LIGHT-CONE COORDINATES

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In special relativity, *light-cone coordinates* are defined by taking the conventional time axis x^0 and one space axis, usually x^1, and defining two new coordinates:

$$\begin{aligned} x^+ &\equiv \frac{1}{\sqrt{2}}\left(x^0 + x^1\right) \\ x^- &\equiv \frac{1}{\sqrt{2}}\left(x^0 - x^1\right) \end{aligned} \tag{1}$$

On a spacetime diagram with the x^0 axis vertical and the x^1 axis horizontal, the x^+axis (corresponding to $x^- = 0$) is the diagonal line $x^0 = x^1$. The x^- axis is the line $x^0 = -x^1$.

We can invert 1 to get

$$\begin{aligned} x^0 &= \frac{1}{\sqrt{2}}\left(x^+ + x^-\right) \\ x^1 &= \frac{1}{\sqrt{2}}\left(x^+ - x^-\right) \end{aligned} \tag{2}$$

The invariant interval in lightcone coordinates becomes

$$-ds^2 = -2dx^+dx^- + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{3}$$

This can be written using the lightcone metric defined as

$$\hat{\eta}_{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

So we have

$$-ds^2 = \hat{\eta}_{\mu\nu}dx^\mu dx^\nu \tag{5}$$

where the components of the infinitesimal displacement are

$$dx^\mu = \left(dx^+, dx^-, dx^2, dx^3\right) \tag{6}$$

Any 4-vector can also be written in lightcone form, as

$$a^+ \equiv \frac{1}{\sqrt{2}}\left(a^0 + a^1\right) \tag{7}$$

$$a^- \equiv \frac{1}{\sqrt{2}}\left(a^0 - a^1\right) \tag{8}$$

Thus we also have

$$a \cdot b = \hat{\eta}_{\mu\nu} a^\mu b^\nu \tag{9}$$

For raising and lowering indexes, we get

$$\begin{aligned} a_+ &= \hat{\eta}_{+\mu} a^\mu = -a^- \\ a_- &= \hat{\eta}_{-\mu} a^\mu = -a^+ \end{aligned} \tag{10}$$

For a particle moving along the x^1 axis with speed β, we have

$$x^1(t) = \beta x^0 \tag{11}$$

Plugging this into the lightcone coordinates gives

$$x^+ = \frac{1+\beta}{\sqrt{2}} x^0 \tag{12}$$

$$x^- = \frac{1-\beta}{\sqrt{2}} x^0 \tag{13}$$

so we have

$$x^- = \frac{1-\beta}{1+\beta} x^+ \tag{14}$$

Thus if the particle is moving at the speed of light ($\beta = 1$) in the $+x^1$ direction, $x^- = 0$, while if it is moving in the $-x^1$ direction, $x^- = \infty$.

The transformation 1 is not a Lorentz transformation. The transformation matrix is

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

where

$$\begin{bmatrix} x^+ \\ x^- \\ x^2 \\ x^3 \end{bmatrix} = C \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{16}$$

If C *were* a Lorentz transformation, it would have to satisfy

$$C^T \eta C = \eta \tag{17}$$

where η is the *original* Minkowski metric. However, a direct calculation shows that

$$C^T \eta C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$= \hat{\eta} \tag{21}$$

Thus the transformation 17 actually transforms the metric from the ordinary metric η into the lightcone metric $\hat{\eta}$.

In light cone coordinates we can represent some typical Lorentz transformations.

(a) A boost in the x^1 direction. In the usual Lorentz transformation, we have

$$\begin{aligned} x'^0 &= \gamma\left(x^0 - \beta x^1\right) \\ x'^1 &= \gamma\left(-\beta x^0 + x^1\right) \end{aligned} \tag{22}$$

so we have

$$x'^{+} = \frac{1}{\sqrt{2}}\left(x'^{0} + x'^{1}\right) \tag{23}$$
$$= \frac{\gamma}{\sqrt{2}}\left(x^0 - \beta x^1 - \beta x^0 + x^1\right) \tag{24}$$
$$= \frac{\gamma(1-\beta)}{\sqrt{2}}\left(x^0 + x^1\right) \tag{25}$$
$$= \gamma(1-\beta)\,x^{+} \tag{26}$$

