# Basis Light-Front QCD Working Notes

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Version 1.9.6, Jan. 12, 2013
Last Modification on Saturday 6th June, 2015

# Contents


**1 Introduction to Light-Front Dynamics** **3**

1.1 Light-Front coordinates . . . 3

1.2 Poincaré algebra . . . 5

1.3 Light-Front Dynamics . . . 6

1.4 Field Theory . . . 8

1.5 Light-Front Quantum Chromodynamics . . . 9

**2 Basis Light-Front Quantized Field Theory** **13**

2.1 Harmonic Oscillator Basis . . . 14

2.2 2-Dimension Harmonic Oscillator Wavefunction . . . 15

2.3 Talmi-Moshinsky Transform . . . 17

2.4 Harmonic Oscillator Basis Function Integrations . . . 18

**3 Light-Front QCD Matrix Elements** **21**

3.1 Spinor Structure . . . 21

3.2 Kinetic Term . . . 21

3.3 Vector Coupling Vertex . . . 23

3.4 Vector Coupling and Triple Gluon Vertex with Instantaneous Gluon . . . 26

3.5 Vector Coupling with Instantaneous Fermion . . . 27

3.6 Vector Coupling with Instantaneous Gluon . . . 28

3.7 Triple Gluon Vertex . . . 31

3.8 Quadruple Gluon Vertex . . . 31

3.9 Triple Gluon Vertices with Instantaneous Gluon . . . 31

**A Conventions** **32**

A.1 General Conventions . . . 32

A.2 Light-Front Variables . . . 32

A.3 Normalization . . . 33

A.4 Spinors . . . 34

A.5 Polarization Vectors . . . 37

A.6 SU(3) Color . . . 38

**B Laguerre Polynomials** **39**

**C Two-body Kinematics** **43**

# Chapter 1

# Introduction to Light-Front Dynamics

## 1.1 Light-Front coordinates

A light front (LF) $\Sigma$ is a hyper-surface defined by the a null vector $\omega^\mu$ and a real constant $\sigma$: $\Sigma = \{x|x\cdot\omega = \sigma, \omega_\mu\omega^\mu = 0, \omega^0 > 0, \sigma \in \mathbb{R}\}$ (see Fig. 1.1). It is easy to see the null vector $\omega^\mu = (|\boldsymbol{\omega}|, \boldsymbol{\omega})$ can be solely determined by a three-vector $\boldsymbol{\omega}$. Different light fronts can be related by rotation and time translation. Given that quantum field theories are often symmetric under these transformations, it is often convenient study one light front (especially in terms of the quantization on the light front), e.g. $\omega^\mu = (1, 0, 0, -1), \sigma = 0$. However, as we shall see later, the transformations relate different light fronts are essentially dynamical, which would impose serious challenges in a practical calculation. Therefore, we should also talk about an arbitrary light front, which shall be referred as a covariant light front.

We shall also introduce the light-front coordinate (See A.2 for more details). Let us start from the special case when $\omega = (1, 0, 0, -1)$. The associated light-front coordinate is defined as

$$x'^0 = x^0 + x^3, \quad x'^1 = x^1, \quad x'^2 = x^2, \quad x'^3 = \tfrac{1}{2}(x^0 - x^3). \tag{1.1}$$

In the literature of light font physics, the temporal and the third spatial components are also denoted as $x'^0 = x^+ = \omega\cdot x$ and $x'^3 = \frac{1}{2}x^-$ respectively, which complies with the notation $x^\pm \equiv x^0 \pm x^3$. The metric tensor becomes,

$$g'_{\mu\nu} = g'^{\mu\nu} = \begin{pmatrix} 0 & & & 1 \\ & -1 & & \\ & & -1 & \\ 1 & & & 0 \end{pmatrix}, \quad \det g' = -1. \tag{1.2}$$

It is also useful to mention the conjugate momentum in the light-front coordinate, which reads,

$$\begin{aligned} &p'^0 = \tfrac{1}{2}(p^0 - p^3), \quad p'^1 = p^1, \quad p'^2 = p^2, \quad p'^3 = p^0 + p^3, \\ &p'_0 = p_0 + p_3, \quad p'_1 = p_1, \quad p'_2 = p_2, \quad p'_3 = \tfrac{1}{2}(p_0 - p_3). \end{aligned} \tag{1.3}$$

It is easy to see, $p'\cdot x' = p\cdot x$, is a Lorentz scalar. Similarly, the literature tends to use the notation, $p^\pm \equiv p^0 \pm p^3, p_\pm \equiv p_0 \pm p_3$ instead of $p'^3, p'^0, p'_0, p'_3$ etc.

Now let us consider the covariant light front coordinate coordinate. We have chosen the temporal component $x'^0 = \omega\cdot x \equiv x^+$. The rest of the components should be chosen on the light fronts, namely,

$$\omega_\mu \frac{\partial x^\mu}{\partial x'^a} = 0, \forall a = 1, 2, 3.$$

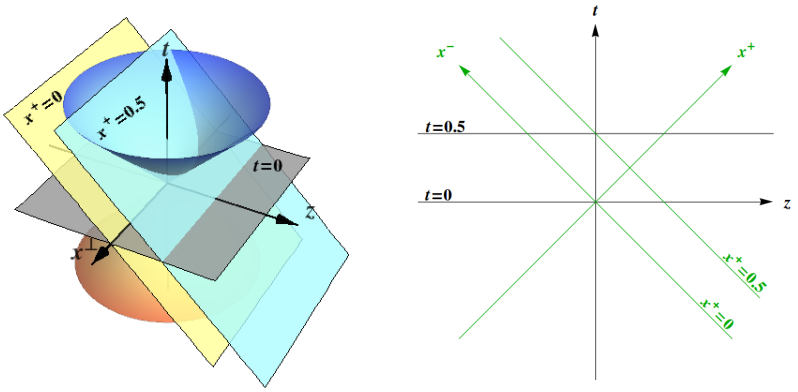



Figure 1.1: Light fronts shown in (2+1) and (1+1) dimension

In this way, we choose an orthogonal coordinate. For a general coordinate transformation,

$$g'_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}, \qquad g_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu} g'_{\lambda\rho}.$$

For the light font coordinate, we require: (1) $\frac{\partial x^\mu}{\partial x'^\nu} \equiv \Theta^\mu_{\ \nu}$ does not depend on $x^\mu$; (2) The metric tensor $g'_{\mu\nu}$ is constant and is given in Eq.(1.2); (3) $\Theta^0_{\ \mu} = \omega_\mu$; (4) $\Theta^3_{\ \mu} \equiv \frac{\partial x'^3}{\partial x^\mu} = \frac{\partial x_\mu}{\partial x'^0}$; We can write the light front coordinate as ($\xi_\mu \equiv \Theta^3_{\ \mu}$),

$$x'^0 = \omega\cdot x, \quad x'^1 = \Theta^1\cdot x, \quad x'^2 = \Theta^2\cdot x, \quad x'^3 = \xi\cdot x, \quad \Theta^\mu_{\ \nu} \equiv \frac{\partial x'^\mu}{\partial x^\nu} = (\omega_\nu, \Theta^1_\nu, \Theta^2_\nu, \xi_\nu)^{\mathrm{T}} \tag{1.4}$$

It is easy to see $\xi\cdot\xi = 0, \xi\cdot\omega = 1, \omega\cdot\Theta^i = 0, \xi\cdot\Theta^i = 0, \Theta^i\cdot\Theta^j = -\delta^{ij}$. In 3+1 dimension, there are 6 free parameters in total. The inverse of $\Theta^\mu_{\ \nu}$ is,

$$\left[\Theta^{-1}\right]^\mu_{\ \nu} \equiv \frac{\partial x^\mu}{\partial x'^\nu} = (\xi, -\Theta^1, -\Theta^2, \omega). \tag{1.5}$$

The components of $\omega$ and $\xi$ are also simple in the light front coordinate, which read $\omega'_\nu = \Theta^\mu_{\ \nu}\omega_\mu = (1,0,0,0)$, $\xi'_\nu = \Theta^\mu_{\ \nu}\xi_\mu = (0,0,0,1)$. Therefore, $\omega'^\mu = g'^{\mu\nu}\omega'_\nu = g'^{0\mu} \equiv g'^{+\mu}$, $\xi'^\mu = g'^{\mu\nu}\xi'_\nu = g'^{3\mu} \equiv g'^{-\mu}$.

The definition of the conjugate momentum $p'$ in the covariant light front coordinate has to preserve the dot product $p'_\mu x'^\mu = p_\mu x^\mu$. Substitute $x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}x^\nu$, and $x^\mu = \frac{\partial x^\mu}{\partial x'^\nu}x'^\nu$, then the old and new momenta are, $p'_\mu \frac{\partial x'^\mu}{\partial x^\nu} = p_\nu$ and $p_\mu \frac{\partial x^\mu}{\partial x'^\nu} = p'_\nu$. In other words,

$$\begin{aligned} &p'_0 = \xi\cdot p, \quad p'_1 = -\Theta^1\cdot p, \quad p'_2 = -\Theta^2\cdot p, \quad p'_3 = \omega\cdot p, \\ &p = p'_0\omega + p'_1\Theta^1 + p'_2\Theta^2 + p'_3\xi. \end{aligned}$$

This is consistent with Eq. (1.3).

**Lorentz Transformation** The Lorentz boost in the longitudinal direction with rapidity $\phi$ (or speed $\beta = \sinh\phi$) is,

$$\begin{aligned} &a^\pm \to e^{\pm\phi}a^+; \\ &\boldsymbol{a}^\perp \to \boldsymbol{a}^\perp; \end{aligned} \tag{1.6}$$

The transverse Lorentz boost with speed $\boldsymbol{\beta}^\perp$,

$$\begin{aligned} &a^+ \to a^+; \\ &\boldsymbol{a}^\perp \to \boldsymbol{a}^\perp + a^+\boldsymbol{\beta}^\perp; \\ &a^- \to a^- + 2\boldsymbol{a}^\perp\cdot\boldsymbol{\beta}^\perp + a^+\boldsymbol{\beta}^{\perp 2} \end{aligned} \tag{1.7}$$

## 1.2 Poincaré algebra

Physical states are the presentation of the Poincaré symmetry ISO(3,1). The 10 generators of the Poincaré symmetry satisfy the following commutation relations,

$$\begin{aligned}
&[P^\mu, P^\nu] = 0;\\
&\left[P^\lambda, M^{\mu\nu}\right] = i\left(g^{\lambda\mu}P^\nu - g^{\lambda\nu}P^\mu\right);\\
&\left[M^{\lambda\rho}, M^{\mu\nu}\right] = i\left(g^{\lambda\nu}M^{\rho\mu} + g^{\rho\mu}M^{\lambda\nu} - g^{\lambda\mu}M^{\rho\nu} - g^{\rho\nu}M^{\lambda\mu}\right);
\end{aligned} \tag{1.8}$$

$J^k \equiv \frac{1}{2}\epsilon^{ijk}M^{ij}$ are the standard angular momentum operators and $K^i \equiv M^{0i}, (i = 1, 2, 3)$ the standard boost generators. It's convenient to introduce the Pauli-Lubanski pseudo-vector $W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\kappa\rho}M_{\nu\kappa}P_\rho$. $W^0 = \boldsymbol{J}\cdot\boldsymbol{P}$, $\boldsymbol{W} = P^0\boldsymbol{J} + \boldsymbol{K}\times\boldsymbol{P}$. It is easy to show that $W$ satisfies,

$$\begin{aligned}
&P_\mu W^\mu = 0;\\
&[P^\mu, W^\nu] = 0;\\
&[M^{\mu\nu}, W^\kappa] = i\left(g^{\kappa\mu}W^\nu - g^{\kappa\nu}W^\mu\right);\\
&[W^\mu, W^\nu] = i\varepsilon^{\mu\nu\kappa\rho}W_\kappa P_\rho
\end{aligned} \tag{1.9}$$

Here $\varepsilon^{\mu\nu\kappa\rho}$ is the Levi-Civita *tensor*. Its relation with Levi-Civita symbol $\epsilon(\mu,\nu,\kappa,\rho)$ is,

$$\varepsilon_{\mu\nu\kappa\rho} = \varepsilon^{\mu\nu\kappa\rho}\sqrt{-\det g} = \epsilon(\mu,\nu,\kappa,\rho), \tag{1.10}$$

and the 4D Levi-Civita symbol is defined as,

$$\epsilon(\mu,\nu,\kappa,\rho) = \begin{cases} +1 & \text{if } \mu,\nu,\kappa,\rho \text{ is an even permutation of } 0,1,2,3;\\ -1 & \text{if } \mu,\nu,\kappa,\rho \text{ is an odd permutation of } 0,1,2,3;\\ 0 & \text{other cases} \end{cases}$$

There are two Casimir operators $W^2 \equiv W_\mu W^\mu, P^2 \equiv P_\mu P^\mu$ that have scalar values in irreducible representations:

$$\begin{aligned}
P^2\,|\mathcal{M}, s\rangle &= \mathcal{M}^2\,|\mathcal{M}, s\rangle\\
W^2\,|\mathcal{M}, s\rangle &= -\mathcal{M}^2 s(s+1)\,|\mathcal{M}, s\rangle
\end{aligned} \tag{1.11}$$

where quantum number $\mathcal{M}$ is called the mass eigenvalue or invariant mass and $s$ the total spin. In the center-of-momentum frame ($\mathcal{M} > 0, \boldsymbol{P} = 0$), the Pauli-Lubanski operator reduces to $W^\mu = \mathcal{M}(0, \boldsymbol{J})$, and $W^2 = -\mathcal{M}^2\boldsymbol{J}^2$, which justifies the name of $s$.

The irreducible representations, also known as the mass eigen-states, can be classified by their mass eigenvalues: (1): $\mathcal{M}^2 > 0$ massive; (2): $\mathcal{M}^2 = 0$ massless; (3): $\mathcal{M}^2 < 0$ tachyonic. The mass eigen-states can be fully identified by the complete set of mutually commuting operators. For Poincaré symmetry, there are at most six of them. It is conventional to take $\{P^2, W^2, \boldsymbol{P}, W^0\}$. Therefore an irreducible representation can be identified as $|\mathcal{M}, \boldsymbol{p}, s, \sigma\rangle \equiv |p, \sigma\rangle$, where $p^2 = \mathcal{M}^2$, $\sigma = -s, s+1, \cdots s-1, s$ is the spin projection. Then the mass eigenstates satisfy the eigenvalue equations,

$$\begin{aligned}
&P^2|p,\sigma\rangle = \mathcal{M}^2|p,\sigma\rangle,\\
&P^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle,\\
&W^2|p,\sigma\rangle = -\mathcal{M}^2 s(s+1)|p,\sigma\rangle,\\
&W^0|p,\sigma\rangle = \mathcal{M}\sigma|p,\sigma\rangle
\end{aligned} \tag{1.12}$$

The Poincaré symmetry can be implemented, e.g., by the Wigner method.

For massive representations ($\mathcal{M}^2 > 0$), one can define a SU(2) algebra $\boldsymbol{S} = \{S^1, S^2, S^3\}$ such that,

$$\left[S^i, S^j\right] = i\varepsilon^{ijk}S^k, \quad \left[S^i, P^j\right] = 0, \quad \boldsymbol{S}^2 \equiv (S^1)^2 + (S^2)^2 + (S^3)^2 = -W^2/P^2. \tag{1.13}$$

Such a vector $\boldsymbol{S}$ is called a relativistic spin operator. However, there is no unique definition of $\boldsymbol{S}$. Different definitions can be related by a Melosh-Wigner rotation.

## 1.3 Light-Front Dynamics

Quantum mechanics postulates the generalized coordinate and momentum operators satisfy the commutation relation at some fixed time, for example $x^0 = 0$, and the states evolves in time according to Schrödinger's equation:

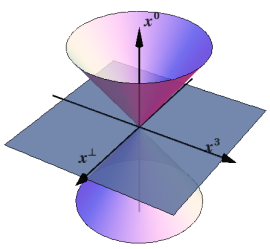
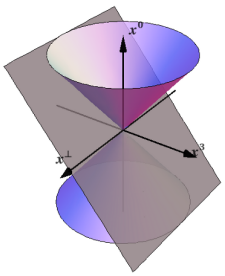
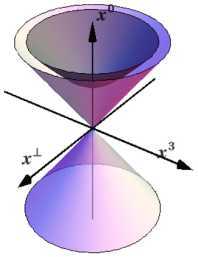
$$i\partial_0|\psi(x^0)\rangle = H|\psi(x^0)\rangle. \tag{1.14}$$

However, the standard time parameter $t$, is by no means unique in a relativistic realization of the quantum theory, as is argued by Dirac [1]. This can be seen from the formal solution of the Schrödinger's equation: $|\psi(x^0)\rangle = U(x^0)|\psi\rangle$, where

$$U(x^0) = \mathcal{T}\exp\left(-i\int \mathrm{d}^4x\mathscr{H}(x)\right), \tag{1.15}$$

where the time-ordering operator $\mathcal{T}$ is not unique (for example $\mathcal{T}_\tau = \mathcal{T}_t = \mathcal{T}_{x^+}$) in the time-like region (inside the light cone). In fact, different time parameters (also known as forms of dynamics) are allowed and should yield the same result as long as the new time parameter is well-defined [2]. Moreover, if we restrict the parametrization by a transitivity condition, namely, kinematical subgroup $G_\Sigma$ of Poincaré symmetry is transitive on the quantization surface $\Sigma \equiv \{x^\mu|\xi(x) = \sigma\}$, there are only *five* inequivalent forms dynamics [3]: (1) $t = \sigma$ (instant form); (2) $\omega_\mu x^\mu = \sigma, (\omega_\mu\omega^\mu = 0, \omega^0 > 0)$ (front form); (3) $x_\mu x^\mu = \sigma^2 > 0$ (point form); (4) $t^2 - z^2 = \sigma^2 > 0$; (5) $t^2 - x^2 - y^2 = \sigma^2 > 0$. Starting from the pioneering work of Dirac [1], the first three forms of dynamics are studied extensively (see Table 1.1). The light-front dynamics is our primary interest in these notes.

Table 1.1: *Forms of Relativistic Dynamics* TODO: update the table to include the other two dynamics.

| | **instant form** | **front form** | **point form** |
|---|---|---|---|
| quantization surface | $t = 0$ | $t + z = 0$ | $t^2 - \boldsymbol{x}^2 = \sigma^2 > 0, t > 0$ |
| |  |  |  |
| kinematical | $\boldsymbol{P}, \boldsymbol{J}$ | $\boldsymbol{P}^\perp, P^+, F^-, E^-, \boldsymbol{E}^\perp$ | $M^{\mu\nu}$ |
| dynamical | $P^0, \boldsymbol{K}$ | $P^-, \boldsymbol{F}^\perp$ | $P^\mu$ |

The kinematical subgroup $G_\Sigma$ is the stabilizer of the quantization surface $\Sigma$, that is, transformations preserve the quantization surface. The associated algebra is also called kinematic. The kinematical subalgebra is important because they can be obtained directly from quantization. Dirac called them the simple ones. The rest of the generators involves time evolution, thus "dynamical" is termed. Dirac called them the difficult ones. In interaction field theories, they involves change of momentum and even particle numbers. Therefore, any truncation results likely the loss of the corresponding symmetries.

As we have mentioned before, the temporal component and third spatial component in light-front dynamics, is conventionally denoted as $x^+$ and $x^-$, respectively. Their conjugate momenta are $p_+ = \frac{1}{2}p^-$ and $p^+$, respectively. The first and second spatial components are collectively called the transverse components $\boldsymbol{x}^\perp = (x^1, x^2)$. We define light-front ration operators as $F^- \equiv M^{12} = J^3, F^i \equiv \epsilon^{ij}M^{j-} =$

$J^i + \epsilon^{ij} K^j$, and boost generators $E^- \equiv \frac{1}{2} M^{+-} = K^3, E^i \equiv M^{+i} = K^i + \epsilon^{ij} J^j, (i, j = 1, 2)$, where $\epsilon^{ij}$ is the 2D Levi-Citiva symbol,

$$\epsilon^{ij} = \begin{cases} +1, & \text{if } i,j \text{ is an even permutation of } 1,2; \\ -1, & \text{if } i,j \text{ is an odd permutation of } 1,2; \\ 0, & \text{others.} \end{cases} \tag{1.16}$$

The 4D Levi-Civita symbol is defined as,

$$\epsilon(\mu,\nu,\kappa,\rho) = \begin{cases} +1 & \text{if } \mu,\nu,\kappa,\rho \text{ is an even permutation of } -,+,1,2; \\ -1 & \text{if } \mu,\nu,\kappa,\rho \text{ is an odd permutation of } -,+,1,2; \\ 0 & \text{other cases} \end{cases}$$

Note that $\sqrt{-\det g} = \frac{1}{2}$, and the above definition guarantees the proper definition of the Levi-Civita *tensor* $\varepsilon^{\mu\nu\kappa\rho}$.

The kinematical generators in the general light-front dynamics are $\{\boldsymbol{P}^\perp, P^+, F^-, \boldsymbol{E}^\perp\}$. If $x^+ = 0$ is chosen as the quantization surface, there is one more kinematical operator $E^-$. That is because the longitudinal boost is merely a scaling. In this case, the number of kinematical generators is maximal.

