(\psi_+, \psi_-)\psi_+ + \eta^-R(\psi_-, \psi_+)\psi_-. \end{aligned}} \tag{4.15}$$

§4.2. The σ-model action on $M^{3|2}$

Since the computations in the nonlinear case are no more difficult than in the linear case, we proceed directly to it. So our superfield (4.6) is a map $\Phi\colon M^{3|2} \to X$ into a Riemannian manifold X. The lagrangian density on $M^{3|2}$ is

$$\boxed{L_0 = |d^3x|\, d^2\theta\, \frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle,} \tag{4.16}$$

where $\langle\cdot,\cdot\rangle$ is the metric on X pulled back to a metric on Φ^*TX. Our goal is to derive the component lagrangian as defined in (1.33). Recall that in this case we use integration over the θ variables, expressed as differentiation in (2.40). The resulting lagrangian *function* on Minkowski space is

$$\check{L}_0 = \int d^2\theta\, \frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle. \tag{4.17}$$

We carry out the integration using (2.40), at intermediate stages using (4.11) and the covariant analog of (2.42), and finally using the definition of the component fields (4.7) and formulas (2.26), (2.28) to restrict to Minkowski space:

(4.18)

$$\begin{aligned}
4\check{L}_0 &= -\epsilon^{ab}i^*D^2\langle D_a\Phi, D_b\Phi\rangle \\
&= i^*\left\{-2\epsilon^{ab}\langle D_a\Phi, \mathcal{D}^2 D_b\Phi\rangle + \epsilon^{ab}\epsilon^{cd}\langle \mathcal{D}_c D_a\Phi, \mathcal{D}_d D_b\Phi\rangle\right\} \\
&= i^*\Big\{-2\epsilon^{ab}\epsilon^{cd}\langle D_a\Phi, \nabla_{bc} D_d\Phi\rangle + \frac{1}{3}\epsilon^{ab}\epsilon^{cd}\langle D_a\Phi, R(D_b\Phi, D_c\Phi)D_d\Phi\rangle \\
&\qquad\qquad + \epsilon^{ab}\epsilon^{cd}\langle \partial_{ca}\Phi - \epsilon_{ca}\mathcal{D}^2\Phi\ ,\ \partial_{db}\Phi - \epsilon_{db}\mathcal{D}^2\Phi\rangle\Big\} \\
&= 2\langle\psi \not{D}\psi\rangle + \frac{1}{3}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle + 2|d\phi|^2 + 2|F|^2.
\end{aligned}$$

Of course, here the Dirac form (2.28) is defined with a covariant derivative. So the final formula is[28]

$$\boxed{\check{L}_0 = \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}_\phi\psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, \phi^*R(\psi_b,\psi_c)\psi_d\rangle + \frac{1}{2}|F|^2.} \tag{4.19}$$

The resulting equations of motion imply that $F = 0$. Putting $F = 0$ in (4.19) we recover (3.47).

§4.3. The potential term on $M^{3|2}$

Let $h\colon X \to \mathbb{R}$ be a real-valued function on X, and set

$$\boxed{L_1 = |d^3x|\, d^2\theta\, \Phi^*(h).} \tag{4.20}$$

[28]We put 'ϕ' and 'ϕ^*' into the notation to remind ourselves that these terms contribute to the equation of motion for ϕ.

The expansion in components is

$$\begin{aligned}\check{L}_1 &= -\frac{1}{2}i^*\epsilon^{ab}D_aD_b\Phi^*(h)\\ &= -\frac{1}{2}i^*\epsilon^{ab}D_a\big(\iota(D_b\Phi)dh\big)\\ &= -\frac{1}{2}i^*\epsilon^{ab}\big(\iota(\mathcal{D}_aD_b\Phi)dh - \iota(D_b\Phi)D_a\Phi^*dh\big)\\ &= \iota(F)\phi^*dh - \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess}h)(\psi_a,\psi_b).\end{aligned} \tag{4.21}$$

In this expression $\operatorname{Hess}h = \nabla dh$ is the covariant hessian of h, a symmetric tensor on X. We include 'ϕ^*' in our equations to make clear the dependence of ϕ. In the total lagrangian function $\check{L}_0 + \check{L}_1$ the field F still enters algebraically, and its equation of motion is

$$\boxed{F = -\phi^*\operatorname{grad}h.} \tag{4.22}$$

Substituting into the lagrangian we obtain

$$\begin{aligned}\check{L}_0 + \check{L}_1 \sim \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi\not{D}_\phi\psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a,\phi^*R(\psi_b,\psi_c)\psi_d\rangle\\ - \frac{1}{2}\phi^*|\operatorname{grad}h|^2 - \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess}h)(\psi_a,\psi_b).\end{aligned} \tag{4.23}$$

The '$\sim$' indicates that we have eliminated auxiliary fields. This is precisely the lagrangian (3.83), and since the superspace model is manifestly supersymmetric we have proved Theorem 3.82(i). The potential energy for the bosons is

$$V = \frac{1}{2}|\operatorname{grad}h|^2. \tag{4.24}$$

It is nonnegative, as we expect in general for a supersymmetric lagrangian. The bilinear form in ψ is a mass term for the fermions, which we abbreviate as

$$\phi^*(\operatorname{Hess}h)(\psi,\psi). \tag{4.25}$$

A *vacuum solution* of a classical field theory on Minkowski space is a field configuration with all fermions set to zero and all scalar fields set to constants which minimize the energy. (If there are gauge fields, then they are chosen to be trivial.) For $\check{L}_0 + \check{L}_1$ this means that $\psi \equiv 0$ and ϕ is a constant which is a critical point of (4.24). Now

$$dV = \langle\operatorname{grad}h, \nabla\operatorname{grad}h\rangle. \tag{4.26}$$

This certainly vanishes at critical points of h. At such points the potential energy vanishes and supersymmetry is *unbroken*. This means that the solution is annihilated by the supersymmetry transformation (4.14). For any vacuum solution (4.14) simplifies to (using (4.22))

$$\begin{aligned}\hat{\xi}\phi &= 0\\ \hat{\xi}\psi_a &= \eta^b\epsilon_{ab}\phi^*\operatorname{grad}h,\end{aligned} \tag{4.27}$$

and clearly $\hat{\xi}\psi = 0$ only at a critical point of h. Thus at critical points of V which are *not* critical points of h, supersymmetry is broken. At such a point $\operatorname{grad}h$ is in the kernel of the mass form (4.25) for the fermions. In other words, at such a point there is a massless fermion, the so-called *Goldstone fermion*.

§4.4. Analysis of the classical theory

We first compute the variation of the lagrangian density $L_0 + L_1$ under an arbitrary *even* variation $\delta\Phi$ of the superfield Φ. One should interpret $L_0 + L_1$ as living on $\mathcal{F}_\Phi \times M^{3|2}$, where $\mathcal{F}_\Phi$ is the supermanifold of superfields Φ. Then 'δ' is the component of the differential along $\mathcal{F}_\Phi$. We have

$$\delta(L_0 + L_1) = |d^3x|\, d^2\theta \left\{ \frac{1}{2}\epsilon^{ab}\langle \delta_\nabla D_a\Phi, D_b\Phi\rangle + \iota(\delta\Phi)dh \right\}, \tag{4.28}$$

where δ_∇ is the extension of the differential δ using the covariant derivative. The first term in braces in (4.28) is

$$\begin{aligned} &|d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\langle \delta_\nabla D_a\Phi, D_b\Phi\rangle \\ &\qquad = |d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\langle \mathcal{D}_a\delta\Phi, D_b\Phi\rangle \\ &\qquad = |d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\Big\{ D_a\langle \delta\Phi, D_b\Phi\rangle - \langle \delta\Phi, \mathcal{D}_a D_b\Phi\rangle \Big\} \\ &\qquad = d\Big\{ \iota(D_a)\big(|d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\langle D_b\Phi, \delta\Phi\rangle\big)\Big\} - |d^3x|\, d^2\theta\ \langle \delta\Phi, \mathcal{D}^2\Phi \end{aligned} \tag{4.29}$$

At the last stage we use the fact that the canonical density $|d^3x|\, d^2\theta$ is invariant under D_a and we use the Cartan formula for the Lie derivative. So altogether

$$\begin{aligned} \delta(L_0 + L_1) = d\Big\{ \iota(D_a)\big(|d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\langle D_b\Phi, \delta\Phi\rangle\big)\Big\} \\ - |d^3x|\, d^2\theta\ \langle \delta\Phi\,,\ \mathcal{D}^2\Phi - \Phi^* \operatorname{grad} h\rangle. \end{aligned} \tag{4.30}$$

From this we read off the equation of motion in superspace

$$\boxed{\mathcal{D}^2\Phi = \Phi^* \operatorname{grad} h} \tag{4.31}$$

and the variational 1-form

$$\boxed{\gamma = -\iota(D_a)|d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\langle D_b\Phi, \delta\Phi\rangle.} \tag{4.32}$$

The appellation '1-form' refers to the fact that γ is a 1-form on the space $\mathcal{F}_\Phi$ of superfields. With respect to $M^{3|2}$ it is an integral density of degree -1, written here as a Berezinian contracted with a vector field, i.e., $\gamma \in \Omega^{1,|-1|}(\mathcal{F}_\Phi \times M^{3|2})$. The differential $\delta\gamma$ is $\omega \in \Omega^{2,|-1|}(\mathcal{F}_\Phi \times M^{3|2})$:

$$\boxed{\omega = \iota(D_a)|d^3x|\, d^2\theta\ \frac{1}{2}\epsilon^{ab}\langle \mathcal{D}_b\delta\Phi \wedge \delta\Phi\rangle.} \tag{4.33}$$

The 2-form ω is a local version of the symplectic form on the space of classical solutions $\mathcal{M} \subset \mathcal{F}_\Phi$; the global symplectic 2-form on $\mathcal{M}$ is obtained by integrating ω over a spacelike hypersurface of codimension $1|0$ in $M^{3|2}$.

This completes the analysis in superspace. Now we expand in components. We begin with the equation of motion (4.31). Restricting to Minkowski space, we

simply recover the equation of motion (4.22) for F. The equation of motion for ψ is found by applying $i^*\mathcal{D}_a$ to (4.31) and using (4.10):

$$\boxed{(\not{D}_\phi\psi)_a = -\epsilon^{bc}\,\phi^*\nabla_{ab}\psi_c = -\frac{1}{3}\epsilon^{bc}\phi^*R(\psi_a,\psi_b)\psi_c + \phi^*(\nabla\,\mathrm{grad}\,h)(\psi_a).} \tag{4.34}$$

The equation of motion for ϕ is a complicated expression which schematically is

$$\boxed{\Box\phi = -\frac{1}{2}\phi^*\,\mathrm{grad}\,|dh|^2 + Q_\phi^{(2)}(\psi) + R_\phi^{(2)}(\psi)d\phi + Q_\phi^{(4)}(\psi),} \tag{4.35}$$

where '$\Box$' is the covariant wave operator

$$\Box = \nabla_{11}\partial_{22} - \nabla_{12}\partial_{12} \tag{4.36}$$

and $Q_\phi^{(2)}$, $R_\phi^{(2)}$, and $Q_\phi^{(4)}$ are forms in ψ whose coefficients are made from the pullback of derivatives of h and derivatives of the curvature R. The precise formula is not of interest to us here. The *bosonic* classical equation for ϕ, obtained by setting the fermion $\psi = 0$, is

$$\Box\phi = -\phi^*\,\mathrm{grad}\,V, \tag{4.37}$$

where V is the potential energy (4.24). Newton's law at last!

Next we compute $\check{\gamma}, \check{\omega}$ which are obtained from γ, ω by integrating out the θ's. Recall from (2.16) that $D_a = \partial_a - \theta^c\partial_{ac}$. Now in (4.32) the term with $\iota(\partial_a)$ drops out after $\int d^2\theta$, so

(4.38)

$$\begin{aligned}
\check{\gamma} &= \iota(\partial_{ac})\,|d^3x|\int d^2\theta\;\frac{1}{2}\epsilon^{ab}\theta^c\langle D_b\Phi,\delta\Phi\rangle \\
&= -\iota(\partial_{ac})\,|d^3x|\left\{\frac{1}{4}\epsilon^{ab}\epsilon^{de}i^*D_dD_e\theta^c\langle D_b\Phi,\delta\Phi\rangle\right\} \\
&= -\iota(\partial_{ac})\,|d^3x|\left\{-\frac{1}{2}\epsilon^{ab}\epsilon^{cd}i^*\left[\langle\mathcal{D}_dD_b\Phi,\delta\Phi\rangle - \langle D_d\Phi,\delta D_b\Phi\rangle\right]\right\} \\
&= -\iota(\partial_{ac})\,|d^3x|\left\{\frac{1}{2}\epsilon^{ab}\epsilon^{cd}\left[\langle\partial_{db}\phi,\delta\phi\rangle + \epsilon_{db}\langle F,\delta\phi\rangle + \langle\psi_b,\delta\psi_d\rangle\right]\right\}.
\end{aligned}$$

Note that '$\delta\psi_d$' is a covariant derivative of ψ_d. Simplifying further we obtain

$$\boxed{\begin{aligned}
\check{\gamma} &= -\iota(\partial_{cd})\,|d^3x|\left\{\frac{1}{2}\epsilon^{bd}\epsilon^{ac}\left[\langle\partial_{ab}\phi,\delta\phi\rangle + \langle\psi_a,\delta\psi_b\rangle\right]\right\} \\
&= -\iota(\partial_\mu)\;|d^3x|\left\{g^{\mu\nu}\langle\partial_\nu\phi,\delta\phi\rangle + \frac{1}{2}\langle\psi\tilde{\Gamma}^\mu\,\delta\psi\rangle\right\}.
\end{aligned}} \tag{4.39}$$

This last expression is written in terms of the basic bilinear forms: the inverse metric $g^{\mu\nu}$ and the paring $\tilde{\Gamma}^\mu$ of (1.2). It agrees with (3.40). As expected, no derivatives of fermions and only one derivative of a boson enter into the formula for $\check{\gamma}$, and the

potential terms do not contribute. We compute $\check{\omega}$ analogously from (4.33), or by simply differentiating (4.39). The results agree:

(4.40)

$$\boxed{\begin{aligned}\check{\omega} &= \iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{bd}\epsilon^{ac}\big[\langle\partial_{ab}\delta\phi\wedge\delta\phi\rangle + \langle\delta\psi_a\wedge\delta\psi_b\rangle + \langle\psi_a\wedge R_\nabla(\delta\phi\wedge\delta\phi)\psi_b\rangle\big]\Big\}\\ &= \iota(\partial_\mu)\;|d^3x|\,\Big\{g^{\mu\nu}\langle\partial_\nu\delta\phi\wedge\delta\phi\rangle + \frac{1}{2}\langle\delta\psi\tilde{\Gamma}^\mu\,\delta\psi\rangle + \frac{1}{2}\langle\psi\tilde{\Gamma}^\mu R_\nabla(\delta\phi\wedge\delta\phi)\psi\rangle\Big\}.\end{aligned}}$$

We compute the global symplectic form Ω on the space of solutions $\mathcal{M}$ by integrating $\check{\omega}$ over a spacelike hypersurface. Using coordinates x^0, x^1, x^2 as in (5.8), with x^0 representing time, and integrating over a constant time, we obtain

(4.41)

$$\boxed{\Omega = \int_{x^0=\text{const}} dx^1dx^2\,\Big\{\langle\delta\dot{\phi}\wedge\delta\phi\rangle - \frac{1}{2}\langle\delta\psi\tilde{\Gamma}^0\delta\psi\rangle + \frac{1}{2}\langle\psi\tilde{\Gamma}^0R_\nabla(\delta\phi\wedge\delta\phi)\psi\rangle\Big\}.}$$

Here '$\dot{\phi}$' denotes the time derivative of ϕ and $\tilde{\Gamma}^0$ is the symmetric form (1.2) on spinors evaluated in the time direction.

We compute the Noether current of the supersymmetry transformation η^aQ_a in two ways. First, we work with superfields—so on $\mathcal{F}_\Phi\times M^{3|2}$—where supersymmetry is manifest. The supersymmetry transformation η^aQ_a induces the vector field

$$\xi = \eta^a\tau_{Q_a} + \widehat{\Xi} \tag{4.42}$$

on $\mathcal{F}_\Phi\times M^{3|2}$, where

$$\widehat{\Xi}\Phi = -\eta^c\tau_{Q_c}\Phi. \tag{4.43}$$

Then

$$J = \iota(\xi)(L_0 + L_1 + \gamma). \tag{4.44}$$

Alternatively, we can work in components—so on $\mathcal{F}_{(\phi,\psi,F)}\times\check{M}^3$—with nonmanifest supersymmetry and compute

$$\check{J} = \iota(\hat{\xi})\check{\gamma} - \alpha, \tag{4.45}$$

where α satisfies

$$\text{Lie}(\hat{\xi})\big(|d^3x|\;(\check{L}_0 + \check{L}_1)\big) = d\alpha \tag{4.46}$$

and may be computed directly from (1.42). Of course, we pass from J to $\check{J}$ by integration (1.32) over the odd variables.

As a first step in computing (4.44) we have

(4.47)

$$\begin{aligned}\iota(\xi)L_0 &= \iota(\eta^c\tau_{Q_c})L_0\\ &= \iota(\eta^c\partial_c + \eta^c\theta^d\partial_{cd})\,|d^3x|\,d^2\theta\;\frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle\\ &= \iota(\partial_c)\,|d^3x|\,d^2\theta\;\frac{1}{4}\epsilon^{ab}\eta^c\langle D_a\Phi, D_b\Phi\rangle + \iota(\partial_{cd})\,|d^3x|\,d^2\theta\;\frac{1}{4}\epsilon^{ab}\eta^c\theta^d\langle D_a\Phi, D_b\end{aligned}$$

Similarly,

$$\iota(\xi)L_1 = \iota(\partial_c)\,|d^3x|\,d^2\theta\;\eta^c\Phi^*(h) + \iota(\partial_{cd})\,|d^3x|\,d^2\theta\;\eta^c\theta^d\Phi^*(h). \tag{4.48}$$

From (4.32) we compute the $(0,|-1|)$ component of $\iota(\xi)\gamma$; the other components play no role.

$$\begin{aligned}\big(\iota(\xi)\gamma\big)^{0,|-1|} &= \iota(\widehat{\Xi})\,\iota(D_a)\,|d^3x|\,d^2\theta\;\frac{1}{2}\epsilon^{ab}\langle D_b\Phi,\delta\Phi\rangle\\ &= \iota(D_a)\,|d^3x|\,d^2\theta\;\Big\{\frac{1}{2}\epsilon^{ab}\langle D_b\Phi,-\eta^c\tau_{Q_c}\Phi\rangle\Big\}\\ &= \iota(\partial_a)\,|d^3x|\,d^2\theta\;\Big\{\frac{1}{2}\epsilon^{ab}\eta^c\langle D_b\Phi,\tau_{Q_c}\Phi\rangle\Big\}\\ &\qquad + \iota(\partial_{ad})\,|d^3x|\,d^2\theta\;\Big\{\frac{1}{2}\epsilon^{ab}\eta^c\theta^d\langle D_b\Phi,\tau_{Q_c}\Phi\rangle\Big\}.\end{aligned} \tag{4.49}$$

The $(0,|-1|)$ component of J is the sum of (4.47), (4.48), and (4.49).

Next, we integrate J over the odd variables using (2.40). The terms with $\iota(\partial_a)$ drop out; only the terms with $\iota(\partial_{ab})$ survive. From (4.47) we have

$$\begin{aligned}\iota(\partial_{cd})\,|d^3x|\;&\int d^2\theta\,\frac{1}{4}\epsilon^{ab}\eta^c\theta^d\langle D_a\Phi,D_b\Phi\rangle\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{8}\epsilon^{ab}\epsilon^{ef}\eta^c i^* D_eD_f\theta^d\langle D_a\Phi,D_b\Phi\rangle\Big\}\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{ab}\epsilon^{de}\eta^c i^*\langle D_eD_a\Phi,D_b\Phi\rangle\Big\}\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{\epsilon^{ab}\epsilon^{de}\eta^c\langle-(\partial_{ae}\phi+\epsilon_{ea}F),\psi_b\rangle\Big\}\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{2}\epsilon^{ab}\epsilon^{de}\eta^c\langle\partial_{ae}\phi,\psi_b\rangle+\frac{1}{2}\epsilon^{db}\eta^c\langle F,\psi_b\rangle\Big\}.\end{aligned} \tag{4.50}$$

From (4.48) we have

$$\begin{aligned}\iota(\partial_{cd})\,|d^3x|\;&\int d^2\theta\,\eta^c\theta^d\Phi^*(h)\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{2}\epsilon^{ab}\eta^c i^* D_aD_b\theta^d\Phi^*(h)\Big\}\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{-\epsilon^{ad}\eta^c i^*\langle\Phi^*\operatorname{grad}h,D_a\Phi\rangle\Big\}\\ &= \iota(\partial_{cd})\,|d^3x|\,\Big\{-\epsilon^{ad}\eta^c\langle\phi^*\operatorname{grad}h,\psi_a\rangle.\Big\}\end{aligned} \tag{4.51}$$

Finally, from (4.49) we compute

(4.52)
$$\begin{aligned}
&\iota(\partial_{cd})\,|d^3x|\int d^2\theta\,\frac{1}{2}\epsilon^{cb}\eta^a\theta^d\langle D_b\Phi,\tau_{Q_a}\Phi\rangle\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{4}\epsilon^{cb}\epsilon^{ef}\eta^a i^*D_eD_f\theta^d\langle D_b\Phi,\tau_{Q_a}\Phi\rangle\Big\}\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{2}\epsilon^{cb}\epsilon^{ef}\eta^a i^*\big[\langle D_eD_b\Phi,\tau_{Q_a}\Phi\rangle-\langle D_b\Phi,D_e\tau_{Q_a}\Phi\rangle\big]\Big\}\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{2}\epsilon^{cb}\epsilon^{ed}\eta^a i^*\big[\langle D_eD_b\Phi,D_a\Phi\rangle+\langle D_b\Phi,D_aD_e\Phi\rangle\big]\Big\}\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{-\frac{1}{2}\epsilon^{ed}(\epsilon^{cb}\eta^a+\epsilon^{ac}\eta^b)i^*\langle D_eD_b\Phi,D_a\Phi\rangle+\epsilon^{cb}\epsilon^{ed}\eta^a i^*\langle\partial_{ea}\Phi,D_b\Phi\rangle\Big\}\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{ed}\epsilon^{ab}\eta^c\langle\partial_{eb}\phi+\epsilon_{ab}F,\psi_a\rangle+\epsilon^{cb}\epsilon^{ed}\eta^a i^*\langle\partial_{ea}\phi,\psi_b\rangle\Big\}.
\end{aligned}$$

Summing (4.50), (4.51), and (4.52) we find after using (4.22) that

$$\check{J}=\iota(\partial_{cd})\,|d^3x|\,\Big\{\epsilon^{bc}\epsilon^{de}\eta^a\langle\partial_{ae}\phi,\psi_b\rangle+\epsilon^{db}\eta^c\langle\phi^*\operatorname{grad}h,\psi_b\rangle\Big\}. \tag{4.53}$$

The supercurrent j_a, which is *minus* the Noether current of the supersymmetry transformation Q_a, is

$$\boxed{j_a=\iota(\partial_{cd})\,|d^3x|\,\Big\{\epsilon^{cb}\epsilon^{de}\langle\partial_{ae}\phi,\psi_b\rangle+\epsilon^{bd}\delta^c_a\langle\phi^*\operatorname{grad}h,\psi_b\rangle\Big\}.} \tag{4.54}$$

As a check, we compute $\check{J}$ directly using (4.45). Using (4.39) we compute the first term of (4.45):

(4.55)
$$\begin{aligned}
\iota(\hat{\xi})\check{\gamma}&=\frac{1}{2}\epsilon^{ac}\epsilon^{bd}\iota(\partial_{cd})\,|d^3x|\,\Big\{\langle\partial_{ab}\phi,-\eta^e\psi_e\rangle+\langle\psi_b,\eta^e(\partial_{ae}\phi-\epsilon_{ae}F)\rangle\Big\}\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{ac}(\epsilon^{db}\eta^e+\epsilon^{de}\eta^b)\langle\partial_{ab}\phi,\psi_e\rangle-\frac{1}{2}\epsilon^{ec}\eta^d\langle\phi^*\operatorname{grad}h,\psi_e\rangle\Big\}.
\end{aligned}$$

Note we use (4.22). Next, we compute α directly from (1.42), setting $\Delta=0$:

(4.56)
$$\begin{aligned}
\alpha&=\iota(\partial_\mu)\,|d^3x|\,\Gamma^\mu_{ab}(-\frac{1}{2}\epsilon^{ef})i^*D_eD_f\eta^a\theta^b\big[\frac{1}{4}\epsilon^{gh}\langle D_g\Phi,D_h\Phi\rangle+\Phi^*(h)\big]\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{ef}\eta^c i^*D_eD_f\theta^d\big[\frac{1}{4}\epsilon^{ab}\langle D_a\Phi,D_b\Phi\rangle+\Phi^*(h)\big]\Big\}\\
&=\iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{ed}\eta^c i^*\big[\epsilon^{ab}\langle D_eD_a\Phi,D_b\Phi\rangle+2\langle\Phi^*\operatorname{grad}h,D_e\Phi\rangle\big]\Big\}\\
&=-\iota(\partial_{cd})\,|d^3x|\,\Big\{\frac{1}{2}\epsilon^{ab}\epsilon^{ed}\eta^c\langle\partial_{ae}\phi,\psi_b\rangle-\frac{1}{2}\epsilon^{ec}\eta^d\langle\phi^*\operatorname{grad}h,\psi_e\rangle\big]\Big\}.
\end{aligned}$$

The first term agrees with (3.29). Combining (4.55) and (4.56) we recover (4.53).

§4.5. Reduction to $M^{2|(1,1)}$

We can reduce the model to $n = 2$ dimensions either in superspace or in components. In both cases we consider the $n = 3$ dimensional theory with fields constrained to be invariant under ∂_{12}. All the formulas worked out in $n = 3$ dimensions hold, but there are simplifications. We already wrote in (4.15) the supersymmetry transformations. We record here the supercharges $\tilde{Q}_+, \tilde{Q}_-$ obtained by integrating the Noether current (4.54) over space:

$$\begin{aligned} \tilde{Q}_+ &= \int_{x^0=\text{const}} \Big\{ \langle \partial_+\phi, \psi_+ \rangle - \langle \phi^* \operatorname{grad} h, \psi_- \rangle \Big\} \, |dx^1| \\ \tilde{Q}_- &= \int_{x^0=\text{const}} \Big\{ \langle \partial_-\phi, \psi_- \rangle + \langle \phi^* \operatorname{grad} h, \psi_+ \rangle \Big\} \, |dx^1|. \end{aligned} \tag{4.57}$$

In the abstract supersymmetry algebra (1.15) the Lie bracket $[Q_+, Q_-]$ of the supersymmetry generators vanishes. However, the Poisson bracket $\{\tilde{Q}_+, \tilde{Q}_-\}$ of the Noether charges is nonzero in this case; it is a locally constant function on the space of classical solutions. We compute it by acting $\widehat{Q}_+$ on $\tilde{Q}_-$, where $\widehat{Q}_+$ is the Hamiltonian vector field which corresponds to the function $\tilde{Q}_+$; its action on fields is given in (4.15):

$$\begin{aligned} \widehat{Q}_+\phi &= -\psi_+ \\ \widehat{Q}_+\psi_+ &= \partial_+\phi \\ \widehat{Q}_+\psi_- &= -\phi^* \operatorname{grad} h \\ \widehat{Q}_+(\phi^* \operatorname{grad} h) &= \nabla_+\psi_- + R(\psi_+, \psi_-)\psi_+. \end{aligned} \tag{4.58}$$

In the following computation most terms drop out after we use the equation of motion (4.34) and the Bianchi identity:

(4.59)

$$\begin{aligned} \frac{1}{2}\{\tilde{Q}_+, \tilde{Q}_-\} &= \frac{1}{2}\int_{x^0=\text{const}} \Big\{ -\langle \partial_-\phi, \phi^* \operatorname{grad} h \rangle + \langle \partial_+\phi, \phi^* \operatorname{grad} h \rangle \Big\} \, |dx^1| \\ &= \int_{x^0=\text{const}} \langle \partial_1\phi, \phi^* \operatorname{grad} h \rangle \, |dx^1| \\ &= \int_{x^0=\text{const}} \partial_1(\phi^* h) \, |dx^1| \\ &= h\big(\phi(\infty)\big) - h\big(\phi(-\infty)\big). \end{aligned}$$

A scalar field $\phi = \phi(x^1)$ depending on the spatial variable has finite energy only if it has limits as $x^1 \to \pm\infty$ which are critical points of h. The Poisson bracket (4.59) measures the difference of the critical values. It induces a central extension of the abstract supersymmetry algebra.

CHAPTER 5
The Supersymmetric σ-Model in Dimension 4

In this chapter we consider the σ-model with 4 supersymmetries, which was described in components in Chapter 3. Here we give a manifestly supersymmetric treatment in superspace. In many respects the treatment parallels that of Chapter 4, so we are briefer here. (One new feature over theories with 2 supersymmetries is the presence of an R-symmetry.) Although the model is well-defined and supersymmetric for any Kähler target manifold X, the superspace model is nicest when X is Hodge. In that case we give a global superspace lagrangian (see (5.22)). In these theories the scalar potential is the norm square of the gradient of a holomorphic function called the *superpotential*. We describe it in §5.3, and in particular prove Theorem 3.82(ii). We do not give a detailed analysis of the classical theory, but simply summarize the most important equations in §5.4.

§5.1. Fields and supersymmetry transformations on $M^{4|4}$

Linear Case. The simplest superfield for which we can write sensible lagrangians is the (linear) *chiral superfield*

$$\Phi\colon M^{4|4} \longrightarrow \mathbb{C}, \tag{5.1}$$

which is required to satisfy

$$\overline{D}_{\dot{a}}\Phi = 0. \tag{5.2}$$

The complex conjugate of a chiral superfield is called an *antichiral multiplet.* Define the component fields as

$$\boxed{\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= \frac{1}{\sqrt{2}} i^* D_a \Phi \\ F &= \frac{1}{2} i^*(-D^2)\Phi. \end{aligned}} \tag{5.3}$$

Component fields for the antichiral superfield $\overline{\Phi}$ are defined by the conjugate equations to (5.3). Note that F is a *complex* scalar field.

Let $\hat{\xi}$ be the vector field on the space $\mathcal{F}_{(\phi,\psi,F)}$ of component fields induced by the supersymmetry transformation $\eta^a Q_a + \bar{\eta}^{\dot{a}} \overline{Q}_{\dot{a}}$. Here $\eta^a, \bar{\eta}^{\dot{a}}$ are independent odd parameters. The action on the components of Φ is computed by similar manipulations as in (4.3)–(4.5). For example, using (2.73) and (2.75) we compute

$$\begin{aligned} \hat{\xi}F &= -i^*(\eta^a D_a + \bar{\eta}^{\dot{a}} \overline{D}_{\dot{a}})(\tfrac{1}{2} D^2)\Phi \\ &= -\bar{\eta}^{\dot{a}} \epsilon^{bc} i^* \partial_{b\dot{a}} D_c \Phi \\ &= -\sqrt{2} \bar{\eta}^{\dot{a}} \epsilon^{bc} \partial_{b\dot{a}} \psi_c \\ &= \sqrt{2} \bar{\eta}^{\dot{a}} (\not{D}\psi)_{\dot{a}}. \end{aligned} \tag{5.4}$$

Collecting the results of similar computations, we display the supersymmetry transformation laws:

$$\boxed{\begin{aligned} \hat{\xi}\phi &= -\sqrt{2}\eta^a \psi_a \\ \hat{\xi}\psi_a &= \sqrt{2}(\bar{\eta}^{\dot{b}} \partial_{a\dot{b}} \phi - \eta^b \epsilon_{ab} F) \\ \hat{\xi}F &= \sqrt{2} \bar{\eta}^{\dot{a}} (\not{D}\psi)_{\dot{a}}. \end{aligned}} \tag{5.5}$$

The transformation laws for the components of $\overline{\Phi}$ are obtained by conjugating the equations in (5.5).

Nonlinear Case. The nonlinear chiral superfield is a map

$$\Phi \colon M^{4|4} \longrightarrow X, \tag{5.6}$$

where X is a complex manifold and Φ is required to pullback holomorphic functions on X to linear chiral superfields on $M^{4|4}$. Another way to state this is that $D_a\Phi$ is a vector field of type $(1,0)$, and so $\overline{D}_{\dot{a}}\overline{\Phi}$ is a vector field of type $(0,1)$. Again the composition of a chiral superfield and a holomorphic function is a chiral superfield. The conjugate conditions define a nonlinear antichiral superfield. To define the component fields we assume that X is Kähler[29] and use the Levi-Civita covariant derivative as in (4.7):

$$\boxed{\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= \frac{1}{\sqrt{2}} i^* D_a \Phi \\ F &= \frac{1}{2} i^*(-\mathcal{D}^2)\Phi. \end{aligned}} \tag{5.7}$$

Note that ψ_a and F have type $(1,0)$. The conjugate fields $\bar{\psi}_{\dot{a}}$ and $\overline{F}$ have type $(0,1)$.

Now the analogs of equations (2.74) and (2.75) for covariant derivatives of nonlinear chiral superfields are easily derived:

$$\mathcal{D}_a \overline{\mathcal{D}}^2 \Phi = \overline{\mathcal{D}}^2 D_a \Phi = 0, \tag{5.8}$$

$$\overline{\mathcal{D}}_{\dot{a}} \mathcal{D}^2 \Phi = -2\epsilon^{bc} \Big\{ \nabla_{b\dot{a}} - \frac{1}{4} R(D_b\Phi, \overline{D}_{\dot{a}}\overline{\Phi}) \Big\} D_c \Phi. \tag{5.9}$$

[29]In any case X must be Kähler to define the σ-model action.

The conjugate equations to (5.8) and (5.9) give information about covariant derivatives of nonlinear antichiral superfields. There is only one curvature term which alters (5.5) in the nonlinear case:

$$\boxed{\begin{aligned} \hat{\xi}\phi &= -\sqrt{2}\eta^a\psi_a \\ \hat{\xi}\psi_a &= \sqrt{2}(\bar{\eta}^{\dot{b}}\partial_{a\dot{b}}\phi - \eta^b\epsilon_{ab}F) \\ \hat{\xi}F &= \sqrt{2}\bar{\eta}^{\dot{a}}\big[(\not{D}\psi)_{\dot{a}} + \frac{1}{4}\epsilon^{bc}R(\psi_b, \bar{\psi}_{\dot{a}})\psi_c\big]. \end{aligned}} \tag{5.10}$$

§5.2. The σ-model action on $M^{4|4}$

Linear Case. Here it is worth computing separately for the linear superfield (5.1). The lagrangian density on $M^{4|4}$ is

$$\mathcal{L}_0 = |d^4x|\, d^4\theta\, \frac{1}{4}\overline{\Phi}\Phi. \tag{5.11}$$

Recall from the discussion following (2.72) that we compute component lagrangians by integrating over the θ's and adding the wave operator. Hence by (2.72)

(5.12)

$$\begin{aligned} 4\check{\mathcal{L}}_0 &= \left\{\int d^4\theta + \Box i^*\right\}\overline{\Phi}\Phi \\ &= \frac{1}{2}i^*(D^2\overline{D}^2 + \overline{D}^2D^2 + 4\Box)\overline{\Phi}\Phi \\ &= \frac{1}{2}i^*\big\{D^2(\overline{D}^2\overline{\Phi}\cdot\Phi) + \overline{D}^2(\overline{\Phi}\cdot D^2\Phi) + 4\Box(\overline{\Phi}\Phi)\big\} \\ &= \frac{1}{2}i^*\big\{(D^2\overline{D}^2 + 4\Box)\overline{\Phi}\cdot\Phi + \overline{\Phi}\cdot(\overline{D}^2D^2 + 4\Box)\Phi + 8\langle d\overline{\Phi}, d\Phi\rangle \\ &\qquad + 2\,\overline{D}^2\overline{\Phi}\cdot D^2\Phi + \epsilon^{ab}D_a\overline{D}^2\overline{\Phi}\cdot D_b\Phi - \epsilon^{\dot{a}\dot{b}}\overline{D}_{\dot{b}}\overline{\Phi}\cdot\overline{D}_{\dot{a}}D^2\Phi\big\} \\ &= \frac{1}{2}\big\{8\langle d\bar{\phi}, d\phi\rangle + 8\overline{F}F + 4\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\psi_b\partial_{a\dot{c}}\bar{\psi}_{\dot{d}} + 4\epsilon^{\dot{a}\dot{b}}\epsilon^{cd}\bar{\psi}_{\dot{b}}\partial_{c\dot{a}}\psi_d\big\}, \end{aligned}$$

where we use (2.74)–(2.77) and the definition (5.7) of component fields in the last step. Finally, by (2.56)

$$\boxed{\check{\mathcal{L}}_0 = \langle d\bar{\phi}, d\phi\rangle + \frac{1}{2}(\bar{\psi}\not{D}\psi + \psi\not{D}\bar{\psi}) + \overline{F}F.} \tag{5.13}$$

The Dirac form in (5.13) is real, but up to an exact term we can write it as the complex form $\bar{\psi}\not{D}\psi$:

$$\frac{1}{2}(\bar{\psi}\not{D}\psi + \psi\not{D}\bar{\psi}) = \bar{\psi}\not{D}\psi - \frac{1}{2}\partial_\mu(\bar{\psi}\Gamma^\mu\psi). \tag{5.14}$$

We drop this exact term in subsequent formulas. As in 3 dimensions, F enters the lagrangian (5.13) as an auxiliary field and can be eliminated through its equation of motion $F = 0$.

Nonlinear Case. Now we carry out the same computation for the nonlinear chiral superfield (5.6) with values in a Kähler manifold X. The superfield expression for the lagrangian depends on the choice of a local Kähler potential K, which is a locally defined real-valued function on X such that the Kähler form is

$$\omega = \sqrt{-1}\,\partial\bar{\partial}K \tag{5.15}$$

and the metric is

$$\langle \cdot, \cdot \rangle = 2\sqrt{-1}\,\omega = 2\bar{\partial}\partial K, \tag{5.16}$$

viewed as a *bilinear* form on $\overline{TX} \otimes TX$. For example, on $\mathbb{C}$ we have a global Kähler potential $K(\bar{z}, z) = \frac{1}{2}\|z\|^2$. We use double bar notation $\|\cdot\|$ throughout for the norm on a hermitian vector space.

The lagrangian density on $M^{4|4}$ is

$$\boxed{\mathcal{L}_0 = |d^4x|\, d^4\theta\, \frac{1}{2} K(\overline{\Phi}, \Phi).} \tag{5.17}$$

Although it depends on a choice of Kähler potential, the component lagrangian makes sense globally by the following argument. Consider a change $K \to K + f + \bar{f}$ for f a holomorphic function on X. Of course, it does not affect the Kähler form (5.15) and is exactly the ambiguity in the choice of Kähler potential. Now since we compute the component lagrangian using the first term of (2.71)—the integral over θ plus the wave operator—and since that operation annihilates chiral and antichiral superfields, such a change in K does not affect the component lagrangian.

There is a global formulation of the lagrangian density $\mathcal{L}_0$ on superspace when X is a *Hodge* manifold. A Hodge manifold is a Kähler manifold X together with a holomorphic hermitian line bundle $L_\omega \to X$ whose curvature is related to the Kähler form by

$$\operatorname{curv}(L_\omega) = \sqrt{-1}\,\omega. \tag{5.18}$$

Recall that if s is a local holomorphic section of $L_\omega \to X$, then

$$\operatorname{curv}(L_\omega) = \bar{\partial}\partial \log \|s\|^2, \tag{5.19}$$

and so $\log \|s\|^2$ is a local Kähler potential. Consider a chiral superfield $\Phi\colon M^{4|4} \to X$. Since Minkowski space is contractible, there is a *global* lift to a chiral superfield[30]

$$\tilde{\Phi}\colon M^{4|4} \longrightarrow \Phi^* L_\omega \tag{5.21}$$

which is a section of $\Phi^* L_\omega \to M^{4|4}$. For any such lift we write the lagrangian (5.17) globally as

$$\boxed{\mathcal{L}_0 = |d^4x|\, d^4\theta\, \frac{1}{2} \log \|\tilde{\Phi}\|^2.} \tag{5.22}$$

[30]The existence of a global lift depends on the fact that

$$H^1(M^{4|4}, \mathcal{C}^*) = 0, \tag{5.20}$$

where $\mathcal{C}^*$ is the sheaf of nonzero chiral superfields. The vanishing (5.20) is obvious in components.

By the argument in the previous paragraph the corresponding component lagrangian is independent of the choice of lift $\tilde{\Phi}$. We remark that it is not unfamiliar to have an action whose lagrangian density depends on an auxiliary map: consider, for example, the usual formulation of the Wess-Zumino-Witten term in two or four dimensions.

As a preliminary to computing the component lagrangian from (5.17), note that the Cartan formula $\mathrm{Lie}(\eta) = d\iota(\eta) + \iota(\eta)d$ for the Lie derivative, valid for all vector fields η, implies the following equations when acting on 1-forms:

$$\iota(\zeta)\iota(\zeta')d = -\mathrm{Lie}(\zeta)\iota(\zeta') - \mathrm{Lie}(\zeta')\iota(\zeta) + \iota([\zeta,\zeta']), \qquad \zeta,\zeta' \text{ odd}; \tag{5.23}$$

$$\iota(\zeta)\iota(\xi)d = \mathrm{Lie}(\xi)\iota(\zeta) - \mathrm{Lie}(\zeta)\iota(\xi) + \iota([\zeta,\xi]), \qquad \zeta \text{ odd}, \ \xi \text{ even}. \tag{5.24}$$

We use these formulas several times in the next computation.

We choose (2.72) to compute the component lagrangian, though (2.71) also leads to a straightforward computation. Thus we expand (5.17):

(5.25)

$$\begin{aligned}
\overline{\mathcal{D}}^2\mathcal{D}^2\Phi^*K &= \frac{1}{2}\epsilon^{ab}\overline{\mathcal{D}}^2 D_a\iota(D_b\Phi)\partial K \\
&= \frac{1}{4}\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}(-\mathcal{D}_a\overline{D}_{\dot{d}} - 2\partial_{a\dot{d}})\iota(D_b\Phi)\partial K \\
&= \frac{1}{4}\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}\Big\{D_a\iota(\overline{D}_{\dot{d}}\overline{\Phi})\iota(D_b\Phi)\bar{\partial}\partial K + 2D_a\iota(\partial_{b\dot{d}}\Phi)\partial K \\
&\qquad\qquad - 2\iota(D_b\Phi)\iota(\partial_{a\dot{d}}\Phi)\bar{\partial}\partial K - 2D_b\iota(\partial_{a\dot{d}}\Phi)\partial K\Big\} \\
&= \frac{1}{4}\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}\Big\{-\frac{1}{2}D_a\langle\overline{D}_{\dot{d}}\overline{\Phi}, D_b\Phi\rangle - \langle\partial_{a\dot{d}}\overline{\Phi}, D_b\Phi\rangle + 4D_a\iota(\partial_{b\dot{d}}\Phi)\partial K\Big\} \\
&= \frac{1}{4}\epsilon^{\dot{c}\dot{d}}\overline{D}_{\dot{c}}\langle\overline{D}_{\dot{d}}\overline{\Phi}, \mathcal{D}^2\Phi\rangle + \epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}D_a\iota(\partial_{b\dot{d}}\Phi)\partial K.
\end{aligned}$$

The first term is

(5.26)

$$\begin{aligned}
\frac{1}{4}\epsilon^{\dot{c}\dot{d}}\overline{D}_{\dot{c}}\langle\overline{D}_{\dot{d}}\overline{\Phi}, \mathcal{D}^2\Phi\rangle &= \frac{1}{2}\langle\overline{\mathcal{D}}^2\overline{\Phi}, \mathcal{D}^2\Phi\rangle - \frac{1}{4}\epsilon^{\dot{c}\dot{d}}\langle\overline{D}_{\dot{d}}\overline{\Phi}, \overline{\mathcal{D}}_{\dot{c}}\mathcal{D}^2\Phi\rangle \\
&= \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\Big\{\langle\overline{D}_{\dot{d}}\overline{\Phi}, \nabla_{a\dot{c}}D_b\Phi\rangle \\
&\qquad - \frac{1}{4}\langle\overline{D}_{\dot{d}}\overline{\Phi}, R(D_a\Phi, \overline{D}_{\dot{c}}\overline{\Phi})D_b\Phi\rangle\Big\} + \frac{1}{2}\langle\overline{\mathcal{D}}^2\overline{\Phi}, \mathcal{D}^2\Phi\rangle.
\end{aligned}$$

The second term in (5.25) is

(5.27)

$$\begin{aligned}
\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}D_a\iota(\partial_{b\dot{d}}\Phi)\partial K &= \epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\Big\{-\mathcal{D}_a\overline{D}_{\dot{c}}\iota(\partial_{b\dot{d}}\Phi)\partial K - 2\partial_{a\dot{c}}\iota(\partial_{b\dot{d}}\Phi)\partial K\Big\} \\
&= \epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\Big\{-\frac{1}{2}D_a\langle\overline{D}_{\dot{c}}\overline{\Phi}, \partial_{b\dot{d}}\Phi\rangle - 2\partial_{a\dot{c}}\iota(\partial_{b\dot{d}}\Phi)\partial K\Big\} \\
&= \epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\Big\{\frac{1}{2}\langle\overline{D}_{\dot{c}}\overline{\Phi}, \nabla_{b\dot{d}}D_a\Phi\rangle + \langle\partial_{a\dot{c}}\overline{\Phi}, \partial_{b\dot{d}}\Phi\rangle \\
&\qquad\qquad - 2\partial_{a\dot{c}}\iota(\partial_{b\dot{d}}\Phi)\partial K\Big\}.
\end{aligned}$$

Combining (5.25)–(5.27) we have

$$\begin{aligned}(5.28)\quad &\frac{1}{2}\big(\overline{\mathcal{D}}^2\mathcal{D}^2 + 2\epsilon^{ab}\epsilon^{\dot c\dot d}\partial_{a\dot c}\iota(\partial_{b\dot d}\Phi)\partial K\big)\big(\frac{1}{2}\Phi^*K\big)\\ &\qquad = \epsilon^{ab}\epsilon^{\dot c\dot d}\Big\{\frac{1}{4}\langle\partial_{b\dot d}\overline{\Phi},\partial_{a\dot c}\Phi\rangle + \frac{1}{4}\langle\overline{D}_{\dot d}\overline{\Phi},\nabla_{a\dot c}D_b\Phi\rangle\\ &\qquad\qquad - \frac{1}{32}\langle\overline{D}_{\dot d}\overline{\Phi},R(D_a\Phi,\overline{D}_{\dot c}\overline{\Phi})D_b\Phi\rangle\Big\} + \frac{1}{8}\langle\overline{\mathcal{D}}^2\overline{\Phi},\mathcal{D}^2\Phi\rangle.\end{aligned}$$

Now we add (5.28) to its complex conjugate. Since

$$(5.29)\qquad 2\epsilon^{ab}\epsilon^{\dot c\dot d}\partial_{a\dot c}\iota(\partial_{b\dot d}\Phi)(\partial K + \bar\partial K) = 2\epsilon^{ab}\epsilon^{\dot c\dot d}\partial_{a\dot c}\partial_{b\dot d}(\Phi^*K) = 4\Box(\Phi^*K),$$

the left hand side restricted to $\check M^4$ is the component lagrangian (as in the second line of (5.12)), and so finally

$$(5.30)\qquad \boxed{\check{\mathcal{L}}_0 = \langle\overline{d\phi},d\phi\rangle + \langle\bar\psi\not{D}_\phi\psi\rangle - \frac{1}{4}\epsilon^{ab}\epsilon^{\dot c\dot d}\langle\bar\psi_{\dot d},\phi^*R(\psi_a,\bar\psi_{\dot c})\psi_b\rangle + \langle\overline{F},F\rangle.}$$

Recall that we have absorbed an exact term (5.14) in writing the Dirac form. With this understood, the lagrangian (5.30) agrees with (3.48) after eliminating the auxiliary field.

We check in Problem FP10(c) of [I-**Homework**] that the lagrangian (5.17) on $M^{4|4}$ reduces to (4.16) in $M^{3|2}$.

A new aspect in four dimensions is the $U(1)$ R-symmetry (2.81). We claim that the σ-model lagrangian is invariant under R. First, in the superspace expression (5.17) (or (5.22)) we note that $d^4\theta = d^2\theta\,d^2\bar\theta$ is invariant. So if R acts trivially on Φ, then $\mathcal{L}_0$ is obviously invariant. When R acts nontrivially it is easier to argue in components. Suppose for simplicity that Φ takes values in $\mathbb{C}$, and let R acts on Φ with weight k. Then

$$(5.31)\qquad \boxed{\begin{array}{ll} R\phi = k\phi, & R\bar\phi = -k\bar\phi\\ R\psi = (k+1)\psi, & R\bar\psi = -(k+1)\bar\psi\\ RF = (k+1)F, & R\overline{F} = -(k+2)\overline{F}.\end{array}}$$

Direct inspection of (5.30) shows that $\check{\mathcal{L}}_0$ is R-invariant.

§5.3. The superpotential term on $M^{4|4}$

As a preliminary, note that if Λ is any chiral superfield, then $|d^4x|\int d^2\theta\,\Lambda$ makes sense as a density on Minkowski space. Similarly, for $\overline{\Lambda}$ antichiral we can define $|d^4x|\int d^2\bar\theta\,\overline{\Lambda}$. One view is that $M^{4|4}$ is a split cs-manifold with a complex two dimensional odd tangent bundle. In fact, there are two such cs-manifolds, one for each of the complex structures on the real four dimensional spin representation (see (2.2)). These cs-manifolds have canonical densities—denoted $|d^4x|\,d^2\theta$ (resp. $|d^4x|\,d^2\bar\theta$)—and global functions on these cs-manifolds are chiral (resp. antichiral) superfields. To compute these chiral integrals we use (2.40).

The distinction between terms in the action of the form $\int|d^4x|\,d^4\theta$ and $\int|d^4x|\,d^2\theta$ is important in $M^{4|4}$ and also in $M^{2|(2,2)}$.

We introduce a superpotential term, and we may as well work in the nonlinear σ-model. Let $W\colon X \to \mathbb{C}$ be a *holomorphic* function, and set

$$\boxed{\mathcal{L}_1 = |d^4x|\ \mathrm{Re}\Big\{d^2\theta\ \Phi^*(W)\Big\}.} \tag{5.32}$$

It lives in the direct sum of the spaces of chiral and antichiral densities. The superpotential term in the component action is the integral of $\mathcal{L}_1$:

$$\int \mathcal{L}_1 = \int |d^4x|\ \frac{1}{2}\left\{\int d^2\theta\ \Phi^*(W) + \int d^2\bar\theta\ \Phi^*(\overline{W})\right\} = \int |d^4x|\ \check{\mathcal{L}}_1. \tag{5.33}$$

We compute $\check{\mathcal{L}}_1$ exactly as in (4.21); recalling our normalizations (5.7) for the component fields we obtain

$$\check{\mathcal{L}}_1 = \iota(F)\phi^*\partial W + \iota(\overline{F})\phi^*\overline{\partial W} - \frac{1}{2}\epsilon^{ab}\phi^*\nabla\partial W(\psi_a,\psi_b) - \frac{1}{2}\epsilon^{\dot a\dot b}\phi^*\nabla\overline{\partial W}(\bar\psi_{\dot a},\bar\psi_{\dot b}). \tag{5.34}$$

We eliminate the auxiliary fields $F, \overline{F}$ from $\check{\mathcal{L}}_0 + \check{\mathcal{L}}_1$ using their equations of motion

$$\boxed{\begin{aligned} F &= -\phi^*\,\mathrm{grad}\,W \\ \overline{F} &= -\phi^*\overline{\mathrm{grad}\,W}, \end{aligned}} \tag{5.35}$$

where now $\overline{\mathrm{grad}\,W}$ is the vector field of type $(0,1)$ corresponding to the $(1,0)$-form ∂W using the hermitian metric on X. The total lagrangian with auxiliary fields eliminated is then

$$\boxed{\begin{aligned} \check{\mathcal{L}}_0 + \check{\mathcal{L}}_1 \sim\ & \langle \overline{d\phi}, d\phi\rangle + \langle\bar\psi \not{D}_\phi \psi\rangle - \frac{1}{4}\epsilon^{ab}\epsilon^{\dot c\dot d}\langle\bar\psi_{\dot d}, \phi^*R(\psi_a,\bar\psi_{\dot c})\psi_b\rangle \\ & - \phi^*\|\,\mathrm{grad}\,W\|^2 - \Big(\frac{1}{2}\epsilon^{ab}\phi^*\nabla\partial W(\psi_a,\psi_b) - \frac{1}{2}\epsilon^{\dot a\dot b}\phi^*\nabla\overline{\partial W}(\bar\psi_{\dot a},\bar\psi_{\dot b})\Big). \end{aligned}} \tag{5.36}$$

This is the lagrangian (3.85), and since the superspace model is manifestly supersymmetric we have proved Theorem 3.82(ii). Of course, the potential energy in (5.36) is

$$V = \|\,\mathrm{grad}\,W\|^2. \tag{5.37}$$

Other aspects of the discussion following (4.23) carry over directly.

Finally, we consider the R-symmetry (2.81). We already noted that it leaves $\mathcal{L}_0$ and $\check{\mathcal{L}}_0$ invariant, even if Φ is assigned a nonzero weight. On the other hand, $d^2\theta$ has weight 2 and so if $\mathcal{L}_1$ is to be invariant we must be able to assign Φ a weight so that $\Phi^*(W)$ has weight -2. (In general, there are several Φ_i with different weights.) Such a W is termed *quasi-homogeneous* in the physics literature.

§5.4. Analysis of the classical theory

We do not give details, having done so exhaustively in §4 for the σ-model on $M^{3|2}$. Rather, we simply record the formulas

$$\boxed{\check\gamma = \iota(\partial_{c\dot d})\,|d^4x|\ \frac{1}{\sqrt{2}}\epsilon^{ac}\epsilon^{\dot b\dot d}\big[\langle\partial_{a\dot b}\phi, \delta\bar\phi\rangle + \langle\partial_{a\dot b}\bar\phi, \delta\phi\rangle + \langle\psi_a, \delta\bar\psi_{\dot b}\rangle + \langle\bar\psi_{\dot b}, \delta\psi_a\rangle\big]} \tag{5.38}$$

for the variational 1-form,

$$\boxed{\begin{aligned}\check{\omega} = \iota(\partial_{c\dot{d}})\,|d^4x|\,\frac{1}{\sqrt{2}}\epsilon^{ac}\epsilon^{\dot{b}\dot{d}}\big[&\langle\partial_{a\dot{b}}\delta\phi\wedge\delta\bar{\phi}\rangle + \langle\partial_{a\dot{b}}\delta\bar{\phi}\wedge\delta\phi\rangle \\ &+ \langle\delta\psi_a\wedge\delta\bar{\psi}_{\dot{b}}\rangle + \langle\delta\bar{\psi}_{\dot{b}}\wedge\delta\psi_a\rangle\big]\end{aligned}} \tag{5.39}$$

for the local symplectic form, and

$$\boxed{j_a = \iota(\partial_{c\dot{d}})\,|d^4x|\,\Big\{\sqrt{2}\epsilon^{cb}\epsilon^{\dot{d}\dot{e}}\langle\partial_{a\dot{e}}\phi,\psi_b\rangle + \sqrt{2}\,\epsilon^{\dot{b}\dot{d}}\delta_a^c\langle\phi^*\,\mathrm{grad}\,W,\bar{\psi}_{\dot{b}}\rangle\Big\}} \tag{5.40}$$

for the supercurrent corresponding to the supersymmetry transformation Q_a

CHAPTER 6
Supersymmetric Yang-Mills Theories

Now we consider theories with gauge fields (connections). In the bosonic case there is a Yang-Mills theory in any dimension. The supersymmetric extension involves a *dual* spinor field as well, and it exists in dimensions $n = 3, 4, 6, 10$ with minimal supersymmetry. Our treatment in §6.1 is based on a special property of spinors which holds only in these cases: the quadratic form $s^* \mapsto \Gamma(s^*, s^*)$ on S^* takes values on the light cone. In §6.2 we describe the supersymmetric extensions of the most general bosonic theory, which is variously called a *gauge theory with matter* or a *gauged σ-model*. We consider theories in dimensions $n = 3, 4, 6$ with minimal supersymmetry. (The only theory in $n = 10$ is pure Yang-Mills.) The basic lagrangians, supersymmetry transformations, and supercurrents are summarized in Theorem 6.33. There are new terms which appear neither in the pure σ-model nor in the pure Yang-Mills theory. There is a superspace formulation of supersymmetric gauge theories in dimensions $n = 3, 4, 6$ (with minimal supersymmetry); we describe some common features in §6.3. The condition on the quadratic form has a geometric significance here as well. Details of the superspace formulation occupy the remaining chapters.

§6.1. The minimal theory in components

The bosonic Yang-Mills theory is determined by the data:

(6.1)
$$\begin{array}{ll} G & \text{compact Lie group} \\ \langle \cdot, \cdot \rangle & \text{bi-invariant scalar product on } \mathfrak{g} \end{array}$$

The lagrangian in n-dimensional Minkowski space is

(6.2)
$$L = -\frac{1}{2}|F_A|^2 \, |d^n x|,$$

where A is a connection on a principal G bundle P. The minimal supersymmetric extension has a single dual spinor field λ with values in the adjoint bundle and lagrangian

(6.3)
$$\boxed{L = \{-\frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \lambda \not{D}_A \lambda \rangle\} \, |d^n x|.}$$

We ask: When is (6.3) supersymmetric? That is, in which dimensions and for what type of spinor field λ is (6.3) supersymmetric? Consider first the abelian theory, which is free. Suppose the spinor field has s components. Upon quantization,[31] the gauge field A has $n-2$ physical degrees of freedom and the spinor field has $s/2$ physical degrees of freedom. With one exception (explained presently) the number of bosonic and fermionic degrees of freedom in a supersymmetric theory must be equal, from which $n = 2+s/2$. This is satisfied only for $n = 3, 4, 6, 10$ with minimal supersymmetry $s = 2, 4, 8, 16$.

Theorem 6.4. *The super Yang-Mills lagrangian* (6.3) *is supersymmetric in* $n = 3, 4, 6, 10$ *dimensions with minimal supersymmetry.*

There is an additional supersymmetric theory—the exception referred to above—in $n = 2$ dimensions with $s = (1, 0)$ supersymmetry; we will not consider it. The quantum argument above shows that (6.3) cannot be supersymmetric in other dimensions and with other amounts of supersymmetry.

Proof. We work in n-dimensional Minkowski space and fix a real spin representation S. Thus we have pairings (1.1), (1.2) which satisfy the Clifford relation (1.3). The spinor field λ takes values in $S^* \otimes \operatorname{ad} P$. The lagrangian (6.3) is

$$\boxed{L = \left(-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\langle F_{\mu\nu}, F_{\sigma\rho}\rangle + \frac{1}{2}\Gamma^{\sigma}_{ab}\langle\lambda^a, \nabla_\sigma\lambda^b\rangle\right)|d^n x|,} \tag{6.5}$$

where $\langle\cdot,\cdot\rangle$ is the invariant inner product on the gauge algebra, the covariant derivative is $\nabla_\sigma = \partial_\sigma + A_\sigma$, and the curvature is

$$F_A = \frac{1}{2}F_{\mu\nu}\,dx^\mu \wedge dx^\nu, \tag{6.6}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{6.7}$$

We work with *real* bases for spinors and vectors. We postulate the induced action of the supersymmetry transformation $\eta^a Q_a$ on the fields to be

$$\boxed{\begin{aligned}(\hat{\xi}A)_\mu &= \eta^a g_{\mu\nu}\Gamma^\nu_{ab}\lambda^b \\ \hat{\xi}\lambda^a &= \frac{1}{2}\eta^b\tilde{\Gamma}^{\mu ac}\Gamma^\nu_{cb}F_{\mu\nu}.\end{aligned}} \tag{6.8}$$

Each side of the first equation is a component of a 1-form with values in the adjoint bundle. These equations are motivated by Lorentz invariance, gauge invariance, parity, and power counting.

There are two things we need to check: (i) $\hat{\xi}$ is a (nonmanifest) symmetry of L; and (ii) the Lie bracket of transformations (6.8) is consistent with (1.30). We will find the conditions under which (i) and (ii) hold.

[31] The free quantum theory is only used here to rule out some cases; otherwise the argument is entirely classical.

First, we investigate $\mathrm{Lie}(\hat{\xi})L$. The variation of the curvature is

$$\boxed{\hat{\xi}F_{\mu\nu} = \eta^a g_{\nu\sigma}\Gamma^\sigma_{ab}\nabla_\mu\lambda^b - \eta^a g_{\mu\sigma}\Gamma^\sigma_{ab}\nabla_\nu\lambda^b,} \tag{6.9}$$

from which we easily compute

$$\hat{\xi}\big(-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\langle F_{\mu\nu}, F_{\sigma\rho}\rangle\big) = -\eta^a g^{\mu\sigma}\Gamma^\nu_{ab}\langle F_{\mu\nu}, \nabla_\sigma\lambda^b\rangle. \tag{6.10}$$

In the variation of the second term in (6.5) we must remember to include the variation of the covariant derivative $\nabla_\sigma = \partial_\sigma + A_\sigma$:

(6.11)

$$\begin{aligned}\hat{\xi}\big(\frac{1}{2}\Gamma^\sigma_{ab}\langle\lambda^a, \nabla_\sigma\lambda^b\rangle\big) &= \frac{1}{2}\Gamma^\sigma_{ab}\langle\eta^c\tilde{\Gamma}^{\mu ad}\Gamma^\nu_{dc}F_{\mu\nu}, \nabla_\sigma\lambda^b\rangle + \frac{1}{2}\Gamma^\sigma_{ab}\langle\lambda^a, \eta^c g_{\sigma\nu}\Gamma^\nu_{cd}[\lambda^d, \lambda^b]\rangle \\ &\qquad + EXACT \\ &= \frac{1}{2}\eta^a\Gamma^\sigma_{bd}\tilde{\Gamma}^{\mu cd}\Gamma^\nu_{ac}\langle F_{\mu\nu}, \nabla_\sigma\lambda^b\rangle - \frac{1}{2}\eta^c g_{\sigma\nu}\Gamma^\sigma_{ab}\Gamma^\nu_{cd}\langle\lambda^a, [\lambda^d, \lambda^b]\rangle \\ &\qquad + EXACT \\ &= \eta^a g^{\mu\sigma}\Gamma^\nu_{ab}\langle F_{\mu\nu}, \nabla_\sigma\lambda^b\rangle - \frac{1}{2}\eta^c g_{\sigma\nu}\Gamma^\sigma_{ab}\Gamma^\nu_{cd}\langle\lambda^a, [\lambda^d, \lambda^b]\rangle \\ &\qquad + EXACT,\end{aligned}$$

where

$$EXACT = \partial_\sigma\big(\frac{1}{4}\eta^a\tilde{\Gamma}^\sigma_{bd}\Gamma^{\mu cd}\tilde{\Gamma}^\nu_{ac}\langle F_{\mu\nu}, \lambda^b\rangle\big). \tag{6.12}$$

In the last step we use the Clifford relation $\Gamma^\mu_{bd}\tilde{\Gamma}^{\sigma cd} + \Gamma^\sigma_{bd}\tilde{\Gamma}^{\mu cd} = 2g^{\mu\sigma}\delta^c_b$ and the Bianchi identity $\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0$. Now the first term in the last line cancels against (6.10), and the second term vanishes if and only if the quartic form

$$Q_{abcd} = g_{\sigma\nu}\big(\Gamma^\sigma_{ab}\Gamma^\nu_{cd} + \Gamma^\sigma_{ad}\Gamma^\nu_{bc} + \Gamma^\sigma_{ac}\Gamma^\nu_{db}\big) \tag{6.13}$$

vanishes. There are two conclusions from this computation. First, if the gauge algebra is abelian, then $\hat{\xi}$ is a symmetry of L in any dimension and with any spin representation S. Second, if the gauge algebra is nonabelian, then $\hat{\xi}$ is a symmetry of L if and only if the quartic form

$$Q\colon s^* \longmapsto |\,\Gamma(s^*, s^*)\,|^2, \qquad s^* \in S^*, \tag{6.14}$$

vanishes identically. This holds for $n = 3, 4, 6, 10$ as in those dimensions the spin group acts transitively on the nonzero spinors. For $n = 3, 4, 6$ the spin representation is $SL(2, \mathbb{F})$ acting on $\mathbb{F}^2$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. For $n = 10$ octonions can be similarly used [I-**Spinors**, §6.5–§6.7].

Next, we investigate the Lie algebra generated by the symmetry transformation (6.8). Let η^a_1, η^b_2 be odd parameters and $\hat{\xi}_i$ the action (6.8) of $\eta^a_i Q_a$. Recall from (1.30) that for any field f we need to check

$$[\hat{\xi}_1, \hat{\xi}_2]f = 2\eta^a_1\eta^b_2\Gamma^\sigma_{ab}\partial_\sigma f \tag{6.15}$$

When acting on the spinor field $f = \lambda$, we replace ∂_σ with the covariant derivative ∇_σ. When acting on the connection $f = A$, we use Lie derivative by the horizontal lift, and it is easy to compute that the action of ∂_σ on A has μ^{th} component equal to $F_{\sigma\mu}$.

We first verify (6.15) for the connection field. Using (6.8) we find:

$$\begin{aligned}\big([\hat\xi_1,\hat\xi_2]A\big)_\mu &= -\frac{1}{2}\eta_1^a\eta_2^b g_{\mu\nu}(\tilde\Gamma^{\rho cd}\Gamma^\nu_{bc}\Gamma^\sigma_{ad} + \tilde\Gamma^{\rho cd}\Gamma^\nu_{ac}\Gamma^\sigma_{bd})F_{\rho\sigma} \\ &= -\frac{1}{2}\eta_1^a\eta_2^b g_{\mu\nu}(4g^{\rho\nu}\Gamma^\sigma_{ab})F_{\rho\sigma} \\ &= 2\eta_1^a\eta_2^b\Gamma^\sigma_{ab}F_{\sigma\mu}.\end{aligned} \tag{6.16}$$

To pass to the second line we added to the first line terms in parentheses with ρ and ν exchanged; then we used the Clifford identity (1.3). This is the desired result. To check the commutation relation (6.15) on the spinor field we need to also impose the equation of motion

$$\Gamma^\sigma_{ab}\nabla_\sigma\lambda^a = 0, \qquad \text{for all } b; \tag{6.17}$$

that is, we only verify the supersymmetry algebra *on-shell*. Then we compute from (6.8)

$$\begin{aligned}[\hat\xi_a,\hat\xi_2]\lambda^c &= -\frac{1}{2}\eta_1^a\eta_2^b g_{\mu\nu}(\tilde\Gamma^{\sigma cd}\Gamma^\nu_{db}\Gamma^\mu_{ae} + \tilde\Gamma^{\sigma cd}\Gamma^\nu_{da}\Gamma^\mu_{be} \\ &\qquad - \tilde\Gamma^{\nu cd}\Gamma^\sigma_{db}\Gamma^\mu_{ae} - \tilde\Gamma^{\nu cd}\Gamma^\sigma_{da}\Gamma^\mu_{be})\nabla_\sigma\lambda^e \\ &= \frac{1}{2}\eta_1^a\eta_2^b g_{\mu\nu}\tilde\Gamma^{\sigma cd}\Gamma^\nu_{de}\Gamma^\mu_{ab}\nabla_\sigma\lambda^e - \frac{1}{2}\eta_1^a\eta_2^b\tilde\Gamma^{\sigma cd}Q_{abde}\nabla_\sigma\lambda^e \\ &\qquad + \frac{1}{2}\eta_1^a\eta_2^b g_{\mu\nu}(\tilde\Gamma^{\nu cd}\Gamma^\sigma_{db}\Gamma^\mu_{ae} + \tilde\Gamma^{\nu cd}\Gamma^\sigma_{da}\Gamma^\mu_{be})\nabla_\sigma\lambda^e.\end{aligned} \tag{6.18}$$

Let 'LT' denote the last term. Applying the Clifford identity we see that the symmetric part in ν,σ vanishes by the equation of motion, whence

$$\begin{aligned}LT &= \frac{1}{2}LT - \frac{1}{4}\eta_1^a\eta_2^b g_{\mu\nu}(\tilde\Gamma^{\sigma cd}\Gamma^\nu_{db}\Gamma^\mu_{ae} + \tilde\Gamma^{\sigma cd}\Gamma^\nu_{da}\Gamma^\mu_{be})\nabla_\sigma\lambda^e \\ &= \frac{1}{2}LT + \frac{1}{4}\eta_1^a\eta_2^b g_{\mu\nu}\tilde\Gamma^{\sigma cd}\Gamma^\nu_{de}\Gamma^\mu_{ab}\nabla_\sigma\lambda^e - \frac{1}{4}\eta_1^a\eta_2^b\tilde\Gamma^{\sigma cd}Q_{abde}\nabla_\sigma\lambda^e.\end{aligned} \tag{6.19}$$

Solving for LT and plugging into (6.18) we have

$$\begin{aligned}[\hat\xi_1,\hat\xi_2]\lambda^c &= \eta_1^a\eta_2^b g_{\mu\nu}\tilde\Gamma^{\sigma cd}\Gamma^\nu_{de}\Gamma^\mu_{ab}\nabla_\sigma\lambda^e - \eta_1^a\eta_2^b\tilde\Gamma^{\sigma cd}Q_{abde}\nabla_\sigma\lambda^e \\ &= 2\eta_1^a\eta_2^b\Gamma^\sigma_{ab}\nabla_\sigma\lambda^c - \eta_1^a\eta_2^b\tilde\Gamma^{\sigma cd}Q_{abde}\nabla_\sigma\lambda^e.\end{aligned} \tag{6.20}$$

Here we use the equation of motion (6.17) to see that the last term in the first line vanishes when σ and ν are exchanged. If the second term vanishes, then this is the desired bracket. Hence the supersymmetry algebra is satisfied on-shell if and only if the quartic form (6.14) vanishes. This applies both to the abelian and nonabelian theories.

This completes the proof that (6.3) is supersymmetric.

We give another interpretation of the quartic form below (6.69).

We now compute the classical equations of motion and the variational 1-form. First, the variation of the curvature (6.7) is

$$\delta F_{\mu\nu} = \nabla_\mu \delta A_\nu - \nabla_\nu \delta A_\mu. \tag{6.21}$$

Now write the lagrangian (6.5) as $L = \check{L}\,|d^n x|$. Then a short computation yields

$$\begin{aligned}\delta \check{L} = \langle \delta A_\rho\,,\, g^{\mu\sigma} g^{\nu\rho} \nabla_\sigma F_{\mu\nu} - \frac{1}{2}\Gamma^\rho_{ab}[\lambda^a, \lambda^b]\rangle + \langle \delta\lambda^a\,,\, \Gamma^\sigma_{ab}\nabla_\sigma \lambda^b\rangle \\ + \partial_\sigma\Big\{-g^{\mu\sigma} g^{\nu\rho}\langle F_{\mu\nu}, \delta A_\rho\rangle + \frac{1}{2}\Gamma^\sigma_{ab}\langle \lambda^a, \delta\lambda^b\rangle\Big\}.\end{aligned} \tag{6.22}$$

This leads to the equations of motion

$$\boxed{\begin{aligned} g^{\mu\sigma} g^{\nu\rho}\nabla_\sigma F_{\mu\nu} &= \frac{1}{2}\Gamma^\rho_{ab}[\lambda^a, \lambda^b] \\ \Gamma^\sigma_{ab}\nabla_\sigma \lambda^b &= 0\end{aligned}} \tag{6.23}$$

and the variational 1-form

$$\boxed{\gamma = \Big(-g^{\mu\sigma} g^{\nu\rho}\langle F_{\mu\nu}, \delta A_\rho\rangle + \frac{1}{2}\Gamma^\sigma_{ab}\langle \lambda^a, \delta\lambda^b\rangle\Big)\,\frac{\partial}{\partial x^\sigma} \otimes |d^n x|.} \tag{6.24}$$

In coordinate-free notation we write the equations of motion (6.23) as

$$\boxed{\begin{aligned} -d_A * F_A &= \frac{1}{2}[\lambda\,\Gamma\,\lambda]\;|d^n x| \\ \not{D}_A \lambda &= 0.\end{aligned}} \tag{6.25}$$

The Noether current associated to $\hat{\xi}$ is computed from (6.12) and (6.24) as

$$\iota(\hat{\xi})\gamma - \Big(\frac{1}{4}\eta^a \Gamma^\sigma_{bd}\tilde{\Gamma}^{\mu cd}\Gamma^\nu_{ac}\langle F_{\mu\nu}, \lambda^b\rangle\,\frac{\partial}{\partial x^\sigma} \otimes |d^n x|\Big) = -\eta^a g^{\mu\sigma}\Gamma^\nu_{ab}\langle F_{\mu\nu}, \lambda^b\rangle\,\frac{\partial}{\partial x^\sigma} \otimes |d^n x|. \tag{6.26}$$

The *supercurrent* j_a is *minus* the Noether current for Q_a:

$$\boxed{j_a = g^{\mu\sigma}\Gamma^\nu_{ab}\langle F_{\mu\nu}, \lambda^b\rangle\,\frac{\partial}{\partial x^\sigma} \otimes |d^n x|.} \tag{6.27}$$

§6.2. Gauge theories with matter

The most general bosonic model without gravity[32] on Minkowski space $\check{M}^n$ is specified by the following data:

(6.28)	G	compact Lie group
	$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on $\mathfrak{g}$
	X	Riemannian manifold on which G acts by isometries
	$V\colon X \longrightarrow \mathbb{R}$	potential function invariant under G

[32] We include only scalar fields and gauge fields. There are also models with p-form fields for $p \geq 2$, for example. The models discussed here cover most *fundamental* (vs. *effective*) lagrangians without gravity.

The fields in the model are

$$\begin{array}{ll} A & \text{connection on some principal } G \text{ bundle } P \longrightarrow \check{M}^n \\ \phi & \text{section of the associated bundle } X^P = P \times_G X \longrightarrow \check{M}^n \end{array} \tag{6.29}$$

The standard fundamental lagrangian is then the "minimally coupled" σ-model lagrangian (3.77) and Yang-Mills lagrangian (6.2):

$$L = \left\{ -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^* V \right\} |d^n x|. \tag{6.30}$$

The moduli space of classical vacua on $\check{M}^n$ is the quotient of the set of minima of V—which we assume occurs at $V = 0$—by the action of constant gauge transformations:

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{6.31}$$

Now we consider supersymmetric extensions of (6.30). As in §3 we work with models in 3, 4, and 6 dimensions with minimal supersymmetry. As in §3 we refer to the model in $3, 4, 6$ dimensions as the $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ model. In these dimensions we have already constructed separately the supersymmetric σ-model and the supersymmetric gauge theory. The next theorem summarizes what happens when we combine them. In these models the bosons are given in (6.29) and the fermions are

$$\begin{array}{ll} \psi & \text{section of } \phi^*\big(P \times_G (TX \otimes_{\mathbb{F}} \Pi\mathbb{F}^2)\big) \\ \lambda & \text{section of } P \times_G (\mathfrak{g} \otimes \Pi\mathbb{F}^{2^*}) \end{array} \tag{6.32}$$

In the $\mathbb{F} = \mathbb{H}$ case we use complex notation for ψ as in (3.49).

A discussion—including an explanation of notation—follows the statement of the theorem.

Theorem 6.33. *Fix*

$$\begin{array}{ll} G & \textit{compact Lie group} \\ \langle \cdot, \cdot \rangle & \textit{bi-invariant scalar product on } \mathfrak{g} \end{array} \tag{6.34}$$

(*i*) ($\mathbb{F} = \mathbb{R}$) *Suppose we are given*

$$\begin{array}{ll} X & \textit{Riemannian manifold with } G \textit{ action by isometries} \\ h\colon X \longrightarrow \mathbb{R} & G\textit{-invariant real-valued function} \end{array} \tag{6.35}$$

Then the lagrangian

$$\begin{aligned} L = \Big\{ & -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 + \frac{1}{2}\langle \lambda \not{D}_A \lambda \rangle + \frac{1}{2}\langle \psi (\not{D}_A)_\phi \psi \rangle - \frac{1}{2}\phi^* |\operatorname{grad} h|^2 \\ & - \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b) + \langle \hat{\lambda}^a_\phi, \psi_a \rangle \\ & + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle \psi_a, \phi^* R(\psi_b, \psi_c)\psi_d \rangle \Big\} |d^3 x|. \end{aligned} \tag{6.36}$$

on $\check{M}^3$ is supersymmetric with the supersymmetry transformation laws

$$\boxed{\begin{aligned}\hat{\xi}\phi &= -\eta^a\,\psi_a\\ (\hat{\xi}A)_{ab} &= \eta^c(\epsilon_{ac}\epsilon_{bd}+\epsilon_{ad}\epsilon_{bc})\lambda^d\\ \hat{\xi}\psi_a &= \eta^b\big((d_A)_{ab}\phi+\epsilon_{ab}\phi^*\operatorname{grad}h\big)\\ \hat{\xi}\lambda^a &= -\eta^b\epsilon^{ac}f_{bc}\end{aligned}} \tag{6.37}$$

The potential energy is

$$\boxed{V=\frac{1}{2}|\operatorname{grad}h|^2} \tag{6.38}$$

and the moduli space of vacua is

$$\boxed{\mathcal{M}_{\text{vac}}=\operatorname{Crit}(h)/G.} \tag{6.39}$$

The supercurrent j_a, which is minus the Noether current for Q_a, is

(6.40)

$$\boxed{\begin{aligned}j_a=\Big\{\epsilon^{cb}\epsilon^{de}\langle\partial_{ae}\phi,\psi_b\rangle+\epsilon^{bd}\delta^c_a\langle\phi^*\operatorname{grad}h,\psi_b\rangle+\frac{1}{2}\epsilon^{bd}\langle f_{ba},\lambda^c\rangle\Big\}\frac{\partial}{\partial y^{cd}}\otimes|d^3x|\\ +\frac{1}{2}\epsilon^{cb}\langle f_{cd},\lambda^d\rangle\frac{\partial}{\partial y^{ab}}\otimes|d^3x|\end{aligned}}$$

(ii) ($\mathbb{F}=\mathbb{C}$) *Suppose we are given*

$$\begin{array}{ll} X & \textit{Kähler manifold with } G \textit{ action}\\ \mu\colon X\longrightarrow\mathfrak{g}^* & \textit{moment map for } G \textit{ action}\\ W\colon X\longrightarrow\mathbb{C} & G\textit{-invariant holomorphic function}\end{array} \tag{6.41}$$

Then the lagrangian

(6.42)

$$\boxed{\begin{aligned}L=\Big\{-\frac{1}{2}|F_A|^2+\langle d_A\bar{\phi},d_A\phi\rangle+\langle\bar{\lambda}\not{D}_A\lambda\rangle+\langle\bar{\psi}(\not{D}_A)_\phi\,\psi\rangle\\ -\phi^*\|\operatorname{grad}W\|^2-2\phi^*|\mu|^2-\operatorname{Re}\big[\epsilon^{ab}\phi^*(\operatorname{Hess}W)(\psi_a,\psi_b)\big]\\ +\sqrt{2}\left(\langle\hat{\lambda}^a_\phi,\psi_a\rangle-\langle\bar{\psi}_{\dot{a}},\hat{\bar{\lambda}}^{\dot{a}}_\phi\rangle\right)-\frac{1}{4}\epsilon^{ac}\epsilon^{\dot{b}\dot{d}}\langle\psi_a,\phi^*R(\psi_c,\bar{\psi}_{\dot{d}})\bar{\psi}_{\dot{b}}\rangle\Big\}\,|d^4x|.\end{aligned}}$$

on $\check{M}^4$ is supersymmetric with the supersymmetry transformation laws

$$\boxed{\begin{aligned}\hat{\xi}\phi &= -\sqrt{2}\,\eta^a\psi_a\\ (\hat{\xi}A)_{a\dot{b}} &= 2\eta^c\,\epsilon_{ac}\epsilon_{\dot{b}\dot{d}}\,\bar{\lambda}^{\dot{d}}+2\bar{\eta}^{\dot{c}}\,\epsilon_{ad}\epsilon_{\dot{b}\dot{c}}\,\lambda^d\\ \hat{\xi}\psi_a &= \sqrt{2}\left[\bar{\eta}^{\dot{b}}(d_A)_{a\dot{b}}\phi+\eta^b\epsilon_{ab}\phi^*\operatorname{grad}W\right]\\ \hat{\xi}\lambda^a &= -\eta^b\epsilon^{ac}F_{bc}-2\eta^a\sqrt{-1}\phi^*\check{\mu}\end{aligned}} \tag{6.43}$$

The potential energy is

$$\boxed{V = \|\operatorname{grad} W\|^2 + 2|\mu|^2} \tag{6.44}$$

and the moduli space of vacua is

$$\boxed{\mathcal{M}_{\text{vac}} = \mu^{-1}(0) \cap \operatorname{Crit}(W) \,/\, G.} \tag{6.45}$$

The supercurrent j_a, which is minus the Noether current for Q_a, is

(6.46)

$$\begin{aligned} j_a = &\left\{\sqrt{2}\epsilon^{cb}\epsilon^{\dot{d}\dot{e}}\langle\partial_{a\dot{e}}\phi, \psi_b\rangle + \sqrt{2}\,\epsilon^{\dot{b}\dot{d}}\delta_a^c\langle\phi^*\operatorname{grad} W, \bar{\psi}_{\dot{b}}\rangle + \frac{1}{2}\epsilon^{bc}\langle F_{ba}, \bar{\lambda}^{\dot{d}}\rangle\right\}\frac{\partial}{\partial y^{c\dot{d}}} \otimes |d^4x| \\ &+ \left\{\frac{1}{2}\epsilon^{\dot{c}\dot{b}}\langle\overline{F}_{\dot{c}\dot{d}}, \bar{\lambda}^{\dot{d}}\rangle - 2\sqrt{-1}\,\phi^*\mu(\bar{\lambda}^{\dot{b}})\right\}\frac{\partial}{\partial y^{a\dot{b}}} \otimes |d^4x| \end{aligned}$$

(*iii*) ($\mathbb{F} = \mathbb{H}$) *Suppose we are given*

$$\begin{array}{ll} X & \text{hyperkähler manifold with } G \text{ action} \\ \mu_{\mathbb{H}}\colon X \longrightarrow \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H} & \text{hyperkähler moment map} \end{array} \tag{6.47}$$

Then the lagrangian

$$\begin{aligned} L = &\left\{-\frac{1}{2}|F_A|^2 + \frac{1}{16}\tilde{h}_{\alpha\beta}\epsilon_{ij}\nu^{abcd}(\partial_{ab}\phi)^{\alpha i}(\partial_{cd}\phi)^{\beta j} + \frac{1}{2}\langle\lambda \not{D}_A\lambda\rangle\right. \\ &+\frac{1}{4}h_{\alpha\beta}\nu^{abcd}\psi_a^\alpha(\phi^*\nabla_{cd})\psi_b^\beta - 2\phi^*|\mu_{\mathbb{H}}|^2 + \sqrt{2}\,\tilde{h}_{\alpha\beta}\epsilon_{ij}(\hat{\lambda}_\phi^{aj})^{\alpha i}\psi_a^\beta \\ &\left.-\frac{1}{24}\nu^{abcd}\phi^*\Omega_{\alpha\beta\gamma\delta}\psi_a^\alpha\psi_b^\beta\psi_c^\gamma\psi_d^\delta\right\}|d^6x|. \end{aligned} \tag{6.48}$$

on $\check{M}^6$ is supersymmetric with the supersymmetry transformation laws

$$\begin{aligned} (\hat{\xi}\phi)^{\alpha i} &= -\sqrt{2}\,\eta^{ai}\psi_a^\alpha \\ (\hat{\xi}A)_{pq} &= 2\eta^{ai}\nu_{pqab}\epsilon_{ij}\lambda^{bj} \\ \hat{\xi}\psi_a^\alpha &= \sqrt{2}\,\epsilon_{ij}\eta^{bj}\big((d_A)_{ab}\phi\big)^{\alpha i} \\ \hat{\xi}\lambda^{ai} &= -\eta^{bi}f_b^a - 2\eta^{aj}\sqrt{-1}\,\phi^*(\check{\mu}_{\mathbb{H}})_j^i. \end{aligned} \tag{6.49}$$

The potential energy is

$$\boxed{V = 2|\mu_{\mathbb{H}}|^2} \tag{6.50}$$

and the moduli space of vacua is the hyperkähler quotient

$$\boxed{\mathcal{M}_{\text{vac}} = \mu_{\mathbb{H}}^{-1}(0) \,/\, G.} \tag{6.51}$$

The supercurrent j_{ai}, which is minus the Noether current for Q_{ai}, is
(6.52)

$$\boxed{\begin{aligned} j_{ai} = &\left\{\frac{1}{\sqrt{2}}\nu^{bcde}\langle\partial_{ac}\phi,\psi_{bi}\rangle + \frac{1}{2}\epsilon_{ij}\langle f_a^e,\lambda^{dj}\rangle\right\}\frac{\partial}{\partial y^{de}}\otimes|d^6x| \\ &- \left\{\frac{1}{2}\epsilon_{ij}\langle f_b^c,\lambda^{bj}\rangle + 2\sqrt{-1}\,\epsilon_{jk}\langle\phi^*(\check{\mu}_{\mathbb{H}})_i^j,\lambda^{ck}\rangle\right\}\frac{\partial}{\partial y^{ac}}\otimes|d^6x|. \end{aligned}}$$

The proof for $\mathbb{F}=\mathbb{R}$ is given in §7.3, and the proof[33] for $\mathbb{F}=\mathbb{C}$ (at least when X is Hodge) in §8.3. In both cases we give a manifestly supersymmetric formulation is superspace. We do not give a proof for $\mathbb{F}=\mathbb{H}$, a case in which there is no known superspace formulation for an arbitrary hyperkähler manifold X, though we have done enough of the computations in components that the reader should have no problem completing the proof directly in components. The reduction of (6.48) to 4 dimensions is discussed in §10.3.

In (6.37) the Hodge dual of the curvature F_A is

$$*F_A = f_{ab}dy^{ab}. \tag{6.53}$$

In (6.43) the curvature F_A has been decomposed into its self-dual and anti-self-dual parts:

$$F_A = \frac{1}{2}(F_{ab}\epsilon_{\dot{c}\dot{d}} + \overline{F}_{\dot{c}\dot{d}}\epsilon_{ab})\,dy^{a\dot{c}}\wedge dy^{b\dot{d}}. \tag{6.54}$$

In (6.49) the curvature 2-form

$$F_A = \frac{1}{8}F_{(ab)(cd)}\,dy^{ab}\wedge dy^{cd} \tag{6.55}$$

has been rewritten in terms of a traceless 4×4 matrix f:

$$f_b^a = \frac{1}{4}\nu^{acde}F_{(bc)(de)}. \tag{6.56}$$

We explain this further in §10.1.

For $\mathbb{F}=\mathbb{C}$ the scalar product on $\mathfrak{g}$ identifies $\mathfrak{g}^*\cong\mathfrak{g}$. We let

$$\check{\mu}\colon X\to\mathfrak{g} \tag{6.57}$$

denote the composition of the moment map with this isomorphism; for $\mathbb{F}=\mathbb{H}$ we have $\check{\mu}_{\mathbb{H}}\colon X\to\mathfrak{g}\otimes\operatorname{Im}\mathbb{H}$. In the last equation in (6.49) we identify $\operatorname{Im}\mathbb{H}$ with traceless hermitian 2×2 matrices:

$$aI+bJ+cK\leftrightarrow\begin{pmatrix} a\sqrt{-1} & -b+c\sqrt{-1} \\ b+c\sqrt{-1} & -a\sqrt{-1}\end{pmatrix}. \tag{6.58}$$

In all three cases there are new terms of the form $\langle\hat{\lambda}_\phi,\psi\rangle$ which appear in the lagrangian. Here $\hat{\lambda}_\phi$ is the odd vector field on $P\times_G X$ corresponding to λ, and the

[33] See Problem ASH4 of [I-**Homework**] for an example, which in particular serves as a check of the constants in (6.42) and (6.43).

inner product is the Riemannian metric on X. For $\mathbb{F} = \mathbb{R}$, in the linear case X is an orthogonal vector space and G acts linearly by an orthogonal representation $\rho\colon G \to O(X)$. Then

$$\hat{\lambda}_\phi = \dot{\rho}(\lambda)\phi, \tag{6.59}$$

where $\dot{\rho}\colon \mathfrak{g} \to \mathfrak{o}(X)$ is the infinitesimal action. This term can be predicted by supersymmetry: the variation in A of $\langle \psi \not{D}_A \psi \rangle$ is canceled by the variation in ϕ of this term. For $\mathbb{F} = \mathbb{C}$ note that ψ_a is a vector field of type $(1,0)$ on X, so only the $(0,1)$-component of $\hat{\lambda}^a_\phi$ and its conjugate enter (6.42). In the linear case G acts by a unitary representation $\rho\colon G \to U(X)$ on a hermitian vector space X. Then the $(1,0)$-component of $\hat{\lambda}_\phi$ is

$$(\hat{\lambda}_\phi)^{1,0} = \dot{\rho}_{\mathbb{C}}(\lambda)\phi, \tag{6.60}$$

where $\dot{\rho}_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}_{\mathbb{C}}(X)$ is the complexification of the differential to ρ. To write a formula for the $(0,1)$-component we identify $\overline{X} \cong X^*$ via the hermitian metric; then

$$(\hat{\lambda}_\phi)^{0,1} = -\dot{\rho}_{\mathbb{C}}(\lambda)^*\phi^* \in X^* \cong \overline{X} \tag{6.61}$$

at the point $\phi^* \in X^*$. (As a real manifold, $X^* \cong \overline{X} \cong X$ so we can identify points of X with points of X^*.) Note that when λ is real, then $\dot{\rho}_{\mathbb{C}}(\lambda)$ is skew-hermitian and (6.61) is the complex conjugate of (6.60), i.e., $\hat{\lambda}_\phi$ is real. Also,[34]

$$\hat{\bar{\lambda}} = \overline{\hat{\lambda}}, \tag{6.62}$$

which also holds in the nonlinear case. This ensures that (6.42) is real.

The quotient in (6.45) is the symplectic (Kähler) quotient, which can often be identified with the complex quotient of $\mathrm{Crit}(W)$ by the complexification of G. The quotient in (6.51) is the hyperkähler quotient. (It was in this context that the hyperkähler quotient was discovered.) For $\mathbb{F} = \mathbb{C}$ the abelian group $\big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)^*$ of infinitesimal characters acts by translations on the space of moment maps. Theories with different choices of moment map are different. Notice that when X is a point we must still choose a moment map, which is now simply an infinitesimal character. The resulting term $-\big(2|\mu|^2\big)$ is usually called a *Fayet-Iliopoulos term*. For $\mathbb{F} = \mathbb{H}$ the moment map is also ambiguous, this time up to $\big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)^* \otimes \mathrm{Im}\,\mathbb{H}$. For X a point this leads to a Fayet-Iliopoulos term.

The action of R-symmetries for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$ is summarized in (5.31), (8.24), and (10.21).

We have not written the equations of motion for these general lagrangians; it seems better to work out the detailed form on a case-by-case basis. However, we indicate here some general features. For the scalar field ϕ we find an equation of the form

$$\Box_A\phi = -\phi^*\,\mathrm{grad}\,V + Q^{(2)}_\phi(\psi,\lambda) + R^{(2)}_\phi(\psi)d\phi + Q^{(4)}(\psi). \tag{6.63}$$

[34]We caution the reader that '$\bar{\lambda}$' denotes the conjugate in the Lie algebra. For example, if $G = \mathbb{T}$ is the circle group, then $\mathfrak{g} \cong \sqrt{-1}\,\mathbb{R}$ and $\mathfrak{g}_{\mathbb{C}} \cong \mathbb{C}$ with conjugation *minus* the usual complex conjugation in $\mathbb{C}$. See Problem ASH4 of [I-**Homework**] for a sample lagrangian in this case.

Here $\Box_A = d_A^* d_A$ is the wave operator on sections of $X^P \to \check{M}^n$. The coefficients of the quadratic form $Q_\phi^{(2)}$ depend on the pullback of the hessian of h or W (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and also on the action of $\mathfrak{g}$ on X; the coefficients of the quadratic form $R_\phi^{(2)}$ depend on the pullback of the Riemann curvature tensor; and the coefficients of the quartic form $Q_\phi^{(4)}$ depend on the pullback of derivatives of the Riemann curvature tensor. Now the action of $\mathfrak{g}$ on X induces a map from $\mathfrak{g}$ to vertical vector fields on X^P, or equivalently a $\mathfrak{g}^*$-valued vector field $\hat{\zeta}$ on X^P. This vector field enters into some equations of motion, for example in the equation

$$\not{D}_A \lambda = -\langle \hat{\zeta}_\phi, \psi \rangle. \tag{6.64}$$

Here $\hat{\zeta}_\phi$ means the pullback of $\hat{\zeta}$ by the section ϕ.

§6.3. Superspace construction

We consider only pure gauge theory. As usual we fix data (6.1). Consider a superspace $M^{n|s}$ in any dimension built out of any real spin representation. Suppose $\mathcal{P} \to M^{n|s}$ is a principal G bundle. Let $\mathcal{A}$ be a connection on $\mathcal{P}$. If $\mathcal{A}$ is unconstrained, then there are too many component fields to obtain a sensible theory, even with n, s small. Hence we impose a constraint. Namely, we require that the curvature $F_\mathcal{A}$ be constrained to vanish along the left-invariant odd distribution τ on $M^{n|s}$:

$$\iota(D_b)\iota(D_a)F_\mathcal{A} = 0. \tag{6.65}$$

In that case the "lowest" piece of the curvature is in the even-odd direction. Let $\mathcal{D}_a$ be the horizontal lift of D_a and ∇_μ the horizontal lift of ∂_μ. Define

$$\boxed{\Lambda^a = \frac{1}{n}\tilde{\Gamma}^{\mu ab}\iota(D_b)\iota(\partial_\mu)F_\mathcal{A} = \frac{1}{n}\tilde{\Gamma}^{\mu ab}[\mathcal{D}_b, \nabla_\mu].} \tag{6.66}$$

(The constant is chosen with hindsight.) Also, we identify G-invariant vertical vector fields with sections of the adjoint bundle; see [**I-Classical Fields**, §4.2].) This is a "projection" of the lowest nonvanishing part of the curvature; Λ^a is an odd section of the adjoint bundle $\operatorname{ad} P = \mathcal{P} \times_G \mathfrak{g} \to M^{n|s}$. There are component fields

$$\begin{aligned} A &= i^*\mathcal{A} && \text{connection on } P = i^*\mathcal{P} \longrightarrow \check{M}^n \\ \lambda^a &= i^*\Lambda^a && \text{dual spinor field on } \check{M}^n \text{ with values in } \operatorname{ad} P \end{aligned} \tag{6.67}$$

and possibly additional component fields as well. To see what other component fields there are, we examine consequences of the Bianchi identity.

First, we apply Bianchi to D_a, D_b, D_c to learn that

$$\text{The symmetrization of } \Gamma^\mu_{ab}[\mathcal{D}_c, \nabla_\mu] \text{ in } a, b, c \text{ vanishes.} \tag{6.68}$$

To interpret this, consider the sequence of maps

$$\begin{array}{ccccc} \operatorname{Sym}^3(S^*) & \xrightarrow{\operatorname{Sym}^3(\Gamma\otimes\mathrm{id})} & V \otimes S^* & \xrightarrow{g\Gamma} & S \\ f_a f_b f_c & \longrightarrow & \operatorname{sym}(\Gamma^\mu_{ab} e_\mu \otimes f_c) & & \\ & & e_\mu \otimes f_c & \longrightarrow & g_{\mu\nu}\Gamma^\nu_{cd} f^d \end{array} \tag{6.69}$$

The composition vanishes if and only if for all s^* in S^*, $\Gamma(s^*, s^*) \otimes s^*$ maps to 0 in S, i.e., if and only if for all $t^* \in S^*$,

$$\langle \text{image of } \Gamma(s^*, s^*) \otimes s^*, t^* \rangle = \langle \Gamma(s^*, s^*), \Gamma(s^*, t^*) \rangle \tag{6.70}$$

vanishes. The last expression is obtained by polarization from $\langle \Gamma(s^*, s^*), \Gamma(s^*, s^*) \rangle$, and the composition vanishes if and only if the quadratic form (6.14) does. As we have seen, this occurs for $n = \dim V = 3, 4, 6, 10$ with minimal supersymmetry $s = \dim S = 2, 4, 8, 16$. Furthermore, in these cases the sequence (6.69) is exact. Assume from now on that we are in one of these situations. Now the dual to (6.69) is

$$S^* \longrightarrow V^* \otimes S \longrightarrow \mathrm{Sym}^3(S). \tag{6.71}$$

The even-odd piece of the curvature can be identified with an element of $\Omega^0(M^{n|s}, \mathrm{ad}\,\mathcal{P}) \otimes V^* \otimes S$;then (6.68) asserts that its image in $\Omega^0(M^{n|s}, \mathrm{ad}\,\mathcal{P}) \otimes \mathrm{Sym}^3(S)$ vanishes. By the exactness of (6.71) we can then identify the even-odd piece of the curvature with an element $\Lambda \in \Omega^0(M^{n|s}, \mathrm{ad}\,\mathcal{P}) \otimes S^*$, as defined in (6.66). In fact, the Clifford identity implies that

$$\begin{aligned} V^* \otimes S &\longrightarrow S^* \\ e^\mu \otimes f^a &\longmapsto \frac{1}{n} \tilde{\Gamma}^{\mu ab} f_b \end{aligned} \tag{6.72}$$

is a splitting of (6.71), whence the formula (6.66) for Λ. To summarize, in the four cases we are considering the even-odd curvature can be expressed in terms of Λ:

$$\boxed{[\mathcal{D}_a, \nabla_\mu] = g_{\mu\nu} \Gamma^\nu_{ab} \Lambda^b.} \tag{6.73}$$

Next, we apply Bianchi to D_a, D_b, ∂_μ. After some manipulation with (6.73) and the Clifford identity we find

$$-2\tilde{\Gamma}^{\mu ac} \Gamma^\nu_{cb} \mathcal{F}_{\mu\nu} = g_{\mu\nu} \tilde{\Gamma}^{\mu ac} \Gamma^\nu_{bd} \mathcal{D}_c \Lambda^d + n \mathcal{D}_b \Lambda^a, \tag{6.74}$$

where the even-even part of the curvature is

$$\mathcal{F}_{\mu\nu} = \iota(\partial_\nu)\iota(\partial_\mu) F_{\mathcal{A}} = -[\nabla_\mu, \nabla_\nu]. \tag{6.75}$$

(The funny sign is due to the fact that we take the bracket as vector fields on P, not as operators on an associated bundle; cf. [I-**Classical Fields**, §4.2].) The first term on the right hand side of (6.74) depends on the particular case, and its precise form determines the structure of the auxiliary fields. In $n = 3, 4, 6$ dimensions we will find a formula of the form

$$\boxed{\mathcal{D}_b \Lambda^a = -\frac{1}{2} \tilde{\Gamma}^{\mu ac} \Gamma^\nu_{cb} \mathcal{F}_{\mu\nu} + \mathcal{E},} \tag{6.76}$$

where $i^*\mathcal{E}$ consists of auxiliary fields. (Compare with the second equation in (6.8).)

We do not learn anything new from the remaining cases of the Bianchi identity. Applied to $D_a, \partial_\mu, \partial_\nu$ we obtain the Jacobi identity for three vector fields $\Lambda^a, \partial_\mu, \partial_\nu$

on $\mathcal{P}$. (Here we identify Λ^a with a G-invariant vertical vector field. Upon restriction to $\breve{M}^n$ this equation gives the supersymmetry transformation law (6.9) for the curvature.) Bianchi applied to $\partial_\mu, \partial_\nu, \partial_\rho$ restricts on $\breve{M}^n$ to the usual Bianchi identity for A.

Set

$$F_{\mu\nu} = i^* \mathcal{F}_{\mu\nu}; \tag{6.77}$$

then

$$F_A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{6.78}$$

is the curvature of A.

From the general formula (1.29) and equations (6.73) and (6.76) we recover the supersymmetry transformation law (6.8), at least if we set $\mathcal{E} = 0$. For example, to compute the action of $\hat{\xi}$ on the connection A, we lift the vector field $\eta^a D_a$ on $M^{n|s}$ to the horizontal vector field $-\eta^a \mathcal{D}_a$ on $\mathcal{P}$ and apply the covariant derivative. From (6.73) the μth component of the answer is

$$\begin{aligned}
(\hat{\xi} A)_\mu &= \iota(\partial_\mu) i^* \operatorname{Lie}(-\eta^a \mathcal{D}_a) A \\
&= i^* \iota(\nabla_\mu) \operatorname{Lie}(-\eta^a \mathcal{D}_a) A \\
&= i^* \Big[\operatorname{Lie}(-\eta^a \mathcal{D}_a) \iota(\nabla_\mu) A - \iota\big([-\eta^a \mathcal{D}_a, \nabla_\mu]\big) A \Big] \\
&= i^* \eta^a [\mathcal{D}_a, \nabla_\mu] \\
&= \eta^a g_{\mu\nu} \Gamma^\nu_{ab} \lambda^b,
\end{aligned} \tag{6.79}$$

which agrees with (6.8). The precise formula for the variation of the fermion λ, which includes the auxiliary fields, must be analyzed on a case-by-case basis.

In §7, §8, and §10 we construct the superspace models for the cases $n = 3, 4$, and 6 (previously referred to as $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$). Supersymmetry is manifest in the superspace formulation, and we determine the auxiliary fields by computing $\mathcal{E}$ in (6.76). In addition, for $n = 3, 4$ we give a superspace formulation of the gauge theory coupled to matter, and so prove Theorem 6.33(i) and Theorem 6.33(ii). There is no known superspace formulation for the $n = 10$ dimensional theory.

There are also superspace models for dimensional reductions of the $n = 3, 4, 6$ theories; particular cases are treated in §9 and §10. We remark here that (6.65) is no longer the correct curvature constraint in superspace; some components of the odd-odd curvature are allowed to be nonzero.

CHAPTER 7
$N = 1$ Yang-Mills Theory in Dimension 3

For the scalar superfield of Chapter 4 and the chiral superfield of Chapter 5 it is easy to see that the supermanifold of superfields is diffeomorphic to the supermanifold of component fields. But for constrained connections the argument is more complicated due to gauge symmetry. The precise statement is the main task of §7.1. In §7.2 and §7.3 we describe the superspace formulation of the Yang-Mills lagrangian and of the lagrangian including matter. There are no surprises, though we do explain where the new term in the component lagrangian (6.36) (coupling the two sorts of spinors) arises from the superspace point of view.

§7.1. Constrained connections on $M^{3|2}$

We specialize the discussion in §6.3 to $M^{3|2}$. Our first task is to evaluate (6.74). To that end we compute using the formulas of §2.3:

$$\begin{aligned} g_{\mu\nu}\tilde{\Gamma}^{\mu ac}\Gamma^{\nu}_{bd} &= g_{\mu\nu}\epsilon^{aa'}\epsilon^{cc'}\Gamma^{\mu}_{a'c'}\Gamma^{\nu}_{bd} \\ &= \frac{1}{2}(\epsilon_{a'b}\epsilon_{c'd} + \epsilon_{a'd}\epsilon_{c'b})(\epsilon^{aa'}\epsilon^{cc'} + \epsilon^{ac'}\epsilon^{ca'}) \\ &= \delta^a_b\delta^c_d + \delta^a_d\delta^c_b. \end{aligned} \tag{7.1}$$

Plugging into (6.74) we obtain

$$-2\tilde{\Gamma}^{\mu ac}\Gamma^{\nu}_{cb}\mathcal{F}_{\mu\nu} = 4\mathcal{D}_b\Lambda^a + \delta^a_b\mathcal{D}_c\Lambda^c. \tag{7.2}$$

Setting $a = b$ and summing we find

$$6\mathcal{D}_c\Lambda^c = -2\epsilon^{aa'}\epsilon^{cc'}\Gamma^{\mu}_{a'c'}\Gamma^{\nu}_{ac}\mathcal{F}_{\mu\nu} = 0, \tag{7.3}$$

since ϵ and $\mathcal{F}$ are skew-symmetric. Thus (7.2) becomes

$$\mathcal{D}_b\Lambda^a = -\frac{1}{2}\tilde{\Gamma}^{\mu ac}\Gamma^{\nu}_{cb}\mathcal{F}_{\mu\nu}. \tag{7.4}$$

This implies $\mathcal{E} = 0$ in (6.76), which means that there are no auxiliary fields.

The multiplet $\{A, \lambda\}$ of component fields is called the *vector multiplet*, and from the vector multiplet we can reconstruct the constrained connection $\mathcal{A}$. The precise statement is the following.

Proposition 7.5. *The category of connections $\mathcal{A}$ on $M^{3|2}$ whose curvature vanishes on the odd distribution τ is equivalent to the category of pairs $\{A, \lambda\}$ consisting of a connection A on $\check{M}^3$ and a dual spinor field λ with values in the adjoint bundle.*

We first comment on the meaning of Proposition 7.5. As odd fields are involved, it is crucial to work over a base S: for any supermanifold S, what Proposition 7.5 claims is that the functor which attaches the component fields to a constrained connection is an equivalence between the following categories: (a) G-torsors on $M^{3|2} \times S$ provided with a relative connection $\mathcal{A}$ satisfying the curvature constraint; (b) G-torsors on $\check{M}^3 \times S$ provided with a relative connection A and an adjoint bundle valued dual spinor field λ. Note that in (b) we have $\check{M}^3 \times S$, not the reduced space of $M^{3|2} \times S$. As usual, in the proof we will keep S silent.

If $\mathcal{P}$ is a G-torsor with a connection $\mathcal{A}$ on $M^{3|2}$ (i.e. on $M^{3|2} \times S$), any automorphism of $(\mathcal{P}, \mathcal{A})$ which induces the identity on the restriction P of $\mathcal{P}$ to $\check{M}^3$(i.e. $\check{M}^3 \times S$) is the identity. Because of this, $(\mathcal{P}, \mathcal{A})$ can be viewed as a structure on P. Concretely, once some choices have been made on $\check{M}^3$ (see the proof of Proposition 7.5 for an example), if $(\mathcal{P}, \mathcal{A})$ is a G-torsor with connection on $M^{3|2}$ with restriction P to $\check{M}^3$, and if s is a trivialization of P, the connection gives rise to a canonical extension of s to a trivialization of $\mathcal{P}$. Given s, then $\mathcal{A}$ becomes a 1-form α with values in the Lie algebra $\mathfrak{g}$ obeying suitable constraints. What Proposition 7.5 claims is that the map from the space of those α's to the space of (A, λ) (A a connection on P, and λ a $\mathfrak{g}$-valued dual spinor field) is bijective. In other words, for P a G-torsor on $\check{M}^3$ the map from (a) the set of isomorphism classes of G-torsors $\mathcal{P}$ on $M^{3|2}$, given with a constrained connection $\mathcal{A}$ and an isomorphism $P \xrightarrow{\sim} \mathcal{P} \mid \check{M}^3$, to (b) the set of (A, λ), is bijective. In (a), the objects considered have no non-trivial isomorphisms. Because of this, the question is local, and one may assume that P is the trivial G-torsor.

Proof. The data on $\mathcal{P}$ of a constrained connection amounts to that of a partial connection in the direction of the odd distribution τ: a G-invariant lift to $\mathcal{P}$ not of all vector fields, but only of those in τ. If $\mathcal{D}_a$ (resp. ∇_{ab}) is the horizontal lift of D_a (resp. ∂_{ab}), the vanishing along τ of F means that

$$\nabla_{ab} = -\frac{1}{2}[\mathcal{D}_a, \mathcal{D}_b]. \tag{7.6}$$

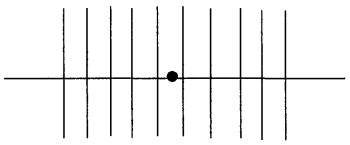
If only the $\mathcal{D}_a$ are given, the corresponding constrained connection is obtained by defining ∇_{ab} by this formula.

The functor implicit in Proposition 7.5 attaches to $(\mathcal{P}, \nabla)$: the restriction to $\check{M}^3$ of $\mathcal{P}$ (denoted P), that of $[\mathcal{D}_a, \mathcal{D}_b]$, and that of $[[\mathcal{D}_a, \mathcal{D}_b], \mathcal{D}_c]$; respectively: a G-torsor on $\check{M}^3$, a connection on it, and a section of the adjoint bundle with vanishing symmetrization in a, b, c. (cf. (6.68)).

We now choose a coordinate system as in §2.3, giving rise to the basis D_1, D_2 of τ. We will use D_1, D_2 to define a retraction q: $M^{3|2} \to \check{M}^3$, and, for $\mathcal{P}$ a torsor with connection on $M^{3|2}$, with restriction P to $\check{M}^3$, an isomorphism of torsors $q^*P \to \mathcal{P}$ (partial gauge fixing).

We first explain an analog, in classical differential geometry of what we will do. Suppose N is of codimension 2 in a variety M, and that D_1, D_2 are vector fields which, on N, span the normal bundle T_M/T_N. In a neighborhood of N, the flow $\exp(tD_1)$ generates from N a codimension 1 subvariety M_1, with a retraction

q_1: $M \to N$ constant on the flow lines. Repeating the process with M_1 and D_2, we obtain in a neighborhood of N a retraction q_2: $M \to M_1$, constant on the flow lines of D_2:



If $\mathcal{P}$ is a torsor on M with connection, with restriction P to N, integrating the connection on the flow lines of D_1, we obtain an isomorphism α_1: $q_1^* P \to \mathcal{P}$ on M_1. Doing the same with D_2, we obtain α_2: $q_2^*(\mathcal{P}|M_1) \tilde{\to} \mathcal{P}$.

In our case of codimension $0|2$, the situation is different in that (a) the vector field D_1 does not integrate to a foliation with leaves $\mathbb{R}^{1|0}$: the obstruction is D_1^2 mod D_1. (b) On $\mathbb{R}^{0|1}$, a connection ∇ does not need to be flat: curvature obstruction ∇_θ^2.

It however remains true that for $N \subset M^{3|2}$ a subvariety and D an odd vector field transversal to N, the vector field D, in fact just its restriction to N, generates from N an embedding $N \times \mathbb{R}^{0|1} \hookrightarrow M^{3|2}$. A function f on $M^{3|2}$ vanishes on the image M_1 if and only if f and Df vanish on N, and the retraction q_1: $M_1 \to N$ is such that for any function u on N, $Dq_1^*(u) = 0$. If $\mathcal{P}$ is a torsor with connection on $M^{3|2}$, with restriction P to N, we have on M_1 a unique isomorphism of torsors $q_1^* P \to \mathcal{P}|M_1$ which is the identity on N and such that on N the liftings of D agree. One should beware that if D is given as a vector field on $M^{3|2}$, it does not need to be tangent to M_1. This tangency to M_1 holds only on N.

We apply this twice, to $\check{M}^3 \subset M^{3|2}$ and D_1, then to M_1 and D_2, obtaining q: $M^{3|2} \to M_1 \to \check{M}^3$ and α: $q^* P \to \mathcal{P}$. The construction of α uses only the connection in the directions of D_1 and D_2: for $\mathcal{P}$ a torsor with a connection along τ, we obtain α: $q^* P \to \mathcal{P}$, and the data of α is equivalent to that of part of the connection, viz. $\mathcal{D}_1$ on $\check{M}^3$ and $\mathcal{D}_2$ on M_1. It remains to check that to complete this partial data to that of a connection ∇ along τ amounts to giving $[\mathcal{D}_a, \mathcal{D}_b]$ on $\check{M}^3$ (a connection on P) and $[[\mathcal{D}_a, \mathcal{D}_b], \mathcal{D}_c]$ on $\check{M}^3$ (a section on $\check{M}^3$ of $\mathrm{ad}\, P$ with vanishing symmetrization in a, b, c, i.e., the data of just $[[\mathcal{D}_1, \mathcal{D}_1], \mathcal{D}_2]$ and $[[\mathcal{D}_2, \mathcal{D}_2], \mathcal{D}_1]$).

Let us choose a local trivialization of P, hence of $\mathcal{P}$. The connection along τ is given by a section a of τ^* with values in the Lie algebra, i.e., by $a_1 = \langle D_1, a \rangle$ and $a_2 = \langle D_2, a \rangle$. That the connection is partially given by α: $q^* P \to \mathcal{P}$ amounts to $a_1 = 0$ on $\check{M}^3$ and $a_2 = 0$ on M_1, i.e. to $a_1 = a_2 = D_1 a_2 = 0$ on $\check{M}^3$.

For the simplicity of notations, let us assume that G is $GL(n)$, i.e., that we are considering a vector bundle. We have then

$$
\begin{aligned}
[\mathcal{D}_1, \mathcal{D}_1] &= -2\partial_{11} + 2D_1 a_1 \quad \text{on} \quad \check{M}^3 \\
[\mathcal{D}_1, \mathcal{D}_2] &= -2\partial_{12} + D_2 a_1 \quad \text{on} \quad \check{M}^3 \\
[\mathcal{D}_2, \mathcal{D}_2] &= -2\partial_{22} + 2D_2 a_1 \quad \text{on} \quad \check{M}^3 \ ,
\end{aligned}
\tag{7.7}
$$

so the data of the first order jet of a_1 and a_2 along $\check{M}^3$ amounts to the data of the connection on P. The remaining freedom: the addition of $\theta_1\theta_2 b_1$ to a_1 and $\theta_1\theta_2 b_2$ to a_2 amounts to the data of $[[\mathcal{D}_i, \mathcal{D}_j], \mathcal{D}_k]$ on $\check{M}^3$. Indeed, by $a_i \mapsto a_i + \theta_1\theta_2 b_i$, $[[\mathcal{D}_1, \mathcal{D}_1], \mathcal{D}_2]$ on $\check{M}^3$ is increased by $-2b_1$ and $[[\mathcal{D}_2, \mathcal{D}_2], \mathcal{D}_1]$ by $-2b_2$.

It is easy in this case to reformulate a constrained connection in terms of an *unconstrained* field. Namely, take the unconstrained field to be a partial connection on $\mathcal{P}$ along the odd distribution τ. Then if $\mathcal{D}_a$ is the horizontal lift of D_a, define the horizontal lift ∇_{ab} of ∂_{ab} to be

$$\nabla_{ab} = -\frac{1}{2}[\mathcal{D}_a, \mathcal{D}_b]. \tag{7.6}$$

In other words, the constraint that the odd-odd components of the curvature vanish uniquely determines the connection $\mathcal{A}$ from the restriction of $\mathcal{A}$ to τ.

It is convenient to lower an index on the spinor field and define[35]

$$\boxed{\Lambda_a = \Lambda^b \epsilon_{ba} = \frac{1}{3}\epsilon^{bc}[\nabla_{ab}, \mathcal{D}_c].} \tag{7.8}$$

Equation (6.73) specializes to

$$[\mathcal{D}_a, \Lambda_{bc}] = \epsilon_{ab}\Lambda_c + \epsilon_{ac}\Lambda_b. \tag{7.9}$$

Note that (7.3) is now

$$\epsilon^{ab}\mathcal{D}_a\Lambda_b = 0. \tag{7.10}$$

We note a useful consequence of (7.10):

$$\mathcal{D}^2\Lambda_a = -\epsilon^{bc}\nabla_{ab}\Lambda_c. \tag{7.11}$$

It is convenient to set

$$f_{ac} = \frac{1}{2}\epsilon^{bd}F_{(ab)(cd)}, \tag{7.12}$$

which is symmetric in a, c. Then

$$*F_A = f_{ab}dy^{ab} \tag{7.13}$$

is the Hodge dual of the curvature (c.f. (2.38)).

In these variables the component fields are

$$\boxed{\begin{aligned} A &= i^*\mathcal{A} \\ \lambda_a &= i^*\Lambda_a \end{aligned}} \tag{7.14}$$

and from (7.3) and (7.10) we have

$$\boxed{f_{ab} = i^*\mathcal{D}_a\Lambda_b = i^*\mathcal{D}_b\Lambda_a.} \tag{7.15}$$

The supersymmetry transformation laws (6.8) specialize to

$$\boxed{\begin{aligned} (\hat{\xi}A)_{ab} &= -\eta^c(\epsilon_{ac}\lambda_b + \epsilon_{bc}\lambda_a) \\ \hat{\xi}\lambda_a &= -\eta^b f_{ab} \end{aligned}} \tag{7.16}$$

and (6.9) specializes to

$$\boxed{\hat{\xi}f_{ab} = \eta^c\left[\nabla_{ca}\lambda_b + \epsilon_{ac}(\not{D}\lambda)_b\right].} \tag{7.17}$$

[35]This goes against our general wish not to raise or lower a single index using a skew-symmetric tensor, since there is a sign ambiguity. However, defining Λ_a facilitates comparison with the physics literature.

§7.2. The Yang-Mills action on $M^{3|2}$

Assume that G is compact. Fix a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. For a constrained connection $\mathcal{A}$ on $M^{3|2}$ the basic lagrangian density is

$$\boxed{L_2 = |d^3x|\, d^2\theta \left\{ -\frac{1}{4}\epsilon^{ab}\langle \Lambda_a, \Lambda_b \rangle \right\}.} \tag{7.18}$$

We compute the component lagrangian function $\check{L}_2$ using (2.40):

$$\begin{aligned} -4\check{L}_2 &= -\epsilon^{ab} i^* D^2 \langle \Lambda_a, \Lambda_b \rangle \\ &= -\epsilon^{ab} i^* \Big\{ 2\langle \mathcal{D}^2 \Lambda_a, \Lambda_b \rangle + \epsilon^{cd} \langle \mathcal{D}_d \Lambda_a, \mathcal{D}_c \Lambda_b \rangle \Big\} \\ &= 2\epsilon^{ab}\epsilon^{cd} \langle \nabla_{ac} \lambda_d, \lambda_b \rangle - \epsilon^{ab}\epsilon^{cd} \langle f_{da}, f_{cb} \rangle. \end{aligned} \tag{7.19}$$

In the last step we use (7.15), (7.11), and (7.14). So from (2.27), (2.28), and the fact that $|{*F_A}|^2 = |F_A|^2$ we conclude

$$\boxed{\check{L}_2 = -\frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \lambda \not{D}_A \lambda \rangle.} \tag{7.20}$$

This is the super Yang-Mills lagrangian (6.3). Thus we do have a superspace formulation of this lagrangian in 3 dimensions.

Next, we begin the analysis of the superspace lagrangian directly in superspace. This is a redundant exercise, since we already have the component expressions in (6.22) and the equations which follow. But it is instructive to see how the computation looks in superspace.

We compute the superspace equation of motion by varying the *unconstrained* partial connection $\mathcal{D}_a$. Notice that a variation $\delta\mathcal{D}_a$ is a G-invariant vertical vector field, so corresponds to a section $\delta\mathcal{A}_a$ of the adjoint bundle. From (7.8), (6.66), and (7.6) we compute

$$\delta\Lambda_a = -\frac{1}{6}\epsilon^{cd}\Big\{ [[\delta\mathcal{D}_a, \mathcal{D}_c], \mathcal{D}_d] + [[\mathcal{D}_a, \delta\mathcal{D}_c], \mathcal{D}_d] - 2[\nabla_{ac}, \delta\mathcal{D}_d] \Big\}. \tag{7.21}$$

We use the invariance of $\langle \cdot, \cdot \rangle$, (7.10), and (7.11) to find

$$\begin{aligned} \delta L_2 &= |d^3x|\, d^2\theta\, \frac{-1}{2}\epsilon^{ab}\langle \delta\Lambda_a, \Lambda_b \rangle \\ &= |d^3x|\, d^2\theta\, \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\Big\{ \langle \delta\mathcal{D}_a, [\mathcal{D}_c, [\mathcal{D}_d, \Lambda_b]] \rangle + \langle \delta\mathcal{D}_c, [\mathcal{D}_a, [\mathcal{D}_d, \Lambda_b]] \rangle \\ &\qquad + 2\langle \delta\mathcal{D}_d, [\nabla_{ac}, \Lambda_b] \rangle \Big\} + EXACT \\ &= |d^3x|\, d^2\theta\, \frac{-1}{2}\epsilon^{ab}\epsilon^{cd}\langle \delta\mathcal{A}_a, \nabla_{bc}\Lambda_d \rangle + EXACT. \end{aligned} \tag{7.22}$$

So the superspace equation of motion is

$$\boxed{\epsilon^{cd}\nabla_{bc}\Lambda_d = 0 \qquad \text{for all } b.} \tag{7.23}$$

The exact term is $-d\gamma$ for the variational 1-form γ. We do not compute the precise expression, since it is unwieldy and not useful. One interesting feature emerges quickly, though. Namely, γ contains a term proportional to

$$\epsilon^{ab}\epsilon^{cd}\iota(D_d)\,|d^3x|\,d^2\theta\,D_c\langle\delta\mathcal{A}_a,\Lambda_b\rangle, \tag{7.24}$$

and so as a differential 1-form on the space of fields γ is *not* linear over functions. (In other words, $\delta\mathcal{A}_a$ is differentiated.)

We extract from (7.23) the equations of motion in components. Restricting (7.23) to $\check{M}^3$ we find

$$\boxed{\not{D}_A\lambda = 0.} \tag{7.25}$$

This agrees with (6.25). To obtain the equation of motion for A, we either apply $i^*\mathcal{D}_e$ to (7.23) or specialize (6.23). In either case we obtain

$$\boxed{\epsilon^{cd}\nabla_{ad}f_{bc} = [\lambda_a,\lambda_b] \qquad \text{for all } a,b.} \tag{7.26}$$

This agrees with (6.23). Finally, the supercurrent (6.27) specializes to

$$\boxed{j_a = \frac{1}{2}\epsilon^{cb}\epsilon^{de}\langle f_{cd},\lambda_e\rangle\,\frac{\partial}{\partial y^{ab}}\otimes|d^3y| + \frac{1}{2}\epsilon^{cd}\epsilon^{be}\langle f_{ca},\lambda_e\rangle\,\frac{\partial}{\partial y^{db}}\otimes|d^3y|.} \tag{7.27}$$

Since we didn't compute the exact term in (7.22), we do not give a superspace computation of the supercurrent.

§7.3. Gauge theory with matter on $M^{3|2}$

In this subsection we prove Theorem 6.33(i). Suppose we are given the data in (6.35): a Riemannian manifold X on which G acts by isometries and a G-invariant function $h\colon X\to\mathbb{R}$. Fix a G bundle $\mathcal{P}\to M^{3|2}$. Then we can form the associated bundle $X^{\mathcal{P}}\to M^{3|2}$ whose typical fiber is X. The superfields in the theory are a constrained connection $\mathcal{A}$ on $\mathcal{P}$ and a section Φ of $X^{\mathcal{P}}$. We view Φ as an equivariant map

$$\Phi\colon\mathcal{P}\longrightarrow X. \tag{7.28}$$

Then the formulas in §4 make sense provided we interpret the vector fields D_a,∂_{ab} as the horizontal lifts to $\mathcal{P}$ of the indicated vector fields on $M^{3|2}$. For example, the component fields are still defined by (4.7), but now $\psi_a = i^*\mathcal{D}_a\Phi$ where $\mathcal{D}_a$ acts by the covariant derivative associated to $\mathcal{A}$. Also, in the definition of the auxiliary field $F = \frac{1}{2}i^*(\mathcal{D}_2\mathcal{D}_1 - \mathcal{D}_1\mathcal{D}_2)\Phi$ the outer covariant derivatives are constructed from both $\mathcal{A}$ and the Levi-Civita connection on X.

The superspace lagrangian for the theory is $L_0+L_1+L_2$, where the individual terms are defined in (4.16), (4.20), and (7.18). The only possible changes to the computation of the component lagrangian are additional curvature terms from the connection $\mathcal{A}$. Recall that $\mathcal{A}$ is constrained to be flat on the odd distribution τ, and this constraint eliminates many such possible terms. With this in mind we examine the computation (4.18). There is a change in the formula (4.11) for $\mathcal{D}^2\mathcal{D}_b\Phi$, which

is used to pass from the second line of (4.18) to the third line. Examining (4.12) we see that there is a new term from the even-odd curvature of $\mathcal{A}$ in passing to the last line. Hence (4.11) is replaced by[36]

$$\mathcal{D}^2\mathcal{D}_a\Phi = \epsilon^{bc}\big(\nabla_{ab} - \frac{1}{6}R(\mathcal{D}_a\Phi, \mathcal{D}_b\Phi)\big)\mathcal{D}_c\Phi + 2(\Lambda_a)_\Phi, \tag{7.29}$$

where $(\Lambda_a)_\Phi$ is the vector field on $X^{\mathcal{P}}$ induced by the action of Λ_a on X. The new term in (7.29) contributes an extra term to (4.18), and the new contribution to the component lagrangian is

$$-\epsilon^{ab}\langle\psi_a, (\hat{\lambda}_b)_\phi\rangle = -\langle\psi_a, \hat{\lambda}^a_\phi\rangle = \langle\hat{\lambda}^a_\phi, \psi_a\rangle. \tag{7.30}$$

The computation (4.21) with the potential term remains unchanged. Altogether, we find the total component lagrangian to be (6.36), and this completes the proof of Theorem 6.33(i).

[36] Recall a tricky sign in [I-**Classical Fields**, §4.2]: An equivariant vertical vector field is identified with a section of the adjoint bundle, and the action on a section of an associated bundle is by *minus* the directional derivative.

CHAPTER 8
$N = 1$ Yang-Mills Theory in Dimension 4

Here we complete the discussion initiated in §6.3 of the supersymmetric formulation of super Yang-Mills theory in dimension 4. This chapter parallels the previous one. One new feature in §8.1 is the appearance of an auxiliary field as well as the existence of an R-symmetry. Another new feature appears in §8.2, where the coupling constant of the Yang-Mills lagrangian is complex. Its imaginary part is proportional to the usual coupling constant, whereas its real part is the coefficient of a Chern-Weil form. (If this term is present, then the theory is not invariant under orientation-reversing symmetries.) In §8.3 we describe the superspace approach to gauge theories with matter in case the target manifold of the scalar field is Hodge. Then there is a geometric explanation for the appearance of the moment map in the component lagrangian (6.42).

§8.1. Constrained connections on $M^{4|4}$

We specialize the discussion in §6.3 to $M^{4|4}$. As explained in §2.4 we work in a complex basis. Our first task is to evaluate (6.74). To that end we compute

$$\begin{aligned} g_{\mu\nu}\tilde{\Gamma}^{\mu a\dot{c}}\Gamma^{\nu}_{b\dot{d}} &= g_{\mu\nu}\epsilon^{aa'}\epsilon^{\dot{c}\dot{c}'}\Gamma^{\mu}_{a'\dot{c}'}\Gamma^{\nu}_{b\dot{d}} \\ &= \frac{1}{2}\epsilon_{a'b}\epsilon_{\dot{c}'\dot{d}}\epsilon^{aa'}\epsilon^{\dot{c}\dot{c}'} \\ &= 2\delta^a_b\delta^{\dot{c}}_{\dot{d}} \end{aligned} \tag{8.1}$$

Plugging into (6.74) we obtain

$$\mathcal{D}_b\Lambda^a = -\frac{1}{2}\tilde{\Gamma}^{\mu ac}\Gamma^{\nu}_{cb}\mathcal{F}_{\mu\nu} - \frac{1}{2}\delta^a_b\overline{\mathcal{D}}_{\dot{c}}\overline{\Lambda}^{\dot{c}}. \tag{8.2}$$

Thus there is an auxiliary field in this theory. Setting $a = b$ and summing, the first term on the right hand side of (8.2) vanishes, and so

$$\mathcal{D}_c\Lambda^c = -\overline{\mathcal{D}}_{\dot{c}}\overline{\Lambda}^{\dot{c}}. \tag{8.3}$$

Define the (auxiliary) field

$$E = -\frac{\sqrt{-1}}{2}\mathcal{D}_c\Lambda^c. \tag{8.4}$$

Equation (8.3) asserts that E is real:

$$E = \overline{E}. \tag{8.5}$$

From (6.74) we also learn that

$$\overline{\mathcal{D}}_{\dot{b}}\Lambda^a = 0; \tag{8.6}$$

in other words, Λ^a is a chiral superfield.

Let A, λ, E be the restrictions of $\mathcal{A}, \Lambda, E$ to $\check{M}^4$. (It should cause no confusion that the restriction of E to $\check{M}^4$ is also denoted 'E'.) The multiplet $\{A, \lambda, E\}$ of component fields is called the *vector multiplet*, and from the vector multiplet we can reconstruct the constrained connection $\mathcal{A}$. The precise statement is the following.

Proposition 8.7. *The category of connections $\mathcal{A}$ on $M^{4|4}$ whose curvature vanishes on the odd distribution τ is equivalent to the category of triples $\{A, \lambda, E\}$ consisting of a connection A on $\check{M}^4$, a dual real spinor field λ with values in the adjoint bundle, and a section E of the adjoint bundle.*

For an elucidation of what Proposition 8.7 claims, we refer to the comments following Proposition 7.5. The following proof contains a description of an unconstrained field equivalent to a constrained connection.

Proof. We first consider the complex analog of Proposition 8.7. For the complex super Minkowski space $M_{\mathbb{C}} = M_{\mathbb{C}}^{4|4}$, instead of a complex structure on the distribution τ, we get a decomposition $\tau = \tau' \oplus \tau''$, and the distributions τ' and τ'' are integrable. Dividing by the corresponding foliations, we obtain

$$M'' \overset{\mathrm{pr}''}{\leftarrow} M_{\mathbb{C}} \overset{\mathrm{pr}'}{\rightarrow} M', \tag{8.8}$$

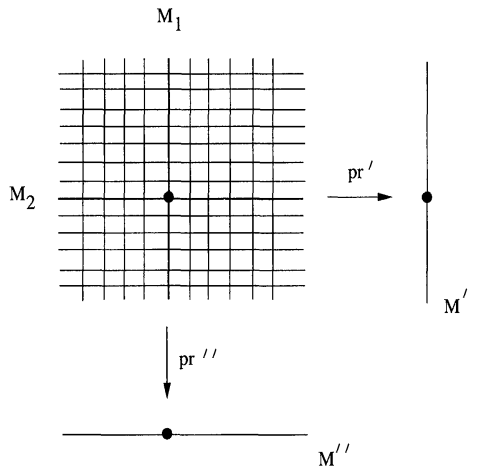
with τ' (resp τ'') being the relative tangent bundle of $M_{\mathbb{C}}/M''$ (resp $M_{\mathbb{C}}/M'$). Let $\check{M}_{\mathbb{C}} \subset M_{\mathbb{C}}$ be the ordinary Minkowski space, with images $\check{M}'$ in M' and $\check{M}''$ in M''.

Put $M_1 = (\mathrm{pr}'')^{-1}(\check{M}'')$ and $M_2 := (\mathrm{pr}')^{-1}(\check{M}')$. The projections pr′ and pr″ induce isomorphisms

$$M_1 \overset{\sim}{\rightarrow} M' \tag{8.9}$$

and

$$M_2 \overset{\sim}{\rightarrow} M''. \tag{8.10}$$



If P is a G-torsor on $M_{\mathbb{C}}$, a constrained connection on $\mathcal{P}$ is determined by its restriction to τ and, arguing as in the proof of Proposition 7.5, one sees that constrained connections correspond one to one to connections in the direction of τ, which are integrable in the directions of τ' and τ''. An integrable connection in the direction of τ'' (resp τ') amounts to a way to descend $\mathcal{P}$ from $M_{\mathbb{C}}$ to M' (resp M''). A constrained connection hence amounts to the data of $\mathcal{P}'$ on M' and $\mathcal{P}''$ on M'' descending $\mathcal{P}$:

$$\mathcal{P} \xrightarrow{\sim} (\mathrm{pr}')^{-1}(\mathcal{P}') \tag{8.11}$$

and

$$\mathcal{P} \xrightarrow{\sim} (\mathrm{pr}'')^{-1}(\mathcal{P}''). \tag{8.12}$$

In the τ' (resp. τ'') direction, the connection is given by (8.12) (resp. (8.11)).

Let P be the restriction of $\mathcal{P}$ to $\check{M}_{\mathbb{C}}$, and let us choose a trivialization of P. From (8.12) (resp (8.11)) we get a trivialization of $\mathcal{P}$ on M_1 (resp M_2): the only one extending the given trivialization and compatible with the connection along τ' (resp τ''). From (8.9) (resp (8.10)) and (8.11) (resp (8.12)), we then get trivializations s' and s'' of $\mathcal{P}'$ and $\mathcal{P}''$.

We keep noting s' and s'' their inverse images in $\mathcal{P}$ and let $g\colon M_{\mathbb{C}} \to G$ be the map such that $s'g = s''$. The map g to G, trivial on M_1 and M_2, is an unconstrained field equivalent to the constrained connection. It can be written $g = \exp(U)$, with U a map to $\mathfrak{g} = \mathrm{Lie}(G)$ vanishing on M_1 and M_2. In terms of g, the data $(\mathcal{P}, \mathcal{A})$ is the trivial torsor, with the trivial connection in the direction of τ'' and, in the direction of τ', the connection for which g is horizontal (given by the 1-form $-g^{-1}dg$).

Let us use coordinates as in §2.4. In those coordinates, M_1 is defined by $\bar{\theta}^{\dot{b}} = 0$, and M_2 by $\theta^a = 0$. Let J be the ideal of functions vanishing on M_1 and M_2, and I be the ideal generated by all θ and $\bar{\theta}$. Then, $(J \cap I^n)/J \cap I^{n+1}$ vanishes for $n \neq 2, 3, 4$ and is spanned respectively by

$$\begin{aligned} n = 2 &: \text{ the } \theta^a \bar{\theta}^{\dot{b}} \\ n = 3 &: \text{ the } \theta^1\theta^2\bar{\theta}^{\dot{b}} \text{ and the } \theta^a\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}} \\ n = 4 &: \ \theta^1\theta^2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}, \end{aligned}$$

the coefficients being functions on $\check{M}_{\mathbb{C}}$. We have

$$\begin{aligned} \mathcal{D}_a &= D_a, \\ \mathcal{D}_{\dot{a}} &= D_a - g^{-1}D_a g. \end{aligned} \tag{8.13}$$

It is then easy to check that $U \pmod{I}^3$ gives the connection on P; that changing U by a section of $(J \cap I^3) \otimes \mathfrak{g}$, taken modulo $J \cap I^4$, gives λ; and changing U by a section of $(J \cap I^4) \otimes \mathfrak{g}$ gives E.

We now consider the real case. The arguments used in the complex case work in just the same way for the cs case, where super Minkowski space is complexified just in the odd direction. As explained in the comments after Proposition 7.5, we have to prove that some map is a bijection. This map is obtained from the

similar map in the cs case by taking the fixed points of complex conjugation, and Proposition 8.7 follows.

In the physics literature the field U is referred to as the vector superfield in "Wess-Zumino gauge".

It is convenient to lower an index on the spinor field and define the chiral superfield[37]

$$\boxed{W_a = \Lambda^b \epsilon_{ba} = \frac{1}{4}\epsilon^{\dot{b}\dot{c}}[\nabla_{a\dot{b}}, \overline{\mathcal{D}}_{\dot{c}}]} \tag{8.14}$$

together with the conjugate antichiral superfield $\overline{W}_{\dot{a}}$. Equation (6.73) specializes to

$$[\mathcal{D}_a, \nabla_{b\dot{c}}] = 2\epsilon_{ab}\overline{W}_{\dot{c}}. \tag{8.15}$$

The auxiliary field E is

$$E = -\frac{\sqrt{-1}}{2}\epsilon^{ab} i^* \mathcal{D}_a W_b. \tag{8.16}$$

The reality condition (8.5) (equivalently (8.3)) is

$$\epsilon^{ab}\mathcal{D}_a W_b + \epsilon^{\dot{a}\dot{b}}\overline{\mathcal{D}}_{\dot{a}}\overline{W}_{\dot{b}} = 0 \tag{8.17}$$

This has the useful consequence

$$\mathcal{D}^2 W_a = -2\epsilon^{\dot{b}\dot{c}}\nabla_{a\dot{b}}\overline{W}_{\dot{c}}. \tag{8.18}$$

It is convenient to write the curvature as

$$F_A := \frac{1}{2}(F_{ab}\epsilon_{\dot{c}\dot{d}} + \overline{F}_{\dot{c}\dot{d}}\epsilon_{ab})\, dy^{a\dot{c}} \wedge dy^{b\dot{d}}, \tag{8.19}$$

where F_{ab} is symmetric in a, b. Our convention is that the curvature is real—$\overline{F}_{\dot{a}\dot{b}} = \overline{F_{ab}}$—for a compact gauge group G. *Warning*: The involution on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is such that the real subalgebra is the Lie algebra of the compact group. Thus for $G = \mathbb{T}$ the circle group the real subalgebra in $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}$ is $i\mathbb{R} \subset \mathbb{C}$.

In these variables the component fields are

$$\boxed{\begin{aligned} A &= i^*\mathcal{A} \\ \lambda_a &= i^* W_a, \end{aligned}} \tag{8.20}$$

and from (8.2) and (8.16) we have

$$\boxed{F_{ab} + \sqrt{-1}\,\epsilon_{ab} E = i^* \mathcal{D}_a W_b.} \tag{8.21}$$

[37]The symbol 'W' is standard in the physics literature.

Let $\hat{\xi}$ be the vector field on the space of component fields corresponding to the supersymmetry transformation $\eta^a Q_a + \bar{\eta}^{\dot{a}} \overline{Q}_{\dot{a}}$. We compute it from the general formula (1.29) and the other formulas in this section. We also can check the result against (6.8) and (6.9) if we set the auxiliary field E to zero. The result is:

$$\boxed{\begin{aligned} (\hat{\xi}A)_{a\dot{b}} &= -2\eta^c \epsilon_{ac} \bar{\lambda}_{\dot{b}} - 2\bar{\eta}^{\dot{c}} \epsilon_{\dot{b}\dot{c}} \lambda_a \\ \hat{\xi}\lambda_a &= -\eta^b F_{ab} + \eta^b \epsilon_{ab}\sqrt{-1}E \\ \hat{\xi}E &= \quad \eta^a \sqrt{-1}(\not{D}\bar{\lambda})_a - \bar{\eta}^{\dot{a}}\sqrt{-1}\,(\not{D}\lambda)_{\dot{a}} \end{aligned}} \tag{8.22}$$

and

$$\boxed{\hat{\xi}F_{ab} = \bar{\eta}^{\dot{c}}\{\nabla_{a\dot{c}}\lambda_b + \nabla_{b\dot{c}}\lambda_a\} + \eta^c\{\epsilon_{ac}(\not{D}\bar{\lambda})_b + \epsilon_{bc}(\not{D}\bar{\lambda})_a\}.} \tag{8.23}$$

Under the R-symmetry (2.81) the component fields transform by (see (8.14) and (8.16))

$$\boxed{\begin{aligned} [R, A_{a\dot{b}}] &= 0 \\ [R, \lambda_a] &= -\lambda_a \\ [R, \bar{\lambda}_{\dot{a}}] &= \bar{\lambda}_{\dot{a}} \\ [R, E] &= 0. \end{aligned}} \tag{8.24}$$

§8.2. The Yang-Mills action on $M^{4|4}$

We assume G compact with $\langle \cdot, \cdot \rangle$ a bi-invariant inner product on $\mathfrak{g}$. Extend $\langle \cdot, \cdot \rangle$ to be a *bilinear* form on $\mathfrak{g}_{\mathbb{C}}$. Fix $\tau \in \mathbb{C}$. Then for a constrained connection on $M^{4|4}$ the basic lagrangian density is

$$\boxed{\begin{aligned} \mathcal{L}_2 &= |d^4x| \left\{ -\frac{1}{16\pi} \operatorname{Im}\Big(d^2\theta\, \tau\, \epsilon^{ab} \langle W_a, W_b \rangle \Big) \right\} \\ &= |d^4x| \, \frac{1}{2i} \Big\{ d^2\theta \, \frac{-\tau}{16\pi} \epsilon^{ab} \langle W_a, W_b \rangle + d^2\bar{\theta} \, \frac{\bar{\tau}}{16\pi} \epsilon^{\dot{a}\dot{b}} \langle \overline{W}_{\dot{a}}, \overline{W}_{\dot{b}} \rangle \Big\}. \end{aligned}} \tag{8.25}$$

Notice that the two terms in $\mathcal{L}_2$ are defined on different cs-manifolds. We compute the contribution of the first to the component lagrangian using (8.20) and (8.21):

(8.26)

$$\begin{aligned} \frac{\tau}{16\pi} \epsilon^{ab} i^* D^2 \langle W_a, W_b \rangle &= \frac{\tau}{16\pi} \epsilon^{ab} i^* \Big\{ 2\langle \mathcal{D}^2 W_a, W_b \rangle + \epsilon^{cd} \langle \mathcal{D}_d W_a, \mathcal{D}_c W_b \rangle \Big\} \\ &= -\frac{\tau}{4\pi} \epsilon^{ab} \epsilon^{\dot{c}\dot{d}} i^* \langle \nabla_{a\dot{c}} \overline{W}_{\dot{d}}, W_b \rangle \\ &\qquad + \frac{\tau}{16\pi} \epsilon^{ab} \epsilon^{cd} \langle F_{da} + \epsilon_{da}\sqrt{-1}E, F_{cb} + \epsilon_{cb}\sqrt{-1}E \rangle \\ &= \frac{\tau}{4\pi} \langle \bar{\lambda} \not{D}_A \lambda \rangle - \frac{\tau}{16\pi} \epsilon^{ab} \epsilon^{cd} \langle F_{ac}, F_{bd} \rangle + \frac{\tau}{8\pi} |E|^2. \end{aligned}$$

The contribution of the second term in (8.25) is similar and we combine them using (2.68). We fix the standard orientation on $\check{M}^4$ to write the result in terms of differential forms:

$$d^4x\, \check{\mathcal{L}}_2 = d^4x \left\{ -\frac{\operatorname{Im}\tau}{8\pi} |F_A|^2 + \frac{\operatorname{Im}\tau}{4\pi} \langle \bar{\lambda} \not{D}_A \lambda \rangle + \frac{\operatorname{Im}\tau}{8\pi} |E|^2 \right\} + \frac{\operatorname{Re}\tau}{8\pi} \langle F \wedge F \rangle. \tag{8.27}$$

Introduce the notation

$$\boxed{\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.} \tag{8.28}$$

Then

$$\boxed{d^4x\, \check{\mathcal{L}}_2 = d^4x\, \frac{1}{g^2}\left\{-\frac{1}{2}|F_A|^2 + \langle \bar{\lambda}\, \not{D}_A\, \lambda\rangle + \frac{1}{2}|E|^2\right\} + \frac{\theta}{16\pi^2}\langle F_A \wedge F_A\rangle.} \tag{8.29}$$

This is essentially the super Yang-Mills lagrangian (6.3). We have written the second term in complex notation and dropped an exact term along the way. Note E enters as an auxiliary field; its equation of motion is $E = 0$. The last term is *topological*; the integral is a characteristic number (for connections of finite action). Note that the first term makes sense as a density whereas to integrate the topological term we need an orientation on $\check{M}^4$. Another way to express (8.29) is to write $F = F^+ + F^-$ in terms of its self-dual and anti-self-dual pieces (see (2.64)). Then the lagrangian function is

$$\boxed{\check{\mathcal{L}}_2 = -\frac{\tau}{4\pi}|F_A^+|^2 + \frac{\bar{\tau}}{4\pi}|F_A^-|^2 + \frac{\operatorname{Im}\tau}{4\pi}\langle \bar{\lambda}\, \not{D}_A \lambda\rangle + \frac{\operatorname{Im}\tau}{8\pi}|E|^2.} \tag{8.30}$$

The orientation on $\check{M}^4$ enters here in distinguishing self-dual from anti-self-dual.

We are following standard physics usage in our choice of constants. The closed 4-form $\langle F_A \wedge F_A\rangle$ represents a real characteristic class of $P \to \check{M}^4$; if the bilinear form $\langle \cdot, \cdot\rangle$ on $\mathfrak{g}$ is such that the real characteristic class $\frac{1}{16\pi^2}\langle F_A \wedge F_A\rangle$ comes from a class in $H^4(BG; \mathbb{Z})$, then θ and $\theta + 2\pi$ yield the same quantum theory.

Next, we consider the action of the R-symmetry (2.81). Now $d^2\theta$ has weight 2 under R and from (8.14) we see that W_a has weight -1. It follows that (8.25) is invariant under R. One can also use (8.24) to check that (8.29) is invariant under the R-symmetry. We remark that even though the classical theory has a $U(1)$ R-symmetry, in the quantum theory this is broken to a finite subgroup by an anomaly (which is computed using the index theorem).

We do not analyze this theory in superspace, but rather simply report the equations of motion:

$$\boxed{\begin{aligned} \epsilon^{cd}\nabla_{d\dot{b}}F_{ac} + \epsilon^{\dot{c}\dot{d}}\nabla_{a\dot{d}}F_{\dot{b}\dot{c}} &= [\lambda_a, \bar{\lambda}_{\dot{b}}] \\ \epsilon^{ab}\nabla_{a\dot{c}}\lambda_b &= 0 \\ E &= 0 \end{aligned}} \tag{8.31}$$

and the supercurrent j_a which corresponds to the supersymmetry transformation Q_a:

$$\boxed{\begin{aligned} j_a = \frac{1}{2}\epsilon^{cd}\epsilon^{\dot{b}\dot{e}}\langle F_{ca}, \bar{\lambda}_{\dot{e}}\rangle\, \frac{\partial}{\partial y^{d\dot{b}}} \otimes |d^4y| \\ + \left\{\frac{1}{2}\epsilon^{\dot{c}\dot{b}}\epsilon^{\dot{d}\dot{e}}\langle \overline{F}_{\dot{c}\dot{d}}, \bar{\lambda}_{\dot{e}}\rangle - \epsilon^{\dot{b}\dot{c}}\langle \sqrt{-1}\, E, \bar{\lambda}_{\dot{c}}\rangle\right\} \frac{\partial}{\partial y^{a\dot{b}}} \otimes |d^4y|. \end{aligned}} \tag{8.32}$$

In this formula we take $g = 1$, $\theta = 0$.

§8.3. Gauge theory with matter on $M^{4|4}$

In this subsection we prove Theorem 6.33(ii). Suppose we are given the data in (6.41), but with the additional proviso that X be Hodge. This assumption is only for exposition, as the computation we review is local and done for arbitrary Kähler manifolds. Let $L_\omega \to X$ be the Hodge line bundle, a hermitian line bundle with curvature $\sqrt{-1}\omega$. In this theory a compact Lie group G acts by isometries on X, and we assume given a lift of the G action to L_ω. This is roughly equivalent to choosing a moment map

$$\mu\colon X \longrightarrow \mathfrak{g}^* \tag{8.33}$$

for the action of G on X, and in fact defines a moment map by the formula

$$\mu(\zeta) = -\sqrt{-1}\,\mathrm{vert}_1(\hat{\hat{\zeta}}), \qquad \zeta \in \mathfrak{g}. \tag{8.34}$$

Here $\mu(\zeta)$ is the function on X obtained by pairing (8.33) with ζ; the vector field $\hat{\hat{\zeta}}$ on L_ω is defined by the lifted action; and 'vert_1' indicates the vertical part with respect to the connection on L_ω, taken at points of unit norm. It is easy to check that μ is a moment map. In other words

$$d\mu(\zeta) = -\iota(\hat{\zeta})\omega, \tag{8.35}$$

where $\hat{\zeta}$ is the vector field on X induced by the G action. In case X is Kähler but not Hodge, we must choose a moment map. We also fix a G-invariant holomorphic function $W\colon X \to \mathbb{C}$.

Fix a G bundle $\mathcal{P} \to M^{4|4}$. The superfields in the theory are a constrained connection $\mathcal{A}$ on $\mathcal{P}$ and a chiral section Φ of the associated bundle $X^{\mathcal{P}}$, which we view as an equivariant chiral map

$$\Phi\colon \mathcal{P} \longrightarrow X. \tag{8.36}$$

As in (5.21) there is a global lift to an equivariant chiral map

$$\tilde{\Phi}\colon \mathcal{P} \longrightarrow \Phi^* L_\omega. \tag{8.37}$$

Then the total superspace lagrangian for the theory is $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$, where the individual terms are defined in (5.22), (5.32), and (8.25). We take the coupling constants in the gauge theory to be $g = 1$ and $\theta = 0$. Our task is to work out the expansion in component fields. Note that component fields are defined by (5.7), though now the vector fields which enter the definitions are the $\mathcal{D}_a$ on $\mathcal{P}$—the horizontal lifts of the D_a defined by the connection $\mathcal{A}$.

We re-examine the computation beginning in (5.25). Since this uses the alternative form (5.17) of the lagrangian in terms of a local Kähler potential K, which we assume is G-invariant, our first job is to express a moment map μ in terms of K. Let $\hat{\zeta}$ be the real vector field on X which corresponds to $\zeta \in \mathfrak{g}$. Then an easy computation from (8.35) shows

$$\langle \check{\mu}, \zeta \rangle = \mu(\zeta) = -\sqrt{-1}\,\hat{\zeta}^{1,0} \cdot K = \sqrt{-1}\,\hat{\zeta}^{0,1} \cdot K \tag{8.38}$$

defines a moment map, where $\hat{\zeta} = \hat{\zeta}^{1,0} + \hat{\zeta}^{0,1}$ is expressed as a sum of its $(1,0)$ and $(0,1)$ components and $\check{\mu}\colon X \to \mathfrak{g}$ is obtained from μ using the inner product on $\mathfrak{g}$.

There are several new terms in (5.25)–(5.27) from the nonzero bracket (8.15). First, in the third line of (5.25) there is a new contribution from the bracket term in (5.24):

(8.39)
$$\begin{aligned} i^*\left(\frac{1}{4}\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}\Big\{2\iota([\mathcal{D}_b, \nabla_{a\dot{d}}])\Phi^*\partial K\Big\}\right) \\ = -2i^*\epsilon^{\dot{c}\dot{d}}\overline{\mathcal{D}}_{\dot{c}}\,\iota(\overline{W}_{\dot{d}})\Phi^*\partial K \\ = -2i^*\epsilon^{\dot{c}\dot{d}}\iota\big([\overline{\mathcal{D}}_{\dot{c}}, \overline{W}_{\dot{d}}]\big)\Phi^*\partial K + 2i^*\epsilon^{\dot{c}\dot{d}}\iota(\overline{\mathcal{D}}_{\dot{c}})\iota(\overline{W}_{\dot{d}})\Phi^*\bar{\partial}\partial K \\ = 4i^*\sqrt{-1}\,\iota\big(-\hat{E}_\Phi\big)\partial K + 2i^*\epsilon^{\dot{c}\dot{d}}\iota(\overline{\mathcal{D}}_{\dot{c}}\Phi)\iota\big(-(\widehat{\overline{W}}_\Phi)_{\dot{d}}\big)\bar{\partial}\partial K \\ = 4\langle\phi^*\check{\mu}, E\rangle + \sqrt{2}\,\epsilon^{\dot{c}\dot{d}}\langle\bar{\psi}_{\dot{c}}, (\hat{\bar{\lambda}}_\phi)_{\dot{d}}\rangle, \end{aligned}$$

where $(\widehat{\overline{W}}_{\dot{d}})_\Phi$ is the vector field on $X^{\mathcal{P}}$ induced by $\overline{W}_{\dot{d}}$ and similarly for E. In the third line we use (5.23) and the fact that $\iota(\overline{\mathcal{D}}_{\dot{c}})\Phi^*\partial K$ is a G-invariant function on $\mathcal{P}$, so its derivative in the fiber direction $\overline{W}_{\dot{d}}$ vanishes; in the fourth the definition (8.16) of E and the fact that vertical vectors on $\mathcal{P}$ act with a minus sign (cf. the footnote in §7.3); and in the fifth (8.38) and (5.16), together with an extra minus sign due to the odd variables.[38] From (5.28) and (5.29) we see that the results of (5.25)–(5.27) enter the component lagrangian with a factor $1/4$ and are added to their complex conjugates. Hence (8.39) leads to a contribution

$$2\langle\phi^*\check{\mu}, E\rangle + \frac{\sqrt{2}}{4}\Big(\epsilon^{cd}\langle(\hat{\lambda}_\phi)_c, \psi_d\rangle + \epsilon^{\dot{c}\dot{d}}\langle\bar{\psi}_{\dot{c}}, (\hat{\bar{\lambda}}_\phi)_{\dot{d}}\rangle\Big) \tag{8.41}$$

to the component lagrangian. Also, in the second line of (5.27) there is a new term

$$\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\Big\{-\mathcal{D}_a\,\iota([\overline{\mathcal{D}}_{\dot{c}}, \nabla_{b\dot{d}}])\Phi^*\partial K\Big\} = -4\epsilon^{ab}\mathcal{D}_a\,\iota(W_b)\Phi^*\partial K, \tag{8.42}$$

which by a similar computation contributes

$$-4\langle\phi^*\check{\mu}, E\rangle - \frac{\sqrt{2}}{2}\Big(\epsilon^{cd}\langle(\hat{\lambda}_\phi)_c, \psi_d\rangle + \epsilon^{\dot{c}\dot{d}}\langle\bar{\psi}_{\dot{c}}, (\hat{\bar{\lambda}}_\phi)_{\dot{d}}\rangle\Big) \tag{8.43}$$

to the component lagrangian.

As in §7.3 there is also a new term due to a revision of (5.9):

$$\overline{\mathcal{D}}_{\dot{a}}\mathcal{D}^2\Phi = -2\epsilon^{bc}\Big\{\nabla_{b\dot{a}}\Phi - \frac{1}{4}R(\mathcal{D}_b\Phi, \overline{\mathcal{D}}_{\dot{a}}\overline{\Phi})\Big\}\mathcal{D}_c\Phi - 4(\widehat{\overline{W}}_{\dot{a}})_\Phi, \tag{8.44}$$

This enters the computation in (5.26), where we pick up

$$\frac{1}{4}\epsilon^{\dot{c}\dot{d}}\langle\overline{\mathcal{D}}_{\dot{d}}\overline{\Phi}, 4(\widehat{\overline{W}}_{\dot{c}})_\Phi\rangle = -\epsilon^{\dot{c}\dot{d}}\langle\overline{\mathcal{D}}_{\dot{c}}\overline{\Phi}, (\widehat{\overline{W}}_{\dot{d}})_\Phi\rangle. \tag{8.45}$$

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[38] For even vectors $\xi_1, \bar{\xi}_2$ of type $(1,0), (0,1)$ respectively, equation (5.16) asserts

$$2\iota(\xi_1)\iota(\bar{\xi}_2)\bar{\partial}\partial K = -2\iota(\bar{\xi}_2)\iota(\xi_1)\bar{\partial}\partial K = \langle\bar{\xi}_2, \xi_1\rangle. \tag{8.40}$$

Now substitute $\xi_i = \eta_i\psi_i$, where ζ_i are odd vectors and η_i are auxiliary odd parameters.

Another new term enters into the last line of (5.27) from commuting $\mathcal{D}_a$ past $\nabla_{b\dot{d}}$:

$$\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\Big\{\frac{1}{2}\langle\overline{\mathcal{D}}_{\dot{c}}\overline{\Phi},[\mathcal{D}_a,\nabla_{b\dot{d}}]\Phi\rangle\Big\} = -2\epsilon^{\dot{c}\dot{d}}\langle\overline{\mathcal{D}}_{\dot{c}}\overline{\Phi},(\widehat{\overline{W}}_{\dot{d}})_\Phi\rangle. \tag{8.46}$$

Taking into account the factor $1/4$ in (5.28), and adding (8.45) and (8.46) to their complex conjugates, we obtain a contribution to the component lagrangian

$$-\frac{3\sqrt{2}}{4}\Big(\epsilon^{cd}\langle(\hat{\lambda}_\phi)_c,\psi_d\rangle + \epsilon^{\dot{c}\dot{d}}\langle\bar{\psi}_{\dot{c}},(\hat{\bar{\lambda}}_\phi)_{\dot{d}}\rangle\Big). \tag{8.47}$$

Combining (8.41), (8.43), and (8.47) we obtain the total new contribution to the component lagrangian:

$$-2\langle\phi^*\check{\mu},E\rangle - \sqrt{2}\Big(\epsilon^{cd}\langle(\hat{\lambda}_\phi)_c,\psi_d\rangle + \epsilon^{\dot{c}\dot{d}}\langle\bar{\psi}_{\dot{c}},(\hat{\bar{\lambda}}_\phi)_{\dot{d}}\rangle\Big) \tag{8.48}$$

The second term appears in (6.42), after using (8.14).

To recover the potential energy term in (6.42) we must eliminate the auxiliary field E. The terms involving E in (8.29) and (8.48) are:

$$\frac{1}{2}|E|^2 - 2\langle\phi^*\check{\mu},E\rangle. \tag{8.49}$$

The classical equation for E is then

$$\boxed{E = 2\phi^*\check{\mu}}, \tag{8.50}$$

and eliminating E we obtain the potential term

$$-2\phi^*|\mu|^2. \tag{8.51}$$

This completes the proof of Theorem 6.33(ii). Note the substitution (8.50) which gives the last equation in (6.43) from the second equation in (8.22).

