Generalized Vector and Dyadic Analysis

APPLIED MATHEMATICS IN FIELD THEORY

SECOND EDITION

Chen-To Tai

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Generalized Vector and Dyadic Analysis

Applied Mathematics in Field Theory

Second Edition



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Contents

Preface to the Second Edition xi

Preface to the First Edition xiii

Acknowledgments for the First Edition xv

1 Vector and Dyadic Algebra 1

1-1 Representations of Vector Functions 1
1-2 Products and Identities 4
1-3 Orthogonal Transformation of Vector Functions 8
1-4 Transform of Vector Products 14
1-5 Definition of Dyadics and Tensors 16
1-6 Classification of Dyadics 17
1-7 Products Between Vectors and Dyadics 19

2 Coordinate Systems 23

2-1 General Curvilinear System (GCS) 23
2-2 Orthogonal Curvilinear System (OCS) 28

2-3 Derivatives of Unit Vectors in OCS 33
2-4 Dupin Coordinate System 35
2-5 Radii of Curvature 37

3 Line Integrals, Surface Integrals, and Volume Integrals 43

3-1 Differential Length, Area, and Volume 43
3-2 Classification of Line Integrals 44
3-3 Classification of Surface Integrals 48
3-4 Classification of Volume Integrals 56

4 Vector Analysis in Space 58

4-1 Symbolic Vector And Symbolic Vector Expressions 58
4-2 Differential Formulas of the Symbolic Expression in the Orthogonal Curvilinear Coordinate System for Gradient, Divergence, and Curl 61
4-3 Invariance of the Differential Operators 65
4-4 Differential Formulas of the Symbolic Expression in the General Curvilinear System 69
4-5 Alternative Definitions of Gradient and Curl 75
4-6 The Method of Gradient 78
4-7 Symbolic Expressions with Two Functions and the Partial Symbolic Vectors 81
4-8 Symbolic Expressions with Double Symbolic Vectors 86
4-9 Generalized Gauss Theorem in Space 91
4-10 Scalar and Vector Green's Theorems 93
4-11 Solenoidal Vector, Irrotational Vector, and Potential Functions 95

5 Vector Analysis on Surface 99

5-1 Surface Symbolic Vector and Symbolic Expression for a Surface 99
5-2 Surface Gradient, Surface Divergence, and Surface Curl 101
5-2-1 Surface Gradient 101

5-2-2 Surface Divergence 102
5-2-3 Surface Curl 103
5-3 Relationship Between the Volume and Surface Symbolic Expressions 104
5-4 Relationship Between Weatherburn's Surface Functions and the Functions Defined in the Method of Symbolic Vector 104
5-5 Generalized Gauss Theorem for a Surface 106
5-6 Surface Symbolic Expressions with a Single Symbolic Vector and Two Functions 111
5-7 Surface Symbolic Expressions with Two Surface Symbolic Vectors and a Single Function 113

6 Vector Analysis of Transport Theorems 116

6-1 Helmholtz Transport Theorem 116
6-2 Maxwell Theorem and Reynolds Transport Theorem 119

7 Dyadic Analysis 121

7-1 Divergence and Curl of Dyadic Functions and Gradient of Vector Functions 121
7-2 Dyadic Integral Theorems 124

8 A Historical Study of Vector Analysis 127

8-1 Introduction 127
8-2 Notations and Operators 129
8-2-1 Past and Present Notations in Vector Analysis 129
8-2-2 Quaternion Analysis 131
8-2-3 Operators 132
8-3 The Pioneer Works of J. Willard Gibbs (1839–1903) 135
8-3-1 Two Pamphlets Printed in 1881 and 1884 135
8-3-2 Divergence and Curl Operators and Their New Notations 138
8-4 Book by Edwin Bidwell Wilson Founded Upon the Lectures of J. Willard Gibbs 141

8-4-1 Gibbs's Lecture Notes 141
8-4-2 Wilson's Book 141
8-4-3 The Spread of the Formal Scalar Product (FSP) and Formal Vector Product (FVP) 146
8-5 ∇ in the Hands of Oliver Heaviside (1850–1925) 149
8-6 Shilov's Formulation of Vector Analysis 151
8-7 Formulations in Orthogonal Curvilinear Systems 152
8-7-1 Two Examples from the Book by Moon and Spencer 152
8-7-2 A Search for the Divergence Operator in Orthogonal Curvilinear Coordinate Systems 154
8-8 The Use of ∇ to Derive Vector Identities 155
8-9 A Recasting of the Past Failures by the Method of Symbolic Vector 157
8-9-1 In Retrospect 159

Appendix A Transformation Between Unit Vectors 161

Appendix B Vector and Dyadic Identities 165

Appendix C Integral Theorems 169

Appendix D Relationships Between Integral Theorems 170

Appendix E Vector Analysis in the Special Theory of Relativity 174

Appendix F Comparison of the Nomenclatures and Notations of the Quantities Used in This Book and in the Book by Stratton 181

References 185

Index 189

Preface to the Second Edition

After the publication of the first edition of this book (IEEE Press, 1992), several professional friends commented that I should have used the new notations for the divergence and the curl of a vector function, namely $\nabla\mathbf{F}$ and $\nabla\mathbf{F}$, instead of preserving Gibbs's notations $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$, commonly used in many books. In this edition, I have added a chapter on the history of vector analysis to point out more emphatically the contradiction and the confusion resulting from the misinterpretation of Gibbs's notations. It seems beyond doubt that the adoption of the new notations is preferable from the logical point of view.

In 1994, I had the opportunity to teach a course at the University of Michigan in which I used the method of symbolic vector and the new notations to teach vector analysis. The reaction from the students was very favorable. Similar views have been communicated to me by colleagues from other institutions. My motivation for revising the book is principally due to these encouragements.

In addition to the overhaul of the notations, the present edition considerably expands the coverage. The method of symbolic vector is now formulated not only in the curvilinear orthogonal system, but also in the general nonorthogonal curvilinear system. The reciprocal base systems, originally introduced by Gibbs, have been used very effectively in the formulation. New vector theorems and vector and dyadic identities have been derived to make the list as complete as possible. The relationship between dyadic analysis and tensor analysis has also been explained. The transformation of electromagnetic field vectors based on the

special theory of relativity is explained by both the conventional method, using differential calculus, and the more sophisticated method due to Sommerfeld, with the aid of four-dimensional vector analysis.