The irreducible representations can be classified by $\{P^2, W^2, \boldsymbol{P}^\perp, P^+, W^+\}$. One of the possible definitions of the light-front spin operator $\boldsymbol{S} = (\boldsymbol{S}^\perp, S^+)$ for a *massive* state is,

$$\begin{aligned} S^+ &= \frac{W^+}{P^+} \equiv S^3 (= S_z), \\ \boldsymbol{S}^\perp &= \frac{1}{\sqrt{P^2}} \left( \boldsymbol{W}^\perp - \boldsymbol{P}^\perp S^+ \right). \end{aligned} \tag{1.17}$$

Note that the transverse rotations are dynamical. As a result, $W^2$ (or rather $\boldsymbol{S}^2$) is dynamical. As both $P^2$ and $W^2$ are both dynamical, they have to be diagonalized simultaneously:

$$P^2 \left| \mathcal{M}, s \right\rangle = \mathcal{M}^2 \left| \mathcal{M}, s \right\rangle \tag{1.18}$$

$$\boldsymbol{S}^2 \left| \mathcal{M}, s \right\rangle = s(s+1) \left| \mathcal{M}, s \right\rangle \tag{1.19}$$

The benefit of the covariant form of light-front dynamics is the angular condition [4]:

$$M^{\mu\nu}_{\text{int}} |p, \sigma, \omega\rangle = i \left( \omega^\mu \frac{\partial}{\partial \omega_\nu} - \omega^\nu \frac{\partial}{\partial \omega_\mu} \right) |p, \sigma, \omega\rangle \tag{1.20}$$

Note that only $\boldsymbol{F}^\perp_{\text{int}}$ is nonzero. If we do evaluate the angular condition Eq.(1.21), around the standard light front $\omega = (1, 0, 0, -1)$,

$$F^i_{\text{int}} |p, \sigma, \omega\rangle = -2i\epsilon^{ij} p_j \frac{\partial}{\partial(\omega \cdot p)} |p, \sigma, \omega\rangle. \tag{1.21}$$

As is mentioned above, the time evolution of a physical state is governed by the time-dependent Schrödinger's equation,

$$i \frac{\partial}{\partial x^+} \left| \Psi(x^+) \right\rangle = P_+ \left| \Psi(x^+) \right\rangle \tag{1.22}$$

The energy eigenstates are obtained from the time-independent Schrödinger's equation:

$$P_+ \left| \Psi_i \right\rangle = E_i \left| \Psi_i \right\rangle \qquad (x^+ = 0) \tag{1.23}$$

For a self-bound system the light-front energy $E_i = \frac{\boldsymbol{P}^{\perp 2} + \mathcal{M}_i^2}{2P^+}$. So the above eigenvalue equation is consistent with the Eq. (1.18). It's often convenient to introduce a "Light-Cone Hamiltonian" $H_{\text{LC}} \equiv P^2 = 2P^+ P_+ - \boldsymbol{P}^\perp$.

## 1.4 Field Theory

A field theory is given in terms of the action,

$$S = \int \mathrm{d}^4x\, \mathscr{L}[\varphi_r(x), \partial\varphi_r(x), x],$$

where $\varphi_r(x), r = 1, 2, \cdots$ are the fields. The Poincaré algebra can be constructed from fields:

$$P^\mu = \int \mathrm{d}^4x\, \mathcal{T}^{\mu\nu}\omega_\nu\delta(\omega\cdot x - \sigma) = \int \mathrm{d}^3x\, \mathcal{T}^{+\mu}, \tag{1.24}$$

$$M^{\mu\nu} = \int \mathrm{d}^4x \left(x^\mu\mathcal{T}^{\lambda\nu} - x^\nu\mathcal{T}^{\lambda\mu}\right)\omega_\lambda\delta(\omega\cdot x - \sigma) = \int \mathrm{d}^3x \left(x^\mu\mathcal{T}^{+\nu} - x^\nu\mathcal{T}^{+\mu}\right). \tag{1.25}$$

where $\mathcal{T}^{\mu\nu}$ is the energy-momentum tensor. According to Noether's theorem, it is defined as,

$$\mathcal{T}^{\mu\nu} = \sum_r \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_r)}\partial^\nu\varphi_r - g^{\mu\nu}\mathscr{L} \equiv \sum_r \pi_r^\mu\partial^\nu\varphi_r - g^{\mu\nu}\mathscr{L}. \tag{1.26}$$

However, this definition may not give a symmetric tensor as required for the energy-momentum tensor. In general, Noether's theorem only determine the current up to a total derivative term: $\mathcal{T}'^{\mu\nu} = \mathcal{T}^{\mu\nu} + \partial_\lambda\mathcal{F}^{\lambda\mu\nu}$, with $\mathcal{F}^{\lambda\mu\nu} = -\mathcal{F}^{\mu\lambda\nu}$. Another prescription to get a symmetric tensor is the minimal coupling (Gauge principle in general relativity),

$$\mathcal{T}^{\mu\nu} = -\frac{2}{\sqrt{-\det g}}\frac{\partial\sqrt{-\det g}\mathscr{L}}{\partial g_{\mu\nu}}. \tag{1.27}$$

The energy-momentum tensor can be split into two pieces, the free part (kinematical) and the interaction part (dynamical). The interaction part of the 4-momentum operator can be written as,

$$P^\mu_{\mathrm{int}} = -\omega^\mu\int \mathrm{d}^4x\delta(\omega\cdot x - \sigma)\mathscr{L}_{\mathrm{int}} = \omega^\mu\int \mathrm{d}^4x\delta(\omega\cdot x - \sigma)\mathscr{H}_{\mathrm{int}} = \omega^\mu\int\frac{\mathrm{d}\tau}{2\pi}e^{-i\tau\sigma}\widetilde{\mathscr{H}_{\mathrm{int}}}(\omega\tau), \tag{1.28}$$

where $\mathscr{H}(x)$ is the Hamiltonian density, and

$$\widetilde{\mathscr{H}}(p) = \int \mathrm{d}^4x\mathscr{H}(x)e^{ip\cdot x}$$

is the Fourier mode of it. The last equation is consistent with the *uncertainty principle*. Similarly, the interaction part of the generalized angular momentum tensor is,

$$M^{\mu\nu}_{\mathrm{int}} = \int \mathrm{d}^4x\delta(\omega\cdot x - \sigma)(\omega^\mu x^\nu - \omega^\nu x^\mu)\mathscr{L}_{\mathrm{int}} = \int \mathrm{d}^4x\delta(\omega\cdot x - \sigma)(x^\mu\omega^\nu - x^\nu\omega^\mu)\mathscr{H}_{\mathrm{int}}. \tag{1.29}$$

## 1.5 Light-Front Quantum Chromodynamics

The Light-Front QCD Hamiltonian in *light cone gauge* $A^+_{\mu a}=0$ is [6, 8]:

$$\begin{aligned}
P_{+\mathrm{LFQCD}} =& \tfrac{1}{2}\int \mathrm{d}^3x\ \bar\psi\gamma^+\frac{(\mathrm{i}\partial^\perp)^2+m^2}{\mathrm{i}\partial^+}\psi - \tfrac{1}{2}\int \mathrm{d}^3x\ A^i_a(\mathrm{i}\partial^\perp)^2 A^i{}_a\\
&+ g\int \mathrm{d}^3x\ \bar\psi\gamma_\mu A^\mu\psi\\
&+ \tfrac{1}{2}g^2\int \mathrm{d}^3x\ \bar\psi\gamma_\mu A^\mu\frac{\gamma^+}{\mathrm{i}\partial^+}\gamma_\nu A^\nu\psi\\
&- \mathrm{i}g^2\int \mathrm{d}^3x\ f^{abc}\bar\psi\gamma^+T^c\psi\frac{1}{(\mathrm{i}\partial^+)^2}\left(\mathrm{i}\partial^+A^\mu_aA_{\mu b}\right)\\
&+ \tfrac{1}{2}g^2\int \mathrm{d}^3x\ \bar\psi\gamma^+T^a\psi\frac{1}{(\mathrm{i}\partial^+)^2}\bar\psi\gamma^+T^a\psi\\
&+ \mathrm{i}g\int \mathrm{d}^3x\ f^{abc}\mathrm{i}\partial^\mu A^{\nu a}A^b_\mu A^c_\nu\\
&- \tfrac{1}{2}g^2\int \mathrm{d}^3x\ f^{abc}f^{ade}\mathrm{i}\partial^+A^\mu_bA_{\mu c}\frac{1}{(\mathrm{i}\partial^+)^2}\left(\mathrm{i}\partial^+A^\nu_dA_{\nu e}\right)\\
&+ \tfrac{1}{4}g^2\int \mathrm{d}^3x\ f^{abc}f^{ade}A^\mu_bA^\nu_cA_{\mu d}A_{\nu e}.
\end{aligned}\tag{1.30}$$

where the fields has been redefined as dynamical ones by removing the constraints. The fields admit free field expansion at $x^+=0$,

$$\psi_i(x)=\sum_{s=\pm\frac{1}{2}}\int\frac{\mathrm{d}^2\boldsymbol{p}^\perp\mathrm{d}p^+}{(2\pi)^32p^+}\vartheta(p^+)\left(b_{si}(p)u_s(p)e^{-\mathrm{i}p\cdot x}+d^\dagger_{si}(p)v_s(p)e^{+\mathrm{i}p\cdot x}\right)\tag{1.31}$$

$$A_{\mu a}(x)=\sum_{\lambda=\pm}\int\frac{\mathrm{d}^2\boldsymbol{k}^\perp\mathrm{d}k^+}{(2\pi)^32k^+}\vartheta(k^+)\left(a_{\lambda a}(k)\varepsilon_\mu(k)e^{-\mathrm{i}k\cdot x}+a^\dagger_{\lambda a}(k)\varepsilon^*_\mu(k)e^{+\mathrm{i}k\cdot x}\right)\tag{1.32}$$

where the momentum basis is normalized as,

$$\left[a_{\lambda a}(k),a^\dagger_{\sigma b}(k')\right]=2k^+\vartheta(k^+)(2\pi)^3\delta^3(k-k')\delta_{\lambda\sigma}\delta_{ab};\tag{1.33}$$

$$\left\{b_{si}(p),b^\dagger_{rj}(p')\right\}=2p^+\vartheta(p^+)(2\pi)^3\delta^3(p-p')\delta_{sr}\delta_{ij};\tag{1.34}$$

$$\left\{d_{si}(p),d^\dagger_{rj}(p')\right\}=2p^+\vartheta(p^+)(2\pi)^3\delta^3(p-p')\delta_{sr}\delta_{ij};\tag{1.35}$$

with $\delta^3(k-k')=\delta(k^+-k'^+)\delta^2(k^\perp-k'^\perp)$. The spinors and polarization vectors are normalized as,

$$\sum_{s=\pm\frac{1}{2}}\bar u_s(p)u_s(p)=\not p+m;\quad \sum_{s=\pm\frac{1}{2}}\bar v_s(p)v_s(p)=\not p-m;\tag{1.36}$$

$$\sum_{\lambda=\pm}\varepsilon^{\mu*}_\lambda(k)\varepsilon^\nu_\lambda(k)=-g^{\mu\nu}+\frac{\omega^\mu k^\nu+\omega^\nu k^\mu}{\omega\cdot k}.\tag{1.37}$$

The momentum states, $|p,\sigma\rangle\equiv a^\dagger_\sigma(p)\,|0\rangle$, are orthonormalized as

$$\begin{aligned}
&\langle p,\sigma|p',\sigma'\rangle=2p^+\vartheta(p^+)(2\pi)^3\delta^3(p-p')\delta_{\sigma,\sigma'};\\
&\sum_\sigma\int\frac{\mathrm{d}^2\boldsymbol{p}^\perp\mathrm{d}p^+}{(2\pi)^32p^+}\vartheta(p^+)\,|p,\sigma\rangle\,\langle p,\sigma|=1.
\end{aligned}\tag{1.38}$$

Different normalization conventions may apply.

**Fock Space Representation** The eigenstates of the $H_{\rm LF}$ also adopt Fock space expansion,

$$|p,\sigma\rangle = \sum_{n=0}^{\infty}\sum_{\sigma_1,\cdots\sigma_n}\int \frac{\mathrm{d}k_1^+\mathrm{d}^2\boldsymbol{k}_1^\perp}{(2\pi)^3 2k_1^+}\vartheta(k_1^+)\frac{\mathrm{d}k_2^+\mathrm{d}^2\boldsymbol{k}_2^\perp}{(2\pi)^3 2k_2^+}\vartheta(k_2^+)\cdots\frac{\mathrm{d}k_n^+\mathrm{d}^2\boldsymbol{k}_n^\perp}{(2\pi)^3 2k_n^+}\vartheta(k_n^+)$$
$$\psi_n^{\sigma_1,\cdots\sigma_n}(k_1,k_2,\cdots;p,\sigma)2p^+\vartheta(p^+)(2\pi)^3\delta^3(k_1+k_2+\cdots k_n-p)a^\dagger_{\sigma_1}(k_1)\cdots a^\dagger_{\sigma_n}(k_n)\,|0\rangle \quad (1.39)$$

where $\psi_n^{\sigma_1,\cdots\sigma_n}(k_1,k_2,\cdots;p,\sigma)$ is called the light-front wavefunction (LFWF). LFWFs are boost-invariant, namely frame independent, following the boost-invariance of the light-cone Hamiltonian and the pure kinematic character of the light-front boosts. LFWFs can be evaluated in any inertial frame, especially in the CM frame where $\boldsymbol{p}^\perp = 0, p^+ = \mathcal{M}$.

Finally, we should point out that

$$2p^+\vartheta(p^+)(2\pi)^3\delta^3(p-k) = \int\frac{\mathrm{d}s}{2\pi}\vartheta(p^+)(2\pi)^4\delta^3(p-k)\delta\left(p^- - \tfrac{\boldsymbol{k}^{\perp\,2}+s}{2k^+}\right) = \int\frac{\mathrm{d}s}{2\pi}\vartheta(p^+)(2\pi)^4\delta^4(p-k),$$
$$\int\frac{\mathrm{d}^2\boldsymbol{k}^\perp\mathrm{d}k^+}{(2\pi)^3 2k^+}\vartheta(k^+) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\vartheta(k^+)2\pi\delta(k^2-s)$$

are both Lorentz invariants. In the light cone, $\vartheta(k^+)$ selects positive energy modes.

**Boost-Invariant coordinates** The light-cone Hamiltonian is boost invariant. So it can be expressed in terms of boost invariant variables. Let $p_i$ be single particle momentum. Momentum conservation reads,

$$\sum_i p_i^+ = P^+; \qquad \sum_i \boldsymbol{p}_i^\perp = \boldsymbol{P}^\perp \qquad (i=1,2,\cdots n).$$

where $n$ is the number of constituents in the Fock sector. We introduce the boost-invariant longitudinal momentum fraction $x_i \equiv \frac{p_i^+}{P^+}$ and transverse momentum $\boldsymbol{k}_i^\perp = \boldsymbol{p}_i^\perp - x_i\boldsymbol{P}^\perp$. The momentum conservation becomes,

$$\sum_i x_i = 1; \qquad \sum_i \boldsymbol{k}_i^\perp = 0. \quad (1.40)$$

The kinetic energy operator (free part of the light-cone Hamiltonian $P^2$),

$$T_n = \sum_i \frac{\boldsymbol{p}_i^{\perp\,2}+m^2}{x_i} - \boldsymbol{P}^{\perp\,2},$$

where $m$ is the mass the constituents. In a two-body system, $T_2 = \frac{\boldsymbol{k}^{\perp\,2}+m^2}{x}$, where $\boldsymbol{k}^\perp = x_2\boldsymbol{p}_1^\perp - x_1\boldsymbol{p}_2^\perp, x = \frac{x_1x_2}{x_1+x_2} = x_1x_2$.

Consider the light-front transverse boost generators at $x^+=0$, $\boldsymbol{E}^\perp = \boldsymbol{M}^{+\perp} = x^+\boldsymbol{P}^\perp - \boldsymbol{x}^\perp P^+ = -\boldsymbol{x}^\perp P^+$. The transverse coordinate operator $\boldsymbol{R}^\perp \equiv \boldsymbol{x}^\perp = -\frac{\boldsymbol{E}^\perp}{P^+}$. Apparently, $[R^i,P^j] = i\delta^{ij}$. The second quantization form it is $\boldsymbol{R}^\perp = -\frac{1}{P^+}\sum_i \boldsymbol{E}_i^\perp = \sum_i \frac{p_i^+}{P^+}\boldsymbol{r}_i^\perp = \sum_i x_i\boldsymbol{r}_i^\perp$. For the two-body case, it's also convenient to define the relative coordinate, $\boldsymbol{b}^\perp = \boldsymbol{r}_1^\perp - \boldsymbol{r}_2^\perp$, such that $[b^i,k^j] = i\delta^{ij}$.

If we scale the transverse momentum with $\sqrt{x}$,

$$\boldsymbol{q}_i^\perp = \frac{\boldsymbol{p}_i^\perp}{\sqrt{x_i}}, \quad \boldsymbol{s}_i^\perp = \sqrt{x_i}\boldsymbol{r}_i^\perp, \quad q_i^+ = p_i^+; \qquad [s_i,q_j] = i\delta_{ij}, \quad (1.41)$$

then the kinetic energy operator becomes,

$$T_n = \sum_i \boldsymbol{q}_i^{\perp\,2} + \frac{m^2}{x_i} - \boldsymbol{P}^{\perp\,2}. \quad (1.42)$$

Table 1.2: the CM and intrinsic momentum and coordinate of a 2-body system

| single-particle | Center-of-Momentum | intrinsic |
|---|---|---|
| $x_i$ | $X = x_1 + x_2,$ | $x = \frac{x_1 x_2}{x_1+x_2}$ |
| $\boldsymbol{p}_i^\perp$ | $\boldsymbol{P}^\perp = \boldsymbol{p}_1^\perp + \boldsymbol{p}_2^\perp$ | $\boldsymbol{p}^\perp = \frac{x_2\boldsymbol{p}_1^\perp - x_1\boldsymbol{p}_2^\perp}{x_1+x_2}$ |
| $\boldsymbol{r}_i^\perp$ | $\boldsymbol{R}^\perp = x_1\boldsymbol{r}_1^\perp + x_2\boldsymbol{r}_2^\perp$ | $\boldsymbol{r}^\perp = \boldsymbol{r}_1^\perp - \boldsymbol{r}_2^\perp$ |
| $\boldsymbol{q}_i^\perp = \frac{\boldsymbol{p}_i^\perp}{\sqrt{x_i}}$ | $\boldsymbol{Q}^\perp = \frac{\boldsymbol{P}^\perp}{\sqrt{X}} = \frac{\sqrt{x_1}\boldsymbol{q}_1^\perp + \sqrt{x_2}\boldsymbol{q}_2^\perp}{\sqrt{x_1+x_2}}$ | $\boldsymbol{q}^\perp = \frac{\boldsymbol{p}^\perp}{\sqrt{x}} = \frac{\sqrt{x_2}\boldsymbol{q}_1^\perp - \sqrt{x_1}\boldsymbol{q}_2^\perp}{\sqrt{x_1+x_2}}$ |
| $\boldsymbol{s}_i^\perp = \sqrt{x_i}\boldsymbol{r}_i^\perp$ | $\boldsymbol{S}^\perp = \sqrt{X}\boldsymbol{R}^\perp = \frac{\sqrt{x_1}\boldsymbol{r}_1^\perp + \sqrt{x_2}\boldsymbol{r}_2^\perp}{\sqrt{x_1+x_2}}$ | $\boldsymbol{s}^\perp = \sqrt{x}\boldsymbol{r}^\perp = \frac{\sqrt{x_2}\boldsymbol{s}_1^\perp - \sqrt{x_1}\boldsymbol{s}_2^\perp}{\sqrt{x_1+x_2}}$ |

Particularly, the two-body kinetic energy operator becomes quadratic: $T_2 = {\boldsymbol{q}^\perp}^2 + \frac{m^2}{x}$, where $\boldsymbol{q}^\perp = \frac{\boldsymbol{p}^\perp}{\sqrt{x}} = \sqrt{x_2}\boldsymbol{q}_1^\perp - \sqrt{x_1}\boldsymbol{q}_2^\perp, x = x_1x_2$. Similarly, we can introduce the scaled relative coordinate $\boldsymbol{s}^\perp = \sqrt{x}\boldsymbol{r}^\perp = \sqrt{x_2}\boldsymbol{s}_1^\perp - \sqrt{x_1}\boldsymbol{s}_2^\perp$.

Introduce a boost-invariant quadratic transverse confining potential,

$$H_{\rm LF} = \sum_i \frac{{\boldsymbol{p}_i^\perp}^2 + m^2}{x_i} - {\boldsymbol{P}^\perp}^2 + \kappa^4(\sum_i x_i {\boldsymbol{r}_i^\perp}^2 - {\boldsymbol{R}^\perp}^2);$$

In the scaled coordinates in a two-body system, it is just a HO potential and a separated longitudinal part,

$$H_{\rm LF} = {\boldsymbol{q}^\perp}^2 + \kappa^4\boldsymbol{s}^2 + \frac{m^2}{x};$$

Let $\{x_i, \boldsymbol{p}_i^\perp, \boldsymbol{r}_i^\perp\}$ be the single particle longitudinal momentum fraction, transverse momentum and transverse coordinate of an arbitrary 2-body system. Table 1.2 gives the the CM and intrinsic (relative, boost invariant) momentum and coordinate.