CHAPTER 9
$N = 2$ Yang-Mills in Dimension 2

The theory we study in this chapter is the reduction to two dimensions of the basic four-dimensional supersymmetric gauge theory studied in §8. Thus we begin in §9.1 with some general remarks about dimensional reduction of ordinary bosonic Yang-Mills. The crucial observation is that in the dimensionally reduced theory we have not only a connection, but also some scalar fields and a nontrivial potential energy. The dimensional reduction of a constrained connection on $M^{4|4}$, which is described in §9.2, may be considered as a connection on $M^{2|4}$ with constraints. But the constraints now allow some nonzero curvature along the odd distribution, unlike the constraint condition in the maximal dimension 4. Recall also that the dimensionally reduced theory has a bigger R-symmetry group. In §9.3 we derive a superspace expression for the Yang-Mills lagrangian.

§9.1. Dimensional reduction of bosonic Yang-Mills

Consider pure Yang-Mills theory on Minkowski space $\check{M}^n$ in n dimensions with gauge group G. Fix a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. The field is a connection A on a principal G bundle $P \to \check{M}^n$, and the lagrangian is

$$L = |d^n x| \, \frac{-1}{2} |F_A|^2. \tag{9.1}$$

Let $x^0, x^1, \ldots, x^{n-1}$ be standard coordinates on $\check{M}^n$ so that the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{9.2}$$

Let T be the group of translations in the direction of the last k basis vectors. Let p be the quotient map $p\colon \check{M}^n \to \check{M}^n/T$. The quotient $\check{M}^n/T$ is a Minkowski space of dimension $n-k$. If $\check{M}^{n-k} \subset \check{M}^n$ is defined by $x^{n-k} = \cdots = x^{n-1} = 0$, the map p identifies $\check{M}^{n-k}$ with $\check{M}^n/T$.

In the dimensional reduction of pure Yang-Mills theory, one considers: (a) a principal G bundle P on $\check{M}^n/T \cong \check{M}^{n-k}$, and (b) a T-invariant connection A on its pullback to $\check{M}^n$. Let a be the restriction of A to $\check{M}^{n-k}$. Being T-invariant, the connection A can be written as

$$A = p^*(a) + \sum_{n-k}^{n-1} p^*(\phi_i) dx^i, \tag{9.3}$$

the ϕ_i being sections of the adjoint bundle.

The lagrangian density $\mathcal{L}_n$ on $\check{M}^n$ for A is T-invariant. In the reduced theory, the lagrangian density $\mathcal{L}_{n-k}$ on $\check{M}^{n-k}$ is obtained by restricting $\mathcal{L}_n$ to $\check{M}^{n-k}$ and contracting it against $\partial_{n-k}, \dots, \partial_{n-1}$: for Λ a lattice in T,

$$\int_{\check{M}^{n-k}} \mathcal{L}_{n-k} = \mathrm{vol}(T/\Lambda)^{-1} \int_{\check{M}^n/\Lambda} \mathcal{L}_n. \tag{9.4}$$

The curvatures F_A of A and F_a of a are related by

$$F_A = p^*(F_a) + \nabla_a(\phi_i) dx^i + \sum_{i<j} [\phi_i, \phi_j]\, dx^i \wedge dx^j, \tag{9.5}$$

giving for the lagrangian density of the reduced theory

$$\mathcal{L}_{n-k} = |d^{n-k}x| \left\{ -\frac{1}{2}|F_a|^2 + \frac{1}{2} \sum_{n-k}^{n-1} |\nabla_a(\phi_i)|^2 + \frac{1}{2} \sum_{i<j} |[\phi_i, \phi_j]|^2 \right\}. \tag{9.6}$$

The field content of the reduced theory is a connection and k sections of the adjoint bundle.

Most relevant to us is the reduction to $n-2$ dimensions. Thus consider a connection A invariant by translations in the x^{n-1}, x^{n-2} directions and write

$$A = a + \phi_1 dx^{n-2} + \phi_2 dx^{n-1}. \tag{9.7}$$

A short computation gives for the reduced lagrangian

$$L = |d^{n-2}x| \left\{ -\frac{1}{2}|F_a|^2 + \langle d_a\bar{\sigma}, d_a\sigma \rangle - \frac{1}{2} \| [\sigma, \bar{\sigma}] \|^2 \right\}, \tag{9.8}$$

where in the last line we set

$$\sigma = \frac{\phi_1 + \sqrt{-1}\phi_2}{\sqrt{2}}. \tag{9.9}$$

Thus σ is a section of the *complexified* adjoint bundle. Therefore, the field content of the reduction of pure Yang-Mills by two dimensions is a connection and a *complex* scalar field in the adjoint representation. There is a nonzero potential energy term for the scalar.

Consider the reduction of the bosonic part of the 4-dimensional supersymmetric Yang-Mills lagrangian (8.29) to 2 dimensions. The kinetic term for the connection reduces as in (9.8). The auxiliary field is a scalar, so the reduction is also a scalar. The topological term reduces nontrivially, and the total reduced bosonic lagrangian is

$$|d^2x|\, \frac{1}{g^2} \left\{ -\frac{1}{2}|F_a|^2 + \langle d_a\bar{\sigma}, d_a\sigma \rangle - \frac{1}{2} \| [\sigma, \bar{\sigma}] \|^2 + \frac{1}{2}|E|^2 \right\} - \sqrt{-1} \frac{\theta}{8\pi^2} \langle F_a, [\sigma, \bar{\sigma}] \rangle. \tag{9.10}$$

§9.2. Constrained connections on $M^{2|(2,2)}$

We use the notation established in (2.107)–(2.119) for $M^{2|(2,2)}$.

Let $\mathcal{P} \to M^{4|4}$ be a principal bundle with structure group G. We suppose that the vector fields $\partial_{1\dot{2}}, \partial_{2\dot{1}}$ on $M^{4|4}$ have been lifted to commuting complex conjugate vector fields $\tilde{\partial}_{1\dot{2}}, \tilde{\partial}_{2\dot{1}}$ on $\mathcal{P}$. Let $\mathcal{A}$ be a constrained connection on $\mathcal{P}$ which is invariant under these translations. All of the equations in §8.1 remain valid for $\mathcal{A}$, and we use them freely. The translation invariance is the equation

$$[\tilde{\partial}_{1\dot{2}}, \mathcal{D}_a] = [\tilde{\partial}_{2\dot{1}}, \mathcal{D}_a] = [\tilde{\partial}_{1\dot{2}}, \overline{\mathcal{D}}_{\dot{a}}] = [\tilde{\partial}_{2\dot{1}}, \overline{\mathcal{D}}_{\dot{a}}] = 0. \tag{9.11}$$

Then the vertical vector field

$$\Sigma = \frac{1}{\sqrt{2}}(\nabla_{2\dot{1}} - \tilde{\partial}_{2\dot{1}}) \tag{9.12}$$

is also translation invariant and corresponds to a section Σ of the complexified adjoint bundle. It is the basic invariant of a translationally invariant constrained connection. The Bianchi identity relates Σ to the superfields considered in §8.1. First, we consider Bianchi for $D_2, D_2, \overline{D}_{\dot{1}}$ and $D_2, \overline{D}_{\dot{1}}, \overline{D}_{\dot{1}}$ to deduce

$$\overline{\mathcal{D}}_{\dot{1}}\Sigma = \mathcal{D}_2\Sigma = 0. \tag{9.13}$$

In other words, the restriction of Σ to $M^{2|(2,2)}$ is *twisted chiral*. Then Bianchi for $D_1, \overline{D}_{\dot{1}}, \overline{D}_{\dot{2}}$ and Bianchi for $D_2, \overline{D}_{\dot{1}}, \overline{D}_{\dot{2}}$ imply

$$\begin{aligned} \sqrt{2}W_1 &= -\overline{\mathcal{D}}_{\dot{1}}\overline{\Sigma} \\ \sqrt{2}W_2 &= \ \overline{\mathcal{D}}_{\dot{2}}\Sigma. \end{aligned} \tag{9.14}$$

Using (9.14) and the results of §8.1 we can express the curvature $F_{\mathcal{A}}$ completely in terms of Σ.

Restrict the connection $\mathcal{A}$ and all derived superfields to $M^{2|(2,2)} \hookrightarrow M^{4|4}$. Let $i\colon \check{M}^2 \hookrightarrow M^{2|(2,2)}$ denote the inclusion, as usual. The component fields of the two-dimensional reduced theory are

$$\boxed{\begin{aligned} A &= i^*\mathcal{A} \\ \sigma &= i^*\Sigma \\ \lambda_+ &= i^*W_+ = -\frac{1}{\sqrt{2}} i^*\overline{\mathcal{D}}_+\overline{\Sigma} \\ \lambda_- &= i^*W_- = \ \frac{1}{\sqrt{2}} i^*\overline{\mathcal{D}}_-\Sigma \end{aligned}} \tag{9.15}$$

We also have

$$\boxed{f_A + [\sigma, \bar{\sigma}] + \sqrt{-1}E = i^*\mathcal{D}_+W_- = \frac{1}{\sqrt{2}} i^*\mathcal{D}_+\overline{\mathcal{D}}_-\Sigma,} \tag{9.16}$$

where the curvature of A is given in (8.19) as

$$F_A = 2f_A\, dy^+ \wedge dy^-. \tag{9.17}$$

The scalar field σ is a section of the complexified adjoint bundle; it is the same field which appears in the previous subsection. So too are f_A, E sections of the adjoint bundle, but both f_A and E are real.

From the point of view of $M^{2|(2,2)}$, we can think of $\mathcal{A}$ as a connection whose curvature is constrained to vanish along the odd distribution *except* for the $D_2, \overline{D}_{\dot{1}}$ direction (and its complex conjugate), where

$$[\mathcal{D}_2, \overline{\mathcal{D}}_{\dot{1}}] = -2\sqrt{2}\,\Sigma. \tag{9.18}$$

There is an important distinction between the abelian and nonabelian cases. In the abelian case the superfield Σ has values in the constant vector space $\mathfrak{g}$ and is unconstrained, except for the requirement that it be twisted chiral. Up to gauge equivalence the connection can be reconstructed from Σ. In the nonabelian case, on the other hand, the superfield Σ is a section of the adjoint bundle, and the twisted chirality condition (9.13) depends on the connection $\mathcal{A}$.

We can compute the transformation laws of the component fields under $\hat{\xi} = \eta^a \tau_{Q_a} + \bar{\eta}^{\dot{a}} \tau_{\overline{Q}_{\dot{a}}}$ directly, or by dimensional reduction from (8.22). The result is

$$\boxed{\begin{aligned}
(\hat{\xi}A)_+ &= -2\eta^- \bar{\lambda}_+ - 2\bar{\eta}^- \lambda_+ \\
(\hat{\xi}A)_- &= 2\eta^+ \bar{\lambda}_- + 2\bar{\eta}^+ \lambda_- \\
\hat{\xi}\sigma &= \sqrt{2}\eta^+ \bar{\lambda}_+ - \sqrt{2}\bar{\eta}^- \lambda_- \\
\hat{\xi}\lambda_+ &= -\eta^-(f_A + [\sigma, \bar{\sigma}] - \sqrt{-1}E) - \sqrt{2}\,\eta^+ \nabla_+ \bar{\sigma} \\
\hat{\xi}\lambda_- &= -\eta^+(f_A + [\sigma, \bar{\sigma}] + \sqrt{-1}E) + \sqrt{2}\,\eta^- \nabla_- \sigma \\
\hat{\xi} f_A &= \eta^- \nabla_- \bar{\lambda}_+ + \eta^+ \nabla_+ \bar{\lambda}_- + \bar{\eta}^+ \nabla_+ \lambda_- + \bar{\eta}^- \nabla_- \lambda_+ \\
\hat{\xi}E &= -\sqrt{-1}(-\eta^+ \nabla_+ \bar{\lambda}_- + \eta^- \nabla_- \bar{\lambda}_+ + \bar{\eta}^+ \nabla_+ \lambda_- - \bar{\eta}^- \nabla_- \lambda_+).
\end{aligned}} \tag{9.19}$$

From (9.12) it is easy to see that the R-symmetries J_+, J_- (see (2.117)) act on Σ by

$$\boxed{[J_\pm, \Sigma] = \mp\Sigma.} \tag{9.20}$$

From this and (9.15) we deduce the action on the component fields:

$$\boxed{\begin{aligned}
[J_\pm, A] = [J_\pm, f_A] = [J_\pm, E] = [J_+, \lambda_+] = [J_-, \lambda_-] = 0 \\
[J_\pm, \sigma] = \mp\sigma \\
[J_\pm, \bar{\sigma}] = \pm\bar{\sigma} \\
[J_+, \lambda_-] = -\lambda_- \\
[J_-, \lambda_+] = -\lambda_+.
\end{aligned}} \tag{9.21}$$

§9.3. The reduced Yang-Mills action

We rewrite the action (8.25) in terms of Σ. First, using (9.14) and (9.13) we have

$$\begin{aligned}
\epsilon^{ab}\langle W_a, W_b\rangle &= 2\langle W_+, W_-\rangle \\
&= -\langle \overline{\mathcal{D}}_+ \overline{\Sigma}, \overline{\mathcal{D}}_- \Sigma\rangle \\
&= -\overline{D}_+ \overline{D}_- \langle \overline{\Sigma}, \Sigma\rangle \\
&= -\overline{D}^2 \|\Sigma\|^2.
\end{aligned} \tag{9.22}$$

Let $\check{\mathcal{L}}'_2$ denote the dimensionally reduced component lagrangian function. From the second line of (8.25) we deduce

(9.23)
$$\begin{aligned}\check{\mathcal{L}}'_2 &= \frac{1}{2\sqrt{-1}}\Big\{\int d^2\theta\, \frac{-\tau}{16\pi}\epsilon^{ab}\langle W_a, W_b\rangle + \int d^2\bar\theta\, \frac{\bar\tau}{16\pi}\epsilon^{\dot a\dot b}\langle \overline{W}_{\dot a}, \overline{W}_{\dot b}\rangle\Big\} \\ &= \frac{1}{2\sqrt{-1}} i^*\Big\{-\frac{\tau}{16\pi} D^2\overline{D}^2\|\Sigma\|^2 + \frac{\bar\tau}{16\pi}\overline{D}^2 D^2\|\Sigma\|^2\Big\} \\ &= -\frac{\operatorname{Im}\tau}{4\pi} i^*\frac{1}{2}(D^2\overline{D}^2 + \overline{D}^2 D^2)\|\Sigma\|^2 - \frac{\operatorname{Re}\tau}{16\pi\sqrt{-1}} i^*\frac{1}{2}(D^2\overline{D}^2 - \overline{D}^2 D^2)\|\Sigma\|^2.\end{aligned}$$

Recall from §2.6 that for terms in a lagrangian in $M^{2|(2,2)}$ involving twisted chiral fields only we compute component lagrangians using exactly the combination of derivatives which occurs in the first term of (9.23). The second term is the topological term in (9.10); it does not seem to reduce to an integral $\int d^4\theta$. So for $\operatorname{Re}\tau = 0$ the $N = 2$ superspace Yang-Mills lagrangian in $M^{2|(2,2)}$ is

(9.24)
$$\boxed{\mathcal{L}'_2 = |d^2y|\, d^4\theta\, \frac{-1}{4g^2}\|\Sigma\|^2.}$$

The corresponding component lagrangian is the dimensional reduction of (8.29), which we compute either directly or with the aid of (9.8). After eliminating the auxiliary field E we obtain

(9.25)
$$\boxed{\begin{aligned}\check{\mathcal{L}}'_2 \sim |d^2y|\, \frac{1}{g^2}\Big\{&-\frac{1}{2}|F_A|^2 + \langle d_A\bar\sigma, d_A\sigma\rangle + \langle \bar\lambda\, \not{D}_A\, \lambda\rangle \\ &- \frac{1}{2}\|\,[\sigma,\bar\sigma]\,\|^2 + \sqrt{2}\,\big(\langle\bar\lambda^-, [\bar\sigma, \lambda^+]\rangle + \langle\bar\lambda^+, [\sigma, \lambda^-]\rangle\big)\Big\}.\end{aligned}}$$

The (classical) theory is invariant under the R-symmetries J_+, J_-, which are defined in (2.117). This follows either in superspace from (9.20) and (9.24) or in components from (9.21) and (9.25).

Another type of topological term may be defined for an invariant trace

(9.26)
$$\langle\cdot\rangle\colon \mathfrak{g} \longrightarrow \mathbb{R},$$

and it has a superspace expression. Define

(9.27)
$$\int d\bar\theta^- d\theta^+\, \ell = i^* D_+\overline{D}_-\ell.$$

The topological term is then

(9.28)
$$\boxed{\mathcal{L}'_3 = |d^2x|\, \frac{1}{\sqrt{2}}\operatorname{Re}\Big(d\bar\theta^- d\theta^+\,\langle\Sigma\rangle\Big).}$$

The component expression is easily seen from (9.16) to be

(9.29)
$$|d^2x|\,\check{\mathcal{L}}'_3 = |d^2x|\,\Big\{\langle f_A\rangle + \sqrt{-1}\langle E\rangle\Big\}.$$

CHAPTER 10
$N = 1$ Yang-Mills in Dimension 6 and $N = 2$ Yang-Mills in Dimension 4

In §10.1 we complete the discussion of the superspace formulation of super Yang-Mills theory in 6 dimensions. As in Chapter 7 and Chapter 8, the main point is to work out the structure of the auxiliary fields and the R-symmetries. The proof that the category of component fields (including the auxiliary fields) is equivalent to the category of superfields is deferred to the next chapter. We did not succeed in finding a manifestly supersymmetric formula for the lagrangian in this case. Section 10.2 is parallel to Chapter 9: We reduce the 6-dimensional pure gauge theory to 4 dimensions, where it has twice the minimal amount of supersymmetry. Neither theories in 4 dimensions with 8 supersymmetries which include matter fields nor theories in 4 dimensions with 16 supersymmetries have superspace formulations which exhibit all of the supersymmetry manifestly. But they can all be written on $M^{4|4}$, where 4 of the supersymmetries are manifest. We do this in §10.3.

§10.1. Constrained connections on $M^{6|(8,0)}$

We specialize the discussion in §6.3 to $M^{6|(8,0)}$. Our first task is to evaluate (6.74). To that end we compute using (2.89)–(2.91):

$$\begin{aligned} g\big(\tilde{\Gamma}(f^{ai}, f^{ck}), \Gamma(f_{bj}, f_{d\ell})\big) &= \frac{1}{2}\epsilon^{ik}\nu^{pqac}\epsilon_{j\ell}\, g(e_{pq}, e_{bd}) \\ &= \epsilon^{ik}\epsilon_{j\ell}\nu^{pqac}\nu_{pqbd} \\ &= 2(\delta^a_b\delta^c_d - \delta^a_d\delta^c_b)\epsilon^{ik}\epsilon_{j\ell}. \end{aligned} \tag{10.1}$$

The curvature term in (6.74) may be expanded using

$$\begin{aligned} \tilde{\Gamma}(f^{ai}, f^{ck}) \otimes \Gamma(f_{ck}, f_{bj}) &= \frac{1}{2}\epsilon^{ik}\nu^{pqac}\epsilon_{kj}\, e_{pq} \otimes e_{cb} \\ &= -\frac{1}{2}\delta^i_j\, \nu^{pqac}\, e_{pq} \otimes e_{cb}, \end{aligned} \tag{10.2}$$

so that the left hand side of (6.74) is

$$-2(-\frac{1}{2}\delta^i_j\nu^{pqac})\mathcal{F}_{(pq)(cb)} = \delta^i_j\, \nu^{pqac}\mathcal{F}_{(pq)(cb)} = \delta^i_j\, \nu^{acpq}\mathcal{F}_{(bc)(pq)}. \tag{10.3}$$

For $a = b$ this vanishes, since ν^{acpq} is symmetric in $(ac), (pq)$ whereas $\mathcal{F}_{(ac)(pq)}$ is skew. Now evaluate (6.74) for $i = j = 1$, then $i = j = 2$, and sum over $a = b$ to learn

$$\mathcal{D}_{ai}\Lambda^{ai} = 0. \tag{10.4}$$

Define

$$E^i_j = -\frac{\sqrt{-1}}{4}\mathcal{D}_{aj}\Lambda^{ai}. \tag{10.5}$$

Then with a bit of work we derive

$$\mathcal{D}_{bj}\Lambda^{ai} = -\frac{1}{2}\tilde{\Gamma}^{\mu(ai)(ck)}\Gamma^{\nu}_{(ck)(bj)}\mathcal{F}_{\mu\nu} + \delta^a_b\sqrt{-1}\,E^i_j, \tag{10.6}$$

which has the desired form (6.76).

To proceed further we remark on some linear algebra. For S_0 of dimension 4, the natural map

$$\bigwedge^2(\bigwedge^2 S_0) \to \bigwedge^2 S_0 \otimes \bigwedge^2 S_0 \to \bigwedge^3 S_0 \otimes S_0, \tag{10.7}$$

which is $u\wedge v \mapsto v\otimes v - v\otimes u$ followed by $(x\wedge y)\otimes(z\wedge t) \mapsto (x\wedge y\wedge z)\otimes t - (x\wedge y\wedge t)\otimes z$, gives rise to a short exact sequence

$$0 \to \bigwedge^2(\bigwedge^2 S_0) \to \bigwedge^3 S_0 \otimes S_0 \to \bigwedge^4 S_0 \to 0. \tag{10.8}$$

For S_0 given with a volume form, $\bigwedge^3 S_0$ is the dual of S_0, the map to $\bigwedge^4 S_0$ becomes the evaluation map, and we get

$$\bigwedge^2(\bigwedge^2 S_0) \cong \{\text{trace zero endomorphisms of } S_0\}. \tag{10.9}$$

This makes it convenient to define

$$f^a_b := \frac{1}{4}\nu^{acde}\mathcal{F}_{(bc)(de)}. \tag{10.10}$$

By (10.9) the tensor f is traceless:

$$f^a_a = 0. \tag{10.11}$$

Also,

$$\mathcal{F}_{(ab)(cd)} = \nu_{ebcd}f^e_a + \nu_{aecd}f^e_b. \tag{10.12}$$

Then using (10.3) we see that (10.6) simplifies to

$$\boxed{\mathcal{D}_{bj}\Lambda^{ai} = \delta^i_j\, f^a_b + \delta^a_b\sqrt{-1}\,E^i_j}. \tag{10.13}$$

We need one additional formula

$$\mathcal{D}_{ai}E^j_k = \sqrt{-1}\,(\epsilon_{ik}\nabla_{ab}\Lambda^{bj} - \delta^j_i\epsilon_{k\ell}\nabla_{ab}\Lambda^{b\ell}), \tag{10.14}$$

which is equivalent to (11.30) and will be explained in Chapter 11.

Now we define the component fields corresponding to a constrained connection. First, we have a connection A on the restriction $P \to \check{M}^6$ of $\mathcal{P}$ to Minkowski space. The odd part of the curvature restricts to a dual spinor field

$$\boxed{i^*\Lambda^{a\imath} = \lambda^{ai}.} \tag{10.15}$$

The restriction of $E_j^\imath$ to $\check{M}^6$, which we continue to denote '$E_j^\imath$', is an auxiliary field. Using (2.95), (2.97), and (6.66) we learn that

$$\overline{E_j^i} = E_i^\jmath, \tag{10.16}$$

By (10.4) we see that E is traceless:

$$E_i^i = 0. \tag{10.17}$$

Thus $\left(E_j^\imath\right)$ has three independent real components; the matrix $\left(\sqrt{-1}\, E_j^i\right)$ is traceless skew-Hermitian (with values in the adjoint bundle). It transforms in the adjoint representation of the $Sp(1)_R$ symmetry (2.102). In other words, the auxiliary fields comprise an imaginary quaternion. The six-dimensional *vector multiplet* is the collection of component fields $\{A, \lambda^{ai}, E_j^\imath\}$.

Proposition 10.18. *The category of connections $\mathcal{A}$ on $M^{6|(8,0)}$ whose curvature vanishes on the odd distribution τ is equivalent to the category of triples $\{A, \lambda, E\}$ consisting of a connection A on $\check{M}^6$, a dual spinor field λ with values in the adjoint bundle, and an imaginary quaternion scalar field E with values in the adjoint bundle.*

The proof of Proposition 10.18 is presented in Chapter 11.

We derive the supersymmetry transformation laws for the components as in previous sections. Denote the restriction of f_b^a to $\check{M}^6$ by 'f_b^a'. Then the supersymmetry transformation $\eta^{ai}Q_{a\imath}$ induces the following transformation on component fields:

$$\boxed{\begin{aligned} (\hat{\xi}A)_{pq} &= 2\eta^{ai}\nu_{pqab}\epsilon_{ij}\lambda^{bj} \\ \hat{\xi}\lambda^{ai} &= -\eta^{bi}f_b^a - \eta^{aj}\sqrt{-1}\, E_j^i \\ \hat{\xi}E_j^\imath &= \eta^{ai}\sqrt{-1}\,(\not{D}\lambda)_{aj} - \eta^{ak}\sqrt{-1}\,\epsilon_{jk}\epsilon^{i\ell}(\not{D}\lambda)_{a\ell}. \end{aligned}} \tag{10.19}$$

and

$$\boxed{\hat{\xi}f_b^a = 2\eta^{ai}(\not{D}\lambda)_{bi} - 2\eta^{c\imath}\epsilon_{ij}\nabla_{bc}\lambda^{aj} - \delta_b^a\eta^{c\imath}(\not{D}\lambda)_{c\imath}.} \tag{10.20}$$

Recall from (2.102) that the R-symmetry group is $SU(2)_R$. The matrix $S = (S_j^i)$ acts on the component fields by

$$\boxed{\begin{aligned} (SA) &= A \\ (S\lambda)^{ai} &= (S^{-1})_{i'}^\imath \lambda^{ai'} \\ (SE)_j^\imath &= (S^{-1})_{i'}^\imath E_{j'}^{i'} S_j^{j'}. \end{aligned}} \tag{10.21}$$

§10.2. Reduction to $M^{4|8}$

The story here is analogous to that in Chapter 9, where we reduce Yang-Mills on $M^{4|4}$ to $M^{2|(2,2)}$. For a worked out example, see Problem ASH8 of [I-**Homework**].

We consider a constrained connection $\mathcal{A}$ on a G-bundle $\mathcal{P} \to M^{6|(8,0)}$ which is "constant" in the $\partial_{12}, \partial_{34}$ directions. That is, ∂_{12} and ∂_{34} are lifted to commuting complex conjugate vector fields $\tilde{\partial}_{12}$ and $\tilde{\partial}_{34}$ on $\mathcal{P}$, and the horizontal vector fields $\mathcal{D}_{a\imath}$ on $\mathcal{P}$ satisfy

$$[\tilde{\partial}_{12}, \mathcal{D}_{ai}] = [\tilde{\partial}_{34}, \mathcal{D}_{ai}] = 0. \tag{10.22}$$

Set

$$\Sigma = \frac{1}{\sqrt{2}}(\nabla_{34} - \tilde{\partial}_{34}). \tag{10.23}$$

Then applying Bianchi to $D_{a\imath}, D_{31}, D_{42}$ we learn

$$\mathcal{D}_{3i}\Sigma = \mathcal{D}_{4i}\Sigma = 0. \tag{10.24}$$

From Bianchi we also find, after some tedious computation, a formula for Λ^{ai} in terms of derivatives of Σ:

$$\sqrt{2}\,\Lambda^{ai} = \begin{cases} J^a_b \epsilon^{ij} \mathcal{D}_{bj}\Sigma, & a = 1,2; \\ J^a_b \epsilon^{ij} \mathcal{D}_{bj}\overline{\Sigma}, & a = 3,4. \end{cases} \tag{10.25}$$

Here J^a_b is the matrix of the pseudoreal structure j described before (2.95). Also, from (10.6) we learn that $\mathcal{D}_{bj}\Lambda^{ai}$ is independent of a, b if $i \neq j$. It follows that

$$\begin{aligned} \mathcal{D}_{11}\mathcal{D}_{21}\Sigma &= \mathcal{D}_{31}\mathcal{D}_{41}\overline{\Sigma} \\ \mathcal{D}_{12}\mathcal{D}_{22}\Sigma &= \mathcal{D}_{32}\mathcal{D}_{42}\overline{\Sigma}. \end{aligned} \tag{10.26}$$

Let $j\colon M^{4|8} \hookrightarrow M^{6|(8,0)}$ be the inclusion, as described in the last paragraph of §2.5. We follow the notation introduced there and also use formula (2.104). Now write $\tilde{\mathcal{A}}, \tilde{D}_{a\imath}, \tilde{\Sigma}, \tilde{\Lambda}^{ai}, \tilde{E}^i_j$ for the fields on $M^{6|(8,0)}$ to distinguish them from their restrictions to $M^{4|8}$. Set

$$\Sigma = j^*\tilde{\Sigma}. \tag{10.27}$$

Then using (2.104), equation (10.24) asserts that Σ is a *chiral* superfield on $M^{4|8}$ in the sense that

$$\overline{\mathcal{D}}_{\dot{a}(\imath)}\Sigma = 0, \qquad a = 1,2, \quad i = 1,2. \tag{10.28}$$

Formula (10.25) expresses $\Lambda^{a\imath} = j^*\tilde{\Lambda}^{a\imath}$ in terms of Σ. We record the result in full:

$$\begin{aligned} \sqrt{2}\,\Lambda^{11} &= -\mathcal{D}^{(2)}_2\Sigma & \sqrt{2}\,\Lambda^{12} &= \mathcal{D}^{(1)}_2\Sigma \\ \sqrt{2}\,\Lambda^{21} &= \mathcal{D}^{(2)}_1\Sigma & \sqrt{2}\,\Lambda^{22} &= -\mathcal{D}^{(1)}_1\Sigma \\ \sqrt{2}\,\Lambda^{31} &= -\overline{\mathcal{D}}_{\dot{2}(1)}\overline{\Sigma} & \sqrt{2}\,\Lambda^{32} &= -\overline{\mathcal{D}}_{\dot{2}(2)}\overline{\Sigma} \\ \sqrt{2}\,\Lambda^{41} &= \overline{\mathcal{D}}_{\dot{1}(1)}\overline{\Sigma} & \sqrt{2}\,\Lambda^{42} &= \overline{\mathcal{D}}_{\dot{1}(2)}\overline{\Sigma} \end{aligned} \tag{10.29}$$

We also work out the auxiliary fields $E_j^i = j^* \tilde{E}_j^i$ using the definition (10.5) together with (10.16), (10.29), and (10.26):

$$
\begin{aligned}
E_1^1 &= -\frac{\sqrt{-1}}{2\sqrt{2}}\left(\mathcal{D}_2^{(1)}\mathcal{D}_1^{(2)}\Sigma - \mathcal{D}_1^{(1)}\mathcal{D}_2^{(2)}\Sigma\right) \\
E_2^1 &= -\frac{\sqrt{-1}}{\sqrt{2}}\mathcal{D}_2^{(2)}\mathcal{D}_1^{(2)}\Sigma \\
&= -\frac{\sqrt{-1}}{\sqrt{2}}\overline{\mathcal{D}}_{\dot{2}(1)}\overline{\mathcal{D}}_{\dot{1}(1)}\overline{\Sigma} \\
E_1^2 &= -\frac{\sqrt{-1}}{\sqrt{2}}\mathcal{D}_1^{(1)}\mathcal{D}_2^{(1)}\Sigma \\
&= -\frac{\sqrt{-1}}{\sqrt{2}}\overline{\mathcal{D}}_{\dot{1}(2)}\overline{\mathcal{D}}_{\dot{2}(2)}\overline{\Sigma} \\
E_2^2 &= -E_1^1.
\end{aligned}
\tag{10.30}
$$

From the point of view of $M^{4|8}$, we can think of $\mathcal{A}$ as a connection whose curvature is constrained to vanish along the odd distribution *except* for the nontrivial curvature

$$
[\overline{\mathcal{D}}_{\dot{1}(2)}, \overline{\mathcal{D}}_{\dot{2}(1)}] = -[\overline{\mathcal{D}}_{\dot{2}(2)}, \overline{\mathcal{D}}_{\dot{1}(1)}] = 2\sqrt{2}\,\Sigma \tag{10.31}
$$

and the complex conjugate.

As in Chapter 9 there is a sharp distinction between the abelian and nonabelian cases—in the abelian case Σ takes values in the constant vector space $\mathfrak{g}$, the only constraint being (10.28); whereas in the nonabelian case Σ is a section of $\operatorname{ad}\mathcal{P}$ and the constraint (10.28) involves the connection $\mathcal{A}$, which therefore cannot be derived solely from Σ.

Formulas (10.29) and (10.30) are more understandable if we restrict to $M^{4|4} \hookrightarrow M^{4|8}$. (The inclusion $M^{4|4} \hookrightarrow M^{6|(8,0)}$ is discussed towards the end of §2.5.) Let $\hat{\mathcal{A}}, \hat{\Sigma}$ denote the restrictions of $\mathcal{A}, \Sigma$ to $M^{4|4}$. Then $\hat{\mathcal{A}}$ is a vector superfield (cf. §8.1)—a constrained connection on the restriction $\hat{\mathcal{P}}$ of $\mathcal{P}$ to $M^{4|4}$—and (10.28) implies that $\hat{\Sigma}$ is a chiral superfield (cf. §5.1) with values in $\operatorname{ad}\hat{\mathcal{P}}$. The pair $(\hat{\mathcal{A}}, \hat{\Sigma})$ is called an $N = 2$ *vector multiplet* on $M^{4|4}$. We claim, but do not write out a proof, that the category of constrained connections $\mathcal{A}$ on $M^{6|(8,0)}$ which satisfy (10.22) is equivalent to the category of pairs $(\hat{\mathcal{A}}, \hat{\Sigma})$ on $M^{4|4}$.

Let $i\colon \breve{M}^4 \hookrightarrow M^{4|8}$ be the inclusion. Motivated by formulas (5.7), and (8.20), (8.4) we define the component fields of the superfields on $M^{4|8}$ to be

$$
\boxed{
\begin{aligned}
A &= i^*\mathcal{A} \\
\sigma &= i^*\Sigma \\
\lambda_a = i^*\epsilon_{ba}\Lambda^{b1} \quad &= -\frac{1}{\sqrt{2}}\, i^*\mathcal{D}_a^{(2)}\Sigma \\
\chi_a = i^*\epsilon_{ba}\Lambda^{b2} \quad &= \frac{1}{\sqrt{2}}\, i^*\mathcal{D}_a^{(1)}\Sigma \\
E = i^*E_1^1 \quad &= -\frac{\sqrt{-1}}{2\sqrt{2}}\, i^*\left(\mathcal{D}_2^{(1)}\mathcal{D}_1^{(2)}\Sigma - \mathcal{D}_1^{(1)}\mathcal{D}_2^{(2)}\Sigma\right) \\
F = -\frac{\sqrt{-1}}{\sqrt{2}}\, i^*E_1^2 &= \frac{1}{2}\, i^*\mathcal{D}_2^{(1)}\mathcal{D}_1^{(1)}\Sigma.
\end{aligned}
}
\tag{10.32}
$$

The indices a, b in these formulas run over $\{1, 2\}$. Also, we use (10.29), (10.30). Following (8.14) we define the dual spinor field

$$\lambda^a = i^*\Lambda^{a1} = -\frac{1}{\sqrt{2}}\, i^*\epsilon^{ab}\mathcal{D}_b^{(2)}\Sigma. \tag{10.33}$$

These component fields group together into $N = 1$ supermultiplets: (A, λ, E) is an $N = 1$ vector multiplet (§8.1) and (σ, χ, F) is an $N = 1$ chiral multiplet (§5.1). The collection $(A, \sigma, \lambda, \chi, E, F)$ of all component fields is called an $N = 2$ *vector multiplet*. The reader should check that (10.32) is consistent with (5.7) and (8.20). The only nontrivial compatibility check is to see that $\tilde{\Lambda}^{a1}$ $(a = 1, 2)$ on $M^{6|(8,0)}$ restricts on $M^{4|4}$ to be the superfield $\hat{\Lambda}^a$, with both $\tilde{\Lambda}$ and $\hat{\Lambda}$ defined by (6.66); we leave this check to the reader. Note that on $M^{4|4}$ we have

$$\hat{W}_a = -\frac{1}{\sqrt{2}}\, i^*\mathcal{D}_a^{(2)}\hat{\Sigma}. \tag{10.34}$$

On $M^{4|4}$ only half of the supersymmetry of the vector multiplet $(\hat{\mathcal{A}}, \hat{\Sigma})$ is manifest. For the rest we write the vector field $\hat{\xi}'$ on the space of $N = 1$ superfields $\mathcal{F}_{(\hat{\mathcal{A}},\hat{\Sigma})}$ which corresponds to the supersymmetry transformation

$$\eta'^a Q_a^{(2)} + \bar{\eta}'^{\dot{a}}\overline{Q}_{\dot{a}(2)}. \tag{10.35}$$

From the formulas in this section we find

$$\boxed{\begin{aligned} (\hat{\xi}'\hat{\mathcal{A}})_a &= 2\sqrt{2}\,\eta'^b\epsilon_{ba}\overline{\hat{\Sigma}} \\ (\hat{\xi}'\hat{\mathcal{A}})_{a\dot{b}} &= \sqrt{2}\big(\eta'^c\epsilon_{ca}\overline{\mathcal{D}}_{\dot{b}}\overline{\hat{\Sigma}} + \bar{\eta}'^{\dot{c}}\epsilon_{\dot{c}\dot{b}}\mathcal{D}_a\hat{\Sigma}\big) \\ \hat{\xi}'\hat{\Sigma} &= \sqrt{2}\,\eta'^a\hat{W}_a \end{aligned}} \tag{10.36}$$

The penultimate line of (6.79) is used to derive the first two of these formulas.

Next, we write all of the supersymmetry transformations on the component fields on $\check{M}^4$. Let $\hat{\xi}$ be the vector field on the space $\mathcal{F}_{(A,\sigma,\lambda,\chi,E,F)}$ of component fields induced by the supersymmetry transformation

$$\eta^a Q_a^{(1)} + \bar{\eta}^{\dot{a}}\overline{Q}_{\dot{a}(1)} + \eta'^a Q_a^{(2)} + \bar{\eta}'^{\dot{a}}\overline{Q}_{\dot{a}(2)}. \tag{10.37}$$

Then from (10.19) (or (8.22) together with (10.36) and the other formulas of this section) we deduce

(10.38)

$$\boxed{\begin{aligned} (\hat{\xi}A)_{a\dot{b}} &= 2\eta^c\epsilon_{ca}\bar{\lambda}_{\dot{b}} + 2\bar{\eta}^{\dot{c}}\epsilon_{\dot{c}\dot{b}}\lambda_a + 2\eta'^c\epsilon_{ca}\bar{\chi}_{\dot{b}} + 2\bar{\eta}'^{\dot{c}}\epsilon_{\dot{c}\dot{b}}\chi_a \\ \hat{\xi}\sigma &= -\sqrt{2}\,\eta^a\chi_a + \sqrt{2}\,\eta'^a\lambda_a \\ \hat{\xi}\lambda_a &= -\eta^b F_{ab} + \eta^b\epsilon_{ab}\sqrt{-1}\,E + \sqrt{2}\,\eta'^b\epsilon_{ab}\overline{F} - \sqrt{2}\,\bar{\eta}'^{\dot{b}}\nabla_{a\dot{b}}\sigma \\ \hat{\xi}\chi_a &= -\sqrt{2}\,\eta^b\epsilon_{ab}F + \sqrt{2}\,\bar{\eta}^{\dot{b}}\nabla_{a\dot{b}}\sigma - \eta'^b F_{ab} - \eta'^b\epsilon_{ab}\sqrt{-1}\,E \\ \hat{\xi}E &= \eta^a\sqrt{-1}\,(\not{D}\bar{\lambda})_a - \bar{\eta}^{\dot{a}}\sqrt{-1}\,(\not{D}\lambda)_{\dot{a}} - \eta'^a\sqrt{-1}\,(\not{D}\bar{\chi})_a + \bar{\eta}'^{\dot{a}}\sqrt{-1}\,(\not{D}\lambda)_{\dot{a}} \\ \hat{\xi}F &= \sqrt{2}\,\eta'^a(\not{D}\bar{\lambda})_a - \sqrt{2}\,\bar{\eta}^{\dot{a}}(\not{D}\chi)_{\dot{a}} \end{aligned}}$$

The R-symmetry group of the dimensionally reduced theory is $U(2)$ (see the end of §2.5). The matrix $S = (S_j^i)$ acts on the component fields by

(10.39)

$$\begin{gathered} S\colon A \longmapsto A \\ S\colon \sigma \longmapsto (\det S)^{-1}\sigma \\ S\colon \begin{pmatrix} \lambda_a \\ \chi_a \end{pmatrix} \longmapsto S^{-1}\begin{pmatrix} \lambda_a \\ \chi_a \end{pmatrix} \\ S\colon \begin{pmatrix} E & -\sqrt{-1}\sqrt{2}\,\overline{F} \\ \sqrt{-1}\sqrt{2}\,F & -E \end{pmatrix} \longmapsto S^{-1}\begin{pmatrix} E & -\sqrt{-1}\sqrt{2}\,\overline{F} \\ \sqrt{-1}\sqrt{2}\,F & -E \end{pmatrix} S. \end{gathered}$$

We remark that it is useful to check (10.38) using equations (5.5), (8.22), and the R-symmetry (10.39) for $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Finally, we write the $N = 2$ Yang-Mills lagrangian in four dimensions in a manifestly supersymmetric way on $M^{4|8}$. As with $M^{4|4}$ (see the beginning of §5.3) there is a "chiral odd integration" on $M^{4|8}$. Namely, if Υ is any superfield on $M^{4|8}$ which is chiral in the sense that $\overline{D}_{\dot{a}(i)}\Upsilon = 0$, then

$$|d^4x|\int d^4\theta\,\Upsilon = |d^4x|\; i^*D_2^{(1)}D_1^{(1)}D_2^{(2)}D_1^{(2)}\Upsilon \tag{10.40}$$

is a well-defined density on $\check{M}^4$. Given a connection $\mathcal{A}$ on $M^{4|8}$ of the type we have been considering, with Σ defined as above, set

$$\boxed{\mathcal{L} = |d^4x|\;\mathrm{Im}\Big\{d^4\theta\,\frac{\tau}{32\pi}\langle\Sigma,\Sigma\rangle\Big\}.} \tag{10.41}$$

We claim that $\mathcal{L}$ is a superspace lagrangian which reproduces in components the dimensional reduction of Yang-Mills that we seek. To verify this we transform it into a lagrangian on $N = 1$ superspace $M^{4|4}$. Namely, the component lagrangian function $\check{\mathcal{L}}$ may be written as

$$\begin{aligned} \check{\mathcal{L}} &= \mathrm{Im}\Big\{\frac{\tau}{32\pi}i^*D_2^{(1)}D_1^{(1)}D_2^{(2)}D_1^{(2)}\langle\Sigma,\Sigma\rangle\Big\} \\ &= \mathrm{Im}\Big\{\frac{\tau}{16\pi}i^*D_2^{(1)}D_1^{(1)}\big(-\langle\mathcal{D}_1^{(2)}\Sigma,\mathcal{D}_2^{(2)}\Sigma\rangle + \langle\mathcal{D}_2^{(2)}\mathcal{D}_1^{(2)}\Sigma,\Sigma\rangle\big)\Big\} \\ &= \mathrm{Im}\Big\{\frac{\tau}{16\pi}i^*D_2^{(1)}D_1^{(1)}\big(-2\langle W_1,W_2\rangle + \langle\overline{\mathcal{D}}_{\dot{2}(1)}\overline{\mathcal{D}}_{\dot{1}(1)}\overline{\Sigma},\Sigma\rangle\big)\Big\} \\ &= i^*D_2^{(1)}D_1^{(1)}\,\mathrm{Im}\Big\{-\frac{\tau\epsilon^{ab}}{16\pi}\langle W_a,W_b\rangle\Big\} + \frac{\mathrm{Im}\,\tau}{16\pi}i^*D_2^{(1)}D_1^{(1)}\overline{D}_{\dot{2}(1)}\overline{D}_{\dot{1}(1)}\langle\overline{\Sigma},\Sigma\rangle \\ &= \int d^2\theta\;\mathrm{Im}\Big\{-\frac{\tau\epsilon^{ab}}{16\pi}\langle W_a,W_b\rangle\Big\} + \frac{\mathrm{Im}\,\tau}{4\pi}\int d^4\theta\,\frac{1}{4}\langle\overline{\Sigma},\Sigma\rangle. \end{aligned} \tag{10.42}$$

In the third equation we use (10.34) and (10.26) (cf. (10.30)); in the fourth we use the fact that Σ is chiral; and in the fifth we use notions of integration on $M^{4|4}$. For the $d^4\theta$ term we use (2.71), so that the last equality holds only up to exact terms, which we ignore. The component expansion of the first term is (8.29). The component expansion of the second is (5.13), but we must remember the extra

terms explained in §8.3. Recall that for this we eliminate the auxiliary field E using (8.50):

$$\boxed{E = \sqrt{-1}\,[\sigma, \bar{\sigma}].} \tag{10.43}$$

Set

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\sqrt{-1}}{g^2}. \tag{10.44}$$

The component lagrangian is then a special case of (6.42):

(10.45)

$$\boxed{\begin{aligned} d^4x\,\check{\mathcal{L}} = d^4x\,\frac{1}{g^2}\Big\{ &-\frac{1}{2}|F_A|^2 + \langle d_A\bar{\sigma}, d_A\sigma\rangle + \langle \bar{\lambda}\not{D}_A\lambda\rangle + \langle \bar{\chi}\not{D}_A\chi\rangle - \frac{1}{2}|\,[\sigma,\bar{\sigma}]\,|^2 \\ &-\sqrt{2}\left(\langle\bar{\sigma},[\lambda^a,\chi_a]\rangle + \langle[\bar{\lambda}^{\dot{a}},\bar{\chi}_{\dot{a}}],\sigma\rangle\right) + \langle\overline{F},F\rangle\Big\} + \frac{\theta}{16\pi^2}\langle F_A\wedge F_A\rangle. \end{aligned}}$$

The $N = 2$ superspace expression (10.41) is convenient for studying quantum corrections to the classical theory (which are explained in Lecture 18 of [II-**Dynamics of QFT**]). Define the *holomorphic prepotential*

$$\mathcal{F}(\Sigma) = \frac{\tau}{2}\langle\Sigma,\Sigma\rangle. \tag{10.46}$$

Then we rewrite (10.41) as

$$\boxed{\mathcal{L} = |d^4x|\ \mathrm{Im}\Big\{d^4\theta\,\frac{\tau}{16\pi}\mathcal{F}(\Sigma)\Big\}.} \tag{10.47}$$

Using (5.17) we see that the last term of (10.42) corresponds to the Kähler potential

$$K(\overline{\Sigma},\Sigma) = \frac{1}{2}\,\mathrm{Im}\big(\overline{\Sigma}\,\frac{\partial\mathcal{F}}{\partial\Sigma}\big). \tag{10.48}$$

We can replace the classical prepotential $\mathcal{F}$ by any holomorphic function of Σ and still obtain a supersymmetric theory; the relation (10.48) between the prepotential $\mathcal{F}$ and the Kähler potential K is unchanged. The computation of the quantum effective action is then the determination of the function $\mathcal{F}$. Kähler metrics of the form (10.48) are called *special* by physicists, and indeed they exhibit many special properties.[39]

§10.3. More theories on $M^{4|4}$ with extended supersymmetry

We record without proof the form of the general $N = 2$ model in 4 dimensions written on the $N = 1$ superspace $M^{4|4}$. Recall (6.27) that the scalars take values in a hyperkähler manifold X. As in §3.6 we fix a Kähler structure on X. Then

[39]For a mathematical treatment of special Kähler geometry, see: D. S. Freed, *Special Kähler manifolds*, Commun. Math. Phys., to appear, `hep-th/9712042`.

the rest of the hyperkähler structure is encoded in a holomorphic symplectic form. The moment map $\mu_{\mathbb{H}}$ of the G action on X is

$$\mu_{\mathbb{H}} = (\mu_{\mathbb{R}}, \mu_{\text{hol}}), \tag{10.49}$$

where $\mu_{\mathbb{R}} \colon X \to \mathfrak{g}^*$ is the moment map for the Kähler structure and $\mu_{\text{hol}} \colon X \to \mathfrak{g}_{\mathbb{C}}^*$ the complex moment map of the holomorphic symplectic structure. The fields in the model are a constrained connection $\mathcal{A}$ on some principal bundle $\mathcal{P} \to M^{4|4}$ and a chiral superfield Φ with values in the associated fiber bundle $\tilde{X}^{\mathcal{P}}$, where

$$\tilde{X} = X \times \mathfrak{g}_{\mathbb{C}}. \tag{10.50}$$

Note that the $\mathfrak{g}_{\mathbb{C}}$-valued scalars are part of the $N = 2$ vector multiplet, as explained in §10.2. The total Kähler moment map is

$$\mu(\phi, \sigma) = \mu_{\mathbb{R}}(\phi) + \frac{\sqrt{-1}}{2}[\sigma, \bar{\sigma}]. \tag{10.51}$$

There is a nonzero superpotential

$$W = \sqrt{2}\, \tilde{\mu}_{\text{hol}} \tag{10.52}$$

coming from the transpose of the holomorphic moment map:

$$\tilde{\mu}_{\text{hol}} \colon X \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}. \tag{10.53}$$

We leave the reader to verify that the component lagrangian (6.42) for these data is the dimensional reduction of (6.48). As mentioned at the end of §3.4, if ζ is a vector field on X which preserves the hyperkähler structure, then we can use ζ to add a term to the superpotential. Namely, if η is the holomorphic symplectic form on X for the particular choice of complex structure we have made, we add to W the holomorphic function W_0 which satisfies

$$\partial W_0 = -\iota(\zeta)\eta. \tag{10.54}$$

In other words, W_0 is a holomorphic Hamiltonian function for the infinitesimal symmetry ζ.

The only $N = 4$ model in 4 dimensions is pure Yang-Mills theory, which is reduced from 10 dimensions. In the description on $M^{4|4}$ there is a constrained connection $\mathcal{A}$ and three chiral superfields $\Sigma_1, \Sigma_2, \Sigma_3$ with values in the complex adjoint bundle. The moment map is

$$\mu(\sigma_1, \sigma_2, \sigma_3) = \frac{\sqrt{-1}}{2}\Big\{[\sigma_1, \bar{\sigma}_1] + [\sigma_2, \bar{\sigma}_2] + [\sigma_3, \bar{\sigma}_3]\Big\}, \tag{10.55}$$

and there is a superpotential

$$W(\sigma_1, \sigma_2, \sigma_3) = \sqrt{2}\, \langle \sigma_1, [\sigma_2, \sigma_3] \rangle. \tag{10.56}$$

Note that the $U(1)$ R-symmetry (2.81) is broken by the superpotential (cf. the end of §5.3). This model does have an $SU(4)$ R-symmetry; the $SU(3)$ subgroup which permutes the chiral superfields is manifest in this formulation.[40]

[40] In components the six *real* scalar fields are permuted by $SU(4) \cong \text{Spin}(6)$.

CHAPTER 11
The Vector Multiplet on $M^{6|(8,0)}$

We prove Proposition 10.18, following the treatment in Koller (see References).

§11.1. Complements on $M^{6|(8,0)}$

As in §2.1 we write the relevant spinor representation S, complexified, as $S_{\mathbb{C}} = S_0 \otimes W$. The real structure comes from pseudoreal structures on S_0 and W.

The following lemma gives a convenient description, compatible with the action of $GL(S_0) \times GL(W)$, of the symmetric and exterior powers of $S_{\mathbb{C}}$. For ρ a representation of the symmetric group S_n, we note $\mathrm{Schur}_\rho(U)$ the corresponding Schur functor

$$\mathrm{Schur}_\rho(U) := \mathrm{Hom}_{S_n}(\rho, U^{\otimes n}). \tag{11.1}$$

One has

$$\bigoplus \mathrm{Schur}_\rho(V) \otimes \rho \xrightarrow{\sim} V^{\otimes n}, \tag{11.2}$$

the sum running over representatives of the isomorphism classes of irreducible representations of S_n. The map (11.2) is an isomorphism of representations of $GL(V) \times S_n$.

Lemma 11.3. *Let U and V be vector spaces. One has*

$$\bigoplus \mathrm{Schur}_\rho(U) \otimes \mathrm{Schur}_{\rho^\vee}(V) \xrightarrow{\sim} \mathrm{Sym}^n(U \otimes V) \tag{11.4}$$

$$\bigoplus \mathrm{Schur}_\rho(U) \otimes \mathrm{Schur}_{\rho^\vee \otimes \mathrm{sgn}}(V) \xrightarrow{\sim} \bigwedge^n(U \otimes V) \tag{11.5}$$

Note that ρ and $\rho^\vee$ are isomorphic. In (11.5) sgn is the sign character of S_n.

Proof. Apply (11.2) to U and V, and take the coinvariants (resp. co-anti-invariants) by S_n.

On $M^{6|(8,0)}$ the tangent bundle is a direct sum $V \oplus \tau$. As vector bundles the structural odd distribution τ is the constant bundle S^*, parity changed, and V is the constant bundle real part of $\bigwedge^2 S_0$. In particular, each s in $S_{\mathbb{C}}^*$ defines a complex odd vector field D_s in $\tau_{\mathbb{C}}$. The bases $\{f_a\}$ of S_0^* and $\{e_i\}$ of W^* define a moving

frame (see §2.5) consisting of the D_{ai} and ∂_{ab} $(a < b)$. For $\partial_{ba} := -\partial_{ab}$ one has $D_{ai} = D_{f_a \otimes e_i}$ and

$$[D_{ai}, D_{bj}] = -2\epsilon_{ij}\partial_{ab}. \tag{11.6}$$

For x in $\mathrm{Sym}^2(S_{\mathbb{C}}^*)$ define $D(x)$ by

$$D(s't') = [D_{s'}, D_{t'}]. \tag{11.7}$$

By the lemma,

$$\mathrm{Sym}^2(S_{\mathbb{C}}^*) = \bigwedge^2 S_0^* \otimes \bigwedge^2 W^* \ \oplus \ \mathrm{Sym}^2(S_0^*) \otimes \mathrm{Sym}^2(W^*), \tag{11.8}$$

and (11.6) gives that $D(x)$ vanishes for x in the second factor. By polarization in w this amounts to saying that for w in W^*, we have $[D_{s_1 \otimes w}, D_{s_2 \otimes w}] = 0$. For $w \neq 0$ fixed in W^*, the $D_{s \otimes w}$ $(s \in S_0^*)$ span an integrable distribution τ_w; its leaves are affine spaces with vector space of translations $S_0 \otimes w$, parity changed. Of course, τ_w is a complex distribution so that it makes sense to speak of its leaves only after complexification, at least in the odd direction (cs spaces). One can as well say that for $s \neq 0$ fixed in S_0^*, the $D_{s \otimes w}$ $(w \in W^*)$ span an integrable distribution τ_s, with leaves affine spaces with vector space of translations $s \otimes W^*$.

The intersection of $s \otimes W^*$ and of $S_0^* \otimes w$ is the line spanned by $s \otimes w$. This line also gives rise to an integrable distribution, with leaves affine spaces of dimension $0|1$, intersections of a leaf of τ_s and a leaf of τ_w.

We can now use the volume forms ν and ϵ to define differential operators

$$D_w^4 := \nu^{1234} D_{f_1 \otimes w} D_{f_2 \otimes w} D_{f_3 \otimes w} D_{f_4 \otimes w} \tag{11.9}$$

$$D_s^2 := \epsilon^{12} D_{s \otimes e_1} D_{s \otimes e_2}. \tag{11.10}$$

They can be viewed as integrals on the leaves of τ_w (resp. τ_s).

Lemma 11.11. $D_w^4 D_s^2 = D_s^2 D_w^4 = 0$.

Proof. It suffices to check it for w and s not zero; if we take a basis with $f_1 = s$ and $e_1 = w$, the lemma reduces to $D_{s \otimes w}^2 = 0$.

Variant. D_w^4 produces a function constant along the leaves of τ_w. The operator D_s^2 is right divisible by $D_{s \otimes w}$. As $s \otimes w$ is in τ_w, the operator D_s^2 annuls the image of D_w^4. Permuting the roles of S_0^* and W^*, one similarly proves that $D_w^4 D_s^2 = 0$.

§11.2. Constrained connections

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. We think to connections ∇ on a G-torsor $\mathcal{P}$ as follows: For each representation R of G, the principal bundle $\mathcal{P}$ gives rise to a vector bundle $R^{\mathcal{P}}$, and ∇ gives connections on the $R^{\mathcal{P}}$, functorially in R and compatibly with the tensor product.

From this point of view, the curvature F is the $\mathfrak{g}^{\mathcal{P}}$-valued 2-form such that for each R, the function $-i_X i_Y F$ acts on $R^{\mathcal{P}}$ (by the action of $\mathfrak{g}^{\mathcal{P}}$ on $R^{\mathcal{P}}$ deduced from the action of $\mathfrak{g}$ on R) by $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The formula

$$-\iota(X)\iota(Y)F = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{11.12}$$

differs by a sign from the one which holds when ∇_X is viewed as a horizontal vector field on $\mathcal{P}$ lifting X. (See (6.75) for an example of the latter formula.)

A constrained connection is one whose curvature vanishes in the τ-direction. Let us write $\mathcal{D}_s$ for $\nabla_{\mathcal{D}_s}$ and $\mathcal{D}_{ai}$ for $\mathcal{D}_{f_a \otimes e_i}$. The constraint is that, in parallel to (11.6),

$$[\mathcal{D}_{ai}, \mathcal{D}_{bj}] = -2\epsilon_{ij}\nabla_{ab}. \tag{11.13}$$

A constrained connection is uniquely determined by its restriction to τ. For this restriction, here is what the constraint becomes. For x in $\mathrm{Sym}^2(S^*_{\mathbb{C}})$, define $\mathcal{D}(x)$ by $\mathcal{D}(s't') = [\mathcal{D}_{s'}, \mathcal{D}_{t'}]$; the constraint is that $\mathcal{D}(x)$ vanishes for x in the second factor (11.8) of $\mathrm{Sym}^2(S_{\mathbb{C}})$. This amounts to saying that for w in W^*, the connection is integrable along the leaves of τ_w. It also amounts to saying that it is integrable along the leaves of τ_s ($s \in S^*_0$). If we define $\mathcal{D}^4_w$ and $\mathcal{D}^4_s$ by (11.9) and (11.10), with D replaced by $\mathcal{D}$, we still have (in any $V^{\mathcal{P}}$)

$$\mathcal{D}^4_w \mathcal{D}^2_s = \mathcal{D}^2_s \mathcal{D}^4_w = 0, \tag{11.14}$$

with the same proof.

§11.3. An auxiliary Lie algebra

Let $\mathcal{L} = \bigoplus_{\nu=0}^{\infty} \mathcal{L}^\nu$ be the graded super Lie algebra (with parity = degree modulo 2) generated in degree 1 by $S^*_0 \otimes W^*$, and with relations given by

$$\mathrm{Sym}^2(S^*_0) \otimes \mathrm{Sym}^2(W^*) \subset \mathrm{Sym}^2(S^*_0 \otimes W^*). \tag{11.15}$$

For any constrained connection, the $\mathcal{D}(s)$ obey the relations of $\mathcal{L}$: for any representation R of G, the Lie algebra $\mathcal{L}$ acts on the local sections of $R^{\mathcal{P}}$. In degree one, we get the $\mathcal{D}(s)$; in degree two, the ∇_{ab}: the connection in the direction of V; in degree three, the components $F_{ab,ck} = [\nabla_{ab}, \mathcal{D}_{ck}]$ of the curvature, in the moving frame of §2.5; in degree ≥ 3 the action of elements of $\mathcal{L}$ is given by sections of $\mathfrak{g}^{\mathcal{P}}$, iterated covariant derivatives of the components of the curvature.

As a preliminary to a description in component fields of constrained connections, we now analyze $\mathcal{L}$ in degree ≤ 5. In degree 1, we have a basis $\mathcal{D}_{ai}$. In degree 2 we have $\mathcal{L}^2 \cong \bigwedge^2 S^*_0$, with basis the ∇_{ab} ($a < b$) defined by (11.13).

Lemma 11.16. *The iterated bracket* $[[\mathcal{D}_{ai}, \mathcal{D}_{bj}], \mathcal{D}_{ck}] = -2\epsilon_{ij}[\nabla_{ab}, \mathcal{D}_{ck}]$ *is antisymmetric in* (a, b, c).

Proof. By (11.13), the iterated bracket B is antisymmetric in (a, b) and in (i, j). As i, j, k take only two values, two of them are equal. We consider the cases $i = j$, $j = k$ and $i = k$ in turn:

$i = j$: one has $B = 0$;
$i = k$: $B = [\mathcal{D}_{ai}, [\mathcal{D}_{bj}, \mathcal{D}_{ck}]]$, which is antisymmetric in (b, c);
$j = k$: B is similarly antisymmetric in (a, c).

In all cases, antisymmetry in (a, b, c) follows.

By the lemma, $[\nabla_{ab}, \mathcal{D}_{ck}]$ defines a morphism $\bigwedge^3 S_0^* \otimes W^* \to \mathcal{L}^3$. As S_0^* is of dimension 4, the volume form of S_0 identifies $\bigwedge^3 S_0^*$ to the dual S_0 of S_0^*. Hence we have a map

$$\Lambda: S_0 \otimes W \longrightarrow \mathcal{L}^3, \tag{11.17}$$

where we take

$$[\mathcal{D}_{ak}, \nabla_{bc}] = 2\epsilon_{kj}\nu_{abcd}\Lambda^{dj}. \tag{11.18}$$

We find it convenient to lower the index on Λ by

$$\Lambda_k^d = \epsilon_{kj}\Lambda^{dj}, \tag{11.19}$$

and so

$$[\mathcal{D}_{ak}, \nabla_{bc}] = 2\nu_{abcd}\Lambda_k^d. \tag{11.20}$$

For any x in $S_0 \otimes W^*$, one easily checks that $\epsilon_{ij}\nu_{abcd}x_k^d$, averaged by cyclic permutations of (ac, bj, ck), vanishes. This gives the Jacobi identity required to check that (11.17) is an isomorphism.

Degree 4

The bracket $[\ \ ]$: $\mathcal{L}^2 \otimes \mathcal{L}^2 \to \mathcal{L}^4$ defines a map

$$\begin{aligned} \textstyle\bigwedge^2\bigwedge^2 S_0^* &\longrightarrow \mathcal{L}^4 \\ (f_a \wedge f_b) \wedge (f_c \wedge f_d) &\longmapsto [\nabla_{ab}, \nabla_{cd}]. \end{aligned} \tag{11.21}$$

The iterated exterior power $\bigwedge^2\bigwedge^2 S_0^*$ is the orthogonal Lie algebra of $\bigwedge^2 S_0^*$, provided with the symmetric bilinear form $h_{ab,cd} = \nu_{abcd}$. The exceptional isomorphism $SL(S_0^*) \xrightarrow{\sim} \mathrm{Spin}(\bigwedge^2 S_0^*)$ identifies it with the Lie algebra of traceless endomorphisms of S_0^*. The map is $f_a^b \mapsto F_{ab,cd}$, with

$$F_{ab,cd} = -\nu_{abef}(\delta_c^e f_d^f - \delta_d^e f_c^f), \tag{11.22}$$

or, moving up the vector index ab by $h^{ab,cd} = \nu^{abcd}$ and multiplying by a suitable factor, we have

$$F_{cd}^{ab} = \text{ antisymmetrization in } (a,b) \text{ and in } (c,d) \text{ of } \delta_c^a f_d^b. \tag{11.23}$$

Note that F, given by this formula, is antisymmetric in (ab, cd) if and only if f is traceless. In particular, we obtain a traceless f_b^a in $\mathcal{L}^4$ with

$$[\nabla_{ab}, \nabla_{cd}] = -\nu_{abef}\left(\delta_c^e f_d^f - \delta_d^e f_c^f\right). \tag{11.24}$$

The degree 4 part of $\mathcal{L}$ is generated by the $[\mathcal{D}_{ai}, \Lambda_j^b]$. They are related to the $F_{ab,cd} := [\nabla_{ab}, \nabla_{cd}]$ by

$$\begin{aligned} -2\epsilon_{k\ell}F_{ab,cd} &= [\nabla_{ab}, [\mathcal{D}_{ck}, \mathcal{D}_{d\ell}]] \\ &= [[\nabla_{ab}, \mathcal{D}_{ck}], \mathcal{D}_{d\ell}] && \text{symmetrized in } (ck, d\ell) \\ &= 2\nu_{abce}[\mathcal{D}_{d\ell}, \Lambda_k^e] && \text{symmetrized in } (ck, d\ell) \\ &= 2\nu_{abef}\delta_c^e[\mathcal{D}_{d\ell}, \Lambda_k^f] && \text{symmetrized in } (ck, d\ell). \end{aligned} \tag{11.25}$$

Let us extract from this formula the parts symmetric and antisymmetric in (k,ℓ). Antisymmetrization gives

$$-4\epsilon_{k\ell}F_{ab,cd} = \nu_{abef}\delta^e_c[\mathcal{D}_{d\ell},\Lambda^f_k] \quad \text{antisymmetrized in } (c,d),\ (k,\ell), \tag{11.26}$$

in other words,

$$4\epsilon_{k\ell}f^a_b = \mathcal{D}_{b\ell}\Lambda^a_k \qquad \text{antisymmetrized in } (k,\ell). \tag{11.27}$$

Symmetrization in (k,ℓ) gives the vanishing of the symmetrization in (c,d), (k,ℓ) and antisymmetrization in (a,b) of $\delta^a_c\mathcal{D}_{d\ell}\Lambda^b_k$. Let us fix k, ℓ and let D^b_d be the symmetrization of $\mathcal{D}_{d\ell}\Lambda^b_k$ in (k,ℓ). For $a=c=1$, $b=2$, $d=1$ we get $D^2_1=0$. This holding in any basis, D^b_d is a multiple of δ^b_a. Hence for some $\mathcal{E}_{ij}$, we have[41]

$$\delta^a_b\mathcal{E}_{ij} = \mathcal{D}_{bj}\Lambda^a_i \qquad \text{symmetrized in } (i,j). \tag{11.29}$$

The morphism from $\bigwedge^2\bigwedge^2 S^*_0 \oplus \mathrm{Sym}^2(W^*)$ to $\mathcal{L}^4$ is an isomorphism. Indeed, if we define $\mathcal{L}^4$ as being $\bigwedge^2\bigwedge^2 S^*_0 \oplus \mathrm{Sym}^2(W^*)$ and the bracket by (11.27), (11.29), we get a Lie algebra. The only new Jacobi identity to check is (11.25), from which (11.27) and (11.29) were derived.

Degree ≥ 5

From the forthcoming description (Proposition 11.33) of constrained connections in components, one can deduce that the part of degree ≥ 2 of the super Lie algebra $\mathcal{L}$ is free, with generators the ∇_{ab}, Λ^a_i and $\mathcal{E}_{ij}$. To prepare for Proposition 11.33, we show here that it is generated by those elements. If $\mathcal{L}_1$ is the subalgebra generated by the ∇_{ab}, Λ^a_i and $\mathcal{E}_{ij}$, it suffices to check that a $\mathcal{D}_{ai}$, applied to any of the generators, is again in $\mathcal{L}_1$. It only remains to show that the $[\mathcal{D}_{ai},\mathcal{E}_{jk}]$ are linear combinations of the $[\nabla_{cd},\Lambda^e_\ell]$. The formula

$$\mathcal{D}_{ai}\mathcal{E}_{jk} = C\epsilon_{ij}\nabla_{ab}\Lambda^b_k \qquad \text{symmetrized in } (j,k) \tag{11.30}$$

for some constant C is the only possible formula compatible with the grading, the actions of $SL(S^*_0)$, $SL(W)$, and with the symmetry in (j,k). (Further computation shows $C=-2$.) Here, we will content ourselves to check that, modulo the $[\nabla,\Lambda]$, all

$$[\mathcal{D}_{ai},[\mathcal{D}_{bj},[\mathcal{D}_{ck},[\mathcal{D}_{d\ell},\mathcal{D}_{em}]]]] = 4\epsilon_{\ell m}\nu_{cdef}[\mathcal{D}_{ai},[\mathcal{D}_{bj},\Lambda^f_k]] \tag{11.31}$$

vanish.

If we antisymmetrize (11.31) in (j,k), we get by (11.27) a $[\mathcal{D},[\nabla,\nabla]]$, which Jacobi turns into a $[\nabla,[\nabla,\mathcal{D}]]$, i.e., a $[\nabla,\Lambda]$. Modulo the $[\nabla,\Lambda]$, this gives the symmetry of (11.31) in (j,k) and, by (11.29), its antisymmetry in (b,c,d,e).

[41] $\mathcal{E}_{ij}$ is related to the E^i_j of Chapter 10 by

$$\mathcal{E}_{ij} = 2\sqrt{-1}\,\epsilon_{ik}E^k_j. \tag{11.28}$$

If a quantity x_{ijk} is symmetric in (j,k), to prove its vanishing, it suffices to check the vanishing of its symmetrization y_{ijk} in (i,j). Indeed, the symmetrization of y in (j,k) gives x_{ijk} back, up to a multiple of the complete symmetrization (of x or y, this amounts to the same):

$$y_{ijk} + y_{jki} = x_{ijk} + (x_{ikj} + x_{jik} + x_{kij}). \tag{11.32}$$

To prove the vanishing of (11.31) modulo the $[\nabla, \Lambda]$, it hence suffices to consider its symmetrization $(11.31)^S$ in (i,j). In $(11.31)^S$, if we symmetrize in (a,b), we get 0: a $[[\mathcal{D}_{ai}, \mathcal{D}_{bj}]\dots]$ symmetrized in (i,j). We conclude that $(11.31)^S$ is antisymmetric in (a,b). Modulo the $[\nabla, \Lambda]$, it is also antisymmetric in (b,c,d,e), and as $\dim S_0^* = 4 < 5$, its vanishing follows.

§11.4. Components of constrained connections

In classical differential geometry, if N is a subvariety of M and $(\mathcal{P}, \nabla)$ a torsor with connection on M, the restriction of $(\mathcal{P}, \nabla)$ to a formal neighborhood of N is determined up to unique isomorphism by (a) the restriction (P, ∇) of $(\mathcal{P}, \nabla)$ to N, and (b) the value on N of all iterated covariant derivatives of the components of the curvature of P. The same holds in the super case. In addition, if N is of codimension $0|s$ in M, the formal neighborhood of N in M is just M itself.

If the codimension is $0|1$, the value on N of the curvature is enough. Indeed, take local coordinates (θ^0, x^i), where the x^i can be even or odd, for which N is defined by $\theta^0 = 0$. A trivialization of P on N extends uniquely to a trivialization s of $\mathcal{P}$ for which $\nabla_0 s = 0$ on N. Given a trivialization of P, the connection on P is given by a $\mathfrak{g}$-valued 1-form α, while $(\mathcal{P}, \nabla)$ is given by a $\mathfrak{g}$-valued 1-form β on M, such that $\langle \partial_0, \beta \rangle = 0$ on N. Let us write $\beta = d\theta^0\, \theta^0 b + \sum dx^i (c^i + \theta^0 d^i)$, with b, c, d functions of the x^i. Then, α gives the c^i, the components $F_{0,i}$ of the curvature, restricted to N, gives the d^i and F_{00} restricted to N gives b.

To handle the case of codimension $0|s$, one chooses $N \subset N_1 \subset \dots \subset M$, each of codimension $0|1$ in the next, and one uses induction on s: on N_i, after recovering $(\mathcal{P}, \nabla)|N_i$, one recovers from the data on N the data on N_i of all covariant derivatives of the components of the curvature, restricted to N_i.

Let $(\mathcal{P}, \nabla)$ be a constrained G-torsor on $M^{6|(8,0)}$. From §11.3, we obtain from it a $\mathfrak{g}^{\mathcal{P}}$-valued spinor Λ_i^a, and $\mathfrak{g}^{\mathcal{P}}$-valued auxiliary fields $\mathcal{E}_{ij}$ By restriction to Minkowski space $\check{M}^6$, we obtain a torsor with connection (P, ∇) on M^6, $\mathfrak{g}^P$-valued spinors λ_i^a and $\mathfrak{g}^P$-valued auxiliary fields E_{ij}. By §11.3, all covariant derivatives of the components of $\mathcal{P}$, restricted to M^6, can be expressed from ∇, λ and $\mathcal{E}$. It follows that $(\mathcal{P}, \nabla)$ is determined up to unique isomorphism by (P, ∇, λ, E). More precisely, one has the following rephrasing of Proposition 10.18.

Proposition 11.33. *The functor $(\mathcal{P}, \nabla) \mapsto (P, \nabla, \lambda, E)$ is an equivalence of categories.*

For a spelling out of what this means, we refer to the text following Proposition 7.5. We will check Proposition 11.33 in the complex setting; the real case results from it as in the proof of Proposition 8.7.

Proof. We have to check that any (P, ∇, λ, E) comes from some constrained $(\mathcal{P}, \nabla)$. The flat P with $\lambda, E = 0$ corresponds to $\mathcal{P}$ flat. We now construct vector fields on the space of connections, tangent to the constrained ones, to get from there to any

(P, ∇, λ, E). Rather, we work with the space of connections along τ. A deformation of a connection $\mathcal{D}$ in the τ direction is given by $\mathfrak{g}^{\mathcal{P}}$-valued $\delta\mathcal{D}_{ai}$. It is tangent to constrained connections if the symmetrization in (a,b) and (i,j) of $[\delta\mathcal{D}_{ai}, \mathcal{D}_{bj}]$ vanishes.

We will need the following operators acting on $\mathfrak{g}^{\mathcal{P}}$:

$$\begin{aligned} &\mathcal{D}_{ab} = \tfrac{1}{2!}\epsilon^{ij}\mathcal{D}_{ai}\mathcal{D}_{bj}, && \text{symmetrized in } (a,b) \\ &\mathcal{D}_{ijk\ell} = \tfrac{1}{4!}\epsilon^{abcd}\mathcal{D}_{ai}\mathcal{D}_{bj}\mathcal{D}_{ck}\mathcal{D}_{d\ell}, && \text{symmetrized in } (i,j,k,\ell). \end{aligned} \tag{11.34}$$

They polarize the $\mathcal{D}_s^2$ and $\mathcal{D}_w^4$ of (11.14), and from (11.14) we get that

$$\mathcal{D}_{ab}\mathcal{D}_{ijk\ell} = \mathcal{D}_{ijk\ell}\mathcal{D}_{ab} = 0 \tag{11.35}$$

(for their action on sections of $\mathfrak{g}^{\mathcal{P}}$).

Fix $\mathfrak{g}^P$-valued fields V^{ij} and consider

$$\delta\mathcal{D}_{ai} := \epsilon^{jj'}\mathcal{D}_{aj}\mathcal{D}_{ij'k\ell}V^{k\ell}. \tag{11.36}$$

If we start from a constrained $\mathcal{D}$, we have

$$\begin{aligned} &[\mathcal{D}_{ai}, \delta\mathcal{D}_{bj}] && \text{symmetrized in } (a,b),\ (i,j) \\ &\quad = \epsilon^{kk'}\mathcal{D}_{ai}\mathcal{D}_{bk}\mathcal{D}_{jk'\ell m}V^{\ell m} && \text{symmetrized in } (a,b) \end{aligned} \tag{11.37}$$

and the (a,b)-symmetrization of $\mathcal{D}_{ai}\mathcal{D}_{bk}$ is alternating in (i,k): it is $2\epsilon_{ik}\mathcal{D}_{ab}$. We get

$$\ldots = -2\mathcal{D}_{ab}\mathcal{D}_{ji\ell m}V^{\ell m} = 0.$$

The $\delta\mathcal{D}_{ai}$ is the promised vector field, tangent to the constrained $\mathcal{D}$.

Let (θ) be the ideal defining M^6 in $M^{6|(8,0)}$. To conclude, it remains to check that:

(a) for V in $(\theta)^6$, the variation of ∇_{ab} on $\breve{M}$ depends only on V modulo $(\theta)^7$, and the ∇_{ab} are changed freely;
(b) for V in $(\theta)^7$, the variation of λ_i^a depends only on V modulo $(\theta)^8$, and the λ_i^a are changed freely;
(c) for V in $(\theta)^8$, the auxiliary fields E are changed freely.

The variation of $[\mathcal{D}_{ai}, \mathcal{D}_{bj}]$ (resp. $[\mathcal{D}_{ai}, [\mathcal{D}_{bj}, \mathcal{D}_{ck}]]$, resp. $[\mathcal{D}_{ai}, [\mathcal{D}_{bj}, [\mathcal{D}_{ck}, \mathcal{D}_{d\ell}]]]$) is given by 6 (resp. 7, resp. 8) operators $\mathcal{D}$ applied to V. For V in $(\theta)^6$ (resp. $(\theta)^7$, resp. $(\theta)^8$), this variation, restricted to M^6, depends only on V modulo the next power of (θ) and, to compute it, one may replace each $\mathcal{D}_{ai}$ by ∂_{ai}. The resulting simplified computation is left to the reader.

REFERENCES

Most of the material discussed here was developed in the physics literature in the '70s and '80s. It is hopeless for us to reference all of the original papers. Our more modest goal is to give the reader an entry into the literature and to point out a few more specialized references we used.

Some standard physics books on the subject (alphabetical by author) are:

P. G. O. Freund, *Introduction to supersymmetry*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1986.

S. J. Gates, M. T. Grisaru, M. Roček, W. Siegel, *Superspace, One thousand and one lessons in supersymmetry*, Frontiers in Physics, vol. 58, Benjamin/Cummings Publishing Co. Inc., Reading, Mass., 1983.

M. F. Sohnius, *Introducing supersymmetry*, Phys. Rep. **128** (1985), 39–204.

J. Wess, J. Bagger, *Supersymmetry and supergravity*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1992.

P. West, *Introduction to supersymmetry and supergravity*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1990.

An expository introduction to some of this material may be found in

D. S. Freed, *Five Lectures on Supersymmetry*, American Mathematical Society, 1999.

The superspace formulation of 6-dimensional Yang-Mills (Chapters 10 and 11) does not seem to be widely known, but it can be found in the literature. See, for example:

P. S. Howe, G. Sierra, P. K. Townsend, *Supersymmetry in six dimensions*, Nucl. Phys. B **221** (1983), 331–348.

J. Koller, *A six-dimensional superspace approach to extended superfields*, Nucl. Phys. B **222** (1983), 319–337.

Sign Manifesto

Pierre Deligne and Daniel S. Freed

§1. Standard mathematical conventions

- *We apply the sign rule relentlessly.*

This means that when passing from ordinary algebra to $\mathbb{Z}/2$-graded, or super, algebra we pick up a sign $(-1)^{|a|\,|b|}$ when permuting homogeneous elements a, b of parity $|a|, |b|$. Structure maps (multiplications, Lie brackets, inner products, ...) are even.

For example, consider a graded complex vector space $V = V^0 \oplus V^1$. A hermitian inner product $\langle \cdot, \cdot \rangle$ satisfies, among other properties,

$$\langle v_1, v_2 \rangle = (-1)^{|v_1|\,|v_2|}\overline{\langle v_2, v_1 \rangle}, \qquad v_1, v_2 \in V \text{ homogeneous.} \tag{1}$$

and from the evenness of the inner product it follows that V^0 is orthogonal to V^1. From (1) we deduce that $\langle v, v \rangle$ is real for v even and pure imaginary for v odd. The adjoint T^* of a homogeneous linear operator $T\colon V \to V$ is characterized by

$$\langle Tv_1, v_2 \rangle = (-1)^{|T|\,|v_1|}\langle v_1, T^* v_2 \rangle. \tag{2}$$

Skew-adjoint operators form a super Lie algebra.

- *Symmetry groups act on the left.*

For example, if $\mathfrak{g}$ is a Lie algebra, then an action of $\mathfrak{g}$ on a vector space V is a *homomorphism* $\mathfrak{g} \to \mathrm{End}(V)$. Brackets are preserved. On the other hand, an action of $\mathfrak{g}$ on a manifold M is an *antihomomorphism* $\mathfrak{g} \to \mathrm{VectorFields}(M)$. The reversal of sign comes from the rule $\xi f = \frac{d}{dt} \exp(t\xi)^* f$ and the fact that $f \mapsto u^* f$ is a *right* action of diffeomorphisms (u) on functions (f).

§2. Choices

- *A hermitian inner product on a complex vector space V is conjugate linear in the first variable:*

$$\langle \lambda_1 v_1, \lambda_2 v_2 \rangle = \overline{\lambda_1}\lambda_2 \langle v_1, v_2 \rangle, \qquad \lambda_i \in \mathbb{C}, \quad v_i \in V. \tag{3}$$

- *If $V = V^0 \oplus V^1$ is a super Hilbert space, then*

$$-i\langle v, v \rangle \geq 0, \qquad v \in V^1. \tag{4}$$

- *We pass from self-adjoint operators to skew-adjoint operators using multiplication by $-i$:*

$$T \text{ self-adjoint} \longleftrightarrow -iT \text{ skew-adjoint.} \tag{5}$$

- *The Lorentz metric g on n dimensional Minkowski space has signature $(1, n-1)$:*

$$\text{Signature}(g) = + - - - \cdots \tag{6}$$

- *The quantum hamiltonian $\hat{H}$ is minus the operator which corresponds to infinitesimal time translation:*

$$\hat{H} = -\hat{P}_0. \tag{7}$$

§3. Rationale

The first choice (3) is not the usual one in mathematics, but it has its merits. For example, since linear operators act on the left, it makes sense to have the commuting scalar multiplication act on the right. In mathematics we do follow this convention for modules over noncommutative rings. With right scalar multiplication (3) reads: $\langle v_1\lambda_1, v_2\lambda_2\rangle = \overline{\lambda_1}\langle v_1, v_2\rangle\lambda_2$. We do not adopt this convention for scalar multiplication, but do adopt (3). Physicists like (3) in view of Dirac's notation '$\langle v_1|T|v_2\rangle$' for '$\langle v_1, Tv_2\rangle$'. Comment: In computations it is often more convenient and safer to work with a bilinear form rather than a sesquilinear form, and so to write the sesquilinear inner product as '$\langle\overline{v_1}, v_2\rangle$'.

From a mathematical point of view it is more natural to quantize with *skew*-adjoint operators, since they form a Lie algebra. We use (5) to convert to *self*-adjoint operators, whose real eigenvalues correspond to physical measurements.

The sign choice in (6) leads to the usual bosonic lagrangian (20) with a plus sign in front of the kinetic energy.

In (7) we assume that time translation is a symmetry of a quantum theory, so that the infinitesimal generator is represented by a self-adjoint operator $\hat{P}_0$ on the quantum (super)Hilbert space. The minus sign gives the standard answer for the hamiltonian of a classical free particle.

§4. Notation

Throughout $i = \sqrt{-1}$.

Let M denote n dimensional affine Minkowski space with associated vector space of translations V and future timelike cone $C \subset V$. We fix linear coordinates $x^0, \ldots, x^{n-1}$ with respect to which the metric is

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^n = (dx^0)^2 - \cdots - (dx^{n-1})^2, \tag{8}$$

and the cone is

$$C = \{x : \langle x, x\rangle \geq 0 \text{ and } x^0 \geq 0\}. \tag{9}$$

Let $\{e_\mu\}$ be the corresponding basis of V and ∂_μ the corresponding vector field on M. The standard density on M is

$$|d^n x| = |dx^0 \ldots dx^{n-1}|. \tag{10}$$

Let S be a real spin representation. Fix a basis $\{f^a\}$ of S and dual basis $\{f_a\}$ of S^*. Then there are symmetric pairings

$$\begin{aligned} \Gamma &: S^* \otimes S^* \longrightarrow V \\ \tilde{\Gamma} &: S \otimes S \longrightarrow V. \end{aligned} \tag{11}$$

We write

$$\begin{aligned} \Gamma(f_a, f_b) &= \Gamma^{\mu}_{ab} e_\mu, \\ \tilde{\Gamma}(f^a, f^b) &= \tilde{\Gamma}^{\mu ab} e_\mu, \end{aligned} \tag{12}$$

where as usual we sum over repeated indices if one is upstairs and the other is downstairs. We raise and lower indices using the metric. The pairings (11) are assumed to satisfy the Clifford relation

$$\tilde{\Gamma}^{\mu ab}\Gamma^{\nu}_{bc} + \tilde{\Gamma}^{\nu ab}\Gamma^{\mu}_{bc} = 2g^{\mu\nu}\delta^a_c \tag{13}$$

and the positivity condition

$$\Gamma(s^*, s^*) \in C \qquad \text{for all } s^* \in S^*. \tag{14}$$

For $v \in C^o$, the form $\langle v, \Gamma(s^*, s^*)\rangle$ is then positive definite. From (13) and (14) it follows that

$$\tilde{\Gamma}(s, s) \in C \qquad \text{for all } s \in S. \tag{15}$$

In a classical field theory we work with a space of fields $\mathcal{F}$, where $f \in \mathcal{F}$ is some sort of function on M. An infinitesimal symmetry is a vector field ξ on $\mathcal{F}$ which preserves the lagrangian in a certain sense. Corresponding to ξ is a Noether current J_ξ, which is a twisted $(n-1)$-form on M. The Noether charge Q_ξ is the integral of J_ξ over a time slice. We usually consider the current and charge only on the space of classical solutions $\mathcal{M}$, which carries a closed 2-form ω. The infinitesimal symmetry and Noether charge are related by

$$dQ_\xi = -\iota(\xi)\omega. \tag{16}$$

For $(\mathcal{M}, \omega)$ symplectic, (16) can be rewritten

$$\xi f = \{Q_\xi, f\} \tag{17}$$

for f a function on $\mathcal{M}$.

Quantization is, in principle, a map

$$Q \longmapsto \hat{Q} \tag{18}$$

from functions on $\mathcal{M}$ to operators on a complex Hilbert space $\mathcal{H}$. We assume

$$\hat{\overline{Q}} = \hat{Q}^*, \tag{19}$$

so that real functions map to self-adjoint operators.

Let P_μ be infinitesimal translation in the Poincaré algebra and Q_a the odd generator of the supersymmetry algebra.[1] Let $\hat{P}_\mu, \hat{Q}_a$ be the corresponding quantum operators.

[1]The notational conflict between the supersymmetry generator and the Noether charge is too ingrained to correct.

§5. Consequences of §2 on other signs

- *The kinetic lagrangian for a scalar field* $\phi\colon M \to \mathbb{R}$ *is*

$$L = \frac{1}{2}|d\phi|^2\ |d^n x| = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\ |d^n x|. \tag{20}$$

The sign of this term is a consequence of (6); it is the main rationale for preferring (6) over the other choice.

- *Suppose* $V = V^0 \oplus V^1$ *is a graded hermitian vector space and* T *an odd skew-adjoint operator. Then*

$$i[T,T] \geq 0. \tag{21}$$

Observe from (2), (3) and (4) that an odd skew-adjoint operator has eigenvalues on the line $i^{-1/2}\mathbb{R} \subset \mathbb{C}$.

- *The bracket in the supersymmetry algebra is*

$$[Q_a, Q_b] = -2\Gamma^\mu_{ab} P_\mu. \tag{22}$$

Because we use left group actions, upon quantization we expect a homomorphism from the supersymmetry algebra to *skew*-adjoint operators. Using (5) we see that the sign in (22) leads to

$$[-i\hat{Q}_a, -i\hat{Q}_b] = -2\Gamma^\mu_{ab}(-i\hat{P}_\mu). \tag{23}$$

Setting $a = b$ we see from (21) that $-2\Gamma^\mu_{aa}\hat{P}_\mu > 0$ for all a. From (14) we see that $\Gamma^\mu_{aa}P_\mu$ has nonnegative norm in V. Except possibly in dimension 2, the positive cone generated by $\{\Gamma^\mu_{aa}P_\mu\}_a$ includes P_0, and so the sign choice in (22) renders the hamiltonian nonnegative (rather than nonpositive), in view of (7).

- *The vector field* $\hat{\xi}_{P_\mu}$ *on* $\mathcal{F}$ *corresponding to infinitesimal translation* P_μ *is*

$$\hat{\xi}_{P_\mu} f = -\partial_\mu f, \qquad f \in \mathcal{F}. \tag{24}$$

This follows since a diffeomorphism $\varphi\colon M \to M$ acts on functions by $(\varphi^{-1})^*$.

- *If* $\mathfrak{g}$ *is a Lie algebra of infinitesimal symmetries, then the vector fields* ξ_λ $(\lambda \in \mathfrak{g})$ *on the space of fields* $\mathcal{F}$ *satisfy*

$$[\xi_{\lambda_1}, \xi_{\lambda_2}] = -\xi_{[\lambda_1,\lambda_2]}. \tag{25}$$

- *The Noether currents satisfy*[2]

$$\{j_{\lambda_1}, j_{\lambda_2}\} = -j_{[\lambda_1,\lambda_2]}. \tag{26}$$

- *The Noether charges satisfy*

$$\{Q_{\lambda_1}, Q_{\lambda_2}\} = -Q_{[\lambda_1,\lambda_2]}. \tag{27}$$

- *The quantum operators satisfy*

$$[-i\hat{Q}_{\lambda_1}, -i\hat{Q}_{\lambda_2}] = -i\hat{Q}_{[\lambda_1,\lambda_2]}. \tag{28}$$

[2]The bracketing operation on Noether currents is defined in [**I-Classical Fields**, §2.6].

Equations (25)–(27) follow from the fact that $\mathfrak{g} \to \text{VectorFields}(\mathcal{F})$ is an antihomomorphism and the standard equations for Poisson brackets which follow from (16). Equation (28) says that $\mathfrak{g} \to \text{End}(\mathcal{H})$ is a homomorphism to skew-adjoint operators, where we use (5).

- *The self-adjoint quantum operators $\hat{Q}_1, \hat{Q}_2$ which correspond to classical functions Q_1, Q_2 on $\mathcal{M}$ satisfy*

$$[\hat{Q}_1, \hat{Q}_2] = -i\hbar\{Q_1, Q_2\}\hat{} \quad \textit{modulo } O(\hbar^2). \tag{29}$$

To the extent that (29) holds exactly, it says that the map

$$Q \longmapsto \frac{-i}{\hbar}\hat{Q} \tag{30}$$

to skew-adjoint operators is an antihomomorphism. The sign in (30) is dictated by (5). The desire to have an *anti*homomorphism is dictated by (27) and (28), and this determines the sign in (29).

- *The Schrödinger equation for the evolution of a state ψ is*

$$\frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}\hat{H}\psi. \tag{31}$$

Evolution through time t for a static hamiltonian $\hat{H}$ is $e^{-it\hat{H}/\hbar}$.

The sign follows from (7) and (5).

- *Let η_1^a, η_2^b be odd parameters and $\hat{\xi}_i$ the even vector field on $\mathcal{F}$ corresponding to $\eta_i^a Q_a$. Then*

$$[\hat{\xi}_1, \hat{\xi}_2]f = 2\eta_1^a\eta_2^b\Gamma_{ab}^{\mu}\partial_\mu f, \qquad f \in \mathcal{F}. \tag{32}$$

To see this, observe from (22) that in the abstract supersymmetry algebra we have

$$[\eta_1^a Q_a, \eta_2^b Q_b] = 2\eta_1^a\eta_2^b\Gamma_{ab}^{\mu}P_\mu. \tag{33}$$

Then (32) follows from (25) and (24); the minus signs in these two equations cancel.

- *If ψ_1, ψ_2 are complex classical odd quantities, then complex conjugation satisfies*

$$\overline{\psi_1\psi_2} = \overline{\psi_1}\,\overline{\psi_2}. \tag{34}$$

This is a consequence of the sign rule if we assume that $\psi \mapsto \overline{\psi}$ is a *$*$-operation* and ψ_1 (super)commutes with ψ_2. (A $*$ operation satisfies $(ab)^* = (-1)^{|a|\,|b|}\,b^*a^*$.) Notice that the classical statement (34) is consistent with the quantum statement (19), since the adjoint operation on linear operators is also a $*$ operation. Notice that the product of real commuting odd quantities is real.

- *The kinetic lagrangian for a spinor field $\psi\colon M \to S$ is*

$$L = \frac{1}{2}\psi\not{D}\psi\;|d^n x| = \frac{1}{2}\tilde{\Gamma}^{\mu ab}\psi_a\partial_\mu\psi_b\;|d^n x|. \tag{35}$$

- *The kinetic lagrangian for a dual spinor field $\lambda\colon M \to S^*$ is*

$$L = \frac{1}{2}\lambda\not{D}\lambda\;|d^n x| = \frac{1}{2}\Gamma_{ab}^{\mu}\lambda^a\partial_\mu\lambda^b\;|d^n x|. \tag{36}$$

The spinor fields are odd. In view of (34), the lagrangians (35) and (36) are real, as they must be in Minkowski space. It is easiest to check the sign in classical mechanics ($n = 1$). Then from (35) we deduce[3] the classical Poisson bracket

$$\{\psi, \psi\} = -1. \tag{37}$$

Upon quantization we know the corresponding operators satisfy (29). The sign in (37) is compatible with (21), and this means that the sign in (35) is correct.

- *The energy-momentum tensor is minus the Noether current of P_μ. The supercurrent is minus the Noether current of Q_a.*

This is a definition and follows by superPoincaré invariance from the definition (7) of the hamiltonian. It means that the charges computed from the energy-momentum tensor are energy and *minus* momentum.

§6. Differential forms

- *When computing with differential forms on superspace, we use a bigraded point of view.*[4]

Objects have a "cohomological" degree, corresponding to a classical (that is, non-super) degree, and a parity. The permutation of objects of parity p_1, p_2 and cohomological degree d_1, d_2 introduces two signs: a classical sign $(-1)^{d_1 d_2}$ and an additional factor $(-1)^{p_1 p_2}$.

- *On $\mathbb{R}^{p|q}$ with coordinates $t^1, \dots t^p, \theta^1, \dots \theta^q$ we have the following table of parities and cohomological degrees:*

quantity	type	parity ($\mathbb{Z}/2\mathbb{Z}$)	coh deg ($\mathbb{Z}$)
t^μ	even coordinate	0	0
θ^a	odd coordinate	1	0
$\partial/\partial t^\mu$, $\iota(\partial/\partial t^\mu)$	even vector field	0	-1
$\partial/\partial\theta^a$, $\iota(\partial/\partial\theta^a)$	odd vector field	1	-1
Lie(ξ)	Lie derivative	$p(\xi)$	0
$dt^\mu, \epsilon(dt^\mu)$	even 1-form	0	1
$d\theta^a, \epsilon(d\theta^a)$	odd 1-form	1	1
$d^p t\, d^q\theta$	berezinian	q (mod 2)	p
$\lvert d^p t\rvert\, d^q\theta$	density	q (mod 2)	0
$\iota(\xi_r)\dots\iota(\xi_1)\, d^p t\, d^q\theta$	integral form	$\sum p(\xi_i) + q$ (mod 2)	$p - r$
$\iota(\xi_r)\dots\iota(\xi_1)\, \lvert d^p t\rvert\, d^q\theta$	integral density	$\sum p(\xi_i) + q$ (mod 2)	$-r$

- *For X a (super)manifold the canonical pairing of vectors and 1-forms is written with the vector on the left.*

Therefore, by the sign rule, for a tangent vector ξ and a 1-form α there is a sign[5] when passing from the canonical pairing $\iota(\xi)\alpha$ to $(-1)^{p(\xi)p(\alpha)}\alpha(\xi)$, where p is the parity.

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[3] See Problem FP2 of [I-**Homework**] for a derivation.

[4] See the appendix to Chapter 1 of [I-**Supersymmetry**].

[5] See [I-**Supersymmetry**, §3.3].

- *For integration over odd variables we have*

$$\int d\theta\ \theta = 1$$

$$\int d\theta^2 d\theta^1\ \theta^1\theta^2 = \int d\theta^2 \left(\int d\theta^1\ \theta^1 \right) \theta^2 = 1.$$

- *For any vector field ξ we have the Cartan formula*

$$\mathrm{Lie}(\xi) = [d, \iota(\xi)]. \tag{38}$$

Both sides of (38) act on differential or integral forms.

§7. Miscellaneous signs

- *Let X be a smooth manifold, ξ a vector field on X, φ_t the one-parameter group of diffeomorphisms generated, and T a tensor field. Then*

$$\mathrm{Lie}(\xi)T = \frac{d}{dt}\Big|_{t=0} \varphi_t^* T = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* T. \tag{39}$$

- *On a Kähler manifold X the Riemannian metric g, Kähler form ω, complex structure J, and a local Kähler potential K are related by the equations*

$$\begin{aligned} \omega(\xi_1, \xi_2) &= g(J\xi_1, \xi_2) \\ \omega &= i\,\partial\bar{\partial} K. \end{aligned} \tag{40}$$

- *Suppose f is a (suitable) function on a vector space V of dimension n. Then f and its Fourier transform $\hat{f}$ on V^* are related by*

$$\begin{aligned} \hat{f}(k) &= \frac{1}{(2\pi)^{n/2}} \int_V e^{-\imath\langle k,x\rangle} f(x)\ |d^n x|, \\ f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{V^*} e^{+i\langle k,x\rangle} \hat{f}(k)\ |d^n k|. \end{aligned} \tag{41}$$

- *Suppose $\mathfrak{g}$ is the Lie algebra of a real Lie group G. Then the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ carries a conjugation whose set of real points is $\mathfrak{g}$.*

For example, starting with the unitary group $G = U(n)$, we obtain the conjugation $A \mapsto -A^*$ on complex $n \times n$ matrices. (A^* is the conjugate transpose of A.)

Part 2

Formal Aspects of QFT

Note on Quantization

Pierre Deligne

The present notes do not pretend to any originality. We have tried to present different aspects of what quantization can mean, including the case of odd variables. Complex polarizations are not considered.

§1.

Let (M,ω) be a symplectic manifold of dimension $2d$. We will use the vague words "big" and "small". For this, we need some notion of size ~ 1 on M. For instance M could carry a Riemannian metric, with curvature of size at most ~ 1 and injectivity radius of size at least ~ 1. The symplectic form should be of size ~ 1, and $\hbar$ is a small number. The 2-form which really matters is $\omega/\hbar$.

Our sign convention for the Poisson bracket corresponding to ω is the following: the Hamiltonian vector field $X(f)$ defined by a function f is given by $X(f)g = \{f,g\}$, and $df = -i_{X(f)}\omega$. If $M = \mathbb{R}^2$ with coordinates p, q and if $\omega = dp\wedge dq$, then $\{p,q\} = 1$.

A quantization of M consists of a complex Hilbert space $\mathcal{H}$ and of a rule to attach to functions f on M operators $f^\wedge$ acting on $\mathcal{H}$. The rule $f \mapsto f^\wedge$ should be $\mathbb{C}$-linear; for f real, $f^\wedge$ should be Hermitian and, in a sense I will not try to make precise, $f \mapsto f^\wedge$ should almost be a homomorphism: $1^\wedge$ should be the identity and for slowly varying functions f and g,

$$(fg)^\wedge \sim f^\wedge g^\wedge \qquad \bmod O(\hbar). \tag{1.1}$$

If (1.1) did hold exactly, the $f^\wedge$ would mutually commute. The symplectic structure controls the failure of commutativity

$$[f^\wedge, g^\wedge] \sim -i\hbar\{f,g\}^\wedge \qquad \bmod O(\hbar^2). \tag{1.2}$$

Formula (1.2) can be rewritten

$$(X(f)g)^\wedge \sim (i/\hbar)[f^\wedge, g^\wedge] \qquad \bmod O(\hbar). \tag{1.2$'$}$$

The symplectic vector field $X(f)$, if it is well behaved at infinity, exponentiates to an automorphism $\exp(X(f))$ of (M,ω), with Taylor's formula reading

$$\exp(X(f))^* g = \exp(\text{derivation } X(f))(g).$$

Formula (1.2)′ should hold in its integrated form

$$(\exp(X(f))^*g)^\wedge \sim \exp((i/\hbar)f^\wedge)g^\wedge \exp(-(i/\hbar)f^\wedge) \qquad \bmod O(\hbar). \tag{1.3}$$

We don't care about the exact rule $f \mapsto f^\wedge$. It matters mainly mod $O(\hbar)$.

From the deformation quantization point of view, (1.1) is the first term in an expansion (*$*$-product*)

$$f^\wedge g^\wedge = (fg)^\wedge + c_1(f,g)^\wedge\hbar + c_2(f,g)^\wedge\hbar^2 + \cdots\ , \tag{1.1}′$$

with $c_i(f,g)$ given by a $\mathbb{C}$-bilinear differential operator in f and g. The formula (1.2) becomes $c_1(f,g) - c_1(g,f) = -i\{f,g\}$. If one considers quantizations $\mathcal{H}(\hbar)$,with $\hbar$ tending to 0, (1,1)′ can be an asymptotic expansion in $\hbar$. If one redefines $f \mapsto f^\wedge$ to be $f^\wedge + d_1(f)^\wedge\hbar + \cdots$ with $d_i(f)$ a linear differential operator in f, (1.1)′ changes to an *equivalent* $*$-product.

Assumption (1.1) means that $\mathcal{H}$ almost localizes on M: an element h of $\mathcal{H}$ can be decomposed as a sum $\sum h_i$ of elements localized in small regions U_i of M: if $x \in U_i$, $f^\wedge h_i$ is approximately $f(x) \cdot h_i$. Because of the noncommutativity (1.2), localization does not make sense for regions smaller than expressed by the uncertainty principle $\Delta p \Delta q \sim 2\pi\hbar$.

If we were only to assume (1.1) and (1.2), a multiple $\mathcal{H}^n$ of a quantization $\mathcal{H}$ of M would again be a quantization. A "finite multiplicity" assumption is that if f is a C^∞ function with compact support on M, the operator $f^\wedge$ is of trace class. If this holds, I would expect that for some integer m (more precisely, one for each connected component of M), one has

$$\mathrm{Tr}(f^\wedge) \sim m \int f(x)dx$$

where, in Darboux local coordinates, dx is the product of the $dp_i\, dq_i/2\pi\hbar$ (Liouville measure). We want $m = 1$:

$$\mathrm{Tr}(f^\wedge) \sim \int f(x)dx \tag{1.4}$$

§2. Example: Cotangent Bundles

The basic example of almost localization, in the sense meant above, is when M is a cotangent bundle T^*V and $\mathcal{H}$ the space of half-densities on V. If f is peaked at x, say in a Gaussian fashion, and if g is a real function, with dg slowly varying, then $fe^{ig/\hbar}$ is localized around (x, dg_x) in M. If f is slowly varying, then $f \cdot e^{ig/\hbar}$ is localized around the section $x \mapsto dg_x$ of $M \to V$, laying over the support of f.

The operator $f^\wedge$ has a natural definition when f is affine linear on the fibers of T^*V over V. If f is the pull-back of a function on V, $f^\wedge$ is multiplication by f. If f is linear on each cotangent space, hence identified with a vector field F on V, then $f^\wedge = -i\hbar\mathcal{L}_F$. For those f, (1.2) holds exactly. To define $f^\wedge$ for more complicated functions requires auxiliary choices.

On T^*V, we have a canonical 1-form α: in local coordinates q^i on V, giving local coordinates (p_i, q^i) on T^*V, α is $\sum p_i dq^i$. The trivial unitary line bundle $\mathcal{L}$, with the connection $-i\alpha/\hbar$: $\nabla f = df - i(\alpha/\hbar)f$, is a prequantization line bundle:

its curvature is $-i\omega/\hbar$, for ω the symplectic form $d\alpha$. A real function g on V defines the Lagrangian section $L(g)\colon x \mapsto (dg)_x$ of T^*V. The pull-back to $L(g)$ of $e^{ig/\hbar}$ is a horizontal section of $\mathcal{L}$.

For any symplectic manifold M, let Λ_M or simply Λ be the fiber space over M whose fiber Λ_m at $m \in M$ is the space of Lagrangian (= maximal isotropic) subspaces of the tangent space T_m of M at m. To take into account the Maslov index story, it is best to take prequantization line bundles as living on Λ. They should satisfy (a) (b) below.

(a) The curvature 2-form R is the pull-back from M of $-i\omega/\hbar$:

$$\nabla^2 = -i\omega/\hbar.$$

In particular, on each fiber Λ_m of $\Lambda \to M$, $(\mathcal{L}, \nabla)$ is flat. One has $\pi_1(\Lambda_m) = \mathbb{Z}$ and

(b) the monodromy of $(\mathcal{L}, \nabla)$ on Λ_m is multiplication by i.

Remarks: (i) In the same way that a Hamiltonian vector field $X(f)$ acts on a prequantization line bundle, with the action depending on f, and not only on $X(f)$, it also acts on $(\mathcal{L}, \nabla)$ as above: $X(f)$ is symplectic, hence lifts to Λ, the question is local on Λ, and locally on Λ $(\mathcal{L}, \nabla)$ is the inverse image of a prequantization line bundle on M. If we exponentiate, we see that $\exp(X(f))$ acts by an automorphism of $(M, \omega, \mathcal{L}, \nabla)$. The infinitesimal action on sections of $\mathcal{L}$ is given by

$$\partial_{X(f)} = \nabla_{X(f)} - if/\hbar \tag{2.1}$$

(ii) To make sense of (b), we should make precise our sign convention identifying $\pi_1(\Lambda_m)$ with $\mathbb{Z}$. For the symplectic vector space $\mathbb{R}^2$, with the form $dp \wedge dq$, the generator is given by the path

$$\theta \longmapsto \text{ line spanned by } (q, p) = (\cos\theta, \sin\theta) \qquad (0 \le \theta \le \pi).$$

(iii) Let V be a symplectic vector space, and Λ_0 be the space of Lagrangian linear subspaces of V. If S_1 and S_2 in Λ_0 intersect transversally, one defines as follows a preferred homotopy class of paths from S_1 to S_2. Identify S_2 to $S_1^\vee$ by $s_2 \mapsto \omega(s_2, s_1)$. Choose a basis e_i of S_1 and let e_1' be the dual basis of S_2. The path is

$$\theta \longmapsto \text{ subspace spanned by } (\cos\theta)e_i + (\sin\theta)e_i' \qquad (0 \le \theta \le \pi/2).$$

It depends only on the quadratic form for which the basis (e_i) is orthonormal. The choice of this quadratic form running over a contractible set, the homotopy class of the path does not depend on the choice. If V is of dimension $2d$, the preferred path from S_1 to S_2, followed by the preferred path from S_2 to S_1, is d times the preferred generator of $\pi_1(\Lambda_0)$.

We now come back to the case where M is a cotangent bundle T^*V. Let $\Lambda^0 \subset \Lambda$ be the subbundle with $\Lambda_m^0 \subset \Lambda_m$ consisting of those Lagrangian subspaces L of T_m intersecting transversally the vertical subspace of T_m^* (tangent to the fibers of $T^*V \to V$). If we pull back the prequantization line bundle on T^*V to Λ^0, it extends uniquely to $\mathcal{L}_\Lambda$ on Λ satisfying the conditions (a), (b).

Let L be a Lagrangian subvariety of M. Let s be the section of Λ over L: $x \mapsto$ tangent space of L at x. The section s has values in Λ^0 if and only if L is locally of the form $L(g)$. As L is Lagrangian, $s^*\mathcal{L}_\Lambda$ is flat and it makes sense to speak of a slowly varying section of $s^*\mathcal{L}$ on L. Generalizing the $f \cdot e^{ig/\hbar}$ considered previously, to a slowly varying half-density u on L with values in $s^*\mathcal{L}$ corresponds $[u]$ in $\mathcal{H}$, with

$$\|[u]\|^2 \sim \int_S \langle u, u \rangle . \tag{2.2}$$

It is localized near $\mathrm{Supp}(u) \subset S \subset M = T^*V$.

A function f defines an automorphism $\exp(X(f))$ of $(M, \mathcal{L}, \nabla)$ (Remark (ii) above), and an automorphism $\exp(-(i/\hbar)f^\wedge)$ of $\mathcal{H}$. Those automorphisms should (almost) preserve the construction $u \mapsto [u]$:

$$[\exp(X(f))^* u] \sim \exp((i/\hbar)f^\wedge)[u]. \tag{2.3}$$

Inner products are given as follows. Fix (S_i, u_i) $(i = 1, 2)$ as above, and assume that S_1 and S_2 intersect transversally. Let x be an intersection point. The Lagrangian subspaces $s_1(x)$ and $s_2(x)$ of T_xM intersect transversally, and the preferred class of paths γ from $s_1(x)$ to $s_2(x)$ (Remark (iii) above) gives an isomorphism $\gamma^*\colon \mathcal{L}_{s_2(x)} \to \mathcal{L}_{s_1(x)}$. The symplectic structure puts $s_1(x)$ and $s_2(x)$ in duality, identifying the line of half-densities of S_1 at x with the dual of the similar line for S_2. The inner product $\langle u_1(x), \gamma^* u_2(x) \rangle$ is hence just a number. The stationary phase approximation gives

$$\langle [u_1], [u_2] \rangle \sim \sum_{x \in S_1 \cap S_2} (2\pi i\hbar)^{d/2} \langle u_1(x), \gamma^* u_2(x) \rangle \tag{2.4}$$

for M of dimension $2d$.

Our convention for inner products $\langle a, b \rangle$ is: antilinear in a, linear in b.

§3.

The flat case is when M is an affine space, with a translation invariant symplectic form. In the flat case, one can define a quantization as the data of $f \mapsto f^\wedge$, just for f a linear function, with (1.2) holding exactly. The purely imaginary linear functions form a Lie algebra for the bracket $-i\hbar\{f, g\}$, and we want a unitary representation $f \mapsto f^\wedge$ of this Lie algebra, with $i^\wedge = i \cdot \mathrm{Id}$. It is more convenient to ask for a unitary representation of the corresponding Lie group. Formula (1.4) becomes a request for irreducibility. Weyl's quantization $f \mapsto f^\wedge$, for a general f, is given by

$$\exp(\ell)^\wedge = \exp(\ell^\wedge) \tag{3.1}$$

for ℓ a purely imaginary linear function, a general $f^\wedge$ being deduced from (3.1) by considering f as a superposition of $\exp(\ell)$, by Fourier transform. For Weyl quantization, formula (1.4) holds exactly.

Understood in those terms, quantization in the flat case is unique, up to isomorphism. If M is the cotangent bundle of an affine space V, §2 gives its Schrödinger model as $L^2(V)$: if V_0 is the vector space of translations of V, one has $M = V_0^\vee \times V$; for f (the pull-back of) a function on V, $f^\wedge$ is multiplication by f. For ℓ a linear form on $V_0^\vee$, identified with a vector $v \in V_0$, $\ell^\wedge$ is $-i\hbar\partial_v$, and $\exp(i\ell)^\wedge$ is $\psi(x) \mapsto \psi(x + \hbar v)$.

This uniqueness assertion depends on the assumption that the flat symplectic variety M is of finite dimension.

Remark. In the flat case, the data on Λ_M of a prequantization line bundle defines a quantization $\mathcal{H}$, as defined above, up to unique isomorphism. This is made plausible by the fact that the group of automorphisms of

$$(\text{affine symplectic space } M, \mathcal{L}_\Lambda, \nabla)$$

and of

$$(\text{affine symplectic space } M, \mathcal{H}, f \mapsto f^\wedge)$$

are the same extension of the affine symplectic group (Sp $\ltimes$ translations) by U(1).

The relation between $\mathcal{L}_\Lambda$ and the quantization $\mathcal{H}$ can be fixed as follows. Let $\mathcal{H}_\infty$ be the C^∞-vectors in $\mathcal{H}$ for the action of the Heisenberg group. In the Schrödinger model these are the Schwartz functions on V. Let $\mathcal{H}_{-\infty} \supset \mathcal{H}$ be the dual of $\mathcal{H}_\infty$. A linear Lagrangian subspace L of M defines a section s_L of Λ_M. A $s_L^*\mathcal{L}_\Lambda$-valued half-density u on L, of the form (translation invariant half-density)$\cdot$ (flat section $s^*\mathcal{L}_\Lambda$), defines $[u]$ in $\mathcal{H}_{-\infty}$, and the inner product formula (2.4) holds exactly, to give the inner product of continuous superpositions of $[u]$. Formula (2.3) holds exactly, for f quadratic and u as above. In particular, if a linear form ℓ vanishes on L, one has $\ell^\wedge[u] = 0$.

§4.

We now go over to the superworld. A super Hilbert space is a mod 2 graded complex vector space H with an even sesquilinear form (u, v), antilinear in u and linear in v, for which

$$(v, u) = (-1)^{p(u)p(v)}(u, v)^-,$$

with a positivity and a completeness condition. As positivity condition, we take

$$\begin{aligned} (u,u) &> 0 \quad \text{for} \quad u \text{ even}, \quad u \neq 0 \\ -i(u,u) &> 0 \quad \text{for} \quad u \text{ odd}, \quad u \neq 0. \end{aligned} \tag{4.1}$$

This positivity condition is stable under tensor product if $(\ ,\ )$ for $H' \otimes H''$ is defined by

$$(u' \otimes u'', v' \otimes v'') = (-1)^{p(u'')p(v')}(u', v')(u'', v''),$$

in accordance with the sign rule. The adjoint $T^\dagger$ of an operator T is defined by

$$(Tu, v) = (-1)^{p(T)p(u)}(u, T^\dagger v). \tag{4.2}$$

The physicists don't like (4.1), and prefer to work with the ordinary mod 2 graded Hilbert space with the inner product $\langle\ ,\ \rangle$ defined by

$$\begin{aligned} (u, v) &= \langle u, v\rangle \quad \text{for} \quad u, v \text{ even} \\ (u, v) &= i\langle u, v\rangle \quad \text{for} \quad u, v \text{ odd}. \end{aligned} \tag{4.3}$$

The adjoint $T^\dagger$ is related to the corresponding ordinary adjoint T^* by

$$\begin{aligned} T^\dagger &= T^* \quad \text{for } T \text{ even} \\ T^\dagger &= iT^* \quad \text{for } T \text{ odd}. \end{aligned} \tag{4.4}$$

The definition (4.2) has the unsettling effect that the eigenvalues of an odd self-adjoint operator are in $i^{1/2}\mathbb{R}$. However, the eigenspaces being neither even nor odd, there is not much wish to consider them.

In a quantization of a supermanifold M, to functions f on M should correspond operators $f^\wedge$, acting on a super Hilbert space $\mathcal{H}$, with $f^\wedge$ of the same parity as f. If f is real, one should have $f^\wedge = f^{\wedge\dagger}$. Because of (3.4), if M is a supermanifold, the physicists will declare to be "real" the real even f, and the $i^{-1/2}\eta$ for η real. With this rule, if α_1 and α_2 are odd and "real", $i\alpha_1\alpha_2$ is "real" (as well as real, being even). Example: on $\mathbb{R}^{1,1}$, with coordinates (x,η), if we put $\theta = i^{-1/2}\eta$, the vector field $\partial_\theta + i\theta\partial_x$ is a multiple of a real vector field:

$$\partial_\theta + i\theta\partial_x = i^{1/2}(\partial_\eta + \eta\partial_x).$$

§5.

Let now M be a super affine space, with a translation-invariant symplectic form. In imitation of §3, we define a quantization as the data of a super Hilbert space $\mathcal{H}$, and of $\ell \mapsto \ell^\wedge$ associating to a linear function on M an operator on $\mathcal{H}$, with ℓ and $\ell^\wedge$ of the same parity. The quantization $\ell \mapsto \ell^\wedge$ should be linear, real: $\ell^\wedge = (\ell^\wedge)^\dagger$ for ℓ real, map 1 to Id, and obey

$$[\ell^\wedge, m^\wedge] = -i\hbar\{f,g\}^\wedge. \tag{5.1}$$

One should also require some irreducibility.

The condition (5.1) requires that for real linear odd functions ℓ, the Poisson bracket $\{\ell,\ell\}$ (a constant) be negative:

$$\{\ell,\ell\} \leq 0. \tag{5.2}$$

Indeed, $[\ell^\wedge,\ell^\wedge] = 2\ell^{\wedge 2} = 2\ell^{\wedge\dagger}\ell^\wedge = 2i\ell^{\wedge *}\ell^\wedge$ must be $-i\hbar\{\ell,\ell\}$. The sign in (5.2) is due to the choices of sign in (4.1) (leading to (4.4)) and (5.1), the latter repeating (1.2).

As in §3, one extends $\ell \mapsto \ell^\wedge$ to polynomial functions by requiring that for a product of even or odd linear functions, one has

$$(\ell_1 \ldots \ell_n)^\wedge = (1/n!)\sum_\sigma \pm \ell^\wedge_{\sigma(1)} \ldots \ell^\wedge_{\sigma(n)} \qquad (\sigma \in S_n)$$

with $\pm$ given by the sign rule. In particular, $\pm$ is $+$ if the ℓ_i are all even, and $\varepsilon(\sigma)$ if they are all odd.

§6.

Let W be a real vector space with a positive definite symmetric bilinear form B. Let M be the affine space defined by ΠW: it is of dimension $(0, \dim W)$, and linear forms on W are odd functions on M. Define a Poisson bracket by

$$\{\ell, m\} = -B^{-1}(\ell, m) \tag{6.1}$$

for $\ell, m \in W^\vee$.

If e_i is an orthonormal basis of W, with dual basis e^i, and if e^i is viewed as an odd function on M, the Poisson bracket (6.1) is given by the super symplectic form

$$\omega = \tfrac{1}{2}\sum de^i de^i. \tag{6.2}$$

On W^*, let Q be the quadratic form $\frac{1}{2}B^{-1}(\ell,\ell)$. If $\mathcal{H}$ is a mod 2 graded module over the (mod 2 graded) Clifford algebra $C(W^*,Q)$, with the module structure written $\mathrm{cl}(\ell)$ for $\ell \in W^*$, (5.1) holds for

$$\ell\hat{} = (i\hbar)^{1/2}\mathrm{cl}(\ell). \tag{6.3}$$

For $\dim W = 1$, one can take $\mathcal{H} = (\mathbb{C}\text{ even})\oplus(\mathbb{C}\text{ odd})$, with $\mathrm{cl}(e^1) = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}$ For the obvious Hilbert space structure of $\mathcal{H}$, and the corresponding super Hilbert space structure, one has then $\mathrm{cl}(e^1)^* = \mathrm{cl}(e^1)$ and for any ℓ in V^*,

$$\ell\hat{} = (\ell\hat{})^\dagger. \tag{6.4}$$

A general W can be written as an orthogonal direct sum of lines, and the corresponding tensor product of super Hilbert spaces is a Clifford module for which (6.4) holds, if $\ell\hat{}$ is defined by (6.3). If $\mathcal{H}_1$ is a graded submodule with the induced Hilbert space structure, (6.4) continues to hold for $\mathcal{H}_1$. This proves the existence of a super Hilbert space $\mathcal{H}$ with a Clifford module structure, irreducible as a graded module, for which (6.4) holds when $\ell\hat{}$ is defined by (6.3).

We now assume W of even dimension. There are the two isomorphism classes of $\mathcal{H}$ as above, exchanged by the parity change $\mathcal{H} \mapsto \mathcal{H}\otimes L$, for L a $(0,1)$-dimensional super Hilbert space. Let $(e^1,\dots,e^n)$ be an orthonormal basis (for B^{-1}), and $[e^1,\dots,e^n]$ be the corresponding density on M. Another orthonormal basis would give the same, or the opposite density. The choice of the density $[e^1,\dots,e^n]$ picks out one of the two isomorphism classes of $\mathcal{H}$: the one for which the following analogue of (1.4) holds. For any f on M, the supertrace $\mathrm{Tr}(f\hat{})$ is given by

$$\mathrm{Tr}(f\hat{}) = \hbar^{n/2}\int [e^1,\dots,e_n]f \tag{6.5}$$

Justification: writing $\mathcal{H}$ as a tensor product, one reduces to the case $n = 2$. For $n = 2$, $\mathrm{cl}(e^1)\mathrm{cl}(e^2)$ has square $-1/4$ and eigenvalues $\pm i/2$. It follows that $(e_1e_2)\hat{} = \frac{1}{2}(e_1\hat{}e_2\hat{} - e_2\hat{}e_1\hat{}) = e_1\hat{}e_2\hat{} = i\hbar\mathrm{cl}(e_1)\mathrm{cl}(e_2)$ has eigenvalues $\mp\hbar/2$, hence supertrace $\pm\hbar$.

§7.

A super symplectic flat space M as in §5 can be decomposed as $M^+ \times M^-$, with M^+ even and M^- odd, as in §6. The tensor product of quantizations of M^+ and M^- gives one for M. If M^+ is the cotangent bundle of an affine space V, and if M^- is obtained as in §6 from an even-dimensional quadratic vector space W, this tensor product can be realized as the space of L^2-densities on V with values in a Clifford module for $W^\vee$, and an analogue of (1.4), (6.5) holds.

§8.

Let V be a manifold and W be an orthogonal vector bundle on V, with a connection ∇ respecting the structural symmetric bilinear form B. We take M to be the fiber product over V of T^*V and ΠW. For any supermanifold S, a S-point of M (Supersymmetry 2.8), that is a morphism from S to M, can be identified with the data of an S-point $f\colon S \to V$ of V, of an even section of $f^*T^*_V$, and of an odd section of f^*W. A local coordinate system (q^i) on V and a local basis (e_α) of W give a local coordinate system (q^i, p_i, ψ^α) on M.

We assume that W is even dimensional, oriented and Spin. It gives rise to a bundle of super Hilbert spaces $\mathcal{H}_V$ on V, reproducing point by point what we got in §6, and the connection ∇ on W gives one on $\mathcal{H}_V$.

Let $\mathcal{H}$ be the super Hilbert space of half-densities on V with values in $\mathcal{H}_V$. To any function f on M, of degree ≤ 1 on the fibers of M/V, when p_i is viewed as of degree 1 and ψ^α as of degree $1/2$, one associates an operator $f^\wedge$ as follows.

(a) For (the pull-back of) a function f on V, $f^\wedge$ is multiplication by f.

(b) For the odd function defined by a section e of $W^\vee$,

$$e^\wedge = (i\hbar)^{1/2}\mathrm{cl}(e)$$

as in (6.3). This is extended as in §5 to define $f^\wedge$ for any function f on ΠW.

(c) If f is a function on T^*V, linear on each fiber of T^*V/V and identified with a vector field F,

$$f^\wedge = -i\hbar\nabla_F.$$

In (c), if x in $\mathcal{H}$ is the product of a section h of $\mathcal{H}_V$ by a half-density v, $\nabla_F(x) := \nabla_F(h)\cdot v + h\cdot\mathcal{L}_F(v)$.

The super vector space of operators $f^\wedge$, for f of degree ≤ 1 on the fibers of M/V, is stable under bracket. By (1.2), the bracket of operators corresponds to a Poisson bracket, which we now compute.

For functions on ΠW, it is as in §6: functions on V commute with functions on ΠW and for ℓ, m odd linear functions, identified with sections of $W^\vee$, one has

$$\{\ell, m\} = -B^{-1}(\ell, m).$$

For f a function on T^*V, linear on the fibers and corresponding to a vector field F on V, $\{f, \quad\}$ is ∇_F on functions on ΠW.

It remains to compute $[f^\wedge, g^\wedge]$ as being of the form $-i\hbar\{f,g\}^\wedge$ when f and g are linear functions on T^*V, corresponding to vector fields F and G. We have

$$\begin{aligned}[f^\wedge, g^\wedge] = [-i\hbar\nabla_F, -i\hbar\nabla_G] &= (-i\hbar)^2\left(\nabla_{[F,G]} + R(F,G)\right)\\ &= -i\hbar\left(\{f,g\}^\wedge_{T^*V} - i\hbar R(F,G)\right)\end{aligned}$$

for $\{\ ,\ \}_{T^*V}$ the Poisson bracket on T^*V and F the curvature 2-form of W, with values in $\mathrm{SO}(W)$ (which acts on $\mathcal{H}_V$).

Let S^β_α be the matrix of $S \in \mathrm{SO}(W)$, relative to a local basis e_α of W: S is $(x^\alpha) \mapsto (S^\alpha{}_\beta x^\beta)$. Let e^α be the dual basis, and move the indices up and down

using B. The action of S on $\mathcal{H}_V$ is then the Clifford multiplication by $S_{\alpha\beta}e^\alpha e^\beta$ (sum on α and β) in the Clifford algebra. This follows from

$$[S_{\alpha\beta}e^\alpha e^\beta, e^\gamma] = S_{\alpha\beta}(e^\alpha[e^\beta, e^\gamma] - [e^\alpha, e^\gamma]e^\beta) \qquad \text{(superbrackets)}$$
$$= \tfrac{1}{2}S_{\alpha\beta}(e^\alpha b^{\beta,\gamma} - b^{\alpha,\gamma}e^\beta) = -S_{\alpha\beta}b^{\alpha\gamma}e^\beta = -S^\gamma\beta e^\beta = S(e^\gamma)$$

for the Lie algbra action.

If we apply this to the curvature, and if ψ^α are the odd functions on M corresponding to the e^α, this gives $[f^\wedge, g^\wedge] = -i\hbar\{f, g\}^\wedge$ for

$$\{f, g\} = \{f, g\}_{T^*V} - R(F, G)_{\alpha,\beta}\psi^\alpha\psi^\beta$$

This Poisson bracket corresponds to the following symplectic 2-form Ω. Let α_{T^*V} be the canonical 1-form pdq on T^*V, as well as its pull-back to M. Let $\alpha_{\Pi W}$ be the 1-form on ΠW whose inverse image by a section ψ is $\frac{1}{2}(\psi, \nabla\psi)$. If (e_α) is a local basis of W, and if the connection is given by the endomorphism-valued 1-form $\gamma^\alpha{}_\beta$:

$$\nabla(x^\alpha e_\alpha) = (dx^\alpha + \gamma^\alpha{}_\beta x^\beta)e_\alpha,$$

one has

$$\alpha_{\Pi W} = \tfrac{1}{2}\psi_\alpha(d\psi^\alpha + \gamma^\alpha{}_\beta\psi^\beta).$$

With these notations,

$$\Omega = d\alpha_{T^*V} + d\alpha_{\Pi V}.$$

If (e_α) is an orthonormal basis, one has

$$d\alpha_{\Pi V} = \tfrac{1}{2}\sum_\alpha (d\psi^\alpha + \gamma^\alpha{}_\beta\psi^\beta)^2 + \sum_{\alpha,\beta} R^\alpha{}_\beta\psi^\alpha\psi^\beta.$$

Introduction to QFT

David Kazhdan

Introduction

In this series of lectures we will present Wightman axioms and show how this set of axioms can be used to derive deep and unexpected results on the behavior of quantum field theories. These axioms constitute a consistent mathematical theory which can serve for a mathematician as a gateway to the strange world of quantum field theories. In spite of the failure to describe the structure of gauge theories, Wightman axioms are very useful for developing an intuition on quantum field theories.

Notes by Roman Bezrukavnikov

LECTURE 1
Wightman Axioms

§1.0. Setup and notations

Let V be a pseudo-Riemannian manifold of signature $(1, d-1)$. In these lectures we will only consider the case $V = \mathbb{R}^{1,d-1}$ with constant quadratic form. We denote by $(\ ,\ )$: $V \times V \to \mathbb{R}$ the corresponding quadratic form and will write v^2 instead of (v, v) for $v \in V$. We will sometimes use the decomposition $V = \mathbb{R} \times \mathbb{R}^{d-1}$, where $\mathbb{R}$ (time) and $\mathbb{R}^{d-1}$ (space) are orthogonal subspaces equipped with constant positive (respectively negative) definite metrics. This decomposition is fixed throughout the lectures. We denote $V_{space} = \{v \in V | v^2 < 0\}$; and $\overline{V}_{space}$ is the closure of V_{space}. The complement $V - \overline{V}_{space}$ consists of 2 connected components, V_+ and V_- (where $V_\pm \subset \mathbb{R}^\pm \times \mathbb{R}^{d-1}$).

For $m \in \mathbb{R}$ we put $\mathcal{O}_m^\pm = \{v \in \overline{V}_\pm | v^2 = m^2,\ v \neq 0\}$. Let dv be the standard measure on V and let $\mu_m = \mathrm{Res}|_{\mathcal{O}_m^+} \frac{dv}{v^2 - m^2}$ be the invariant volume form on $\mathcal{O}_m^+$.

Let $G = \mathrm{Spin}(V)$ be the double (universal for $d > 3$) cover of the connected component of the identity of the group $\mathrm{SO}(1, d-1)$. The semidirect product $P = G \ltimes V$ is called the Poincaré group. It acts on V by isometries. We denote by $\tau \in G$ the nontrivial element of the kernel of the map $G \to \mathrm{SO}(1, d-1)$.

We will use the quadratic form to identify V^* with V, but we will distinguish between vectors and covectors by reserving the letters v, w, etc. for vectors, and p, q, etc. for covectors.

Note that the cone V_+ is self-dual (i.e. $\{p \in V | (p, v) > 0\ \forall\, v \in V_+\} = V_+$). For a function f on V we will denote by $\mathcal{F}(f)$ its Fourier transform; so we have

$$\mathcal{F}(f)(p) = \frac{1}{(2\pi)^{d/2}} \int\limits_V e^{-\imath(p,v)} f(v) dv.$$

Let U: $V \to \mathrm{Aut}(\mathcal{H})$ be a unitary representation of the vector group V in a Hilbert space $\mathcal{H}$.

Definition. The representation U is called *positive* if the support of the corresponding spectral measure lies in $\overline{V}_+$.

If U: $P \to \mathrm{Aut}(\mathcal{H})$ is a unitary representation, then we say that U is positive if $U|_V$ is positive.

For a G-module H we will write $H = H^{even} \oplus H^{odd}$, where H^{even} (respectively H^{odd}) is the space of invariants (respectively anti-invariants) of τ. We will endow

H with the structure of superspace given by this decomposition (see [**I-Supersymmetry**, §2.1] for the definition of a superspace).

Let $R = R^{even} \oplus R^{odd}$ be a finite dimensional real representation of G. Since P acts transitively on V and the stabilizer of 0 is G, we can associate to R a P-equivariant vector bundle $\mathcal{R}$ on V.

Let $\mathcal{S}^R = \mathcal{S}^R_{even} \oplus \mathcal{S}^R_{odd}$ be the space of Schwartz sections of this bundle. (Note that our bundle is trivialized equivariantly with respect to translations, and $\mathcal{S}^R$ is identified with the space of smooth rapidly decreasing R-valued functions on V.) We will often abbreviate $\mathcal{S} = \mathcal{S}^R$.

Finally we need some preparations on signs.

We say that a complex superspace $H = H^{even} \oplus H^{odd}$ is a super-Hermitian space if it is endowed with an even $\mathbb{R}$-bilinear pairing $(\ ,\ ) : H \times H \to \mathbb{C}$ such that $(\lambda x, y) = (x, \overline{\lambda} y) = \lambda(x, y)$ and $(x, y) = (-1)^{|x|\cdot|y|}\overline{(y, x)}$. Note that since $(\ ,\ )$ is even we have $(x, y) = 0$ for $x \in H^{even}$, $y \in H^{odd}$.

H is called a super-Hilbert space if $(H^{even}, (\ ,\ ))$ and $(H^{odd}, -i(\ ,\ ))$ are Hilbert spaces. For $T \in \mathrm{End}(H)$ the (super)adjoint operator T^* is defined by $(Tx, y) = (-1)^{|x|\cdot|T|}\ (x, T^* y)$. An operator is super-Hermitian symmetric if $T^* = T$. We will often omit the word super.

Now we are ready to formulate

§1.1. Wightman axioms

Data:

1) A finite dimensional real representation ρ: $G \to \mathrm{Aut}\, R$.
2) A positive unitary representation U: $P \to \mathrm{Aut}(\mathcal{H})$ in a (super)Hilbert space $\mathcal{H}$.
3) A dense P-invariant subspace $\mathcal{D} \subset \mathcal{H}$, and a P-invariant vector $\Omega \in \mathcal{D}$, $\|\Omega\| = 1$.
4) A linear map ϕ: $\mathcal{S}^R \to \mathrm{End}(\mathcal{D})$, called *the field map* such that

α) ϕ is P-equivariant, where P acts on $\mathrm{End}(\mathcal{D})$ by conjugation.

β) $\phi(f)$ is super-Hermitian symmetric.

γ) For any $\psi_1, \psi_2 \in \mathcal{D}$ the map $f \to \langle \psi_1, \phi(f)\psi_2 \rangle$ from $\mathcal{S}$ to $\mathbb{C}$ is continuous ($\mathcal{S} = \mathcal{S}^R$ is equipped with the Schwartz topology).

δ) The space $\mathcal{D}$ is generated by vectors $\phi(f_1)\phi(f_2)\cdots\phi(f_n)\Omega$, for $f_1, \dots, f_n \in \mathcal{S}$.

ε) (space-locality) If $f_1, f_2 \in \mathcal{S}$ are such that for any $v_1 \in \mathrm{supp}(f_1)$, $v_2 \in \mathrm{supp}(f_2)$ we have $(v_1 - v_2) \in V_{space}$ (such f_1, f_2 are called *space-like separated*), then $[\phi(f_1), \phi(f_2)] = 0$ where $[\ ,\ ]$ stands for supercommutator.

Remark. It is often assumed that the space of V-invariants in $\mathcal{H}$ is 1-dimensional (uniqueness of vacuum).

Such a set of data satisfying the above properties is called QFT in the Wightman axioms (or simply Wightman QFT).

§1.2. Wightman functions

Let us denote $V_n = V \times V \times \cdots \times V$ (n times); $\mathcal{R}_n = \mathcal{R}^{\boxtimes n}$ (this is a bundle on V_n). The space of Schwartz sections of $\mathcal{R}_n$ will be denoted by $\mathcal{S}_n$.

For $n \geq 1$ the Schwartz theorem and the property (γ) imply the existence of unique tempered $\mathbb{C}$-valued distributions $\mathcal{W}_n \in \mathcal{S}'_n$ such that

$$\mathcal{W}_n(f_1 \times f_2 \times \cdots \times f_n) = \langle \Omega, \phi(f_1)\phi(f_2)\dots\phi(f_n)\Omega \rangle$$

for $f_1, \dots, f_n \in \mathcal{S}$.

These $\mathcal{W}_n$ are called *Wightman functions*, or correlation functions of the theory. (We will use the word "function" for a section of an appropriate vector bundle when it causes no confusion).

§1.3. Reconstruction of QFT from Wightman functions

We are going to show that a QFT is uniquely determined by its Wightman functions. So we assume that a representation ρ of G and a collection of distributions $\mathcal{W}_n \in \mathcal{S}_n$ are given, and that $\mathcal{W}_n$ are the Wightman functions of some Wightman QFT $(\rho; \mathcal{H}, \mathcal{D}, \Omega; \phi)$. We want to reconstruct $\mathcal{H}, \mathcal{D}, \Omega$ and ϕ.

Let us put $\widetilde{\mathcal{S}} = \bigoplus_n \mathcal{S}_n$, $\mathcal{W} = \prod_n \mathcal{W}_n \in \widetilde{\mathcal{S}}'$ (where we use the standard notation L' for the topological dual of a topological vector space L). The space $\widetilde{\mathcal{S}}$ carries an algebra structure: $f_1 \cdot f_2 = f_1 \boxtimes f_2$. For $f \in \mathcal{S}_n$ we define $f^* \in \mathcal{S}_n$ by $f^*(v_1, \dots, v_n) = \pm f(v_n, \dots, v_1)$ (the sign is determined by the super-rule as usual). Consider a bilinear form $\{\ ,\ \}$ on $\widetilde{\mathcal{S}}$ given by

$$\{f_1, f_2\} = \mathcal{W}(f_1^* \cdot f_2).$$

Claim. $\{f_1, f_2\} = \langle \phi(f_1)\Omega, \phi(f_2)\Omega \rangle$.

Proof. Clear.

We can recover $\mathcal{H}$ as the Hilbert space completion of $(\widetilde{\mathcal{S}} \otimes \mathbb{C}, \{\ ,\ \}')$, where $\{\ ,\ \}'$ is the unique Hermitian scalar product on $\widetilde{\mathcal{S}} \otimes \mathbb{C}$, such that $\{\ ,\ \}'|_{\widetilde{\mathcal{S}}} = \{\ ,\ \}$. Then we can take $\mathcal{D}$ to be the image of the obvious map $\widetilde{\mathcal{S}} \otimes \mathbb{C} \to \mathcal{H}$ and define Ω as the image of $1 \in \widetilde{\mathcal{S}}$ under this map. Since P acts on $\widetilde{\mathcal{S}}$ preserving $\{\ ,\ \}$ this action defines an action of P on $\mathcal{H}$. Finally, we get an algebra map ϕ: $\widetilde{\mathcal{S}} \to \mathrm{End}(\mathcal{D})$.

Before we formulate the next Claim let us fix some notations. Let $\Delta: V \to V_n$ be the diagonal embedding, and $\overline{V}_n = V_n/\Delta$ be the quotient space. Since $\mathcal{R}_n$ is P-equivariant it defines a bundle $\overline{\mathcal{R}}_n$ on $\overline{V}_n$. Let $\overline{\mathcal{S}}_n$ be the space of Schwartz sections of $\overline{\mathcal{R}}_n$. For $f \in \mathcal{S}_n$ let $\bar{f} \in \overline{\mathcal{S}}_n$ be its averaging over $\Delta(V)$, and let pr^* be the dual map on distributions.

We identify $\overline{V}_n \tilde{\to} V_{n-1}$ by

$$(v_1, v_2, \dots, v_n) \to (v_2 - v_1, \dots, v_n - v_{n-1}).$$

Claim. *The $\mathcal{W}_n$ satisfy the following properties*

1) *$\mathcal{W}_n$ is P-invariant for all n.*
   In particular, $\mathcal{W}_n$ is $\Delta(V)$-invariant; hence $\mathcal{W}_n = \mathrm{pr}^*(W_n)$ for a unique $W_n \in \overline{\mathcal{S}}'_n$.
2) $\mathrm{supp}\, \mathcal{F}(W_n) \subset (\overline{V}_+)^{n-1}$.

3) $\mathcal{W}_n(f^*) = \overline{\mathcal{W}_n(f)}$.

4) $\mathcal{W}(v_1, \dots, v_i, v_{i+1}, \dots, v_n) = \pm \mathcal{W}(v_1, \dots, v_{i+1}, v_i, \dots, v_n)$ *if* $v_i - v_{i+1} \in V_{space}$. *(Here the sign comes from the* $\mathbb{Z}/2$ *grading on the space of test functions* $\mathcal{S}_n = \mathcal{S}_n^+ \oplus \mathcal{S}_n^-$.*)*

5) $\mathcal{W}(f^* \cdot f) \geq 0$.

Proof. 1) and 3)–4) are clear. Let us prove 2).

We have to check that $W_n(f) = 0$ if for some i $\operatorname{supp}(\mathcal{F}(f))$ lies in the domain $\{(v_1, .., v_n) \in V_n | v_i \in V - \overline{V}_+\}$. For such f we can find $g \in \mathcal{S}_{n+1}$ with $\bar{g} = f$ satisfying $\operatorname{supp} \mathcal{F}(g) \subset \{(v_1, \dots, v_{n+1}) \in V_{n+1} | v_1 + \dots + v_{i+1} \in V - \overline{V}_+\}$. So we have to show for such g that $\mathcal{W}_n(g) = 0$. We can assume $g = g_1 \otimes g_2$ for $g_1 \in \mathcal{S}_{i+1}, g_2 \in \mathcal{S}_{n-i}$, and $\operatorname{supp} \mathcal{F}(g_1) \subset \{(v_1, \dots, v_{i+1}) \in V_{i+1} | v_1 + \dots + v_{i+1} \subset V - \overline{V}_+\}$. We have $\mathcal{W}_n(g) = \langle \phi(g_1)^* \Omega, \phi(g_2) \Omega \rangle$. But the spectral support of $\phi(g_1^*)\Omega$ is contained in the set $\{v_1 + \dots + v_{i+1} | (v_1, \dots, v_{i+1}) \in \operatorname{supp} \mathcal{F}(g_1)\}$. On the other hand, it is contained in $\overline{V}_+$ by the positivity condition. Hence, it is empty, and $\phi(g_1^*)\Omega = 0 \Rightarrow \mathcal{W}_n(g) = 0$.

Remark. By a standard theorem from analysis property 2) is equivalent to the following

2′) W_n is a boundary value of an analytic function on $V_{n-1} + i(V_+)^{n-1}$.

It is not hard to see the following

Theorem. *The above constructions provide a bijection between the following sets of data:*

a) ρ; $\mathcal{H} \supset \mathcal{D} \ni \Omega$; ϕ *satisfying Wightman's axioms;*

b) *representation* ρ *of* G *and a collection of distributions* $\{\mathcal{W}_n\}$ *satisfying properties 1) – 5) above.*

Remark. Sometimes one considers theories satisfying weaker axioms. Namely, let $\mathcal{S}_n^0 \subset \mathcal{S}_n$ be the subspace of sections which vanish on diagonals together with all partial derivatives. A QFT satisfying the weak Wightman Axioms is (a representation ρ together with) a collection of $\mathcal{W}_n \in (\mathcal{S}_n^0)'$ satisfying 1), 2′), 3), 4) above.

Note that the essential property 5) cannot be formulated if $\mathcal{W}_n$ are defined only on $\mathcal{S}_n^0$ because $f^* \cdot f$ does not lie in $\mathcal{S}^0$ for $f \neq 0$.

§1.4. Spin-statistics Theorem

This is a statement that our convention on the parity of the basic superspace R is the only right one. More precisely, we have

Theorem. *Consider the Wightman axioms with the sign in the axiom of space-locality changed. Any theory satisfying this modified set of axioms is trivial. (The field map* $\phi = 0$.*)*

Proof. Let us take a 2-dimensional subspace $U \subset V$ of signature (1,1). Let σ' be the element of $\mathrm{SO}(d-1,1)$ which acts by -1 on U and by 1 on the orthogonal complement; let σ be an element in the complexification of $\mathrm{Spin}(d-1,1)$ which projects to σ'. Then $\sigma^2 = \tau$ is the spin element. Also note that the eigenspaces of σ in the space $R \otimes \mathbb{C}$ are defined over $\mathbb{R}$. If τ acts trivially in R then this is obvious because σ is real; otherwise τ acts by -1 and we use the identity $\bar{\sigma} = \tau\sigma$ to see that the $\pm i$ eigenspace of τ is invariant under the conjugation.

In the next lecture (see §2.2) we will show that W_n extends to an analytic $G_{\mathbb{C}}$-equivariant function on a certain open domain in the complexification $\overline{V}_n \otimes \mathbb{C}$. In particular W_2 is analytic in the domain $\widetilde{\mathcal{T}} := \{v \mid v^2 \notin \mathbb{R}^{\geq 0}\}$. This analytic function is also denoted W_2.

Let now $r \in R$ be a σ eigenvector; let v be a point in $U \otimes \mathbb{C} \cap \widetilde{\mathcal{T}}$. We claim that $\langle r \otimes r, W_2(v)\rangle = 0$. Indeed, we have $\langle r \otimes r, W_2(v)\rangle = \langle \sigma(r) \otimes \sigma(r), W_2(-v)\rangle = \mathrm{spin}(R) \cdot \langle r \otimes r, W_2(-v)\rangle$ where $\mathrm{spin}(R)$ is the τ eigenvalue on R; here the first equality follows from σ-equivariance, and the second is obvious from $\sigma^2 = \tau$.

On the other hand by our assumption

$$\langle r_1 \otimes r_2, W_2(v)\rangle = -\mathrm{spin}(R) \cdot \langle r_2 \otimes r_1, W_2(-v)\rangle$$

for $v \in V^{space}$. Since $V^{space} \subset \widetilde{\mathcal{T}}$, we get $\langle r \otimes r, W_2(v)\rangle = 0$ for $v \in V^{space}$, and hence for all $v \in \widetilde{\mathcal{T}}$.

For any real valued Schwartz function f on V we have $\langle \phi(f \cdot r)\Omega, \phi(f \cdot r)\Omega\rangle = \int \langle f \cdot r \times f \cdot r, \mathcal{W}_2\rangle = \int \langle \mathcal{F}(f) \cdot r \times \mathcal{F}(f) \cdot r, \mathcal{F}(\mathcal{W}_2)\rangle = \int \mathcal{F}(f) \cdot \overline{\mathcal{F}(f)}\, \langle r \otimes r, \mathcal{F}(W_2)\rangle$. This shows that the distribution $\langle r \otimes r, \mathcal{F}(W_2)\rangle$ is nonnegative.

Now let us take $v \in V^+$. Since $iv \in \widetilde{\mathcal{T}}$, we get

$$0 = \langle r \otimes r, W_2(iv)\rangle = \int \exp(-v, p)\, \langle r \otimes r, \mathcal{F}(W_2)\rangle\,.$$

Since $\exp(-v, p)$ is positive this implies that $\langle r \otimes r, \mathcal{F}(W_2)\rangle = 0$, hence $(\phi(f \cdot r)\Omega)^2 = 0$, so the theory is trivial.

We finish with some elementary properties of Wightman functions, and their relation to properties of the field map.

§1.5. Mass spectrum of a theory

Let us denote $\Delta_+(p, m^2) = \mathcal{F}(\delta_{\mathcal{O}_m^+})(p)$.

Claim. *For any scalar field theory ϕ: $\mathcal{S}(V) \to \mathrm{End}(\mathcal{D})$ there exists a positive measure $d\rho$ on $[0, \infty)$ such that*

$$W_2(p) = \int \Delta_+(p, m) d\rho(m)\,.$$

Proof. We have to show that $\mathcal{F}(W_2) = \int d\rho(m)\delta_{\mathcal{O}_m^+}$ for a positive measure $d\rho$. Since $\mathcal{O}_m^+$ is an orbit of the group G it is obvious that any G-invariant distribution supported in $\sqcup \mathcal{O}_m^+ = \overline{V_+}$ can be written down in such a form for some distribution ρ. In order to see that ρ is positive we have to show that $\mathcal{F}(W_2)$ is positive.

For any $f \in \mathcal{S}$ we have $\mathcal{W}_2(f \boxtimes f) = \langle \phi(f)\Omega, \phi(f)\Omega\rangle \geq 0$. Hence for $g = \mathcal{F}(f)$ we have $\mathcal{F}(\mathcal{W}_2)\,(g \boxtimes g) \geq 0$.

We know that $\mathcal{F}(\mathcal{W}_2)$ is concentrated on the anti-diagonal and

$$\mathcal{F}(\mathcal{W}_2)(p, -p) = \mathcal{F}(W_2)(p)$$

Now note that $f \in \mathcal{S}$ iff $g = \mathcal{F}(f) \in \mathcal{S}^{\mathbb{C}}$ and $g(-p) = \overline{g(p)}$. Thus for any such g

$$\mathcal{F}(W_2)(g \cdot \overline{g}) = \mathcal{F}(\mathcal{W}_2)(g \boxtimes g) \geq 0$$

which means that $\mathcal{F}(W_2)$ is positive.

Definition. The *mass spectrum* of a QFT is the support of the measure $d\rho$. The *mass* of the theory is the infimum of its mass spectrum with 0 removed.

§1.6. Asymptotics of Wightman functions

The next statement is a baby example of the relation between the analytic behavior of Wightman functions and properties of the field map. Estimates of this kind will be used essentially in the lecture on scattering theory (see §4.3, Proposition 1 and Lemma 1).

Claim.

a) Uniqueness of vacuum in a QFT *is equivalent to the equality*

$$\lim_{\lambda\to\infty} \mathcal{W}_n(v_1,\dots,v_i,v_{i+1}+\lambda a,\dots,v_n+\lambda a) = \mathcal{W}_i(v_1,\dots,v_i)\mathcal{W}_{n-i}(v_{i+1},\dots,v_n)$$

being true for all i, n and a such that $a^2<0$.

b) Assume that the mass of the theory is $m_0>0$. Then $W_2(v)\sim e^{-m_0\sqrt{-v^2}}$ as $v^2\to-\infty$.

Proof. a) For any $f_1,\dots,f_n\in\mathcal{S}$ we have

$$\mathcal{W}_n(f_1,\dots,f_i,T_{\lambda a}(f_{i+1}),\dots,T_{\lambda a}(f_n)) = \langle\phi(f_i)\dots\phi(f_1)\Omega,$$
$$\phi(T_{\lambda a}(f_{i+1}))\dots\phi(T_{\lambda a}(f_n))\Omega\rangle = \langle\phi(f_i)\dots\phi(f_1)\Omega\,,\,U(T_{\lambda a})(\phi(f_{i+1})\dots\phi(f_n)\Omega)\rangle$$

where $T_{\lambda a}$ stands for translation by λa. From the positivity condition it follows that the weak limit $\lim\limits_{\lambda\to\infty} U(T_{\lambda a})$ exists as a bilinear form (i.e. all matrix coefficients converge), and equals the projection on $\mathcal{H}^V$.

Hence

$$\lim_{\lambda\to\infty}\langle\phi(f_i)\dots\phi(f_1)\Omega, U(T_{\lambda a})\,(\phi(f_{i+1})\dots\phi(f_n)\Omega)\rangle =$$
$$\langle \mathrm{pr}(\phi(f_i)\dots\phi(f_1)\Omega), \mathrm{pr}(\phi(f_{i+1})\dots\phi(f_n)\Omega)\rangle$$

where pr is the projection on $\mathcal{H}^V$. The statement follows.

b) We have

$$W_2(v) = \int\limits_V \mathcal{F}(W_2)(p)e^{i(v,p)}dp = \int\limits_0^\infty d\rho(m)\int\limits_{\mathcal{O}_m^+} e^{(iv,p)}d\mu_m(p)$$

where μ_m is the invariant measure on $\mathcal{O}_m^+$. For $v\in\mathbb{R}^{d-1}-\{0\}$ it can be rewritten as

$$\text{const} + \int\limits_{m_0}^\infty d\rho(m)\int\limits_{\mathbb{R}^{d-1}} e^{i(v,q)}dq/\sqrt{m^2-q^2}$$

(Recall that $q^2=(q,q)$ is negative for $q\in\mathbb{R}^{d-1}-\{0\}$).

The integrand is analytic in the region $Re(m_0^2-q^2)>0$; it also decreases as $(\vec{p})^2\to-\infty$ for fixed p_0, where $p=(\vec{p},p_0)$ is the decomposition into space and time components. Hence we can shift the contour of integration from $\mathbb{R}^{d-1}$ to $\Gamma\subset\mathbb{C}^{d-1}$, where Γ is $\mathbb{R}^{d-1}+im_0v/\sqrt{-v^2}$ squeezed slightly towards $\mathbb{R}^{d-1}$ in a small neighborhood of the singular point $im_0v/\sqrt{-v^2}$ to avoid the singularity. Then the estimate becomes obvious.

We will usually assume that uniqueness of vacuum holds for the theory under consideration.

References and cross references

R. Streater, A. Wightman, *PCT, spin and statistics and all that,* §§*3.1, 3.4*, Benjamin, NY, 1964.

LECTURE 2
Euclidean Formulation of Wightman QFT

§2.1. Analytic continuation of Wightman functions

In Lecture 1 we were working with Minkowski space-time V. However it is often desirable to be able to work with Euclidean (Riemannian rather than pseudo-Riemannian) space-time as well. In this lecture we explain what data over the Euclidean space-time $E = \mathbb{R}^d$ corresponds to a QFT in the Wightman axioms.

More precisely, let $V_{\mathbb{C}} = V \otimes \mathbb{C}$, $V_n^{\mathbb{C}} = V_n \otimes \mathbb{C}$, and consider the Euclidean affine space $E = i\mathbb{R} \times \mathbb{R}^{d-1} \subset V_{\mathbb{C}}$.

Going between QFT data on Minkowski space-time $V = \mathbb{R} \times (\mathbb{R})^{d-1}$ and data on $E = i\mathbb{R} \times (\mathbb{R})^{d-1}$ is often called *Wick rotation*.

The plan is to obtain Euclidean counterparts of Wightman functions by analytically continuing $\mathcal{W}_n$ to an open domain in $V_n^{\mathbb{C}}$; this domain contains the complement to diagonals in $E^n \subset V_n^{\mathbb{C}}$, so we can restrict these analytic functions to get the desired functions on E^n without diagonals.

We need some notations. Let $G_{\mathbb{C}}$ be the complexification of G. We set $\mathcal{T} = V - iV_+ \subset V_{\mathbb{C}}$ (the *backward tube*); $\mathcal{T}_n = (\mathcal{T})^n \subset V_n^{\mathbb{C}}$ (the *extended tube*); $\widetilde{\mathcal{T}}_n = G_{\mathbb{C}}(\widetilde{\mathcal{T}}_n) \subset V_n^{\mathbb{C}}$ (the *permuted extended tube*) ; $J_n = \widetilde{\mathcal{T}}_n \cap V_n$. Recall that we identified $V_n = \overline{V}_{n+1}$, so we have an action of the symmetric group Σ_{n+1} on V_n.

Proposition.

a) *Assume $d > 2$. Then for any $\sigma \in \Sigma_{n+1}$ the subset $J_n \cap \sigma(J_n) \subset V_n$ is open and nonempty.*

b) *If $d > 2$ then for any $\sigma \in \Sigma_{n+1}$ the subset $\widetilde{\mathcal{T}}_n \cap \sigma(\widetilde{\mathcal{T}}_n)$ is connected.*

c) *Put $E = i\mathbb{R} \times \mathbb{R}^{d-1} \subset V_{\mathbb{C}}$; $E_n = E \times \cdots \times E \subset V_n^{\mathbb{C}}$. Also let $E_n^0 \subset E_n$ be the complement of all diagonals, and $\bar{E}_n = E_n/\Delta(E)$; $\bar{E}_n^0 = E_n^0/\Delta(E)$. Then $\bigcup_{\sigma \in \Sigma_{n+1}} \sigma(\widetilde{\mathcal{T}}_n) \supset \overline{E_{n+1}^0}$.*

Proof. We start with the following result.

Lemma. *(Jost's theorem)* $J_n = \{(v_1, \ldots, v_n) \in V_n | \forall (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n - \{0\}$: $(\lambda_1 v_1 + \cdots + \lambda_n v_n) \in V_{space}\}$.

Proof of the Lemma. Assume that $(v_1, \ldots, v_n) \in J_n$. Then for some $g \in G_{\mathbb{C}}$ we have $g(v_1), g(v_2), \ldots, g(v_n) \in \mathcal{T}$, hence $g(\lambda_1 v_1 + \cdots + \lambda_n v_n) \in \mathcal{T}$ for $(\vec{\lambda}) \in \mathbb{R}_+^n - \{0\}$. Let us write $\sum \lambda_k v_k = \xi - i\eta$, where $\eta \in V_+$. Then $(\sum \lambda_k v_k)^2 = (\xi)^2 - (\eta)^2 - 2i(\xi, \eta)$. The L.H.S. is real, hence $(\xi, \eta) = 0$, and $(\xi)^2 \leq 0$, i.e. $(\sum \lambda_k v_k)^2 < 0$.

Let us prove the converse statement. So assume that the (convex) cone generated by $v_1, \dots, v_n$ does not intersect $\overline{V}_+ \cup \overline{V}_-$. Then this cone can be separated from $V_\pm$ by a hyperplane tangent to $\overline{V}_\pm$. This means that there exist $\alpha \in \overline{V}_+$ and $\beta \in \overline{V}_-$ such that $(\alpha)^2 = (\beta)^2 = 0$, $(\alpha, v_k) < 0$, $(\beta, v_k) < 0$ for $k = 1, \dots, n$ and $(\alpha, w) > 0$ for $w \in V_+$; $(\beta, w) > 0$ for $w \in V_-$. Then $(\alpha, \beta) \neq 0$, and it is easy to show that there exists $g \in G_{\mathbb{C}}$ such that $g(\alpha) = i\alpha$, $g(\beta) = -i\beta$ and g is the identity on the subspace orthogonal to $\langle \alpha, \beta \rangle$. This g satisfies the condition $g(v_k) \in \mathcal{T}$ for $k = 1, \dots, n$.

Proof of the Proposition. a) It is obvious that this subset is open. So we must show it is nonempty, i.e. we are to find a point $\underline{v} = (v_1, \dots, v_n)$ such that $(v_1, \dots, v_n)$ and $(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ satisfy the conditions of the last lemma. We can choose two linearly independent vectors $\ell_1, \ell_2 \in \mathbb{R}^{d-1} \subset V$ (since $d > 2$) and put $\underline{v} = (\ell_1, \dots, \ell_1) + \sigma^{-1}(\ell_2, \dots, \ell_2)$. Then it is immediate to check the conditions.

b) It is enough to show that $\sigma(\widetilde{\mathcal{T}}_n) \cap \mathcal{T}_n$ is connected for any σ.

We have a standard

Lemma. *Generic element of $G_{\mathbb{C}}$ can be written as $g = g' \cdot g_{\mathbb{R}}$, where $g_{\mathbb{R}} \in G$ and g' preserves an orthogonal decomposition of $V_{\mathbb{C}}$ into the sum of subspaces of dimension not greater than 2 defined over $\mathbb{R}$. Here by generic we mean an element belonging to an open (in fact real Zariski open) dense subset.*

Let us now finish the proof of b). Fix $\sigma \in \Sigma_{n+1}$ and consider $S = \{(g \in G_{\mathbb{C}}, \underline{v} \in V_n^{\mathbb{C}}) \mid g(\sigma(\underline{v})) \in \mathcal{T}_n\}$. Take a point $(g, \underline{v}) \in S$.

Moving $(g, \underline{v})$ slightly inside S we can achieve that g is generic as in the last lemma. We can now take g', $g_{\mathbb{R}}$ as in the Lemma. Obviously $(g', g_{\mathbb{R}}^{-1}(\underline{v}))$ and $(g, \underline{v})$ lie in one connected component of S. Set $\underline{v}' = g_{\mathbb{R}}^{-1}(\underline{v})$.