I am most grateful to Professor Phillip Alexander of the University of Windsor and to my colleague, Mr. Richard Carnes of the University of Michigan, for technical assistance and manuscript editing. For all their help and encouragement, I want to thank Dr. John H. Bryant, Professor Fawwaz T. Ulaby, Professor John H. Kraus, Mr. Jui-Ching Cheng, Mrs. Carol Truszkowski, Ms. Patricia Wolfe, Dr. Roger DeRoo, and Dr. Jian Gong.

I owe my gratitude to Prof. Donald G. Dudley, Editor of the IEEE Press/OUP Series on Electromagnetic Wave Theory, and to Mr. Dudley Kay, IEEE Press, and his staff, particularly, Ms. Denise Phillip, for valuable suggestions in the production of this book by the IEEE Press.

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The IEEE Press and the Editor of the IEEE Press/OUP Series on Electromagnetic Wave Theory, Donald Dudley, would like to thank Associate Editor, Professor Ehud Heyman, for coordinating the reviews for this book. We would also like to thank the anonymous reviewers for their helpful and incisive reviews.

Preface to the First Edition

Mathematics is a language.
The whole is simpler than its parts.
Anyone having these desires will make these researches.
—J. Willard Gibbs

This monograph is mainly based on the author's recent work on vector analysis and dyadic analysis. The book is divided into two main topics: Chapters 1–6 cover vector analysis, while Chapter 7 is exclusively devoted to dyadic analysis. On the subject of vector analysis, a new symbolic method with the aid of a symbolic vector is the main feature of the presentation. By means of this method, the principal topics in vector analysis can be developed in a systematic way. All vector identities can be derived by an algebraic manipulation of expressions with two partial symbolic vectors without actually performing any differentiation. Integral theorems are formulated under one roof with the aid of a generalized Gauss theorem. Vector analysis on a surface is treated in a similar manner. Some basic differential functions on a surface are defined; they are different from the surface functions previously defined by Weatherburn, although the two sets are intimately related. Their relations are discussed in great detail. The advantage of adopting the surface functions advocated in this work is the simplicity of formulating the surface integral theorems based on these newly defined functions.

The scope of topics covered in this book on vector analysis is comparable to those found in the books by Wilson [21], Gans [4], and Phillips [11]. However, the topics on curvilinear orthogonal systems have been treated in great detail. One important feature of this work is the unified treatment of many theorems and formulas of similar nature, which includes the invariance principle of the differential operators for the gradient, the divergence, and the curl, and the relations between various integral theorems and transport theorems. Some quite useful

topics are found in this book, which include the derivation of several identities involving the derivatives of unit vectors, and the relations between the unit vectors of various coordinate systems based on a method of gradient.

Tensor analysis is outside the scope of this book. There are many excellent books treating this subject. Since dyadic analysis is now used quite frequently in engineering sciences, a chapter on this subject, which is closely related to tensor analysis in a three-dimensional Euclidean space, may be timely.

As a whole, it is hoped that this book may be useful to instructors and students in engineering and physical sciences who wish to teach and to learn vector analysis in a systematic manner based on a new method with a clear picture of the constituent structure of this mature science not critically studied in the past few decades.

Acknowledgments for the First Edition

Without the encouragement which I received from my wife and family, and the loving innocent interference from my grandchildren, this work would never have been completed. I would like to express my gratitude to President Dr. Qian Wei-Chang for his kindness in inviting me as a Visiting Professor at The Shanghai University of Technology in the Fall of 1988 when this work was started. Most of the writing was done when I was a Visiting Professor at the Chung Cheng Institute of Technology, Taiwan, in the Spring of 1990. I am indebted to President Dr. Chen Chwan-Haw, Prof. Bor Sheau-Shong, and Prof. Kuei Ching-Ping for the invitation.

The assistance of Prof. Nenghang Fang of The Nanjing Institute of Electronic Technology, China, currently a Visiting Scholar at The University of Michigan, has been most valuable. His discussion with me about the Russian work on vector analysis was instrumental in stimulating my interest to formulate the symbolic vector method introduced in this book. Without his participation in the early stage of this work, the endeavor could not have begun. He has kindly checked all the formulas and made numerous suggestions. I am grateful to many colleagues for useful information and valuable comments. They include: Prof. J. Van Bladel of The University of Gent, Prof. Jed Z. Buchwald of The University of Toronto, Prof. W. Jack Cunningham of Yale University, Prof. Walter R. Debler and Prof. James F. Driscoll of The University of Michigan, Prof. John D. Kraus and Prof. H. C. Ko of The Ohio State University, and Prof. C. Truesdell of The Johns Hopkins University. My dear old friend Prof. David K. Cheng of Syracuse University kindly edited the manuscript and suggested the title of the book. The teachings of Prof.

Chih-Kung Jen of The Johns Hopkins Applied Physics Laboratory, formerly of Tsing Hua University, and Prof. Ronold W. P. King of Harvard University remain the guiding lights in my search for knowledge. Without the help of Ms. Bonnie Kidd, Dr. Jian-Ming Jin, and Dr. Leland Pierce, the preparation of this manuscript would not have been so professional and successful.

I wish to thank Prof. Fawwaz T. Ulaby, Director of the Radiation Laboratory, for providing me with technical assistance. The speedy production of this book is due to the efficient management of Mr. Dudley Kay, Executive Editor, and the valuable technical supervision of Ms. Anne Reifsnyder, Associate Editor, of the IEEE Press. Some major changes have been made in the original manuscript as a result of many valuable suggestions from the reviewers. I am most grateful to these reviewers.

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Chapter 1

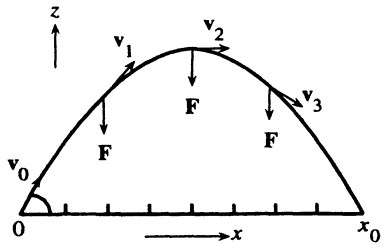
Vector and Dyadic Algebra

1-1 Representations of Vector Functions

A vector function has both magnitude and direction. The vector functions that we encounter in many physical problems are, in general, functions of space and time. In the first five chapters, we discuss only their characteristics as functions of spatial variables. Functions of space and time are covered in Chapter 6, dealing with a moving surface or a moving contour.