**Normalization of the scaled coordinates** The momentum states are normalized by the othonormality and completeness conditions,

$$\begin{aligned} &\langle p|p'\rangle = A_p(2\pi)^3\delta^3(p-p') \\ &\int \frac{\mathrm{d}^3p}{(2\pi)^3A_p}\,|p\rangle\langle p| = 1. \end{aligned} \tag{1.43}$$

where $A_p$ is some function of $p$ that defines the normalization. Consider $\hat{\boldsymbol{q}}^\perp|p\rangle = \frac{\hat{p}^\perp}{\sqrt{\hat{x}}}|p\rangle = \frac{\boldsymbol{p}^\perp}{\sqrt{x}}|p\rangle$. So $|q\rangle = C|p\rangle$, because $|p\rangle$ are not degenerate. The scaled momentum states are also normalized through

$$\begin{aligned} &\langle q|q'\rangle = B_q(2\pi)^3\delta^3(q-q') \\ &\int \frac{\mathrm{d}^3q}{(2\pi)^3B_q}\,|q\rangle\langle q| = 1. \end{aligned} \tag{1.44}$$

where $B_q$ is the function that defines the normalization. Substitute $|q\rangle = C|p\rangle$ into the above normalization relations, we obtain $|C|^2 = xB_q/A_p$. The most convenient choice of the proportionality constant is $C = \sqrt{x}$, that gives $B_q = A_p$, regardless of the specific form of $A_p$ and $B_q$. In BLFQ, the longitudinal coordinate is confined in a box $-\frac{L}{2} < x_+ < \frac{L}{2}$ with periodic or anti-periodic boundary condition. It is convenient to take $A_p = B_q = \frac{1}{L}$. Note the correspondence:

$$\int \mathrm{d}x_+ \quad\to\quad \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x_+; \qquad \int \frac{\mathrm{d}k^+}{2\pi} \quad\to\quad \frac{1}{L}\sum_{k^+}; \qquad 2\pi\delta(k^+ - k'^+) \quad\to\quad L\cdot\delta_{k^+,k'^+}$$

Then the orthonormality and completeness conditions for the scaled coordinate become

$$\begin{aligned} &\langle q|q'\rangle = (2\pi)^2\delta^2(q-q')\delta_{q^+,q'^+} \\ &\sum_{q^+}\int \frac{\mathrm{d}^2q}{(2\pi)^2}\,|q\rangle\langle q| = 1. \end{aligned} \tag{1.45}$$

**Advantages of light-front dynamics**

- *simpler formulation*
- *simpler dynamics, kinematic boosts*
- *trivial vacuum*

# Chapter 2

# Basis Light-Front Quantized Field Theory

Basis Light-Front Quantized Field Theory (BLFQ) is an *ab inito* light-front Hamiltonian approach to quantum field theory. It solves the time-independent Schrödinger equation,

$$P^2 \left|\Phi\right\rangle = \mathcal{M}^2 \left|\Phi\right\rangle .$$

In practice, Hamiltonian methods adopt finite basis and expand the eigenstates and Hamiltonian,

$$\left|\Phi\right\rangle = \sum_i^n c_i \left|\phi_i\right\rangle , \quad H_{ij} = \left\langle\phi_i\right| P^2 \left|\phi_j\right\rangle$$

The problem becomes a standard eigenvalue problem,

$$\sum_j H_{ij} c_j = \lambda c_i.$$

Currently BLFQ adopts but not limited to the basis function,

$$\begin{aligned} f_{nm}(x, \boldsymbol{p}^\perp) \equiv & \mathscr{N} e^{\mathrm{i}m\theta} \rho^{|m|} e^{-\frac{\rho^2}{2x}} L_n^{|m|}(\rho^2/x), \\ = & \Psi_n^m(\boldsymbol{p}/\sqrt{x}) \\ \rho = |\boldsymbol{p}^\perp|/b, & \theta = \arg \boldsymbol{p}^\perp \end{aligned} \tag{2.1}$$

It is a 2D transverse harmonic oscillator in the $\sqrt{x}$-scaled momentum and 1D longitudinal plane wave function.

The 1D longitudinal momentum $p^+$ is discretized with *periodic boundary condition* for bosons and *anti-periodic boundary condition* for fermions. Therefore,

$$p^+ = \frac{2\pi l}{L}, l = \begin{cases} \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots & \text{for fermions} \\ 0, 1, 2, 3, \cdots & \text{for bosons} \end{cases}$$

where $L$ is the length of the longitudinal box. In BLFQ $P^+$ is fixed as $P^+ = \frac{2\pi\mathcal{K}}{L}$. Therefore, the momentum fraction $x_i = \frac{l_i}{\mathcal{K}}$ does not depend on $L$. $\mathcal{K}$ serves as the resolution longitudinal momentum fraction. Note that

$$\sum_i x_i = 1 \quad \Rightarrow \quad \sum_i l_i = \mathcal{K};$$

BLFQ also imposes truncation ($\mathcal{N}_{\max}$ truncation) on the 2D transverse direction:

$$\sum_i [2n_i + |m_i| + 1] \leq \mathcal{N}_{\max}; \tag{2.2}$$

BLFQ retains exact symmetries of the system except for $\boldsymbol{S}^2$ operator. Since $\boldsymbol{S}^2$ is dynamics, any Fock sector truncation will violate the rational symmetry. In BLFQ instead of the commonly used mutually commuting operator set $(P^2, P^+, \boldsymbol{P}, \boldsymbol{S}^2, S_z)$, we use $(P^2, P^+, H_{\rm CM}, L_z, J_z)$, to identify the irreducible representations. Here $H_{\rm CM} = \boldsymbol{P}^2 + b^4\boldsymbol{R}^2$ and $L_z$ is the orbital angular momentum operator. It differs from $J_z$ only by spin projections. It's easy to see the finite truncated basis space $\mathscr{B}(\mathcal{N}_{\max}, \mathcal{K})$ is an invariant subspace of $P^+$, $L_z$ and $J_z$. BLFQ imposes a constraint on $J_z$,

$$\sum_i [m_i + \sigma_i] = M_J;$$

where $\sigma$ is the parton helicity in $z$ direction. For fermions, $\sigma$ is the same as their spin projections.

It's not obvious that $\mathscr{B}(\mathcal{N}_{\max}, \mathcal{K})$ is indeed an invariant subspace of $H_{\rm CM}$. Recall the Talmi-Moshinsky transform,

$$\Psi^{m_1}_{n_1}(\boldsymbol{q}_1)\Psi^{m_2}_{n_2}(\boldsymbol{q}_2) = \sum_{NMnm} (NMnm|n_1m_1n_2m_2)\,\delta_{E_1+E_2,E+\varepsilon}\delta_{M+m,m_1+m_2}\Psi^M_N(\boldsymbol{Q})\Psi^m_n(\boldsymbol{q})$$

where $E_i = 2N_i + |M_i| + 1, E = 2N + |M| + 1, \varepsilon = 2n + |m| + 1$. It can be easily generalized to arbitrary number of particle system,

$$\Psi^{m_1}_{n_1}(\boldsymbol{q}_1)\Psi^{m_2}_{n_2}(\boldsymbol{q}_2)\cdots\Psi^{m_k}_{n_k}(\boldsymbol{q}_k) = \sum_{NM\nu_1\mu_1\cdots\nu_{k-1}\mu_{k-1}} (\nu_1\mu_1\cdots\nu_{k-1}\mu_{k-1}NM|n_1m_1n_2m_2\cdots n_km_k)$$
$$\delta_{E_1+E_2+\cdots E_k,E+\varepsilon_1+\cdots\varepsilon_{k-1}}\delta_{M+\mu_1+\cdots\mu_{k-1},m_1+m_2+\cdots m_k}\Psi^M_N(\boldsymbol{Q})\Psi^{\mu_1}_{\nu_1}(\boldsymbol{\zeta}_1)\cdots\Psi^{\mu_{k-1}}_{\nu_{k-1}}(\boldsymbol{\zeta}_{k-1})$$

where $\boldsymbol{\zeta_i}$ are boost invariant Jacobe variables. The r.h.s. $\Psi^M_N(\boldsymbol{Q})\Psi^{\mu_1}_{\nu_1}(\boldsymbol{\zeta}_1)\cdots\Psi^{\mu_{k-1}}_{\nu_{k-1}}(\boldsymbol{\zeta}_{k-1})$ are the eigenstates of $H_{\rm CM}$ with eigenvalues $(2N + |M| + 1)\hbar^2b^2$. So states with the same $\mathcal{N}_{\max}$ are in the invariant subspace of $H_{\rm CM}$. It's clear that under the $\mathcal{N}_{\max}$ truncation, the basis state $\mathscr{B}$ is the invariant subspace of $H_{\rm CM}$. As a result, the LFWF within $\mathscr{B}$ is a direct product of a CM part and an intrinsic part:

$$|\Phi\rangle = |\Phi_{\rm CM}\rangle \otimes |\varphi\rangle, \quad \text{where } |\Phi_{\rm CM}\rangle = \sum_{N,M} c_{NM}\,|NM\rangle$$

We are only interested in the intrinsic part. It is useful to add a Lagrange multiplier term to the Hamiltonian to break down the degeneracy in the CM part and only take the $(N, M) = (0, 0)$:

$$P^2 \to P^2 + \lambda_{\rm CM}(H_{\rm CM} - 2\hbar^2b^2), \quad \lambda_{\rm CM} > 0$$

## 2.1 Harmonic Oscillator Basis

The basis function for light-front dynamics is defined as the transverse Harmonic Oscillator basis in the scaled coordinates with the **discretized** longitudinal plane wave. In other words,

$$\langle n, m, x|\boldsymbol{q}^\perp, x\rangle = \Psi^m_n(\boldsymbol{q}^\perp) \quad \Leftrightarrow \quad a_{\lambda c}(n, m, x) = \int \frac{\mathrm{d}^2q}{(2\pi)^2}\Psi^m_n(\boldsymbol{q}^\perp)a_{\lambda c}(\boldsymbol{q}^\perp, x) \tag{2.3a}$$

Note that $|\boldsymbol{q}^\perp, x\rangle = \sqrt{x}\,|\boldsymbol{p}^\perp, x\rangle$ or $a_{\lambda c}(\boldsymbol{q}^\perp, x) = \sqrt{x}a_{\lambda c}(\boldsymbol{p}^\perp, x)$.

$$\langle n, m, x| = \int \frac{\mathrm{d}^2\boldsymbol{q}^\perp}{(2\pi)^2}\Psi^m_n(\boldsymbol{q}^\perp)\,\langle \boldsymbol{q}^\perp, x| \tag{2.3b}$$

Completeness and orthonormality:

$$\begin{aligned}&\langle n', m', x'|n, m, x\rangle = \delta_{n,n'}\delta_{m,m'}\delta_{x,x'};\\ &\sum_{n,m,x} |n, m, x\rangle\,\langle n, m, x| = \mathbf{1}.\end{aligned} \tag{2.4}$$

**Phenomenonalogical Confining Potential**

$$V_{\text{CON}} = \kappa^4 x \boldsymbol{r}^2 = \kappa^4 \boldsymbol{s}^2 \tag{2.5}$$

In $N$-fermion sector,

$$\begin{aligned} V_{\text{CON}} &= \kappa^4 \sum_i x_i \boldsymbol{r}_i^2 - \kappa^4 \chi \boldsymbol{R}^2 = \kappa^4 \sum_{i<j} x_i x_j (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 \\ &= \kappa^4 \sum_i \boldsymbol{s}_i^2 - \kappa^4 \boldsymbol{S}^2 = \kappa^4 \sum_{i<j} \boldsymbol{s}_{ij}^2 \end{aligned} \tag{2.6}$$

where $\boldsymbol{R} = \sum_i x_i \boldsymbol{r}_i$ is the CM coordinate, $\boldsymbol{S} = \sqrt{\chi}\boldsymbol{R}$, $\chi = \sum_i x_i$ is the CM longitudinal momentum fraction; $\boldsymbol{s}_{ij} = \sqrt{x_i x_j}(\boldsymbol{r}_i - \boldsymbol{r}_j)$ is the scaled relative coordinate.

## 2.2 2-Dimension Harmonic Oscillator Wavefunction

Harmonic Oscillator (HO) basis states $|n, m\rangle$ or $|\Psi_n^m\rangle$ are the eigenstates of the non-relativistic HO Hamiltonian,

$$H_{\text{HO}} \equiv \frac{1}{2M}\boldsymbol{P}^2 + \frac{1}{2}M\Omega^2\boldsymbol{R}^2 = \tfrac{1}{2}\hbar\Omega\left[\boldsymbol{p}^2 + \boldsymbol{r}^2\right], \tag{2.7}$$

where $\boldsymbol{p} = \boldsymbol{P}/\sqrt{\hbar M\Omega}, \boldsymbol{r} = \boldsymbol{R}/\sqrt{\hbar/M\Omega}$. We introduce basis energy parameter $b = \sqrt{\hbar M\Omega}$ and HO natural length $\ell = \sqrt{\frac{\hbar}{M\Omega}}$. In natural units, $b = \ell^{-1}$.

In momentum space the 2-Dimension HO wavefunction is (up to a global phase),

$$\langle \boldsymbol{p}|\Psi_n^m\rangle \equiv \Psi_n^m(\boldsymbol{p}) = \ell\sqrt{\frac{4\pi\cdot n!}{(n+|m|)!}}e^{\mathrm{i}m\varphi}\rho^{|m|}e^{-\rho^2/2}L_n^{|m|}(\rho^2), \quad (n, m \in \mathbb{Z}, n \geq 0) \tag{2.8}$$

where $\rho = |\boldsymbol{p}|/b, \varphi = \arg\boldsymbol{p}$, $L_n^\alpha(x)$ generalized Laguerre polynomials (See Appendix B). Correspondingly, the 2D HO wavefunction in coordinate space is similar but with an additional phase $e^{\mathrm{i}\frac{\pi}{2}(2n+|m|)} = \mathrm{i}^{|m|}(-1)^n$ (See Appendix B for more details),

$$\begin{aligned} \langle \boldsymbol{r}|\Psi_n^m\rangle \equiv \widetilde{\Psi}_n^m(\boldsymbol{r}) &= \int \frac{\mathrm{d}^2p}{(2\pi)^2}\, e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\Psi_n^m(\boldsymbol{p}), \quad (n, m \in \mathbb{Z}, n \geq 0) \\ &= e^{\mathrm{i}\frac{\pi}{2}(2n+|m|)}b\sqrt{\frac{n!}{\pi(n+|m|)!}}e^{\mathrm{i}m\theta}\gamma^{|m|}e^{-\gamma^2/2}L_n^{|m|}(\gamma^2), \\ &= \frac{(-1)^{n+\frac{1}{2}|m|}}{2\pi}\Psi_n^m(\boldsymbol{\gamma}) \end{aligned} \tag{2.9}$$

where $\gamma = |\boldsymbol{r}|/\ell, \theta = \arg\boldsymbol{r}$.

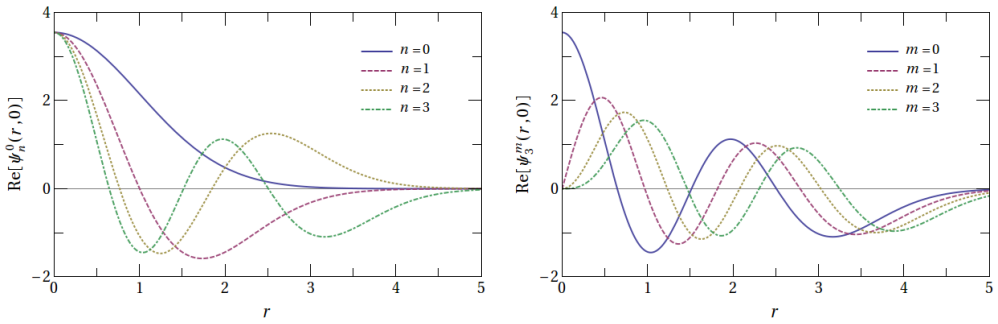
The orthonormality and completeness read,

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\Psi_n^{m*}(p)\Psi_n^m(p') \to (2\pi)^2\delta^2(p-p'); \qquad \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\widetilde{\Psi}_n^{m*}(r)\widetilde{\Psi}_n^m(r') \to \delta^2(r-r'); \tag{2.10}$$

$$\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\Psi_n^{m*}(p)\Psi_{n'}^{m'}(p) = \delta_{n,n'}\delta_{m,m'}; \qquad \int \mathrm{d}^2r\,\widetilde{\Psi}_n^{m*}(r)\widetilde{\Psi}_{n'}^{m'}(r) = \delta_{n,n'}\delta_{m,m'}. \tag{2.11}$$

**Recurrence Relations** Let $p = p_x + \mathrm{i}p_y$ be *complex representation* of $\boldsymbol{p}^\perp$, $b = 1, \vartheta(0) = 1$.

$$p\cdot\Psi_n^m(p) = \begin{cases} \sqrt{n+|m|+1}\Psi_n^{m+1}(p) - \vartheta(n-1)\sqrt{n}\Psi_{n-1}^{m+1}(p) & (m \geq 0) \\ \sqrt{n+|m|}\Psi_n^{m+1}(p) - \sqrt{n+1}\Psi_{n+1}^{m+1}(p) & (m < 0) \end{cases} \tag{2.12}$$




(a) $\mathrm{Re}\Psi_n^0(p), n = 0, 1, 2, 3$ (b) $\mathrm{Re}\Psi_3^m(p), m = 0, 1, 2, 3$

Figure 2.1: Image of HO wavefunctions at $\theta = 0$, $b = 1$.

**Exponential Generating Function** The exponential generating function of $\Psi_n^m(\boldsymbol{p})$ is,

$$e^{-\frac{1}{2}\boldsymbol{p}^2+2\boldsymbol{p}\cdot\boldsymbol{z}-\boldsymbol{z}^2} = b\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(-1)^n}{\sqrt{4\pi(n+|m|)!n!}}\Psi_n^m(\boldsymbol{p})e^{-\mathrm{i}m\theta}z^{2n+|m|} \tag{2.13}$$

where, $z = |\boldsymbol{z}|/\ell, \theta = \arg \boldsymbol{z}$, where $b$ is the basis energy parameter.

**Integrations** Let $\boldsymbol{p}$ be 2D vector. Let $p$ be the complex representation of $\boldsymbol{p}$. Define,

$$f_{n,m,n',m'} = \begin{cases} \delta_{n,n'}\sqrt{n+|m+1|} - \delta_{n',n-1}\sqrt{n} & (m \geq 0) \\ \\ \delta_{n,n'}\sqrt{n+|m|} - \delta_{n',n+1}\sqrt{n+1} & (m < 0) \end{cases} \tag{2.14}$$

$$\int \frac{\mathrm{d}^2p}{(2\pi)^2}\ \Psi_n^m(p) = b\cdot\frac{(-1)^n}{\sqrt{\pi}} \tag{2.15}$$

$$\int \frac{\mathrm{d}^2p}{(2\pi)^2}\ p\Psi_n^m(p) = b^2\cdot 2(-1)^n\sqrt{\frac{n+1}{\pi}} \tag{2.16}$$

$$\int \frac{\mathrm{d}^2p}{(2\pi)^2}\ p\Psi_{n'}^{m'*}(p)\Psi_n^m(p) = b\cdot\delta_{m',m+1}f_{n,m,n',m'}. \tag{2.17}$$

$$\begin{aligned}&\int \frac{\mathrm{d}^2p}{(2\pi)^2}\ \boldsymbol{p}^2\Psi_{n'}^{m'*}(p)\Psi_n^m(p) \\ &= b^2\cdot\delta_{m,m'}\left[\delta_{n,n'}(2n+|m|+1) - \delta_{n',n+1}\sqrt{n'(n'+|m'|)} - \delta_{n,n'+1}\sqrt{n(n+|m|)}\right]\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\int \mathrm{d}^2r\ \boldsymbol{r}^2\ \widetilde{\Psi}_{n'}^{m'*}(r)\widetilde{\Psi}_n^m(r) \\ &= \ell^2\cdot\delta_{m,m'}\left[\delta_{n,n'}(2n+|m|+1) + \delta_{n',n+1}\sqrt{n'(n'+|m'|)} + \delta_{n,n'+1}\sqrt{n(n+|m|)}\right]\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\ \boldsymbol{p_1}\cdot\boldsymbol{p_2}\ \Psi_{n_1}^{m_1}(p_1)\Psi_{n_2}^{m_2}(p_2)\Psi_{n_1'}^{m_1'*}(p_1)\Psi_{n_2'}^{m_2'*}(p_2) \\ =&\tfrac{1}{2}b^2\left[f_{n_1,m_1,n_1',m_1'}f_{n_2',m_2',n_2,m_2}\delta_{m_1+1,m_1'}\delta_{m_2'+1,m_2} + f_{n_1',m_1',n_1,m_1}f_{n_2,m_2,n_2',m_2'}\delta_{m_1'+1,m_1}\delta_{m_2+1,m_2'}\right]\end{aligned} \tag{2.20}$$

$$\begin{aligned}&\int \mathrm{d}^2r_1\mathrm{d}^2r_2\ \boldsymbol{r_1}\cdot\boldsymbol{r_2}\ \widetilde{\Psi}^{m_1}_{n_1}(r_1)\widetilde{\Psi}^{m_2}_{n_2}(r_2)\widetilde{\Psi}^{m_1'*}_{n_1'}(r_1)\widetilde{\Psi}^{m_2'*}_{n_2'}(r_2)\\&=(-1)^{n_1+n_2-n_1'-n_2'+\frac{1}{2}(|m_1|+|m_2|-|m_1'|-|m_2'|)}\\&\tfrac{1}{2}\ell^2\left[f_{n_1,m_1,n_1',m_1'}f_{n_2',m_2',n_2,m_2}\delta_{m_1+1,m_1'}\delta_{m_2'+1,m_2}+f_{n_1',m_1',n_1,m_1}f_{n_2,m_2,n_2',m_2'}\delta_{m_1'+1,m_1}\delta_{m_2+1,m_2'}\right]\end{aligned}\tag{2.21}$$

$$\begin{aligned}L_z&=\sum_a x_a\varepsilon^{ij}s_a^iq_a^j+\sum_{a<b}\sqrt{x_ax_b}\varepsilon^{ij}(s_a^iq_b^j+s_b^iq_a^j)\\&=\tfrac{i}{2}\sum_a x_a(s_aq_a^*-s_a^*q_b)+\tfrac{i}{2}\sum_{a<b}\sqrt{x_ax_b}(s_aq_b^*-s_a^*q_b+s_bq_a^*-s_b^*q_a)\end{aligned}\tag{2.22}$$

$$\begin{aligned}&\langle n,m,x|\varepsilon^{ij}s^iq^j|n',m',x'\rangle\\&=\tfrac{i}{2}\delta_{m,m'}\sum_{n''}i^{|m+1|-|m|}[(-1)^{n-n''}f_{n,-m,n'',m''}f_{n',-m,n'',m''}-(-1)^{n'-n''}f_{n,m,n'',m''}f_{n',m,n'',m''}]\end{aligned}\tag{2.23}$$

## 2.3 Talmi-Moshinsky Transform

Consider the generating function of 2D HO wavefunctions $\Psi_n^m(\boldsymbol{q})$,

$$e^{-\frac{1}{2}\boldsymbol{q}^2+2\boldsymbol{q}\cdot\boldsymbol{z}-\boldsymbol{z}^2}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(-1)^n}{\sqrt{4\pi(n+|m|)!n!}}\Psi_n^m(\boldsymbol{q})e^{-\mathrm{i}m\theta}z^{2n+|m|}\tag{2.24}$$

where, $z=|\boldsymbol{z}|/\ell,\theta=\arg\boldsymbol{z}$. If we define,

$$\begin{aligned}\boldsymbol{Q}&=\boldsymbol{q}_1\cos\xi+\boldsymbol{q}_2\sin\xi,\quad \boldsymbol{q}=\boldsymbol{q}_1\sin\xi-\boldsymbol{q}_2\cos\xi,\\\boldsymbol{Z}&=\boldsymbol{z}_1\cos\xi+\boldsymbol{z}_2\sin\xi,\quad \boldsymbol{z}=\boldsymbol{z}_1\sin\xi-\boldsymbol{z}_2\cos\xi\end{aligned}\tag{2.25}$$

where $\xi=\arctan\sqrt{\frac{x_2}{x_1}}$. Then there exists an identity,

$$\begin{aligned}&(-\tfrac{1}{2}\boldsymbol{q}_1^2+2\boldsymbol{q}_1\cdot\boldsymbol{z}_1-\boldsymbol{z}_1^2)+(-\tfrac{1}{2}\boldsymbol{q}_2^2+2\boldsymbol{q}_2\cdot\boldsymbol{z}_2-\boldsymbol{z}_2^2)=\\&\qquad(-\tfrac{1}{2}\boldsymbol{Q}^2+2\boldsymbol{Q}\cdot\boldsymbol{Z}-\boldsymbol{Z}^2)+(-\tfrac{1}{2}\boldsymbol{q}^2+2\boldsymbol{q}\cdot\boldsymbol{z}-\boldsymbol{z}^2)\end{aligned}\tag{2.26}$$

If $\boldsymbol{q}$'s are the scaled momentum, namely $\boldsymbol{q}_i=\frac{\boldsymbol{p}_i}{\sqrt{x_i}}$, then

$$\boldsymbol{Q}=\frac{\sqrt{x_1}\boldsymbol{q}_1+\sqrt{x_2}\boldsymbol{q}_2}{\sqrt{x_1+x_2}}=\frac{\boldsymbol{p}_1+\boldsymbol{p}_2}{\sqrt{x_1+x_2}},\qquad \boldsymbol{q}=\frac{\sqrt{x_2}\boldsymbol{q}_1-\sqrt{x_1}\boldsymbol{q}_2}{\sqrt{x_1+x_2}}=\frac{\frac{x_2}{x_1+x_2}\boldsymbol{p}_1-\frac{x_1}{x_1+x_2}\boldsymbol{p}_2}{\sqrt{\frac{x_1x_2}{x_1+x_2}}}\tag{2.27}$$

are proportional to the two-body scaled CM momentum and scaled relative momentum, respectively.