Let $V_0 \subset V$ be the unique block in the decomposition corresponding to g' on which $(\ ,\ )$ is not negative, and let $\wp : V_n^{\mathbb{C}} \to (V_0^{\mathbb{C}})^n$ be the orthogonal projection. One sees immediately that for $\underline{v} \in \mathcal{T}_n$ the segment connecting $\underline{v}$ with $\wp(\underline{v})$ lies in $\mathcal{T}_n$. Also $\wp$ commutes with g' and σ; it follows that this segment also lies in $\sigma^{-1}(g'^{-1}(\mathcal{T}_n))$, hence $(g', \underline{v}')$ and $(g', \wp(\underline{v}'))$ are connected inside S.

If $\dim(V_0) = 1$ then $g'|_{V_0} = \pm Id$. So in that case we have $\wp(\underline{v}') \in \mathcal{T}_n$, $\pm\sigma(\wp(\underline{v}')) \in \mathcal{T}_n$, which is only possible if $\sigma = 1$ or $\sigma = w_0$ is the "long" permutation ($w_0(i) = n + 2 - i$). But in these two cases $\sigma(\widetilde{\mathcal{T}}_n) \supset \mathcal{T}_n$, so there is nothing to prove anyway.

Suppose now that $\dim(V_0) = 2$. Set $G^0 = \mathrm{Spin}(V_0)$, $G^0_{\mathbb{C}} = \mathrm{Spin}(V_0 \otimes \mathbb{C})$, $\mathcal{T}^0_n = V_0 + iV_0^+$. Then $G^0_{\mathbb{C}}/G^0$ is the circle S^1, and one can show that for $\sigma \neq 1, w_0$ the set $\{g \in G^0_{\mathbb{C}}/G^0 \mid g(\sigma(\mathcal{T}^0_n)) \cap \mathcal{T}^0_n \neq \emptyset\}$ coincides with $S^1 - \{1, -1\}$. For given g the set $g(\sigma(\mathcal{T}^0_n)) \cap \mathcal{T}^0_n$ is convex and hence connected; thus the set $S_{V_0} := \{(g \in G^0_{\mathbb{C}}, \underline{v} \in \mathcal{T}^0_n \mid g(\sigma(\underline{v})) \in \mathcal{T}_n\}$ has 2 connected components. It is easily seen that these 2 components are permuted by conjugation by an element of G, hence they lie in the same connected component of S. But we have shown that any point in S is connected to a point in $(Stab_{V_0} \times V_0) \cap S$, and hence to a point in $S_{V_0} \subset S$. (Here we use that the action of g' on V_0 has determinant 1, because otherwise it has eigenvalues 1, -1, which does not happen for generic g.)

Since G transitively permutes 2-dimensional subspaces of V of signature $(1, 1)$ we see that S is connected, which implies the statement.

c) For $v \in V_{\mathbb{C}}$ let us write $v = (v^0, \vec{v})$, $v_0 \in \mathbb{C}$, $\vec{v} \in \mathbb{C}^{d-1}$. It is clear that for $(v_1, \dots, v_{n+1}) \in E^0_{n+1}$ there exists $g \in \mathrm{SO}(n+1)$ such that $(gv_i)^0 \neq (gv_j)^0$ for $i \neq j$. Then we can permute $(v_1, \dots, v_{n+1})$ so as to get $(v_i)^0 > (v_j)^0$ for $i > j$. But this implies that the corresponding vector in $\overline{V}_{n+1}$ lies in $\widetilde{\mathcal{T}}_n$.

Now we are ready to prove

Theorem. *Let $d > 2$. Then W_n extends to an analytic function on $\bigcup\limits_{\sigma \in \Sigma_{n+1}} \sigma(\widetilde{\mathcal{T}}_n)$. These functions are Σ_{n+1} and $G_{\mathbb{C}}$-equivariant. (The Σ_{n+1}-equivariance is understood as usual in the "super" sense.)*

Proof. Since $\mathcal{F}(W_n)$ is supported in V_+ it follows that W_n extends to an analytic function on $\mathcal{T}$ (which will be denoted by the same letter). Since W_n is G-equivariant, this analytic function is G-equivariant.

To ensure that W_n extends to a single-valued analytic function on $\widetilde{\mathcal{T}}$, which then is automatically $G_{\mathbb{C}}$-equivariant, it is enough to check that $W_n(x) = W_n(g(x))$ for $g \in G_{\mathbb{C}}$, $x \in \mathcal{T}_n$ and $g(x) \in \mathcal{T}_n$.

If $g^{-1}\mathcal{T}_n \cap \mathcal{T}_n \neq \emptyset$ then also $g^{-1}\mathcal{T} \cap \mathcal{T} \neq \emptyset$. Let us take $v \in g^{-1}\mathcal{T} \cap \mathcal{T}$. It is easy to see that for $v \in \mathcal{T}$ the set $\{h \in G_{\mathbb{C}} | h(v) \in \mathcal{T}\}$ is connected. Hence there exists a neighborhood $U \subset \mathcal{T}_n \cap g^{-1}\mathcal{T}_n$ of the point $(v, \dots, v)$ such that g^{-1} lies in the connected component of 1 of $\{h \in G_{\mathbb{C}} | h(U) \subset \mathcal{T}\}$. From G-invariance of W_n it follows that for fixed $x \in V_n^{\mathbb{C}}$ the function $W_n(g(x))$ defined on some open subset of $G_{\mathbb{C}}$ is locally constant. Hence $W_n(x) = W_n(g(x))$ for $x \in U$. Since U is open in the connected set $g^{-1}\mathcal{T}_n \cap \mathcal{T}_n$ we can conclude that $W_n(x) = W_n(g(x))$ for all $x \in g^{-1}\mathcal{T}_n \cap \mathcal{T}_n$, so analyticity of W_n in $\widetilde{\mathcal{T}}$ is proved.

Now by Proposition a) for any $\sigma \in \Sigma_{n+1}$ the set $\sigma^{-1}(\widetilde{\mathcal{T}}) \cap \widetilde{\mathcal{T}}$ contains an open non-empty subset of V_n. Moreover, from Jost's theorem it follows that

$$\sigma^{-1}(\widetilde{\mathcal{T}}) \cap \widetilde{\mathcal{T}} \cap V_n \subset (V_{space})^n.$$

But $W_n\big|_{(V_{space})^n}$ is Σ_{n+1} (super)equivariant. Hence W_n and $\sigma^*(W_n)$ agree on a neighborhood of the set of real points in $\sigma^{-1}(\widetilde{\mathcal{T}}) \cap \widetilde{\mathcal{T}}$, and hence on the whole of $\sigma^{-1}(\widetilde{\mathcal{T}}) \cap \widetilde{\mathcal{T}}$ by Proposition b). So the collection of functions $\sigma^*(W_n)$ for all $\sigma \in \Sigma_n$ give a well-defined analytic function on $\bigcup\limits_{\sigma \in \Sigma_{n+1}} \sigma(\widetilde{\mathcal{T}}_n)$, which is obviously Σ_{n+1}-(super)equivariant. The theorem is proven.

The Theorem together with part c) of the Proposition imply that $\overline{E^0_n}$ lie in the domain of analyticity of W_n. We denote the restriction of this function to $\overline{E^0_n}$ by $\overline{\mathcal{S}}_n$. We also put $\mathcal{S}_n = \pi^*(\overline{\mathcal{S}}_n)$ where $\pi : E^0_n \to \overline{E^0_n}$ is the projection.

Definition. The functions $\mathcal{S}_n$ are called *Schwinger functions.*

From the theorem it follows that $\mathcal{S}_n$ is a real-analytic $\mathrm{SO}(n) \ltimes E$ and Σ_n equivariant function.

Remark. The theorem (and its proof) relies only on the weak Wightman axioms. Hence Schwinger functions can be constructed from a set of $\mathcal{W}_n \in \widetilde{\mathcal{S}}'_n$ which satisfy the weak Wightman axioms. Also note that the proof of analyticity of $\mathcal{W}_n$ in $\widetilde{\mathcal{T}}$ did not use the space-locality axiom as well.

Remark. The assumption that $d > 2$ is not essential in the Theorem. A method which allows to treat also the case $d = 2$ and furhter extend the analyticity domain (the so called "edge of the wedge Theorem") is explained e.g. in [Streater-Wightman], §3.3.

Remark. It was explained that QFT has two formulations: Minkowski and Euclidean. It is more convenient to work technically with the Euclidean picture, though the basic notions are naturally formulated in the Minkowski picture.

Remark. The analytic Schwinger functions on $\Sigma_{n+1}(\widetilde{\mathcal{T}}_n)$ seem to be the correct objects of consideration. So it is natural to ask whether they can be further extended to some larger domain.

The domain $\Sigma_{n+1}(\widetilde{\mathcal{T}}_n)$ is not holomorphically convex.

It may be worthwhile to describe its convex hull and look at the extension of Schwinger functions to it.

§2.2. Euclidean formulation of Wightman QFT

Recall that the decomposition $E = (i\mathbb{R}) \times \mathbb{R}^{d-1}$ is fixed. Let θ denote reflection in the hyperplane $\{0\} \times \mathbb{R}^{d-1}$; put $E^+ = (i\mathbb{R}_{>0}) \times \mathbb{R}^{d-1}$; $E_n^+ = E^+ \times E^+ \times \cdots \times E^+ \subset E_n$. Let $\mathcal{E}_n$ denote the space of Schwartz sections of $\mathcal{R}_n^{\mathbb{C}}|_{E_n}$, and $\mathcal{E}_n^0 \subset \mathcal{E}_n$ be the subspace of sections with support in E_n^0.

Claim.

a) *Schwinger functions satisfy the following properties*

i) *$\mathfrak{S}_n$ is a real-analytic $\widehat{P}$-invariant and Σ_n (super) invariant real valued function on E_n^0, where $\widehat{P} = \widehat{G} \ltimes E$, $\widehat{G} = \mathrm{Spin}(E)$;*
If moreover the $\mathfrak{S}_n$ come from a strong (as opposed to weak) QFT then they satisfy

ii) *$\mathfrak{S}_n$ is of moderate growth; i.e. $\mathfrak{S}_n$ defines a tempered distribution on $\mathcal{E}_n^0$.*

iii) *Reflection positivity.*
Put $\mathcal{E}_n^+ = \{F \in \mathcal{E}_n^0 | \mathrm{supp}\, F \subset E_n^+\}$. Also let $\mathcal{E}^+ = \oplus \mathcal{E}_n^+$, $\widetilde{\mathcal{E}} = \oplus \mathcal{E}_n^0$, and $\widetilde{\mathfrak{S}} = \Pi \mathfrak{S}_n \in (\widetilde{\mathcal{E}})'$. Then for $F \in \mathcal{E}^+$ we have $\widetilde{\mathfrak{S}}(\theta(F) \boxtimes F) \geq 0$.

iv) *Let $\overline{\mathcal{E}}_n$ be the space of Schwartz sections of $\overline{\mathcal{R}_n^{\mathbb{C}}}|_{\overline{E}_n}$, and let $\overline{\mathcal{E}_n^+} \subset \overline{\mathcal{E}}_n$ be the subspace of sections with support in $E_{n-1}^+ \subset \overline{E}_n \cong E_{n-1}$. For $r \in \mathbb{R}^+$ and a differential operator D consider the seminorm on $\overline{\mathcal{E}_n^+}$ defined by*

$$|f|_{r,D} = \sup_{x \in (\mathbb{R}_{>0} \times \mathbb{R}^{d-1})^n} |(1 + |x|^2)^{r/2} D\mathcal{F}(f)(x)|$$

(here for $x \in V$, $x = (x_0, \vec{x})$ we put $|x|^2 = x_0^2 + |\vec{x}^2|$). Then $\overline{\mathfrak{S}}_n$ defines a distribution on $\overline{\mathcal{E}_n^+}$ which is continuous with respect to the seminorms $|\ |_{r,D}$.

b) *The construction of §2.1 provides a bijection between Wightman QFT's and collections of functions $\mathfrak{S}_n$ satisfying the conditions i)–iv) above.*

Proof. See [Osterwalder-Schrader(1975)].

It should be mentioned that this theorem is not really useful, because the property iv) of the list is impossible to check in practice.

A more useful list is obtained by replacing the property iv) by the following axiom

Axiom iv'). (Linear growth condition) There exist $\alpha, \beta, r \in \mathbb{R}_{>0}$ such that

$$|\mathfrak{S}_n(f)| < \alpha(n!)^\beta |f|_{nr}$$

for all $f \in \mathcal{E}_n^0$ where the seminorm $|f|_{nr}$ is given by $|f|_{nr} = \sup_{x \in E_n^0, D} |(1-x^2)^{r/2} Df(x)|$ and D runs over the set of differential monomials of degree less than nr.

This property is more tractable, but it is more restrictive than iv).

Some remarks are in order. For Schwinger functions constructed from a (strong) Wightman QFT the reflection positivity property is equivalent to the positivity property (5) of the Wightman axioms. However, to formulate it we do not have to assume anything about the behavior of $\mathfrak{S}_n$ on diagonals. (Note that if $F_1, F_2 \in \mathcal{E}^+$ then $\theta(F_1) \boxtimes F_2$ automatically vanishes on diagonals, so $\theta(F_1) \boxtimes F_2 \in \widetilde{\mathcal{E}}$). So it can be formulated for Schwinger functions of a weak QFT. It is not clear how to restate this property directly in the Minkowski picture (which is more intuitive from the physical point of view) for a weak Wightman QFT.

§2.3. Schwinger functions and measures on the map-spaces

In fact, the following picture should stand behind the formalism of Schwinger functions. Assume we have a (quasi) measure ν on $\mathcal{E}'$ (where $\mathcal{E}$ is the space of Schwartz sections of $\mathcal{R}^{\mathbb{C}}|_E$), such that if $F_1, \dots, F_n \in \mathcal{E}$ and $\operatorname{supp}(F_i) \cap \operatorname{supp}(F_j) = \emptyset$ for $i \neq j$ then the function $\alpha \to \alpha(F_1) \dots \alpha(F_n)$ is measurable. Then we can define a distribution in $(\mathcal{E}_n^0)'$ for any n by requiring that its value on $F_1 \boxtimes \dots \boxtimes F_n$ equals $\int_{\mathcal{E}'} \alpha(F_1) \dots \alpha(F_n)$. This distribution should coincide with $\mathfrak{S}_n$; thus $\mathfrak{S}_n$ describes the n-th moment of ν. If the measure ν exists, it is uniquely determined by its moments $\mathfrak{S}_n$. However it does not necessarily exist for given $\mathfrak{S}_n$ coming from a Wightman QFT.

So working with $\mathfrak{S}_n$ is a way to talk about the properties of a measure when the measure itself is not defined.

The formulation of QFT involving a measure on $\mathcal{E}'$ has another serious drawback. In general one is interested in field theories where the fields are not sections of a vector bundle, but rather maps from a manifold to another manifold (no linear structure on the target space). Then we cannot even say on what space the measure ν is defined. Indeed, in the linear case the measure lives on the space of generalized sections of a vector bundle. But there is no notion of a generalized map from a manifold to another manifold. The formalism of Schwinger functions also does not work in this setting: to define moments of a measure one needs a linear structure on the underlying space.

The difficulty can be overcome if $d = 1$. In this case for a linear QFT the measure is actually defined and is concentrated on the subspace of continuous sections. So one gets a reasonable theory by considering the space of continuous maps from $V = \mathbb{R}$ to M and introducing a measure on this space. This is illustrated in the next two examples.

Example 1. (Wightman QFT for $d = 1$) In this case QFT is equivalent to usual quantum mechanics. Precisely, let us put $\mathcal{H} = L^2(\mathbb{R})$, let A be a Hamiltonian. We assume that A is of the form $A = -\frac{d^2}{du^2} + r(u)$, where $r(u) > m^2u^2$ for some positive constant m^2 and large $|u|$. Then A has a smallest eigenvalue, and the corresponding eigenspace is one-dimensional. Assume that the smallest eigenvalue of A is 0. Let Ω be the corresponding eigenfunction. Also put $\mathcal{D} = \mathcal{S}(R)$; then $\Omega \in \mathcal{D} \subset \mathcal{H}$. Finally, let $\widehat{u} \in \mathrm{End}(\mathcal{D})$ be the operator of multiplication by the function $f(u) = u$. We define the action of $P = \mathbb{R}$ on $\mathcal{H}$ by $t \to e^{\imath At}$ and put

$$\phi(f) = \int f(t)e^{iAt}\widehat{u}e^{-\imath At}dt.$$

Let us describe the corresponding measure on $\mathcal{S}'(\mathbb{R})$. We start with Gaussian measure ν_0 with covariance $(-(\frac{d}{dt})^2+m^2)^{-1}$, where $m^2 > 0$ is such that $r(u)-m^2u^2 > 0$ for $|u| \gg 0$. (So for $f \in \mathcal{S}$ we have $\mathcal{F}(\nu_0)(f) = e^{((-(d/dt)^2+m^2)^{-1}f,f)}$.) We look for ν of the form $\nu = a\nu_0$ where a is a function. It is enough to define a on a set of measure 1. We define a by the formula

$$a(\alpha) = \exp(\int_{-\infty}^{\infty} (r(\alpha(t)) - m^2\alpha(t)^2)dt)$$

Using the Sobolev theorem one can show that ν_0 is essentially concentrated on the set of continuous functions α such that the above integral converges. Hence the measure $\nu = a\nu_0$ is correctly defined.

The fact that the measure ν and the field operators defined above give the same set of Schwinger functions follows from the so-called Feynman-Kac formula, see [**II-CFT**, §1.3].

Example 2. (Abstract QFT for $d = 1$) Let M be an arbitrary compact Riemannian manifold, $\mathcal{H} = L^2(M)$, Δ is the Laplacian, $A = -\Delta + r(m)$. In this case one still can describe a measure ν on the space of continuous maps $\mathrm{Map}(\mathbb{R}, M)$ (one can get rid of the compactness assumption by introducing requirements on the asymptotic behavior of r; Example 1 is a particular case of this situation).

The measure can be characterized as follows. Let $t_1, \dots, t_k$ be points in $\mathbb{R}$, and $U_1, \dots, U_k \subset M$ be open subsets. Put $X^{t_1,\dots,t_k}_{U_1,\dots,U_k} = \{f \in \mathrm{Hom}(\mathbb{R}, M) | f(t_i) \in U_i \text{ for } i = 1, \dots, k\}$. Let $K_t(m_1, m_2)$, $m_1, m_2 \in M$, $t \in \mathbb{R}_{>0}$ be the fundamental solution of the differential equation $\frac{df}{dt} = Af$. Then ν is the unique measure such that

$$\nu(X^{t_1,\dots,t_k}_{U_1,\dots,U_k}) = \int_{U_1\times\dots\times U_k} \prod_{i=1}^{k-1} K_{t_{i+1}-t_i}(m_i, m_{i+1})dm_1 \dots dm_k.$$

But there is no easy way to translate it to a Wightman-type picture.

We now apply the analytic continuation of Wightman functions obtained in §2.1 to get more information on the structure of Wightman axioms.

§2.4. PCT Theorem

Let a *nonlocal Wightman QFT* mean the same set of data as in Wightman QFT satisfying all the Wightman axioms except for (maybe) space-locality. Arguing

as in §1.3 we see that such a theory is encoded by a set of Wightman functions $\mathcal{W}_n$ satisfying properties 1)–3), 5) (or 1), 2'), 3), 5)) of §1.3, but not necessarily satisfying property 4).

Let us say that a point $(v_1, \dots, v_n) \in V_n$ is a *Jost point* if $\pi(v_1, \dots, v_n) \in J_{n-1}$ where $\pi : V_n \to V_{n-1}$ is the standard projection.

Consider the following *weak space-locality* condition

4') We have $\mathcal{W}_n(v_1, \dots, v_n) = \mathcal{W}_n(v_n, \dots, v_1)$ provided $(v_1, \dots, v_n)$ is a Jost point.

For simplicity we deal below only with the case of a scalar QFT in even dimensions.

PCT Theorem. *Assume that d is even. Then for a nonlocal scalar QFT weak space-locality is equivalent to the following PCT (parity-charge-translation symmetry) condition*

$$\mathcal{W}_n(v_1, \dots, v_n) = \mathcal{W}_n(-v_n, \dots, -v_1)$$

being true for all $v_1, \dots, v_n \in V$.

Proof. The proof of analytic continuation of W_n to the extended tube $\widetilde{\mathcal{T}}$ did not use space-locality, so it still works in the present setting. Thus we have generalized functions W_n on V_n such that $\mathcal{W}_n = \pi^*(W_{n-1})$, and $G_{\mathbb{C}}$-equivariant analytic functions on $\widetilde{\mathcal{T}}_n$ denoted again by W_n such that $W_n(v) = \lim_{\epsilon \to +0} W_n(v + i\epsilon w)$ for $v \in V_n$, $w \in V_n^+$.

From $G_{\mathbb{C}}$-invariance of $\mathcal{W}_n$ we see that $\mathcal{W}_n(-v) = \mathcal{W}_n(v)$ for v in the domain of analyticity, because $-Id \in \mathrm{SO}$ lies in the image of $G_{\mathbb{C}}$ (but not in the image of G). Since $\mathcal{W}_n$ is analytic at Jost points, we see that PCT implies weak space-locality.

Conversely, suppose that the weak space-locality holds. By property 3) of §1.3

$$\overline{\mathcal{W}_n(v_1, \dots, v_n)} = \mathcal{W}_n(v_n, \dots, v_1).$$

If $(v_1, \dots, v_n)$ is a Jost point then we have

$$\overline{\mathcal{W}_n(v_1, \dots, v_n)} = \mathcal{W}_n(v_n, \dots, v_1) = \mathcal{W}_n(v_1, \dots, v_n),$$

thus $\mathcal{W}_n$ is analytic and takes real values on the real domain of Jost points. It follows that

$$\overline{\mathcal{W}_n(v_1, \dots, v_n)} = \mathcal{W}_n(\overline{v_1}, \dots, \overline{v_n}).$$

We can now derive the PCT property. Fix $(w_1, \dots, w_n) \in V_n$ such that $\pi(w_1, \dots, w_n) \in V_n^+$. We have

$$\mathcal{W}_n(v_1, \dots, v_n) = \lim_{\epsilon \to +0} \mathcal{W}_n(v_1 + i\epsilon w_1, \dots, v_n + i\epsilon w_n) =$$

$$\lim_{\epsilon \to +0} \mathcal{W}_n(-v_1 - i\epsilon w_1, \dots, -v_n - i\epsilon w_n) = \overline{\lim_{\epsilon \to +0} \mathcal{W}_n(-v_1 + i\epsilon w_1, \dots, -v_n + i\epsilon w_n)} =$$

$$\overline{\mathcal{W}_n(-v_1, \dots, -v_n)} = \mathcal{W}_n(-v_n, \dots, -v_1)$$

We finish with a technical variation on the definition of Wightman functions which will be used later.

§2.5. Time-ordering

Let $\theta_0 \in \mathrm{SO}(1, d-1)$ be reflection in the hyperplane $\mathbb{R}^{d-1} \subset V$ (change of time direction). Then θ_0 can be lifted to an element $\theta \in G_{\mathbb{C}}$, but not to an element of G. Hence the analytic continuations of $\mathcal{W}_n$ we constructed in section 1 are θ-invariant, though Wightman functions themselves are not.

We will now present a variant of the definition of $\mathcal{W}_n$ which does not have this drawback.

Namely, the time-ordered Wightman function $\mathcal{W}_n^T$ is given by

$$\mathcal{W}_n^T(v) = \lim_{\varepsilon \to +0} \mathfrak{S}_n(v + i\varepsilon v).$$

For $\mathrm{pr}(v) \in J_{n-1}$ we have $\mathcal{W}_n^T(v) = \mathcal{W}_n(v)$, where pr is the projection $V_n \to \overline{V}_n$. The function $\mathcal{W}_n^T$ is θ and Σ_n-invariant (unlike W_n).

Like a usual Wightman function, $\mathcal{W}_n^T$ is lifted from a unique function W_n^T on $\overline{V}_n$.

Let us describe W_2^T for the free scalar QFT of mass m; we will denote it by $\Delta_F(\cdot, m^2)$, where F stands for Feynman. It can be expressed through functions $\Delta_+(\cdot, m^2) = \mathcal{F}(\delta_{\sigma_m^+})$, and $\Delta_-(\cdot, m^2) = \mathcal{F}(\delta_{\sigma_m^-})$.

Claim. *We have*

a) $\Delta_F(v, m^2) = \Delta_\pm(v, m^2)$ *for* $v \in \overline{V}_\pm$; $\Delta_F(v, m^2) = \Delta_+(v, m^2) = \Delta_-(v, m^2)$ *for* $v \in V_{space}$;

b) $(-\Box + m^2)(\Delta_F(\cdot, m^2)) = 2\pi i\, \delta_0$, *i.e.* W_2^T *is the fundamental solution of* $(-\Box + m^2)$;

c) $\mathcal{F}(\Delta_F(\cdot, m^2)) = \frac{1}{2\pi\imath(p^2+m^2+i0)} \overset{\text{def}}{=} \lim_{\varepsilon \to +0} \frac{1}{2\pi i(p^2+m^2+i\varepsilon)}$ *(this is a well-defined distribution).*

[Just to remind us how such things look, we recall that $\frac{1}{x+i0}$ and $\frac{1}{x-i0}$ are well-defined distributions on $\mathbb{R}$ and we have $\frac{1}{2\pi i}\left(\frac{1}{x-i0} - \frac{1}{x+i0}\right) = \delta_0$ and $\frac{1}{2}\left(\frac{1}{x+i0} + \frac{1}{x-i0}\right) = \left(\frac{1}{x}\right)_{\text{p.v.}}$, where $\left\langle \left(\frac{1}{x}\right)_{\text{p.v.}}, f \right\rangle \overset{\text{def}}{=} \lim_{\varepsilon \to +0} \int_{-\infty}^{-\varepsilon} \frac{1}{x} f(x)dx + \int_{\varepsilon}^{\infty} \frac{1}{x} f(x)dx$.]

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LECTURE 3
Free Field Theories

§3.1. Some examples of free classical field theories

As was explained in Bernstein's lectures, for any classical field theory the space Sol of stationary points of the Lagrangian carries a canonical symplectic form, or at least a closed 2-form (see [I-**Classical Fields**, §2.5]).

The theory is called free if Sol is isomorphic to a linear space with a constant 2-form and linear action of the Poincaré group. Let us consider some examples.

Example 1. A scalar bosonic field in dimension $(1, d-1)$.
A classical field is a real-valued function on $\mathbb{R}^{1,d-1}$. The Lagrangian is given by

$$\mathcal{L}(\varphi) = |d\varphi|^2 - m^2\varphi^2.$$

The space of stationary points is $\mathrm{Sol} = \{\varphi | \Box\varphi + m^2\varphi = 0\}$. It is linear, the symplectic form is constant. It is given by $[\varphi_1, \varphi_2] = \int\limits_\Gamma *(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$ for $\varphi_1, \varphi_2 \in \mathrm{Sol}$, where Γ is a space-like hypersurface in V. By a space-like hypersurface we mean a complete smooth oriented hypersurface such that the normal vector at each point lies in V_+ (for example the hyperplane $\{t = \text{const}\}$ is space-like). Since $*(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$ is closed and sufficiently rapidly decaying the integral does not depend on the choice of a hypersurface. (The condition of Schwartz decay in space directions is tacitly imposed on the classical solutions under consideration throughout the lecture).

Example 2. Spinors. Let V be 10-dimensional, or of any dimension $n = 8k + 2$. Then the spinor representation S decomposes as $S = S^+ \oplus S^-$, and S^+, S^- are real. There exists a unique symmetric pairing π: $S^+ \times S^+ \to V$ (see [I-**Spinors**, §6.1]). We also have a pairing $(\ ,\ )$: $S^+ \times S^- \to \mathbb{R}$ and the action map $s : S^\pm \otimes V \to S^\mp$. The space of classical fields is defined to be the space of maps, $\mathrm{Map}(V, \mathbb{R}^{0,1} \otimes S^+) = \Pi\overline{\mathrm{Sol}}$, where $\overline{\mathrm{Sol}} = \mathrm{Map}(V, S^+)$. We define the Lagrangian by $\mathcal{L}(\psi) = (\partial\psi, \psi)$. Here $d\psi \in \Omega^1(V, \mathbb{R}^{0,1} \otimes S^+) = \Gamma(V, \mathbb{R}^{0,1} \otimes S^+ \otimes V)$, and $\partial\psi \stackrel{\text{def}}{=} sd\psi \in \Gamma(V, \mathbb{R}^{0,1} \otimes S^-)$. We have $\mathrm{Sol} = \{\psi | \partial\psi = 0\}$, it is a purely odd vector space. Giving a symplectic form on Sol is equivalent to giving a symmetric form on $\overline{\mathrm{Sol}}$. The symmetric form is constant and is given by $(\psi_1, \psi_2) = \int\limits_\Gamma \pi(\psi_1, \psi_2)$ where Γ is a space-like hypersurface, and we use the isomorphism $V \simeq \Lambda^{d-1}(V^*)$ provided by the volume form on V to interpret $\pi(\psi_1, \psi_2)$ as a $(d-1)$-form on V.

Example 3. Free electrodynamics in any dimension.
Let M be the space of 1-forms on V modulo the exact forms. The Lagrangian $\mathcal{L}\colon M \to \Omega^d(V)$ is as follows

$$\mathcal{L}(\alpha) = \alpha \wedge (d * d)\alpha.$$

We have $\mathrm{Sol} = \{\alpha | d * d\alpha = 0\}/\{\text{exact 1-forms}\}$. We will write down the symplectic form using the Fourier transform. The Fourier transform of the exact sequence

$$0 \to \Omega^0(V)/\mathbb{R} \xrightarrow{d} \Omega^1(V) \xrightarrow{*d*d} \Omega^1(V)$$

looks as follows

$$0 \to \mathcal{S}(V) \xrightarrow{\nu} \mathcal{S}(V) \otimes V \xrightarrow{\eta} \mathcal{S}(V) \otimes V$$

where $\mathcal{S}(V)$ is the Schwartz space, and

$$\nu(f)(p) = f(p)p \text{ for } p \in V,\ f \in \mathcal{S}(V),$$
$$\eta(G)(p) = (G(p), p)p - (p, p)G(p).$$

Let $\mathcal{O}_0 = \{p | p^2 = 0,\ p \neq 0\}$. Let $\mathcal{N}$ denote a vector bundle over $\mathcal{O}_0$, such that $\mathcal{N}_p = p^\perp/\mathbb{R}p$.

Lemma. a) *For any $G \in \operatorname{Ker}\eta$ we have $G(p) \in p^\perp$ for all $p \in \mathcal{O}_0$.*

b) *The map $\operatorname{Ker}\eta \to \Gamma(\mathcal{O}_0, \mathcal{N})$ given by $G \to G(p)/\mathbb{R}p$ is an isomorphism between* Sol *and $\{\gamma \in \Gamma(\mathcal{O}_0, \mathcal{N}) | \gamma(-p) = \overline{\gamma(p)}\}$. The symplectic form [,] on* Sol *is constant and is given by $[\gamma_1, \gamma_2] = \int\limits_{\mathcal{O}_0^+} \operatorname{Im} \gamma_1 \overline{\gamma_2}$.*

Remark. It is a useful exercise to find an expression for the symplectic form [,] in terms of the Fourier transform in examples 1,2 also.

§3.2. Clifford module

Quantization of a free classical field theory is based on the following linear algebra construction.

Let H be a real superspace, equipped with an even super-symplectic form $[\ ,\ ] : H \times H \to \mathbb{R}$. Let Cl^H be the corresponding Clifford (=Heisenberg) algebra.

A complex polarization of H is an automorphism $I\colon H \to H$, preserving [,] and such that $I^2 = -\mathrm{id}$.

Define the maps $i_\pm\colon H \to H_{\mathbb{C}}$ by $i_\pm(v) = v \pm i^{-1}Iv$. Let $H_\pm = \operatorname{Im} i_\pm$. Then $H_\pm$ is a complex isotropic subspace. We also have an antilinear isomorphism $^-\colon H_+ \to H_-$. Define $(\ ,\ )\colon H_+ \times H_+ \to \mathbb{C}$ by $(h_1, h_2) = \frac{i}{2}[h_1, \bar{h}_2]$.

Claim. a) $(\ ,\ )$ *is a super-hermitian form on H_+.*

b) $(\ ,\ )$ *is positive iff $[h, h] > 0$ for $h \in H^{\mathrm{odd}} - \{0\}$ and $[Ih, h] > 0$ for $h \in H^{\mathrm{even}} - \{0\}$.*

Proposition. *Let $(H, [\ ,\ ], I)$ be as before such that $(\ ,\ )$ is positive. Then*

a) *there exists a unique up to unique isomorphism triple $(j_I, \mathcal{D}, \Omega)$, where j_I is a complex representation of Cl^H on $\mathcal{D}$ and $\Omega \in \mathcal{D} - \{0\}$ is such that*
(1) $j_I(\mathrm{Cl}^H)(\Omega) = \mathcal{D}$
(2) $j_I(H_-)(\Omega) = 0$.

b) *There exists a unique positive Hermitian form $(\ ,\ )_{\mathcal{D}}$ on $\mathcal{D}$ such that $(\Omega, \Omega) = 1$ and $j_I(h)$ is an anti-Hermitian operator on $\mathcal{D}$ for $h \in H$.*

Remark. We have an embedding $\mathrm{Sym}(H_+) \hookrightarrow \mathrm{Cl}^H$, which induces an isomorphism $j_{H_+}\colon \mathrm{Sym}(H_+) \xrightarrow{\sim} \mathcal{D}$, where $j_{H_+}(h) = j_I(h)(\Omega)$. The algebra Cl^H acts on $\mathcal{D} = \mathrm{Sym}(H_+)$ by differential operators.

Remark. If H is infinite-dimensional, then the representation depends essentially on the complex polarization I. For finite dimensional H the L^2-completions of representations constructed via different polarizations are isomorphic.

Remark. The above construction for the finite-dimensional case is discussed in [I-**Quantization**] §§3–7.

§3.3. Examples of free QFT's

Scalars. Let H be the space of real solutions of $\Box\varphi + m^2\varphi = 0$ and $[\ ,\ ]$ be the form on H given by $[\varphi_1, \varphi_2] = \int_\Gamma *(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$ (Example 1 above).

To $f \in \mathcal{S}(V)$ there corresponds a classical observable, i.e. a function on H, given by $h \to \langle h, f\rangle_{L^2(V)}$. So in quantum theory an operator $\phi(f)$ should be defined for $f \in \mathcal{S}(V)$.

Let us construct a positive complex polarization I on $(H, [\ ,\ ])$. We describe I in terms of the Fourier transform. The Fourier transform gives an isomorphism $H = \{f \in \mathcal{S}_{\mathbb{C}}(\mathcal{O}_m) | f(-p) = \overline{f(p)}\}$, where $\mathcal{O}_m = \{p \in V - \{0\} | p^2 = m^2\}$. We have $\mathcal{O}_m = \mathcal{O}_m^+ \sqcup \mathcal{O}_m^-$ (union of 2 connected components). Restriction to $\mathcal{O}_m^+$ defines an isomorphism $H \xrightarrow{\sim} \mathcal{S}_{\mathbb{C}}(\mathcal{O}_m^+)$. Since $\mathcal{S}_{\mathbb{C}}(\mathcal{O}_m^+)$ has a natural complex structure and $\mathcal{S}_{\mathbb{C}}(\mathcal{O}_m^+) \subset \mathcal{S}_{\mathbb{C}}(\mathcal{O}_m) = H_{\mathbb{C}}$ is an isotropic subspace, we get a positive complex polarization of H. Consider the "canonical" representation $j_I\colon \mathrm{Cl}^H \to \mathrm{End}(\mathcal{D})$ described above. Let $\mathcal{H} = \overline{\mathcal{D}}$ be the Hilbert space completion with respect to the invariant Hermitian form (see Proposition). Since the action of P on H commutes with I, we get an action of P on $\mathcal{H}$.

We define the field operators $\phi\colon \mathcal{S}(V) \to \mathrm{End}(\mathcal{D})$ as the composition

$$\mathcal{S}(V) \ni f \to \mathcal{F}(f) \to \mathcal{F}(f)|_{\mathcal{O}_m} \xrightarrow{j_I} \mathrm{End}(\mathcal{D})$$

where we consider the restriction $\mathcal{F}(f)|_{\mathcal{O}_m}$ as an element of H.

Let us finally analyze the support of the spectral measure of $\mathcal{H}$ considered as a V-module. We obviously have $\mathrm{Spec}\,\rho\big|_V = \mathcal{O}_m^+$, where ρ denotes the action of P on H_+. Recall that $j_{H_+}\colon \mathrm{Sym}\, H_+ \xrightarrow{\sim} \mathcal{D}$ is an isomorphism.

Hence $\mathrm{Spec}\, U\big|_V$ is the closure of the set $\{0\} \cup \mathcal{O}_m^+ \cup \mathcal{O}_m^+ + \mathcal{O}_m^+ \cup \ldots$, where we use the notation $A + B = \{a + b \,|\, a \in A, b \in B\}$, and U stands for the representation of P on $\mathcal{H}$. It is easy to see that $\mathrm{Spec}\, U\big|_V = \{0\} \sqcup \mathcal{O}_m^+ \sqcup \{p \in V_+ | p^2 \geq 2m\}$.

Spinors. We keep the notations of Example 2 above. Let $H = \Pi(\{\psi | \partial\psi = 0\})$. (Recall that Π denotes change of parity of a superspace).

A classical observable is associated with $f \in \mathcal{S}^{S^-}(V)$, and is given by $h \to \int_V (f(v), h(v))\, dv$.

In Fourier coordinates the classical field equation reads as follows

$$s(p, \mathcal{F}(\psi)(p)) = 0.$$

Since $\mathcal{D}^2 = \Box$ we see that, for any $\psi \in H$, $\mathrm{supp}(\mathcal{F}(\psi)) \subset \mathcal{O}_0$.

Let $\mathcal{A}$ be a subbundle of the trivial bundle over $\mathcal{O}_0$ with fiber S^+, such that $\mathcal{A}_p \subset S^+ = \mathrm{Ker}\, s(p)$.

Then $H \cong \{f \in \mathcal{S}_{\mathbb{C}}(\mathcal{O}_0, \mathcal{A}) | f(-p) = \overline{f(p)}\}$. A symplectic form on Sol is the same as a symmetric pairing on H; the canonical form on Sol corresponds to the symmetric form on H given by $(f_1, f_2) = \int\limits_{\mathcal{O}_0^+} \frac{\pi(f_1(p), f_2(-p))}{p} \mu_0$, where we use linear dependence of the vectors $\pi(f_1(p), f_2(-p))$, p, and denote by $\frac{\pi(f_1(p), f_2(-p))}{p}$ the coefficient of proportionality. (This formula is equivalent to the one given in Example 2).

Using the decomposition $\mathcal{O}_0 = \mathcal{O}_0^+ \sqcup \mathcal{O}_0^-$, we obtain a polarization and the canonical representation j_I: $\mathrm{Cl}^H \to \mathrm{End}(\mathcal{D})$. If U is, as usual, the action of P on $\mathcal{D}$, then $\mathrm{Spec}\, U\big|_V = \overline{V}_+$.

The field operators are also defined similarly to the above. Namely, for $f \in \mathcal{S}(V, S^-)$ the section $\varkappa(f)$ such that $\varkappa(f)(p) = s(p, \mathcal{F}(f)\big|_{\mathcal{O}_0}(p))$ lies in H because $s(p)^2 = 0$ for $p \in \mathcal{O}_0$. We define $\phi(f) = j_I(\varkappa(f)) \in \mathrm{End}\, \mathcal{D}$.

Photons (Free quantum electrodynamics = free abelian gauge theory). Here $\mathrm{Sol} \cong \{f \in \mathcal{S}(\mathcal{O}_0, \mathcal{N}) | f(-p) = \overline{f(p)}\}$. Polarization comes again from the decomposition $\mathcal{O}_0 = \mathcal{O}^+ \sqcup \mathcal{O}^-$. Thus we obtain the canonical representation j_I: $\mathrm{Cl}^H \to \mathrm{End}\, \mathcal{D}$.

Recall that in the classical situation (Example 3) the space Sol is a subspace of $\Omega^1(V)/\, d\Omega^0(V)$. So to any 2-form $\omega \in \Omega^2(V)$ there corresponds a classical gauge-invariant observable

$$\Omega^1/d\Omega^0 \ni \alpha \to \int\limits_V d\alpha \wedge *\omega$$

Consider the map $\varkappa$: $\Omega^2(V) = \mathcal{S}^{\Lambda^2 V} \to H = S_{\mathbb{C}}(\mathcal{O}_0, \mathcal{N})$ given by $\varkappa(\omega)(p) = i_p \mathcal{F}(\omega)(p) \in p^\perp$. Now we define the field map $\phi(\omega) = j_I(\varkappa(\omega))$.

§3.4. Free QFT of arbitrary spin

A QFT is called free of mass m if the field operator $\phi(f)$ depends only on $\mathcal{F}(f)|_{\mathcal{O}_m}$. (In other words, the classical field equation $\Box + m^2 = 0$ is satisfied on the quantum level).

All examples considered in the previous section are of this kind.

In this section we give a construction of a free QFT of mass m which generalizes these examples. To describe it we need some preliminaries.

Recall that $G = \mathrm{Spin}(d-1, 1)$, and $\tau \in G$ is the spin-element. We fix the mass of the theory $m \geq 0$. Let St denote the stabilizer of a point $p \in \mathcal{O}_m^+$.

If $m > 0$ then $St = \mathrm{Spin}(p^\perp) \cong \mathrm{Spin}(d-1)$. If $m = 0$ then St has nontrivial unipotent radical N and $St/N = \mathrm{Spin}(p^\perp/\mathbb{R}p) \cong \mathrm{Spin}(d-2)$. Let G' denote a maximal semisimple subgroup of St. (So $G' = St$ if $m > 0$, and $G' = St/N$ if $m = 0$).

We fix a real representation R of G, and ρ of G'. We will view R, ρ as representations of real algebraic groups defined over $\mathbb{R}$.

ρ can be also viewed as a representation of the group St. We fix a nonzero morphism $i : R|_{St} \to \rho$.

We do not have to assume that R, ρ are irreducible, but it is convenient to require that τ acts on R and ρ by a constant, i.e. R and ρ are sums of irreducible representations of equal spin. This constant will be denoted by $(-1)^{\mathrm{spin}(R)}$, $(-1)^{\mathrm{spin}(\rho)}$ respectively. Obviously $(-1)^{\mathrm{spin}(R)} = (-1)^{\mathrm{spin}(\rho)}$.

Let $\widetilde{St}$ denote the subgroup $\{g \in G | g(p) = \pm p\}$; let $\widetilde{G'}$ denote a maximal semisimple subgroup of $\widetilde{St}$. (So $\widetilde{G'} = \widetilde{St}$ if $m > 0$, and $\widetilde{G'} = \widetilde{St}/N$ if $m = 0$). Clearly G' is a subgroup of $\widetilde{G'}$ of index 2.

More precisely, if $m > 0$ then $\widetilde{G'} \cong \mathrm{Pin}(n)$. If $m = 0$ then $\widetilde{G'} \cong (\mathrm{Spin}(d-2) \times \mathbb{Z}_4)/(\tau, -1)$.

$\widetilde{G'}$ carries the structure of algebraic group over $\mathbb{R}$. Note that the nontrivial connected component of $\widetilde{G'}$ has *no real points.*

We assume that complexification of ρ extends to a representation of $\widetilde{G'}$ in such a way that the map i is $\widetilde{G'}$-equivariant. (The representation extends automatically if $m = 0$ or if d is even).

Let K be the normalizer in $\widetilde{G'}(\mathbb{C})$ of the subgroup of real points $\widetilde{G'}(\mathbb{R}) = G'(\mathbb{R})$. Then K is a compact Lie subgroup in the Lie group $\widetilde{G'}(\mathbb{C})$; moreover, K has 2 connected components and the connected component of 1 is $G'(\mathbb{R})$. We have $K = \{g \in \widetilde{G'}(\mathbb{C}) | \pi(g) \in \mathrm{SO}(d-1,1)(\mathbb{R})\}$ where $\pi : \mathrm{Spin} \to \mathrm{SO}$ is the projection.

Note that K is *not* the set of real points of $\widetilde{G'}$. More precisely, for $g \in K - G'(\mathbb{R})$ we have $\overline{g} = g \cdot \tau$. Since K is compact, the space of $\rho_{\mathbb{C}}$ carries a K-invariant Hermitian form $\langle\ ,\ \rangle$. It also yields a quadratic form $(\ ,\ )$, where $(x, y) = \langle x, \overline{y} \rangle$. The quadratic form restricted to the subspace of real vectors (the space of ρ) is real and positive definite.

Now let us return to QFT.

To ρ there corresponds a G-equivariant real vector bundle $\mathcal{A}_\rho$ on $\mathcal{O}_m^+$; its complexification will be denoted by $\mathcal{A}_\rho^{\mathbb{C}}$. Let H be the space of Schwartz sections of $\mathcal{A}_\rho^{\mathbb{C}}$ viewed as a real vector space.

As usual we get the structure of a superspace on H by declaring the parity of H to be $(-1)^{\mathrm{spin}(\rho)}$.

The bundle $\mathcal{A}_\rho^{\mathbb{C}}$ carries G-equivariant Euclidean and Hermitian metrics, obtained from metrics $(\ ,\ )$, $\langle\ ,\ \rangle$ on the space of $\rho_{\mathbb{C}}$ in the standard way. We will abuse notations by denoting the metrics on the vector bundle by $(\ ,\ )$, $\langle\ ,\ \rangle$ also. Of course $(\ ,\ )|_{\mathcal{A}_\rho}$ is real and positive.

A super-symplectic form on H is given by $[h_1, h_2] = \int\limits_{\mathcal{O}_m^+} \mathrm{Im} \langle h_1, \overline{h_2} \rangle$ if H is even, and $[h_1, h_2] = \int\limits_{\mathcal{O}_m^+} \mathrm{Re} \langle h_1, \overline{h_2} \rangle$ if H is odd. The obvious complex structure on H yields the positive polarization, to be denoted I. Thus we get the canonical representation $\mathcal{D} \ni \Omega$ and its Hilbert space completion $\mathcal{H}$.

To provide the field map $\phi : \mathcal{S}^R \to \mathrm{End}(\mathcal{D})$ we first construct a map $\varkappa : \mathcal{S}^R \to H$ and then take $\phi = j_I \circ \varkappa$.

To the morphism $i : R|_{St} \to \rho$ there corresponds a map $\iota : R \to \Gamma(\mathcal{A}_\rho)$, where $\Gamma(\mathcal{A}_\rho)$ is the space of sections.

Now we can define $\varkappa$. For $f \otimes r \in \mathcal{S}^R = \mathcal{S} \otimes R$ we put $\varkappa(f \otimes r) = \mathcal{F}(f)|_{\mathcal{O}_m^+} \cdot \iota(r)$ where $\mathcal{F}$ stands for Fourier transform as usual.

The data of QFT is constructed.

Remark. In fact the maps $\varkappa$, ϕ are defined on a larger function space than the space of Schwartz sections $\mathcal{S}^R$. Indeed, let F be a Schwartz function on $\mathbb{R}^{d-1}$. Let f be a generalized function on $V = \mathbb{R} \times \mathbb{R}^{d-1}$ of the form $f = \delta_{t=t_0} \times F$ where t is the coordinate on $\mathbb{R}$. Then it is easy to see that $\mathcal{F}(f)|_{\mathcal{O}_m^+}$ is a Schwartz function on

the hyperboloid $\mathcal{O}_m^+$. Thus $\varkappa$ and ϕ are defined on the generalized sections of $\mathcal{R}$ of the form $f \cdot r$ for f as above and $r \in \mathbb{R}$.

Let us now check the axiom ϵ) of Wightman QFT (the space-locality property).

We need the following

Key lemma. *For any r_1, $r_2 \in R$ the real-valued function on $\mathcal{O}_m^+$ defined by*

$$P_{r_1,r_2}(p) = (\iota(r_1), \iota(r_2))|_p$$

extends to a polynomial function on $\mathcal{O}_m$. This polynomial function is even if R and ρ have trivial spin, and odd otherwise.

Proof of the Lemma. We can view $\mathcal{A}_\rho$ as an algebraic vector bundle on the real algebraic variety $\mathcal{O}_m$, and $\iota(r)$ for $r \in R$ as an algebraic section of this vector bundle.

Since the metric on $\mathcal{A}_\rho$ is algebraic as well, the first part of the claim is clear.

For any $g \in G(\mathbb{C})$ such that $g(p) = -p$ we have by the definition

$$(\iota(r_1), \iota(r_2))_{-p} = (\iota(g(r_1)), \iota(g(r_2)))_p = (i(g(r_1)), i(g(r_2)))$$

In particular the last equality holds for any $g \in K - G'(\mathbb{R})$.

As was mentioned above for $g \in K - G'(\mathbb{R})$ we have $g^{-1}\overline{g} = \tau$. So for such g

$$(i(g(r_1)), i(g(r_2))) = \langle i(g(r_1)), \overline{i(g(r_2))}\rangle = \langle (i(r_1)), \overline{g}(i(r_2))\rangle = \langle i(r_1), g^{-1}\overline{g}(i(r_2))\rangle$$
$$= \langle i(r_1), \tau(i(r_2))\rangle = (-1)^{\mathrm{spin}(\rho)}\langle i(r_1), i(r_2)\rangle = (-1)^{\mathrm{spin}(\rho)}(\iota(r_1), \iota(r_2))_p$$

So for r_1, $r_2 \in R$ we proved that $(\iota(r_1), \iota(r_2))_{-p} = (-1)^{\mathrm{spin}(\rho)}\,(\iota(r_1), \iota(r_2))_p$. Since the map $(r_1, r_2) \to P_{r_1,r_2}$ is G-equivariant it follows that the same holds for any $p' \in \mathcal{O}_m^+$, and the Lemma is proven.

Proof of space-locality. It is enough to check that if $\varphi_1, \varphi_2 \in \mathcal{S}^R$ are space-like separated then the scalar product $[\varkappa(\varphi_1), \varkappa(\varphi_2)]$ vanishes.

Let us take F_1, $F_2 \in \mathcal{S}(\mathbb{R}^{d-1})$ such that $\mathrm{supp}(F_1) \cap \mathrm{supp}(F_2) = \emptyset$. Let $f_j = \delta_{t=0} \times F_j$ for $j = 1, 2$. For r_1, $r_2 \in R$ the expression $f_j \cdot r_j$ is a generalized section of $\mathcal{R}$. As was mentioned above $\varkappa$ is actually defined on such sections.

Since $\varkappa$ and $[\,,\,]$ are P-equivariant it sufficies to check that $[\varkappa(f_1 \cdot r_1), \varkappa(f_2 \cdot r_2)] = 0$ for f_j, r_j as above.

Put $g_j = \mathcal{F}(f_j)$. Then the functions g_j satisfy $g_j(p + (t, \vec{0})) = g_j(p)$, so we can write $g_j(t, q) = g_j(q)$ where $t \in \mathbb{R}$, $q \in \mathbb{R}^{d-1}$. We also have $g_j(-p) = \overline{g_j(p)}$ since f_j is a real function. So

$$[\varkappa(f_1 \cdot r_1), \varkappa(f_2 \cdot r_2)] = \mathrm{const} \cdot \int_{\mathcal{O}_m^+} \left(g_1\overline{g_2} - (-1)^{\mathrm{spin}(\rho)} g_2\overline{g_1}\right)(\iota(r_1), \iota(r_2))\, d\mu$$
$$= \mathrm{const} \cdot \int_{\mathbb{R}^{d-1}} \left(g_1(q)\overline{g_2(q)} - (-1)^{\mathrm{spin}(\rho)} g_2(q)\overline{g_1(q)}\right) P_{r_1,r_2}(|q|, q) \cdot |q|^{-1} dq$$

(notations of the Key Lemma). It can be rewritten as

$$\int_{\mathbb{R}^{d-1}} \left(g_1(q)\overline{g_2(q)} - (-1)^{\mathrm{spin}(\rho)} g_1(-q)\overline{g_2(-q)}\right) P_{r_1,r_2}(|q|, q) \cdot |q|^{-1} dq$$
$$= \int_{\mathbb{R}^{d-1}} (g_1(q)\overline{g_2(q)}) \left(P_{r_1,r_2}(|q|, q) - (-1)^{\mathrm{spin}(\rho)} P_{r_1,r_2}(|q|, -q)\right) \cdot |q|^{-1} dq$$

By the Key Lemma

$$P_{r_1,r_2}(|q|,q)-(-1)^{\mathrm{spin}(\rho)}P_{r_1,r_2}(|q|,-q)=P_{r_1,r_2}(|q|,q)-P_{r_1,r_2}(-|q|,q)$$

and $P_{r_1,r_2}(|q|,q)$ is a polynomial function.

But then it follows that the function $Q(q)=|q|^{-1}(P_{r_1,r_2}(|q|,q)-P_{r_1,r_2}(-|q|,q))$ is again polynomial.

Fourier transform of multiplication by a polynomial function is a differential operator, in particular it is local. Thus we have

$$[\varkappa(f_1\cdot r_1),\varkappa(f_2\cdot r_2)]=\int\limits_{\mathbb{R}^{d-1}}(g_1(q)\overline{g_2(q)})Q(q)dq=(Q(q)\cdot g_1,g_2)_{L^2(\mathbb{R}^{d-1})}$$
$$=(\widehat{Q}(F_1),F_2)_{L^2(\mathbb{R}^{d-1})}=0$$

where $\widehat{Q}$ is the differential operator with constant coefficients corresponding to Q and the last equality is obvious because F_1, F_2 have separated supports.

Space-locality is proved.

§3.5. Wightman functions of a free field theory; truncated Wightman functions

The next statement characterizes free theories in terms of Wightman functions.

Claim. *Let $\mathcal{W}_n$ be the Wightman functions of the free scalar* QFT *of mass m. Then we have $\mathcal{W}_2(v_1,v_2)=\mathcal{F}(\delta_{\mathcal{O}_m^+})\ (v_1-v_2)$; $\mathcal{W}_n=0$ if n is odd, and*

$$\mathcal{W}_{2k}(v_1,\dots,v_{2k})=\sum\mathcal{W}_2(v_{i_1},v_{j_1})\mathcal{W}_2(v_{i_2},v_{j_2})\dots\mathcal{W}_2(v_{i_k},v_{j_k}),$$

where the sum is taken over all partitions of $\{1,\dots,n\}$ in pairs $(i_1,j_1),\dots,(i_k,j_k)$, and $i_r<j_r$.

Proof is straightforward.

This statement can be easily generalized to other free QFT's.

It also can be reformulated using the so-called *truncated Wightman functions.*

Since these objects will be used in the next lecture, let us give their definition. We set

$$\mathcal{W}_n^{\mathrm{tr}}\overset{\mathrm{def}}{=}\sum_{\pi=(\pi_1,\dots,\pi_k)}(-1)^{k-1}(k-1)!\prod_{\ell=1}^{k}\mathcal{W}_{|\pi_\ell|}(v_{i_1},v_{i_2},\dots,v_{i_{|\pi_\ell|}})$$

Here the sum is over all decompositions of $\{1,\dots,n\}$ into the union of disjoint nonempty subsets $\pi_1,\dots,\pi_k$, (the order of $\pi_1,\dots,\pi_k$ does not matter), and $i_1<i_2\dots<i_{|\pi_\ell|}$ are the elements of π_ℓ.

An equivalent equality is

$$\mathcal{W}_n=\sum_{\pi=(\pi_1,\dots,\pi_k)}\prod_{\ell=1}^{k}\mathcal{W}^{\mathrm{tr}}_{|\pi_\ell|}(v_{i_1},\dots,v_{i_{|\pi_\ell|}})$$

Corollary. *For a free QFT we have $W_n^{\mathrm{tr}}=0$ for $n>2$.*

§3.6. Gaussian measures

One can also easily describe free field theory in Euclidean formulation.

For a free QFT one can make sense of the measure ν on the space of tempered distributions discussed in §2.3; the corresponding measure is Gaussian. Here is the formal definition for the case of the scalar theory.

Claim. *Let $\mathfrak{S}_n(e_1,\dots,e_n)$ be the Schwinger functions of the free Euclidean scalar* QFT *of mass m. Then $\mathfrak{S}_n(e_1,\dots,e_n)$ corresponds to the n-th moment of a Gaussian measure ν_m on the space of tempered distributions $\mathcal{S}'(E)$ with covariance $(-\Box_E+m^2)^{-1}$ (cf. the first paragraph of §2.3). In other words, ν_m is determined by requiring that for $f\in\mathcal{S}$ we have $\mathcal{F}(\nu_m)(f)=e^{((-\Box_E+m^2)^{-1}f,f)}$; and for all $F_1,\dots,F_n\in\mathcal{S}$ we have*

$$\langle\mathfrak{S}_n, F_1\boxtimes\dots\boxtimes F_n\rangle=\int_{\mathcal{S}'}\alpha(F_1)\dots\alpha(F_n)d\nu_m(\alpha).$$

Remark. Gaussian measure on a finite dimensional real vector space H with covariance $\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ is a nondegenerate quadratic form on H, is characterized by the equalities

$$\nu(1)=1;\quad \nu\left(\frac{\partial P}{\partial h}\right)=-h^\vee\nu(P)$$

for any $h\in H$, and $P\in\mathrm{Sym}(H')$ a polynomial function on H. (Here $\frac{\partial}{\partial h}$ is a constant vector field, and $h^\vee=\langle h,\ \rangle$ is a linear functional on H.)

In our setting of kernel spaces ($H=\mathcal{S}'$, $H'=\mathcal{S}$ and $\langle f,f\rangle=\big((-\Box+m^2)f,f\big)$) this is still so, if one requires that the equality holds for all h for which it makes sense, namely for all $h\in\mathcal{S}\subset\mathcal{S}'$.

Reconstruction of the Hilbert space of a free QFT in terms of the Gaussian measure is based on the next Lemma.

Let I be a positive polarization of H, $[\ ,\ ]$ and let $\langle h_1,h_2\rangle=[Ih_1,h_2]$. Then we have an imbedding $\mathcal{D}=\mathrm{Sym}(H_+)\hookrightarrow A_H$ defined by the decomposition $H_{\mathbb{C}}=H_+\oplus H_-$.

Lemma. *For $F_1,F_2\in\mathcal{D}$ we have $(F_1,F_2)_{\mathcal{D}}=\nu(F_1\overline{F}_2)$*

We finish the lecture with a discussion of

§3.7. Normal ordering

In general in QFT one is interested in the behavior of correlation functions near diagonals. Naively speaking, we would like to consider along with an operator-valued generalized function $\phi(v)$ also an operator-valued generalized function $\phi^n(v)$. As usual such operator-valued functions should be encoded in correlation functions on V_k whose "value at $(v_1,\dots,v_k)$" equals $(\Omega,\phi^{n_1}(v_1)\cdots\phi^{n_k}(v_k)\Omega)$. If the naive definition would work, these correlators would obviously be the restrictions of the original Wightman function to the corresponding diagonal. Of course this is not possible, since Wightman functions are singular along diagonals. Techniques to overcome this difficulty are discussed in Lecture 3 of [I-**Perturbative QFT**]; here we only present a simple device, which leads to an answer in the free theory case (see also [I-**Perturbative QFT**, §3.2], [I-**Perturbative QFT**, §3.5]).

Let $(\mathcal{H},\phi,\mathcal{D},\Omega)$ be the free scalar QFT. (The assumption that the theory is scalar is not essential, and made for notational convenience).

For any function $f \in \mathcal{S}$ let us write $f = f_+ + f_-$, where $f_\pm \in \mathcal{S}_{\mathbb{C}}$ and $\mathcal{F}(f_\pm)|_{\mathcal{O}_m^\mp} = 0$.

Then define normally ordered powers of the field operator by

$$:\phi(f)^n: \stackrel{\text{def}}{=} \sum \binom{n}{i} \phi(f_+)^i \phi(f_-)^{n-i}.$$

Claim.

a) *For any $v \in V$, $x, y \in \mathcal{D}$ and $f \in \mathcal{S}$ the limit $\lim\limits_{f \to \delta_v} (:\phi(f)^n:(x), y)$ exists; we will write $(:\phi^n(v):(x), y) \stackrel{\text{def}}{=} \lim\limits_{f \to \delta_v} (:\phi(f)^n: x, y)$. Thus the correlator $(\phi(v_1), .., :\phi^n(v_i):, .., \phi(v_n))$ is a well-defined distribution.*

b) *If $d = 2$, then for $f \in \mathcal{S}$ there exists a well-defined endomorphism $:\phi^n(f):\in \mathrm{End}(\mathcal{H})$ such that $(:\phi^n(f):(x), y) = \int f(v)\,(:\phi^n(v):(x), y)\,dv$.*

In fact Claim b) is valid for a much wider class of 2-dimensional theories than free ones; see §8.6 of [Glimm-Jaffe].

One can define normal ordering more algebraically as follows.

Recall that we have $H = \{\varphi | \Box\varphi + m^2\varphi = 0\}$, $[\varphi_1, \varphi_2] = \int *(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$. Consider a Lagrangian subspace $L \subset H$, $L = \left\{\varphi: \frac{\partial}{\partial t}\varphi\big|_{t=0} = 0\right\}$. Since L is isotropic we have an embedding $\mathrm{Sym}(L) \hookrightarrow \mathrm{Cl}^H$. Then the map $F \to j_I(F)\Omega$ induces an isomorphism $j_L: \mathrm{Sym}(L) \xrightarrow{\sim} \mathcal{D}$ (because $H_{\mathbb{C}} = L \oplus H_-$). Also, the projection along H_- induces an isomorphism $u: H_+ \xrightarrow{\sim} L$.

Now consider the endomorphism of $\mathrm{Sym}\, L$ given by the composition of the arrows

$$\begin{array}{ccc} \mathrm{Sym}\, L & \xrightarrow{u^*} & \mathrm{Sym}\, H_+ \\ & {\scriptstyle j_L^{-1}}\nwarrow \quad \swarrow{\scriptstyle j_{H_+}} & \\ & \mathcal{D} & \end{array}$$

We denote this endomorphism by $F \to :F:$.

Lemma.

a) *For $\ell \in L$ we have $:\phi^n(\ell) := j_I(:\ell^n:)$. (The LHS is defined by the Remark in §3.4).*

b) *$F - :F: \in \bigoplus\limits_{k=0}^{n-1} \mathrm{Sym}^k(L)$ for $F \in \mathrm{Sym}^n L$.*

c) *$(j_L(:F:), j_L(G))_{\mathcal{D}} = 0$ for $F \in \mathrm{Sym}^n L$, $G \in \mathrm{Sym}^k(L)$, $k < n$.*

Lemma. *For any $F_1, F_2 \in \mathrm{Sym}(L)$ we have*

$$(j_L(F_1), j_L(F_2))_{\mathcal{D}} = \nu_{\langle\ ,\ \rangle\big|_L}(F_1 \cdot \overline{F}_2)$$

Let $\varkappa \in \mathrm{End}(\mathrm{Sym}\, L)$ be the second order differential operator corresponding to the form $\langle\ ,\ \rangle$. (So we have $\varkappa(a^k) = k(k-1)\,\langle a, a\rangle\, a^{k-2}$ for $a \in L$.)

Lemma. $:F: = \exp\left(-\frac{\varkappa}{2}\right) F$.

Corollary. $:\exp(\ell^\vee) := \exp\left(-\frac{(\ell,\ell)}{2}\right) \exp(\ell^\vee)$, *for $\ell \in L$.*

References and cross references

B. Simon, *The $P(\phi)_2$ Euclidean (quantum) field theory;* §*II.5*, Princeton University Press, 1974.

For a discussion of the Gaussian measure approach to free QFT see Chapter 3 of *loc. cit.* or

J. Glimm, A. Jaffe, *Quantum Physics. (A functional integral point of view),* §*6.2*, Springer-Verlag, 1987.

LECTURE 4
Scattering Theory

§4.1. Introduction

This important branch of functional analysis is not as popular as it should be; one of the reasons is that the phenomena studied in scattering theory have no finite-dimensional analog.

The situation is roughly as follows.

Let A_0 be a self-adjoint operator in a Hilbert space H (the function space). Let λ be a point in the *continuous* spectrum of A_0 and let f_λ^0 be the corresponding "eigenfunction" (quotation marks are put to remind us that f_λ^0 lies not in the Hilbert space H, but rather in an appropriate larger function space, i.e. f_λ is a linear functional on a dense subspace in H.)

We consider a "small" perturbation $A = A_0 + B$ of our self-adjoint operator A_0, where B is a self-adjoint operator such that $(\sqrt{1+A_0^2})^{-1}B$ is compact, and we look for a λ-eigenfunction f_λ of the operator A.

Remark. This problem makes sense only if λ lies in the continuous spectrum of A_0; in particular it never makes sense in finite dimensions. Indeed, the spectrum of an operator moves as we perturb the operator; so an isolated point of the spectrum in general does not lie in the spectrum of the perturbed operator.

We look for an eigenfunction of the form $f_\lambda = f_\lambda^0 + g$. We have

$$Af_\lambda = \lambda f_\lambda \Leftrightarrow (A-\lambda)g = Bf_\lambda^0, \quad \text{so formally}$$
$$g = (A-\lambda)^{-1}B(f_\lambda^0)\ .$$

Of course the last equality makes no sense since $(A-\lambda)$ is not invertible. Nevertheless in some cases one can show existence of the limits $g_\pm = \lim\limits_{\varepsilon\to\pm 0}(A-\lambda\pm i\varepsilon)^{-1}B(f_\lambda^0)$. Then we put $f_\pm^\lambda = f_\lambda^0 + g_\pm$.

Let us consider a concrete example of the above situation.

In this example one can actually show existence of the limits

$$\lim_{\varepsilon\to\pm 0}(A-\lambda\pm i\varepsilon)^{-1}B(f_\lambda^0).$$

We will present another way to obtain the two eigenfunctions $f_\pm^\lambda$. We leave it as an exercise to the reader to check that the two procedures lead to the same answer.

We take $H = L^2(E)$, where E is a d-dimensional Euclidean space. Let $A_0 = -\Delta + m^2$, and let B be multiplication by a function $b \in C_c^\infty(E)$.

Claim. *For $f \in C_c^\infty(E)$ we have*

$$\|Be^{\imath tA_0} f\| \sim |t|^{-d/2} \qquad \text{for} \qquad t \to \pm\infty \ .$$

Corollary. *For any $f \in C_c^\infty(E)$ the L^2-limits $\lim_{t\to\pm\infty} e^{itA} e^{itA_0} f$ exist provided $d > 2$.*

Proof of Corollary. Denote $f^t = e^{itA} e^{-itA_0} f$. Then we have

$$\begin{aligned} \left\| \tfrac{\partial f^t}{\partial t}|_{t_0} \right\| &= \|iAe^{it_0 A} e^{-\imath t_0 A_0} f - ie^{it_0 A} A_0 e^{-\imath t_0 A_0} f\| \\ &= \|e^{\imath t_0 A} Be^{-it_0 A_0} f\| = \|Be^{-t_0 A_0} f\| \ . \end{aligned}$$

So by the previous claim $f_+ = \int_0^\infty \left(\frac{\partial f^t}{\partial t}\right) dt$ converges if $d > 2$. The proof for f_- is analogous.

Obviously $\|f_\pm\|_{L^2} = \|f\|_{L^2}$, hence there exist well-defined operators $W_\pm \colon H \to H$, such that $W_\pm(f) = f_\pm$. Moreover, we see that

$$\|(AW_\pm - W_\pm A_0)(f)\| = \lim_{t\to\pm\infty} \|e^{itA} Be^{-itA_0}(f)\| = 0 \ ,$$

hence $AW_\pm = W_\pm A_0$.

If the image of W_+ is equal to the image of W_-, we define the *S-matrix* by $S = W_+^* W^-$. Thus $S \in \mathrm{End}(H)$, and S commutes with A_0.

§4.2. System of n particles (potential scattering)

This is a generalization of the last problem. We take $H = L^2(E^n)$, $A = -\sum_{i=1}^n \Delta_i + \sum B_{ij}$, where B_{ij} is multiplication by a function $b(v_i - v_j)$, for $b \in C_c^\infty(E)$, and Δ_i is the Laplace operator along the ith copy of E.

Let π be a partition of $\{1, \dots, n\}$ in a union of k disjoint sets $\pi_1, \dots, \pi_k$ ("clusters").

Consider the decomposition $E^{\pi_\ell} \simeq E \times \bar{E}_{\pi_\ell}$, where $\bar{E}_{\pi_\ell} = \{(v_{i_1}, \dots, v_{i_{|\pi_\ell|}}) \in E^{\pi_\ell} | \sum v_i = 0\}$, and the isomorphism is given by $(v_{i_1}, \dots, v_{i_{|\pi_\ell|}}) \to (w_\ell = \frac{1}{|\pi_\ell|} \sum v_i;$ $(v_{i_1} - w_\ell, v_{i_2} - w_\ell, \dots v_{i_{|\pi_\ell|}} - w_\ell))$.

We have obvious decompositions $E = \prod_{\ell=1}^k E^{\pi_\ell}$, and the corresponding decomposition of the function space

$$H = L^2(E^n) = \bigotimes_{\ell=1}^k L^2(E^{\pi_\ell}) = \bigotimes_{\ell=1}^k \left(L^2(E) \otimes L^2(\bar{E}_{\pi_\ell})\right)$$

The operator A_0 obviously decomposes as $A_0 = \sum_{\ell=1}^k A_\ell$, where A_ℓ acts along E^{π_ℓ}; and furthermore $A_\ell = A'_\ell + A''_\ell$, where A'_ℓ is minus the Laplace operator acting along E, and A''_ℓ acts along $\bar{E}_{\pi_\ell}$. Let $H_\ell = \oplus H_\ell^\lambda \subset L^2(\bar{E}_{\pi_\ell})$ be the sum of eigenspaces of A''_ℓ, where λ runs over the discrete spectrum of A''_ℓ. Put $H_0(\pi) = \bigotimes_{\ell=1}^k (L^2(E) \otimes H_\ell) \subset L^2(E^n) = H$. Let $J_\pi \colon H_0(\pi) \hookrightarrow H$ denote the natural isometric inclusion; thus $J_\pi(H_0(\pi))$ is an A_0-invariant subspace in H. We denote $J = \oplus J_\pi \colon \bigoplus_\pi H_0(\pi) \to H$.

Theorem 1. a) *There exist strong limits* $W_\pm = \lim_{t\to\pm\infty} e^{\imath tA} J e^{-\imath tA_0}$.

b) $W_\pm$ *is an isometric imbedding and we have* $A \circ W_\pm = W_\pm \circ A_0$.

c) $W_\pm$ *is an isomorphism.*

We omit the proof. Just remark that the proof of a) and b) is routine, while c) is a difficult theorem, the full proof of which was obtained only recently.

§4.3. Haag-Ruelle theory

We now apply the method of scattering theory to QFT.

Claims and methods of scattering theory discussed below apply only to the Minkowski, not to the Euclidean picture.

We will describe the following construction.

Let $(U, \mathcal{H} \supset \mathcal{D} \ni \Omega_i)$ be a scalar QFT. Suppose that

a) $\mathcal{H}^V = \mathbb{C}\Omega$ (uniqueness of vacuum).

b) There exists a P-invariant Hilbert subspace $H \subset \mathcal{H}$ with a P-equivariant isomorphism $H \simeq L^2(\mathcal{O}_m^+)$, $m > 0$.

c) The mass spectrum of $\mathcal{H}/H \oplus \mathbb{C}\Omega$ does not contain 0 and m.

d) The span of $\langle \phi(f)\Omega \rangle$, $f \in \mathcal{S}$ is not contained in $H^\perp$.

Let $(U_0, \mathcal{H}_0 \supset \mathcal{D}_0, \phi_0)$ be the free field theory of mass m. Then one can construct two P-equivariant isometric imbeddings $W_\pm \colon \mathcal{H}^0 \hookrightarrow \mathcal{H}$.

Remark. We cannot hope that $W_\pm$ is an isomorphism in general, because $\Box\big|_{\mathcal{H}}$ can have discrete spectrum outside of $\{0, m^2\}$ while for the free theory it does not. An ideal theorem would assert that a free theory constructed on the discrete part of the spectral decomposition of $\Box$ on $\mathcal{H}$ maps isomorphically to $\mathcal{H}$ (compare with potential scattering described in the previous section). Unfortunately there is no known way to formulate a precise theorem of this sort.

Remark. The construction can be carried out under weaker assumptions. Namely one can omit condition d), and replace condition c) by the requirement that the theory has positive mass, and m^2 is an isolated point of the discrete spectrum of $\Box$.

However, the assumption that 0 is an isolated point of $\mathrm{Spec}(U\big|_V)$ is essential. Application of scattering theory to a massless QFT confronts much more serious difficulties than in the massive case.

Remark. The construction does not work for small d. Consider the case $d = 1$. Then QFT reduces to quantum mechanics. The spectrum of the Hamiltonian of the free theory, $\mathrm{Spec}\, H_0 = m\mathbb{Z}_+$, is totally discrete, thus scattering theory is not applicable. One can prove a slightly weaker statement in the case $d = 2$. Our argument will work for $d \geq 4$.

We now turn to construction of the imbeddings $W_\pm$. We try to imitate the construction of the previous section (potential scattering).

Let $\Lambda \subset V^+$ be a G-invariant neighborhood of $\mathcal{O}_m^+$, such that $\Lambda \cap \mathrm{Spec}\, U\big|_{H^\perp} = \emptyset$. For any function f on V such that $\mathrm{supp}\, \mathcal{F}(f) \subset \Lambda$ and $t \in \mathbb{R}$ let $f^t \in \mathcal{S}(V)$ be the function such that

$$\mathcal{F}(f^t)(p_0, \vec{p}) = e^{it\left(p_0 - \sqrt{m^2 - \vec{p}^2}\right)} \mathcal{F}(f)(p_0, \vec{p})\,.$$

Note that $f^t = f$ provided $\mathrm{supp}(\mathcal{F}(f)) \subset \mathcal{O}_m^+$.

In the rest of the lecture we prove the following

Theorem 2. a) *If $d \geq 4$ then for any $f_1, \dots, f_n \in \mathcal{S}(V)$ such that $\operatorname{supp}(\mathcal{F}(f_i)) \subset \Lambda$ the limits $\lim_{t \to \pm\infty} \phi(f_1^t) \dots \phi(f_n^t)\Omega$ exist.*

b) *There exists a unique isometric imbedding $W_\pm : \mathcal{H}_0 \hookrightarrow \mathcal{H}$, such that for any $f_1, \dots, f_n \in \mathcal{S}(V)$ with $\operatorname{supp}(\mathcal{F}(f)) \subset \lambda$ we have*

$$W_\pm(\phi_0(f_1) \dots \phi_0(f_n)\Omega) = \lim_{t \to \pm\infty} \phi(f_1^t) \dots \phi(f_n^t)\Omega$$

The imbeddings $W_\pm$ are P-equivariant.

Remark. For a free field theory the vector $\phi_0(f_1^t) \dots \phi_0(f_n^t)\Omega$ does not depend on t. Indeed, the field operator $\phi_0(f)$ for a free field theory depends on $\phi(f)\big|_{\mathcal{O}_m^+}$ only, and $\mathcal{F}(f^t)\big|_{\mathcal{O}_m^+}$ does not depend on t.

Remark. Let L be the space of functions satisfying the conditions of Lemma 1. Fourier transform identifies L with the space of Schwartz functions on $\mathcal{O}_m^+$. Let χ be a Schwartz function of one variable such that $\mathcal{F}(\chi)$ is supported in a small neighborhood of 0. For $f \in L$ let us put $f_\chi(x) = \chi(x_0) \cdot f(x)$ where $x = (x_0, \vec{x})$. Then $f_\chi \in \mathcal{S}(V)$, and the Fourier transform of f_χ is concentrated in a neighborhood of $\mathcal{O}_m^+$, being the convolution of the function $\mathcal{F}(f)$ concentrated on $\mathcal{O}_m^+$ and a function concentrated in a small neighborhood of 0 on the time-axis.

In this way we can fix an initial imbedding j of (a dense subspace of) $\mathcal{H}_0$ to $\mathcal{H}$ such that $j(\phi_0(f_1) \cdots \phi_0(f_k)\Omega) = \phi(f_{1\chi}) \cdots \phi(f_{k\chi})\Omega$ for any functions $f_1, .., f_k \in L$. Then one checks directly that

$$\phi(f_{1\chi}^t) \cdots \phi(f_{k\chi}^t)\Omega = U((t, \vec{0}))j(U_0((-t, \vec{0}))\phi_0(f_1) \cdots \phi_0(f_k)\Omega).$$

Thus the situation is quite analogous to the potential scattering considered in the previous section.

Proof of the Theorem. We start with two auxiliary estimates needed in the proof. The first is a standard fact from analysis.

Lemma 1 (locality for free fields). *Let $f \in C^\infty(V)$ be a solution of the Klein-Gordon equation, i.e. $\operatorname{supp} \mathcal{F}(f) \subset \mathcal{O}_m^+$. Assume that $f(0, \vec{x})$ and $\frac{\partial f}{\partial x_0}(x_0, \vec{x})\big|_{x_0=0}$ are Schwartz functions on $\mathbb{R}^{d-1}$. Then for some constant C we have*

a) $\sup_{\vec{x} \in \mathbb{R}^{d-1}} |f(x_0, \vec{x})| \leq C(1 + |x_0|)^{-(d-1)/2}$

b) $\int_{\mathbb{R}^{d-1}} |f(x_0, \vec{x})| d\vec{x} \leq C(1 + |x_0|)^{(d-1)/2}$

We omit the proof; it is an exercise in the stationary phase method (see §1.6, Claim 1b) for a similar argument).

The second ingredient is the following estimate on truncated Wightman functions (see §3.5 for the definition).

Proposition 1. *There exist a differential operator with constant coefficients D, a C^0 function of tempered growth F, and a polynomial function P on V_n such that $\mathcal{W}_n^{\mathrm{tr}} = D(F)$ and the following estimate holds.*

Let $[1, \dots, n] = I \cup J$ be a partition of $[1, \dots, n]$ into union of 2 disjoint subsets; let $a_1, \dots, a_n \in \mathbb{R}^{d-1} \subset V$ be such that $a_i \neq a_j$ for $i \in I$, $j \in J$. Then for all $t_1, ..., t_n \in \mathbb{R} \subset V$ we have

$$|F(t_1 + a_1, \dots, t_n + a_n)| < |P(t_1 + a_1, \dots, t_n + a_n)| e^{-m \min_{i \in I, j \in J} (\sqrt{-(a_i - a_j)^2} - |t_i - t_j|)}.$$

Remark. The particular case $n = 2$ of the Proposition is implied by §1.6, Claim 1b). Also §1.6, Claim 1a) guarantees that at least $\mathcal{W}_n^{\mathrm{tr}}(v_1, .., v_i, v_{i+1} + \lambda a, .., v_n + \lambda a) \to 0$ for any $v_1, \dots, v_n$ provided $a^2 < 0$, which obviously follows from the Proposition.

The statement we will actually need is the following immediate

Corollary. *Let i_t denote the embedding of $(\mathbb{R}^{d-1})^n$ in V_n, $\prod \vec{v}_i \to \prod(t, \vec{v}_i)$, and let $\pi : (\mathbb{R}^{d-1})^n \to (\mathbb{R}^{d-1})^{n-1} = (\mathbb{R}^{d-1})^n/\Delta$ be the projection to the quotient modulo diagonal. For a Schwartz function h on V_n the restriction $i_t^*(h * \mathcal{W}_n^{\mathrm{tr}})$ is a smooth function invariant under diagonal translations, hence $i_t^*(h*\mathcal{W}_n^{\mathrm{tr}}) = \pi^*(F_t^h)$ for some function F_h^t on $(\mathbb{R}^{d-1})^{n-1}$. Then $F_h^t \in L^1\left((\mathbb{R}^{d-1})^{n-1}\right)$ for any t, and $\left|F_t^h\right|_{L^1}$ is bounded by a constant independent of t.*

Proof of the Proposition. We start with an auxiliary result. Recall from the first lecture that the Fourier transform of Wightman functions has the property $\mathcal{F}(\mathcal{W}_n(p_1, \dots, p_n)) = 0$ if $p_1 + \dots + p_k \notin \overline{V}_+$ for some k. One can easily show the following.

Lemma 2. *Let $V_+(m)$ be $\{p \in V_+ | p^2 > m\}$. Then we have $\mathcal{F}(\mathcal{W}_n^{\mathrm{tr}}(p_1, \dots, p_n)) = 0$ provided $p_1 + \dots + p_k \notin \overline{V_+(m)}$ for some k, where m is the mass of the theory. Also $\mathcal{F}(\mathcal{W}_n^{\mathrm{tr}}(p_1, \dots, p_n))$ is concentrated on the hyperplane $\sum_{i=1}^{n} p_i = 0$.*

We are ready to prove the Proposition.

Let us denote $\mathcal{W}'(v_1, \dots, v_n) = \mathcal{W}^{\mathrm{tr}}(v_{i_1}, \dots, v_{i_k}; v_{j_1}, v_{j_2}, \dots, v_{j_{n-k}})$ and $\mathcal{W}''(v_1, \dots, v_n) = \mathcal{W}^{\mathrm{tr}}(v_{j_1}, \dots, v_{j_{n-k}}; v_{i_1}, \dots, v_{i_k})$ where $i_1 < i_2 < \dots < i_k$ are the elements of I, and $j_1 < \dots < j_{n-k}$ are the elements of J. From the Lemma it follows that

$$\mathcal{F}(\mathcal{W}')(p_1, \dots, p_n) = 0 \text{ if } p_{i_1} + \dots + p_{i_k} \notin \overline{V_+(m)};$$
$$\mathcal{F}(\mathcal{W}'')(p_1, \dots, p_n) = 0 \text{ if } -(p_{i_1} + \dots + p_{i_k}) \notin \overline{V_+(m)}$$

Let $\widetilde{\chi}$ be a smooth function of one variable, such that $\widetilde{\chi}(x) = 1$ for $x > m$, $\widetilde{\chi}(x) = -1$ for $x < -m$, and its Fourier transform has asymptotics $\widetilde{\varphi} = \mathcal{F}(\widetilde{\chi}(t)) \sim e^{-m|t|}$ as $t \to \pm\infty$. [It is a standard fact from calculus that such a function exists].

Let us put $\chi(p_1, \dots, p_n) = \widetilde{\chi}(p_{i_1}^{(0)} + \dots + p_{i_k}^{(0)})$, where $p_i = (p_i^{(0)}, \vec{p}_i)$ for $p_i^{(0)} \in \mathbb{R}$, $\vec{p}_i \in \mathbb{R}^{d-1}$ is the usual decomposition. Then we see that $\chi\mathcal{F}(\mathcal{W}') = \mathcal{F}(\mathcal{W}')$, $\chi\mathcal{F}(\mathcal{W}'') = -\mathcal{F}(\mathcal{W}'')$. Let φ be the Fourier transform of χ; so φ is concentrated on a 1-dimensional subspace. Then we have $\varphi * \mathcal{W}' = \mathcal{W}'$, $\varphi * \mathcal{W}'' = -\mathcal{W}''$, where $*$ stands for convolution.

Since $\mathcal{W}_n$ is a tempered distribution it can be represented as $\mathcal{W}_n = D(F)$ where F is a continuous function of polynomial growth and D is a differential operator with constant coefficients. We can assume that D is invariant under the action of the symmetric group Σ_{n+1}. Then the functions $F' = F(v_{i_1}, .., v_{i_k}; v_{j_1}, .., v_{j_{n-k}})$ and $F'' = F(v_{j_1}, .., v_{j_{n-k}}; v_{i_1}, .., v_{i_k})$ satisfy $D(F') = \mathcal{W}'$ and $D(F'') = \mathcal{W}''$. Then obviously $\varphi * F' = F'$ and $\varphi * F'' = -F''$.

The space-locality property implies that $F'(v_1, \dots, v_n) = F''(v_1, \dots, v_n) = F(v_1, \dots, v_n)$ provided $(v_i - v_j)^2 < 0$ for $i \in I$, $j \in J$.

So for $|(a_i - a_j)^2| > (t_i - t_j)^2$ we have

$$F(t_1 + a_1, \dots, t_n + a_n) = \tfrac{1}{2}(F'(t_1 + a_1, \dots, t_n + a_n) + F''(t_1 + a_1, \dots, t_n + a_n))$$
$$= \tfrac{1}{2}\varphi * (F' - F'') = \tfrac{1}{2}\int_{-\infty}^{\infty} (F' - F'')((t_1 + a_1)^t, (t_2 + a_2)^t, \dots, (t_n + a_n)^t)\widetilde{\varphi}(t)dt$$

where we used the notation $(t_i + a_i)^t = t_i + a_i - t$ if $i \in I$, and $(t_j + a_i)^t = t_j + a_j$ for $j \in J$. The expression in the integral vanishes if $t < \min(\sqrt{-(a_i - a_j)^2} - |t_i - t_j|)$. So since F is of polynomial growth we get the desired upper bound for the integral.

The crucial step of the proof is the following

Lemma 3.

$$|\mathcal{W}_n^{\mathrm{tr}}(g_1^t, \dots, g_n^t)| \leq C(1 + |t|)^{(1-d)(n-2)/2}$$

for some C and all t, where $g_j^t = f_j^t$ or $g_j^t = \frac{\partial f_j^t}{\partial t}$ or complex conjugate for each j.

Proof of the Lemma. Let us work in Fourier coordinates; let pr be the projection $\mathrm{pr} : V_n \to (\mathbb{R}^{d-1})^n$.

Of course $\mathcal{W}_n^{\mathrm{tr}}(g_1^t, \dots, g_n^t) = \int \mathcal{F}(\mathcal{W}_n^{\mathrm{tr}})(-p)\mathcal{F}(g_1^t \times \cdots \times g_n^t) = \int \mathrm{pr}_*(\mathcal{F}(W_n^{\mathrm{tr}})(-p) \cdot \mathcal{F}(g_1^t \times \cdots \times g_n^t))$. Let us look more closely at the last integral.

For this purpose it is convenient to write $\mathcal{F}(f_i)$ (respectively $\mathcal{F}(\bar{f}_i)$) in the form $\tilde{\psi}_i \cdot \tilde{\eta}_i$ where $\tilde{\psi}_i$ is a Schwartz function with support in Λ (respectively in $-\Lambda$), and $\tilde{\eta}_i$ is the pull-back of a Schwartz function under the projection on $\mathbb{R}^{d-1}$.

Let η_i be given by $\mathcal{F}(\eta_i) = \delta_{\mathcal{O}_m^+} \cdot \tilde{\eta}_i$ (respectively $\mathcal{F}(\eta_i) = \delta_{\mathcal{O}_m^-} \cdot \tilde{\eta}_i$).

Using the projection formula we can rewrite

$$\mathrm{pr}_*\left(\mathcal{F}(\mathcal{W}_n)(-p) \cdot \mathcal{F}(\prod g_i^t)\right) = \mathrm{pr}_*[T_t\left(\mathcal{F}(\mathcal{W}_n)(-p) \cdot \prod \tilde{h}_i\right)] \cdot \mathrm{pr}_*[T_{-t}\left(\mathcal{F}(\prod \eta_i)\right)]$$

Here $T_t(f)(p) = \exp(it \cdot p_0)f(p)$ and $\tilde{h}_i = \tilde{\psi}_i$ if $g_i^t = f_i^t$ or complex conjugate, and $\tilde{h}_i = \tilde{\psi}_i \cdot \pm i(p_0 - \sqrt{m^2 - \vec{p}^2})$ if $g_i^t = \frac{\partial f_i^t}{\partial t}$ or complex conjugate.

Let h denote $\mathcal{F}(\prod \tilde{h}_i)$. Then the integral we need to estimate equals

$$\int_{(\mathbb{R}^{d-1})^n} \mathrm{pr}_*\left[T_t\left(\mathcal{F}(\mathcal{W}_n)(-p) \cdot \mathcal{F}\left(\prod \tilde{h}_i\right)\right)\right] \cdot \mathrm{pr}_*\left[T_t\left(\overline{\mathcal{F}\left(\prod \eta_i\right)}\right)\right]$$
$$= \int_{(\mathbb{R}^{d-1})^n} i_t^*(F * h) \cdot i_{-t}^*\left(\prod \eta_i\right)$$

By the Corollary to the Proposition, $i_t^*(\mathcal{W}_n * h) = \pi^*(F_t^h)$ for some function $F_t^h \in L^1((\mathbb{R}^{d-1})^{n-1})$, and $|F_h^t|_{L^1} < C$ where C does not depend on t.

Thus we have

$$\int_{(\mathbb{R}^{d-1})^n} |\mathrm{pr}^* i_t^*(\mathcal{W}_n * h) \cdot \mathrm{pr}_1 i_{-t}^*(\eta_1)| \leq C \cdot \int_{\mathbb{R}^{d-1}} |j_t^*(\eta_1)| \leq C \cdot (1 + |t|)^{(d-1)/2}$$

by Lemma 1b), where j_t is the imbedding $\mathbb{R}^{d-1} \hookrightarrow V$, $\vec{v} \to (t, \vec{v})$), and pr_i : $(\mathbb{R}^{d-1})^n \to \mathbb{R}^{d-1}$ is the projection to the i-th component. Hence

$$\int_{(\mathbb{R}^{d-1})^n} i_t^*(\mathcal{W}_n * h) \cdot i_{-t}^* \left(\prod \eta_i \right) \leq C(1+|t|)^{(d-1)/2}$$

$$\cdot \max_{(\mathbb{R}^{d-1})^n} \prod_{k=2}^{n} \mathrm{pr}_k^* j_{-t}^*(\eta_j) \leq C(1+|t|)^{-(n-2)(d-1)/2}$$

where the last inequality follows from Lemma 1a). Lemma 3 is proved.

We are now ready to prove the Theorem. Denote $v^t = \phi(f_1^t) \dots \phi(f_n^t)\Omega$. To show that the limit $\lim\limits_{t\to\infty} v^t$ exists it suffices to check that the integral $\int\limits_0^\infty \left| \frac{\partial v^t}{\partial t} \right| dt$ converges. We have

$$\left(\tfrac{\partial v^t}{\partial t}\right)^2 = \left(\sum_i \phi(f_1^t) \dots \phi(\dot{f}_i^t) \dots \phi(f_n)\Omega, \sum_j \phi(f_1^t) \dots \phi(\dot{f}_j^t) \dots \phi(f_n)\Omega \right)$$

$$= \sum_{i,j} \mathcal{W}_{2n} \left(\bar{f}_n^t, \bar{f}_{n-1}^t, \dots, \bar{\dot{f}}_i^t, \dots, \bar{f}_1^t, f_1^t, \dot{f}_j^t, \dots, f_n \right)$$

Let us decompose $\mathcal{W}_{2n}$ as a sum of products of truncated Wightman functions, and consider the corresponding decomposition of each summand in the last sum.

Each factor containing $\mathcal{W}_1^{\mathrm{tr}}$ obviously vanishes. By the previous lemma each factor containing $\mathcal{W}_i^{\mathrm{tr}}$ is bounded by const $\cdot (1+|t|)^{(i-2)(1-d)/2}$; in particular each factor is bounded.

Let us look at a factor of the form $\mathcal{W}_k^{\mathrm{tr}}(g_1^t, ..., g_k^t)$ where at least one of g_i^t is a time derivative of f_ℓ^t (or complex conjugate). If $k = 2$ then such a factor vanishes. Indeed, if say $g_1^t = \overline{f_i^t}$ and $g_2^t = \dot{f}_j^t$ then $\mathcal{W}_2^{\mathrm{tr}}(g_1^t, g_2^t) = \left\langle \phi(f_i^t)\Omega, \phi(\dot{f}_j^t)\Omega \right\rangle = 0$ since $\phi(f_j^t)\Omega$ does not depend on t. (Here we use that $\operatorname{supp} \mathcal{F}(f_j) \cap \operatorname{supp} U\big|_V = \mathcal{O}_m^+$.) The other cases are absolutely parallel.

We now see that each non-zero summand contains a factor of the form $\mathcal{W}_k^{\mathrm{tr}}(g_1^t, \dots, g_k^t)$ with $k \geq 3$. Furthermore, it either contains at least 2 factors $\mathcal{W}_k^{\mathrm{tr}}$ with $k \geq 3$ or at least one factor $\mathcal{W}_k$ with $k \geq 4$, because the term has the form $\mathcal{W}_{k_1}^{\mathrm{tr}} \dots \mathcal{W}_{k_\ell}^{\mathrm{tr}}$ with $k_1 + \dots + k_\ell = 2n$ being even, and each $k_i \geq 2$. In any case it is bounded by const $\cdot (1+|t|)^{(1-d)/2 \cdot \sum (k_i - 2)} \leq$ const $\cdot (1+|t|)^{1-d}$. Thus

$$|\dot{v}^t| \leq \text{const} \cdot (1+|t|)^{(1-d)/2},$$

and $\int\limits_{-\infty}^{\infty} \dot{v}^t \, dt$ converges absolutely if $d \geq 4$.

Proof of b). Let us check that if $v^t = \phi(f_1^t) \dots \phi(f_k^t)\Omega$, $w^t = \phi(g_1^t) \dots \phi(g_\ell^t)\Omega$ for f_i, g_j as above then $\lim\limits_{t \to \pm\infty} (v^t, w^t) = (\phi_0(f_1) \dots \phi_0(f_k)\Omega, \phi_0(g_1) \dots \phi_0(g_\ell)\Omega)$. Let us again decompose $(v^t, w^t) = \mathcal{W}_{\ell+k}(\bar{f}_k^t, \bar{f}_{k-1}^t, \dots, \bar{f}_1^t, g_1^t, \dots, g_\ell^t)$ into a sum of products of truncated Wightman functions. From the above arguments we see that every term containing $\mathcal{W}_1^{\mathrm{tr}}$ vanishes, and every term containing $\mathcal{W}_k^{\mathrm{tr}}$ for $k \geq 3$ tends

to 0 as $t \to \pm\infty$. Besides, $\mathcal{W}_2^{\mathrm{tr}}(\bar{f}, g) = (\phi(f)\Omega, \phi(g)\Omega) = (\phi_0(f)\Omega, \phi_0(g)\Omega)$. Thus we have

$$\lim_{t\to\pm\infty}(\phi(f_1^t)\dots\phi(f_k^t)\Omega, \phi(g_1^t)\dots\phi(g_\ell^t)\Omega) = \lim_{t\to\pm\infty}(\phi_0(f_1^t)\dots\phi_0(f_k^t)\Omega,$$
$$\phi_0(g_1^t)\dots\phi_0(g_\ell^t)\Omega) = (\phi_0(f_1)\dots\phi_0(f_k)\Omega, \phi_0(g_1)\dots\phi_0(g_\ell)\Omega)$$

We can now finish the proof. For free field theory the space $\mathcal{D}$ is generated by the vectors $\phi_0(f_1)\dots\phi_0(f_n)\Omega$ for f_i as above, because $\phi_0(f)x$ depends only on $\mathcal{F}(f)\big|_{\mathcal{O}_m^+}$ for $f \in \mathcal{S}$, $x \in \mathcal{H}$.

For $x \in \mathcal{H}$ let us choose functions f_i such that $x = \sum \phi_0(f_{i_1})\dots\phi_0(f_{i_s})\Omega$, and consider the limit $\lim_{t\to\pm\infty}\sum\phi(f_{i_1}^t)\dots\phi(f_{i_s}^t)\Omega$. We proved already that for any $y \in \mathcal{H}$, and functions g_j such that $y = \sum\phi_0(g_{j_1})\dots\phi_0(g_{j_r})\Omega$ we have

$$\left(\lim_{t\to\pm\infty}\sum\phi(f_{i_1}^t)\dots\phi(f_{i_s}^t)\Omega, \lim_{t\to\pm\infty}\sum\phi(g_{j_1}^t)\dots\phi(g_{j_r}^t)\Omega\right) = (x, y).$$

Hence the limits do not depend on the choice of f_i, g_j, and give a well defined isometric imbedding. The Theorem is proved.

§4.4. Scattering matrix

Proposition 2. *We have* $\mathrm{Im}(W_+) = \mathrm{Im}(W_-)$.

Thus we can define the S-matrix $S = W_+^* W_- \in \mathrm{End}(\mathcal{H}_0)$. It is interesting that S can be described in terms of the Wightman functions of the theory.

The precise formulation is as follows. Let $\tilde{f}$ denote the partial Fourier transform of f in space variables. A vector $\vec{v} \in \mathbb{R}^{d-1}$ is called a velocity of f if we have $\vec{v} = \vec{p}/\sqrt{\vec{p}^2 + m^2}$ for $\vec{p}$ such that $(p_0, \vec{p}) \in \mathrm{supp}(\tilde{f})$ for some real number p_0.

Theorem. (Lehmann-Symanzik-Zimmermann)

a) *The residue* $\mathcal{F}(\mathcal{W}_n^T)\big|_{(\mathcal{O}_m^+)^n}$ *is a well-defined distribution on* V_n *supported on* $(\mathcal{O}_m^+)^n$, *where* $\mathcal{W}_n^T$ *is the time-ordered Wightman function (see §2.4 for the definition).*

b) *Assume* $f_i \in \mathcal{S}$ *and the set of velocities of* f_i *and* f_j *does not intersect for* $i \neq j$. *Then*

$$\sum_{I\subset(1,\dots,n)}(-1)^{|I|}\left\langle W_-\left(\prod_{i\in I}\phi_0(f_i)\Omega\right), W_+\left(\prod_{j\notin I}\phi_0(f_j)\Omega\right)\right\rangle =$$
$$= \left\langle \mathrm{Res}(\mathcal{F}(\mathcal{W}_n^T)), \mathcal{F}(f_1)\times\dots\times\mathcal{F}(f_n)\right\rangle .$$

It is an elementary combinatorial problem to deduce an expression for the matrix coefficients of the S-matrix $\langle W_-\prod\phi_0(f_1)\Omega, W_+\prod\phi_0(f_j)\Omega\rangle$ in terms of $\mathcal{F}(\mathcal{W}^T)$ from the last formula.

References and cross references

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LECTURE 5
Feynman Graphs

§5.1. Feynman graph expansion

The technique of Feynman graphs allows one to write down an asymptotic series for the functional integral of a QFT in a neighborhood of a free QFT (see Lecture 1 of [**I-Perturbative QFT**]). Nevertheless, the technique itself can be applied to a purely finite dimensional integral. In this lecture we will discuss this application only.

Let V be a finite dimensional real vector space, let V' be its dual. We will view elements of the symmetric algebra $\mathrm{Sym}(V')$ as polynomial functions on V. We fix a nondegenerate positive definite quadratic form $b \in \mathrm{Sym}^2(V')$. Let $b^{-1} \in \mathrm{Sym}^2(V)$ be the corresponding quadratic form on V'.

Let $\mu_0 = e^{-b^{-1}/2} dv$ be the Gaussian measure on V'; we have $\mathcal{F}(\mu_0) = e^{-b/2}$.

Let P be a polynomial function on V'. We want to study the "perturbed" measure $\mu = e^{\varepsilon P}\mu_0$. This is a well-defined measure if $\varepsilon > 0$ and P is negative.

We are going to write down an asymptotic expansion for the Fourier transform $\mathcal{F}(\mu)$.

More precisely, we will obtain a formal Taylor series $\widetilde{\mu} \in \mathbb{C}[\![V', \varepsilon]\!]$ in the following way.

Let $D(P)$ be the differential operator with constant coefficients on V corresponding to P. More precisely, D is a homomorphism of the algebra of polynomial functions on V' to the algebra of differential operators with constant coefficients on V, such that $D(x) = i\partial/\partial x$ for $x \in V$.

We write

$$\mathcal{F}(e^{\varepsilon P}\mu_0) = e^{\varepsilon D(P)}(e^{-b/2}) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\varepsilon^n D(P)^n(e^{-b/2}) \ .$$

This is obviously an element of $\mathbb{C}[\![V', \varepsilon]\!]$.

This formal power series is connected with the original analytical problem via the notion of Borel summability. However in this lecture we will be interested in the formal expression only.

It is notationally convenient to perform "Wick rotation". So we consider the series

$$\widetilde{\mu}(v) = \mathcal{F}(e^{\varepsilon P}\mu_0)(iv) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\varepsilon^n \widehat{P}^n(e^{b/2}) \ .$$

where $\widehat{P} = D(P(-ix))$, so that $\widehat{x} = \partial/\partial x$.

The series $\widetilde{\mu}$ is defined for an arbitrary nondegenerate quadratic form b, and a polynomial function P over any field of characteristic 0. (The condition on b to be positive definite is of course irrelevant in this setting.)

There exists a combinatorial way to "compute" $\widetilde{\mu}$. To describe it we first fix some notations.

By a graph we will mean the data consisting of two sets Γ_1 (edges) and Γ_0 (vertices) and a map $j\colon \Gamma_1 \to \Gamma_0 \times \Gamma_0$. We also assume an involution σ of Γ_1 is fixed such that $j(\gamma) = (x, y) \Leftrightarrow j(\sigma(\gamma)) = (y, x)$, and σ has no fixed points. We will only consider graphs with no isolated vertices (i.e. $\mathrm{pr}_1 \circ j$ is surjective). An edge of a graph is an element of Γ_1/σ. For a vertex $\gamma_0 \in \Gamma_0$ we define its "star" as $\Gamma(\gamma_0) = \{\gamma_1 \in \Gamma_1 |\ \mathrm{pr}_2 \circ j(\gamma_1) = \gamma_0\}$. The set of external vertices is $\Gamma_{\mathrm{ex}} = \{\gamma_0 \in \Gamma_0 | \#(\Gamma(\gamma_0)) = 1\}$, and the set of inner vertices is $\Gamma_{\mathrm{in}} = \Gamma_0 - \Gamma_{\mathrm{ex}}$. We put $L(\gamma_0) = \mathrm{Sym}^{\#(\Gamma(\gamma_0))}(V)$; $\widetilde{L}(\Gamma) = \underset{\gamma_0 \in \Gamma_0}{\otimes} L(\gamma_0)$; $L(\Gamma) = \underset{\gamma \in \Gamma_{\mathrm{in}}}{\otimes} L(\gamma)$.

Assume that for any $\gamma_0 \in \Gamma_0$ we are given an element $s(\gamma_0) \in L(\gamma_0)$. We then take $\ell_s = \underset{\gamma_0 \in \Gamma_0}{\otimes} s(\gamma_0) \in \widetilde{L}(\Gamma)$. For any map $\tilde{s}\colon \Gamma_0 \to V$ we define $s_{\tilde{s}}(\gamma_0) = (\tilde{s}(\gamma_0))^{\#(\Gamma(\gamma_0))} \in L(\gamma_0)$, and denote $\ell_{\tilde{s}} = \ell_{s_{\tilde{s}}}$.

Lemma 1. *There exists a unique linear function $\widetilde{\tau}_b^{\Gamma}$ on $\widetilde{L}(\Gamma)$ such that for any map $\tilde{s}\colon \Gamma_0 \to V$ we have*

$$\widetilde{\tau}_b^{\Gamma}(\ell_{\tilde{s}}) = \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{\gamma_1 \in \Gamma_1/\sigma} b\left(\tilde{s}(\mathrm{pr}_1 \circ j(\gamma_1)), \tilde{s}(\mathrm{pr}_2 \circ j(\gamma_1))\right) .$$

The proof is clear.

We can view $\widetilde{\tau}_b^{\Gamma}$ as a map $\tau_b^{\Gamma}\colon L(\Gamma) \to \mathrm{Sym}^{\#\Gamma_{\mathrm{ex}}}(V')$.

Example. Γ_n is a disjoint union of n copies of the graph: •——• . (The nonoriented edge on the picture corresponds to two elements of Γ_1 permuted by σ.) Then $L(\Gamma) = \mathbb{C}$ canonically, because Γ_{in} is empty. We have

$$\tau_b^{\Gamma_n}(1) = \frac{b^n}{2^n n!} .$$

Note the equality $\sum_{n=0}^{\infty} \tau_b^{\Gamma_n}(1) = e^{b/2}$ where Γ_0 is the empty graph.

Let P, $\widetilde{\mu}$ be as above. We write $P = \sum_{n \geq 2} P_n/n!$ where P_n is homogeneous of degree n (thus we assume for convenience that P contains no constant or linear term).

Theorem. *We have an equality of formal series*

$$\widetilde{\mu} = \left(\sum_{\Gamma \in \mathfrak{g}} \varepsilon^{\#\Gamma_{\mathrm{in}}} \tau_b^{\Gamma}(s_P)\right) e^{b/2}$$

Here $\mathfrak{g}$ is the set of isomorphism classes of graphs and $s_P(\gamma_0) = P_{\#\Gamma(\gamma_0)}$.

Proof. [Another proof appears in [I-**Perturbative QFT**, §1.3]].

The idea of the proof is as follows. We want to prove an equality between two expressions, which are power series in ε and in P_n's. We will show that both the left hand side and the right hand side satisfy the same system of linear differential equations in P_n's as variables, and then we will check that the initial conditions coincide.

So, let Γ be a graph and let $\lambda \subset \Gamma_{ex}$ be such that $|\lambda| = n$. We denote by Γ^λ the "cone over λ". This means that Γ^λ is a graph such that $\Gamma^\lambda_0 = \Gamma_0 \cup \{*\} - \lambda$ and $\Gamma^\lambda_1 = \Gamma_1 - \{\text{all edges of } \Gamma \text{ with an end in } \lambda\} \cup \{\text{all edges from } * \text{ to vertices connected with } \lambda\}$. (We add all edges connecting $*$ to a vertex in λ, and then erase the vertices belonging to λ.) Assume now that $n > 1$. Then $*$ is an internal vertex of Γ^λ.

For any $d \in S^n(V)$ and $s \in L(\Gamma)$ we define $s^d_\lambda = d \otimes s \in L(\Gamma^\lambda)$.

Lemma 2. *For any $d \in S^n(V)$, $\Gamma \in \mathfrak{g}$ and $s \in L(\Gamma)$ we have*

$$\frac{\hat{d}}{n!}(\tau_b(\Gamma)(s)) = \sum_{\lambda \subset \Gamma_{ex}, |\lambda|=n} \tau_b(\Gamma^\lambda)(s^d_\lambda)$$

(Here $\hat{d}$ is the differential operator with constant coefficients on V corresponding to d.)

Now we can prove the theorem. Let $l(P)$ denote the left hand side and let $r(P)$ be the right hand side of the equality. We want to write down a certain system of differential equations in P_n's as variables, which will be satisfied by both sides. It is easy to see from the definitions that $l(P)$ satisfies the system of equations

$$\frac{\partial l(P)}{\partial P_n}(Q) = (\varepsilon \widehat{Q}/n!) l(P)$$

where Q is an arbitrary polynomial of degree n on V' and $\widehat{Q}$ is the corresponding differential operator on V; on the left hand side Q is considered as a tangent vector to the space of polynomials of degree n.

But now Lemma 2 tells us precisely that the right hand side of this expression is equal to $\frac{\partial r(P)}{\partial P_n}(Q)$.

Therefore $l(P)$ and $r(P)$ satisfy the same system of differential equations. Let's now check the initial conditions. By the definition, $l\big|_{P=0} = e^{b/2}$. On the other hand, one also has $r\big|_{P=0} = e^{b/2}$ by the above Example. Hence we have $l(P) = r(P)$ for any P, so the theorem is proved.

Remark 1. The theorem and its proof along with all the statements below can be immediately generalized in the following way. We can take P to be an element of $\mathbb{C}[V][\![\varepsilon]\!]$ rather than a polynomial, i.e. $P = \sum_{m=0}^\infty \varepsilon^m P_{(m)}$ is a Taylor series in ε with polynomial functions on V' as coefficients.

Remark 2. This way of writing an asymptotic expansion for a measure uses Fourier transform, and hence the linear structure of V heavily. For example when one applies the Feynman graph expansion to gauge theory, V being the space of connections modulo gauge transformations, then V has no natural linear structure.

In this case one still can write an asymptotic expansion for the functional integral, but an individual term assigned to a particular graph is not canonically defined.

We denote $Z = \sum_{\Gamma \in \mathfrak{g}} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\text{in}}}$.

One of the difficulties in using this expansion is that there are "too many" terms; so we try to reduce it somehow. We denote $F = \sum_{\Gamma \in \mathfrak{g}_{\text{con}}} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\text{in}}}$, where $\mathfrak{g}_{\text{con}}$ is the set of connected nonempty graphs.

Note that F (and all series that will be introduced below) lie in a subring $\mathbb{C}[V']][\varepsilon]] \subset \mathbb{C}[[V', \varepsilon]]$.

Claim 1. $F = \log Z$.

Proof. Decomposing a graph into the union of its irreducible components one can identify the two combinatorial expressions

$$Z = \sum_{\Gamma \in \mathfrak{g}} \tau_b^\Gamma = \sum_{\Gamma_1,..,\Gamma_k \in \mathfrak{g}_{\text{con}}; n_1,..,n_k} \frac{1}{n_1! n_2! .. n_k!} \tau_b^{\Gamma_1} \dots \tau_b^{\Gamma_k} = \exp(F)$$

where the summation is over all sets of distinct elements $\Gamma_i \in \mathfrak{g}_{\text{con}}$.

We next want to study the "quasi-classical approximation" to our integral. So we introduce another variable $\hbar$ and consider the expression

$$F_\hbar(v) = \log \mathcal{F}\left(e^{(b^{-1}/2 + \varepsilon P)/\hbar} dv\right)(\hbar v) \ .$$

As an immediate corollary to the Theorem, we get

Lemma 3. $F_\hbar = \hbar^{-1} \sum_{\Gamma \in \mathfrak{g}_{\text{con}}} \hbar^{h^1(\Gamma)} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\text{in}}}$ *where* $h^1(\Gamma) = \dim(H^1(\Gamma)) = \#\Gamma_1/\sigma - \#\Gamma_0$.

Proof. Apply the same formal procedure to the quadratic form $\tilde{b} = \hbar b$ and polynomial $\widetilde{P} = \hbar^{-1} P$, and note that for a connected graph Γ we have $\tau_{\hbar b}^\Gamma(s_{P/\hbar})(\hbar v) = \hbar^{h^1(\Gamma)-1} \tau_b^\Gamma(s_P)(v)$.

§5.2. Quasi-classical (low-loop) approximations

Recall that the "classical" approximation to the Fourier transform is the Legendre transform. An appropriate version of the definition of Legendre transform is as follows.

If f is a function on V', then we can view its differential df as a map $df\colon V' \to V$.

If df is an isomorphism then the Legendre transform of f is defined by

$$L(f)(v) = \langle v, (df)^{-1}(v) \rangle - f\left((df)^{-1}(v)\right)$$

i.e. $L(f)(v)$ is the critical value of the function $v - f$, where v is considered as a linear function on V'.

Let now $G \in \mathbb{C}[[V, \varepsilon]]$ be a formal power series. Then its differential dG (in the V'-direction) is an element of $\mathrm{Hom}(V', V) \otimes \mathbb{C}[[V, \varepsilon]]$.

Assume that $G = g_0 + g_1 + \varepsilon G_1$, where g_0 is a nondegenerate quadratic form on V', and g_1 contains no terms of degree less then 3 in V. Then dG is invertible,

i.e. there exists $H = (dG)^{-1} \in \mathrm{Hom}(V, V') \otimes \mathbb{C}[\![V, \varepsilon]\!]$ with no constant term such that $dG \circ H = id$ and $H \circ dG = id$. In this situation the Legendre transform $L(G) \in \mathbb{C}[\![V', \varepsilon]\!]$ can be defined by the same formula.

To describe the next term of the asymptotics we define for any function f on V' as above a new function $H(f)$ on V as follows. Let $\mathrm{Hess}_f(p) = \det\big(\frac{\partial^2 f}{\partial v_i \partial v_j}\big)|_p$ be the Hessian of f (we assume that a constant volume form on V is fixed so that the determinant of the quadratic form is taken with respect to that volume form).

We put $H(f)(v) = \mathrm{Hess}\big((df)^{-1}(v)\big)$.

For $G = g_0 + g_1 + \varepsilon G_1 \in \mathbb{C}[\![V', \varepsilon]\!]$ as above we can define $\mathrm{Hess}(G)$ and $H(G) \in \mathbb{C}[\![V', \varepsilon]\!]$ by the same formula.

Claim 2. a) *Let* $F_0 = \sum_{\Gamma \in T} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\mathrm{in}}}$, *where* T *is the set of (nonempty) trees. Then* F_0 *is the Legendre transform of* $\frac{b^{-1}}{2} - \varepsilon P$.

b) *Let* $F_1 = \sum\limits_{\Gamma \in \mathfrak{g}_1} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\mathrm{in}}}$ *where* $\mathfrak{g}_1$ *is the set of one-loop connected graphs (i.e. graphs with* $h^0(\Gamma) = h^1(\Gamma) = 1$. *Then* $F_1 = \log\big(H\big(\frac{b^{-1}}{2} - \varepsilon P\big)^{-1}\big)$. (We assume that the background volume form is such that $\det(\frac{b^{-1}}{2}) = 1$. Then $H(\frac{b^{-1}}{2} - \varepsilon P) \in 1 + \varepsilon\mathbb{C}[\![V', \varepsilon]\!] + V'\mathbb{C}[\![V', \varepsilon]\!]$, and $\log(H(\frac{b^{-1}}{2} - \varepsilon P))$ is a well-defined element of $\mathbb{C}[\![V', \varepsilon]\!]$).

Proof of the Claim. From Lemma 3 we see that $F_\hbar(v) = F_0(v)/\hbar + F_1(v) + O(\hbar)$. Thus the claim follows from the stationary phase approximation applied to the integral

$$\mathcal{F}(f)(iv) = \frac{1}{(2\pi)^{d/2}} \int\limits_{V'} e^{(v,p)+(\varepsilon P - b^{-1}/2)(p)}\, dp$$

One can also directly show the equality of the two combinatorial expressions

$$\begin{aligned} F_0\Big(d(\frac{b^{-1}}{2} - \varepsilon P)|_p\Big) &= \frac{b^{-1}}{2}(p) - \varepsilon \sum_n \frac{P_n}{(n-1)!}(p) + \varepsilon \sum_n \frac{P_n}{n!}(p) \\ &= \langle p, d\Big(\frac{b^{-1}}{2} - \varepsilon P\Big)|_p\rangle - (b^{-1}/2 - \varepsilon P)\Big(d(\frac{b^{-1}}{2} - \varepsilon P)\Big) \end{aligned} \tag{5.1}$$

and

$$F_1\Big(d(\frac{b^{-1}}{2} - \varepsilon P)|_p\Big) = -\log \mathrm{Hess}\Big(\frac{b^{-1}}{2} - \varepsilon P\Big) \tag{5.2}$$

Here (5.1) is equivalent to $F_0 = L\big(\frac{b^{-1}}{2} - \varepsilon P\big)$ and (2) is equivalent to $F_1 = \log\big(H\big(\frac{b^{-1}}{2} - \varepsilon P\big)^{-1}\big)$.

Let us sketch the combinatorial proof of (5.1) and (5.2).

Let us identify V and V' by means of b and use an equality

$$\varepsilon^{\#\Gamma_{\mathrm{in}}}\tau_b^\Gamma(s_P) \circ d\big(\frac{b^{-1}}{2} - \varepsilon P\big) = \sum (-1)^{c_i} \varepsilon^{\#\Gamma^i_{\mathrm{in}}} \tau_b^{\Gamma^i}(s_P)$$

where the graph Γ^i is obtained from the graph Γ by adding c_i vertices of arbitrary valencies, each one of which is connected with an external vertex of Γ in such a way that any external vertex of Γ is connected with no more then one new vertex.

Now it is not hard to check that in the expansion of $F_0 \circ \left(d\left(\frac{b^{-1}}{2} - \varepsilon P\right)\right)$ all terms with ε^i for $i \geq 2$ cancel, and identify the sum of remaining terms with the RHS of (5.1).

Likewise in the expansion of $F_1 \circ \left(d\left(\frac{b^{-1}}{2} - \varepsilon P\right)\right)$ only the terms with one-particle irreducible one-loop graphs ("circles") do not cancel; the sum of these terms is identified with $\sum_{n=1}^{\infty} \varepsilon^i \mathrm{tr}(dP)^n/n = -\log\det(Id - dP)$ which coincides with the RHS of (5.2).

§5.3. Effective potential

Let us call a nonempty graph 1-*particle irreducible* (or just *1-irreducible*) if it is connected and remains connected after removal of any internal edge. To any connected graph there corresponds a unique tree with a 1-particle irreducible graph assigned to each vertex together with an identification of the set of edges coming to a vertex of a tree with the set of external vertices of the corresponding graph. Thus "computation" of F_0 can be reduced to summation over trees and over 1-particle irreducible graphs separately. Denote $F_{1-\mathrm{irr}} = \sum_{\Gamma \in \mathfrak{g}_{1-\mathrm{irr}}} \varepsilon^{\#\Gamma_{\mathrm{in}}} \tau_b^{\Gamma}(s_P)$ where $\mathfrak{g}_{1-\mathrm{irr}}$ is the set of 1-particle irreducible graphs. (Notice that the graph •——• is 1-particle irreducible by definition).

Claim 3. *$F_{1-\mathrm{irr}} = b - b^*(L(F))$ where b is viewed as a map from V to V' and L is the Legendre transform.*

Proof. By Remark 1 after the proof of the theorem we can apply Claim 2 to $P' = \varepsilon^{-1}(b^{-1})^*(F_{1-\mathrm{irr}} - b/2) \in \mathbb{C}[V'][\![\varepsilon]\!]$.

From the above combinatorial observation we see that $F = F_P = (F_0)_{P'}$. Hence claim 2 implies that $F = L(b^{-1}/2 - \varepsilon P') = L\left((b^{-1})^*(b - F_{1-\mathrm{irr}})\right)$. Since the Legendre transform is involutive we get the claim.

$F_{1-\mathrm{irr}}$ is called the effective potential of the theory.

References and cross references

See [I-**Perturbative QFT**, §1.3] for the introduction to Feynman graphs technique, and Lecture 1 of [I-**Perturbative QFT**]–Lecture 4 of [I-**Perturbative QFT**] for examples of its applications in QFT.

Perturbative Quantum Field Theory

Edward Witten

Introduction

Having studied quantum field theory in a rather formal way up to this point, we would now like to get some insight about what examples actually exist. For this we must look more closely at the perturbation theory – the Feynman diagram expansion – which has been formally introduced in Kazhdan's lectures and will be reviewed at the outset of my first lecture. Making sense of the perturbation expansion requires a process of "renormalization," which works for some theories but not others. Understanding which theories are renormalizable, when enriched in the important borderline cases by the renormalization group and asymptotic freedom, is the basis for most of what we know about what quantum field theories should be expected, or in some cases proved, to exist.

My first two lectures this fall will be devoted to explaining perturbative renormalization. The way I organize the discussion is that in lecture one we select a concrete question (the two-point function or expectation value of the product of two fields), and after reviewing what the analytic behavior should be on axiomatic grounds, we then calculate, in a concrete theory. In the process, we find that some renormalization must be made; we then discuss the renormalization process systematically in the second lecture.

A systematic exploration of the perturbative renormalization theory also lays the groundwork for exhibiting, in perturbation theory, important formal properties whose validity is believed (and sometimes proved) to extend beyond perturbation theory. Examples range from the renormalization group, which will be introduced by Gross, to the operator product expansion, which will be the subject of my third lecture. Thorough understanding of perturbation theory is also a necessary foundation for practically any attempt to understand nonperturbative dynamical properties – the subject I will lecture on in the spring – roughly as, in many areas of mathematics, some local understanding is a necessary foundation for global results.

Lecture four in this series has a slightly different theme; it aims to make the concept of the S matrix – a basic notion of quantum field theory – more accessible by describing the S matrix in nonrelativistic quantum mechanics.

The ability to compare quantum field theory to experiment depends largely on perturbation theory. In the electroweak part of the standard model, there are small dimensionless parameters, and perturbation theory is compared directly to experiment. For the strong interactions, things are harder, and one must enrich

perturbation theory with the aid of asymptotic freedom. A schematic account of what this entails is the subject of lecture five.

LECTURE 1
Renormalization of Feynman Diagrams

This lecture is a very elementary introduction to renormalization of Feynman integrals. I begin by discussing properties of the two-point function from an axiomatic point of view, partly because the statements are important and partly so that when we get to an actual calculation, we will have some idea of what to expect.

§1.1. Perturbative expansion of a two-point correlation function

Consider a quantum field theory with one scalar bosonic field $\phi(x)$ on a Minkowski space $(V,(,))$ of signature $(1,n-1)$ (see Lecture 1 of [I-**Axiomatics**]). Let V_+, V_- be the closures of the upper and the lower part of the full cone of time-like vectors. Let $\mathcal{H}$ be the Hilbert space of the theory, $\mathcal{D} \subset \mathcal{H}$ a dense subspace, and $\varphi : \mathcal{S}(V) \to \mathrm{End}(\mathcal{D})$ be the quantization map. We assume that the triple $(\mathcal{H}, \mathcal{D}, \varphi)$ satisfies the Wightman axioms. In this case we can define quantum fields $\phi(x) := \varphi(\delta_x)$, which are distributions on V with values in $\mathrm{End}(\mathcal{D})$. When it does not cause confusion, we will treat them as usual $\mathrm{End}(\mathcal{D})$-valued functions.

We can assume without loss of generality that the 1-point Wightman function of the theory vanishes. Indeed, the 1-point function is a constant C, and we can redefine the map φ by setting $\varphi'(f) = \varphi(f) - C\int f dv$. The map φ' satisfies the Wightman axioms as well and has a zero 1-point function.

We look at the time ordered 2-point Wightman function

$$\mathcal{W}_2^T(x,y) = \langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle \tag{1.1}$$

Here the time ordering T means the following. By definition,

$$T(\phi(x)\phi(y)) = \phi(y)\phi(x)$$

if $x - y \in V_-$, and

$$T(\phi(x)\phi(y)) = \phi(x)\phi(y)$$

otherwise. Since $\phi(x)$ commutes with $\phi(y)$ when $x-y$ is space-like, the function (1.1) is even. Because of the Poincaré invariance axiom, (1.1) is a function of $x-y$. Denote this function by $W(x)$. It follows from the Poincaré invariance that the function $W(x)$ actually depends only on x^2.

Let $\mathcal{H}_1$ be the closed subspace of $\mathcal{H}$ spanned by vectors of the form $\phi(f)\Omega$, where f is a Schwartz function on V. The space $\mathcal{H}_1$ is a representation of the

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Poincaré group, and all its irreducible components have spin zero, i.e. have the form $L^2(\mathcal{O}_m^+)$, where $\mathcal{O}_m^+$ is the upper sheet of the two-sheeted hyperboloid $k^2 = m^2$ in the dual space V^*. This happens because we have a homomorphism of representations $\mathcal{S}(V) \to \mathcal{H}_1$, given by $f \to \phi(f)\Omega$, whose image is dense in $\mathcal{H}_1$.

Let $\rho_1 : P \to \mathrm{Aut}(\mathcal{H}_1)$ be the action of the Poincaré group in $\mathcal{H}_1$. If $x \notin V_-$, we can write $W(x)$ as

$$W(x) = \lim_{\varepsilon \to 0} \langle \rho_1(x) v_\varepsilon, v_\varepsilon \rangle, v_\varepsilon = \varphi(\delta_\varepsilon)\Omega, \tag{1.2}$$

where δ_ε is a family of smooth Schwartz functions tending to the δ-function.

Assume first that $\mathcal{H}_1$ is an irreducible representation, i.e.

$$\mathcal{H}_1 = L^2(\mathcal{O}_m^+).$$

In this case, it is very easy to evaluate (1.2) explicitly. Indeed, the operator $\rho_1(x)$ in this case is just the operator of multiplication by the function

$$e^{i(k,x)} \ (k \in \mathcal{O}_m^+ \subset V^*).$$

It is easy to see that there exists a limit as $\varepsilon \to 0$ of v_ε in the sense of distributions. Because of P-invariance, this limit is an $O(V)$-invariant distribution on $\mathcal{O}_m^+$. Therefore, this limit equals a constant function on $\mathcal{O}_m^+$. We can always normalize this constant to 1, by rescaling the map φ. Then from (1.2) we will get

$$W(x) = W_m(x) := \int_{\mathcal{O}_m^+} e^{i(k,x)} dk, x \notin V_-; W(-x) = W(x), \tag{1.3}$$

where dk is the $O(V)$-invariant measure on $\mathcal{O}_m^+$: if $k = (k_0, k_1)$, $k_0 \in \mathbb{R}$, $k_1 \in \mathbb{R}^{n-1}$, then $dk = dk_1/\sqrt{k_1^2 + m^2}$, and dk_1 is the Lebesgue measure on $\mathbb{R}^{n-1}$.

Let δ_m^+ denote the delta function of the upper sheet of the hyperboloid $k^2 = m^2$, and δ_m^- the delta function of its lower sheet. Then (1.3) says that $W_m(x)$ is equal to the Fourier transform of δ_m^+ in V_+, of δ_m^- in V_-, and to both of them in the rest of V (these Fourier transforms are equal outside of $V_+ \cup V_-$).

It is clear from this description that it is more convenient to work with the Fourier transform of the function $W_m(x)$ than with this function itself. The Fourier transform of $W_m(x)$ is obtained by a direct computation, similar to one from Lecture 3 of [I-**Axiomatics**]. The answer is called the Klein-Gordon propagator:

$$\tilde{W}_m(k) = w_m(k^2) = \frac{i}{k^2 - m^2 + i\varepsilon}, \tag{1.4}$$

where by definition $\frac{1}{k^2 - m^2 + i\varepsilon}$ is the distribution on V^* obtained as the weak limit of $\frac{1}{k^2 - m^2 + ia}$ as $a \to +0$.

Now we turn to the general case, when $\mathcal{H}_1$ is not an irreducible representation of P. We will define the spectral measure μ of $\mathcal{H}_1$ (on $\mathbb{R}$) by the following rule. Let $\alpha \in \mathcal{S}(\mathbb{R})$ be a positive function. We define the integral $\int \alpha(s) d\mu(s)$ by

$$\int \alpha(s) d\mu(s) = \langle \varphi(f), \varphi(f) \rangle, \tag{1.5}$$

where f is any function in $\mathcal{S}(V)$ such that $\int_{\mathcal{O}_s^+} |\tilde{f}|^2 dk = \alpha(s)$, (here $\tilde{f}$ is the Fourier transform of f). It is easy to show that the r.h.s. of (1.5) does not depend on the choice of f, so the measure $\mu(s)$ is well defined. By the definition of $\mu(s)$, the function $W(x)$ can be expressed in the form

$$W(x) = \int W_s(x) d\mu(s).$$

Therefore, the Fourier transform of $W(x)$ has the form

$$\tilde{W}(k) = w(k^2) = \int \frac{id\mu(s)}{k^2 - s^2 + i\varepsilon}.$$

It follows from scattering theory (See Lecture 4 of [I-**Axiomatics**]) that if $\mathcal{H}$ has a subrepresentation of P isomorphic to $L^2(\mathcal{O}_m^+)$ then it has components of the form $L^2(\mathcal{O}_s^+)$, occurring as continuous spectrum, for all $s \geq 2m$. Therefore, one expects that for a sufficiently generic quantum field theory, the space $\mathcal{H}_1$ will already contain some of this spectrum, and therefore it will be possible to see it by looking at the measure μ, i.e. at the function $W(x)$.

In the case when the field theory φ is a small perturbation of the theory φ_0 of a free scalar field of mass m_0, it is expected that $\mathcal{H}$ has a P-invariant subspace isomorphic to $L^2(\mathcal{O}_m^+)$ for m close to m_0, a continuous spectrum from $2m$ to ∞, and, possibly, a finite number of discrete components. This assumption can be tested by looking at μ: it would mean that μ is supported at $\{m\} \cup [2m, \infty)$, has an atom at $s = m$, and is absolutely continuous with respect to the Lebesgue measure for $s \geq 2m$ (the new discrete spectrum makes an exponentially small contribution to μ, with respect to the deformation parameter, and so it is not seen in the perturbation expansion).

In terms of the function $w(s)$, this means the following. Let us analytically continue $w(s)$ to a complex analytic function. Then, the conjectural behavior of the spectrum that we described above would mean that $\tilde{w}(s)$ has a pole at $s = m^2$ and a cut from $4m^2$ to $+\infty$, with jump $-2\pi d\mu/ds$ when crossing the cut from up to down at the point s.

Now we will take a concrete quantum field theory and compute a perturbative expansion of the function $w(k^2)$, in order to find out if it really has such analytic properties.

§1.2. The ϕ^3-theory

Now we consider the quantum field theory with the Lagrangian

$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{g}{3!}\phi^3 \right) d^n x. \tag{1.6}$$

This theory is a perturbation of the theory of a free scalar field of mass m with respect to a small parameter g. It is called the ϕ^3-theory.

Remark. From a physical point of view, the ϕ^3-theory is unsatisfactory, since the energy in this theory is not bounded below, for any finite nonzero value of g. However, one can consider this theory perturbatively, i.e. regard g as an infinitesimal formal parameter, which in algebraic terms means that we work over the ring $\mathbb{C}[[g]]$. Using this theory as an example, we will do some Feynman diagram computations, which are done in a similar manner in more complicated but more physically meaningful theories.

According to the rules of quantum mechanics, if the ϕ^3-theory actually existed, the correlation functions of this theory would be given by the formal expression

$$\langle \Omega | \phi(x_1) ... \phi(x_N) | \Omega \rangle = \frac{1}{Z} \int \phi(x_1) ... \phi(x_N) e^{i\mathcal{L}(\phi)} D\phi, \tag{1.7}$$

where

$$Z = \int e^{i\mathcal{L}(\phi)} D\phi \tag{1.8}$$

is the partition function. As usual, these "formulas" do not a priori make sense, as the formal expression $e^{i\mathcal{L}(\phi)}D\phi$ does not represent a measure on the space of fields. However, if $g = 0$ (the free theory), we can use (1.7) as a definition: define the integral

$$\int \phi(x_1)\dots\phi(x_N)e^{i\mathcal{L}(\phi)}D\phi \tag{1.9}$$

to be equal to $\langle\Omega|\phi(x_1)\dots\phi(x_N)|\Omega\rangle$. Further, for $g \neq 0$, we can expand (1.7) in a formal series in powers of g, and successive coefficients will be expressed as finite-dimensional integrals of expressions of the form (1.9). If we can compute these finite-dimensional integrals, we can get a natural definition of (1.7). This computation is done using the Feynman diagrammatic techniques. Unfortunately, it turns out that some of these integrals are divergent and need to be renormalized.

Remark. Strictly speaking, in ϕ^3-theory as we stated it, the 1-point function does not vanish. However, as we explained before, this problem can be removed by shifting the quantization map φ. In the language of functional integrals, it corresponds to adding an auxiliary linear term to the Lagrangian, and in the language of Feynman diagrams it corresponds to ignoring graphs with one external vertex, and all graphs which contain such graphs as subgraphs. In fact, this shifting procedure is not so trivial, since the integral representing the 1-point function is divergent for $n \geq 1$. Thus the shifting procedure requires renormalization of some graphs with 1 external vertex. Therefore, we will come back to this topic at the end of the lecture. Until then, we will assume that the 1-point function has been normalized to zero.

§1.3. Perturbative expansion of Feynman integrals

In this part of the lecture we will recall how to compute the perturbative expansion of Feynman integrals. For simplicity consider the finite-dimensional case (Lecture 5 of [I-**Axiomatics**]). Suppose we have a finite dimensional real vector space $\mathcal{S}$ with a positive definite symmetric bilinear form B. Let dv be a Lebesgue measure on $\mathcal{S}$ such that

$$\int_{\mathcal{S}} e^{-B(v,v)/2} dv = 1. \tag{1.10}$$

We want to learn to compute the integral

$$\int_{\mathcal{S}} P(v) e^{-B(v,v)/2} dv, \tag{1.11}$$

where $P : \mathcal{S} \to \mathbb{R}$ is a polynomial. This integral is a sum of integrals of the form

$$\langle f_1 \dots f_N\rangle_0 := \int_{\mathcal{S}} f_1(v)\dots f_N(v) e^{-B(v,v)/2} dv, \tag{1.12}$$

where $f_1, \dots, f_N \in \mathcal{S}^*$. It is clear that if N is odd, this integral vanishes, as the integrand is an odd function. Thus, it is enough to consider the case when $N = 2K$. In this case, the answer is given by the following formula, which is called the Wick formula.

Proposition 1.1.

$$\langle f_1 \dots f_{2N} \rangle_0 = \sum_{s \in S_{2K}/\sim} B^{-1}(f_{s(1)}, f_{s(2)}) \dots B^{-1}(f_{s(2K-1)}, f_{s(2K)}), \tag{1.13}$$

where B^{-1} is the inverse form to B on $\mathcal{S}^*$, S_{2K} is the symmetric group, and $s_1 \sim s_2$, $s_1, s_2 \in S_{2K}$ if they define the same term in (1.13).

The proof is obvious: Since both sides of (1.13) are symmetric polylinear functions, the proposition reduces to the case $f_1 = \dots = f_{2N} = f$, which is straightforward.

Remark. The set $S_{2K}/\sim$ is the set of pairings, i.e. splittings of the set $\{1, ..., 2K\}$ into pairs. Thus, terms of the r.h.s. of (1.13) are in 1-1 correspondence with pairings. In particular, the number of these terms is equal to $(2m)!/2^m m!$.

Now consider a perturbation of this situation. Let

$$Q(v) = \sum_{n \geq 1} g_m Q_m(v^{\otimes m})/m!,$$

where g_m are formal variables and $Q_m \in S^m \mathcal{S}^*$. Consider the integral

$$\langle f_1 ... f_N \rangle = \int_{\mathcal{S}} f_1(v) \dots f_N(v) e^{-B(v,v)/2 + Q(v)} dv \in \mathbb{C}[[g_1, g_2, g_3, ...]]. \tag{1.14}$$

This integral can be computed using Feynman graphs, as follows.

Let $\mathfrak{n} = \{n_1, n_2, n_3, n_4, ...\}$ be any sequence of nonnegative integers which is eventually zero. Let $G(N, \mathfrak{n})$ be the set of equivalence classes of all graphs which have N 1-valent vertices labeled by $1, ..., N$, and n_i unlabeled i-valent vertices, $i \geq 1$. The labeled vertices are called external, and the unlabeled ones internal.

Any graph $\Gamma \in G(N, \mathfrak{n})$ defines a polylinear function F_Γ of $f_1, ..., f_N$ which is defined as follows. At the 1-valent vertex of Γ labeled by i one places the vector f_i, at every m-valent vertex one places the tensor Q_m, and then takes contractions along edges using the form B^{-1} (see Lecture 5 of [I-**Axiomatics**]). Then one has

Theorem 1.2.

$$\langle f_1 ... f_N \rangle = \sum_{\mathfrak{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G(N, \mathfrak{n})} |Aut(\Gamma)|^{-1} F_\Gamma(f_1, ..., f_N), \tag{1.15}$$

where $Aut(\Gamma)$ denotes the group of automorphisms of Γ which fix the external vertices.

The proof is easy. First one observes that (1.13) is a special case of (1.15), when $\mathfrak{n} = 0$. Next, one can think of each i-valent vertex of a graph Γ as a collection of i 1-valent vertices which are situated very close to each other. Then it is clear that any graph $\Gamma \in G(N, \mathfrak{n})$ defines as many as $|\mathrm{Aut}(\Gamma)|^{-1} \prod i!^{n_i} \prod n_i!$ different pairings of such 1-valent vertices. Thus, formula (1.13) implies (1.15).

§1.4. Computation of a Feynman integral over functions on a Minkowski space

Now we will try to compute the function $w(k^2)$. Using formula (1.7), we get

$$W(x) = Z^{-1} \int T(\phi(x)\phi(0)) e^{i\mathcal{L}(\phi)} D\phi \tag{1.16}$$

Applying formula (1.15), we express the r.h.s. of (1.16) as a sum over graphs.The function $Q(\phi)$, which was considered above for the finite dimensional case, is of the form $\frac{g}{3!}Q_3(\phi)$, where Q_3 is a cubic form given by

$$Q_3(\phi) = \langle \phi(a) \otimes \phi(b) \otimes \phi(c), i\delta_{a=b=c} \rangle.$$

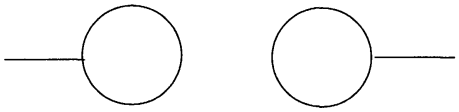
Thus, we get a sum over all graphs with two external 1-valent vertices, and a number of internal trivalent vertices.

Graphs which have components without external vertices will not occur in this sum, since we have divided by Z. So there are two remaining types of graphs: connected and disconnected. Disconnected graphs have two components, each having one external vertex.

It is easy to see that the sum over all disconnected graphs gives a contribution to the two-point function that is just the square of the one-point function, that is, it equals

$$\langle \Omega | \phi(0) | \Omega \rangle^2 = 0.$$

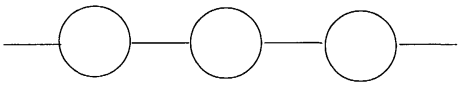
(A typical disconnected graph is pictured; it is the disjoint union of two graphs with one external 1-valent vertex, each of which would enter in computing the one-point function of ϕ.) Therefore, disconnected graphs can be ignored.



We see that the function $w(k^2)$ can be written as a sum over connected graphs with two external vertices. So, we can represent $w(k^2)$ in the form

$$w(k^2) = \sum_{j=0}^{\infty} w^{(j)}(k^2), \tag{1.17}$$

where $w^{(j)}(k^2)$ is the sum over all connected graphs which are chains of j 1-particle irreducible graphs. (Such a chain is pictured for $j = 3$.)



We have $w^{(0)}(k^2) = w_m(k^2) = \frac{i}{k^2 - m^2 + i\varepsilon}$ (the free propagator), and

$$w^{(j)}(k^2) = w^{(0)}(k^2)\left(\frac{w^{(1)}(k^2)}{w^{(0)}(k^2)}\right)^j. \tag{1.18}$$

Set

$$\Sigma(k^2) = i\frac{w^{(1)}(k^2)}{w^{(0)}(k^2)^2}.$$

($\Sigma(k^2)$ is a real function k^2 is sufficiently small or negative). Summing the geometric progression in (1.17), we get

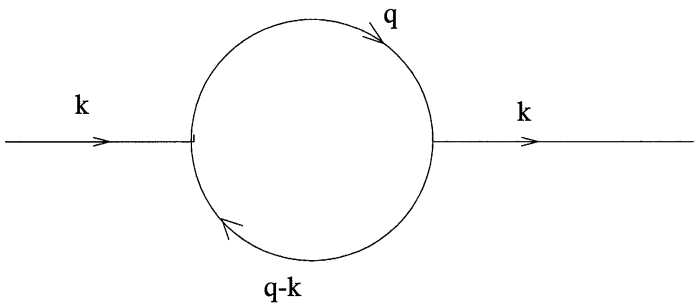
$$w(k^2) = \frac{i}{k^2 - m^2 - \Sigma(k^2) + i\varepsilon}.$$

Σ is often called the self-energy of the ϕ particle.

Thus, it remains to compute $\Sigma(k^2)$. We hope to find that $\Sigma(k^2)$ is analytic near $k^2 = m^2$, and has a cut at $k^2 \geq 4m^2$. This would confirm the expected analytic behavior of $w(k^2)$.

In our setup, the bilinear form B^{-1} from the previous section has the kernel $\frac{i}{(k_1-k_2)^2-m^2+i\varepsilon}$, and the 3-tensor Q_3 has the kernel $i\delta(k_1+k_2+k_3)$. Therefore, the term corresponding to each graph Γ in the sum for $\Sigma(k^2)$ is an integral of a rational function over the space $H^1(\Gamma,\mathbb{R})\otimes V^*$. So the number of integrations is proportional to the number of loops in the graph. Therefore, these integrals, over loop parameters, are called loop integrals.

Now we will compute the first nontrivial coefficient of the g-expansion of Σ. We have $\Sigma = g^2\Sigma_2 + O(g^3)$, where Σ_2 is the sum over all connected graphs with two trivalent vertices (there is no graph with only one trivalent vertex). It is easy to see that there is only one such graph, namely the following one:



So we must put an integration parameter $q \in V^*$ on the only loop of the graph. Thus, the function $\Sigma_2(k^2)$ can be represented as a single loop integral

$$\Sigma_2(k^2) = \frac{i}{2}\int_V \frac{d^n q/(2\pi)^n}{(q^2-m^2+i\varepsilon)((q-k)^2-m^2+i\varepsilon)}. \tag{1.19}$$

(division by 2 arises from the automorphism group of Γ, which has order 2; the subscript on Σ_2 refers to the fact that $\Sigma = g^2\Sigma_2 + O(g^3)$).

Remark 1. When we compute the group of automorphisms of a graph in ϕ^3 theory, we do not take into account the orientation of edges. The arrows are put on edges arbitrarily, in order to give sense to the momentum attached to a line; if the orientation of a line is reversed, its momentum changes sign. Because of the structure of the 3-tensor Q_3, the distribution of momenta should satisfy the condition that the sum of incoming momenta at each internal vertex should equal the sum of outgoing momenta.

Remark. It may appear that the integral in (1.19) is real, but in fact it is not. It is easy to see that for real negative k^2 the integral is imaginary, so the r.h.s. of (1.19) is real. This happens because of addition of $i\varepsilon$ in the denominator.

Now we try to compute the integral (1.19). The integrand is a fraction, where the denominator is a product of two different factors. For a general graph, the integrand will have a product of many different factors in the denominator, which causes inconvenience. There is an elementary but extremely convenient trick, which allows one to convert this integral into an integral of a function whose denominator is a power of a single factor. This trick is the "Feynman famous formula":

Proposition 1.3.

$$\frac{1}{A_1\ldots A_N} = \int_{\substack{\alpha_1+\ldots+\alpha_N=1\\ \alpha_i\geq 0}} \frac{1}{(\sum \alpha_i A_i)^N} d\sigma, \tag{1.20}$$

where $d\sigma$ is the Lebesgue measure on the simplex with volume 1.

Proof. We prove the statement by induction on N. For $N = 1$, the statement is obvious. Let $N > 1$.

Denote the r.h.s. of (1.20) by $I_N(A_1, ..., A_N)$. Then from a homogeneity argument it follows that

(1.21)
$$\int_{s\le\alpha_1+\ldots+\alpha_N\le t} \frac{1}{(\sum\alpha_i A_i)^N} d\alpha_1\wedge\ldots\wedge d\alpha_N = \frac{1}{(N-1)!} I_N(A_1,\ldots,A_N)\ln(t/s).$$

Now observe that the N-form under the integral in (1.21) has the form $d\omega$, where

$$\omega = \sum_{j=1}^{N} \frac{(-1)^j}{N(N-1)A_j(\sum\alpha_i A_i)^{N-1}} d\alpha_1\wedge\ldots\wedge\widehat{d\alpha_j}\wedge\ldots\wedge d\alpha_N. \tag{1.22}$$

Therefore, using Stokes' formula, we get

$$\begin{aligned}&\sum_{j=1}^{N} \frac{1}{N(N-1)A_j}\frac{1}{(N-2)!} I_{N-1}(A_1,\ldots,\hat{A}_j,\ldots,A_N)\ln(t/s)\\ &= \frac{1}{(N-1)!} I_N(A_1,\ldots,A_N)\ln(t/s).\end{aligned} \tag{1.23}$$

(the integrals of ω over simplices $\sum\alpha_i = s, t$ cancel each other, and only the integrals over N truncated simplices remain). Using the induction assumption, we obtain (1.20).□

We will use formula (1.20) for $N = 2$. In this case it has the form

$$\frac{1}{AB} = \int_0^1 \frac{d\alpha}{(\alpha A + (1-\alpha)B)^2}. \tag{1.24}$$

Applying this formula to (1.19), and shifting the integration variable, we obtain

$$\Sigma_2 = \frac{i}{2}\int_0^1\int_V \frac{1}{(q^2+\alpha(1-\alpha)k^2 - m^2 + i\varepsilon)^2}\frac{d^n q}{(2\pi)^n} d\alpha. \tag{1.25}$$

It is convenient now to perform a Wick rotation of the integration cycle. Write any vector $q \in V$ as (q_0, q_1), $q_0 \in \mathbb{R}$, $q_1 \in \mathbb{R}^{n-1}$, so that $q^2 = q_0^2 - q_1^2$ (where q_1^2 is the usual squared norm). Consider the 1-parameter family of integration cycles, $C(t) = \{(e^{\pi i t/2}q_0, q_1) \in V_{\mathbb{C}} | (q_0, q_1) \in V\}$, $0 \le t \le 1$. The integral (1.25) is over $C(0)$. It is clear that during the deformation $C(t)$ we do not pick up any poles of the integrand in (1.25) (since $\varepsilon > 0$). Therefore, if (1.25) converges, the integral over $C(0)$ (which is Σ_2) is equal to the integral over $C(1)$.

The cycle $C(1)$ is a real subspace of $V_{\mathbb{C}}$ which carries a natural positive definite metric, $|q|^2 = -q^2$. Thus, integral (1.25) can be written as

$$\Sigma_2(k^2) = -\frac{1}{2}\int_0^1\int_{\mathbb{R}^n} \frac{1}{(|q|^2 - \alpha(1-\alpha)k^2 + m^2)^2}\frac{d^n q}{(2\pi)^n} d\alpha. \tag{1.26}$$

(The $i\varepsilon$ is no longer necessary, as the integrand is now smooth for negative k^2).

It is now an exercise, which is recommended to the reader, to compute this integral explicitly in elementary functions. The qualitative properties of the answer can be seen even without doing so. From formula (1.26) it is obvious that

1) Σ_2 is analytic near $k^2 = m^2$.

2) Σ_2 has a cut at the set of those values of k^2 for which the integrand is singular for some q.

Since the function $\alpha(1-\alpha)$ varies between 0 and $1/4$ as α varies from 0 to 1, the cut occurs at $k^2 \geq 4m^2$. Thus, the function Σ_2 has the expected analytic behavior.

Remark. We could have considered from the very beginning a Euclidean theory. This means that V is a Euclidean space, and the Lagrangian is

$$\int (\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3)d^n x, \tag{1.27}$$

The correlation functions are

$$\langle\phi(x)\phi(0)\rangle = \int_{S(V)} \phi(x)\phi(0)e^{-\mathcal{L}(\phi)}D\phi, \tag{1.28}$$

the propagator is $\frac{1}{k^2+m^2}$, the cubic functional corresponding to a trivalent vertex is $-\frac{1}{3!}\delta(k_1+k_2+k_3)$, and the Fourier transform of (1.28) is $\frac{1}{k^2+m^2+\Sigma}$, where $-\Sigma$ is computed as a sum over 1-particle irreducible connected graphs. If we try to compute Σ_2, we will get exactly the same answer as given by (1.26), i.e. $\Sigma_2^{\text{Euclidean}}(|k|^2) = \Sigma_2^{\text{Minkowski}}(-k^2)$. This is a general phenomenon: the Wick rotated answer for a theory in the Minkowski space coincides with the answer for the corresponding Euclidean theory.

§1.5. Renormalization of divergent graphs

Unfortunately, the integrals (1.25), (1.26) are divergent for $n \geq 4$, since the integrand does not decay rapidly enough at infinity. Renormalization techniques help to give meaning to these integrals anyway.

From now until the end of the lecture we consider the case $n = 4$, and the Euclidean picture. Pragmatically, to make sense of (1.26) we could simply differentiate with respect to k^2. If we differentiate under the integral sign, we will obtain a convergent integral, and then we can integrate it back, which will define Σ_2 up to a constant. This process will give the right answer, in this particular case, but to proceed in higher orders it is very helpful to develop the following more systematic point of view.

The first step is to regularize the divergent integrals, i.e. to introduce a parameter that makes the integrals converge. There are many ways to do this, and they all lead in the end to the same theory. In the present case,[1] the most elementary form of regularization is simply to replace the propagator $\frac{1}{k^2+m^2}$ with a more rapidly decreasing propagator depending on a parameter Λ, of the form $P_\Lambda = \frac{\chi(k^2,\Lambda^2)}{k^2+m^2}$, where $\chi(k^2,\Lambda^2)$ is a smooth function with sufficiently rapid decay at $k^2 \to \infty$ for a fixed Λ, which tends to 1 at $\Lambda \to \infty$ (it is called the cutoff function). For instance, one can take $\chi(k^2,\Lambda^2) = \frac{\Lambda^{2l}}{(k^2+\Lambda^2)^l}$, $l \in \mathbb{N}$. Computing Σ_2 for this new propagator, we will obtain a convergent integral for each finite value of Λ, and the answer will depend on Λ in the following way:

$$\Sigma_2(k^2,\Lambda,m) = -A\ln(\Lambda/m) + O(1), \Lambda \to \infty, \tag{1.29}$$

where A is a positive constant. Of course, the limit of (1.29) as $\Lambda \to \infty$ (which would be the value of (1.26)) does not exist: one says that the integral is logarithmically divergent.

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[1]For a more sophisticated form of regularization, see [**I-Dim'l Regularization**] as well as [**I-Renorm Groups**, §1.6].

The parameter m that appeared in the Lagrangian (in the mass term $\frac{1}{2}m^2\phi^2$) does not necessarily coincide with the actual mass of the ϕ particle, that is, with the parameter which we also called m in our axiomatic discussion of two-point functions and representation theory at the beginning of this lecture. It will henceforth be important to distinguish them. We will write m_0 (the bare mass) for the parameter that appears in the Lagrangian, and m_R (the renormalized mass) for the actual mass of the particle; this means, as we know from the earlier discussion, that $k^2 = -m_R^2$ is the position of the pole of the propagator. At $g = 0$, m_0 and m_R are equal, but this will not be true for $g \neq 0$. Given Λ, we will adjust the bare mass m_0 of the theory in such a way that the propagator $(k^2 + m_0^2 + \Sigma(k^2, \Lambda, m_0, g))^{-1}$ for the theory with this mass and the propagator P_Λ has a pole exactly at $k^2 = -m_R^2$. That is, define $m_0(m_R, g, \Lambda) = m_R + O(g)$ by the equation

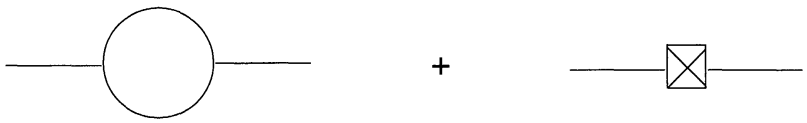
$$m_R^2 = m_0^2 + \Sigma(-m_R^2, \Lambda, m_0, g).$$

It is clear the solution of this equation (modulo g^3) has the form $m_0^2 = m_R^2 - g^2\Sigma_2(-m_R^2, \Lambda, m_0)$, so by (1.29) it has the form $m_0^2 = m_R^2 + g^2 A \ln(C\Lambda/m_R)$, where C is a constant which depends only on the cutoff function χ. It is easy to see that there exists a limit

$$\Sigma_2^R(k^2, m_R, g) = \lim_{\Lambda\to\infty} \Sigma(k^2, \Lambda, m_0(m_R, g, \Lambda), g).$$

This limit is called the renormalized Σ-function. Of course, it depends on the choice of m_R.

Let us now understand what this renormalization does to the Feynman diagram expansion. In constructing perturbation theory, we understand that the Lagrangian is exactly as before, but the mass term in the Lagrangian is now called $\frac{1}{2}m_0^2\phi^2$ and m_0 is regarded as a function of m_R, g, and Λ. In the one-loop approximation, the mass term in the new Lagrangian of the theory differs from $\frac{1}{2}m_R^2\phi^2$ (which one would expect in a classical Lagrangian) by the additional quadratic term $\frac{1}{2}Ag^2\ln(C\Lambda/m_R)\phi^2$. Perturbation theory is to be computed by expanding in powers of g, keeping m_R and Λ fixed; after making the expansion to a given order, we hope to take the limit as $\Lambda \to \infty$. Therefore, according to the rules of Feynman calculus, the g^2-term of the perturbation expansion will now be the sum of terms corresponding to two Feynman diagrams:



where the crossed square in the second diagram denotes a counterterm. At the 2-valent vertex of the second diagram, we put the tensor

$$-Ag^2\ln(C\Lambda/m_R)\delta(k_1 - k_2)$$

which comes from the g^2 part of the mass term. This gives us an extra additive constant contribution in $\Sigma_2(k^2, \Lambda, m_0)$, which compensates the divergence and ensures that there exists a limit Σ_2^R of

$$\Sigma_2(k^2, \Lambda, m_0(m_R, g, \Lambda))$$

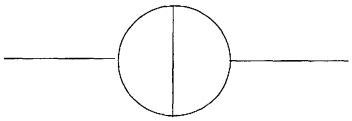
as $\Lambda \to \infty$.

Remark. We have chosen our renormalization in such a way that the function $w(k^2)$ (modulo g^3) has a pole at $k^2 = -m_R^2$. This is the reason that the constant

m_R is called the renormalized mass. Alternatively, one could have given some other definition of m_R and still carried out the renormalization procedure by adjusting m_0 while Λ is taken to infinity so as to keep m_R fixed. As long as m_R is a measureable quantity that should be well-defined for $\Lambda \to \infty$, and is sufficiently generic as to represent a "good coordinate" on the space of theories, it can be used in the renormalization procedure. A concrete example which actually is often useful in practice is to define m_R by saying that $m_R^2 = w(k=0)^{-1}$. Such an m_R is often called a "mass parameter" (rather than a mass; it is not the mass of any particle).

§1.6. Renormalization in higher orders

Now let us consider the terms of our Feynman diagrams expansion which come with a power of g higher than 2. For example, consider the following graph,



which occurs with g^4. It is easy to see that this graph defines a convergent integral if the spacetime dimension is $n = 4$. We will see that this is also the case for more complicated graphs. Namely, for $n = 4$, the more vertices a graph has, the better is the rate of convergence of the corresponding integral.

Let us analyze the situation for arbitrary n. It is convenient to introduce the following definitions.

Definition. The superficial divergence index $div(\Gamma)$ of a graph Γ is the difference of the degrees of the numerator and denominator in the integrand of the corresponding integral.

Remark. By the definition, the degree of the integration variables q_i and their differentials dq_i equals 1.

Definition. A graph is called superficially divergent if $div(\Gamma) \geq 0$ and superficially convergent if $div(\Gamma) < 0$.

Remark. Γ is called logarithmically, linearly, quadratically,... divergent if $div(\Gamma) = 0, 1, 2, ...$

Proposition 1.4. *In the ϕ^3-theory, $div(\Gamma) = (n-6)b_1 + 6 - 2E$, where b_1 is the number of loops, and E the number of external vertices.*

Proof. Let M be the number of internal edges, and N the number of external vertices. Since Γ is connected and has only trivalent internal vertices, we have $M = \frac{3N-E}{2}$, $b_1 = \frac{N+2-E}{2}$. It is easy to see that the degree of the numerator in the integral corresponding to Γ is nb_1 (b_1 loop integrations over $\mathbb{R}^n$), and the degree of the denominator is $2M$ (M quadratic factors). Thus, $div(\Gamma) = nb_1 - 2M = (n-6)b_1 + 6 - 2E$.

Proposition 1.4 implies the following concerning diagrams with at least two external lines:[2]

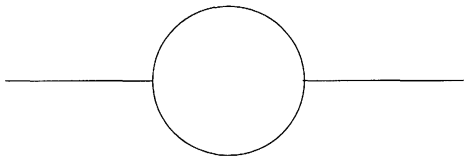
1) If $n > 6$, divergence worsens as the number of vertices grow.

[2]We noted earlier that we will temporarily simply declare the one-point function of ϕ to be zero, and not calculate the diagrams with one external lines. And, as we discussed, diagrams with no external lines are "disconnected" and cancel out when one computes correlation functions.

2) If $n = 6$, all graphs with $E = 2, 3$ are equally bad (have $div(\Gamma) = 6 - 2E$), while for $E \geq 4$ they are superficially convergent.

3) For $n \leq 3$, all graphs with $E \geq 2$ are superficially convergent.

4) For $n = 4$, the only superficially divergent graph with $E \geq 2$ is Γ_2:

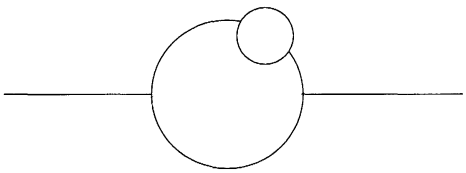


5) For $n = 5$, there are finitely many superficially divergent graphs.

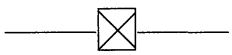
It is clear that superficial convergence of a graph is necessary but not sufficient for convergence of the corresponding integral. Indeed, a superficially convergent integral may have a sub-integral that diverges. However, one can formulate a sufficient condition for actual convergence in terms of superficial convergence. This condition is given by Weinberg's theorem.

Weinberg's Theorem. *Let Γ be a graph such that the integral of the corresponding function over any subset of the set of loops of Γ is superficially convergent. Then the integral corresponding to Γ is convergent.*

Weinberg's theorem guarantees that in the ϕ^3-theory all integrals are convergent for $n \leq 3$. Let us see what happens if $n = 4$. In this case, we have one superficially divergent graph Γ_2. Of course, there are infinitely many superficially convergent but still divergent graphs, namely, all graphs which contain Γ_2 as a subgraph, e.g.



However, we have renormalized the graph Γ_2, i.e. compensated its divergence by adding another auxiliary vertex of the form



As a result, all graphs which are divergent because they contain Γ_2 will be renormalized automatically and become convergent. That is, the divergence of each graph containing Γ_2 will be compensated by a counterterm, in which Γ_2 will be replaced by the above auxiliary graph. The fact that this ensures convergence in all orders follows from the "Strong Weinberg theorem", which states, roughly, that if all graphs at orders $\leq N$ (in g) have been renormalized, then all superficially convergent graphs at order $N + 1$ are actually convergent. Thus, after mass renormalization all correlation functions become well defined.