A vector function is denoted by **F**. Geometrically, it is represented by a line with an arrow in a three-dimensional space. The length of the line corresponds to its magnitude, and the direction of the line represents the direction of the vector function. The convenience of using vectors to represent physical quantities is illustrated by a simple example shown in Fig. 1-1, which describes the motion of a mass particle in a frictionless air (vacuum) against a constant gravitational force. The particle is thrown into the space with an initial velocity $\mathbf{v}_0$, making an angle θ_0 with respect to the horizon. During its flight, the velocity function of the particle changes both its magnitude and direction, as shown by $\mathbf{v}_1$, $\mathbf{v}_2$, and so on, at subsequent locations. The gravitational force that acts on the particle is assumed to be constant, and it is represented by **F** in the figure. A constant vector function means that both the magnitude and the direction of the function are constant, being independent of the spatial variables, x and z in this case.

The rule of the addition of two vectors **a** and **b** is shown geometrically by Fig. 1-2a, b, or c. Algebraically, it is written in the same form as the addition of



$$z = x\left(1 - \frac{x}{x_0}\right)\tan\theta_0$$

$$x_0 = \frac{1}{g}v_0^2 \sin 2\theta_0$$

g = gravitational constant

Figure 1-1 Trajectory of a mass particle in a gravitational field showing the velocity v and the constant force vector F at different locations.

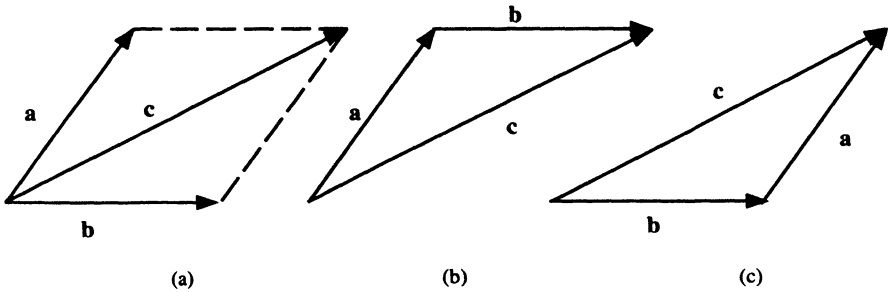


Figure 1-2 Addition of vectors, $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

two numbers of two scalar functions, that is,

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{1.1}$$

The subtraction of vector **b** from vector **a** is written in the form

$$\mathbf{d} = \mathbf{a} - \mathbf{b}. \tag{1.2}$$

Now, $-\mathbf{b}$ is a vector that has the same magnitude as **b**, but of opposite direction; then (1.2) can be considered as the addition of **a** and $(-\mathbf{b})$. Geometrically, the meaning of (1.2) is shown in Fig. 1-3. The sum and the difference of two vectors obey the associative rule, that is,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{1.3}$$

and

$$\mathbf{a} - \mathbf{b} = -\mathbf{b} + \mathbf{a}. \tag{1.4}$$

They can be generalized to any number of vectors.

The rule of the addition of vectors suggests that any vector can be considered as being made of basic components associated with a proper coordinate system. The

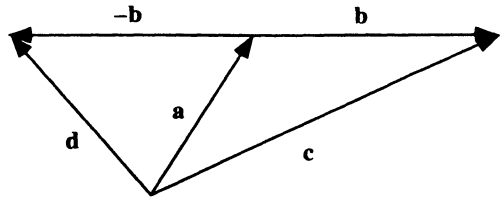


Figure 1-3 Subtraction of vectors, $\mathbf{a} - \mathbf{b} = \mathbf{d}$.

most convenient system to use is the Cartesian system or the rectangular coordinate system, or more specifically, a right-handed rectangular system in which, when x is turning to y, a right-handed screw advances to the z direction. The spatial variables in this system are commonly denoted by x, y, z. A vector that has a magnitude equal to unity and pointed in the positive x direction is called a *unit vector* in the x direction and is denoted by $\hat{x}$. Similarly, we have $\hat{y}, \hat{z}$. In such a system, a vector function $\mathbf{F}$ that, in general, is a function of position, can be written in the form

$$\mathbf{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}. \tag{1.5}$$

The three scalar functions F_x, F_y, F_z are called the *components* of $\mathbf{F}$ in the direction of $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively, while $F_x\hat{x}$, $F_y\hat{y}$, and $F_z\hat{z}$ are called the *vector components* of $\mathbf{F}$. The geometrical representation of $\mathbf{F}$ is shown in Fig. 1-4. It is seen that F_x, F_y, and F_z can be either positive or negative. In Fig. 1-4, F_x and F_z are positive, but F_y is negative.

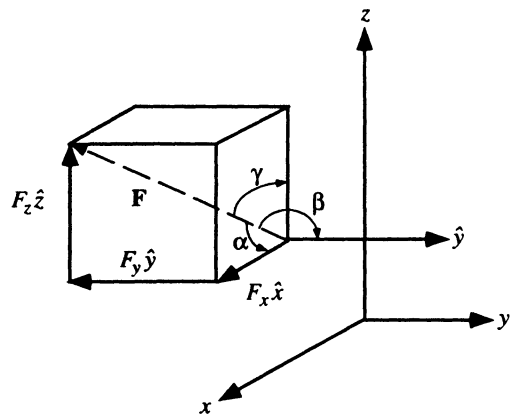


Figure 1-4 Components of a vector in a Cartesian system.