Thanks to the generating function Eq.(2.24),

$$\begin{aligned}&\sum_{n_1,m_1,n_2,m_2}\frac{(-1)^{n_1+n_2}\Psi_{n_1}^{m_1}(\boldsymbol{q_1})\Psi_{n_2}^{m_2}(\boldsymbol{q_2})}{4\pi\sqrt{(n_1+|m_1|)!n_1!(n_2+|m_2|)!n_2!}}e^{-\mathrm{i}m_1\theta_1-\mathrm{i}m_2\theta_2}z_1^{2n_1+|m_1|}z_2^{2n_2+|m_2|}=\\&\qquad\sum_{N,M,n,m}\frac{(-1)^{N+n}\Psi_N^M(\boldsymbol{Q})\Psi_n^m(\boldsymbol{q})}{4\pi\sqrt{(N+|M|)!N!(n+|m|)!n!}}e^{-\mathrm{i}M\Theta-\mathrm{i}m\theta}Z^{2N+|M|}z^{2n+|m|}\end{aligned}\tag{2.28}$$

where, $z_i=|\boldsymbol{z}_i|,Z=|\boldsymbol{Z}|,z=|\boldsymbol{z}|,\theta_i=\arg\boldsymbol{z}_i,\Theta=\arg\boldsymbol{Z},\theta=\arg\boldsymbol{z}$. We express $Z,z,e^{\mathrm{i}\Theta},e^{\mathrm{i}\theta}$ in terms of $z_1,z_2,e^{\mathrm{i}\theta_1},e^{\theta_2}$ on the right-hand side of the identity and identify each term from the left-hand side by equating their coefficients. In fact, $Z,z,e^{\mathrm{i}\Theta},e^{\mathrm{i}\theta}$ are (generalized) polynomials of variable $z_1,z_2,e^{\mathrm{i}\theta_1},e^{\theta_2}$.

Hence, one can expand the them in terms of $z_1, z_2, e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2}$ using the (generalized) multinomial theorem. We obtain,

$$\Psi^{m_1}_{n_1}(\boldsymbol{q}_1)\Psi^{m_2}_{n_2}(\boldsymbol{q}_2) \equiv \sum_{NMnm} \mathcal{M}^{NMnm}_{n_1m_1n_2m_2}\big|_\xi \, \delta_{2n_1+|m_1|+2n_2+|m_2|,2N+|M|+2n+|m|}\delta_{m_1+m_2,M+m} \Psi^M_N(\boldsymbol{Q})\Psi^m_n(\boldsymbol{q}) \quad (2.29)$$

where, the Talmi-Monshinsky coefficients $\mathcal{M}^{NMnm}_{n_1m_1n_2m_2}\big|_\xi \equiv (NMnm|n_1m_1n_2m_2)|_\xi$ are

$$\mathcal{M}^{N_1M_1N_2M_2}_{n_1m_1n_2m_2}\big|_\xi = (-1)^{N_1-n_1-n_2+\frac{1}{2}(|M_2|+M_2)} \cdot \sqrt{\frac{(n_1+|m_1|)!(n_2+|m_2|)!n_1!n_2!}{(N_1+|M_1|)!(N_2+|M_2|)!N_1!N_2!}} \cdot$$
$$(\sin\xi)^{2n_2+|m_2|}(\cos\xi)^{2n_1+|m_1|}(\tan\xi)^{M_2} \cdot \sum_{\gamma_1=0}^{v_1}\sum_{\gamma_2=0}^{v_2}\sum_{\beta_1=0}^{\gamma_1}\sum_{\beta_2=0}^{\gamma_2}\sum_{\beta_3=0}^{V_2}\sum_{\beta=0}^{M_2}(-\tan\xi)^{\beta_1-\beta_2+\beta}$$
$$\binom{M_1}{v_1-v_2+m_1+\gamma_2-\gamma_1-\beta}\binom{M_2}{\beta}\binom{V_2}{\beta_1,\beta_2,\beta_3,\beta_4}\binom{v_1+v_2-V_2}{\gamma_1-\beta_1,\gamma_2-\beta_2,v_1-\gamma_1-\beta_3,\beta_5} \quad (2.30)$$

where $v_i = n_i + \frac{1}{2}(m_i - |m_i|), V_i = N_i + \frac{1}{2}(M_i - |M_i|), i = 1, 2$. The multinomial coefficients $\binom{n,}{m_1,m_2,\cdots,m_k}$ satisfies $m_1 + m_2 + \cdots m_k = n$, $0 \le m_i \le n, i = 1, 2, \cdots k$. So, $\beta_4 = V_2 - \beta_1 - \beta_2 - \beta_3, \beta_5 = v_2 - \gamma_2 - \beta_4$. The binomial coefficients however are the generalized ones, $\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!}, m \ge 0$ and $\binom{n}{0} = 1$.

The identified series is finite, because the degrees of polynomials on the left-hand side and a typical term of right-hand side are $2n_1 + |m_1| + 2n_2 + |m_2|$ and $2N + |M| + 2n + |m|$ respectively. So relation $0 \le 2N + |M| + 2n + |m| \le 2n_1 + |m_1| + 2n_2 + |m_2|$ limits the values of $N$ and $|M|$.

Kronecker deltas, $\delta_{2N_1+|M_1|+2N_2+|M_2|,2n_1+|m_1|+2n_2+|m_2|}$ and $\delta_{M_1+M_2,m_1+m_2}$, represent energy and angular momentum conservation respectively. They also suffice to estimate if a TM coefficient is zero. The accidental zeros is $\le 5\%$ for all $\mathcal{M}^{N_1M_1N_2M_2}_{n_1m_1n_2m_2}$ that satisfies $2n_1 + |m_1| + 2n_2 + |m_2| \le N_{\max}$ for all $N_{\max}$ (See Fig. 2.2).

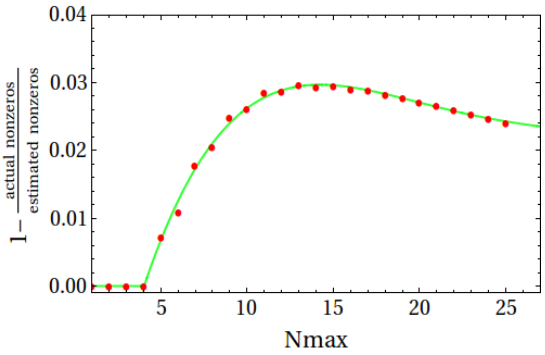



Figure 2.2: Efficiency of the estimate of nonzero coefficients using the two deltas for all TM coefficients $\mathcal{M}^{N_1M_1N_2M_2}_{n_1m_1n_2m_2}$ that satisfies $2n_1 + |m_1| + 2n_2 + |m_2| \le N_{\max}$

## 2.4 Harmonic Oscillator Basis Function Integrations

Assume longitudinal momentum conservation: $\sum_i x_i = \sum_j x'_j$. Let $n, m, k$ be integers, $q = q_x + \mathrm{i}q_y$ be the complex representation of 2D vector $\boldsymbol{q} = q_x\boldsymbol{e}_x + q_y\boldsymbol{e}_y$ and $\mathrm{d}^2p \equiv \frac{\mathrm{d}^2p}{(2\pi)^2}$. $\ell = \sqrt{\frac{\hbar}{M_{\mathrm{HO}}\Omega_{\mathrm{HO}}}}$ is the natural length of the HO basis wavefunctions $\Psi^m_n$, $b = \ell^{-1}$.

Recall

$$f_{nmn'm'} = \begin{cases} \sqrt{n+|m+1|}\delta_{n,n'} - \sqrt{n}\delta_{n',n-1} & m \geq 0 \\ \sqrt{n+|m|}\delta_{n,n'} - \sqrt{n+1}\delta_{n',n+1} & m < 0. \end{cases} \tag{2.31}$$

**Wavefunction Integrals of $q_1 + q_2 \to q'$:**

$$\begin{aligned} &\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q'\ (2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x'}q')\cdot q_1 \cdot \Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{m'*}_{n'}(q') \\ =&\frac{\delta_{m_1+m_2,m'-1}}{\ell^2 x'^{\frac{3}{2}}\pi^{\frac{1}{2}}}\left[\frac{}{\sqrt{x_1}}\sum_{N=\max(n'-1,0)}^{\min(\nu,n'+1)}\mathcal{M}^{N,m'-1,\nu-N,0}_{n_1,m_1,n_2,m_2}(-1)^{\nu-N}f_{N,m'-1,n',m'} + 2\vartheta(n)\sqrt{x_2}\mathcal{M}^{n',m',n,-1}_{n_1,m_1,n_2,m_2}(-1)^n\sqrt{n+1}\right] \\ &\nu = n_1+n_2+\tfrac{1}{2}(|m_1|+|m_2|-|m_1+m_2|), n = n_1+n_2-n'+\tfrac{1}{2}(|m_1|+|m_2|-1-|m'|)\geq 0 \end{aligned} \tag{2.32a}$$

$$\begin{aligned} &\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q'\ (2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x'}q')\cdot q_2 \cdot \Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{m'*}_{n'}(q') \\ =&\frac{\delta_{m_1+m_2,m'-1}}{\ell^2 x'^{\frac{3}{2}}\pi^{\frac{1}{2}}}\left[\frac{}{\sqrt{x_2}}\sum_{N=\max(n'-1,0)}^{\min(\nu,n'+1)}\mathcal{M}^{N,m'-1,\nu-N,0}_{n_1,m_1,n_2,m_2}(-1)^{\nu-N}f_{N,m'-1,n',m'} - 2\vartheta(n)\sqrt{x_1}\mathcal{M}^{n',m',n,-1}_{n_1,m_1,n_2,m_2}(-1)^n\sqrt{n+1}\right] \\ &\nu = n_1+n_2+\tfrac{1}{2}(|m_1|+|m_2|-|m_1+m_2|), n = n_1+n_2-n'+\tfrac{1}{2}(|m_1|+|m_2|-1-|m'|)\geq 0 \end{aligned} \tag{2.32b}$$

$$\begin{aligned} &\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q'\ (2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x'}q')\cdot q' \cdot \Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{m'*}_{n'}(q') \\ =&\frac{\delta_{m_1+m_2,m'-1}}{\ell^2 x'\pi^{\frac{1}{2}}}\sum_{N=\max(n'-1,0)}^{\min(\nu,n'+1)}\mathcal{M}^{N,m'-1,\nu-N,0}_{n_1,m_1,n_2,m_2}(-1)^{\nu-N}f_{N,m'-1,n',m'} \\ &\nu = n_1+n_2+\tfrac{1}{2}(|m_1|+|m_2|-|m_1+m_2|) \end{aligned} \tag{2.32c}$$

$$\begin{aligned} &\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q'\ (2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x'}q')\cdot \Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{m'*}_{n'}(q') \\ =&\vartheta(n)(-1)^n\frac{\delta_{m_1+m_2,m'}}{\ell x'\pi^{\frac{1}{2}}}\mathcal{M}^{n',m',n,0}_{n_1,m_1,n_2,m_2} \\ &n = n_1+n_2-n'+\tfrac{1}{2}(|m_1|+|m_2|-|m_1+m_2|)\geq 0 \end{aligned} \tag{2.32d}$$

**Wavefunction Integrals of $q_1 + q_2 + q_3 \to q'$ and $q_1 + q_2 \to q_1' + q_2'$:**

$$\begin{aligned} &\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q_3\,\text{đ}q'\ (2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 + \sqrt{x_3}q_3 - \sqrt{x'}q')\Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{m_3}_{n_3}(q_3)\Psi^{m'*}_{n'}(q') \\ =&\frac{\delta_{m_1+m_2+m_3,m'}}{\ell^2 x'\pi}(-1)^{\xi+\chi}\sum_{N=\max(0,-\chi)}^{\xi}\mathcal{M}^{N,m_1+m_2,\xi-N,0}_{n_1,m_1,n_2,m_2}\mathcal{M}^{n',m',\chi+N,0}_{N,m_1+m_2,n_3,m_3} \end{aligned} \tag{2.33a}$$

$\xi = n_1+n_2+\frac{1}{2}(|m_1|+|m_2|-|m_1+m_2|), \chi = n_3-n'+\frac{1}{2}(|m_1+m_2|+|m_3|-|m'|), \delta_1 = \arctan\sqrt{\frac{x_2}{x_1}}, \delta_2 = \arctan\sqrt{\frac{x_3}{x_1+x_2}}$.

$$\begin{aligned} &\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q_1'\,\text{đ}q_2'\ (2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x_1'}q_1' - \sqrt{x_2'}q_2')\Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{m_1'*}_{n_1'}(q_3')\Psi^{m_2'*}_{n_2'}(q_2') \\ =&\frac{\delta_{m_1+m_2,m_1'+m_2'}}{\ell^2(x_1+x_2)\pi}(-1)^{\chi+\xi}\sum_{N=0}^{\min(\xi,\chi)}\mathcal{M}^{N,m_1+m_2,\xi-N,0}_{n_1,m_1,n_2,m_2}\mathcal{M}^{N,m_1+m_2,\chi-N,0}_{n_1',m_1',n_2',m_2'} \\ &\xi = n_1+n_2+\tfrac{1}{2}(|m_1|+|m_2|-|m_1+m_2|), \chi = n_1'+n_2'+\tfrac{1}{2}(|m_1'|+|m_2'|-|m_1'+m_2'|); \end{aligned} \tag{2.33b}$$

**Interaction Structure of QCD** Structure of QCD interaction can help to simplify basis light front QCD vertices. QCD interactions are boost invariants. Since transverse boosts are kinematical, boost invariant operators can only depend on boost invariant quantities, i.e. the boost invariant relative momenta. Furthermore, angular momentum $J = L + S$ is conserved. Since gluon is spin-1, quarks are spin-$\frac{1}{2}$, $|\Delta L| \leq 1 \cdot \hbar$. In fact, $|\Delta L| = 1$ only happens in vertex `missing Feynman diagram`and vertex `missing Feynman diagram`There are three types of vertices,

$$\begin{aligned}&\langle n', m'|V|n_1, m_1; n_2, m_2\rangle\\ =&\frac{1}{x'}\vartheta(n)\mathcal{M}^{n',m',n,m}_{n_1,m_1,n_2,m_2}\int \mathrm{d}^2q\, \bar{V}(q)\Psi^m_n(q)\end{aligned} \tag{2.34}$$

where $\bar{V} \equiv V/(2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x'}q'), q = \frac{\sqrt{x_2}q_1 - \sqrt{x_1}q_2}{\sqrt{x_1+x_2}}$. $m = m_1 + m_2 - m', n = n_1 + n_2 - n' + \frac{1}{2}(|m_1| + |m_2| - |m'| - |m|) \geq 0$. Constraints on external quantum numbers: $|m_1 + m_2 - m'| \leq 1$; $2n_1 + |m_1| + 2n_2 + |m_2| \geq 2n' + |m'| + |m_1 + m_2 - m'| \geq 2n' + |m'|$.

$$\begin{aligned}&\langle n'_1, m'_1; n'_2, m'_2|V|n_1, m_1; n_2, m_2\rangle\\ &= \frac{1}{x_1 + x_2}\sum_{N,M}\mathcal{M}^{N,M,n,m}_{n_1,m_1,n_2,m_2}\mathcal{M}^{N,M,n',m'}_{n'_1,m'_1,n'_2,m'_2}\int \mathrm{d}^2q\,\mathrm{d}^2q' \cdot \bar{V}(q, q')\Psi^m_n(q)\Psi^{m'*}_{n'}(q')\end{aligned} \tag{2.35}$$

where $\bar{V} \equiv V/(2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 - \sqrt{x'_1}q'_1 - \sqrt{x'_2}q'_2), q = \frac{\sqrt{x_2}q_1 - \sqrt{x_1}q_2}{\sqrt{x_1+x_2}}, q' = \frac{\sqrt{x'_2}q'_1 - \sqrt{x'_1}q'_2}{\sqrt{x'_1+x'_2}}$. $m = m_1 + m_2 - M$, $m' = m'_1 + m'_2 - M$, $n = n_1 + n_2 - N + \frac{1}{2}(|m_1| + |m_2| - |M| - |m|) \geq 0$, $n' = n'_1 + n'_2 - N + \frac{1}{2}(|m'_1| + |m'_2| - |M| - |m'|) \geq 0$. Constraints on external quanta: $|m_1 + m_2 - m'_1 - m'_2| \leq 2$.

$$\begin{aligned}&\langle n', m'|V|n_1, m_1; n_2, m_2; n_3, m_3\rangle\\ &= \frac{1}{x'}\sum_{NM}\mathcal{M}^{N,M,n,m}_{n_1,m_1,n_2,m_2}\mathcal{M}^{n',m',n'',m''}_{N,M,n_3,m_3}\int \mathrm{d}^2q\,\mathrm{d}^2q'' \cdot \bar{V}(q, q'')\Psi^m_n(q)\Psi^{m''}_{n''}(q'')\end{aligned} \tag{2.36}$$

where $\bar{V} \equiv V/(2\pi)^2\delta^2(\sqrt{x_1}q_1 + \sqrt{x_2}q_2 + \sqrt{x_3}q_3 - \sqrt{x'}q'), q = \frac{\sqrt{x_2}q_1 - \sqrt{x_1}q_2}{\sqrt{x_1+x_2}}, q'' = \frac{\sqrt{x_1x_3}q_1 + \sqrt{x_2x_3}q_2 - (x_1+x_2)q_3}{\sqrt{x_1+x_2}\sqrt{x_1+x_2+x_3}}$. $m = m_1 + m_2 - M, m'' = M + m_3 - m'$, $n = n_1 + n_2 - N + \frac{1}{2}(|m_1| + |m_2| - |M| - |m|) \geq 0$, $n'' = N + n_3 - n' + \frac{1}{2}(|M| + |m_3| - |m'| - |m''|) \geq 0$. Constraints on external quanta: $|m_1 + m_2 + m_3 - m'| \leq 2$, $2n_1 + |m_1| + 2n_2 + |m_2| + 2n_3 + |m_3| \geq 2n' + |m'|$.

Furthermore, if the vertex is essentially a 2-body operator, it may only depend on the two-body relative kinematics. In that case, the expression can be further simplified (See App. C for example).