Remark. As we explained above, the procedure of setting the one-point function to zero in ϕ^3-theory also requires renormalization. Namely, we have two graphs with one external vertex,



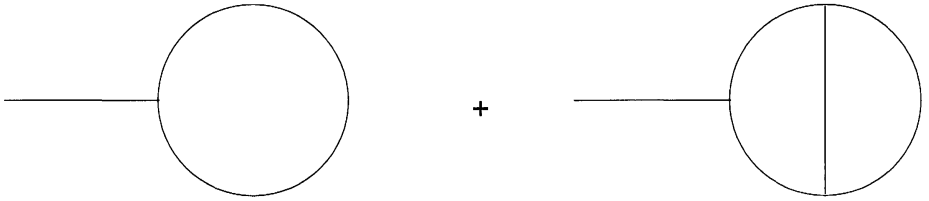
among which the first is quadratically divergent and the second logarithmically. Consider a new Lagrangian of the form

$$\mathcal{L}' = \mathcal{L} + B(m_R, g, \Lambda) \int \phi,$$

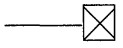
where

$$B(m_R, g, \Lambda) = a(m_R, g)\Lambda^2 + b(m_R, g)\Lambda + c(m_R, g)\ln(\Lambda/m_R) + d(m_R, g),$$

and choose the constants a, b, c, d in such a way that the limit of the one-point function for this Lagrangian, computed for the cutoff propagator at Λ, tends to 0. The constants a, b, c, d are determined by this condition uniquely, but they of course depend on the cutoff function χ. In the language of Feynman graphs, this corresponds to compensating the divergence in the sum



by adding a third summand



$B \cdot \delta(k)$

where at the vertex we put the linear functional $B \cdot \delta(k)$. This reduces the problem of renormalization of the ϕ^3 theory to renormalization of the graph Γ_2, which was done above.

References and cross references

A more detailed discussion of the basics of field theory is in [I-**Axiomatics**]. Other examples of Feynman diagram computations are in Problems 12, 13, and 18 of [I-**Homework**]. For basic material on perturbation theory, consult any textbook on quantum field theory; some of the most recent are

S. Weinberg, *The Quantum Theory Of Fields*, Vols. 1 and 2 (Cambridge University Press, 1995).

M. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory* (Addison-Wesley, 1995).

LECTURE 2
Perturbative Renormalizability of Field Theories

§2.1. Renormalizability of quantum field theories

In Lecture 1 we considered the ϕ^3 theory in dimension n and found that its renormalizability (i.e. how many graphs are divergent, and how badly divergent they are) depends on whether $n < 6$, $n = 6$, or $n > 6$. Now we will see how to find that the critical value of n in the ϕ^3 theory is 6 without considering any graphs at all.

In order to find the critical value of n, we will define and compute the homogeneity degree of all terms in the Lagrangian. We recall that the classical Lagrangian is

$$\int (\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3)d^n x, \tag{2.1}$$

(we are considering the Euclidean theory; in the Minkowski version, the story is the same). Consider the dilation $x \to t^{-1}x$, $t \in \mathbb{R}$, $x \in V$. Under this transformation $d^n x \to t^{-n}d^n x$, $\nabla \to t\nabla$. If we want the integral (2.1) to be invariant under this dilation, we must impose the following three transformation laws: $(\nabla\phi)^2 \to t^n(\nabla\phi)^2$, $m^2\phi^2 \to t^n m^2\phi^2$, $g\phi^3 \to t^n g\phi^3$. From the first law we get $\phi \to t^{\frac{n-2}{2}}\phi$, from the second $m \to tm$, and from the third $g \to t^{\frac{6-n}{2}}g$. In the future we will write these scaling laws like this:

$$[x] = -1, [d^n x] = -n, [\nabla] = 1, [\phi] = \frac{n-2}{2}, [m] = 1, [g] = \frac{6-n}{2}. \tag{2.2}$$

In other words, any homogeneous quantity a scales as $a \to t^{[a]}a$ under the dilation, and the number $[a]$ is called the dimension of a.

Remark. It is not difficult to define the notion of dimension completely formally (using the tensor calculus on the tangent bundle of V), but it is more illuminating to illustrate it on examples.

We see that the number 6 appears in the scaling law for g. More precisely, we see that the behavior of the ϕ^3 theory is determined by whether $[g]$ is positive, zero, or negative. Namely, we have three cases.

1. $[g] > 0$ ($n < 6$): there are finitely many superficially divergent graphs (by graphs we always mean connected graphs).

Notes by Pavel Etingof and David Kazhdan

2. $[g] = 0$ ($n = 6$): the number of superficially divergent graphs is infinite, but $div(\Gamma)$ is bounded from above.

3. $[g] > 0$ ($n > 6$): there are infinitely many graphs with any number of external vertices and arbitrarily high $div(\Gamma)$.

Now we will explain why this method works, and prove that it can be used to compute the critical value of the dimension of the spacetime in a general field theory. In general, we can consider the following setup. Suppose we have a quantum field theory with fields $\phi_1, \dots, \phi_N$, and the Lagrangian

$$\mathcal{L} = \int \left(\sum_i Q_i(\phi_i) + \sum_k g_k I_k(\phi_1, \dots, \phi_N) \right) d^n x, \tag{2.3}$$

where Q_i is a free (quadratic) part, and I_k are interaction (coupling) terms (differential monomials in fields, cubic and higher). Each interaction term comes with a small parameter g_k, which is called the coupling constant.

Definition. An interaction I_k is called subcritical if $[g_k] > 0$, and critical if $[g_k] = 0$. [1] A field theory is called superrenormalizable if all interaction terms in the Lagrangian are subcritical, and is called renormalizable, or critical, if all interaction terms are critical or subcritical, but not all of them are subcritical. Otherwise, the theory is called unrenormalizable.

For example, the ϕ^3 theory is superrenormalizable for $n < 6$, renormalizable for $n = 6$, and unrenormalizable for $n > 6$.

We will assume that our theory is a perturbation of a free theory (for the definition of a free theory, see Lecture 3 of [I-**Axiomatics**]). In a free theory, one can easily see that the dimension of bosonic fields is $\frac{n-2}{2}$, and of fermionic fields is $\frac{n-1}{2}$. Since dimensions of fields are determined from the quadratic part of the Lagrangian, these dimensions will be the same in the perturbed (classical) theory as well.

We will also assume that $n \geq 2$. (In the quantum mechanical case $n = 1$ renormalization theory is not necessary). Then $[\phi_i] \geq 0$.

Theorem 2.1. (i) *If a theory is superrenormalizable, there are finitely many superficially divergent graphs in its Feynman diagram expansion.*

(ii) *If a theory is renormalizable then the number of superficially divergent graphs is infinite, but $div(\Gamma)$ is bounded from above.*

(iii) *If a theory is unrenormalizable then there are infinitely many graphs with any number of external vertices and arbitrarily high $div(\Gamma)$.*

Proof. Let Γ be a graph in the Feynman diagram expansion of our theory. Types of internal vertices of such a graph correspond to interaction terms in the Lagrangian, and types of its external vertices and edges correspond to fields ϕ_i. Let e_i be the number of external vertices of type ϕ_i, and v_k be the number of internal vertices of type I_k. Then it is easy to show that

$$div(\Gamma) = n - \sum_i e_i[\phi_i] - \sum_k v_k[g_k]. \tag{2.4}$$

Statements (i)-(iii) of the theorem follows immediately from (2.4).

[1] Critical interactions are often called renormalizable ones, and subcritical interactions are called superrenormalizable. I am trying to avoid over-using the word "renormalizable," especially until we become thoroughly familiar with the concepts.

Definition. A theory of the form (2.3) is called classically scale invariant if the n-form under the integral in the Lagrangian is invariant under dilations.

For example, the theory of a free scalar field is scale invariant iff it is massless.

It is clear that a massless theory is scale invariant if and only if it is purely critical, i.e. $[g_k] = 0$ for all k.

Remark. In practice, scale invariant classical theories are always conformal. Indeed, the Lagrangian is always written naturally in terms of the metric on the spacetime, and depends only on the fields and their first derivatives. This implies that a scale invariant Lagrangian has to be invariant under conformal changes of metric. Quantum mechanically it is suspected (but has not been proved in general) that a scale invariant theory that obeys all the usual axioms is also conformally invariant.

§2.2. Critical dimensions of some field theories

Now we will compute the critical dimension for several important field theories.

Example 1. Sigma-models

Let M be a Riemannian manifold. The Lagrangian of the sigma-model on $\mathbb{R}^n$ with the target space M is

$$\mathcal{L}(\phi) = \int d^n x g_{ij}(\phi) \nabla\phi^i \cdot \nabla\phi^j. \tag{2.5}$$

Since this theory is conformal in two dimensions, 2 must be the critical dimension. Let us show by a direct computation that this theory is not renormalizable for $n > 2$.

Let ϕ_i be coordinates on M near some point. If the metric g_{ij} is constant in these coordinates, the theory is free. Consider a non-constant metric of the form

$$g_{ij}(\phi) = \delta_{ij} + a_{ijk}\phi^k + r_{ijkl}\phi^k\phi^l + ... \tag{2.6}$$

(If we chose normal coordinates, we could get rid of a, but not of r, as r in normal coordinates is the Riemann curvature tensor). Substituting this into the Lagrangian, we find that $[r] = 2-n$. This shows that sigma-model is not renormalizable beyond $n \geq 2$, unless the metric is flat.

Example 2. Gravity

In the theory of gravity, from the standpoint of perturbation theory, the spacetime is the space $\mathbb{R}^n$ with a metric of the form $g_{ij} = \delta_{ij} + h_{ij}\sqrt{G}$, where G is Newton's constant (for our present purposes, we treat it as any other interaction constant). The Lagrangian of the theory is

$$\mathcal{L}(g) = \frac{1}{16\pi G}\int R(g) d^n x, \tag{2.7}$$

where $R(g)$ is the scalar curvature of the metric. In terms of h, this Lagrangian can be rewritten schematically as

$$\mathcal{L} = \frac{1}{16\pi}\int d^n x((\nabla h)^2 + \sqrt{G}h(\nabla h)^2 + ...), \tag{2.8}$$

so we get $[h] = \frac{n-2}{2}$, $[G] = 2 - n$. Thus, as in the previous example, the theory is unrenormalizable for $n > 2$ and critical for $n = 2$. In particular, for $n = 4$ – General

Relativity in ordinary four-dimensional spacetime – we have difficulties, which in fact has been the most important single motivation for developing string theory.

Remark. In the last equation, $h(\nabla h)^2$ stands for an expression which is linear in h and quadratic in ∇h. It is straightforward to compute what it is exactly, but it does not matter to us, since we are only interested in the dimension. So we will content ourselves with this schematic description.

Example 3. Gauge theory

In gauge theory in Euclidean space, the fields are connections in a fixed principal G-bundle on the space $\mathbb{R}^n$, where G is a compact Lie group. The Lagrangian has the form

$$\mathcal{L}(\tilde{A}) = \frac{1}{2e^2}\int \mathrm{Tr}(F_{\tilde{A}} \wedge *F_{\tilde{A}}), \tag{2.9}$$

where $F_{\tilde{A}}$ is the curvature of the connection $\tilde{A}$, and Tr is an invariant non-degenerate bilinear form on the Lie algebra $\mathfrak{g}$ of G.

In the computation of dimension, we will assume that our G-bundle is trivial, so a connection is represented by a 1-form A: $\tilde{A} = d + A$. Then $F = dA + \frac{1}{2}A \wedge A$.

Consider the field $B = A/e$. In terms of B, the Lagrangian takes the form

$$\mathcal{L} = \int ((\nabla B)^2 + eB^2\nabla B + e^2B^4)d^nx$$

We have $[B] = \frac{n-2}{2}$, so $[B^2\nabla B] = \frac{3n}{2} - 2$, and $[e] = \frac{4-n}{2}$. Thus, if the group G is noncommutative, the theory is superrenormalizable for $n < 4$, renormalizable for $n = 4$, and unrenormalizable for $n > 4$. In $n = 4$, the theory is conformal. If G is commutative, the theory is free for any n.

Remark. Dimensions of fields in our computation agree with geometric dimensions. Indeed, in geometry, since $d \to td$ under the dilation $x \to t^{-1}x$, we have $[d] = 1$, so we must have $[A] = 1$. This coincides with our result: $[A] = [Be] = [B] + [e] = \frac{n-2}{2} + \frac{4-n}{2} = 1$.

Example 4. Gauge theory with a scalar bosonic field or with a fermionic field

Consider the setting of Example 3, and let V be a finite-dimensional representation of G. Let ϕ be a section of the corresponding vector bundle on $\mathbb{R}^n$ (associated with the above G-bundle). The Lagrangian of the gauge theory with ϕ is

$$\mathcal{L}'(\tilde{A}, \phi) = \mathcal{L}(\tilde{A}) + \frac{1}{2}\int (\nabla_{\tilde{A}}\phi)^2 d^nx, \tag{2.10}$$

where $\mathcal{L}(\tilde{A})$ is the Lagrangian (2.9). Writing $\tilde{A}$ in the form $\tilde{A} = d + A$, $A = eB$, we get

$$\mathcal{L}' = \int ((\nabla B)^2 + eB^2\nabla B + e^2B^4 + \frac{1}{2}(\nabla\phi)^2 + e(\nabla\phi, B\phi) + \frac{e^2}{2}(B\phi)^2)d^nx$$

For this formula we get $[B] = [\phi] = \frac{n-2}{2}$, and (even when G is commutative!) $[e] = \frac{4-n}{2}$. Thus, the theory is superrenormalizable in $n < 4$, renormalizable in $n = 4$, and nonrenormalizable for $n > 4$ (regardless of the commutativity of G).

The same answer applies if we have a fermionic field. Let ψ be a section of $V \otimes S$, where S is the spin bundle over the spacetime. The Lagrangian of the gauge theory with ψ is

$$\mathcal{L}''(\tilde{A}, \psi) = \mathcal{L}(\tilde{A}) + \int (\psi, D_{\tilde{A}}\psi) d^n x, \tag{2.11}$$

where $D_{\tilde{A}}$ is (i times) the Dirac operator along the connection $\tilde{A}$. The critical dimension in this theory, as before, is 4, for any nontrivial compact group G.

Remark. If $G = U(1)$ and V is the standard 1-dimensional representation of G, this theory is called quantum electrodynamics (QED). It was the first quantum field theory to be discovered (since both quantum mechanics and special relativity were discovered in studying electromagnetic interactions!), and renormalization theory was developed around 1950 to understand it – in the critical case $n = 4$, of course.

Example 5. Theory of a scalar bosonic field

Consider the Lagrangian

$$\mathcal{L}(\phi) = \int (\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + Q(\phi)) d^n x,$$

where $Q(\phi) = \sum g_k \phi^k$. We already considered this type of Lagrangian in Lecture 1. In particular, the ϕ^3-theory is a special case of this situation, when Q is a cubic polynomial. We have $[g_k] = n - k\frac{n-2}{2}$. Thus, for $n = 2$ all terms in the Taylor expansion of $Q(\phi)$ are subcritical. For $n = 3$, the term ϕ^k is subcritical for $k < 6$, critical for $k = 6$, and unrenormalizable for $k > 6$. For $n = 4$, the term ϕ^k is subcritical for $k < 4$, critical for $k = 4$, and unrenormalizable for $k > 4$. For $n = 5, 6$, the terms ϕ^k, $k \geq 4$ are unrenormalizable, and the term ϕ^3 is subcritical for $n = 5$ and critical for $n = 6$.

In particular, the following theories are critical: the ϕ^3 theory in 6 dimensions, the ϕ^4-theory in 4 dimensions, and the ϕ^6 theory in 3 dimensions.

Example 6. Yukawa interaction

Consider a theory with a scalar bosonic field ϕ and a fermionic field ψ. We will consider the Lagrangian

$$\mathcal{L}(\phi, \psi) = \int ((\nabla\phi)^2 + (\psi, D\psi) + g\phi[\psi, \psi]) d^n x \tag{2.12}$$

(ψ takes values in a vector bundle W which is a direct sum of several copies of the spin bundle; $(\ ,\ )$,$[\ ,\ ]$ are a symmetric and a skew-symmetric form on W which are invariant under gauge transformations). The cubic term $\phi[\psi, \psi]$ is called the Yukawa interaction. Let us compute the dimension of its coefficient g.

We have $[\phi] = \frac{n-2}{2}$, $[\psi] = \frac{n-1}{2}$, so $[g] = \frac{4-n}{2}$. So this theory is critical in dimension 4, superrenormalizable in dimension < 4, and unrenormalizable in dimension > 4.

We observe that in dimension 4 all interactions except ϕ^3, ϕ^4 and $\phi[\psi, \psi]$ are "bad" (unrenormalizable).

Example 7. Standard model

Let us now try to write down the most general renormalizable theory that lives in our 4-dimensional physical world. According to the above examples, we cannot include gravity or the σ-model, but we can include connections, bosons with terms up to degree 4, and fermions with Yukawa interaction. If we only take

these fields, these are the only renormalizable terms we can write. Thus the most general Lagrangian we can write in dimension 4 giving a renormalizable theory is

$$\mathcal{L}(A,\phi,\psi) = \int (e^{-2}F_A^2 + (\psi, D\psi) + (\nabla\phi)^2 + g_1\phi^4 + g_2\phi\psi^2 + \text{lower terms})d^4x. \tag{2.13}$$

The Standard Model is a theory which belongs to this family, with the group G being a form of $SU(3) \times SU(2) \times U(1)$. The scalars transform trivially under $SU(3)$ and in the two-dimensional representation of $SU(2)$ (with a certain character of $U(1)$), and the fermions are in a representation of $SU(3) \times SU(2) \times U(1)$ that at first sight looks rather unwieldy.

§2.3. Perturbative renormalization of critical theories

From now on we will consider only critical theories. As a model example we can consider ϕ^4-theory in 4 dimensions, which has one critical interaction ϕ^4, or its extension containing fermions, which has an additional critical interaction $\phi\psi^2$ (the Yukawa interaction).

Consider the Lagrangian of ϕ^4-theory:

$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right)d^n x. \tag{2.14}$$

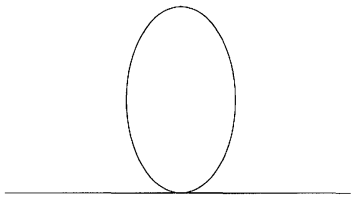
As before, we want to study the N-point Schwinger function $\langle\phi(x_1)...\phi(x_N)\rangle$ (it is enough to consider only even N, as the Schwinger function vanishes for odd N because the theory has a symmetry under $\phi \to -\phi$). As usual, it is more convenient to consider the Fourier transform. This Fourier transform has the form $G_N(k_1,...,k_N)\delta(k_1+...+k_N)$, where G_N is a function on the hyperplane $k_1+...+k_N = 0$.

Consider the Feynman diagram expansion of the function G_N. In ϕ^4 theory we have to sum over graphs whose internal vertices have 4 edges. As usual, we can restrict ourselves to connected graphs, as the sum over disconnected graphs can be expressed via Schwinger functions with fewer insertion points. In fact, as shown in Lecture 5 of [**I-Axiomatics**], we can always restrict to 1-particle irreducible graphs (i.e. those which cannot be split by cutting one edge). So from now on we only consider connected, 1-particle irreducible graphs.

From formula (2.4) we get that the superficial divergence index of any graph equals $4 - E$, where E is the number of external edges. Thus, any graph with 2 external edges is quadratically divergent, any graph with 4 external edges is logarithmically divergent, and any graph with > 4 external edges is superficially convergent.

Now we will explain how to renormalize all N-point functions at all orders in g.

First of all, we can exclude graphs with 2 external edges which connect to the same internal vertex, e.g.



and all those which contain such a loop-like graph as a subgraph. Indeed, these graphs produce a constant function of k^2, so they can be removed by renormalization of mass. More precisely, there exists a function $P(\Lambda, m, g) = gP_1(\Lambda, m) + g^2P_2(\Lambda, m) + ...$, such that the sum of terms corresponding to all graphs for the theory with mass m^2, computed with the cutoff propagator (see Lecture 1) equals the sum of terms corresponding only to graphs without loop-like subgraphs, but with mass $M(m, \Lambda)$ such that $M^2 = m^2 + P(\Lambda, g, m)$. So we can assume from the beginning that we have a theory with mass $M(m, \Lambda)$ and not worry about loop-like graphs.

Now we have no divergent graphs with one internal vertex, so we have no corrections to make in the first order in g. Let us look at the second order in g. In this case we have the following bad graphs:



Denote the first graph by Γ_2 and the second by Γ_4.

Thus, in order g^2 we have a problem in the 2-point and the 4-point functions. The problem in the 2-point function is created by Γ_2. The graph Γ_2 diverges quadratically. Therefore, the term corresponding to Γ_2 computed using the cutoff propagator is a function of k^2, Λ of the form $g^2\Sigma_2(k^2, \Lambda)$, where

$$\Sigma_2(k^2, \Lambda) = Ak^2 \ln(\Lambda/\mu) + B(\Lambda) + O(1), B(\Lambda) = B_0\Lambda^2(1 + o(1)), \Lambda \to +\infty \tag{2.15}$$

This asymptotics follows from the fact that the first derivative of (2.15) with respect to k^2 is logarithmically divergent, and the second one is convergent.

Another problem we have is in the 4-point function, created by the graph Γ_4. This graph is logarithmically divergent. This means that, if we compute the term corresponding to this graph using the cutoff propagator, we will get a function of k_i, Λ of the form $g^2\Theta_2(k_1, k_2, k_3, \Lambda)$, where

$$\Theta_2(k_1, k_2, k_3, \Lambda) = C \ln(\Lambda/\mu) + O(1), \Lambda \to +\infty \tag{2.16}$$

This asymptotics follows from the fact that the derivative of the integral corresponding to Γ_4 with respect to k_i is convergent.

In order to renormalize the 2- and 4- point functions in order g^2, we choose renormalized functions $\Sigma_2^R(k^2)$, $\Theta_2^R(k_1, k_2, k_3)$. Here Σ_2^R is an arbitrary function of k^2 whose second derivative is given by the (convergent!) integral obtained by applying $\left(\frac{d}{d(k^2)}\right)^2$ to the quadratically divergent integral corresponding to Γ_2. Analogously, Θ_2^R is an arbitrary function of k_1, k_2, k_3 whose derivatives are given by the convergent integrals obtained by differentiating the logarithmically divergent integral corresponding to Γ_4. It is clear that the function Σ_2^R is defined uniquely up to addition of a function of k^2 of the form $ak^2 + b$, and the function Θ_2^R is defined uniquely up to addition of a constant.

Just as in the example we considered in Lecture 1, the somewhat *ad hoc* renormalization procedure of the last paragraph gives the right answer to this order, but to develop a systematic approach it helps to formulate the "subtractions" made in the last paragraph as resulting from the presence of additional terms in the Lagrangian, which are called "counterterms." In the present approximation, we make

second order corrections to the coefficients of $(\nabla\phi)^2$, ϕ^2 and ϕ^4 in the Lagrangian. Namely, we will consider a new Lagrangian of the form

$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{\alpha(\Lambda)}{2}(\nabla\phi)^2 + \frac{\beta(\Lambda)}{2}\phi^2 + \frac{\gamma(\Lambda)}{4!}\phi^4 \right) d^n x, \tag{2.17}$$

α, β, and γ may depend on m – this point will be analyzed later – but this is not indicated in the notation. Let $\Sigma_2'(k^2, \Lambda, \alpha, \beta, \gamma)$, $\Theta_2'(k_1, k_2, k_3, \Lambda, \alpha, \beta, \gamma)$ be the functions Σ_2, Θ_2 for this Lagrangian and the cutoff propagator. We will choose the functions α, β, γ in such a way that

$$\begin{aligned} \lim_{\Lambda\to\infty} \Sigma_2'(k^2, \Lambda, m, \alpha(\Lambda), \beta(\Lambda), \gamma(\Lambda)) &= \Sigma_2^R(k^2), \\ \lim_{\Lambda\to\infty} \Theta_2'(k_1, k_2, k_3, \Lambda, m, \alpha(\Lambda), \beta(\Lambda), \gamma(\Lambda)) &= \Theta_2^R(k_1, k_2, k_3), \end{aligned} \tag{2.18}$$

with Σ_2^R and Θ_2^R as in the previous paragraph. As we are working modulo g^3, we can choose α, β, γ in the form $\alpha = g^2\alpha_2$, $\beta = g^2\beta_2$, $\gamma = g^2\gamma_2$, where $\alpha_2, \beta_2, \gamma_2$ are independent of g. It is easy to check that in order for (2.18) to hold, the functions $\alpha_2, \beta_2, \gamma_2$ should have the following asymptotics:

$$\begin{aligned} \alpha_2 = A\ln(\Lambda/m) + D_1 + o(1), \beta_2 = -B(\Lambda) + D_2 + o(1), \\ \gamma_2 = C\ln(\Lambda/m) + D + o(1), \Lambda \to +\infty, \end{aligned} \tag{2.19}$$

where D_1, D_2 depend on the choice of Σ_2^R, and D depends on the choice of Θ_2^R. Of course, there are many ways to choose such functions, but they are unique up to adding terms $o(1)$, $\Lambda \to \infty$.

Thus, we have renormalized the graphs Γ_2, Γ_4. This removes the divergences in all correlation functions modulo g^3. Thus, all correlation functions of our theory are now defined modulo g^3.

Now we proceed inductively in the order of g. Suppose we have removed divergences and defined all correlation functions modulo g^K. Consider the $2N$-point function (for the deformed Lagrangian and the cutoff propagator) modulo g^{K+1}:

$$F_{2N} = \sum_{j=0}^{K-1} g^j F_{2N,j}^R + g^K F_{2N,K} \tag{2.20}$$

(the superscript R means that the corresponding coefficient has already been renormalized). The term $F_{2N,K}$ is represented by the sum over all graphs with K internal vertices. This sum has superficial divergence index $4 - 2N$. Therefore, the second derivative of F_2 by k^2, the first partial derivatives of F_4, and F_{2N}, $N \geq 3$, are superficially convergent. The crucial fact for renormalization theory, which follows from the "Strong Weinberg Theorem" (see Lecture 1), is

Proposition 2.2. *There exist finite limits, as* $\Lambda \to \infty$, *of the functions* $F_{2N,K}(k_1, \ldots, k_{2N-1}, \Lambda)$, $\nabla_k F_{4,K}(k_1, k_2, k_3, \Lambda)$, *and* $\left(\frac{d}{d(k^2)}\right)^2 F_{2,K}(k^2, \Lambda)$, $N \geq 3$.

Remark. For ϕ^4 theory, this proposition holds for the contribution of each individual graph,[2] but in general (for example, for theories with gauge fields) this is not the case: the sum over all graphs may have a meaning while each individual graph

[2]To be more precise, the proposition holds for the contribution of any graph that is made from the vertices in the classical Lagrangian together with additional graphs that arise by replacing divergent subdiagrams of the given graph with vertices that arise from the counterterms that were used to renormalize the subdiagrams.

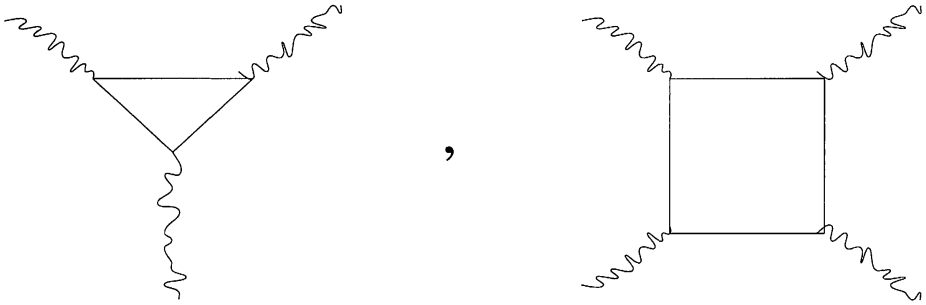
does not. However, an analogue of Proposition 2.2 for the sum over all graphs holds in any renormalizable theory.

Proposition 2.2 allows us to fulfill the induction step. It shows that the function F_2 (in the limit) is defined up to adding $ak^2 + b$, the function F_4 is defined up to adding a constant, and F_{2N}, $N \geq 3$, is defined uniquely. So one can choose renormalized functions $F^R_{2N,K}$ and make corrections in the Lagrangian, $\alpha \to \alpha + g^K\alpha_K$, $\beta \to \beta + g^K\beta_K$, $\gamma \to \gamma + g^K\gamma_K$, to compensate the divergence in $F_{2N,K}$ and obtain $F^R_{2N,K}$ instead of it. This procedure is completely analogous to the one for order g^2. In this way we will complete the renormalization in order K.

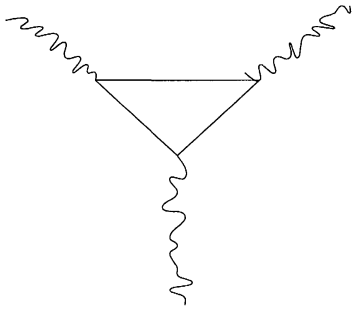
Remark 1. At every step of our renormalization procedure we had to choose 3 constants of integration. This may create an impression that we get a family of theories parametrized by 3 infinite sequences of constants. However, it is easy to see that in fact we get a family of theories parametrized by only 3 constants. This means that any 4 invariants attached to the theory (for example, values of the 2-point function at 4 points in spacetime) are linked by a universal functional relation.

Remark 2. Even if in the original theory certain critical or subcritical interactions were not present, they may appear in the process of renormalization. In general, renormalization brings in all missing critical and subcritical terms, unless there is a symmetry which prevents it from doing so. Let us demonstrate this by a few examples.

Example 1. In the process of renormalization of the Lagrangian (2.12) in 4 dimensions we will be forced to introduce the subcritical term ϕ^3 and the critical term ϕ^4, in order to remove logarithmic divergence in the graphs



where wavy lines correspond to bosons and straight ones to fermions. However, the critical term $\phi^2\, d\phi$ will not appear, since there is no Poincaré invariant expression of this form. In terms of graphs, this means that the graph



whose superficial divergence is linear, in fact diverges only logarithmically, because of cancellations in the integrand caused by Poincaré symmetry; so there is no linear divergence to compensate and hence no need for $\phi^2\, d\phi$ to appear.

Example 2. In ϕ^4 theory in 4 dimensions, the subcritical term ϕ^3 does not appear in renormalization, since it is not preserved by the symmetry $\phi \to -\phi$ of the original theory. In the language of graphs, this is clear: there are no graphs with 3 external edges, so there is no divergence to compensate by ϕ^3.

In a general field theory, every type (in terms of external edges) of a superficially divergent graph corresponds to a number of critical and subcritical terms in the Lagrangian, which should be renormalized in order to compensate the divergence in the corresponding graph. More precisely, divergent terms which are quadratic in k correspond to terms in the Lagrangian which have two derivatives by x, linear terms in k correspond to terms with one derivative, and divergences independent of k correspond to terms without derivative.

Remark. It follows from formula (2.4) that in a renormalizable theory, all divergences in $N \geq 2$-point correlation function are no worse than quadratic. So the coefficient of k^2 in the 2-point function is divergent at most logarithmically. Therefore, if the quadratic forms Q_i have to be renormalized (like in ϕ^4 theory), they will be multiplied by coefficients which depend on the cutoff parameter Λ at worst logarithmically. This shows that the dimension of Q_i and hence of ϕ_i survives renormalization to all finite orders in the asymptotic expansion.

References and cross references

A discussion of renormalizability from the renormalization group point of view is in Lecture 1 of [I-**Renorm Groups**]. On a proof of renormalizability which does not use graph techniques see Lecture 5 of [I-**Renorm Groups**]. For basic material on renormalization, the reader may also wish to consult standard textbooks on quantum field theory, such as those cited at the end of Lecture 1.

LECTURE 3

Composite Operators and Operator Product Expansion

In this lecture we will discuss composite operators, their renormalization and operator product expansion (OPE).

§3.1. Local functionals in a classical field theory

Consider an n-dimensional classical field theory with spacetime V and a Lagrangian $\mathcal{L}$ of the form (2.3), with fields $\phi_1, ..., \phi_N$. Let X be the space of classical solutions for this theory. We want to consider functions on X called local functionals, which are defined as follows.

Let $x \in V$ be a point in the spacetime.

Definition. A local functional at x is a function of the fields and finitely many derivatives of the fields, evaluated at x.

Example. In the theory of a scalar bosonic field,

$$\phi^l(x), \phi'(x)^2, \phi^2(x)\phi'(x)^2$$

are local functionals (here l is an integer and ϕ' denotes a derivative of ϕ with respect to one of the coordinates), but $\phi(x) + \phi(2x)$ is not.

From the previous lecture it is clear how to define dimension of a local functional. We want to consider only homogeneous functionals of finite dimension and their finite linear combinations. Therefore, if a field has positive dimension (which is always the case if $n > 2$), we only consider polynomial functionals of this field. However, in 2-dimensional theories, where bosonic fields are dimensionless, it is reasonable and useful to consider more general functions of them (for example, the Lagrangians of Toda theories contain expressions of the form e^{ϕ}).

If $n > 2$, the space of all functionals of a given dimension is finite-dimensional, but if $n = 2$, it is infinite-dimensional.

Since elements of X satisfy the classical field equations, the same local functional can be written in different ways. For example, in ϕ^4-theory the classical field equation is

$$\Delta\phi = m^2\phi + \frac{g}{3!}\phi^3, \tag{3.1}$$

Notes by Pavel Etingof and David Kazhdan

so the left and the right hand sides of (3.1) evaluated at $x \in V$ represent the same local functionals of ϕ. Thus, local functionals are all possible differential expressions in ϕ modulo the classical field equations.

Remark 1. One should be careful to distinguish between two different notions of field dimension which arise in field theory. The first is "the engineering dimension" and says in what units the field is measured (if the units are centimeters^{-d} then the engineering dimension is d). The second is "the scaling dimension", which is the more significant concept for our purposes and is what we generally mean by "dimension." These two dimensions are not always the same. For example, in the theory of one field ϕ defined by (3.1), the engineering dimension of the field $m^2\phi$ is 3 (it is measured in cm^{-3}), while the scaling dimension is the same as that of ϕ, i.e. 1.

All meaningful expressions and equations in physics are homogeneous with respect to the engineering dimension. In particular, this is true for the field equations in field theory (e.g. (3.1)), so the engineering dimension defines a grading on the space of local functionals. On the other hand, whenever a field theory is not scale invariant, its field equations are not homogeneous with respect to the scaling dimension, so scaling dimension defines a filtration, rather than a grading, on the space of local functionals. This means that the space of all local functionals is a union of subspaces of functionals of dimension $\leq d$, and a functional is said to be of dimension d if it is of dimension $\leq d$ but not $\leq d-1$. On the other hand, if the theory happens to be scale invariant, scaling dimension does define a grading on the space of local functionals.

In this lecture, we will not use engineering dimension, and the word "dimension" will always mean scaling dimension.

Remark 2. From a local functional $\mathcal{O}$ one can obtain more general functionals on X of the form $\int \mathcal{O}(\phi, x)d\mu(x)$, where $d\mu(x)$ is a (generalized) density on V. Using this operation, one can obtain from any local functional all derivatives of thus functional (it is enough to take for $d\mu(x)$ all possible densities supported at x). Therefore, without loss of information we could consider local functionals modulo the image of derivatives. However, for the purposes of this lecture this is not necessary.

§3.2. Quantization of local functionals in a free theory

We have seen (see Chapter 2 of [I-**Classical Fields**]) that the space X of classical solutions of a meaningful classical field theory always carries a natural closed 2-form Ω. If this form is nondegenerate, X is a symplectic manifold. In this case, suitable functions on X form a Poisson algebra.

In the quantum theory, the space X should be quantized, and the Poisson algebra of functions on X should become the algebra of operators (observables) in some Hilbert space of states. In particular, we should be able to assign an operator to every local functional.

If V is a Minkowski space, and the field theory satisfies the Wightman axioms (for example, the free theory), the Hilbert space of states $\mathcal{H}$ is constructed as described in Lecture 1 of [I-**Axiomatics**]. In this case, Wightman fields $\phi(x)$ are distributions on V with values in the space of operators on the subspace $\mathcal{D} \subset \mathcal{H}$

of multiparticle states. This means, for any Schwartz function f on V there is an honest operator $\varphi(f)$ on $\mathcal{D}$.

However, we would like to have more general operators of the form $\phi^N(x)$, $(\nabla\phi)^2(x)$, etc. which correspond to local functionals in the classical theory. That is, for any Schwarz function f we want to have operators $\varphi^N(f)$, $(\nabla\varphi)^2(f)$, etc.

Unfortunately, such operators are not automatically defined. For example, even $\varphi^2(f)$ does not, in general, make sense. Indeed, let $\delta_\varepsilon = e^{-|x|^2/\varepsilon}(2\pi/\varepsilon)^{n/2}$ be the smooth approximation to the δ-function, and let $\varphi \otimes \varphi$ be the operator-valued distribution on $V \times V$ given by $\varphi\otimes\varphi(f_1 \otimes f_2) = \varphi(f_1)\varphi(f_2)$. The natural definition of $\varphi^2(f)$ would be that $\varphi^2(f)$ is the limit, as $\varepsilon \to 0$, of the operator $\varphi^2_\varepsilon(f) := \varphi \otimes \varphi(f(x)\delta_\varepsilon(x-y))$ (in the sense of convergence of matrix elements). However, it is easy to see that this limit does not usually exist, because of singularities of the Wightman functions on the diagonals.

One way to deal with this problem is to say that an "operator" $\mathcal{O}$ is just a collection of its matrix elements, i.e. a collection of correlation functions

$$\langle\phi(y_1)\dots\phi(y_l)|\mathcal{O}(x)|\phi(z_1)\dots\phi(z_r)\rangle.$$

(Here the ϕ's are used to prepare rather general initial and final states, and we consider the matrix element of $\mathcal{O}$ between those states; x, y_i, z_j denote distinct points in spacetime.) If we accept this point of view, we might as well forget about the Hilbert space of states, and consider Wightman functions instead; and for a further technical simplification, we can perform a Wick rotation from Minkowski space to Euclidean space, and consider Schwinger functions instead of Wightman functions.

So from now on we will consider only the Euclidean situation, in which we will mean by an operator $\mathcal{O}$ a collection of functions

$$\langle\mathcal{O}(x)\phi(z_1)...\phi(z_r)\rangle$$

with certain properties. Here we are considering what we will regard as a Schwinger function or correlation function of $r+1$ operators, one of which is $\mathcal{O}(x)$, rather than, as in the previous paragraph, a matrix element of $\mathcal{O}$ between initial and final states; and as we are in Euclidean space, the ordering of operators will be immaterial (any two points are spacelike separated). Roughly, an operator is just a symbol which can be inserted in a correlation function, obeying certain properties. The properties that we want correspond roughly to the Wightman axioms except that we will permit more severe singularities along diagonals than the Wightman axioms would permit; the allowed behavior of the singularities will be codified in the Operator Product Expansion. (While allowing more severe singularities that the Wightman axioms do, the Operator Product Expansion also places restrictions on the behavior near diagonals that are not obvious consequences of the Wightman axioms.)

Remark. The information we get from studying $\mathcal{O}$ in the Euclidean situation can be used (after Wick rotation) for studying the Minkowski situation, and in particular one can extract from the correlation functions what one interprets as matrix elements of $\mathcal{O}(x)$ between initial and final states in suitable dense subspaces of Hilbert space (analogous to Fock spaces). But $\mathcal{O}(x)$ does not act as an operator (or an operator-valued distribution) on any reasonable subspace of Hilbert space.

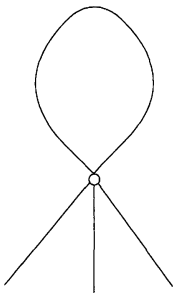
Consider now the problem of quantization of local functionals. By definition, in order to quantize the local functional $\mathcal{O}(x)$, we should say what the functions $\langle \mathcal{O}(x)\phi(y_1)...\phi(y_r)\rangle$ are. Here the y_i are distinct from each other and from x.

Consider the theory of a scalar bosonic field in $n > 2$ dimensions, with the Lagrangian

$$\mathcal{L}(\phi) = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + Q(\phi)\right) d^n x, \tag{3.2}$$

We will give all definitions and constructions for this example. In other field theories, everything is done in a similar manner.

We first consider the case when the theory is free ($Q = 0$). In a free theory, quantization is done with the help of normal ordering, as follows. Consider a local functional $\mathcal{O}$ of degree J in ϕ. According to Feynman rules, in order to compute the correlation function $\langle \phi(y_1)...\phi(y_r)\mathcal{O}(x)\rangle$, we should consider all graphs with r external vertices and only one internal vertex v, which has J edges. We should put a certain function at the vertex v, and compute the term (amplitude) corresponding to this graph as usual in Feynman calculus. If $J > 1$, we will run into trouble: we will get some graphs with loops going from v to itself (an example is pictured for the operator ϕ^5, represented by a five-valent vertex), and the integration over the loops is divergent. The easiest remedy is to ignore all such graphs. Then we obtain certain correlation functions, which define some operator. This operator is denoted by $:\mathcal{O}(x):$ and called the normal ordering of $\mathcal{O}(x)$. A priori, $\mathcal{O}(x)$ does not make sense as an operator, while $:\mathcal{O}(x):$ does. We call operators of the form $:\mathcal{O}(x):$ composite operators. When no confusion is possible, we will drop the dots and write $\mathcal{O}(x)$.



Thus, we have assigned canonically to each differential polynomial $\mathcal{O}$ in ϕ an operator $:\mathcal{O}:$ in the free theory. However, recall that two different polynomials $\mathcal{O}_1, \mathcal{O}_2$ might define the same local functional. So does the map $\mathcal{O} \to :\mathcal{O}:$ actually define canonically a quantization of the local functional represented by $\mathcal{O}$? The answer turns out to be affirmative: whenever polynomials $\mathcal{O}_1$, $\mathcal{O}_2$ define the same local functional, the operators $:\mathcal{O}_1:, :\mathcal{O}_2:$ coincide. To check this is an easy exercise. For example, the polynomials $\mathcal{O}_1 = \phi\Delta\phi$, $\mathcal{O}_2 = m^2\phi^2$ define the same local functional, and the operators $:\phi\Delta\phi:, :m^2\phi^2:$ coincide.

After considering some further questions in the free theory, we will discuss how to define local operators in the non-free case.

§3.3. Multiplication of composite operators

It is easy to multiply composite operators supported at different points. That means, given two composite operators $\mathcal{O}$, $\mathcal{O}'$, and points $x, x' \in V$, $x \neq x'$, it is easy to define an operator $\mathcal{O}(x)\mathcal{O}'(x')$, using the Feynman diagram expansion. All

this means is that we want to define correlation functions containing an $\mathcal{O}(x)$ and an $\mathcal{O}'(x')$ (together with elementary fields $\phi(y_i)$. To define such a correlation function $\langle \mathcal{O}(x)\mathcal{O}'(x')\phi(y_1)\dots\phi(y_s)\rangle$ (with all points distinct) via Feynman diagrams, we simply consider diagrams with s external lines, for the ϕ fields, and two vertices v and v', associated with $\mathcal{O}(x)$ and $\mathcal{O}'(x')$, respectively. We sum over all ways of connecting the external lines and the two vertices which involve no loops from a vertex to itself. However, multiplication of two composite operators $\mathcal{O}, \mathcal{O}'$ at the same point x cannot be defined in the same way: if $x = x'$, then propagators connecting v and v' must be evaluated on the diagonal, where they are singular.

To avoid this problem, one can try to smear the operators, i.e. consider the operator $\mathcal{O}(f) = \int f(x)\mathcal{O}(x)d^n x$, where f is a compactly supported smooth function on V. Such smeared operators can sometimes be multiplied: for example, we can define the product $\varphi(f_1)...\varphi(f_s)$. (see Lecture 1 of [**I-Axiomatics**]).

However, most composite operators cannot be multiplied, even after smearing. Indeed, it can be deduced from reflection positivity that for any (real) operator $\mathcal{O}$, one has $\langle \mathcal{O}(x)\mathcal{O}(x')\rangle \sim |x - x'|^{-2[\mathcal{O}]}$, $x \to x'$ where $[\mathcal{O}]$ is the dimension of $[\mathcal{O}]$. (Given a complex-valued operator, its real and imaginary parts behave in the same way.) On the other hand, in order for the product $\mathcal{O}(f)\mathcal{O}(f')$ to be defined, it is necessary and sufficient that the functions $\langle \phi(y_1)...\phi(y_r)\mathcal{O}(x)\mathcal{O}(x')\rangle$ be distributions. The function $|x|^{-l}$ defines a distribution (without regularization) if and only if it is locally L^1, i.e. iff $l < n$. Thus, the product $\mathcal{O}(f)\mathcal{O}(f')$ is automatically defined iff $[\mathcal{O}] < n/2$. Otherwise, it takes some further discussion to make sense of this product.

Remark. In the Minkowski situation, the product $\mathcal{O}(f)\mathcal{O}(f')$ exists (in the sense of correlation functions) if and only if the "operator" $\mathcal{O}(x)$ is an operator-valued distribution, so that $\mathcal{O}(f)$ is actually an honest operator (i.e. matrix elements of $\mathcal{O}(f)$ are matrix elements of some operator on $\mathcal{D}$). Indeed, the "if" part is clear. The "only if" part: for any $f_1, ..., f_N$ we can regard the product $X = \mathcal{O}(f)\varphi(f_1)...\varphi(f_N)$ as a linear functional on $\mathcal{D}$. The fact that $\mathcal{O}(f)^2$ is defined implies that this functional has finite L^2-norm, as

$$||X||^2 = \langle \varphi(f_N)\dots\varphi(f_1)\mathcal{O}(f)^2\varphi(f_1)\dots\varphi(f_N)\rangle.$$

Therefore, X corresponds to a vector in $\mathcal{H}$, Q.E.D.

§3.4. Operator product expansion (OPE) in the free theory

Composite operators in a reasonable field theory have an important algebraic property, which is called the operator product expansion (OPE). This is a property of algebraic nature, saying roughly that composite operators form an algebra of a certain kind. It does not follow obviously from the Wightman axioms, but on the other hand is often satisfied when the Wightman axioms fail. (In fact, the Wightman axioms correspond to fields that are operator-valued distributions, and as we have just seen, fields of dimension greater than $n/2$ do not have that property.) The operator product expansion is extremely useful in practice.

We will now define and compute the OPE for the free theory of a scalar bosonic field, before discussing the non-free case. The idea is to formalize the behavior of the product $\mathcal{O}(x)\mathcal{O}'(x')$ as $x \to x'$. In the classical theory, this is accomplished by the usual Taylor series. Indeed, we have

$$\mathcal{O}(x)\mathcal{O}'(x') = \mathcal{O}(x')\mathcal{O}'(x') + (x - x')_\mu \partial_\mu \mathcal{O}(x')\mathcal{O}'(x') + ... \tag{3.3}$$

Thus, for any two local functionals $\mathcal{O}, \mathcal{O}'$ the product $\mathcal{O}(x)\mathcal{O}'(x')$ decomposes in a series in products of powers of $(x-x')_\mu$ whose coefficients are other local functionals.

In the quantum theory, the product $: \mathcal{O}(x) : : \mathcal{O}'(x') :$, as we have seen, is singular at $x = x'$. Thus, we have to compute the asymptotic expansion of this product near $x = x'$.

We will assume that $\mathcal{O}, \mathcal{O}'$ are represented by some monomials in ϕ and its derivatives. According to Feynman rules, we should assign to both $\mathcal{O}, \mathcal{O}'$ a vertex, say v and v'. The numbers of edges at v, v' are J, J', where J, J' are the degrees of $\mathcal{O}, \mathcal{O}'$ as monomials of ϕ. At both v, v', we have to place symmetric distributions in J, J' variables corresponding to $\mathcal{O}, \mathcal{O}'$. Now we have to sum amplitudes over all ways of connecting some edges going from v with some edges going from v'. The amplitude of each particular graph is computed as follows: at each edge going from v to v' we put the function $D(y-z)$, the two-point Schwinger function of the free theory (the Green's function for the Helmholtz equation). On each remaining external edge we put $\phi(y)$. Then we compute formally the product of things on vertices and edges, and put it under the normal ordering sign. Since for each number K of connecting edges, we have only one possible graph, we should just compute the amplitude of this graph, and then multiply it by the number of ways to get this graph, i.e. by $J!J'!/K!(J-K)!(J'-K)!$.

This will give us a formula of the form

$$: \mathcal{O}(x) :: \mathcal{O}'(x') := \sum_{k=1}^{L} : \tilde{\mathcal{O}}_k(x)\tilde{\mathcal{O}}'_k(x') : E_k(x-x'), \tag{3.4}$$

where $: \tilde{\mathcal{O}}_k :, : \tilde{\mathcal{O}}'_k :$ are some local functionals, and $E_k(y)$ are polynomial differential operators in $D(y)$. Here $: \mathcal{O}(x)\mathcal{O}'(x') :$ is the amplitude of the graph in which the vertices v and v' are not connected.

Now we can use the Taylor formula (3.3) inside of the normal ordering, to get the following (already infinite) expansion:

$$: \mathcal{O}(x) :: \mathcal{O}(x') := \sum_{s} : \mathcal{O}_s(x') : D_s(x-x'), \tag{3.5}$$

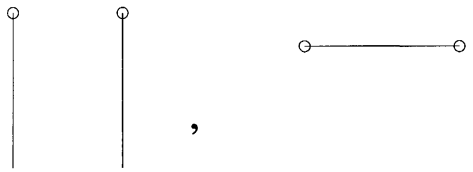
where $\mathcal{O}_s$ is a local functional, and $D_s(y)$ is a product of the function $E_k(y)$ for some k and a polynomial in y.

Definition. Expansion (3.5) is called the operator product expansion (OPE).

Now recall that the Green's function $D(x)$ has the following behavior at $x = 0$: $D(x) \sim |x|^{2-n}$. Therefore, all functions $d_k(x)$ have the property $|d_k(x)| \leq C_k|x|^{r_k}$ for small x, where the numbers r_k are almost all positive. This proves an important property of the OPE: only finitely many terms are singular. However, these singular terms carry the most interesting information in many situations.

Remark. We see that the operator product expansion (3.5) is in fact the quantum analogue of the Taylor expansion (3.3). Indeed, it is an expansion of the form similar to (3.3), but also involving finitely many singular terms at $x = x'$.

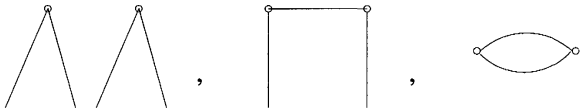
Let us now consider examples of OPE (we will always write the finite formula (3.4)). Let us first compute $\phi(x)\phi(x')$. Using the above rules, we get two graphs:



which yields the formula

$$\phi(x)\phi(x') =: \phi(x)\phi(x') : + D(x - x'). \tag{3.6}$$

Now we compute $: \phi^2(x)\phi^2(x') :$. This product gives us three graphs:



so we get the formula

$$: \phi^2(x) :: \phi^2(x') :=: \phi^2(x)\phi^2(x') : + 4 : \phi(x)\phi(x') : D(x - x') + 2D^2(x - x'). \tag{3.7}$$

In general, it is easy to prove that

$$\begin{aligned} : \phi^J(x) :: \phi^{J'}(x') := \sum_{K=0}^{\min(J,J')} & \frac{J!J'!}{K!(J-K)!(J'-K)!} \\ & \times D(x - x')^K : \phi^{J-K}(x)\phi^{J'-K'}(x') : \end{aligned} \tag{3.8}$$

The last formula can be written more nicely using generating functions:

$$: e^{\alpha\phi(x)} :: e^{\beta\phi(x')} := e^{\alpha\beta D(x-x')} : e^{\alpha\phi(x)} e^{\beta\phi(x')} : , \tag{3.9}$$

where α, β are formal parameters.

If we take α, β to be any differential operators on V with formal coefficients, this formula remains valid. (α acts on x, β on x'). In this form, formula (3.7) represents the most general OPE for the free theory of a scalar bosonic field.

Remark. Dimensions of fields in quantum theory may differ from the dimensions of their classical analogues. For example, in a (2-dimensional) classical field theory, if a field ϕ is 0-dimensional, so is $f(\phi)$, where $f : \mathbb{R} \to \mathbb{R}$ is any function. However, in quantum theory a renormalized composite operator $f(\phi)$ may acquire a nontrivial (anomalous) dimension. For example, in the free 2-dimensional theory of a massless scalar bosonic field, $D(y) = -\ln|y|$, so we have from (3.9):

$$\langle : e^{i\alpha\phi(x)} :: e^{-i\alpha\phi(y)} : \rangle = |x - y|^{-\alpha^2}.$$

But we know that $\langle \mathcal{O}(x)\mathcal{O}^*(y) \rangle \sim |x - y|^{-2[\mathcal{O}]}$. Thus, the scaling dimension of $e^{i\alpha\phi}$ is $d(\alpha) = \alpha^2/2$. Observe that the function d is not linear in α, so for quantum dimensions, in contrast with classical dimensions,

$$[: \mathcal{O}_1\mathcal{O}_2 :] \neq [: \mathcal{O}_1 :] + [: \mathcal{O}_2 :].$$

The dimension function $d(\alpha) = \alpha^2/2$ appears in the theory of vertex operator algebras, namely, in the Frenkel-Kac vertex operator construction, and is extremely important in string theory.

In an interacting theory, even polynomial fields can have different dimensions quantum-mechanically than they do classically. This is essential, however, mostly in the non-perturbative setting. In the perturbative setting we can count dimensions as in the classical theory, since dimensional anomalies are infinitesimally small.

§3.5. Normal ordering and renormalization

Now we will reformulate the definition of normal ordering in the free theory in terms of renormalization theory. This reformulation will be crucial in understanding what is the analogue of normal ordering in interacting theories.

At the beginning of this lecture, we defined normal ordering of a local functional by formally throwing away graphs with self-loops; these graphs of course produce divergences. Another way of dealing with self-loops is renormalizing them, along the lines of what we have learned in the two previous lectures. Suppose we want to renormalize all local functionals of dimension $\leq d$. For this purpose, we introduce a regularization; for instance, we replace the standard propagator $\frac{1}{k^2+m^2}$ with a cutoff propagator $\frac{1}{k^2+m^2}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^l$, where l is sufficiently big. Now all the loop integrals converge, and for any local functional $\mathcal{O}$ of degree $\leq d$ we can consider the cutoff correlation function $\langle\phi(y_1)...\phi(y_r)\mathcal{O}(x)\rangle_\Lambda$. If we take $\Lambda \to \infty$, we will of course find that the limit does not exist. However, it is not difficult to prove the following.

Let A_d be the space of local functionals of dimension $\leq d$.

Proposition 3.1. *There exists a Λ-dependent linear map $R_\Lambda : A_d \to A_d$, strictly triangular with respect to the filtration of A_d by dimension, such that for any local functional $\mathcal{O} \in A_d$ there exists a limit*

$$\lim_{\Lambda\to\infty} \langle\phi(y_1)\dots\phi(y_r)R_\Lambda\mathcal{O}(x)\rangle_\Lambda, \tag{3.10}$$

for all r and all points $y_1, \dots, y_r$ distinct from each other and from x.

Thus, Proposition 3.1 allows us to assign to every local functional $\mathcal{O}$ an operator $\tilde{\mathcal{O}}$, which is, by definition, the operator whose matrix elements are the limits of the corresponding matrix elements of $R_\Lambda\mathcal{O}$. However, given $\mathcal{O}$, the operator $\tilde{\mathcal{O}}$ is defined a priori non-canonically, as the map R_Λ is not unique: it is defined up to left multiplication with a Λ-independent strictly triangular map.

That is, $\tilde{\mathcal{O}}$ is defined uniquely up to adding composite operators of lower dimension. This shows that the space of composite operators is naturally a filtered object (by dimension), and not a graded object.

Remark. Of course, if the theory is not classically scale invariant, we saw that there is no grading on functionals already at the classical level. The statement here is that even for a classically scale-invariant theory, where the space of classical functionals automatically has a grading by scaling dimension, the grading is usually lost in the process of quantization. An exception is a free scale-invariant theory, where there is a canonical method of quantization by normal ordering, and therefore the grading survives quantization.

Thus, in general we may be able to quantize naturally the space of local functionals of dimension $\leq d$, but not every functional separately. We shall give a hint of the proof of the proposition presently, after restating it in the non-free case. Note that the strict triangularity in the above statements holds only for free theories.

§3.6. Composite operators in an interacting critical theory

Now consider an interacting renormalizable field theory. As a model example we consider the Lagrangian (3.2) and take $Q(\phi) = \frac{g}{4!}\phi^4$. Let $\mathcal{O}$ be any local functional

represented by a monomial. To quantize $\mathcal{O}$, we proceed as in the free case, but we will formulate everything in a slightly different language.

We want to consider correlation functions $\langle\phi(y_1)...\phi(y_r)\mathcal{O}(z)\rangle$. These correlation functions can be viewed as the ε-coefficients in the usual Schwinger functions for a perturbed Lagrangian, of the form

$$\mathcal{L}_\varepsilon(\phi) = \mathcal{L}(\phi) + \varepsilon\mathcal{O}(\phi, z), \tag{3.11}$$

where ε is a formal variable such that $\varepsilon^2 = 0$. In the language of Feynman diagrams, this means that we are introducing an additional vertex v corresponding to $\mathcal{O}$, and summing over all graphs which contain exactly one such vertex (with no self-loops at this vertex) and are otherwise as usual.

Remark. One should remember that there is no momentum conservation at the vertex v.

In general, such an alteration will worsen the divergence properties of the Feynman graphs. More precisely, now the superficial divergence of a graph with E external edges is given by $div(\Gamma) = [\mathcal{O}] - E$. However, we can renormalize these divergences, using the cutoff propagator considered in the previous section. Then, analogously to Proposition 3.1, one can prove the following.

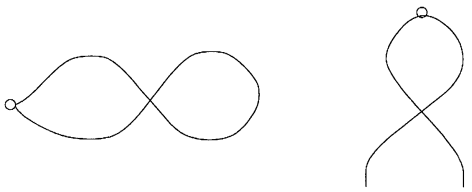
Proposition 3.2. *In a renormalizable theory, there exists in perturbation theory a Λ-dependent linear map $R_\Lambda : A_d \to A_d$, triangular (in general, not strictly) with respect to the filtration of A_d by dimension, such that for any local functional $\mathcal{O} \in A_d$ there exists a limit*

$$\lim_{\Lambda\to\infty} \langle\phi(y_1)\dots\phi(y_r)R_\Lambda\mathcal{O}(x)\rangle_\Lambda, \tag{3.12}$$

As in the free theory, this Proposition allows us to quantize the space of local functionals of dimension $\leq d$, but in general there is no canonical quantum analogue for each classical local functional. This non-uniqueness is not only due to the non-uniqueness of renormalization, but also due to the non-uniqueness of representation of a given functional by a differential polynomial.

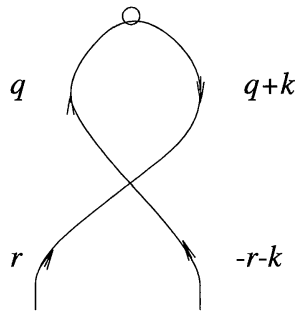
We will not give a proof of the proposition. But we note that the proof follows by applying the techniques of renormalization theory, as developed in the previous lecture, to the generalized Lagrangian of equation (3.11). The triangular structure arises because, once one adds to the Lagrangian the term $\epsilon\mathcal{O}(x)$, renormalization theory requires us to add all terms $\epsilon f_i\mathcal{O}_i(x)$, with all operators $\mathcal{O}_i(x)$ of equal or lesser dimension, and with the f_i being some functions of the parameters in the Lagrangian. The triangular structure claimed in the proposition depends on the renormalizability of the theory, which enters because it gives a bound on the degree of divergence of diagrams that one encounters in perturbation theory. In an unrenormalizable theory, there is no analogous proposition (with a uniform bound on the dimension of the operators $\mathcal{O}'$) that holds to all orders of perturbation theory. Once we learn about the renormalization group, the intuitive reason for this will be clear. To make sense of an unrenormalizable theory, one must find an ultraviolet fixed point, and consider operators and dimensions appropriate at that fixed point, rather than comparing to the classical field theory and using classical dimensions as the starting point, as we have been doing here.

Example. Let $\mathcal{O} = \phi^2/2$. Let us compute the renormalization of $\mathcal{O}$ of order g. The only divergent graphs we have in this order are



Let us call the first graph Γ_0 and the second Γ_2. The graph Γ_0 is quadratically divergent. If we replace the usual propagator with the cutoff propagator, the integral will converge to a Λ-dependent constant of the form gC_Λ, where C_Λ grows quadratically in Λ.

Now consider the graph Γ_2. As usual, it is more convenient to work in the momentum space, i.e. consider the Fourier transform of the term corresponding to this graph. Let k_1, k_2, k be the corresponding momentum variables. Then the Fourier transform of the term corresponding to Γ_2 is of the form $F_2\delta(k_1+k_2+k)$, where F_2 is a function on the plane $k_1+k_2+k=0$. Set $k_1=r$, then $k_2=-r-k$ (on this plane). Thus, $F_2=F_2(r,k)$, and its order g correction is the amplitude of the Feynman diagram



This amplitude is given by the integral

$$I(k) = -\frac{g}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)((q+k)^2+m^2)}. \tag{3.13}$$

This integral is logarithmically divergent. If we replace the usual propagator with the cutoff propagator, the integral will converge to a function $I_\Lambda(k)$, which has the asymptotic behavior

$$I_\Lambda(k) = -\frac{gA}{2}\ln(\Lambda/\mu) + O(1), \Lambda\to\infty$$

where A is a constant.

This shows that if instead of $\phi^2/2$ we use the renormalized functional

$$\left(\frac{\phi^2}{2}\right)_\Lambda := \frac{\phi^2}{2}(1+\frac{gA}{2}\ln(\Lambda/\mu)) - gC_\Lambda, \tag{3.14}$$

the matrix elements will have a finite limit modulo g^2. Observe that the constant μ can be chosen arbitrarily, so the renormalization is not canonical already in this case.

§3.7. Stability of the classical field equations under quantization

In a free theory, we know that the classical field equations are also satisfied quantum-mechanically. This means that if the classical equations are $P\phi=0$, where P is a linear differential operator, then the correlation functions in the quantum theory satisfy the equation $P_x\langle\phi(y_1)...\phi(y_r)\phi(x)\rangle = 0$ outside of the diagonals $x=y_i$. For

example, the Green's function $D(x-y) = \langle \phi(x)\phi(y) \rangle$ is the fundamental solution of the equation $Pf = 0$.

In an interacting theory, there are some problems. Namely, since the classical field equations for an interacting theory are nonlinear (e.g. (3.1)), they do not make sense quantum-mechanically in the setting of the Wightman axioms. However, in the OPE setting they make sense in a suitable interpretation, and one can show that they are satisfied.

Remark. We will say that the equation $F(\phi) = 0$ is satisfied in quantum theory (where F is a renormalized local functional) if the function $\langle \phi(y_1)...\phi(y_r)F(\phi(x)) \rangle$ vanishes outside of the diagonals $x = y_i$.

We will show that the classical field equations are satisfied in ϕ^4 theory. Consider the space of classical local functionals of dimension ≤ 3 which are Poincaré-invariant and odd under the symmetry $\phi \to -\phi$. This space is 2-dimensional: it has 3 generators ϕ^3, $\Delta\phi$, and ϕ, but they are linearly dependent, since they satisfy relation (3.1).

Now consider the quantum theory. Consider composite operators of dimension ≤ 3, with the same invariance properties as above. In this space we have renormalized operators $\Delta\phi$, ϕ, ϕ^3. Our goal is to show that, like in the classical theory, they are linearly dependent. This is equivalent to the statement that ϕ satisfies the field equation $C\phi^3 = A\Delta\phi + B\phi$.

We will first prove the validity of the field equation for the theory with the cutoff propagator, since in this theory we do not have divergence problems. In the cutoff theory, the classical field equation is

$$P_\Lambda \phi = \frac{g}{3!}\phi^3, P_\Lambda = (\Delta - m^2)\left(1 - \frac{\Delta}{\Lambda^2}\right)^l. \tag{3.15}$$

Proposition 3.3. *Equation (3.15) holds in the quantum theory with the cutoff propagator, where ϕ and ϕ^3 are regarded as composite operators.*

Proof. Consider the correlation function $\langle \phi(y_1) \dots \phi(y_r) P_\Lambda \phi(x) \rangle$. Consider the graph decomposition of this function. We should consider all possible graphs with r external edges, any number of 4-valent internal vertices, and a special vertex v with 1 outgoing edge. These graphs can be of two kinds: 1) graphs in which v connects to an external vertex; 2) graphs in which v connects to an internal vertex. The sum over graphs of the first type is the corresponding correlation function for the free theory, so it is supported on the union of diagonals $x = y_i$. Thus, outside of the diagonals we have to sum only over graphs of the second kind. But graphs of the second kind with N external edges are in 1-1 correspondence with usual Feynman graphs (without v) with $N+3$ external edges (this correspondence is obtained by biting off the vertex v and the vertex it connects to). This shows that the sum over graphs of the second kind is just $\frac{g}{3!}\langle \phi(y_1)...\phi(y_r)\phi^3(x) \rangle$. Thus, the field equation is satisfied, Q.E.D.

Proposition 3.3 shows that for any Λ, the composite operator $\frac{g}{3!}\phi^3$ is linearly dependent on composite operators which are linear in ϕ. This property has to be preserved in the limit $\Lambda \to \infty$. This shows that in the renormalized ϕ^4 theory we have an equation $C\phi^3 = P\phi$, where P is a linear differential operator with constant coefficients, and C is a constant. Since the r.h.s. of this equation can only have terms of dimension ≤ 3, the operator P has to be of the form $A\Delta + B$. Thus in

the quantum theory we have the equation

$$C\phi^3 = A\Delta\phi + B\phi \tag{3.16}$$

Remark. Of course, the constants A, B, C are not uniquely defined, as they depend on the choice of the renormalization. The statement is only that for any choice of renormalization, some nontrivial equation of the form (3.16) holds. In other words, the statement is that the space of Poincaré invariant, odd composite operators of dimension ≤ 3 is 2-dimensional, i.e. has the same dimension as the corresponding space of classical local functionals.

In general, one can consider the space of local functionals of dimension $\leq N$. Denote its dimension by $d(N)$. One can prove the following proposition.

Proposition 3.4. *The dimension of the space of composite operators of dimension $\leq N$ is also equal to $d(N)$.*

Thus, all differential equations which are satisfied classically have quantum analogues.

§3.8. Operator product expansion in an interacting theory

In a free theory, once we defined composite operators, it was no problem to define the product of several of them supported at different points. The same is true in the interacting theory. Namely, if $\mathcal{O}_1, ..., \mathcal{O}_N$ are renormalized composite operators, and $x_1, ..., x_N \in V$ distinct points, then the correlation function $\langle \phi(y_1)...\phi(y_r)\mathcal{O}_1(x_1)...\mathcal{O}_n(x_N)\rangle$ is just the coefficient of $\varepsilon_1...\varepsilon_N$ in the usual Schwinger function $\langle \phi(y_1)...\phi(y_r)\rangle$ for the Lagrangian

$$\mathcal{L}_{\varepsilon_1,...,\varepsilon_N}(\phi) = \mathcal{L}(\phi) + \sum_{i=1}^{N} \varepsilon_i \mathcal{O}_i(\phi, x_i), \tag{3.17}$$

where $\varepsilon_i\varepsilon_j = 0$. This Schwinger function is, by definition, the limit of the corresponding function computed with the cutoff propagator. It can be seen, by looking at Feynman diagrams, that this limit is always finite.

Remark. Roughly speaking, this means that once we renormalized each of the operators $\mathcal{O}_i$, we have automatically renormalized the product

$$\mathcal{O}_1(x_1)...\mathcal{O}_N(x_N), \quad x_i \neq x_j.$$

The reason is that, as long as the points x_i are distinct, having more than one composite operator does not introduce any new divergences. Thus, now we are in as good shape as we were in the free theory before we defined OPE. It turns out that we can define OPE in the interacting theory as well. Namely, for any two composite operators $\mathcal{O}, \mathcal{O}'$ and for any $r \in \mathbb{R}$ there exists a finite collection of composite operators $\mathcal{O}_1, ..., \mathcal{O}_l$ such that

$$\mathcal{O}(x)\mathcal{O}'(x') = \sum_{s=1}^{l} \mathcal{O}_s(x')D_s(x - x') + O(|x - x'|^r).$$

We will show how to partially compute the OPE on examples.

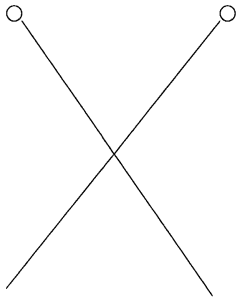
First we consider (in ϕ^4 theory) the product $\phi(x)\phi(x')$, and compute its asymptotics as $x \to x'$. In this case the expansion is of the form:

$$\phi(x)\phi(x') = \phi_R^2(x')f(x - x') + h(x - x') + \text{terms vanishing for } x \to x', \tag{3.18}$$

where ϕ^2_R is the renormalized operator ϕ^2 (recall that ϕ^2_R is defined non-uniquely; because of the symmetry under sign change of ϕ, the non-uniqueness is only the possibility of adding a multiple of the identity operator). Let us find the functions f, h.

The function $h(z)$ can be found by looking at the vacuum expectation value of the operator $\phi(x)\phi(x')$. Indeed, since $\langle\phi^2_R(x)\rangle = 0$ by definition, we get $h(x-x') = \langle\phi(x)\phi(x')\rangle$. If we compute the answer modulo g^2, we get no corrections to the free theory answer, so $h(z) = D(z)$ (see Lecture 2).

Now let us find $f(z)$ (we recall that it is defined up to scaling $f \to a(g)f$, where $a = 1 \bmod g$). This function is found from the 2-point function of $\phi(x)\phi(x')$. We have $f(z) = 1 + gf_1(z) + O(g^2)$. The only graph that contributes to $f_1(z)$ is



The amplitude of this graph (in the position space) is

$$\begin{aligned} & g\int D(x-z)D(x'-z)D(y_1-z)D(y_2-z)dz \\ & = -\frac{gA}{2}\ln|\frac{x-x'}{\mu}|D(y_1-x')D(y_2-x') \\ & \quad + \text{ regular part} \end{aligned} \tag{3.19}$$

(A is the same constant as in (3.17)). This shows that $f_1(z) = -\frac{gA}{2}\ln|\frac{x-x'}{\mu}|$, and thus

$$\begin{aligned} \phi(x)\phi(x') = \phi^2_R(x')(1 - \frac{gA}{2}\ln|\frac{x-x'}{\mu}|) + D(x-x') \\ + \text{ regular part } + O(g^2). \end{aligned} \tag{3.20}$$

Now consider a more complicated operator product, for example

$$\phi^2_R(x)\phi^2_R(x').$$

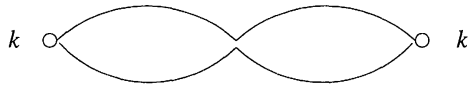
In this case the expansion is of the form

$$\begin{aligned} \phi^2_R(x)\phi^2_R(x') = \phi^4_R(x')f_1(x-x') + (\nabla\phi^2_R)(x')f_2(x-x') \\ + \phi^2_R(x')f_3(x-x') + f_4(x-x') \\ + \text{ regular part }, \end{aligned} \tag{3.21}$$

where the subscript R means "renormalized composite operator". Of course the functions $f_1, ..., f_4$ will depend on the choice of the renormalization, but they are well defined up to an upper triangular linear transformation.

For the sake of brevity we will only compute the expansion modulo $O(|x-x'|^{-1})$. Thus, the functions f_1, f_2 can be ignored, and we only have to compute f_3, f_4 modulo $O(|x-x'|^{-1})$.

The function f_4 is defined canonically up to multiplication by a scalar. As before, this function is computed using the vacuum expectation value: $f_4(x-x')= \langle \phi_R^2(x)\phi_R^2(x')\rangle$. The only graph that contributes to the first order in g of f_4 is



This shows that

$$f_4(z) = 2D(z)^2 + g\widehat{S(k)} + \text{ regular part } + O(g^2),$$

where $S(k)$ is the amplitude (in the momentum space) of the above graph, and hat denotes the Fourier transform. The function $S(k)$ equals $T(k)^2$, where $T(k)$ is the renormalized integral

$$T(k) = C\int_R \frac{dq}{(q^2+m^2)((k-q)^2+m^2)},$$

Remark. Since $T(k)$ is defined up to adding a constant, $f_4(z)$ is defined up to adding a multiple of $g\widehat{T(k)}$. Since $\widehat{T(k)}$ is proportional to $D^2(z)$ when $z \neq 0$, adding such a multiple is equivalent to multiplying f_4 by $1+cg$. This freedom is natural, as f_4 is defined up to multiplying by $1+cg$.

Now let us try to compute $f_3(z)$, which is also defined canonically up to a scalar. For this purpose we should consider the 2-point function of

$$\phi_R^2(x)\phi_R^2(x'),$$

i.e.

$$\langle \phi(y_1)\phi(y_2)\phi_R^2(x)\phi_R^2(x')\rangle.$$

We will work with the Fourier transform of this function, which we denote by $F_2(p_1,p_2,q,q')$ (here p_1,p_2,q,q' are the dual variables to y_1,y_2,x,x').

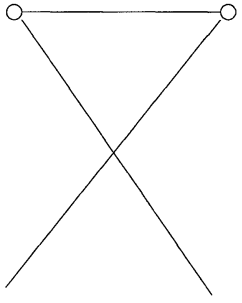
We will try to compute the function $f_3(z)$ by looking at the asymptotic expansion of $F_2(p_1,p_2,q,r-q)$ as $|q|\to\infty$, and using the following fact from calculus:

Claim. Let f be an L^1-function on an n-dimensional Euclidean space V whose Fourier transform $\hat{f}$ satisfies the inequality $|\hat{f}(q)| \leq C|q|^{-n-N-\varepsilon}$, $\varepsilon > 0$. Then $f \in C^n(V)$.

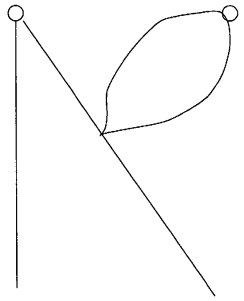
By doing so we will be able to find $f_3(z)$ modulo C^∞-functions, which is all we want at this point.

We have $f_3(z) = 4D(z) + gf_3^1(z)$. Looking at the Feynman diagram expansion of the 2-point function in the first order of g, we see terms of two kinds: 1) terms where the special vertices v, v' are directly connected with an edge; 2) terms where v, v' are not connected.

The only graph of the first kind that contributes to f_3^1 is



and the only graph of the second kind contributing to f_3^1 is



(this graph occurs 4 times, with 4 different labelings of vertices). It is easy to check directly that both graphs contribute to $\hat{f}_3(q)$ a term of the form $Cq^{-2}\ln|q/\mu| + O(q^{-4+\varepsilon})$, where C is a constant. Thus, $f_3^1(z)$ is the Fourier transform of the function $C\ln|q/\mu|/q^2$ modulo $O(|x-x'|)^{-1}$ and $O(g^2)$. As before, the constant μ depends on the choice of renormalization; changing this constant is equivalent to adding a multiple of $D(z)$ to $f_3^1(z)$, which is the same as multiplying $f_3(z)$ by a scalar of the form $1+cg$.

Remark: Operator product expansion in conformal field theory

In a 2-dimensional conformal field theory, the OPE of composite operators has an especially simple form. In this case, $V=\mathbb{C}$, and the space of functions on $V\setminus 0$ has a "bigrading": the function $z^a\bar{z}^b$ has bidegree (a,b) $(a-b\in\mathbb{Z})$. The space of composite operators also has a bigrading: to any homogeneous operator $\mathcal{O}$ one assigns two numbers – the holomorphic dimension $d(\mathcal{O})$ and antiholomorphic dimension $\bar{d}(\mathcal{O})$ $(d-\bar{d}\in\mathbb{Z})$. Therefore, if $\mathcal{O}(z)\mathcal{O}'(z') = \sum_k D_k(z-z')\mathcal{O}_k(z')$, then $D_k(z-z')$ has degree $(d_k,\bar{d}_k)$, where $d_k = d(\mathcal{O})+d(\mathcal{O}')-d(\mathcal{O}_k)$, $\bar{d}_k = \bar{d}(\mathcal{O})+\bar{d}(\mathcal{O}')-\bar{d}(\mathcal{O}_k)$. This implies that $D_k(z) = C_k z^{d_k}\bar{z}^{\bar{d}_k}$. Also, the action of the Virasoro algebra allows one to reduce the problem of computing OPE of arbitrary operators to the problem of computing OPE for primary fields only, i.e. for fields which are highest weight vectors for the Virasoro algebra. In a rational conformal field theory, one has a chiral algebra of symmetries (for example, an affine Lie algebra) which is so big that there are only finitely many fields which are primary with respect to this algebra. This fact allows one to treat a rational 2-dimensional conformal field theory in a purely algebraic setting, and reduce many of its problems to problems in algebraic geometry.

References and cross references

More on renormalization of operators and the renormalization group flow for them is in Lecture 3 of [I-**Renorm Groups**]. On ambiguity in operator products, see §5.1 of this series. On operator product expansion in conformal field theory, see Lecture 2 of [II-**CFT**], and [II-**Chiral Algebras**] (on the mathematical understanding of the operator product expansion). There is an extensive use of OPE in [II-**Strings**].

LECTURE 4
Scattering Theory

Abstract: This lecture is an introduction to Scattering Theory in Quantum Mechanics. I hope this will help the listener, or reader, get a better intuition about the S matrix, which is so important in quantum field theory. After introducing the asymptotic conditions and explaining their interpretation in terms of observable quantities, we introduce the Lippmann-Schwinger equation and the Born approximation. We then discuss Feynman diagrams and some differences between the non-relativistic and relativistic propagation of signals and particles.

§4.1. Nonrelativistic scattering theory: the asymptotic conditions

We start by considering a particle of mass m moving in $\mathbb{R}^3$ in the time-dependent potential $V(\vec{x},t)$ (which will have to vanish sufficiently fast at spatial infinity). The evolution of the wave function $\psi(\vec{x},t)$ is determined by the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = H\psi(\vec{x},t), \tag{4.1}$$

where the Hamiltonian is defined by $H = -\frac{1}{2m}\Delta_x + V$. We will assume for a while that $m = 1$.

We want to analyze the solutions of (4.1) with a certain behavior in the far past and the far future. In the far past, we require that the solutions approach, as $t \to -\infty$, a solution of the Schrödinger equation corresponding to the *free* Hamiltonian $H_0 = -\frac{1}{2}\Delta$. Solutions of the free Schrödinger equation have the general form

$$\psi(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \frac{k^2}{2}t)}, \tag{4.2}$$

where $f(\vec{k})$ represents the distribution of momenta at time $t = 0$. More specifically, we want to consider those solutions of (4.2) corresponding approximately to an incoming particle with velocity $\vec{v}$, which means that $f(\vec{k})$ is localized near $\vec{k}_0 = \vec{v}$ (since $m = 1$); a possible choice for f is

$$f(\vec{k}) = \exp(-\frac{1}{\alpha}(k - k_0)^2) \tag{4.3}$$

Notes by Radu Constantinescu

with small α. For large t, the integral in (4.2) is highly oscillatory so the main contribution comes from stationary phase; varying the exponent with respect to k gives $\vec{k} = \vec{x}/t$; if α is small then $\vec{k}$ has to be close to $\vec{k}_0$, and the condition $\vec{x} = \vec{k}_0 t$ does indeed describe a freely moving particle with velocity $\vec{k}_0$.

In the far future, we can no longer expect the wave function ψ to be localized near a definite momentum because of the scattering effect of the potential, so we will instead look for solutions of (4.1) which behave asymptotically like a sum of a localized solution of (4.2) and an outgoing spherical wave.

In order to find solutions of (4.1) we will first solve an eigenvalue problem for the Hamiltonian. For each $\vec{k}$ we will search for a solution $\psi_{\vec{k}}(\vec{x})$ of the equation $H\psi_{\vec{k}}(\vec{x}) = \frac{k^2}{2}\psi_{\vec{k}}(\vec{x})$ with the following asymptotic behavior at large distances:

$$\psi_{\vec{k}}(\vec{x}) \overset{x\to\infty}{\longrightarrow} e^{i\vec{k}\cdot\vec{x}} + \frac{1}{|x|}e^{i|k|\,|x|}F_{\vec{k}}(\Omega) + O\left(\frac{1}{|x|^2}\right). \tag{4.4}$$

The plus sign in the exponent of the second term is important, it will be needed in the construction of an *outgoing* wave solution of (4.1). The complex-valued function $F_{\vec{k}}$ is defined on the sphere of radius 1 and the notation Ω is a shorthand for a vector $\hat{x}$ on the sphere. Alternatively we can regard $F_{\vec{k}}$ as a function $F(\vec{k}', \vec{k})$ defined for $|\vec{k}'| = |\vec{k}|$.

The functions $\psi_{\vec{k}}$ can be used to construct exact solutions of the time-dependent Schrödinger equation, namely

$$\psi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k})\psi_{\vec{k}} e^{-i\frac{k^2}{2}t}. \tag{4.5}$$

Let us use (4.4) and (4.5) to see how ψ looks at infinity if f is assumed to be of the form (4.3). For $|t| \to \infty$ and $|x| \to \infty$, the two terms on the right hand side contribute to the stationary phase; one of them is $\exp i(\vec{k}\cdot\vec{x} - k^2 t/2)$ and has already been analyzed – the stationary phase condition is $\vec{x} = \vec{k}_0 t$. The other one has, for large $|\vec{x}|$, the phase $|\vec{k}|\cdot|\vec{x}| - k^2 t/2 + \arg F$; varying with respect to $|k|$ we find that the stationarity condition is

$$|x| = |k|t + \frac{\partial}{\partial|k|}\arg F_{\vec{k}} \tag{4.6}$$

For large t, this is approximately $|x| = |k|t$, which cannot be satisfied for negative t, in agreement with the fact that we only want to have an *outgoing* spherical wave.

The extra term in the right-hand side is a finite time advance or delay describing the fact that the moment when the scattered particles arrive at a detector is affected by the interaction with the target. The stationarity condition shows that the advance/delay is the radial derivative of the phase of F, called the *phase shift.*

§4.2. Relation with experiments

It is now time to relate the above constructions to quantities which are actually measurable in scattering experiments. The function $F(\Omega)$ is called the *scattering amplitude.* The reason for this is that the measure $|F(\Omega)|^2 d^2\Omega$ on the sphere $S_1 = \{|x| = 1\}$ can be interpreted as the number of scattered particles per unit solid angle if there is one incident particle per unit area ($d^2\Omega$ denotes the usual measure

on the 2-sphere). Notice that only the absolute value of F is relevant here. The integral

$$\sigma = \int_{S_1} |F(\Omega)|^2 d^2\Omega \tag{4.7}$$

is called the *total cross-section* and represents the total number of scattered particles (per unit incoming particle in unit area). As such, σ can also be viewed as the *effective cross-sectional area* of the target: if the target is thought to remove a fraction of the incoming particles of momentum $\vec{k}_0$, then σ is the total number of removed particles if there is one particle per unit area. The relationship between σ and F is usually written as

$$\frac{d\sigma}{d\Omega} = |F(\Omega)|^2. \tag{4.8}$$

§4.3. The Lippmann-Schwinger equation

We have so far used the eigenfunctions of the Hamiltonian to solve the time-dependent Schrödinger equation. Now we ask how to find those eigenfunctions. One possibility is to start from the free Hamiltonian $H_0 = -\frac{1}{2}\Delta$ whose eigenfunctions are $\exp(i\vec{k}\cdot\vec{x})$ (corresponding to the eigenvalues $k^2/2$). We will search for eigenfunctions $\psi_{\vec{k}}$ of H of the form $\psi_{\vec{k}} = \exp(i\vec{k}\cdot\vec{x}) + \alpha_{\vec{k}}$. Since $H = H_0 + V$ we have

$$0 = \left(H - \frac{k^2}{2}\right)\psi_{\vec{k}} = V\psi_{\vec{k}} + \left(H_0 - \frac{k^2}{2}\right)\alpha_{\vec{k}} \tag{4.9}$$

which implies

$$\alpha_{\vec{k}} = -\frac{1}{H_0 - \frac{k^2}{2}} V\psi_{\vec{k}}, \tag{4.10}$$

where the operator $1/(H_0 - k^2/2)$ is supposed to be a right inverse to $H_0 - k^2/2$.

Although the operator $H_0 - k^2/2$ is not invertible, we still can construct a right-inverse of it in momentum space; for instance, the operator $H_0 - k^2/2$ is roughly speaking a multiplication operator

$$\left(H_0 - \frac{k^2}{2}\right) e^{i\vec{q}\cdot\vec{x}} = \frac{1}{2}(q^2 - k^2)e^{i\vec{q}\cdot\vec{x}} \tag{4.11}$$

so we should have

$$\frac{1}{H_0 - \frac{k^2}{2}} = \frac{1}{\frac{1}{2}(q^2 - k^2)}. \tag{4.12}$$

Because of the pole, there are several ways of making sense of the above; one possibility, in the usual notation, is:

$$\frac{1}{H_0 - \frac{k^2}{2}} = \frac{1}{\frac{1}{2}(q^2 - k^2) - i\epsilon}. \tag{4.13}$$

The conversion to position space is realized through the following

Exercise. Show that the integral kernel (in position space) of the operator defined by (4.13) is

$$G(\vec{x}, \vec{y}) = \frac{1}{2\pi} \frac{e^{i|k|\cdot|x-y|}}{|x-y|}. \tag{4.14}$$

By using the exercise, (4.10) leads to the *Lippmann-Schwinger equation*

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} - \frac{1}{2\pi}\int d^3\vec{y}\frac{e^{i|k|\cdot|x-y|}}{|x-y|}V(\vec{y})\psi_{\vec{k}}(\vec{y}). \tag{4.15}$$

If the potential V has compact support then any solution of (4.15) has the boundary conditions required by the asymptotic condition built in. This can be seen by using the fact that for large $|x|$ and $\vec{y}$ in a compact set we have $|x-y| = |x| - \hat{x}\cdot\vec{y} + O(1/|x|)$, where $\hat{x} = \vec{x}/|x|$. Indeed, if we use this approximation then (4.15) becomes

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} - \frac{1}{2\pi}\frac{e^{i|k|\cdot|x|}}{|x|}\int d^3\vec{y}e^{-i|k|\,\hat{x}\cdot\vec{y}}V(\vec{y})\psi_{\vec{k}}(\vec{y}) + O\left(\frac{1}{x^2}\right), \tag{4.16}$$

which shows the existence of a scattered wave of the promised type.

Remark. The procedure used in (4.13) for going around the pole is chosen precisely in order to guarantee the existence of the outgoing spherical wave.

Notice that (4.16) gives an expression for the scattering amplitude: for any point $\Omega = \hat{x}$ on the sphere of radius 1 we have

$$F(\Omega) = -\frac{1}{2\pi}\int d^3\vec{y}e^{-i|k|\,\hat{x}\cdot\vec{y}}V(\vec{y})\psi_{\vec{k}}(\vec{y}). \tag{4.17}$$

Remark. We can reinterpret the function F as a function of k' and k such that $|k'| = |k|$, in agreement with the idea that after scattering the particle moves in the $\hat{x}$ direction with the same speed as before.

Although the Lippmann-Schwinger equation cannot in general be solved exactly, it can be used to compute $\psi_{\vec{k}}$ in perturbation theory as an expansion in the powers of the potential (assumed to be sufficiently weak). This can be done by an iterative procedure: we first calculate $\psi_{\vec{k}}$ to order V by using the free eigenfunction $\exp(i\vec{k}\cdot\vec{y})$ in the right-hand side of (4.15) instead of $\psi_{\vec{k}}(\vec{y})$; once we know $\psi_{\vec{k}}$ to order V we plug it back into (4.15) to get the answer to order V^2 and so on. For any smooth V that vanishes fast enough at infinity, this expansion is valid at large energies.

§4.4. The Born approximation

We have seen above an expression of the scattering amplitude in terms of the eigenfunctions $\psi_{\vec{k}}$. The *first Born approximation* is the computation of the scattering amplitude with $\psi_{\vec{k}}$ replaced in (4.17) by the free plane wave $\exp(i\vec{k}\cdot\vec{y})$. We have seen that the scattered wave travels with the same speed as the incoming one so we can write, for $|\vec{k}'| = |\vec{k}|$,

$$F(\vec{k}',\vec{k}) = -\frac{1}{2\pi}\int d^3\vec{y}e^{-i\vec{k}'\cdot\vec{y}}V(\vec{y})\psi_{\vec{k}}(\vec{y}). \tag{4.18}$$

In the first Born approximation this becomes

$$F(\vec{k}',\vec{k})_{Born} = -\frac{1}{2\pi}\int d^3\vec{y}e^{-i(\vec{k}-\vec{k}')\cdot\vec{y}}V(\vec{y}). \tag{4.19}$$

If we introduce the *momentum transfer* $\vec{q} = \vec{k}' - \vec{k}$ we see that $F(\vec{k}',\vec{k})_{Born}$ is the Fourier transform of the potential in the $\vec{q}$ variable.

Remark. This fact is important since it shows that the less smooth V is, the less rapidly does $F(\vec{k}', \vec{k})$ decay as a function of $\vec{k}' - \vec{k}$ (for fixed $\vec{k}$). A classical analog of this fact led Rutherford to postulate the existence of the atomic nucleus; he observed a high probability of scattering with large momentum transfer, and guessed that the potential was not smooth. Later, a somewhat similar observation in high energy scattering of electrons from protons (with multiparticle production) led to the insight that protons are composite (made of quarks and gluons).

Remark. Instead of considering a particle incident on a target we can regard our previous discussion as a description of the interaction of two particles. The center of mass decouples and the same results go through for the relative motion of the particles.

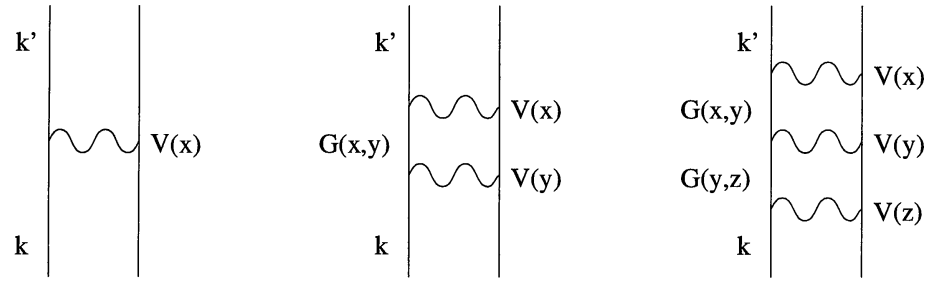
Remark. We can also generalize the preceding arguments for the case of n interacting particles. Let us note that this time we won't be able to simultaneously normalize the masses to 1 and that we need one spherical wave centered at each interaction point. The exact solutions of the Schrödinger equation $\psi_{\vec{k}_1,\ldots,\vec{k}_n}$ will be sums of free and spherical waves away from the diagonals (i.e. whenever $\vec{k}_i \neq \vec{k}_j$ for $i \neq j$).

§4.5. Feynman diagrams

Now we consider the higher Born approximations. Starting from (4.17) and the Lippmann-Schwinger equation, we can compute the transition amplitude as an infinite sum

$$\begin{aligned} F(\vec{k}', \vec{k}) &= -\frac{1}{2\pi}\int d^3\vec{x} e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} V(\vec{x}) \\ &\quad + \frac{1}{4\pi^2}\int d^3\vec{x} d^3\vec{y} e^{-i\vec{k}'\cdot\vec{x}} V(\vec{x}) G(\vec{x},\vec{y}) V(\vec{y}) e^{i\vec{k}\cdot\vec{y}} \\ &\quad - \frac{1}{8\pi^3}\int d^3\vec{x} d^3\vec{y} d^3\vec{z} e^{-\imath\vec{k}'\cdot\vec{x}} V(\vec{x}) G(\vec{x},\vec{y}) V(\vec{y}) G(\vec{y},\vec{z}) V(\vec{z}) e^{\imath\vec{k}\cdot\vec{z}} \\ &\quad + \cdots \end{aligned}$$

Given our experience with Feynman diagrams, it is easy to see that the terms of the sum can be represented graphically (for instance, the graphs corresponding to the first three terms in the sum are shown below).



Each curly line represents an interaction due to the potential V. In between interactions, the particles move freely (hence the free propagator $G(x, y)$). The incoming and outgoing particles have definite momentum ($\vec{k}, \vec{k}'$, respectively). The diagrams can be thought to encode either the relative motion of two particles or the evolution of one particle scattered by a fixed target. Notice that only very

simple (ladder) diagrams appear, corresponding to the fact that there are no creation/annihilation phenomena in this non-relativistic description.

§4.6. Relativistic versus non-relativistic scattering theory: propagation of particles

We will now amplify on that last statement.

Going back to the time-dependent Schrödinger equation (4.1), we can try to solve it directly by imitating the method used in 1.3 to derive the Lippmann-Schwinger equation. We can use the plane wave solutions $\exp i(\vec{k}\cdot\vec{x} - \frac{k^2}{2}t)$ of the *free* Schrödinger equation to transform the Schrödinger equation for H into an integral equation: any solution of the perturbed Schrödinger equation satisfies the following analogue of Lippmann-Schwinger:

$$\psi = e^{\imath(\vec{k}\cdot\vec{x}-\frac{k^2}{2}t)} - \frac{1}{i\frac{\partial}{\partial t} - \frac{1}{2}\Delta} V\psi. \tag{4.20}$$

As in Subsection 1.3., the easiest way of making sense of the inverse of $i\frac{\partial}{\partial t} - \frac{1}{2}\Delta$ is in momentum space. Since

$$\left(i\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) e^{i(\vec{q}\cdot\vec{x}-Et)} = \left(E - \frac{q^2}{2}\right) e^{i(\vec{q}\cdot\vec{x}-Et)}, \tag{4.21}$$

an inverse (in momentum space) can be found by prescribing the way to go around the pole $E = q^2/2$. For instance, we could use

$$\frac{1}{i\frac{\partial}{\partial t} - \frac{1}{2}\Delta} e^{i(\vec{q}\cdot\vec{x}-Et)} = \frac{1}{E - \frac{q^2}{2} + i\epsilon} e^{\imath(\vec{q}\cdot\vec{x}-Et)}. \tag{4.22}$$

The integral kernel of the chosen inverse, in position space, is given by the inverse Fourier transform

$$G(\vec{x},t;\vec{x}',t') = \int \frac{d^3\vec{q}}{(2\pi)^4} e^{\imath\vec{q}\cdot\vec{x}} \int_{-\infty}^{\infty} dE \frac{e^{-\imath E(t-t')}}{E - \frac{q^2}{2} + i\epsilon}. \tag{4.23}$$

The only pole of the E integral is at $E = \frac{q^2}{2} - i\epsilon$ and so, *if* $t - t' < 0$ we get $G = 0$ (because we can then avoid the pole by closing the integration contour in the upper half-plane). This result has an important implication: *particles can only travel forward in time.*

This is no longer true in a relativistic context: we have seen that the typical propagator of a particle of mass m is

$$\frac{1}{q_0^2 - \vec{q}^2 + m^2 + i\epsilon}$$

in momentum space; in position space, the inverse Fourier transform gives (for $x = (\vec{x},t)$)

$$G(x,x') = \int \frac{d^3\vec{q}}{(2\pi)^4} e^{i\vec{q}\cdot\vec{x}} \int_{-\infty}^{\infty} dE \frac{e^{-iE(t-t')}}{q_0^2 - \vec{q}^2 + m^2 + i\epsilon}. \tag{4.24}$$

No matter whether we close the integration contour in the upper or lower half-plane, we cannot avoid both poles, therefore it is no longer true that G vanishes if the time coordinates of the points x and x' satisfy $t < t'$. As a consequence, particles can make zig-zags in time, a phenomenon which is interpreted as the creation or annihilation of particle/antiparticle pairs.

§4.7. Relativistic versus non-relativistic scattering theory: propagation of signals

Non-relativistically, interactions are instantaneous. However, this is no longer true in the relativistic case.

Let us consider the example of the electromagnetic field; the interaction is transmitted by photons traveling at the speed of light.

The photon propagator in momentum space equals $1/(q_0^2 - \vec{q}^2 + i\epsilon)$, an expression whose non-relativistic limit (given by $q_0 \to 0$) is formally $-1/\vec{q}^2$.

In a scattering process involving nonrelativistic particles, the energy transfer between particles is always much less than the momentum transfer, so the propagator $1/(q_0^2 - \vec{q}^2 + i\epsilon)$ can be approximated by $-1/\vec{q}^2$. The (four-dimensional!) Fourier transform of $-1/\vec{q}^2$ equals

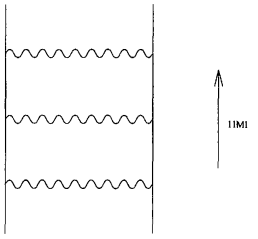
$$-\int \frac{d^4q}{(2\pi)^4} \frac{e^{iq\cdot y}}{\vec{q}^2} = -\delta(t)\frac{1}{|\vec{y}|}, \tag{4.25}$$

which is precisely a delta-function in time multiplied by the Coulomb potential. Therefore the non-relativistic limit of the relativistic propagator corresponds to an *instantaneous* interaction via the Coulomb potential. Relativistically,

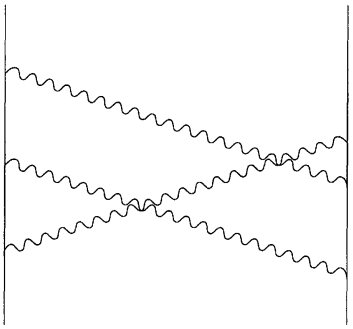
$$-\int \frac{d^4q}{(2\pi)^4} \frac{e^{iq\cdot y}}{q_0^2 - \vec{q}^2 + i\epsilon} \tag{4.26}$$

is no longer supported at a fixed point in time, so the interaction is not instantaneous. (It is transmitted instead at the speed of light, which is almost instantaneous from a nonrelativistic point of view!)

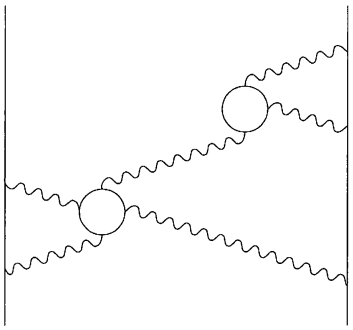
We can also illustrate the results on the propagation of particles and interactions in perturbation theory. It was shown above that non-relativistically only ladder diagrams are encountered; intuitively, if time flows in the vertical direction in the last illustration, then these diagrams represent particles moving forward in time with the horizontal curly lines being instantaneous interactions.



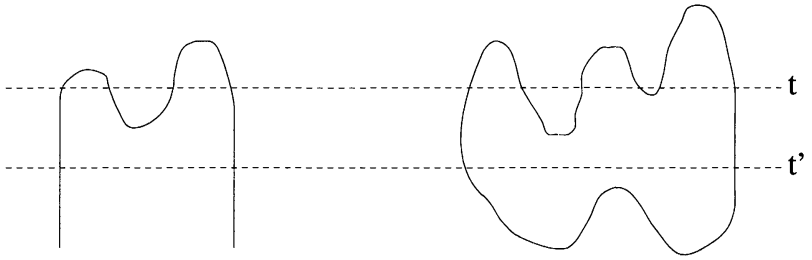
By contrast, in the relativistic case interactions travel at the speed of light and particle/antiparticle pairs can appear, so more complicated Feynman diagrams such as the ones below have to be considered.



This diagram illustrates the fact that photons traveling at the speed of light replace the non-relativistic instantaneous interaction. The curly line which represents the interaction in the non-relativistic case is relativistically a photon. In the second diagram we show the creation and annihilation of particle pairs. (Of course there are also diagrams in which both effects are present.)



This diagram also illustrates another important fact: in a local theory, the occurrence of pair creation makes it impossible to count "the total number of particles in the universe." Electrons zig-zagging back and forth in time can be interpreted in terms of continual creation and annihilation of electron-positron pairs.



References and cross references

A mathematical discussion of relativistic scattering is in Lecture 4 of [**I-Axiomatics**]. Another discussion of scattering is in Lecture 3 of [**I-Faddeev**]. An example of computation of a nontrivial relativistic scattering matrix is in Lecture 4 of [**II-Dynamics of QFT**]. For the multiparticle version of the nonrelativistic scattering theory described in this lecture, allowing for possible bound states in initial and final states, see M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, 1964).

LECTURE 5
Remarks on Renormalization and Asymptotic Freedom

§5.1. Ambiguity in operator products

In Lecture 3 we considered Lagrangians of the form

$$\mathcal{L} = \int d^4x (\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + \sum_i \varepsilon_i(x)\mathcal{O}_i(x)),$$

where $\mathcal{O}_i$ are local functionals. We saw that defining correlation functions of such Lagrangians to the first order in $\varepsilon_i(x)$ (i.e. imposing $\varepsilon_i\varepsilon_j = 0$) is equivalent to defining composite operators corresponding to $\mathcal{O}_i$. Now let us work to the second order in ε_i (we impose now $\varepsilon_i\varepsilon_j\varepsilon_k = 0$). This corresponds to considering products of composite operators. Previously we considered products of composite operators at non-coinciding points x, y, and saw that such a product $\mathcal{O}_i(x)\mathcal{O}_j(y)$ is defined automatically once we have defined $\mathcal{O}_i$ and $\mathcal{O}_j$. Now we will allow the points x and y to coincide. Then, as we know, the product is not automatically defined, and its definition requires additional renormalization. This means that we have to introduce counterterms in the Lagrangian, i.e. consider a new Lagrangian of the form

$$\mathcal{L}' = \mathcal{L} + \sum_{i,j,k} L_k\varepsilon_i(x)R_k\varepsilon_j(x)W_k(\Lambda, g)\mathcal{O}_k(x),$$

where L_k, R_k are differential operators (in x) with constant coefficients, and W_k are some functions which diverge as $\Lambda \to \infty$. The functions W_k are usually not uniquely determined and cannot be chosen canonically.

Let us consider an example. In Problem FP18 of [I-**Homework**] we compute the 1-loop correction to the 1-particle irreducible bosonic 2-point function Σ in QED. We discussed that this correction can be computed (in momentum space) as

$$\int e^{\imath kx}\langle J_\mu(x)J_\nu(0)\rangle dx,$$

where $J_\mu(x) = \bar{\psi}\gamma_\mu\psi$ is the operator of current, and the correlator in this equation is in the theory of a free fermion. Now, as follows from the above discussion, the product $J_\mu(x)J_\nu(x)$ is defined up to operators of lower order, and the expectation value $\langle J_\mu(x)J_\nu(0)\rangle$ is not uniquely defined, which causes an ambiguity in

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the computation of (1). However, this non-uniqueness occurs only at $x = 0$. The non-uniqueness involves terms proportional to derivatives of $\delta(x)$ of order at most two. In momentum space, the nonuniqueness involves only a quadratic polynomial in the momentum vector k; by Lorentz invariance this polynomial is of the form $Ak_\mu k_\nu + Bg_{\mu\nu}k^2 + Cg_{\mu\nu}$. (Here $g_{\mu\nu}$ is the spacetime metric, and A, B, and C are constants.)

In QED, to preserve gauge invariance, it is necessary to choose this constant in such a way that the condition $k^\mu \Sigma_{\mu\nu}(k) = 0$ is satisfied. This gives $B = -A$ and $C = 0$. The ambiguous coefficient A corresponds to the fact that one could add to the Lagrangian a multiple of the gauge field kinetic energy $\int F \wedge *F$, which would correct the two-point function of the gauge field just as the correlation function discussed in the last paragraph does. One should only expect that correlation function to be well-defined modulo this sort of renormalization.

This is a particularly simple example, but in general in renormalizing a product of local operators at possibly coincident points, one meets all the complexities of renormalization theory in a situation with interactions of very high dimension.

§5.2. Symmetry breaking

A symmetry that exists in a classical field theory may be lost in a particular renormalization scheme for the corresponding quantum theory. Of course it is possible that there exists a better scheme which preserves this symmetry, but it is also possible that there exists no such scheme, i.e. the symmetry is not valid or is, as one says, explicitly broken[1] at the quantum level. For example, consider N free massless fermions $\psi_1, ..., \psi_N$, with Lagrangian $\sum(\bar{\psi}_i, D\psi_i)$, $\psi_i \in S_+$, $\bar{\psi}_i \in S_-$. In this theory we have a $U(N)$ symmetry. Let us add interactions in such a way that part of this symmetry is preserved. For example, add a gauge field A (i.e. regard the fermions together to make a section ψ of the vector bundle $S_+ \otimes E$, where E is an N-dimensional Hermitian vector bundle over the spacetime), which takes values in the Lie algebra of a subgroup $H \subset U(N)$, and consider the Lagrangian $(\bar{\psi}, D_A\psi)$, where D_A is the covariant Dirac operator defined by the connection A. The classical symmetry of this theory is the centralizer $Z(H)$ of H. In quantum theory, however, this symmetry may fail for topological reasons. In other words, a topological anomaly may appear. This will be discussed in the spring semester.

§5.3. An oversimplified version of experimental confirmation of asymptotic freedom

Consider a field theory with electromagnetic and strong interactions, which contains electrons (which interact only electromagnetically), and quarks (which interact both electromagnetically and strongly). The Lagrangian of such a theory can be written as follows. The fields are:

(i) An $SU(3)$-connection A_g (the field of strong interactions),
(ii) A $U(1)$-connection A_e (the electromagnetic field),
(iii) Quarks q_i and an electron ε (which are fermions with values in $\mathbb{C}^3$ and $\mathbb{C}$ respectively).

[1]This is to be distinguished from a situation we will meet next spring, where a symmetry is valid quantum mechanically but is "spontaneously broken."

The Lagrangian is

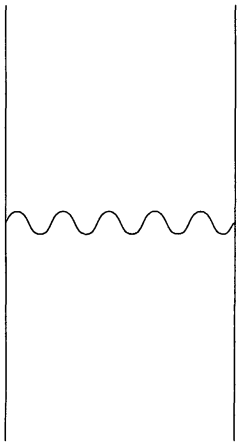
$$\int d^4x \left(\sum_j \bar{q}_j (iD + A_g + A_e - m_j) q_j + \bar{\varepsilon}(iD + A_e - m)\varepsilon + \frac{1}{e^2} F_{A_e}^2 + \frac{1}{g^2} F_{A_g}^2 \right).$$

Here e is the charge of the electron and g is the coupling of the strong interaction.

Now suppose that we scatter two electrons against each other with momenta p_1, p_2, and measure the amplitude of the event that after scattering they will have momenta q_1, q_2. As we know, this amplitude is defined by the 4-point function $\Gamma_4(p_1, p_2, q_1, q_2)$. Let us try to compute this function and thus predict the result of measurement.

First of all, we can use the fact that e^2 is small. This means that we can trust the perturbative expansion in powers of e.

To order e^2, we can assume that the electrons, during scattering, exchange only one photon, which does not interact while it moves from one electron to the other. This corresponds to the following Feynman diagram:



Thus we have

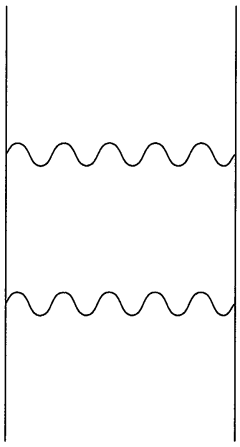
$$\Gamma_4(p_1, p_2, q_1, q_2) = e^2(G_2((p_1 - q_1)^2) + G_2((p_1 - q_2)^2)),$$

where $G_2(p^2)$ is the free photon propagator. Here G_2 is regarded as an operator from $S \otimes S$ to $S \otimes S$, where S is the space of spinors.

Remark. In principle, we should include the diagrams where one of the electrons exchanges a photon with itself, but we will not consider them, regarding them as absorbed in the electron propagator.

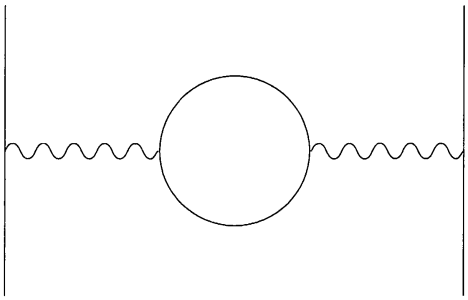
To order e^4, we have two possibilities.

1) The electrons could exchange two non-interacting photons. The amplitude of the corresponding 1-loop diagram

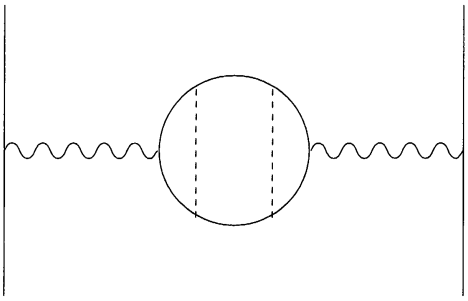


can be computed within the framework of QED.

2) The electrons could exchange only one photon, but on its way it could split in an electron and positron, or in a quark and an antiquark. The first splitting scenario

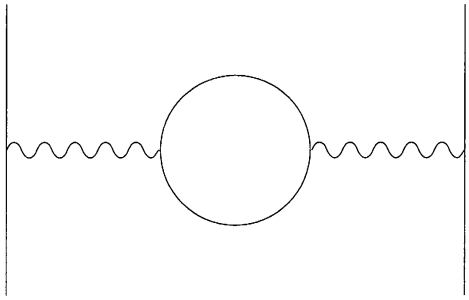


is harmless, since it gives only one 1-loop diagram with no strong interactions, and we can compute the amplitude of this diagram as in QED. (Notice that to calculate this diagram, one must evaluate the free field 1-loop diagram for $\langle J_\mu J_\nu \rangle$ that was discussed at the beginning of this lecture; renormalizing this diagram will bring in the possible F^2 renormalization that we also discussed.) However, the second scenario (with quarks) really gives us trouble. Indeed, the coupling constant g of the strong interaction is not small, so we cannot trust the perturbation expansion in g and thus have to take into account infinitely many Feynman diagrams with any number of loops:



(here dotted lines denote the field of strong interaction).

However, we know that the theory of strongly interacting quarks is asymptotically free (when the number of the quarks is not too large). Thus, we should expect that the perturbative expansion in the effective coupling $g_{eff} \sim g(\ln p^2)^{-1/2}$ should be valid at high momenta. This would mean that at high momenta we can restrict to the 1-loop diagrams, which involve no strong interactions. Since we have one such diagram for each type of quark,



the total amplitude of these diagrams is $\sum e_i^2 \Sigma_i(p)$, where e_i are charges of quarks, and Σ_i are amplitudes of the corresponding diagrams for a particle with charge 1. Of course, the functions $\Sigma_i(p)$ depend on the not well measured masses m_i of quarks, but at high momenta masses are irrelevant, and all functions Σ_i are approximately equal to each other. Thus, at high p the amplitude is $\Sigma(p) \cdot \sum e_i^2$, where $\Sigma(p)$ is a universal function computed from QED (as in one of the homework exercises).

It follows from asymptotic freedom that the (relative) error of this computation is of order $1/\ln p^2$. If we compute the 2-loop correction (still working to order e^4), we will get an additional term of the order $1/\ln p^2$, and the error will be of order $1/\ln^2 p^2$. More generally, if we take into account N-loop diagrams, the error will be of order $1/\ln^N p^2$. Thus, we get an asymptotic series, with very slowly decaying terms, but at very high p one can hope that it gives a reasonably good approximation to the 4-point scattering amplitude. This approximation (to the 0-th order) could in principle be checked experimentally, and can be regarded as a prediction of asymptotic freedom.

Remark. In practice, it is not practical to test asymptotic freedom precisely as I have described. The reason for this is that high energy scattering cannot be measured precisely enough. Actual tests of asymptotic freedom are however similar in concept to what I have said.

References and cross references

A detailed discussion of various types of symmetry breaking is in [II-**Dynamics of QFT**], especially in Lecture 1. On the theory of asymptotic freedom, see [I-**Renorm Groups**, §§2,3].

Index of Dirac Operators

Edward Witten

Introduction

In these lectures, we will discuss some applications of path integrals and quantum field theory to topology. In the first lecture, we describe the path integral proof of the Atiyah-Singer index theorem for the Dirac operator. We also describe a perturbation of the Dirac operator (involving a Killing vector field) that provides useful background for the second lecture.

In the second lecture, we consider Dirac operators not on an ordinary finite-dimensional manifold M, but on a loop space $\mathcal{L}M$. We show that the supercharge of the supersymmetric nonlinear sigma model in two dimensions can be understood as a Dirac-like operator on $\mathcal{L}M$ – with a certain crucial perturbation relative to a standard Dirac operator. (This viewpoint is indeed rather close to the line of thinking that led P. Ramond to introduce in 1971 what is now called world-sheet supersymmetry.) We use path integrals to explain why the index of the Dirac operator on $\mathcal{L}M$ has modular properties, and show that the study of the Dirac operator on $\mathcal{L}M$ gives a natural framework for theorems concerning possible circle actions on the finite-dimensional spin manifold M.

Notes by John Morgan

LECTURE 1
The Dirac Operator in Finite Dimensions

§1.1. Introduction

One mathematical approach to the proof of the Hirzebruch signature theorem – and generalizations such as the Atiyah-Singer index theorem – is through the heat equation. One considers, for each $t > 0$, the operator $e^{-t\Delta}$ acting on the differential forms on a closed, oriented Riemannian manifold Y. A direct spectral argument shows that the trace of this operator (or more precisely the difference of its trace on the self-dual forms and on the anti-self-dual forms) is independent of t and limits as $t \to \infty$ to the index of the manifold. On the other hand, the limit as $t \to 0$ of this operator can be computed in terms of local curvature invariants of the manifold. One can even explicitly evaluate these local expressions, and in this manner recover the result that the signature of the manifold is given by the L-polynomials of Hirzebruch. In a related context, that of equivariant signatures when there is a group action, one sees a localization taking place in the Atiyah-Hirzebruch formula for the equivariant signature in terms of local data (characteristic classes) at the fixed points. These localizations are reminiscent of localizations in quantum field theory, e.g., perturbation expansions around the classical solutions of Lagrange's equations, and raise the question of whether there are quantum field theory arguments establishing these and similar results.

The answer to this question is yes, and it is our purpose here to show how to establish such results (for Dirac operators on a spin manifold) by doing a perturbation expansion of an appropriate path integral or Hamiltonian eigenstates. The explanation of the quantum field theories that do indeed lead to index theorems for the Dirac operator on a spin manifold (and its equivariant analogue) are the first subject of these lectures. Once these classical cases have been covered, we turn in the next lecture to more sophisticated applications of the same techniques. These involve the index of Dirac operators on the loop spaces of spin manifolds. While the results on loop spaces were first discovered by the path integral approach that we take here, eventually, mathematical proofs were also found for these results; see the references at the end of Lecture 2.

Before we get into the details of the computations, one general comment is in order. So far the perturbative expansions we have encountered have been around an isolated, nondegenerate critical point. That is to say the space of classical solutions

of lowest energy (if we are using a Hamiltonian approach) or of lowest action (if we are using a path integral approach with Euclidean signature)[1] is a single point and the Hessian (the quadratic term in the energy or action) is nondegenerate at that point. The expansions that we encounter here will be of a more general type. The set of classical solutions will be a compact, finite dimensional smooth manifold (or supermanifold). The appropriate nondegeneracy condition (which will hold in our context) is that the restriction of the quadratic term in the Lagrangian to the normal bundle to the critical submanifold is positive definite. The same techniques will work for oscillatory integrals as long as the Hessian in the normal directions is nondegenerate (the Chern-Simons functional in three dimensions gives examples of this type). Of course, one could meet degenerate situations where the critical set is singular and the Hessian has null-space which is not of constant rank. Such situations require techniques that are not discussed here.

In the nondegenerate situation where the critical set is a supermanifold M and at each point of the critical set the Hessian has null space equal to the tangent space of the critical set, the basic approach is to apply the usual perturbative expansion using Feynman diagrams and regularization in the normal direction where there is an isolated, nondegenerate critical point to obtain a result, and then integrate this result over the critical set. Of course, it may well happen that the individual expansions over the normal spaces have no intrinsic meaning and depend on choices, but we expect the resulting integral over M to be well-defined. Since what we are interested in here are topological invariants, it will suffice to compute only the constant term in the expansions. These of course will depend only on the quadratic term in the Lagrangian and will be written in terms of the determinant of the Hessian operator (or in physical terminology, the kinetic energy operator) for the bosons and Pfaffian of the Hessian (or kinetic operator) of the fermions integrated over the normal directions.

§1.2. The Dirac operator on a spin manifold

In this section we shall give a path integral derivation of the index theorem for the Dirac operator on a spin manifold. Fix for this section a compact, oriented, $2n$-dimensional spin manifold Y with a Riemannian metric.

§1.2.1. The Lagrangian formulation

Let $\mathbf{R}^{1|1}$ denote the super Euclidean space with one even variable and one odd variable; i.e., $C^\infty(\mathbf{R}^{1|1}) = C^\infty(\mathbf{R}) \otimes \wedge^*(\mathbf{R})$. We denote by $t, \theta \in C^\infty(\mathbf{R}^{1|1})$ the natural even and odd variables, respectively. We consider a quantum theory of maps

$$X\colon \mathbf{R}^{1|1} \to Y.$$

The Lagrangian we take is

$$\mathcal{L} = -\int_{\mathbf{R}^{1|1}} dtd\theta \frac{1}{2} g_{IJ}(X) \frac{dX^I}{dt} DX^J = -\int_{\mathbf{R}^{1|1}} \frac{1}{2} dtd\theta \langle \dot{X}, DX \rangle,$$

where as usual g_{IJ} denotes the metric tensor on Y and D is the vector field

$$D = \frac{\partial}{\partial\theta} - \theta \frac{\partial}{\partial t}$$

[1] Or the space of classical solutions relevant to the problem at hand, if we are using a path integral approach with Lorentz signature.

on $\mathbf{R}^{1|1}$.

Let us recall some of the material from Problem FP2 of [I-**Homework**]. We can view X as a map x from $\mathbf{R}^1$ to Y together with a section ψ of the odd vector bundle $\Pi(x^*TY)$ over $\mathbf{R}^1$. We write $X = x(t) + \theta\psi(t)$ with this understanding as to the meanings of x and ψ. Then we can rewrite the Lagrangian as

$$\mathcal{L} = -\int_{\mathbf{R}^{1|1}} dtd\theta \frac{1}{2}\langle \dot{x}(t) + \theta\nabla_t\psi(t), \psi(t) - \theta\dot{x}(t)\rangle. \tag{1.1}$$

If one prefers an expression in local coordinates, doing the θ-integration yields:

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{R}^1} (\langle \dot{x}(t), \dot{x}(t)\rangle - \langle \nabla_t\psi(t), \psi(t)\rangle)\, dt.$$

The classical solutions to this Lagrangian form a supermanifold of maps $X\colon \mathbf{R}^{1|1} \to Y$ satisfying

$$\nabla_t DX = 0.$$

We can also write this equation as a pair of equations:

$$\begin{aligned} \nabla_t\psi &= 0 \\ F(\psi, \dot{x})(\psi) &= \nabla_t(\dot{x}), \end{aligned}$$

where F is the curvature of Y. Evaluating at a fixed time t_0 identifies this supermanifold with the space of "initial conditions" at time t_0. The initial conditions are simply a pair – a tangent vector $\dot{x}_0$ at some point x_0 of Y and an element $\psi_0 \in \Pi TY_{x_0}$, a copy of TY_{x_0} considered as an supervector space of odd parity. That is to say, the space of all initial conditions is identified with the odd vector bundle $p^*\Pi(TY)$ over TY, where $p\colon TY \to Y$ is the natural projection mapping. The symplectic structure on this space of classical solutions is given by

$$\omega = \omega_{T^*Y} + \frac{1}{2}(d\psi, d\psi),$$

where the first term is the canonical symplectic form on the cotangent bundle which is identified with the tangent bundle using the Riemannian metric on Y.

§1.2.2. The quantization

We wish to apply canonical quantization to this supermanifold of classical solutions. Were the space of classical solutions simply TY with the usual symplectic structure, then canonical quantization would yield the Hilbert space $L^2(Y)$ of complex-valued L^2-functions on Y. (More precisely, we would get the space of L^2 half-densities.) The functions on Y act by multiplication and sections of TY (i.e., vector fields) act by differentiation. But in our case where the space of classical solutions is a supermanifold, we must mix this canonical quantization of TY with a quantization of the vector spaces which are the fibers of the odd bundle $p^*(\Pi TY) \to TY$.

Let us consider a real, even dimensional super vector space V which is purely odd with respect to the $\mathbf{Z}/2\mathbf{Z}$-grading. Suppose further that V has a positive definite inner product, an orientation, and a spin structure. We quantize this space by taking as the Hilbert superspace the complex spin bundle $S(V)$, with the $\mathbf{Z}/2\mathbf{Z}$-grading given by the usual decomposition of $S(V)$ into $S^+(V) \oplus S^-(V)$. The linear forms on V are quantized to operators on this space which act by Clifford multiplication:

$$V^* \otimes S(V) \to S(V).$$

Now we must mix together the two constructions. We have the odd vector bundle $p^*(\Pi TY)$ over the symplectic manifold TY. The Hilbert space is the space of $L^2(S(Y))$, the space of L^2-sections of the spin bundle $S(Y)$, again tensored by the half-densities on Y. The functions on Y act as usual by multiplication; the linear forms in TY act by covariant differentiation, and the linear forms on ΠTY act by Clifford multiplication. This space of sections of $S(Y)$ is a super Hilbert space – the $\mathbf{Z}/2\mathbf{Z}$-grading is induced with the usual decomposition of spinors in even dimensions $S(Y) = S^+(Y) \oplus S^-(Y)$.

The Hamiltonian operator for this quantization is given by quantizing the Hamiltonian function h associated with the flow $\partial/\partial t$ on the theory:

$$dh = -i_{\partial/\partial t}\omega.$$

In this case it is easy to see (cf, Problem FP2 of [I-**Homework**]) that

$$h = \frac{1}{2}|\dot{x}|^2.$$

In quantizing this function there is in general an ambiguity in the resulting second order operator (adding zero and first order operators). In the absence of supersymmetry, there would be no natural way to specify the "lower order terms."[2] Supersymmetry changes things because in our problem, the vector field

$$Q = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$$

on $\mathbf{R}^{1|1}$ (whose square is $\partial/\partial t$) is an (odd) vector field which should be quantized to give an (odd) first order differential operator. Moreover, the symbol of this operator is the symbol of the Dirac operator, and there are no lower order terms of appropriate dimension constructed only from the Riemannian geometry of Y that can be added to the Dirac operator. Thus any reasonable supersymmetric quantization will equate Q with the Dirac operator $\not{\partial}$, and since the supersymmetry algebra gives $H = Q^2$, H will be quantized as the square of the Dirac operator or in other words the Laplacian acting on spinors.

Thus, we have now chosen to have the Hamiltonian operator associated with infinitesimal time evolution given by

$$H = \not{\partial}^2$$

acting on the L^2-sections of the spin bundle $S(Y)$ thought of as a super vector bundle over Y. Fix $\beta > 0$ and let $S^{1|1}$ denote the supercircle of length β obtained by dividing $\mathbf{R}^{1|1}$ by a translation of distance β. The partition function for the resulting supersymmetric sigma model of maps of $S^{1|1}$ into Y is the supertrace of the operator of time evolution from $t = 0$ to $t = \beta$; that is to say,

$$\mathrm{tr}_s \exp(-\beta H) = \mathrm{tr}_s\left(\exp(-\beta\not{\partial}^2)\right) \tag{1.2}$$

where the action is on the L^2-sections of the spin bundle $S(Y)$. Of course, by the usual spectral analysis (as alluded to in the introduction in the case of the heat

[2]The possibility of such terms reflects the fact that such terms could be added to the underlying Lagrangian, so that one cannot expect the quantum theory to be unique. Moreover in general, the matching of classical theories and quantum theories depends on the details of a quantization procedure.

equation proof of the Hirzebruch signature theorem), we see that the supertrace of this operator is independent of $\beta > 0$ and as $\beta \to \infty$ it approaches

$$\dim\left(\operatorname{Ker} \not\partial^2 | \Gamma(S^+(Y))\right) - \dim\left(\operatorname{Ker} \not\partial^2 | \Gamma(S^-(Y))\right)$$

which is the index of the Dirac operator

$$\not\partial^+ : \Gamma(S^+(Y)) \to \Gamma(S^-(Y)).$$

Our strategy is to compute a perturbative expansion for this partition function near $\beta = 0$. In fact, as we mentioned in the introduction, we will only compute the constant term in this expansion.

§1.2.3. The path integral

Now we wish to compute a power series expansion near $\beta = 0$ for the supertrace in Equation 1.2. Of course, as was explained in [I-**Faddeev**], the basic connection between the Lagrangian and Hamiltonian formulations is given by the Feynman-Kac formula. For the current application, the formula must be worked out in the context of spinor fields instead of scalar fields. This was done in detail in [J-M. Bismut, *The Atiyah-Singer theorems: a probabilistic approach. II. The Lefschetz fixed point formulas.*, J. Funct. Anal. 57 (1984), no. 3, pp. 329–348.] In any event, the perturbative power series expansion that we shall compute by path integrals over Feynman diagrams is in fact a power series that is asymptotic to the above supertrace. We shall ignore all terms which are positive powers of β, thus computing the constant term in this power series. Since we know that the series is asymptotic to a constant function, this will suffice to compute the constant term in the series.

Recall that we are considering the space of maps of the supercircle $S^{1|1}$ of length β into Y. The super Lagrangian is

$$\mathcal{L}(X) = -\int_{\mathbf{S}^{1|1}} dt d\theta \frac{1}{2} g_{IJ}(X) \frac{dX^I}{dt} DX^J = \int_{\mathbf{S}^{1|1}} dt d\theta \frac{1}{2} \langle \frac{dX}{dt}, DX \rangle.$$

The supermanifold of classical solutions is the supermanifold of maps of $S^{1|1} \to Y$ which are critical points for this Lagrangian. The underlying geometric manifold of this supermanifold is the space of closed geodesics in Y. The geodesics of minimum action are simply the constant maps of the circle to Y (which we regard as a degenerate case of geodesics). Nonconstant geodesics have strictly positive action and contribute terms that vanish exponentially fast for $\beta \to 0$. The supermanifold M of critical points that minimize the Lagrangian is identified with $\Pi TY \to Y$, an odd vector bundle over the space of constant maps from S^1 to Y. We shall do our perturbation expansion around M.

By Equation 1.1 we can rewrite this path integral as

$$\int_{x \in \operatorname{Maps}(S^1, Y); \psi \in \Gamma(x^* \Pi TY)} \mathcal{D}x \mathcal{D}\psi \exp\left(-\int_0^\beta dt \left[\frac{g_{IJ}}{2} \left(\frac{dx^I}{dt} \frac{dx^J}{dt} - \psi^I \frac{D\psi^J}{Dt} \right) \right] \right).$$

Rescale by setting $t' = \beta t$ and $\psi' = \frac{1}{\sqrt{\beta}} \psi$. Doing this and then rewriting the equation dropping the primes from the notation yields

(1.3)

$$\int_{x \in \operatorname{Maps}(S^1, Y); \psi \in \Gamma(x^* \Pi TY)} \mathcal{D}x \mathcal{D}\psi \exp\left(-\frac{1}{\beta} \int_0^1 dt \left[\frac{g_{IJ}}{2} \left(\frac{dx^I}{dt} \frac{dx^J}{dt} - \psi^I \frac{D\psi^J}{Dt} \right) \right] \right).$$

This change of variables on the circle introduces a change in the measure of integration by a formally infinite constant. The regularization procedure will determine a finite value for this constant, which depends only on the dimension of Y. In fact it is of the form C^{2n} for some universal constant. (Recall that the dimension of Y is $2n$.) We shall not try to explicitly evaluate this constant nor others like it which arise later in the argument, though it is possible to do so. Thus, our final answer must be taken up to a multiplicative constant of the form $C^{\dim Y}$.

As we have already remarked, a minimum for the Lagrangian is a pair (x_0, ψ_0) where $x_0 \in Y$ is regarded as the constant map of S^1 to Y and $\psi_0(t)$ is a constant section of ΠTY_{x_0}. We expand to first order around this point by taking $x = x_0 + a$, where $a(t) \in TY_{x_0}$ satisfies

$$\int_0^1 a(t) = 0 \tag{1.4}$$

and $\psi = \psi_0 + \eta$, where $\eta(t) \in \Pi TY|_{x_0}$ satisfies

$$\int_0^1 \eta dt = 0. \tag{1.5}$$

These variations describe the normal bundle to M at (x_0, ψ_0). To project this parameterization of the normal bundle to a submanifold in the space of maps transverse to M we consider

$$x(s,t) = \exp_{x_0}(sa(t))$$

as a map of $S^1 \times (-\epsilon, \epsilon)$ into Y and

$$\psi(s,t) = P_{(s,t)}\left(\psi_0 + \eta(t)\right),$$

where $P_{s,t}$ denotes parallel translation in ΠTY from x_0 along the path $x(s',t)$ as s' runs from 0 to s. Thus, $\psi(s,t) \in \Pi TY_{x(s,t)}$, so that $x(s,t) + \theta\psi(s,t)$ is a map from $S^{1|1}$ to Y.

Since we are at a critical point, the first-order variation of the Lagrangian with respect to a and η is zero. We are interested in the quadratic term, which is all that is needed to compute the constant term in the β-expansion of the path integral in Equation 1.3. To compute the quadratic term with respect to a and η in the Lagrangian we take the second derivative at a minimum (x_0, ψ_0). We find that the quadratic term is

$$I = -\frac{1}{2\beta}\int_0^1 dt\left[|\dot{a}|^2 + \langle \eta, \dot{\eta}\rangle - \frac{1}{2}\psi_0^I\psi_0^J R_{IJKL}a^K\dot{a}^L\right].$$

As noted above the higher order terms in a and η in the action will contribute to positive powers of β in the asymptotic expansion. In this expression the term $\dot{\eta}$ is the ordinary derivative of η with respect to t, which makes sense because for all t, the image $\eta(t)$ is contained in the vector space ΠTY_{x_0}. The term R_{IJKL} is the curvature of Y. Notice that in this integral there is no mixing in the quadratic terms between the a's and the η's. Thus, I is a sum $I_1 + I_2$, where

$$I_1(x,\psi) = -\frac{1}{2\beta}\int_0^1 dt\left(|\dot{a}|^2 - \frac{1}{2}\langle a, \psi_0^I\psi_0^J R_{IJ}\dot{a}\rangle\right)$$

and

$$I_2 = -\frac{1}{2\beta}\int_0^1 dt\langle \eta, \dot{\eta}\rangle.$$

Hence, the path integral is a product

$$\int_a \mathcal{D}a \, \exp(-I_1) \cdot \int_\eta \mathcal{D}\eta \, \exp(-I_2).$$

Also notice that the η integral

$$\int_\eta \mathcal{D}\eta \exp\left(-\frac{1}{2\beta}\langle \eta, \dot{\eta}\rangle\right)$$

is independent of x_0, a and ψ, since this integral depends only on TY_{x_0} with its inner product and the isomorphism class of this finite dimensional positive definite inner product space depends only on the dimension of Y. This path integral is purely quadratic and is identified with the Pfaffian of $\frac{d}{dt}$ acting on the Hilbert space of maps η. This is a constant which depends only on the dimension. In fact it is of the form C^{2n} for some universal constant C. Notice also that, in the regularization that we are performing, these purely quadratic integrals are independent of β.

Thus, ignoring such multiplicative constants, to compute the constant term of the expansion in β, we must compute

$$\int_a \mathcal{D}a \, \exp\left(-\frac{1}{2}\int_0^1 dt \left(|\dot{a}|^2 - \frac{1}{2}\langle a, \psi_0^I \psi_0^J R_{IJ} \dot{a}\rangle\right)\right)$$

By integrating by parts from 0 to 1 we can rewrite the integrand of the functional integral as the exponential of

$$-\frac{1}{2}\int_0^1 \langle D_B(a), a\rangle$$

where the kinetic operator D_B depends on (x_0, ψ_0) and is given by

$$D_B(x_0, \psi_0) = -\frac{d^2}{dt^2} - \frac{1}{2}\psi_0^I \psi_0^J R_{IJ}\frac{d}{dt}.$$

Thus, modulo β, the path integral over the normal space to the supermanifold M of minima at the point (x_0, ψ_0) of I_a gives

$$\frac{1}{\sqrt{\det_a D_B(x_0, \psi_0)}}$$

where the determinant is computed over the space of a for fixed (x_0, ψ_0) and is regularized using the ζ-function regularization. The final formula for the path integral is then

$$\int \mathcal{D}x\mathcal{D}\psi \, \exp(-I_a - I_\eta) \equiv C^{2n} \int_{\Pi TY} \frac{1}{\sqrt{\det D_B(x, \psi)}} \pmod{\beta}. \tag{1.6}$$

Here C is a universal constant (explicitly calculable with a little effort).

It is convenient to realize that this integral over the supermanifold ΠTY can be reinterpreted as an ordinary integral of a top dimensional form over the underlying manifold Y. To see this note that, as explained Chapter 2 of [I-**Supersymmetry**], the C^∞-functions on ΠTY are simply the C^∞-sections of $\wedge^*(T^*Y)$ over Y with the usual $\mathbf{Z}/2\mathbf{Z}$-grading; that is to say, the C^∞-functions on ΠTY are exactly the C^∞-differential forms on Y with the $\mathbf{Z}/2\mathbf{Z}$-grading being the degree modulo two. Let f be such a C^∞-function and let ω be the corresponding differential form on Y. We have an orientation on Y. Since $\mathrm{Ber}(TY)$ and $\mathrm{Ber}(T^*Y)$ are naturally dual line bundles, so that their product is canonically trivial, it follows that the Berezinian of the supermanifold ΠTY is canonically trivial. This means that the objects we

can integrate over ΠTY are ordinary functions. For a function $f \in C^\infty(\Pi TY)$ the integral

$$\int_{\Pi TY} f$$

is simply the integral over the fundamental cycle of Y of the corresponding differential form ω.

Thus, we have the path integral expression for the index of the Dirac operator:

$$\text{index}(\not{\partial}^+) = (C')^n \int_Y \frac{1}{\sqrt{\det D_B(x,\psi)}},$$

where the determinant is computed using the ζ-function regularization, where C' is a universal constant, and, as discussed above, the ψ-dependence turns the integrand into a differential form (with components of various degrees).

§1.2.4. Evaluation of the determinant

Our next task is to compute this determinant as a differential form. Let us fix first a element $g \in so(TY_{x_0})$ and consider the operator

$$D_g = -\frac{d^2}{dt^2} - \frac{1}{2} g \frac{d}{dt}$$

acting on the space of maps $a\colon S^1 \to TY_{x_0}$ which satisfy the constraint given in Equation 1.4. We can decompose TY_{x_0} into two-dimensional spaces invariant under g such that on the $i\,th$ space g is given by the matrix

$$\begin{pmatrix} 0 & x_i \\ -x_i & 0 \end{pmatrix}$$

with respect to an orthonormal basis. The determinant of D_g will be the product over the blocks in this decomposition of the determinants associated to the individual two-dimensional pieces associated with each block. This allows us to reduce to the two-dimensional case. (This step is similar in spirit to the reduction step to Riemann surfaces in some other proofs of the same result.) Fix a two-dimensional subspace V of TY_{x_0} on which g has the form

$$g = \begin{pmatrix} 0 & x \\ -x & 0. \end{pmatrix}$$

We use the metric and orientation of TY_{x_0} to induce a complex structure on V. With respect to this structure g is simply multiplication by $-ix$. We can decompose the complexification of the space of maps of S^1 into V into two-dimensional spaces of the form $\exp(2\pi ikt)a$ for $a \in V$. Notice that by the constraint given in Equation 1.4, this decomposition is over $k \neq 0$. It is easy to see that the eigenvalues of D_g on the two-dimensional subspace indexed by k are

$$(2\pi k)^2 \pm (2\pi k) ix/2.$$

Thus,

$$\sqrt{\det D_g} = \prod_{\pm} \prod_{k \neq 0} \sqrt{((2\pi k)^2 \pm i(2\pi k)x/2)}.$$

Grouping together the four terms for each pair $k, -k$ we get

$$\sqrt{\det D_g} = \prod_{k=1}^{\infty} (2\pi k)^4 \left(1 + \left(\frac{x}{4\pi k}\right)^2\right).$$

As in Problem FP15 of [I-**Homework**], the ζ-function regularization sets the infinite factor

$$\prod_{k=1}^{\infty}(2\pi k)^4$$

equal to C^2 for some universal constant C. Thus, the result is that

$$\sqrt{\det D_g} = C^2 \prod_{k=1}^{\infty}\left(1+\left(\frac{x}{4\pi k}\right)^2\right) = C^2\left(\frac{x/2}{\sinh(x/2)}\right)^{-1}.$$

This computation was done under the assumption that TY was two-dimensional. In the general case when TY is a sum of two-dimensional spaces of the above form we find

$$\sqrt{\det D_g} = C^{2n}\prod_{i=1}^{n}\left(\frac{x_i/2}{\sinh(x_i/2)}\right)^{-1}.$$

This power series can be rewritten using the elementary symmetric functions of the x_i. But by the definition of the $\hat{A}$ genus, what we obtain is

$$(\det D_g)^{-1/2} = C^{2n}\hat{A}(g).$$

Now let us return to $\det D_B^{-1/2}$ as a function on ΠTY, or as we have seen, as a differential form on Y. Consider the function $\sum_{I,J} R_{IJ}(x_0)\psi_0^I\psi_0^J$ on ΠTY with values in $\mathrm{ad}(TY)$. As indicated above this function is identified with a two-form on Y with values in $\mathrm{ad}(TY)$. Not surprisingly, under this identification the sum becomes the curvature two-form $R(Y)$ of Y. Thus,

$$\det(D_B)^{-1/2} = C^{2n}\hat{A}(R(Y)).$$

We have now established that

$$\mathrm{index}\,(\not{\partial}^+) = C^{2n}\int_Y \hat{A}(R(Y))$$

where C is a universal constant. Of course, the usual formula of Atiyah-Singer is

$$\mathrm{index}\,(\not{\partial}^+) = \int_Y \hat{A}(R(Y)).$$

In this derivation we have been careless with universal constants depending only on dimension (infinite but regularized in some manner). To evaluate the constant C one can either follow through the regularization procedure or simply evaluate it on one example where the index is nontrivial.

§1.3. The case of a circle action

In this section Y is a compact $2n$-dimensional spin manifold with an isometry $g\colon Y \to Y$. We suppose that this automorphism lifts to the bundle of spinors. We define the g-index of the Dirac operator to be

$$\mathrm{Tr}_{\mathcal{H}_+} g - \mathrm{Tr}_{\mathcal{H}_-} g$$

where $\mathcal{H}_\pm$ are the plus and minus harmonic spinors. Equivalently, we can compute this as

$$\mathrm{tr_s}\left(g\,\exp(-\beta H)\right),$$

where H is the square of the Dirac operator. As before, there is a path integral computation of a power series expansion in β asymptotic to this supertrace at $\beta = 0$. The constant term in this series computes this g-index. In fact, this procedure works

for any isometry g which lifts to the spin bundle (see Problem FE1 of [**I-Homework**]). But here we shall suppose that g lies in a connected compact group of isometries, which is equivalent to assuming that it lies in a circle of symmetries.

So we consider an action $S^1 \times Y \to Y$ by isometries. At the possible expense of doubling the circle, this action automatically lifts to the bundle of spinors. We take a Killing vector field V for this action (the vector field generating the action), and denote by $\mathcal{L}_V$ the Lie derivative of this vector field on the spinors. We consider

$$F(\theta) = \mathrm{tr}_s\left(\exp(i\theta\mathcal{L}_V)\exp(-\beta H)\right) = \sum_{n\in\mathbf{Z}} \exp(in\theta)\mathrm{ind}_n\not{\partial}^+,$$

where $\mathrm{ind}_n\not{\partial}^+$ is the index of the Dirac operator from the subspace $\mathcal{H}_n^+$ of plus spinors on which the character of S^1 is the nth power to the corresponding subspace $\mathcal{H}_n^-$ of minus spinors. In what follows we denote $\mathrm{ind}_n\not{\partial}^+$ by a_n.

Our goal is to prove the theorem of Atiyah-Hirzebruch.

Theorem 1.3.0.1. *If $S^1 \times Y \to Y$ is a nontrivial isometric circle action on a spin manifold Y then* $\mathrm{index}_n(\not{\partial})^+ = a_n = 0$ *for all n.*

§1.3.1. Proof of Atiyah-Hirzebruch Theorem using the Hamiltonian formalism

Instead of computing this via path integrals as we did in the case of a trivial action above, we introduce a Hamiltonian approach. (The main reason for explaining this is that in the process we will become familiar with things we will need in the next lecture.) We perturb the Dirac operator to an operator depending on a real parameter t defined by

$$\not{\partial}_t(\psi) = \not{\partial}(\psi) + tV\cdot\psi$$

on spinors. (Here $V\cdot\psi$ is Clifford multiplication.) It is a formally self-adjoint operator. The Hamiltonian is

$$H_t = (\not{\partial}_t)^2 = (\not{\partial})^2 + t^2|V|^2 + t\left(2\nabla_V + \mathrm{div}(V) + dV^*\cdot(\ )\right),$$

where V^* is the one-form dual under the metric to V and the last term is Clifford multiplication by dV^*. Once again this is a formally self-adjoint operator. Since the circle action preserves the metric, $\mathrm{div}(V) = 0$. Thus, we have

$$H_t = (\not{\partial}_t)^2 = (\not{\partial})^2 + t^2|V|^2 + t\left(2\nabla_V + dV^*\cdot(\ )\right).$$

We also have a formula for the Lie derivative $\mathcal{L}_V$ as follows:

$$\mathcal{L}_V = \nabla_V + \frac{1}{4}dV^*\cdot(\ ),$$

so that we can rewrite the Hamiltonian as

$$H_t = (\not{\partial}_t)^2 = (\not{\partial})^2 + t^2|V|^2 + t\left(2\mathcal{L}_V + \frac{1}{2}dV^*\cdot(\ )\right).$$

Since $\mathcal{L}_V$ generates a circle action on the spin bundle, its eigenvalues are integers. We denote by $\mathcal{H}_n$ the eigenspace for $\mathcal{L}_V$ of eigenvalue n in the Hilbert space of all L^2-sections of the spin bundle. The vector field V generates a symmetry of H and in fact of H_t for every $t \in \mathbf{R}$. Hence, $\mathcal{L}_V$ commutes with H_t. The index a_n we are trying compute is simply the index of the restriction of the operator defined by H (or H_t) from the invariant space $\mathcal{H}_n^+$ to the invariant subspace $\mathcal{H}_n^-$; or put another way, using self-adjointness of H_t it is the dimension of the kernel of H_t on $\mathcal{H}_n^+$ minus the dimension of the kernel of H_t on $\mathcal{H}_n^-$. We shall prove the theorem by

showing that if $tn \geq 0$ and $|t| >> 1$ then H_t is positive on $\mathcal{H}_n$ and in particular has no kernel on this subspace. Since, for any given n, we can take t positive or negative as we wish, it follows immediately that $a_n = 0$ for all n. (When we consider in the next lecture the Dirac operator on loop space, we will be in a situation where we can consider only positive t or only negative t, since there is a discontinuity at $t = 0$. Hence a similar argument will prove vanishing of a_n only for $n \geq 0$ or only for $n \leq 0$.

For large $|t|$ the spectrum of H_t restricted to the subspace $\mathcal{H}_n$ can be calculated in an asymptotic expansion in powers of $1/|t|$ by expanding near the minima of the potential. These minima are the zeros of V. We shall work first in the case when the S^1-action has isolated fixed points. We consider the more general case briefly at the end of this section. Let us consider an isolated fixed point $\alpha \in Y$. We decompose the linearization of the S^1-action at this fixed point into a sum of two-planes:

$$TY_\alpha = \sum_j W_j$$

where the restriction of V on W_j is given in orthonormal coordinates (x_{2j-1}, x_{2j}) by

$$k_j \left(x_{2j-1} \frac{\partial}{\partial x_{2j}} - x_{2j} \frac{\partial}{\partial x_{2j-1}} \right), k_j \in \mathbf{Z}^+.$$

We use the orientation induced on W_j by these coordinates. The condition that the S^1-action lifts to the spin bundle implies that

$$\sum_j k_j \equiv 0 \pmod 2. \tag{1.7}$$

To leading approximation in $1/|t|$ the restriction of H_t near α is given by

$$H_t^{\mathrm{loc}}(\alpha) = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} t^2 k_j^2 (|x_{2j-1}|^2 + |x_{2j}|^2) - t \sum_{j=1}^{n} k_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2t\mathcal{L}_V.$$

Ignoring the last term and splitting along the two-dimensional subspaces for the moment we consider the operator

$$\begin{aligned} H_t^{\mathrm{loc},j}(\alpha) = &-\frac{\partial^2}{\partial x_{2j-1}^2} - \frac{\partial^2}{\partial x_{2j}^2} \\ &+ t^2 k_j^2 (|x_{2j-1}|^2 + |x_{2j}|^2) - t k_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 \leq j \leq n. \end{aligned} \tag{1.8}$$

With these definitions we have

$$H_t^{\mathrm{loc}}(\alpha) = \sum_{j=1}^{n} H_t^{\mathrm{loc},j}(\alpha) + 2t\mathcal{L}_V.$$

Let us consider the spectrum of the harmonic oscillator which is the sum of all but the last term in the expression for $H_t^{\mathrm{loc},j}(\alpha)$:

$$-\frac{\partial^2}{\partial x_{2j-1}^2} - \frac{\partial^2}{\partial x_{2j}^2} + t^2 k_j^2 (|x_{2j-1}|^2 + |x_{2j}|^2).$$

Its ground state Ω has eigenvalue $2|t||k_j|$, and its spectrum is

$$\{(4\ell + 2)|t||k_j\} \quad \ell = 0, 1, 2, \cdots$$

with each eigenvalue being of multiplicity two.

The last term in Expansion 1.8 commutes with the harmonic oscillator and has eigenvalues $\pm tk_j$. Thus, it splits each of the two-dimensional eigenspaces into two one-dimensional spaces with distinct eigenvalues. Thus, the lowest eigenvalue for $H_t^{\mathrm{loc},j}(\alpha)$ is $|t||k_j|$, and its spectrum is

$$\{(2r+1)|t||k_j|\},\ r=0,1,2,\cdots.$$

Though we do not need it now, it will be important later to understand how $\mathcal{L}_V$ acts on these eigenstates. A direct computation in these local coordinates shows that the eigenstate of $H_t^{\mathrm{loc},j}(\alpha)$ with eigenvalue $(2r+1)k_j|t|$ is an eigenstate for $\mathcal{L}_V$ with eigenvalue $(r+1/2)k_j$. Also notice that if $n>0$ then $\mathrm{Ker}H_t\cap\mathcal{H}_n$ consists of plus spinors and if $n<0$ it consists of minus spinors.

Summing up over the two-planes and using the definition of $\mathcal{H}_n$, we see that the lowest eigenvalue for $H_t^{\mathrm{loc}}(\alpha)$ on $\mathcal{H}_n$ is

$$\sum_j |t||k_j| + 2tn.$$

As $|t|\to\infty$, any element of norm one in the kernel of the global Hamiltonian operator must localize around the fixed points. In fact it must converge at each fixed point to an element in the kernel of $H_t^{\mathrm{loc}}(\alpha)$. But we have already seen that for $tn\geq 0$ the local operators are positive and have no kernel. It follows that if $t>>0$ and $tn\geq 0$ there is no kernel for the restriction of H_t to $\mathcal{H}_n$.

This proves the Atiyah-Hirzebruch theorem stated above, at least in the case when the circle action has isolated fixed points. We postpone momentarily a discussion of the more general case.

§1.3.2. A local integral for the index of the Dirac operator

Even though we have established the Atiyah-Hirzebruch theorem that the a_n vanish, it is still interesting to give a different computation of the a_n in terms of local data around the fixed points of the circle action. This computation is particularly relevant when we get to the case of twisted Dirac operators where the equivariant indices do not all vanish.

The function $F(\theta)=\sum_n a_n e^{in\theta}$ is computed as the supertrace of the circle action on the kernel of the operator H_t for any t. The convergence of the operators H_t to the sum of the local operators H_t^{loc} for large $|t|$ implies that in fact we can do the computation by using the kernels of the local operators at the fixed points. Let us fix $t<0$ and compute the kernel of $H_t^{\mathrm{loc}}(\alpha)$. Note that the argument in the last subsection did not show that this kernel vanished, only that the restriction of H_t to the $\mathcal{H}_n$ with $tn\geq 0$ had no kernel.

As we have already seen, the spectrum of the operator $H_t^{\mathrm{loc},j}(\alpha)$ is

$$\{(2r+1)|k_j||t|\},\ r=0,1,\cdots,$$

with the eigenspaces of dimension one, generated by λ_r, say. Furthermore, the vector field V is the infinitesimal generator of an S^1-action on the state λ_r whose character is given by $e^{(r+1/2)k_j\theta}$. Thus, V is the infinitesimal generator of an S^1-action whose character on the sum of the states is given by

$$e^{k_j i\theta/2}+e^{3k_j i\theta/2}+\cdots=\frac{e^{k_j i\theta/2}}{1-e^{-i\theta k_j}}=\frac{i}{2}\frac{1}{\sin(k_j\theta/2)}. \tag{1.9}$$

Next we consider the sum of all possible tensor products over j of eigenspaces of the $H_t^{\mathrm{loc},j}$. The vector field V is the infinitesimal generator of an S^1-action on this sum of tensor products whose character is given by

$$F_\alpha(\theta) = \prod_{j=1}^{n} \left(\frac{i}{2} \frac{1}{\sin(k_j \theta/2)} \right). \tag{1.10}$$

Notice also that since $t < 0$, the computation above of the action of $\mathcal{L}_V$ on the eigenstates of the $H_t^{\mathrm{loc},j}$ immediately implies that all these tensor product states are in the kernel of our local operator $H_t^{\mathrm{loc}}(\alpha)$ and are plus spinors. It is clear that this accounts for the entire kernel of the local operator. Notice that even though the k_j are not necessarily even, all the exponents that appear in the above product expansion are of the form

$$((2r_1 + 1)k_1 + \cdots + (2r_d + 1)k_d)\,\theta/2,$$

and by the spin condition (Equation 1.7) are integral multiples of θ. Hence, $F_\alpha(\theta)$ is a character of an action of the original S^1, not some covering of it. It is the character of the action of the circle on the kernel of $H_t^{\mathrm{loc}}(\alpha)$. Thus, $F_\alpha(\theta)$ is the local contribution at α of the S^1-equivariant index.

Summing up over the various isolated fixed points α gives

$$F(\theta) = \mathrm{tr}_s \left(e^{\imath\theta\mathcal{L}_V} e^{-\beta H} \right) = \left(\frac{i}{2} \right)^n \sum_\alpha (-1)^{\sigma_\alpha} \prod_{j=1}^{n} \frac{1}{\sin(k_{\alpha,j}\theta/2)}.$$

where α ranges over the fixed points of the circle action (still assumed to be isolated), and σ_α compares the orientation on TY_α coming from the ambient orientation of Y with the orientation induced from the direct sum of the orientations on the various two-planes $V_{\alpha,j}$. (The $k_{\alpha,j}$, $j = 1, \ldots n$, are the rotation numbers of the circle action at the fixed point α.)

Of course, in this case we have shown that this sum is in fact zero. This is a nontrivial statement about the nature of the S^1-action at the fixed points. (Recall that we are assuming in the discussion so far that the S^1-action has only isolated fixed points.)

§1.3.3. Dirac Operators with values in a bundle

If we pass from the case considered above of the pure Dirac operator to the case of a Dirac operator twisted by a bundle with a connection, then the vanishing theorem of Atiyah-Hirzebruch does not generalize. Nevertheless, the second argument computing the equivariant index in terms of local data at the fixed points does generalize. It produces the Atiyah-Bott formula.

Let $\xi \to Y$ be a vector bundle over a compact, even dimensional spin manifold. We consider the Dirac operator twisted by this bundle. The local analysis around the fixed points α involves the two-plane bundles $W_{\alpha,j}$ at α as well as the restriction $\xi|_\alpha$. The formula is (assuming that the S^1-action has only isolated fixed points)

$$F_\xi(\theta) = \mathrm{ind}_{S^1}(\not{\partial} \otimes \xi) = \left(\frac{i}{2} \right)^n \sum_\alpha (-1)^{\sigma_\alpha} \prod_{j=1}^{n} \frac{1}{\sin(k_{\alpha,j}\theta/2)} \mathrm{Tr}_{\xi|_\alpha} \left(e^{i\theta\mathcal{L}_V} \right).$$

If ξ is a trivial bundle then we are back to the case of the pure Dirac operator and we know that the S^1-index $F_\xi(\theta)$ is identically 0. There is an important generalization of this result for a certain class of spin bundles $\xi \to Y$. Let us consider the case

when $\xi = S(Y)$, the spin bundle of Y. While it is no longer true that $F_\xi(\theta) = 0$, it is true is that $F_\xi(\theta)$ is a constant function. The reason is that $S(Y) \otimes S(Y)$ can be identified with the differential forms and hence the kernel spaces are identified with the harmonic forms. Of course, $\mathcal{L}_V$ acts trivially on these spaces. One could ask a more general question here: Let $P_Y \to Y$ be the principal Spin$(2n)$-bundle of Y and let $\mathcal{R}$ be a (finite dimensional) representation of Spin$(2n)$. We define

$$\xi = P \times_{\mathrm{Spin}(2n)} \mathcal{R}$$

and consider the character-valued index function $F_\xi(\theta)$ as above. The question is: For which W constructed in this manner is the function $F_\xi(\theta)$ equal to a constant? We shall see in the next lecture that there is an infinite series of such representations $\mathcal{R}$ with this property. By constructing enough examples Landweber and Stong (cited at the end of Lecture 2) showed that the infinite series that we construct includes all the representations with this property.

§1.3.4. The Case of Non-isolated Fixed Points

Let us finish this section with a brief discussion of the case when the S^1-action does not have isolated fixed points. Suppose that the fixed point set is $\coprod_r M_r$, with the M_r being connected. Again we consider the deformation

$$\not{\partial}_t(\ ) = \not{\partial}(\ ) + tV(\cdot).$$

Exactly as before, taking $|t| \to \infty$ localizes the low-lying eigenstates of H_t around the fixed components M_r of the circle action. The local Hamiltonian H_t^{loc,M_r} at the fixed component M_r has harmonic oscillators in the normal direction, analogous to the one encountered when the fixed points were isolated, and is the square of the Dirac operator along M_r. The decomposition of TY_α into two-dimensional spaces $V_{\alpha,j}$, which occurred in the case of isolated fixed points, is replaced in this case by a decomposition of the normal bundle V_r to M_r as a direct sum of subbundles $V_{r,j} \to M_r$ of dimension $2d_j$, say, on which the circle action is given by the character which is raising to the $k_j\,th$-power. We orient the $V_{r,j}$ so that $k_j > 0$, hence inducing an orientation on V_r and consequently on M_r. (If M_r is a point, then this orientation differs from the natural one on the point by the sign $(-1)^{\sigma_\alpha}$ encountered before.) The argument given above generalizes easily to this case to show that for $tn \geq 0$ there is no kernel for H_t^{loc,M_r} on the subspace $\mathcal{H}_n$. Thus, we see in this case as well that H_t has no kernel on $\mathcal{H}_n$ for $|t| >> 1$ and $tn \geq 0$, establishing the Atiyah-Hirzebruch theorem in the case of general circle actions.

One can also generalize the second argument to produce a formula in terms of fixed point data for the equivariant index. Once again it suffices to consider $|t| >> 1$ and to find the kernel of H_t^{loc,M_r}. Let us suppose first, for the purposes of illustration, that the component M_r is of codimension two and is a spin submanifold. Let k be the character of the circle action in the normal direction. Of course, our spin hypotheses imply that $k \equiv 0 \pmod 2$. For $t < 0$ we find, exactly as before, one state in the kernel of the normal operator restricted to $\mathcal{H}_n$ for $n = (2s+1)k/2$, $s = 0, 1, 2, \ldots$. Each of these states varies with the point of M_r and hence forms a complex bundle over M_r. We then have line bundles $L_1, L_3, \ldots$ with L_{2s+1} having S^1-character $(2s+1)k/2$. Let us relate the L_{2s+1} to the normal bundle V_r (which is a complex line bundle). Since the eigenstate φ_s with character $(2s+1)k/2$ is homogeneous of degree $2s+1$ in the spin bundle associated to V_r, it follows that

$L_{2s+1} = \sqrt{V_r}^{\otimes(2s+1)}$, the $2s+1$ tensor power of the square root of the normal bundle of M_r. (The fact that M_r is spin implies that V_r has a square root and even picks out one, so that these bundles are well-defined.) Thus, the spaces of vacua in the normal direction fit together to form a sum of S^1-equivariant complex line bundles. The local contribution around this component of the fixed point set is given by a sum like the one given by Equation 1.9 with $ik\theta/2$ replaced by $(x + ik\theta)/2$ where $x = c_1(V_r)$. Of course, along the fixed point set M_r, the operator H_t^{loc,M_r} is the square of the Dirac operator. The usual index theorem for the Dirac operator then computes the local contribution at this component M_r to the equivariant index to be

$$\int_{M_r} \hat{A}(M_r) \frac{1}{2} \frac{1}{\sinh((x + ik\theta)/2)}.$$

If M_r is codimension-two but not spin, then two things break down in the above argument: There are no spin bundles on M_r and hence no Dirac operator on M_r, and we cannot form the square root line bundle to V_r. Nevertheless, the description of the kernel states in the normal direction is still valid locally. What happens is that the obstruction to defining the spin bundles and Dirac operator globally on M_r is exactly canceled by the obstruction to forming the square root line bundle to V_r. Thus, the local descriptions fit together to define an operator which is locally described as a Dirac operator with values in a line bundle but is globally twisted. The index theorem still applies in this twisted situation and gives the same formula. (Now, of course, $x/2$ is only a rational cohomology class.)