In addition to the representation by (1.5), it is sometimes desirable to express $\mathbf{F}$ in terms of its magnitude, denoted by $|\mathbf{F}|$, and its directional cosines, that is,

$$\mathbf{F} = |\mathbf{F}|\left(\cos\alpha\hat{x} + \cos\beta\hat{y} + \cos\gamma\hat{z}\right). \tag{1.6}$$

α, β, and γ are the angles $\mathbf{F}$ makes, respectively, with $\hat{x}$, $\hat{y}$, and $\hat{z}$, as shown in Fig. 1-4. It is obvious from the geometry of that figure that

$$|\mathbf{F}| = \left(F_x^2 + F_y^2 + F_z^2\right)^{1/2} \tag{1.7}$$

and

$$\cos\alpha = \frac{F_x}{|\mathbf{F}|}, \qquad \cos\beta = \frac{F_y}{|\mathbf{F}|}, \qquad \cos\gamma = \frac{F_z}{|\mathbf{F}|}. \tag{1.8}$$

Furthermore, we have the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{1.9}$$

In view of (1.9), only two of the directional cosine angles are independent. From the previous discussion, we observe that, in general, we need three parameters to specify a vector function. The three parameters could be F_x, F_y, and F_z or $|\mathbf{F}|$ and two of the directional cosine angles. Representations such as (1.5) and (1.6) can be extended to other orthogonal coordinate systems, which will be discussed in a later chapter.

1-2 Products and Identities

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b}$, and it is defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta, \tag{1.10}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, as shown in Fig. 1-5. Because of the notation used for such a product, sometimes it is called the *dot product*. By applying (1.10) to three orthogonal unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$, one finds

$$\hat{u}_i \cdot \hat{u}_j = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right\}, \qquad i, j = 1, 2, 3. \tag{1.11}$$

The value of $\mathbf{a} \cdot \mathbf{b}$ can also be expressed in terms of the components of $\mathbf{a}$ and $\mathbf{b}$ in any orthogonal system. Let the system under consideration be the rectangular system, and let $\mathbf{c} = \mathbf{a} - \mathbf{b}$; then

$$|\mathbf{c}|^2 = |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta.$$

Hence

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta &= \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2}{2} \\ &= \frac{a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - (a_x - b_x)^2 - \left(a_y - b_y\right)^2 - (a_z - b_z)^2}{2} \\ &= a_x b_x + a_y b_y + a_z b_z. \end{aligned} \tag{1.12}$$

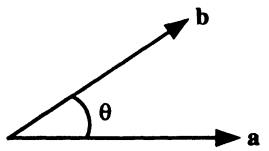


Figure 1-5 Scalar product of two vectors, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

By equating (1.10) and (1.12), one finds

$$\begin{aligned}\cos\theta &= \frac{1}{|\mathbf{a}|\,|\mathbf{b}|}\left(a_x b_x + a_y b_y + a_z b_z\right) \\ &= \cos\alpha_a \cos\alpha_b + \cos\beta_a \cos\beta_b + \cos\gamma_a \cos\gamma_b,\end{aligned} \tag{1.13}$$

a relationship well known in analytical geometry. Equation (1.12) can be used to prove the validity of the distributive law for the scalar products, namely,

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}. \tag{1.14}$$

According to (1.12), we have

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} &= (a_x + b_x)\,c_x + (a_y + b_y)\,c_y + (a_z + b_z)\,c_z \\ &= \left(a_x c_x + a_y c_y + a_z c_z\right) + \left(b_x c_x + b_y c_y + b_z c_z\right) \\ &= \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}.\end{aligned}$$

Once we have proved the distributive law for the scalar product, (1.12) can be verified by taking the sum of the scalar products of the individual terms of **a** and **b**.

The vector product of two vector functions **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is defined by

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\hat{u}_c, \tag{1.15}$$

where θ denotes the angle between **a** and **b**, measured from **a** to **b**; $\hat{u}_c$ denotes a unit vector perpendicular to both **a** and **b** and is pointed to the advancing direction of a right-hand screw when we turn from **a** to **b**. Figure 1-6 shows the relative position of $\hat{u}_c$ with respect to **a** and **b**. Because of the notation used for the vector product, it is sometimes called the *cross product*, in contrast to the dot product or the scalar product. For three orthogonal unit vectors in a right-hand system, we have $\hat{u}_1\times\hat{u}_2 = \hat{u}_3$, $\hat{u}_2\times\hat{u}_3 = \hat{u}_1$, and $\hat{u}_3\times\hat{u}_1 = \hat{u}_2$. It is obvious that $\hat{u}_i\times\hat{u}_i = 0$, $i = 1, 2, 3$. From the definition of the vector product in (1.15), one finds

$$\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}. \tag{1.16}$$

The value of $\mathbf{a}\times\mathbf{b}$ as described by (1.15) can also be expressed in terms of the components of **a** and **b** in a rectangular coordinate system. If we let $\mathbf{a}\times\mathbf{b} = \mathbf{v} =$

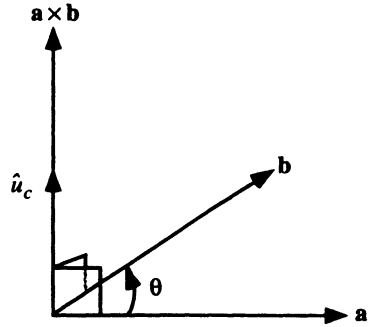


Figure 1-6 Vector product of two vectors, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{u}_c$; $\hat{u}_c \perp \mathbf{a}$, $\hat{u}_c \perp \mathbf{b}$.

$v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$, which is perpendicular to both **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{v} = a_x v_x + a_y v_y + a_z v_z = 0, \tag{1.17}$$

$$\mathbf{b} \cdot \mathbf{v} = b_x v_x + b_y v_y + b_z v_z = 0. \tag{1.18}$$

Solving for v_x/v_z and v_y/v_z, from (1.17) and (1.18) we obtain

$$\frac{v_x}{v_z} = \frac{a_y b_z - a_z b_y}{a_x b_y - a_y b_x}, \qquad \frac{v_y}{v_z} = \frac{a_z b_x - a_x b_z}{a_x b_y - a_y b_x}.$$

Thus,

$$\frac{v_x}{a_y b_z - a_z b_y} = \frac{v_y}{a_z b_x - a_x b_z} = \frac{v_z}{a_x b_y - a_y b_x}.$$

Let the common ratio of these quantities be denoted by c, which can be determined by considering the case with $\mathbf{a} = \hat{x}$, $\mathbf{b} = \hat{y}$; then $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \hat{z}$; hence from the last ratio, we find $c = 1$ because $v_z = 1$ and $a_x = b_y = 1$, while $a_y = b_x = 0$. The three components of **v**, therefore, are given by

$$\left.\begin{aligned} v_x &= a_y b_z - a_z b_y \\ v_y &= a_z b_x - a_x b_z \\ v_z &= a_x b_y - a_y b_x \end{aligned}\right\}, \tag{1.19}$$

which can be assembled in a determinant form as

$$\mathbf{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{1.20}$$

We can use (1.20) to prove the distributive law of vector products, that is,

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}. \tag{1.21}$$

To prove (1.21), we find that the x component of $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ according to (1.20) is equal to

$$(a_y + b_y)\, c_z - (a_z + b_z)\, c_y = (a_y c_z - a_z c_y) + (b_y c_z - b_z c_y). \tag{1.22}$$

The last two terms in (1.22) denote, respectively, the x component of $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$. The equality of the y and z components of (1.21) can be proved in a similar manner.