# Chapter 3

# Light-Front QCD Matrix Elements

## 3.1 Spinor Structure

The elementary spinor vertices of LFQCD are summarized in Table A.2. Any other spinor structure can be derived from the elementary ones by using crossing symmetries (See A.4),

$$v_s(p) = u_{\text{-}s}(\text{-}p); \tag{3.1}$$

$$\varepsilon^{\mu *}_{\lambda}(k) = \varepsilon^{\mu}_{\text{-}\lambda}(k) = \varepsilon^{\mu}_{\text{-}\lambda}(\text{-}k). \tag{3.2}$$

Table 3.1: Elementary Spinor Vertices. $\omega$ is the light-front normal, $m_f$ fermion mass, $\delta^{\pm}_{\lambda} \equiv \delta_{\lambda,\pm}$.

| | | $s=+$ | $s=-$ |
|---|---|---|---|
| $\bar{u}_{s'}(p')\not{\epsilon}_{\lambda}(k)u_s(p)$ | $s'$=+ | $2\left[\left(\frac{k^R}{k^+}-\frac{p^R}{p^+}\right)\delta^+_{\lambda}+\left(\frac{k^L}{k^+}-\frac{p'^L}{p'^+}\right)\delta^-_{\lambda}\right]$ | $\sqrt{2}\,m_f\,\delta_{\lambda,+}\left(\frac{1}{p'^+}-\frac{1}{p^+}\right)$ |
| a diagram | $s'$=− | $-\sqrt{2}\,m_f\,\delta_{\lambda,-}\left(\frac{1}{p'^+}-\frac{1}{p^+}\right)$ | $2\left[\left(\frac{k^R}{k^+}-\frac{p'^R}{p'^+}\right)\delta^+_{\lambda}+\left(\frac{k^L}{k^+}-\frac{p^L}{p^+}\right)\delta^-_{\lambda}\right]$ |
| $\bar{u}_{s'}(p')\not{\omega}u_s(p)$ | $s'$=+ | 2 | 0 |
| a diagram | $s'$=− | 0 | 2 |
| $\bar{u}_{s'}\not{\epsilon}^*_{\lambda'}\not{\omega}\not{\epsilon}_{\lambda}u_s$ | $s'$=+ | $4\delta_{\lambda,-}\delta_{\lambda',-}$ | 0 |
| a diagram | $s'$=− | 0 | $4\delta_{\lambda,+}\delta_{\lambda',+}$ |

## 3.2 Kinetic Term

The kinetic energy terms are,

$$P^-_0 = \int \mathrm{d}^3x\ \bar{\psi}(x)\gamma^+\frac{(\mathrm{i}\partial^{\perp})^2+m_f^2}{\mathrm{i}\partial^+}\psi(x)\bigg|_{x^+=0} - \int \mathrm{d}^3x\ A^i_c(x)(\mathrm{i}\partial^{\perp})^2 A_{ic}(x)\bigg|_{x^+=0} \tag{3.3}$$

In Fock space, the kinetic energy operator can be written as,

$$\begin{aligned} P_0^2 =& \sum_{i=1}^{N_f+N_b} \frac{\boldsymbol{p}_i^2 + M_i^2}{x_i} - \boldsymbol{P}_{\rm CM}^2 \\ P_\lambda^2 =& \lambda_{\rm CM} \left[\boldsymbol{P}_{\rm CM}^2 + b^4 \boldsymbol{R}_{\rm CM}^2 - 2b^2\right] \\ P_{\rm CON}^2 =& \kappa^4 \sum_{i=1}^{N_f} x_i(1-x-x_i)\boldsymbol{r}_i^2 - 2\kappa^4 \sum_{i<j}^{N_f} x_i x_j \ \boldsymbol{r}_i \cdot \boldsymbol{r}_j \end{aligned} \tag{3.4}$$

where $N_f$ ($N_b$) is number of fermions (bosons); $M_i$ are particle masses, $M_i = m_f$ for fermions and $M_i = \mu \to 0$ for bosons. $\boldsymbol{P}_{\rm CM} = \sum_{i=1}^{N_f+N_b} \boldsymbol{p}_i$ is the center-of-mass momentum; $\boldsymbol{R}_{\rm CM} = \sum_{i=1}^{N_f+N_b} x_i \boldsymbol{r}_i$ is the center-of-mass coordinate; $b$ is the basis energy parameter, $x = \sum_i^{N_b} x_i$. We have added a Lagrange multiplier term (controlled by $\lambda_{\rm CM}$) and a confining potential term (controlled by $\kappa$) only on *fermions*.

**The Matrix Element in Harmonic Oscillator Basis** Denote the integral 2.4 as,

$$\begin{aligned} \mathbb{I}^{n_1 m_1 n_2 m_2}_{n_1' m_1' n_2' m_2'} =& \delta_{m_1+1,m_1'} \delta_{m_2'+1,m_2} f_{n_1,m_1,n_1',m_1'} \ f_{n_2',m_2',n_2,m_2} \\ &+ \delta_{m_1'+1,m_1} \delta_{m_2+1,m_2'} f_{n_1',m_1',n_1,m_1} \ f_{n_2,m_2,n_2',m_2'} \end{aligned} \tag{3.5}$$

with

$$f_{n,m,n',m'} = \begin{cases} \delta_{n,n'}\sqrt{n+|m+1|} - \delta_{n',n-1}\sqrt{n} & (m \geq 0) \\ \delta_{n,n'}\sqrt{n+|m|} - \delta_{n',n+1}\sqrt{n+1} & (m < 0) \end{cases}$$

$$\begin{aligned} P_0^2 =& b^2 \sum_{i=1}^{N_f+N_b} \sum_{\bar\alpha_i} \left[(1-x_i)(2n_i + |m_i| + 1) + \frac{M_i^2/b^2}{x_i}\right] c_i^\dagger(\bar\alpha_i) c_i(\bar\alpha_i) \\ &- b^2 \sum_{i=1}^{N_f+N_b} \sum_{\bar\alpha_i} (1-x_i)\sqrt{(n_i+1)(n_i+1+|m_i|)} \left[c_i^\dagger(\bar\alpha_i) c_i(\bar\alpha_i + \bar 1_n) + h.c.\right] \\ &+ b^2 \sum_{i<j}^{N_f+N_b} \sum_{\bar\alpha_i \bar\alpha_j \bar\alpha_i' \bar\alpha_j'} \sqrt{x_i x_j}\ \mathbb{I}^{n_i m_i n_j m_j}_{n_i' m_i' n_j' m_j'} \delta_{\alpha_i,\alpha_i'} \delta_{\alpha_j,\alpha_j'} c_i^\dagger(\bar\alpha_i') c_j^\dagger(\bar\alpha_j') c_j(\bar\alpha_j) c_i(\bar\alpha_i) \end{aligned} \tag{3.6}$$

where, $c$ is an annihilation operator of any type (fermion, boson, anti-fermion etc.). $\alpha = (\text{color}, x, s), \bar\alpha = (\alpha, n, m) = (\text{color}, x, s, n, m), \bar\alpha + \bar 1_n \equiv \bar\alpha + (\text{color} = 0, x = 0, s = 0, n = 1, m = 0)$.

$$\begin{aligned} P_\lambda^2 =& \lambda_{\rm CM} b^2 \sum_{i=1}^{N_f+N_b} \sum_{\bar\alpha_i} 2x_i(2n_i + |m_i| + 1) c_i^\dagger(\bar\alpha_i) c_i(\bar\alpha_i) \\ &+ \lambda_{\rm CM} b^2 \sum_{i<j}^{N_f+N_b} \sum_{\bar\alpha_i \bar\alpha_j \bar\alpha_i' \bar\alpha_j'} \left[1 + (-1)^{n_i+n_j-n_i'-n_j'+\frac{1}{2}(|m_i|+|m_j|-|m_i'|-|m_j'|)}\right] \\ &\qquad \cdot \sqrt{x_i x_j}\ \mathbb{I}^{n_i m_i n_j m_j}_{n_i' m_i' n_j' m_j'} \delta_{\alpha_i,\alpha_i'} \delta_{\alpha_j,\alpha_j'} c_i^\dagger(\bar\alpha_i') c_j^\dagger(\bar\alpha_j') c_j(\bar\alpha_j) c_i(\bar\alpha_i) \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
P^2_{\rm CON} =&\kappa^4\ell^2\sum_{i=1}^{N_f}\sum_{\bar\alpha_i}(1-x-x_i)(2n_i+|m_i|+1)c_i^\dagger(\bar\alpha_i)c_i(\bar\alpha_i)\\
&+\kappa^4\ell^2\sum_{i=1}^{N_f}\sum_{\bar\alpha_i}(1-x-x_i)\sqrt{(n_i+1)(n_i+1+|m_i|)}\left[c_i^\dagger(\bar\alpha_i)c_i(\bar\alpha_i+\bar 1_n)+h.c.\right]\\
&-\kappa^4\ell^2\sum_{i<j}^{N_f}\sum_{\bar\alpha_i\bar\alpha_j\bar\alpha_i'\bar\alpha_j'}(-1)^{n_i+n_j-n_i'-n_j'+\frac12(|m_i|+|m_j|-|m_i'|-|m_j'|)}\\
&\qquad\cdot\sqrt{x_ix_j}\,\mathbb{I}^{n_im_in_jm_j}_{n_i'm_i'n_j'm_j'}\delta_{\alpha_i,\alpha_i'}\delta_{\alpha_j,\alpha_j'}c_i^\dagger(\bar\alpha_i')c_j^\dagger(\bar\alpha_j')c_j(\bar\alpha_j)c_i(\bar\alpha_i)
\end{aligned}
\tag{3.8}
$$

## 3.3 Vector Coupling Vertex

$$
P_1^- = 2g\int \mathrm{d}^3x\;\bar\psi(x)\gamma^\mu T^a\psi(x)A_{\mu a}\Big|_{x^+=0}
\tag{3.9}
$$

**1.** $q+g\to q$ $(p_1+k=p_2)$

$$
\begin{aligned}
P^2_{1,1}=V_{qg\to q} =&b^2\sum_{\bar\alpha_1\bar\alpha_2\bar\gamma}\frac{-gT^a_{i_2i_1}}{\pi\sqrt{2\mathcal{K}x_k}}(-1)^{n'}\vartheta(n')\mathcal{M}^{n_2m_2n'm'}_{n_1m_1nm}f^{s_2}_{s_1\lambda}\\
&\cdot\delta_{x_1+x_k,x_2}b^\dagger_{i_2}(\bar\alpha_2)b_{i_1}(\bar\alpha_1)a_a(\bar\gamma)
\end{aligned}
\tag{3.10}
$$

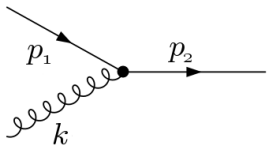



Figure 3.1: Vertex Diagram of process: $q+g\to q$

where $m'=s_2-s_1-\lambda, n'=n_1+n-n_2+\frac12(|m_1|+|m|-|m_2|-|m'|)$.
$s_1,s_2=\pm\frac12,\lambda=\pm1$, $\delta=\arctan\sqrt{\frac{x_k}{x_1}}$.

$$
f^{s_2}_{s_1\lambda}=\begin{cases}
2\sqrt{n'+1}\left(\delta_{\lambda,+1}+\frac{x_1}{x_2}\delta_{\lambda,-1}\right) & s_1=+,s_2=+;\\
2\sqrt{n'+1}\left(\frac{x_1}{x_2}\delta_{\lambda,+1}+\delta_{\lambda,-1}\right) & s_1=-,s_2=-;\\
\frac{m_f}{b}\sqrt{\frac{x_k}{x_2}}\left(\sqrt{\frac{x_1}{x_2}}-1\right)\delta_{\lambda,-1} & s_1=+,s_2=-;\\
\frac{m_f}{b}\sqrt{\frac{x_k}{x_2}}\left(1-\sqrt{\frac{x_1}{x_2}}\right)\delta_{\lambda,+1} & s_1=-,s_2=+.
\end{cases}
$$

**2.** $q\to q+g$ $(p_1=k+p_2)$

$$
\begin{aligned}
P^2_{1,2}=V_{q\to qg} =&b^2\sum_{\bar\alpha_1\bar\alpha_2\bar\gamma}\frac{-gT^a_{i_2i_1}}{\pi\sqrt{2\mathcal{K}x_k}}(-1)^{n'}\vartheta(n')\mathcal{M}^{n_1m_1n'm'}_{n_2m_2nm}f^{s_2\lambda}_{s_1}\\
&\cdot\delta_{x_2+x_k,x_1}b^\dagger_{i_2}(\bar\alpha_2)b_{i_1}(\bar\alpha_1)a^\dagger_a(\bar\gamma)
\end{aligned}
\tag{3.11}
$$

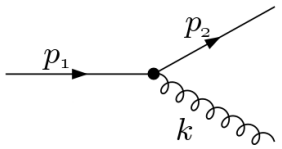



Figure 3.2: Vertex Diagram of process: $q\to q+g$

where $m'=s_1-s_2-\lambda, n'=n_2+n-n_1+\frac12(|m_2|+|m|-|m_1|-|m'|)$.
$s_1,s_2=\pm\frac12,\lambda=\pm1$, $\delta=\arctan\sqrt{\frac{x_k}{x_2}}$.

$$
f^{s_2\lambda}_{s_1}=\begin{cases}
2\sqrt{n'+1}\left(\delta_{\lambda,+1}+\frac{x_2}{x_1}\delta_{\lambda,-1}\right) & s_1=+,s_2=+;\\
2\sqrt{n'+1}\left(\frac{x_2}{x_1}\delta_{\lambda,+1}+\delta_{\lambda,-1}\right) & s_1=-,s_2=-;\\
\frac{m_f}{b}\sqrt{\frac{x_k}{x_1}}\left(1-\sqrt{\frac{x_2}{x_1}}\right)\delta_{\lambda,+1} & s_1=+,s_2=-.\\
\frac{m_f}{b}\sqrt{\frac{x_k}{x_1}}\left(\sqrt{\frac{x_2}{x_1}}-1\right)\delta_{\lambda,-1} & s_1=-,s_2=+;
\end{cases}
$$

**3.** $\bar{q}+g\to\bar{q}$ $(p_1+k=p_2)$

$$P^2_{1,3}=V_{\bar{q}g\to\bar{q}}=b^2\sum_{\bar{\alpha}_1\bar{\alpha}_2\bar{\gamma}}\frac{gT^a_{i_1i_2}}{\pi\sqrt{2\mathcal{K}x_k}}(-1)^{n'}\vartheta(n')\mathcal{M}^{n_2m_2n'm'}_{n_1m_1nm}f^{s_2}_{s_1\lambda}\cdot\delta_{x_1+x_k,x_2}d^\dagger_{i_2}(\bar{\alpha}_2)d_{i_1}(\bar{\alpha}_1)a_a(\bar{\gamma}) \tag{3.12}$$

where $m'=s_2-s_1-\lambda, n'=n_1+n-n_2+\frac{1}{2}(|m_1|+|m|-|m_2|-|m'|)$. $s_1,s_2=\pm\frac{1}{2},\lambda=\pm1,\ \delta=\arctan\sqrt{\frac{x_k}{x_1}}$.

$$f^{s_2}_{s_1\lambda}=\begin{cases}2\sqrt{n'+1}\left(\delta_{\lambda,+1}+\frac{x_1}{x_2}\delta_{\lambda,-1}\right) & s_1=+,s_2=+;\\ 2\sqrt{n'+1}\left(\frac{x_1}{x_2}\delta_{\lambda,+1}+\delta_{\lambda,-1}\right) & s_1=-,s_2=-;\\ \frac{m_f}{b}\sqrt{\frac{x_k}{x_2}}\left(\sqrt{\frac{x_1}{x_2}}-1\right)\delta_{\lambda,-1} & s_1=+,s_2=-;\\ \frac{m_f}{b}\sqrt{\frac{x_k}{x_2}}\left(1-\sqrt{\frac{x_1}{x_2}}\right)\delta_{\lambda,+1} & s_1=-,s_2=+.\end{cases}$$

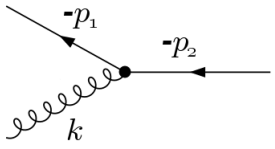



Figure 3.3: Vertex Diagram of process: $\bar{q}+g\to\bar{q}$

**4.** $\bar{q}\to\bar{q}+g$ $(p_1=p_2+k)$

$$P^2_{1,4}=V_{q\to qg}=b^2\sum_{\bar{\alpha}_1\bar{\alpha}_2\bar{\gamma}}\frac{gT^a_{i_2i_1}}{\pi\sqrt{2\mathcal{K}x_k}}(-1)^{n'}\vartheta(n')\mathcal{M}^{n_1m_1n'm'}_{n_2m_2nm}f^{s_2\lambda}_{s_1}\cdot\delta_{x_2+x_k,x_1}d^\dagger_{i_2}(\bar{\alpha}_2)d_{i_1}(\bar{\alpha}_1)a^\dagger_a(\bar{\gamma}) \tag{3.13}$$

where $m'=s_1-s_2-\lambda, n'=n_2+n-n_1+\frac{1}{2}(|m_2|+|m|-|m_1|-|m'|)$. $s_1,s_2=\pm\frac{1}{2},\lambda=\pm1,\ \delta=\arctan\sqrt{\frac{x_k}{x_2}}$.

$$f^{s_2\lambda}_{s_1}=\begin{cases}2\sqrt{n'+1}\left(\delta_{\lambda,+1}+\frac{x_2}{x_1}\delta_{\lambda,-1}\right) & s_1=+,s_2=+;\\ 2\sqrt{n'+1}\left(\frac{x_2}{x_1}\delta_{\lambda,+1}+\delta_{\lambda,-1}\right) & s_1=-,s_2=-;\\ \frac{m_f}{b}\sqrt{\frac{x_k}{x_1}}\left(1-\sqrt{\frac{x_2}{x_1}}\right)\delta_{\lambda,+1} & s_1=+,s_2=-.\\ \frac{m_f}{b}\sqrt{\frac{x_k}{x_1}}\left(\sqrt{\frac{x_2}{x_1}}-1\right)\delta_{\lambda,-1} & s_1=-,s_2=+;\end{cases}$$

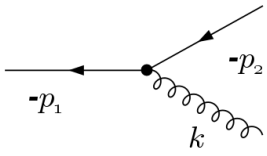



Figure 3.4: Vertex Diagram of process: $\bar{q}\to\bar{q}+g$

**5.** $q+\bar{q}\to g$ $(p_1+p_2=k)$

$$P^2_{1,5}=V_{q\bar{q}\to g}=b^2\sum_{\bar{\alpha}_1\bar{\alpha}_2\bar{\gamma}}\frac{gT^a_{i_2i_1}}{\pi\sqrt{2\mathcal{K}x_k}}(-1)^{n'}\vartheta(n')\mathcal{M}^{nmn'm'}_{n_1m_1n_2m_2}f^{\lambda}_{s_1s_2}\cdot\delta_{x_1+x_2,x_k}d_{i_2}(\bar{\alpha}_2)b_{i_1}(\bar{\alpha}_1)a^\dagger_a(\bar{\gamma}) \tag{3.14}$$

where $m'=\lambda-s_1-s_2, n'=n_1+n_2-n+\frac{1}{2}(|m_1|+|m_2|-|m|-|m'|)$. $s_1,s_2=\pm\frac{1}{2},\lambda=\pm1,\ \delta=\arctan\sqrt{\frac{x_2}{x_1}}$.

$$f^{\lambda}_{s_1s_2}=\begin{cases}2\sqrt{n'+1}\left(-\frac{x_2}{x_k}\delta_{\lambda,+1}+\frac{x_1}{x_k}\delta_{\lambda,-1}\right) & s_1=-,s_2=+;\\ 2\sqrt{n'+1}\left(+\frac{x_1}{x_k}\delta_{\lambda,+1}-\frac{x_2}{x_k}\delta_{\lambda,-1}\right) & s_1=+,s_2=-;\\ +\frac{m_f}{b}\left(\sqrt{\frac{x_1}{x_k}}+\sqrt{\frac{x_2}{x_k}}\right)\delta_{\lambda,+1} & s_1=+,s_2=+;\\ -\frac{m_f}{b}\left(\sqrt{\frac{x_1}{x_k}}+\sqrt{\frac{x_2}{x_k}}\right)\delta_{\lambda,-1} & s_1=-,s_2=-.\end{cases}$$

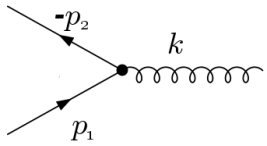



Figure 3.5: Vertex Diagram of process: $q+\bar{q}\to g$

**6.** $g\to q+\bar{q}$ $(k=p_1+p_2)$

$$P^2_{1,6}=V_{g\to q\bar{q}}=b^2\sum_{\bar{\alpha}_1\bar{\alpha}_2\bar{\gamma}}\frac{gT^a_{i_1i_2}}{\pi\sqrt{2\mathcal{K}x_k}}(-1)^{n'}\vartheta(n')\mathcal{M}^{nmn'm'}_{n_1m_1n_2m_2}f^{s_1s_2}_{\lambda}\cdot\delta_{x_1+x_2,x_k}b^\dagger_{i_1}(\bar{\alpha}_1)d^\dagger_{i_2}(\bar{\alpha}_2)a_a(\bar{\gamma}) \tag{3.15}$$

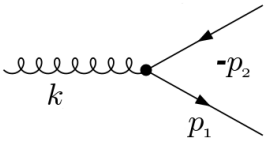



Figure 3.6: Vertex Diagram of process: $g\to q+\bar{q}$

where $m' = \lambda - s_1 - s_2, n' = n_1 + n_2 - n + \frac{1}{2}(|m_1| + |m_2| - |m| - |m'|)$.
$s_1, s_2 = \pm\frac{1}{2}, \lambda = \pm 1$, $\delta = \arctan\sqrt{\frac{x_2}{x_1}}$.

$$f_\lambda^{s_1 s_2} = \begin{cases} 2\sqrt{n'+1}\left(+\frac{x_1}{x_k}\delta_{\lambda,+1} - \frac{x_2}{x_k}\delta_{\lambda,-1}\right) & s_1 = +, s_2 = -; \\ 2\sqrt{n'+1}\left(-\frac{x_2}{x_k}\delta_{\lambda,+1} + \frac{x_1}{x_k}\delta_{\lambda,+1}\right) & s_1 = -, s_2 = +; \\ +\frac{m_f}{b}\left(\sqrt{\frac{x_1}{x_k}} + \sqrt{\frac{x_2}{x_k}}\right)\delta_{\lambda,+1} & s_1 = +, s_2 = +; \\ -\frac{m_f}{b}\left(\sqrt{\frac{x_1}{x_k}} + \sqrt{\frac{x_2}{x_k}}\right)\delta_{\lambda,-1} & s_1 = -, s_2 = -. \end{cases}$$

## 3.4 Vector Coupling and Triple Gluon Vertex with Instantaneous Gluon

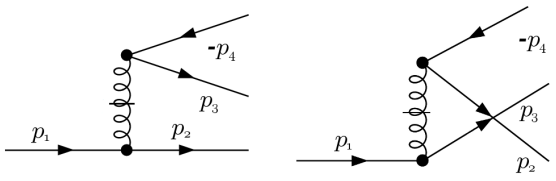
$$P_2^- = -2g^2 \mathrm{i} f^{abc} \int \mathrm{d}^3x \; \bar{\psi}\gamma^+ T^c \psi \frac{1}{(\mathrm{i}\partial^+)^2} \left(\mathrm{i}\partial^+ A_a^\mu A_{\mu b}\right)\Big|_{x^+=0} \tag{3.16}$$