More generally, when the normal bundle V_r to M_r splits as a sum of complex line bundles invariant under the S^1-action, the space of vacua is a sum of terms which are tensor products of the above type, and the normal contribution is a product of terms of the above type:

$$\prod_j \frac{e^{(x_j + ik_j\theta)/2}}{1 - e^{x_j + ik_j\theta}} = \prod_j \frac{1}{2\sinh((x_j + ik_j\theta)/2)},$$

where x_j is the first Chern class of the $j\,th$ line bundle and k_j is the weight of the S^1-action on this line bundle. In the more general case when V_r does not split as a sum of complex line bundles invariant under the S^1-action case we get the expression for the normal contribution by using the splitting principle. Suppose that the normal bundle V_r to M_r splits as $\oplus_j V_{r,j}$ as described before. Recall that $d_j = \dim_{\mathbf{C}}(V_{r,j})$. The resulting S^1-equivariant Chern character is

$$\mathrm{ch}_\theta(V_r) = 2^{-\dim(V_r)/2} \prod_j \prod_{\ell=1}^{d_j} \frac{1}{\sinh((x_{\ell,j} + ik_j\theta)/2)}$$

with the $x_{\ell,j}$ being the formal roots whose elementary symmetric functions are the Chern classes of $V_{r,j}$. (Of course, as always, this expression must be rewritten in terms of the elementary symmetric functions in order to find the expression in terms of the Chern classes of $V_{r,j}$.) This describes the bundle of vacua over M_r for the operator H_t^{loc,M_r}. Of course, so far we have been describing this operator only in the normal directions to M_r. Along M_r it is the square of the usual Dirac operator. Thus, the index of H_t^{loc,M_r} is simply that of the Dirac operator with values in the bundle of vacua in the normal direction. (As before, if M_r is not a spin submanifold, then neither the spin bundles along M_r nor the spin bundle in the normal direction can be independently defined, but the index formula works

exactly as if they could.) The usual index theorem now evaluates the S^1-index of this operator to be

$$F_{M_r}(\theta) = \int_{M_r} \hat{A}(M_r)\mathrm{ch}_\theta(V_r).$$

Adding up the above expression for the local S^1-indices over all the components of the fixed point set yields the following formula for the S^1-index of H:

$$F(\theta) = \sum_r \int_{M_r} \hat{A}(M_r)\mathrm{ch}_\theta(V_r). \tag{1.11}$$

This formula together with the Atityah-Hirzebruch theorem that the equivariant signature vanishes can then be viewed as giving nontrivial information about the nature of the fixed point submanifolds of the S^1-action.

More generally, if we are considering a Dirac operator coupled to a connection on an S^1-vector bundle $\xi \to Y$ then we replace V_r in Equation 1.11 by

$$V_r \otimes \left(\oplus_b e^{i\theta b}\xi_{r,b}\right)$$

where

$$\xi | M_r = \oplus_b \xi_{r,b}$$

and the character of the circle action on $\xi_{r,b}$ is the $b\,th$-power. With this substitution for $\mathcal{V}_r$, the same equation gives the index of the coupled Dirac operator as a function on S^1.

§1.4. σ-models in $1+1$ dimensions

To finish up for today, let us study in a rough qualitative way the case of maps of a two-dimensional supermanifold into a compact manifold Y of dimension $2n$. (We will be more quantitative next week.) We shall consider several contexts, but always with an action given by the same formula. First, let us consider the supermanifold

$$\mathrm{Maps}(\mathbf{R}^{2|2}, Y),$$

with the action given by

$$\mathcal{L} = \int_{\mathbf{R}^{2|2}} du\, dv\, d^2\theta\ g_{IJ} D_+ X^I D_- X^J$$

where the even coordinates on $\mathbf{R}^{2|2}$ are (u, v), the odd coordinates are θ_+, θ_-, and

$$D_+ = \frac{\partial}{\partial\theta_+} - \theta_+ \frac{\partial}{\partial u}$$

$$D_- = \frac{\partial}{\partial\theta_-} - \theta_- \frac{\partial}{\partial v}$$

are vector fields on $\mathbf{R}^{2|2}$. Notice that a map of $X\colon \mathbf{R}^{2|2} \to Y$ can be written as

$$x + \psi_+\theta_+ + \psi_-\theta_- + F\theta_+\theta_-$$

where x is a map $x\colon \mathbf{R}^2 \to Y$, ψ_+, ψ_- are sections of the pullback $x^*\Pi TY$ of the tangent bundle of Y, considered as an odd bundle and F is a section of x^*TY considered as an even bundle. For any super Riemann surface Σ (supermanifold of

type (2|2)) we have this action, but invariantly the $\psi_\pm$ are sections of $S^\pm(\Sigma)\otimes x^*TY$. With this identification, we can rewrite the action as

$$\mathcal{L} = \int_{\mathbf{R}^2} du\, dv \left[g_{IJ}\partial_u X^I \partial_v X^J + i\psi_+\partial_v\psi_+ \right. \\ \left. + i\psi_-\partial_u\psi_- + R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L + g_{IJ}F^IF^J \right].$$

Since the field F enters the Lagrangian purely quadratically and only algebraically (no derivatives of F appear), it is easy to integrate out the F dependence. When we do this Gaussian integration in F we are left with the constraint $F = 0$ and the same action (without the $|F|^2$ term) to integrate over the space of fields $x, \psi_\pm$.

We shall study to some degree this Lagrangian formulation of this supersymmetric σ-model, but we shall also need a Hamiltonian formulation. In the Hamiltonian formulation we study a quantum theory of maps of the supercylinder $\mathbf{R}^1/S \times \mathbf{R}^{1|2}$ into Y where S is translation by a fixed amount. Equivalently, we consider the space of maps of $\mathbf{R}^{1|2}$ into the loop space $\mathcal{L}Y$ of Y. Letting t be the geometric variable in $\mathbf{R}^{1|2}$ and letting s be the loop variable in $\mathcal{L}Y$, we identify $t = u + v$ and $s = u - v$. Under this identification, the Euler-Lagrange equation derived from the action above becomes a flow equation: it is not the usual analogue of the geodesic flow equation for maps of $\mathbf{R}^{1|2}$ into $\mathcal{L}Y$ since there are extra potential terms. Nevertheless, the space of classical solutions is given by the initial conditions at $t = 0$, where as before $F = 0$ is a constraint. That is to say, the space of classical solutions is identified with the odd vector bundle $\Pi T\mathcal{L}Y \oplus \Pi T\mathcal{L}Y$ over $T\mathcal{L}Y$ which is two copies of the tangent bundle to the loop space. Quantizing this bundle yields the super Hilbert space of sections of $\mathrm{Sym}^*\Pi T(\mathcal{L}Y)$, which should be viewed as the differential forms on $\mathcal{L}Y$.

There is one important feature we wish to discuss today. There is a $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$-symmetry of the Lagrangian given by

$$\sigma \cdot \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} -\psi_+ \\ \psi_- \end{pmatrix}$$

and

$$\tau \cdot \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} \psi_+ \\ -\psi_-. \end{pmatrix}$$

We wish to discuss whether these symmetries lift to the quantum field theory.

Before addressing this problem directly, let us consider the analogous question for the space of maps of $\mathbf{R}^{1|1}$ into a finite, even dimensional spin manifold Z. Quantizing the odd vector bundle ΠTZ over an even dimensional spin manifold Z yields the Hilbert space of L^2-sections of the spin bundle $S(Z)$, and the field ψ (analogous to say ψ_+ above) acts by Clifford multiplication. The analogue of the symmetry σ above is then the symmetry that multiplies ψ by -1. It is implemented on the Hilbert space by the involution ϵ whose ± 1-eigenspaces are the sections of $S^\pm(Z)$. By this we mean

$$\mathrm{Cl}(-\psi_+) = \epsilon \circ \mathrm{Cl}(\psi_+) \circ \epsilon,$$

where $\mathrm{Cl}(\chi)$ denotes Clifford multiplication by χ. Likewise in the case of maps of $\mathbf{R}^{1|2} \to Z$ one can implement both of the symmetries σ and τ using the two copies of the spin bundle associated to the fields $\psi_\pm$. Thus, it is clear that in this finite

dimensional analogue the symmetries σ and τ lift to the quantum level when Z is an even dimensional spin manifold.

Now consider the case of mapping $\mathbf{R}^{1|2}$ into an finite, even dimensional manifold Z, no longer assumed to be spin. Even though the manifold is not spin, as we indicated above, we can quantize the direct sum of the two copies of ΠTZ resulting in the differential forms on Z with their mod 2 grading. (Of course, if Z is spin we got the space of L^2-sections of the tensor product of $S(Z) \otimes S(Z)$, but this is naturally identified with the space of L^2 differential forms on Z.) With this quantization, it turns out that, provided that Z is oriented, it is still the case that the symmetries σ and τ lift to the realization. They are given by

$$(-1)^{\deg} = \sigma\tau$$

and

$$* = \sigma,$$

where $*$ is the Hodge $*$-operator multiplied by a factor of $i^{\deg}$ in order to make it an involution. Thus, in this finite dimensional analogue, the $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$-symmetry group lifts to the quantum theory of maps of $\mathbf{R}^{1|2}$ into Z as long as Z is orientable.

We claim that the answer in the infinite dimensional case is the same, with a fairly natural definition of the notion of $\mathcal{L}Y$ being orientable. We shall examine this question via path integrals, considering the theory of all maps of super Riemann surfaces Σ (supermanifolds of type $(2|2)$) into Y. Since σ and τ are symmetries of the Lagrangian, the question of whether they lift is the same as asking whether the symmetries are symmetries of the formal measure of integration $\mathcal{D}X\mathcal{D}\psi_+\mathcal{D}\psi_-$. It turns out that just as in the finite dimensional case $\sigma\tau$ always lifts to a symmetry of the quantum theory. Let us consider σ. First, we claim that σ is a symmetry of the measure of integration if and only if $\dim \mathrm{Ker}\mathcal{D}_+ \equiv 0 \pmod 2$ for all maps $x\colon \Sigma \to Y$. Here $\mathcal{D}_+$ is the Dirac operator on $S^+(\Sigma)$ coupled with covariant differentiation on the pullback x^*TY. The reason is that σ clearly preserves the $\mathcal{D}X\mathcal{D}\psi_-$ and σ will be a symmetry of $\mathcal{D}_+$ provided that the space of sections of the pullback bundle $S^+(\Sigma) \otimes x^*TY$ is even dimensional. The idea is that σ is acting by -1 on these fields and thus is preserving the orthogonal form. The issue is whether formally σ preserves the orientation. (In finite dimensions -1 is a symmetry of the integration measure if and only if it is orientation-preserving, which is equivalent to the statement that the space is even-dimensional.) Of course, in our context the space is infinite dimensional, so in order to determine its dimension we have to regularize it. The way this is done is the following. We have a skew-symmetric pairing on this bundle of sections given by

$$\langle \chi_1, \chi_2 \rangle = \int_\Sigma \langle \chi_1, \mathcal{D}_+(\chi_2) \rangle.$$

This pairing has a finite dimensional null space which is exactly the kernel of the Dirac operator $\mathcal{D}_+$. Given a skew-pairing A on a finite dimensional vector space V we know that $\dim(V) \equiv \dim(\mathrm{Ker}(A)) \pmod 2$. In the infinite dimensional theory that we are studying here, this is the natural notion of dimension mod 2. Thus, for the dimension of the space of sections of $S^+(\Sigma) \otimes x^*(TY)$ modulo two we use the dimension of the kernel of $\mathcal{D}_+$. Since TY is real this coupled Dirac operator is skew adjoint. Its spectrum is purely imaginary and symmetric about zero. Thus, its algebraic spectral flow is zero, and the dimension of its kernel modulo two is a topological invariant (unchanged under deformation of the operators through

elliptic skew adjoint operators). In fact, there is a mod two index theorem for the dimension of the kernel. Since TY is even dimensional, the mod two index theorem identifies the dimension of the kernel with $w_2(x^*TY)$. Thus, we see that σ will lift to an involution on the quantum field theory for all maps of all super Riemann surfaces Σ into Y if and only if $w_2(TY) = 0$.

At least in the case when $\pi_1(Y) = \{1\}$, it is natural to view the condition that $w_2(Y) = 0$ as the statement that the loop space $\mathcal{L}(Y)$ is orientable. (Of course, in analogy with finite dimensions, we do not expect to implement the Hodge star operator unless the manifold in question is orientable. To put it differently, in the next lecture we will use this theory of maps of $\mathbf{R}^{1|2}$ into $\mathcal{L}Y$ to compute equivariant signatures of the tautological circle action on $\mathcal{L}Y$. In order for these signatures to be defined, we must assume that $\mathcal{L}Y$ is orientable.) Let us explain how $w_2(Y) = 0$ should be viewed as equivalent to the orientability of $\mathcal{L}Y$. The tangent space of $\mathcal{L}(Y)$ at a map $x\colon S^1 \to Y$ is the space of sections of $x^*(TY)$. This vector bundle has a natural skew pairing on it given by

$$\omega_x(\alpha, \beta) = \langle \alpha, \beta \rangle = \int_0^1 (\alpha, \nabla_\theta \beta) d\theta$$

where the inner product on the right-hand side is induced from the metric on Y. It is clear that the radical of this pairing is the space of covariantly constant sections of $x^*(TY)$. In particular, the radical of this pairing is always finite-dimensional. As before with the coupled Dirac operator, it is easy to see that the spectrum of the skew adjoint operator associated to this pairing is discrete with all eigenvalues of finite multiplicity. Furthermore, the eigenvalues are purely imaginary and symmetric with respect to the origin. Given a closed path x_t in $\mathcal{L}(Y)$, then a local orientation of $\mathcal{L}(Y)$ at the initial point x_0 will be preserved around the loop if and only if the spectral flow of the family T_{x_t} from negative to positive is even. (By skew-symmetry the full spectral flow is algebraically zero: here we are looking at half of it – the half from negative to positive.) This spectral flow modulo two is easily identified with $w_2(Y)$ evaluated on the map of the torus $S^1 \times S^1$ into Y given by $(t, s) \to x_t(s)$. Thus, provided that Y is simply connected, we have that $w_2(Y) = 0$ if and only if $\mathcal{L}Y$ is orientable if and only if the kernel of $\mathcal{D}_+$ is even dimensional for every map of a super Riemann surface into Y.

If $w_2(Y) \neq 0$ this doesn't mean that we cannot perform path integrals for maps x with $x^*w_2(Y) \neq 0$. It simply means that the results of such path integrals will be odd under σ, as the measure is. To get something nonzero we need an operator which is a product of local operators $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$, $x_i \in \Sigma$, which is $\sigma\tau$ invariant but σ anti-invariant. Then we can calculate the expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle.$$

The symmetry σ tells us that we get zero contribution at the minima (constant maps) and the nonzero contributions come only from maps $x\colon \Sigma \to Y$ with the property that $x^*(w_2(Y)) \neq 0$. Fix a homotopy class of maps with this property. Of course, since the maps must be homotopically nontrivial, we see that

$$\int_\Sigma |dx|^2$$

is bounded away from zero. Furthermore, the maps minimizing this integral are exactly the harmonic maps in the given homotopy class. That is to say, we need to find minimal area surfaces $S \subset Y$ and then take holomorphic maps $\Sigma \to S \subset Y$.

We consider $\Sigma = \mathbf{C}P^1$, or alternatively delete a point and consider $\Sigma = \mathbf{R}^2$. Thus we are considering maps $X\colon \mathbf{R}^2 \to Y$ which converge at infinity to a constant map. The expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$$

will be of the order $\exp(-tM)$ where M is the energy of the harmonic map. (Notice that we have not normalized this expectation value by dividing by the partition function Z. To do this, we must recall that Z is the sum over all maps $x\colon \Sigma \to Y$, not just those that pull back w_2 nontrivially. The leading order term in this partition function will come from the constant maps and in particular is not exponentially suppressed by any large classical action.) The critical points of the Lagrangian will not be isolated. Even with the boundary condition at infinity of $\mathbf{R}^2$, there are four obvious parameters for such maps: rotation, translation and scaling of $\mathbf{R}^2$. There may well be others in addition to these universal ones. Thus, we will be computing an expression of the form

$$\int_{\mathcal{M}} (\text{perturbation expansion})$$

where $\mathcal{M}$ is the instanton moduli space, i.e., the space of harmonic maps in the given homotopy class. This time the propagator is more complicated than it was when we computed the Dirac index at the beginning of this lecture; and the linearization around the instanton moduli space gives only an asymptotic expansion, not an exact formula such as we obtained for the index.

References and cross references

The basic references for this material are:

L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer index theorem*, Comm. Math. Phys. **90** (1983), pp. 161-173.

S. Friedan and P. Windey, *Supersymmetric derivation of the Atiyah-Singer index theorem and the chiral anomaly*, Nucl. Phys. B **235** (1984), pp. 385-416.

M. Goodman, *Proof of character-valued index theorems*, Comm. Math. Phys. **107** (1986), pp. 391-409.

M. Goodman and E. Witten, *Global symmetries in four and higher dimensions*, Nucl. Phys. B **271** (1986), pp. 21-52.

A discussion of the Berezinian of ΠTM and the equivalence of integrating function on ΠTM and differential forms on M is contained in Chapter 2 of [I-**Supersymmetry**].

Basic material on the space of maps of $\mathbf{R}^{1|1}$ to a manifold and its Lagrangian can be found in Problem FP2 of [I-**Homework**]. A proof of the fixed point formula for general isometries of spin manifolds (not necessarily connected to the identity) is contained in Problem FE1 of [I-**Homework**].

A proof of the Atiyah-Bott formula and the Atiyah-Hirzebruch Theorem can be found, respectively, in

M. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: II, Applications*, Ann. Math **88** (1968), pp. 451-491;

M. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, in *Essays dédié à Georges deRham* (ed. A. Haefliger and R. Narasimhan), 1970, Springer-Verlag, pp. 18-28.

LECTURE 2
The Dirac Operator on Loop Space

§2.1. Introduction

In this lecture we shall extend the discussion from last time to the case of signature operators and Dirac operators on loop spaces. One consequence of this study will be a purely finite dimensional result about the constancy of certain indices for spin manifolds with circle actions.

§2.2. The Lagrangian formulation: σ-models in two dimensions

We shall consider the space (or rather the superspace) of maps of $\mathbf{R}^{2|2}$ to a finite dimensional, complete Riemannian manifold M of even dimension. Let u, v be the coordinates on the underlying space $\mathbf{R}^2$ of $\mathbf{R}^{2|2}$ and let θ_+, θ_- be the odd coordinates on this super manifold. The metric on the underlying $\mathbf{R}^2$ is the flat Minkowski metric $ds^2 = du\, dv$. On $\mathbf{R}^{2|2}$ we have the vector fields D_+ and D_- given by

$$D_+ = \frac{\partial}{\partial \theta_+} - \theta_+ \frac{\partial}{\partial u},$$

and

$$D_- = \frac{\partial}{\partial \theta_-} - \theta_- \frac{\partial}{\partial v}.$$

The Lagrangian that we consider is

$$\mathcal{L} = \frac{1}{2} \int_{\mathbf{R}^{2|2}} d^2\theta du dv \; g_{IJ} D_+ X^I D_- X^J,$$

or written invariantly using the metric $\langle \ , \ \rangle_M$ on TM

$$\mathcal{L} = \frac{1}{2} \int_{\mathbf{R}^{2|2}} d^2\theta du dv \langle D_+ X, D_- X \rangle_M = \frac{1}{2} \int_{\mathbf{R}^2} du dv \, (D_- D_+ \langle D_+ X, D_- X \rangle_M) \, |_{\theta=0}.$$

Let us write the superfield out in coordinates as

$$X = x + \theta_+ \psi_- + \theta_- \psi_+ + \theta_+ \theta_- F$$

where $x = X|_{\theta=0}$, $\psi_\pm = D_\pm(X)|_{\theta=0}$, and $F = D_-D_+(X)|_{\theta=0}$. Integrating out the odd variables yields

$$\mathcal{L} = \int_{\mathbf{R}^2} dudv \frac{1}{2}\Big(\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\rangle + \langle\psi_+, D_u\psi_+\rangle + \langle\psi_-, D_v\psi_-\rangle + \frac{1}{4}R_{IJKL}\psi_-^I\psi_-^J\psi_+^K\psi_+^L + \langle F, F\rangle\Big).$$

Of course, since F appears purely quadratically and with no derivatives, in any evaluation of path integrals we can simply integrate it out. Henceforth we simply drop the F^2 term.

The presence of the indefinite form $\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\rangle$ indicates that neither u nor v can be used as a time coordinate on $\mathbf{R}^2$. We switch coordinates letting (time coordinate) $t = (u+v)/2$ and (space coordinate) σ be $(u-v)/2$ with the metric $ds^2 = dt^2 - d\sigma^2$. Under this change of coordinates we have

$$\frac{d}{du} = \frac{d}{dt} + \frac{d}{d\sigma}$$
$$\frac{d}{dv} = \frac{d}{dt} - \frac{d}{d\sigma}$$

and analogously for covariant derivatives.

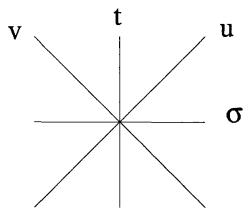


Figure 1.

We also allow the possibility that the spatial direction parametrized by σ has been made periodic so that it is a circle instead of a copy of $\mathbf{R}$.

Rewriting the Lagrangian in these variables yields

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{R}^2} dtd\sigma\Big(\langle\dot{x},\dot{x}\rangle + \langle\psi_+, \dot{\psi}_+\rangle + \langle\psi_-, \dot{\psi}_-\rangle - \langle\frac{\partial x}{\partial\sigma}, \frac{\partial x}{\partial\sigma}\rangle + \langle\psi_+, \frac{\partial\psi_+}{\partial\sigma}\rangle - \langle\psi_-, \frac{\partial\psi_-}{\partial\sigma}\rangle + \frac{1}{4}R_{IJKL}\psi_-^I\psi_-^J\psi_+^K\psi_+^L\Big).$$

The first three terms, those involving the time derivatives, are the kinetic energy terms. The other terms are potential terms.

As usual, we have symmetries of the Lagrangian associated with the right invariant vector fields

$$Q_+ = \frac{\partial}{\partial\theta^+} + \theta^+\frac{\partial}{\partial u}$$
$$Q_- = \frac{\partial}{\partial\theta^-} + \theta^+\frac{\partial}{\partial v}.$$

Of course we have

$$Q_+^2 = \frac{\partial}{\partial u}$$
$$Q_-^2 = \frac{\partial}{\partial v}.$$

§2.3. Quantization

Now that we have a time direction t it makes sense to quantize in the Hamiltonian framework. In order to quantize we restrict to the slice $t = 0$. The fields $\psi_\pm$ should be viewed as spinors or, equivalently, as half-densities on the slice $t = 0$ with values in $x^*(TM)$. Since we are taking periodic boundary values for the $\psi_\pm$ the spin bundles $S_\pm$ are trivial over the circle (i.e., give the nonbounding spin structure on S^1.) We can write

$$\int_{t=0} (\psi_+, \psi_+)$$

as a well-defined quadratic form on the space of sections on the slice. Quantizing this infinite dimensional quadratic space yields a spin space. Since we are in infinite dimensions now, the spin space depends not only on the underlying quadratic space but also on a polarization (i.e., a self-annihilating subspace in the complexification). The subspace we choose is given by the operator $D/D\sigma$ acting on the spinors. Since space is now a circle, this operator has discrete spectrum. For the spinors ψ_+ we use the polarization coming from the negative eigenspaces plus a polarization of the zero eigenspace. The choice of this latter polarization is irrelevant since it is finite dimensional. For the spinors ψ_- we choose the polarization consisting of the positive eigenspaces of $D/D\sigma$ plus a polarization for the zero eigenspace. The reason for these choices will be apparent presently. We use the name S_R for the spin space of right-moving spinors (the ψ_+) and S_L for that of left-moving spinors (the ψ_-). The Hilbert space after quantizing is the space of sections of the tensor product $S_R \otimes S_L$ over the free loop space $\mathcal{L}M$. This gives us a model for the differential forms on the loop space $\Omega^*(\mathcal{L}M)$. They are a type of middle dimensional forms in the sense that they are of neither finite dimension nor codimension.

The Hamiltonian operator has pieces quadratic in fermions

$$H_{\psi_+} = -\int_{S^1} (\psi_+, \frac{D}{D\sigma}\psi_+)$$

and

$$H_{\psi_-} = \int_{S^1} (\psi_-, \frac{D}{D\sigma}\psi_-).$$

Since we need these two operators to be bounded below, the choices we have made are forced upon us (up to a finite dimensional change which does not affect the resulting spin space).

The supersymmetries $Q_\pm$ discussed above become after quantization operators satisfying the same supersymmetry algebra they did before quantization. In addition, the operators $Q_\pm$ commute with the Hamiltonian H, which is associated with the vector field

$$\frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right).$$

Associated to the vector fields $Q_\pm$ are conserved quantities, also denoted $Q_\pm$, given by

$$Q_+ = \int_{S^1} (\psi_+, D_u x)$$

$$Q_- = \int_{S^1} (\psi_-, D_v x).$$

Writing things out in local coordinates on M, we have

$$Q_+ = \int_{S^1} d\sigma g_{IJ}\psi_+^I(\sigma)\left(\frac{\partial x^J}{\partial t} + \frac{\partial x^J}{\partial \sigma}\right).$$

Under quantization $\partial x^I/\partial t$ becomes $-ig^{IJ}\delta/\delta x^J(t)$, so that

$$Q_+ = \int_{S^1} d\sigma\left(-i\psi_+^I\frac{\delta}{\delta x^I(t)} + (\psi, \frac{dx}{d\sigma})\right).$$

The first term in this last formula for Q_+ is formally the Dirac operator for the free loop space $\mathcal{L}M$. The second term is needed to make the operator well-behaved quantum mechanically. Recall that in the last lecture (in the finite dimensional S^1-equivariant case) we had the operator $D+t\Gamma\cdot V$ where D was the Dirac operator and V was the Killing vector field of a circle action (t is a real number and Γ represents Clifford multiplication). In the case of the free loop space, there is a natural circle action which rotates the parameterization on the domain circle. Its Killing vector field V is $\frac{\partial}{\partial\sigma}$ so that the analogue of $t\Gamma\cdot V$ in this case is the second term in the last expression for Q_+. Notice that there is no parameter here playing the role of t in the finite dimensional discussion. If we tried to vary our operators replacing D/Du by $D/Dt + \epsilon D/D\sigma$, this would work fine as long as $\epsilon > 0$ (varying ϵ would change the speed of light, but that need not concern us here), but when $\epsilon < 0$ our Hamiltonian operators would no longer be bounded below. To achieve this we would have to pick the opposite polarizations for the spinors, so our operators and index theory would jump discontinuously in passing through $\epsilon = 0$. This is the basic reason that the vanishing results we will get for index theory on free loop space are weaker than the finite dimensional vanishing theorem. Half of the S^1-indices vanish in the loop space case rather than all of them as in the finite dimensional case. In the language of our discussion from the last lecture, we are forced to keep $t > 0$ and hence can only show that half the indices vanish.

Of course, for a spin manifold M of finite, even dimension we have that the differential forms can be decomposed in terms of spinors (which we denote $S(M)$) by $\Omega^*(M) = S(M)\otimes S(M)$. In general, in Riemannian geometry, there is a natural Dirac operator acting on sections of $S(M)\otimes E$, where E is any vector bundle with connection. By setting $E = S(M)$ and letting the Dirac operator act on either of the factors, we have two Dirac operators acting on $\Omega^*(M)$. These coupled Dirac operators commute and are given by $D = d + d^*$, using the Dirac operator on the first factor, and $\tilde{D} = d - d^*$, using the Dirac operator on the second factor. If M has a circle action generated by a Killing vector field V, then we can form the operators

$$Q_+ = D + \Gamma\cdot V$$
$$Q_- = \tilde{D} - \tilde{\Gamma}\cdot V,$$

where Γ and $\tilde{\Gamma}$ represent Clifford multiplication in the first and second factor respectively. These operators commute and furnish finite-dimensional prototypes for the supersymmetry generators $Q_\pm$ in the loop space case.

§2.4. The index of Q_+

Our goal is to calculate index(Q_+) on the free loop space. Of course, this index must be interpreted properly since the free loop space is of infinite dimension. One thing we need to do in this infinite dimensional context is define the "degree" of our

"forms" on the loop space, at least modulo two. How to do this was discussed at the end of the last lecture. Recall from that discussion that in our situation there is a group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ with generators acting via

$$\psi_+ \mapsto -\psi_+; \quad \psi_- \mapsto \psi_-$$

and

$$\psi_+ \mapsto \psi_+; \quad \psi_- \mapsto -\psi_-.$$

As we saw, if M is spin then the full group of symmetries exists in the quantum theory; in any event, the diagonal copy of $\mathbf{Z}/2\mathbf{Z}$ always exists as a symmetry of the quantum theory. It turns out that using the diagonal $\mathbf{Z}/2\mathbf{Z}$ produces an index which plays the role of the Euler characteristic in finite dimensions. (After all, in our finite dimensional analogy, the diagonal $\mathbf{Z}/2\mathbf{Z}$ acts with $(S^+ \otimes S^+) \oplus (S^- \otimes S^-)$ as the fixed space and $(S^+ \otimes S^-) \oplus (S^- \otimes S^+)$ as the minus one eigenspace. Under the isomorphism of $S \otimes S$ with $\Omega^*(M)$, this $\mathbf{Z}/2\mathbf{Z}$-grading corresponds to the degree modulo two. So it is not surprising that in the context of the loop space the index graded by the diagonal involution gives an Euler characteristic-type object.) But if we assume that M is spin, then we can take one of these involutions, say the involution ρ sending ψ_+ to $-\psi_+$. In the finite dimensional situation this involution is the Hodge $*$-operator, so it is reasonable to view the index computed using this involution as a generalization the usual signature of a finite dimensional manifold. This proves to be much more interesting and is the index that we shall study.

But just using the above-stated involution is not enough. In order to get finite answers, we need to use the circle symmetry on the free loop space in addition to the involution. The way we set things up is to consider the index as a formal character of the circle. That is to say we consider

$$\operatorname{index} Q_+ = \operatorname{Tr}_{\operatorname{Ker}^+(Q_+)} q^P - \operatorname{Tr}_{\operatorname{Ker}^-(Q_+)} q^P$$

where q is a formal variable, the operator P is given by

$$P = -\frac{\partial}{\partial \sigma},$$

and $\operatorname{Ker}^{\pm}(Q_+)$ refer to the decomposition of $\operatorname{Ker}(Q_+)$ into ± 1-eigenspaces under the involution ρ. We will compute this index by quantizing the space of maps of a $2|2$ supermanifold whose underlying geometric manifold is $(S^1 \times \mathbf{R})$ into a compact even dimensional spin manifold M.

§2.5. The computation around the fixed points of the S^1-action

As usual the computation localizes around the fixed points of the supersymmetry Q_+. Let us see what these are. The operator

$$Q_+^2 = H + P$$

is the Laplacian on forms plus $\mathcal{L}_V + |V|^2$, where V is the vector field generating the natural circle action on the free loop space. Thus, as in the last lecture, we compute the index by localizing around the fixed points of V. The zeros of the vector field V acting on the loop space are easy to determine: They are simply the constant loops. Together they make a copy of the manifold M, embedded as

a subspace of $\mathcal{L}M$. Thus, by an infinite dimensional analogue of the Atiyah-Bott fixed point theorem we expect the computation to yield an answer of the form

$$F(q) = \int_M \hat{A}(M)\mathrm{ch}(\mathcal{R})$$

where $\mathcal{R}$ is a bundle (or more precisely, a formal sum of bundles) that will be found presently by a computation on the normal bundle to M in the free loop space.

We made a similar computation in the last lecture for circle actions on finite dimensional manifolds. The method was to find the kernels of the Hamiltonian operators in the normal direction to the fixed point set. That is what we do here as well. Let us begin the discussion by parameterizing this normal bundle. We consider a constant loop at $x_0 \in M$. Then any nearby x can be written as

$$x(\sigma) = x_0 + \exp(\varphi(\sigma))$$

where $\varphi(\sigma) \in TM_{x_0}$ satisfies $\int_{S^1} d\sigma\varphi(\sigma) = 0$ and $\varphi(\sigma)$ satisfies the free wave equation:

$$\left(\frac{d^2}{dt^2} - \frac{d^2}{d\sigma^2}\right)\varphi = 0.$$

This means that we can decompose $\varphi(\sigma)$ uniquely as

$$\varphi(\sigma) = \varphi_+(\sigma) + \varphi_-(\sigma),$$

with φ_+ being left-moving and φ_- being right-moving. This decomposition is unique since we have required that the zero Fourier mode for φ vanishes.

Since we are considering "differential forms" on the loop space as being the tensor product of two spin bundles $S_R \otimes S_L$, we also have to consider the fermions

$$\psi_-(v), \psi_+(u).$$

If we want to consider the Dirac operator Q_+ that acts on spinors S_R, then the roles of S_R and S_L are not symmetrical. S_R is the bundle of "spinors on free loop space," while S_L is an auxiliary bundle on free loop space that just happens to be a spin bundle constructed with a different polarization. (In fact, at the end of the lecture we will replace S_L by a more general vector bundle on free loop space.) Let us then focus on the role of S_R, which is obtained by quantizing ψ_+. By quantizing the zero modes of ψ_+, we get the spinors on M (regarded as the fixed point subset of $\mathcal{L}M$), while quantization of the nonzero modes of ψ_+ gives the spinors on the normal bundle N to M. The tensor product is $S_R = S(M) \otimes S(N)$, where the two factors are $S(M)$, the spinors on M, obtained by quantizing the constant modes of ψ_+, and $S(N)$, the spinors of the normal bundle, obtained by quantizing the part of ψ_+ that is orthogonal to the zero modes. Thus $\psi_+ = \psi_{+,0} + \psi_{+,\perp}$, where $\psi_{+,0}$ is constant and

$$\int_{S^1} d\sigma\psi_{+,\perp}(u) = 0.$$

$S(N)$ is obtained by quantizing $\psi_{+,\perp}$.

First suppose that M is the flat manifold $\mathbf{R}^{2n}$. Then the theory would have no interaction terms and the action would be purely quadratic. The resulting Hilbert space would be the tensor product of spinor-valued L^2 functions on $\mathbf{R}^{2n}$ (obtained by quantizing x_0, $\dot{x}_0$, and $\psi_{+,0}$) with a tensor product Hilbert space

$$\chi = \chi^+ \otimes \chi^-,$$

where χ^+ and χ^- are Hilbert spaces of right-movers and left-movers, respectively. Concretely χ^+ comes from quantizing φ_+ and $\psi_{+,\perp}$, and χ^- comes from quantizing φ_-, ψ_-. (Because the Dirac operator whose index we will compute is Q_+ – a choice that breaks the symmetry between left and right – it is very helpful to separate out the zero modes from ψ_+, as we have done, but there is no need to similarly separate out the zero modes from ψ_-.) χ^+ is a Fock space since, for $M = \mathbf{R}^{2n}$, the theory is free. χ^- is almost a Fock space; it is actually the tensor product of a copy of $S(M)$ (obtained by quantizing the zero modes in ψ_-) with a Fock space obtained by quantizing the nonzero modes of ψ_- and φ_-. The Dirac operator Q_+ of the free model is the sum of an operator that acts in χ^+ and an operator that acts on $(\psi_{+,0}, \dot{x}_0)$; this last operator is interpreted as an ordinary Dirac operator on $\mathbf{R}^{2n}$) acting on the spinors on $\mathbf{R}^{2n}$.

Now suppose that M is not flat, but scale up the metric of M by a large factor so that the curvature is everywhere very small. Working near the constant loops in $\mathcal{L}M$, we can think of a neighborhood of M in $\mathcal{L}M$ as being fibered over M. Using the fact that the flat model is everywhere a good approximation, we make a fiberwise version of the description in the last paragraph. The Hilbert space of the theory consists of spinor-valued L^2 sections of a certain bundle of Fock-like spaces over M. The bundle over M is a tensor product $\chi = \chi^+ \otimes \chi^-$ where χ^+ and χ^- are constructed exactly as in the flat model. (We call the factors Fock-like only because χ^- is not quite a Fock space.) The idea is to compute the index of the Dirac operator by using the fibration of a neighborhood of $M \subset \mathcal{L}M$ over M; as usual the first step is to solve the Dirac equation along the fibers, which are copies of the normal bundle to M in $\mathcal{L}M$.

We write $Q_{+,\perp}$ for the Dirac operator along the fibers, the part of Q_+ that acts in χ. It is easily described. The operator $Q_{+,\perp}$ acts in χ^+ as the tensor algebra of its action on the space generated by $\varphi_+(n), \psi_+(n)$ for $n > 0$. (As usual we make Fourier expansions $\varphi_+ = \sum_n e^{in\sigma}\varphi_+(n)$, and likewise for other fields.) In each such space, $Q_{+,\perp}$ is equivalent to an operator we studied in the last lecture and has a one-dimensional kernel. So the kernel of $Q_{+,\perp}$ in the full Fock space χ^+ is the infinite tensor product L of the one-dimensional invariant subspaces. Hence its kernel in χ is identified with χ^- tensored with L:

$$\mathrm{Ker}(Q_{+,\perp}) = L \otimes \chi^-.$$

The involution that we are using acts nontrivially on χ^+ but trivially on L and trivially on χ^-. It then follows that the involution acts trivially on the kernel of $Q_{+,\perp}$. To understand better that kernel, let us examine χ^- in more detail. We have fields

$$\varphi_-(\sigma) = \sum_{n \neq 0} \varphi_n e^{in\sigma}$$

$$\psi_-(\sigma) = \sum_n \psi_n e^{in\sigma}.$$

These produce creation and annihilation operators in the Fock space χ^- which satisfy:

$$[\varphi_n^i, \varphi_m^j] = \frac{1}{m}\delta_{n+m}\delta^{ij}$$

$$\{\psi_n^i, \psi_m^j\} = \delta^{ij}\delta_{n+m},$$

where δ^{ij} is the metric in the tangent space to M at x_0. We quantize so that H is bounded below. In particular, the bosons can be quantized with a vacuum state $|\Omega\rangle$, such that

$$\varphi_n|\Omega\rangle = 0, \quad n > 0,$$

and hence the bosonic states in the Fock space are

$$\prod_{\ell=1}^{\infty} \varphi_{-\ell}^{\epsilon_\ell}|\Omega\rangle$$

of energy $\sum_\ell \ell\epsilon_\ell$.

For the fermions ψ_- the zero mode is allowed so that the quantization of ψ_0 gives a copy of the spin bundle S of the manifold M. In particular, there is not a unique vacuum, since any state of S is of zero energy. We use $|s\rangle$ for such zero energy states. As before, we want $\psi_n|s\rangle = 0$, $n > 0$ and for any zero energy state $|s\rangle$. Thus, the states in the Fock space are

$$\prod_{n=1}^{\infty} \psi_{-n}^{\delta_n}|s\rangle$$

of energy $\sum_{n=1}^{\infty} n\delta_n$. Now we are in position to describe $\mathrm{Ker}(Q_+) = \chi^-$ as an S^1-module. Clearly, from the above description we see that it is given by

$$\chi^- = S \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TM) \otimes_{n=0}^{\infty} \Lambda_{q^n}(TM).$$

Here we use the abbreviations

$$\mathrm{Sym}_{q^n}(TM) = \oplus_{m=0}^{\infty} q^{nm}\mathrm{Sym}^m(TM),$$

and similarly

$$\Lambda_{q^n}(TM) = \oplus_{m=0}^{\infty} q^{nm}\Lambda^m(TM),$$

with Sym^m and Λ^m denoting as usual the $m\,th$ symmetric or exterior powers. Finally, a factor q^n in front of a vector bundle is to indicate that the action of the circle on this bundle is by the $n\,th$ power of the standard representation.

After solving the Dirac equation along the fibers to get the kernel $L \otimes \chi^-$, the next step is to solve the Dirac equation along M with values in this kernel. Thus, formally at least, we have

$$F(q) = q^{-\dim M/24} \int_M \hat{A}(M)\mathrm{ch}\left(S \otimes_{n-1}^{\infty} \mathrm{Sym}_{q^n}(TM) \otimes_{m=1}^{\infty} \Lambda_{q^m}(TM)\right),$$

where $F(q)$ is Signature$(\mathcal{L}M)(q)$ as defined by the index of Q_+. Several comments are in order about this formula. First of all the power of q out front comes from the action of the circle on the fixed complex line L; we will explain later why this is the correct factor. Secondly, notice that only positive powers of q appear under the integral sign. The vanishing of negative powers of q is the loop space analog of the Atiyah-Hirzebruch theorem which, as we discussed in the last lecture, says that *all* nonzero characters vanish for the equivariant signature in finite dimension. (As noted earlier, for loop space we get only half-vanishing, since we cannot interpolate from positive to negative t.) Vanishing of the negative powers means we have a function of q that is holomorphic near $q = 0$ except for the prefactor. We shall see shortly that setting $q = e^{2\pi i\tau}$, this function extends to a holomorphic function of τ in the entire upper half-plane, a function that is modular for the subgroup $\Gamma_0(2)$ of $SL_2(\mathbf{Z})$.

§2.6. Path integral approach

In the path integral approach to computing the index under discussion we consider, for real θ, the expression

$$\mathrm{Tr}\left(e^{-\beta H}e^{i\theta P}(-1)^{F_\rho}\right)$$

where ρ is the involution sending ψ_+ to $-\psi_+$ and ψ_- to ψ_-. Since $Q_+^2 = H+P = 0$ on $\mathrm{Ker}(Q_+)$, on this kernel $H = -P$. This trace is equal to

$$\mathrm{Tr}_{\mathrm{Ker}^+(Q_+)}e^{-(\beta+\imath\theta)H} - \mathrm{Tr}_{\mathrm{Ker}^-(Q_+)}e^{-(\beta+\imath\theta)H}.$$

This of course is simply the supertrace

$$F(q) = \mathrm{str}_{\mathrm{Ker}(Q_+)}q^H$$

for $q = e^{-(\beta+\imath\theta)}$. Notice that $|q| < 1$ since $\beta > 0$. The path integral has a propagator $e^{-\beta H}$ (so that time is Euclidean) for fields on a circle. The term $e^{i\theta P}$ says that before we glue the ends of the cylinder together we rotate by an angle θ. This yields a Riemann surface diffeomorphic to T^2. As we have seen from the Hamiltonian discussion previously, there is a power series expression for this supertrace which is a holomorphic function of $q = \exp(2\pi i\tau)$ for τ in the upper half-plane. One can also prove directly using path integrals that this path integral is a holomorphic function of τ in the upper half-plane. (The key step in the proof is to write certain components of the stress tensor in the form $\{Q_+, \dots\}$; one then uses this to prove that the $\overline{\tau}$ derivative of the index vanishes.)

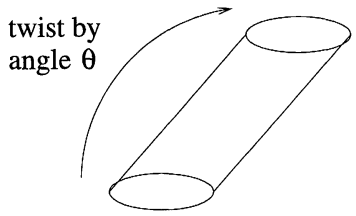


Figure 2.

In this path integral description, the fields $\psi_\pm$ are sections over T^2 of the spin bundles $S_\pm$. Since there are four spin structures on T^2, there are 16 different possibilities for the pair of spin bundles $S^\pm$ and hence 16 different possibilities for the path integral

$$\int_{X\colon T^2\to M} \mathcal{D}X\mathcal{D}\psi_+\mathcal{D}\psi_- e^{-\mathcal{L}}.$$

We need to know which spin structures we should use. These are of course determined by the boundary conditions, or gluing conditions on the fields $\psi_\pm$. In the spatial direction for both ψ_+ and ψ_- we are using the ordinary periodicity and hence the spin structure in that direction is the nonbounding one for both types of spinors. In the time direction for ψ_+ we are computing the supertrace; this means that the spin structure for S^+ in that direction is the nonbounding one. In the time direction for ψ_- we are computing the usual trace and hence the spin structure for S^- in that direction is the bounding one. Letting '+' denote the nonbounding spin structure on S^1 and '−' the bounding one we see that the spin structures are given as pictured below:

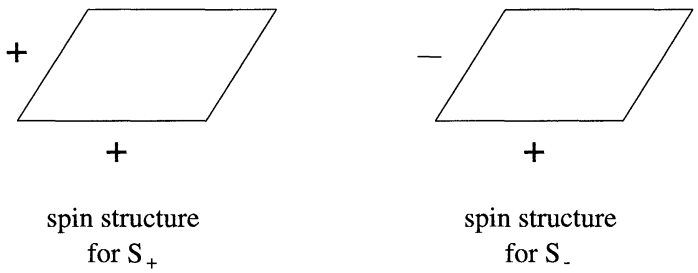


Figure 3.

Let us consider the action of $SL_2(\mathbf{Z})$ on these spin structures. First of all, the action of an element in $SL_2(\mathbf{Z})$ on a spin structure depends only on its reduction modulo two. The spin structure for S_+ is in fact invariant under the full group $SL_2(\mathbf{Z})$ whereas the subgroup leaving the second one invariant is $\Gamma_0(2)$, which consists of matrices whose lower left entry is zero modulo two. This is a subgroup of index three in $SL_2(\mathbf{Z})$ reflecting the fact that the orbit of this spin structure under $SL_2(\mathbf{Z})$ has three elements. The fundamental domain for $\Gamma_0(2)$ acting on the upper half-plane is:

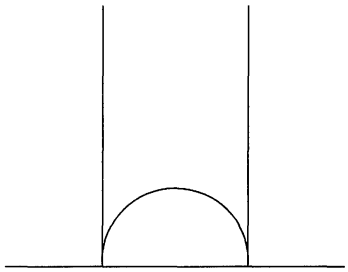


Figure 4. Fundamental Domain for $\Gamma_0(2)$.

Notice that the quotient of the upper half-plane by $\Gamma_0(2)$ has three cusps. The one represented at ∞ in the upper half-space corresponds to $\beta \to \infty$. Computing in this limit shows that the path integral reduces to the supertrace on $\mathrm{Ker}(Q_+)$ of q^H, as discussed above. The other cusps correspond to different computations. The cusps are interchanged by the action of $SL_2(\mathbf{Z})$ and hence computing in the limit at one of these means computing the same path integrals but with different spin structures.

As we have already mentioned, the action of $SL_2(\mathbf{Z})$ leaves invariant the spin bundle S_+, so that when we compute at the other cusps ψ_+ is still a section of the same spin bundle. What changes is the spin bundle S_-. Computing at one of the two other cusps means taking ψ_- as a section of

$$x^*TM \otimes \epsilon$$

where ϵ is a real line bundle on T^2 which is non-trivial over the the spatial circle. The two other cusps differ by whether ϵ is trivial in the time direction; let us consider the case that it is.

The previous analysis can be repeated to determine the index that corresponds to a path integral with this spin structure. Namely, the Hilbert space of states χ for the theory is a tensor product $\chi = \chi_+ \otimes \chi_-$ of Fock spaces. Nothing has changed with χ_+, but let us examine χ_-. As before, we have, for example, the

Fourier decomposition

$$\varphi_-(\sigma) = \sum_{n \neq 0} \varphi_n e^{in\sigma}.$$

The difference is that because of the change in the spin structure S_- around the spatial circle, the Fourier expansion for ψ_- is now

$$\psi_-(\sigma) = \sum_{m \in \mathbf{Z}+\frac{1}{2}} \psi_n e^{im\sigma}.$$

In particular, in the Fock space there is no copy of S, since there is no zero Fourier coefficient for ψ_-. (So things are now a bit simpler: $\chi^{\pm}$ are true Fock spaces, while before this was not quite so for χ^-.) Thus, the expansion at this cusp for the index is

$$F(q) = q^{-\dim M/24} \int_M \hat{A}(M) \mathrm{ch}\left(\otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TM) \otimes_{m=1/2,3/2,\ldots} \Lambda_{q^m}(TM)\right).$$

The third cusp is the image of the second cusp under the map $\tau \longrightarrow \tau + 1$. Thus, the expansion at the third cusp differs from this one simply by changing the sign of $q^{1/2}$ and thus, in the last formula, changing the sign of q^m for all m. (Recall that m is in $\mathbf{Z} + \frac{1}{2}$.)

Now there is one important point that we have not yet discussed. This is the source of the factor $q^{-\dim M/24}$ in front of these integral expressions. This comes from the S^1 action on

$$L = \mathrm{Ker}(Q_{+,\perp})$$

In fact, L is an infinite tensor product of one dimensional spaces which are obtained by quantizing $\varphi_+(n), \psi_+(n)$ for $n = 1, 2, 3, \ldots$. On the $n\,th$ such space, the circle acts with character

$$\frac{n}{2} \dim M.$$

Thus, on the tensor product the circle action contributes a multiplicative factor

$$q^{\frac{1}{2}\sum_{n=1}^{\infty} n \dim M}.$$

Of course, this factor is regularized in the renormalization procedure. It becomes

$$q^{\frac{1}{2}\zeta(-1)\dim M} = q^{-\dim M/24}.$$

As an alternative to the Hamiltonian approach that we have followed, the index of Q_+ can be evaluated by directly computing the path integral. One gets equivalent formulas, of course. Scaling up the metric of M reduces the evaluation of the path integral to one-loop computations. One requires computing Pfaffians for D_+ and D_- on spinors in a nontrivial but flat bundle over T^2 and computing the determinant of the Laplacian for the bosons. We will not explore this direction.

§2.7. Bundles whose coupled signature or Dirac operator has constant character

Let us examine the formulas we get for the index of Q_+ at the various cusps. For one cusp we get a formula of the form

$$F(q) = \int_M \hat{A}(M) \mathrm{ch}\left(S \otimes (\oplus_{n=0}^{\infty} q^n R_n)\right).$$

At the other cusps we get a formula of the form

$$F(q) = \int_M \hat{A}(M)\mathrm{ch}(\oplus_{m\in\mathbf{Z}/2} q^m \tilde{R}_m).$$

Both the bundles R_n and $\tilde{R}_m$ are SO-bundles associated to the tangent bundle of M by certain representations which we can make explicit. We claim that these bundles have a special property.

Theorem 2.7.0.1. *Suppose that M has an $U(1)$-symmetry G: $S^1 \times M \to M$, with generating vector field K. Then*

$$F(q,\alpha) = \mathrm{Tr}_{\mathrm{Ker}^+(Q_+)} q^H e^{i\alpha K} - \mathrm{Tr}_{\mathrm{Ker}^-(Q_+)} q^H e^{i\alpha K}$$

is independent of α.

Of course, this is really a statement about the signature operator coupled with the finite dimensional vector bundles R_m (or the Dirac operator with $\tilde{R}_m$) on a finite dimensional spin manifold with circle action. The statement is that for the special bundles arising from consideration of the loop space index, the indices of these operators, viewed as characters on the circle, are constant.

To establish this result, let us consider the path integrals where instead of using maps of $T^2 \to M$ we consider sections of a bundle over an M-bundle over T^2 where as we pass from time $t = 0$ to $t = \beta$ we identify M with itself by twisting by $\exp(\alpha K)$. (Of course, we could do the same sort of twisting in the space direction as well. That we will do presently.) We expand the resulting index as

$$F(q,\alpha) = q^{-\dim M/24} \sum_{k,m} q^k e^{im\alpha} c_{k,m}$$

for constants $c_{k,m} \in \mathbf{Z}$. We have already seen that

$$c_{k,m} = 0 \quad \text{for } k < 0.$$

Next we claim that:

$$c_{k,m} = c_{k+m,m}.$$

If we can establish this claim, then together with the first fact, it immediately follows that $c_{k,m} = 0$ for $m \neq 0$, which is exactly the constancy of $F(q,\alpha)$ with respect to α.

It remains to establish the above claim. To do this, let us introduce another $\gamma \in S^1$, and let us consider the Dirac operator Q_+ on a twisted loop space $\mathcal{L}_\gamma M$, the M bundle over the spatial circle obtained by gluing the ends together using the automorphism $\exp(\gamma K)$. The σ-model makes sense as a function of γ, and $F(q,\alpha)$, being an index, is independent of the continuous parameter γ. On $\mathcal{L}_\gamma M$ we have

$$e^{2\pi \frac{d}{d\theta}} = e^{\gamma K}.$$

Let us define subspaces $\mathcal{H}_{\gamma,n,m}$ in the Hilbert space of states by the conditions

$$\mathcal{H}_{\gamma,n,m} = \left\{\psi | K\psi = \psi \text{ and } P\psi = \left(n - \frac{\gamma m}{2\pi}\right)\psi\right\}.$$

The dimension of the space $\mathcal{H}_{\mathrm{Id},m,n}$ is the index $c_{n,m}$. Once again, we know that the dimension of these spaces is independent of γ (the Fredholm property of the operators). As we let γ go from 0 to 2π this has the effect of shifting the $c_{n,m}$ to $c_{n+m,m}$. This, then, establishes the claim and hence the constancy statement.

In this way we have found an infinite sequences of bundles $\tilde{R}_m$ and R_n associated to the tangent bundle of a compact even dimensional spin manifold M with

the following property. If M has an isometric circle action, then the index of the Dirac (or signature) operator coupled to these bundles is a constant character of S^1. The first $\tilde{R}$'s are:

$$\begin{aligned} \tilde{R}_0 &= \mathcal{O}; \text{ the trivial bundle} \\ \tilde{R}_{1/2} &= TM \\ \tilde{R}_1 &= TM \oplus \wedge^2 TM. \end{aligned}$$

By computing relevant bordism groups Landweber-Stong showed that for general spin manifolds the $\tilde{R}_m$ are the only bundles naturally associated to the tangent bundle with this constancy property.

§2.8. Generalization to vector bundles over the loop space

There is an analogue of this theory where one considers spinors on the loop space with values in certain vector bundles $\mathcal{V}$ over the loop space. The relevant $\mathcal{V}$'s are constructed from ordinary finite-rank vector bundles $V \to M$. Physically, this analogue is very important because it arises in heterotic string theory. Mathematically, one finds that the index of the Dirac operator with values in certain bundles made from V (rather than from the tangent bundle of M) is a constant character of the circle if M has a circle action obeying certain conditions. Let us sketch briefly the relevant version of the nonlinear sigma model and its mathematical application.

We use "chiral" superfields; we set θ_- and ψ_- to zero and consider maps

$$X\colon \mathbf{R}^{2|1} \to M,$$

where in our previous notation $\mathbf{R}^{2|1}$ is parametrized by u, v, θ_+ only. Of the two supersymmetries of the previous discussion, only Q_+ survives. The Lagrangian we consider is

$$\int d\theta_+ du dv \left((D_v X, D_{\theta_+} X\right) = \int du dv \left(g_{IJ}\partial_u x^I \partial_v x^J + \psi_+ D_v \psi_+\right).$$

Quantization of this theory would give us spinors on loop space (sections of what we earlier called S_R) rather than differential forms (sections of $S_L \otimes S_R$). To get spinors with values in a vector bundle over loop space, we add another superfield

$$\Lambda \in \Gamma\left(\mathbf{R}^{2|1}, S_- \otimes x^*(V)\right),$$

where $V \to M$ is a real $SO(k)$-bundle with a connection. We express Λ in components as $\Lambda = \lambda + \theta_+ G$. We form a new Lagrangian

$$\int d\theta_+ du dv \left((D_v X, D_{\theta_+} X) + (\Lambda, D_{\theta_+}\Lambda)\right)$$

which after performing the θ_+ integral and integrating out the auxiliary field G becomes schematically

$$\int du dv \left(g_{IJ}\partial_u X^I \partial_v x^J + \psi_+ D_v \psi_+ + \lambda D_u \lambda + F\psi_+\psi_+\lambda\lambda\right).$$

Here F is the curvature of the connection on V.

In this theory we have the operator Q_+ as before, but no longer do we have Q_-. When we go to evaluate path integrals we find that for a fixed choice of the Riemann surface Σ and the map $X : \Sigma \to M$, the fermion path integral takes values in a product of determinant line bundles $\det D_+(\lambda) \cdot \det D_-(\lambda)$. In order for the

path integral to be a real number, and not just a section of a line bundle, we need to trivialize this product of determinant line bundles. The first question then is a topological one: when is this product of determinant line bundles topologically trivial? The answer depends a bit on what choices of spin structures one wishes to use. Assuming that we want to be able to make independent choices of spin structure for both ψ_+ and λ (this is what one needs to do in the heterotic string theory, and it leads to the most interesting mathematical results), one needs the theory to be invariant under the full $\mathbf{Z}_2 \times \mathbf{Z}_2$ of independent sign changes for ψ_+ and λ. For this, we need M to be a spin manifold and V a spin bundle. The triviality of the product of determinant line bundles for any Riemann surface and any spin structures gives further

$$(*) \qquad \frac{p_1(TM)}{2} = \frac{p_1(V)}{2}.$$

(Here p_1 is the first Pontryagin class, which is naturally divisible by two for spin manifolds and spin bundles.)

The next question is to find a flat connection on this product of determinant line bundles. Mathematically, there is a standard definition of a connection on the determinant line bundle. Though it is not flat, it can be modified to be flat if one is given a 3-form H on M such that

$$dH = \frac{\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F}{8\pi^2}.$$

Here, R is the curvature of the Riemannian manifold M and F the curvature of V. The point of the formula is that the right hand side is the four-form that appears in the standard mathematical formula for the curvature of the usual connection. The equality of $p_1(TM)$ and $p_1(V)$ ensures that this four-form equals dH for some (of course not unique) H. If one is given such an H, then one modifies the usual connection on the determinant line bundle as follows. The Dirac operators whose product of determinants we wish to trivialize are defined when one is given a choice of a Riemann surface Σ and a map $X : \Sigma \to M$. A one-parameter family of such data give a three-manifold $W = \Sigma \times [0, 1]$ and a map $X_W : W \to M$. We multiply the usual parallel transport from 0 to 1 in the determinant line by the phase factor

$$\exp(i \int_W X_W^*(H)).$$

The usual local formula for the curvature of the determinant line bundle ensures that the modified connection is flat; its global holonomy is also trivial if the formula $(*)$ holds integrally and not just at the level of differential forms.

This construction is related to some very important physics. Physically, H is the curvature of the two-form field that is usually called B, and the inclusion of the H-dependent phase factor is part of the Green-Schwarz anomaly cancellation mechanism.

To give a mathematical application, we suppose that M and V have S^1-actions and we ask, as before, if the index is a constant character for the S^1. Here a new question arises: the action of S^1 induces a symmetry of the Lagrangian, but is it a symmetry of the quantum field theory? For this, we want not just that the product of determinant lines considered before should be trivial, but that if $X : \Sigma \to M$ is an S^1-invariant map, the S^1 action on the fiber of the product of determinant lines over X should be trivial. Via some further index theory, one can show that a

sufficient condition for this is that $(*)$ must hold in the S^1-equivariant cohomology, not just in ordinary integral cohomology:

$$(**) \qquad \frac{(p_1)_{S^1}(V)}{2} - \frac{(p_1)_{S^1}(TM)}{2} = 0.$$

(Here, $(p_1)_{S^1}$ refers to the Pontryagin class in S^1-equivariant cohomology.) Once we have this, the arguments as in the previous case apply to show that the Dirac index on M with values in $\tilde{R}_m(V)$ or in $S(V) \otimes R_n(V)$ is constant as a character of the circle.

One can actually make a somewhat similar statement if the left hand side of $(**)$ is not zero but is the pullback of a class in $H^4(BG)$. For if this is so, then by adding or in the sense of K theory subtracting from V a trivial bundle with nontrivial S^1 action, one can reduce to the case that $(**)$ is satisfied.

References and cross references

The basic reference for this lecture

E. Witten, *The index of the Dirac operator in loop space* in Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), 161–181, Lecture Notes in Math., 1326, 1988.

A mathematical approach to this material is contained in

R. Bott and C. Taubes, *On the rigidity theorems of Witten*, J. Amer. Math. Soc. 2 (1989), pp. 137–186.

The Landweber-Stong result is contained in

P. Landweber and R. Stong, *Circle actions on Spin manifolds and characteristic numbers*, Topology **27** (1988), pp. 145–161.

For the original path integral computation of the loop space index (or actually of an anomaly closely related to it), see

K. Pilch, A. N. Schellekens,and N. P. Warner, *Path Integral Calculation Of String Anomalies*, Nucl. Phys. **B287** (1987) 362.

Elementary Introduction to Quantum Field Theory

Ludwig Faddeev

Introduction

The invitation to participate in the Program "Quantum Field Theory", organized at IAS by Professors Witten, Deligne and Kazhdan, was unexpected by me. The definite cause for me to accept this invitation was the "Manifest" written by Professor MacPherson stating the goals of the program. In particular, he stressed that these goals were not the providing of the rigorous proofs for the results, already known by physicists (which is called Mathematical Physics by some people), but rather the conveying of the intuition developed by physicists to the minds of specialists, whose intuition is based on algebra and geometry. This for me is an important part of genuine Mathematical Physics and so I was immediately convinced to participate in the Program. In fact, I understood that my role was rather modest: I was to give a no nonsense introduction to the apparatus of QFT. I already have done this once in my lectures in Les Houches in 1975 and so decided to repeat the same approach 20 years later. Not too much has changed in the foundations of QFT during these years, so I hope that my course was not too conservative.

As in 1975, I have chosen the formalism of functional integrals to describe the quantum dynamics. The field can be considered as a collection of interacting oscillators, so that the formalism can be introduced on the example of just one oscillator, with evident generalization to the field theory. Only the considerations of locality and manifest Lorentz invariance are to be added. I am sure that this approach is the shortest way to get the basics of S-matrix theory and Feynman diagrams, beginning from nothing. (In my lectures it was done in 4 hours.) I have a textbook, published with A. Slavnov, where this type of exposition is used. As I learned during my stay in Princeton, using the excellent library of IAS, this approach is used nowadays by several authors of numerous new monographs on QFT.

Two more lectures were devoted to describing the specifics of the geometrical fields such as the Yang-Mills field and gravity and corresponding degenerate Lagrangians. This also closely follows the monograph with A. Slavnov.

Notes by Lisa Jeffrey

I decided to add to this now standard exposition a less popular explanation of the specifics of the scattering theory for the field theoretical models with conservation of momentum, which elucidates the renormalization of mass. A simple model, illustrating the renormalization of charge, was also presented in the same lecture.

In lecture 6 I presented a point of view on anomalies. This subject being huge, it was not possible to expose it with the same level of detail as in previous lectures. So I decided not to include it in the published text.

The publication of the first five lectures is made possible due to the indispensable help of Professor Lisa Jeffrey, who kindly agreed to write and edit my oral presentation. I was very fond of our collaboration and hope that it was not too much burden for her. I decided against heavy editing of notes, prepared by Professor Jeffrey, so they have the flavor of my improvisation and her effective putting of things in order. Of course, I have full responsibility for the possible misprints in formulas.

In conclusion, I want to acknowledge the atmosphere of genuine interest of the participants, which was characteristic for all lectures which I was able to attend or present. I hope that the publication of these volumes will play an important role for the implementation of the goal, stated by Professor MacPherson.

LECTURE 1

Basics of Quantum Mechanics and Canonical Quantization in Hilbert Space

§1.1. Observables and states

I decided to start my lecture by presenting a suitable and reasonably natural framework into which QM satisfactorily fits. The framework itself is an abstraction of the thoughts and work of many people. I mention P. Dirac, H. Weyl, J. Neumann, I. Segal and G. Mackey. I shall be very cryptic.

A description of physical reality is made in terms of two sets of objects: observables and states. A set of observables A, $B, \ldots$ will be denoted by $\mathfrak{A}$ and states ω, $\mu, \ldots$ by Ω. Each state ω assigns to each observable A its probability distribution $\omega_A(\lambda)$ on the real line. The pairing (mean value)

$$\langle \omega | A \rangle = \int \lambda d\omega_A(\lambda)$$

defines a duality between A and Ω. It is natural to assume the separability (completeness) of states: two observables having equal pairings with all states are equal. This immediately introduces a real linear structure on $\mathfrak{A}$, whereas Ω is a convex set. The next structure is a notion of functional dependence of observables. The observable B is a function $f(A)$ of the observable A if for any state

$$\langle \omega | B \rangle = \int f(\lambda) d\omega_A(\lambda).$$

This definition of a function is rather unwieldy, so a technical assumption is added that A lies in an algebra with product AB such that the notion of function is compatible with this product.

Now comes dynamics: each observable generates a motion of $\mathfrak{A}$, that is, a one-parameter group of automorphisms of $\mathfrak{A}$ via the differential equation

$$dA/dt = \{B, A\},$$

where $\{\ ,\ \}$ is a bracket, which is necessarily of Lie type and is a derivation of the product

$$\{A, BC\} = \{A, B\}C + B\{A, C\}.$$

This essentially finishes the description of the framework.

Both classical and quantum mechanics fit perfectly in it and give concrete and nonequivalent realizations.

All features of CM are reconstructed as soon as one assumes that $\mathfrak{A}$ is commutative. It follows that $\mathfrak{A}$ can be realized as an algebra of functions on a symplectic manifold (phase space) M: states are normalized measures on the phase space and $\{\ ,\ \}$ is a Poisson bracket, defined by the symplectic structure on it. In particular

$$\omega_A(\lambda) = \int \theta(\lambda - f_A(p,q))\, d\mu_\omega,$$

where $\theta(t)$ is a step function, $f_A(p,q)$ is a function corresponding to the observable A and μ_ω is a measure corresponding to the state ω. Pure (extremal) states are defined by atomistic measures concentrated in one point of M. All observables have exact values (zero dispersions) in all pure states.

In the case of QM, the set $\mathfrak{A}$ consists of real elements of a complex associative algebra with involution, which is usually realized as an algebra of linear operators in some complex Hilbert space H. The states are the positive trace class operators with unit trace. The distribution $\omega_A(\lambda)$ is given by

$$\omega_A(\lambda) = \operatorname{tr}(MP_A(\lambda)),$$

where M (density matrix) defines the state and $P_A(\lambda)$ is a special projector function of the operator A. The dynamical bracket is defined in terms of the product

$$\{A, B\} = (i/\hbar)(AB - BA),$$

where $i = (-1)^{1/2}$ and the normalizing factor $1/\hbar$ here employs the famous Planck constant $\hbar$. Its introduction here is necessitated just from dimensional considerations.

The pure states are given by one-dimensional projectors and a given observable is exact (dispersionless) only in pure states, defined by its eigenvectors. This feature, which is reflected in the "uncertainty" inequality

$$\Delta_\omega A \Delta_\omega B \geq (\hbar/2)|\langle\omega|\{A, B\}\rangle|,$$

where

$$\Delta_\omega A = (\langle\omega|(A - \langle\omega|A\rangle)^2\rangle)^{1/2}$$

is the main physical distinction between QM and CM. By no means does it preclude the measurability of observables in QM. Indeed, the collection of eigenstates is enough to characterize any observable. On the contrary, the exactness of observables in all pure states in CM is a kind of unnecessary luxury.

It is instructive to compare the CM and QM on some common playing field. We shall consider the simplest mechanical system with *one degree of freedom*: in other words the phase space is $M = \mathbf{R}^2$, and points in phase space are denoted (p, q). The Hilbert space is

$$\mathcal{H} = L^2(\mathbf{R}) = \{\psi(q) \in \mathbf{C} : \int_{q=-\infty}^{\infty} |\psi(q)|^2 dq < \infty\}.$$

Quantum mechanically we replace the basic observables (*canonical* coordinates) p and q by operators P and Q acting on $\mathcal{H}$: these are defined by

$$(P\psi)(q) = \frac{\hbar}{i}\frac{d}{dq}\psi(q), \tag{1.1}$$

$$(Q\psi)(q) = q\psi(q); \tag{1.2}$$

this is the so called *coordinate realization.* An observable A is characterized as an integral operator defined by its *kernel*

$$(A\psi)(q'') = \int_{-\infty}^{\infty} A(q'', q')\psi(q')dq'. \tag{1.3}$$

Alternatively we may define the observable A by means of its *symbol*, which is a function on the phase space M: the symbol f corresponds to the integral kernel A_f by means of the *Weyl quantization*

$$A_f(q'', q') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} f(p, \frac{q'' + q'}{2})e^{ip(q''-q')/\hbar}dp. \tag{1.4}$$

There is no unique way to substitute operators for functions because the function pq is equal to the function qp while the operator PQ does not equal the operator QP: one must choose a prescription for the ordering of the operators. The Weyl quantization corresponds to one choice of ordering, which is to completely symmetrize over the operators P and Q. The Weyl quantization is not a homomorphism from the algebra of functions to the algebra of operators: if one associates the operators A_f and A_g to the functions f and g respectively, one may define the kernel of the product $A_f A_g$ by

$$A_f A_g(q'', q') = \int_{-\infty}^{\infty} A_f(q'', q)A_g(q, q')dq. \tag{1.5}$$

This corresponds to a product $f \star_\hbar g$ on the algebra of functions which equals

$$f \star_\hbar g = fg + \frac{i\hbar}{2}\{f, g\} + O(\hbar^2), \tag{1.6}$$

so that

$$(i/\hbar)(f * g - g * f) = \{f, g\} + O(\hbar).$$

We see that both CM and QM can be realized in terms of the same objects (functions) but the structure constants of the main operations (product, Lie bracket) of QM are defined as a series in positive powers of $\hbar$, the zero order term being the structure constants of CM. So we can say that QM is a deformation of CM in the framework described above. The Planck constant is the corresponding deformation parameter. This is for me the most concise formulation of the correspondence principle and explains what is meant by quantization.

Now comes the notion of equivalent deformation and that of stability. Beautiful results, which I learned from A. Lichnerowicz, M. Flato and D. Sternheimer, allow one to say that CM is unstable and that QM is essentially a unique deformation of it into a nonequivalent stable structure. The degeneracy of CM (exactness of pure states) is intimately connected with its instability. So it is only natural that the passage from CM to QM was prompted by the experimental activity of physicists. The stability of QM, on the contrary, shows that an analogous modification of it is less feasible. In this case one has first to modify also all the general framework, as emphasized above.

§1.2. Dynamics

Now I turn to discussion of dynamics. The goal is to make sense of the evolution operator

$$U(t) = e^{-iHt/\hbar},$$

which defines the solution of the dynamical equation

$$\frac{dA(t)}{dt} = \{H, A(t)\}_\hbar, \ A(t)|_{t=0} = A \tag{1.7}$$

in the form

$$A(t) = U^{-1}(t)AU(t). \tag{1.8}$$

From now on we take the normalization $\hbar = 1$.

It is convenient to consider arbitrary initial and final times, so we want to construct the kernel of the evolution operator

$$U = e^{-iH(t''-t')} \tag{1.9}$$

in terms of the symbol corresponding to the Hamiltonian $h(p,q)$. Because Weyl quantization is not a homomorphism taking the product of functions to the product of operators, one cannot substitute the kernel for the Hamiltonian into the Taylor series for the exponential. Nonetheless if $\triangle$ is a small interval, one may make the approximation

$$U_\triangle = e^{-\imath H\triangle} \cong 1 - iH\triangle \tag{1.10}$$

and then if $t'' - t' = N\triangle$ one may approximate

$$e^{-iH(t''-t')} = (U_\triangle)^N. \tag{1.11}$$

We write the integral kernel for $U_\triangle$ as

$$\begin{aligned} U_\triangle(q'', q') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{ip(q''-q')}\Big(1 - ih(p, \frac{q''+q'}{2})\triangle\Big) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{ip(q''-q')-ih(p,\frac{q''+q'}{2})\triangle} + O(\triangle^2). \end{aligned} \tag{1.12}$$

To obtain the integral kernel for $e^{-iH(t''-t')}$ we compose N copies of $U_\triangle$ and integrate over the $N-1$ intermediate variables q_j using (1.5):

$$\begin{aligned} U(q'', q'; t''-t') = \int\cdots\int e^{ip_N(q_N-q_{N-1})+\imath p_{N-1}(q_{N-1}-q_{N-2})+\cdots+ip_1(q_1-q_0)} \\ \times e^{-ih(p_N,\frac{q_N+q_{N-1}}{2})-\cdots-\imath h(p_1,\frac{q_1+q_0}{2})}\frac{dp_1dq_1}{2\pi}\cdots\frac{dp_{N-1}dq_{N-1}}{2\pi}\frac{dp_N}{2\pi}. \end{aligned} \tag{1.13}$$

In the limit as N tends to infinity while $q_N - q_{N-1}$ is proportional to $1/N$, we formally obtain an integral over trajectories $(p(t), q(t))$ in phase space:

$$U(q'', q'; t''-t') = \int \exp i\Big\{\int_{t'}^{t''}(p(t)\dot{q}(t) - h(p(t), q(t)))dt\Big\}\prod_t \frac{dp(t)dq(t)}{2\pi}. \tag{1.14}$$

One recognizes

$$\int_{t'}^{t''}(p(t)\dot{q}(t) - h(p(t), q(t))\,dt \tag{1.15}$$

as the *action functional*, while $\prod_t \frac{dp(t)dq(t)}{2\pi}$ is the Liouville measure on the space of paths.

A *real polarization* of a symplectic manifold or phase space of dimension $2n$ is a surjective map π to a manifold of dimension n such that the fibres are Lagrangian submanifolds (in other words the fibres are of dimension n and the symplectic form

restricts to zero on them). The boundary condition on the trajectories in (1.14) is that $q(t'') = q', q(t') = q'$ while $p(t'')$ and $p(t')$ are unconstrained. This condition says that when t takes the extremal value t' the endpoint of the path $(p(t), q(t))$ lies in a prescribed fibre $L' \stackrel{\text{def}}{=} \pi^{-1}(q')$ of the real polarization

$$\pi : (p, q) \mapsto q \tag{1.16}$$

of the phase space, while at the other extremal value $t = t''$ the path lies in a different fibre

$$L'' = \pi^{-1}(q'')$$

of the polarization.

Remark: In our example, a phase space arising from a physical system with one degree of freedom is $\mathbf{R}^2$. Other interesting examples arise where the phase space is S^2. No real polarization exists in those cases. A complex polarization leads to a finite dimensional representation of the group $O(3)$, so that this phase space corresponds to a spin degree of freedom.

§1.3. Quantization

If the Hamiltonian is given by

$$h(p, q) = \frac{1}{2}p^2 + v(q), \tag{1.17}$$

the change of variables

$$p(t) \mapsto p'(t) = p(t) + \dot{q}(t), \; q(t) \mapsto q(t) \tag{1.18}$$

transforms the integrand in (1.15) to a form in which no terms appear which involve both $p(t)$ and $q(t)$ (in other words these variables separate):

$$p'\dot{q}' - h(p', q') = -\frac{1}{2}(p')^2 + \frac{1}{2}(\dot{q})^2 - v(q) \tag{1.19}$$

We may then integrate over $p(t)$ (to obtain a constant which is independent of $q(t)$) and recover the more usual form of the action which depends only on $q(t)$:

$$U(q'', q'; t'' - t') = \text{const} \int \exp\left\{ i \int_{t'}^{t''} (\frac{1}{2}\dot{q}^2 - v(q))dt \right\} \prod_t dq(t) \tag{1.20}$$

Here what appears is the usual Lagrangian, which depends on q and $\dot{q}$ but not on p. This change of variables thus transforms the formula (1.14) for the kernel of the time evolution operator U into a path integral over trajectories in *configuration* space (the space of possible values of the position) rather than over trajectories in *phase* space (the space of possible values of position and momentum, whose dimension is twice that of the configuration space). We remark that this method of reducing from paths in phase space to paths in configuration space works only when the action is quadratic in the momentum p. However, more general actions quadratic in p can also be treated by this method: for instance, one may use a general Riemannian metric on the configuration space rather than the flat metric p^2, which simply gives rise to a factor of the Riemannian volume element.

Let us now consider an example where the form of the potential $v(q)$ is given explicitly: the *harmonic oscillator*

$$h(p, q) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \tag{1.21}$$

Here we intentionally keep here the frequency parameter ω. This will be instructive when we turn to the free fields, which are nothing but a collection of oscillators with different frequencies. The classical equation of motion corresponding to this Hamiltonian is the equation for simple harmonic motion

$$\frac{d^2}{dt^2}q(t) + \omega^2 q(t) = 0. \tag{1.22}$$

In the approach to quantization which we have described, one finds that the eigenvectors of the Hamiltonian are defined in terms of certain orthogonal polynomials in q, the Hermite polynomials.

The use of special functions can be avoided if one alternatively considers the *holomorphic* quantization of the harmonic oscillator: this will complement the treatment given in [II-**CFT**]. We introduce complex coordinates a and a^* on the phase space, defined by

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \tag{1.23}$$

$$a^* = \frac{1}{\sqrt{2\omega}}(\omega q - ip) \tag{1.24}$$

so that

$$q = \frac{1}{\sqrt{2\omega}}(a^* + a). \tag{1.25}$$

After quantization, a and a^* become operators $\hat{a}$ and $\hat{a}^*$, which satisfy the commutation relation

$$[\hat{a}, \hat{a}^*] = 1. \tag{1.26}$$

The Hilbert space $\mathcal{H}$ now consists of entire functions in a^*: it is the closure of the space of polynomials in a^* in the inner product

$$(f, g) = \int_{\mathbf{R}^2} \bar{f}(a) g(a^*) e^{-a^* a} \frac{da^* da}{2\pi i}. \tag{1.27}$$

One finds for instance that the elements

$$\psi_n = \frac{(a^*)^n}{\sqrt{n!}} \tag{1.28}$$

in $\mathcal{H}$ are orthonormal with respect to the inner product $(\cdot, \cdot)$. The operators $\hat{a}$ and $\hat{a}^*$ are represented on $\mathcal{H}$ by

$$\hat{a} f(a^*) = \frac{df(a^*)}{da^*}, \tag{1.29}$$

$$\hat{a}^* f(a^*) = a^* \cdot f(a^*). \tag{1.30}$$

The Hamiltonian may be written as

$$H = \omega \hat{a}^* \hat{a}, \tag{1.31}$$

and it has the ψ_n as eigenvectors: we have

$$H\psi_n = n\omega\psi_n. \tag{1.32}$$

In particular, we have $\psi_0 = 1$, for which $H\psi_0 = 0$; one thus sees that

$$\psi_n = \frac{(\hat{a}^*)^n}{\sqrt{n!}}\psi_0. \tag{1.33}$$

The operators on $\mathcal{H}$ may be represented by integral kernels $A((a^*)'', a')$, which are analytic functions of two independent variables $(a^*)''$ and a'. The composition of two operators corresponds to the following operation on their kernels A and B:

$$(AB)((a^*)'', a') = \int A((a^*)'', \alpha)B(\alpha, a')e^{-\alpha^*\alpha}\frac{d\alpha^* d\alpha}{2\pi i}. \tag{1.34}$$

Occasionally we shall drop the primes on a^* and a in cases when this does not lead to confusion.

Another possibility is to use *normal symbols*: if we write the operator A as

$$A = \sum_{n,m} K_{mn}(\hat{a}^*)^n \hat{a}^m, \tag{1.35}$$

then the corresponding normal symbol is

$$K(a^*, a) = \sum_{n,m} K_{mn}(a^*)^n a^m. \tag{1.36}$$

It is an exercise to see that

$$K(a^*, a) = e^{-a^* a}A(a^*, a). \tag{1.37}$$

If we now turn to dynamics, the kernel of the evolution operator for a small time interval $\triangle$ may be written in terms of the normal symbol of the Hamiltonian $h(a^*, a)$ as

$$U_\triangle = e^{-a^* a - ih(a^*, a)\triangle}. \tag{1.38}$$

To write the time evolution operator for the time interval $t'' - t'$, we compose N copies of $U_\triangle$ as before:

$$U(a^*, a) = \int \cdots \int e^{a_N^* a_{N-1} - a_{N-1}^* a_{N-1} + a_{N-1}^* a_{N-2} - \cdots - a_1^* a_1 + a_1^* a_0} \times e^{-i\{h(a_N^*, a_{N-1}) + \cdots + h(a_1^*, a_0)\}\triangle} \prod_{k=1}^{N-1} \frac{da_k^* da_k}{2\pi i}. \tag{1.39}$$

In this context the analogue of the condition that q_N and q_0 are fixed is that a_N and a_0 are fixed:

$$a_N = a^*, \ a_0 = a. \tag{1.40}$$

In the limit as $N \to \infty$ we obtain

$$U = \int \exp\left\{a^* a|_{t=t''} + \int_{t'}^{t''} \{-a^*\dot{a} - ih(a^*, a)\}dt\right\} \prod_{t' \le t \le t''} \frac{da^*(t)da(t)}{2\pi i}. \tag{1.41}$$

For values of t that are not equal to the extremal values t' and t'', we assume that $a^*(t) = \overline{a(t)}$, but we do *not* assume this on the boundary: we impose the condition that $a^*(t'') = a^* \in \mathbf{C}$ (this is the value that was denoted by a^* in (1.40)) there and similarly $a^*(t) = a \in \mathbf{C}$ (this is the value that was denoted by a in (1.40)). The variables a^*, a are the arguments of the integral kernel in (1.39): they are arbitrary complex numbers, in particular we do *not* assume that $a^* = \bar{a}$.

If one now substitutes $h = \omega a^* a$, the integral is Gaussian; using the equations of motion

$$\dot{a} + i\omega a = 0, \ \dot{a}^* - i\omega a^* = 0 \tag{1.42}$$

with the boundary conditions $a_N = a^*$, $a_0 = a$ (where a^* and a are arbitrary complex numbers which are not required to be complex conjugate) we then see that

$$a(t) = ae^{-i\omega(t-t')}, \tag{1.43}$$

$$a^*(t) = a^*e^{i\omega(t-t')}. \tag{1.44}$$

Note that $a(t)$ and $a^*(t)$ are not complex conjugates of each other. Substituting in (1.41) we obtain

$$U_0(a^*, a, t'' - t') = \exp\left\{ia^*ae^{-i\omega(t''-t')}\right\}. \tag{1.45}$$

Applying this to an element ϕ of $\mathcal{H}$, we find

$$(U_0\phi)(a^*) = \int \exp\left(a^*\alpha e^{-i\omega(t''-t')}\right)\phi(\alpha^*)e^{-\alpha^*\alpha}\frac{d\alpha d\alpha^*}{2\pi i} \tag{1.46}$$

$$= \phi(e^{-i\omega t}a^*).$$

This shows that the evolution operator is obtained by simply substituting the same argument rotated *backward* in time.

References and cross references

Another discussion of quantization is given in [I-**Quantization**]; see also Lecture 1 of [II-**CFT**].

LECTURE 2
The Harmonic Oscillator and Free Fields

§2.1. The harmonic oscillator

We revisit the functional integral formalism in holomorphic quantization used to describe the harmonic oscillator. We make use of operators $\hat{a}$ and $\hat{a}^*$ satisfying $[\hat{a}, \hat{a}^*] = 1$. States are represented by analytic functions $\phi(a^*)$ of one complex variable a^*; operators are represented by integral kernels $A(a^{*\prime\prime}, a')$ depending on two *independent* complex variables $\hat{a}^{*\prime\prime}$ and a'. The operator whose integral kernel is A may equivalently be represented by its *normal symbol*

$$K(a^{*\prime\prime}, a') = e^{-a^{*\prime\prime}a'} A(a^{*\prime\prime}, a'). \tag{2.1}$$

We compose operators by writing the integral kernel of the composition of the operators with integral kernels A and B as

$$AB(a^{*\prime\prime}, a') = \int_{\mathbf{R}^2} A(a^{*\prime\prime}, a) e^{-a^* a} B(a^*, a') \frac{da^* da}{2\pi i}. \tag{2.2}$$

The operator corresponding to the normal symbol is written as

$$\mathcal{K} = K(\hat{a}^*, \hat{a}) \tag{2.3}$$

where the ordering of the variables has been changed in such a way that all starred variables appear to the left of all unstarred variables.

The evolution operator $U_\triangle = e^{-iH\triangle}$ was written

$$U_\triangle(a^{*\prime\prime}, a') = \exp(-a^{*\prime\prime}a' - ih(a^{*\prime\prime}, a')\triangle).$$

We pause to point out that we have been regarding $\phi(a^*)$ (which represents a state) as an analytic *function* of one complex variable. In geometric quantization it is actually a *section* of a line bundle L over the phase space $\mathbf{R}^2$. We denote the section of the line bundle as

$$\tilde{\phi}(a^*, a) = e^{-\frac{1}{2}a^* a} \phi(a^*); \tag{2.4}$$

the object $\tilde{\phi}$ (the "dressed" analytic function) is a section of a line bundle L over the phase space $\mathbf{R}^2$. The dressed kernel $\tilde{A}(a^{*\prime\prime}, a')$ is a section of the pullback of two copies of L over $\mathbf{R}^2 \times \mathbf{R}^2$:

$$\tilde{A}(a^{*\prime\prime}, a') = e^{-\frac{1}{2}a^{*\prime\prime}a''} A(a^{*\prime\prime}, a') e^{-\frac{1}{2}a^{*\prime}a'}. \tag{2.5}$$

Dressed kernels are composed by integrating over $\mathbf{R}^2$. One may also recover the ordinary kernel from the dressed kernel by integration:

$$A(a^{*\prime\prime}, a') = \int e^{a^{*\prime\prime}a - \frac{1}{2}a^*a} \tilde{A}(a^*, a|b^*, b) e^{-\frac{1}{2}b^*b} e^{b^*a'} \frac{da^*da}{2\pi i} \frac{db^*db}{2\pi i}. \tag{2.6}$$

The time evolution operator becomes

$$\begin{aligned} U(a^{*\prime\prime}, a') = \int & e^{a^{*\prime\prime}a_N - \frac{1}{2}a_N^*a_N} e^{a_N^*a_{N-1} - a_{N-1}^*a_N + \cdots - a_1^*a_1 + a_1^*a_0} \\ & \times e^{-i[h(a_N^*, a_{N-1}) + \ldots h(a_1^*, a_0)]\triangle} \prod_{k=0}^{N} e^{-\frac{1}{2}a_0^*a_0 + a_0^*a'} \frac{da_k^* da_k}{2\pi i} \\ = \int \exp\Bigg\{ & a^{*\prime\prime}a(t'') - \frac{1}{2}a^*(t'')a(t'') + i\int_{t'}^{t''} \left[\frac{\dot{a}^*a - a^*\dot{a}}{2i} - h(a^*, a)\right]\Bigg\} dt \end{aligned} \tag{2.7}$$

$$-\frac{1}{2}a^*(t')a(t') + a^*(t')a'\Bigg\} \prod_{t' \le t \le t''} \frac{da^*(t)da(t)}{2\pi i}. \tag{2.8}$$

This formula is better than (1.41) in Lecture I. All integrations here are done over a real plane R^2, in which $a^*(+)$ and $a(+)$ are complex coordinates. If one takes the explicit Hamiltonian $h(a^*, a) = \omega a^* a$, this integral is Gaussian: we may compute it by "completing the square", or equivalently observing that the integrand is an inhomogeneous quadratic and so the integral may be obtained by replacing the value of the integrand by its value at the (complex) critical point of the integrand. Explicitly, if A is a matrix with determinant 1 we have

$$\int \exp\left\{\frac{1}{2}(Az, z) + (b, z)\right\} dz = e^{-\frac{1}{2}(A^{-1}b, b)}. \tag{2.9}$$

The critical points of the integrand (2.8) are the solutions of the equation of motion

$$\dot{a}^* - i\omega a^* = 0, \ a^*(t'') = a^{*\prime\prime}$$

$$\dot{a} + i\omega a = 0, \ a(t') = a'.$$

We obtain as in Lecture 1

$$U_0(a^{*\prime\prime}, a'; t'' - t') = \exp(a^{*\prime\prime}a' e^{i\omega(t''-t')}), \tag{2.10}$$

which implies

$$U_0(t'')\phi(a^*) = \phi(e^{-i\omega t'}a^*).$$

§2.2. Perturbations

Now let us consider a more complicated example: we perturb the Hamiltonian by adding a term involving an external force $\eta(t)$ (which should be thought of as a real valued function), which depends on the time t. If we introduce

$$q = \frac{1}{\sqrt{2\omega}}(a^* + a),$$

the new Hamiltonian takes the form

$$H = \omega a^* a + \eta(t)q(t)$$

and is explicitly time dependent. The equations of motion for a and a^* acquire corrections:

$$\dot{a}^* - i\omega a^* - i\eta(t) = 0, \tag{2.11}$$

$$\dot{a}^* + i\omega a^* + i\eta(t) = 0. \tag{2.12}$$

The time evolution operator now depends on t'' and t' and not only on their difference: we obtain

$$\begin{aligned} U(a^*, a; t'', t') = \exp\Big\{ & a^* a e^{-i\omega(t''-t')} - i a^* e^{-i\omega t''} \int_{t'}^{t''} e^{i\omega s} \frac{\eta(s)}{\sqrt{2\omega}} ds \\ & - i a e^{i\omega t'} \int_{t'}^{t''} e^{-i\omega s} \frac{\eta(s)}{\sqrt{2\omega}} ds - \int_{t=t'}^{t''} \frac{\eta(t)}{\sqrt{2\omega}} \int_{s=t}^{t''} e^{-i\omega t} e^{i\omega s} \frac{\eta(s)}{\sqrt{2\omega}} dt ds \Big\}, \end{aligned} \tag{2.13}$$

(where from now on the primes on the variables a and a^* will be suppressed). We note that the expression for U contains both terms linear in η and terms quadratic in η.

Suppose $\eta = 0$ for t outside some interval $[\tau_1, \tau_2]$ (in other words the external force is turned on at time τ_1 and turned off at time τ_2). We would like to study how the motion of the system deviates from its motion in the absence of an external force: this is described by the normal symbol for the *transition operator* T, defined as follows.

$$T(a^*, a) = \lim_{t''\to\infty,\ t'\to-\infty} U_0(t'')^{-1} U(t'', t') U_0(t'). \tag{2.14}$$

Recalling the expression for U_0 given in (2.10), the transition operator is obtained from the expression (2.13) for U by removing the initial term $a^* a e^{-i\omega(t''-t')}$ from the argument of the exponential, and also removing the factors $e^{-i\omega t''}$ and $e^{i\omega t'}$ from the second and third terms in the argument of the exponential in (2.13). Notice that the integration variable s in (2.13) is subject to the constraint $s > t$; this enables us to rewrite the formula for the normal symbol of T in the form

$$T(a^*, a)_{\text{symb}} = \exp\Big\{ -i \int_{-\infty}^{\infty} q(t)\eta(t) dt + \frac{i}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\omega|t-s|}}{2i\omega} \eta(t)\eta(s) dt ds \Big\}. \tag{2.15}$$

To obtain this formula we have symmetrized over t and s.

Here,

$$q(t) = \frac{1}{\sqrt{2\omega}} (a^* e^{i\omega t} + a e^{-i\omega t}) \tag{2.16}$$

is a solution of the free equation of motion

$$\ddot{q} + \omega^2 q = 0,$$

or

$$Dq = 0.$$

The operator D is defined by

$$D = \frac{d^2}{dt^2} + \omega^2.$$

The kernel

$$G(t, s) = G(t - s) = \frac{e^{-i\omega|t-s|}}{2i\omega}$$

is the *Green's function* for the operator D, or the fundamental solution to the equation

$$(\frac{d^2}{dt^2} + \omega^2)G(t,s) = \delta(t-s). \tag{2.17}$$

Since G depends only on the difference $t-s$, we shall from now on regard G as a function of one variable t. We may describe it in terms of its Fourier transform as

$$G(t) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\imath kt}}{-k^2 + \omega^2 - i\epsilon} dk, \tag{2.18}$$

which is often denoted as

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\imath kt}}{-k^2 + \omega^2 - i0} dk.$$

The integral over k should be completed to a contour integral and '$-i0$' tells us how we have moved the poles in the integrand (which were on the real axis). The concrete choice of Green's function corresponds to the domain of definition $\mathcal{O}$ of the operator D: $\mathcal{O}$ consists of functions $f(t)$ which are asymptotic to $ae^{\imath\omega t}$ as $t \to -\infty$ and to $be^{-\imath\omega t}$ as $t \to \infty$ (where a and b are constants).

§2.3. Quantum field theory

We now pass from quantum mechanics to quantum field theory, replacing $\mathbf{R}$ by Minkowski space V of dimension $d+1$ with signature $(+,-,\dots,-)$ (in other words d space dimensions and one time dimension; sometimes this will be denoted as $V = V' \oplus \mathbf{R}$, where we have denoted d-dimensional space by V'.) We shall usually index the coordinates of V by $\mu = 0,\dots,d$ while the coordinates of space V' are indexed by $j = 1,\dots,d$. Thus a point x in V is denoted $x = (x^0, \vec{x})$; its coordinates are $x^\mu = (x^0, x^1, \dots, x^d)$.

Let us list the formulas from quantum mechanics which we shall generalize:

$$h = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \ = \omega a^* a \tag{2.19}$$

$$\ddot{q} + \omega^2 q = 0. \tag{2.20}$$

$$[\hat{a}, \hat{a}^*] = 1 \tag{2.21}$$

$$\phi(a^*) = \frac{1}{\sqrt{n!}}(a^*)^n, \ H\phi = n\omega\phi \tag{2.22}$$

$$q = \frac{a^* + a}{\sqrt{2\omega}} \tag{2.23}$$

$$T(a^*, a) = \exp\left\{-i \int q(t)\eta(t)dt + \frac{i}{2} \int G(t-s)\eta(t)\eta(s)dtds\right\}. \tag{2.24}$$

The analogue of the coordinate $q(t)$ is a scalar field (denoted by φ) which depends on $x \in V$. The field φ satisfies the classical equation of motion, which is the *Klein-Gordon equation*

$$\Box\varphi + m^2\varphi = 0, \tag{2.25}$$

where we have introduced

$$\Box = \partial_\mu \partial^\mu = \partial_0^2 - \triangle.$$

To see the analogy with (2.20) we introduce the operator $\Omega^2 = -\triangle + m^2$, so that (2.25) becomes

$$\ddot{\varphi} + \Omega^2 \varphi = 0.$$

A free field φ should be thought of as a collection of oscillators with different frequencies, which are eigenvalues of the operator Ω: Ω^2 can be diagonalized by Fourier transforms and we get for the eigenfrequencies

$$\omega^2 = \vec{k}^2 + m^2, \tag{2.26}$$

or

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}.$$

The space coordinate (or equivalently its Fourier transform, the momentum coordinate $\vec{k}$) labels the oscillators. Quantization now is evident. In the holomorphic quantization the basis elements, denoted by $\Phi_n(a^*)$, are given by

$$\Phi_n(a^*) = \prod_{j=1}^{n} a^*(\vec{k}_1) \dots a^*(\vec{k}_n); \tag{2.27}$$

at times we shall use the notation $(\vec{k})_n$ to denote the n-tuple of *unordered* points $(\vec{k}_1, \dots, \vec{k}_n)$ in $(V')^*$. The Hilbert space corresponding to n particles is the space of *symmetric* functions of n momentum variables. Proper states are given by

$$\int f((\vec{k})_n) a^*(\vec{k}_1) \dots a^*(\vec{k}_n) d\vec{k}_1 \dots d\vec{k}_n. \tag{2.28}$$

The Hamiltonian becomes

$$H = \int \omega(\vec{k}) \hat{a}^*(\vec{k}) \hat{a}(\vec{k}) d\vec{k}. \tag{2.29}$$

The state $\Phi((\vec{k})_n)$ is a generalized eigenstate for the Hamiltonian

$$H\Phi((\vec{k})_n) = \left(\sum_{j=1}^{n} \omega(\vec{k}_j) \right) \phi((\vec{k})_n). \tag{2.30}$$

and the momentum operator $\vec{P}$:

$$\vec{P}\Phi((\vec{k})_n) = \left(\sum_{j=1}^{n} \vec{k}_j \right) \Phi((\vec{k})_n). \tag{2.31}$$

Thus we obtain a description of the spectrum in terms of particles. In a one-particle subspace, the components of the momentum operator form a complete set of commuting operators, and the Hamiltonian is a function $\omega(\vec{k})$ of them.

In field theory one requires multiparticle states in order to obtain scattering, and one must also allow the possibility of creating and annihilating particles. A Hilbert space accommodating all possible numbers of particles and on which the creation and annihilation operators may act is the *Fock space* $\mathcal{H}_{\text{Fock}}$, which should be thought of as a linear combination of all the n-particle states, or as the result of tensoring together a collection of Hilbert spaces representing one-particle systems and symmetrizing over the action of the permutation group.

If we introduce the variable $\pi(\vec{x})$ which is canonically conjugate to the field operator $\varphi(\vec{x})$, the free field Hamiltonian becomes (compare with (2.19))

$$H_0 = \frac{1}{2}\int_{V'} \left(\pi^2(\vec{x}) + (\nabla\varphi, \nabla\varphi) + m^2\varphi^2(\vec{x})\right) d\vec{x}. \tag{2.32}$$

The initial data analogous to (2.23) is

$$\phi(\vec{x}) = \frac{1}{(2\pi)^{d/2}} \int \left(a^*(\vec{k})e^{-i(\vec{k},\vec{x})} + a(\vec{k})e^{i(\vec{k},\vec{x})}\right) \frac{d\vec{k}}{\sqrt{2\omega}}. \tag{2.33}$$

We may perturb the Hamiltonian H_0 by adding a term:

$$H = H_0 + \int \eta(\vec{x}, t)\varphi(\vec{x}, t)d\vec{x}.$$

The normal symbol of the corresponding transition operator is then (by analogy with (2.24))

(2.34)

$$T_\eta(a^*(\vec{k}), a(\vec{k})) = \exp\left\{-i\int_V \eta(x)\varphi(x)dx + \frac{i}{2}\int_{V\times V} G(x-y)\eta(x)\eta(y)dxdy\right\},$$

where the Green's function satisfies

$$(\Box + m^2)G(x, y) = \delta^{d+1}(x - y); \tag{2.35}$$

it is given in terms of its Fourier transform by

$$G(x) = \frac{1}{(2\pi)^{d+1}}\int_{V^*} \frac{e^{-i(k,x)}}{k^2 - m^2 + i0}dk. \tag{2.36}$$

The field operator φ generalizing (2.16) is

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^{d/2}} \int \left\{a^*(\vec{k})e^{-i(\vec{k},\vec{x})}e^{i\omega t} + a(\vec{k})e^{i(\vec{k},\vec{x})}e^{-i\omega t}\right\} \frac{d\vec{k}}{\sqrt{2\omega}}. \tag{2.37}$$

Let us now consider what happens when we perturb the Hamiltonian by adding, not a term corresponding to an external force, but a term corresponding to an interaction between particles. We suppose for example that

$$H = H_0 + \lambda\int_{V'} \mathcal{V}(\varphi(\vec{x}))d\vec{x}, \tag{2.38}$$

where $\mathcal{V}(\varphi)$ is a polynomial in φ in which all terms are higher than quadratic; we could for example take

$$\mathcal{V}(\varphi) = \frac{\varphi^3}{3!}. \tag{2.39}$$

We shall expand in terms of powers of the *coupling constant* λ, which is assumed small. The term $\mathcal{V}$ is independent of the time; it should be regarded as analogous to a potential.

§2.4. S-matrix and Feynman diagrams

We shall compute the *scattering operator* or *S-matrix*

$$\mathcal{S}' = \lim_{t''\to\infty,\ t'\to-\infty} e^{\imath H_0 t''} e^{-\imath H(t''-t')} e^{-\imath H_0 t'}. \tag{2.40}$$

Repeating the previous formalism we obtain a functional integral for the time evolution operator where the integrand contains a factor

$$\exp\left\{-i\lambda \int \mathcal{V}(\varphi(x))dx\right\}. \tag{2.41}$$

This factor is contained in the integrand for the time evolution operator U in the path integral which is the generalization to $d+1$ dimensions of the integral given in (2.8). We now use the identity

$$\begin{aligned}&\exp\left\{-i\lambda \int \mathcal{V}(\varphi(x))dx\right\}\\ &\qquad = \exp\left\{-i\lambda \int \mathcal{V}(\frac{1}{i}(\frac{\delta}{\delta\eta(x)})dx\right\}\exp\left\{i\int \varphi(x)\eta(x)dx\right\}|_{\eta=0}\end{aligned} \tag{2.42}$$

to enable us to use our formula (2.34) for the transition operator to extract a formula for the S-matrix.

We replace integration of functions involving $\mathcal{V}(\varphi(x))$ by differentiation with respect to η of the formula (2.15) for the transition operator. The purpose of this device is to replace integration (which involves the fields φ in a nonlinear manner) by differentiation with respect to the parameter $\eta(x)$, which pairs linearly with $\varphi(x)$ The formula (2.34) for T_η may be thought of as a *generating functional* for the S-matrix when one perturbs the Hamiltonian by adding a potential term involving terms of higher than quadratic order in φ.

We obtain

$$\begin{aligned}\mathcal{S}'(a^*,a) &= \exp\left\{-i\lambda \int \mathcal{V}(\frac{\delta}{\delta\eta(x)})dx\right\}\\ &\exp\left\{i\int_V \eta(x)\varphi(x)dx + \frac{i}{2}\int_{V\times V} G(x-y)\eta(x)\eta(y)dxdy\right\}.\end{aligned} \tag{2.43}$$

We are applying $\delta/\delta\eta(x)$ to the exponential of an expression which contains one term linear in η and another term quadratic in η. One application of $\delta/\delta\eta(x)$ produces terms of the form $\varphi(x)$ or of the form $\int_V G(x-y)\eta(y)dy$. Differentiating twice with respect to η produces factors of $G(x-y)$. We may represent the combinatorics of the resulting expression for the S-matrix diagrammatically: this is the formalism of *Feynman diagrams*. To obtain the term of order λ^N in the formula for the S-matrix one should proceed as follows. We restrict for the moment to the potential $\mathcal{V}(\varphi) = \varphi^3/3!$ The factor $\mathcal{V}(\varphi)$ is represented by a vertex with three edges emanating from it (*trivalent* vertex) , which represent the three derivatives $\delta/\delta\eta(x)$ applied to the expression (2.34). Each vertex carries a factor of the coupling constant λ, and the j-th vertex is labeled by a coordinate $x_j \in V$. The combination $(\delta/\delta\eta)^2$ yields a term $G(x_j - x_k)$; diagrammatically this is represented by a line connecting the j-th and k-th vertices. (Here, j need not be distinct from k, so the line could start and end at the same vertex.)

Diagrammatically the term of order λ^N in the asymptotic expansion of $\mathcal{S}'$ is obtained by the following procedure:

1. Draw all diagrams with N trivalent vertices. Label the j-th vertex by the coordinate $x_j \in V$.
2. A line connecting the j-th and k-th vertices (*internal* line) carries a factor $G(x_j - x_k)$ (the *propagator*).
3. A line emanating from the j-th vertex but not connected to any other vertex carries a factor $\varphi(x_j)$.
4. The expression thus obtained must be multiplied by $(-i\lambda)^N/N!(3!)^N$ and integrated over the $x_1, \ldots, x_N$.

We note that many of the diagrams thus obtained will differ from each other only by permutations of the integration variables $x_1, \ldots, x_N$ or of the edges leaving any given vertex; these permutations are compensated by the factor $1/(3!)^N N!$ One obtains the same answer by summing only over one representative of each equivalence class under such diagrams and omitting the factor $1/(3!)^N N!$, provided one divides by the automorphism group of the diagram (the group of relabelings of the variables $x_1, \ldots, x_N$ and permutations of the lines leaving each vertex which leaves the diagram invariant).

Of course if the potential $\lambda\varphi^3/3!$ were replaced by a different order polynomial in φ, for example $\lambda\varphi^4/4!$, one would obtain an analogous formula involving vertices with a larger number of edges (in this case four) emanating from them.

We note that one must include contributions from disconnected diagrams, and also from diagrams where an edge starts and ends at the same vertex.

References and cross references

The treatment given here closely follows Sections 2.2 and 2.3 of

L.D. Faddeev, A.A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, Addison-Wesley (Frontiers in Physics vol. 83), (second edition), 1991.

For another treatment of free fields, the S-matrix, and Feynman diagrams see Lecture 3 of [I-**Axiomatics**]. Feynman diagrams and perturbation theory are developed in [I-**Perturbative QFT**]. The time evolution of a harmonic oscillator is computed from a path integral in Problem FP11 of [I-**Homework**] (see also Problem FP15 of [I-**Homework**]).

LECTURE 3
Comments on Scattering

§3.1. The S-matrix

We first describe how to rewrite the expression for the S-matrix (equation (2.43) from Lecture 2)

$$\begin{aligned}\mathcal{S}'(\varphi_0) = \exp\Big\{-i\lambda \int \mathcal{V}(\frac{\delta}{\delta\eta(x)})dx\Big\} \\ \exp\Big\{i\int_V \eta(x)\varphi(x)dx + \frac{i}{2}\int_{V\times V} G(x-y)\eta(x)\eta(y)dxdy\Big\}.\end{aligned} \tag{3.1}$$

Here, the argument of $\mathcal{S}'$ is the solution of the free Klein-Gordon equation

$$\varphi_0(x) = \frac{1}{(2\pi)^{3/2}}\int \left(a^*(\vec{k})e^{ik\cdot x} + a(\vec{k})e^{-ik\cdot x}\right)\frac{d\vec{k}}{\sqrt{2k_0}}, \tag{3.2}$$

for $k_0 = \omega(\vec{k})$. We shall rewrite (3.1) as a path integral of the form

$$\mathcal{S}'(\varphi_0) = \int_\varphi \exp\Big\{i\int L(x)dx\Big\}\prod_x d\varphi(x), \tag{3.3}$$

where we integrate over fields φ tending asymptotically to certain asymptotic values φ_{in} (resp. φ_{out}) as $t \to -\infty$ (resp. $t \to +\infty$). Starting from the formula (3.3) one may derive the Feynman rules for an expansion of the S-matrix in terms of Feynman diagrams, where we expand the Lagrangian L about its stationary point φ_{cl}.

§3.2. Mass renormalization

We now pass to several examples which illustrate the notion of renormalization. First we shall consider an instructive example which shows that the mass renormalization is not necessarily associated with infinities. Rather it is connected with the proper definition of the one-particle states. As an example let us consider a Hamiltonian

$$H = H_0 + \lambda\sum_{n,m\geq 0} v_{n,m}. \tag{3.4}$$

Here H_0 is the free scalar field Hamiltonian

$$H_0 = \int \omega(\vec{k}) a^*(\vec{k}) a(\vec{k}) d\vec{k}, \tag{3.5}$$

λ is a coupling constant and $v_{n,m}$ are polynomials in the creation and annihilation operators:

$$\begin{aligned} v_{n,m} = \int \mathcal{V}\Big((\vec{k})_n, (\vec{k}')_m\Big) a^*(\vec{k}_1) \dots a^*(\vec{k}_n) a(\vec{k}'_1) \dots \\ a(\vec{k}'_m) \delta(\vec{k}_1 + \dots + \vec{k}_n - \vec{k}'_1 - \dots - \vec{k}'_m) (d\vec{k})_n (d\vec{k}')_m. \end{aligned} \tag{3.6}$$

The delta function in (3.6) implies that the Hamiltonian commutes with the momentum operators:

$$[H, \vec{P}] = 0, \tag{3.7}$$

where

$$\vec{P} = \int \vec{k} a^*(\vec{k}) a(\vec{k}) d\vec{k}.$$

We shall make frequent use of the commutation relations

$$[a(\vec{k}), a^*(\vec{k}')] = \delta(\vec{k} - \vec{k}'), \tag{3.8}$$

$$[a(\vec{k}), a(\vec{k}')] = [a^*(\vec{k}), a^*(\vec{k}')] = 0. \tag{3.9}$$

We can consider that the v_{11} term is absent in (3.4); otherwise it is to be included into H_0 from the beginning.

Now if terms $v_{n,0}$ are present then the Fock vacuum Ω for which

$$a(\vec{k})\Omega = 0$$

is not an eigenvector of H. So the physical vacuum is distinct from Ω. Furthermore terms $v_{n,1}$, even in the absence of terms $v_{n,0}$, preclude the one-particle states

$$\phi_1(\vec{k}) = a^*(\vec{k})\Omega$$

from being eigenvectors of H. On the other hand a Hamiltonian of the form (3.4) but summing only over n and m such that $n \geq 2$ and $m \geq 2$ has the same vacuum and one-particle states as H_0. One can apply to it the usual formalism of scattering theory.

We shall show (in the framework of formal series expansions) that it is possible to find a unitary transformation $R = e^{iQ}$, where Q is given by an expression

$$Q = Q_1 + Q_1^* + Q_2 + Q_2^*$$

and

$$\begin{aligned} Q_1 = \sum_n Q_{n,0}, \quad Q_1^* = \sum_m \bar{Q}_{0,m} \\ Q_2 = \sum_n Q_{n,1}, \quad Q_2^* = \sum_m \bar{Q}_{1,m} \end{aligned}$$

with terms of the type $(n,0)$, $(0,m)$, $(n,1)$ and $(1,m)$, and corresponding coefficient functions $q((\vec{k})_n)$, $\bar{q}((\vec{k}')_m)$, $q((\vec{k})_n, \vec{k}')$, and $\bar{q}(\vec{k}, (\vec{k}')_m)$, such that after the conjugation

$$\tilde{H} = RHR^{-1}$$

all unwanted terms can be canceled. Of course, each Q is a power series in λ:

$$Q = \sum \lambda^j Q^j$$

Alternatively we can say that in a new representation of canonical variables

$$b^{\#} = Ra^{\#}R^{-1}$$

the Hamiltonian H has the same vacuum and one-particle states as a free Hamiltonian $\tilde{H}_0$, which however is different from H_0.

We shall define $\tilde{H}^{(j)} = RHR^{-1} + O(\lambda^{j+1})$, solving in such a way that all the $\tilde{H}^{(j)}$ only include terms with $n \geq 2$ and $m \geq 2$. We see that

(3.10)

$$e^{iQ}He^{-iQ} = H_0 + \lambda(i[Q^1, H_0] + \sum_{n,m} v_{n,m}) + \lambda^2(-[Q^2, H_0] + i[Q^1, \sum_{n,m} v_{n,m}]) + \dots$$

We shall decompose this equation into its components at every order in λ, successively canceling the terms $v_{n,0}$, $v_{0,m}$, $v_{n,1}$ and $v_{1,m}$.

To leading order in λ, we need to solve

$$i[Q^1, H_0] + \sum_{n\geq 0}\sum_{m\geq 0} v_{n,m} = \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} v^{(1)}_{n,m}, \tag{3.11}$$

with some $v^{(1)}_{n,m}$ (in other words we arrange to cancel the terms $v_{n,m}$ with $n \leq 1$ or $m \leq 1$). We shall show how to choose Q^1 to satisfy (3.11); then to second order in λ the equation (3.10) implies

$$-[Q^2, H_0] + i[Q^1, \sum_{n\geq 0}\sum_{m\geq 0} v_{n,m}] = \sum_{n\geq 2}\sum_{m\geq 2} v^{(2)}_{n,m} \tag{3.12}$$

for some $v^{(2)}_{n,m}$ with $n \geq 2$ and $m \geq 2$. It will then be necessary to solve (3.12) for Q^2, and so on; this process enables us successively to cancel all the terms $v_{n,1}, v_{1,m}, v_{n,0}, v_{0,m}$.

We now describe how to solve equation (3.11) for Q^1: exactly the same procedure will then suffice to solve (3.12) for Q^2, and so on. Since (3.11) expresses Q^1 as a linear function of $\sum_{n,m} v_{n,m}$, it suffices to solve the equations

$$[Q^1_1, H_0] = iv_{n,0}, \tag{3.13}$$

$$[(Q^1_1)^*, H_0] = iv_{0,m}, \tag{3.14}$$

$$[Q^1_2, H_0] = iv_{n,1}, \tag{3.15}$$

$$[(Q^1_2)^*, H_0] = iv_{1,m} \tag{3.16}$$

for individual values of n and m.

Notice that for any operator $r_{n,m}$ of type n, m,

$$[r_{n,m}, H_0] = \sum_{n,m} \int d\vec{k}_1 \dots d\vec{k}_n d\vec{k}'_1 \dots d\vec{k}'_m \delta(\vec{k}_1 + \dots + \vec{k}_n - \vec{k}'_1 - \dots - \vec{k}'_m) \times$$

$$r_{n,m}((\vec{k})_n, (\vec{k}')_m) a^*(\vec{k}_1) \dots a^*(\vec{k}_n) a(\vec{k}'_1) \dots a(\vec{k}'_m) (\sum_j \omega(\vec{k}_j) - \sum_j \omega(\vec{k}'_j)),$$

so that the equations (3.13) - (3.16) are solved simply by dividing the coefficient functions $v_{n,0}$, $v_{0,m}$, $v_{n,1}$, $v_{1,m}$ by factors $\sum_j \omega(\vec{k}_j)$, $-\sum_j \omega(\vec{k}'_j)$, $\sum_j \omega(\vec{k}_j) - \omega(\vec{k}'_1)$, $\omega(\vec{k}_1) - \sum_j \omega(\vec{k}'_j)$ respectively. Assuming that the one-particle energy is *convex*, in other words that

$$\omega(\vec{k}_1) + \omega(\vec{k}_2) \geq \omega(\vec{k}_1 + \vec{k}_2)$$

(which means that a particle cannot decay into two or more particles while conserving momentum) we see that the factors above are nonvanishing and there is no difficulty in dividing by them.

However this method cannot be generalized to cancel terms $v_{n,m}$ where both $n \geq 2$ and $m \geq 2$, for then the factor appearing in the integral for $[Q_1, H_0]$ involves $\sum_{j=1}^{n} \omega(\vec{k}_j) - \sum_{l=1}^{m} \omega(\vec{k}'_l)$ which will in general become zero for some values of $\vec{k}_j$ and $\vec{k}'_l$, so it is no longer possible to divide through by this factor in order to solve the equation.

The equations of higher order are solved analogously. As a result we achieve our goal of canceling the unwanted terms.

In the course of this procedure a nonvanishing vacuum energy (constant terms in the expansion) appears, and the one-particle energy $\omega(\vec{k})$ changes. In particular, in the relativistic case this amounts to the renormalization of mass: in the new Hamiltonian $\tilde{H}$, the function $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ which multiplies $a(\vec{k})a^*(\vec{k})$ in the Hamiltonian H is replaced by $\omega_{ren}(\vec{k}) = \sqrt{\vec{k}^2 + M^2}$ for an appropriate value of M. What we have done may thus be described as *mass renormalization.*

§3.3. Charge renormalization

Our second example of renormalization is "charge renormalization". Its origin can be illustrated by a simple example. We begin with a Hamiltonian

$$H = -\frac{1}{2}\triangle + \lambda\delta^3(x),$$

where $\delta^3(x)$ is the Dirac delta distribution. After Fourier transforming the equation $H\hat{\psi}(x) = E\hat{\psi}(x)$ (in position space) becomes

$$\vec{p}^{\,2}\psi(\vec{p}) + \lambda \int \psi(\vec{p})d^3\vec{p} = \vec{k}^2\psi(\vec{p}) \tag{3.17}$$

where the eigenvalue of the energy is $E = \vec{k}^2$. One may rewrite this as

$$\psi(\vec{p}) = \delta^3(\vec{p} - \vec{k}) - \frac{\lambda}{\vec{p}^{\,2} - \vec{k}^2 - i0} \int \psi(\vec{p}\,')d\vec{p}\,'. \tag{3.18}$$

Putting $t = \int \psi(\vec{p}\,')d\vec{p}\,'$, we obtain, integrating both sides over $\vec{p}$,

$$t = 1 - \lambda t \int \frac{d\vec{p}}{\vec{p}^{\,2} - \vec{k}^{\,2} - i0}. \tag{3.19}$$

The integral diverges; to regularize it we introduce a cutoff L and integrate only over $|\vec{p}\,| \leq L$. We find

$$\int_{|p|\leq L} \frac{d\vec{p}}{|p|^2 - |\vec{k}|^2 - i0} = 4\pi \int_0^L \frac{|\vec{p}\,|^2 d|\vec{p}\,|}{|\vec{p}\,|^2 - |\vec{k}\,|^2 - i0}. \tag{3.20}$$

This leads to

$$\begin{aligned} t &= 1 - \lambda t \int_{|p|\leq L} \frac{d\vec{p}}{|p|^2 - \vec{k}^{\,2} - i0} \\ &= 1 - \lambda t\Big(4\pi L + |\vec{k}^{\,2} \int \frac{d|p|}{|p|^2 - k^2 - i0}\Big). \end{aligned} \tag{3.21}$$

This may be rewritten as

$$\lambda t = \frac{\lambda}{1 + \lambda(4\pi L + f(\vec{k}))},$$

where

$$f(\vec{k}) = 4\pi(|\vec{k}|^2) \int_0^\infty \frac{d|\vec{p}|}{|\vec{p}|^2 - |\vec{k}|^2 - i0} = 2\pi^2 i|\vec{k}|.$$

We may rewrite this equation in terms of a "renormalized" parameter λ_{ren}, by writing

$$\lambda t = \frac{\lambda_{ren}}{1 + \lambda_{ren} f(\vec{k})}$$

where

$$\lambda_{ren} = \frac{\lambda}{1 + 4\pi\lambda L}$$

or

$$\frac{1}{\lambda_{ren}} = \frac{1}{\lambda} + 4\pi L. \tag{3.22}$$

We shall now choose λ_{ren} (depending on L and λ) in such a way that λ_{ren} may remain finite as L tends to infinity. It is clear from (3.22) that this is only possible if λ is *negative* (in other words it describes an attracting force, not a repelling one): then one may allow λ to depend on L in such a way that $\frac{1}{\lambda}$ tends to $-\infty$ as L tends to ∞, while the value of the renormalized constant λ_{ren} remains finite.

References and cross references

The discussion here follows closely that of

L.D. Faddeev, A.A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, Addison-Wesley (Frontiers in Physics vol. 83), (second edition), 1991.

Renormalization is discussed further in the three lectures in [I-**Perturbative QFT**].

LECTURE 4
Singular Lagrangians

In other lectures, considerable attention has been focused on how to construct the symplectic structure on the space of classical solutions, starting from the Lagrangian. In this lecture we will instead focus on the fixed time Hamiltonian formalism, and we will see that the transformation which maps the Lagrangian into the Hamiltonian is simply a change of variables. The lecture will chiefly treat the case of singular Lagrangians. Our aim is to present the first order formalism, in which the Lagrangian depends linearly on time derivatives of the dynamical variables.

§4.1. Lagrangian and Hamiltonian formalisms

In the usual Lagrangian formalism, we begin with the configuration space M: coordinates on M will be denoted q. Coordinates on the tangent bundle TM are then given by q and $\dot{q}$, and the Lagrangian is a function $\mathcal{L}(q, \dot{q})$. The action is then defined by

$$A = \int \mathcal{L}(q(t), \dot{q}(t))\, dt.$$

Applying the standard variational principle we see that the extrema of the action are given by the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0. \tag{4.1}$$

This is ordinarily a system of equations of *second order* since the Euler-Lagrange equation involves the term $\ddot{q}\partial^2\mathcal{L}/\partial\dot{q}\partial\dot{q}$. The Lagrangian is said to be *nondegenerate* if the Hessian $\partial^2\mathcal{L}/\partial\dot{q}\partial\dot{q}$ is invertible.

In the Hamiltonian formalism, on the other hand, we work with $T^*M \times \mathbf{R}$ (where T^*M is the cotangent bundle of configuration space, or the phase space). On this there is a natural 1-form

$$pdq - H(p, q)dt, \tag{4.2}$$

where we have introduced (q, p) as natural coordinates on T^*M. The corresponding Lagrangian is written as

$$l = p\dot{q} - H(p, q), \tag{4.3}$$

and the action is

$$A = \int (p\dot{q} - H(p, q))dt.$$

The Euler-Lagrange equation applied to A then gives Hamilton's equations:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}. \tag{4.4}$$

Note that the Lagrangian l is linear in time derivatives, and the Hessian of l is identically zero. Thus our system is of first order. In the standard fashion, we have converted a system of second order differential equations into a system of first order differential equations, at the expense of increasing the number of variables. To explicitly see how the Hamiltonian formalism is derived from the Lagrangian formalism, we introduce $v = \dot{q}$ and treat q and v as independent variables. The Lagrangian then depends on q, v, $\dot{q}$ and $\dot{v}$, and has the following form:

$$l = \frac{\partial \mathcal{L}}{\partial v}(\dot{q} - v) + \mathcal{L}(q, v). \tag{4.5}$$

Comparing with (4.3) one finds that the Hamiltonian is given by

$$H = v\frac{\partial \mathcal{L}}{\partial v} - \mathcal{L},$$

and we obtain the Hamiltonian formalism by performing the Legendre transformation

$$(q, v) \mapsto (q, p)$$

where we have defined

$$p = \frac{\partial \mathcal{L}}{\partial v}.$$

§4.2. Constraints

On a general manifold Γ with coordinates ξ, we would write the Lagrangian as

$$l = \sum_a f_a(\xi)\dot{\xi}^a - \phi(\xi). \tag{4.6}$$

The one-form on $\Gamma \times \mathbf{R}$ corresponding to (4.2) would be written as

$$\sum_a f_a(\xi)d\xi^a - \phi dt = \omega - \phi dt,$$

and it gives rise to a two-form $\Omega = d\omega$. (Here d denotes the de Rham differential on Γ, rather than on $\Gamma \times \mathbf{R}$.) The equation of motion becomes

$$\Omega\dot{\xi} = \frac{\partial \phi}{\partial \xi}.$$

The condition that the Lagrangian is nonsingular is equivalent to the condition that the two-form Ω is invertible: for this to hold, it is necessary that Γ be even dimensional. For important Lagrangians in physics (for instance the Yang-Mills Lagrangian and the Lagrangian for electrodynamics) the two-form Ω is in fact not invertible. Let us investigate this situation in more detail.

Invoking the Darboux theorem, on Γ we write the coordinates ξ as (p, q, z) in such a way that the one-form ω becomes

$$\omega = pdq + d\theta.$$

The total derivative $d\theta$ may be discarded, and after this the one-form corresponding to the Lagrangian becomes

$$l = pdq - \phi(p, q, z)dt,$$

where $\frac{\partial \phi}{\partial z} = 0$. The variables z may be divided into "excludable" variables (which will ultimately be discarded) and others which enter linearly. After eliminating the excludable variables, the Lagrangian becomes

$$l = pdq - \phi(p,q)dt - \sum_\alpha \lambda^\alpha \varphi_\alpha(p,q). \tag{4.7}$$

Here, the λ^α play the role of *Lagrange multipliers* while the $\varphi_\alpha(p,q)$ are constraints. Among the equations of motion we now obtain the constraints

$$\varphi_\alpha(p,q) = 0,$$

so we may substitute $p = p(\eta)$, $q = q(\eta)$ where η are coordinates on the manifold

$$\Gamma^l = \{(p,q) : \varphi_\alpha(p,q) = 0\}.$$

We thus obtain a new phase space Γ^l on which the excludable variables and the Lagrange multipliers have been eliminated, so that the variables z no longer appear and we are left with a subspace parameterized by the variables (p,q) where the constraints $\varphi_\alpha(p,q) = 0$ have been imposed. The new phase space Γ^l may have a nondegenerate symplectic form, and then our Hamiltonian reduction is done. If not, we repeat the above process until Lagrange multipliers no longer appear, in which case we have reduced to a phase space with a nondegenerate symplectic form.

Explicit solution of the constraints could be complicated, so it is instructive to understand what properties of the singular Lagrangian (4.7) guarantee that it will become nonsingular after the constraints are imposed. There are two important instances of this. We can distinguish two different types of constraints. A collection of constraints $\{\varphi_a\}$ are called *first class* if the Poisson brackets of all constraints vanish on the zero locus of the constraints: in other words, if

$$\{\varphi_a, \varphi_b\}|_{\varphi=0} = 0. \tag{4.8}$$

We claim that if $\{\phi, \varphi_a\}|_{\varphi=0} = 0$ for all a, then in the next step of the procedure one obtains a proper nondegenerate Lagrangian. More precisely, the submanifold $\Gamma|_{\varphi=0}$ may be fibered so that the base is symplectic and the fibers are isotropic with respect to the Poisson bracket $\{\cdot,\cdot\}$. One may summarize the procedure by saying that "first class constraints kill twice": when one restricts to the zero locus of the constraints φ_a, one must also set to zero the variables which are canonically conjugate to the φ_a, in other words one must set to zero all Poisson brackets with the φ_a. If there are N constraint equations φ_a, $a = 1, \ldots, N$, then imposing the constraints reduces the dimension of the phase space by $2N$.

The second type of constraints φ_a are those for which $\det\{\varphi_a, \varphi_b\} \neq 0$: a system of constraints for which this condition holds is called a system of *second class constraints*. If a Lagrangian gives rise to a system of second class constraints, the next step of reduction leads to a nondegenerate Lagrangian. The number of ϕ_a must be $2N$ and the dimension of the phase space is then reduced by N through imposition of the constraints.

§4.3. Examples

Example 1. The free scalar field

Given a field $\varphi(t, \vec{x})$, and the free field Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2, \tag{4.9}$$

canonical quantization normally specifies that one should introduce the canonically conjugate momentum $\pi(t,\vec{x}) = \partial_0\varphi(t,\vec{x})$, where $x_0 = t$ and $(x_1,x_2,x_3) = \vec{x}$. The points $\vec{x}$ in space should be thought of as labeling the field. Notice that the momentum π is not Lorentz invariant.

We may alternatively replace the Lagrangian (4.9) (which is second order in time derivatives) by a first order Lagrangian: we do this by introducing a vector field φ_μ (in other words a collection of four fields $(\varphi_0,\varphi_1,\varphi_2,\varphi_3)$ which transform according to the standard action of the Lorentz group on $\mathbf{R}^4$) and write

$$l = (\partial^\mu\varphi)\varphi_\mu - \frac{1}{2}(\varphi_\mu)^2. \tag{4.10}$$

Of course under the substitution $\varphi_\mu = \partial\varphi/\partial x^\mu$ the Lagrangian in (4.10) reduces to that in (4.9): however if we regard φ_μ as independent of φ, (4.10) is first order in time derivatives. (We have introduced the Einstein summation convention, summing over pairs of repeated indices μ, where $\mu = \{0,k\}$ and $k = 1,2,3$ correspond to space coordinates while $\mu = 0$ corresponds to the time coordinate.) We may rewrite (4.10) as

$$l = (\partial_0\varphi)\varphi^0 - (\partial^k\varphi)\varphi_k - \frac{1}{2}\varphi_0^2 + \frac{1}{2}\varphi_k^2. \tag{4.11}$$

Under the substitution $\pi = \varphi_0$, and solving for φ_k, $\varphi_k = \partial_k\varphi$, we obtain

$$\mathcal{L} = (\partial^0\varphi)\pi - \frac{1}{2}(\pi^2 + (\partial_k\varphi)^2), \tag{4.12}$$

which indeed shows that π and φ are canonically conjugate and that the Hamiltonian H has the usual form

$$H = \frac{1}{2}(\pi^2 + (\partial_k\varphi)^2.)$$

The variables φ_k can be eliminated because they enter (4.10) quadratically and without time derivatives.

Example 2. Electromagnetism

In this example it is important that spacetime has dimension 4. We start with a vector field $A_\mu(x)$ (the photon field), which should be thought of as a 1-form or a connection:

$$A = \sum_\mu A_\mu dx^\mu.$$

The associated curvature is

$$F = dA,$$

or as a two-form $F_{\mu\nu}dx^\mu \wedge dx^\nu$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.13}$$

The Lagrangian is

$$\mathcal{L} = \int |F|^2, \tag{4.14}$$

which is second order in time derivatives. A trick due to J. Schwinger enables us to replace it with a Lagrangian which is first order in time derivatives: this is to regard A and F as independent variables. By analogy with our procedure in Example 1, we write

$$\mathcal{L} = (\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} - \frac{1}{2}F_{\mu\nu}^2. \tag{4.15}$$

Of course (4.15) reduces to (4.14) by restoring the functional dependence (4.13) of F on A. However the Lagrangian in (4.15) is linear in time derivatives of the fields, and also introduces some constraints.

We rewrite (4.15) as

$$\mathcal{L} = (\partial_0 A_k)F^{0k} + A_0(\partial_k F^{0k}) - F^{ik}(\partial_i A_k - \partial_k A_i) - \frac{1}{2}(F^{0k})^2 + \frac{1}{2}(F^{ik})^2. \tag{4.16}$$

Here, $F^{0k} = E^k$ are the components of the *electric field* $\vec{E}$, while A_k are the components of the *vector potential.* Up to the Hodge star operator $*$ which converts two-forms on $\mathbf{R}^3$ into one-forms, the quantity F^{ik} is the *magnetic field* $\vec{H}$: we substitute

$$\vec{H} = \nabla \times \vec{A}, \tag{4.17}$$

or $H_i = \epsilon_{ijk}\nabla_j A_k$ in terms of the totally antisymmetric tensor ϵ in three indices which represents the Hodge star operator in $\mathbf{R}^3$. Thus the Lagrangian becomes

$$\mathcal{L} = (\partial_0 A_k)E^k - \frac{1}{2}(\vec{E}^2 - \vec{H}^2) + A_0 \partial_k E^k, \tag{4.18}$$

where we already excluded the independent variables $\vec{H}$, so that $\vec{H}$ in (4.18) is given by (4.17). On the other hand, A_k and E^k are canonically conjugate variables (appearing in a term which is first order in time derivatives), and A_0 is a Lagrange multiplier multiplying the constraint

$$G = \nabla \cdot E = \partial_k E^k. \tag{4.19}$$

The constraint $G = 0$ is called *Gauss's law*: it tells us that the divergence of the electric field is zero. This is a first class constraint, $\{G(\vec{x}), G(\vec{y})\} = 0$, and we have also

$$\{G, \vec{E}\} = \{G, \vec{H}\} = 0. \tag{4.20}$$

For any function λ on $\mathbf{R}^3$ we have

$$\{A_k, \int G(\vec{x})\lambda(\vec{x})d\vec{x}\} = \partial_k \lambda(\vec{x});$$

equivalently G is a generator of $U(1)$ gauge transformations: the gauge group $\mathcal{G}$ is the group of maps from $\mathbf{R}^3$ to $U(1)$, and its Lie algebra is the group of maps from $\mathbf{R}^3$ to $\mathbf{R}$. An element $\lambda \in \mathrm{Lie}(\mathcal{G})$ sends A_k to $A_k + \partial_k \lambda$. The imposition of Gauss's law may be viewed as symplectic reduction with respect to the action of the $U(1)$ gauge group $\mathcal{G}$.

Since the constraint G is of first class, it reduces the degrees of freedom from three functions $A_k(\vec{x})$ to two. Thus (in physical language) light has two polarizations.

The above treats the situation of the electromagnetic field in the absence of sources. If on the other hand the electric field interacts with a charged field J^μ, Gauss's law becomes

$$G = \nabla \cdot E + J_0(\vec{x}) \tag{4.21}$$

where J_0 is the *charge density.* If we define

$$Q = \int_{\mathbf{R}^3} J_0 dx$$

and if we express Q as a flux of the electromagnetic field via a surface in $\mathbf{R}^3$ (for instance a sphere $S^2(R)$ of radius R) we obtain

$$Q = \lim_{R\to\infty} \int_{S^2(R)} \epsilon_{ikj} E^k dS_{ij}.$$

Thus the equation (4.21) leads to *Coulomb's law*, according to which the electromagnetic force from a point charge at distance r is proportional to $1/r^2$.

Example 3. Gravitation

In this example we introduce a metric $g^{\mu\nu}$ with the associated *affine connection* Γ^k_{ij} and the scalar curvature (we omit indices for clarity)

$$R = \partial\Gamma + \Gamma^2.$$

To write a Lagrangian that is first order in time derivatives, we take g and Γ as independent variables: more precisely we write

$$h^{\mu\nu} = \sqrt{g} g^{\mu\nu},$$

and write the Lagrangian as

$$\mathcal{L} = h(\partial\Gamma + \Gamma^2). \tag{4.22}$$

The connection Γ contains first order time derivatives of h, so $\partial\Gamma$ contains second order time derivatives of h. This may be altered by integrating by parts, so that we obtain

$$\mathcal{L} = -\Gamma\partial h + h\Gamma^2. \tag{4.23}$$

This Lagrangian is not manifestly covariant; however, it is this form which is to be used in the so called *asymptotically flat* case.

The definition of asymptotic flatness can be given on different levels of sophistication; we shall use the most naïve but practical way of introducing the admissible coordinates. In such coordinates we have in the vicinity of spatial infinity

$$g = g_0 + O(\frac{1}{r}),$$

$$\partial g = O(\frac{1}{r^2})$$

$$\Gamma = O(\frac{1}{r^2})$$

where g_0 is the Minkowski metric. An admissible change of variables is of the form

$$X_\mu \to \Lambda_{\mu\nu} X_\nu + a_\mu + b_\mu(x), \tag{4.24}$$

where $\Lambda_{\mu\nu}$ is a Lorentz rotation, a_μ defines a translation and

$$b_\mu(X) = O\left(\frac{1}{r}\right), \quad \partial b = O\left(\frac{1}{r^2}\right).$$

In these coordinates the density (4.23) is of order $O\left(\frac{1}{r^4}\right)$ and the action

$$A = \int \mathcal{L}\, dx$$

is invariant with respect to the change of coordinates (4.24).

Inspection of the density (4.23), cubic in the variables (h, Γ), shows that out of $10 + 40 = 50$ variables, the Γ^μ_{00} enter linearly and 26 other components Γ enter quadratically without time derivatives. So the Lagrangian (4.23) takes the form

(4.7); however, the constraints do not commute with the corresponding ϕ. So we must solve them explicitly. Fortunately this is possible, and altogether solving the constraints and eliminating the excludable variables allows one to express 30 components of Γ via the metric $h^{\mu\nu}$ and 6 components Γ^0_{ik}. Let

$$\begin{aligned} q^{ik} &= h^{00}h^{ik} - h^{in}h^{nk} \\ \Pi_{ik} &= \frac{\Gamma^0_{ik}}{h^{00}} \end{aligned}$$

be a (density of weight +2) of the first quadratic form and (a density of weight -1) of the second quadratic form of the surface $x_0 = 0$ embedded into space-time. After solving for Γ, mentioned above, the Lagrangian (4.23) reduces to the form

$$\Pi_{ik}\partial_0 q^{ik} + \frac{1}{h^{00}}C_0(\Pi, q) + \frac{h^{0k}}{h^{00}}C_k(\Pi, q) + \partial_i\partial_k q^{ik}$$

where C_0 and C_k are some functionals of Π, q and derivatives. The term $\partial_i\partial_k q^{ik}$ cannot be discarded in spite of the fact that it is a pure divergence as its flux through infinity does not vanish. Indeed, this flux

$$\mathcal{M} = \lim_{R\to\infty}\int_{S_R} \epsilon_{k\ell m}\partial_{ik}q^{ik}dS^{\ell n}$$

defines the full energy $\mathcal{M}$ of the gravitational field and a very nontrivial theorem states that $\mathcal{M}$ is positive if the metric is not flat.

Four constraints C_0, C_k are of the first class; let $\eta_k(x)$, $\eta_0(x)$ be the vector field and the function on the spatial slice. We put

$$\begin{aligned} C_0(\eta_0) &= \int C_0(\Pi(x), q(x))\eta_0(x)d^3x \\ C(\eta) &= \int C_k(\Pi(x), q(x))\eta^k(x)d^3x \end{aligned}$$

to get the brackets

$$\{C(\eta''), C(\eta')\} = C([\eta^n, \eta'])$$

where [,] is the commutator of vector fields. This means that $C(\eta)$ are infinitesimal generators of the group of diffeomorphisms of our spatial slice. Furthermore

$$\{C(\eta), C_0(\eta_0)\} = C_0(\eta\eta_0)$$

where $\eta\eta_n$ is the result of the action of the vector field η on the function η_0. Finally

$$\{C_0(\eta_0''), C_0(\eta_0')\} = C(\hat{\eta}),$$

where the vector field $\hat{\eta}$ is given by

$$\hat{\eta}^n = q^{in}(\partial_i\eta_0''\eta_0' - \eta_0''\partial_i\eta_0')$$

What about the Hamiltonian? Taking into account the asymptotic condition, it is more natural to consider $\left(\frac{1}{h^{0n}} + 1\right)$ rather than $1/h^{nm}$ to be a Lagrange multiplier; thus, the Hamiltonian

$$H = C_0 + \partial_i\partial_n q^{in}$$

differs from the constraint C_0 only by a term which is a total divergence. The explicit expression for C_0 shows that H is quadratic in Π and first derivatives of q, like all other Hamiltonians in previous examples. The 4 constraints kill 4 degrees of freedom from the 6 in q^{ik}, so gravitons (like photons above) have two polarizations.

Example 4. The Dirac equation In this lecture, lack of time prevents us from treating the case of the Dirac equation, in which the field is a spin 1/2 fermion, and the Lagrangian is first order in time derivatives from the beginning and is nondegenerate.

References and cross references

Constrained quantization was originally considered by Dirac in

P.A.M. Dirac, *Proc. Roy. Soc.* **A 246** (1958), p. 326.
P.A.M. Dirac, *Lectures on Quantum Mechanics*, New York (Yeshiva University), 1964.

The reduction above is associated with the names of Dirac and Arnowitt-Deser-Misner [ADM]. The explicit forms of all expressions above in the coordinates q^{ik}, Π_{ik} can be found in my survey

L.D. Faddeev, Sov. Phys. Uspekhi **25** (1982), p. 130.

Some of the topics discussed here are also treated in [**I-Classical Fields**].

LECTURE 5
Quantization of Yang-Mills Fields

This lecture treats Yang-Mills theory, an analogue of the field theory for electromagnetism (Example 2 of Lecture 4) in which the gauge group $C^\infty(\mathbf{R}^4, U(1))$ is replaced by $\mathcal{G} = C^\infty(\mathbf{R}^4, G)$ where G is a nonabelian compact Lie group. In this case Feynman discovered the disturbing fact that if one attempts to treat this theory following the approaches which worked in the case of the electromagnetic field, one obtains an S-matrix which is not unitary. Here we shall see that the reason for this is a difference of the geometry of the gauge orbits, which are linear in the electromagnetic case but nonlinear for the case of the nonabelian gauge field.

§5.1. The physical variables

We begin much as in our treatment of electromagnetism in Lecture 4. As in electromagnetism, we have a field $A_\mu(\vec{x}, t)$ which is a vector representation of the Lorentz group (in other words, it is a one-form on $V = \mathbf{R}^4$); however each component A_0, A_1, A_2, A_3 now takes values in the Lie algebra Lie(G), so the one-form $A = A_\mu dx^\mu$ takes values in Lie(G). (More generally it could be taken to have values in an arbitrary unitary representation of G, but for simplicity we take the representation to be Lie(G) endowed with the adjoint representation.) An element g of the gauge group $\mathcal{G} = C^\infty(\mathbf{R}^4, G)$ acts on A by the transformation

$$A \mapsto A^g \stackrel{\text{def}}{=} gAg^{-1} + (\partial_\mu g)g^{-1} \tag{5.1}$$

The curvature is given by

$$F = dA + A \wedge A, \tag{5.2}$$

where if $A = \sum_a X_a \otimes \alpha_a$ for a basis X_a of Lie(G) and a collection of one-forms α_a, we define

$$A \wedge A = \sum_{a,b} [X_a, X_b] \alpha_a \wedge \alpha_b.$$

The curvature (which is a Lie(G)-valued two-form on $\mathbf{R}^4$) transforms under the gauge transformation g as

$$F \mapsto F^g = gFg^{-1}. \tag{5.3}$$

As in the abelian case, the Lagrangian is

$$\mathcal{L} = \frac{1}{\epsilon^2}\,\mathrm{Tr}(F^2), \tag{5.4}$$

where if $F = \sum_a X_a \otimes \beta_a$ for a collection of two-forms β_a, we define

$$\mathrm{Tr}(F^2) = \sum_{a,b} < X_a, X_b > \beta_a \wedge *\beta_b. \tag{5.5}$$

(A symmetric quadratic form on the space of two-forms on $\mathbf{R}^4$ is defined in the same way, and is also denoted by Tr.) Here, we have introduced an Ad-invariant inner product $< \cdot, \cdot >$ on Lie(G), which in the case of matrix groups is given by the trace Tr. We have also introduced a coupling constant ϵ.

This Lagrangian contains terms of the form $(dA)^2$ as well as terms cubic in A (of order 1 in derivatives) and terms quartic in A (with no derivatives). There are thus two types of vertices, a 3-valent vertex (corresponding to the term cubic in A) and a 4-valent vertex (corresponding to the term quartic in A). By studying the corresponding perturbation expansion, Feynman found that the S-matrix was not unitary.

To proceed further, we need to find the physical variables. We write the Lagrangian in a form that is first order in derivatives, by the device introduced in Lecture 4 where we regarded A and F as independent variables. Thus we see that the Lagrangian is (suppressing the overall multiplicative constant $1/\epsilon^2$ for the moment)

$$\mathcal{L} = \mathrm{Tr}(dA + A^2, F) - \frac{1}{2}\,\mathrm{Tr}(F^2). \tag{5.6}$$

As in Lecture 4, we have ten variables

$$A_0,\ A_k,\ F_{0k} = E_k,\ F_{ik} = \epsilon_{ikl} H_l.$$

Exactly as in Lecture 4, the variables A_k and E_k are canonically conjugate, while the F_{ik} (or equivalently the H_l) are excludable variables and A_0 is a Lagrange multiplier multiplying a constraint

$$\mathfrak{G} = \nabla_k E^k = \partial_k E^k + [A_k, E^k]. \tag{5.7}$$

As in Lecture 4 we obtain an equation of motion (*Gauss's Law*)

$$\mathfrak{G} = 0;$$

the difference from the case of abelian G (electromagnetism) is the presence in (5.7) of the quadratic term $[A_k, E_k]$.

The canonical commutation relations are

$$\left\{A_k^a(\vec{x}), E_b^k(\vec{y})\right\} = \delta_l^a \delta_b^l \delta^{(3)}(\vec{x} - \vec{y}). \tag{5.8}$$

Here, a and b index a basis $\{X_a\}$ of the Lie algebra Lie(G) and we have decomposed $A_k = \sum_a A_k^a \otimes X_a$ for functions A_k^a which form a one-form A^a on $\mathbf{R}^3$.

We introduce a source term $\mathfrak{G}(\eta) = \int \mathrm{Tr}\mathfrak{G}(\vec{x})\eta(\vec{x})d\vec{x}$ (where $\eta : \mathbf{R}^3 \to \mathrm{Lie}(G)$ is an element of the Lie algebra of the gauge group). We find that the Poisson brackets of the source terms are

$$\{\mathfrak{G}(\eta_1), \mathfrak{G}(\eta_2)\} = \mathfrak{G}([\eta_1, \eta_2]), \tag{5.9}$$

in other words the map $\eta \mapsto \mathfrak{G}(\eta)$ (which may be regarded as the momentum map for the gauge group action) is a homomorphism of Lie algebras from the Lie

algebra of the gauge group (equipped with the Lie bracket induced by [,]) to the Lie algebra of functions (equipped with the Poisson bracket). Equivalently, the constraints are a representation of the Lie algebra of the gauge group. The constraint $\mathfrak{G}$ generates gauge transformations through the Poisson bracket:

$$\{\mathfrak{G}(\eta), A_k(x)\} = \partial_k \eta + [A_k, \eta]. \tag{5.10}$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(E^2 + H^2).$$

As in the case of electromagnetism, the variables F_{ik} are excludable. Likewise as in the case of electromagnetism, the constraint equations are first class: we must set $\mathfrak{G} = 0$ and also the Poisson brackets $\{\mathfrak{G}, f\}$ (for arbitrary functions f) must all be set to zero.

The observables of the reduced system are thus functionals

$$\phi(A, E)|_{\mathfrak{G}=0} \tag{5.11}$$

subject to the condition

$$\{\mathfrak{G}, \phi\} = 0. \tag{5.12}$$

§5.2. Gauge conditions in the functional integral

Reduction and solving constraints break the manifest Lorentz invariance. So if we wish to retain this invariance, we must learn how to describe dynamics (the functional integral) without explicitly solving the constraints. To understand how to treat a simpler case, let us return to a finite dimensional phase space Γ, and recall the treatment of this situation as at the beginning of Lecture 4. We assume we have a Lagrangian

$$l = \sum_{i=1}^{N} p_i dq_i - H + \sum_{\alpha=1}^{M} \lambda_\alpha \varphi^\alpha; \tag{5.13}$$

here we assume that $M < N$ (so that the dimension $2(N-M)$ of the reduced phase space is positive). To parameterize the physical phase space, we take

$$\tilde{\Gamma} = \{(p, q) \in \Gamma : \varphi^\alpha(p, q) = 0 \text{ for all } \alpha\}/\mathcal{G} \tag{5.14}$$

where $\mathcal{G}$ is the gauge group generated by the constraints φ^α through the Poisson bracket. (In the Yang-Mills example the action of the constraints via Poisson bracket actually does exponentiate to a group action, though this is not always true more generally.) To parameterize the physical degrees of freedom we introduce a slice

$$\chi_\beta(p, q) = 0, \ \alpha = 1, \ldots, M \tag{5.15}$$

which intersects each orbit of $\mathcal{G}$ exactly once: the slice (5.15) is called a *subsidiary condition.* We choose the subsidiary conditions χ_α to satisfy

$$\{\chi_\alpha, \chi_\beta\} = 0. \tag{5.16}$$

In order to obtain a nondegenerate symplectic structure, we require also that

$$\det(\{\varphi_\alpha, \chi^\beta\}) \neq 0. \tag{5.17}$$

Imposing the subsidiary conditions (5.15) and also the constraint equations

$$\varphi_\alpha(p, q) = 0$$

we find a smaller phase space Γ^*. We may take local coordinates on the big phase space Γ given by

$$q = (\chi^\alpha, q^*)$$

and

$$p = (p^\alpha, p^*);$$

here p^* and q^* are the physical variables, while p^α and χ^α must be excluded. Let us write the Liouville measure as

$$dp^* dq^* = \delta(\chi)\delta(\varphi)\det(\{\varphi, \chi\})dpdq; \tag{5.18}$$

if we were able to solve

$$\varphi(p, q) = 0 \Longleftrightarrow p^\alpha = p^\alpha(p^*, q^*) \tag{5.19}$$

this would correspond to the transformation of the delta function under a change of variables,

$$\delta(\varphi) = (\partial\varphi/\partial p)\delta(p - p(p^*, q^*)) \tag{5.20}$$

which would give rise to the determinant in (5.18).

We additionally make use of the fact that

$$\delta(\varphi) = \int e^{i\lambda\varphi} d\lambda,$$

so that we can omit the delta function $\delta(\varphi)$ if the action already contains a term $\sum_\alpha \lambda_\alpha \varphi^\alpha$. Thus the functional integral giving rise to the evolution operator is

$$\int \exp \frac{i}{\hbar} \int (p\dot{q} - H - \lambda\varphi)dt \prod_t \frac{dp(t)dq(t)d\lambda(t)}{2\pi\hbar} \delta(\chi)\det(\{\chi, \varphi\}). \tag{5.21}$$

To pass to the S-matrix only the *physical* degrees of freedom are fixed at the asymptotic values of t, while the Lagrange multiplier and the values of the constraints are allowed to vary freely. Returning to the Yang-Mills case, we get the path integral

$$\int \exp\left\{ \frac{i}{\hbar} \int_{\mathbf{R}^4} \left((\partial_0 A)E - \frac{1}{2}(E^2 + H^2) - A_0(\nabla \cdot E) \right) dx \right\} \prod_t \prod_{\vec{x}} \delta(\partial_k A_k)(\det M_C) dA_\mu dE_k, \tag{5.22}$$

where as our subsidiary condition we have taken the *Coulomb gauge* condition,

$$\partial_k A_k = 0, \tag{5.23}$$

and have introduced the operator M_C on Lie(G)-valued functions on $\mathbf{R}^3$, defined by

$$M_C \eta = -\partial_k \partial_k \eta - [A_k, \eta]. \tag{5.24}$$

The only objects appearing in (5.22) which are not Lorentz invariant are $\det M_C$ and $\partial_k A_k$.

We can explicitly integrate in (5.22) over the variables E to get

$$\int \exp\left\{ \frac{i}{\hbar} \int_{\mathbf{R}^4} (F_{\mu\nu})^2) dx \right\} \prod_x dA_\mu(x) \prod_t \prod_{\vec{x}} \delta(\partial_k A_k) \det M_C(t). \tag{5.25}$$

We interpret this expression as an integral over $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is a space of connections and $\mathcal{G}$ is a gauge group. Our subsidiary condition (5.23) is the choice of a point on the orbit of $\mathcal{G}$: the determinant $\det M_C$ gives the correct measure.

To check that we have indeed obtained the correct measure on the space of $\mathcal{G}$ orbits reduced from the measure $\prod_x \prod_\mu dA_\mu(x)$, it suffices to check that

$$\int \delta(\partial_k A_k^g) \prod_x \prod_{g \in \mathcal{G}} dg_x|_{\partial_k A_k = 0} = \frac{1}{\det M_C}; \tag{5.26}$$

here dg_x is the Haar measure on G at the point g_x. We can linearize the action of $\mathcal{G}$ so that infinitesimally we obtain

$$\int \delta(M_C \eta) \prod_x d\eta_x = \frac{1}{\det M_C}$$

where we have defined the operator M_C by (5.24). After this understanding we can proceed to choose a different subsidiary condition, in particular a manifestly Lorentz invariant one

$$\partial_\mu A_\mu = f, \tag{5.27}$$

where f is an arbitrary scalar function. The reduced measure is given by

$$\det M_L \prod_x \delta(\partial_\mu A_\mu - f(x)) \prod dA_\mu(x)$$

where $\det M_L$ is obtained by averaging the δ-function

$$\frac{1}{\det M_L} = \int \delta(\partial_\mu A_\mu^g - f(x)) \prod_x dg(x).$$

Here, we have defined the operator M_L (compare the definition (5.24) of M_C) by

$$M_L \eta = \Box \eta + \partial_\mu [A_\mu, \eta]. \tag{5.28}$$

The path integral now becomes

$$\int \exp\left\{\frac{i}{\hbar} \int (F_{\mu\nu}^2 dx)\right\} \prod_x \delta(\partial_\mu A_\mu - f)(\det M_L) \prod_x dA_\mu(x). \tag{5.29}$$

The final step is to average over $c(x)$. Ultimately the expression (5.29) becomes

(5.30)

$$\int \exp\left\{\frac{i}{\hbar} \int_{\mathbf{R}^4} (F_{\mu\nu})^2 dx\right\} \prod_x \delta(\partial_\mu A_\mu(x) - f)(\det M_L) \prod dA_\mu(x) \prod_{f^2} e^{i\alpha \int f^2 dx}$$

$$= \int \exp\left\{\frac{i}{\hbar} \int_{\mathbf{R}^4} (F_{\mu\nu})^2 + \alpha(\partial_\mu A_\mu)^2) dx\right\} \prod_x (\det M_C) \prod dA_\mu(x). \tag{5.31}$$

Here, α is an arbitrary constant. The integrand in (5.30) is not local (because the determinant of the operator M_C is not local): to transform it into a local expression we introduce Grassmann variables $c = (c_1, \ldots, c_r)$ and $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_r)$, in other words odd variables in a Grassmann algebra, satisfying the anticommutation relations

$$c_i \bar{c}_j + \bar{c}_j c_i = 0,$$
$$c_i^2 = \bar{c}_i{}^2 = 0$$

and recall the identity

$$\det A = \int e^{\bar{c} \mathfrak{A} c} dc^* dc \tag{5.32}$$

where A is an endomorphism of a vector space of dimension r and we have introduced the corresponding endomorphism of a vector space of dimension $2r$,

$$\mathfrak{A} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}. \tag{5.33}$$

We now replace the Grassmann variables c and $\bar{c}$ (which took values in a finite dimensional vector space) by Grassmann variables c and $\bar{c}$ taking values in the infinite dimensional space of functions on $\mathbf{R}^4$. Rewriting the determinant in (5.30) by formally applying (5.32), (5.30) is transformed into the form

(5.34)

$$\int \exp\left\{\frac{i}{\hbar\epsilon^2}\operatorname{Tr}\int_{\mathbf{R}^4}\Big((F_{\mu\nu})^2+\alpha(\partial_\mu A^\mu)^2+(\partial_\mu\bar{c},\partial_\mu c+[A_\mu,c]\Big)dx\right\}\prod dA_\mu dc d\bar{c}.$$

We have now transformed the path integral into an integral of the form $\int \exp(iS)$, where the action S is local and has a nondegenerate quadratic form, and we integrate over all fields (which are assumed to satisfy appropriate boundary conditions). Here c and $\bar{c}$ are fields which should be regarded as carrying charge (because of the terms mixing A_μ, c and $\bar{c}$) and which should be regarded as fermionic (because they are odd elements in a superalgebra or Grassmann algebra). The expression (5.34) is the generating functional from which the S-matrix of Yang-Mills theory may be extracted. There are still the 3-valent and 4-valent vertices in which all the lines emanating from the vertex correspond to the propagator of A. Additionally we now have another 3-valent vertex with one A propagator, one c propagator and one $\bar{c}$ propagator: the c and $\bar{c}$ propagators may be distinguished by writing an arrow on the c propagator pointing toward the vertex and an arrow on the $\bar{c}$ propagator pointing away from the vertex.

Near a classical solution of the equations of motion for the Yang-Mills fields, the correction terms entering the perturbation expansion are proportional to $F_{\mu\nu}^2$ (recall that a term of the form $F_{\mu\nu}^2$ appeared in our original Lagrangian). The term $\frac{1}{\epsilon^2}(F_{\mu\nu}^2)$ in the Lagrangian is renormalized to

$$\left(\frac{1}{\epsilon^2}+a\ln(\Lambda)\right)F_{\mu\nu}^2$$

for an appropriate cutoff parameter Λ and a suitable constant a. We notice that a must be negative in order for the renormalization procedure to succeed: this property is referred to in the literature as "asymptotic freedom".

References and cross references

For a related discussion of the functional integral with gauge conditions, see Lecture 4 of [II-**Strings**]. There, and also in Lecture 3 of [II-**Dynamics of QFT**], the BRST quantization is discussed.

Renormalization Groups

David Gross

Introduction

As we have seen in other contexts, the behavior of a physical system depends on a scale (of energies, distances, momenta, etc.) at which this behavior is studied. The change of behavior when the scale is changed is described by the renormalization group equation. In quantum field theory, the dependence of the behavior on the scale is often expressed mathematically by the fact that in order to regularize (i.e. render finite) Feynman diagram integrals one must introduce auxiliary scales, cutoffs, etc. The effect of these choices on the physics is encoded into the renormalization group equation. This equation then becomes an important tool for the study of physical theories. The purpose of this series of lectures is to introduce the basic notions and techniques of renormalization and the renormalization group equation (or flow) and to apply these techniques in some simple but important examples.

Lecture 1 considers the effect of an energy scale on physics and introduces the crucial idea that physics at one energy scale largely decouples from the physics at higher energy scales. This idea of essential decoupling is then made precise in the Wilsonian picture of the renormalization flow on the space of theories. The first lecture ends with a reminder of the basic methods of regularizing integrals over Feynman diagrams. There is a discussion of various types of cutoffs that can be used to regularize these integrals, as well as a discussion of the more abstract dimensional regularization.

Lecture 2 treats methods of finite renormalization in the dimensional regularization scheme for renormalizable Lagrangian theories of a certain type. This is a method of determining, by fixing a finite number of physically measurable quantities, the coupling constants and masses in the Lagrangian provided that we work at a mass scale μ. The dependence of the coupling constants and masses on μ is encoded in the renormalization group equation. It is actually a vector field—i.e., a flow equation—on the space of the renormalizable theories, which are indexed by the coupling constants and masses appearing in the Lagrangian, times the space of mass scales (a copy of $\mathbf{R}^+$). This flow equation is the equation that tells us how to vary the coupling constants and masses in the Lagrangian as a function of the

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mass scale μ in order to keep the physically measurable quantities constant. Then the β-function is defined and the notion of asymptotic freedom is introduced.

In Lecture 3 we take a closer look at the β-function. The importance of the sign of the first Taylor coefficient of β at zero is discussed and the opposite cases of asymptotic freedom and infrared freedom are explored. The β-function for QED, to one-loop accuracy, is discussed and it is shown that QED is not asymptotically free. Then the one-loop β-function for nonabelian gauge theories is discussed. The precise conditions under which these nonabelian gauge theories are asymptotically free is given. Then we present the Coleman-Gross theorem, which shows that in dimension four no reasonable theory which does not involve nonabelian gauge fields can be asymptotically free. Renormalization of operators and anomalous dimensions are also discussed.

After an introductory discussion of the general topic of dynamical mass generation, Lecture 4 applies the general machinery of the renormalization group equation to the Gross-Neveu model. This is a two-dimensional field theory with only fermionic fields. It is shown that its Lagrangian is renormalizable and asymptotically free (in fact the theory is known to exist non-perturbatively) and the theory exhibits dynamical symmetry breaking and spontaneous mass generation.

The last lecture returns to the Wilsonian renormalization group picture and uses it to generate what is known as the exact renormalization equation. This equation and asymptotic freedom can be used to produce nonperturbative field theories.

LECTURE 1
Introduction to Renormalization Groups

§1.1. What is renormalization group?

In these lectures we will consider the method of renormalization group and its applications to quantum field theory.