In addition to the scalar product and the vector product introduced before, there are two identities involving the triple products that are very useful in vector analysis. They are

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{1.23}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}. \tag{1.24}$$

Identities described by (1.23) can be proved by writing $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ in a determinant form:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

According to the theory of determinants,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The last two determinants represent, respectively, $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ and $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$; hence we have the validity of (1.23). To prove (1.24), we observe that the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ lies in the plane containing $\mathbf{b}$ and $\mathbf{c}$, so we can treat $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as being made of two components $\alpha\mathbf{b}$ and $\beta\mathbf{c}$, as shown in Fig. 1-7, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \alpha\mathbf{b} + \beta\mathbf{c}. \tag{1.25}$$

Because

$$\mathbf{a} \cdot [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] = 0,$$

hence

$$\alpha\,(\mathbf{a} \cdot \mathbf{b}) + \beta\,(\mathbf{a} \cdot \mathbf{c}) = 0.$$

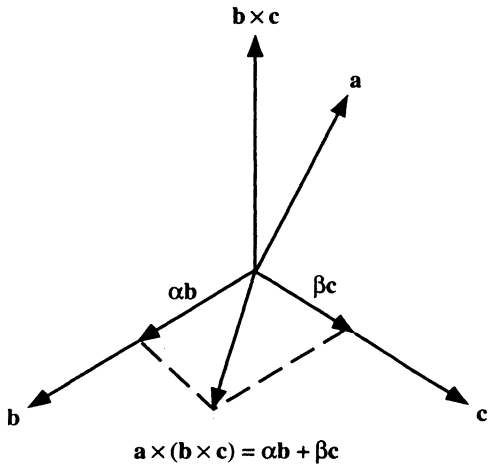


Figure 1-7 Orientation of various vectors in $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Equation (1.25), therefore, can be written in the form

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \alpha \left[\mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{c}} \mathbf{c} \right] = \alpha' \left[(\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \right], \tag{1.26}$$

where α' is a constant to be determined. By considering the case $\mathbf{a} = \hat{y}$, $\mathbf{b} = \hat{x}$, $\mathbf{c} = \hat{y}$, we have

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \hat{x}, \\ (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} &= \hat{x}, \\ (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} &= 0. \end{aligned}$$

Hence $\alpha' = 1$. All other choices of **a**, **b**, and **c** yield the same answer. The validity of (1.23) and (1.24) is independent of the choice of the coordinate system in which these vectors are represented.

1-3 Orthogonal Transformation of Vector Functions

A vector function represented by (1.5) in a specified rectangular system can likewise be represented in another rectangular system. To discuss the relation between these two representations, we must first show the geometry of the two coordinate systems. The relative orientation of the axes of these two systems can be formed by three successive rotations originally due to Leonhard Euler (1707–1783).

Let the coordinates of the original system be denoted by (x, y, z). We first rotate the (x, y) axes by an angle ϕ_1 to form the (x_1, y_1) axes, keeping $z_1 = z$ as shown in Fig. 1-8; then the coordinates of a point (x, y, z) change to (x_1, y_1, z_1) with

$$x_1 = x\cos\phi_1 + y\sin\phi_1, \tag{1.27}$$

$$y_1 = -x\sin\phi_1 + y\cos\phi_1, \tag{1.28}$$

$$z_1 = z. \tag{1.29}$$

Now we turn the (y_1, z_1) axes by an angle ϕ_2 to form the (y_2, z_2) axes with $x_2 = x_1$; then,

$$y_2 = y_1\cos\phi_2 + z_1\sin\phi_2, \tag{1.30}$$

$$z_2 = -y_1\sin\phi_2 + z_1\cos\phi_2, \tag{1.31}$$

$$x_2 = x_1. \tag{1.32}$$

Finally, we rotate the (z_2, x_2) axes by an angle ϕ_3 to form the (z_3, x_3) axes with $y_3 = y_2$; then,

$$z_3 = z_2\cos\phi_3 + x_2\sin\phi_3, \tag{1.33}$$

$$x_3 = -z_2\sin\phi_3 + x_2\cos\phi_3, \tag{1.34}$$

$$y_3 = y_2. \tag{1.35}$$

By expressing (x_3, y_3, z_3) in terms of (x, y, z) and changing the letters (x, y, z) and (x_3, y_3, z_3), respectively, to the unprimed and primed indexed letters (x_1, x_2, x_3) and (x_1', x_2', x_3'), we obtain

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad i = 1, 2, 3, \tag{1.36}$$