## 3.5 Vector Coupling with Instantaneous Fermion

$$P_3^- = g^2 \int \mathrm{d}^3x \; \bar{\psi} \not{A} \frac{\gamma^+}{\mathrm{i}\partial^+} \not{A} \psi \Big|_{x^+=0} \tag{3.17}$$

## 3.6 Vector Coupling with Instantaneous Gluon

$$P_4^- = g^2 \int \mathrm{d}^3x\ \bar{\psi}\gamma^+ T^a \psi \frac{1}{(\mathrm{i}\partial^+)^2} \bar{\psi}\gamma^+ T^a \psi \Big|_{x^+=0} \tag{3.18}$$



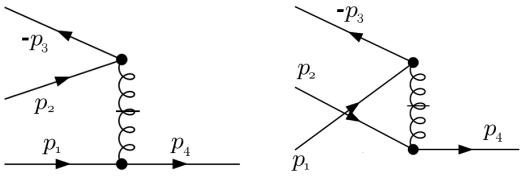

(a) channel $a$ (b) channel $b$

Figure 3.7: Vertex diagram of the process $q \to q + q + \bar{q}$

**1.** $q \to q + q + \bar{q}$ $(p_1 = p_2 + p_3 + p_4)$

$$\begin{aligned} P^2_{4,1} = V_{q\to qq\bar{q}} =& b^2 \sum_{\bar{\alpha}_1\langle\bar{\alpha}_2\bar{\alpha}_3\rangle\bar{\alpha}_4} \frac{g^2}{\pi^2\mathcal{K}} \sqrt{\frac{x_2x_3x_4}{x_1}} \left[ \frac{T^c_{i_2i_1}T^c_{i_3i_4}}{(x_3+x_4)^2}\delta_{s_1,s_2}\delta_{s_3,-s_4} - \frac{T^c_{i_3i_1}T^c_{i_2i_4}}{(x_2+x_4)^2}\delta_{s_3,s_1}\delta_{s_2,-s_4} \right] \\ &\cdot(-1)^{\chi+\xi} \sum_{N=\max(0,-\chi)}^{\xi} \mathcal{M}^{N,m_2+m_3,\xi-N,0}_{n_2,m_2,n_3,m_3} \mathcal{M}^{n_1,m_1,\chi+N,0}_{N,m_2+m_3,n_4,m_4} \\ &\cdot\delta_{m_1,m_2+m_3+m_4} \cdot \delta_{x_1,x_2+x_3+x_4} \cdot b^\dagger_{i_2}(\bar{\alpha}_2) b^\dagger_{i_3}(\bar{\alpha}_3) d^\dagger_{i_4}(\bar{\alpha}_4) b_{i_1}(\bar{\alpha}_1) \end{aligned} \tag{3.19}$$

where $\xi = n_2 + n_3 + \frac{1}{2}(|m_2| + |m_3| - |m_2 + m_3|), \chi = n_4 - n_1 + \frac{1}{2}(|m_2 + m_3| + |m_4| - |m_1|)$;
$\delta_1 = \arctan\sqrt{\frac{x_3}{x_2}}, \delta_2 = \arctan\sqrt{\frac{x_4}{x_2+x_3}}$.



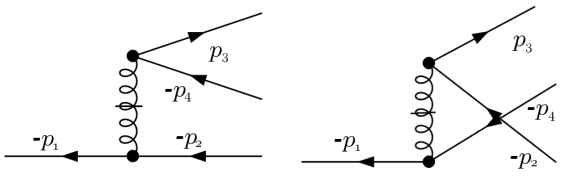

(a) channel $a$ (b) channel $b$

Figure 3.8: Vertex diagram of the process $q + q + \bar{q} \to q$

**2.** $q + q + \bar{q} \to q$ $(p_1 + p_2 + p_3 = p_4)$

$$\begin{aligned} P^2_{4,2} = V_{qq\bar{q}\to q} =& b^2 \sum_{\langle\bar{\alpha}_1\bar{\alpha}_2\rangle\bar{\alpha}_3\bar{\alpha}_4} \frac{g^2}{\pi^2\mathcal{K}} \sqrt{\frac{x_1x_2x_3}{x_4}} \left[ \frac{T^c_{i_4i_1}T^c_{i_3i_2}}{(x_2+x_3)^2}\delta_{s_1,s_4}\delta_{-s_3,s_2} - \frac{T^c_{i_4i_2}T^c_{i_3i_1}}{(x_1+x_3)^2}\delta_{s_2,s_4}\delta_{s_1,-s_3} \right] \\ &\cdot(-1)^{\chi+\xi} \sum_{N=\max(0,-\chi)}^{\xi} \mathcal{M}^{N,m_1+m_2,\xi-N,0}_{n_1,m_1,n_2,m_2} \mathcal{M}^{n_4,m_4,\chi+N,0}_{N,m_1+m_2,n_3,m_3} \\ &\cdot\delta_{m_1+m_2+m_3,m_4} \cdot \delta_{x_1+x_2+x_3,x_4} \cdot b^\dagger_{i_4}(\bar{\alpha}_4) d_{i_3}(\bar{\alpha}_3) b_{i_2}(\bar{\alpha}_2) b_{i_1}(\bar{\alpha}_1) \end{aligned} \tag{3.20}$$

where $\xi = n_1 + n_2 + \frac{1}{2}(|m_1| + |m_2| - |m_1 + m_2|), \chi = n_3 - n_4 + \frac{1}{2}(|m_1 + m_2| + |m_3| - |m_4|)$;
$\delta_1 = \arctan\sqrt{\frac{x_2}{x_1}}, \delta_2 = \arctan\sqrt{\frac{x_3}{x_1+x_2}}$.



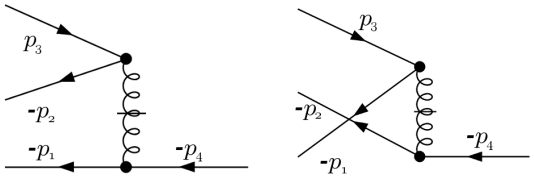

(a) channel $a$ (b) channel $b$

Figure 3.9: Vertex diagram of the process $\bar{q} \to \bar{q} + \bar{q} + q$

**3.** $\bar{q} \to q + \bar{q} + \bar{q}$ $(p_1 = p_2 + p_3 + p_4)$

$$\begin{aligned} P_{4,3}^2 =& V_{\bar{q}\to\bar{q}\bar{q}q} \\ =& b^2 \sum_{\bar{\alpha}_1\bar{\alpha}_3\langle\bar{\alpha}_2\bar{\alpha}_4\rangle} \frac{g^2}{\pi^2\mathcal{K}} \sqrt{\frac{x_2x_3x_4}{x_1}} \left[ \frac{T^c_{i_3i_4}T^c_{i_1i_2}}{(x_3+x_4)^2}\delta_{s_3,-s_4}\delta_{s_1,s_2} - \frac{T^c_{i_3i_2}T^c_{i_1i_4}}{(x_2+x_3)^2}\delta_{s_3,-s_2}\delta_{s_1,s_4} \right] \\ & \cdot(-1)^{\chi+\xi} \sum_{N=\max(0,-\chi)}^{\xi} \mathcal{M}^{N,m_2+m_3,\xi-N,0}_{n_2,m_2,n_3,m_3} \mathcal{M}^{n_1,m_1,\chi+N,0}_{N,m_2+m+3,n_4,m_4} \\ & \cdot\delta_{m_1,m_2+m_3+m_4} \cdot \delta_{x_1,x_2+x_3+x_4} \cdot b^\dagger_{i_3}(\bar{\alpha}_3) d^\dagger_{i_2}(\bar{\alpha}_2) d^\dagger_{i_4}(\bar{\alpha}_4) d_{i_1}(\bar{\alpha}_1) \end{aligned} \tag{3.21}$$

where $\xi = n_2 + n_3 + \frac{1}{2}(|m_2| + |m_3| - |m_2 + m_3|), \chi = n_4 - n_1 + \frac{1}{2}(|m_2 + m_3| + |m_4| - |m_1|)$;
$\delta_1 = \arctan\sqrt{\frac{x_3}{x_2}}, \delta_2 = \arctan\sqrt{\frac{x_4}{x_2+x_3}}$.



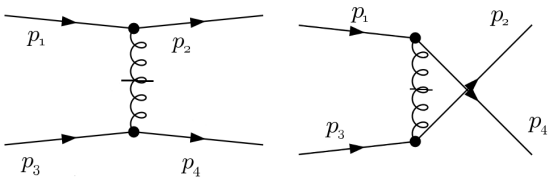

(a) channel $a$ (b) channel $b$

Figure 3.10: Vertex diagram of the process $q + \bar{q} + \bar{q} \to \bar{q}$

**4.** $q + \bar{q} + \bar{q} \to \bar{q}$ $(p_1 + p_2 + p_3 = p_4)$

$$\begin{aligned} P_{4,4}^2 =& V_{q\bar{q}\bar{q}\to\bar{q}} \\ =& b^2 \sum_{\langle\bar{\alpha}_1\bar{\alpha}_2\rangle\bar{\alpha}_3\bar{\alpha}_4} \frac{g^2}{\pi^2\mathcal{K}} \sqrt{\frac{x_1x_2x_3}{x_4}} \left[ \frac{T^c_{i_2i_3}T^c_{i_1i_4}}{(x_2+x_3)^2}\delta_{s_3,-s_2}\delta_{s_1,s_4} - \frac{T^c_{i_1i_3}T^c_{i_2i_4}}{(x_1+x_3)^2}\delta_{s_3,-s_1}\delta_{s_2,s_4} \right] \\ & \cdot(-1)^{\xi+\chi} \sum_{N=\max(0,-\chi)}^{\xi} \mathcal{M}^{N,m_1+m_2,\xi-N,0}_{n_1,m_1,n_2,m_2} \mathcal{M}^{n_4,m_4,\chi+N,0}_{N,m_1+m_2,n_3,m_3} \\ & \cdot\delta_{m_1+m_2+m_3,m_4} \cdot \delta_{x_1+x_2+x_3,x_4} \cdot d^\dagger_{i_4}(\bar{\alpha}_4) d_{i_2}(\bar{\alpha}_2) d_{i_1}(\bar{\alpha}_1) b_{i_3}(\bar{\alpha}_3) \end{aligned} \tag{3.22}$$

where $\xi = n_1 + n_2 + \frac{1}{2}(|m_1| + |m_2| - |m_1 + m_2|), \chi = n_3 - n_4 + \frac{1}{2}(|m_1 + m_2| + |m_3| - |m_4|)$;
$\delta_1 = \arctan\sqrt{\frac{x_2}{x_1}}, \delta_2 = \arctan\sqrt{\frac{x_3}{x_1+x_2}}$.



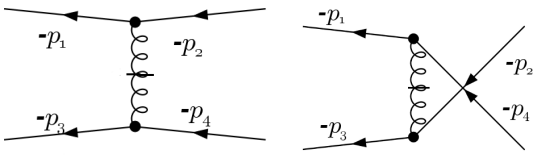

(a) $t$-channel (b) $u$-channel

Figure 3.11: Vertex diagram of the process $q + q \to q + q$

**5.** $q + q \to q + q$ $(p_1 + p_3 = p_2 + p_4)$

$$\begin{aligned} P^2_{4,5} =& V_{qq\to qq} \\ =& b^2 \sum_{\langle\bar\alpha_1\bar\alpha_3\rangle\langle\bar\alpha_2\bar\alpha_4\rangle} \frac{g^2}{\pi^2\mathcal{K}} \frac{\sqrt{x_1x_2x_3x_4}}{x_1+x_3} \left[ \frac{T^c_{i_2i_1}T^c_{i_4i_3}}{(x_1-x_2)^2}\delta_{s_1,s_2}\delta_{s_3,s_4} - \frac{T^c_{i_4i_1}T^c_{i_2i_3}}{(x_1-x_4)^2}\delta_{s_1,s_4}\delta_{s_3,s_2} \right] \\ & \cdot(-1)^{\xi+\chi} \sum_{N=0}^{\min(\xi,\chi)} \mathcal{M}^{N,m_1+m_3,\xi-N,0}_{n_1,m_1,n_3,m_3} \mathcal{M}^{N,m_2+m_4,\chi-N,0}_{n_2,m_2,n_4,m_4} \\ & \cdot\delta_{m_1+m_3,m_2+m_4} \cdot \delta_{x_1+x_3,x_2+x_4} \cdot b^\dagger_{i_2}(\bar\alpha_2) b^\dagger_{i_4}(\bar\alpha_4) b_{i_3}(\bar\alpha_3) b_{i_1}(\bar\alpha_1) \end{aligned} \tag{3.23}$$

where $\xi = n_1 + n_3 + \frac{1}{2}(|m_1| + |m_3| - |m_1 + m_3|), \chi = n_2 + n_4 + \frac{1}{2}(|m_2| + |m_4| - |m_2 + m_4|)$; $\delta_1 = \arctan\sqrt{\frac{x_3}{x_1}}, \delta_2 = \arctan\sqrt{\frac{x_4}{x_2}}$.



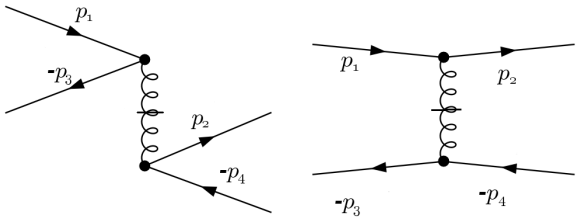

(a) $t$-channel (b) $u$-channel

Figure 3.12: Vertex diagram of the process $\bar q + \bar q \to \bar q + \bar q$

**6.** $\bar q + \bar q \to \bar q + \bar q$ $(p_1 + p_2 = p_3 + p_4)$

$$\begin{aligned} P^2_{4,6} =& V_{\bar q\bar q\to \bar q\bar q} \\ =& b^2 \sum_{\langle\bar\alpha_1\bar\alpha_3\rangle\langle\bar\alpha_2\bar\alpha_4\rangle} \frac{g^2}{\pi^2\mathcal{K}} \frac{\sqrt{x_1x_2x_3x_4}}{x_1+x_3} \left[ \frac{T^c_{i_1i_2}T^c_{i_3i_4}}{(x_1-x_2)^2}\delta_{s_1,s_2}\delta_{s_3,s_4} - \frac{T^c_{i_1i_4}T^c_{i_3i_2}}{(x_1-x_4)^2}\delta_{s_1,s_4}\delta_{s_3,s_2} \right] \\ & \cdot(-1)^{\xi+\chi} \sum_{N=0}^{\min(\xi,\chi)} \mathcal{M}^{N,m_1+m_3,\xi-N,0}_{n_1,m_1,n_3,m_3} \mathcal{M}^{N,m_2+m_4,\chi-N,0}_{n_2,m_2,n_4,m_4} \\ & \cdot\delta_{m_1+m_2,m_3+m_4} \cdot \delta_{x_1+x_3,x_2+x_4} \cdot d^\dagger_{i_2}(\bar\alpha_2) d^\dagger_{i_4}(\bar\alpha_4) d_{i_3}(\bar\alpha_3) d_{i_1}(\bar\alpha_1) \end{aligned} \tag{3.24}$$

where $\xi = n_1 + n_3 + \frac{1}{2}(|m_1| + |m_3| - |m_1 + m_3|), \chi = n_2 + n_4 + \frac{1}{2}(|m_2| + |m_4| - |m_2 + m_4|)$; $\delta_1 = \arctan\sqrt{\frac{x_3}{x_1}}, \delta_2 = \arctan\sqrt{\frac{x_4}{x_2}}$.




(a) $s$-channel (b) $t$-channel

Figure 3.13: Vertex diagram of the process $q + \bar{q} \to q + \bar{q}$

**7.** $q + \bar{q} \to q + \bar{q}$ $(p_1 + p_3 = p_2 + p_4)$

$$\begin{aligned} P_{4,7}^2 =& V_{qq\to\bar{q}\bar{q}} \\ =& b^2 \sum_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4} \frac{g^2}{\pi^2\mathcal{K}} \frac{\sqrt{x_1x_2x_3x_4}}{x_1+x_3} \left[ \frac{T^c_{i_3i_1}T^c_{i_2i_4}}{(x_1+x_3)^2}\delta_{s_1,-s_3}\delta_{s_2,-s_4} - \frac{T^c_{i_2i_1}T^c_{i_3i_4}}{(x_1-x_2)^2}\delta_{s_1,s_2}\delta_{s_3,s_4} \right] \\ & \cdot(-1)^{\xi+\chi} \sum_{N=0}^{\min(\xi,\chi)} \mathcal{M}^{N,m_1+m_3,\xi-N,0}_{n_1,m_1,n_3,m_3} \mathcal{M}^{N,m_2+m_4,\chi-N,0}_{n_2,m_2,n_4,m_4} \\ & \cdot\delta_{m_1+m_2,m_3+m_4} \cdot \delta_{x_1+x_3,x_2+x_4} \cdot b^\dagger_{i_2}(\bar{\alpha}_2) d^\dagger_{i_4}(\bar{\alpha}_4) d_{i_3}(\bar{\alpha}_3) b_{i_1}(\bar{\alpha}_1) \end{aligned} \tag{3.25}$$

where $\xi = n_1 + n_3 + \frac{1}{2}(|m_1| + |m_3| - |m_1 + m_3|), \chi = n_2 + n_4 + \frac{1}{2}(|m_2| + |m_4| - |m_2 + m_4|)$; $\delta_1 = \arctan\sqrt{\frac{x_3}{x_1}}, \delta_2 = \arctan\sqrt{\frac{x_4}{x_2}}$.

## 3.7 Triple Gluon Vertex

$$P_5^- = 2g\,\mathrm{i}f^{abc}\int \mathrm{d}^3x\,\mathrm{i}\partial^\mu A^\nu_a A_{\mu b} A_{\nu c}\Big|_{x^+=0} \tag{3.26}$$

## 3.8 Quadruple Gluon Vertex

$$P_6^- = -\frac{g^2}{2}\int \mathrm{d}^3x\,\mathrm{i}f^{cc_1c_2}\mathrm{i}f^{cc_3c_4} A^\mu_{c_1} A^\nu_{c_2} A_{\mu c_3} A_{\nu c_4}\Big|_{x^+=0} \tag{3.27}$$

## 3.9 Triple Gluon Vertices with Instantaneous Gluon

$$P_7^- = g^2\int \mathrm{d}^3x\,\mathrm{i}f^{abc}\,\mathrm{i}f^{ade}\,\mathrm{i}\partial^+ A^\mu_b A_{\mu c}\frac{1}{(\mathrm{i}\partial^+)^2}\left(\mathrm{i}\partial^+ A^\nu_d A_{\nu e}\right)\Big|_{x^+=0} \tag{3.28}$$

# Appendix A

# Conventions

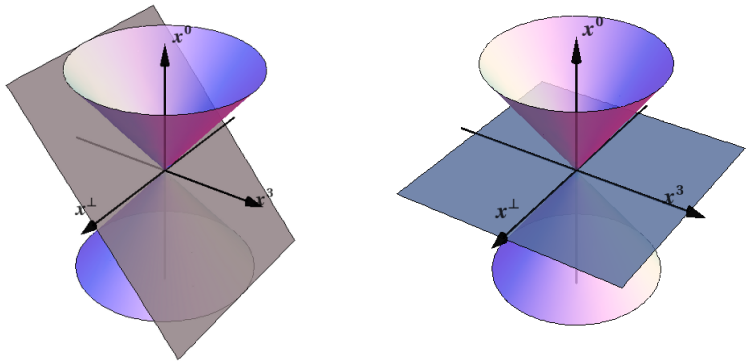
## A.1 General Conventions

Throughout this set of notes, we adopt natural units, $\hbar = c = 1$. Let $x = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$ be regular form of spacetime coordinates. The corresponding *Minkowski* space metric is defined as,

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{A.1}$$

## A.2 Light-Front Variables

A 4-vector $\omega$ is called null if $\omega^2 = 0$. A plane is called light-front (LF) if its normal vector is null. A Light-Front Quantum Field Theory (LFQFT) is a field theory quantized on a light-front. It's convenient to introduce light-front variables. In most cases (except for the Covariant Light-Front Dynamics) we take $\omega^\mu = (1, 0, 0, -1)$ (See. Fig. ??).




(a) light-front quantization (b) equal-time quantization

Figure A.1: Left: quantization on a Light-Front with normal vector $\omega = (1, 0, 0, 1)$.

Define $v^\pm = v^0 \pm v^3$. A 4-vector can be represented with light-front variables $v = (v^+, v^-, v^1, v^2) =$

$(v^+, v^-, \mathbf{v}^\perp)$. The *Minkowski* space metric in light-front variables is,

$$g^{\mu\nu} = \begin{pmatrix} & 2 & & \\ 2 & & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{and} \quad g_{\mu\nu} = \begin{pmatrix} & \frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{A.2}$$

A Lorentz invariant bilinear inner product $a\cdot b = a^+b_+ + a^-b_- + a^1b_1 + a^2b_2 = \frac{1}{2}a^+b^- + \frac{1}{2}a^-b^+ - a^1b^1 - a^2b^2$. Note that $a_+ = \frac{1}{2}a^-$.

The coordinate space integration measure is defined as $\int \mathrm{d}^3x \equiv \int \mathrm{d}x_+\mathrm{d}^2x^\perp = \frac{1}{2}\int \mathrm{d}x^-\mathrm{d}^2x^\perp$.

**Light-Front Time and Longitudinal Momentum Fraction** Similar with the equal-time case, $x^+$, the coordinate perpendicular to the quantization surface is called light-front time. The momentum $p_+ = \frac{1}{2}p^-$ conjugate to $x^+$ is called light-front energy. Momentum $p^+$ is called longitudinal momentum. A momentum fraction of a particle is defined as $x = \frac{p^+}{P^+}$ where $p^+$ is the particle's longitudinal momentum, $P^+$ is the longitudinal momentum of the system. Momentum fraction is a Lorentz invariant and sits in $[0, 1]$.