Very generally speaking, the method of renormalization group is a method designed to describe how the dynamics of some system changes when we change the scale (distances, energies) at which we probe it.

Before describing more precisely what exactly this method does, recall two basic facts about physics:

1. Scale dependence. Physics is scale-dependent. For example, consider a fluid. At each scale of distances, we need a different theory to describe its behavior:

At $\sim$ 1cm – classical continuum mechanics (Navier-Stokes equations)
At $\sim 10^{-5}$cm – theory of granular structure
At $\sim 10^{-8}$cm – theory of atom (nucleus + electronic cloud)
At $\sim 10^{-13}$cm – nuclear physics (nucleons)
At $\sim 10^{-13} - 10^{-18}$cm quantum chromodynamics (quarks)
At $\sim 10^{-33}$cm string theory

At each scale, we have different degrees of freedom and different dynamics.

2. Decoupling. Physics at a larger scale (largely) decouples from the physics at a smaller scale. For example, to describe the behavior of fluid at the scale $\sim$ 1cm, we don't need to know about the granular structure, nor about atoms or nucleons. The only things we need to know are the viscosity and the density of the fluid. Of course, these values can be computed from the physics at a smaller scale, but if we found them out in some way (for example, measurement), we can do without smaller scale theories at all. Similarly, if we want to describe atoms, we don't need to know anything about the nucleus except its mass and electric charge.

Thus, a theory at a larger scale remembers only finitely many parameters from the theories at smaller scales, and throws the rest of the details away. More precisely, when we pass from a smaller scale to a larger scale, we average over irrelevant degrees of freedom. Mathematically this means that they become integration variables and thus disappear in the answer.

This decoupling is the reason why we are able to do physics. If there was no decoupling, it would be necessary for Newton to know string theory to describe the motion of a viscous fluid.

Remark. This transition of information from smaller scale theories to larger scale theories through finitely many parameters is of course irreversible, since in such transition we forget (=integrate out) many irrelevant degrees of freedom. For example, it is possible to reconstruct the thermodynamics of a gas from molecular theory (by averaging over all configurations), but it is impossible to reconstruct the behavior of molecules from the macroscopic behavior of the gas itself.

The general aim of the renormalization group method is to explain how this decoupling takes place and why exactly information is transmitted from scale to scale through finitely many parameters.

Remark 1. In quantum theory, decoupling of scales is not at all obvious. Indeed, because of the uncertainty principle, we have to work at all scales at once. The renormalization group explains why decoupling survives in quantum theory.

Remark 2. In classical mechanics, there are 3 basic units of measurement (distance D, time T, mass M), and all others can be expressed through them. Thus, in classical mechanics we deal with three scales. In nonrelativistic quantum theory and in classical relativity there remains only two of them, as in the first case we can express M through T and D using the Planck constant, and in the second T can be expressed via D using the speed of light. Thus, in relativistic quantum theory we only have one scale – the scale of distances. Equivalently we can use the inverse scale – the scale of momenta. Thus we have:

$$\text{SMALL distances, times} = \text{LARGE momenta, energies, masses .}$$

§1.2. The general scheme of the method of renormalization group

As we already mentioned, in quantum physics in general, in particular in any quantum field theory, we must consider arbitrarily short scales. Thus, in order to understand the theory at a given scale, we should know something about its behavior at smaller scales. The theory is called renormalizable if all the information we need from smaller scales can be absorbed in finitely many parameters (renormalized masses and couplings). In this case, we can measure these parameters (or some functions of them), and thus have a complete picture of the theory. However, if the theory is not renormalizable, the number of parameters coming from smaller scales is infinite, so the theory is unmanageable. This is why we can successfully work at all scales only with renormalizable theories.

Now we make a general review of renormalization group theory. To think about the renormalization group, one can use the following general picture, due to Wilson.

The central object in Wilson's picture is "the space of theories" S. In general, this object is extremely ill-defined. However, in special cases it may be possible to give a precise definition of S. In quantum field theory we can model this space as the space of all possible Lagrangians of certain types of fields.

Wilson's procedure is the following. Let M be the set of scales of momenta. Mathematically, M is a principal homogeneous space (torsor) of $\mathbb{R}_+$. (This means that M is isomorphic to $\mathbb{R}_+$, but noncanonically; in particular, there is no distinguished element in M). Let $\Lambda, \mu \in M$ be scales of momenta, and $\mu < \Lambda$. For each theory $\mathcal{L} \in S$, one finds another theory, $R_{\Lambda\mu}\mathcal{L} \in S$ which is the effective theory, at the scale μ, for the original theory $\mathcal{L}$ at the scale Λ. Thus, we have a map $R_{\Lambda\mu}\colon S \to S$, and $R_{\Lambda_1\Lambda_2}R_{\Lambda_2\Lambda_3} = R_{\Lambda_1\Lambda_3}$. Thus, we have an action of the semigroup

$\mathbb{R}_+^{\leq 1}$ on the space $S \times M$, by $\lambda \circ (\mathcal{L}, \Lambda) = (R_{\Lambda,\lambda\Lambda}\mathcal{L}, \lambda\Lambda)$. Abusing mathematical terminology, physicists call this semigroup the renormalization group.

§1.3. Wilsonian scheme for the theory of a scalar field: a mathematical description (an interpretation of D.Gross' lectures by the preparers of these lecture notes)

Let us give a precise mathematical description of the renormalization group in the special case of theories of a scalar bosonic field in 4 dimensions.

Let V be a 4-dimensional Euclidean space, and $\mathcal{S}(V)$ the space of Schwartz functions on V. By a homogeneous interaction of degree k we mean a continuous, homogeneous, Poincare invariant polynomial functional $J\colon \mathcal{S}(V) \to \mathbb{C}$ of degree k. That is,

$$J(\phi) = \int K_J(x_1, \ldots, x_k)\phi(x_1)\ldots\phi(x_k)dx_1\ldots dx_k,$$

where K_J is a Poincaré invariant tempered distribution.

Definition. (i) A homogeneous interaction J is called local if K_J is supported on the diagonal.

(ii) J is called quasilocal if K_J has rapid decay at infinity, i.e. if the Fourier transform $\tilde{K}_J$ of K_J has the form

$$\tilde{K}_J(p_1, \ldots, p_k) = \delta(p_1 + \ldots + p_k)\tilde{F}_J(p_1, \ldots, p_{k-1}),$$

where $\tilde{F}_J$ is a smooth function.

(iii) We say that a quasilocal interaction J is of order $\geq s$ if the function $\tilde{F}_J$ has order $\geq s$ at the origin. We will write $O(\nabla^s)$ for a quasilocal interaction of order $\geq s$.

It is clear that a local interaction is the same thing as a differential polynomial in ϕ, and the order of a differential monomial is the number of derivatives in it. This statement can be generalized to the case of quasilocal interactions, as follows.

Proposition 1. *Let J be a quasilocal interaction of degree $k \geq 1$. Then for any $j \geq 0$ there exists a unique homogeneous differential operator D_j of order j on V^{k-1} with constant coefficients, such that for any $\phi \in \mathcal{S}(V)$ and any $s \geq 0$ the functional $J(\phi)$ is given by a Taylor formula*

$$J(\phi) = \sum_{j=0}^{s-1} \int_V D_j[\phi(x_1)\ldots\phi(x_{k-1})]|_{x_1=\ldots=x_{k-1}=x}\phi(x)dx + J^s(\phi),\ J^s(\phi) = O(\nabla^s).$$

Proof. We have

$$J(\phi) = \int_{(V^*)^{(k-1)}} \tilde{F}_J(p_1, \ldots, p_{k-1})\tilde{\phi}_1(p_1)\ldots\tilde{\phi}(p_{k-1})\tilde{\phi}_k(-p_1-\ldots-p_{k-1})dp_1\ldots dp_{k-1},$$

where $\tilde{\phi}$ is the Fourier transform of ϕ. Since $\tilde{F}_J$ is smooth, it has a unique Taylor expansion

$$\tilde{F}_J(p_1, \ldots, p_{k-1}) = \sum_{j=0}^{s-1} D_j(p_1, \ldots, p_{k-1}) + O(|p|^s).$$

This implies the proposition.

Now let τ be a formal variable. We will work over the ring $\mathbb{R}[[\tau]]$.

Let X be the space of all Lagrangians of ϕ with coefficients in $\mathbb{R}[[\tau]]$, invariant under $\phi \to -\phi$, of the form

$$\mathcal{L} = \int \left(\frac{1}{2}(a(\nabla\phi)^2 + m^2\phi^2) + I(\phi) \right) d^4x, \tag{1.2}$$

where $I(\phi) = \sum_{k\geq 2} I_k(\phi)$ are interaction terms (I_k has degree k), such that

(i) $a, m^2 \in \mathbb{R}_+$ and $I_k = O(\tau)$ for all k;

(ii) For any $N \geq 0$, $I_k = O(\tau^N)$ for almost all k;

(iii) I_k are quasilocal.

Consider the space $S = X \times M$. The space S is the relevant space of theories in which Wilson's flow will be defined.

Let F_k be the space of smooth functions on the hyperplane $\sum_{i=1}^k v_i = 0$ in $(V^*)^{\oplus k}$ (we identify this hyperplane with $(V^*)^{(k-1)}$ by throwing out the last component v_k). Let $F = \prod_{i=1}^\infty F_{2i}$.

In the next few paragraphs we will define a map $\mathcal{G}\colon S \to F$, $\mathcal{G} = (\mathcal{G}_2, \mathcal{G}_4, \dots)$. The value of $\mathcal{G}_k$ at $(\mathcal{L}, \Lambda)$ is called the k-point correlation function of the Lagrangian $\mathcal{L}$ at scale (cutoff) Λ.

Let $P(r)$ be a smooth function on $\mathbb{R}_+$ which equals 1 for $r \leq 1/2$, 0 for $r > 2$, and varies between 0 and 1 in between. Define the cutoff propagator (in momentum space) to be $\tilde{G}_\Lambda(p) = \frac{P(p^2/\Lambda^2)}{p^2+m^2}$.

Let $(\mathcal{L}, \Lambda) \in S$. By definition, the algorithm for computing $\mathcal{G}_k(\mathcal{L}, \Lambda)$ is as follows. $\mathcal{G}_k$ is the sum of amplitudes of all Feynman diagrams with k external edges. Computation of the amplitude is as usual (by Feynman rules), except that on each internal edge, instead of the free propagator $1/(p^2+m^2)$ one should put the cutoff propagator $\tilde{G}_\Lambda(p)$. On external edges one should put the free propagator.

One can also define connected and 1-particle irreducible correlation functions, which are obtained by summing the amplitudes of connected and 1-particle irreducible diagrams, respectively (see [I-**Axiomatics**]).

For example, if $\Lambda = 0$, the cutoff propagator vanishes, so that connected correlation functions are the same as interactions.

Now let μ, Λ be arbitrary scales. Then one can make the following definition.

Definition. A Lagrangian $\mathcal{L}_e$ of the form (1.2) is called an effective Lagrangian at the scale μ of a Lagrangian $\mathcal{L}$ at the scale Λ if $\mathcal{G}(\mathcal{L}_e, \mu) = \mathcal{G}(\mathcal{L}, \Lambda)$.

Proposition 2. *For any $\mathcal{L} \in X$ and any μ, Λ there exists a unique effective Lagrangian $\mathcal{L}_e$ at the scale μ of $\mathcal{L}$ at the scale Λ.*

Proof. It is easy to check that a Lagrangian is uniquely determined by its correlation functions. This implies the uniqueness of $\mathcal{L}_e$, if it exists. The existence of $\mathcal{L}_e$ is proved by giving an explicit construction. The construction: the function $\tilde{F}_{J_k}$ for $\mathcal{L}_e$ is defined to be the sum of amplitudes of all connected Feynman diagrams with k external edges, computed for $\mathcal{L}$ with the propagator $\tilde{G}_{\Lambda\mu}(p) := \tilde{G}_\Lambda(p) - \tilde{G}_\mu(p)$. It is not difficult to check that $\mathcal{L}_e$ is an effective Lagrangian at μ for $\mathcal{L}$ at Λ.

We will denote the Lagrangian $\mathcal{L}_e$ by $R_{\Lambda\mu}\mathcal{L}$. This defines a map $R_{\Lambda\mu}\colon X \to X$ for any two scales Λ, μ. It is clear that $R_{\Lambda_1\Lambda_2}R_{\Lambda_2\Lambda_3} = R_{\Lambda_1\Lambda_3}$, and $R_{\Lambda\Lambda} = 1$.

Thus we have a flow on the space S given by the formula $t(\mathcal{L}, \Lambda) = (R_{\Lambda,t\Lambda}\mathcal{L}, t\Lambda)$, $t \in \mathbb{R}^+$. This flow is called the renormalization group flow.

Remark 1. It is easy to see that if a Lagrangian $\mathcal{L}$ is free (i.e. $I_k = 0$) then $R_{\Lambda\mu}\mathcal{L} = \mathcal{L}$, since in this case we have no internal vertices and no nontrivial diagrams.

Remark 2. In principle, instead of the described "smeared cutoff", we could use the most obvious cutoff, i.e. define a cutoff propagator $\tilde{G}^*_\Lambda(p) := \frac{1_\Lambda}{p^2+m^2}$, where 1_Λ is the characteristic function of the ball $|p| < \Lambda$. In the language of functional integrals, this means the following: the k-point correlation function (in position space) is obtained by integration of $\phi(x_1)\dots\phi(x_n)$ not over all fields $\phi(x)$, but only over fields whose Fourier transform is supported in the region $|p| < \Lambda$. Similarly, the effective Lagrangian $\mathcal{L}_e$ at μ for $\mathcal{L}$ at Λ is given (for $\mu < \Lambda$) by the formula

$$e^{-\mathcal{L}_e(\phi)} = \int e^{-\mathcal{L}(\phi+\eta)} D\eta \tag{1.3}$$

where ϕ is a field whose Fourier transform is supported in the ball $|p| \le \mu$, and the functional integral is taken over all fields η whose Fourier transform is supported in $\mu \le |p| \le \Lambda$.

This picture has a nice physical interpretation: in order to compute $R_{\Lambda\mu}$, we integrate out degrees of freedom – the amplitudes of harmonics with frequencies between Λ and μ.

Remark 3. The map $R_{\Lambda\mu}$, although it is formally defined (by Feynman rules) for any Λ, μ, makes physical sense only for $\Lambda > \mu$. This is well illustrated by Remark 2, as there $R_{\Lambda\mu}$ by definition reduces to integrating out degrees of freedom. In fact, beyond perturbation theory the map $R_{\Lambda\mu}$ is usually not defined for $\Lambda < \mu$.

Remark 4. Unfortunately, the sharp cutoff $\tilde{G}^*_\Lambda$ defined in Remark 2 is not quite satisfactory, since the corresponding renormalization group flow on the space of Lagrangians does not preserve quasilocality (quasilocality is necessary for formulation of Theorem 4). Indeed, let us consider a Lagrangian

$$\mathcal{L} = \int \left(\frac{1}{2}(a(\nabla\phi)^2 + m^2\phi^2) + \frac{g}{4!}\phi^4\right) d^4x, \tag{1.4}$$

where $g \sim \tau$. It is easy to see that at the order τ^2 we will get a correction to the quadratic term of the form

$$\int \phi(x)\phi(y)(G^*_\Lambda(x-y) - G^*_\mu(x-y))dxdy,$$

where $G^*_\Lambda(x)$ is the cutoff propagator in position space. Since the cutoff propagator in momentum space is not smooth, this term is not quasilocal.

This is why it is better to use the smooth cutoff $\tilde{G}_\Lambda$ instead of the sharp cutoff $\tilde{G}^*_\Lambda$.

Now we will describe the main property of the renormalization group. Very roughly, this property is that the operator $R_{\Lambda\mu}$ for small μ/Λ makes all non-renormalizable terms in the Lagrangian irrelevant, so in the limit $\Lambda \to \infty$ only

renormalizable terms matter. If there is only finitely many renormalizable terms (which is usually the case), this means that the renormalization group flow has a finite dimensional attractor in the space of theories – the space of renormalizable, or critical theories. This property is the reason why we actually never have to consider non-renormalizable theories in the perturbative setting.

Let us describe this property more carefully. First recall that in the theory of a scalar field, the only renormalizable Lagrangians are of the form (1.4). Let us denote the set of all renormalizable Lagrangians by X_r.

Let $T_0(X)$ be the tangent space to X at any point (all of them are naturally isomorphic).

For any $\mathcal{L} \in T_0(X)$, denote by $P_s(\mathcal{L})$ the part of the Lagrangian $\mathcal{L}$ whose coefficients have scaling dimension $\leq -s$. That is, if $\mathcal{L}(\phi) = \sum_k J_{2k}(\phi)$, then by definition $P_s(\mathcal{L})(\phi) = \sum J_{2k}^{s-2k+4}(\phi)$, where J^l for $l > 0$ was defined in Proposition 1, and by convention for nonpositive l $J^l(\phi) = J(\phi)$.

We will call a Lagrangian $\mathcal{L} \in T_0(X)$ purely non-renormalizable if $\mathcal{L} = P_1(\mathcal{L})$. It is clear that $\mathcal{L}$ is purely nonrenormalizable if and only if $\tilde{F}_{J_2}(0) = 0$, $\frac{\partial \tilde{F}_{J_2}}{\partial p^2}|_{p=0} = 0$, and $\tilde{F}_{J_4}(0,0,0) = 0$. Denote by X_n the set of all purely non-renormalizable Lagrangians.

It is obvious that any Lagrangian can be written uniquely as a sum $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_n$, where $\mathcal{L}_r \in X_r$, $\mathcal{L}_n \in X_n$. Thus, $X = X_r \times X_n$.

Now fix an element $\mathcal{L} \in X_r$, and consider the intersection $(\mathcal{L} + X_n) \cap R_{\Lambda\mu} X_r \subset X$.

Lemma 3. *The intersection $(\mathcal{L} + X_n) \cap R_{\Lambda\mu} X_r$ consists of 1 point.*

Proof. The Lemma is obvious modulo τ, and to higher orders in τ it is established by induction (the implicit function theorem).

Let $B_{\Lambda\mu}(\mathcal{L})$ be the unique point in $(\mathcal{L} + X_n) \cap R_{\Lambda\mu} X_r$.

For any $\nu \in \mathbb{R}^+$, define the linear rescaling map $Q_\nu \colon T_0(X) \to T_0(X)$ by $Q_\nu(\mathcal{L})(\phi(x)) = \mathcal{L}(\nu^{-1}\phi(\nu^{-1}x))$. For example, $Q_\nu(\int \phi^6(x)d^4x) = \nu^{-2} \int \phi^6(x)d^4x$.

The following theorem represents the main properties of the renormalization group.

Theorem 4.

(i) (Convergence) *There exists a limit $B_\mu(\mathcal{L}) = \lim_{\Lambda\to\infty} B_{\Lambda\mu}(\mathcal{L})$. That is, all interactions converge (in the weak sense).*

(ii) (Rate of convergence) *The rate of this convergence is determined by the formula*

$$P_s(B_{\Lambda\mu}(\mathcal{L})) - P_s(B_\mu(\mathcal{L})) = O(\Lambda^{-s+\varepsilon}), \Lambda \to \infty.$$

for any $s \geq 1$ and $\varepsilon > 0$.

(iii) (Stability) *Let $Y_r(\mu) = B_\mu(X_r)$. Let $\Delta\mathcal{L} \in X_n$, Then the set $(\mathcal{L} + X_n) \cap R_{\Lambda\mu}(X_r + Q_{\Lambda/\mu}(\Delta\mathcal{L}))$ consists of only one point, $B_{\Lambda\mu}^{\Delta\mathcal{L}}(\mathcal{L})$. Moreover, there exists a limit $\lim_{\Lambda\to\infty} B_{\Lambda\mu}^{\Delta L}(\mathcal{L}) = B_\mu^{\Delta\mathcal{L}}(\mathcal{L})$, which belongs to $Y_r(\mu)$.*

(iv) (Set convergence) *Let $X(s) \subset X$ be the set of Lagrangians which are polynomials of degree $\leq s$ in ϕ. Then for $s \geq 4$, $Y_r(\mu) = \lim_{\Lambda\to\infty} R_{\Lambda\mu}(X)(S)$.*

The proof of this theorem will be partially discussed later. It is essentially contained in the paper by J. Polchinski "Renormalization and effective Lagrangians", Nucl. Phys. B231(1984) 269-295.

Theorem 4 shows that in our case the renormalization group flow has an attractor – a three-dimensional family of trajectories Y_r, which can be parametrized by the values of critical and subcritical coefficients in the Lagrangian. Thus, the only things we (effectively) inherit from smaller scales are the values of these 3 parameters.

We see that $Y_r(\mu)$ is defined completely canonically, and $R_{\mu_1\mu_2}Y_r(\mu_1) = Y_r(\mu_2)$. The space $Y_r(\mu)$ has the meaning of the space of exact (renormalized) quantum field theories, i.e. those which are obtained by perturbative renormalization as described in [I-**Perturbative QFT**]. These are the only theories that we will care about.

This very simple example demonstrates the main principle of the method of renormalization group.

§1.4. Applications of renormalization group theory to phase transitions

As we mentioned, one of the main applications of the renormalization group in quantum field theory is that it allows us to absorb all short-scale information in finitely many constants. However, there are other applications as well. One of them is to the theory of phase transitions in many-body systems (statistical mechanics). For example, consider a magnetic chain on a lattice L. Each vertex v of the lattice has a spin $s(v)$, 1 or -1. The Lagrangian of the theory, for example, can be

$$\mathcal{L} = \frac{c}{T} \sum_{v--v'} (s(v) - s(v'))^2,$$

where T is the temperature, and $v - -v'$ means that v is a neighbor of v'. For low T the correlation is very strong, so there are large (macroscopic) clusters of aligned spin (magnetism). For high T the spins are almost uncorrelated, so there are only very small clusters of aligned spins (no magnetism). The transition from one situation to the other is happening at some fixed temperature T_c called the critical temperature. At this temperature we expect to see clusters of all sizes, and the theory to be scale invariant (conformal). Hence this theory should correspond to a "fixed point" (=an attracting trajectory) of the renormalization group flow on the relevant space of theories. We may try to find this trajectory if we have some description of the relevant space of theories and of the renormalization group flow, and thus understand the behavior of the magnet at the critical temperature.

Remark. In general (for example, in the theory of a scalar field), the renormalization group does not have an attracting trajectory, but has an attracting finite-dimensional family of trajectories – the space of critical (renormalizable) theories. In this case we have to make finitely many measurements in order to determine at which point of this space we are. For instance, in the theory of a scalar field we would have to measure three parameters, which determine a point on the 3-dimensional variety Y_r. All other physical quantities can then be computed, i.e. expressed via the results of these measurements.

§1.5. Reminder of renormalization theory

Before we discuss how the renormalization group works in quantum field theory, we recall the basics of classical renormalization theory. Since there were other lectures on this subject, we will do it briefly.

In quantum field theory, we consider Lagrangians depending on certain kinds of fields (scalar fields, fermionic fields, gauge fields). For example, we can consider a Lagrangian of the form (1.1). We are interested in correlation functions of the form $\langle T(\phi_1(x_1)...\phi_N(x_N))\rangle$, where T means time ordering. It is usually easier to compute the Fourier transform of this function. Feynman rules allow us to do it perturbatively, i.e. to obtain the answer in the form of a power series in coupling constants. This answer comes with two problems:

1. Every term of the series is given by a divergent integral.
2. Even if all terms are convergent, the series is divergent in any neighborhood of zero.

Renormalization theory helps to overcome the first problem. The second problem has to do with the (difficult) question whether the theory actually exists nonperturbatively. If it does, this problem can be solved by regarding the series expansion not as a Taylor expansion but as an asymptotic expansion of a certain function.

Renormalization theory works in two steps: 1) regularization; 2) renormalization.

Regularization is introducing some kind of cutoff, which makes the integral convergent. After regularization, the integral is convergent, but depends on the regularization parameter Λ. If Λ tends to ∞ (i.e. when the cutoff is removed), the integral tends to infinity. This problem is removed by the second step – renormalization. Namely, we find a way to change the parameters in the Lagrangian in such a Λ-dependent way that when Λ and the parameters in the Lagrangian are varied simultaneously, with $\Lambda \to \infty$, the integral converges to a finite limit. This limit is then proclaimed to be the value of the integral.

The most naive cutoff scheme is suppressing large momenta by changing the propagator, for example $\frac{1}{k^2+m^2} \to \frac{1_{|k|\leq\Lambda}}{k^2+m^2}$ (where 1_A is the characteristic function of a set A), or $\frac{1}{k^2+m^2} \to \frac{1}{k^2+m^2}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^l$. This scheme has some nice features (e.g. it can be applied in a non-perturbative setting), but is not always good enough (for example, in gauge theory), as it does not preserve all symmetries. Namely, it preserves Lorentz symmetry but does not preserve gauge symmetries or conformal symmetries. So if we use this scheme, we have no gauge or conformal symmetry for each finite Λ, and may not recover this symmetry even in the limit.

There is a more ingenious regularization scheme, which is called dimensional regularization. This scheme preserves all symmetries, but it is only valid in the perturbation expansion. Now we will describe what dimensional regularization is.

§1.6. Dimensional regularization

The idea of dimensional regularization is to compute the integral corresponding to each Feynman graph in the "space" of dimension D, where D is a generic complex number. If the graph is divergent, the answer turns out to be a meromorphic function of D, usually having a pole at D equal to the physical dimension d. The renormalization technique usually used with this kind of regularization is minimal subtraction: we subtract the pole part (this is called "minimal subtraction") and proclaim the rest (evaluated at $D = d$) to be the value of the integral.

Let us now describe the algorithm of dimensional regularization. For simplicity we assume that our theory includes only scalar massive fields.

Suppose we have a Feynman graph G with M external edges, and momenta $k_1, ..., k_M$ at them ($k_1 + ... + k_M = 0$). Let $q_1, ..., q_{N-M}$ be the loop variables over which we are integrating. To uniformize the notation, we set $q_{N-M+i} = k_i$. Then the amplitude of the graph is (up to a factor) given by the formula

(1.5)
$$\Sigma^G(q_{N-M+1}, \dots, q_N) = \int_{V^{N-M}} g(q) dq_1 \dots dq_{N-M}, \; g(q) = \frac{Q(q)}{\prod_j (l_j(q)^2 + m_j^2)},$$

where Q is a polynomial, $q = (q_1, \dots, q_N)$ and l_j: $V^N \to V$ are given by linear forms on $\mathbb{R}^N$.

Unfortunately, the integral (1.5) is divergent, so we will explain how to regularize it using dimensional regularization. As does any regularization scheme, dimensional regularization works inductively in the number of loops (or internal vertices) of the graph. So we assume that the coefficients in the Lagrangian have been adjusted at the previous step of the process in such a way that all proper subgraphs of G are convergent.

In order to regularize the integral corresponding to G, we pretend that it exists and, using the formula

(1.6)
$$\int_0^\infty e^{-at} dt = a^{-1},$$

transform it to the form

(1.7)
$$\begin{aligned} \Sigma^G &= \int Q(q) \left[\int_{t_j \in [0,\infty)} e^{-\sum t_j l_j(q)^2 - \sum t_j m_j^2} dt \right] dq \\ &= \int_{t_j \in [0,\infty)} e^{-\sum t_j m_j^2} \left[\int Q(q) e^{-\sum t_j l_j(q)^2} dq \right] dt \, . \end{aligned}$$

The q-integral in brackets in the last expression is a Gaussian integral, so it can be computed according to the standard rules. Namely, from Poincaré invariance it is clear that the answer is of the form $\psi(t, A, d) \det(B(t))^{-d/2}$, where $B(t)$ is a quadratic form on $\mathbb{R}^{N-M}$ such that $B(t)_{ij} = (\sum t_s l_s^2)_{ij}$, $A = (a_{ij}) = (k_i \cdot k_j)$, and ψ is a polynomial in d.

Thus we get

(1.8)
$$\Sigma^G = \Sigma^G(A) = \int_{t_j \in [0,\infty)} e^{-\sum t_j m_j^2} \psi(t, A, d) (\det B(t))^{-d/2} dt.$$

Of course, this integral is also divergent. However, since it does not contain integration with respect to q, we can replace d in it with a generic complex number D. It is easy to see that the obtained integral $\Sigma_D^G(A)$ is convergent for $\mathrm{Re} D << 0$, and one can prove that it continues meromorphically to the whole complex plane (this is a nontrivial theorem).

Definition. $\Sigma^G(A)$ is the value of the regular part of $\Sigma_D^G(A)$ at $D = d$. That is,

(1.9)
$$\Sigma^G(A) = \frac{1}{2\pi i} \oint_{|D-d|=\varepsilon} \Sigma_D^G(A) d \ln(D - d).$$

Now, if the analytic continuation of integral (1.8) is singular at $D = d$, we should adjust the Lagrangian by adding to it a counterterm $C_G(D)$ (infinite at $D = d$) in such a way that the singular part cancels and the regular part remains unchanged. After this we can go to the next step of the process.

Remark. The only divergences that produce a pole at $D = d$ are logarithmic divergences. Pure power divergences produce poles at smaller values of D, and do not produce a singularity at $D = d$.

References and cross references

For a more detailed description of the general theory of integration in a D-dimension space, where D is a complex number, see [I-**Dim'l Regularization**].

LECTURE 2
The Renormalization Group Equation

§2.1. Finite renormalization

As we saw in the previous lecture and in [II-**CFT**], [I-**Perturbative QFT**], renormalizable quantum field theories always come in families (which may even be infinite-parameter in 2 dimensions), and we need to specify certain quantities in order to pinpoint a specific theory among such a family. The only ways to do this are to measure these quantities or to set their values artificially. But when we know them, we can (in principle) compute all other quantities in the theory as functions of them.

The procedure of picking a theory by setting its parameters to the results of measurements (or artificially) is called finite renormalization, as opposed to "infinite" renormalization which we discussed above. While infinite renormalization deals with removing divergences (infinities), finite renormalization adjusts the finite part of the coefficients in the Lagrangian (as functions of the cutoff Λ), which infinite renormalization leaves ambiguous.

Thus, a prescription of finite renormalization is a way to define a system of coordinates (or a parametrization) of the space of renormalizable theories of a given type.

It is clear that the number of quantities to be specified (i.e. the dimension of the space of renormalizable theories) is equal to the number of coefficients of the Lagrangian of nonnegative dimension. However, these coefficients themselves are, in general, not suitable as coordinates on the space of theories, as they depend on the cutoff Λ and are infinite at $\Lambda \to \infty$. Therefore, people use other systems of coordinates.

For example, for the space Y of renormalizable theories of a scalar field, we could use the following parametrizations:

1. $T \to (\mathcal{G}_2(0), \frac{\partial \mathcal{G}_2}{\partial p^2}(0), \mathcal{G}_4(0,0,0))$, where $\mathcal{G}_i$ are connected correlation functions.
2. $T \to (\mathcal{G}_2(\mu^2), \frac{\partial \mathcal{G}_2}{\partial p^2}(\mu^2), \mathcal{G}_4(p_1,p_2,p_3))$, where $\mu \in M$ is some scale, and $p_1, p_2, p_3, p_4 = -p_1 - p_2 - p_3$ are vertices of a regular tetrahedron such that $p_i^2 = \mu^2$.

Notice that the first recipe defines one fixed parametrization of the space of theories, while the second one is in fact a family of such parametrizations, depending

on a scale $\mu \in M$. In the future, such families of parametrizations will be more important to us than fixed parametrizations.

Remark. The coordinates in both of the above parametrizations are not measurable quantities, since momenta of all particles which can be generated in a lab satisfy the condition $p^2 = -m^2$, where m is the mass of the particle (people say that "p is on the mass shell"). In the case of ϕ^4 theory, it is possible to make up a system of coordinates purely out of measurable quantities. (Of course, here we make the (invalid) assumption that the ϕ^4 theory describes a real physical process.) Let us explain how to do this.

Suppose that we have an elementary particle ϕ which obeys a renormalizable ϕ^4-theory. We have seen that renormalizable ϕ^4-theories live in a 3-parameter family Y. For $T \in Y$, let σ_T be the residue of the 2-point function of T (in momentum space) at its pole. The dimensionless quantity σ_T is called the scale of ϕ in T.

Measurable quantities are masses of particles and scattering amplitudes, which are scale-of-ϕ-independent objects. So let us be interested only in scale-of-ϕ-independent characteristics of a given theory. In this case we can restrict our attention to the 2-parameter subfamily of theories $Y_1 \subset Y$ in which the scale of ϕ equals 1. This causes no loss of generality, since any theory $T \in Y$ can be normalized to a theory $T_1 \in Y_1$ by a transformation $\phi \to \alpha\phi$, in such a way that all scale-of-ϕ-independent quantities are unchanged.

Now our space of theories is 2-dimensional, so we need to measure 2 things in order to specify a concrete theory. One of the possible prescriptions is the following. The two measured things are:

(i) The mass m of the field ϕ. Mathematically, it is determined from the condition that $-m^2$ is the pole of the 2-point function of the theory, regarded as a function of the square of momentum k^2 (in the Euclidean picture).

(ii) The effective coupling g – the scattering amplitude for two particles at the same point (p_1, p_2, p_3, p_4) "on shell", i.e. with $p_i^2 = -m^2$ and $\sum p_i = 0$. One of the possible choices is to take p_i to be vertices of a regular tetrahedron lying on some 3-dimensional subspace of the dual spacetime V^* (it is clear that all such configurations can be obtained from each other by rotation). The effective coupling measures how strong the interaction is.

However, as we will see in the future, one often needs to consider parametrizations of the space of theories in which parameters are not necessarily measurable or have any physical meaning. An example of such physically meaningless, but very useful parametrization is the dimensional regularization prescription, which is described in the next section.

§2.2. The dimensional regularization prescription of finite renormalization

Now let us describe a prescription of finite renormalization which uses dimensional regularization.

Recall from Lecture 1 that given a renormalizable ϕ^4-Lagrangian, we can compute its correlation functions using dimensional regularization and minimal subtraction. At first sight it seems that the answer is completely canonical and does not depend on anything. However, after careful consideration it becomes obvious that the answer depends on the choice of unit of length. This happens because the

dimensions of couplings in the Lagrangian, in general, depend on the dimension D of the spacetime, so that they should be regarded not as functions but rather as sections of some line bundle over the space of values of D. In order to work with functions, we should trivialize this bundle.

The correct way of computing correlation functions with dimensional regularization is, in general, the following. We should choose a scale of momentum μ, and for any coupling (or mass) κ in the Lagrangian find the dimensionless quantity κ_d such that $\kappa = \kappa_d \mu^{\dim(\kappa,d)}$, where $\dim(\kappa, D)$ is the dimension of κ when the dimension of spacetime is D (this is some function of the form $aD + b$), and d is the physical dimension of the spacetime. Now we should replace all masses and couplings κ^i in the Lagrangian with the expressions $\kappa^i(D) = \kappa^i_d \mu^{\dim(\kappa^i,D)}$, and perform dimensional regularization – minimal subtraction with this new Lagrangian, as described in Section 1.6 of Lecture 1.

Remark. One should remember that the Feynman integrals used to define correlation functions will have d-dependent dimensions, so they have to be multiplied by appropriate powers of μ to make their dimensions constant, before the subtraction procedure is applied.

It is clear that the answer (correlation functions) heavily depends on the choice of μ. The reason is that some couplings κ will nontrivially depend on D (for fixed κ_d), and their first derivatives by D (which contain $\ln \mu$) will influence the regular part of the correlation functions at $D = d$.

For example, in the case of ϕ^4-theory, we have a Lagrangian of the form

$$\mathcal{L} = \int d^4x \left(\frac{a}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right), \tag{2.1}$$

and $m(D) = m_d\mu, a(D) = a_d, g(D) = g_d\mu^{4-D}$.

Now, the dimensional regularization prescription of finite renormalization is: for any $\mu \in M$, $T \to (m, a, g)$, where (m, a, g) are the parameters of the ϕ^4-Lagrangian of the form (2.1) which gives the correlation functions of T if renormalized with the help of dimensional regularization, using the scale μ.

The parameters m, a, g have the meaning of "the effective mass at scale μ", "the inverse of the effective scale of ϕ at μ", and "the effective coupling at μ" (we will later explain why).

§2.3. Scale-dependence of finite renormalization prescriptions

In physics we are interested in the behavior of a system at different scales, and how it changes when going from scale to scale. Therefore, we are interested in scale-dependent parametrizations of the space of theories, such that at each scale μ the corresponding parametrization represents in some way the behavior of the system at μ. Among the prescriptions 1,2 given in Section 2.1, only 2 satisfies this property, while 1 is scale-independent.

The dimensional regularization prescription also satisfies this property. That is, given a theory $T \in Y$, the parameters of the Lagrangian $\mathcal{L}_\mu(T)$ which reproduces the correlation functions of T if renormalized by dimensional regularization using μ represent the behavior of the theory at scale μ. This is not immediately obvious, since the dimensional regularization procedure is formal and has no physical meaning.

The reason is that for large μ the mass can be neglected, and thus the function Γ_4 is a function of p_i/μ, which does not explicitly depend on μ. Therefore, the coupling $g(\mu)$, which is formally obtained from the dimensional regularization procedure, characterizes the magnitude of Γ_4 at $p_i \sim \mu$.

Let us illustrate this by considering an example.

Given a ϕ^4-Lagrangian $\mathcal{L}$ of the form (2.1), its 1-particle irreducible 4-point function, computed by dimensional regularization using the scale μ, in the 1-loop approximation equals

$$\Gamma_4(p_i, \mu, m, a, g) = -g + \frac{g^2}{a^2}[f((p_1+p_2)^2) + f((p_1+p_3)^2) + f((p_1+p_4)^2)], \tag{2.2}$$

where $f(k^2)$ is the amplitude of the corresponding 1-loop diagram. This amplitude was computed Lecture 3 of [II-**CFT**] (the formula for $\hat{I}^{\Lambda}_{4,ren}(k)$), and the answer is

$$f(k^2) = -\frac{A}{6}\int_0^1 \ln\left(\alpha(1-\alpha)\frac{k^2}{\mu^2} + \frac{m^2}{a\mu^2}\right) d\alpha + const, \tag{2.3}$$

where A is some positive numerical constant.

For large μ the term $m^2/a\mu^2$ can be neglected, and we get

$$f(k^2) \sim -\frac{A}{6}\int_0^1 \ln\left(\alpha(1-\alpha)\frac{k^2}{\mu^2}\right) d\alpha + const = -\frac{A}{6}\ln(k^2/\mu^2) + const, \tag{2.4}$$

In particular, if p_1, p_2, p_3, p_4 are vertices of a regular tetrahedron such that $p_i^2 = \mu^2$ (a point representing the scale μ), then for large μ in the 1-loop approximation we get

$$\Gamma_4 = -g + \frac{g^2}{a^2}C,$$

where C is some numerical constant independent of μ. In particular, the scale-of-ϕ-independent quantity $\frac{\Gamma_4}{a^2}$ has the form

$$\frac{\Gamma_4}{a^2} = -g_* + g_*^2 C, g_* = g/a^2$$

If g_* is small, $\Gamma_4/a^2 \sim -g_*$. So the "effective coupling constant" g_* does indeed characterize the intensity of interaction at the scale μ.

§2.4. The renormalization group flow corresponding to a scale-dependent renormalization prescription

Given a scale-dependent finite renormalization prescription, we can define the renormalization group flow in the space of parameters, which is given by transition functions from the coordinates at one scale to coordinates at another scale. This is done as follows.

Let Y denote the space of renormalizable theories of a single scalar field ϕ. For any scale $\mu \in M$ we have an injective map $\pi_\mu \colon Y \to \tilde{P}$, where $\tilde{P}$ is a fixed (μ-independent) space of parameters. We will assume that the image $P_\mu = \mathrm{Im}\pi_\mu$ is independent of μ. Denote this image by P.

Now we can fix a theory T and look at the dynamics of $\pi_\mu(T)$ as μ changes. To do this, for any $\mu_1, \mu_2 \in M$ define a transformation $R_{\mu_1\mu_2}\colon P \to P$ by $R_{\mu_1\mu_2} = \pi_{\mu_2} \circ \pi_{\mu_1}^{-1}$. This allows us to define a flow on $P \times M$: $t(z, \mu) = (R_{\mu,t\mu}z, t\mu)$. The flowlines of this flow are of the form $(\pi_\mu(T), \mu)$, where $T \in Y$ is a fixed theory. This flow is called the renormalization group flow associated to the corresponding family π_μ of renormalization prescriptions.

It is convenient to represent the renormalization group flow as a vector field. Let $\pi_\mu(T) = (m_\mu(T), a_\mu(T), g_\mu(T))$, where $m = m_\mu, a = a_\mu, g = g_\mu$ are the effective mass, inverse scale, and coupling at μ for some renormalization prescription. Then the vector field of the renormalization group looks like

$$W = \mu\frac{\partial}{\partial\mu} + \tilde{\gamma}m\frac{\partial}{\partial m} + \tilde{\beta}\frac{\partial}{\partial g} + \tilde{\delta}a\frac{\partial}{\partial a}, \tag{2.5}$$

where $\tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ are functions on $P \times M$.

It is clear that the renormalization group commutes with the group G_1 of rescalings of ϕ: $a \to ta, m \to t^{1/2}m, g \to t^2 g, \mu \to \mu$, as well as with the group G_2 of changing the units of measurement: $a \to a, m \to tm, g \to g, \mu \to t\mu$. The invariants of these actions are $g_* = g/a^2$ and $m_* = m/\mu\sqrt{a}$. This shows that $\tilde{\gamma} = \hat{\gamma}(g_*, m_*)$, $\tilde{\beta} = \hat{\beta}(g_*, \mu_*)a^2$, and $\tilde{\delta} = \delta(g_*, m_*)a$.

It is convenient to rewrite W in the coordinates μ, g_*, m_*, a. The result is

$$W = \mu\frac{\partial}{\partial\mu} + (\gamma - 1)m_*\frac{\partial}{\partial m_*} + \beta\frac{\partial}{\partial g_*} + \delta a\frac{\partial}{\partial a}, \tag{2.6}$$

where $\gamma := \hat{\gamma} - \delta/2$, and $\beta = \hat{\beta} - 2g_*\delta$.

Using this formula, we can write down a differential equation for correlation functions. Let $\Gamma_j(p_i, \mu, m_*, a, g_*)$ be the j-point function of the theory with parameters m_*, a, g_* computed at scale μ. Since the "engineering" (i.e. classical) dimension of Γ_j is $4 - j$, we have

$$\left(\mu\frac{\partial}{\partial\mu} + s\frac{\partial}{\partial s} + j - 4\right)\Gamma_j(sp_i, \mu, m_*, a, g_*) = 0. \tag{2.7}$$

Subtracting from (2.7) the equation $W\Gamma_j = 0$, we get

$$\left(s\frac{\partial}{\partial s} - (\gamma - 1)m_*\frac{\partial}{\partial m_*} - \beta\frac{\partial}{\partial g_*} - \delta a\frac{\partial}{\partial a} + j - 4\right)\Gamma_j(sp_i, \mu, m_*, a, g_*) = 0. \tag{2.8}$$

This equation allows one to study the dynamics of Γ_j under the scaling of external momenta.

Now consider the renormalization group flow in the coordinates g_*, m_*, a. According to formula (2.6), integral curves of this flow are given by the formulas $g_* = g_*(\mu), m_* = m_*(\mu), a = a(\mu)$, where

$$g_*' = \beta(g_*, m_*), m_*' = (\gamma(g_*, m_*) - 1)m_*, a' = \delta(g_*, m_*)a. \tag{2.9}$$

and prime denotes the derivative with respect to $\ln\mu$. From these formulas it is clear that a expresses trivially through g_*, μ_*:

$$a(\mu) = a_0 e^{\int_{\mu_0}^{\mu} \delta(g_*(s), m_*(s))d\ln s}. \tag{2.10}$$

However, the equations for g_* and m_* cannot be easily solved, even if β and γ are known, unless β and γ have some special form.

However, suppose that the vector field W has a third one-parameter symmetry group: $\mu \to \mu, a \to a, m_* \to tm_*, g_* \to g_*$ (mass rescaling). In this case we will call our renormalization prescription mass-independent. Then β, γ, δ are independent on m_*, and equations (2.9) are easily solved. Namely, $g_*(\mu)$ is represented implicitly by

$$\frac{\mu}{\mu_0} = e^{\int_{g_*(\mu_0)}^{g_*(\mu)} \frac{dg}{\beta(g)}}, \tag{2.11}$$

and m_* is expressed via g_* by

$$m_*(\mu) = \frac{\mu_0}{\mu} m_*(\mu_0) e^{\int_{\mu_0}^{\mu} \gamma(g_*(s)) d\ln s}. \tag{2.12}$$

Unfortunately, most renormalization prescriptions are mass-dependent. This is the case, for example, for prescription 2 from Section 2.1. However, this prescription is "asymptotically mass-independent", i.e. almost mass-independent at high momenta. Indeed, the correlation functions, which are used as parameters in this prescription, do not significantly depend on m at high momenta. Therefore, although β and γ depend on m_*, there exist limits $\beta(g_*) := \lim_{m_* \to 0} \beta(g_*, m_*)$, $\gamma(g_*) := \lim_{m_* \to 0} \gamma(g_*, m_*)$, so that formulas (2.11), (2.12) become valid for very large μ_0, μ, and, in particular, are applicable for studying the asymptotics of solutions at $\mu \to \infty$.

This "asymptotic mass independence" is characteristic of all prescriptions π_μ which are based on correlation functions at μ. But all of them are mass-dependent at finite μ.

Nevertheless, mass-independent renormalization prescriptions exist. The most convenient of them is the dimensional regularization prescription.

Let us now explain why the dimensional regularization prescription is mass-independent. We will work in the spacetime dimension D (a generic complex number), with $\epsilon = 4 - D$. For a renormalizable Lagrangian $\mathcal{L}$ of the form (2.1) with parameters m, a, g, let $\Gamma_i^\epsilon(p_j, m, a, g)$ be the correlation functions of $\mathcal{L}$ in dimension D, computed by plain integration in D dimensions as described in Lecture 1, and $(\Gamma_i^{\epsilon,sub}(p_j, \mu, m, a, g)$ be the correlation functions in dimension D, computed using scale μ, with pole parts subtracted. The functions Γ_i^ϵ are singular at $\epsilon = 0$, while $\Gamma_i^{\epsilon,sub}$ are regular at $\epsilon = 0$, and the functions $\Gamma_i := \lim_{\epsilon \to 0} \Gamma_i^{\epsilon,sub}$ are the true correlation functions of the theory, computed at scale μ as in Section 2.2.

Let m^b, a^b, g^b be the bare parameters, defined by the condition

$$\Gamma_i^\epsilon(p_j, m^b, a^b, g^b) = \Gamma_i^{\epsilon,sub}(p_j, \mu, m, a, g)$$

(b stands for "bare"). These bare parameters are functions of renormalized parameters, ϵ, and μ, of the form

$$m^b = m^b(m, a, g, \epsilon, \mu), a^b = a^b(m, a, g, \epsilon, \mu), g^b = g^b(m, a, g, \epsilon, \mu).$$

From dimensional analysis (using the groups G_1 and G_2) it follows that the functions m^b, a^b, g^b are defined by the equations

$$a^b = aZ_a(m_*, g_*, \epsilon), m^b(a^b)^{-1/2} = \mu m_* Z_m(m_*, g_*, \epsilon), g^b(a^b)^{-2} = \mu^\epsilon g_* Z_g(m_*, g_*, \epsilon),$$

where

$$Z_i(m_*, g_*, \epsilon) = 1 + \sum_{k \geq 1} Z_i^k(g_*, m_*)\epsilon^{-k}.$$

(no positive powers of ϵ, as we used minimal subtraction).

The key point now is that Z_i do not actually depend on m_*. Indeed, the functions Z_i are sums of counterterms introduced in the process of renormalization. These counterterms at each step of the process are just sums of pole parts (in ϵ) of amplitudes of all diagrams with a given number of loops. These pole parts are determined purely by the asymptotic expansion of the amplitudes at $|p_i| \to \infty$. From this one can deduce that these pole parts are independent of m. Thus, $Z_i = Z_i(g_*, \epsilon)$.

Let W_ϵ be the vector field of the form

$$W_\epsilon = \mu\frac{\partial}{\partial\mu} + (\gamma_\epsilon - 1)m_*\frac{\partial}{\partial m_*} + \beta_\epsilon\frac{\partial}{\partial g_*} + \delta_\epsilon a\frac{\partial}{\partial a},$$

which preserves the functions $\Gamma_i^{\epsilon,sub}(p_j, \mu, m, a, g)$. It is clear that such W_ϵ exists and is unique, and $\lim_{\varepsilon\to 0} W_\varepsilon = W$. On the other hand, using the definition of $\Gamma_i^{\epsilon,sub}$, we get that W_ε is defined by the condition $W_\epsilon m_*^b = 0, W_\epsilon a^b = 0, W_\epsilon g_*^b = 0$. Writing these equations in components, we get

$$\beta_\epsilon Z_m' + \gamma_\epsilon Z_m = 0,\ \beta_\epsilon Z_a' + \delta_\epsilon Z_a = 0,\ \epsilon g_* + \beta_\epsilon\frac{(g_* Z_g)'}{g_* Z_g} = 0$$

(prime denotes the derivative by g_*). From these equations we find:

$$\beta_\epsilon = -\frac{\varepsilon g_*^2 Z_g}{(g_* Z_g)'},\ \gamma_\epsilon = -\beta_\epsilon\frac{Z_m'}{Z_m}, \delta_\epsilon = -\beta_\epsilon\frac{Z_a'}{Z_a}.$$

Using the facts that Z_i has a Laurent expansion in inverse powers of ϵ, and the fact that β, γ, δ are by definition analytic at $\varepsilon = 0$, we get the following formulas:

$$\beta_\epsilon(g_*) = -g_*\epsilon + g_*^2(Z_g^1)',\ \gamma_\epsilon(g_*) = g_*(Z_m^1)', \delta_\epsilon(g_*) = g_*(Z_a^1)'$$

In the limit $\varepsilon \to 0$, these formulas give

$$\beta(g_*) = g_*^2(Z_g^1)',\ \gamma(g_*) = g_*(Z_m^1)', \delta(g_*) = g_*(Z_a^1)'$$

The last formula shows that β, γ, δ depend only on g_*, and gives a simple and useful rule of evaluating them.

In particular we see that although counterterms in dimensional regularization can have poles of arbitrary order, the contribution to the renormalization group vector field comes only from residues of these poles. Thus, in principle the renormalization group vector field should be more elementary than the complete system of counterterms. This is one of the reasons why it is useful to write down the renormalization group equation.

We have seen that the asymptotic properties of our field theory at $\mu \to \infty$ depend on the properties of the function $\beta(g_*)$, since this function determines the dynamics of the coupling $g_*(\mu)$. This function is called the beta-function. Later we will see that the sign of this function near $g_* = 0$ is especially important.

§2.5. Computation of the renormalization group flow in the 1-loop approximation

In this section we will compute the vector field W in the 1-loop approximation. Let $m_* = m_*(\mu)$, $g_* = g_*(\mu)$, $a = a(\mu)$ be an integral curve of the flow. We will express the derivatives m'_*, g'_*, a' from the equations $\mu\frac{\partial}{\partial\mu}\Gamma_2(p^2, m_*, a, g_*, \mu) = 0$, $\mu\frac{\partial}{\partial\mu}\Gamma_4(p_i, m_*, a, g_*, \mu) = 0$.

First of all, the 1-loop correction to the 2-point function can be absorbed in mass renormalization: $m(\mu) = m_0(1+b\frac{g}{a^2}\ln(\mu/\mu_0))+O(g^2)$, where b is a numerical constant. Thus, $a' = 0, m'_* = -m_*(1+bg_*)$.

In order to compute g'_*, we differentiate equation (2.2) with respect to $\ln\mu$. This yields, modulo g_*^3, the following expression:

$$g'_* = Ag_*^2. \tag{2.13}$$

Thus, we get in the one-loop approximation: $\beta(g_*) = Ag_*^2, A > 0; \gamma(g_*) = bg_*; \delta(g_*) = 0$.

Remark. A more careful calculation gives: $A = \frac{3}{16\pi^2}, b = \frac{1}{32\pi^2}$.

Now consider the renormalization group equation (2.13). The general solution of this equation is

$$g_* = \frac{g_*^0}{1 - Ag_*^0\ln(\mu/\mu_0)}. \tag{2.14}$$

Thus, we have found an approximation to the function $g_*(\mu)$ based on the 1-loop approximation to the β-function.

Now we face the following fundamental questions. In what respect is expression (2.14), obtained from the 1-loop approximation to the beta-function, any better than the expression

$$g_* = g_*^0 + A(g_*^0)^2\ln(\mu/\mu_0), \tag{2.15}$$

obtained directly from (2.2)? (Note that (2.15) coincides with (2.14) modulo $(g_*^0)^3$!) In what sense does (2.14) better represent "reality"than (2.15)? In other words, why did we care to introduce the renormalization group, beta-function, and so on, instead of just getting (2.15) directly?

The answer is as follows. Assume that ϕ^4 theory exists in the non-perturbative setting (it is not believed to be the case, but never mind; you can think instead of another field theory). Then formula (2.15) has a chance to be valid only when $g_*^0\ln(\mu/\mu_0)$ is small, since the series for g_* will contain terms of the form $(g_*^0)^{k+1}\ln(\mu/\mu_0)^k$. On the other hand, formula (2.13) is valid for small g_*, since it is the beginning of the Taylor expansion of β. So in the region where g_* is small but $g_*^0\ln(\mu/\mu_0)$ is not (for example, $Ag_*^0\ln(\mu/\mu_0) = 1/2, g_*^0 << 1$), formula (2.14) is reliable but (2.15) is not.

§2.6. Asymptotic freedom

So far we have always worked with quantum field theories in the perturbative setting, primarily because this was the only setting in which we could define and study

them. However, physics lives over $\mathbb{R}$ rather than $\mathbb{R}[[\tau]]$, so we should investigate when we can use perturbative results to judge about the actual process. One necessary condition is clear: all couplings should be small, so that expansions in formal series with respect to them have a chance to make sense.

In particular, we can ask whether we can trust the rules of perturbative renormalization, which were discussed in the previous lecture and in [I-**Perturbative QFT**]. As you remember, these rules tell how to handle divergences arising at high momenta (ultraviolet divergences). Thus, in order to trust the theory of perturbative renormalization we should be sure that interactions are small at high momenta.

A (Euclidean) quantum field theory (in the non-perturbative sense; for example, in the sense of Wightman axioms) is called asymptotically free if its interactions are small at high momenta. That is, a theory is asymptotically free if its correlation functions in momentum space are asymptotically (at high momenta) close to those of a free theory. Thus, in order to use perturbative renormalization legitimately, we should be sure that our theory is asymptotically free.

Of course, in the perturbative setting we cannot even show that a quantum field theory exists, much less that it is asymptotically free. It is therefore a pleasant surprise that if we assume that a theory with a given perturbation expansion exists, its asymptotic freedom can be checked perturbatively, usually already in the 1-loop approximation. Later we will discuss it in more detail, but now we will just demonstrate it for ϕ^4-theory.

We will only consider renormalizable theories, i.e. points of the space Y. It is clear that such a theory $T \in Y$ is free (i.e. quadratic) if the effective coupling g_* vanishes. Thus, the condition for asymptotic freedom is that the coupling g_* vanishes asymptotically, at high momenta. This means that $\lim_{\mu\to\infty} g_*(\mu) = 0$.

It is clear that we cannot see whether $g_*(\mu)$ approaches zero or not by looking at $g_*(\mu)$ in the perturbation expansion. This is clear from the example of the function $f(\mu) = \frac{1}{1+\tau \ln(\mu/\mu_0)}$, which vanishes at infinity for any finite τ, but diverges at infinity if reduced modulo τ^N for each N. What we can see, however, is whether, as μ increases, the value of $g_*(\mu)$ increases or decreases.

Let us do this for ϕ^4 theory. In this case, the physically meaningful region of values of g_* is $g_* > 0$. So we should find out whether g'_* is positive or negative when g_* is in this region.

Now it is very easy for us to answer this question. Namely, if g_* is small, we can trust formula (2.13): $g'_* = Ag_*^2$. This means that g_* increases, i.e. gets away from zero. Of course, we cannot claim on the basis of (2.13) that g_* increases all the time (remember that (2.13) is only valid for small g_*), but what we can definitely guarantee on the basis of (2.13) is that $\lim_{\mu\to\infty} g_*(\mu) \neq 0$. Thus *the ϕ^4 theory is not asymptotically free*, and its perturbation expansion is not valid at high momenta. This is a non-perturbative result which we obtained using only perturbation theory.

Thus, the renormalization group gives us a subtle way of partial summation of the perturbation series, allowing us to extend the region of validity of perturbation theory, and thus pushing us closer toward the world of non-perturbative field theories.

References and cross references

The Les Houches volume

> Roger Balian, Jean Zinn-Justin (eds.), *Methods in field theory*, Les Houches, XXVIII (1975), American Elsevier Pub. Co., 1976.

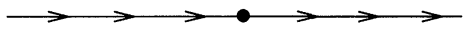
is a basic physics reference for the renormalization group equation.

For more about the renormalization group equation, see Problem FP20 of [I-**Homework**] and [II-**CFT**, §3.4-3.8]. For an example of a nontrivial fixed point see Lecture 3 of [II-**N=1 Dynamics**]. For a practical use of the renormalization group equation see Lecture 2 of [II-**N=1 Dynamics**].

LECTURE 3
A Closer Look at the Renormalization Group Equation

§3.1. Dynamical patterns of the renormalization group flow

In the last lecture we saw that the renormalization group equation for the ϕ^4-theory in the one-loop approximation had the form $g'_* = Ag_*^2$. This equation has a fixed point at $g_* = 0$, which is attracting for $g_* < 0$ and repelling for $g_* > 0$, so it looks like



The fact that this fixed point is repelling for $g_* > 0$ allowed us to show that ϕ^4 theory is not asymptotically free in the region $g_* > 0$.

Now consider what happens for an arbitrary renormalizable field theory with a single renormalizable coupling λ. Let us write down the renormalization group equation for λ based on some mass-independent finite renormalization prescription:

$$\mu \frac{d\lambda}{d\mu} = \beta(\lambda).$$

This equation always has a fixed point at $\lambda = 0$ (the free theory). The behavior of solutions of this equation near the fixed point can be determined by looking at the sign of the first nontrivial term of the Taylor expansion of $\beta(\lambda)$. There can be two patterns of behavior.

1. An attracting, or ultraviolet (UV) stable fixed point ($\beta(\lambda)$ and λ have opposite signs for small λ):

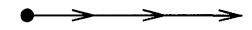


In this case, $\lambda(\mu)$ is driven to the fixed point $\lambda = 0$ in the UV limit $\mu \to \infty$, which implies asymptotic freedom. More precisely, if $\beta(\lambda) \sim A\lambda^{k+1}$, $\lambda \to 0$, then, as one finds by solving the renormalization group equation,

$$|\lambda(\mu)| \sim (|A|k)^{-1/k} |\ln(\mu/\mu_0)|^{-1/k}, \mu \to \infty. \tag{3.1}$$

However, in the limit $\mu \to 0$, $\lambda(\mu)$ is driven away from the fixed point.

2. A repelling, or infrared (IR) stable fixed point ($\beta(\lambda)$ has the same sign as λ for small λ).

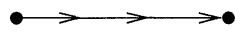


In this case $\lambda(\mu)$ is driven away from the fixed point $\lambda = 0$ in the UV limit $\mu \to \infty$, so there is no asymptotic freedom. However, in the IR limit $\mu \to 0$ the coupling $\lambda(\mu)$ approaches the point $\lambda = 0$, at the rate given by (3.1).

Now we face a question: what happens if $\lambda(\mu)$ is driven away from the fixed point? What will be its behavior when λ is not very small?

The answer to this question cannot be given within the framework of perturbation theory. If the field theory which we are considering actually exists non-perturbatively, then there could be two options:

Option 1. $\lambda(\mu)$ is driven to a fixed point $\lambda_0 \neq 0$:



Option 2. $\lambda(\mu)$ is driven to infinity.

Remark. In general, it could happen that the differential equation $\lambda' = \beta(\lambda)$ has no nonzero solutions which are defined globally for all μ. (Example: $\beta(\lambda) = A\lambda^2$). This means that the coupling $\lambda(\mu)$ becomes infinite at some finite value of μ (e.g. $\lambda = \frac{\lambda_0}{1 - A\lambda_0 \ln(\mu/\mu_0)}$). This kind of singularity is called a renormalon. The appearance of a renormalon does not necessarily mean that the theory does not exist but may just mean that we are doing perturbation expansion around a wrong vacuum state (see the next lecture).

Let us consider some examples.

Example 1: ϕ^4 theory. In this case, as we saw above, the fixed point $g = 0$ is UV stable for $g < 0$ and IR stable for $g > 0$, so we have asymptotic freedom in the unphysical region $g < 0$. For $g > 0$, there is no asymptotic freedom, and in the 1-loop approximation $\lambda(\mu)$ becomes infinite at finite μ: $\mu = \mu_0 e^{\frac{1}{Ag_*^0}}$. Whether this singularity survives non-perturbatively, provided that ϕ^4-theory exists non-perturbatively – cannot be determined in perturbation theory.

Example 2: QED. In QED, the beta-function has the form

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5), \tag{3.2}$$

where e is the dimensionless electric charge. This can be inferred from computing the 1-loop correction to the 3-point function $\langle \psi(x) A(y) \psi(z) \rangle$ where A is the electromagnetic field and ψ represents a charged particle, e.g. an electron (see also the solution to Problem FP18 of [I-**Homework**]). Therefore, the free fixed point $e = 0$ is IR stable in both regions $e < 0$, $e > 0$ (which are both physically meaningful). Thus QED is not asymptotically free.

So what happens in the UV limit? Solving the renormalization group equation in the 1-loop approximation, we get

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2} \ln(\mu/\mu_0)}. \tag{3.3}$$

Thus, if we trust the 1-loop approximation, the electric charge grows as $\mu \to \infty$, and becomes infinite at $\mu = \mu_0 \exp(6\pi^2/e_0^2)$ (the so-called Landau singularity).

One can wonder whether this singularity exists nonperturbatively. This is a rather academic question, since QED is not proved and generally not believed to exist nonperturbatively. However, let us estimate how big μ is. We have $e_0^2/4\pi \sim 1/137$, and $\mu_0 \sim 1\text{MeV}$. So we get $\mu \sim e^{3\pi\times 137/2}\mu_0 \sim 10^{250}\text{MeV}$. Such energy is not contained in the whole observable part of the universe. So even if the singularity occurred at μ, it does not concern us, since we are studying physics at much lower energy scales. This is the reason why, although QED is not asymptotically free, and its perturbation expansion is not valid at high energies, it gives a good approximation to reality in a certain range of energies, and can be used to make predictions which can be checked experimentally.

In formula (3.3), e_0 has the meaning of the observed (effective) charge when the measurements are made at an experimentally accessible scale of energies μ_0, while e has the meaning of the actual charge (i.e. the charge computed on the basis of a very precise theory which describes physics at a very high scale of energies μ). Passage from e to e_0 by moving down with the renormalization group flow from μ to μ_0 has the meaning of averaging out degrees of freedom between μ_0 and μ. The inequality $e_0 < e$ for $\mu_0 < \mu$, which follows from (3.3), has the meaning of *screening* of charges: the observed charge is less than the actual charge.

Remark. This screening phenomenon is known already in classical physics (screening of charges by medium): $e_0 = e/\varepsilon$, where $\varepsilon > 1$ is the dielectric permeability of the medium. To explain this, one should model the medium as a collection of very small dipoles. In the presence of a charged particle with charge e (say $e > 0$) they will align in such a way that the negative parts will face the particle. If we draw a sphere S centered at the particle, it will cut some of the dipoles in half, so that the negative parts will be inside of the sphere and the positive parts outside. Now we can use Newton's theorem: the field of a uniformly charged sphere with charge q is zero inside of the sphere, and is the same as the field of a point charge q situated at the center, outside of the sphere. This theorem implies that the total contribution of the outside parts of the dipoles to the electric field at a point of S is zero, while the total contribution of the negative parts is the same as that of a negative point charge q at the center. So the effective value of the charge is $e - q$. (We should note that q depends on e, since the number of dipoles which will align depends on the strength of the electric field. In fact, q is proportional to e.)

Example 3: Nonabelian gauge theory in 4 dimensions. Before the 1970-s, it was believed that all field theories are IR stable but not asymptotically free. However, in the 70-s it was found that pure nonabelian gauge theories are asymptotically free. Moreover, they remain asymptotically free even if fermions are added, as long as the number of fermions does not exceed a certain limit.

Namely, one can show that the β-function for a nonabelian gauge theory with gauge group G (a compact simple Lie group) and fermions ψ_i has the following form in the 1-loop approximation:

$$\beta(g) = -\frac{1}{48\pi^2}\Big(11C_{\mathfrak{g}} - 4\sum_i C_{V_i}\Big)g^3, \tag{3.4}$$

where $\mathfrak{g}$ is the Lie algebra of G, V_i are the representations of G where the fermions ψ_i live, and C_V is the Casimir C of $\mathfrak{g}$ acting in a representation V. Here the Casimir is normalized in such a way that $C_{\mathfrak{g}} = h^\vee$, where $h^\vee$ is the dual Coxeter number of

$\mathfrak{g}$ ($h^\vee = N$ for $SU(N)$). The proof of formula (3.4) can be found it textbooks on QFT, for example in P.Ramond's textbook.

Formula (3.4) tells us that the pure gauge theory is asymptotically free, so that $g(\mu)$ is a decreasing function for small g (so-called "antiscreening"), and gives us the bound on the number of fermions which can be added without destruction of asymptotic freedom.

For example, consider QCD (quantum chromodynamics), which is believed to describe strong interactions. In QCD, $G = SU(3)$, and $V_i = \mathbb{C}^3$, so $C_{\mathfrak{g}} = 3$ and $C_{V_i} = 1/2$, so we get

$$\beta(g) = -\frac{1}{48\pi^2}(33 - 2N_f)g^3,$$

where N_f is the number of fermions. So, in order for this theory to be asymptotically free, we have to have $N_f \leq 16$.

Asymptotic freedom of QCD can be checked experimentally. One of the possible experiments is described in a simplified way in Lecture 5 of [I-**Perturbative QFT**].

§3.2. Are there any asymptotically free theories without nonabelian gauge fields?

Example 3 in the previous section raises the question whether there can be an asymptotically free theory without nonabelian gauge fields. The answer is the following.

In two dimensions, there are many asymptotically free theories without nonabelian gauge fields. For example, the 2-dimensional theory with N fermions and interaction $\sum_j(\bar{\psi}_j\psi_j)^2$, which we will consider in the next lecture, is asymptotically free.

In four dimensions, this is not the case. Namely, there is the following result.

Coleman-Gross Theorem. *Let $\mathcal{L}(\phi_i, \psi_j)$ be a Poincaré invariant, renormalizable Lagrangian of scalar fields ϕ_k and spinor fields ψ_j:*

$$\mathcal{L} = \int d^4x \left(\frac{1}{2}\sum_k (\nabla\phi_k)^2 + \sum_j (\psi_j, (iD + M_j)\psi_j) + V(\phi) + Y(\phi, \psi) \right), \tag{3.5}$$

where $V(\phi)$ is a quartic polynomial, and $Y(\phi, \psi)$ are Yukawa interaction terms of the form $\phi\psi\psi$. Suppose that the function $V(\phi)$ be bounded from below. Then the quantum field theory defined by $\mathcal{L}$ is not asymptotically free.

It is clear that this theorem generalizes to the case where $\mathcal{L}$ is allowed to contain Abelian gauge fields: in example 2 we saw that this would make the theory even "less" asymptotically free.

The condition that $V(\phi)$ is bounded from below cannot be dropped: we have seen that the ϕ^4 theory with $g < 0$ is asymptotically free. However, one can show that such a theory cannot exist nonperturbatively. Namely, one can show that if such a theory had existed, its spectrum of energy would not be bounded from below, which contradicts the existence of the theory (the energy positivity axiom would be violated).

Let us show this for the case of ϕ^4 theory. We consider the ϕ^4 theory with negative coupling constant g. In this case, the classical potential $V_0(\phi) = \frac{g\phi^4}{4!} + \frac{m^2}{2}\phi^2$ is not bounded from below. A priori, it does not imply that the theory

makes no sense, since this problem could be cured in the quantum theory. In order to see what happens in the quantum theory, we should compute the effective potential, $V_{\text{eff}}(\phi)$, which represents the effective classical theory for the quantum theory defined by $V_0(\phi)$. If this potential is not bounded from below, we can conclude that the theory does not exist quantum-mechanically.

As we know, the effective classical action is computed as a sum of amplitudes of 1-particle irreducible Feynman diagrams (see the end of Lecture 5 of [I-**Axiomatics**]):

$$S_{\text{eff}}(\phi) = \sum_{k\geq 1} \int_{\sum p_i = 0} \Gamma_{2k}(p_1, ..., p_{2k}) \tilde{\phi}(p_1) ... \tilde{\phi}(p_{2k}) dp, \tag{3.6}$$

where Γ_{2k} are 1-particle irreducible correlation functions, and $\tilde{\phi}$ denotes the Fourier transform of ϕ. By definition, the effective potential $V_{\text{eff}}(\phi)$ is the function of one variable defined by the formula

$$V_{\text{eff}}(\phi) = S_{\text{eff}}(\phi), \tag{3.7}$$

where ϕ is a constant function. That is,

$$V_{\text{eff}}(\phi) = \sum_{k\geq 2} \Gamma_{2k}(0) \phi^{2k}. \tag{3.8}$$

In general, it is not easy to see whether this function is bounded from below or not. However, the fact that the ϕ^4 theory with $g < 0$ is asymptotically free allows to figure this out. The demonstration that $V_{\text{eff}}(\phi)$ is not bounded from below is contained in Weinberg's textbook in Chapter 18, Section 18.2. This unboundedness implies that at large scales of ϕ there exist states of arbitrarily low energy, i.e. that the corresponding quantum theory in fact does not exist.

Thus, we have seen that a sensible 4-dimensional quantum field theory which is asymptotically free must contain nonabelian gauge fields. But theories which are not asymptotically free are usually not believed to exist nonperturbatively. Does this mean that theories without nonabelian gauge fields, like QED, do not make any physical sense? (This would really be very strange, since QED was used to make some predictions, which were in a very good agreement with experiment).

The answer of course is no. Even though QED may not exist nonperturbatively as a separate theory, its Lagrangian may be a part of a bigger Lagrangian, which contains other fields (in particular, nonabelian gauge fields), and define a theory which is asymptotically free and perhaps exists nonperturbatively. However, at low energies, when considering electromagnetic interactions, we can ignore all the additional terms in the Lagrangian, and use the Lagrangian of QED.

In fact, the real situation is the following. There is the so called "Standard model", which is a field theory with many different fields, and includes QED as a subtheory. This theory is believed to exist nonperturbatively, and to describe all interactions except gravity in the real world. Now, if we want to compute the scattering data for two electrons in this theory, the answer will be of the order e^2, where e is the charge of the electron, and corrections due to terms in the Lagrangian not contained in the Lagrangian of QED are of order e^4 and higher (see Lecture 5 of [I-**Perturbative QFT**]). Therefore, at low energies, where e is small, QED in the 1-loop approximation is still valid.

§3.3. Renormalization group equations with many couplings

So far we have considered the renormalization group flow only for theories with one critical coupling. Similarly one can define the renormalization group flow in the case when there are many such couplings, $g_1, ..., g_n$. We will assume that the renormalization group flow is defined by a mass-independent family of renormalization prescriptions π_μ parametrized by $\mu \in M$ (for example, dimensional regularization – minimal subtraction). In this case, subcritical couplings do not enter in the renormalization group equation for g_i, so it has the form

$$\mu \frac{dg_i}{\partial \mu} = \beta_i(g_1, \dots, g_n). \tag{3.9}$$

In general, we know nothing about the flow defined by (3.9), except that the point $g_i = 0$ (the free theory) is a fixed point of this flow. However, there is the following Conjecture, which says that this flow behaves as a gradient flow.

Conjecture 3.1. *There exists a function $F(g_1, \dots, g_n)$ such that its total derivative with respect to μ is positive at any nonsingular point of the flow (3.9).*

Remark. Here we, of course, assume that the theory we are considering exists nonperturbatively.

Conjecture 3.1 prohibits various complicated dynamical patterns which could occur in a multidimensional dynamical system (limiting cycles, ergodicity, attractors, etc.), and implies that any trajectory is either driven to a fixed point or to infinity.

The intuition behind conjecture 3.1 is the Wilsonian point of view that the (backward) renormalization group flow comes from erasing degrees of freedom. The meaning of the function F is "the measure of the number of degrees of freedom". Unfortunately, it is unknown how to make mathematical sense of this idea in general, so conjecture 3.1 remains open. It is proved, however, in many special cases, for example, in 2-dimensional field theories.

Remark. In 2-dimensional theories with bosonic fields the space of couplings is infinite-dimensional, since the classical dimension of elementary bosonic fields vanishes (couplings are arbitrary functions). Therefore, the renormalization group equations are equations in the space of functions.

§3.4. The renormalization group equation for composite operators

Suppose we are given a renormalizable field theory T given by some Lagrangian $\mathcal{L}$. Recall from Lecture 3 of [I-**Perturbative QFT**] that every local operator $\mathcal{O}(x)$ in the classical field theory defined by $\mathcal{L}$ can be quantized (non-canonically), and that the space $A_l^0(T)$ of classical operators of scaling dimension $\leq l$ admits a canonical quantization, which we will denote by $A_l(T)$.

Recall that for any operator $\mathcal{O} \in A_l(T)$, the correlation functions of $\mathcal{O}$ are defined to be the ε-coefficients of the correlation functions of the Lagrangian

$$\mathcal{L}_\varepsilon = \mathcal{L} + \varepsilon \mathcal{O}(y), \tag{3.10}$$

where $\varepsilon^2 = 0$.

Let $Z = \cup_{l \geq 0} Z_l$ be a subspace in the space of differential polynomials of fields (with complex coefficients), invariant under the orthogonal group, and the groups G_1, G_2 of scaling the values of fields and scalings of spacetime, and such that the natural map $Z_l \to A_l^0(T)$ is an isomorphism for all theories T of a given type. (Here Z_l is the space of elements of Z of scaling dimension $\leq l$.) We will call V the set of representative differential polynomials, and will always represent a classical local functional by its unique preimage in Z_l. Abusing language, we will call elements of Z "operators".

Remark. The need to introduce Z is caused by the fact that in any field theory, there are nontrivial relations between differential monomials (field equations), and therefore each local functional has many representations as a differential polynomial of elementary fields.

Now we will define the renormalization group flow on the space of operators. We will use the notations of Lecture 2 of this series and Lecture 3 of [I-**Perturbative QFT**].

Let Y be the space of renormalizable theories of the corresponding type, and Y_l be the space of Lagrangians of the form (3.10), where $\mathcal{O}$ is of dimension $\leq l$. The space Y_l is a vector bundle over Y: $Y_l \to Y$, whose fiber at a point $T \in Y$ is the space $A_l(T)$ of operators of dimension $\leq l$ for the theory T.

Let $\pi_\mu \colon Y \to P$ be a renormalization prescription. Let $P_l = P \times Z_l$ be the total space of the trivial bundle over P with fiber Z_l, and and $\pi_\mu^l \colon Y_l \to P_l$ be an isomorphism of vector bundles which descends to $\pi_\mu \colon Y \to P$. π_μ^l is called an extension of the renormalization prescription π_μ to operators of dimension $\leq l$ (this extension is not, in general, defined by π_μ, and has to be defined additionally; below we will give one of such definitions). Let $R^l_{\mu_1\mu_2} = \pi_{\mu_2} \circ \pi_{\mu_1}^{-1} \colon P_l \to P_l$. The transformation $R^l_{\mu_1\mu_2}$ is called the renormalization group transformation for composite operators for the renormalization prescription π_μ^l. It defines a flow on $P_l \times M$ by $t(z,\mu) = (R_{\mu,t\mu} z, \mu)$.

As before, it is convenient to represent the renormalization group flow as a vector field. Denote this vector field on P_l by W_l. It is clear that W_l projects to W under the projection $P_l \to P$, and preserves the structure of a vector bundle on P_l. Therefore,

$$W_l(z,\mu) = W(z,\mu) + \tilde{\gamma}_l(z,\mu), \tag{3.11}$$

where $\tilde{\gamma}_l \colon P \times M \to \mathrm{End} Z_l$ is an operator-valued function (regarded as a vertical vector field on P_l).

If $\mathcal{O}_i$ is a basis of Z_l, homogeneous with respect to both G_1, G_2, and $\mathcal{O} = \sum f_i \mathcal{O}_i$, then the renormalization group equations can be written explicitly as

$$\mu \frac{df_i(z,\mu)}{d\mu} = \sum_j \tilde{\gamma}_l^{ij}(z,\mu) f_j(z,\mu), \tag{3.12}$$

where $\tilde{\gamma}_l^{ij}$ is the matrix of the operator $\tilde{\gamma}_l$ in the basis $\mathcal{O}_i$, and $z = z(\mu)$ is a solution of the renormalization group equation for parameters of the theory.

Now we should explain how to extend the map π_μ to π_μ^l. As usual, there are many ways to do it. However, if π_μ is the dimensional regularization prescription, there is the following natural way of extending π_μ to π_μ^l.

We will consider the ϕ^4 theory. Let $T \in Y$, $\pi_\mu(T) = \mathcal{L}$ be the Lagrangian which describes T at scale μ (in the sense of dimensional regularization) and $\mathcal{O} \in A_l(T)$.

Proposition 3.2. *There exists a unique differential polynomial $F_{\mathcal{O}} \in Z_l$ such that the correlation functions of the operator $\mathcal{O}$ in the theory T are equal to the ε-coefficients of the correlation functions of the Lagrangian $\mathcal{L}_\varepsilon = \mathcal{L} + \varepsilon F_{\mathcal{O}}$ computed using dimensional regularization and minimal subtraction at scale μ (see Section 2.2 of Lecture 2).*

Proof. The statement follows from the fact that classical field equations survive quantization, which was explained (for ϕ^4 theory), and checked in the simplest case in [I-**Perturbative QFT**, §3.7].

Now we define the map π^l_μ by the formula $\pi^l_\mu(T, \mathcal{O}) = (\pi_\mu(T), F_{\mathcal{O}})$. As was explained above, this defines a renormalization group flow on the space of operators. Moreover, since our renormalization prescription is mass-independent, the function $\tilde{\gamma}_l(z, \mu)$ does not explicitly depend on μ.

Remark. The mass independence of the dimensional regularization prescription implies that the map $\tilde{\gamma}_l(z)$ respects not only the filtration of the space Z_l of operators by classical scaling dimension, but also the grading by scaling dimension. That is, operators mix only with operators of the same dimension. This may be false for a mass-dependent prescription.

Let us now consider the example of the ϕ^4 theory. In this case, equation (3.12) has the form

$$\mu \frac{df_i}{d\mu} = \sum_j \tilde{\gamma}_l^{ij}(g_*(\mu), a(\mu)) f_j \tag{3.13}$$

(there is no dependence on m_* due to mass independence of the renormalization prescription). Let Δ_i be the degree of $\mathcal{O}_i$ with respect to scaling of ϕ. It is clear from the scale-of-ϕ-invariance of (3.13) that $\gamma_l^{ij}(g_*, a) = \hat{\gamma}_l^{ij}(g_*) a^{(\Delta_i - \Delta_j)/2}$. Therefore, if we conjugate equation (3.13) by $a^{-D_\phi/2}$, where D_ϕ is the scale-of-ϕ-degree operator, and introduce functions $h_i = f_i a^{-\Delta_i/2}$, we will get the following equations:

$$\mu \frac{dh_i}{d\mu} = \sum_j \gamma_l^{ij}(g_*(\mu)) h_j, \tag{3.14}$$

where

$$\gamma_l = \hat{\gamma}_l - \frac{D_\phi \delta}{2}, \tag{3.15}$$

where δ is defined in the previous lecture. The operator γ_l is already a function only of one variable g_*.

§3.5. Anomalous dimension

Definition. The operator γ_l is called the operator of anomalous dimensions at g_*, for operators of degree $\leq l$. The matrix $\gamma_l^{ij}(g_*)$ is called the matrix of anomalous dimensions at g_*.

For the simplest operators one can define scalar-valued anomalous dimension. Namely, Let $F \in Z_l[a^{\pm 1/2}]$ be a generalized eigenvector of $\gamma_l(g_*)$ with eigenvalue

$\gamma_F(g_*)$ for all g_*. In this case we will say that $\gamma_F(g_*)$ is the anomalous dimension of the operator F at g_*. In general, any quantity X depending on μ and obeying the equation $\mu\frac{dX}{d\mu} = \gamma_X(g_*(\mu))X$, where γ_X is a scalar function, will be said to have anomalous dimension γ_X.

Now let us explain the name "anomalous dimension". Let F be a (real) operator (with a-dependent coefficients) of engineering dimension d_F, scale-of-ϕ degree 0, and scalar anomalous dimension $\gamma_F(g_*)$. For simplicity we assume that it is a true (not only generalized) eigenvector of γ_l. Let

$$\langle F(x)F(0)\rangle = G_2^F(x^2, \mu, m_*, a, g_*) \tag{3.16}$$

be the 2-point function of F in position space. Using the renormalization group flow, we conclude that

$$\begin{aligned}\mu \frac{dG_2^F(x^2, \mu, m_*(\mu), a(\mu), g_*(\mu))}{d\mu} \\ = 2\gamma_F(g_*(\mu))G_2^F(x^2, \mu, m_*(\mu), a(\mu), g_*(\mu)) \ .\end{aligned} \tag{3.17}$$

But the engineering dimension of G_2^F is $2d_F$, so (3.17) can be rewritten in the form

$$\begin{aligned}s\frac{d}{ds}G_2^F((x/s)^2, \mu_0, m_*(s\mu_0), a(s\mu_0), g_*(s\mu_0)) \\ = 2(d_F + \gamma_F(g_*(s\mu_0)))G_2^F((x/s)^2, \mu_0, m_*(s\mu_0), a(s\mu_0), g_*(s\mu_0)) \ .\end{aligned} \tag{3.18}$$

Integrating this equation, we get

$$\begin{aligned}G_2^F((x/s)^2, \mu_0, m_*(s\mu_0), a(s\mu_0), g_*(s\mu_0)) \\ = s^{2(d_F + \int_{\mu_0}^{s\mu_0} \gamma_F(g_*(r\mu_0))d\ln r)} G_2^F(x^2, \mu_0, m_*(\mu_0), a(\mu_0), g_*(\mu_0)) \ .\end{aligned} \tag{3.19}$$

If m_*, a, g_* did not depend on μ for large μ, we would get

$$G_2^F(x^2) \sim |x|^{-2(d_F+\gamma_F)}, \tag{3.20}$$

which shows that the operator F has "quantum scaling dimension" $d_F + \gamma_F$. Thus, anomalous dimension is just the difference between the quantum and the classical scaling dimension.

In general, formula (3.20) does not hold, because there is a nontrivial renormalization group flow on the space of parameters. If this flow does not have an UV stable fixed point, there is usually no hope to derive a scaling law of the form (3.20). However, if the flow has an UV stable fixed point g_*^f, and $\gamma_F(g_*^f) = \gamma_F$, then up to logarithmic factors formula (3.20) is true.

Furthermore, if there is an UV stable fixed point g_*^f, one can replace matrix anomalous dimensions by scalar ones. Namely, one should bring the matrix $\gamma_l(g_*^f)$ to Jordan normal form. The generalized eigenvectors of $\gamma_l(g_*^f)$ with eigenvalue ν are then called "operators of pure anomalous dimension ν".

Unfortunately, the only type of an UV stable fixed point that we can study successfully in perturbation theory is the fixed point $g_*^f = 0$ for an asymptotically free theory. In this case, the anomalous dimensions $\gamma_F(g_*^f)$ in the above sense vanish, because $\gamma_l(0) = 0$. However, for g_* close to zero anomalous dimensions are nonzero, and the knowledge of them helps to get a very sharp UV asymptotics of operators and correlation functions. The most remarkable thing is that in order to compute the exact asymptotics of operators, up to multiplication by a constant, it is enough to know anomalous dimensions only in the first order approximation, which are obtained by an easy explicit calculation. Let us show how this works.

We will consider ϕ^4 theory with $g < 0$ (never mind that it does not exist nonperturbatively; the results will apply to any asymptotically free theory).

Let F be a scale-of-ϕ-independent (i.e. G_1-invariant) eigenvector of γ_l with eigenvalue $\gamma_F(g_*)$, and

$$\gamma_F(g_*) = \gamma_F^1 g_* + O(g_*^2), \tag{3.21}$$

We will assume that F is scale of ϕ-independent. We have seen above that because of asymptotic freedom for large μ one has $g_*(e^t\mu_0) = -\frac{16\pi^2}{3t} + o(t^{-2+\alpha})$, for any $\alpha > 0$. Therefore, using the fact that a is constant modulo g_*^2, we obtain from formula (3.19):

$$G_2^F(x^2) \sim \text{const}|x|^{-2d_F}|\ln|x||^{32\pi^2\gamma_F^1/3}. \tag{3.22}$$

This remarkable formula is actually the exact asymptotics (!!!), except that in order to compute the constant on the right hand side we of course have to know the function γ_F to all orders. However, if we are not interested in the constant, we only need to know the first order approximation.

Formula (3.22) can be generalized to the case of an arbitrary correlation function. Namely, if $F_1, ..., F_n$ are any scale-of-ϕ-independent operators which are eigenvectors of the operator γ_l^1, with eigenvalues $\gamma_{F_1}^1, ..., \gamma_{F_1}^n$, then

$$\langle F_1(e^{-t}x_1)...F_n(e^{-t}x_n)\rangle \sim C(x_1, ..., x_n)e^{-t(\sum d_{F_i})}t^{16\pi^2(\sum \gamma_{F_i}^1)/3}. \tag{3.23}$$

In order to compute the anomalous dimension of an operator, it is enough to study the dynamics of its 2-point function. Let us show how to do it in examples.

Examples. 1. $F = \phi$. The anomalous dimension of ϕ is zero in the 1-loop approximation, since there is no 1-loop contributions to the 2-point function at $x \neq 0$. However, in the 2-loop approximation, computing the only nontrivial 2-loop diagram, one gets a nontrivial correction. Of course, this correction does not enter in formula (3.23), so that correlation functions scale asymptotically by the naive classical rule.

In fact, it is easy to see that

$$\gamma_\phi = -\delta/2 \tag{3.24}$$

This is true because by the definition $\hat{\gamma}_\phi = 0$, since Schwinger functions do not change along the renormalization group flow.

2. $F = \phi^2$. In this case, analyzing the function $\langle\phi^2(x)\phi^2(0)\rangle$, which was computed in Lecture 3 of [I-**Perturbative QFT**], it is easy to find that in the 1-loop approximation

$$\gamma_{\phi^2}(g_*) = -\frac{g_*}{16\pi^2} + O(g_*^2). \tag{3.25}$$

In fact, it is easy to see that $\gamma_{\phi^2} = -2\gamma - \delta$. This follows from the fact that the anomalous dimension of the Lagrangian by definition vanishes. Therefore, the anomalous dimension of $m^2\phi^2/2$ vanishes, because it occurs in the Lagrangian, and cannot mix with other terms. Thus, $\gamma_{\phi^2} = -2\gamma_m$. Since $m = m_* a^{1/2}\mu$, $\gamma_m = \gamma_{m_*} + \gamma_a/2 = \gamma + \delta/2$, as desired.

Since we know that $\gamma = \frac{g_*}{32\pi^2} + O(g_*^2)$, we obtain another proof of (3.25).

Formula (3.25) allows us to write down the asymptotics of the 2-point function of ϕ^2 in the asymptotically free case. Indeed, using (3.22), we get

$$G_2^{\phi^2}(x^2) \sim \text{const}|x|^{-4}|\ln|x||^{-2/3}. \tag{3.26}$$

Finally, let us discuss to what extent the notion of anomalous dimension is canonical. The problem is, we have made a lot of choices (renormalization prescription, the space Z), and of course the anomalous dimension operator γ_l will depend on them. So what is choice independent? The answer: 1) if g_*^f is a nonzero UV stable fixed point, then the eigenvalues of $\gamma_*^f(g_*)$ are invariantly defined, and they are anomalous dimensions of fields in the quantum theory corresponding to g_*^f; 2) the eigenvalues of the first order coefficients of γ_l at $g_* = 0$ are also invariantly defined; in the asymptotically free case they give the logarithmic part of the short-distance asymptotics of the correlation functions.

§3.6. The canonical part of the β-function

Suppose we have a quantum field theory with a single renormalizable coupling λ. In this case, we saw that the renormalization group flow is defined by the equation

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda), \tag{3.27}$$

where $\beta(\lambda) = A\lambda^{k+1} + \ldots$ is the beta-function. Clearly, we can redefine the coupling λ by using a change of variable $\lambda_1 = f(\lambda)$. This redefinition should be trivial for small values of the coupling in order to make physical sense, so we should impose the condition $f'(0) = 1$; that is, $f(\lambda) = \lambda + a_2\lambda^2 + \ldots$ Of course, under this transformation the beta-function will change. So to what extent is the beta-function defined canonically?

This question reduces to the following mathematical problem from the theory of "normal forms".

Problem. Let S_A^k be the set of all formal vector fields of the form $A\lambda^{k+1}\frac{d}{d\lambda} + \ldots$ with real coefficients ($k > 0$), and G be the group of formal diffeomorphisms of the form $\lambda \to f(\lambda)$, $f'(0) = 1$. Compute the quotient S_A^k/G, where G acts by conjugation. In other words, find the normal form of a vector field $\beta(\lambda)\frac{d}{d\lambda}$, where β is as above.

This problem is easily solved by recursion, and the answer is the following.

Answer. $S^k_A/G = \{(A\lambda^{k+1} + B\lambda^{2k+1})\frac{d}{d\lambda}, B \in \mathbb{R}\}$. In particular, any vector field of the form $\beta(\lambda)\frac{d}{d\lambda}$ as above can be reduced by conjugation to the normal form $(A\lambda^{k+1} + B\lambda^{2k+1})\frac{d}{d\lambda}$, while different normal forms are not equivalent.

Thus, we see that it is not only the coefficient A that is defined canonically in the beta-function, but also the coefficient B. In particular, in ϕ^4-theory, the beta-function has the form $A\lambda^2 + B\lambda^3 + \ldots$, and in QED the form $A\lambda^3 + B\lambda^5 + \ldots$, where A, B are defined canonically.

Let us explain the meaning of the coefficient B on the example of ϕ^4 theory. Consider this equation in the region $\lambda < 0$, where we have asymptotic freedom. Solving the equation $\lambda' = A\lambda^2 + B\lambda^3 + \ldots$, we get

$$\lambda(t) = \frac{1}{At} + \frac{B}{A^2 t^2}\ln t + O(1/t^2), \tag{3.28}$$

where $t = \ln(\mu/\mu_0)$. Thus, the coefficient B encodes the second term of the asymptotic expansion of $\lambda(t)$, which characterizes the singularity of λ, as a function of $z = 1/t$, at $z = 0$.

References and cross references

For an experiment that verifies the asymptotic freedom of a nonabelian gauge theory see [II-**Dynamics of QFT**, §5.3]. For renormalization of operators and anomalous dimension, see also Problem FP19 of [I-**Homework**], Problem FE2 of [I-**Homework**], and [II-**CFT**, §3.4-3.8].

LECTURE 4
Dynamical Mass Generation and Symmetry Breaking in the Gross-Neveu Model

§4.1. Dynamical mass generation

In the previous three lectures we developed a general formalism of the method of renormalization group, derived the renormalization group equations, and considered its applications. In particular, we considered the notion of asymptotic freedom. In this lecture, we will actually consider an example of a good quantum theory which exists nonperturbatively and is asymptotically free. But before this let us finish the discussion of general properties of the renormalization group equation.

Consider a theory given by a Lagrangian which classically has a symmetry prohibiting masses (e.g. QED with a chiral fermion with no antichiral partner; such a theory has a chiral symmetry which prohibits the mass of a fermion). Let us ask the question: can a physical mass be dynamically generated in the corresponding quantum theory? Of course, this can only happen if the prohibiting symmetry is broken quantum mechanically.

Let m_p be the physical mass of the particle in this theory. That is, m_p is the pole of the 2-point function in momentum space.

Since the Lagrangian can have no mass term, there is no bare masses. Therefore, it is clear from counting of dimensions that the physical mass of the particle is given by

$$m_p = \mu f(g), \tag{4.1}$$

where f is a scalar function and g is a dimensionless coupling.

As we made clear in the previous lectures, the main property of the renormalization group vector field on the space $P \times M$ (where M is the set of momentum scales, and P the space of parameters) is that any physical parameters – actual (not bare!) masses of particles, scattering amplitudes – are CONSTANT along the trajectories of this vector field. Thus, the renormalization group equation says

$$(\mu\partial_\mu + \beta\partial_g)\mu f(g) = 0, \tag{4.2}$$

which yields

$$m_p = C\mu e^{-\int_{g_0}^{g} \frac{dx}{\beta(x)}}. \tag{4.3}$$

Consider the behavior of this function for weak coupling: $g \to 0$. As usual, this behavior is determined from the behavior of the beta-function, which can be obtained from a 1-loop calculation. For example, assume that in our theory $\beta(g) \sim \pm g^3$ (e.g. QED,QCD). Then

$$m_p \sim \tilde{C}\mu e^{\mp 1/g^2}. \tag{4.4}$$

Thus, for an asymptotically free theory we have $m_p \sim \mu e^{-1/g^2}$. This is a reasonable behavior, since $m_p \to 0$ as $g \to 0$, so the theory is massless in the UV limit, as it should be. Also, if Λ is a cutoff and g_Λ the coupling at Λ, we have $m_p \sim \Lambda e^{-1/g_\Lambda^2}$. In particular, $g_\Lambda^2 \sim 1/\ln\Lambda$ as $\Lambda \to \infty$, which can also be seen from the renormalization group equation.

Note that (4.4) goes to 0 as $g \to 0$ faster than any polynomial; so it is a nonperturbative correction; in particular, all perturbative (Feynman diagram) approximations to $m_p(g)$ vanish. This is not surprising, since the Feynman diagrams are written from the Lagrangian in which there is no room for mass corrections.

Of course, it is possible that the proportionality coefficient in (4.4) vanishes, in which case $m_p = 0$ even in the nonperturbative quantum theory. This can happen for example if the prohibiting symmetry is unbroken (e.g. supersymmetry).

However, if the theory is not asymptotically free, but IR free, we get $m_p \sim \mu e^{1/g^2}$, which is absurd. Thus, in this case, mass generation is not possible.

Finally, let us explain why theories which are not asymptotically free are usually not believed to exist as well-defined nonperturbative quantum theories. For this purpose consider the dependence of the bare coupling g_0 of the cutoff ε (in the dimensional regularization – minimal subtraction scheme); this dependence can be derived from the renormalization group equation. Namely, we have (see Lecture 2)

$$g_0(\varepsilon, g) = g e^{\int_0^g \frac{\beta(x)dx}{x(\varepsilon x/2 - \beta(x))}}, \tag{4.5}$$

where $\varepsilon = 4 - d$.

Let $\beta(x) = \pm x^2 + \ldots$. Then formula (4.5) can be written in the form

$$g_0 = g + \sum_{m=1}^{\infty} g^{m+1} P_m(1/\varepsilon), \tag{4.6}$$

where P_m is a polynomial of degree m with zero free term. This expression yields counterterms in all orders of perturbation theory.

Let us see, however, whether (4.5) and (4.6) make sense nonperturbatively. If the sign in the beta-function is minus (asymptotic freedom), then the integral in (4.5) is well defined, at least for small enough g, since the denominator is positive. However, if the sign is plus (no asymptotic freedom), then the denominator vanishes for $x \sim \varepsilon$, so if ε is small then the integral gets a singularity even for small x. This is usually regarded as a sign that the theory does not exist, i.e. that its perturbation expansion does not correspond to any nonperturbative theory. In particular, the lack of asymptotic freedom for ϕ^4 theory is one of the reasons people believe this theory does not exist.

Remark. In spite of this, physicists often use Lagrangians which are not asymptotically free and even nonrenormalizable for describing quantum field theories. In this case, it is usually meant that the theory is considered with a permanent UV cutoff, which is not sent to infinity; thus the theory only makes sense as a "low energy effective theory". See Lecture 1 of [II-**Dynamics of QFT**] for more details.

§4.2. The Gross-Neveu model

Now we are ready to consider a concrete field theory, on whose example we will be able to illustrate the method of renormalization group. This is a 2-dimensional, purely fermionic theory, which is called the $(\bar\psi\psi)^2$-model, or the Gross-Neveu model.

This model exhibits the following nice properties:

1. It is described by a local Lagrangian and is both renormalizable and asymptotically free.

2. It exists nonperturbatively; this is a nontrivial fact which can be proved by applying a nonperturbative version of the renormalization group method.

3. It has dynamical symmetry breaking and dynamical mass generation (these things go together, as the symmetry which is broken is the one prohibiting mass).

4. This theory is in 2 dimensions, but we can formally continue its large N expansion to $2+\varepsilon$ (using dimensional regularization techniques), and the obtained amplitudes will be regular at $\varepsilon = 1$. Thus we obtain a 3-dimensional "theory" which is nonrenormalizable but exists.

5. This theory is soluble (integrable). This means that its S-matrix can be computed exactly for any fixed N and the Green's functions can be computed exactly in the large N limit (where N is the number of particles). Computation of the S-matrix leads to the theory of quantum groups (Yangians) – see Lecture 4 of [II-**Dynamics of QFT**], where a similar model is studied.

6. It is closely related to 2-dimensional nonlinear sigma-model with target a homogeneous space of positive curvature (see [II-**Dynamics of QFT**]).

Let us now define the Gross-Neveu model. It is a theory of N massless Dirac fermions in two dimensions with a quartic interaction. Namely, let V be a complex N-dimensional vector space with a positive definite Hermitian form $(,)$: $\bar V \times V \to \mathbb{C}$, and let ψ be a Dirac spinor on $\mathbb{R}^{1,1}$ with coefficients in V. Consider the Lagrangian in Minkowski space

$$L(\psi) = \int d^2x \Big(i(\bar\psi, D\psi) + \frac{g^2}{2}(\bar\psi,\psi)^2 \Big). \tag{4.7}$$

Let us determine the symmetries of this Lagrangian. Let S_+, S_- be the spin bundles over $\mathbb{R}^{1,1}$ (these are real, 1-dimensional representations of $Spin(1,1)$ with spins $1/2$ and $-1/2$. Let $S = S_+ \oplus S_-$. By definition, ψ is a section of $\Gamma(S)\otimes V$, so $\psi = (\psi_+, \psi_-)$. In terms of the chiral components (ψ_+, ψ_-), the Lagrangian (4.7) can be written as

$$L(\psi) = \int d^2x \Big(i(\bar\psi_+, D\psi_+) + i(\bar\psi_-, D\psi_-) + g^2((\bar\psi_+, \psi_-) + (\bar\psi_-, \psi_+)) \Big). \tag{4.8}$$

We see that (4.8) has an obvious global $U(N)$ symmetry (unitary transformations of V), as well as a chiral symmetry $\psi_\pm \to \pm\psi_\pm$. This chiral symmetry is the one that prevents a mass term $m(\bar\psi,\psi)$. We will see that, in quantum theory, the chiral symmetry will be broken and the mass will be dynamically generated.

Recall that in 2 dimensions spinor fields have scaling dimension $1/2$. This implies that the theory defined by (4.7) is purely renormalizable (i.e. classically conformally invariant), with a single renormalizable coupling g.

Remark. One can show that the $U(N)$ symmetry survives in quantum theory, and so no new couplings will appear in the process of renormalization.

We will see below that the Gross-Neveu model is asymptotically free for $g^2 > 0$ and IR free for $g^2 < 0$. So we could hope to have a nice theory only for $g^2 > 0$. Note that unlike ϕ^4-theory, the positivity of the coefficient at the quartic term in Minkowski signature is not a problem, since our theory is fermionic and positivity of g^2 does not create a divergence in the functional integral.

To study the Gross-Neveu model, it will be useful to introduce an auxiliary real scalar field σ (with values in $\mathbb{R}$), and consider the Lagrangian

$$L_\sigma(\psi, \sigma) = \int d^2x (i(\bar{\psi}, D\psi) + g\sigma(\bar{\psi}, \psi) - \frac{1}{2}\sigma^2). \tag{4.9}$$

It is clear that this Lagrangian is equivalent to (4.7): indeed, (4.7) is obtained from (4.9) by integrating out σ (i.e. setting σ to equal its stationary point, $\sigma = g(\bar{\psi}, \psi)$.

§4.3. The large N limit

The Gross-Neveu model becomes especially simple in the limit $N \to \infty$. More precisely, in this limit the theory is exactly solvable. This means that we can compute the asymptotic expansion of the Green's functions in this theory in terms of the variable $1/N$. This is a kind of perturbation expansion which is also useful for many other theories. The advantage of this expansion versus perturbation expansions with respect to other parameters (like g) is that $1/N$ is not changed by the renormalization group flow.

We will solve the Gross-Neveu model in the large N limit in two ways: 1) by explicitly summing the perturbation series; and 2) by using the presentation (4.9).

Remark. It is not claimed that there is a well-defined quantum field theory at $N = \infty$; we just have a family of theories depending on N whose Green's functions have a nice asymptotic expansion in N as $N \to \infty$.

We start with summing the perturbation series. We note that the perturbation series for bosonic theories (with a cutoff) usually diverges, and one has to apply Borel summation to get a nonperturbative result. However, in purely fermionic theories (like the Gross-Neveu model) the perturbation theory may converge, defining an analytic (not only C^∞) function of the perturbation parameter. To understand why this is so, it is best to look at two integrals

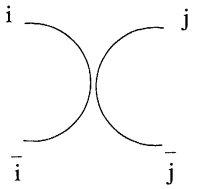
$$I_1(h) = \int_{\mathbb{R}^n} e^{-\frac{v^2 + (v^2)^2}{h}} dv, I_2(h) = \int_{\mathbb{R}^{0,2n}} e^{-\frac{(\psi, A\psi) + (\psi, A\psi)^2}{h}} d\psi$$

where $\mathbb{R}^{0,2n}$ is an odd vector space, and A a skewsymmetric nondegenerate matrix. Clearly, the first integral gives a divergent perturbation series, while the second gives a polynomial. For infinite dimensional integrals, convergence is not obvious even for purely fermionic theories, but one can show that for the Gross-Neveu model the perturbation series indeed converges. (see the paper of Gawedzki and Kupiainen, Comm. Math. Physics, v.102(1),p.1-30).

Consider Lagrangian (4.7) for large N. It is clear that the first (kinetic) term is of order N and the second (quartic) term is of order N^2. Therefore, if set $g_N = \overline{g} N^{-1/2}$, where $\overline{g}$ is fixed, and consider the sequence of theories with N-dimensional V and coupling constant g_N, then both terms in the Lagrangian scale

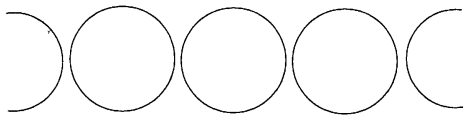
in the same way for large N (they are both of order N). Thus, the large N limit for our theory is similar to its quasiclassical limit (i.e. $1/N$ plays the role of the Planck constant).

Let us consider the Feynman rules for our theory. For this we will choose an orthonormal basis e_i in V and write $\psi = \sum \psi_j e_j$, where ψ_j are complex valued spinors. The Feynman rules are as follows: the fermion propagator (in momentum space of Minkowski signature) is $\frac{ip}{p^2}$ (as usual), and the 4-vertex corresponding to the term $(\bar{\psi}_j, \psi_{j'})(\bar{\psi}_k, \psi_{k'})$ carries the factor $ig^2\delta_{jj'}\delta_{kk'}$. This makes it natural to draw the 4-vertex as two separate (oriented) arcs:



(bars over i, j remember the orientation). In particular, every Feynman diagram is a union of lines and circles, like

(4.10)

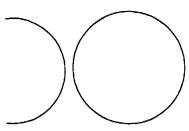


For large N, each circle gives a factor of N (because of summation over the index that lives on this circle) and every "tangency" of two circles (a 4-vertex) gives a factor of g^2. Thus the order of the contribution from such a diagram is N^K, where K is the number of circles minus the number of tangencies.

Now consider the perturbation expansion of the 1-particle irreducible 2-point function Γ_2 for ψ. This means we should consider diagrams with one line and any number of circles. For convenience let us close up the line (this will only give us an extra factor of N). Then we get diagrams with circles only. For any such diagram D, consider the graph $\Gamma(D)$ in which vertices are the circles of D and edges are tangencies of circles. We see that $\Gamma_2 = \sum_D A_D$, where A_D is the amplitude of D, and for large N one has $A_D \sim N^{-n}$, where n is the number of cycles in $\Gamma(D)$. (Note that $\Gamma(D)$ is always connected).

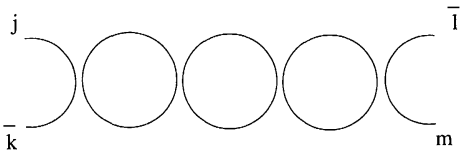
Thus, the leading term of the large N expansion of our theory is given by "tree" diagrams, i.e. diagrams for which $\Gamma(D)$ (not D itself!) is a tree.

Now, it is easy to see that the amplitude of the diagram



vanishes (it contains the factor $\int_{|p|\leq\Lambda} \frac{ip}{p^2} d^2p$ which is zero by symmetry arguments). Therefore, there are no nontrivial trees which give a contribution of order N^0 to the 2-point function (indeed, such a tree would have to have only one 1-valent vertex, which is impossible for a nontrivial tree); thus the propagator does not get a correction in perturbation theory in the top order in N.

Let us now consider the 4-point function $\langle \psi_j \bar{\psi}_k \bar{\psi}_l \psi_m \rangle$. Arguing similarly to the 2-point case, we see that all leading contributions (in N) come from diagrams D for which $\Gamma(D)$ is a tree with only two 1-valent vertices. Any such tree must be a chain. Thus all diagrams of importance in the leading order are the following diagrams, which we call D_n and D'_n (where n is the number of bubbles):



(The picture shows the diagram D_n (for $n = 3$). The diagram D'_n is a similar chain oriented vertically, with the same labels at the ends; thus the semiscircles in D'_n connect j with l, and k with m).

Thus, computing the leading term of the 4-point function reduces to summing a geometric series. Namely, we have

$$\Gamma_4^{jklm}(p_1, p_2, p_3, p_4)\big|_{N=\infty} = \delta_{jk}\delta_{lm}F(s) + \delta_{jl}\delta_{km}F(t), \tag{4.11}$$

where

$$s = (p_1 + p_2)^2, t = (p_1 + p_3)^2, \tag{4.12}$$

j, k, l, m are the labels of the components of ψ for which the 4-point function is computed, and p_1, p_2, p_3, p_4 are the momenta corresponding to labels j, k, l, m. In this formula, the first summand comes from diagrams D_n and the second from D'_n. Thus,

$$F(s) = \sum_{n \geq 0} A_{D_n}. \tag{4.13}$$

The amplitude of D_n is easily computed. Since the Feynman integrals are IR convergent, it suffices to introduce an ultraviolet cutoff Λ. Then

$$A_{D_n} = (ig^2)^{n+1} A^n, \tag{4.14}$$

where

$$A = \frac{iN}{\pi} \ln(-\Lambda^2/s) \tag{4.15}$$

(remember that a fermionic loop contributes a minus sign). Thus, after introducing a counterterm $Cg^4 \ln(\Lambda/\mu)(\bar{\psi}, \psi)^2$ in the Lagrangian, using some scale μ, we will get by summing the geometric series:

$$F(s) = \frac{ig^2}{1 + \frac{g^2 N}{\pi} \ln(-s/\mu^2)}. \tag{4.16}$$

Formula (4.16) implies that the coupling g has the meaning of the strength of the interaction at the scale μ: indeed, if we take p_i such that $(p_i + p_j)^2 = -\mu^2$ then $\Gamma_4^{jklm} = ig^2(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km})$.

Another consequence of formula (4.16) is that our theory is asymptotically free for large N. Indeed, as $s \to -\infty$, the function (4.16) behaves as $1/(\ln(-s))$, i.e. goes to 0. Moreover, we can compute the β-function exactly: from the renormalization group equation and (4.16) we get:

$$\beta(g) = -\frac{g^3 N}{\pi} + O(1),\ N \to \infty. \tag{4.17}$$

This is the exact β-function to leading order in $1/N$.

Thus, we should expect that our theory is well defined in the UV limit.

Let us now consider what happens to this theory in the infrared limit. Then we notice that formula (4.16) has the unfortunate property that the function $F(s)$ has a pole at $s = -\mu^2 e^{-\pi/g^2 N}$. Thus, the Γ-function has a pole outside of 0 for "Euclidean" momenta. This cannot happen in a well defined quantum field theory: it would mean that the theory contains a "tachyon" – particle of imaginary mass.

So what is wrong with our theory? There could be two things: either the theory is "sick" (does not exist) or we have been doing perturbation theory around a "wrong" (unstable) vacuum.

In fact, the situation (which can be expected if we believe the theory exists) is as follows. This theory classically has no mass gap, so the quasiclassical approximation is not enough to determine what it does quantum mechanically. What in fact happens is that the theory has two different realizations, which are permuted by the chiral symmetry (see Lecture 1 of [**II-Dynamics of QFT**]). In particular, the chiral symmetry is broken quantum mechanically. These two vacuum states are massive (i.e. mass is dynamically generated), and the expectation value of $(\bar{\psi}, \psi)$ at them is nonzero: we have two opposite values for two different vacua. However, these expectation values are exponentially small (in terms of the coupling), so they cannot be distinguished in perturbation theory.

Thus, in order to find the quantum vacuum states, we have to use nonperturbative methods. We will use formula (4.9), which contains an auxiliary bosonic field σ.

Consider the Euclidean path integral with Lagrangian (4.9) (remember that we have to change the signs when going from Minkowski to Euclidean space, and remove the i in front of the Dirac operator):

$$\int D\psi D\sigma e^{-\int d^2x((\bar{\psi}, D\psi) - g\sigma(\bar{\psi},\psi) + \frac{1}{2}\sigma^2)}. \tag{4.18}$$

In this integral, ψ has $4N$ real components, and integration is carried out over all of them. Remembering the asymptotics of the terms of the Lagrangian in terms of N, we see that it is convenient to replace σ with $\sigma N^{1/2}$ and rewrite this path integral as

$$\int D\psi D\sigma e^{-N\int d^2x(\frac{1}{N}(\bar{\psi}, D\psi) - \frac{g}{\sqrt{N}}\sigma(\bar{\psi},\psi) + \frac{1}{2}\sigma^2)} \tag{4.19}$$

Now, since the path integral (4.19) is Gaussian in ψ, we can integrate ψ out. We get (up to an overall factor)

$$\int D\sigma \det{}^N(D - g\sqrt{N}\sigma) e^{-N\int d^2x\sigma^2/2} =$$

$$\int D\sigma e^{N\mathrm{Tr}\ln(D - g\sqrt{N}\sigma) - N\int d^2x\sigma^2/2}. \tag{4.20}$$

The large N limit of last integral can be treated by the stationary phase method. In particular, the expectation value of σ for large N is such that it makes the functional

$$I(\sigma) = \mathrm{Tr}\ln(D - g\sqrt{N}\sigma) - \int d^2x\sigma^2/2 \tag{4.21}$$

stationary.

We look for constant functions σ. Then the stationarity equation for (4.21) is

$$\sigma = -\mathrm{Tr}\frac{g\sqrt{N}}{D - g\sqrt{N}\sigma} = -\int \frac{d^2p}{(2\pi)^2}\frac{g\sqrt{N}}{iP - g\sqrt{N}\sigma}, \tag{4.22}$$

where P is the operator of Clifford multiplication by p. Replacing σ with $-i\rho$, we get

$$\rho = Tr\int \frac{d^2p}{(2\pi)^2}\frac{g\sqrt{N}}{P + g\sqrt{N}\rho}, \tag{4.23}$$

It can be shown that this equation has two opposite nonzero solutions as well as the zero solution. To find the nonzero solutions, the right hand side, which is divergent, needs to be renormalized in the UV, using some scale μ. The renormalized equation has the form

$$\rho = \lim_{\Lambda\to\infty}\left(\int_{|p|=\Lambda}\frac{d^2p}{(2\pi)^2}\frac{g\sqrt{N}}{P + g\sqrt{N}\rho} - \frac{g\sqrt{N}}{2\pi}\ln(\Lambda/\mu)\right). \tag{4.24}$$

Simplifying (4.24), we get

$$\begin{aligned}\rho &= 2\lim_{\Lambda\to\infty}\frac{g^2N\rho}{4\pi}\ln\left[\frac{\mu^2}{g^2N\rho^2}(1 + g^2N\rho^2/\Lambda^2)\right] \\ &= 2\frac{g^2N\rho}{4\pi}\ln\left[\frac{\mu^2}{g^2N\rho^2}\right],\end{aligned} \tag{4.25}$$

(the factor 2 in front is due to the fact that P is a 2 by 2 matrix) which implies that

$$\rho = \frac{\mu}{g\sqrt{N}}e^{-\pi/g^2N}. \tag{4.26}$$

Thus, the expected physical mass of the fermions is

$$m_p = g\sqrt{N}\rho = \mu e^{-\pi/g^2N}. \tag{4.27}$$

Another way to find the explicit expression for the nonzero solutions of equation (4.23) is to compute the effective potential for ρ. The computation is analogous to the above computation of the 4-point function, and yields (up to scaling

$$V_{eff}(m) = \frac{m^2}{2} + \frac{g^2Nm^2}{4\pi}\left(\ln\frac{m^2}{\mu^2} - 1\right). \tag{4.28}$$

The critical points of this function are 0 and $\pm m_p$, where m_p is given by (4.27). However, the point $m = 0$ is unphysical (=does not define a quantum vacuum): at this point $V''_{eff}(m) = -\infty$, so the point is a local maximum of the effective potential.

Thus, we have shown that our theory exhibits dynamical mass generation. This is a special case of so-called "dimensional transmutation" – a phenomenon in which a bare dimensionless coupling (g) transforms dynamically into a dimensionful measurable quantity (m).

References and cross references

The original paper describing the Gross-Neveu model is:

D. J. Gross, A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, Phys. Rev. D **10** (1974), 3235.

For more on the Gross-Neveu model, see Problem FE4 of [I-**Homework**]. About dimensional transmutation, see [II-**N=1 Dynamics**, §1.2].

LECTURE 5
Wilsonian renormalization group equation

In this lecture we will return to the Wilsonian point of view on the method of renormalization group, which was discussed in the first lecture. Namely, we will discuss the exact renormalization group equation, which describes the renormalization group flow in the space of all Lagrangians. This equation together with asymptotic freedom provides a tool for constructing nonperturbative field theories.

We follow J.Polchinski's paper "Renormalization and effective Lagrangians", Nuclear Physics B231, 1984, p. 269-295.

For simplicity we will consider the theory of a scalar bosonic field in 4 dimensions, as in the first lecture. We consider the theory with a cutoff propagator $\frac{P(p^2/\Lambda^2)}{p^2+m^2}$, where P is a smooth positive nonincreasing function, such that $P(a) = 1$ for $a \leq 1$ and $P(a) \to 0$ sufficiently fast as $a \to \infty$.

The free action for a massive scalar associated to this cutoff can be written in momentum space as

$$S_0(\phi, \Lambda_0) = \frac{1}{2} \int \hat{\phi}(p)\hat{\phi}(-p) \frac{p^2 + m^2}{P(p^2/\Lambda_0^2)} \frac{d^4p}{(2\pi)^4} \tag{5.1}$$

(hat denotes the Fourier transform). This action is very big if high Fourier modes in ϕ are present, so it "suppresses" high Fourier modes in the path integral. Also, when $\Lambda_0 \to \infty$, this action obviously tends to the standard action for a massive scalar.

Introduce a perturbation of the action S_0 by "interaction" terms:

$$S_I^0(\phi) = \int (\frac{1}{2} g_1^0 \phi^2 + \frac{1}{2} g_2^0 (\nabla\phi)^2 + \frac{g_3^0}{4!}\phi^4) d^4x, \tag{5.2}$$

where g_i^0 are formal variables, and consider the total action $S(\phi, \Lambda_0) = S_0(\phi, \Lambda_0) + S_I^0(\phi)$. Consider the theory whose Green's functions are defined by the path integral

$$Z(J) = \int D\phi \; e^{-S(\phi,\Lambda_0) + \int J\phi d^4x}. \tag{5.3}$$

Because we are considering a cutoff theory, all Feynman diagrams are well defined, so we get a perturbation series.

Now consider some smaller scale $\Lambda_R < \Lambda_0$. We are interested in physics at the scale Λ_R. This means that we assume that $m << \Lambda_R$ and we are interested in values of $Z(J)$ for such J that $\hat{J}(p) = 0$ for $p^2 > \Lambda_R^2$.

The physics at the scale Λ_R will be described by an effective Lagrangian, which is obtained from the original Lagrangian by "integrating out" degrees of freedom between Λ_R and Λ_0.

To study the effective Legrangian, we introduce an arbitrary interaction term $S_I(\phi, \Lambda)$ and consider the integral

$$Z(J, S, \Lambda) = \int D\phi \; e^{-S_0(\phi,\Lambda)+S_I(\phi,\Lambda)+\int J\phi d^4x} \tag{5.4}$$

Let us look for $S_I(\phi, \Lambda)$ that $Z(J, S, \Lambda)$ is independent of Λ, and write down a differential equation for such $S_I(\phi, \Lambda)$.

The independence of Z on Λ yields

$$\frac{d}{d\Lambda} \int D\phi e^{-S_0(\phi,\Lambda)+S_I(\phi,\Lambda)+\int J\phi d^4x} = 0. \tag{5.5}$$

Differentiating, we get

$$\int D\phi \left(\frac{1}{2} \int \hat{\phi}(p)\hat{\phi}(-p)(p^2+m^2) \frac{d}{d\Lambda} P(p^2/\Lambda^2) \frac{d^4p}{(2\pi)^4} - \frac{dS_I(\phi,\Lambda)}{d\Lambda} \right) \times$$
$$e^{-S_0(\phi,\Lambda)-S_I(\phi,\Lambda)+\int \hat{J}(p)\hat{\phi}(-p)\frac{d^4p}{(2\pi)^4}} = 0. \tag{5.6}$$

From this one can get

$$\frac{\partial S_I}{\partial \Lambda} = -\frac{1}{2} \int d^4p (2\pi)^4 (p^2+m^2)^{-1} \frac{\partial P(p^2/\Lambda^2)}{\partial \Lambda} \times$$
$$\left(\frac{\delta S_I}{\delta\phi(-p)} \frac{\delta S_I}{\delta\phi(p)} + \frac{\delta^2 S_I}{\delta\phi(-p)\delta\phi(p)} \right). \tag{5.7}$$

Indeed, substituting (5.7) into (5.6), we get the integral of a complete derivative, i.e. zero; here we heavily use the fact that $\hat{J}(p)$ has support disjoint from that of $\frac{\partial P(p^2/\Lambda^2}{\partial \Lambda}$. And it can be shown that (5.7) follows from (5.6).

Note that equation (5.7) corresponds to a linear equation for e^{S_I}:

$$\frac{\partial e^{S_I}}{\partial \Lambda} = -\frac{1}{2} \int d^4p (2\pi)^4 (p^2+m^2)^{-1} \frac{\partial P(p^2/\Lambda^2)}{\partial \Lambda} \frac{\delta^2 e^{S_I}}{\delta\phi(-p)\delta\phi(p)}. \tag{5.8}$$

Equation (5.8) can be regarded as an infinite-dimensional heat equation. It is called the exact renormalization group equation. If $S_I(\phi, \Lambda)$ is the solution of this equation such that $S_I(\phi, \Lambda_0) = S_I^0(\phi)$, then $S_I(\phi, \Lambda_R)$ is the effective action at the scale Λ_R. This equation allows one to prove the renormalizability of ϕ^4 theory without the use of graph techniques, in particular of Weinberg's theorem (see Polchinski's paper).

References and cross references

The basic paper on the Wilsonian renormalization group equation is

J.Polchinski, *Renormalization and effective Lagrangians*, Nuclear Physics **B231** (1984), 269-295.

Note on Dimensional Regularization

Pavel Etingof

These are comments on dimensional regularization of Feynman integrals, which was discussed in the lectures of K.Gawedzki and D.Gross. This material can be found in the physics literature, but in a slightly different form.

We will first define a certain remarkable operator (the D-dimensional integral), which acts on functions of nonnegative symmetric bilinear forms, and then show how it applies to dimensional regularization.

I am grateful to Pierre Deligne, who helped me improve this text.

§1. The D-dimensional integral

Let E be a real vector space of dimension N with a volume element dv_E. Denote by S^2E^* the space of symmetric bilinear forms on E, and by $S^2_+E^* \subset S^2E^*$ the set of nonnegative symmetric bilinear forms. We think of elements of S^2E^* as selfadjoint operators $E \to E^*$. For any such operator A, we can canonically define its determinant $\det(A) \in \mathbb{R}$.

Let dA be the volume element on S^2E^* defined by dv_E. That is: identify E^* with E by any positive operator $X : E \to E^*$ with determinant 1, regard $A \in S^2E^*$ as a selfadjoint operator on E using this identification, and define dA to be the volume element on S^2E^* corresponding to the norm $||A||^2 = \mathrm{Tr}(A^2)$. (It is clear that this definition does not depend on X).

Fix a complex number D with $\mathrm{Re}D > N - 1$.

Lemma 1. *For any $B \in S^2E$ such that $B > 0$*

$$\int_{S^2_+E^*} (\det A)^{(D-N-1)/2} e^{-\mathrm{tr}(AB)} dA = \Gamma_N(D/2)(\det B)^{-D/2},$$

$$\Gamma_N(D/2) = (2\pi)^{N(N-1)/4} \prod_{l=0}^{N-1} \Gamma\left(\frac{D+l}{2}\right). \tag{1}$$

Proof. Denote this integral by $f(B)$. It is easy to see that for any $S \in GL(E)$, $f(S^*BS) = f(B)|\det S|^{-D}$. From this it follows that $f(B) = C(D)(\det B)^{-D/2}$.

It remains to compute $C(D)$. Let us pick B such that $\det B = 1$, and identify E^* with E using the isomorphism $B : E^* \to E$. Then E becomes a Euclidean space

with the inner product B^{-1}. Thus the constant $C(D)$ has the form

$$\int_{A:E\to E, A^*=A, A\geq 0} (\det A)^{(D-N-1)/2} e^{-\mathrm{tr}A} dA$$

Since this integral is invariant under conjugation by $SO(E)$, we can reduce it to the quotient space $S^2_+E^*/SO(E) = [0,\infty)^N/S_N$. It is known from the theory of symmetric spaces that the pushforward of dA from $S^2_+E^*$ to $S^2_+E^*/SO(E)$ has the form $b_N \prod_{1\leq i<j\leq N} |s_i - s_j| ds_1 ... ds_N$, where s_i are the eigenvalues of $A \in S^2_+E^*/SO(E)$, and

$$b_N = \frac{2^{N(N-1)/4}\pi^{N(N+1)/4}}{\prod_{l=1}^N \Gamma(l/2)},$$

Thus, $C(D)$ has the form

$$b_N N!^{-1} \int_{[0,\infty)^N} \prod_i t_i^{(D-N-1)/2} \prod_i e^{-t_i} \prod_{1\leq i<j\leq N} |t_i - t_j| dt_1 ... dt_N. \tag{2}$$

To compute this integral, we use (a special case of) Selberg's integral formula:

$$\int_{0\leq t_i\leq 1} \prod_i t_i^a \prod_i (1-t_i)^b \prod_{1\leq i<j\leq N} |t_i - t_j| dt_1 ... dt_N =$$

$$\prod_{l=1}^N \frac{\Gamma(a+\frac{l+1}{2})\Gamma(b+\frac{l+1}{2})\Gamma(\frac{l}{2}+1)}{\Gamma(a+b+\frac{2N-l+3}{2})\Gamma(\frac{3}{2})}. \tag{3}$$

Let us set $s_i = bt_i$ in Selberg's integral, and send b to $+\infty$. In the limit, using Stirling's formula, and setting $a = (D-N-1)/2$, we get

$$C(D) = b_N N!^{-1} \prod_{l=1}^N \frac{\Gamma(\frac{D+l-N}{2})\Gamma(\frac{l}{2}+1)}{\Gamma(\frac{3}{2})} =$$

$$b_N \pi^{-N/2} \prod_{l=1}^N \Gamma\left(\frac{D+l-N}{2}\right)\Gamma\left(\frac{l}{2}\right) = \Gamma_N(D/2). \tag{4}$$

The Lemma is proved.

Now consider the function ρ_D^E on S^2E^* which is supported on $S^2_+E^*$ and is given by the formula

$$\rho_D^E(A) = \pi^{ND/2}\Gamma_N(D/2)^{-1}(\det A)^{(D-N-1)/2}. \tag{5}$$

By Lemma 1, we have

$$\int_{S^2_+E^*} \rho_D^E(A) e^{-\mathrm{tr}(AB)} dA = \pi^{ND/2}(\det B)^{-D/2}, \tag{6}$$

$\mathrm{Re}D > N-1$.

If $\mathrm{Re}D > N-1$, the function $\rho_D^E(A)$ is a locally L^1-function, so it defines a distribution on S^2E^*. Let us compute the Fourier transform of $\rho_D^E(A)$, i.e.

$$\hat{\rho}_D^E(B) = \int_{S^2E^*} \rho_D^E(A) e^{-i\mathrm{tr}(AB)} dA,$$

$B \in S^2E$ (the integral is understood in the sense of distributions).

Lemma 2.

$$\hat{\rho}_D^E(B) = \lim_{J \to +0} \pi^{ND/2} \det(iB + J)^{-D/2}. \tag{7}$$

where $J \in S^2E, J > 0$.

Remark. Here we use the branch of the function $z^{-D/2}$ which is equal to $e^{-D \ln z/2}$ for real positive z.

Proof. By analytic continuation, formula (6) holds for arbitrary $B \in (S^2E)_{\mathbb{C}}$ with $\mathrm{Re}B > 0$. In particular, we can replace B with $iB + J$, where $B \in S^2E$. In the limit when $J \to +0$, we obtain formula (7).

Let $\mu : S^2E \to \mathrm{Hom}(E^*, E)$ be the tautological embedding. Let Δ be the polynomial of degree N on S^2E defined by the equation $\Delta(B) = \det(\mu(B))$. We regard Δ as a differential operator of order N on S^2E^* with constant coefficients.

Corollary 3. *If* $ReD > N + 1$, *then*

$$\Delta \rho_D^E(A) = \pi^N \rho_{D-2}^E(A). \tag{8}$$

Proof. The proof is obvious after passing to Fourier transforms.

Let $\mathcal{S}(S^2E^*)$ be the space of complex-valued Schwartz functions on S^2E^*. For any $f \in \mathcal{S}(S^2E^*)$ and $\mathrm{Re}D > N - 1$, define

$$I_D^E(f) = \int_{S^2E^*} \rho_D^E(A) f(A) dA. \tag{9}$$

Corollary 4. $I_D^E(f)$ *extends to an entire function of* D.

Proof. For $ReD > N - 1 - 2K$, $K \in \mathbb{Z}_+$, we can define $I_D^E(f)$ by the formula

$$I_D^E(f) = (-\pi)^{-KN} \int_{S^2E^*} \rho_{D+2K}^E(A) \Delta^K f(A) dA. \tag{10}$$

According to Corollary 3, if $\mathrm{Re}D >> 0$, formula (10) defines the same function as (9). Thus, (10) defines an analytic continuation of I_D^E.

Proposition 5. *Let* $D \geq 0$ *be a nonnegative integer, and* V *be a vector space of dimension* D, *with a positive definite symmetric bilinear form* β. *Then for any* $f \in \mathcal{S}(S^2E^*)$ *one has*

$$I_D^E(f) = \int_{\mathrm{Hom}(E,V)} f(x^*(\beta)) dx, \tag{11}$$

where $x^*(\beta)$ *denotes the inverse image of* β *under* x *(a nonnegative symmetric bilinear form on* E*).*

Proof. It is enough to check that (11) holds for functions $e^{-\mathrm{tr}(AB)}1_{S^2_+E^*}(A)$ where 1_Y is the characteristic function of the set Y, and $\mathrm{Re}(B) > 0$. This follows from the fact that the function $f(A)$ admits a Fourier representation.

In the case

$$f(A) = e^{-\mathrm{tr}(AB)}1_{S^2_+E^*}(A), \tag{12}$$

evaluating the Gaussian integral on the right hand side of (11), we get $\pi^{ND/2}(\det B)^{-D/2}$. On the left hand side of (11), by Lemma 1, we get the same. The proposition is proved.

Remark. From Proposition 5 it is clear why formula (9) works only for $\mathrm{Re}D > N-1$. Indeed, for integer D between 0 and $N-1$ the distribution $\rho_D(A)dA$ is concentrated on forms of rank $\leq D$, so it is a purely singular distribution. Thus, the measure $\rho_D(A)dA$ is not absolutely continuous with respect to dA and hence cannot be represented as a function times dA. In fact, it turns out that this distribution is singular (not locally L^1) for all D with $\mathrm{Re}D \leq N-1$.

Proposition 5 motivates the following definition.

Definition. *$I^E_D(f)$ is called the D-dimensional integral of f.*

Another way of computing the D-dimensional integral is using Fourier transforms. By the Plancherel formula, it follows from (9) that

$$I^E_D(f) = (2\pi)^{-N(N+1)/2}\int_{S^2E^*}\hat{f}(B)\overline{\hat{\rho}^E_D(B)}dB, \tag{13}$$

where $\hat{f}$ is the Fourier transform of f. (It is clear that $\overline{\hat{\rho}^E_D(B)} = \lim_{J\to+0}\pi^{ND/2}\det(-iB+J)^{-D/2}$).

For $\mathrm{Re}D < 2$, this formula can be written as

$$I^E_D(f) = 2^{-N(N+1)/2}\pi^{N(D-N-1)/2}\int_{S^2E^*}\hat{f}(B)e^{\frac{\pi i D}{4}\sigma(B)}|\det(B)|^{-D/2}dB, \tag{14}$$

where $\sigma(B)$ is the signature of B (the number of pluses minus the number of minuses). If $\mathrm{Re}D \geq 2$, the distribution $\overline{\hat{\rho}^E_D(B)}$ is singular, so in order to define $I^E_D(f)$ by a direct integration formula, we need to use the operator Δ. Namely, from Lemma 1 it follows that

$$\Delta\hat{\rho}^E_D(B) = (-1)^N p_N(D)\hat{\rho}^E_{D+2}(B), \quad p_N(D) = \pi^{-N}\prod_{l=1}^{N}\frac{D+l-N}{2} \tag{15}$$

So if $\mathrm{Re}D < 2+2K$, we can define I^E_D by the formula

$$I^E_D(f) = 2^{-N(N+1)/2}\pi^{N(D-N-1)/2}\prod_{J=1}^{K}p_N(D-2J)^{-1}\times$$
$$\int_{S^2E^*}\Delta^K\hat{f}(B)e^{\frac{\pi i(D-2K)}{4}\sigma(B)}|\det(B)|^{(-D+2K)/2}dB, \tag{16}$$

Example. If $N = 1$, the D-dimensional integral of f is (up to normalization) just the Mellin transform of f:

$$I_D^E(f) = \pi^{D/2}\Gamma(D/2)^{-1}\int_0^\infty x^{\frac{D}{2}-1}f(x)dx.$$

Let $\mathcal{S}(S_+^2E^*)$ be the space of smooth functions f on $S_+^2E^*$, such that all the derivatives of f are rapidly decaying. It is easy to see that the restriction map $\mathcal{S}(S^2E^*) \to \mathcal{S}(S_+^2E^*)$ is surjective. So, I_D^E defines a linear functional $\mathcal{S}(S_+^2E^*) \to \mathbb{C}$.

§2. D-dimensional integral with parameters

Now we will extend our construction to a more general situation. Let $F \subset E$ be a subspace of dimension M with a volume element dv_F. Any symmetric bilinear form A on E defines a form A_F on F (the restriction of E).

Also, for any symmetric bilinear form B on E^* such that $B_{F^\perp}$ is nondegenerate we can define a form B^{F^*} on F^* as follows. Let $(F^\perp)^{\perp_B}$ be the B-orthogonal complement of $F^\perp$. As $B_{F^\perp}$ is nondegenerate, this space is naturally identified with $F^* = E^*/F^\perp$. By definition, the form B^{F^*} is the restriction of B to $(F^\perp)^{\perp_B} = F^*$.

Let us find an explicit expression of B^{F^*}. Choose a positive definite inner product on E and thus identify E^* with E. Then we have a decomposition $E = F \oplus F^\perp$, and B becomes a selfadjoint operator $E \to E$ with a decomposition

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $B_{ij}^* = B_{ji}$. Now the operator corresponding to $B_{F^\perp}$ is B_{22}, and the operator corresponding to B^{F^*} is $B_{11} - B_{12}B_{22}^{-1}B_{21}$. From this formula it is obvious that for this operator to be defined, it is sufficient for B_{22} to be invertible.

The following Lemma is a generalization of Lemma 1.

Lemma 6. *For $ReD > N - 1$ and any positive $B \in S^2E$, $C \in S^2F^*$*

$$\int_{S_+^2E^*} \delta(A_F - C)(\det A)^{(D-N-1)/2}e^{-\mathrm{tr}(AB)}dA =$$

$$\frac{\Gamma_N(D/2)}{\Gamma_M(D/2)}e^{-\mathrm{tr}(CB^{F^*})}(\det C)^{(D-M-1)/2}(\det B_{F^\perp})^{-D/2}, \tag{17}$$

Remark. The space S^2F^* inherits a volume element from F, so the delta-function $\delta(A_F - C)$ is well defined.

Proof. Denote the left hand side of (17) by $I(B, C)$. Let $S \in GL(E)$ be an automorphism which preserves F. Make a change of variable $A \to S^*AS$. Then $A_F \to S_F^*A_FS_F$, where S_F is the restriction of S to F. Thus we get

$$I(B, C) = |\det S|^{N-M}I(SBS^*, (S_F^*)^{-1}CS_F^{-1}). \tag{18}$$

Now choose B with $\det B_{F^\perp} = 1$, and identify E^* with E using the map $B : E \to E^*$. Then E becomes a Euclidean space with the inner product B^{-1}. Now we can identify F^* with F using the map $B^{F^*} : F^* \to F$; thus F is also a Euclidean space.

After these identifications, integral (17) takes the form

$$\int_{A\in S^2_+E^*}\delta(A_F-C)(\det A)^{(D-N-1)/2}e^{-\mathrm{tr}A}dA. \tag{19}$$

Using the decomposition $E=F\oplus F^\perp$, we can write A in the form

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}, \tag{20}$$

where $A^*_{ij}=A_{ji}$. This allows us to rewrite (19) as

$$\int_{A\in S^2_+E^*:A_{11}=C}(\det A)^{(D-N-1)/2}e^{-\mathrm{tr}A}dA. \tag{21}$$

To compute this integral, let us use the identity

$$\det A=\det A_{11}\det(A_{22}-A_{21}A_{11}^{-1}A_{12}).$$

Since we are integrating over such A that $A_{11}=C$, we get

$$\det A=\det C\det(A_{22}-A_{21}C^{-1}A_{12}).$$

After the change of variable $X=A_{22}-A_{21}C^{-1}A_{12}$, integral (21) takes the form

$$(\det C)^{(D-N-1)/2}e^{-\mathrm{tr}\,C}\int_{X\geq 0}(\det X)^{(D-N-1)/2}e^{-\mathrm{tr}X}dX \int_{\mathrm{Hom}(F^\perp,F)}e^{-\mathrm{tr}A^*_{12}C^{-1}A_{12}}dA_{12} \tag{22}$$

The second integral in (22) is Gaussian and equals

$$(2\pi)^{M(N-M)/2}(\det C)^{(M-N)/2}. \tag{23}$$

The first integral in (22) was computed in Lemma 1 and equals $\Gamma_{N-M}((D-M)/2)$. Using the identity

$$\frac{\Gamma_N(D/2)}{\Gamma_M(D/2)}=(2\pi)^{M(N-M)/2}\Gamma_{N-M}((D-M)/2). \tag{24}$$

we obtain the statement of the Lemma.

Now consider the distribution on S^2E^* (depending on C as a parameter)

$$\rho_D^{EF}(A,C)=\delta(A_F-C)\frac{\rho_D^E(A)}{\rho_D^F(C)}=$$

$$\pi^{(N-M)D/2}\frac{\Gamma_M(D/2)}{\Gamma_N(D/2)}\delta(A_F-C)(\det C)^{-(D-M-1)/2}(\det A)^{(D-N-1)/2}, \tag{25}$$

for $\mathrm{Re}D>N-1$.

For $f\in\mathcal{S}(S^2_+E^*)$, define a function $I_D^{EF}(f)$ on $C>0$ by setting

$$I_D^{EF}(f)(C)=\int_{S^2_+E^*}\rho_D^{EF}(A,C)f(A)dA. \tag{26}$$

Since $(\det C)^{-1}\leq(\det A)^{-1}(\mathrm{tr}A)^{N-M}$, the function $I_D^{EF}(f)$ extends continuously to $C\geq 0$, i.e. defines a continuous, rapidly decaying function on $S^2_+F^*$.

Remark. Although the power of $(\det C)^{-1}$ in (25) is higher than that of $\det A$, for small $\det C$ the intersection of the region of integration in (26) with any fixed ball has area $\sim (\det C)^{(N-M)/2}$ (because of the presence of the delta-function), which helps compensate the negative exponent of $\det C$.

Using Lemma 6, it is easy to compute the Fourier transform $\hat{\rho}_D^{EF}(B,C)$ of $\rho_D^{EF}(A,C)$ with respect to A.

Lemma 7.

$$\hat{\rho}_D^{EF}(B,C) = \lim_{J\to+0} \pi^{(N-M)D/2} e^{-i\mathrm{tr}(CB^{F^*})} \det(iB_{F^\perp} + J)^{-D/2}. \tag{27}$$

Proof. Analogous to the proof of Lemma 2.

Let Δ_2 be the operator Δ acting on $\mathcal{S}(S^2F^\perp)$.

Corollary 8. *If $ReD > N+1$, then*

$$\Delta_2 \rho_D^{EF}(A,C) = \pi^N \rho_{D-2}^{EF}(A,C). \tag{28}$$

(here Δ_2 acts on A).

Proof. The proof is immediate after passing to Fourier transforms.

Corollary 9. *For any $f \in \mathcal{S}(S_+^2 E^*)$, $I_D^{EF}(f)$ is a Schwartz function of C, and for any $C \geq 0$ the function $I_D^{EF}(f)(C)$, regarded as a function of D, analytically continues to an entire function.*

Proof. Similar to the Proof of Corollary 4.

Proposition 10. *Let $D \geq 0$ be a nonnegative integer, and V be a vector space of dimension D with a positive definite symmetric bilinear form β. Then for any $f \in \mathcal{S}(S_+^2 E^*)$ and $y : F \to V$ one has*

$$I_D^{EF}(f)(y^*(\beta)) = \int_{x\in \mathrm{Hom}(E,V):x|_F=y} f(x^*(\beta))dx. \tag{29}$$

Proof. As in Proposition 5, it is enough to consider $f(A) = e^{-\mathrm{tr}(AB)}$. In this case, on the r.h.s. of (19) is a Gaussian integral, whose value is

$$\pi^{(N-M)D/2} e^{-\mathrm{tr}(y^*(\beta)B^{F^*})} (\det B_{F^\perp})^{-D/2}.$$

Thus, the Proposition follows from Lemma 6.

Definition. *I_D^{EF} is called the D-dimensional integral with external parameters.*

As before, the D-dimensional integral can be computed using Fourier transform. Let $f \in \mathcal{S}(S^2E^*)$. By the Plancherel formula,

$$I_D^{EF}(f)(C) = (2\pi)^{-N(N+1)/2} \int_{S^2E^*} \hat{f}(B)\overline{\hat{\rho}_D^{EF}(B,C)}dB, \tag{30}$$

where $\hat{f}$ is the Fourier transform of f. Here

$$\overline{\hat{\rho}_D^{EF}(B,C)} = \lim_{J\to+0} \pi^{(N-M)D/2} e^{i\mathrm{tr}(CB^{F^*})} \det(-iB_{F^\perp} + J)^{-D/2}.$$

For $\mathrm{Re}D < 2$, this formula can be written as

(31)
$$I_D^{EF}(f)(C) =$$
$$2^{-\frac{N(N+1}{2}} \pi^{\frac{(N-M)D-N(N+1)}{2}} \int_{S^2E^*} \hat{f}(B) e^{\frac{\pi i D}{4}\sigma(B_{F^\perp})} e^{i\mathrm{tr}(CB^{F^*})} |\det(B_{F^\perp})|^{-D/2} dB.$$

If $\mathrm{Re}D \geq 2$, the distribution $\overline{\hat{\rho}_D^{EF}(B,C)}$ is singular, so in order to define $I_D^{EF}(f)$ by a direct integration formula, we need to use the operator Δ. Namely, from Lemma 6 it follows that

$$(\det C)^{-1} \Delta \hat{\rho}_D^{EF}(B,C) = (-1)^{N-M} p_{NM}(D) \rho_{D+2}^{EF}(B),$$

(32)
$$p_{NM}(D) = \pi^{M-N} \prod_{l=M+1}^{N} \frac{D+l-N}{2}$$

So if $\mathrm{Re}D < 2 + 2K$, we can define I_D^{EF} by the formula

$$I_D^{EF}(f)(C) = 2^{-N(N+1)/2} \pi^{((N-M)D-N(N+1))/2} (\det C)^{-K} \prod_{J=1}^{K} p_N(D-2J)^{-1} \times$$
$$\int_{S^2E^*} \Delta^K \hat{f}(B) e^{i\mathrm{tr}(CB^{F^*})} e^{\frac{\pi i(D-2K)}{4}\sigma(B)} |\det(B)|^{(-D+2K)/2} dB.$$

Thus we have defined an operator $I_D^{EF} : \mathcal{S}(S_+^2E^*) \to \mathcal{S}(S_+^2F^*)$. It is clear that $I_D^{FG} I_D^{EF} = I_D^{EG}$ if $E \supset F \supset G$, and $I_D^{E0} = I_D^E$.

Now we can define the D-dimensional integral of functions with polynomial growth. Let f be a smooth function on S^2E^* such that there exists $a \in \mathbb{R}$ for which f and all its derivatives satisfy the condition $\phi(A) = O(||A||^a)$, $||A|| \to \infty$ (for any norm on S^2E^*). In this case, the integral $I_D^{EF}(f)$ is defined for $\mathrm{Re}D << 0$. Indeed, the condition on f implies that the Fourier transform $\hat{f}$ is a distribution defined on functions with sufficiently many continuous derivatives which have polynomial growth at infinity. Thus, we can use formula (30) to define $I_D^{EF}(f)$ for $\mathrm{Re}D << 0$.

Remark. The operator I_D^{EF} has rather strange properties. For example, if P is any polynomial, then $I_D^E(P(A)) = 0$. Indeed, the Fourier transform of P is a distribution supported at 0, so for sufficiently negative D formula (14) gives 0. Thus, I_D^E is not continuous in any topology in which polynomials are dense.

Example. Let $N = 1$, $M = 0$, and f be a rational function, regular at $x \geq 0$. Write f as a sum of simple fractions: $f(x) = f_0(x) + \sum_i \alpha_i (x + a_i)^{-n_i}$, where f_0 is a polynomial, $a_i \in \mathbb{C} \setminus \mathbb{R}_-$, and n_i are positive integers. Then

$$I_D^E(f) = \pi^{D/2} \sum_i \alpha_i \frac{\Gamma(n_i - D/2)}{\Gamma(n_i)} a_i^{-n_i + \frac{D}{2}}.$$

Thus we see that if f satisfies the above condition of polynomial growth then $I_D^{EF}(f)(C)$ is holomorphic in D for $\mathrm{Re}D << 0$. In general, we cannot expect that this function will analytically continue to the whole complex plane. However, for functions of a certain algebraic nature, it extends meromorphically to the whole complex plane.

§3. D-dimensional integral of functions arising from Feynman diagrams

Definition. *A function f on $S^2_+E^*$ is called a function of Feynman type if it has the form*

$$f(A) = \frac{P(A)}{\prod_{j=1}^{l}(tr(AB_j) + m_j^2)},$$

where P is a polynomial, $B_j \geq 0$, such that $(\sum t_i B_i)_{F^\perp}$ is nondegenerate for $t_i > 0$, and $m_j \neq \overset{*}{0}$ are real numbers.

Proposition 11. *If f is of Feynman type then $I_D^{EF}(f)$ extends meromorphically to the whole complex plane.*

Proof. We will prove the statement for $P = 1$. For general P, the proof is similar to the case $P = 1$.

Using the formula $b^{-1} = \int_0^\infty e^{-tb}dt$, we can rewrite $I_D^{EF}(f)$ in the form

$$I_D^{EF}(f) = \int_{t_1,\ldots,t_l \in [0,\infty)} e^{-\sum t_j m_j^2} I_D^{EF}(e^{-\mathrm{tr}(A(\sum t_j B_j))})dt. \tag{33}$$

Set $B(t) = \sum t_j B_j$. Using Lemma 6, we can rewrite this in the form

$$I_D^{EF}(f)(C) = \pi^{(N-M)/2} \int_{t_1,\ldots,t_l \in [0,\infty)} e^{-\sum t_j m_j^2 - \mathrm{tr}(CB^{F^*}(t))} (\det B_{F^\perp}(t))^{-D/2} dt. \tag{34}$$

Now we will use the following theorem of Bernstein, which is proved with the help of the theory of $\mathcal{D}$-modules.

Bernstein's theorem. *Let $Q(t)$ be any polynomial in l variables. Then there exists a polynomial differential operator $L(D)$ in l variables, with coefficients depending polynomially on D, and a polynomial q in D, such that $L(D)Q^{-D/2} = q(D)Q^{-1-D/2}$.*

This theorem implies the following statement, which is a special case of a result of Bernstein.

Corollary. *Assume that Q takes positive values for $t_j > 0$, and let g be a rapidly decaying smooth function defined for $t_i \geq 0$, such that its derivatives are also rapidly decaying. Then the integral*

$$I(D, g) = \int_{t_i \in [0,\infty)} g(t)Q(t)^{-D/2}dt, \tag{35}$$

convergent for $ReD << 0$, extends meromorphically to the whole complex plane.

The corollary is proved by induction on l (the number of t_j). The base of the induction is trivial. Now, using Bernstein's theorem and integration by parts, we can write

$$I(D+2, g) = q(D)^{-1} \int g(t)L(D)(Q(t)^{-D/2})dt =$$

$$q(D)^{-1} \int (L(D)^* g(t))Q(t)^{-D/2}dt + C(D) = q(D)^{-1} I(D, L(D)^* g) + C(D), \tag{36}$$

where $C(D)$ is the integral over the boundary, which is meromorphic by the assumption of induction.

Now we apply the corollary to our situation, and obtain the proof of the Proposition for $P = 1$.

Remark 1. Using Hironaka's theorem about the existence of a smooth resolution of singularities for a compact projective variety, it is possible to prove that I_D^{EF} is meromorphic in D in a much more general situation than in Proposition 11. For example, as was explained to me by M.Kontsevich, one can prove that if f is a rational function without singularities in $S_+^2 E^*$, then $I_D^{EF}(f)$ is meromorphic.

It is also possible to show that for rational f as above the poles of $I_D^{EF}(f)$ are all rational, and belong to a finite number of arithmetic progressions.

§4. Dimensional regularization of Feynman integrals

Proposition 11 allows one to define the procedure of dimensional regularization of Feynman integrals in quantum field theory.

Suppose that we have a quantum field theory given by some Lagrangian, and we want to compute its correlation functions. According to the Feynman rules, such a correlation function equals the sum of amplitudes of all Feynman diagrams (graphs), which are given by some finite-dimensional integrals.

We will work in the Euclidean setting and in momentum space. For simplicity we will assume that our theory contains only scalar massive fields. Then the amplitude of any Feynman graph Γ is given by the formula

$$\int_{x\in \mathrm{Hom}(E,V):x|_F=y} g(x)dx, y \in \mathrm{Hom}(F,V), \tag{37}$$

where V is the momentum space (a Euclidean space of dimension d with inner product β), $E = H_c^1(\Gamma \setminus \partial\Gamma, \mathbb{R})$ (the cohomology with compact supports), F is the image of the map $H^0(\partial\Gamma, \mathbb{R}) \to H_c^1(\Gamma \setminus \partial\Gamma, \mathbb{R})$ in the long exact sequence of cohomology, and g is a function on $\mathrm{Hom}(E,V)$ of the form $g(z) = f(z^*(\beta))$, where f is a certain function of Feynman type attached to Γ.

Remark. It is clear from the long exact sequence that $E/F = H^1(\Gamma, \mathbb{R})$.

Since cohomology groups are defined over $\mathbb{Z}$, they have natural volume elements. This defines the volume element dx in (37).

The problem is that integral (37) is often divergent, since the function g does not decay rapidly enough.

Dimensional regularization is a method to give meaning to the integral (37). This is done as follows. Recall that since f is of Feynman type, $I_D^{EF}(f)$ is meromorphic in D.

Definition. Dimensional regularization of the integral (37) is the regular part of $I_D^{EF}(f)$ at $D = d$, i.e. $\frac{1}{2\pi i}\oint_{|D-d|=\varepsilon} I_D^{EF}(f) d\ln(D-d)$.

Remark. In general, we have to consider functions f which (even before the regularization) depend meromorphically on D. This is caused by the fact that our graph Γ may contain divergent proper subgraphs, which must be regularized before we start working with Γ. Since regularization is performed by adding D-dependent counterterms, we will have to work with functions which depend meromorphically on D.

§5. D-dimensional Stokes formula

Let $F \subset E$ be vector spaces with volume elements, $\dim E = N$, $\dim F = M$, and D be any complex number. Define the space Ω^{top}_{EF} of (Schwartz) top degree forms to be the space $\mathcal{S}(S^2_+E^*)$, and the space $\Omega^{\text{top}-1}_{EF}$ of forms of degree top -1 to be the space of Schwarz functions $\omega : S^2_+E^* \to \text{Hom}(E/F, E)$. That is, $\Omega^{\text{top}}_{EF} = \text{Hom}(E/F, E) \otimes \mathcal{S}(S^2_+E^*)$.

If D is an integer, and V is a Euclidean space of dimension D with inner product β, then we have a linear map $\theta : \text{Hom}(E, V) \to \text{Hom}(F, V)$ given by restriction, and any $f \in \Omega^{\text{top}}_{EF}$ indeed defines a top differential form $f(x^*(\beta))dx$ on each fiber of θ (dx is the natural volume form on the fiber).

Also, any $\omega \in \Omega^{\text{top}-1}_{EF}$ defines a (top -1)-form on the fiber. Indeed, let $z : (E/F)^* \to V$ be a linear map. We regard z as a constant 1-form on the fibers of θ. Then the (top -1)-form corresponding to Ω is defined by the formula $(\omega \wedge z)(x) = \text{tr}(\omega\beta(x, z))dx$, where $\beta(x, z) \in \text{Hom}(E \otimes (E/F)^*, \mathbb{R})$ is regarded as an operator $E \to E/F$.

Recall that the group $GL(E)$ acts on $S^2_+E^*$ by $A \to S^*AS$, $S \in GL(E)$. Thus, any element X of the Lie algebra $\text{End}(E)$ of $GL(E)$ defines a vector field on $S^2_+E^*$: $L_X f(A) = \frac{d}{dt}|_{t=0} f(e^{X^*t}Ae^{Xt})$.

For any $\omega = \sum X_i \otimes f_i \in \Omega^{\text{top}-1}_{EF}$, $X_i \in \text{Hom}(E/F, E)$, $f_i \in \mathcal{S}(S^2_+E^*)$, define the form $d\omega \in \Omega^{\text{top}}_{EF}$ by the formula

$$d\omega = \sum_i (L_{X_i} f_i + D\text{tr}(X_i) f_i), \tag{38}$$

where X_i are regarded as elements of $\text{End}(E)$. Thus, we have defined a linear map $d : \Omega^{\text{top}-1}_{EF} \to \Omega^{\text{top}}_{EF}$. It is easy to see that if D is an integer, d coincides with the usual differential.

Proposition 12. *(Stokes formula.)* $I^{EF}_D(d\omega) = 0$ *for any* $\omega \in \Omega^{\text{top}-1}_{EF}$.

Proof. It is enough to prove the formula for $\text{Re}D >> 0$; for other values of D the formula follows by analytic continuation.

For $\text{Re}D >> 0$, the formula follows by integration by parts from the definition (26) of I^{EF}_D, as $L_X((\det A)^{(D-N-1)/2}dA) = D\text{tr}(X)(\det A)^{(D-N-1)/2}dA$.

References and cross references

Bernstein's theorem can be found in the paper by J.Bernstein, in Funct.Anal.Appl., v.6, issue 4, p.273,1972. Dimensional regularization was discussed in Lecture 1 of [**I-Renorm Groups**] and Lecture 3 of [**II-CFT**].

Homework

Edward Witten

Introduction

We collect here many of the problem sets given throughout the special year at IAS. (Other problem sets appear in [II-**Strings**], [II-**Dynamics of QFT**], and in [II-**N=1 Dynamics**].) The problems are divided into four sections:

Problem Sets from Fall Term
Fall Exam
Superhomework
Addendum to Superhomework

Problems in the Fall Term are mostly related to the lecture series in Part 2, as are the problems in the Fall Exam. The Superhomework and Addendum were given in the spring of the special year in conjunction with [II-**Dynamics of QFT**]. The solutions to the homework problems were written over a long time period by different people, so there is much variation in the style and notation. The solutions to the superhomework may be found in [I-**Supersolutions**].

Solutions in Chapter 2 by Pierre Deligne, Daniel Freed, Lisa Jeffrey, and Siye Wu

CHAPTER 1
Problems

Problem sets from fall term

The problems below are supplements to the lectures in Part 2 (Formal Aspects of Quantum Field Theory)

Problem FP1. Let M be a fixed Riemannian manifold, and let $X : \mathbb{R} \to M$ be a map from the t-line $\mathbb{R}$ (endowed with the metric $(dt)^2$) to M. Let

$$L = \frac{1}{2} \int dt g_{IJ} \frac{dX^I}{dt} \frac{dX^J}{dt}. \tag{1}$$

(a) Describe the space W of critical points of L. (You should find that it is closely related to the space of geodesics on M.)

(b) Show that given any choice of a point $t_0 \in \mathbb{R}$, W has a natural identification with T^*M. (Use the metric on M to identify TM and T^*M.)

(c) Compute the symplectic structure ω on W. Show that for any choice of t_0, W becomes identified (under the identification in (b)) with T^*M with its natural symplectic structure.

(d) The vector field d/dt on $\mathbb{R}$ induces a vector field v on W. The "Hamiltonian" is a function H on W such that

$$dH = i_v \omega. \tag{2}$$

This is usually expressed by saying that "H generates time translations via Poisson brackets." Compute H.

Problem FP2. Consider $\mathbb{R}^{1|1}$ with even and odd coordinates t and θ and the odd vector field

$$D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}. \tag{3}$$

Thus, $D^2 = -\partial/\partial t$. As in the last exercise, let M be a fixed Riemannian manifold. Let X be a map from $\mathbb{R}^{1|1}$ to M. Consider the Lagrangian

$$L = \frac{1}{2} \int_{\mathbb{R}^{1|1}} g_{IJ} \frac{\partial X^I}{\partial t} DX^J. \tag{4}$$

(a) Make sense of the definition of L by interpreting the integrand as a section of the Berezinian of the tangent bundle of $\mathbb{R}^{1|1}$.

(b) Setting $X^I = x^I + \theta\psi^I$, write L explicitly as an ordinary integral, over the t-line $\mathbb{R}^{1|0}$, of a function of x^I and ψ^I.

(c) Describe the space Y of critical points of L. You should get a description of the following kind. Describe the reduced space $Y_{\rm red}$ as an ordinary manifold, and the normal bundle to $Y_{\rm red}$ in Y as ΠV, where V is a vector bundle over $Y_{\rm red}$. What are the "initial data" at a given point $t = t_0$ in $\mathbb{R}^{1|1}$ needed to determine a critical point? (This is the analog of problem 1(b) above.)

(d) Compute the symplectic structure on Y.

(e) The vector fields

$$\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial t} \tag{5}$$

induce vector fields on Y. Find the functions Q and H that generate these vector fields by Poisson brackets. Explain why $\{Q, Q\} = 2H$.

Problem FP3. Consider the free electromagnetic field on $\mathbb{R}^n$ (with a metric of signature $(+ - - \cdots -)$). Consider the free electromagnetic field – a connection A with curvature (or field strength $F = dA$) and Lagrangian

$$I = -\frac{1}{4e^2}\int_{\mathbb{R}^n} F \wedge *F \tag{6}$$

regarded as a functional of A. (e is a constant, related to the charge of the electron; it has been written here in its usual place but in this classical exercise you can set it to 1 by rescaling A.)

(a) Derive the classical field equations and verify that they are second order differential equations for A.

(b) Compute the closed two-form ω_0 on the space M_0 of solutions of the classical field equations.

(c) Let G be the group of gauge transformations, which act by $A \to A + d\phi$, where ϕ is a function on $\mathbb{R}^n$. Let $M = M_0/G$ be the space of solutions of the equations modulo gauge transformations. Show that ω_0 is the pullback to M_0 of a closed two-form ω on M, and show that ω is non-degenerate.