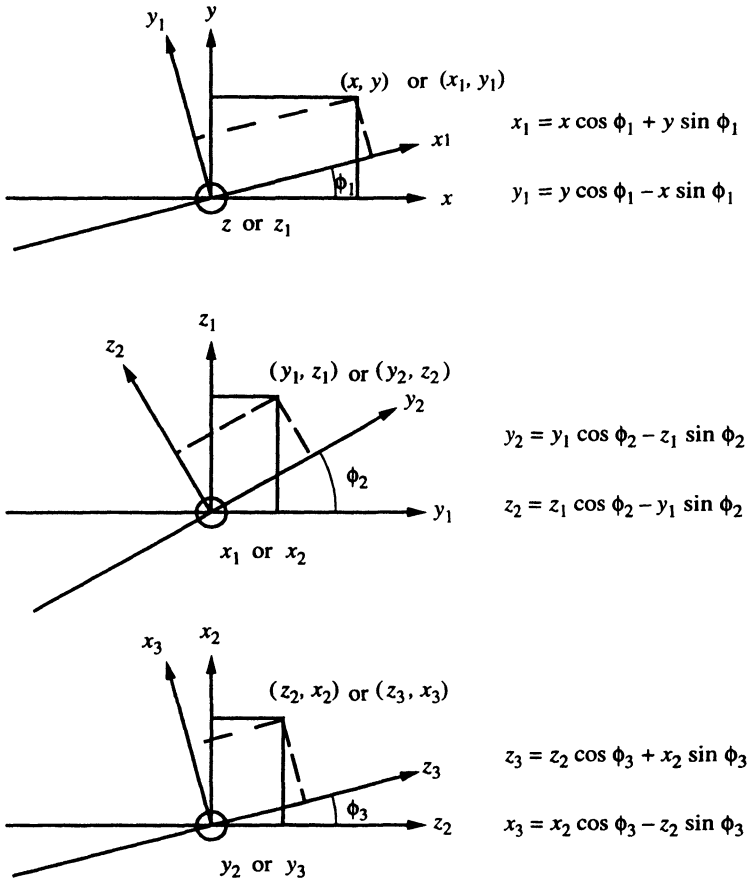


Figure 1-8 Sequences of rotations of the axes of a rectangular coordinate system.

where

$$
\begin{aligned}
a_{11} &= \cos\phi_1 \cos\phi_3 - \sin\phi_1 \sin\phi_2 \sin\phi_3, \\
a_{12} &= \sin\phi_1 \cos\phi_3 + \cos\phi_1 \sin\phi_2 \sin\phi_3, \\
a_{13} &= -\cos\phi_2 \sin\phi_3, \\
a_{21} &= -\sin\phi_1 \cos\phi_2, \\
a_{22} &= \cos\phi_1 \cos\phi_2, \\
a_{23} &= \sin\phi_2, \\
a_{31} &= \cos\phi_1 \sin\phi_3 + \sin\phi_1 \sin\phi_2 \cos\phi_3, \\
a_{32} &= \sin\phi_1 \sin\phi_3 - \cos\phi_1 \sin\phi_2 \cos\phi_3, \\
a_{33} &= \cos\phi_2 \cos\phi_3.
\end{aligned}
\tag{1.37}
$$

The coefficients a_{ij} correspond to the directional cosines between the x_i' and x_j axes, that is,

$$a_{ij} = \cos \beta_{ij}, \tag{1.38}$$

where β_{ij} denotes the angle between the two axes.

If we solve (x, y, z) in terms of (x_3, y_3, z_3) or x_j $(j = 1, 2, 3)$ in terms of x_i' $(i = 1, 2, 3)$ from (1.27) to (1.32), we obtain

$$x_j = \sum_{i=1}^{3} a_{ij} x_i', \qquad j = 1, 2, 3, \tag{1.39}$$

where a_{ij} denotes the same coefficients defined in (1.37). It should be observed that the summation indices in (1.36) and (1.39) are executed differently in these two equations. For example,

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \tag{1.40}$$

but

$$x_1 = a_{11}x_1' + a_{21}x_2' + a_{31}x_3', \tag{1.41}$$

and $a_{ij} \neq a_{ji}$ when $j \neq i$. Henceforth, whenever a summation sign is used, it is understood that the running index goes from 1 to 3 unless specified otherwise. A more efficient notation is to delete the summation sign in (1.36) and (1.39). When the summation index appears in two terms, we write (1.36) in the form

$$x_i' = a_{ij}x_j. \tag{1.42}$$

The single index (i) means $i = 1, 2, 3$ and the double index (j) represents a summation of the terms from $j = 1$ to $j = 3$. Such a notation, originally due to Einstein, can be applied to more than three variables. In this book we will use the summation sign in order to convey the meaning more vividly, particularly when several summation indices are involved in an equation. The summation index will be placed under the sign for one summation or several summations separated by a comma. The linear relations between the coordinates x_i' and x_j, as stated by (1.36) and (1.39), apply equally well to two sets of unit vectors $\hat{x}_i'$ and $\hat{x}_j$ and also to the components of a vector **A**, denoted respectively by A_i' and A_j in the two systems. This is evident from the processes by which the primed system is formed from the unprimed system. To recapitulate these relations, it is convenient to construct a 3×3 square matrix, as shown in Table 1-1. We identify i, the first subscript of a_{ij}, as the ordinal number of the rows, and j, the second subscript, as the ordinal number of the columns. The quantities involved in the transformation are listed at the side and the top.

The matrix can be used either horizontally or vertically, for example,

$$A_3' = a_{31}A_1 + a_{32}A_2 + a_{33}A_3,$$

Table 1-1: The Matrix of Transformation $[a_{ij}]$ for Quantities Defined in Two Orthogonal Rectangular Systems

$i \backslash j$	x_j or $\hat{x}_j$ or A_j		
x_i' or	a_{11}	a_{12}	a_{13}
$\hat{x}_i'$ or	a_{21}	a_{22}	a_{23}
A_i'	a_{31}	a_{32}	a_{33}

and

$$A_2 = a_{12}A_1' + a_{22}A_2' + a_{32}A_3',$$

which conform with (1.36) and (1.39) after x_i' and x_j are replaced by A_i' and A_j. For convenience, we will designate the a_{ij}'s as the directional coefficients and its matrix by $[a_{ij}]$. There are several important properties of the matrix that must be shown. In the first place, the determinant of $[a_{ij}]$, denoted by $|a_{ij}|$, is equal to unity for a right-hand system under consideration. In such a system, when one turns $\hat{x}_1$ to $\hat{x}_2$ using a right-hand screw, it advances to the $\hat{x}_3$ direction. To prove

$$|a_{ij}| = 1, \tag{1.43}$$

we consider a cubic made of $\hat{x}_i'$ with $i = 1, 2, 3$. Its volume is equal to unity, that is,

$$\hat{x}_1' \cdot (\hat{x}_2' \times \hat{x}_3') = 1. \tag{1.44}$$

The expression on the left side of (1.44) is given by

$$\sum_l a_{il}\hat{x}_l \cdot \left(\sum_m a_{jm}\hat{x}_m \times \sum_n a_{kn}\hat{x}_n \right) = |a_{ij}|, \tag{1.45}$$