**Transverse Momenta** $\mathbf{p}^\perp = (p^1, p^2)$ are called transverse momenta. It will be convenient to introduce complex representation, $p = p^1 + \mathrm{i}p^2, p^{L/R} = \frac{p^1 \mp p^2}{\sqrt{2}}$, for transverse vectors. Throughout the notes, we will drop the superscript $\perp$ for transverse vectors. An italic font vector normally refers to either a 4-vector or the complex representation of its transverse components depending on the context.

**Alternative Conventions** Another popular convention of light-front variables is $v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3)$. Under this convention, the *Minkowski* space metric in light-front variables is,

$$g^{\mu\nu} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix} \quad \text{and} \quad g_{\mu\nu} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix} \tag{A.3}$$

Thoughout these notes, we primarily use the first convention.

## A.3 Normalization

We define Fourier transform,

$$f(p) = \int \mathrm{d}^d x \; e^{\mathrm{i}p\cdot x} f(x) \tag{A.4}$$

$$f(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \; e^{-\mathrm{i}p\cdot x} f(p). \tag{A.5}$$

The corresponding orthonormality relations of the coordinate and momentum space are,

$$\langle p'|p\rangle = (2\pi)^d \delta^d(p - p') \qquad \langle x'|x\rangle = \delta^d(x - x') \tag{A.6}$$

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \; |p\rangle \langle p| = \mathbb{I}_d \qquad \int \mathrm{d}^d x \; |x\rangle \langle x| = \mathbb{I}_d \tag{A.7}$$

For light-front variables, the appropriate Lorent invariant phase space measure is

$$\begin{aligned}&\int \frac{\mathrm{d}^4p}{(2\pi)^4}\theta(p^+)2\pi\delta(p^2-m^2)\\&=\int \frac{\mathrm{d}^2p\mathrm{d}p^+}{(2\pi)^3 2p^+}\theta(p^+)=\int\frac{\mathrm{d}^2p}{(2\pi)^2}\int_0^1\frac{\mathrm{d}x}{2\pi 2x}\end{aligned} \tag{A.8}$$

where $x$ is momentum fraction. The momentum states are normalized as,

$$\langle p|p'\rangle=(2\pi)^3 2p^+\theta(p^+)\delta^3(p-p')=(2\pi)^3 2x\delta(x-x')\delta^2(\mathbf{p}-\mathbf{p}') \tag{A.9}$$

For discrete variables, however, it is more convenient to normalize to unity (Kronecker delta). For example, for a countable set of basis $|f_{nml}\rangle\,,(n,m,l=1,2,\cdots)$, the proper normalization is,

$$\langle f_{n'm'l'}|f_{nml}\rangle=\delta_{nn'}\delta_{mm'}\delta_{ll'}. \tag{A.10}$$

## A.4 Spinors

Introduce $4\times 4$ gamma matrices:

$$\gamma^0=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\quad \gamma^3=\begin{pmatrix}0&\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\quad \gamma^1=\begin{pmatrix}-\mathrm{i}\sigma^2&0\\0&\mathrm{i}\sigma^2\end{pmatrix}\quad \gamma^2=\begin{pmatrix}\mathrm{i}\sigma^1&0\\0&-\mathrm{i}\sigma^1\end{pmatrix} \tag{A.11}$$

with the conventional Pauli matrices read,

$$\sigma^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad \sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \sigma^2=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{A.12}$$

The $\gamma$-matrices defined here furnish a representation of the Clifford algebra:

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}. \tag{A.13}$$

Then, $S^{\mu\nu}\equiv\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]$ furnishes the spinor representation of the Lorentz group.

We also introduce several definitions, corresponding to similar individuals in equal-time formalism.

**Definition:**

- front-form: $\gamma^\pm\equiv\gamma^0\pm\gamma^3, \gamma^{L/R}\equiv\frac{\gamma^1\mp\mathrm{i}\gamma^2}{\sqrt{2}}$;
- projections: $\Lambda^\pm\equiv\Lambda_\pm\equiv\frac{1}{2}\gamma^0\gamma^\pm$;
- parity matrix: The parity matrix $\beta$ should satisfy: $\beta^2=1, \beta\vec{\gamma}\beta=-\vec{\gamma}, \beta\gamma^0\beta=\gamma^0$. Under our convention of the metric tensor $g_{\mu\nu}$, it is convenient to choose $\beta=\gamma^0$.
- Miscellaneous: $\gamma^5=\gamma_5\equiv\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{a}\equiv\gamma_\mu a^\mu$, $\bar{\psi}\equiv\psi^\dagger\beta$ (for spinor vector) and $\bar{A}=\beta A^\dagger\beta$ (for spinor matrix).

Under these conventions,

$$\gamma^+=\begin{pmatrix}0&0\\2\mathrm{i}&0\end{pmatrix}\quad \gamma^-=\begin{pmatrix}0&-2\mathrm{i}\\0&0\end{pmatrix},\quad \Lambda^+=\begin{pmatrix}1&\\&0\end{pmatrix}\quad \Lambda^-=\begin{pmatrix}0&\\&1\end{pmatrix}$$

$$\gamma^L=\sqrt{2}\begin{pmatrix}\sigma^L&0\\0&-\sigma^L\end{pmatrix}\quad \gamma^R=\sqrt{2}\begin{pmatrix}-\sigma^R&0\\0&\sigma^R\end{pmatrix},\quad \sigma^L=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad \sigma^R=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$\gamma_5$ and $S_z$ are diagonalized,

$$\gamma^5=\begin{pmatrix}\sigma^3&\\&-\sigma^3\end{pmatrix}\qquad S_z=S^{12}=\frac{1}{2}\begin{pmatrix}\sigma^3&\\&\sigma^3\end{pmatrix}. \tag{A.14}$$

**Corollary:**

- $\gamma^\pm\gamma^\pm = 0; \gamma^+\gamma^-\gamma^+ = 4\gamma^+, \gamma^-\gamma^+\gamma^- = 4\gamma^-; \gamma^0\gamma^\pm = \gamma^\mp\gamma^0$
- $\frac{1}{4}\gamma^\pm\gamma^\mp = \Lambda^\mp$;
- $\Lambda_\pm^2 = \Lambda_\pm, \Lambda^\pm\Lambda^\mp = 0, \Lambda^+ + \Lambda^- = 1,$ (i.e., $\Lambda^\pm$ are projections);

**Spinors** Spinor $u_s(p) = u(p,s) = u(p^\perp, \alpha), v_s(p) = v(p,s) = v(p^\perp, \alpha)$ are solutions of Dirac equations:

$$(\not{p} - m)u_s(p) = 0; \qquad (\not{p} + m)v_s(p) = 0, \tag{A.15}$$

We define their helicity along $z$ direction,

$$\begin{aligned} &S_z u_\pm(\mathbf{0}) = \pm\tfrac{1}{2}u_\pm(\mathbf{0}); \\ &S_z v_\pm(\mathbf{0}) = \mp\tfrac{1}{2}v_\pm(\mathbf{0}). \end{aligned} \tag{A.16}$$

The two-component spinors is used throughout these notes,

$$\begin{aligned} u_s(p) &= \frac{1}{2p^+}(\not{p} + m)\gamma^+\chi_s = \frac{1}{p^+}(p^+ + \boldsymbol{\alpha}^\perp\cdot\boldsymbol{p}^\perp + \beta m)\chi_s; \\ v_s(p) &= \frac{1}{2p^+}(\not{p} - m)\gamma^+\chi_{-s} = \frac{1}{p^+}(p^+ + \boldsymbol{\alpha}^\perp\cdot\boldsymbol{p}^\perp - \beta m)\chi_{-s}; \end{aligned} \tag{A.17}$$

where $\chi_\pm$ are the eigenvectors of $\Lambda_+$ with eigenvalue 1. Specifically, $\chi_+ = \binom{\xi_+}{0} = (1,0,0,0)^\intercal; \chi_- = \binom{\xi_-}{0} = (0,1,0,0)^\intercal$, with $\chi_s^\dagger\chi_{s'} = \delta_{ss'}$.

As a comparison, we also introduce Lepage-Brodsky's spinors:

$$\begin{aligned} u_s^{\rm LB}(p) &= \frac{1}{\sqrt{p^+}}(p^+ + \boldsymbol{\alpha}^\perp\cdot\boldsymbol{p}^\perp + \beta m)\chi_s^{\rm LB} \\ v_s^{\rm LB}(p) &= \frac{1}{\sqrt{p^+}}(p^+ + \boldsymbol{\alpha}^\perp\cdot\boldsymbol{p}^\perp - \beta m)\chi_{-s}^{\rm LB}; \end{aligned} \tag{A.18}$$

where the $\chi_s^{\rm LB}$-spinors, again, the eigenvectors of the projector $\Lambda_+^{\rm LB}$ with eigenvalue 1. Because Lepage and Brodsky uses Dirac's gamma matrices, the entry of their $\chi_s^{\rm LB}$-spinors are $\chi_+^{\rm LB} = \frac{1}{\sqrt{2}}(1,0,1,0)^\intercal, \chi_-^{\rm LB} = \frac{1}{\sqrt{2}}(0,1,0,-1)^\intercal$.

As we can see, the two-component spinor formalism differs from L-B convention only by a $\sqrt{p^+}$.

**Corollary:**

$$\bar u_s(p)u_{s'}(p) = \frac{2m\delta_{ss'}}{p^+}, \quad \bar v_s(p)v_{s'}(p) = -\frac{2m\delta_{ss'}}{p^+}, \quad \bar u_s(p)v_{s'}(p) = 0; \tag{A.19}$$

$$\sum_{s=\pm} u_s(p)\bar u_s(p) = \frac{\not{p} + m}{p^+}, \quad \sum_{s=\pm} v_s(p)\bar v_s(p) = \frac{\not{p} - m}{p^+}; \tag{A.20}$$

$$u_s(p) = v_{-s}(-p), \quad \bar u_s(p) = \bar v_{-s}(-p); \tag{A.21}$$

$$2m\bar u_{s'}(p')\gamma^\mu u_s(p) = \bar u_{s'}(p')(p+p')^\mu u_s(p) - 2\mathrm{i}\bar u_{s'}(p')S^{\mu\nu}(p-p')_\nu u_s(p); \tag{A.22}$$

$$2m\bar v_{s'}(p')\gamma^\mu v_s(p) = -\bar v_{s'}(p')(p+p')^\mu v_s(p) + 2\mathrm{i}\bar v_{s'}(p')S^{\mu\nu}(p-p')_\nu v_s(p); \tag{A.23}$$

$$2m\bar u_{s'}(p')\gamma^\mu v_s(p) = \bar u_{s'}(p')(p'-p)^\mu v_s(p) + 2\mathrm{i}\bar u_{s'}(p')S^{\mu\nu}(p'+p)_\nu v_s(p); \tag{A.24}$$

$$\bar u_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu\delta_{ss'}/p^+, \quad \bar v_s(p)\gamma^\mu v_{s'}(p) = 2p^\mu\delta_{ss'}/p^+; \tag{A.25}$$

$$\bar u_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) = 0; \tag{A.26}$$

$$\bar u_s(p)\gamma^+ u_{s'}(p') = 2\delta_{ss'}, \quad \bar v_s(p)\gamma^+ v_{s'}(p') = 2\delta_{ss'}; \tag{A.27}$$

**Trace of Gamma Matrices** Because gamma matrices satisfy anti-commutation relations, the trace of a string of gamma matrices follows *Wick theorem*:

*The trace of a string of gamma matrices equals the sum of all possible contractions with the corresponding permutation signatures included.*

$$\mathrm{tr}\{\text{ product of odd number } \gamma\} = 0, \quad \mathrm{tr}\{\gamma_5 \cdot \text{ product of odd number } \gamma\} = 0, \tag{A.28}$$

$$\mathrm{tr}\{\not{a}\not{b}\} = 4(a\cdot b), \quad \mathrm{tr}\{\gamma_5\not{a}\not{b}\} = 0 \tag{A.29}$$

$$\mathrm{tr}\{\not{a}\not{b}\not{c}\not{d}\} = 4\left((a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right) \tag{A.30}$$

$$\mathrm{tr}\{\gamma_5\not{a}\not{b}\not{c}\not{d}\} = 8\,\varepsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma \tag{A.31}$$

$$\mathrm{tr}\{\gamma_5\not{a}_1\not{a}_2\not{a}_3\not{a}_4\not{a}_5\not{a}_6\} = \sum_{\langle i,j\rangle}(-1)^{\mathrm{Sgn}(i,j)}(a_i\cdot a_j)\mathrm{tr}\{\gamma_5\not{a}_1\cdots\not{a}_6 \text{ without } \not{a}_i \text{ or } \not{a}_j\} \tag{A.32}$$

Note that, because $\varepsilon$ is not a tensor, but $\frac{\varepsilon}{\sqrt{-g}}$ is, the second last one gets an extra factor $-2\mathrm{i}$. Its indices run through "$+,-,L,R$", not "$0,1,2,3$". The new Levi-Civita symbol is defined such that $\varepsilon^{+-LR} = -1$, which can be checked by computing $8\varepsilon^{+-LR} = \mathrm{tr}\gamma_5\not{n}^+\not{n}^-\not{n}^L\not{n}^R = 4\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\not{n}^+_\mu\not{n}^-_\nu\not{n}^L_\rho\not{n}^R_\sigma(\mu,\nu,\rho,\sigma = 0,1,2,3)$. Here $n^\mu{}_\nu = \delta^\mu_\nu$ is a unit vector in $\mu$ direction, esp. $n^+ = \omega$.

**Spinor Structures** The spinor parts in LFQCD Hamiltonian are in three forms:

$$\bar{u}_{s'}(p')\not{\epsilon}_\lambda(k)u_s(p) \tag{A.33}$$

$$\bar{u}_{s'}(p')\gamma^+u_s(p) \tag{A.34}$$

$$\bar{u}_{s'}(p')\not{\epsilon}_{\lambda'}(k')\gamma^+\not{\epsilon}_\lambda(k)u_s(p) \tag{A.35}$$

Systematically, they can be computed via taking the trace of gamma matrices:

$$\begin{aligned}\mathbf{V}_n =&\bar{u}_{s'}(p')\not{a}_1\not{a}_2\cdots\not{a}_n u_s(p)\\ =&\frac{1}{4p^+p'^+}\bar{\chi}_{s'}\gamma^+(\not{p}'+m)\not{a}_1\cdots\not{a}_n(\not{p}+m)\gamma^+\chi_s\\ =&\frac{1}{2p^+p'^+}\mathrm{tr}\left[(\not{p}'+m)\not{a}_1\cdots\not{a}_n(\not{p}+m)\not{\omega}\xi_{ss'}\right]\end{aligned} \tag{A.36}$$

where, $\omega = (1,0,0,-1)$ is the light-front normal, $\omega\cdot a = a^+$,

$$\xi_{s,s'} = \begin{cases}\frac{1+\gamma_5}{2} & s=+,s'=+\\ -\frac{\gamma^R}{\sqrt{2}} & s=+,s'=-\\ \frac{\gamma^L}{\sqrt{2}} & s=-,s'=+\\ \frac{1-\gamma_5}{2} & s=-,s'=-\end{cases} \tag{A.37}$$

The spinor structures can be summarized as following,

Table A.1: Elementary Spinor Vertices. $\omega$ is the light-front normal, $m_f$ fermion spinor mass.

| **spinor vertex** | | $s=+$ | $s=-$ |
|---|---|---|---|
| $\bar{u}_{s'}(p')\not{\varepsilon}_\lambda(k)u_s(p)$ | $s'{=}+$ | $2\left[\left(\frac{k^R}{k^+}-\frac{p^R}{p^+}\right)\varepsilon_\lambda^L+\left(\frac{k^L}{k^+}-\frac{p'^L}{p'^+}\right)\varepsilon_\lambda^R\right]$ | $\sqrt{2}\,m_f\,\varepsilon_\lambda^L\left(\frac{1}{p'^+}-\frac{1}{p^+}\right)$ |
| a diagram | $s'{=}-$ | $-\sqrt{2}\,m_f\,\varepsilon_\lambda^R\left(\frac{1}{p'^+}-\frac{1}{p^+}\right)$ | $2\left[\left(\frac{k^R}{k^+}-\frac{p'^R}{p'^+}\right)\varepsilon_\lambda^L+\left(\frac{k^L}{k^+}-\frac{p^L}{p^+}\right)\varepsilon_\lambda^R\right]$ |
| $\bar{u}_{s'}(p')\not{\omega}u_s(p)$ | $s'{=}+$ | 2 | 0 |
| a diagram | $s'{=}-$ | 0 | 2 |
| $\bar{u}_{s'}\not{\varepsilon}^*_{\lambda'}\not{\omega}\not{\varepsilon}_\lambda u_s$ | $s'{=}+$ | $4\varepsilon_\lambda^R\varepsilon_{\lambda'}^R$ | 0 |
| a diagram | $s'{=}-$ | 0 | $4\varepsilon_\lambda^L\varepsilon_{\lambda'}^L$ |

Note that $\varepsilon_\lambda^L=\delta_{\lambda,+1},\varepsilon_\lambda^R=\delta_{\lambda,-1}$.

Table A.2: More Spinor Vertices. $\omega$ is the light-front normal, $m_f$ fermion spinor mass.

| **spinor vertex** | $s_1$ | $s_2$ | $s_1'$ | $s_2'$ | |
|---|---|---|---|---|---|
| $\bar{u}_{1'}\gamma^\mu u_1\bar{u}_{2'}\gamma_\mu u_{2'}$ | + | + | + | + | $2m_f^2\left(\frac{1}{x_1x_1'}+\frac{1}{x_2x_2'}\right)+2\left(\frac{p_1}{x_1}-\frac{p_2'}{x_2'}\right)\left(\frac{p_1'^*}{x_1'}-\frac{p_2^*}{x_2}\right)$ |
| a diagram | − | − | − | − | |

## A.5 Polarization Vectors

Define the polarization vectors:

$$\varepsilon_\lambda^\mu(k)=\left(0,\frac{2\epsilon_\lambda^\perp\cdot k^\perp}{k^+},\epsilon_\lambda^\perp\right), \tag{A.38}$$

where $\epsilon_\pm^\perp=\frac{1}{\sqrt{2}}(1,\pm\mathrm{i})$. In fact, $\varepsilon_\lambda^L=\delta_{\lambda,+},\varepsilon_\lambda^R=\delta_{\lambda,-}$.

This definition satisfies the light-cone gauge $A^+=0$ and Lorenz condition [1] $\partial_\mu A^\mu=0$,

$$\varepsilon^+=0; \tag{A.39}$$

$$k_\mu\varepsilon_\lambda^\mu(k)=0 \tag{A.40}$$

[1] It should be metioned that using Lorenz condition will lose a class of solutions, though also simplifies the calculation greatly. For original derivation, see Brodsky *et.al.* [6]; for discussion of this condition see W. M. Zhang and A. Harindranath [7].

**Corollary:**

$$\varepsilon^{\mu}_{\lambda}(k)\varepsilon^{*}_{\lambda'\mu}(k) = -\delta_{\lambda,\lambda'} \tag{A.41}$$

$$\sum_{\lambda=\pm} \varepsilon^{i*}_{\lambda}(k)\varepsilon^{j}_{\lambda}(k) = \delta^{ij} \tag{A.42}$$

$$\sum_{\lambda=\pm} \varepsilon^{\mu*}_{\lambda}(k)\varepsilon^{\nu}_{\lambda}(k) = -g^{\mu\nu} + \frac{\omega^{\mu}k^{\nu} + \omega^{\nu}k^{\mu}}{\omega \cdot k} \tag{A.43}$$

$$\varepsilon^{\mu*}_{\lambda}(k) = \varepsilon^{\mu}_{-\lambda}(k) = \varepsilon^{\mu}_{-\lambda}(-k) \tag{A.44}$$

where $\omega^{\mu} = (1, 0, 0, -1)$, is the null normal vector of light-front, $\omega \cdot v = v^{+}$.

## A.6 SU(3) Color

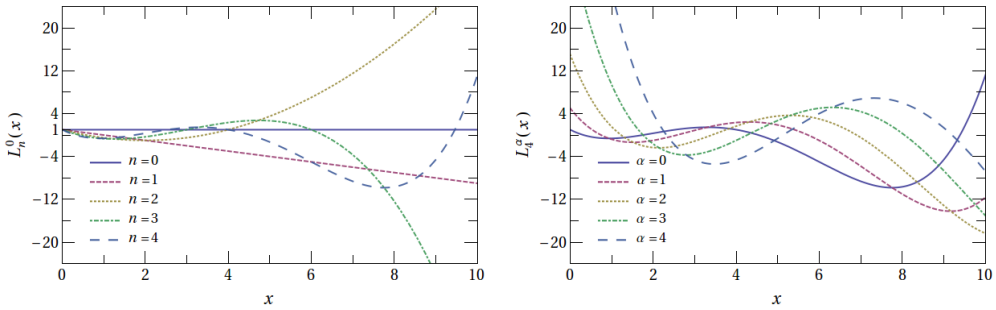
**$\mathfrak{su}_3$ Lie algebra and its representations**

**Gell-Mann Matrices**

**Colorsinglet** Young tableaux and CG decomposition

# Appendix B

# Laguerre Polynomials




(a) $L_n^0(x)$ (b) $L_4^\alpha(x)$

Figure B.1: Laguerre Polynomials

Laguerre Polynomial $L_n^\alpha(x)$ is the solution of differential equation,

$$xy'' + (\alpha + 1 - x)y' + ny = 0, \tag{B.1}$$

with $L_0^\alpha(x) = 1$. $L_n(x) \equiv L_n^0(x)$ is also called Laguerre polynomial, whereas $L_n^\alpha(x)$ termed generalized Laguerre polynomial. Laguerre polynomials admit series expansion,

$$L_n^\alpha(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!}. \tag{B.2}$$

**Orthonormality** Laguerre polynomials are a class of orthogonal polynomials defined on interval $[0, +\infty)$. They satisfy the orthonormality relation, with the weight function $e^{-x}x^a$,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n!}{(n+\alpha)!} x^\alpha e^{-x} L_n^\alpha(x) L_n^\alpha(y) &\to \delta(x-y) \\ \frac{n!}{(n+\alpha)!} \int_0^\infty \mathrm{d}x\, x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) &= \delta_{nm} \end{aligned} \tag{B.3}$$

Finite number of Laguerre polynomials follow Christoffel-Darboux Identity,

$$\sum_{n=0}^{N} \frac{n!}{(n+\alpha)!} x^\alpha e^{-x} L_n^\alpha(x) L_n^\alpha(y) = \frac{(N+1)!}{(N+\alpha)!} x^\alpha e^{-x} \frac{L_N^\alpha(x) L_{N+1}^\alpha(y) - L_{N+1}^\alpha(x) L_N(y)^\alpha}{x-y} \tag{B.4}$$

**Zeros** The largest zero of Laguerre polynomial $L_n^\alpha(x)$ is smaller than $2\cdot(2n+\alpha+1)$ [15]. In fact, the $m$-th zero $\frac{j_{\alpha,m}^2}{2k_n} < x_m^{(n)} = \frac{j_{\alpha,m}^2}{2k_n}\left(1+\frac{2(\alpha^2-1)+j_{\alpha,m}^2}{12k_n^2}\right)+\mathcal{O}(n^{-5}) < \frac{k_m}{k_n}(k_m+\sqrt{k_m^2+\frac{1}{4}-\alpha^2})$, where $k_n = 2n+(\alpha+1)$, $j_{\alpha,m}$ is the $m$-th positive zero of the Bessel function $J_\alpha(x)$.