Problem FP4. In this problem, we will consider the free electromagnetic field in three dimensions, say on $\mathbb{R}^3$ with flat metric of signature $(+ - -)$. The interest in three dimensions is that it is possible to add an extra term to the Lagrangian, which becomes

$$I = -\frac{1}{4e^2}\int F \wedge *F + \mu\int A \wedge F. \tag{7}$$

(If you prefer, set e to 1 and μ to ± 1 by rescaling of A and dilation of $\mathbb{R}^3$.) Note that the second term is invariant under $A \to A + d\phi$, where ϕ has compact support.

(a) Write the equations of motion in Fourier space and show that the space of solutions mod gauge transformations is supported not on a cone $p^2 = 0$ but on a certain hyperboloid H of the form $p^2 =$ constant.

(b) The space of solutions of the equations mod gauge transformations is a line bundle over H to which the action of $SO(2,1)$ lifts. Describe this line bundle. What happens if you change the sign of μ?

Problem FP5. In the next few exercises, we develop facts and supersymmetric models that are important in string theory.

(a) Consider in two dimensions $\mathbb{R}^2$ with coordinates t, x and metric $dt^2 - dx^2$ the Lagrangian of a free massless boson ϕ

$$L = -\frac{1}{2}\int_{\mathbb{R}^2} d\phi \wedge *d\phi. \tag{8}$$

Show that the general solution of the equations of motion is of the form $\phi(t,x) = \phi_R(t-x)+\phi_L(t+x)$, where the decomposition is unique up to adding constants to ϕ_R and ϕ_L. We call them right- and left-moving waves. Show that the decomposition into right and left- moving waves is compatible with the symplectic structure. (This means that the symplectic structure paired with one left-moving variation of ϕ and one right-moving one gives zero.)

(b) Consider the Dirac equation in two dimensions

$$\Gamma \cdot \partial\psi = 0 \tag{9}$$

where ψ is a section of the spin bundle S of $\mathbb{R}^2$. Show that, under the decomposition $S = S_+ \oplus S_-$, ψ decomposes as a sum of a right-moving wave with values in S_+ and a left-moving wave with values in S_-.

Problem FP6. Now we get closer to physics. First we construct a supersymmetric model in two dimensions in which all odd variables are right-moving. We start with a $(2|1)$ supermanifold W with a one-dimensional even distribution B_- generated by a vector field ∂_- and a one-dimensional odd distribution generated by a vector field D; we require that TW is generated by ∂_-, D, and $\partial_+ = -D^2$. A "flat" model is $\mathbb{R}^{2|1}$ with even coordinates u, v, odd coordinate θ, and

$$\begin{aligned} \partial_- &= \frac{\partial}{\partial u} \\ D &= \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial v} \\ \partial_+ &= \frac{\partial}{\partial v}. \end{aligned} \tag{10}$$

Notice that in this problem (unlike some previous exercises) one is given only the distributions, and not specific vector fields generating them.

Let M be a Riemannian manifold with metric g and $X : W \to M$ a map. The Lagrangian we want is

$$L_X = \int_W g_{IJ}\partial_- X^I D X^J. \tag{11}$$

(a) Show that the Lagrangian is well-defined (and invariant under all automorphisms of W that preserve the distributions) by interpreting the integrand as a section of the Berezinian.

(b) By performing the theta integral, reduce this to an ordinary Lagrangian. Show that (i) the reduction of the Lagrangian (throwing away odd variables) is the Lagrangian for a harmonic map to M, (ii) the equations of motion are such that the odd variables are right-moving, that is functions of v and not u.

(c) Thus the phase space (=space of classical solutions) Y is a bundle over $Y_{\rm red}$, where $Y_{\rm red}$ is the space of harmonic maps and the normal bundle to $Y_{\rm red}$ in Y is ΠV, where V is some vector bundle over $Y_{\rm red}$. From (b) you should know that $Y_{\rm red}$ is the cotangent bundle of $Hom(\mathbb{R}, M)$, where we identify a classical solution with the initial conditions on the spacelike hypersurface $u = -v$. Show that V is the tangent bundle of $Hom(\mathbb{R}, M)$. Thus, a quantization of this system gives (formally) a spin bundle of $Hom(\mathbb{R}, M)$, together with a Dirac operator. To be precise, the Dirac operator is the one that arises upon quantization of the Hamiltonian function associated with the motion on the phase space that comes from the vector field on W

$$\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial v}. \tag{12}$$

(d) Consider the flat model with $W = \mathbb{R}^{2|1}$ as above. Let $u = t + x$, $v = t - x$, Restrict the Lagrangian to maps X that are invariant under translations in x. The quotient of $\mathbb{R}^{2|1}$ by such translations is $\mathbb{R}^{1|1}$. Show that for these translation-invariant maps, L reduces to

$$L' = \int dt\, d\theta g_{IJ}\frac{dX^I}{dt}DX^J. \tag{13}$$

Recall from a previous problem set that quantization of L' gives the spin bundle and Dirac operator of M. Our previous result is thus essentially a consequence of the present more general one upon specializing to the translation-invariant maps.

Problem FP7. Now we will consider the Dirac operator on $Hom(\mathbb{R}, M)$ with values in a certain kind of vector bundle. Let Q be a finite dimensional rank n real vector bundle over M with structure group $SO(n)$, and a choice of metric and connection. Let $B_-^{1/2}$ be a square root of the even distribution B_- of the previous exercise. Let Λ be an odd section of $B_-^{-1/2} \otimes X^*(Q)$. The Lagrangian for Λ is

$$L_\Lambda = \int_W (\Lambda, D\Lambda). \tag{14}$$

For the coupled system of the map X and Λ, one simply adds the two Lagrangians to get $L = L_X + L_\Lambda$.

(a) Show that this is well-defined (and invariant under all of the diffeomorphisms of W that preserve the distributions).

(b) Writing $\Lambda = \lambda + \theta G$, perform the θ integral, to show that G is a non-propagating "auxiliary field" (whose field equations have a unique solution in terms of the other variables) while the odd variables λ are "left-moving." Keeping X fixed, show that the space of critical points of L_Λ is a vector bundle $P = \Gamma(\mathbb{R}, X^*(Q))$ over $Hom(\mathbb{R}, M)$. Thus, quantization of this system will give a spin bundle derived from P, and the operator that in the quantum theory is derived from (12) is a Dirac operator on $Hom(\mathbb{R}, M)$ with values in P.

(c) Show that in the special case that $Q = TM$ is the tangent bundle to M, P is the tangent bundle of $Hom(\mathbb{R}, M)$, so quantization of $L_X + L_\Lambda$ will give for the quantum Hilbert space a spin bundle S of $Hom(\mathbb{R}, M)$ with values in another spin bundle S'. (Unlike the finite-dimensional case, S and S' are not isomorphic as they are constructed using different polarizations: one set of odd variables is left-moving and one is right-moving.)

In finite dimensions, $S \otimes S'$ is isomorphic to the de Rham complex. (An explanation of that is in exercise 6(b) below.) Thus, this theory in the special case $V = TM$ is formally related to some sort of differential forms on $Hom(\mathbb{R}, M)$. One might suspect that this special case has additional symmetry, if one takes the metric on V to be the Riemannian metric. That is true and will be pursued later.

Problem FP8. This is really more an explanation than an exercise.

(a) Let M be an n-dimensional differentiable manifold (no metric). The rank $2n$ bundle $W = TM \oplus T^*M$ has a natural quadratic form (in which TM and T^*M are isotropic and paired with each other) which can be interpreted as a symplectic form on ΠW. Quantize ΠW to get the associated bundle $S(W)$ of spinors of W. Identify it with the de Rham complex $\Omega^*(M)$. Therefore, $TM \oplus T^*M$ must act on $\Omega^*(M)$. Show that a section of $TM \oplus T^*M$ of the form $v \oplus \alpha$ (with v and α a vector field and one-form respectively) acts on a differential form θ by $\theta \to i_v\theta + \alpha \wedge \theta$. Deduce the classical formula $(i_v\alpha + \alpha i_v) = \alpha(v)$.

(b) If a metric is introduced on M, then ΠTM (or equivalently ΠT^*M) can be quantized to get the spin bundle $S(M)$. (For present purposes ignore possible global obstructions to this.) Suppose for simplicity that M is even-dimensional. Explain from the present point of view the standard relation between the de Rham complex and the spin bundle: $\Omega^*(M) \cong S(M) \otimes S(M)$ (over $\mathbb{C}$).

(c) Consider a supermanifold N. To keep things simple, take $N = \mathbb{R}^{0|n}$, with odd coordinates θ_i, $i = 1, \dots, n$. $X = \Pi TN \oplus \Pi T^*N$ is a $2n$-dimensional vector space with a natural symplectic form, in which ΠTN and ΠT^*N are isotropic and paired with each other. As in (a) above, one can try to quantize X to get "the de Rham complex of N." The difference from (a) is that such a quantization will be infinite dimensional. Moreover, the only natural ($Sp(2n, \mathbb{R})$-invariant) quantization of such a symplectic vector space gives a Hilbert space of certain L^2 functions.

However, an algebraic quantization can be obtained if one picks a polarization, that is, a maximal isotropic subspace Y of X, and asks that there should be a vector annihilated by Y. The quantization of X – that is, the "differential forms on N" – is then the space of polynomials on X/Y. So the Clifford algebra of X (and in particular the subalgebra $\Lambda^*\Pi T^*N$) acts on $S^*(X/Y)$ for any Y. (But there is no way to identify $S^*(X/Y)$ for different Y.)

Show that the case $Y = \Pi TN$ corresponds to what Bernstein described as the "differential forms of finite order" on N while the case $Y = \Pi T^*N$ corresponds to the "differential forms of finite codimension."

Problem FP9. Here we will study the theory related to "differential forms on the loop space of a manifold."

We let $W^{2|2}$ be a supermanifold of the indicated dimension with one-dimensional odd distributions Λ_+, Λ_- generated by vector fields D_+, D_- such that D_+, D_-, D_+^2, D_-^2 generate the tangent bundle of $W^{2|2}$. The flat model is $\mathbb{R}^{2|2}$ with even coordinates u, v, and odd coordinates θ_-, θ_+, with

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta_+} - i\theta_+\frac{\partial}{\partial v} \\ D_- &= \frac{\partial}{\partial\theta_-} - i\theta_-\frac{\partial}{\partial u}. \end{aligned} \tag{15}$$

$\mathbb{R}^{2|2}$ has an infinite dimensional supergroup of automorphisms, roughly $G \times G$, where G is the symmetry group of the supermanifold $\mathbb{R}^{1|1}$ (with even and odd coordinates t, θ and distribution generated by $\partial_\theta + i\theta\partial_t$). For instance, generators are

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta_+} + i\theta_+\frac{\partial}{\partial v} \\ Q_- &= \frac{\partial}{\partial\theta_-} + i\theta_-\frac{\partial}{\partial u}. \end{aligned} \tag{16}$$

Let M be a fixed Riemannian manifold with metric g. Let $\Phi : W^{2|2} \to M$ be a map, and let

$$L = \int_{W^{2|2}} g_{IJ} D_+\Phi^I D_-\Phi^J. \tag{17}$$

(a) By now it should be routine to interpret the integrand as a section of the sheaf of volume forms and thereby show that the integral makes sense and that L is invariant under all of the symmetries of $W^{2|2}$.

Show that this has the following generalization: if B is a two-form on M, we can add another term

$$L' = \int_{W^{2|2}} B_{IJ} D_+\Phi^I D_-\Phi^J \tag{18}$$

which still preserves the symmetries of $W^{2|2}$. (Of course we can write a lot of other terms, some of which will be discussed later, if we pick a volume form on $W^{2|2}$.) Adding L' will give a theory that involves not ordinary differential forms on the path or loop space of M, but differential forms with values in a certain line bundle. Show that (18) is unchanged (up to a total derivative) if one replaces B by $B' = B + d\Lambda$.

(b) To understand more concretely what this theory describes, compute in the flat $\mathbb{R}^{2|2}$ and set

$$\begin{aligned} X &= \Phi|_{\theta_\pm=0} \\ \psi_+ &= D_+\Phi|_{\theta_\pm=0} \\ \psi_- &= D_-\Phi|_{\theta_\pm=0} \\ F &= -iD_+D_-\Phi|_{\theta_\pm=0}. \end{aligned} \tag{19}$$

(So X is a map $X : W_{\text{red}} \to M$, while $\psi_\pm$ and F take values in the obvious line bundles on W_{red} tensored with $X^*(TM)$, and the definition of F uses the connection on $X^*(TM)$ as well as the explicit $D_\pm$.)

Calculate $L = \int du\, dv\, d\theta_+\, d\theta_- S$ (with S as in (17) above – omit the generalization using the two-form) by writing it as $L = \int du\, dv\, D_+D_-S|_{\theta_\pm=0}$. Don't be too careless in differentiating the metric – note the appearance of the Riemann tensor!

You should see that ψ_+ are right-moving fermions, with values in $X^*(TM)$, and ψ_- are left-movers, again with values in $X^*(TM)$. So as in **FP6** and **FP7**, quantization of ψ_+ and ψ_- will each give spin bundles on the loop or path space $\mathcal{L}M$ of M. Call these spin bundles $S_L(\mathcal{L}M)$ and $S_R(\mathcal{L}M)$. S_L and S_R are not isomorphic, unlike the finite-dimensional case, because the positive energy condition gives opposite polarizations for left and right movers.

Nonetheless, the tensor product $S_L \otimes S_R$ would in finite dimensions be isomorphic to the de Rham complex $\Omega^*(\mathcal{L}M)$, so this theory gives a version of Hodge - de Rham theory of $\mathcal{L}M$.

Show that this theory is $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded, where one $\mathbb{Z}_2$ acts by $\theta_+ \to -\theta_+$ and the other by $\theta_- \to -\theta_-$. The two $\mathbb{Z}_2$'s would correspond for a finite-dimensional manifold to the Hodge $*$ and the operator that acts on p-forms by $(-1)^p$. The $\mathbb{Z}$-valued grading of differential forms is lost in this (good!) version of Hodge theory of loop space.

(c) Let $u = t - x$, $v = t + x$, and consider only Φ's that are invariant under translations of x. The theory thus reduces to a theory on $\mathbb{R}^{1|2}$ with coordinates t, θ_+, θ_-, and

$$L = \int dt d\theta_+ d\theta_- g_{IJ} D_+\Phi^I D_-\Phi^J. \tag{20}$$

Q_+ and Q_- in this theory correspond to $d + *d*$ and $i(d - *d*)$ acting on differential forms.

Note that in reducing from $\mathbb{R}^{2|2}$ to $\mathbb{R}^{1|2}$ we have lost the infinite-dimensional symmetry algebra of $\mathbb{R}^{2|2}$. Show however, that one new symmetry appeared in the reduction – an $SO(2)$ that rotates θ_+, θ_-. Show that this corresponds in the quantum theory to the $\mathbb{Z}$-grading of differential forms. (A $\mathbb{Z}_2$ -subgroup of $SO(2)$, which acts by sign change of both θ_+ and θ_-, is present in the two-dimensional version, as seen in (b).)

(d) Now we want to compare this theory to a theory on $\mathbb{R}^{2|1}$ which is obtained from $\mathbb{R}^{2|2}$ by setting $\theta_- = 0$.

We set $\Phi' = \Phi|_{\theta_-=0}$, and $\Lambda_- = D_-\Phi|_{\theta_-=0}$. Write $\int du\, dv\, d\theta_+\, d\theta_-\, S = \int du\, dv\, d\theta_+\, D_-S|_{\theta_-=0}$ to convert the Lagrangian on $\mathbb{R}^{2|2}$ to an equivalent Lagrangian on $\mathbb{R}^{2|1}$.

Compare with the theory of "spinors on $\mathcal{L}M$ with values in a vector bundle" that was in **FP7**, and show that – as befits "differential forms on $\mathcal{L}M$" – the theory discussed here is a special case of "spinors on $\mathcal{L}M$ with values in a vector bundle."

(e) So far we have seen that the theories of Dirac and Hodge-de Rham operators on a manifold M as customarily studied are the one-dimensional versions of two-dimensional theories. (The physics that goes with this is of course string theory.) For Dirac, that is as far as we can go – the chain that leads in this way to the Dirac operator on a manifold begins in two dimensions. But I want to point out that the Hodge-de Rham theory can come by reduction of a three-dimensional theory (generally less important in physics, so far at least, as it is less well-behaved quantum mechanically).

Here we start with a supermanifold $W^{3|2}$ of the indicated dimension, with a two-dimensional odd distribution A which is generic in the sense that A and $\{A, A\}$ generate the tangent bundle of $W^{3|2}$. A sharp difference from the two-dimensional case is that we need a volume form ϵ on A. Letting $\Phi : W^{3|2} \to M$ be a map as before to a fixed Riemannian manifold M, and D_1, D_2 a basis of A such that $\epsilon(D_1, D_2) = 1$, the Lagrangian is

$$L = \int g_{IJ} D_1\Phi^I \, D_2\Phi^J. \tag{21}$$

Show that this is well-defined, and invariant under all symmetries that preserve the distribution and volume form.

The flat model is $\mathbb{R}^{3|2}$ with odd coordinates θ^A, $A = 1, 2$ and even coordinates y^{AB} (with $y^{AB} = y^{BA}$ so there are three y's) and

$$D_A = \frac{\partial}{\partial\theta^A} - i\theta^B \frac{\partial}{\partial y^{AB}}. \tag{22}$$

Given a real structure and volume form on the space of the θ^A's the group that can act linearly on the θ^A's is $SL(2, \mathbb{R})$. This is the double cover of $SO(2, 1)$, which acts on the y's, preserving a metric of signature $+ - -$. The model lacks the infinite-dimensional symmetry group of the two-dimensional case, but has the following finite-dimensional group of symmetries. Let

$$Q_A = \frac{\partial}{\partial\theta^A} + i\theta^B \frac{\partial}{\partial y^{AB}}. \tag{23}$$

Show that the Q_A preserve the distribution, and that together with the $\{Q_A, Q_B\}$ and the $SL(2, \mathbb{R})$, they generate three-dimensional super-Poincaré. Together with some orientation and time-orientation reversing symmetries, this is the full symmetry group of $\mathbb{R}^{3|2}$ with the volume form.

Show that, if we impose invariance under translation of one of the y's, the model reduces to the two-dimensional model introduced above.

If one imposes invariance under translations of *two* of the y's – in fact, pick coordinates in which the metric is $dt^2 - dx_1^2 - dx_2^2$ and impose invariance under translation of x_1, x_2 – then we get the theory on $\mathbb{R}^{1|2}$ that is related upon quantization to ordinary differential forms on M with the d and d^* operators. This theory therefore has an $SO(2)$ symmetry of rotation of the $x_1 - x_2$ plane – it is the $SO(2)$ noted above that rotates θ_+ and θ_- in the original presentation; the generator acts on p-forms by multiplication by p (or $p - \frac{1}{2}\dim M$).

Problem FP10. On differential forms on M there acts the $SO(2)$ just noted, commuting with the Laplacian. It comes by dimensional reduction from a $1+2=3$-dimensional theory. Differential forms on a generic M cannot come from reduction of a four-dimensional theory, as the $SO(2)$ - which is better called $Spin(2)$ - would then have to extend to a $Spin(3)$.

Likewise, the Dirac equation on a general M cannot come by reduction from a theory above two dimensions. If it came from a three-dimensional theory, we would need to find a $Spin(2)$ action on the spin bundle, commuting with the Laplacian and acting non-trivially on the Dirac operator. So $d=2$ is the "starting point" in this sense for the Dirac equation.

But suppose M is a Kähler manifold. Then the $Spin(2)$ action on differential forms *does* extend to a $Spin(3)=SU(2)$ action (sometimes called $SL(2)$, but $SU(2)$ is the group that acts on the *Hilbert space* of differential forms preserving the structure). d and d^* can be decomposed into four operators ∂, $\bar\partial$, ∂^*, $\bar\partial^*$. These are (complex) linear combinations of four self-adjoint operators that transform as the four-dimensional real spinor representation of $Spin(3)$. This suggests that differential forms on a Kähler manifold M can come from a theory in $1+3=4$ dimensions.

Likewise if M is hyper-Kähler then the $Spin(3)$ extends further to a $Spin(5)$ with d and d^* generating eight self-adjoint operators that transform as the eight-dimensional real spinor of $Spin(5)$. This suggests that differential forms on M can come from a theory in $1+5=6$ dimensions.

Recall that, according to Nahm's theorem, six is the maximum dimension for supersymmetric sigma models. The supersymmetric theory in six dimensions of maps to a hyper-Kähler manifold (plus odd variables) does exist – but like all the limiting cases allowed by Nahm's theorem (the others being gauge theory in ten dimensions and gravity in eleven dimensions) there is no known construction via supermanifolds.

For the four-dimensional case, however, with M a Kähler manifold, a superspace construction does exist. For the time being, we will only describe it locally. (It takes a few more steps to describe a more global version.)

We consider a $(4|4)$ supermanifold W with the structure described in Bernstein's lectures – two complex conjugate integrable distributions A_+ and A_- of dimension $(0|2)$ such that $\{A_+, A_-\}$ generates $TM/(A_+ \oplus A_-)$; A_+ and A_- are endowed with volume forms. A chiral superfield is a field ϕ that is a function on W/A_-. If M is a complex manifold, a chiral map $\Phi : W \to M$ is one such that, for f a local holomorphic function on M, $\Phi^*(f)$ is a chiral function on W. In that case, $\Phi^*(\bar f)$ is antichiral, that is, it is a function on W/A_+.

Let K be a function on W and consider the Lagrangian

$$L = \int_W d^4y d^4\theta\, \Phi^*(K). \tag{24}$$

$d^4y\, d^4\theta$ is the section of the Berezinian coming from the volume forms on $A_\pm$.

(a) Show that L is invariant under $K \to K + f + \bar f$, where f is a local holomorphic function on W.

(b) Recall that any K determines the two-form $\omega = -i\bar\partial\partial K$ which for a suitable class of K's is the Kähler class of a Kähler metric g on M. Of course, ω is invariant under the transformation considered in (a).

Show that L can be written just in terms of ω (or equivalently g) and that the "reduced" part of L is the harmonic map action, specialized to maps to a Kähler manifold:

$$L = \int_{W_{\rm red}} d^4y g_{i\bar{j}}(dX^i, dX^{\bar{j}}) \tag{25}$$

where X^i are local holomorphic coordinates on M.

(c) Specialize now to the flat model with $W = \mathbb{R}^{4|4}$. A_+ is generated by

$$D_A = \frac{\partial}{\partial\theta^A} - i\sigma^m_{A\dot{A}}\bar{\theta}^{\dot{A}}\frac{\partial}{\partial y^m} \tag{26}$$

where θ^A are odd coordinates transforming as "positive" spinors of $Spin(1,3) = SL(2,\mathbb{C})$, $\bar{\theta}^{\dot{A}}$ are their complex conjugates, and y^m are even coordinates. σ is the isomorphism $S_+ \otimes S_- = V$. A_- is generated by

$$D_{\dot{A}} = \frac{\partial}{\partial\theta^{\dot{A}}} - i\sigma^m_{A\dot{A}}\bar{\theta}^{A}\frac{\partial}{\partial y^m}. \tag{27}$$

Impose invariance under a space-like translation of W, say a translation of y^3 (the metric being $(dy^0)^2 - \sum_{i=1}^3 (dy^i)^2$ to get a model on $\mathbb{R}^{3|4}$. Show that this is equivalent to the $\mathbb{R}^{3|2}$ σ-model described in **FP**9(e) (but specialized to the case that M is Kähler). Here is an attempt to explain the strategy: under $Spin(1,2)$, the two spinor representations of $Spin(1,3)$ become isomorphic. So after the reduction, A_+ and A_- are naturally isomorphic. One can in an $SO(1,2)$-invariant way take the "real" combination $\chi^A = \theta^A + \bar{\theta}^A$ of the θ's and $\bar{\theta}$'s. The $\mathbb{R}^{3|2}$ we want has even coordinates y^i, $i = 0,1,2$, and odd coordinates χ^A. By "integrating over the fibers" of a map from $\mathbb{R}^{3|4}$ to $\mathbb{R}^{3|2}$, one reduces (24) (specialized to y^3-independent maps) to the $\mathbb{R}^{3|2}$ Lagrangian of **FP**9(e).

Problem FP11. Consider a simple harmonic oscillator in one dimension. The Lagrangian is

$$L = \int dt \left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}m\omega^2 x^2 \right). \tag{28}$$

The Hamiltonian is accordingly

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{29}$$

where

$$p = -i\frac{d}{dx} \tag{30}$$

acting on $L^2(\mathbb{R})$.

Calculate the kernel of the operator e^{-itH}, that is, the matrix element

$$\langle b|e^{-\imath Ht}|a\rangle, \tag{31}$$

using path integrals.

That is, find the classical path $x_0(t)$ that starts at a at time 0 and ends at b at time t. Write $x(t) = x_0(t) + \tilde{x}(t)$. By substituting $x(t)$ in L, write the action as a quadratic function of $\tilde{x}(t)$ plus a known function of a, b and t. Do the path integral over $\tilde{x}(t)$ by interpreting it as a determinant which you must study somehow.

In fact after dividing out formally by an infinite but ω-independent constant – or if you prefer, upon using something like a zeta function definition of the determinant – you should be able to reduce to an infinite product that will look something like

$$\prod_{n=1}^{\infty} \left(1 + \frac{p^2}{n^2}\right) \tag{32}$$

for some p. This product in fact defines a certain elementary function of p.

Problem FP12. Consider the scalar field theory with Lagrangian

$$L = \int d^n x \left(\frac{1}{2}|d\phi|^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3\right) \tag{33}$$

in n-dimensional Euclidean space.

Calculate the contribution of order g^3 to the three-point function $\langle\phi(x)\phi(y)\phi(z)\rangle$ (It is best to work in momentum space, that is, to set $z = 0$ and take the Fourier transform with respect to x and y.)

(a) Demonstrate that the result is finite in $n \leq 6$, and for $n = 6$, describe the precise form of the divergence – that is, introduce a cutoff Λ, and describe just what sort of Λ-dependence you get for large Λ.

Renormalize the result for $n = 6$. That is, carry out the necessary renormalization and write down the renormalized, finite correction to the three-point function.

(b) For $n < 6$ make the result as explicit as you can. Use Feynman's identity

$$\frac{1}{ABC} = \int_0^1 \int_0^1 \int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(\alpha + \beta + \gamma - 1) \frac{1}{(\alpha A + \beta B + \gamma C)^3}. \tag{34}$$

After completing the square, you should be able to reduce the integral over the loop momentum q to

$$\int\int\int d\alpha\, d\beta\, d\gamma\, \delta(\alpha + \beta + \gamma - 1) \int d^n q\, \frac{1}{(q^2 + b^2)^3} \tag{35}$$

for some b (which is a polynomial in the three external momenta and in α and β). Set $n = 4$ and carry out the q integral explicitly.

For extra credit, get as far as you reasonably can with the α and β integrals.

Problem FP13. Now consider a theory in n-dimensional Euclidean space of a scalar ϕ and a spinor ψ with Lagrangian

$$L = \int d^n x \left(\frac{1}{2}|d\phi|^2 + \frac{1}{2}M^2\phi^2 + \frac{i}{2}(\psi, i(D + m)\psi) + \frac{g}{2}\phi(\psi, \psi)\right). \tag{36}$$

(ψ takes values in a representation of $Spin(n)$ that is associated with a real representation of $Spin(1, n-1)$. The precise details depend a bit on n, but this should not affect the calculation you will have to do. (,) is a quadratic form on the spin representation. D is the Dirac operator.)

The free propagator of the ψ field is in momentum space $1/(\Gamma \cdot k + im)$, where $\Gamma \cdot k$ is the physicist's notation for Clifford multiplication by k.

The free propagator of ϕ is $1/(k^2 + M^2)$.

L is written so that there is only one interaction vertex, a cubic vertex with two ψ's and one ϕ. The factor to be included at such a vertex in a Feynman diagram is just $-g$ (times the identity matrix or natural bilinear form (,) on the fermion ψ, depending on how you look at it).

Compute the correction of order g^2 to the fermion two-point function $\langle \psi(x)\psi(0)\rangle$ (Again, it is best to take the Fourier transform. Also, it is best to write the two-point function as $1/(\Gamma \cdot k + im + i\Sigma)$ and think of computing a correction to Σ, which will be given by an "amputated one-loop diagram." Σ will be of the form $w(k^2)\Gamma \cdot k + iv(k^2)$ and the problem is to study both functions w and v.)

(a) Show that the result is finite for $n < 4$.

(b) Show that (for $n < 4$) the result is analytic in k^2 for $k^2 > -(m+M)^2$. (We are writing the formulas in Euclidean signature here; in Lorentz signature with metric $+ - - - \cdots$ this would read $k^2 < (m+M)^2$.) How does this compare to what you would have expected? In particular, your result is holomorphic near $k^2 = -m^2$, so that the correction changes the value of the fermion mass (by an amount that is small if g is small) but not the existence of a discrete state.

(c) For $n = 4$, what renormalizations are necessary to remove the infinity that you have found? Write a formula for the renormalized, finite, correction to the two-point function, that is, to Σ. For $n = 4$, does the renormalized two-point function have the same analytic behavior found in (b) for $n < 4$?

Problem FP14. Consider motion of a non-relativistic particle in one dimension with a delta function potential. The Hamiltonian is

$$H = \frac{p^2}{2m} + \lambda\delta(x) \tag{37}$$

with $p = -id/dx$ and m, λ being constants. Find a solution of the Schrodinger equation $H\psi = k^2\psi$ which is asymptotic for $x \to -\infty$ to $e^{ikx} + R(k)e^{-ikx}$ and for $x \to +\infty$ to $T(k)e^{ikx}$. R and T are called the reflection and transmission amplitudes; show that $|R|^2 + |T|^2 = 1$.

Does this problem have a bound state? That is, is there an L^2 solution of the equation $H\psi = -\epsilon\psi$ with $\epsilon > 0$? (The answer may depend on λ!)

Problem FP15. The following is a continuation of **FP**11. We considered the simple harmonic oscillator in one dimension with Lagrangian

$$L = \int dt\ \left(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2\right) \tag{38}$$

and Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{39}$$

and we computed the kernel $K(b,a;t) = (b, e^{-tH}a)$ by a path integral

$$K(b,a;t) = \int_{X(0)=a}^{X(t)=b} DX(t')\; e^{-L} \tag{40}$$

over maps X of $I = [0,t]$ to $\mathbb{R}$ with $X(0) = a$ and $X(t) = b$. (In this exercise we will be computing e^{-tH} with real t so I is "Euclidean" and we use e^{-L} rather than e^{iL} in the path integral.)

Now we'd like to compute $\operatorname{Tr} e^{-tH} = \int da K(a,a;t)$. To do so, we simply set $b = a$ in (40) and integrate over a. But a map $X : I \to \mathbb{R}$ that has the same value a at the endpoints is the same as a map $X : S \to \mathbb{R}$, where S is the circle (of circumference t) obtained by gluing together the ends of I. Moreover, in the formula (40) for K we are already, once we set $b = a$, integrating over all maps $S \to \mathbb{R}$ with value a at the former endpoint; once we also integrate over a to get $\operatorname{Tr} e^{-Ht}$, we are simply integrating over all maps $S \to \mathbb{R}$.

So the upshot is a formula

$$\operatorname{Tr} e^{-Ht} = \int dX\, e^{-L} \tag{41}$$

where the integral is over all real-valued functions on S and L is the Lagrangian evaluated for such an X.

(a) As L is a homogeneous quadratic function on the linear space $Maps(S,\mathbb{R})$, the integral can be evaluated formally as $1/\sqrt{\det D}$ where D is a differential operator on S. Calculate the determinant (using the same sort of methods you needed in last week's problem) and compare to a standard formula for $\operatorname{Tr} e^{-Ht}$.

(b) Now let J be the operation $X \to -X$. We want to compute by path integrals $\operatorname{Tr} Je^{-Ht}$. This is $\int da K(-a,a;t)$ so it can be computed exactly as before except that when we glue the endpoints of I to make the circle S, we have to require that the boundary values obey $b = -a$. This means that X is not a real-valued function on S, but a section of an unorientable real line bundle $\mathcal{L}$ over S. With $\mathcal{L}$ understood to have a metric (its structure group in particular is $\mathbb{Z}_2$ and not $\mathbb{R}^*$) the Lagrangian L makes sense for X a section of $\mathcal{L}$ and we have

$$\operatorname{Tr} J\, e^{-Ht} = \int' DX\, e^{-L} \tag{42}$$

where now the integral runs over the space of sections of $\mathcal{L}$. As this space is again a linear space on which L is a quadratic function, the path integral is again a Gaussian. Evaluate the determinant and compare to another evaluation of $\operatorname{Tr} J\, e^{-Ht}$.

Problem FP16. Now we will do an analogous computation for fermions, to demonstrate by path integrals that quantization of the Lagrangian

$$L = \int_M dt\, i \sum_{j=1}^{n} \psi_j \frac{d\psi_j}{dt} \tag{43}$$

leads to the spin representation. Here M is an oriented one-manifold and the ψ_j are n real fermions. Actually, to be precise, we will take n to be even; there are then two spin representations S_+ and S_-, and we will see them both appear.

First of all, in (43) we might take $M = \mathbb{R}$ in which case we quantize and get a Hilbert space $\mathcal{H}$. On this Hilbert space there would be in general a Hamiltonian, but in the case at hand—since (43) is invariant under reparametrizations of t—the Hamiltonian is $H = 0$.

Rather than trying to determine by canonical quantization what $\mathcal{H}$ would be, here we will study the situation with path integrals. For this we take M to be S^1, as in the last exercise. To find the dimension of $\mathcal{H}$, we want $\mathrm{Tr}_{\mathcal{H}} e^{-tH}$ which (as $H = 0$) is $\mathrm{Tr}_{\mathcal{H}} 1 = \dim \mathcal{H}$. By naively imitating the last exercise, you would think we would have

$$\mathrm{Tr}_{\mathcal{H}} 1 \stackrel{??}{=} \int D\psi(t) e^{-L} \tag{44}$$

which in turn could be computed as a Pfaffian, rather as for the bosonic harmonic oscillator.

Once we are on a circle, though, we note that ψ is best understood as a section of the spin bundle of the circle. There are two such spin bundles, the trivial and non-trivial $\mathbb{Z}_2$ bundles over S^1 (tensored with $\mathbb{R}$), so we have two candidate path integrals. One spin structure can be obtained from the other by twisting by an element of $H^1(S^1, \mathbb{Z}_2)$. What that means is that if we are building S^1 by gluing together the ends of an interval I, then (given a recipe for the gluing to make one spin structure) the second can be made by multiplying all fermions by -1 prior to the gluing.

(a) If the path integral with one of the two spin structures computes $\mathrm{Tr}_{\mathcal{H}} 1$, the path integral with the other spin structure computes $\mathrm{Tr}_{\mathcal{H}} \Gamma$, where Γ is an operator in the quantum theory such that $\Gamma \psi_j = -\psi_j \Gamma$ for all j. Using your knowledge of canonical quantization and the spin representation, what is Γ? And by considering whether the Pfaffian is zero or not, can you see which spin structure must go with $\mathrm{Tr}_{\mathcal{H}} 1$ and which computes $\mathrm{Tr}_{\mathcal{H}} \Gamma$?

(b) But our main interest here is in the following generalization. In gluing the ends of the interval I to make S^1, we could "twist" by an arbitrary element g of the symmetry group $SO(n)$. Depending on what spin structure we use, the path integral will compute $\mathrm{Tr}_{\mathcal{H}} g$ or $\mathrm{Tr}_{\mathcal{H}} g\Gamma$.

By introducing such a g, one builds an $SO(n)$ bundle over S^1. ψ is now a section of $\epsilon \otimes F_g$, where ϵ is a spin bundle of S^1 and F_g is now understood to be the rank n flat vector bundle with structure group $SO(n)$ made by twisting by g. The Lagrangian makes sense for such ψ's and should compute $\mathrm{Tr}_{\mathcal{H}} g$ or $\mathrm{Tr}_{\mathcal{H}} g\Gamma$ as the case may be.

The problem is now simply to do the computation and get the appropriate formulas. First of all, the fermion path integral gives the Pfaffian of the one-dimensional Dirac operator D/Dt (which acts on sections of $\epsilon \otimes F_g$), Mathematically, this Pfaffian is not a number but a section of a determinant line (better called a Pfaffian line). The relevant Pfaffian line bundle is real, but is in fact a non-trivial line bundle with structure group $\mathbb{Z}_2$. In other words, if one considers the family of flat bundles F_g parametrized by $g \in SO(n)$, we get a family of Dirac operators parametrized by the group $SO(n)$, and the Pfaffian line bundle is a non-trivial real line bundle over $SO(n)$.

If one takes the double cover of $SO(n)$, replacing it by $Spin(n)$, then the Pfaffian line becomes trivial and can be trivialized. Once a trivialization is picked,

the Pfaffian of the Dirac operator becomes a function on $Spin(n)$. Of course, we get two such functions, say f_ϵ, depending on the choice of spin bundle ϵ.

Compute the f_ϵ's by actually writing down the eigenvalues of the Dirac operator and taking their product (with a regularization analogous to what you needed in the bosonic case). The functions f_ϵ are obviously invariant under conjugation in $Spin(n)$ and so are linear combinations of characters of that group. But in fact, using your knowledge of spinors, describe precisely what characters here appear and compare to your identification above of the operator Γ.

Problem FP17. In free scalar field theory in four dimensions, with mass m, calculate the operator product expansion $\phi^2(x)\phi^2(x')|_{x\to x'}$ up to and including all operators of dimension four or less. (This means, relative to what we did in class, working out contributions of some operators with derivatives. This should not be a long exercise.)

Problem FP18. In this exercise we will compute the one-loop renormalization of $U(1)$ gauge theory with fermions in four dimensions. It is of considerable theoretical importance, and also is part of the computation of the "Lamb shift" which was one of the historic tests of QED around 1950.

The Lagrangian is

$$L = \frac{1}{4e^2}\int d^4x F_{ij}F^{ij} + \int d^4x(\psi,(D+im)\psi). \tag{45}$$

Here the gauge field is $A = A_i dx^i$—a connection on a line bundle $\mathcal{L}$—the curvature is $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$, and ψ is a charged spinor field (section of $(S_+ \oplus S_-)\otimes\mathcal{L}$).

As explained in [I-**Faddeev**], to quantize the theory one must "pick a gauge." One can pick a gauge in which the kinetic energy for $B = A/e$ is just

$$L' = \frac{1}{2}\int d^4x \sum_{i,j}(\partial_i B_j)^2. \tag{46}$$

This means that the free propagator of B is

$$\langle B_i(x)B_j(0)\rangle = \delta_{ij}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}. \tag{47}$$

The inverse of the free propagator is therefore, in momentum space, $D^0_{ij} = \delta_{ij}k^2$. The inverse of the exact propagator of B is $D_{ij} = D^0_{ij} + \Sigma_{ij}$, where Σ_{ij}, the "self-energy," is to be computed from loops.

A very fundamental property of Σ_{ij} is that in momentum space $k^i\Sigma_{ij}(k) = 0$. This implies that

$$\Sigma_{ij}(k) = (k^2\delta_{ij} - k_i k_j)F(k^2), \tag{48}$$

with a scalar function F.

The problem here is to compute F in the one-loop approximation. (More exactly, you will compute Σ_{ij}, find it has the form (48), and compute F, after some renormalization.) The importance is twofold. (1) The sign of F—which can

be predicted without computation, but in this exercise we will just compute it—is the basic quantity that determines the different behavior of abelian and nonabelian quantum gauge theories in four dimensions (this is a slight simplification, as the definition of the analog of F is a bit more elaborate in the nonabelian case). (2) The renormalized F is observable and, as I mentioned above, its calculation is part of the computation of the Lamb shift.

To compute F in one-loop approximation, just write down the one-fermion-loop diagram. The fermion propagator is in momentum space $1/(\Gamma \cdot k + im)$ where $\Gamma \cdot k$ is Clifford multiplication by k. In the one-loop diagram for Σ_{ij}, there are two vertices on the fermion loop; at these vertices one multiplies by $e\Gamma_i$ or $e\Gamma_j$ respectively (the e comes from using $B = A/e$). The loop gives a supertrace over the spin representation of $SO(4)$; don't forget that it is four-dimensional as both S_+ and S_- are included here! (To avoid confusion, in this context supertrace just means -1 times the ordinary trace, since the objects propagating in the loop are all fermions. No distinction is made between S_+ and S_-. We explained in a previous homework session where the -1 comes from.) You will have to rationalize the fermion propagator, writing $1/(\Gamma \cdot k + im) = (\Gamma \cdot k - im)/(k^2 + m^2)$, introduce Feynman propagators to combine the denominators, perform the k integral, and at that point you will know that you are on the right track when you demonstrate that Σ is of the form claimed in (48). Before you demonstrate (48) it will appear that you are dealing with a possibly quadratically divergent integral, but once you've confirmed (48) it should be clear that there is only a logarithmic divergence which you'll renormalize in the by now familiar fashion (just subtracting the value of the integral at any given value of k, such as $k = 0$).

Problem FP19. **(a)** Show that in ϕ^4 theory the anomalous dimension of ϕ vanishes in the one-loop approximation and that the one-loop beta function is therefore given by a one-loop diagram that we have computed before. The one-loop beta function is in particular $\beta(\lambda) = A\lambda^2$ where λ is the ϕ^4 coupling, and A is the coefficient of the one-loop divergence.

(b) Convince yourself that the sign of the logarithmic divergence in this one-loop diagram (whose significance we by now appreciate) can be determined without any computation; it is determined by the fact that in *free* scalar field theory the two-point function $\langle \phi^2(x)\phi^2(0)\rangle$ is positive. (This is actually a point that has been made previously.)

Problem FP20. Consider a four-dimensional scalar field theory with n scalar fields ϕ_i and a Lagrangian

$$L = \int d^4x \left(\frac{1}{2} \sum_{i,\alpha} \left(\frac{\partial \phi_i}{\partial x^\alpha} \right)^2 + \sum_{ijkl} \lambda_{ijkl} \phi^i \phi^j \phi^k \phi^l + \dots \right), \tag{49}$$

where the ... are possible quadratic and cubic terms in ϕ_j that are irrelevant for the present purposes.

The renormalization group operator will read

$$D = \mu \frac{\partial}{\partial \mu} + \sum_{ijkl} \beta_{ijkl} \frac{\partial}{\partial \lambda_{ijkl}} + \dots \tag{50}$$

(the ... are anomalous dimension terms) with some functions β_{ijkl}.

(a) Write down the one-loop approximation to β_{ijkl}. You are not expected to do any real calculation here; write down the β_{ijkl} in terms of the λ_{ijkl} and the coefficient called A in **FP**19.

(b) Show that in this approximation the renormalization group flow

$$\frac{d\lambda_{ijkl}}{dt} = \beta_{ijkl} \tag{51}$$

is a gradient flow (with respect to the obvious metric on the space of couplings).

As Gross mentioned, proving that this is true in general (or at least that there are no closed orbits) is a longstanding open problem. The special case of this problem for two-dimensional sigma models is actually quite significant in string theory. (More precisely, what one wants there is that if g is the metric on a sigma model target space, then the renormalization group flow for g should be, after possibly redefining g in a local fashion by a substitution such as $g_{ij} \to g_{ij} + R_{ij} + \dots$, with R_{ij} the Ricci tensor and ... analogous terms of higher degree, of the form

$$\frac{dg}{dt} = \frac{\delta}{\delta g}\Gamma \tag{52}$$

with Γ the integral over the target of a polynomial in the Riemann tensor and its covariant derivatives.)

At the end of one of Gawedzki's lectures, when I asserted that locally there is no obstruction to perturbing a Ricci-flat metric to give a zero of β, I omitted to mention that this is so if and only if it is true in sigma models that the renormalization group flow is a gradient flow in the above sense. Otherwise, there is an obstruction coming from the Bianchi identity.

Problem FP21. Consider in four dimensions a Yukawa theory with a spin one-half field ψ and a scalar ϕ. The part of the Lagrangian involving ψ is

$$L = \int d^4x \left(\bar{\psi}(i\Gamma \cdot \partial - \phi)\psi\right). \tag{53}$$

Compute the anomalous dimension of the operator $\bar{\psi}\psi$ in the one-loop approximation.

Problem FP22. The goal here is to explain the claim that the magnetic coupling is responsible for the negative beta function of nonabelian gauge theories.

We consider scalar, spinor, and vector fields in some representation R of a compact gauge group G, on a four-manifold M. The Laplacian for a scalar field, a spinor field, or a section of the tangent bundle will be called ∇_0, $\nabla_{1/2}$, or ∇_1, respectively. The operator governing small fluctuations of the scalar, spinor, or

vector field will be called X_0, $X_{1/2}$, or X_1. In each case $X_j = \nabla_j + G_j$ where G is a term of order zero (absent for $j = 0$) which Gross called the magnetic moment coupling. In the case of spin one, the vector fields are really gauge fields, and the operator X_1 is accompanied by some operators X_0 for ghost fields that are obtained upon quantization. (This was explained in [I-**Faddeev**].)

The one-loop beta function comes from the logarithmic divergence in the determinant of X_j. As we are interested in terms that depend on the gauge field and not the metric of M, we may as well take M to be flat. In this case, $\nabla_{1/2}$ is just four copies of ∇_0, and ∇_1 is likewise just four copies of ∇_0 (as the spinor and tangent bundles of M are trivial rank four bundles with flat Levi-Civita connection—at least, let's agree to consider the full rank four spin bundle).

Here is how to determine the X_j for $j = 1/2, 1$.

(a) For $j = 1/2$, X_j is the square of the Dirac operator D. (The log of the determinant of D is conveniently studied as one-half the log of the determinant of $X_{1/2} = D^2$.) A standard calculation shows that

$$X_{1/2} = D^2 = \nabla_{1/2} + \Gamma \cdot F \tag{54}$$

where F is the curvature of the gauge connection and $\Gamma \cdot F$ is Clifford multiplication. If the second "magnetic moment" term were absent, the log determinant of $X_{1/2}$ would be four times that of X_0 but, because one want a minus sign for the fermion loop, the contribution of the fermions to the beta function would be negative. The $\Gamma \cdot F$ term makes a contribution of the opposite sign, which is larger in magnitude, so fermions actually make a positive contribution.

(b) Now consider the gauge fields.

Maxwell's equations read

$$d * dA = 0. \tag{55}$$

You might therefore believe that if we have a nonabelian gauge connection B and we write $A = B + a$, with a a small fluctuation, and we write d_B for the covariant derivative with the connection B, the linearization of the Yang-Mills equations would be

$$d_B * d_B a = 0. \tag{56}$$

Not only is that not so, but the equations (56) are inconsistent, in the sense that they imply the first order constraint $d_B d_B (* d_B a) = 0$ which is trivial when $B = 0$ but generically non-trivial; thus the space of solutions is not "flat" (if I may risk the expression) in a deformation away from $B = 0$. That certainly corresponds to a pathology. By linearizing the Yang-Mills equations you get not (56) but

$$(d_B * d_B)a + (* F_B) \wedge a = 0. \tag{57}$$

(Verify this!) Equations (57) have a "flat" behavior (the constraint gotten by applying d_B to the equations is trivial) if B obeys the Yang-Mills equations. (Verify this!) More fundamentally, (57) has the gauge invariance $a \to a + d_B \epsilon$ (for any $\epsilon \in \Omega^0(M, \mathrm{ad}\, G)$) if B obeys the Yang-Mills equations. (Verify this too!)

In gauge fixing the theory one would have introduced a gauge fixing term, as Fadde'ev explained. This can conveniently be taken to be $(d_B^* a)^2$, and if so the gauge-fixed version of (57) is

$$\nabla_1 a + (*F_B) \wedge a = 0. \tag{58}$$

In (58) we see finally the gauge-fixed operator X_1 of the vector field. The "magnetic coupling" $a \to (*F_B) \wedge a$ is the term responsible for asymptotic freedom, in the sense that without this term, X_1 would be four copies of X_0 and would make a positive contribution to the beta function.

Fall exam

Problem FE1. Prove via path integrals the character-valued index theorem for the Dirac operator. That is, let M be a spin manifold and g an isometry of M with a lift to the spin bundles. Calculate

$$\operatorname{Tr} g(-1)^F \tag{59}$$

via path integrals.

That is, write down the path integral for sections of a suitable M bundle over a circle. Taking the circumference of the circle to be small, compute the integral explicitly and obtain the Atiyah-Bott formula for (59).

Problem FE2. Compute in ϕ^4 theory in four dimensions the anomalous dimension of the operator ϕ^2, in one-loop approximation.

Problem FE3. Determine the asymptotic behavior of perturbation theory in soft or cutoff scalar field theories, as follows. (Fermions and gauge fields introduce different behavior, not to mention other types of theories such as sigma models.)

(a) Consider the integral

$$Z(g) = \int_{\infty}^{\infty} d\phi \, \exp(-\tfrac{1}{2}\phi^2 - g\phi^4/4!). \tag{60}$$

Expand in an asymptotic series in g

$$Z(g) = \sum_k a_k g^k \tag{61}$$

and determine the large k behavior of a_k. *Hint:* Write an integral representation for a_k and find a method of computing the integral for large k. For instance, where is the maximum of the integrand?

(b) Interpret the answer to (a) as a statement about the number of graphs of a particular kind, weighted according to their automorphism groups.

(c) Now consider the path integral

$$Z(g) = \int D\phi\, e^{-L} \tag{62}$$

in a scalar field theory in n dimensions, with

$$L = \int d^n x \left(\tfrac{1}{2}|d\phi|^2 + \tfrac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4 . \right). \tag{63}$$

Write down, again, an integral representation for the coefficients a_k in the asymptotic expansion of $Z(g)$ and determine the large k behavior, at least at a physical level of rigor, by finding the maximum of the integrand.

You should get a non-linear differential equation, and you should be able to prove that it has a spherically symmetric solution; it is believed that this solution does determine the large k behavior of the a_k in a cutoff ϕ^4 theory or in dimensions below four. In four dimensions, the much more complicated renormalization, which you've presumably ignored in the computation above, changes things (L must be redefined by incorporation of counterterms and the expansion in powers of g is not made as simply as you've presumably done above).

Problem FE4. Consider in two dimensions a theory of n fermi fields ψ^i, $i = 1, \dots, n$. Each of the ψ^i is the sum of a section of S_+ and a section of S_-, those being the two spin bundles, which we recall are real if the two-dimensional world has Lorentz signature. (But you will actually compute with Euclidean signature.)

The Lagrangian is

$$L = n \int d^2x \left(\tfrac{1}{2} \sum_i (\psi^i, D\psi^i) + \tfrac{1}{2} m_0 (\psi^i, \psi^i) - \frac{g_0}{8} \left(\sum_i (\psi^i, \psi^i) \right)^2 \right). \tag{64}$$

Notice the crucial factor of n in front of the Lagrangian. Here D is the Dirac operator. In what follows, you can think of the fermion propagator as being

$$\frac{1}{\Gamma \cdot p + i m_0}. \tag{65}$$

Expressions such as (ψ, ψ) (which physicists write as $\bar{\psi}\psi$) are the only possible invariants of the given kind.

Consider the operator $\Sigma = \frac{1}{n} \sum_i (\psi^i, \psi^i)$.

(a) Consider the problem of computing the one-point function $\langle \Sigma \rangle$ (which is an abbreviation for the vacuum expectation value $\langle \Omega | \Sigma | \Omega \rangle$ in perturbation theory. In k-loop order, which Feynman diagrams survive as $n \to \infty$?

(b) Write down an integral equation that formally sums all the Feynman diagram contributions to $\langle \Sigma \rangle$ that survive for $n \to \infty$.

(c) The integral equation in (b) needs some regularization and renormalization. Do this and obtain the renormalized equation obtained as the cutoff goes to infinity.

(d) If one had set $m_0 = 0$ in (64), one would have an extra $\mathbb{Z}_2$ symmetry that is broken by the mass. What is this? Is the operator Σ even or odd under this symmetry?

The renormalization in (c) can be done so that one of the parameters, a renormalized mass m, is odd under the $\mathbb{Z}_2$, while the other parameters are even. Hopefully you've done so!

If m is zero, how many solutions are there to the integral equation found in (c)? Are they invariant under the $\mathbb{Z}_2$?

Problem FE5. QCD is an $SU(3)$ gauge theory with fermions (quarks) which describes the nuclear forces. For this problem you only need to know a few facts. For instance, the theory contains a "current" operator, which is simply a Lorentz vector J^i (that is, a local operator of spin one) that is conserved, $d_i J^i = 0$ (or $d * J = 0$), has zero anomalous dimension (we'll learn why later), and has the property that its two-point function needs an additive c-number renormalization.

The renormalized two-point function of J is therefore the limit as a cutoff Λ goes to infinity of

$$\int d^4x e^{ik\cdot x}\langle J^i(x)J^j(0)\rangle \; - \; A(\Lambda/\mu, g)(k^i k^j - \delta^{ij} k\cdot k). \tag{66}$$

Here g is the (dimensionless) coupling constant, Λ the cutoff, μ a mass that must be introduced in renormalization, and A is a function of the indicated arguments which must be "subtracted" to make the two-point function finite. Also, the theory is asymptotically free and for present purposes you can ignore quark bare masses and assume that there are no parameters in the theory except g, μ, and Λ (and of course Λ is eventually eliminated by taking it to infinity). There is no multiplicative renormalization of J in (66) because the anomalous dimension vanishes. Because J is conserved, the expression in (66) is of the form

$$(k^i k^j - \delta^{ij} k^2) B(k^2, g, \mu) \tag{67}$$

where Λ is omitted as we want to consider the limit $\Lambda \to \infty$.

I suppose it also helps to know that J is built in terms of quarks as $J = \bar{q}\Gamma^i q$, so that the one-loop approximation to $\langle JJ\rangle$ is given by a one-loop diagram with a fermion loop that we computed in homework. But all you really need to know is that in the free field theory, that is, if $g = 0$, $A = a\ln(\Lambda/\mu)$, where a is a constant, independent of g.

(a) Write down the renormalization group equation obeyed by B.

(b) Use this equation, together with the fact that QCD is asymptotically free, to determine the large k^2 behavior of B.

This large k^2 behavior, after including some additional theoretical wrinkles to put it in a form that can be compared with experiment, is one of the classic tests of QCD.

Problem FE6. Let $\mathcal{O}(x)$ be a local operator which annihilates the vacuum:

$$\mathcal{O}(x)|\Omega\rangle = 0 \tag{68}$$

Show, at least at a physical level of rigor, that $\mathcal{O}(x) = 0$.

Now extend this slightly: suppose that for some distinct points x and y $\langle\Omega|\mathcal{O}^*(y)\mathcal{O}(x)|\Omega\rangle = 0$. ($\mathcal{O}^*$ is the hermitian adjoint of $\mathcal{O}$.) Prove that $\mathcal{O} = 0$.

Problem FE7. Prove that a $U(1)$ gauge theory in four dimensions (without magnetic monopoles) cannot be conformally invariant unless it is free.

For present purposes, a $U(1)$ gauge theory is any theory in four dimensions that has a two-form valued local field F which obeys the Bianchi identity

$$dF = 0. \tag{69}$$

Now in four dimensions with Euclidean signature we can decompose $F = F_+ + F_-$ where F_+ and F_- are selfdual and antiselfdual, respectively.

(a) Work in four-dimensional (flat) Euclidean space $\mathbb{R}^4$ and show that conformal invariance implies vanishing of the two-point function $\langle F_+(x)F_-(y)\rangle$ for $x \neq y$.

(Conformal invariance means that the correlation functions can be extended to be defined for points in the one-point compactification S^4 of $\mathbb{R}^4$ and that when so extended the correlation functions are invariant under the group of conformal motions of S^4, which is $SO(5,1)$. Use $SO(5,1)$ to prove vanishing of $\langle F_+(x)F_-(y)\rangle$.)

(b) Given this, use the Bianchi identity to prove that $\langle d^*F(x)d^*F(y)\rangle = 0$.

(c) To conclude the argument, use the result of (6) to deduce that $d^*F = 0$, which (together with the Bianchi identity $dF = 0$) implies that F is a free field.

Superhomework[1]

The purpose of the following is to describe facts that we will need in order to study supersymmetric quantum field theories.

In doing so, we will mainly be interested in flat models (for the supermanifolds on which the quantum field theory is formulated), and in any event our supermanifolds will always be "split" (because we will not include supergravity) which generally means that one won't lose much that is essential by considering the flat model.

(Even without doing quantum field theory, one of the important things one learns from these exercises is that the interesting relativistic field theories have supersymmetric generalizations, with some restriction on the dimension of spacetime.)

We make one small change in notation. A manifold with p even and q odd coordinates will be said to be of dimension $(p|q)$. This frees up the comma for other purposes; for instance, if the reduced space has a metric of signature (n,m) (with $n+m=p$) we might call this a manifold of dimension $(n,m|q)$.

I haven't tried to explain everything below. You will have to consult Wess and Bagger, or other references, especially for the four-dimensional gauge theory. However, if you understand these problems, you are probably familiar with the classical facts that are the most important background for beginning to discuss supersymmetric quantum theories.

Problem SH1. The first part is review from last fall. We consider a $(3|2)$ supermanifold $\mathbb{R}^{(3|2)}$ (or one can dimensionally reduce to $\mathbb{R}^{(n|2)}$ with $n < 3$ by imposing invariances under some translations). We call the fermionic coordinates θ^A, $A = 1, 2$

[1]During the special year at IAS, this problem set was distributed in the spring as preparation for the later lectures in [II-**Dynamics of QFT**]. Solutions to the Superhomework are incorporated into [I-**Supersolutions**].

and the bosonic coordinates y^{AB} (symmetric in A and B). The supersymmetry generators are

$$Q_A = \frac{\partial}{\partial \theta^A} + i\theta^B \frac{\partial}{\partial y^{AB}} \tag{70}$$

and commute with

$$D_A = \frac{\partial}{\partial \theta^A} - i\theta^B \frac{\partial}{\partial y^{AB}} \tag{71}$$

which is used in writing Lagrangians.

The symbol ϵ^{AB} $(= -\epsilon^{BA})$ will denote a translation-invariant volume form on what Bernstein called the odd distribution – the odd subspace of the tangent bundle generated by the D's.

The super-Poincaré group is generated by the odd vector fields Q_A, their anti-commutators $\partial/\partial y^{AB}$, and an $SU(2) \cong SO(3)$ for which the θ^A and the y^{AB} are in the two- and three- dimensional representations, respectively.

A superfield is just a function Φ on $\mathbb{R}^{(3|2)}$. Let X be a fixed Riemannian manifold with metric g. Let Φ be a map from $\mathbb{R}^{(3|2)}$ to X. Picking local coordinates on X, we describe Φ via functions Φ^I and write simply g_{IJ} for the pullback via Φ.

(a) Then as we hopefully remember from last fall we can make the supersymmetric Lagrangian

$$L_0 = \int d^3y\, d^2\theta\, \epsilon^{AB} g_{IJ} D_A \Phi^I D_B \Phi^J. \tag{72}$$

Refresh your recollections; show that this is equivalent to an ordinary sigma model with target space X plus fermions and describe the fermions.

Determine the conserved supercurrent.

(b) Now let h be a smooth function on X and let $L = L_0 + L_1$, where

$$L_1 = \int d^3y\, d^2\theta\, \Phi^*(h). \tag{73}$$

Identify the reduced model. In particular, what is the potential energy and what is the mass term for the fermions?

Suppose that X is compact and that h has only isolated critical points. Can you state a lower bound for the number of classical minima of the energy? (After dimensional reduction to $\mathbb{R}^{(1|2)}$ this observation was of course the starting point for my article on Morse theory in 1982. Even if h does not have isolated critical points, there is a sense in which the same lower bound on the number of vacua holds.)

(c) If X is noncompact it may be the case that there is no critical point of h and moreover that the set with $|dh| < c$ is compact for any constant c. In this case the energy is bounded strictly above zero. This is a theory in which supersymmetry is spontaneously broken. As you can read in Wess and Bagger, unbroken supersymmetry means that the energy of the vacuum is zero.

Consider an arbitrary local minimum of the classical energy function at which the energy is not zero. Show that there is a massless fermion—called the Goldstone fermion. What you have just checked is a special case of the analog of Goldstone's theorem for supersymmetry.

Problem SH2. Now we want to do gauge theory, but again in $\mathbb{R}^{(3|2)}$ to begin with.

First we will do pure gauge theory. G is a compact gauge group and A is a connection on a G-bundle P over $\mathbb{R}^{(3|2)}$. The curvature F is defined in the usual way. One can freely write down all kinds of supersymmetric actions by integrating over $\mathbb{R}^{(3|2)}$ an appropriate polynomial in F and its derivatives. However, such constructions (like the generic classical field theory) are "wrong," that is, the bosonic part does not have positive energy, for instance. We will verify the analogous statement later in a case that may be a bit easier.

To get a good theory, we consider connections such that $F(D_A, D_B) = 0$. (Bernstein expressed this somewhat differently, I think.) An important fact is that this condition on the connection can be solved locally, without imposing any differential equations (but reducing the number of independent functions in the description of the connection).

Let $\mathcal{D}_A$ be the gauge-covariant extension of D_A and $\mathcal{D}_{AB}$ the gauge-covariant extension of $\partial/\partial y^{AB}$.

The basic curvature invariant is then

$$\lambda_A = \epsilon^{BC} [\mathcal{D}_A, \mathcal{D}_{BC}]. \tag{74}$$

Show that the whole curvature can be expressed in terms of λ and its covariant derivatives.

The basic gauge theory Lagrangian is then

$$L_2 = \int d^3y\, d^2\theta\, \epsilon^{AB} \mathrm{Tr}\, \lambda_A \lambda_B. \tag{75}$$

Here Tr should be understood simply as a gauge-invariant quadratic form on the Lie algebra.

(a) Show that this Lagrangian has for its bosonic reduction ordinary Yang-Mills theory. Identify what the fermions are. (Hint: you might use the relation between $d\theta^A$ and $\mathcal{D}_A$.)

You should see that this model is equivalent to one of the simplest models in three dimensions of Yang-Mills plus fermions; we might have studied it as a simple example of such without noticing it was supersymmetric.

What is the conserved supercurrent?

(b) Suppose that G acts on the manifold X considered in Problem SH1 of [I-**Homework**] above and reinterpret Φ as a section of $X \times_G P$. L_0 and L_1 are defined by the same formulas as in Problem SH1 of [I-**Homework**] above.

Show that, in terms of ordinary fields, L_0 and L_1 are now the obvious gauge-covariant extensions of what you had before.

Problem SH3. Now we move on to four bose dimensions. We cannot work in $\mathbb{R}^{(4|2)}$ because although $SO(3,1)$ has two-dimensional spinor representations, they are not real; we would not be able to get real Lagrangians in Minkowski space.

So we introduce spinors of both kinds and work on $\mathbb{R}^{(4|4)}$. The following will be written in Euclidean signature without imposing any reality condition. We write

θ^A, $A = 1, 2$ and $\bar{\theta}^{\dot{A}}$, $\dot{A} = 1, 2$ for the two kinds of spinors[2] and $x^{A\dot{A}}$ for the bose coordinates. Under the action of $SO(4) \cong SU(2) \times SU(2)$, the undotted indices A, B, C transform in the two-dimensional representation of the first $SU(2)$ and the dotted indices $\dot{A}, \dot{B}, \dot{C}$ transform in the two-dimensional representation of the second.

The supersymmetry generators are

$$Q_A = \frac{\partial}{\partial \theta^A} + i\bar{\theta}^{\dot{A}} \frac{\partial}{\partial x^{A\dot{A}}}$$
$$\bar{Q}_{\dot{A}} = \frac{\partial}{\partial \bar{\theta}^{\dot{A}}} + i\theta^A \frac{\partial}{\partial x^{A\dot{A}}} \tag{76}$$

and commute with

$$D_A = \frac{\partial}{\partial \theta^A} - i\bar{\theta}^{\dot{A}} \frac{\partial}{\partial x^{A\dot{A}}}$$
$$\bar{D}_{\dot{A}} = \frac{\partial}{\partial \bar{\theta}^{\dot{A}}} - i\theta^A \frac{\partial}{\partial x^{A\dot{A}}} \tag{77}$$

which are used in writing Lagrangians.

Now, the most obvious kind of field, imitating what we did in Problem SH1 of [I-**Homework**] above, would appear to be a field Φ which is simply a function on $\mathbb{R}^{(4|4)}$. This is called an ordinary superfield.

However, *it is impossible to write a sensible Lagrangian for such a field*, where again sensible means that the bosonic part of the energy is positive.

(a) Write some supersymmetric Lagrangians for an ordinary superfield and show that (if you specialize to Lorentz signature) the bosonic part of the energy is not positive.

Instead, we use a *chiral superfield* Φ, which is a function on $\mathbb{R}^{(4|4)}$ such that

$$\bar{D}_{\dot{A}} \Phi = 0. \tag{78}$$

Likewise $\bar{\Phi}$ is called an antichiral superfield if

$$D_A \bar{\Phi} = 0. \tag{79}$$

If X is a complex manifold, a chiral map from $\mathbb{R}^{(4|4)}$ to X is a map such that local holomorphic functions on X pull back to chiral superfields, and antiholomorphic functions pull back to antichiral superfields.

If K is a real analytic function on X, and $\Phi : \mathbb{R}^{(4|4)} \to X$ is a chiral map, then one defines a Lagrangian

$$\mathcal{L}_0 = \int d^4x \, d^4\theta \; \Phi^*(K). \tag{80}$$

(I will use $\mathcal{L}$ for four-dimensional Lagrangians, and L for somewhat similar three-dimensional ones.)

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[2] $\bar{\theta}^A$ and $\theta^{\dot{A}}$ will never appear (except as a result of a possible misprint—in that case remove bars from undotted θ's and add them to dotted ones). In Lorentz signature if one imposes a reality condition, then $\bar{\theta}^{\dot{A}}$ is the complex conjugate of θ^A. In Euclidean signature there is no reality condition and no relation between these two. Similar statements hold for other objects introduced below such as the W's.

(b) Show that this is invariant under $K \to K + f + \bar{f}$, where f is a holomorphic function on X.

(c) Show that $\mathcal{L}_0$ is equivalent to an ordinary sigma model Lagrangian on $\mathbb{R}^4$, with target X—the metric on X being a Kähler metric with K as the Kähler potential—plus fermion terms. Describe the fermion terms.

Notice that $\mathcal{L}_0$, after doing all (or as you should be able to see, just half) of the θ integrals, makes sense globally for maps to any Kähler manifold X—even though the description here required the choice, and in particular the existence, of a Kähler potential, which of course exists locally but may not exist globally. Problem: find a good and economical global construction of $\mathcal{L}_0$. (To some extent this can be done using gauge theory as developed below.)

(d) Show that this theory has a $U(1)$ symmetry (classically—it may be anomalous) coming from a symmetry of $\mathbb{R}^{(4|4)}$ that acts only on the odd coordinates and acts by an outer automorphism of the super-Poincaré algebra.

A symmetry that acts by outer automorphisms of super-Poincaré and in superspace acts only on the odd coordinates is called an R symmetry, a term one will often encounter.

(e) Show that if one imposes invariance under one translation in $\mathbb{R}^4$ (by a vector that is not a null vector—if you are doing Lorentz signature it should be a spatial translation, here and in other cases below), $\mathcal{L}_0$ reduces to L_0 as considered in Problem SH1 of [I-**Homework**] above—for the special case that the target is actually Kähler.

It is therefore true, given also **FP**10, that if one imposes invariance under three translations in $\mathbb{R}^4$, to reduce to $\mathbb{R}^{(1|4)}$, one gets the Lagrangian whose quantization gives the Hodge theory of differential forms on X.

Deduce that there is an action of $SU(2) \times U(1)$ on differential forms on a Kähler manifold, commuting with the Hamiltonian, where the $U(1)$ was seen in (d) and the $SU(2)$ comes from spatial rotations of $\mathbb{R}^4$.

(f) The following is something that we will need to know about if we really want to do Donaldson theory.

Suppose that X is hyper-Kähler. Then the Lagrangian $\mathcal{L}_0$ has twice as much supersymmetry as for a Kähler X. It can be shown comparatively easily as follows:

After writing down "in components" the sigma model action, show that the $U(1)$ found in (d) actually extends to $SU(2)$ in case X is hyper-Kähler. (The $SU(2)$ only acts on fermions. This should be obvious—there is no universal action of $SU(2)$ on a hyper-Kähler manifold.)

This $SU(2)$ does *not* act by outer automorphisms on the super-Poincaré algebra of $\mathbb{R}^{(4|4)}$. Can you describe what algebra is generated by the $SU(2)$ together with the automorphisms of $\mathbb{R}^{(4|4)}$? Hint: it can be naturally realized geometrically as a supergroup of automorphisms of an $\mathbb{R}^{(4|8)}$. It is called the $N = 2$ supersymmetry algebra in four dimensions, another term that is likely to be encountered frequently.

Unfortunately, there is no reasonable way known to make a superspace construction starting from $\mathbb{R}^{(4|8)}$. That is why I asked you to construct this $N = 2$ theory by looking at it as an $N = 1$ theory that just happens to have extra supersymmetry. The above computation is thus analogous to the way that mathematicians usually find the extra symmetries of Hodge theory on a Kähler manifold "by hand," not by starting with a geometrical construction of maps from $\mathbb{R}^{(1|4)}$.

(g) Deduce from the above that there is an action of $SU(2) \times SU(2)$ on the differential forms on a hyper-Kähler manifold, commuting with the Hamiltonian. Actually, this extends to an action of $Sp(2) = Spin(5)$. That is because there is a supersymmetric model in six dimensions in which the data consist of maps of $\mathbb{R}^6$ to a hyper-Kähler manifold plus fermions. This theory is one of the limiting cases permitted by Nahm's theorem (that is, six is the maximum dimension for sigma models according to Nahm's theorem). It does not have a known superspace construction.

Write down the Lagrangian for this model by using the fact that it must reduce in four dimensions to a Lagrangian found above. (For extra credit do the following not very hard exercise: Verify supersymmetry of this Lagrangian.)

Problem SH4. Going back to the case that X is only Kähler, let W be a holomorphic function on X—called the superpotential. Let

$$\mathcal{L}_1 = \int d^4x\, d^2\theta \Phi^*(W) + \int d^4x\, d^2\bar{\theta} \Phi^*(\bar{W}). \tag{81}$$

(a) Show that this is supersymmetric.

(b) Reduce to an ordinary Lagrangian and compute the potential energy function on X and fermion mass terms.

(c) Upon imposing invariance under translation of one coordinate, show that $\mathcal{L}_1$ becomes equivalent to L_1 as constructed in Problem SH1 of [I-**Homework**](b) above with $h = \mathrm{Re}(W)$.

(d) Under what conditions on X and W does the R symmetry—the outer automorphism of super-Poincaré that comes from a symmetry of $\mathbb{R}^{(4|4)}$, as found above, lift to a symmetry of this theory? (That is, a classical symmetry—quantum mechanically it may be anomalous.)

Problem SH5. Now we want to do gauge theory. Once again, we introduce a compact connected Lie group G and G-bundle P over $\mathbb{R}^{(4|4)}$. A is a connection on P and F is the corresponding curvature.

One can write all kinds of supersymmetric Lagrangians for such a general connection, but they are not of much interest—the usual classical pathology is nonpositive energy for the bosons.

For the good construction, one restricts A by requiring the vanishing of some components of the curvature, in fact $F(D_A, D_B) = F(D_A, \bar{D}_{\dot{A}}) = F(\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}})$. (Or more succinctly, F vanishes when evaluated on any two fermionic D's.)

(a) Show that these conditions can be solved locally, without differential equations.

(b) With these conditions, the basic curvature invariants are

$$W_A = \epsilon^{\dot{B}\dot{C}} [\bar{\mathcal{D}}_{\dot{B}}, \mathcal{D}_{A\dot{C}}], \tag{82}$$

and

$$\bar{W}_{\dot{A}} = \epsilon^{BC} [\mathcal{D}_B, \mathcal{D}_{C\dot{A}}]. \tag{83}$$

where as before $\mathcal{D}$ is the gauge-covariant extension of D.

Show that W_A is a chiral superfield (with values in one of the two spin bundles[3]) and $\bar{W}_{\dot{A}}$ is antichiral, and that all components of the curvature can be expressed in terms of the W's and their covariant derivatives by using the Bianchi identity.

(c) The basic Lagrangian is

$$\mathcal{L}_2 = \tau \int d^4x\, d^2\theta \epsilon^{AB} \operatorname{Tr} W_A W_B + \bar{\tau} \int d^4x\, d^2\bar{\theta} \epsilon^{\dot{A}\dot{B}} \operatorname{Tr} \bar{W}_{\dot{A}} \bar{W}_{\dot{B}}. \tag{84}$$

τ is a complex conjugate and (to get a real Lagrangian in Lorentz signature) $\bar{\tau}$ is its complex conjugate.

Reduce this to an ordinary Lagrangian. It is equivalent to a simple special case of four-dimensional gauge theory coupled to fermions. Which special case?

In order to do this, you may have made contact with the formalism in Wess and Bagger, Chapter 6, where a field that they call V is introduced. It will probably be necessary to make contact with that in order to solve an important part of a later exercise.

(d) Show that this theory at the classical level possesses the R symmetry that was discussed in the previous exercise.

Now the one quantum question in these exercises. Using your knowledge of determinant line bundles, do you believe that this R symmetry is valid quantum mechanically? If you claim that there is an anomaly breaking the $U(1)$ R symmetry to a finite subgroup of $U(1)$, can you identify the subgroup in question?

(e) If you impose invariance under one of the spatial translations, this theory does *not* reduce to L_2 as formulated in the three-dimensional case in Problem SH2 of [I-**Homework**] above. However, it does reduce to one of the problems of gauge theory plus matter formulated in Problem SH2 of [I-**Homework**] (that is, a Lagrangian of the form $L_0 + L_2$). Can you identify which special case?

Problem SH6. Now we want to consider a Kähler manifold X with a G action. To begin with, we suppose that $X = \mathbb{C}^n$ for some n with a linear action of G on X, leaving fixed the origin. We pick a G-invariant Kähler potential K on X.

We consider a theory in which the data are a connection A on a G-bundle P (obeying the curvature constraints discussed in the last problem) and a chiral section Φ of $X \times_G P$.

(a) Explain what it means for Φ to be chiral.

(b) Convince yourself that the Lagrangian

$$\mathcal{L}_0 = \int d^4x d^4\theta\, \Phi^*(K) \tag{85}$$

makes sense in this situation and is, in fact, equivalent to the obvious gauge covariant extension of $\mathcal{L}_0$ as defined above.

[3]In this flat model the spin bundle has a flat connection. A chiral superfield with values in a flat bundle is defined in the obvious way.

(c) Consider the Lagrangian $\mathcal{L}_0 + \mathcal{L}_2$ (with $\mathcal{L}_2$ as defined above). Determine the moduli space of classical states of minimum energy. You should meet some notions of "symplectic quotient" (related to geometric invariant theory) and you should find that the moment map you get vanishes at the origin in $\mathbb{C}^n$.

In fact, it is possible (in case G has $U(1)$ factors—if G is semi-simple the following generalization is vacuous) to add an additional supersymmetric interaction that adds a constant to the moment map.

In the notation of Wess and Bagger, for $G = U(1)$, the additional interaction is

$$\mathcal{L}_3 = r \int d^4x\, d^4\theta\ V. \tag{86}$$

Here r is a real constant. ($\mathcal{L}_3$ is commonly called the Fayet-Iliopoulos term or D term and is introduced by Wess and Bagger starting on p. 52.) Show that this is supersymmetric (a trivial consequence of what in my edition of Wess and Bagger is eqn. 6.4). I do not know in four dimensions a more intrinsic way to write (86)–my apologies for using a formula that depends on Wess-Zumino gauge (which leads to the definition of V). After reduction to two dimensions there is a nice formula that we will see later.

Show that replacing $\mathcal{L}_0 + \mathcal{L}_2$ by $\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3$ has the effect of shifting the moment map by a constant r. The space of classical zeroes of the energy (or supersymmetric vacua) is the symplectic quotient with the new moment map. Of course, if G has several $U(1)$ factors, we add a term like (86) for each such factor, so we can get a general moment map.

(d) The global generalization of the above is as follows. Let X be any Kähler manifold with G action (and Kähler metric invariant under G) and such that there is a moment map for the G action; pick a moment map. For each such choice, an appropriate global version of $\mathcal{L}_0 + \mathcal{L}_3$ exists. (There is no canonical $\mathcal{L}_0$ just as in ordinary geometry there is no canonical choice of a moment map. There is the canonical family of $\mathcal{L}_0 + \mathcal{L}_3$'s with the variable constant r in the definition of $\mathcal{L}_3$.) The verification of this is in a paper by Bagger and Witten. I do not know a natural global construction in superspace. (The above is not even a universal local construction since in identifying X at least locally with $\mathbb{C}^n$ with linear action of G we assumed that G has a fixed point. The supersymmetric theory exists for any G action on X, even if there is no fixed point, as long as there is a moment map.)

(e) Now, let W be a G-invariant holomorphic function on X and consider the superpotential term $\mathcal{L}_1$. Convince yourself that $\mathcal{L}_1$ makes sense in this gauge theory setting and describe the moduli space of classical vacua in the theory with Lagrangian $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ (the inclusion of the $\mathcal{L}_3$ term with a constant r in $\mathcal{L}_3$ just means, of course, that you are to work with a general moment map).

(f) (We will need this when we get to Donaldson theory.) Consider the case that G is n-dimensional and $X = \mathbb{C}^n$ with G acting on X via the adjoint representation. Consider the Lagrangian $\mathcal{L}_0 + \mathcal{L}_2$. By writing it out in terms of ordinary fields, show that the $U(1)$ R symmetry extends to to an $SU(2)$ symmetry which does *not* act by outer automorphisms of super-Poincaré, so that super-Poincaré is extended to a super-algebra with twice as many odd generators. (In fact, in this case, unlike the sigma model case, there is a rather nice realization via gauge theory on $\mathbb{R}^{(4|8)}$.)

More generally, if $X = \mathbb{C}^n \times Y$, where (i) G acts on $\mathbb{C}^n$ via the adjoint representation; (ii) Y is hyper-Kähler; (iii) G acts on Y preserving the metric and with a hyper-Kähler moment map, then (for every choice of all this data including the moment map) there is a gauge-invariant sigma model with the data being the connection A and a section of $X \times_G P$. One again needs this generalization for Donaldson theory; however, I won't explain it here. (The moduli space of classical supersymmetric vacua is the hyper-Kähler quotient of Y by G, which therefore must naturally be a hyper-Kähler manifold; that is how the hyper-Kähler quotient was discovered.)

Problem SH7. The supersymmetric theories that I will lecture about in two dimensions are actually all obtained by dimensional reduction from $\mathbb{R}^{(4|4)}$, that is, by imposing invariance under two translations to get a theory on $\mathbb{R}^{(2|4)}$. However, some properties that appear upon reduction to $\mathbb{R}^{(2|4)}$ deserve special discussion.

First of all, the rotation of the two coordinates that are removed in going from four to two dimensions acts as an outer automorphism of the two-dimensional super-Poincaré. Since there already was such an outer automorphism in four dimensions, there are two of them in two dimensions, making up a group $U(1) \times U(1)$ of R symmetries. (It is not the case that all the theories we can write admit the full $U(1) \times U(1)$ even classically, since in four dimensions we had the possibility of breaking the R symmetry classically with the choice of W.)

The bosonic coordinates in two dimensions, say, x^+ and x^-, are of weights 1 and -1 under Lorentz boosts. The fermionic coordinates consist of two of weight $1/2$ and two of weight $-1/2$.

(a) Show—just using the geometrical properties of $\mathbb{R}^{(4|4)}$—that of the R symmetry group $U(1) \times U(1)$ in two dimensions, one $U(1)$ acts only on the weight $1/2$ odd coordinates and the other $U(1)$ acts only on the weight $-1/2$ odd coordinates.

I label the weight $1/2$ odd coordinates as θ^+, $\bar{\theta}^+$ where one of the two $U(1)$'s in the R symmetry group acts by $\theta^+ \to e^{i\alpha}\theta^+$, $\bar{\theta}^+ \to e^{-i\alpha}\bar{\theta}^+$. Likewise I label the weight $-1/2$ odd coordinates as θ^-, $\bar{\theta}^-$ with analogous action of the second $U(1)$. (As you know well, in Lorentz signature the barred objects are complex conjugates of the unbarred ones, but in Euclidean signature, as always, there is no reality condition.)

For the supersymmetries I write

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+\frac{\partial}{\partial x^+} \\ \bar{Q}_+ &= \frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\frac{\partial}{\partial x^+}. \end{aligned} \tag{87}$$

Thus $Q_+^2 = \bar{Q}_+^2 = 0$ and $\{Q_+, \bar{Q}_+\}$ is a multiple of $\partial/\partial x^+$. Q_- and $\bar{Q}_-$ are defined in the same way with $+ \to -$.

The supersymmetries commute with vector fields D_+, $\bar{D}_+$, D_-, $\bar{D}_-$ that are defined by the same formulas with $i \to -i$. (Or if you've suppressed the i's, then reverse the sign of $\partial/\partial x^\pm$.)

Mirror symmetry is the automorphism of $\mathbb{R}^{(2|4)}$ that exchanges θ^- and $\bar{\theta}^-$ while doing nothing to other odd or even coordinates. Thus it reverses one of the R symmetries and commutes with the other. More precisely, two field theories

defined in $\mathbb{R}^{(2|4)}$ are said to be mirrors if the stated automorphism of $\mathbb{R}^{(2|4)}$ lifts to an isomorphism between the given theories.

(b) A chiral superfield is a function Φ on superspace such that $\bar{D}_+\Phi = \bar{D}_-\Phi = 0$. An antichiral superfield $\bar{\Phi}$ obeys $D_\pm\bar{\Phi} = 0$.

A twisted chiral superfield is a function Σ on superspace such that $\bar{D}_+\Sigma = D_-\Sigma = 0$. A twisted antichiral superfield $\bar{\Sigma}$ obeys $D_+\bar{\Sigma} = \bar{D}_-\bar{\Sigma} = 0$.

The sigma model of a chiral superfield is our friend

$$\mathcal{L}_0' = \int d^2x\, d^4\theta\, K(\Phi, \bar{\Phi}) \tag{88}$$

and the sigma model of a twisted chiral superfield is defined exactly analogously. One can also straightforwardly define sigma models containing both chiral and twisted chiral superfields, but we will not study them.

Likewise, if W is a holomorphic function of a chiral superfield, we have

$$\mathcal{L}_1' = \int d^2x d^2\theta \Phi^*(W) + \int d^2x d^2\bar{\theta}\Phi^*(\bar{W}), \tag{89}$$

as in four dimensions.

(c) Some novelties occur if one considers gauge theory. We want to consider gauge theory with chiral superfields. For this, we consider a connection A on a G-bundle P such that the curvature F obeys $F(u, v) = 0$ where u, v are either both linear combinations of D_+ and D_- or both linear combinations of $\bar{D}_+$ and $\bar{D}_-$. (If we wanted to use twisted chiral superfields, then we would pair D_+ and $\bar{D}_-$ in the constraint on the curvature.)

For a connection of this kind, the basic curvature invariants are $\Sigma = F(\bar{D}_+, D_-)$ (or equivalently $\Sigma = \{\bar{\mathcal{D}}_+, \mathcal{D}_-\}$ where the $\mathcal{D}$'s are gauge-covariant extensions of the D's) and $\bar{\Sigma} = F(D_+, \bar{D}_-)$.

Show that all components of the curvature can be expressed in terms of Σ, $\bar{\Sigma}$, and their covariant derivatives.

Show that Σ is a twisted chiral superfield and that $\bar{\Sigma}$ is a twisted antichiral superfield.

(d) The basic Lagrangian for the gauge multiplet (that is, the "supermultiplet" of fields that includes the connection) is

$$\mathcal{L}_2' = \frac{1}{4e^2}\int d^2x d^4\theta\ \mathrm{Tr}\,\Sigma\bar{\Sigma}. \tag{90}$$

Here Tr is to be understood as any invariant quadratic form on the Lie algebra.

Either by reducing this to an ordinary Lagrangian or in any other way of your choosing, show that this is equivalent to the reduction to two dimensions of our four-dimensional Lagrangian $\mathcal{L}_2$.

In particular, it describes gauge fields, fermions, *and scalars*, in contrast to the three- and four-dimensional gauge actions L_2 and $\mathcal{L}_2$ constructed earlier, which describe gauge fields and fermions only.

(e) This will be very important in applications. If $G = U(1)$, one has the additional possible term

$$\mathcal{L}_3' = r\int d^2x d\theta^+ d\bar{\theta}^- \Sigma + r\int d^2x d\theta^- d\bar{\theta}^+ \bar{\Sigma} \tag{91}$$

with r a constant. More generally, for any G one has such a term for any $U(1)$ factor in G. In other words, Σ can be replaced by any gauge-invariant linear function of the Σ's. (One can more generally use any gauge invariant holomorphic function of the Σ's, obviously.)

$\mathcal{L}_3'$ arises by dimensional reduction from the four-dimensional interaction $\mathcal{L}_3$, but as you can see, in two dimensions a much nicer and more intrinsic description is possible.

This also enables one to give a nicer global description of the sigma model of maps to a Kähler manifold in two dimensions, at least if the Kähler class is integral.

The supersymmetric Lagrangians we will study in two dimensions will be special cases of $\mathcal{L}_0' + \mathcal{L}_1' + \mathcal{L}_2' + \mathcal{L}_3'$.

Addendum to Superhomework[4]

Recall that unitary irreducible representations of super-Poincaré are constructed from certain equivariant vector bundles supported on a cone or hyperboloid in momentum space. The equivariant vector bundles in question are classified by representations of the subgroup H_p of super-Poincaré that leaves fixed a chosen vector p on the cone or hyperboloid. Moreover, one can replace H_p by its maximal reductive subgroup, whose bosonic part is $Spin(2)$ if we are on the cone and $Spin(3)$ if we are on the hyperboloid.

The $Spin(2)$ Lie algebra has one generator which we call J; its eigenvalues are in $\mathbb{Z}/2$. The $Spin(3)$ Lie algebra has one representation R_n of every positive dimension n.

The $N = 1$ super-Poincaré Lie algebra in four dimensions is (with notation as in the superhomework)

$$\{Q_A, \bar{Q}_{\dot{A}}\} = P_{A\dot{A}} \tag{92}$$

with other anticommutators vanishing and Poincaré acting on the odd generators Q in the usual fashion.

Upon fixing a vector on the cone or hyperboloid, the $P_{A\dot{A}}$ become numbers, say $p_{A\dot{A}}$. Unitary representations of (92) are analyzed in a fashion we should know: Q's that are in the kernel of the quadratic form in (92) are represented by zero, and the remaining Q's generate a Clifford algebra.

Problem ASH1. Show that if p is lightlike, the quadratic form in (92) has a two-dimensional null space, and that quantization of the Clifford algebra gives a two-dimensional representation with J eigenvalues $j, j+1/2$, for arbitrary $j \in \mathbb{Z}/2$. We will call the $0, 1/2$ and $-1/2, 0$ representations the chiral representations C_+

[4]In the solutions to these problems we follow the notation and sign conventions of Part 1, which differ from that in the statement of the problems given here.

and C_- and the $1/2, 1$ and $-1, -1/2$ representations the vector representations V_+ and V_-.

Show that quantization of a chiral superfield gives a sum of C_- and C_+ for the one-particle states, and quantization of a vector superfield gives similarly a sum of V_+ and V_-.

Problem ASH2. Show that if p is timelike, the quadratic form is nondegenerate. Its irreducible representation is four-dimensional and is (as a $Spin(3)$ representation) $C' = R_1 \oplus R_1 \oplus R_2$. An irreducible representation of H_p is then $C' \otimes R$, where R is any irreducible representation of $Spin(3)$.

We call C' the massive chiral representation and we call $V' = C' \otimes R_2$ the massive vector representation.

Problem ASH3. Show that when a null vector p is perturbed to be timelike, $C_+ \oplus C_-$ can be deformed to C'. Now do a bit of field theory: show that the representation theory fact just stated corresponds to the fact that it is possible to add a bare mass for a massless chiral superfield.

Problem ASH4. The analogous statement for the vector representation is a bit more complex (as we should expect from our study of the Higgs mechanism in the bosonic case). Show that the representation $V_+ \oplus V_- \oplus C_+ \oplus C_-$ can be perturbed to V'.

Now show how this can be realized in field theory.[5] Consider a theory with a vector superfield and a chiral superfield Φ of charge 1. Take the Lagrangian to be the kinetic energy for both superfields plus the "Fayet-Iliopoulos D-term"

$$r \int d^4x D \tag{93}$$

(related to the constant term in the moment map) discussed in the superhomework.

Show that (a) for $r = 0$, we get the representation $V_+ \oplus V_- \oplus C_+ \oplus C_-$ at tree level, (b) for $r \neq 0$ one gets at tree level depending on the sign of r either supersymmetry breaking, or a supersymmetric vacuum with mass gap and representation V'. The occurrence of the latter is called the super-Higgs effect.

Problem ASH5. Now we consider the $N = 2$ supersymmetry algebra. In the absence of a central term, the algebra reads

$$\{Q_A^I, \bar{Q}_{\dot{A}J}\} = \delta_J^I P_{A\dot{A}}, \quad I, J = 1, 2 \tag{94}$$

with other anticommutators vanishing.

Classify the unitary representations as follows:

(a) In the massless case, the "helicities" or J eigenvalues in an irreducible representation are $j, j + 1/2, j + 1/2$, and $j + 1$ for any $j \in \mathbb{Z}/2$.

[5] I write the following in a slightly naive way; the model presented here is actually anomalous. One can add additional fields to cancel the anomaly without affecting the occurrence of the super-Higgs mechanism. (The supersymmetry breaking that occurs in one case in the text may be destroyed.)

(b) In the superhomework you met two kinds of $N = 2$ fields. Show that the massless hypermultiplet gives on quantization a representation H which is the sum of two copies of the $j = -1/2$ representation (helicities from $-1/2$ to $+1/2$; there are two copies because of the quaternionic structure of the hypermultiplet) while the vector multiplet gives a representation V that is the sum of a representation with $j = -1$ (helicities from -1 to 0) and $j = 0$ (helicities from 0 to 1). We call the $j = -1/2$ representation H_0.

(c) In the massive case, the basic representation H', obtained by quantizing the nondegenerate Clifford algebra in (94), has helicities ranging from -1 to 1 with multiplicities $1, 4, 6, 4, 1$.[6] Any irreducible unitary representation of the little group H_p is $H' \otimes R$ with R an irreducible representation of $Spin(3)$.

The above facts might seem to imply, for instance, that a bare mass for the hypermultiplet is impossible (why?) but it is in fact possible to add a supersymmetric bare mass term. In the presence of the bare mass, the $N = 2$ super-Poincaré is deformed to the most general possibility allowed by the Haag-Sohnius-Lopuszański theorem (the subject of Bernstein's first lecture). In the process, the R symmetry group of outer automorphisms, isomorphic to $U(2)$ (and discussed in the superhomework), is reduced to $SU(2)$.

Concretely, the deformation of the algebra is as follows. (94) is unchanged, but instead of the other anticommutators vanishing one has

$$\{Q_A^I, Q_B^J\} = \epsilon^{IJ}\epsilon_{AB}Y, \tag{95}$$

where the ϵ's are the antisymmetric tensors invariant under the action of Poincaré and of the $SU(2)$ group of R symmetries. Y is a "central charge," that isn an operator that commutes with the whole super-Poincaré, including its adjoint $\overline{Y}$ which appears in the remaining anticommutator:

$$\{\bar{Q}_{\dot{A}I}, \bar{Q}_{\dot{B}J}\} = \epsilon_{AB}\epsilon_{IJ}\overline{Y}. \tag{96}$$

Problem ASH6. Consider a representation of mass M and $Y = y$ (y is a complex number; as Y is central, in an irreducible representation it can be set to a constant). We have already analyzed the representations for $y = 0$. Show that more generally the representations with $M > |y|$ are similar (for instance, deformable to) the massive representations with $y = 0$ and the representations with $M = |y|$ are likewise similar to the representations with $M = y = 0$.

Problem ASH7. There are two important cases of this in practice. One is the hypermultiplet, for which it is in fact possible to add a bare mass. Recall from the superhomework that a single hypermultiplet has for its bosonic fields a map to the flat hyper-Kähler manifold $\mathbb{H} = \mathbb{R}^4$. $\mathbb{H}$ has a group $SU(2)$ of symmetries that preserve the hyper-Kähler structure (and so commute with super-Poincaré). Show that it is possible to add a bare mass, in such a way that Y is equal to an arbitrary element of the $SU(2)$ Lie algebra. (This is easy if you have the right frame of mind, but perhaps not otherwise. In any event it is less immediately necessary than the results stated in the next exercise, which are essential for doing Donaldson theory.)

[6] In the massive case, by helicity we mean the weight under the generator of a maximal torus of $Spin(3)$.

Problem ASH8. The other important case is the vector multiplet. First we have to discuss the Higgs mechanism for $N = 2$. We recall that the bosonic fields of a vector multiplet for gauge group G are a connection A and a complex scalar ϕ in the adjoint representation. There may also be charged scalars that transform in hypermultiplets; this means that one has for bosonic fields not only the vector multiplet but also a map from Minkowski space to a hyper-Kähler manifold X (on which G acts).

There are two basic cases of the Higgs mechanism for $N = 2$. We describe these in a language that is tied to weak coupling (so we can speak informally of the expectation value of a charged field):

(i) The scalar fields that have vacuum expectation values are all in the vector multiplet; this means that the vacuum is determined by a point in X that is G-invariant.

(ii) The vacuum corresponds to a non-trivial orbit in the action of G on X, for instance a free orbit.

In this context:

(a) Consider case (i), which is really essential for Donaldson theory. You may as well assume there are no hypermultiplets. Then because of the term $\mathrm{Tr}[\phi, \bar{\phi}]^2$ term in the potential (recall the superhomework) the unbroken subgroup of G has the same rank as G and is in fact, for generic ϕ, a maximal torus. By going from $\phi = 0$ to $\phi \neq 0$, most of the representations get mass, but the representations do not change in size; the massive representations we get are obviously representations with $M = |Y|$, as those are the massive representations to which the massless vector representation can be deformed. The relevant Y is computed in the Olive-Witten paper that was distributed previously.

(b) In this same situation, magnetic monopoles, which arise from solutions of the BPS equation $F = *D\phi$, are likewise in "small" representations of supersymmetry. This is shown as follows: (1) The classical solution is invariant under half of the Q's. (2) The other half of the Q's give fermion zero modes, four of them to be precise. There are no other fermion zero modes. (3) Quantization of four fermion zero modes gives a four-dimensional Clifford algebra with a four-dimensional irreducible representation, which is what we called above the representation H_0. The monopole *of fixed electric charge*[7] transforms in the representation H_0. The anti-monopole (obtained by quantizing the solution of $F = - * D\phi$, and with opposite electric charge) gives another copy of H_0. Together they make the hypermultiplet representation H; this will be important in Donaldson theory.

(c) Now show that a Higgs effect of type (ii) gives a representation of $M > |Y|$. To show that this is so generically, it is enough to consider a special case, and I would recommend working through in detail the following example. Take $G = U(1)$, with a linear action on the quaternions $\mathbb{H}$ (preserving the hyper-Kähler structure) and let $X = \mathbb{H}^2$. A Higgs mechanism is possible, since the hyper-Kähler quotient of X by $U(1)$ is non-trivial and in fact is four-dimensional. (What is it?) For a generic classical vacuum, show that the spectrum at tree level is as follows: a massless hypermultiplet and a copy of the basic massive $Y = 0$ representation H'. We recall that the latter has helicities running from -1 to 1 with multiplicities $1, 4, 6, 4, 1$.

[7] We have seen in [II-**Dynamics of QFT**, §9.6] that monopoles have an "arbitrary" electric charge. We recall that the electric charge of the monopole is obtained by quantizing a bosonic zero mode.

(As a $Spin(3)$ representation it is the sum of R_3, four copies of R_2, and five copies of R_1.)

CHAPTER 2
Solutions to Selected Problems

As we stated earlier, the solutions to the Superhomework are incorporated into [I-**Supersolutions**]. The contributors to the solutions printed here are:

Pierre Deligne: **FP**3, **FP**6

Pierre Deligne and Daniel Freed: **FP**2, **FP**9, **FP**11

Daniel Freed: **FP**1, **FP**4–5, **FP**7, **FP**10, **FP**12, **FP**14–19, **FE**2–7, **ASH**1–8

Lisa Jeffrey and Siye Wu: **FE**1

Solution to Problem FP1

We denote the Riemannian metric on M by $\langle \cdot, \cdot \rangle$ and so write the lagrangian density as

$$\mathcal{L} = \tfrac{1}{2}\langle \dot{x}, \dot{x} \rangle dt \ .$$

here $\dot{x} = \frac{dx}{dt}$ is the tangent to the curve.

(a) As explained in [I-**Classical Fields**], we do calculus on $\mathcal{F}_x \times \mathbb{R}$, where $\mathcal{F}_x$ is the space of maps $x\colon \mathbb{R} \to M$. Let δ be the differential on $\mathcal{F}_x$ and d the differential on $\mathbb{R}$. Let ∇ denote the Levi-Civita connection on M, and d_∇, δ_∇ the differentials on $\Omega^\bullet(\mathcal{F}_x \times \mathbb{R}, e^*TM)$, where e is the evaluation map $e\colon \mathcal{F}_x \times \mathbb{R} \to M$. Then we have the standard Riemannian geometry computation

$$\begin{aligned} \delta\mathcal{L} &= \frac{1}{2}\delta\langle \dot{x}, \dot{x} \rangle \, dt \\ &= \langle \dot{x}, \delta_\nabla \dot{x} \rangle \wedge dt \\ &= \langle \dot{x}, \delta_\nabla dx \rangle \\ &= -\langle \dot{x}, d_\nabla \delta x \rangle \\ &= d\langle \dot{x}, \delta x \rangle + \langle d_\nabla \dot{x}, \delta x \rangle \\ &= -d\langle \dot{x}, \delta x \rangle - \langle \nabla_t \dot{x}, \delta x \rangle \wedge dt. \end{aligned} \tag{1.1}$$

In the third equation we use the fact that d, δ anticommute and the fact that ∇ is torsionfree.[8] From (1.1) we read off the variational 1-form

$$\gamma = \langle \dot{x}, \delta x \rangle \tag{1.2}$$

and the equation of motion

$$\nabla_t \dot{x} = 0. \tag{1.3}$$

The space of classical solutions, which are the critical points of the action, is then

$$W = \{x\colon\ \nabla_t \dot{x} = 0\}.$$

The image of any such x is a geodesic; it is parametrized at constant speed.

(b) We assume that M is complete. Then the standard theory of ordinary differential equations shows that there is a unique solution x to $\nabla_t \dot{x} = 0$ given an initial position and velocity, i.e., a point of TM.

(c) To compute the symplectic structure we differentiate (1.2):

$$\omega = \delta\gamma = \langle \delta_\nabla \dot{x} \wedge \delta x \rangle. \tag{1.4}$$

Now the identification of W with TM is the map

$$\begin{aligned} W &\longrightarrow TM \\ x &\longmapsto (x(t_0), \dot{x}(t_0)). \end{aligned}$$

If we identify TM and T^*M using the metric, it is easy to see that ω is the pullback of the standard symplectic form on T^*M.

(d) We follow the sign conventions in [I-**Classical Fields**]. Then the vector field v induced by d/dt is

$$v \cdot x = \iota(v)\delta x = -\dot{x}.$$

So using the equation of motion (1.3) we find

$$\begin{aligned} \iota(v)\omega &= \langle \nabla_t \dot{x}, \delta x \rangle - \langle \delta_\nabla \dot{x}, -\dot{x} \rangle \\ &= \langle \delta_\nabla \dot{x}, \dot{x} \rangle \\ &= -\delta j \end{aligned} \tag{1.5}$$

for

$$j = -\frac{1}{2}|\dot{x}|^2. \tag{1.6}$$

Equation (1.5) asserts that j is the function which generates v by Poisson brackets. But with these sign conventions the hamiltonian is minus the Noether charge of time translation (see [I-**Signs**]), and so

$$H = \frac{1}{2}|\dot{x}|^2.$$

[8] If $\phi\colon N \to M$ for any N, and ∇ is an arbitrary connection on M, then $D_\nabla D\phi = \phi^* T_\nabla$, where T_∇ is the torsion of ∇ and D the differential on N. In (1.1) we apply this to $N = \mathcal{F}_x \times \mathbb{R}$ and $D = \delta + d$. For a different interpretation of torsion, see the appendix to this solution.

Alternatively, we can compute j using the general formula for the Noether current:

$$\begin{aligned} j &= \iota(v + d/dt)(\mathcal{L} + \gamma) \\ &= \iota(v)\gamma + \iota(d/dt)\mathcal{L} \\ &= -|\dot{x}|^2 + \frac{1}{2}|\dot{x}|^2 \\ &= -\frac{1}{2}|\dot{x}|^2. \end{aligned}$$

Notice in this problem that we obtain for Noether currents constant functions on $\mathbb{R}$. The Noether charge is the constant value.

Appendix to FP1: On torsion (by Pierre Deligne)

An affine space A gives rise to a vector space V, the translations of A, and it is convenient (barycentric calculus) to view A and V as living in the same vector space B of dimension one more:

$$0 \to V \to B \xrightarrow{m} \mathbb{R} \to 0, \qquad A = m^{-1}(1) \ .$$

If A is an affine space bundle over a manifold M (structural group the affine group $\mathrm{GL}(n) \ltimes \mathbb{R}^n$), a connection ∇ on A allows us to define $\nabla_x a$, for a a section of A, as a section of the associated vector bundle (vector bundle of translations of A), structural group $\mathrm{GL}(n)$, obtained by applying $\mathrm{GL}(n) \ltimes \mathbb{R}^n \to \mathrm{GL}(n)$.

Let T^{aff} be the tangent bundle T to M, viewed as a bundle of affine spaces. The vector bundle T is the vector bundle associated to T^{aff}. A connection ∇ on T defines as follows a connection ∇^{aff} on T^{aff}: ∇^{aff} induces ∇ on T and for z the zero section of T^{aff}, one has $\nabla^{\mathrm{aff}}_x z = -x$ [connexion sans glissement].

The torsion of ∇ is the translation part of the curvature of ∇^{aff}. View x as a map of $\mathbb{R}^2$ to the zero section of the tangent bundle of M. If ∇ is torsion-free, vanishing of the translation part of the curvature gives

$$\begin{aligned} \nabla_t \nabla^{\mathrm{aff}}_u x - \nabla_u \nabla^{\mathrm{aff}}_t x = 0 \ , \qquad &\text{i.e.} \\ -\nabla_t \partial_u x + \nabla_u \partial_t x = 0 \end{aligned}$$

Solution to Problem FP2

For the special case where $M = \mathbb{R}$ (in a different notation!), see [I-**Supersolutions**, §1.3].

The space of maps $X\colon \mathbb{R}^{1|1} \to M$ is some kind of infinite dimensional supermanifold $\mathcal{F}_X$. We will not try to define it as a topological space provided with a sheaf of $\mathbb{R}$-algebras. We will only define it as a functor: For B a supermanifold, a map from B to $\mathcal{F}_X$ is by definition a morphism $\mathbb{R}^{1|1} \times B \to M$. It is also called a B-point of $\mathcal{F}_X$, and it is viewed as a family of points parametrized by B. In the exercise, the phrase "let X be a map from $\mathbb{R}^{1|1}$ to M" is an abuse of language for "let X be a map from $\mathbb{R}^{1|1}$ to M, depending on some parameter $b \in B$, for some superspace B", i.e. "let B be a supermanifold and let X be a map from $\mathbb{R}^{1|1} \times B$ to M".

This kind of interpretation of what a space of maps is does not apply only in the superworld. If we were to consider (as in **FP**1) the space of maps from $\mathbb{R}$ to M, one could try to put on it the structure of an infinite dimensional manifold but, for the kind of problem we are considering, this is extra baggage. What is important is to know, for B a space "of parameters" (now an ordinary manifold), what is a family of maps $\mathbb{R} \to M$ parametrized by B.

"Working in components" means reinterpreting the functor $B \mapsto B$-points of $\mathcal{F}_X$ as follows. A morphism $X\colon \mathbb{R}^{1|1} \times B \to M$ gives us

- a restriction $x\colon \mathbb{R} \times B \to M$. We view here $\mathbb{R}$ as the subvariety $\theta = 0$ of $\mathbb{R}^{1|1}$;
- an odd section $\psi := \frac{\partial}{\partial\theta} X\big|_{\mathbb{R}\times B}$ of the pullback by x of the tangent bundle T of M.

The construction $X \mapsto (x, \psi)$ is a bijection. Note that x^*T is an "even vector bundle" in the sense that it is of rank $(*, 0)$. This does not prevent it from having odd sections,for instance the product of the pullback of a section of T with an odd function on B.

This will usually be expressed with the abuse of language consisting in omitting B: A basis B is tacitly assumed to always be there, and one will say that to give a morphism X from $\mathbb{R}^{1|1}$ to M is the same thing as to give its restriction x to $\mathbb{R}$ and its derivative $\psi = \frac{\partial}{\partial\theta} X$ on $\mathbb{R}$ which is an odd section of x^*T. Taken literally, this statement would be useless: With no B tacitly assumed, the even vector bundle x^*T would have no nonzero odd section. When a B is present, one should beware that to give $x\colon \mathbb{R} \times B \to M$ is not just giving a morphism from $\mathbb{R} \times B_{\mathrm{red}}$ to M.

The lagrangian density is a map from $\mathcal{F}_X := \underline{\mathrm{Hom}}(\mathbb{R}^{1|1}, M)$ to the space of densities on $\mathbb{R}^{1|1}$. This is to be interpreted in its functorial meaning. A family of densities on $\mathbb{R}^{1|1}$, parametrized by B, is by definition a section on $\mathbb{R}^{1|1} \times B$ of the pullback of the line bundle of densities on $\mathbb{R}^{1|1}$. The lagrangian density attaches, functorially in B, to any family of maps from $\mathbb{R}^{1|1}$ to M parametrized by B, a family of densities on $\mathbb{R}^{1|1}$ parametrized by B.

"Families of densities" are also called "relative densities". An integration map is defined, which to a relative density on $V \times B$ over B with support proper over B attaches a function on B. If V is a supermanifold of dimension (p, q), the line bundle of relative densities is of dimension $(1, 0)$ or $(0, 1)$, depending on the parity of q, and the integration map is even.

The space of critical points of the lagrangian is a subspace of the space of all maps from $\mathbb{R}^{1|1}$ to M. As such, it is first to be defined as a functor: One has to define what it means for a map $X\colon B \to \underline{\mathrm{Hom}}(\mathbb{R}^{1|1}, M)$, i.e. $\mathbb{R}^{1|1} \times B \to M$, to be a map to the subspace S of $\underline{\mathrm{Hom}}(\mathbb{R}^{1|1}, M)$. The following two definitions are equivalent, with the second being closer to what is done in computations.

(i) For any supermanifold V, pointed by $0 \in V$, and any $X_V\colon B \times V \to \underline{\mathrm{Hom}}(\mathbb{R}^{1|1}, M)$, such that (a) it extends X: X is X_V restricted to $B \times 0$, (b) outside a compact region of $\mathbb{R}^{1|1}$, X_V coincides with X (X_V is a family of deformations of X, with compact support, parametrized by V), the following holds. Consider $\mathcal{L}(X_V) - \mathcal{L}(X)$, where $\mathcal{L}(X)$ is abuse of language for "pullback of $\mathcal{L}(X)$ by $\mathbb{R}^{1|1} \times B \times V \to \mathbb{R}^{1|1} \times B$. It is a relative density on $\mathbb{R}^{1|1} \times B \times V$ with support proper over $B \times V$. Integrate it, to get a function $\Delta \int \mathcal{L}$ on $B \times V$. One requires that the derivative of this function in the V-direction be zero on $B \times \{0\} \subset B \times V$.

(ii) In the second definition, one requires that after any change of base $B' \to B$, statement (i) holds for $V = (\mathbb{R}, 0)$.

This defines the space of stationary points as a functor. In the exercise at hand, and assuming M to be complete, this functor is representable: the space of stationary points "is" an ordinary supermanifold of dimension $(2 \dim M, \dim M)$, as we will see.

(a) The coordinates (t, θ) on $\mathbb{R}^{1|1}$ define a density $|dt|\, d\theta$. We prefer instead to orient the line and work with the Berezinian $dt\, d\theta$. The lagrangian "density" considered is then

$$\mathcal{L} = -\tfrac{1}{2}\, dt\, d\theta \langle \dot{X}, DX \rangle. \tag{2.1}$$

The minus sign is to make (2.5) below reasonable.

(b) The fastest computation follows the techniques developed in [I-**Supersolutions**]. Let $i\colon \mathbb{R} \hookrightarrow \mathbb{R}^{1|1}$ be the inclusion. Define the component fields

$$\begin{aligned} x &= i^* X \\ \psi &= i^* DX. \end{aligned}$$

Then the component lagrangian density is

$$\begin{aligned} \mathcal{L}' &= i^* D\big(-\frac{1}{2}\langle \dot{X}, DX \rangle\big)\, dt \\ &= \Big\{ i^*\big(-\frac{1}{2}\langle \nabla_D \dot{X}, DX \big)\rangle + \frac{1}{2} i^* \langle \dot{X}, \dot{X} \rangle \Big\}\, dt \\ &= \frac{1}{2}|\dot{x}|^2\, dt + \frac{1}{2}\langle \psi, \nabla_t \psi \rangle\, dt. \end{aligned}$$

In the second line we use $\nabla_D D = -\partial_t$.

There are many equivalent ways to compute. For example, if we choose local coordinates $\tilde{x}^i$ on M, a map $X\colon \mathbb{R}^{1|1} \to M$ can be written

$$X = (x^i(t) + \theta \psi^i(t)),$$

meaning $X^* \tilde{x}^i = x^i(t) + \theta \psi^i(t)$ with $x^i(t)$ (resp. $\psi^i(t)$) an even (resp. odd) function on $\mathbb{R}$, pulled back to $\mathbb{R}^{1|1}$ by $(t, \theta) \mapsto t$. Independently of the local coordinates, x is a map from $\mathbb{R}$ to M and ψ is an odd section of x^*T, for T the tangent bundle of M. The meaning of this, and in particular of "odd section of x^*T", was explained in the preface. The philosophy is that we tacitly work over a base B, functorially in B. The map x is the restriction of X to the subvariety $\mathbb{R}$ of $\mathbb{R}^{1|1}$ defined by $\theta = 0$, while ψ is the restriction of $\partial_\theta X$ to $\mathbb{R}$. On $\mathbb{R}$, the vector fields ∂_θ and D are equal, and it will often be more convenient to describe ψ as the restriction of DX to $\mathbb{R}$. We will compute the density $\mathcal{L}$ on $\mathbb{R}^{1|1}$ in term of (x, ψ).

In local coordinates, $\mathcal{L}$ is $-dt\, d\theta$ times

$$\tfrac{1}{2} X^*(g_{ij}) \partial_t (X^*(\tilde{x}^i)) D(X^*(\tilde{x}^j)). \tag{2.2}$$

As $\theta^2 = 0$, $X^*(g_{ij})$ is given by a finite Taylor series

$$X^*(g_{ij}) = g_{ij}(x(t) + \theta\psi(t)) = g_{ij} + \theta \psi^k \partial_k g_{ij}$$

with g_{ij}, $\partial_k g_{ij}$ and ψ^k evaluated at $x(t)$ and t. Expanding (2.2) gives

$$\begin{aligned}(2.3)\qquad &\tfrac{1}{2}(g_{ij}+\theta\psi^k\partial_k g_{ij})(\dot{x}^i+\theta\dot{\psi}^i)(\psi^j-\theta\dot{x}^j)\\ &\qquad=\tfrac{1}{2}g_{ij}\dot{x}^i\psi^j+\tfrac{1}{2}\theta(g_{ij}\dot{\psi}^i\psi^j+\partial_k g_{ij}\dot{x}^i\psi^k\psi^j-g_{ij}\dot{x}^i\dot{x}^j).\end{aligned}$$

As $\psi^k\psi^j=-\psi^j\psi^k$, we have $\partial_k g_{ij}\psi^k\psi^j=-\partial_j g_{ik}\psi^k\psi^j$. As $g_{ij}=g_{ji}$, we similarly have $\partial_i g_{jk}\psi^k\psi^j=0$, and

$$\partial_k g_{ij}\psi^k\psi^j=\tfrac{1}{2}(\partial_k g_{ij}-\partial_j g_{ik}+\partial_i g_{jk})\psi^k\psi^j=g_{\ell j}\Gamma^\ell_{ik}\psi^k\psi^j,$$

where the Γ are the Christoffel symbols defining the Levi-Civita connection. In (2.3), this allows us to rewrite

$$g_{ij}\dot{\psi}^i+\partial_k g_{ij}\dot{x}^i\psi^k=g_{ij}\dot{\psi}^i+g_{\ell j}\Gamma^\ell_{ik}\dot{x}^i\psi^k=g_{ij}(\nabla_t\psi)^i,$$

giving for (2.2)

$$\begin{aligned}(2.4)\qquad &\tfrac{1}{2}g_{ij}\dot{x}^i\psi^j+\tfrac{1}{2}\theta(g_{ij}(\nabla_t\psi)^i\psi^j-g_{ij}\dot{x}^i\dot{x}^j)\\ &\qquad=\tfrac{1}{2}(\dot{x},\psi)+\tfrac{1}{2}\theta((\nabla_t\psi,\psi)-(\dot{x},\dot{x})).\end{aligned}$$

Here is a shortcut from (2.3) to (2.4). Write (2.2) in the form $f(t)+\theta g(t)$. We have to compute $f(t_0)$ and $g(t_0)$ for each t_0. We now remember that we always work over a space B of parameters. The point t_0 is in fact a section of $\mathbb{R}\times B\to B$. The local coordinates $\tilde{x}_i$ used can be taken to vary with the parameters. This makes it possible to choose them so that at $x(t_0)$, i.e. along the section $x(t_0)$ of $M\times B\to B$, the metric tensor g_{ij} has vanishing first derivatives. The Christoffel symbols then vanish at $x(t_0)$, and (2.4) at t_0 results immediately from (2.3).

Here is a coordinate-free derivation of (2.4). We use that for any function F on $\mathbb{R}^{1|1}$, if we write $F(t,\theta)=f_1(t)+\theta f_2(t)$, we have

$$\begin{aligned}f_1&=F\text{ restricted to }\mathbb{R}\\ f_2&=DF\text{ restricted to }\mathbb{R}\ .\end{aligned}$$

We have $\psi=DX$ restricted to $\mathbb{R}$, so that for $F=\langle\dot{X},DX\rangle$, f_1 is $\langle\dot{x},\psi\rangle$, while f_2 is the restriction to $\mathbb{R}$ of

$$D\langle\dot{X},DX\rangle=\langle\nabla_D\dot{X},DX\rangle+\langle\dot{X},\nabla_D DX\rangle.$$

The Levi-Civita connection is torsion-free, and D and ∂_t commute, while $D^2=\frac{1}{2}[D,D]=-\partial_t$. It follows that

$$\begin{aligned}\nabla_D\dot{X}&=\nabla_t DX\\ \nabla_D DX&=\tfrac{1}{2}(\nabla_D DX+\nabla_D DX)=-\partial_t X\end{aligned}$$

and $D\langle\dot{X},DX\rangle$, restricted to $\mathbb{R}$, is

$$\langle\nabla_t\psi,\psi\rangle+\langle\dot{x},-\dot{x}\rangle.$$

This agrees with (2.4).

Let $\mathcal{L}'$ be the density on $\mathbb{R}$ deduced from $\mathcal{L}$ by integrating along the fibres of the projection from $\mathbb{R}^{1|1}$ to $\mathbb{R}$ given by $(t,\theta)\mapsto t$. From (2.4), we get

$$\mathcal{L}' = \tfrac{1}{2}\langle \dot{x},\dot{x}\rangle dt - \tfrac{1}{2}\langle \nabla_t\psi,\psi\rangle dt, \tag{2.5}$$

which agrees with our first computation.

The space of critical points is the same for $\mathcal{L}$ and $\mathcal{L}'$: it is the space of $X=(x,\psi)$ such that, for any deformation with compact support $X(u)$ of X, one has $\int \partial_u\mathcal{L}=0$ (resp. $\int \partial_u\mathcal{L}'=0$) at $u=0$, and the density $\partial_u\mathcal{L}'$ on $\mathbb{R}$ is deduced by θ-integration from the density $\partial_u\mathcal{L}$ on $\mathbb{R}^{1|1}$.

(c) We compute both in superspace and directly in components. In superspace we work on $\mathcal{F}_X\times\mathbb{R}^{1|1}$ with differential $\delta+d$. Recalling from [I-**Supersymmetry**, §1.11] that $dt\,d\theta$ has odd parity and cohomological degree one, we compute

$$\begin{aligned}\delta\mathcal{L} &= \delta\Big\{-\frac{1}{2}dt\,d\theta\,\langle\dot{X},DX\rangle\Big\}\\ &= dt\,d\theta\,\Big\{\frac{1}{2}\langle\delta_\nabla\dot{X},DX\rangle+\frac{1}{2}\langle\dot{X},\delta_\nabla DX\rangle\Big\}\\ &= dt\,d\theta\,\Big\{-\langle\delta X,\nabla_t DX\rangle+\frac{1}{2}\partial_t\langle\delta X,DX\rangle+\frac{1}{2}D\langle\dot{X},\delta X\rangle\Big\}.\end{aligned} \tag{2.6}$$

In this computation we use the fact that ∇ is torsion-free. So the Euler-Lagrange equation is

$$\nabla_t DX = 0. \tag{2.7}$$

The variational 1-form γ satisfies: $\delta\mathcal{L}+d\gamma$ is linear over functions. It is computed from the last two terms of (2.6):

$$\begin{aligned}&dt\,d\theta\,\Big\{\frac{1}{2}\partial_t\langle\delta X,DX\rangle+\frac{1}{2}D\langle\dot{X},\delta X\rangle\Big\}\\ &\qquad= dt\,d\theta\,\Big\{\mathrm{Lie}(\partial_t)\frac{1}{2}\langle\delta X,DX\rangle+\mathrm{Lie}(D)\frac{1}{2}\langle\dot{X},\delta X\rangle\Big\}\\ &\qquad= \mathrm{Lie}(\partial_t)dt\,d\theta\,\frac{1}{2}\langle\delta X,DX\rangle-\mathrm{Lie}(D)dt\,d\theta\,\frac{1}{2}\langle\dot{X},\delta X\rangle\\ &\qquad= d\Big\{\iota(\partial_t)dt\,d\theta\,\frac{1}{2}\langle\delta X,DX\rangle-\iota(D)dt\,d\theta\,\frac{1}{2}\langle\dot{X},\delta X\rangle\Big\}.\end{aligned}$$

In the last step the expression in braces is an *integral form* of degree 0 (see [I-**Supersymmetry**, §3.12]). The variational 1-form is minus the expression in braces on the last line:

$$\gamma = -\iota(\partial_t)dt\,d\theta\,\frac{1}{2}\langle\delta X,DX\rangle+\iota(D)dt\,d\theta\,\frac{1}{2}\langle\dot{X},\delta X\rangle. \tag{2.8}$$

Now the equation of motion for ψ is the restriction of (2.7) to $\mathbb{R}$:

$$0 = i^*\nabla_t DX = \nabla_t i^*DX = \nabla_t\psi. \tag{2.9}$$

The equation of motion for x is the restriction of D applied to (2.7):

$$\begin{aligned} 0 &= i^*\nabla_D\nabla_t DX \\ &= i^*\nabla_D\nabla_D\partial_t X \\ &= -i^*\nabla_t\dot{X} + i^*\frac{1}{2}R(DX,DX)\dot{X} \\ &= -\nabla_t\dot{x} + \frac{1}{2}R(\psi,\psi)\dot{x}. \end{aligned} \tag{2.10}$$

Here we use the equation for the Riemannian curvature

$$R(D,D) = 2\nabla_D\nabla_D + 2\nabla_t.$$

The Bianchi identity asserts

$$R(\psi,\psi)\dot{x} = 2R(\psi,\dot{x})\psi,$$

so we can rewrite (2.9) as

$$\nabla_t\dot{x} = R(\psi,\dot{x})\psi. \tag{2.11}$$

Alternatively, we work directly in components, where we can borrow (1.1) from **FP**1. The other computation we need is

$$\begin{aligned} \delta\big(\frac{1}{2}\langle\psi,\nabla_t\psi\rangle\,dt\big) &= \delta\big(\frac{1}{2}\langle\psi,d_\nabla\psi\rangle\big) \\ &= \frac{1}{2}\langle\delta_\nabla\psi\wedge d_\nabla\psi\rangle + \frac{1}{2}\langle\psi,\delta_\nabla d_\nabla\psi\rangle. \end{aligned} \tag{2.12}$$

Integrating by parts in the second term, we have

$$\begin{aligned} \langle\psi,\delta_\nabla d_\nabla\psi\rangle &= -\langle\psi,d_\nabla\delta_\nabla\psi\rangle + \langle\psi,R(\delta x\wedge dx)\psi\rangle \\ &= -d\big(\langle\psi,\delta_\nabla\psi\rangle\big) + \langle d_\nabla\psi\wedge\delta_\nabla\psi\rangle + \langle\psi,R(\delta x\wedge dx)\psi\rangle \\ &= -d\big(\langle\psi,\delta_\nabla\psi\rangle\big) + \langle\delta_\nabla\psi\wedge d_\nabla\psi\rangle + \langle\delta x\wedge R(\psi,\psi)dx\rangle. \end{aligned} \tag{2.13}$$

Here we have used the symmetries of the Riemann curvature in passing to the last equation. Collecting our results we have

$$\delta\mathcal{L} = -d\gamma' + \langle\delta x, -\nabla_t\dot{x} + \frac{1}{2}R(\psi,\psi)\dot{x}\rangle\wedge dt + \langle\delta_\nabla\psi,\nabla_t\psi\rangle\wedge dt, \tag{2.14}$$

where

$$\gamma' = \langle\dot{x},\delta x\rangle + \frac{1}{2}\langle\psi,\delta_\nabla\psi\rangle. \tag{2.15}$$

Thus we recover the equations of motion (2.9) and (2.10).

Next, we show how to recover (2.15) by integrating (2.8) over the odd coordinate θ. Recall that this integration is accomplished by applying i^*D. More precisely, we integrate over the fibers of the projection

$$\pi\colon \mathcal{F}_X\times\mathbb{R}^{1|1}\longrightarrow\mathcal{F}_{\{x,\psi\}}\times\mathbb{R}^1.$$

Here $\mathcal{F}_{\{x,\psi\}}$ is the space of component fields, and it may be identified with $\mathcal{F}_X$. For the second term of (2.8) we use $D = \partial_\theta - \theta\partial_t$. The term with $\iota(\partial_\theta)$ does not contribute to the integral over θ, and so

$$\begin{aligned}\int_\pi \gamma &= -i^*D\Big\{\frac{1}{2}\langle \delta X, DX\rangle\Big\} + i^*D\Big\{\frac{1}{2}\theta\langle \dot X, \delta X\rangle\Big\}\\ &= i^*\Big\{-\frac{1}{2}\langle \delta_\nabla DX, DX\rangle + \frac{1}{2}\langle \delta X, \dot X\rangle\Big\} + i^*\Big\{\frac{1}{2}\langle \dot X, \delta X\rangle\Big\}\\ &= -\frac{1}{2}\langle \delta_\nabla \psi, \psi\rangle + \langle \delta x, \dot x\rangle,\end{aligned}$$

which agrees with (2.15).

The space Y of critical points of $\mathcal{L}$ or $\mathcal{L}'$ is hence the space of solutions (x,ψ) of (2.9), (2.10). The reduced space $Y_{\rm red}$ is obtained by imposing $\psi = 0$: It is the space found in **FP**1, the space of solutions $x(t)$ of $\nabla_t \dot x = 0$.

The normal bundle of $Y_{\rm red}$ in Y is found by linearizing (2.9), (2.10) around $\psi = 0$: it is the odd vector bundle whose (even) sections are the odd sections ψ of x^*T obeying $\nabla_t \psi = 0$.

The Cauchy data for (2.9) (2.10) are the data at some t_0 of $x(t_0)$, $\dot x(t_0)$ and $\psi(t_0)$. In term of X: X, $\dot X$ and DX at $(t_0, 0)$. If M is complete, Y maps isomorphically to the space of Cauchy data. This gives a description of Y as the vector bundle $T\times \Pi T$ over M (viewed as a space). In particular, we get a description of Y as an odd vector bundle over $Y_{\rm red}$. Warning: This structure, and even the corresponding map $Y \to Y_{\rm red}$, depend on the choice of t_0. (See the solution to **FP**6 for further discussion.)

(d) To compute in superspace we begin by differentiating (2.8):

$$\omega = \delta\theta = \iota(\partial_t) dt\, d\theta\, \frac{1}{2}\langle \delta X \wedge \delta_\nabla DX\rangle + \iota(D) dt\, d\theta\, \frac{1}{2}\langle \delta_\nabla \dot X \wedge \delta X\rangle, \tag{2.16}$$

where we use the fact that ∇ is torsionfree ($\delta_\nabla \delta X = 0$). Hence

$$\omega' = i^*D\Big\{\frac{1}{2}\langle \delta X \wedge \delta_\nabla DX\rangle + \frac{1}{2}\theta\langle \delta_\nabla \dot X \wedge \delta X\rangle\Big\}. \tag{2.17}$$

The second term is straightforward; the first is

$$\begin{aligned} &i^*\Big\{\frac{1}{2}\langle \delta_\nabla DX \wedge \delta_\nabla DX\rangle + \frac{1}{2}\langle \delta X \wedge \nabla_D \delta_\nabla DX\rangle\Big\}\\ &\qquad = i^*\Big\{\frac{1}{2}\langle \delta_\nabla DX \wedge \delta_\nabla DX\rangle + \frac{1}{2}\langle \delta X \wedge \delta_\nabla \nabla_D DX\rangle\\ &\qquad\qquad + \frac{1}{2}\langle \delta X \wedge R(DX, \delta X) DX\rangle\Big\}\\ &\qquad = \frac{1}{2}\langle \delta_\nabla \psi \wedge \delta_\nabla \psi\rangle + \frac{1}{2}\langle \delta_\nabla \dot x \wedge \delta x\rangle + \frac{1}{2}\langle \delta x \wedge R(\psi, \delta x)\psi\rangle.\end{aligned} \tag{2.18}$$

Hence

$$\omega' = \langle \delta_\nabla \dot x \wedge \delta x\rangle + \frac{1}{2}\langle \delta_\nabla \psi \wedge \delta_\nabla \psi\rangle + \frac{1}{4}\langle \psi, R(\delta x \wedge \delta x)\psi\rangle, \tag{2.19}$$

where we use the symmetries of R in the last step.

Alternatively, we can compute ω' directly in components by differentiating (2.15) In that computation we need the equation

$$\delta_\nabla \delta_\nabla = \frac{1}{2} R(\delta x \wedge \delta x).$$

(e) In superspace $\mathbb{R}^{1|1}$ we compute the Noether current for the vector field $\tau_Q = \partial_\theta + \theta\partial_t$ as

$$j = \iota(\tau_Q + w)(\mathcal{L} + \gamma), \tag{2.20}$$

where w is the *odd* vector field on $\mathcal{F}_X$ given as

$$w \cdot X = \iota(w)\delta X = -\tau_Q X.$$

We use the fact that τ_Q and D commute to compute

$$\iota(w)\gamma = -\iota(\partial_t)dt\, d\theta\, \frac{1}{2}\langle \tau_Q X, DX\rangle - \iota(D)dt\, d\theta\, \frac{1}{2}\langle \dot{X}, \tau_Q X\rangle, \tag{2.21}$$

and

$$\iota(\tau_Q)\mathcal{L} = -\frac{1}{2}\iota(\tau_Q)dt\, d\theta\, \langle \dot{X}, DX\rangle. \tag{2.22}$$

Now we integrate out θ using the fact that τ_Q and D agree along $\mathbb{R} \subset \mathbb{R}^{1|1}$. The partial integral of (2.21) is

$$\begin{aligned} i^* D\Big\{-\frac{1}{2}\langle \tau_Q X, DX\rangle - \frac{1}{2}\theta\langle \dot{X}, \tau_Q X\rangle\Big\} \\ = i^*\Big\{\frac{1}{2}\langle \nabla_{\tau_Q} DX, DX\rangle - \frac{1}{2}\langle \tau_Q X, \dot{X}\rangle - \frac{1}{2}\theta\langle \dot{X}, \tau_Q X\rangle\Big\} \\ = -\frac{3}{2}\langle \psi, \dot{x}\rangle. \end{aligned}$$

Similarly, the partial integral of (2.22) is $\frac{1}{2}\langle \psi, \dot{x}\rangle$, so summing up we find

$$j = -\langle \psi, \dot{x}\rangle.$$

The supercurrent Q is minus the Noether current (see [I-**Signs**]) so

$$Q = \langle \psi, \dot{x}\rangle.$$

The computation of $H = \frac{1}{2}|\dot{x}|^2$ as *minus* the Noether charge associated to infinitesimal time translation is similar, and we omit it.

The equation $\{Q, Q\} = -2H$ for Poisson brackets reflects the vector field bracket $[\tau_Q, \tau_Q] = 2\partial_t$; the signs are explained in [I-**Signs**].

Solution to Problem FP3

We will write $\mathbb{R}^{d+1}$ for $\mathbb{R}^n$, coordinates $t = x_0, x_1, \ldots, x_d$.

(a) In the Minkowski signature, $*$ is defined as follows:

1. Extend the given bilinear form on V to $\wedge^p V$, by

$$(v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p) = \det(v_i, w_j).$$

   Same for V^*, identified with V.
2. An orientation has to be chosen. Together with the metric, it defines a form ω of maximal degree: in standard coordinates, $dt \wedge dx_1 \wedge \ldots \wedge dx_d$.
3. The operation $*$ is defined by $\alpha \wedge * \beta = (\alpha, \beta) \cdot \omega$.

The lagrangian density $\mathcal{L}$ is

$$-\tfrac{1}{4e^2}\,(F, F) \cdot (\text{standard density } dv)$$

In this way of writing, no orientation is needed. However, in the computations which follow, we choose an orientation to identify densities and forms of maximal degree. The variation of the lagrangian density by $A \mapsto A + \delta A$ is

$$\begin{aligned} -\tfrac{1}{2e^2}\,(d\delta A, dA)dv &= -\tfrac{1}{2e^2}\, d\delta A \wedge * dA \\ = d\left(-\tfrac{1}{2e^2}\,\delta A \wedge * dA\right) &- \tfrac{1}{2e^2}\,\delta A \wedge d * dA; \end{aligned}$$

the variational derivative $\delta\mathcal{L}$ is $(-1/2e^2)\,\delta A \wedge d * dA$, giving as Euler-Lagrange equation for the extremals (a.k.a. equation of motion)

$$d * dA = 0,$$

i.e., for d^* the adjoint of d, $d^* dA = 0$.

(b) The general rule for defining ω_0 is as follows:

1. Pick a space-like hypersurface Γ (e.g. $t = 0$).
2. Let a be the d-form whose exterior derivative is used to justify the integration by parts leading to the Euler-Lagrange equations. Here,

$$a = -\tfrac{1}{2e^2}\,\delta A \wedge * dA.$$

   it is a 1-form on the space of fields, with values in d-forms, and $\delta A \mapsto a(\delta A)$ is linear over C^∞-functions.
3. Define a 1-form α_Γ on the space of fields by integrating a on Γ.
4. Restrict α_Γ to the space of extremals, and define

$$\omega_0 = d\alpha_\Gamma\ .$$

For this to make sense, one needs some control at spatial infinity. it is not unreasonable to consider only A which vanish at spatial infinity: we will see in (c) that if at $t = 0$, A and $\dot{A}$ vanish for $\left(\sum x_i^2\right)^{1/2} > R$, and if A obeys the equations of motion $d * dA = 0$, then some gauge transform of A vanishes for $\left(\sum x_i^2\right)^{1/2} > R + |t|$.

On the space of extremals vanishing at spatial infinity, the 1-form α_Γ makes sense. Its exterior derivative is

$$\omega_0(\delta_1 A, \delta_2 A) = \int_\Gamma -\tfrac{1}{2e^2}\,(\delta_2 A \wedge * d\delta_1 A - \delta_1 A \wedge * d\delta_2 A).$$

The space of extremals is an affine space, and ω_0 is translation invariant. The general lagrangian formalism ensures that this 2-form is independent of Γ.

(c) Gauge transformations $A \mapsto A + d\phi$ preserve the lagrangian density, hence the space of extremals. Of course, if one imposes on A to vanish at spatial infinity, one has on impose to ϕ to be locally constant at infinity (hence constant at spatial infinity if $d+1 \neq 2$). Suppose $d+1 \neq 2$: one can impose to ϕ to be zero at spatial infinity, by subtracting a constant. The formula for ω_0 makes it clear that if $A \mapsto A + \delta_1 A$ is an infinitesimal gauge transformation with support disjoint from Γ, then $\omega_0(\delta_1 A, \delta_2 A) = 0$. As Γ can be chosen as one pleases, it follows that ω_0 comes from the quotient by gauge transformations: decompose an infinitesimal gauge transformation δA_1 in gauge transformations with support in $t < 1$, $-1 < t < 1$ and $-1 < t$, and for each evaluate $\omega_0(\delta_1 A, \delta_2 A)$ using a Γ for which it obviously vanishes.

Direct proof: If B is a tangent vector to the space of extremals: $d * dB = 0$ and B vanishing at spatial infinity, then

$$\int_\Gamma (d\phi) \wedge * dB + B \wedge * d(d\phi) = \int_\Gamma d(\phi * dB) = 0\ .$$

To investigate the quotient of the space of extremals by the group of gauge transformation, we now fix a decomposition of space-time as space $\times$ time, coordinates $x_0 = t$, x_i. Using $\phi = \int_0^t A_0(x,\tau)d\tau$ (vanishing at spatial infinity if A does), one gauge transforms A into a new A with $A_0 = 0$. The equation $d * dA = 0$ becomes

$$\partial_0 \left(\sum \partial_i A_i\right) = 0 \tag{3.1}$$

$$\Box A = -d \sum \partial_i A_i \tag{3.2}$$

if one gives A at $t = 0$, (3.1) gives $f := \sum \partial_i A_i$ at all times: it is independent of t. If $\dot{A}$ is given at $t = 0$, one can then solve $\Box A = -df$ with the given Cauchy data $(A, \dot{A})$ at $t = 0$ (hyperbolicity of $\Box$). The support of the resulting A is contained in what can be reached by a time-like vector from the support of $(A, \dot{A})$ at $t = 0$.

We have

$$\begin{aligned} &\sum \partial_i A_i = f \quad \text{at } t = 0,\ f \text{ independent of } t \\ &\Box A_i = -\partial_i f \end{aligned}$$

It follows that

$$\Box\left(\sum \partial_i A_i - f\right) = \sum \partial_i \Box A_i - \Box f = \sum -\partial_i \partial_i f - \Box f = Df - Df = 0$$

and if $\frac{d}{dt}\,(\sum \partial_i A_i - f) = 0$ at $t = 0$, i.e. if (3.1) holds at $t = 0$, then $\sum \partial_i A_i - f = 0$ always, i.e. (3.1) holds.

It follows that we have a bijection

$$A \longmapsto (A, \dot{A}) \quad \text{at} \quad t = 0$$

from the space of extremals A (vanishing at spatial infinity), with $A_0 = 0$, to the space of Cauchy data $(A, \dot{A})$ at $t = 0$, vanishing at infinity with $A_0 = 0$, $\dot{A}_0 = 0$, $\sum \partial_i \dot{A}_i = 0$. The last condition is what keeps making sense, on the Cauchy data, of the equation of motion $d * dA = 0$.

Transported to the affine space of Cauchy data, the 2-form ω_0 becomes the translation invariant 2-form

$$(A, \dot{A}), (B, \dot{B}) \longmapsto \int (A, \dot{B}) - (\dot{A}, B) \tag{3.3}$$

(up to a constant). It is degenerate, as we still have to divide by the time-independent gauge transformations ϕ: we have vanishing for $(A, \dot{A}) = (d\phi, 0)$. Conversely, if $(A, \dot{A})$ is in the kernel, we have $\int (\dot{A}, B) = 0$ for any compactly supported B, hence $\dot{A} = 0$, and $\int (A, \dot{B}) = 0$ if $d^* \dot{B} = 0$. One can take $\dot{B} = d^* C$ and integration by parts gives $\int (dA, C) = 0$, hence $dA = 0$ and $A = d\phi$ for some ϕ.

This can be presented as follows: on the space of all $(A, \dot{A})$ at $t = 0$ ($d^* \dot{A} = 0$ not required), we have the symplectic structure (3.3). We took the symplectic quotient $L^\perp / L$ by the isotropic linear subspace L of the $(d\phi, 0)$.

Geometrically, the hyperbolicity modulo gauge of the equation of motion $d * dA = 0$ has the following meaning.

One thinks of the 1-form A as being a connection ∇ on the trivial $\mathbb{R}$-torsor (= principal $\mathbb{R}$-bundle) T. For electromagnetism, it might be better to use rather $\mathbb{R}/\mathbb{Z} = U(1)$, but if one wants a change of trivializations to be given by a function ϕ with values in $\mathbb{R}$, not $\mathbb{R}/\mathbb{Z}$, one needs $\mathbb{R}$. The gauge transform $A + d\phi$ of A is the 1-form describing the same connection ∇, in the trivialization ϕ of the torsor T.

Call "space" the hypersurface $t = 0$ and let U be a region of space. Suppose given a 1-jet P_1 of $\mathbb{R}$-torsor along U. If s_1 is a trivialization of P_1, any other trivialization can be written as $s_1 + \phi_1$, for ϕ_1 a 1-jet of function along U: $\phi_1 = (\phi, \dot{\phi})$ on U. Suppose a connection ∇_1 is given on P_1. Given a trivialization of P_1, it is the data of A_0, A_i, $\dot{A}_i$ ($i \neq 0$) in U (hence at $t = 0$). Suppose finally that what makes sense of the equation of motion holds. In coordinates:

$$\sum_{i \neq 0} \partial_i^2 A_0 - \partial_i \dot{A}_i = 0$$

This is the "Cauchy data".

Let V be the region of space-time which, in terms of causality

for $t > 0$: is influenced only by U;

for $t < 0$: influences only U.

Otherwise said: v is in V if all time-like trajectories through v hit $t = 0$ in U. For U the ball $\left(\sum x_i^2\right)^{1/2} < R$ of space, i.e. at $t = 0$, V is the region $\left(\sum x_i^2\right)^{1/2} < R - |t|$.

Hyperbolicity gives that the Cauchy data (P_1, ∇_1) extends uniquely to a torsor with connection obeying the equation of motion on V: an extension exists, any two extensions are isomorphic by an isomorphism whose first jet along U is the identity, and such an isomorphism is unique.

Example. If U is the region $\sum x_i^2 > R$, then V is the region $\sum x_i^2 > R + |t|$. If (P, ∇) obeys the equation of motion and has a 1-jet along U isomorphic to the trivial flat torsor, it follows that (P, ∇) is isomorphic to the trivial flat torsor on V.

In the preceding stories, one can replace "space" $t = 0$ by a space-like hypersurface. Proofs are similar. One begins by mapping V to U by sending $v = (t, x)$ to the point of U with the same x-coordinates, and by using the connection along the time lines to write the torsors with connection on V as pull-back, as torsors, of a torsor on U ("gauge transformation making $A_0 = 0$").

In this geometric language, the restriction to A which vanish at spatial infinity, means considering only (P, ∇) such that P admits a trivialization which is horizontal at spatial infinity: flatness at spatial infinity, and, for $d = 2$, trivial monodromy at spatial infinity and vanishing of the integral of the curvature on a space-like hypersurface, hence on all of them.

Solution to Problem FP4

The point of this exercise is that in three dimensions a Chern-Simons term gives a mass to gauge fields.

We follow the procedures developed in [I-**Classical Fields**]. We replace the affine space of circle connections modulo gauge equivalence by the vector space of (real) 1-forms modulo addition of an exact function.

(a) The lagrangian is

$$L = -\frac{1}{4e^2} d\alpha \wedge *d\alpha + \mu\alpha \wedge d\alpha.$$

Its variation is

$$\begin{aligned}\delta L &= \frac{1}{2e^2} d\delta\alpha \wedge *d\alpha + \mu\delta\alpha \wedge d\alpha + \mu\alpha \wedge d\delta\alpha \\ &= d\Big\{\frac{1}{2e^2}\delta\alpha \wedge *d\alpha - \mu\alpha \wedge \delta\alpha\Big\} + \delta\alpha \wedge \Big\{-\frac{1}{2e^2} d * d\alpha + 2\mu d\alpha\Big\}.\end{aligned}$$

So the variational 1-form (which we do not need in this problem) is

$$\gamma = -\frac{1}{2e^2}\delta\alpha \wedge *d\alpha + \mu\alpha \wedge \delta\alpha,$$

and the equation of motion is

$$d * d\alpha = 4\mu e^2 d\alpha.$$

This means that in some gauge

$$*d\alpha = 4\mu e^2 \alpha. \tag{4.1}$$

Choosing usual coordinates x^0, x^1, x^2 on Minkowski 3-space, we write

$$\alpha(x) = \alpha_0(x)dx^0 + \alpha_1(x)dx^1 + \alpha_2(x)dx^2, \tag{4.2}$$

and in terms of the the Fourier transform $\hat{\alpha}$ equation (4.1) is

$$4\mu e^2 \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = i \begin{pmatrix} 0 & -p_2 & p_1 \\ -p_2 & 0 & p_0 \\ p_1 & -p_0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix}.$$

Here $i = \sqrt{-1}$. Let I be the identity matrix and A the matrix on the right hand side of (4.2). Then solutions to (4.1) exist only where $\det(4\mu e^2 I - iA) = 0$, or equivalently

$$p_0^2 - p_1^2 - p_2^2 = (4\mu e^2)^2.$$

So the solutions form a massive representation of Poincaré with mass $4\mu e^2$.

(b) The line bundle is described explicitly by (4.2). We do the usual "little group" analysis. Consider first $\mu > 0$. Then at $(p_1, p_1, p_2) = (4\mu e^2, 0, 0)$ the matrix $4\mu e^2 I - iA$ is

$$4\mu e^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix}.$$

It has kernel spanned by

$$4\mu e^2 \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.$$

The little group is the group of rotations in the (p_1, p_2) plane, and rotation through angle θ acts as multiplication by $e^{-i\theta}$. Now if $\mu < 0$ we consider the point $(p_0, p_1, p_2) = (-4\mu e^2, 0, 0)$, and the same analysis shows that rotation through angle θ acts as multiplication by $e^{i\theta}$. So reversing the sign of μ gives the inverse bundle.

The one-particle Hilbert space is the space of sections of this complex line bundle over the positive mass hyperboloid. The little group has two nonisomorphic representations of spin 1, and both are realized here by including the Chern-Simons term. The sign of μ distinguishes the two representations. Also, both representations are allowed by CPT.

Solution to Problem FP5

(a) Here $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is a real scalar field. The lagrangian is

$$L = \frac{1}{2} d\phi \wedge *d\phi$$

with variation

$$\begin{aligned} \delta L &= -d\delta\phi \wedge *d\phi \\ &= -d(\delta\phi \wedge *d\phi) - \delta\phi \wedge d * d\phi. \end{aligned}$$

So the variational 1-form is

$$\gamma = \delta\phi \wedge *d\phi \tag{5.1}$$

and the equation of motion is

$$d * d\phi = 0.$$

This is the ordinary wave equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0. \tag{5.2}$$

The assertion that the general solution is

$$\phi(t,x) = \phi_R(t-x) + \phi_L(t+x) \tag{5.3}$$

follows from the fact that in terms of the Fourier transform $\hat{\phi}(\omega,k)$, equation (5.2) is

$$(\omega^2 - k^2)\hat{\phi}(\omega,k) = 0. \tag{5.4}$$

So the Fourier transform of a solution is supported on $\omega = \pm k$, which is (5.3). The local symplectic form is

$$\omega = \delta\gamma = \delta\phi \wedge *d\delta\phi,$$

and the global symplectic form Ω is the integral of ω over a spacelike hypersurface, say the set of points with $t = 0$. As the space of solutions $\mathcal{M}$ is linear, we identify tangent vectors to $\mathcal{M}$ with elements of $\mathcal{M}$. Then for two such elements ϕ_1, ϕ_2 we have

$$\Omega(\phi_1,\phi_2) = \int_{t=0} (\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1)\, dx,$$

where the dot denotes time derivative. For $\phi_1 = f(t-x)$, $\phi_2 = g(t+x)$ of compact support, we find

$$\Omega(\phi_1,\phi_2) = \int \left[f(-x)\frac{dg}{dx}(x) - \frac{df}{dx}(-x)g(x)\right] dx = \int d\left[f(-x)g(x)\right] = 0.$$

(b) We follow [I-**Supersolutions**, §2.6]. Introduce coordinates

$$y^+ = \frac{t+x}{2},$$
$$y^- = \frac{t-x}{2}.$$

Now a real spinor field

$$\psi(y^+,y^-) = \begin{pmatrix} \psi_+(y^+,y^-) \\ \psi_-(y^+,y^-) \end{pmatrix}$$

consists of two real functions, since the spin group $\mathrm{Spin}(1,1)$ has two inequivalent real representations. The Dirac operator is

$$\not{D}\psi = \begin{pmatrix} 0 & -\partial_+ \\ \partial_- & 0 \end{pmatrix}\psi,$$

where $\partial_\pm = \partial/\partial y^\pm$. So a solution is of the form

$$\psi(y^+,y^-) = \begin{pmatrix} \psi_+(y^+) \\ \psi_-(y^-) \end{pmatrix};$$

it consists of a left-moving wave ψ_+ and a right-moving wave ψ_-. (Note that with the conventions of [I-**Supersolutions**], a *dual* spinor field λ which satisfies the Dirac equation decomposes into a right-moving wave λ^+ and a left-moving wave λ^-.)

Solution to Problem FP6

(a) Let P be the subbundle of the tangent bundle T of W generated by D, and P_2 be generated by D and $D^2 := \frac{1}{2}[D, D]$. We have

$$B_- \oplus P_2 \xrightarrow{\sim} T$$

and by the Frobenius pairing, $P_2/P \simeq P \otimes P$. The filtration $T \supset P_2 \supset P$ of T, with successive quotients B_-, P_2/P and P, gives a canonical isomorphism

$$\begin{aligned}\mathrm{Ber}(T) &\simeq \mathrm{Ber}(B_-) \otimes \mathrm{Ber}(P_2/P) \otimes \mathrm{Ber}(P) \\ &\simeq B_- \otimes P_2/P \otimes P^\vee.\end{aligned}$$

As $P_2/P \simeq P^{\otimes 2}$, this gives

$$\mathrm{Ber}(T)^\vee = B_-^\vee \otimes P^\vee$$

and the lagrangian density

$$\left\langle dX|_{B^-}, dX|_P \right\rangle$$

is canonically a section of the bundle of volume elements.

(b) Not all W are locally isomorphic to the flat model given. Indeed, P_2 and B_- are integrable distributions, giving locally a decomposition of W as

$$W^{1|1} \times W^1.$$

The odd distribution P is in the tangent space of the fibers of the projection pr_2 to W^1. All fibers are identified with $W^{1|1}$ by the projection to $W^{1|1}$ and the given flat model corresponds to the case where the induced distributions on the fibers $\mathrm{pr}_2^{-1}(t) \xrightarrow{\sim} W^{1|1}$ are all equal.

Let us compute the Euler-Lagrange equations for the extremals for the "flat" W given. Because $D^2 = \partial_+$, the relevant trivialization of $\mathrm{Ber}(T)$ is the one corresponding to the basis $(\partial_-, \partial_+, D)$ or, what amounts to the same, corresponding to the basis $(\partial_u, \partial_v, \partial_\theta)$ (triangular change of basis). The lagrangian density is

$$\langle \partial_- X, DX \rangle \cdot (\text{volume form } [u, v, \theta] \text{ given by the coordinates}).$$

If we let X depend on an additional even parameter a, the variation $\frac{d}{da}$ of the lagrangian density is given as follows at $a = 0$: let X be what X is at $a = 0$, and δX be $\delta_a X$: a section of $X^* T_M$. Defining derivatives of vectors using the Levi-Civita connection, using that this connection preserves $\langle\ ,\ \rangle$ and that it is torsion free (cf. **FP**1), we get at $a = 0$ that

$$\begin{aligned}\tfrac{d}{da}\mathcal{L} &= (\langle \partial_a \partial_- X, DX \rangle + \langle \partial_- X, \partial_a DX \rangle) \quad [u, v, \theta] \\ &= (\langle \partial_- \partial_a X, DX \rangle + \langle \partial_- X, D\partial_a X \rangle) \quad [u, v, \theta] \\ &= (\langle \partial_- \delta X, DX \rangle + \langle \partial_- X, D\delta X \rangle) \quad [u, v, \theta] \\ &= (\partial_- \langle \delta X, DX \rangle + D \langle \partial_- X, \delta X \rangle) \quad [u, v, \theta] \\ &\quad - (\langle \delta X, \partial_- DX \rangle - \langle D\partial_- X, \delta X \rangle) [u, v, \theta]\end{aligned}$$

where the first term is

$$d\left(\langle \delta X, DX\rangle\, \partial_- \perp [u,v,\theta] + \langle \partial_- X, \delta X\rangle\, D\perp [u,v,\theta]\right),$$

because the Lie derivative $\mathcal{L}_D([u,v,\theta])$ vanishes, and the second is

$$\delta\mathcal{L} = -2\,\langle \delta X, \partial_- DX\rangle\,[u,v,\theta].$$

This gives the equation of motion

$$\partial_- DX = 0. \tag{6.1}$$

If one works over a purely even parameter space B, a map $X\colon W \to M$ factors as

$$W \to W_{\mathrm{red}} \xrightarrow{x} M,$$

the first map being $(u,v,\theta) \to (u,v)$. Indeed, the even functions on $W \times B$ are just the functions on $W_{\mathrm{red}} \times B$, as $W \times B$ is of dimension $(*,1)$. The equation (6.1) amounts to

$$\partial_-\partial_+ x = 0 \tag{6.1$_{\mathrm{red}}$}$$

for x: the reduced space Y_{red} of the space Y of extremals is the space of "harmonic" maps from W_{red} (viewed as a quotient of W) to M. Beware that "harmonicity" is of a hyperbolic character, as we are in a Minkowski space setting.

Over a general parameter space B, a map $X\colon W \to M$ (i.e. a map from $W \times B$ to M) can be described by

- its restriction x to the subvariety $\theta = 0$ of W (i.e. its restriction to $W_{\mathrm{red}} \times B$ (not $(W \times B)_{\mathrm{red}}$);
- its derivative $\psi = \partial_\theta X$, restricted to $\theta = 0$. It is an odd section of x^*T_M.

If we use local coordinates on M, this means writing

$$X^i = x^i + \theta\psi^i,$$

with x^i and ψ^i depending only on u and v.

Let $\overline{\mathcal{L}}$ be obtained from the lagrangian density by integrating along the fibers of $(u,v,\theta) \to (u,v)$. It is

$$\overline{\mathcal{L}} = \left(-\langle \partial_- x, \partial_+ x\rangle + \langle \nabla_-\psi, \psi\rangle\right) du\, dv\ .$$

We can view $\overline{\mathcal{L}}$ as a lagrangian density for the space of fields (x,ψ) on the (u,v)-space W_{red}: x a map to M, ψ an odd section of x^*T_M. For general reasons, the lagrangian density $\overline{\mathcal{L}}$ gives rise to the same space of extremals as $\mathcal{L}$ does, and to the same 2-form on the space of extremals.

As in **FP**2, the Euler-Lagrange equations for (x,ψ) are of the form

$$\nabla_-\partial_+ x = \text{ quadratic in } \psi \text{ (involving the curvature)}, \tag{6.2}$$

$$\nabla_-\psi = 0. \tag{6.3}$$

If the parameter space B is purely even, the odd section ψ of x^*T_M (on $W_{\mathrm{red}} \times B$) is zero, and one finds again that Y_{red} is a space of "harmonic" maps.

Over a general B, if I is the ideal defining B_{red} and if one works modulo I^2, (6.2) again reduces to $\nabla_-\partial_+ x = 0$. This gives the normal bundle of Y_{red} in Y: at x in Y_{red}, it is ΠV, for V the space of sections s of x^*TM such that $\nabla_- s = 0$.

Because of the quadratic terms in (6.2), I don't know any natural way to map Y to Y_{red}, to view Y as a bundle over Y_{red}. Taking $(x,\psi) \mapsto x = (x,0)$ will not do.

However, the quadratic terms in (6.2) don't spoil the kind of Cauchy data for which (6.2) (6.3) is a well posed system of equations. With the notations of (d): $u = t + x$, $v = t - x$, a Cauchy data consists in x, $\dot{x}$ and ψ at $t = 0$. In terms of $X\colon W \to M$, this is X and its derivatives $\dot{X}$ and $\partial_\theta X$ at $t = \theta = 0$. The projection $(x,\dot{x},\psi) \mapsto (x,\dot{x})$: Cauchy data for $Y_{\text{red}} \to$ Cauchy data for Y, gives a projection $Y \to Y_{\text{red}}$. It is, however, not "natural" in the sense that it is not invariant by time evolution: it depends on having chosen the line $t = 0$.

Suppose the parameter space B is of dimension $(*, b)$. Let I be the ideal defining B_{red} in B. To solve (6.2) (6.3) for a given Cauchy data, one can proceed as follows:

1. Solve (6.2) (6.3) over B_{red}. After pull-back to B_{red}, we have $\psi = 0$ for parity reasons and one has to solve for a "harmonic" map: $\nabla_-\partial_+ x_0 = 0$.
2. One can view x_0 as being over B/I^2: modulo I^2, the even part of $C^\infty(W \times B)$ is $C^\infty(W \times B_{\text{red}})$. One solves (6.2) (6.3) modulo I^2, using x_0 given by (6.2) and solving (6.3). This amounts to solving an equation $\nabla_-\psi = 0$ on $W_{\text{red}} \times B_{\text{red}}$, for ψ a section of $x_0^*T_M \otimes \text{pr}_2^*(I/I^2)$. We now have (x_1,ψ_1) modulo I^2.
3. If we already have (x_r,ψ_r) which is a solution $\mod I^{r+1}$, i.e. an r-jet of solution along $B_{\text{red}} \subset B$, we extend it in any way to (x^0,ψ^0) modulo I^{r+2}. Then correct (x^0,ψ^0) to a solution (x_{r+1},ψ_{r+1}), by a correction trivial modulo I^{r+1}. At each step, this amounts to solving on $W_{\text{red}} \times B_{\text{red}}$ an equation

$$\nabla_-\partial_+ x = \text{second member}, \qquad \text{or}$$
$$\nabla_-\psi = \text{second member},$$

with x or ψ in $x_0^*T_M \otimes \text{pr}_2^* I^{r+1}/I^{r+2}$. This process terminates when $r = b$.

Assuming M complete, and some control at spatial infinity, we can now identify the space Y of extremals with the space of Cauchy data.

Solution to Problem FP7

As this problem is a continuation of **FP6**, so are these brief comments.

(a) Following the solution to **FP6**(a), we see that the lagrangian $\langle\Lambda, D\Lambda\rangle$ is canonically a section of

$$\left(B_-^{1/2}\right)^\vee \otimes \left(P^\vee \otimes (B_-^{1/2})^\vee\right) \cong P^\vee \otimes B_-^\vee,$$

so is a volume form.

(b) We work in the flat model. Let $i\colon \mathbb{R}^2 \hookrightarrow \mathbb{R}^{2|1}$ be the inclusion. Define component fields

$$\lambda = i^*\Lambda$$
$$G = i^*D\Lambda.$$

The integral over θ of the lagrangian is

$$\begin{aligned} i^*D\big(\langle\Lambda, D\Lambda\rangle\big) &= i^*\big(\langle D\Lambda, D\Lambda\rangle - \langle\Lambda, \partial_-\Lambda\rangle\big) \\ &= \langle G, G\rangle - \langle\lambda, \partial_-\lambda\rangle. \end{aligned}$$

So G is an auxiliary field—its equation of motion is the algebraic equation $G = 0$—and the equation of motion for λ is $\partial_-\lambda = 0$. A solution is determined by its initial value, which is a section of $X^*(Q)$ over $\mathbb{R}$.

Solution to Problem FP9

It should have been added to the assumptions on $(W^{2|2}, \Lambda_+, \Lambda_-)$ that the Frobenius pairing

$$[\quad,\quad]\colon \Lambda_+ \otimes \Lambda_- \to T/(\Lambda_+ \oplus \Lambda_-)$$

vanishes. If this condition holds, there are local coordinates $(u, v, \theta_+, \theta_-)$ in which Λ_+ and Λ_- are generated by D_+, D_- as in the given formulae, with the "i" omitted. (We use the sign conventions of [I-**Signs**].)

Proof. If D_+ generates Λ_+, D_+ and D_+^2 generates an integrable distribution, as results from the identity $[D_+, D_+^2] = 0$. The distributions $\langle D_+, D_+^2\rangle$ and $\langle D_-, D_-^2\rangle$ are integrable and, by assumption, transversal. They correspond locally to a decomposition

$$W^{2|2} = W_+^{1|1} \times W_-^{1|1}\ .$$

The assumption that $[\Lambda_+, \Lambda_-]$ be zero modulo $\langle D_+, D_+^2, D_-\rangle$ amounts to saying that, locally, Λ_- comes from a distribution on the factor $W_-^{1|1}$. Hence assume that Λ_+ and Λ_- come from distributions, still denoted Λ_+ and Λ_-, on $W_+^{1|1}$ and $W_-^{1|1}$.