where (l, m, n) and $(i, j, k) = (1, 2, 3)$ in cyclic order. The identity between (1.44) and (1.45) yields (1.43). A second identity relates the directional coefficients a_{ij} with the cofactors or the signed minors of $[a_{ij}]$. If we solve (x_1, x_2, x_3) in terms of (x_1', x_2', x_3') from (1.36) based on the theory of linear equations, we find

$$x_j = \frac{1}{|a_{ij}|} \sum_i A_{ij}x_i', \tag{1.46}$$

where A_{ij} denotes the cofactor or the signed minor of $[a_{ij}]$ obtained by eliminating the ith row and jth column. By comparing (1.46) with (1.39) and because $|a_{ij}| = 1$, we obtain

$$A_{ij} = a_{ij}. \tag{1.47}$$

An alternative derivation is to start with the relation

$$\hat{x}_j = \sum_i A_{ij}\hat{x}_i', \tag{1.48}$$

which is the same as (1.46) with $|a_{ij}| = 1$, and to replace x_j, x_i' with $\hat{x}_j$ and $\hat{x}_i$; then the scalar product of (1.48) with $\hat{x}_i'$ yields

$$a_{ij} = A_{ij}. \tag{1.49}$$

As an example, let $i = 1$, $j = 2$; then

$$a_{12} = -(a_{21}a_{33} - a_{23}a_{31}). \tag{1.50}$$

The validity of (1.49) can also be verified by using the expressions of a_{ij} defined in terms of the Eulerian angles listed in (1.37); that is,

$$\begin{aligned} -a_{21}a_{33} + a_{23}a_{31} &= \sin\phi_1 \cos\phi_2 \cos\phi_2 \cos\phi_3 \\ &\quad + \sin\phi_2(\cos\phi_1 \sin\phi_3 + \sin\phi_1 \sin\phi_2 \cos\phi_3) \\ &= \cos\phi_1 \sin\phi_2 \sin\phi_3 + \sin\phi_1 \cos\phi_3 \\ &= a_{12}. \end{aligned}$$

Equation (1.47) is a very useful identity in discussing the transformation of vector products.

Because the axes of the two coordinate systems x_i' and x_j or the unit vectors $\hat{x}_i'$ and $\hat{x}_j$ are themselves orthogonal, then

$$\hat{x}_i' \cdot \hat{x}_j' = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{1.51}$$

and similarly,

$$\hat{x}_i \cdot \hat{x}_j = \delta_{ij}, \tag{1.52}$$

where δ_{ij} denotes the Kronecker δ function defined in (1.51). In terms of the unprimed unit vectors, (1.51) becomes

$$\sum_m a_{im}\hat{x}_m \cdot \sum_n a_{jn}\hat{x}_n = \delta_{ij}; \tag{1.53}$$

because

$$\hat{x}_m \cdot \hat{x}_n = \delta_{mn},$$

(1.53) reduces to

$$\sum_m a_{im}a_{jm} = \delta_{ij}, \tag{1.54}$$

and similarly, by expressing (1.52) in terms of $\hat{x}_m'$ and $\hat{x}_n'$, we obtain

$$\sum_m a_{mi}a_{mj} = \delta_{ij}. \tag{1.55}$$

Either (1.54) or (1.55) contains six identities. Looking at the rotational relations between the unprimed and the primed coordinates, we observe that the three Eulerian angles or parameters generate nine coefficients. Only three of them are therefore independent, provided they are not the triads of

$$\sum_j a_{ij}^2 = 1, \qquad i = 1, 2, 3, \tag{1.56}$$

or

$$\sum_i a_{ij}^2 = 1, \qquad j = 1, 2, 3. \tag{1.57}$$

The remaining six coefficients are therefore dependent coefficients that are related by (1.54) or (1.55) or a mixture of six relations from both of them. For example, if a_{11}, a_{12}, and a_{23} have been specified, then we can determine a_{13} from the equation

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,$$

and subsequently the coefficient a_{33} from

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1.$$

The remaining four coefficients a_{21}, a_{22}, a_{31}, a_{32} can be found from the equations

$$a_{11}\underline{a}_{21} + a_{12}\underline{a}_{22} + a_{13}a_{23} = 0,$$

and

$$a_{11}\underline{a}_{31} + a_{12}\underline{a}_{32} + a_{13}a_{33} = 0.$$

We underline the unknowns by placing a bar underneath these coefficients. They must also satisfy

$$a_{23}\underline{a}_{21} + a_{33}\underline{a}_{31} + a_{13}a_{11} = 0,$$

and

$$a_{23}\underline{a}_{22} + a_{33}\underline{a}_{32} + a_{13}a_{12} = 0.$$

For convenience, we summarize here the important formulas that have been derived:

$$x_i' = \sum_j a_{ij}x_j, \qquad i = 1, 2, 3, \tag{1.58}$$

$$x_j = \sum_i a_{ij}x_i', \qquad j = 1, 2, 3, \tag{1.59}$$

$$\hat{x}_i' = \sum_j a_{ij}\hat{x}_j, \qquad i = 1, 2, 3, \tag{1.60}$$

$$\hat{x}_j = \sum_i a_{ij}\hat{x}_i', \qquad j = 1, 2, 3, \tag{1.61}$$

$$A_i' = \sum_j a_{ij}A_j, \qquad i = 1, 2, 3, \tag{1.62}$$

$$A_j = \sum_i a_{ij}A_i', \qquad j = 1, 2, 3, \tag{1.63}$$

$$|a_{ij}| = 1, \tag{1.64}$$

$$a_{ij} = A_{ij}, \tag{1.65}$$

$$\sum_m a_{im}a_{jm} = \delta_{ij}, \tag{1.66}$$

$$\sum_m a_{mi}a_{mj} = \delta_{ij}. \tag{1.67}$$

In expressions (1.58) through (1.67), all the summation signs are understood to be executed from 1 to 3. The conglomerate puts these relations into a group.