Similarly

$$x'^{-} = \frac{1}{\sqrt{2}}\left(x'^{0} - x'^{1}\right) \tag{27}$$
$$= \frac{\gamma}{\sqrt{2}}\left(x^0 - \beta x^1 + \beta x^0 - x^1\right) \tag{28}$$
$$= \frac{\gamma(1+\beta)}{\sqrt{2}}\left(x^0 - x^1\right) \tag{29}$$
$$= \gamma(1+\beta)\,x^{-} \tag{30}$$

(b) A rotation by θ in the x^1x^2 plane has the ordinary Lorentz transformation of (with $c \equiv \cos\theta$ and $s \equiv \sin\theta$):

$$\begin{aligned} x'^{1} &= cx^1 - sx^2 \\ x'^{2} &= sx^1 + cx^2 \end{aligned} \tag{31}$$

In lightcone coordinates, 3 coordinates will change, since all but x^3 involve x^1 or x^2. We have

$$x'^{+} = \frac{1}{\sqrt{2}}\left(x'^{0} + x'^{1}\right) \tag{32}$$
$$= \frac{1}{\sqrt{2}}\left(x^{0} + cx^{1} - sx^{2}\right) \tag{33}$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(x^{+} + x^{-}\right) + c\left(\frac{1}{\sqrt{2}}\left(x^{+} - x^{-}\right)\right) - sx^{2}\right) \tag{34}$$
$$= \frac{1}{2}\left(1+c\right)x^{+} + \frac{1}{2}\left(1-c\right)x^{-} - \frac{1}{\sqrt{2}}sx^{2} \tag{35}$$
$$x'^{-} = \frac{1}{\sqrt{2}}\left(x'^{0} - x'^{1}\right) \tag{36}$$
$$= \frac{1}{\sqrt{2}}\left(x^{0} - cx^{1} + sx^{2}\right) \tag{37}$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(x^{+} + x^{-}\right) - c\left(\frac{1}{\sqrt{2}}\left(x^{+} - x^{-}\right)\right) + sx^{2}\right) \tag{38}$$
$$= \frac{1}{2}\left(1-c\right)x^{+} + \frac{1}{2}\left(1+c\right)x^{-} + \frac{1}{\sqrt{2}}sx^{2} \tag{39}$$
$$x'^{2} = sx^{1} + cx^{2} \tag{40}$$
$$= \frac{s}{\sqrt{2}}\left(x^{+} - x^{-}\right) + cx^{2} \tag{41}$$

(c) A boost in the x^3 direction has ordinary Lorentz transformations of

$$\begin{aligned} x'^{0} &= \gamma\left(x^{0} - \beta x^{3}\right) \\ x'^{3} &= \gamma\left(-\beta x^{0} + x^{3}\right) \end{aligned} \tag{42}$$

Therefore, we have

$$x'^{+} = \frac{1}{\sqrt{2}}\left(x'^{0} + x'^{1}\right) \tag{43}$$

$$= \frac{1}{\sqrt{2}}\left(\gamma\left(x^0 - \beta x^3\right) + x^1\right) \tag{44}$$

$$= \frac{\gamma}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(x^+ + x^-\right) - \frac{\gamma\beta}{\sqrt{2}}x^3 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(x^+ - x^-\right) \tag{45}$$

$$= \frac{\gamma+1}{2}x^+ + \frac{\gamma-1}{2}x^- - \frac{\gamma\beta}{\sqrt{2}}x^3 \tag{46}$$

$$x'^{-} = \frac{1}{\sqrt{2}}\left(x'^{0} - x'^{1}\right) \tag{47}$$

$$= \frac{1}{\sqrt{2}}\left(\gamma\left(x^0 - \beta x^3\right) - x^1\right) \tag{48}$$

$$= \frac{\gamma}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(x^+ + x^-\right) - \frac{\gamma\beta}{\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(x^+ - x^-\right) \tag{49}$$

$$= \frac{\gamma-1}{2}x^+ + \frac{\gamma+1}{2}x^- - \frac{\gamma\beta}{\sqrt{2}}x^3 \tag{50}$$

$$x'^{3} = \gamma\left(-\beta x^0 + x^3\right) \tag{51}$$

$$= -\beta\gamma\frac{1}{\sqrt{2}}\left(x^+ + x^-\right) + \gamma x^3 \tag{52}$$

$$= -\frac{\beta\gamma}{\sqrt{2}}x^+ - \frac{\beta\gamma}{\sqrt{2}}x^- + \gamma x^3 \tag{53}$$

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