To find local coordinates in which Λ_+ and Λ_- are as given, one can consider separately $(W_+^{1|1}, \Lambda_+)$ and $(W_-^{1|1}, \Lambda_-)$. Write (W, Λ) for $(W_+^{1|1}, \Lambda_+)$ or $(W_-^{1|1}, \Lambda_-)$. It remains to choose an odd vector field D generating Λ and to apply the

Lemma. *Consider the systems (W, D): W of dimension $(1|1)$, D an odd vector field such that D and D^2 span the tangent bundle T. They are all locally isomorphic.*

Proof. The Lie algebra F freely generated by an odd generator x is spanned by x and x^2, as is shown by the identity $[x, x^2] = 0$. Let G be the corresponding simply connected group. The Lie algebra of infinitesimal diffeomorphisms of W is the Lie algebra of vector fields, with the opposite of the usual bracket. Let ρ be the morphism $F \to \operatorname{Lie}\operatorname{Diff}(W)$ for which $\rho(x) = D$. It integrates to a local action of G on W. For w a point of W, the map $g \mapsto gw$ is an isomorphism from a neighborhood of e in G to one of w in W. By construction, D corresponds to the right-invariant vector field x on G.

Remark. The formulae given, in local coordinates, for D_+ and D_- show that D_+ and D_- commute: the distributions Λ_+ and Λ_- are locally generated by commuting vector fields D_+ and D_-. Other commuting vector fields D_+, D_- generating Λ_+ and Λ_- are of the form f_+D_+, f_-D_-, with $D_-f_+ = D_+f_- = 0$.

Remark. The discussion up to now works as well in a relative setting, over a base B. What comes next does not: As S is of dimension $(*,1)$, we have a canonical projection $W \to W_{\mathrm{red}}$. The quotient $\mathcal{O}_{\mathrm{red}}$ of $\mathcal{O}_W$ is an isomorphic image of the subsheaf $\mathcal{O}^+$ of $\mathcal{O}$. For $W^{2|2} = W^{1|1} \times W^{1|1}$, this gives a canonical retraction

$$p\colon W^{2|2} \to W_{\mathrm{red}}^{2|2} .$$

By integration along the fibers of p, a density on $W^{2|2}$ projects to a density on $W_{\mathrm{red}}^{2|2}$. In local coordinates, it is given by

$$p_*(f \cdot du\, dv(d\theta_+ d\theta_-)^{-1}) = (D_- D_+ f) du\, dv$$

where $D_-\ D_+\ f$ is to be restricted to $W_{\mathrm{red}}^{2|2}$ (coordinates u, v), defined in $W^{2|2}$ by the vanishing of θ_+ and θ_-.

Of course, once we have p, it persists after change of base: we have a morphism induced by p from $W^{2|2} \times B$ to $W_{\mathrm{red}}^{2|2} \times B$. The formula for p_* remains valid.

To give $\phi\colon W^{2|2} \to M$ amounts to giving its restriction X to $W_{\mathrm{red}}^{1|1}$, $\psi_\pm = D_\pm X$: odd sections of the pull-back X^*TM of the tangent bundle of M, and $F = \nabla_{D^+} D_- X = -D_- \nabla_{D^+} X$, an even section of this pull-back. For the abuse of language used, see **FP**2. Here D_+ and D_- are commuting generators of C_+ and D_-. If $\Lambda_\pm^0$ is the restriction of $\Lambda_\pm$ to $W_{\mathrm{red}}^{1|1}$, independently of a choice of $D_\pm$, one can see $\psi_\pm$ and F as even sections of $(\Lambda_\pm)^\vee \otimes X^*TM$ and $(\Lambda_+ \otimes \Lambda_-)^\vee \otimes X^*TM$. Using X, $\psi_\pm$ and F instead of ϕ, and $p_*\mathcal{L}$ instead of $\mathcal{L}$, is working "in components".

Remark to (b). If to a lagrangian density $\mathcal{L}(\phi)$ is added a density of the form $d\mathcal{M}(\phi)$, with $\mathcal{M}(\phi)$ a codimension 1 integral form attached locally to ϕ, the space of stationary points does not change: we are to consider deformations with compact support ϕ_u of ϕ, and

$$\int d\mathcal{M}(\phi_u) - d\mathcal{M}(\phi) = \int d(\mathcal{M}(\phi_u) - \mathcal{M}(\phi)) = 0$$

as $\mathcal{M}(\phi_u) - \mathcal{M}(\phi)$ has compact support.

The 1-form attached to a space-like hypersurface Γ changes, but the 2-form on the space of stationary points does not: the 1-form changes by

$$\delta\phi \longmapsto \int_\Gamma \partial_u \mathcal{M}(\phi + \delta\phi) ,$$

the differential of $\phi \mapsto \int_\Gamma \mathcal{M}(\phi)$.

After these preliminaries, we are ready to solve the problem

(a) The bundle of volume forms is canonically isomorphic to $\Lambda_+^{-1} \otimes \Lambda_-^{-1}$. To see this, note that the Frobenius pairing is an isomorphism

$$\mathrm{Fr}\colon \Lambda_+ \otimes \Lambda_+ \oplus \Lambda_- \otimes \Lambda_- \longrightarrow TW/\Lambda_+ \oplus \Lambda_- ,$$

and from the exact sequence

$$0 \to \Lambda_+ \oplus \Lambda_- \to TW \to TW/\Lambda_+ \oplus \Lambda_- \to 0$$

we conclude

$$\begin{aligned}\mathrm{Vol}(TW) &\simeq \mathrm{Vol}(\Lambda_+ \oplus \Lambda_-) \otimes \mathrm{Vol}(TW/\Lambda_+ \oplus \Lambda_-)\\ &\simeq (\Lambda_+ \otimes \Lambda_-) \otimes \mathrm{Vol}(\Lambda_+^{\otimes 2} \oplus \Lambda_-^{\otimes 2})\\ &\simeq (\Lambda_+ \otimes \Lambda_-) \otimes (\Lambda_+^{\otimes -2} \otimes \Lambda_-^{\otimes -2})\\ &\simeq \Lambda_+^{-1} \otimes \Lambda_-^{-1}\end{aligned}$$

The lagrangian density is

$$\mathcal{L} = (D_+\Phi, D_-\Phi) D_+^{-1} \otimes D_-^{-1} \ .$$

The same argument shows that for a 2-form B,

$$\mathcal{L}'_B = B(D_+\Phi, D_-\Phi) D_+^{-1} \otimes D_-^{-1}$$

is well-defined.

For any vector field ξ we have $\mathrm{Lie}(\xi) = [d, \iota(\xi)] = d\iota(\xi) + i_\xi d$, relating the Lie derivative and the contraction, so if λ is an even 1-form, then

$$\begin{aligned} d\lambda(\xi_1, \xi_2) &= \pm \iota(\xi_2)\iota(\xi_1) d\lambda = \pm \iota(\xi_2)\left(\mathrm{Lie}(\xi_1)\lambda - d\iota(\xi_1)\lambda\right)\\ &= \left[\mathrm{Lie}(\xi_1)\iota(\xi_2) - \iota([\xi_1, \xi_2])\lambda\right] \mp \mathrm{Lie}(\xi_2)\iota(\xi_1)\lambda \end{aligned} \tag{9.1}$$

Here the sign $\pm$ is $-$ if both ξ_1, ξ_2 are odd and $+$ otherwise. We apply this to $\mathcal{L}'_B$ for $B = d\Lambda$:

$$\begin{aligned} \mathcal{L}'_{d\Lambda} &= d\Lambda(D_+\Phi, D_-\Phi)\Delta\\ &= -\iota(D_-)\iota(D_+)\Phi^*\Lambda \cdot \Delta\\ &= \left\{\mathrm{Lie}(D_+)\left(\iota(D_-)\Phi^*\Lambda\right) + \mathrm{Lie}(D_-)\left(\iota(D_+)\Phi^*\Lambda\right)\right\}\Delta\\ &= \mathrm{Lie}(D_+)\left(\iota(D_-)\Phi^*\Lambda \cdot \Delta\right) + \mathrm{Lie}(D_-)\left(\iota(D_+)\Phi^*\Lambda \cdot \Delta\right)\\ &= -d\Big\{\iota(D_-)\Phi^*\Lambda \cdot \iota(D_+)\Delta + \iota(D_+)\Phi^*\Lambda \cdot \iota(D_-)\Delta\Big\} \ . \end{aligned} \tag{9.2}$$

In this computation $\Delta = D_+^{-1} \otimes D_-^{-1}$, and we made use of the fact that $\mathrm{Lie}(D_\pm)(\Delta) = 0$.

(b) Let i: $M^2 \hookrightarrow M^{2|2}$ be the inclusion. One has $i^*(\theta^\pm) = 0$. If p is the projection $(u, v, \theta_+, \theta_-) \to (u, v)$ of $M^{2|2}$ to M^2, and Δ the density $du\, dv(d\theta_+ d\theta_-)$, we have

$$p_*(f\Delta) = \iota^*(D_+D_-f) du\, dv.$$

For $\mathcal{L}$ we have, omitting $du\, dv$ for convenience:

$$\begin{aligned} p_*(\mathcal{L}) &= i^*D_+D_-\mathcal{L} = i^*D_+D_-(D_+\Phi, D_-\Phi)\\ &= i^*D_+\left\{(\nabla_-D_+\Phi, D_-\Phi) - (D_+\Phi, \nabla_-D_-\Phi)\right\}\\ &= i^*\{(\nabla_+\nabla_-D_+\Phi, D_-\Phi) + (\nabla_-D_+\Phi, \nabla_+D_-\Phi)\\ &\qquad - (\nabla_+D_+\Phi, \nabla_-D_-\Phi) + (D_+\Phi, \nabla_+\nabla_-\Phi)\}\\ &= i^*\{(R(D_+\Phi, D_-\Phi)D_+\Phi, D_-\Phi) - (\nabla_-\nabla_+D_+\Phi, D_-\Phi) +\\ &\quad + (\nabla_+D_-\Phi, \nabla_+D_-\Phi) - (\nabla_+D_+\Phi, \nabla_-D_-\Phi) + (D_+\Phi, \nabla_+\nabla_-D_-\Phi)\}\\ &= (R(\psi_+, \psi_-)\psi_+, \psi_-) + (i^*\nabla_-\partial_v\Phi, \psi_-) + (F, F) - (\partial_v X, \partial_u X)\\ &\qquad - (\psi_+, i^*\nabla_+\partial_u\Phi)\\ &= (R(\psi_+, \psi_-)\psi_+, \psi_-) + (\nabla_v\psi_-, \psi_-) + (F, F) - (\partial_v X, \partial_u X) - (\psi_+, \nabla_u\psi_+) \end{aligned}$$

Here we modified the notation of the problem to remove the factors of $\sqrt{-1}$:

$$D_+ = \frac{\partial}{\partial\theta_+} - \theta_+ \frac{\partial}{\partial v}$$
$$D_- = \frac{\partial}{\partial\theta_-} - \theta_- \frac{\partial}{\partial u}$$

and we expressed ϕ in components:

$$\begin{aligned} X &= i^*\Phi \\ \psi_+ &= i^* D_+ \Phi \\ \psi_- &= i^* D_- \Phi \\ F &= i^* \nabla_+ D_- \Phi \ . \end{aligned}$$

Also, $\nabla_\pm$ denotes the covariant derivative with respect to $D_\pm$.

We compute how $\mathcal{L}$ changes by an *even* deformation of Φ:

$$\begin{aligned} \delta\mathcal{L} = \delta(D_+\Phi, D_-\Phi) &= (\nabla_\delta D_+\Phi, D_-\Phi) + (D_+\Phi, \nabla_\delta D_-\Phi) \\ &= (\nabla_+\delta\Phi, D_-\Phi) + (D_+\Phi, \nabla_-\delta\Phi) \\ &= D_+(\delta\Phi, D_-\Phi) - D_-(D_+\Phi, \delta\Phi) - 2(\nabla_+ D_-\Phi, \delta\Phi) \end{aligned}$$

The first two terms are exact, and the equation of motion is

$$\nabla_+ D_-\Phi = 0 \ . \tag{9.3}$$

We evaluate the components of (9.3) by applying i^*, $i^*\nabla_+$, $i^*\nabla_-$, and $i^*\nabla_+\nabla_-$. The results are:

$$\begin{aligned} i^*(\nabla_+ D_-\Phi) &= F \ . \\ i^*\nabla_+(\nabla_+ D_-\Phi) &= R(\psi_+, \psi_-)\psi_+ + \nabla_v\psi_- \\ i^*\nabla_-(\nabla_+ D_-\Phi) &= R(\psi_-, \psi_+)\psi_- + \nabla_u\psi_+ \end{aligned}$$

We give details for the last of these computations:

$$\begin{aligned} i^*\nabla_+\nabla_-(\nabla_+ D_-\Phi) &= i^*\nabla_+ \{R(D_-\Phi, D_+\Phi)D_-\Phi - \nabla_+\nabla_- D_-\Phi\} \\ &= i^*\nabla_+ \left\{-\tfrac{1}{2} R(D_-\Phi, D_-\Phi)D_+\Phi - \nabla_+\nabla_- D_-\Phi\right\} \\ &= i^*\{(\nabla_+ R)(D_-\Phi, D_+\Phi)D_+\Phi - R(\nabla_+ D_-\Phi, D_-\Phi)D_+\Phi \\ &\quad - \tfrac{1}{2} R(D_-\Phi, D_-\Phi)\nabla_+ D_+\Phi + \nabla_u\partial_v\Phi - \tfrac{1}{2} R(D_+\Phi, D_+\Phi)\partial_v\Phi\} \\ &= (\nabla_{\psi_+} R)(\psi_-, \psi_-)\psi_+ - R(F, \psi_-)\psi_+ - \tfrac{1}{2} R(\psi_-, \psi_-)\partial_v X \\ &\qquad + \nabla_u\partial_v X - \tfrac{1}{2} R(\psi_+, \psi_+)\partial_v X \ . \end{aligned}$$

In the second line we use the Bianchi identity and in the third we use

$$\nabla_+\nabla_+ = -\nabla_u + \tfrac{1}{2} R(D_+, D_+) \ .$$

Of course, we can set $F = 0$ in the final expression because of the first equation of motion. Collating the results, we obtain the equations of motion in components:

$$\begin{aligned} F &= 0 \\ \nabla_v\psi_- &= -R(\psi_+, \psi_-)\psi_+ \\ \nabla_u\psi_+ &= -R(\psi_-, \psi_+)\psi_- \\ \nabla_u\partial_v X &= \tfrac{1}{2}\{R(\psi_-, \psi_-)\partial_v X + R(\psi_+, \psi_+)\partial_u X\} - (\nabla_{\psi_+} R)(\psi_-, \psi_-)\psi_+ \end{aligned}$$

(d) Let $j\colon M^{2|1} \hookrightarrow M^{2|2}$ so that j^* sets $\theta_- = 0$. Then

$$\begin{aligned} j^*D_-\mathcal{L} &= j^*D_-(D_+\Phi, D_-\Phi) \\ &= j^*\{(\nabla_-D_+\Phi, D_-\Phi) - (D_+\Phi, \nabla_-D_-\Phi)\} \\ &= j^*\{(\nabla_+D_-\Phi, D_-\Phi) + (D_+\Phi, \partial_u\Phi)\} \\ &= (\nabla_+\Lambda, \Lambda) + (D_+\Phi', \partial_u\Phi') \end{aligned}$$

where

$$\begin{aligned} j^*\Phi &= \Phi' \\ j^*D_-\Phi &= \Lambda\ . \end{aligned}$$

Solution to Problem FP10

(a) Let $W^{\pm}$ denote the $4|2$ supermanifold with complex odd directions ("cs-manifold") obtained by dividing W by $A_{\pm}$. Let $p_{\pm}\colon W \to W_{\pm}$ be the projection. If Φ is chiral and f holomorphic, then $\Phi^*f = p_+^*g$ for some function g on W^+. Let Δ denote the volume form on W induced from the volume forms on A_+ and A_-. (Note that the Frobenius pairing identifies $A_+ \otimes A_- \cong TM \,/\, (A_+ \oplus A_-)$.) Then

$$(p_+)_*(p_+^*g)\Delta = g\,(p_+)_*\Delta = 0.$$

(b) The $\partial\bar\partial$ lemma in Kähler geometry and part (a) imply that L is a function only of ω.

The computation of the reduced action in case M is Minkowski spacetime is given in [I-**Supersolutions**, §5.2].

(c) We follow the notation and normalizations of [I-**Supersolutions**]. As explained in [I-**Supersolutions**, §2.4], to compute the component lagrangian associated for the σ-model on $\mathbb{R}^{4|4}$ we use the formula

$$i^*\frac{1}{2}\big(D_1\overline{D}_{\dot 1}\overline{D}_{\dot 2}D_2 + D_2\overline{D}_{\dot 2}\overline{D}_{\dot 1}D_1\big),$$

where $i\colon \mathbb{R}^4 \hookrightarrow \mathbb{R}^{4|4}$ is the inclusion. This expression is real and can be rewritten as

$$i^*\frac{1}{4}\big(-D_1\overline{D}_{\dot 2}(\overline{D}_{\dot 1}D_2) - \overline{D}_{\dot 1}D_2(D_1\overline{D}_{\dot 2}) + D_2\overline{D}_{\dot 1}(D_1\overline{D}_{\dot 2} + 2\partial_{1\dot 2}) + \overline{D}_{\dot 2}D_1(\overline{D}_{\dot 1}D_2 + 2\partial_{2\dot 1})\big).$$

According to [I-**Supersolutions**, §5.2], the lagrangian of the $\mathbb{R}^{4|4}$ σ-model is $|d^4x|\,d^4\theta\,\frac{1}{2}\Phi^*(K)$. Now

$$\begin{aligned} D_1\overline{D}_{\dot 2}\Phi^*K &= D_1\iota(\overline{D}_{\dot 2}\overline{\Phi})\bar\partial K \\ &= -\iota(D_1\Phi)\iota(\overline{D}_{\dot 2}\overline{\Phi})\partial\bar\partial K + \iota\big([D_1\Phi, \overline{D}_{\dot 2}\overline{\Phi}]\big)\bar\partial K \\ &= -\frac{1}{2}\langle\overline{D}_{\dot 2}\overline{\Phi}, D_1\Phi\rangle - 2\iota(\partial_{1\dot 2}\Phi)\bar\partial K, \end{aligned}$$

and similarly

$$\overline{D}_{\dot{1}} D_2 \Phi^* K = \frac{1}{2}\langle \overline{D}_{\dot{1}} \overline{\Phi}, D_2 \Phi\rangle - 2\imath(\partial_{2\dot{1}}\Phi)\partial K.$$

Now restrict to fields invariant under $\partial/\partial x^3$. Then by [I-**Supersolutions**, §2.4], using the hat is used to denote vector fields on $\mathbb{R}^{3|2}$, we have

$$\begin{aligned} D_a = \overline{D}_{\dot{a}} &= \sqrt{2}\hat{D}_a \qquad \text{on } \mathbb{R}^{3|2}, \\ \partial_{1\dot{2}} = \partial_{2\dot{1}} &= \hat{\partial}_{12} \qquad \text{on } \mathbb{R}^{3|2}. \end{aligned}$$

Recalling $d = \partial + \bar{\partial}$ we find

$$\frac{1}{2} i^* \big(D_1 \overline{D}_{\dot{1}} \overline{D}_{\dot{2}} D_2 + D_2 \overline{D}_{\dot{2}} \overline{D}_{\dot{1}} D_1\big) \frac{\Phi^*(K)}{2} = i^* \frac{1}{2} (\hat{D}_2 \hat{D}_1 - \hat{D}_1 \hat{D}_2) \langle \hat{D}_1 \Phi, \hat{D}_2 \Phi\rangle.$$

According to [I-**Supersolutions**, §4.2] this is twice the lagrangian in $\mathbb{R}^{3|2}$. We expect the factor of 2 due to the different normalization of lagrangians for real and complex scalar fields.

Solution to Problem FP11

As a preliminary we evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{\imath\lambda x^2/2}, \qquad \lambda \in \mathbb{R}, \tag{11.1}$$

by viewing x as a complex variable and rotating the contour of integration. Now $e^{\imath\lambda x^2/2}$ decays at ∞ in the first and third quadrants if $\lambda > 0$, and it decays at ∞ in the second and fourth quadrants if $\lambda < 0$. Supposing $\lambda > 0$ we set $x = x' e^{\imath\pi/4}$ to obtain

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{\imath\lambda x^2/2} = e^{i\pi/4} \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}}\, e^{-\lambda x'^2/2} = \frac{e^{\imath\pi/4}}{\sqrt{\lambda}}. \tag{11.2}$$

For $\lambda < 0$ we set $x = x' e^{-\imath\pi/4}$ to obtain

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{\imath\lambda x^2/2} = \frac{e^{-\imath\pi/4}}{\sqrt{|\lambda|}}. \tag{11.3}$$

Since (11.1) is not a convergent integral, we can view (11.2) and (11.3) as definitions. An equivalent definition: Observe that the Fourier transform (in the sense of distributions) is a continuous function, and define the integral in terms of the value of the Fourier transform at 0. Combining (11.2) and (11.3), now for a quadratic form Q in n variables, we obtain

$$\int \frac{d^n x}{(2\pi)^{n/2}}\, e^{\imath Q(x,x)/2} = \frac{e^{(\pi\imath/4)\operatorname{sign} Q}}{\sqrt{|\det Q|}}. \tag{11.4}$$

Note that if Q is a quadratic form on a real vector space V, then $\det Q$ is an element in $(\operatorname{Det} V^*)^{\otimes 2}$, the square of the determinant line. To perform the integral we need

to trivialize this line, i.e., choose a particular Lebesgue measure on V. Of course, the signature sign Q is defined by Q without any additional choices.

Turning to the problem at hand, we first solve for the classical path $x_0(t)$, $0 \leq t \leq T$, with boundary conditions $x_0(0) = a$, $x_0(T) = b$. The Euler-Lagrange equation (Newton's law) is

$$\ddot{x} + \omega^2 x = 0.$$

So

$$x_0(t) = Ae^{\imath\omega t} + Be^{-\imath\omega t}$$

for some constants A, B. We find

$$A = \frac{e^{-i\omega T}a + b}{e^{i\omega T} - e^{-i\omega T}}, \qquad B = \frac{e^{i\omega T}a - b}{e^{i\omega T} - e^{-i\omega T}}.$$

The pole signals that for T a multiple of π/ω, a classical path may not exist: for a classical path $x(t)$, one has $x(t + \pi/\omega) = -x(t)$. Then

(11.5)
$$\begin{aligned} L(x_0) &= \int_0^T dt\ \frac{-m\omega^2}{2}\Big\{(Ae^{i\omega t} - Be^{-i\omega t})^2 + (Ae^{\imath\omega t} + Be^{-\imath\omega t})^2\Big\} \\ &= \frac{m}{2T}\Big\{(a^2 + b^2)\frac{\omega T}{\tan\omega T} - (2ab)\frac{\omega T}{\sin\omega T}\Big\}. \end{aligned}$$

Write $x = x_0 + \tilde{x}$, where $\tilde{x}(0) = \tilde{x}(T) = 0$. Then since x_0 is a critical point, and L is quadratic in x,

(11.6)
$$\begin{aligned} L(x_0 + \tilde{x}) &= L(x_0) + L(\tilde{x}) \\ &= L(x_0) + Q(\tilde{x}, \tilde{x})/2, \end{aligned}$$

where Q is the quadratic form

$$Q(\tilde{x}, \tilde{x}) = \int_0^T dt\ \tilde{x}A\tilde{x}, \qquad A = m\big(-\frac{d^2}{dt^2} - \omega^2\big).$$

The expression for $L(\tilde{x})$ follows from integration by parts, since $\tilde{x}(0) = \tilde{x}(T) = 0$.

The Feynman-Kac formula says

$$K = \langle b|e^{-iTH}|a\rangle = \int dx\ e^{iL(x)},$$

where the path integral on the right hand side is over paths x with $x(0) = a$ and $x(T) = b$. We understand this—at least formally—as an analytic continuation of the (Feynman-Kac) formula for the heat kernel as a path integral. From (11.6) we have

(11.7)
$$K = e^{iL(x_0)} \int d\tilde{x}\ e^{iQ(\tilde{x},\tilde{x})/2}.$$

We easily diagonalize A using the functions $\phi_n = \sin n\pi t/T$, $n = 1, 2, \ldots$:

$$\begin{aligned} A\phi_n &= \lambda_n\phi_n, \\ \lambda_n &= m\big(\frac{\pi^2 n^2}{T^2} - \omega^2\big). \end{aligned}$$

Note that there is a finite number of $\lambda_n < 0$. We assume that no $\lambda_n = 0$; if there were such a λ_n, then ωT would be an integer multiple of π and the classical solution x_0 would be ill-defined. Now we claim

$$\operatorname{sign} Q = -2 \cdot \#\{n : \lambda_n < 0\}.$$

This follows since the "total number" of λ_n, counted using a ζ-function, vanishes.[9] Then from (11.4),

$$\begin{aligned} \int d\tilde{x}\, e^{iQ(\tilde{x},\tilde{x})/2} &= \frac{e^{-\pi i/2 \cdot \#\{n:\lambda_n<0\}}}{\sqrt{|\det Q|}} \\ &= \frac{\pm 1}{\sqrt{\det Q}}. \end{aligned} \tag{11.8}$$

We do not see how to compute the sign without keeping track of the number of negative eigenvalues. A more accurate statement is that the square of the Gaussian integral in (11.8) is $1/\det Q$. In any case,

$$\begin{aligned} \det Q &= \prod_{n=1}^{\infty} m\Big(\frac{\pi^2 n^2}{T^2} - \omega^2\Big) \\ &= \prod_{n=1}^{\infty} \Big(\frac{m\pi^2}{T^2} n^2\Big) \prod_{n=1}^{\infty} \Big(1 - \Big(\frac{\omega T}{n\pi}\Big)^2\Big) \\ &= \frac{T}{\pi\sqrt{m}}\, 2\pi\, \frac{\sin \omega T}{\omega T}. \end{aligned} \tag{11.9}$$

We use the values $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$ of the Riemann ζ-function to evaluate the divergent product in the second line of (11.9). Putting together (11.5), (11.7), (11.8), and (11.9) we have

$$K = \frac{m^{1/4}}{\sqrt{2}} \frac{1}{\sqrt{T}} \Big(\frac{\omega T}{\sin \omega T}\Big)^{1/2} \exp\left[\frac{im}{2T}\Big\{(a^2+b^2)\frac{\omega T}{\tan \omega T} - (2ab)\frac{\omega T}{\sin \omega T}\Big\}\right].$$

This differs slightly from the correct (Mehler's) formula for the kernel of e^{-itH}: the initial constant $m^{1/4}/\sqrt{2}$ should be replaced by $\sqrt{m/2\pi}$.

Solution to Problem FP12

We solve this problem using dimensional regularization. The dimensions of the fields and parameters are

$$\begin{aligned} [\phi] &= \tfrac{n-2}{2} \\ [m] &= 1 \\ [g] &= \tfrac{6-n}{2} \end{aligned}$$

[9] If λ_n are the eigenvalues of a second order elliptic operator Δ in d dimensions, and

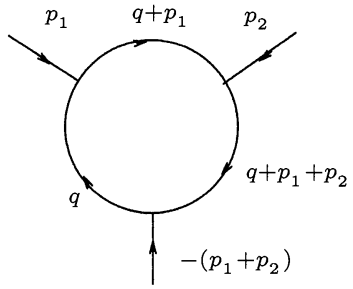
$$\zeta_\Delta(s) = \sum |\lambda_n|^{-s}$$

is the corresponding ζ-function, then $\zeta_\Delta(0) = 0$ if d is odd. We define $\operatorname{sign} Q$ using such ζ-functions to count positive and negative eigenvalues.

Introduce a parameter μ with the dimension of mass and a dimensionless parameter λ. Then the lagrangian density is

$$\mathcal{L} = \left\{ \tfrac{1}{2}\left(|d\phi|^2 + m^2\phi^2\right) + \lambda\mu^{\frac{6-n}{2}}\,\phi^3/3! \right\} d^n x\ .$$

To order λ^3 the 3-point function is computed by the diagram shown.



Here we are working in momentum space and label the edges of the diagram with momenta. Applying the Feynman rules for Euclidean space, we evaluate the diagram as

$$\Gamma = -\lambda^3\mu^{\frac{3}{2}(6-n)} \int \tfrac{d^n q}{(2\pi)^n} \tfrac{1}{(q^2+m^2)((q+p_1)^2+m^2)((q+p_1+p_2)^2+m^2)}$$

Now use Feynman parameters to rewrite the integral as

(12.1)

$$\begin{aligned}\Gamma_n &= -\lambda^3\mu^{\frac{3}{2}(6-n)} \int \tfrac{d^n q}{(2\pi)^n} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \tfrac{1}{[(1-\alpha-\beta)q^2+\alpha(q+p_1)^2+\beta(q+p_1+p_2)^2+m^2]^3} \\ &= -\lambda^3\mu^{\frac{3}{2}(6-n)} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \tfrac{d^n q}{(2\pi)^n} \tfrac{1}{[q^2+A^2]^3},\end{aligned}$$

where

$$A^2 = m^2 + \alpha(1-\alpha)p_1^2 + \beta(1-\beta)(p_1+p_2)^2 - 2\alpha\beta p_1(p_1+p_2). \tag{12.2}$$

the interchange of integrals is justified for n sufficiently small (here $n < 6$). Note that we made a shift in the integration variable q. Now the rules for integrals in n dimensions (n complex!) give

$$\begin{aligned}\int \tfrac{d^n q}{(2\pi)^n} \tfrac{1}{[q^2+a^2]^3} &= \tfrac{1}{(2\pi)^n} \tfrac{\pi^{n/2}}{\frac{1}{2}\Gamma(n/2)} \int_0^\infty dr\, r^{n-1} \tfrac{1}{[r^2+A^2]^3} \\ &= \tfrac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)} \cdot \tfrac{1}{2} \tfrac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(3-\frac{n}{2}\right)}{\Gamma(3)} A^{n-6} \\ &= \tfrac{\Gamma\left(3-\frac{n}{2}\right)}{2^{n+1}\pi^{n/2}} A^{n-6}.\end{aligned} \tag{12.3}$$

Here $\frac{\pi^{n/2}}{\frac{1}{2}\Gamma(n/2)}$ is the volume of the unit sphere in n dimensions, and for any N,

$$\begin{aligned}\int_0^\infty dr\, r^{n-1} \tfrac{1}{[r^2+A^2]^N} &= \int_0^\infty dr\, r^{n-1} \tfrac{1}{\Gamma(N)} \int_0^\infty dt\, e^{-(r^2+A^2)t} t^{N-1} \\ &= \tfrac{A^{n-2N}}{\Gamma(N)} \int_0^\infty \tfrac{dt}{t} \int_0^\infty dr\, e^{-r^2 t} r^{n-l} e^{-t} t^N \\ &= \tfrac{A^{n-2N}}{2\Gamma(N)} \int_0^\infty \tfrac{dt}{t} \int_0^\infty \tfrac{du}{u} e^{-u} u^{\frac{n}{2}} e^{-t} t^{N-n/2} \\ &= \tfrac{1}{2} \tfrac{\Gamma\left(\frac{n}{2}\right)\Gamma(N-n/2)}{\Gamma(N)} A^{n-2N}\ .\end{aligned}$$

The interchange of integrations is valid for n small. Now (12.3) is finite for $n < 6$ and has a simple pole at $n = 6$. Collecting (12.1) and (12.3), we find

$$\begin{aligned}\Gamma_n &= -\tfrac{\lambda^3}{2^{n+1}\pi^{n/2}}\Gamma\left(\tfrac{6-n}{2}\right)\mu^{\frac{3}{2}(6-n)}\int_0^1 d\alpha\int_0^{1-\alpha} d\beta A^{n-6}\\ &= -\tfrac{\lambda^3 B_n}{2^{n+1}\pi^{n/2}}\mu^{\frac{3}{2}(6-n)}\Gamma\left(\tfrac{6-n}{2}\right),\end{aligned}$$

where

$$B_n = \int_0^1 d\alpha\int_0^{1-\alpha} d\beta\left[m^2+\alpha(1-\alpha)p_1^2+\beta(1-\beta)(p_1+p_2)^2-2\alpha\beta p_1(p_1+p_2)\right]^{\frac{n-6}{2}}.$$

Note $B_6 = 1/2$ and B_n is a holomorphic function of n near $n = 6$. Introduce $\varepsilon = 3 - n/2$ and write

$$\Gamma_n = -\lambda^3\mu^\varepsilon\left\{\tfrac{1}{2(4\pi)^3}\Gamma(\varepsilon)(4\pi\mu^2)^\varepsilon B_n\right\}$$

and then expand around $\varepsilon = 0$:

$$\begin{aligned}\Gamma_n = -\lambda^3\mu^\varepsilon\cdot\tfrac{1}{2(4\pi)^3}\Big\{&\tfrac{1}{2\varepsilon}-\tfrac{\gamma}{2}\\ &-\int_0^1 d\alpha\int_0^{1-\alpha} d\beta\ln\Big(\tfrac{m^2+\alpha(1-\alpha)p_1^2+\beta(1-\beta)(p_1+p_2)^2-2\alpha\beta p_1(p_1+p_2)}{4\pi\mu^2}\Big)\Big\}+O(\varepsilon),\end{aligned}$$

where γ is Euler's constant.

To renormalize for $n = 6$, we introduce the new lagrangian density

$$\mathcal{L}' = \left\{\tfrac{1}{2}\left(|d\phi|^2+m^2\phi^2\right)+\lambda\left(1+\tfrac{\lambda^2}{2(4\pi)^3}\tfrac{1}{6-n}\right)\mu^{\frac{6-n}{2}}\tfrac{\phi^3}{3!}\right\}d^nx\ .$$

This introduces a diagram which cancels the pole in Γ_n at $n = 6$, and the new 3-point function to one loop is the finite part of Γ_n at $n = 6$:

$$\tfrac{\lambda^3}{2(4\pi)^3}\left\{\tfrac{\gamma}{2}+\int_0^1 d\alpha\int_0^{1-\alpha} d\beta\ln\left(\tfrac{m^2+\alpha(1-\alpha)p_1^2+\beta(1-\beta)(p_1+p_2)^2-2\alpha\beta p_1(p_1+p_2)}{4\pi\mu^2}\right)\right\}\ .$$

Solution to Problem FP14

For simplicity consider the equation $H\psi = \dfrac{k^2}{2m}\psi$ by changing the definition of k. Then

$$\psi(x) = \begin{cases} e^{ikx}+R(k)e^{-ikx}, & x<0;\\ T(k)e^{ikx}, & x>0.\end{cases} \tag{14.1}$$

is a solution for $x \neq 0$ with the desired asymptotic behavior. Continuity at $x = 0$ requires

$$1+R = T. \tag{14.2}$$

The limit of $\psi'(x)$ at $x = 0$ is different from the left and the right:

$$\lim_{x \nearrow 0} \psi'(x) = ik(1 - R)$$
$$\lim_{x \searrow 0} \psi'(x) = ik(1 + R)$$

So ψ'' is a smooth function plus the distribution $2ikR\delta(x)$. Imposing the Schrodinger equation at $x = 0$ we find, since $\psi(0) = 1 + R$,

$$-\frac{ikR}{m} + \lambda(1 + R) = 0. \tag{14.3}$$

Solving (14.2) and (14.3) we obtain

$$R(k) = \frac{m\lambda}{ik - m\lambda},$$
$$T(k) = \frac{ik}{ik - m\lambda}.$$

It is straightforward to check $|R|^2 + |T|^2 = 1$.

If $H\psi = -\epsilon\psi$ for $\epsilon > 0$, then the L^2 inner product satisfies $\langle H\psi, \psi\rangle < 0$. But

$$\langle H\psi, \psi\rangle = \int_{-\infty}^{\infty} \frac{1}{2m}|\psi'(x)|^2\, dx + \lambda|\psi(0)|^2$$

is nonnegative for $\lambda > 0$. So there is no bound state for $\lambda > 0$.

To solve $H\psi = -\epsilon\psi$ with $\epsilon > 0$ we try our solution (14.1) with $k = ia$ imaginary:

$$\psi(x) = \begin{cases} e^{a|x|} + R(ia)e^{-a|x|}, & x < 0; \\ T(ia)e^{-a|x|}, & x > 0. \end{cases}$$

This is never L^2 as written, but now divide by $T(ia)$. Then we obtain an L^2 solution precisely when $1/T(ia) = 0$ and $a > 0$. In other words, a bound state occurs when k is at a pole of $T(k)$:

$$k = ia = -im\lambda.$$

For this value of k, we find $\epsilon = -m\lambda^2/2$.

Solution to Problem FP15

(a) We should compute $\operatorname{Tr} e^{-TH}$ for

$$\frac{1}{2m}\left(-\frac{\partial^2}{\partial x^2} + m^2\omega^2 x^2\right).$$

First, we can compute directly in the Hamiltonian picture. Introduce

$$A = \frac{1}{\sqrt{2m}}\left(\frac{\partial}{\partial x} + m\omega x\right)$$
$$A^* = \frac{1}{\sqrt{2m}}\left(-\frac{\partial}{\partial x} + m\omega x\right)$$

Then $H = A^*A + \frac{\omega}{2}$ and $[A, A^*] = \omega$. So the spectrum of H is

$$\operatorname{Spec} H = \left\{\frac{\omega}{2} + n\omega\right\}_{n\in\mathbb{Z}^{\geq 0}}$$

and

$$\begin{aligned}\operatorname{Tr} e^{-TH} &= e^{-\omega T/2}\sum_{n=0}^{\infty} e^{-n\omega T} \\ &= e^{-\omega T/2}\frac{1}{1-e^{-\omega T}} \\ &= \frac{1}{2\sinh \omega T/2}\end{aligned} \tag{15.1}$$

To compute the heat kernel trace as a path integral we use the Feynman-Kac formula

$$\operatorname{Tr} e^{-TH} = \int e^{-L(X)} dX, \tag{15.2}$$

where $X\colon S^1(T) \to \mathbb{R}$ is a function on the circle $S^1(T) = \mathbb{R}/T\mathbb{Z}$, and

$$L(X) = \frac{m}{2}\int_0^T \left\{\dot{X}(t)^2 + \omega^2 X(t)^2\right\} dt$$

Since X is periodic we can integrate by parts to obtain

$$L(X) = (QX, X)/2 \tag{15.3}$$

$$Q = m\left(-\frac{\partial^2}{\partial t^2} + \omega^2\right) \tag{15.4}$$

where the inner product in (15.3) is the L^2 inner product on $S^1(T)$. So the path integral (15.2) is formally

$$\operatorname{Tr} e^{-TH} = \frac{1}{\sqrt{\det Q}} \tag{15.5}$$

Now Q has eigenfunctions $\{e^{2\pi i n t/T}\}_{n\in\mathbb{Z}}$ with corresponding eigenvalues $\left\{m\left(\frac{4\pi^2 n^2}{T^2} + \omega^2\right)\right\}_{n\in\mathbb{Z}}$. Therefore, the determinant is the divergent product

$$\begin{aligned}\det Q &= \prod_{n\in\mathbb{Z}} m\left(\frac{4\pi^2 n^2}{T^2} + \omega^2\right) \\ &= \prod_{n\in\mathbb{Z}^+}\left(\frac{4\pi^2 m}{T^2} n^2\right)^2 \prod_{n=1}^{\infty}\left(1 + \left(\frac{\omega T}{2\pi n}\right)^2\right)^2 \cdot m\omega^2 \\ &= \prod_{n\in\mathbb{Z}^+}\left(\frac{4\pi^2 m}{T^2} n^2\right)^2 \cdot \frac{\sinh^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2} \cdot m\omega^2 \ .\end{aligned}$$

We evaluate the infinite constants using the Riemann ζ-function:

$$\prod_{n\in\mathbb{Z}^+}\left(\frac{4\pi^2 m}{T^2}n^2\right)^2 = \left(\frac{4\pi^2 m}{T^2}\right)^{2\zeta(0)} e^{-4\zeta'(0)}$$
$$= \frac{T^2}{4\pi^2 m}4\pi^2 ,$$

since $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\ln 2\pi$. Altogether we have

$$\det Q = \left(2\sinh\frac{\omega T}{2}\right)^2 ,$$

and plugging in (15.5) we obtain agreement with (15.1).

(b) We first evaluate the path integral. We must diagonalize (15.4) with antiperiodic boundary conditions. So the eigenfunctions are $\{e^{2\pi i n t/T}\}_{n\in\mathbb{Z}+1/2}$ with corresponding eigenvalues $\left\{m\left(\frac{4\pi^2 n^2}{T^2}+\omega^2\right)\right\}_{n\in\mathbb{Z}+1/2}$. So

$$\det Q = \prod_{n\in\mathbb{Z}+1/2} m\left(\frac{4\pi^2 n^2}{T^2}+\omega^2\right)$$
$$= \prod_{n\in\mathbb{Z}+1/2}\left(\frac{4\pi^2 m}{T^2}n^2\right)\prod_{n\in\mathbb{Z}+1/2}\left(1+\left(\frac{\omega T}{2\pi n}\right)^2\right)^2 \tag{15.6}$$
$$= 4\cosh^2\frac{\omega T}{2} ,$$

from which the path integral is

$$\frac{1}{\sqrt{\det Q}} = \frac{1}{2\cosh\frac{\omega T}{2}}. \tag{15.7}$$

In (15.6) we use the fact that the ζ-normalized number of integers is zero to compute

$$\prod_{n\in\mathbb{Z}+1/2}\left(\frac{4\pi^2 m}{T^2}n^2\right) = \prod_{n\in\mathbb{Z}+1/2} n^2 = \prod_{n\ \text{odd}} n^2 = \frac{\prod_{n\neq 0} n^2}{\prod_{\substack{n\ \text{even}\\ n\neq 0}} n^2} = \frac{\prod_{n\neq 0} n^2}{\prod_{n\neq 0} 4n^2} = 4. \tag{15.8}$$

In the last step we use the fact that the ζ-normalized number of *nonzero* integers is -1.

On the Hilbert space the operator J is the action on functions induced by $x \to -x$. Notice that J anticommutes with A^*, so that its action on the nth eigenfunction $\frac{(A^*)^n}{n!}\Omega$ is multiplication by $(-1)^n$. Hence

$$\operatorname{Tr} Je^{-TH} = e^{-\omega T/2}\sum_{n=0}^{\infty}(-1)^n e^{-n\omega T}$$
$$= e^{-\omega T/2}\frac{1}{1+e^{-\omega T}}$$
$$= \frac{1}{2\cosh\omega T/2}$$

This agrees with (15.7).

Solution to Problem FP16

We first carry out canonical quantization to obtain the spin representation. Then we do the path integral over S^1 and compare with the Hamiltonian picture.

Let V be a finite-dimensional real inner product space. The theory in question has a single fermi field

$$\psi\colon M^1 \longrightarrow \Pi V$$

for a 1-manifold M^1, which we variously take to be $\mathbb{R}$, the circle S^1, and an interval I. The lagrangian density

$$L = \frac{1}{2}\langle \psi, \dot{\psi} \rangle \, |dt| \tag{16.1}$$

is the lagrangian (2.5) of **FP2** with the bose field x set to zero.

We follow the sign conventions of Part 1.

Canonical quantization

See [I-**Quantization**] for another discussion.

For the canonical quantization we study the lagrangian (16.1) with $M^1 = \mathbb{R}$. Adopting the results in **FP2**—or doing the easy computations directly—we see that the space of classical solutions is canonically the odd vector space ΠV, which appears as the space of constant fields $\psi\colon \mathbb{R} \to \Pi V$. The symplectic form is (see (2.19))

$$\omega = \frac{1}{2}\langle \delta\psi \wedge \delta\psi \rangle. \tag{16.2}$$

A function on ΠV is an element of $(\operatorname{Sym} \Pi V)^* \cong \operatorname{Sym} \Pi V^* \cong \Pi \Lambda V^*$, and for two linear functions $f, g \in \Pi V^*$ the Poisson bracket is

$$\{f, g\} = -\langle f, g \rangle. \tag{16.3}$$

(We blur the notation a bit in identifying elements of V^* and ΠV^*.)

As usual, to quantize a symplectic vector space we choose a polarization. Here there are no real polarizations, and a complex polarization only exists if $\dim V$ is even, which we now assume. Then a polarization

$$\Pi V_{\mathbb{C}} = \Pi W \oplus \Pi \overline{W} \tag{16.4}$$

is given by choosing a decomposition $V_{\mathbb{C}} \cong W \oplus \overline{W}$ into the sum of complex conjugate maximal isotropic subspaces. The complexified inner product identifies $\overline{W}$ with W^*. The quantum Hilbert space is the space of "L^2 functions" on ΠW, which we identify with

$$\mathcal{H} = (\operatorname{Sym} \Pi W)^* \cong \Lambda W^* \cong \Lambda \overline{W}. \tag{16.5}$$

The $\mathbb{Z}/2$-graded exterior algebra $\Lambda \overline{W}$ is a familiar realization of the spin representation.

Quantization maps complex functions on ΠV to operators on $\mathcal{H}$; real functions map to self-adjoint operators. Elements of $(\Pi W)^* \cong \Pi \overline{W}$ act by multiplication;

elements of $(\Pi\overline{W})^* \cong \Pi W$ act by contraction. In physical language, the line $\Lambda^0\overline{W} \subset \mathcal{H}$ is the vacuum, elements of $\Pi\overline{W}$ act as creation operators, and elements of ΠW act as annihilation operators. The quantum operators $\hat{f}, \hat{g}$ which correspond to linear functions f, g must satisfy

$$[\hat{f}, \hat{g}] = -i\{f, g\}\hat{} = i\langle f, g\rangle. \tag{16.6}$$

The first equation is explained in §5 of [I-**Signs**], the second is (16.3), and the right hand side is multiplied by the identity operator on $\mathcal{H}$. We have set $\hbar = 1$. Let f, g be *real*. Consider the Clifford algebra on V^* defined by requiring that Clifford multiplication c satisfy

$$[c(f), c(g)] = -\langle f, g\rangle. \tag{16.7}$$

On any complex Clifford module $\mathcal{H}$ we choose an inner product invariant under the Pin group; then the action of f, also denoted $c(f)$, is skew-adjoint in the usual sense. Viewing $\mathcal{H}$ as a $\mathbb{Z}/2$-graded Hilbert space, we have a different notion of self-adjointness for odd operators, as explained in §1 of [I-**Signs**]. Take $\mathcal{H}$ to be the basic complex Clifford module

$$\mathcal{H} = S^+ \oplus S^-. \tag{16.8}$$

Then we identify the quantum self-adjoint operators in terms of Clifford multiplication

$$\hat{f} = i^{-1/2}c(f). \tag{16.9}$$

Remark. We could choose a + sign in (16.7) instead. Then $c(f)$ is self-adjoint in the usual sense, and in (16.9) we replace $i^{-1/2}$ with $i^{1/2}$. In the quantization of the model considered in **FP**1, if the target M is oriented then these two possibilities lead to the same quantum Hilbert space of spinor fields on M. (There is a choice of spin structure to be fixed.) However, if M is nonorientable then these two possibilities correspond to the two different notions of pin structure.

Let Γ be the (self-adjoint) grading operator on $\mathcal{H}$, which is ± 1 on $S^{\pm}$. Set $\dim V = n = 2m$. Then

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}\Gamma &= 0, \\ \mathrm{Tr}_{\mathcal{H}}1 &= 2^m. \end{aligned} \tag{16.10}$$

An element $g \in \mathrm{Spin}(n)$ acts on $\mathcal{H}$ by a unitary operator which preserves the grading. Its trace over $\mathcal{H}$ without/with Γ is the sum/difference of the characters of the half-spin representations. The computation is standard. Write

$$g = \exp(x_1 e^1 e^2 + x_2 e^3 e^4 + \dots) \tag{16.11}$$

relative to some orthonormal basis $\{e^i\}$ of V^*. This corresponds to the block diagonal decomposition

$$g = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{16.12}$$

By taking products we reduce to the two-dimensional case $g = \exp(xe^1e^2)$. Quantize quadratic functions symmetrically:

$$\begin{aligned}(e^1e^2)\hat{} &= \frac{1}{2}(\hat{e}^1\hat{e}^2 - \hat{e}^2\hat{e}^1)\\ &= \hat{e}^1\hat{e}^2\\ &= i^{-1}c(e^1)c(e^2).\end{aligned} \tag{16.13}$$

The basic Clifford module is $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$, and we take

$$c(e^1) = \begin{pmatrix} & 1/\sqrt{2} \\ 1/\sqrt{2} & \end{pmatrix}, \qquad c(e^2) = \begin{pmatrix} & -i/\sqrt{2} \\ i/\sqrt{2} & \end{pmatrix}.$$

Then

$$\begin{aligned}(xe^1e^1)\hat{} &= i^{-1}x\begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}\\ &= \begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix}.\end{aligned}$$

Exponentiating and taking the product, we find

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}} g &= 2^m \prod_{j=1}^{m} \cosh\frac{x_j}{2},\\ \mathrm{Tr}_{\mathcal{H}} g\Gamma &= 2^m \prod_{j=1}^{m} \sinh\frac{x_j}{2}.\end{aligned} \tag{16.14}$$

Path integral on S^1

There are two choices of spin structure on the oriented manifold S^1: the *bounding* spin structure (the spin bundle L is nontrivial) and the *nonbounding* spin structure (the spin bundle L is trivial). The classical field ψ takes values in $\Pi(L \otimes V)$; it is a spinor field on the circle. The *Euclidean* path integral is formally the pfaffian of the quadratic form in the lagrangian (16.1):

$$\int D\psi\; e^{-\int_{S^1} \frac{1}{2}\langle\psi,\dot{\psi}\rangle\,|d\tau|} = \mathrm{pfaff}\left(\frac{D}{D\tau}\right). \tag{16.15}$$

The operator $D/D\tau$ has a kernel for the nonbounding spin structure and is invertible for the bounding spin structure. Comparing with (16.10),

$$\begin{aligned}\text{nonbounding spin structure} &\longleftrightarrow \mathrm{Tr}_{\mathcal{H}}\Gamma,\\ \text{bounding spin structure} &\longleftrightarrow \mathrm{Tr}_{\mathcal{H}} 1.\end{aligned} \tag{16.16}$$

Taking the length of S^1 to be 2π, the operator $D/D\tau$ is

$$\frac{D}{D\tau} \longleftrightarrow \begin{cases} \dfrac{d}{d\tau}, & \text{nonbounding spin structure;}\\ \dfrac{d}{d\tau} + \dfrac{i}{2}, & \text{bounding spin structure.}\end{cases} \tag{16.17}$$

The computation of the pfaffian (16.15) is a special case of the next computation.

Consider now the twisted bundle $V_g \to S^1$ with holonomy $g \in SO(n)$. Consider first the nonbounding spin structure, so L is trivial. Topologically V_g is untwisted, but the connection is twisted and $D/D\tau$ may be identified with the operator

$$\frac{d}{d\tau} + \begin{pmatrix} 0 & -x_1/2\pi & & \\ x_1/2\pi & 0 & & \\ & & & \ddots \end{pmatrix} \tag{16.18}$$

on V-valued functions. (As above, we take S^1 to have length 2π.) In case $\dim V = 2$ the operator (16.18) has eigenvalues

$$\left\{ i(n + \frac{x}{2\pi}),\ i(n - \frac{x}{2\pi}) \right\}_{n\in\mathbb{Z}}, \tag{16.19}$$

and so the determinant is

(16.20)

$$\prod_{n\in\mathbb{Z}} -\left(n^2 - \left(\frac{x}{2\pi}\right)^2\right) = \left(\frac{x}{2\pi}\right)^2 \prod_{n\in\mathbb{Z}^{\neq 0}} n^2 \prod_{n\in\mathbb{Z}^{\neq 0}} \left(1 - \left(\frac{x}{2\pi n}\right)^2\right)$$
$$= 4\sin^2\frac{x}{2}.$$

(In the first step we use the fact that the ζ-normalized cardinality of $\mathbb{Z}$ is zero.) For the bounding spin structure replace $d/d\tau$ in (16.18) with $d/d\tau + 1/2$, or what amounts to the same, x_j with $x_j + \pi$. Then in (16.20) sin gets replaced with cos. We cannot determine the sign of the pfaffian, which involves a square root. Take the product over the 2×2 blocks in (16.18) to obtain

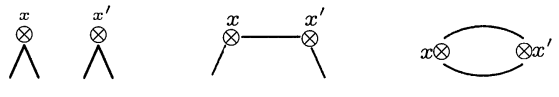
$$\operatorname{pfaff}\frac{D}{D\tau} = \begin{cases} 2^m \prod_{j=1}^{m} \sin\frac{x_j}{2}, & \text{nonbounding spin structure;} \\ 2^m \prod_{j=1}^{m} \cos\frac{x_j}{2}, & \text{bounding spin structure.} \end{cases} \tag{16.21}$$

The indeterminacy of the sign in (16.21) means that pfaff $D/D\tau$ is better interpreted as an element of a real line Pfaff $D/D\tau$. If we consider the family of operators parametrized by $g \in SO(n)$, then pfaff $D/D\tau$ is a section of a real line bundle over $SO(n)$. The line bundle is nontrivial, so to make sense of pfaff $D/D\tau$ as a function we must pass to the double cover $\mathrm{Spin}(n)$.

Finally, to compare to the canonical picture (16.14) we need to rotate from imaginary time to real time. Substituting $\tau = it$ in (16.18) we find n in (16.19) replaced by n/i, or equivalently (since the cardinality of $\mathbb{Z}$ is zero) x replaced by ix. hence the trigonometric functions in (16.21) become hyperbolic functions as appear in (16.14).

Solution to Problem FP17

We represent $\phi^2(x)$ by  and then there are 3 diagrams in $\phi^2(x)\phi^2(x')$:



I II III

The first diagram is the ordinary Taylor series, which for operators of dimension four is simply

$$\mathrm{I} = \phi(x)^4 + \cdots$$

The second diagram occurs 4 times and is

$$\begin{aligned}\mathrm{II} &= 4G(x-x')\phi(x)\phi(x') \\ &= 4G(x-x')\Big[\phi(x)^2 + (x'-x)^\mu \phi(x)\partial_\mu\phi(x) \\ &\qquad + \tfrac{1}{2}(x'-x)^\mu (x'-x)^\nu \phi(x)\partial_\mu\partial_\nu\phi(x) + \cdots\Big].\end{aligned}$$

The third diagram occurs twice and is simply

$$\mathrm{III} = 2G(x-x')^2.$$

Since $G(x-x') \sim \frac{1}{|x-x'|^2}$ as $x' \to x$, we have the operator product expansion up to operators of dimension four:

$$\begin{aligned}\phi^2(x)\phi^2(x') \underset{x'\to x}{\sim} &\tfrac{2}{|x-x'|^4} + \tfrac{4\phi(x)^2}{|x-x'|^2} + 4\tfrac{(x-x')^\mu}{|x-x'|^2}\phi(x)\partial_\mu\phi(x) \\ &+ \Big\{\phi(x)^4 + 4\tfrac{(x-x')^\mu(x-x')^\nu}{|x-x'|^2}\phi(x)\partial_\mu\partial_\nu\phi(x)\Big\} \\ &+ \cdots\end{aligned}$$

Solution to Problem FP18

We work in Euclidean space.

Write $A = eB$ and add to the lagrangian density a gauge fixing term

$$\tfrac{\alpha}{2}\left|\sum_i \partial_i B_i\right|^2 d^4x,$$

where α is a real number to be determined. Then the free part of the lagrangian for the field B is

$$\begin{aligned}&\tfrac{1}{4}\sum_{i\neq j}\|\partial_i B_j - \partial_j B_i\|^2 + \tfrac{\alpha}{2}\|\sum_i \partial_i B_i\|^2 \\ &\qquad = \tfrac{1}{2}\sum_{i\neq j}\|\partial_i B_j\|^2 - \tfrac{1}{2}\sum_{i\neq j}\langle \partial_i B_j, \partial_j B_i\rangle + \tfrac{\alpha}{2}\sum_i \|\partial_i B_i\|^2 \\ &\qquad\qquad + \tfrac{\alpha}{2}\sum_{i\neq j}\langle \partial_i B_i, \partial_j B_j\rangle\,.\end{aligned} \tag{18.1}$$

Here we write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the integrated inner products. Take $\alpha = 1$. Then the second and fourth terms in (18.1) cancel using integration by parts, and the free lagrangian for B is

$$\tfrac{1}{2}\sum_{i,j}\|\partial_i B_j\|^2.$$

So the free propagator for B in momentum space is

$$(D^0_{ij})^{-1}(k) = \tfrac{\delta_{ij}}{k^2}. \tag{18.2}$$

The total lagrangian density (after canceling the terms in (18.1)) is now

$$\mathcal{L}' = \Big\{ \tfrac{1}{2} \sum_{i,j} |\partial_i B_j|^2 + (i(\Gamma \cdot \partial + m)\psi, \psi) + e(\Gamma \cdot B\psi, \psi) \Big\} d^4x. \tag{18.3}$$

Here Γ^μ is Clifford multiplication in the μth coordinate direction, and $i\Gamma \cdot \partial = i\Gamma^\mu \partial_\mu$ is the flat space Dirac operator. Our convention is

$$\Gamma^\mu \Gamma^\nu + \Gamma^\mu \Gamma^\nu = 2\delta^{\mu\nu}. \tag{18.4}$$

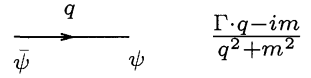
The Γ^μ are *hermitian* operators, so that $i\Gamma \cdot \partial$ is hermitian, while im is skew-Hermitian. Note that ψ is an odd variable so that $(i\Gamma \cdot \partial\psi, \psi)$ is purely imaginary, while $(im\,\psi, \psi)$ is real.[10] The fermionic propagator in momentum space is

$$\tfrac{1}{\Gamma \cdot k + im} = \tfrac{\Gamma \cdot k - im}{k^2 + m^2}. \tag{18.5}$$

We remark that our inner product is linear in the first variable and conjugate linear in the second variable. Think of A as a connection in a complex line bundle L, so that the spinor fields are sections of $S \otimes L$, where S is the (complexified) spin space. We view

$$Q(\psi, \bar{\psi}) = (i(\Gamma \cdot \partial + m)\psi, \bar{\psi}) \tag{18.6}$$

as a bilinear form, where ψ is a section of $S \otimes L$ and $\bar{\psi}$ a section of $S \otimes \bar{L}$. The hermitian inner product on the right hand side of (18.6) is written as linear in both variables. (The spin space S is quaternionic in the Euclidean signature, so there is an identification $S \cong \bar{S}$. Note that in Minkowski space the spinors are real.) The ordering of the variable is important, since the spinor fields are odd (anticommuting). The same holds for the propagator Q^{-1}, and so fermion lines in Feynman diagrams have a definite orientation:



$$\tfrac{\Gamma \cdot q - im}{q^2 + m^2}$$

Any loop in a Feynman diagram requires a trace, and by general principles in a theory with fermions it should be a *supertrace*. The reasoning is as follows. The edges around the loop are propagators, which are bilinear forms. The vertices represent polynomials and we are instructed to contract the polynomials with the propagators. After some contractions we can view the loop as a composition of operators, where when we close the loop we must take a trace. In general, if V is a graded vector space we agree to contract V^* and V by letting V^* operate on the left. An operator on V is identified (without signs) with an element of $V \otimes V^*$: the

[10] Continued to Minkowski space the lagrangian is real. Note that we use the physics convention about conjugation of spinors in this problem.

operator T corresponds to $Te_i \otimes e^i$, where $\{e_i\}$ is a basis of V and $\{e^i\}$ the dual basis of V^*. The natural contraction of an operator is the composition

$$\operatorname{End} V \xrightarrow{\simeq} V \otimes V^* \longrightarrow V^* \otimes V \longrightarrow \mathbb{C}$$

which is the supertrace.

Since we will use dimensional regularization, we compute the scaling dimensions of the fields in d-dimensional space:

$$[B] = \tfrac{d-2}{2}, \quad [\psi] = \tfrac{d-1}{2}, \quad [m] = 1, \quad [e] = \tfrac{4-d}{2}$$

so we replace e by a dimensionless constant λ and a mass parameter μ (i.e., $[\mu] = [m] = 1$):

$$e \to \lambda \mu^{\frac{4-d}{2}}.$$

It is convenient to define

$$\epsilon = \tfrac{4-d}{2},$$

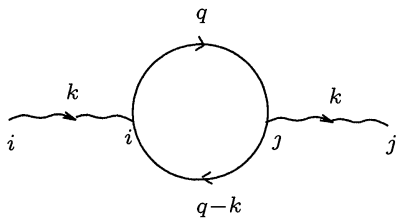
so that the interaction term in the lagrangian density is

$$\lambda \mu^{\epsilon} (\Gamma \cdot B\psi, \psi).$$

Now the 1-loop correction to the 2-point function of B comes by evaluating the graph shown. Denote its value as $-\Sigma_{ij}(k)$. Summing over all chains made out of this one-particle irreducible graph we see that the corrected 2-point function is D_{ij}^{-1}, where

$$D_{ij} = D_{ij}^0 + \Sigma_{ij}. \tag{18.7}$$

(See Witten's lectures for an explanation.)



We remark that if V is an inner product space and Q: $V \to \operatorname{Sym}^2 V^*$ a quadratic form which is equivariant under the orthogonal group $O(V)$, then

$$Q_v(\xi_1, \xi_2) = E(|v|^2)(\xi_1, \xi_2) + F(|v|^2)(v, \xi_1)(v, \xi_2), \quad v, \xi_1, \xi_2 \in V,$$

for some functions E, F. Also, if $Q_v(v, \xi) = 0$ for all ξ, then

$$Q_v(\xi_1, \xi_2) = F(|v|^2)\{(v, \xi_1)(v, \xi_2) - |v|^2(\xi_1, \xi_2)\}.$$

This is the form taken by Σ.

At last we evaluate the diagram in d dimensions as

$$-\Sigma_{ij}(k) = \lambda^2 \mu^{2\varepsilon} \int \tfrac{d^d q}{(2\pi)^d} \left[-\operatorname{Tr} \left\{ \Gamma^i \tfrac{\Gamma \cdot (q-k) - im}{(q-k)^2 + m^2} \Gamma^j \tfrac{\Gamma \cdot q - im}{q^2 + m^2} \right\} \right]. \tag{18.8}$$

We first expand the numerator of the expression in braces:

$$\begin{aligned}&\mathrm{Tr}\Big[\Gamma^i(\Gamma^\mu(q_\mu-k_\mu)-im)\Gamma^j(\Gamma^\nu q_\nu-im)\Big]\\&\qquad=\mathrm{Tr}\Big[(q_\mu-k_\mu)q_\nu\Gamma^i\Gamma^\mu\Gamma^j\Gamma^\nu-m^2\Gamma^i\Gamma^j\Big]\end{aligned}\tag{18.9}$$

Here we use the fact that the trace of an odd product of Γ matrices vanishes.* (Note that $\mathrm{Tr}(1)=4$ and $\mathrm{Tr}\,\Gamma^i=\mathrm{Tr}\,\Gamma^i\Gamma^j=\mathrm{Tr}\,\Gamma^i\Gamma^j\Gamma^k=\mathrm{Tr}\,\Gamma^1\Gamma^2\Gamma^3\Gamma^4=0$ if i, j,k are distinct.) For the moment we leave (18.9) unevaluated and introduce Feynman parameters into (18.8):

$$\begin{aligned}\tfrac{1}{(q-k)^2+m^2}\tfrac{1}{q^2+m^2}&=\int_0^1 d\alpha\,\tfrac{1}{[\alpha(q-k)^2+(1-\alpha)q^2+m^2]^2}\\&=\int_0^1 d\alpha\,\tfrac{1}{[(q-\alpha k)^2+m^2+\alpha(1-\alpha)k^2]^2}\end{aligned}$$

Let

$$M^2=m^2+\alpha(1-\alpha)k^2\tag{18.10}$$

and make the shift of integration $q\to q+\alpha k$ in (18.8). Then using (18.9) we obtain

(18.11)

$$-\Sigma_{ij}(k)=-\lambda^2\mu^{2\varepsilon}\int_0^1 dx\int\tfrac{d^dq}{(2\pi)^d}\,\tfrac{\mathrm{Tr}\big[(q_\mu+(\alpha-1)k_\mu)(q_\nu+\alpha k_\nu)\Gamma^i\Gamma^\mu\Gamma^j\Gamma^\nu-m^2\Gamma^i\Gamma^j\big]}{(q^2+M^2)^2}$$

Now it is easy to compute that for any (co)vectors v and w,

$$v_\mu w_\nu\mathrm{Tr}(\Gamma^i\Gamma^\mu\Gamma^j\Gamma^\nu)=4(v_iw_j+v_jw_i-\delta_{ij}v\cdot w),$$

and

$$\mathrm{Tr}(\Gamma^i\Gamma^j)=4\delta_{ij}.$$

Observe that $\int d^dqF(q)=0$ for an odd function of q, simply by changing variables $q\to -q$. Thus the $\int d^dq$ in (18.11) reduces to

$$\begin{aligned}&4\int\tfrac{d^dq}{(2\pi)^d}\,\tfrac{(2q_iq_j-\delta_{ij}q^2)+\alpha(\alpha-1)(2k_ik_j-\delta_{ij}k^2)-\delta_{ij}m^2}{(q^2+M^2)^2}\\&\qquad=-8\alpha(1-\alpha)(k_ik_j-\delta_{ij}k^2)\int\tfrac{d^dq}{(2\pi)^d}\,\tfrac{1}{(q^2+M^2)^2}\\&\qquad\quad+4\int\tfrac{d^dq}{(2\pi)^d}\left\{\tfrac{2q_iq_j}{(q^2+M^2)^2}-\tfrac{\delta_{ij}}{q^2+M^2}\right\}\end{aligned}\tag{18.12}$$

The last integral vanishes for $i\neq j$ (take $q_j\to -q_j$) and is independent of i for $i=j$, so reduces to

$$4\delta_{ij}\int\tfrac{d^dq}{(2\pi)^d}\left\{\tfrac{\frac{2}{d}q^2}{(q^2+M^2)^2}-\tfrac{1}{q^2+M^2}\right\}.\tag{18.13}$$

*We evaluate the Γ matrices in 4 dimensions before continuing the integral to d dimensions.

We claim that under dimensional regularization (18.13) vanishes. To see this consider

$$I(\lambda) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{(\lambda q^2 + M^2)}.$$

Then by scaling $I(\lambda) = \lambda^{-d/2} I(1)$, and differentiating in λ and taking $\lambda = 1$ we find $I'(1) = -(d/2)I(1)$. This implies (18.13) vanishes. Now we evaluate the remaining integral in (18.12) in d dimensions as

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + M^2)^2} = \frac{\Gamma(\epsilon)}{(4\pi)^{d/2}} \frac{1}{(M^2)^\epsilon}.$$

(See problem set 4, problem 2 for the derivation of a similar formula.)

Finally, plugging back into (18.11) we have

$$-\Sigma_{ij}(k) = (k_i k_j - \delta_{ij} k^2) F(k^2), \tag{18.14}$$

where

$$F(k^2) = \frac{\lambda^2}{2\pi^2} \Gamma(\epsilon) (4\pi\mu^2)^\epsilon \int_0^1 d\alpha\, \alpha(1-\alpha)(m^2 + \alpha(1-\alpha)k^2)^{-\epsilon}.$$

This is finite for $\epsilon > 0$ (which is $d < 4$) and has a simple pole at $\epsilon = 0$. Using the expansion

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon), \qquad \gamma = \text{ Euler's constant},$$

we have

$$F(k^2) = \frac{\lambda^2}{2\pi^2} \left\{ \frac{1}{6\epsilon} - \frac{\gamma}{6} - \int_0^1 d\alpha\, \alpha(1-\alpha) \ln\left(\frac{m^2 + \alpha(1-\alpha)k^2}{4\pi\mu^2} \right) + O(\epsilon) \right\}. \tag{18.15}$$

The $\int_0^1 d\alpha$ can be evaluated in terms of elementary functions, but the result is complicated and unilluminating.

To renormalize we must adjust the fields and parameters in the lagrangian (18.3), which we write now including the gauge fixing term:

(18.16)

$$\mathcal{L} = \left\{ \tfrac{1}{2} \Sigma_{i,j} |\partial_i B_j|^2 + \tfrac{\alpha - 1}{2} |\Sigma_i \partial_i B_i|^2 + (i(\Gamma \cdot \partial + m)\psi, \psi) + e(\Gamma \cdot B\psi, \psi) \right\} dx^4.$$

The inverse propagator, written in terms of the Fourier variable k, is

$$D_{ij} = k^2 \delta_{ij} + (\alpha - 1) k_i k_j.$$

The renormalization scheme defines the bare field B_0 and bare coupling α_0 as

$$\begin{aligned} B_0 &= (1 + P)^{1/2} B \\ \alpha_0 &= \frac{\alpha}{1 + P} \end{aligned} \tag{18.17}$$

so that the bare inverse propagator is

$$\begin{aligned} D^0_{ij} &= (1 + P) k^2 \delta_{ij} + (\alpha - 1 - P) k_i k_j \\ &= D_{ij} - \Sigma_{ij} \end{aligned}$$

Comparing to (18.14) we have for $\alpha = 1$ to 1-loop accuracy, using (18.15),

$$P = -\tfrac{\lambda^2}{12\pi^2}\tfrac{1}{\epsilon} + \text{ finite.}$$

Finally, we renormalize the charge $e = \lambda\mu^\epsilon$ in (18.16) by setting

$$\begin{aligned} e_0 &= \lambda\mu^\epsilon \cdot (1+P)^{-1/2} \\ &\approx \lambda\mu^\epsilon \left(1 + \tfrac{\lambda^2}{24\pi^2}\tfrac{1}{\epsilon} + \text{ finite}\right). \end{aligned} \tag{18.18}$$

The β-function is

$$\beta = \mu\tfrac{\partial\lambda}{\partial\mu}\big|_{\epsilon=0}$$

with e_0 fixed. Using minimal subtraction we set the finite part to zero and differentiate (18.18):

$$0 = \epsilon\left(\lambda + \tfrac{\lambda^3}{24\pi^2}\tfrac{1}{\epsilon}\right) + \beta\left(1 + \tfrac{\lambda^2}{8\pi^2}\tfrac{1}{\epsilon}\right) \tag{18.19}$$

So to leading order in ϵ we have $\beta = -\epsilon\lambda$, and then taking the regular part of (18.19) at $\epsilon = 0$ we find

$$\mu\tfrac{d\lambda}{d\mu} = \beta = \tfrac{\lambda^3}{12\pi^2}. \tag{18.20}$$

This ODE has solution

$$\tfrac{1}{\lambda^2(\mu)} - \tfrac{1}{\lambda^2(\mu_0)} = -\tfrac{1}{6\pi^2}\ln(\mu/\mu_0)$$

for some constant μ_0. Equivalently,

$$\lambda^2(\mu) = \frac{\lambda^2(\mu_0)}{1 - \frac{\lambda^2(\mu_0)}{6\pi^2}\ln(\mu/\mu_0)}$$

which exhibits a (Landau) pole at

$$\mu = \mu_0 \exp\left(\tfrac{6\pi^2}{\lambda^2(\mu_0)}\right).$$

Finally, we give an *a priori* argument that the coefficient of the divergence in $F(k^2)$ is positive (see (18.15)). The same argument in a simpler situation appears in **FP**19. Here we identify the one-particle irreducible diagram we computed as minus the two-point function of the *current* J_i in the free theory:

$$-\overset{\vee}{\Sigma}_{ij}(x) = -\lambda^2\left\langle J_i(x)\overline{J_j(0)}\right\rangle_{\text{free}}.$$

Here $\overset{\vee}{\Sigma}_{ij}$ is the inverse Fourier transform of Σ_{ij}, and the current is

$$J_i(x) = \Gamma^i\psi_{(x)}\overline{\psi_{(x)}},$$

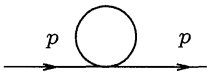
where the product of the spinors is taken to be a vector. The minus sign is due to the fact that conjugation reverses the sign of J_i, since we interchange ψ and $\bar{\psi}$. Now reflection positivity implies that $\left\langle J_i(x)\overline{J_j(0)}\right\rangle_{\text{free}}$ is a nonnegative quadratic form for $x \neq 0$. The behavior of the Fourier transform $-\Sigma_{ij}(k)$ as $k \to \infty$ is determined by the behavior of $-\overset{\vee}{\Sigma}_{ij}(x)$ as $x \to 0$, and it follows easily that $-\Sigma_{ij}(k)$ is nonpositive for large k. Using (18.14) this translates into $F(k^2) > 0$ for large k. This is consistent with (18.15).

Solution to Problem FP19

(a) The anomalous dimension of ϕ gets contributions from 2-point function diagrams. All such (connected) diagrams have naive quadratic divergences. Let p be the momentum flowing in (and by conservation flowing out) and m the mass. Then with a cutoff Λ the diagram diverges as

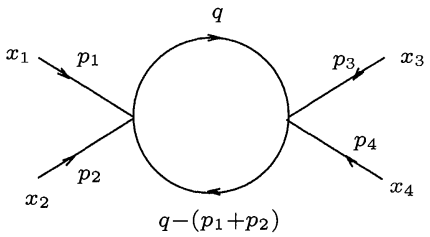
$$f\left(\frac{p}{m}\right)\Lambda^2 + g\left(\frac{p}{m}\right)\log\left(\frac{\Lambda}{m}\right)p^2 + h\left(\frac{p}{m}\right)\log\left(\frac{\Lambda}{m}\right)m^2$$

for some functions f, g, h. This follows by dimensional analysis. (In fact, since the second derivative of a diagram with respect to p is finite, we see that f, g, h are simple polynomials.) Now the only 1-loop diagram is shown, and the loop integral is independent of p. For this diagram, then, $g \equiv 0$ and there is no renormalization



to the coefficient of $|d\phi|^2$ and so no contribution to the anomalous dimension of ϕ.

The diagram which contributes to the 1-loop β-function is the integral



$$\hat{D}(p_1,p_2,p_3,p_4) = \lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2+m^2)((q-p_1-p_2)^2+m^2)} \cdot \delta(p_1+p_2+p_3+p_4)$$

which has a leading divergence $A\lambda^2 \log \frac{\Lambda}{m}$ if an ultraviolet cutoff Λ is imposed. (Here $\hat{\Gamma}$ is a distribution supported on the subspace $\{p_1+p_2+p_3+p_4=0\}$.)

(b) We want to understand *a priori* that $A > 0$. For this it is best to Fourier transform to position space. This is straightforward and leads to

$$D(x_1,x_2,x_3,x_4) = G(x-y)^2\delta(x_1-x)\delta(x_2-x)\delta(x_3-y)\delta(x_4-y),$$

where G is the propagator. (So D is supported on the subspace $x_1 = x_2$, $x_3 = x_4$ and its value there is the square of the Green's function.) In general, the position space prescription for computing a *one-particle irreducible* Feynman diagram as part of the effective action is the same as usual — we attach propagators to the legs and integrate over the positions in the internal vertices — except that we replace the propagators on the *external* legs by δ-functions. For convenience we take $y = 0$ and write $D(x) = G(x)^2$.

Now introduce the composite operator $\mathcal{O}(x) = \phi^2(x)$. Then it is immediate that the two-point function $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{\text{free}}$ in the *free* theory is computed by the diagram shown.

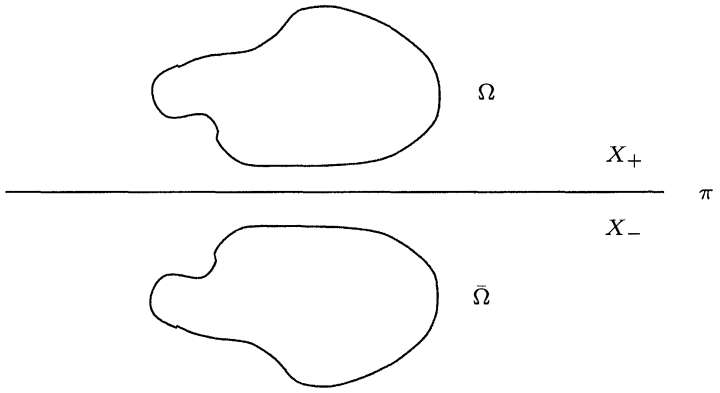


We evaluate this diagram as $G(x)^2$, so it is precisely equal to $D(x)$. It is manifest that $D(x) \geq 0$, of course, since it is written as the square of a real function. But we would like to invoke instead *reflection positivity* since the argument is useful in more general situations.

Reflection positivity asserts that if $\mathcal{O}_\Omega$ is a (complex) operator in a Euclidean field theory, supported in some set Ω lying on one side of a hyperplane, and if $\bar{\Omega}$ denotes the reflection of Ω in the hyperplane, then

$$\left\langle \mathcal{O}_\Omega \overline{\mathcal{O}_{\bar{\Omega}}} \right\rangle \geq 0.$$

If $\mathcal{O}_\Omega$ takes values in a vector space V, then the assertion is that $\left\langle \mathcal{O}_\Omega \overline{\mathcal{O}_{\bar{\Omega}}} \right\rangle$ is a nonnegative hermitian form on V^*.



We can understand this heuristically by writing a path integral over the half-space. Namely, suppose for simplicity that we are dealing with a complex scalar field ϕ with action $S(\phi)$. Then for ϕ_0 a function on π we set

$$Z_{X_+}(\phi_0) = \int\limits_{\substack{\phi\colon X_+ \to \mathbb{C} \\ \phi\big|_\pi = \phi_0}} \mathcal{O}_\Omega(\phi) e^{-S_{X_+}(\phi)} D\phi.$$

This should be viewed as a vector in some Hilbert space of functions of ϕ_0. Now $X_- \cong \bar{X}_+$, where $\bar{X}_+$ is X_+ with the reversed orientation. The Euclidean action has the property that $S_{\bar{Y}} = \overline{S_Y}$ for any spacetime Y: orientation reversal conjugates the action. It follows that

$$\begin{aligned}
\overline{Z_{X_+}(\phi_0)} &= \int\limits_{\substack{\phi\colon X_+ \to \mathbb{C} \\ \phi\big|_\pi = \phi_0}} \overline{\mathcal{O}_\Omega(\phi)} e^{-\overline{S_{X_+}(\phi)}} D\phi \\
&= \int\limits_{\substack{\phi\colon X_- \to \mathbb{C} \\ \phi\big|_\pi = \phi_0}} \overline{\mathcal{O}_\Omega(\phi)} e^{-S_{X_-}(\phi)} D\phi
\end{aligned}$$

The norm square of $Z_{X_+}(\phi)$ in $\mathcal{H}$ is computed by integrating over ϕ_0:

$$\begin{aligned}
\|Z_{X_+}(\phi_0)\|^2_{\mathcal{H}} &= \int\limits_{\phi_0\colon \pi \to \mathbb{C}} Z_{X_+}(\phi_0) \overline{Z_{X_+}(\phi_0)} D\phi_0 \\
&= \int\limits_{\phi\colon X \to \mathbb{C}} \mathcal{O}_\Omega(\phi) \overline{\mathcal{O}_{\bar{\Omega}}(\phi)} e^{-S_X(\phi)} D\phi \\
&= \left\langle \mathcal{O}_\Omega \overline{\mathcal{O}_{\bar{\Omega}}} \right\rangle,
\end{aligned}$$

where the measure is normalized so that the partition function is one. The left hand side is obviously positive, hence so is the right.

In our example $\mathcal{O}(x) = \phi^2(x)$ is real, and to show that $D(x) = \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{\text{free}} \geq 0$ we use reflection positivity for the perpendicular bisector of the line segment connecting 0 and x. Now $D(x)$ diverges as $x \to 0$, and since $D(x) \geq 0$ the coefficient of the divergence is positive. This implies that the divergence in D with a momentum cutoff Λ has a positive coefficient.

Solution to Problem FE1

We use a supersymmetric quantum field theory where the fields are maps $X : \mathbb{R}^{1,1} \to M$ satisfying a periodicity condition. We write the maps as $X(t,\theta)$ where t is the even variable and θ the odd variable: decomposing into bosonic and fermionic fields we write

$$X(t,\theta) = x(t) + \theta\psi(t).$$

The Lagrangian is

$$L(X) = \int dt d\theta g_{IJ} \frac{dX^I}{dt} DX^J$$

where we have introduced a metric g_{IJ} on M and the operator

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}.$$

In components the action becomes

$$L(X) = \int dt \left\{ g_{IJ}\frac{dx^I}{dt}\frac{dx^J}{dt} + \frac{i}{2} g_{IJ}\psi^I \frac{D\psi^J}{Dt} \right\},$$

where $D\psi^J/Dt$ denotes the covariant derivative (with respect to the Levi-Civita connection) in the direction of the tangent vector $x_*(d/dt)$ in M. The supersymmetry operator is

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}.$$

Upon quantization the space of states is identified with the sections $\mathcal{H}_\pm$ of the spin bundle $\mathcal{S}_\pm$ on M and the operator Q is identified with the Dirac operator $\mathcal{D}$ on M. We introduce the operator $(-1)^F$ on $\mathcal{H}_\pm$ which is defined to have the values $+1$ on $\mathcal{H}_+$ (states with an even number of fermions) and (-1) on $\mathcal{H}_-$ (states with an odd number of fermions). The index of the Dirac operator $\mathcal{D}$ is then

$$\text{Index}(\mathcal{D}) = \text{Trace}(-1)^F = \text{Trace } (-1)^F e^{-\beta H}, \tag{1.1}$$

where we have introduced the Hamiltonian H and a positive parameter β. Notice that since the Dirac operator $\mathcal{D}$ gives an isomorphism between the eigenspaces $\mathcal{H}_+^\lambda$ of H on $\mathcal{H}_+$ with a positive eigenvalue λ and the corresponding eigenspaces $\mathcal{H}_-^\lambda$ of the action of H on $\mathcal{H}_-$, the trace of $(-1)^F e^{-\beta H}$ is in fact independent of β. This can be computed by the path integral

$$\text{Trace } (-1)^F e^{-\beta H} = \int \mathcal{D}X e^{-\int_0^\beta L(X)dt}, \tag{1.2}$$

where we integrate over fields $X(t)$ satisfying periodic boundary conditions $X(t+1) = X(t)$.

The corresponding character valued index is

$$\text{Index}_g(\mathcal{D}) = \text{Trace}(-1)^F g = \text{Trace}\ (-1)^F g e^{-\beta H}. \tag{1.3}$$

Since the isometry g commutes with the Dirac operator, the trace $\text{Trace}(-1)^F g e^{-\beta H}$ is again independent of β. It is given by the path integral

$$\text{Trace}(-1)^F g e^{-\beta H} = \int \mathcal{D}X e^{-\int_0^\beta L(X)dt}, \tag{1.4}$$

which is the same as (1.2) but with the boundary conditions on the fields changed to

$$x(t+1) = gx(t),\ \psi(t+1) = g\psi(t). \tag{1.5}$$

We now rescale the time variable t by replacing it by $t' = \beta t$ where $t \in [0,1]$: this is equivalent to replacing the action $\int_0^\beta L(X)dt$ by

$$\frac{1}{\beta}\int_0^1 \left\{ g_{IJ}\frac{dx^I}{dt}\frac{dx^J}{dt} + \frac{i}{2}g_{IJ}\psi^I\frac{D\psi^J}{Dt}\right\}, \tag{1.6}$$

where we have also rescaled the fermionic variables ψ^I by multiplying them by a factor $1/\sqrt{\beta}$.

We must first identify the solutions of the Euler-Lagrange equation. This equation is

$$dX^I/dt = 0;$$

given the twisted periodicity condition (1.5) the only solutions are constant paths $x_0(t) = x_0$ which send $\mathbb{R}$ to a point x_0 in the fixed point set $Z = M^g$ of g, and constant paths $\psi_0(t) = \psi_0$ which send $\mathbb{R}$ to a point ψ_0 in the tangent space $\Pi T_{x_0}M$.

If $X_0 = (x_0, \psi_0)$ is a solution of the Euler-Lagrange equation, we write an infinitesimal variation δX at X_0 as

$$\delta X = (\alpha, \eta)$$

(where α is the bosonic piece, a path in $T_{x_0}M$, and η is the fermionic piece, a path in $\Pi T_{x_0}M$). To avoid overcounting infinitesimal variations δX which are simply deformations within the class of solutions X_0 of the Euler-Lagrange equation, we impose the conditions

$$\int_0^1 \alpha(t)dt = 0,\ \int_0^1 \eta(t)dt = 0. \tag{1.7}$$

We decompose the infinitesimal variations (α, η) further into

$$\alpha = \alpha^{\|} + \alpha^{\perp},\ \eta = \eta^{\|} + \eta^{\perp}$$

where $\alpha^{\|}$ is a path in the tangent space $T_{x_0}Z$ at x_0 to the fixed point set Z, while $\alpha^{\perp}$ is a path in the fibre $\mathcal{N}_{x_0}$ of the normal bundle $\mathcal{N}$ to Z in M. The fermionic

piece η is similarly decomposed into a path $\eta^{\|}$ in the tangent space $T_{x_0}Z$ and a path $\eta^{\perp}$ in the fibre $\mathcal{N}_{x_0}$ of the normal bundle. These components separately satisfy the condition (1.7). The endpoint conditions become

$$\alpha^{\|}(1) = \alpha^{\|}(0),\ \eta^{\|}(1) = \eta^{\|}(0) \tag{1.8}$$

$$\alpha^{\perp}(1) = g\alpha^{\perp}(0),\ \eta^{\perp}(1) = g\eta^{\perp}(0) \tag{1.9}$$

where in (1.9) g acts as an automorphism of the fibres of the normal bundle to Z in M.

Expanding the Lagrangian to second order in (α, η) we find $L = L^{\|} + L^{\perp}$, where

$$L^{\|} = \frac{1}{\beta}\int_0^1 dt\left(\frac{1}{2}|\frac{d\alpha^{\|}}{dt}|^2 - \frac{i}{4}\psi_0^I\psi_0^J R_{IJKL}(\alpha^{\|})^K\frac{d(\alpha^{\|})^L}{dt} + \frac{i}{2}(\eta^{\|}, \frac{d\eta^{\|}}{dt})\right) \tag{1.10}$$

while

$$L^{\perp} = \frac{1}{\beta}\int_0^1 dt\left(\frac{1}{2}|\frac{d\alpha^{\perp}}{dt}|^2 - \frac{i}{4}\psi_0^I\psi_0^J R_{IJKL}(\alpha^{\perp})^K\frac{d(\alpha^{\perp})^L}{dt} + \frac{i}{2}(\eta^{\perp}, \frac{d\eta^{\perp}}{dt})\right). \tag{1.11}$$

(Because the isometry g preserves the Levi-Civita connection, the fixed point set Z is totally geodesic. The solutions ψ_0 to the Euler-Lagrange equation are tangent to the fixed point set Z. Thus, in (1.10) the indices I, J, K, L run over a basis for the tangent bundle to the fixed point set Z, while in (1.11) the indices I, J run over a basis for the tangent space to Z while the indices K, L run over a basis for the normal bundle $\mathcal{N}$ to the fixed point set.)

The bosonic parts of these Lagrangians are given by a differential operator $(1/\beta)D_B$ of order 2 acting on $\alpha^{\|}$ (resp. $\alpha^{\perp}$), where D_B is defined by

$$(D_B)_{KL} = -\frac{d^2}{dt^2}\delta_{KL} - F_{KL}\frac{d}{dt}, \tag{1.12}$$

where $F = (F_{KL})$ is the curvature $(i/2)\psi_0^I\psi_0^J R_{IJKL}$ of the Levi-Civita connection on M. Similarly the fermionic Lagrangians are given by a differential operator $(1/\beta)\ D_F$ (where $D_F = i\frac{D}{Dt}$ has order 1) acting on $\eta^{\|}$ (resp. $\eta^{\perp}$). (At this point we may use normal coordinates to identify the covariant derivative D/Dt with the ordinary derivative d/dt.)

We compute the path integral (1.4) perturbatively. Since the result is independent of β, we may evaluate the integral in the limit $\beta \to 0$: however, since the propagator is multiplied by a factor β, this means that in the limit $\beta \to 0$ the exact answer is given by the leading order term in perturbation theory (i.e. the ratio of determinants arising from the Gaussian integral). In other words, the path integral (1.4) becomes

$$\frac{\mathrm{Pf}(D_F)}{\sqrt{\det D_B}}. \tag{1.13}$$

Since the operators D_B and D_F preserve the tangent bundle to Z and the normal bundle to Z (in other words D_B decomposes into operators

$$D_B = {D_B}^{\|} \oplus {D_B}^{\perp},$$

and similarly for D_F: this will follow from the decomposition (1.10)-(1.11) of the Lagrangian given below) our path integral decomposes as

$$\int_Z \frac{\mathrm{Pf}({D_F}^{\|})}{\sqrt{\det {D_B}^{\|}}} \frac{\mathrm{Pf}({D_F}^{\perp})}{\sqrt{\det {D_B}^{\perp}}}. \tag{1.14}$$

The computation of

$$\frac{\mathrm{Pf}({D_F}^{\|})}{\sqrt{\det {D_B}^{\|}}}$$

is identical to the computation presented in the lecture and yields

$$\hat{A}(F^{\|} = \frac{1}{(2\pi)^{\dim Z/2}} \prod_{a=1}^{\dim Z/2} \frac{F_a^{\|}/2}{\sinh F_a^{\|}/2}. \tag{1.15}$$

$\hat{A}(Z) = \hat{A}(F^{\|}/(2\pi))$ represents the A-hat genus of the fixed point set Z. Here, we have decomposed the curvature $F^{\|}$ of the Levi-Civita connection of Z (which is a matrix valued two-form on Z with values in skew-symmetric square matrices of dimension $\dim Z$) into its Chern roots: in other words we have skew-diagonalized it and written it in the form

$$F^{\|} = \begin{pmatrix} 0 & F^1 & 0 & 0 & 0\ldots & 0 \\ -F^1 & 0 & 0 & 0 & 0\ldots & 0 \\ 0 & 0 & 0 & F^2 & 0\ldots & 0 \\ 0 & 0 & -F^2 & 0 & 0\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \ldots \end{pmatrix} \tag{1.15}$$

The computation of the determinants for ${D_B}^{\perp}$ and ${D_F}^{\perp}$ differs from the computation presented in the lecture only because new periodicity conditions (1.5) for the fields have been introduced. We first observe that since g is orientation preserving, the normal bundle $\mathcal{N}$ to each component of the fixed point set Z of g is even-dimensional. Moreover, the multiplicity of the eigenvalue -1 is even. Therefore the fiber $\mathcal{N}_{x_0}$ over $x_0 \in Z$ decomposes into subspaces V_b invariant under g, $b = 1, \ldots, (\dim_{\mathbb{R}} \mathcal{N})/2$. Each V_b has real dimension 2 and g acts on V_b as

$$g = \begin{pmatrix} \cos\vartheta_b & \sin\vartheta_b \\ -\sin\vartheta_b & \cos\vartheta_b \end{pmatrix} \tag{1.16}$$

Here a pair of eigenspaces of eigenvalue -1 corresponds to one V_b with $\vartheta_b = \pi$. (Alternatively, identifying each eigenspace V_b with $\mathbb{C}$ the isometry g acts as multiplication by $e^{i\vartheta_b}$. Of course, here the ϑ_b need not be distinct.) Since the isometry g preserves the Levi-Civita connection, the basis of the normal bundle which skew-diagonalizes the curvature (as in (1.15)) may be chosen to coincide with the basis

of eigenspaces of g. Thus we may compute the action of ${D_B}^{\perp}$ and ${D_F}^{\perp}$ acting on paths $\alpha_b^{\perp}, \eta_b^{\perp}$ taking values in one eigenspace V_b.

In fact, it is convenient to replace $\alpha_b^{\perp}$ and $\eta_b^{\perp}$ by paths $\alpha_{b,\pm}^{\perp}, \eta_{b,\pm}^{\perp}$ in the complexification $V_b \otimes \mathbb{C}$: this decomposes naturally as $V_b \otimes \mathbb{C} = V_b^+ \oplus V_b^-$, where g acts on $V_b^{\pm}$ with the eigenvalue $e^{\pm i\vartheta_b}$. Thus we may convert the endpoint conditions (1.5) to periodic endpoint conditions by writing

$$\alpha_{b,\pm}^{\perp}(t) = \exp(\pm it\vartheta_b)\zeta_b^{\pm}(t), \tag{1.17}$$

$$\eta_{b,\pm}^{\perp}(t) = \exp(\pm it\vartheta_b)\xi_b^{\pm}(t), \tag{1.18}$$

where $\zeta_b^{\pm} : \mathbb{R} \to V_b^{\pm}$ and $\xi_b^{\pm} : \mathbb{R} \to \Pi V_b^{\pm}$ satisfy the usual periodic boundary conditions $\zeta_b^{\pm}(t+1) = \zeta_b^{\pm}(t)$, $\xi_b^{\pm}(t+1) = \xi_b^{\pm}(t)$.

If we substitute (1.17) into the definition of D_B we find that the operator D_B acting on the variables $\zeta_b^{\pm}$ becomes instead (using the decomposition of $F^{\perp}$ corresponding to (1.15), where the eigenspaces are the eigenspaces $V_b^{\pm}$: the subspaces which diagonalize the curvature coincide with the eigenspaces of g because the isometry g is assumed to preserve the Riemannian metric on M)

$$(D_B)_b^{\pm} = -(\frac{d}{dt} \pm i\vartheta_b)^2 \pm F_b(\frac{d}{dt} \pm i\vartheta_b). \tag{1.19}$$

Since the eigenvalues of $\frac{d}{dt}$ on the ζ_b are $2\pi in$ for $n \in \mathbb{Z}$, we find that the eigenvalues are given by

$$(2\pi in \pm i\vartheta_b)^2 \pm F_b(2\pi in \pm i\vartheta_b). \tag{1.20}$$

To compute the determinant of D_B we take the product over all the eigenvalues.

The computation of the character-valued index differs from the ordinary index in that the fermionic Pfaffian now depends on the values of the eigenvalues ϑ_b for the isometry. Substituting (1.18) into $D_F = id/dt$ we find that the eigenvalues of D_F acting on the periodic fermionic paths $\xi_b^{\pm}$ are

$$2\pi n \pm \vartheta_b.$$

We shall make use of the identity

$$\prod_{n \geq 1}(1 - \frac{z^2}{4\pi^2 n^2}) = \frac{\sin z/2}{z/2} \tag{1.21}$$

Combining all factors we find

$$\det D_B = \prod_{\pm} \prod_{n \in \mathbb{Z}} \prod_{b} (2\pi in + i\vartheta_b)^2 \pm F_b(2\pi in \pm i\vartheta_b) \tag{1.22}$$

while the fermionic pfaffian is given by

$$(\operatorname{pfaff} D_F)^2 = \prod_{\pm} \prod_{b} \prod_{n} (2\pi n \pm \vartheta_b). \tag{1.23}$$

Combining (1.22) and (1.22) we see that (up to a multiplicative constant from regularization)

(1.24)
$$\frac{(\text{pfaff}\, D_F)^2}{\det D_B} = \prod_{n\in\mathbb{Z}} \prod_{b,\pm} \frac{1}{2\pi n \pm (\vartheta_b + iF_b)},$$
$$= \prod_b \frac{1}{\vartheta_b + iF_b} \prod_{\pm} \prod_{n\geq 1} \frac{1}{4\pi^2 n^2 - (\vartheta_b + iF_b)^2} = \prod_b \frac{1}{2\sin\Big((\theta_b + iF_b)/2\Big)}.$$

The final answer for the character-valued index of the Dirac operator is then

(1.25)
$$\text{Index}_g(\mathcal{D}) = \frac{1}{(2\pi)^{\dim Z/2}} \int_Z \hat{A}(F^{\|}) \prod_{b=1}^{\dim_{\mathbb{R}} \mathcal{N}/2} \frac{1}{2\sin\Big((\vartheta_b + iF_b)/2\Big)}$$
$$= \int_Z \hat{A}(Z) \prod_{b=1}^{\dim_{\mathbb{R}} \mathcal{N}/2} \frac{1}{2\sin\Big((\vartheta_b + iF_b/(2\pi))/2\Big)}.$$

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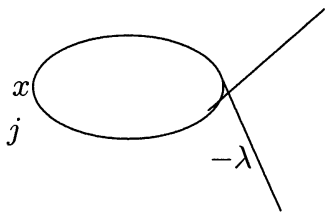
Solution to Problem FE2

Consider the Lagrangian density in Euclidean space

$$\mathcal{L} = \frac{1}{2}|d\phi|^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - j\phi^2.$$

The coefficient j is renormalized to 1-loop by the diagram shown,



which contributes to the correlation function $\langle\phi^2\phi\phi\rangle$. Evaluating in momentum space, we obtain

$$\frac{-j\lambda}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2+m^2)\left((k-q)^2+m^2\right)},$$

where the factor of 2 comes from the symmetry of the diagram. This is logarithmically divergent, and cutting off at Λ we compute the divergence as $\Lambda \longrightarrow \infty$ to be

$$\frac{-j\lambda}{2} \cdot \frac{1}{(2\pi)^4} \cdot \text{vol}\, S^3 \cdot \ln \Lambda/\mu = \frac{-j\lambda}{16\pi^2} \ln \Lambda/\mu,$$

where μ is a fixed constant.

To cancel this divergence, we renormalize j to

$$\hat{j}_0 = j \left(1 + \frac{\lambda}{16\pi^2} \ln \Lambda/\mu\right).$$

The anomalous dimension of j is computed as $\gamma_j = \frac{\mu}{j} \frac{\partial j}{\partial \mu}$ with $\hat{j}_0$ fixed and $\Lambda \longrightarrow \infty$. Thus

$$0 = \gamma_j \left(1 + \frac{\lambda}{16\pi^2} \ln \Lambda/\mu\right) - \frac{\lambda}{16\pi^2} + \frac{\beta}{16\pi^2} \ln \Lambda/\mu,$$

where $\beta = \mu \frac{\partial \lambda}{\partial \mu}$ is the β-function. Since $\gamma_j = O(\lambda)$ and $\beta = O(\lambda^2)$ we find $\gamma_j = \frac{\lambda}{16\pi^2}$. Now the anomalous dimension of the Lagrangian vanishes, and so the anomalous dimension γ_{ϕ^2} of ϕ^2 equals $-\gamma_j$, or

$$\gamma_{\phi^2} = -\frac{\lambda}{16\pi^2}.$$

Solution to Problem FE3

(a) We write

$$Z(g) = \int_{-\infty}^{\infty} d\phi\; e^{-\phi^2/2 - g\phi^4/4!} \sim \sum_{k=0}^{\infty} a_k g^k$$

so that by Taylor's formula

$$a_k = \frac{(-1)^k}{k!(4!)^k} \int_{-\infty}^{\infty} d\phi\; e^{-\phi^2/2}\, \phi^{4k}. \tag{3.1}$$

We can evaluate the integral in (3.1) exactly: set

$$\begin{aligned} F(J) &= \int_{-\infty}^{\infty} d\phi\; e^{-\phi^2/2 + J\phi} \\ &= \int_{-\infty}^{\infty} d\phi\; e^{-(\phi+J)^2/2}\; e^{J^2/2} \\ &= \sqrt{2\pi}\; e^{J^2/2}. \end{aligned}$$

The integral in (3.1) is the derivative

$$F^{(4k)}(J=0) = \sqrt{2\pi}\; \frac{(4k)!}{(2k)!2^{2k}}, \tag{3.2}$$

so from (3.1) we have

$$a_k = \frac{(-1)^k (4k)!\ \sqrt{2\pi}}{k!(24)^k (2k)!\ 2^{2k}}. \tag{3.3}$$

Now we apply Stirling's formula

$$n! \sim \sqrt{2\pi}\ \left(\frac{n}{e}\right)^n\ \sqrt{n}$$

to deduce

$$a_k \sim (-1)^k\ \left(\frac{2k}{3e}\right)^k\ \sqrt{\frac{2}{k}}. \tag{3.4}$$

Alternatively, write the integral in (3.1) as

$$\int_{-\infty}^{\infty} d\phi\ \phi^{4k}\ e^{-\phi^2/2} = \int_{-\infty}^{\infty} d\phi\ e^{-\phi^2/2 + 2k \ln \phi^2} \tag{3.5}$$

and find the maximum of the exponent at $\phi = \pm\sqrt{2k}$. At these points the second derivative of the exponents is -2, so the integral to a Gaussian approximation at either of these maxima is

$$\begin{aligned}\int_{-\infty}^{\infty} d\phi\ e^{-\phi^2/2 + 2k \ln \phi^2} &\sim (4k)^{2k}\ e^{-2k} \int_{-\infty}^{\infty} d\tilde{\phi}\ e^{-2\tilde{\phi}^2/2} \\ &= (4k)^{2k}\ e^{-2k}\ \sqrt{\pi}.\end{aligned}$$

Since there are two such maxima we find from (3.1) that

$$a_k \sim \frac{(-1)^k\, 2(4k)^{2k}\ e^{-2k}\ \sqrt{\pi}}{k!(24)^k}.$$

Using Stirling's formula we obtain agreement with (3.4).

(b) According to the Feynman calculus $|a_k|/\sqrt{2\pi}$ counts the number of equivalence classes of graphs with k 4-valent vertices, each graph Γ counted with a factor of $1/\sharp \mathrm{Aut}\Gamma$. We can check this directly by counting. Start with k 4-valent vertices, with the vertices numbered $1, 2, ..., k$ and the edges of each vertex labeled 1,2,3,4. Then each edge has a number ab attached, where a is the vertex number and b the label of the edge at that vertex. Now there is an obvious ordering of all edges (11, 12, 13, 14, 21, 22, 23, 24, etc.). We must attach the edges. For the first edge there are $(4k-1)$ choices, for the next available edge $(4k-3)$ choices, etc. So the total number of labeled graphs is

$$1 \cdot 3 \cdot 5 \cdot \cdots \cdot (4k-1) = \frac{(4k)!}{(2k)!\ 2^{2k}},$$

and since there are $k!$ automorphisms of the vertex numberings and $(4!)^k$ automorphisms of the edge labelings, our counting is consistent with (3.1).

(c) Now for the path integral we have, analogously to (3.1),

$$a_k = \frac{(-1)^k}{k!(4!)^k} \int D\phi \; e^{-\frac{1}{2}\int d^n x(|d\phi|^2+m^2\phi^2)} \left(\int d^n x \, \phi^4\right)^k. \tag{3.6}$$

As in (3.5) we rewrite the integrand in (3.6) as

$$\exp\left\{-\frac{1}{2}\int d^n x \, (|d\phi|^2 + m^2\phi^2) + k \ln \int d^n x \; \phi^4\right\}.$$

To find the maximum of the exponent we differentiate at ϕ in the direction $\dot{\phi}$ and set the result to zero:

$$0 = \int d^n x \, (-\Delta\phi - m^2\phi, \; \dot{\phi}) + \frac{4k \int d^n x \, \phi^3 \dot{\phi}}{\int d^n x \, \phi^4}$$

which implies

$$(\Delta + m^2)\phi = \frac{4k \, \phi^3}{\int d^n x \, \phi^4}. \tag{3.7}$$

(Our Laplacian Δ is nonnegative.)

We search for a spherically symmetric solution $\phi = \phi(r)$ for r the radial variable, so

$$\Delta\phi = -\frac{1}{r^{n-1}} \frac{d}{dr} \left(r^{n-1} \frac{d\phi}{dr}\right).$$

Thus (3.7) becomes

$$-\frac{d^2\phi}{dr^2} - \frac{n-1}{r} \frac{d\phi}{dr} + m^2\phi = \frac{k\phi^3}{\pi^2 \int_0^\infty \phi^4(r) r^3 dr}.$$

By rescaling ϕ it suffices to solve the equation

$$-\frac{d^2\phi}{dr^2} - \frac{n-1}{r} \frac{d\phi}{dr} + m^2\phi = k\phi^3; \tag{3.8}$$

namely, we replace the solution ϕ to (3.8) by

$$\left[\pi^2 \int_0^\infty \phi^4(r) r^3 dr\right]^{-2} \phi.$$

Since we want $\int_0^\infty \phi^4(r) \, r^3 dr$ to be finite, we require

$$\lim_{r\to\infty} \phi(r) = 0. \tag{3.9}$$

Also, to make the first derivative term meaningful at $r = 0$, we require

$$\lim_{r\to 0} \phi'(r) = 0. \tag{3.10}$$

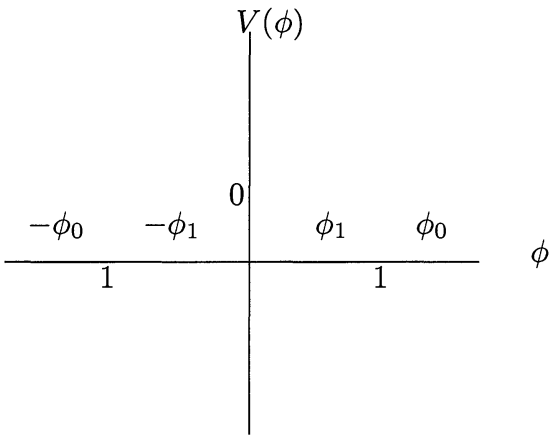
We now argue that a solution to (3.8) satisfying (3.9) and (3.10) exists.

For $n = 1$ equation (3.8) is Newton's equation

$$\frac{d^2\phi}{dr^2} = -V'(\phi) \tag{3.11}$$

for the potential

$$V(r) = \frac{k}{4}\phi^4 - \frac{m^2}{2}\phi^2.$$



There is a solution $\phi(r)$ to (3.11) with the desired boundary conditions where $\phi(0) = \phi_0$ is the positive zero of V. Picture a ball released in this potential at $\phi = \phi_0$. It rolls down and back up to $\phi = 0$, where it is at rest. For $n > 1$, we have

$$\frac{d^2\phi}{dr^2} = -V'(\phi) - \frac{n-1}{r}\frac{d\phi}{dr}.$$

In terms of our mechanical model, the ball feels an additional acceleration when moving to the left and an additional deceleration when moving to the right. The effect dies off with time (r) and is proportional to the speed. Also, the effect is bounded for small time if we have zero initial velocity (as prescribed by (3.9)). So now if we release the ball at $\phi = \phi_0$, it overshoots $\phi = 0$. Of course, if we release the ball at $\phi = \phi_1$, the minimum of the potential, it stays at rest. By continuity there is some value $\bar{\phi}$ between ϕ, and ϕ_0 where the ball released at $\phi = \bar{\phi}$ will roll exactly up to $\phi = 0$ as desired.

Solution to Problem FE4

In this solution we follow the sign conventions of Part 1 (see [I-**Signs**]), which are different than those used in the statement of the problem. In particular, the expression $\langle\bar{\psi}\psi\rangle$ is purely imaginary. All lagrangians are in Euclidean space. For a closely related discussion, see Lecture 4 of [I-**Renorm Groups**].

Let ψ be the vector with components ψ^i, so that ψ is a section of $(S^+\oplus S^-)\otimes\mathbb{C}^n$. After rescaling ψ we write the lagrangian as

$$L(\psi) = \frac{1}{2}\langle\bar{\psi}(\not{D} + im_0)\psi\rangle - \frac{g_0}{8n}\langle\bar{\psi}\psi\rangle^2, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ is the usual bilinear form on $\mathbb{C}^n$ and we omit writing '$|d^2x|$'. Introduce a real scalar field σ and consider the lagrangian

$$L'(\psi, \sigma) = \frac{1}{2}\langle \bar{\psi}(\not{D} + im_0)\psi \rangle + \frac{i}{4}\sqrt{\frac{g_0}{n}}\sigma \langle \bar{\psi}\psi \rangle - \frac{\sigma^2}{8}. \tag{4.2}$$

The field σ is nondynamical and appears quadratically, so can be integrated out using its equation of motion $\sigma = i\langle \bar{\psi}\psi \rangle / \sqrt{n}$. Thus we see that (4.2) is equivalent to (4.1). We are asked to investigate the one-point function $\sigma_0 = \langle \sigma \rangle / \sqrt{n}$ in the large n limit.

We integrate out ψ and expand the effective action around the unknown σ_0, which is defined by asking that there be no linear term in the effective potential (so that σ_0 is a critical point of the effective potential). Thus set

$$\sigma = \sigma' + \sqrt{n}\,\sigma_0,$$

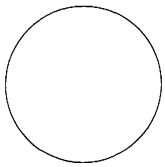
and substitute into (4.2):

$$L''(\psi, \sigma') = \frac{1}{2}\langle \bar{\psi}\,(\not{D} + im_0 + \frac{i\sqrt{g_0}\,\sigma_0}{4})\,\psi \rangle + \frac{i}{4}\sqrt{\frac{g_0}{n}}\,\sigma'\,\langle \bar{\psi}\psi \rangle \tag{4.3}$$
$$- \frac{\sigma'^2}{8} - \frac{\sqrt{n}}{4}\sigma_0\,\sigma' - \frac{n}{8}\sigma_0^2.$$

The linear term vanishes if

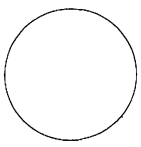
$$\sigma_0 = -\frac{4I}{\sqrt{n}}, \tag{4.4}$$

where I is the value of the Feynman diagram:



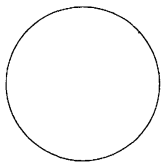
We use solid lines for the field ψ and dotted lines for σ.

The value of σ_0 determined implicitly by (4.4) is not the full quantum expectation value $\langle \sigma \rangle / \sqrt{n}$. Rather, the effective potential for σ'—in particular, the term linear in σ'—receives quantum corrections from perturbation theory in σ', but all such corrections vanish as $n \to \infty$. In fact, the terms in the effective lagrangian for σ' which depend on the qth power of the field are computed from the diagram



with q external legs. Such a diagram is $O(n^{1-q/2})$, since the loop contributes n and each vertex contributes $n^{-1/2}$. Thus every vertex in the effective theory for σ' contributes a strictly negative power of n. This means that in the large n limit, σ_0 does indeed compute the expectation value $\langle\sigma\rangle/\sqrt{n}$. Note that the first correction to the large n limit we compute is given by the diagram

in the σ' theory, which is the diagram



in the theory (4.3). It is $O(n^{-1/2})$.

We evaluate I. The ψ propagator in momentum space is

$$\frac{1}{\not{q} + i(m_0 + \frac{\sqrt{g_0}\,\sigma_0}{4})},$$

where $\not{q}$ denotes Clifford multiplication by the momentum q. The vertex is $\frac{i}{4}\sqrt{\frac{g_0}{n}}$. The loop instructs us to compute $\int \frac{d^2q}{(2\pi)^2}\,\mathrm{Tr}$. Set

$$M = m_0 + \frac{\sqrt{g_0}\,g\sigma_0}{4}.$$

Using $\not{q}^2 = -|q|^2$ (see [**I-Classical Fields**, §7.4]) and $\mathrm{Tr}\not{q} = 0$, we find

$$\frac{4}{\sqrt{g_0}} I = \frac{i}{\sqrt{n}} \int \overline{d^2q (2\pi)^2} \frac{-iMn}{-|q^2| - M^2} = -M\sqrt{n} \int_0^\infty \frac{d\rho}{2\pi}\, \frac{\rho}{\rho^2 + M^2},$$

or

$$-\frac{4}{M\sqrt{g_0 n}}\, I = \int_0^\infty \frac{d\rho}{2\pi}\, \frac{\rho}{\rho^2 + M^2}.$$

We regulate this integral by subtracting $\frac{1}{4\pi}\log\Lambda^2/\mu^2$ for a cutoff Λ and a scale μ. Then the renormalized value is

$$-\frac{4}{M\sqrt{g_0 n}} I = \frac{1}{4\pi}\log\frac{\mu^2}{M^2}$$

after taking $\Lambda \to \infty$. So from (4.4) we find the desired equation for σ_0:

$$\sigma_0 = \frac{M\sqrt{g_0}}{4\pi}\log\frac{\mu^2}{M^2}.$$

When $m_0 = 0$ we find the solution

$$\sigma_0 = \frac{4\mu}{\sqrt{g_0}} e^{-8\pi/g_0}.$$

The fermion mass is

$$\frac{\sqrt{g_0}}{4}\sigma_0 = \mu e^{-8\pi/g_0}.$$

The $\mathbb{Z}/2$ symmetry of (4.2) exchanges $\psi \leftrightarrow \bar{\psi}$ and $\sigma \leftrightarrow -\sigma$. The nonzero value of σ_0 breaks this symmetry.

Solution to Problem FE5

First, a reminder about the renormalization group equation. A perturbative theory depends on some coupling constants g and masses m. (For simplicity we assume there is just one of each.) Renormalization introduces a scale μ. The renormalization group operator is

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma_m m \frac{\partial}{\partial m},$$

where β and γ_m (the *anomalous* dimension of mass) are in general functions of g and $\frac{m}{\mu}$. They describe how the parameters of the theory flow when μ is varied. Their precise form depends on the renormalization procedure; for dimensional regularization/minimal subtraction, they are independent of $\frac{m}{\mu}$. Now any operator $\mathcal{O}$ in the theory has an anomalous dimension $\gamma_{\mathcal{O}}$ (which is also a function of g and $\frac{m}{\mu}$). Since the operators are renormalized they also depend on μ, and the meaning of the anomalous dimension is that they satisfy the renormalization group equation

$$\mathcal{D}\mathcal{O} = \gamma_{\mathcal{O}}\mathcal{O}.$$

In general the operators mix under renormalization: the renormalization of a given operator $\mathcal{O}_0$ involve operators of equal and lower dimension. Thus $\gamma_{\mathcal{O}}$ is a triangular matrix acting on operators of dimension $\leq k$ for some fixed k. Of course, the mixing only occurs among operators with the same symmetries. In the renormalization group equation $\mathcal{D}\mathcal{O} = \gamma_{\mathcal{O}}\mathcal{O}$, then, $\mathcal{O}$ is a vector of operators and γ_0 a matrix (of functions of g and $\frac{m}{\mu}$). Similar remarks apply to equation (5.1) below, though we write it as if there is no mixing. Notice that this holds for the elementary operators (i.e., the fields which appear in the Lagrangian) as well as for composite operators. What defines β, γ_m, and $\gamma_{\mathcal{O}}$ is that correlation functions are preserved under the renormalization group flow. That is, for operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ and *distinct* points $x_1, \ldots, x_n$, we have

$$\mathcal{D}\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle = (\gamma_{\mathcal{O}_1} + \cdots + \gamma_{\mathcal{O}_n})\ \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle. \tag{5.1}$$

The correlation function blows up as some points x come together, and if we consider coinciding points this necessitates further renormalization. Such renormalizations will introduce additional terms in (5.1).

(a) The function B is defined by

$$\lim_{\Lambda\to\infty}\left\{\int_{|x|\geq\frac{1}{\Lambda}} d^4x\, e^{-ik\cdot x}\,\langle J^i(x)J^j(0)\rangle - A\left({}^{\Lambda}/_{\mu}, g_{n}/_{\mu}\right)\,(k^ik^j-\delta^{ij}k^2)\right\}$$

$$= B(k^2,g,\mu)\,(k^ik^j-\delta^{ij}k^2), \tag{5.2}$$

where $g_{\Lambda/\mu}$ is the effective coupling constant with initial value g_1. Since there are no masses the renormalization group operator is

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}.$$

We are given $\gamma_J = 0$, so differentiating both sides of (5.2) using (5.1), we obtain

$$\mathcal{D}B = -\lim_{\Lambda\to\infty}\mathcal{D}A \tag{5.3}$$

as the renormalization group equation of B. There is no operator mixing since J is the lowest dimensional vector operator.

Now asymptotic freedom means $\lim_{\Lambda\to\infty} g_\Lambda = 0$ and so to leading order

$$\lim_{\Lambda\to\infty}\mathcal{D}A = \lim_{\Lambda\to\infty}\mathcal{D}A(\Lambda/\mu,0) = \lim_{\Lambda\to\infty}\mu\frac{\partial}{\partial\mu}\,(a\ln\Lambda/\mu) = -a.$$

Hence (5.3) becomes

$$\mathcal{D}B = a. \tag{5.4}$$

(b) It follows from (5.2) that the naive engineering dimension of B is zero. Thus simple scaling combined with (5.4) gives

$$\begin{aligned} B(e^{2t}k_0^2,\, g_1, \mu) &= B(k_0^2,\, g_1,\, e^{-t}\mu) \\ &= B(k_0^2,\, g_{e^t},\, \mu) - at. \end{aligned} \tag{5.5}$$

Now take $k_0^2 = 1$ and set $k^2 = e^{2t}$. Then taking $t\to\infty$ we learn

$$B(k^2,g_1,\mu) \sim -\frac{a}{2}\ln k^2/\mu^2 \qquad \text{as} \qquad k\longrightarrow\infty.$$

Observe that this is the answer for the free theory. In other words, in the absence of anomalous dimensions the high momentum behavior of a quantity in an asymptotically free theory is the same as the high momentum behavior in the free theory.

Solution to Problem FE6

We recall that in a quantum field theory the correlation function (better, distribution) $F(x_1,\dots,x_n) = \langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle$ is initially defined for $x_1,\dots,x_n$ distinct points in Minkowski space V. It is the boundary value of an *analytic* function on a connected domain $\mathcal{D}\subset(V_{\mathbb{C}})^{\times n}$ where $\mathcal{D}$ includes:

(1) The subspace of $E^{\times n}$ consisting of n distinct points where E is Euclidean space; and

(2) an open set in $V^{\times n}$ consisting of n spacelike separated points.

(The precise description of some admissible open sets in (2) is given in the lemma of Jost.) The restriction of F to either of the two sets described determines F.

Now suppose $\mathcal{O}$ is an operator which annihilates the vacuum: $\mathcal{O}(x)|\Omega\rangle = 0$. We would like to show that $\mathcal{O} = 0$, which means that any correlation function involving $\mathcal{O}$ vanishes identically. For this we restrict the correlation function to a set of the form (2) above, and then

$$\begin{aligned}\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}(x) \cdots \mathcal{O}_n(x_n)\rangle &= \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\mathcal{O}(x)\rangle \\ &= \langle \Omega | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\mathcal{O}(x)|\Omega\rangle \\ &= 0.\end{aligned}$$

For the second assertion, we suppose $\langle \overline{\mathcal{O}(y)}\mathcal{O}(x)\rangle = 0$ for distinct points x, y in Euclidean space; we want to show that any correlation function with $\mathcal{O}$ vanishes (in Euclidean space). Let π denote the perpendicular bisecting hyperplane to the segments joining x and y, and denote by $\overline{p}$ the reflection of p in π. Thus $y = \overline{x}$. Now let

$$\begin{aligned}\mathcal{O}' &= \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n), \\ \overline{\mathcal{O}'} &= \overline{\mathcal{O}_1(\overline{p}_1)} \cdots \overline{\mathcal{O}_n(\overline{p}_n)},\end{aligned}$$

denote some product of operators. We must show $\langle \mathcal{O}(x)\mathcal{O}'\rangle = 0$. Consider

$$\langle (\overline{\mathcal{O}(\overline{x})} + \lambda\mathcal{O}')\ (\mathcal{O}(x) + \lambda\overline{\mathcal{O}'})\rangle.$$

By reflection positivity this is nonnegative. Set

$$\lambda = \frac{-\mathrm{Re}\langle \mathcal{O}(x)\mathcal{O}'\rangle}{\langle \mathcal{O}'\overline{\mathcal{O}'}\rangle}$$

to conclude

$$(\mathrm{Re}\langle \mathcal{O}(x)\mathcal{O}'\rangle^2 \leq \langle \overline{\mathcal{O}(\overline{x})}\mathcal{O}(x)\rangle\ \langle \mathcal{O}'\overline{\mathcal{O}'}\rangle = 0.$$

Thus $\mathrm{Re}\langle \mathcal{O}(x)\mathcal{O}'\rangle = 0$. The arguments applied to $\sqrt{-1}\mathcal{O}'$ show the imaginary part vanishes as well.

Solution to Problem FE7

(a) Let $\Lambda_\pm(x)$ denote the space of (anti-)self-dual 2-forms at $x \in S^4$. The two-point function $\langle F_+(x)\, F_-(y)\rangle$ takes values in $\Lambda_+(x) \otimes \Lambda_-(y)$. We claim that any conformally-invariant tensor on S^4 with values in $\Lambda_+(x) \otimes \Lambda_-(y)$ vanishes. For we can use the conformal group to move x to 0 and y to ∞. Then $\langle F_+(0)\, F_-(\infty)\rangle \in \Lambda_+(0) \otimes \Lambda_-(\infty)$ must be invariant under the group SO_4 of rotations. But the representation of SO_4 on $\Lambda_+(0) \otimes \Lambda_-(\infty)$, expressed in terms of the double cover $\mathrm{Spin}_4 \cong SU_2 \times SU_2$, is the tensor product of the three-dimensional representations of each of the SU_2 factors. As such, it is irreducible.

(b) The Bianchi identity $dF = 0$ implies $dF_+ = -dF_-$. Now $d^* = - * d *$ acting on 2-forms, and since $*F_\pm = \pm F_\pm$ by definition, we easily find

$$d^*F = 2 * dF_- = -2 * dF_+.$$

Thus

$$\langle d^*F(x)\, d^*F(y)\rangle = -4\langle *dF_+(x)\ *dF_-(y)\rangle.$$

This vanishes by differentiating $\langle F_+(x)\, F_-(y)\rangle = 0$ using $*d$ in both variables.

(c) Since the operator d^*F is real, we deduce from $\langle d^*F(x)\, d^*F(y)\rangle = 0$ and **FE**6 that $d^*F = 0$. Together with the Bianchi identity this means that F obeys the equations of the free field. To illustrate why this implies that F *is* a free field, we sketch the argument for a scalar field ϕ. That is, assume that ϕ is a scalar field which satisfies the Klein-Gordon equation $\Delta_m\phi = 0$, where $\Delta_m = \Delta + m^2$ is the nonnegative laplacian plus m^2. (Recall that we work in Euclidean space.) Then we argue that ϕ is a free field.

Consider first the 2-point function

$$G(x) = \langle \phi(x)\, \phi(0)\rangle,$$

which is defined for $x \neq 0$. By hypothesis, $\Delta_m G(x) = 0$ for $x \neq 0$. On the other hand, we have the spectral representation of the 2-point function, which asserts that the Fourier transform of G is

$$\hat{G}(k) = \int_0^\infty d\mu(M)\, \frac{1}{k^2 + M^2}$$

for some positive measure μ. (Here we are assuming that G is a Schwartz distribution.) Now

$$\widehat{\Delta_m G}(k) = \int_0^\infty d\mu(M)\, \frac{k^2 + m^2}{k^2 + M^2}$$

must be a polynomial function of k, and imposing the condition that some finite derivative vanish it is clear that this forces $d\mu(M)$ to be concentrated at $M = m$ and hence a multiple of the δ-function at m. This means that G is the Green's function (up to a multiple, which we absorb by rescaling ϕ). Of course, this is the 2-point function of the free field.

Next, we argue that the 4-point function is the 4-point function of a free field. Fix distinct points y_1, y_2, y_3 and consider

$$\begin{aligned} g(x) = \langle \phi(x)\phi(y_1)\phi(y_2)\phi(y_3)\rangle &- \langle \phi(x)\phi(y_1)\rangle\langle \phi(y_2)\phi(y_3)\rangle \\ &- \langle \phi(x)\phi(y_2)\rangle\langle \phi(y_1)\phi(y_3)\rangle - \langle \phi(x)\phi(y_3)\rangle\langle \phi(y_1)\phi(y_2)\rangle. \end{aligned}$$

This is defined for x distinct from y_i. We need to show that g vanishes. By hypothesis we have $\Delta_m g(x) = 0$ for $x \neq y_i$. The cluster decomposition property implies $g \to 0$ as $x \to \infty$. Now as $x \to y_1$, say, we use the operator product expansion to see that the leading singularity of $\langle \phi(x)\phi(y_1)\phi(y_2)\phi(y_3)\rangle$ cancels against the leading singularity of $\langle \phi(x)\phi(y_1)\rangle\langle \phi(y_2)\phi(y_3)\rangle$. Since this is the singularity of the Green's function $G(x - y_1)$, we conclude that $g(x)/G(x - y_1) \to 0$ as $x \to y_1$. To conclude that $g \cong 0$ we invoke the following lemma.

Lemma. *Suppose g is a Schwartz distribution on $\mathbb{R}^n$ which is smooth outside the origin; $g(x) \to 0$ and $dg(x) \to 0$ as $x \to \infty$; $\Delta_m g(x) = 0$ for $x \neq 0$; and $g(x)/G(x) \to 0$ as $x \to 0$, where G is the Green's function. Then g is identically zero.*

Proof. It suffices to show $\Delta_m g = 0$ for g considered as a distribution on $\mathbb{R}^n$. For then by elliptic regularity g is smooth, and the hypothesis that $g, dg \to 0$ at infinity implies that g vanishes. Now we are given that $\Delta_m g$ is supported at the origin, so $\Delta_m g$ is a finite linear combination of the δ-function and its derivatives. Thus g is a finite linear combination of G and its derivatives plus a smooth function f which satisfies $\Delta_m f = 0$. But any derivative H of G satisfies $H(x)/G(x) \to \infty$ as $x \to 0$, and of course $G(x)/G(x) \to 1$ as $x \to 0$. Since we are given $g(x)/G(x) \to 0$ as $x \to 0$, the only possibility is $\Delta_m g = 0$.

We treat higher correlation functions by an inductive argument.

Solution to Problem ASH1

For the solutions to the **Addendum to Superhomework** we follow the conventions of Part 1. In particular, we use the coordinates and Lie brackets given in [I-**Supersolutions**, §§2.4, 2.5].

We begin with some general remarks. A positive energy irreducible unitary representation of the Poincaré group is a space of sections of an irreducible equivariant vector bundle over an orbit of the Lorentz group acting on the dual to the vector space of translations of Minkowski space. Such a vector bundle is determined by an irreducible finite dimensional representation of the *little group*, which is the reductive part of the stabilizer of a point on the orbit. In n dimensions the little group is $\mathrm{Spin}(n-1)$ for massive representations (a timelike orbit) and $\mathrm{Spin}(n-2)$ for massless representations (a lightlike orbit). A deformation of a representation is given by an equivariant vector bundle over a family of orbits. Below we consider deformations of massless representations to massive representations, so construct equivariant vector bundles over a family of orbits including the lightcone.

In the super case the little group is a Lie supergroup. In that case the representation we seek is only projective; the nontrivial brackets of the odd generators lead to a central extension. The story with deformations is again told by equivariant vector bundles over a family of orbits.

Consider the lightlike vector p whose coordinates in the standard coordinate system x^0, x^1, x^2, x^3 are $(1, 1, 0, 0)$. Its coordinates in the $y^{a\dot{b}}$-coordinate system are $y^{1\dot{1}} = 1$, $y^{2\dot{2}} = y^{1\dot{2}} = y^{2\dot{1}} = 0$. Now according to [I-**Supersolutions**, §2.4] the translation part of the abstract super Poincaré algebra is generated by basis elements $e_{a\dot{b}}; f_a; \bar{f}_{\dot{a}}$, where the indices range over the set $\{1, 2\}$ and the nontrivial bracket is

$$[f_a, \bar{f}_{\dot{b}}] = -2e_{a\dot{b}}. \tag{1.1}$$

(The entire super Poincaré algebra has Lorentz generators as well.) In a representation of super Poincaré the lightlike "vector" p should be viewed as a linear *functional* via the Lorentz inner product, and a translation vector v in Minkowski space acts as multiplication by $e^{i\langle p,v\rangle}$ in the representation. (We use $i = \sqrt{-1}$.) Hence the infinitesimal translation $e_{a\dot{b}}$ acts as multiplication by $i\langle p, e_{a\dot{b}}\rangle$. For the specific p described above this means that if $\hat{f}_a, \widehat{\bar{f}}_{\dot{b}}$ are operators which represent the odd generators, then

$$[\hat{f}_1, \widehat{\bar{f}}_{\dot{1}}] = -2i \tag{1.2}$$

with all other brackets zero. As explained in the solution to **FP**16 and [I-**Quantization**], if we write

$$\hat{f}_a = i^{-1/2} F_a, \tag{1.3}$$

then the $F_a, \overline{F}_{\dot{a}}$ satisfy a Clifford relation

$$[F_1, \overline{F}_{\dot{1}}] = 2 \tag{1.4}$$

with all other brackets zero. A complex representation of this degenerate Clifford algebra is a graded vector space

$$W = W^0 \oplus W^1. \tag{1.5}$$

An irreducible representation has $\dim W^0 = \dim W^1 = 1$.

The generator J of the stabilizing $\mathfrak{spin}(2)$ algebra is one half infinitesimal rotation in the x^2-x^3 plane. Its brackets with $f_a, \bar{f}_{\dot{a}}$ are

$$\begin{aligned} [J, f_1] &= \frac{i}{2} f_1, & [J, \bar{f}_{\dot{1}}] &= -\frac{i}{2} \bar{f}_{\dot{1}}, \\ [J, f_2] &= -\frac{i}{2} f_2, & [J, \bar{f}_{\dot{2}}] &= \frac{i}{2} \bar{f}_{\dot{2}}. \end{aligned} \tag{1.6}$$

So the action of J on $W = W^0 \oplus W^1$ is diagonal, and $\mathfrak{spin}(2)$ acts via representations labeled by $j, j + 1/2$ for some $j \in \frac{1}{2}\mathbb{Z}$.

We summarize the representations of interest to us in the chart below, where we indicate the dimension of the spin j representation which occurs in W.

(1.7)

	-1	$-1/2$	0	$1/2$	1
C_+			1	1	
C_-		1	1		
V_+				1	1
V_-	1	1			
$C_1 \oplus C_-$		1	2	1	
$V_+ \oplus V_-$	1	1		1	1

CPT invariance in 4 dimensions requires that the helicities of a representation be symmetric about 0. Thus only the last two representations can occur in a quantum field theory.

The lagrangian for a massless chiral superfield is described in [I-**Supersolutions**, §5.2], where it is referred to as the "linear case". The physical degrees of freedom in components are a complex scalar field ϕ and a spinor field ψ. The lagrangian—after eliminating an auxiliary field—is the free lagrangian for these fields. The one-particle states are obtained by solving the (wave) equation of motion, complexifying, and polarizing, as described in Lecture 3 of [I-**Axiomatics**]. The representations of Poincaré obtained are

(1.8)

	-1	$-1/2$	0	$1/2$	1
ϕ			2		
ψ		1		1	

The sum of these representations *is* a representation of super Poincaré, since the lagrangian is supersymmetric, and comparison with (1.7) shows it must be $C_+ \oplus C_-$.

The lagrangian for the vector multiplet is given in [**I-Supersolutions**, §8.2]. (Take the gauge group $G = U(1)$ with coupling constant $\tau = 4\pi i$—equivalently, $g = 1$, $\theta = 0$.) After eliminating an auxiliary field we obtain the free lagrangian for a $U(1)$ connection A and a dual spinor field λ. Upon quantization these give the representation of Poincaré described by

(1.9)

	-1	$-1/2$	0	$1/2$	1
A	1				1
λ		1		1	

By supersymmetry of the lagrangian the sum is a representation of super Poincaré, and comparison with (1.7) shows it must be $V_+ \oplus V_-$.

Solution[11] to Problems ASH2 and ASH3

We continue the exposition in **ASH**1.

Now take $p = (2, 0, 0, 0)$ in the x^0, x^1, x^2, x^3 coordinate system, which is $y^{1\dot{1}} = 1$, $y^{2\dot{2}} = 1$, $y^{1\dot{2}} = y^{2\dot{1}} = 0$. Then the operators corresponding to the supersymmetry generators satisfy (see (1.2))

$$[\hat{f}_1, \widehat{\overline{f}}_{\dot{1}}] = [\hat{f}_2, \widehat{\overline{f}}_{\dot{2}}] = -2i \tag{2.1}$$

with other brackets zero. After scaling as in (1.3) we obtain a *nondegenerate* Clifford algebra with 4 generators, so an irreducible complex Clifford module $W = W^0 \oplus W^1$ has $\dim W^0 = \dim S^1 = 2$.

The infinitesimal Lorentz transformation J described in (1.6) sits in the stabilizing $\mathfrak{spin}(3)$ algebra, and can be taken to be the infinitesimal generator of the maximal torus. It follows from the bracketing relations (1.6) that the eigenvalues of J on an *irreducible* W are $i = \sqrt{-1}$ times $j, j, j + 1/2, j - 1/2$ for some $j \in \frac{1}{2}\mathbb{Z}$. For this to be part of a representation of $\mathfrak{spin}(3)$ we must have $j = 0$. Other representations of the semidirect product of $\mathfrak{spin}(3)$ and the Clifford algebra are the tensor product of the irreducible W with a representation of $\mathfrak{spin}(3)$. The relevant cases for us are

(2.2)

	-1	$-1/2$	0	$1/2$	1
C'		1	2	1	
V'	1	2	2	2	1

Both C', V' satisfy CPT invariance.

It is easy to realize C' in terms of fields. As described in [**I-Supersolutions**, §5.3] we can add a potential term to the free chiral superfield via a *superpotential*, which is a holomorphic function $w\colon \mathbb{C} \to \mathbb{C}$. To obtain a free lagrangian we choose

$$W(z) = \frac{1}{2}mz^2, \tag{2.3}$$

[11]Solution by Daniel Freed

for $m \in \mathbb{C}$. The corresponding component lagrangian, after eliminating the auxiliary field, is that of a massive complex scalar ϕ and a massive spinor ψ. Upon quantization we obtain C'.

It is clear from this field theory description that $C_+ \oplus C_-$ in (1.7) deforms to C': simply let the mass m vary in the real interval $[0, \delta]$ for some $\delta > 0$. In terms of the direct analysis of representations of super Poincaré, we consider the curve of vectors p_δ with standard coordinates

$$p_\delta = (1, 1 - 2\delta, 0, 0). \tag{2.4}$$

The bracketing relations of the supersymmetry generators are then (see (1.2), (2.1))

$$[\hat{f}_1, \widehat{\overline{f}}_{\dot{1}}] = -2i(1 - \delta), \qquad [\hat{f}_2, \widehat{\overline{f}}_{\dot{2}}] = -2i\delta, \tag{2.5}$$

with all other brackets vanishing. For $\delta > 0$ we obtain a nondegenerate Clifford algebra, and it degenerates at $\delta = 0$. The little group representations we described above—for $C_+ \pm C_-$ if $\delta = 0$ and C' if $\delta \neq 0$—fit together smoothly and determine an equivariant vector bundle over the union of the lightcone and the positive timelike vectors. This is the geometric version of the assertion that $C_+ \oplus C_-$ deforms to C'.

For an explicit model, let $\Lambda^\bullet$ be the exterior algebra on 2 complex generators a_1, a_2. Then the representation of the odd generators is

$$\begin{aligned} \hat{f}_1 &= i^{-1/2}\sqrt{2(1-\delta)}\,\epsilon(a_1), & \widehat{\overline{f}}_{\dot{1}} &= i^{-1/2}\sqrt{2(1-\delta)}\,\iota(a_1), \\ \hat{f}_2 &= i^{-1/2}\sqrt{2\delta}\,\epsilon(a_2), & \widehat{\overline{f}}_{\dot{2}} &= i^{-1/2}\sqrt{2\delta}\,\iota(a_2) \end{aligned} \tag{2.6}$$

Here ϵ is exterior multiplication and ι is interior multiplication. J acts trivially on $\Lambda^0 \oplus \Lambda^2$, as $i/2$ on a_1, and as $-i/2$ on a_2. Hence $\Lambda^\bullet$ has its usual grading into even and odd subspaces. The group Spin(3) acts on Λ^1 via the spin representation and trivially on $\Lambda^0 \oplus \Lambda^2$.

Solution to Problem ASH4

We continue with the exposition of the previous problems.

The assertion that $V_+ \oplus V_- \oplus C_+ \oplus C_-$ deforms to V' again amounts to the existence of an equivariant vector bundle over the union of the lightcone and positive timelike vectors. For an explicit presentation, we work with the curve of vectors p_δ in (2.4) and simply tensor the family of representations $\Lambda^\bullet$ constructed in (2.6) with the fixed two dimensional representation R_2 of Spin(3). (The odd generators act trivially on R_2.)

The field theory described in the problem is a special case of the general model discussed in [I-**Supersolutions**, §8.3]. The relevant component lagrangian appears in [I-**Supersolutions**, §6.2], where in our case the gauge group is $G = U(1)$, the Kähler manifold is $X = \mathbb{C}$ with the usual $U(1)$ action, the superpotential W is zero, and the moment map $\mu\colon \mathbb{C} \to i\mathbb{R}$ is

$$\mu(z) = \frac{i}{2}(|z|^2 - r), \qquad r \in \mathbb{R}. \tag{4.1}$$

The fields in the model are

(4.2)
ϕ a complex scalar field which is a section of a complex line bundle
ψ a spinor field[12] which is a section of the same line bundle
λ a (dual) spinor field which we take to be real[13]
A a $U(1)$ gauge field

The lagrangian (function) is

$$L = -\frac{1}{2}|F_A|^2 + d_A\bar{\phi}\, d_A\phi + \bar{\lambda}\not{D}\lambda + \bar{\psi}\not{D}_A\psi - \frac{1}{2}(|\phi|^2 - r)^2 - \sqrt{2}(\lambda^a\bar{\phi}\psi_a + \bar{\lambda}^{\dot{a}}\phi\bar{\psi}_{\dot{a}}). \tag{4.3}$$

Note that the potential energy $V\colon \mathbb{C} \to \mathbb{R}$ is

$$V(z) = \frac{1}{2}(|z|^2 - r)^2. \tag{4.4}$$

The theory is *not* free: in addition to the quartic term in the potential, the lagrangian contains cubic terms which couple the scalar field to the spinor fields. The analysis of the representation of super Poincaré at tree level means that we ignore the cubic and quartic interaction terms and quantize the resulting free lagrangian.

A general reference for the remainder of this discussion, in particular the Higgs mechanism, is [I-**Classical Fields**, §2.10].

For $r = 0$ there are no mass terms and the representation of Poincaré agrees with $V_+ \oplus V_- \oplus C_+ \oplus C_-$. Since the lagrangian is supersymmetric, this is also the representation of super Poincaré obtained.

For $r < 0$ there is no classical solution with zero energy. The same is true for the quantum theory at tree level. On the other hand, any vacuum invariant under the super Poincaré group has zero energy, so in this case there is no supersymmetric vacuum—supersymmetry is broken in any vacuum.

For $r > 0$ there is a circle of vacua $\{|z|^2 = r\}$, all of which are gauge equivalent. Expand around $z_0 = \sqrt{r}$. Write $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. The real scalar field ϕ_2 can be gauged away, and to compute the mass of the field ϕ_1 we expand the potential around $\phi_1 = \sqrt{r}$:

$$\frac{1}{2}\big((\phi_1/\sqrt{2} - \sqrt{r})^2 + (\phi_2/\sqrt{2})^2 - r\big) = r\phi_1^2 + \cdots. \tag{4.5}$$

So ϕ_1 has mass square $2r$. The cubic interaction terms contribute

$$-\sqrt{2r}\,(\lambda^a\psi_a + \bar{\lambda}^{\dot{a}}\bar{\psi}_{\dot{a}}) \tag{4.6}$$

to the quadratic lagrangian at z_0; (4.6) is a mass term for the spinor fields. We compute the mass by solving the equations of motion, using the formulas of [I-**Supersolutions**, §2.4] as an aide. We find

$$\begin{aligned} \epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\partial_{a\dot{c}}\psi_b &= -\sqrt{2r}\,\bar{\lambda}^{\dot{d}} \\ \partial_{e\dot{d}}\bar{\lambda}^{\dot{d}} &= \sqrt{2r}\,\psi_e, \end{aligned} \tag{4.7}$$

[12]We use the standard notation for spinors on Minkowski 4-space. Thus if S is the real 4-dimensional spin representation of the Lorentz group, then the complexification is $S_{\mathbb{C}} \cong S' \oplus S''$ with $S'' \cong \overline{S'}$. Then 'ψ' denotes the S' component of the real spinor field. Note that S is self-dual.

[13]For a general gauge group G, this field is $\mathfrak{g}$-valued, so in our case pure imaginary. We take λ to be $-\sqrt{-1}$ times that imaginary field.

from which

$$\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}\partial_{e\dot{d}}\partial_{a\dot{c}}\psi_b = -2r\psi_e. \tag{4.8}$$

Multiply both sides of (4.8) by a formal variable E^e and use the identity $\epsilon^{ab}E^e + \epsilon^{be}E^a + \epsilon^{ea}E^b = 0$. We find

$$(\Box + 2r)\psi = 0, \tag{4.9}$$

and so ψ has mass square $2r$. Evidently λ has the same mass square. The gauge field A acquires mass from the $d_A\bar{\phi}\, d_A\phi$ term, which contributes $r|A|^2$ to the quadratic lagrangian at z_0. Thus its mass square is also $2r$. The real scalar, massive spinors, and massive vector together comprise the field content of the representation V' with mass square $2r$, and since the vacuum is supersymmetric, this is indeed the realized representation of super Poincaré at tree level.

Solution to Problem ASH5

The bracketing relations in the $N = 2$ supersymmetry algebra appear in [I-**Supersolutions**, §2.5].

The argument follows **ASH**1 closely. A basis for the odd generators is $\{f^i_a, \bar{f}_{\dot{a}i}\}$, where all indices run over $\{1, 2\}$. At $p = (1, 1, 0, 0)$ the corresponding operators have nontrivial brackets

$$[\hat{f}^1_1, \hat{\bar{f}}_{\dot{1}1}] = [\hat{f}^2_1, \hat{\bar{f}}_{\dot{2}2}] = -2i. \tag{5.1}$$

The minimal complex Clifford module $W = W^0 \oplus W^1$ in this case has $\dim W^0 = \dim W^1 = 2$. The bracket relations (1.6) generalize easily—the extra index goes along for the ride—and it is easy to see that the helicities are $j, j, j + 1/2, j - 1/2$ for some $j \in \frac{1}{2}\mathbb{Z}$. Recall that physically relevant representations are constrained by CPT.

The representations of interest are

(5.2)

	−1	−1/2	0	1/2	1
H_0		1	2	1	
H		2	4	2	
V	1	2	2	2	1

The representation H_0 is not realized in field theory. The representation H is realized in field theory by the *hypermultiplet*, which is constructed in [I-**Supersolutions**, §3.1] using a quaternionic-valued scalar field and two spinor fields.[14] The representation V is realized in field theory by the *vector multiplet*—a complex scalar, a spinor field, a dual spinor field, and a $U(1)$ gauge field—which is described in [I-**Supersolutions**, §10.2].

In the massive case the odd generators satisfy bracket relations which generalize (2.1) in **ASH**2. The minimal complex Clifford module $W = W^0 \oplus W^1$ has

[14] In fact, the discussion in [I-**Supersolutions**, §3.1] takes place in 6 dimensions. The model we want is the dimensional reduction to 4 dimensions.

$\dim W^0 = \dim W^1 = 8$. One model is the exterior algebra $\Lambda^\bullet$ on 4 generators which correspond to the f_a^i (cf. equation (2.6) in **ASH**2). Under J the generators f_1^i have weight $1/2$ while f_2^i have weight $-1/2$. So if Λ^0 has weight j, then the helicities which occur in $\Lambda^\bullet$ are

(5.3)

$j-1$	$j-1/2$	j	$j+1/2$	$j+1$
1	4	6	4	1

We are interested in $j = 0$, which we need by CPT if we are to have a representation with only 16 physical degrees of freedom:

(5.4)

	-1	$-1/2$	0	$1/2$	1
H'	1	4	6	4	1

This representation includes a massive vector, so we expect that in field theory it is realized using the Higgs mechanism. We will see this in **ASH**8.

Solution to Problem ASH6

As described in [I-**Supersolutions**, §§2.2, 2.5] we can understand the $N = 2$ supersymmetry group $P^{4|8}$ in 4 dimensions by dimensional reduction from the minimal supersymmetry group $P^{6|(8,0)}$ in 6 dimensions. In 6 dimensions we use coordinates $\{\tilde{y}^{pq}\}_{p<q}$ on Minkowski space, and in the dimensional reduction we set $\tilde{y}^{12} = \tilde{y}^{34} = 0$. (Note $\overline{\tilde{y}^{12}} = \tilde{y}^{34}$.) Dually, we set the corresponding translation generators $\tilde{e}_{12}, \tilde{e}_{34}$ in the supersymmetry algebra to zero. If instead we let $\tilde{e}_{12}$ be a central element Y, and so $\tilde{e}_{34}$ its conjugate $\overline{Y}$, we obtain the indicated central extension of the $N = 2$ superalgebra in 4 dimensions. The nontrivial brackets of the odd generators are

$$
\begin{aligned}
[f_a^i, \bar{f}_{\dot{b}j}] &= -2\delta_j^i e_{a\dot{b}} \\
[f_a^i, f_b^j] &= -2\epsilon^{ij}\epsilon_{ab} Y \\
[\bar{f}_{\dot{a}i}, \bar{f}_{\dot{b}j}] &= -2\epsilon_{\dot{a}\dot{b}}\epsilon_{ij}\overline{Y}.
\end{aligned}
\tag{6.1}
$$

Here ϵ is the skew tensor with $\epsilon^{12} = \epsilon_{12} = 1$. The $SU(2)$ R-symmetry group of the 6-dimensional algebra persists in the dimensional reduction with central extension. (For $Y = 0$ it is enlarged to a $U(2)$ R-symmetry group.)

We can analyze the representations using (6.1), but there is a simpler conceptual argument based on the description of the algebra as a dimensional reduction. Namely, for the 6-dimensional algebra we know that we are led to a Clifford algebra with 4 generators for a lightlike 6-vector and a Clifford algebra with 8 generators for a timelike 6-vector. Let M denote the 4-dimensional mass square. Then the 6-dimensional mass square becomes $M^2 - |Y|^2$ in the reduced algebra with central extension. So the assertions about its representations follow.

Solution to Problem ASH7

As a warmup, consider a massless scalar field ϕ in n dimensions with values in a real inner product space X. Suppose J is a complex structure on X, i.e., a skew-symmetric endomorphism with $J^2 = -1$. We construct a theory in $n-1$ dimensions

not by taking fields constant in some spatial direction, but rather by restricting to fields ϕ which satisfy

$$\frac{\partial\phi}{\partial x^{n-1}} = aJ\phi \tag{7.1}$$

for some $a \in \mathbb{R}$. (We use standard coordinates on Minkowski n-space.) Then the free massless lagrangian in n dimensions reduces to the lagrangian in $n-1$ dimensions for a scalar field with mass square a^2. If $X = \mathbb{C}$ and J is multiplication by $i = \sqrt{-1}$, then the one-particle Hilbert space consists of two copies of the spin 0 representation of the $(n-1)$-dimensional Poincaré group with mass square a^2. The Noether charge of $\partial/\partial x^{n-1} = aJ$ is a times what is usually called the total charge Q, and the subspace of the one-particle Hilbert space of total charge $Q = 1$ is an irreducible representation of Poincaré.

In the problem at hand, we take advantage of our description in **ASH**6 of the centrally extended $N = 2$ algebra. Namely, start with the 6-dimensional supersymmetric linear σ-model described in [I-**Supersolutions**, §3.1]. The fields are a quaternion-valued scalar $\phi\colon M^6 \to \mathbb{H}$ and a spinor field $\psi\colon M^6 \to \Pi\mathbb{H}^2$. (Here M^6 is Minkowski 6-space.) The usual dimensional reduction, in which we consider fields invariant under the translation vector fields $\partial/\partial x^4, \partial/\partial x^5$, leads to the massless hypermultiplet in 4 dimensions. Instead, restrict to fields ϕ, ψ which satisfy

$$\begin{aligned} \frac{\partial\phi}{\partial x^4} &= aJ\phi, & \frac{\partial\psi}{\partial x^4} &= aJ\psi, \\ \frac{\partial\phi}{\partial x^5} &= -bJ\phi, & \frac{\partial\psi}{\partial x^5} &= -bJ\psi \end{aligned} \tag{7.2}$$

for $a, b \in \mathbb{R}$ and J a unit imaginary quaternion. (We take the same J for both translations since $\partial/\partial x^4$ and $\partial/\partial x^5$ commute.) The reduction of the 6-dimensional lagrangian to 4 dimensions gives mass $a^2 + b^2$ to all fields. Since the reduced lagrangian is $N = 2$ supersymmetric, the field content is that of the representation H in (5.2). Furthermore, as the representation is massive, it must be a small (BPS) representation of the central extension. The mass bound for the central extension implies that the central charge y satisfies $|y|^2 = a^2+b^2$. From the warmup discussion we see that the Noether charge of the central element Y is $a + ib$ times a standard charge Q, and on the subspace of the one particle Hilbert space with $Q = 1$ this Noether charge acts as $y = a + ib$, which is an arbitrary complex number. Comparing with the list of BPS representations (5.2), we see that the subrepresentation with $Q = 1$ is a massive deformation of H_0. Its sum with the subrepresentation on which $Q = -1$, and so $Y = -(a + ib)$, is a massive deformation of the hypermultiplet H.

Solution to Problem ASH8

We only comment on part (c). Problem **ASH**4, which treats the analogous Higgs mechanism in $N = 1$ supersymmetry, is helpful preparation for this slightly more complicated case.

The general $N = 2$ model in 4 dimensions is described in [I-**Supersolutions**, §10.3] by a lagrangian on the $N = 1$ superspace $M^{4|4}$. Let G be the gauge group and X a hyperkähler manifold with a G action. To define the associated

$N = 1$ model we must fix a Kähler structure on the hyperkähler manifold X. Then the rest of the hyperkähler structure is encoded in a holomorphic symplectic form. The moment map $\mu_{\mathbb{H}}$ of the G action on X is

$$\mu_{\mathbb{H}} = (\mu_{\mathbb{R}}, \mu_{\text{hol}}), \tag{8.1}$$

where $\mu_{\mathbb{R}}: X \to \mathfrak{g}^*$ is the moment map for the Kähler structure and $\mu_{\text{hol}}: X \to \mathfrak{g}_{\mathbb{C}}^*$ the complex moment map of the holomorphic symplectic structure. The manifold of scalars in the model is

$$\tilde{X} = X \times \mathfrak{g}_{\mathbb{C}}. \tag{8.2}$$

The total Kähler moment map is

$$\mu(\phi, \sigma) = \mu_{\mathbb{R}}(\phi) + \frac{i}{2}[\sigma, \bar{\sigma}]. \tag{8.3}$$

There is a contribution to the superpotential

$$W = \sqrt{2}\, \tilde{\mu}_{\text{hol}} \tag{8.4}$$

coming from the transpose of the holomorphic moment map:

$$\tilde{\mu}_{\text{hol}}: X \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}. \tag{8.5}$$

In the case at hand, $G = U(1)$ and $X = \mathbb{H}^2$ with its standard linear $U(1)$ action by multiplication. Identify $\mathbb{H} \cong \mathbb{C} \oplus \overline{\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{C}^*$ as usual, so that $X \cong T^*\mathbb{C}^2$ with its standard Kähler and holomorphic symplectic structures. Write a point in $T^*\mathbb{C}^2$ as $(z^1, z^2; p_1, p_2)$. Then the moment map is[15]

$$\begin{aligned} \mu_{\mathbb{R}} &= \frac{i}{2}\Big\{\sum_j \big(|z^j|^2 - |p_j|^2\big) - r\Big\} \\ \mu_{\text{hol}} &= -i z^j p_j + s + it \end{aligned} \tag{8.6}$$

for some real constants r, s, t. These indeterminate constants, which appear for groups with nontrivial abelian characters, are encoded by physicists in *Fayet-Iliopoulos terms* (see [I-**Supersolutions**, §6.2]).

We take $r = 1$, $s = t = 0$.

The hyperkähler quotient is $T^*\mathbb{CP}^1$, the cotangent space to the complex projective line. It is given by the equations

$$\begin{aligned} |z|^2 &= |p|^2 + 1 \\ z \cdot p &= 0, \end{aligned} \tag{8.7}$$

where $z = (z^1, z^2)$, $p = (p_1, p_2)$, and we mod out by the action of $U(1)$. To see this, note that $z \neq 0$ by the first equation. To each nonzero z we attach the quotient $H_z = \mathbb{C}^2/\mathbb{C} \cdot z$. the set of p which satisfy the second equation is H_z^*. Given a hyperplane H and $p \in H^*$ we use the first equation to fix the norm of z

[15]Since $\mathfrak{g} = i\mathbb{R}$, the "real" moment map $\mu_{\mathbb{R}}$ is pure imaginary.

with $H_z = H$. The indeterminacy of the phase of z is wiped out by taking the $U(1)$ quotient.

The lagrangian for this theory is a special case of the general lagrangian in [I-**Supersolutions**, §6.2] for the "$\mathbb{F} = \mathbb{C}$" case. The fields are:

(8.8)

A	a $U(1)$ gauge field
λ	a (dual) spinor field which we take to be real[16]
χ	a spinor field
σ	a complex scalar field
$\phi = (z^\jmath; p_\jmath)$	a complex scalar field with values in $X \cong T^*\mathbb{C}^2$
$\psi = (\alpha^\jmath; \beta_\jmath)$	a spinor field with values in $TX \cong T^*\mathbb{C}^2$

The superpotential, according to (8.4) and (8.6), is

$$W = -\sqrt{2}\, i\sigma z^\jmath p_\jmath. \tag{8.9}$$

The lagrangian is

$$\begin{aligned} L = &-\frac{1}{2}|F_A|^2 + \bar\lambda \not{D}\lambda + \langle d\bar\sigma, d\sigma\rangle + \bar\chi \not{D}\chi \\ &+ \sum_\jmath \Big(\langle d_A \bar z^\jmath, d_A z^\jmath\rangle + \langle d_A \bar p_\jmath, d_A p_\jmath\rangle + \bar\alpha^\jmath \not{D}_A \alpha^\jmath + \bar\beta_\jmath \not{D}_A \beta_\jmath \Big) \\ &- \frac{1}{2}\Big[\sum_j (|z^j|^2 - |p_\jmath|^2) - 1\Big]^2 - 2|z^\jmath p_\jmath|^2 - 2|\sigma|^2(|z|^2 + |p|^2)^2 \\ &+ 2\sqrt{2}\ \mathrm{Re}\Big[i\epsilon^{ab}\big(\sigma \alpha^\jmath_a \beta_{\jmath b} + z^\jmath \chi_a \beta_{\jmath b} + p_\jmath \chi_a \alpha^\jmath_b\big)\Big] \\ &- \sqrt{2}\sum_\jmath \big(\lambda^a \bar z^\jmath \alpha^\jmath_a - \lambda^a \bar p_\jmath \beta_{\jmath a} + \bar\lambda^{\dot a} z^\jmath \bar\alpha^\jmath_{\dot a} - \bar\lambda^{\dot a} p_\jmath \bar\beta_{\jmath \dot a}\big). \end{aligned} \tag{8.10}$$

A vacuum solution is at a zero of the scalar potential (minus the third line of (8.10)), and so satisfies

$$\begin{aligned} \sum_\jmath (|z^\jmath|^2 - |p_\jmath|^2) - 1 &= 0 \\ z^\jmath p_\jmath &= 0 \\ \sigma(|z|^2 + |p|^2) &= 0. \end{aligned} \tag{8.11}$$

We must also divide out by the action of the gauge group $U(1)$. The classical moduli space of vacua contains the hyperkähler quotient of X by $U(1)$, and indeed we are asked to consider vacua with $\sigma = 0$, which exactly form the hyperkähler quotient. To see the tree level behavior, we expand the lagrangian about the point

$$\sigma = 0, \quad z^1 = 1, \quad z^2 = p_1 = p_2 = 0 \tag{8.12}$$

[16] See the footnote in **ASH4**

and examine the quadratic terms.

The scalar potential around this point is

$$V = 2(\operatorname{Re} z^1)^2 + 2|\sigma|^2 + 2|p_1|^2 + \cdots . \tag{8.13}$$

So the scalar fields $\operatorname{Re} z^1, \sigma, p_1$ all have mass square 2 and the scalar fields z^2, p_2 are massless. (The field $\operatorname{Im} z^1$ can be gauged away.) Hence there are 5 real scalars of mass square 2 and 2 complex massless scalars. The gauge field A acquires mass from the $d_A \bar{z}^1 \, d_A z^1$ term, and its mass square is 2. The expansion of the cubic terms in the last two lines of (8.10) about the point (8.12) is

$$\sqrt{2}i\big(\epsilon^{ab}\chi_a\beta_{1b} - \epsilon^{\dot{a}\dot{b}}\bar{\chi}_{\dot{a}}\bar{\beta}_{1\dot{b}}\big) - \sqrt{2}\big(\lambda^a\alpha_a^1 + \bar{\lambda}^{\dot{a}}\bar{\alpha}_{\dot{a}}^1\big) + \cdots . \tag{8.14}$$

Thus (see (4.6) in **ASH**4) the spinors $\lambda, \chi, \alpha^1, \beta_1$ all have mass square 2 and the spinors α^2, β_2 are massless. Altogether we have the field content of a hypermultiplet $(z^2, p_2, \alpha^2, \beta_2)$ and an H' representation (see (5.4) in **ASH**5) consisting of $(\operatorname{Re} z^1, \sigma, p_1, \lambda, \chi, \alpha^1, \beta_1, A)$. Since supersymmetry is unbroken, we obtain these representations of the (central extension of the) $N = 2$ supersymmetry group.

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Quantum Fields and Strings: A Course for Mathematicians

Pierre Deligne, Pavel Etingof, Daniel S. Freed,
Lisa C. Jeffrey, David Kazhdan, John W. Morgan,
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