### Connections with Other Special functions

$$L_n^\alpha(x) = \binom{n+\alpha}{n} M(-n,\alpha+1,x) \tag{B.5}$$

$$L_n^{-\frac{1}{2}}(x) = \frac{(-1)^n}{n!2^{2n}} H_{2n}(\sqrt{x}) \tag{B.6}$$

$$L_n^{\frac{1}{2}}(x) = \frac{(-1)^n}{n!2^{2n+1}\sqrt{x}} H_{2n+1}(\sqrt{x}) \tag{B.7}$$

$$L_n^\alpha(x) = \frac{e^x x^{-\frac{\alpha}{2}}}{n!}\int_0^\infty \mathrm{d}t\; e^{-t}t^{n+\frac{\alpha}{2}}J_\alpha(2\sqrt{tx}) \tag{B.8}$$

### Generating Functions

$$(1-z)^{-\alpha-1}e^{-\frac{xz}{z-1}} = \sum_{n=0}^\infty L_n^\alpha(x)z^n \tag{B.9}$$

$$(xz)^{-\frac{1}{2}\alpha}e^z J_\alpha(2\sqrt{xz}) = \sum_{n=0}^\infty \frac{1}{\Gamma(n+\alpha+1)} L_n^\alpha(x)z^n \tag{B.10}$$

### Asymptotic Behaviors

$$\lim_{n\to\infty}\left[\left(\frac{\sqrt{x}}{n}\right)^\alpha L_n^\alpha(x/n)\right] = J_\alpha(2\sqrt{x}) \tag{B.11}$$

$$\lim_{x\to\infty} L_n^\alpha(x) \sim \frac{(-1)^n}{n!}x^n \tag{B.12}$$

### Recurrence Relations

$$L_n^\alpha(x) = \left(2+\frac{\alpha-1-x}{n}\right)L_{n-1}^\alpha(x) - \left(1+\frac{\alpha-1}{n}\right)L_{n-2}^\alpha(x) \tag{B.13}$$

$$\frac{x^\beta}{\beta!}L_n^\alpha(x) = \sum_{i,j=0}^\beta \frac{(-1)^{i+j}}{n!}\frac{\binom{n+\alpha+i}{\beta}}{\binom{n+\alpha+i}{n+\alpha}}\binom{\beta}{i}\binom{\beta}{j}L_{n+i-j}^\alpha(x) \tag{B.14}$$

$$L_n^{\alpha+1}(x) = \sum_{k=0}^n L_k^\alpha(x) \tag{B.15}$$

$$L_n^\alpha(x) = \sum_{k=0}^n \binom{\gamma+k-1}{k} L_{n-k}^{\alpha-\gamma}(x) \tag{B.16}$$

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x) \tag{B.17}$$

$$L_n^\alpha(x) = \sum_{k=0}^\beta (-1)^k\binom{\beta}{k}L_{n-k}^{\alpha+\beta}(x) \tag{B.18}$$

**Sum Rules**

$$\sum_{k=0}^{n} L_k^{\alpha}(x) = L_n^{\alpha+1}(x) \tag{B.19}$$

$$\sum_{k=0}^{n} L_k^{\alpha}(x) L_{n-k}^{\beta}(y) = L_n^{\alpha+\beta+1}(x+y) \tag{B.20}$$

**Multiplication Rules**

$$L_n^{\alpha}(\mu x) = \sum_{k=0}^{n} \binom{n+\alpha}{k} \mu^{n-k}(1-\mu)^k L_{n-k}^{\alpha}(x) \tag{B.21}$$

**Rodrigues' Formula**

$$L_n^{\alpha}(x) = \frac{1}{n! x^{\alpha} e^{-x}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} x^{\alpha+1} e^{-x} \tag{B.22}$$

**Differentials**

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} L_n^{\alpha}(x) = (-1)^k L_{n-k}^{\alpha+k}(x) \tag{B.23}$$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{\alpha} L_n^{\alpha}(x) = k! \binom{n+\alpha}{k} L_n^{\alpha-k}(x) \tag{B.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} L_n^{\alpha}(x) = \frac{n}{x} L_n^{\alpha}(x) - \frac{n+\alpha}{x} L_{n-1}^{\alpha}(x) \tag{B.25}$$

**Integrations**

$$\int_x^{\infty} \mathrm{d}t\; e^{-t} L_n^{\alpha}(t) = e^{-x}(L_n^{\alpha}(x) - L_{n-1}^{\alpha}(x)) \tag{B.26}$$

$$\int_0^{x} \mathrm{d}t\; (x-t)^{\beta} t^{\alpha} L_n^{\alpha}(t) = \frac{(n+\alpha)!\beta!}{(n+\alpha+\beta)!} x^{\alpha+\beta+1} L_n^{\alpha+\beta+1}(x) \tag{B.27}$$

$$\int_0^{x} \mathrm{d}t\; L_m(t) L_n(x-t) = \int_0^{x} \mathrm{d}t\; L_{m+n}(t) = L_{m+n}(x) - L_{m+n+1}(x) \tag{B.28}$$

$$\int_0^{\infty} \mathrm{d}t\; t^{\beta-1} e^{-t} L_n^{\alpha}(t) = \frac{n+\alpha-\beta)!(\beta-1)!}{n!(\alpha-\beta)!} \tag{B.29}$$

$$\int_0^{\infty} \mathrm{d}t\; t^{\alpha+1} e^{-t} \left[L_n^{\alpha}(t)\right]^2 = \frac{(n+\alpha)!}{n!}(2n+\alpha+1) \tag{B.30}$$

$$\int_0^{\infty} \mathrm{d}t\; t^{\alpha+\beta} e^{-t} \left[L_n^{\alpha}(t)\right]^2 = \frac{(n+\alpha)!\beta!}{n!} \sum_{k=0}^{\beta} \binom{n+k}{k} \binom{n+\alpha}{\beta-k} \binom{\beta}{k} \tag{B.31}$$

$$\begin{aligned} &\int_0^{\infty} dx\; x^{\frac{\alpha+\beta}{2}+\gamma} e^{-x} L_a^{\alpha}(x) L_b^{\beta}(x) \qquad (\alpha > \beta > 0, \gamma \geq 0; \frac{\alpha+\beta}{2} + \gamma \in \mathbb{Z}) \\ =&(-1)^{a+b} \left\{ \vartheta\left(\frac{\alpha-\beta}{2} - \gamma\right) \sum_{k=\max\{0,a-b\}}^{\min\{a,\xi+a-b\}} (-1)^k \binom{\Delta+k-1}{k} \binom{\xi}{b-a+k} \frac{(\frac{\alpha+\beta}{2}+\gamma+a-k)!}{(a-k)!} \right. \\ &\left. + \vartheta\left(\gamma - \frac{\alpha-\beta}{2}\right) \sum_{k=\max\{0,a-b\}}^{\min\{a,\xi+a-b,\Delta\}} \binom{\Delta}{k} \binom{\xi}{b-a+k} \frac{(\frac{\alpha+\beta}{2}+\gamma+a-k)!}{(a-k)!} \right\} \end{aligned} \tag{B.32}$$

where $\xi = \frac{\alpha-\beta}{2} + \gamma; \Delta = \left|\frac{\alpha-\beta}{2} - \gamma\right|$.

**Application: Harmonic Oscillator Wavefunction in Coordinate Space** Once we fix the wavefunction in momentum space, its form in coordinate space is also fixed by a Fourier transform. The 2D Harmonic Oscillator (HO) wavefunction in momentum space reads,

$$\Psi_n^m(p) \equiv \ell\sqrt{\frac{4\pi\cdot n!}{(n+|m|)!}}e^{\mathrm{i}m\varphi}\rho^{|m|}e^{-\rho^2/2}L_n^{|m|}(\rho^2), \tag{B.33}$$

where $\rho = |p|/b, \varphi = \arg p, b = \sqrt{\hbar M_{\mathrm{HO}}\Omega_{\mathrm{HO}}}$, $L_n^\alpha(x)$ the $n$-th generalized Laguerre polynomial of order $\alpha$.

$$\langle r|\Psi_n^m\rangle \equiv \Psi_n^m(r) = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\, e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Psi_n^m(p) \tag{B.34}$$

Since $\Psi_n^{-m}(p) = \Psi_n^m(-p)$, we assume $m > 0$ without loss of generality. Then

$$\langle r|n,m\rangle = \mathcal{N}\int_0^{2\pi}\mathrm{d}\varphi\int_0^\infty \rho\mathrm{d}\rho\, e^{\mathrm{i}\rho\gamma\cos(\varphi-\theta)}\rho^m e^{\mathrm{i}m\varphi}e^{-\rho^2/2}L_n^m(\rho^2)$$

where $\gamma = |r|/\ell, \theta = \arg r, \mathcal{N} = \frac{b}{(2\pi)^2}\sqrt{\frac{4\pi n!}{(n+m)!}}$.

The angular part: subtitute $\beta = \varphi - \theta$,

$$\begin{aligned}&\int_0^{2\pi}\mathrm{d}\varphi\, e^{\mathrm{i}\rho\gamma\cos(\varphi-\theta)+\mathrm{i}m\varphi}\\ =&e^{\mathrm{i}m\theta}\int_0^{2\pi}\mathrm{d}\beta\, e^{\mathrm{i}\rho\gamma\cos\beta+\mathrm{i}m\beta}\\ =&2\pi e^{\mathrm{i}m\theta}e^{\mathrm{i}\frac{\pi}{2}m}J_m(\rho\gamma)\end{aligned}$$

where $J_m(x)$ is Bessel function of the first kind. We have used Jacobi-Anger identity (Abramowitz and Stegun 9.1.42 [16]),

$$e^{\mathrm{i}z\cos\phi} = \sum_{n=-\infty}^{\infty} e^{\mathrm{i}\frac{\pi}{2}n}J_n(z)e^{\mathrm{i}n\phi}.$$

$J_m(x)$ is a generating function of $L_n^m$ (Abramowitz and Stegun. Ch. 22 [13]),

$$(xz)^{-\alpha/2}e^z J_\alpha(2\sqrt{xz}) = \sum_{n=0}^\infty \frac{L_n^\alpha(x)z^n}{(n+\alpha)!}.$$

We take $z = \frac{\gamma^2}{2}, x = \frac{\rho^2}{2}$,

$$\langle r|n,m\rangle = 2\pi\mathcal{N}e^{\mathrm{i}m\theta+\mathrm{i}\frac{\pi}{2}m}\int_0^\infty \rho\mathrm{d}\rho\, e^{-\frac{1}{2}\rho^2}L_n^m(\rho^2)e^{-\frac{1}{2}\gamma^2}\left(\frac{\rho\gamma}{2}\right)^m\rho^m\sum_{k=0}^\infty\frac{L_k^m(\frac{\rho^2}{2})}{(k+m)!}\left(\tfrac{1}{2}\gamma^2\right)^k$$

One useful recurrence relation of Laguerre Polynomial can be found in last section is

$$L_n^m(2t) = \sum_{i=0}^n(-1)^i 2^{n-i}\binom{n+m}{i}L_{n-i}^m(t).$$

Also recall the series definition of Laguerre polynomial

$$L_n^\alpha(x) = \sum_{k=0}^n(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}$$

Therefore,

$$\Psi_n^m(r) \equiv e^{\mathrm{i}\frac{\pi}{2}(2n+|m|)}b\sqrt{\frac{n!}{\pi(n+|m|)!}}e^{\mathrm{i}m\theta}\gamma^{|m|}e^{-\frac{1}{2}\gamma^2}L_n^{|m|}(\gamma^2), \tag{B.35}$$

where $\gamma = |r|/\ell, \theta = \arg r$.

# Appendix C

# Two-body Kinematics

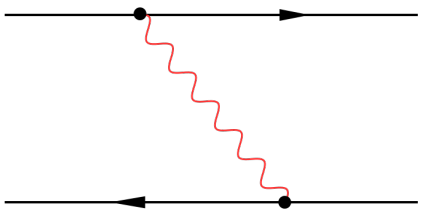


Figure C.1

**Application to Energy Denominator Integral** The Feynman diagram of the energy denominator operator is shown in Fig. C.1.

$$I=\int \text{đ}q_1\,\text{đ}q_2\,\text{đ}q_1'\,\text{đ}q_2'\,\frac{(2\pi)^2\delta^2(\sqrt{x_1}q_1+\sqrt{x_2}q_2-\sqrt{x_1'}q_1'-\sqrt{x_2'}q_2')}{\frac{x_1\mathbf{q}_1^2+m_f^2}{x_1}-\frac{x_1'\mathbf{q}_1'^2+m_f^2}{x_1'}-\frac{(\sqrt{x_1}\mathbf{q}_1-\sqrt{x_1'}\mathbf{q}_1')^2}{x_1-x_1'}}\Psi_{n_1}^{m_1}(q_1)\Psi_{n_2}^{m_2}(q_2)\Psi_{n_1'}^{m_1'*}(q_1')\Psi_{n_2'}^{m_2'*}(q_2') \tag{C.1}$$

with the longitudinal momentum conservation $X\equiv x_1+x_2=x_1'+x_2'\equiv X'$. Introduce a boost invariant $\mathbf{k}\equiv\sqrt{x_1'}\mathbf{q}_1-\sqrt{x_1}\mathbf{q}_1'$. It can be shown the energy denominator can be written as,

$$D\equiv\frac{x_1\mathbf{q}_1^2+m_f^2}{x_1}-\frac{x_1'\mathbf{q}_1'^2+m_f^2}{x_1'}-\frac{(\sqrt{x_1}\mathbf{q}_1-\sqrt{x_1'}\mathbf{q}_1')^2+\mu^2}{x_1-x_1'}=-\frac{\mathbf{k}^2}{x_1-x_1'}+\frac{m_f^2}{x_1}-\frac{m_f^2}{x_1'}-\frac{\mu^2}{x_1-x_1'}.$$

Hence the energy denominator is also a boost invariant. On the other hand, within CM and intrinsic parameters, the only boost invariant momenta are $\mathbf{q},\mathbf{q}',\mathbf{Q}-\mathbf{Q}'$. By TM transform,

$$\begin{aligned}I=&\sum_{NMnm}\sum_{N'M'n'm'}\mathcal{M}^{NMnm}_{n_1m_1n_2m_2}\mathcal{M}^{N'M'n'm'}_{n_1'm_1'n_2'm_2'}\\&\cdot\int \text{đ}Q\,\text{đ}q\,\text{đ}Q'\,\text{đ}q'\,\frac{(2\pi)^2\delta^2(\sqrt{X}\mathbf{Q}-\sqrt{X'}\mathbf{Q}')}{D(\mathbf{q},\mathbf{q}',\mathbf{Q}-\mathbf{Q}')}\Psi_N^M(Q)\Psi_n^m(q)\Psi_{N'}^{M'*}(Q')\Psi_{n'}^{m'*}(q')\end{aligned} \tag{C.2}$$

Note that $X=X'$, hence $\mathbf{Q}=\mathbf{Q}'$, $D=D(\mathbf{q},\mathbf{q}')$. In fact it is easy to show $\mathbf{k}=\frac{1}{\sqrt{X}}(\sqrt{x_1'x_2}\mathbf{q}-\sqrt{x_1x_2'}\mathbf{q}')$. So,

$$\begin{aligned}I=&\sum_{NMnm}\sum_{N'M'n'm'}\mathcal{M}^{NMnm}_{n_1m_1n_2m_2}\mathcal{M}^{N'M'n'm'}_{n_1'm_1'n_2'm_2'}\delta_{N,N'}\delta_{M,M'}\\&\cdot\int \text{đ}q\,\text{đ}q'\,\frac{1}{-\frac{(\sqrt{x_1'x_2}\mathbf{q}-\sqrt{x_1x_2'}\mathbf{q}')^2}{x_1-x_1'}+\frac{m_f^2X}{x_1}-\frac{m_f^2X}{x_1'}-\frac{\mu^2X}{x_1-x_1'}}\Psi_n^m(q)\Psi_{n'}^{m'*}(q')\end{aligned}$$

Finally, we introduce "CM" and "relative" coordinates

$$\mathbf{P} \equiv \frac{\sqrt{x_1 x_2'}\mathbf{q} + \sqrt{x_1' x_2}\mathbf{q}'}{\sqrt{x_1 x_2' + x_1' x_2}}$$
$$\mathbf{p} \equiv \frac{\sqrt{x_1' x_2}\mathbf{q} - \sqrt{x_1 x_2'}\mathbf{q}'}{\sqrt{x_1 x_2' + x_1' x_2}}$$

and do TM transform to the new coordinates,

$$\begin{aligned} I = &\sum_{NMnm} \sum_{N'M'n'm'} \sum_{N''M''n''m''} \mathcal{M}^{NMnm}_{n_1 m_1 n_2 m_2} \mathcal{M}^{N'M'n'm'}_{n_1' m_1' n_2' m_2'} \mathcal{M}^{N'',M'',n'',m''}_{n,m,n',-m'} \delta_{N,N'} \delta_{M,M'} \\ &\cdot \int \text{đ}P\ \Psi^{M''}_{N''}(P) \int \text{đ}p\ \frac{1}{-\frac{x_1 x_2' + x_1' x_2}{x_1 - x_1'}\mathbf{p}^2 + (x_1 + x_2)\left(\frac{m_f^2}{x_1} - \frac{m_f^2}{x_1'} - \frac{\mu^2}{x_1 - x_1'}\right)} \Psi^{m''}_{n''}(p) \end{aligned} \tag{C.3}$$

The phases for each TM transform are $\delta = \arctan\sqrt{\frac{x_2}{x_1}}, \delta' = \arctan\sqrt{\frac{x_2'}{x_1'}}, \delta'' = \arctan\sqrt{\frac{x_1' x_2}{x_2' x_1}}$ respectively.

$$I^m_n(\Delta) = \int \text{đ}p\ \frac{1}{\mathbf{p}^2 + \Delta} \Psi^m_n(p) = \frac{\delta_{m,0}}{\sqrt{4\pi}} \int \mathrm{d}t \frac{1}{t + \Delta} e^{-t/2} L_n(t) \tag{C.4}$$

where $L_n(t)$ is the Laguerre polynomial of the n-th order. Using the series expansion of Laguerre polynomials, the above expression can also be rewritten as,

$$\begin{aligned} I^m_n(\Delta) =& \delta_{m,0} \frac{e^{\Delta/2}}{\sqrt{4\pi}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^k \Gamma\left(-k, \frac{\Delta}{2}\right) \\ =& \delta_{m,0} \frac{e^{\Delta/2}}{\sqrt{4\pi}} \int_{\frac{\Delta}{2}}^{\infty} \mathrm{d}t\ e^{-t} \left(1 - \frac{\Delta}{t}\right)^n \frac{1}{t} \end{aligned} \tag{C.5}$$

where, $\Gamma(s,z) = \int_z^\infty \mathrm{d}t\ e^{-t} t^{s-1}$ is the incomplete gamma function. Therefore, the energy denominator integral is,

$$\begin{aligned} I = &\delta_{m_1+m_2, m_1'+m_2'} \frac{x_1' - x_1}{x_1 x_2' + x_1' x_2} \frac{e^{\Delta/2}}{2\pi} \sum_{m=\frac{m_1+m_2-\varepsilon}{2}}^{\frac{m_1+m_2+\varepsilon}{2}} \sum_{N=0}^{\min\{\mu,\nu\}} \sum_{N''=0}^{\mu+\nu+|m|-2N} \sum_{k=0}^{n''} \binom{n''}{k} (-1)^{k+N''} \\ &\cdot \mathcal{M}^{N,m_1+m_2-m,\mu-N,m}_{n_1,m_1,n_2,m_2} \mathcal{M}^{N,m_1'+m_2'-m,\nu-N,m}_{n_1',m_1',n_2',m_2'} \mathcal{M}^{N'',0,n'',0}_{\mu-N,m,\nu-N,-m} \Delta^k \Gamma\left(-k, \tfrac{\Delta}{2}\right) \end{aligned} \tag{C.6}$$

where, $E = 2(n_1+n_2)+|m_1|+|m_2|, E' = 2(n_1'+n_2')+|m_1'|+|m_2'|, \varepsilon = \min\{E, E'\}$, $\mu = (E-|m_1+m_2-m|-|m|)/2, \nu = (E'-|m_1'+m_2'-m|-|m|)/2$, $n'' = \mu+\nu+|m|-2N-N''$. $\Delta = \frac{x_1+x_2}{x_1 x_2' + x_2 x_1'}\left(m_f^2 \frac{(x_1-x_1')^2}{x_1 x_1'} - \mu^2\right)$.

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