Equations (1.62) and (1.63) are two important equations or requirements for the transformation of the components of a vector in two rectangular systems rotated with respect to each other. These relations also show that a vector function has an invariant form, namely,

$$\mathbf{A} = \sum_i A_i \hat{x}_i = \sum_j A'_j \hat{x}'_j \tag{1.68}$$

and

$$\mathbf{A} \cdot \mathbf{A} = \sum_i A_i^2 = \sum_j A'^2_j. \tag{1.69}$$

Equation (1.69) shows that the magnitude of a vector is an invariant scalar quantity, independent of the defining coordinate system. The speed of a car running 50 miles per hour is independent of its direction. However, its direction does depend on the reference system that is being used, namely, either $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ or $(\hat{x}'_1, \hat{x}'_2, \hat{x}'_3)$. The vector functions that transform according to (1.62) and (1.63) are called *polar vectors*, to be distinguished from another class of vectors that will be covered in the next section.

1-4 Transform of Vector Products

A vector product formed by two polar vectors **A** and **B** in the unprimed system and their corresponding expressions **A**′ and **B**′ in the primed system is

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{1.70}$$

or

$$\mathbf{C}' = \mathbf{A}' \times \mathbf{B}'. \tag{1.71}$$

According to the definition of a vector product, (1.19), its expression in a right-handed rectangular system is

$$C'_k = A'_i B'_j - A'_j B'_i \tag{1.72}$$

with $i, j, k = 1, 2, 3$ in cyclic order. Now,

$$A'_i = \sum_m a_{im} A_m, \tag{1.73}$$

$$B'_j = \sum_n a_{jn} B_n. \tag{1.74}$$

Hence

$$\begin{aligned} C'_k = A'_i B'_j - A'_j B'_i &= \sum_m \sum_n (a_{im} a_{jn} - a_{jm} a_{in}) A_m B_n \\ &= \sum_m \sum_n a_{im} a_{jn} (A_m B_n - A_n B_m). \end{aligned} \tag{1.75}$$

It appears that the components $A_m B_n - A_n B_m$ in (1.75) do not transform like the components of a polar vector as in (1.62). However, if we inspect the terms in (1.75), for example, with $k = 1$, $i = 2$, $j = 3$, then

$$\begin{aligned} C_1' = {} & (a_{22}a_{33} - a_{23}a_{32})(A_2B_3 - A_3B_2) \\ & + (a_{23}a_{31} - a_{21}a_{33})(A_3B_1 - A_1B_3) \\ & + (a_{21}a_{32} - a_{22}a_{31})(A_1B_2 - A_2B_1). \end{aligned} \tag{1.76}$$

The terms involving the directional coefficients are recognized as the cofactors A_{11}, A_{12}, and A_{13} of $[a_{ij}]$ and, according to (1.65), they are equal to a_{11}, a_{12}, and a_{13}; hence

$$\begin{aligned} C_1' &= A_{11}C_1 + A_{12}C_2 + A_{13}C_3 \\ &= a_{11}C_1 + a_{12}C_2 + a_{13}C_3 \\ &= \sum_j a_{ij}C_j. \end{aligned} \tag{1.77}$$

Equation (1.77) obeys the same rule as the transformation of two polar vectors. Thus, in a three-dimensional Euclidean space, the vector product does transform like a polar vector even though its origin stems from the vector product of two polar vectors. From the physical point of view, the vector product is used to describe a quantity associated with rotation, such as the angular velocity of a rotating body, the moment of force, the vorticity in hydrodynamics, and the magnetic field in electrodynamics. For this reason such a vector was called a *skew vector* by J. Willard Gibbs (1837–1903), one of the founders of vector analysis. Nowadays, it is commonly called an *axial vector*. From now on, the word *vector* will be used to comprise both the polar and the axial vectors in a three-dimensional or Euclidean space. In a four-dimensional manifold as in the theory of relativity, the situation is different. In that case, we have to distinguish the polar vector, or the four-vector, from the axial vector, or the six-vector. This topic will be briefly discussed after the subjects of dyadic and tensor analysis are introduced. Even though the transformation rule of a polar vector applies to an axial vector, we must remember that we have defined an axial vector according to a right-hand rule. In a left-hand coordinate system, obtained by an inversion of the axes of a right-hand system, the components of a polar vector change their signs; then we must use a left-hand rotating rule to define a vector product to preserve the same rule of transformation between a polar vector and an axial vector. We would like to mention that in a left-hand system the determinant of the corresponding directional coefficients is equal to -1.

Before we close this section, we want to point out that as a result of the identical rule of transformation of the polar vectors and the axial vectors, the characteristics of the two triple products $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can be ascertained. The scalar triple product is, indeed, an invariant scalar because

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A}' \cdot (\mathbf{B}' \times \mathbf{C}'). \tag{1.78}$$

For the vector triple product, it behaves like a vector because by decomposing it into two terms using the vector identity (1.24),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}, \tag{1.79}$$

we see that $\mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A} \cdot \mathbf{B}$ are invariant scalars, and $\mathbf{B}$ and $\mathbf{C}$ are vectors; the sum at the right side of (1.79) is therefore again a vector. This synthesis may appear to be trivial but it does offer a better understanding of the nature of these quantities. The reader should practice constructing these identities when the vectors are defined in a left-hand coordinate system.

1-5 Definition of Dyadics and Tensors

A vector function $\mathbf{F}$ in a three-dimensional space defined in a rectangular system is represented by

$$\mathbf{F} = \sum_i F_i \hat{x}_i. \tag{1.80}$$

If we consider three independent vector functions denoted by

$$\mathbf{F}_j = \sum_i F_{ij} \hat{x}_i, \qquad j = 1, 2, 3, \tag{1.81}$$

then a dyadic function can be formed that will be denoted by $\bar{\bar{F}}$ and defined by

$$\bar{\bar{F}} = \sum_j \mathbf{F}_j \hat{x}_j. \tag{1.82}$$

The unit vector $\hat{x}_j$ is juxtaposed at the posterior position of $\mathbf{F}_j$. By substituting the expression of $\mathbf{F}_j$ into (1.82), we obtain

$$\bar{\bar{F}} = \sum_i \sum_j F_{ij} \hat{x}_i \hat{x}_j = \sum_{i,j} F_{ij} \hat{x}_i \hat{x}_j. \tag{1.83}$$

Equation (1.83) is the explicit expression of a dyadic function defined in a rectangular coordinate system. Sometimes the name *Cartesian dyadic* is used. A dyadic function, or simply a dyadic, therefore, consists of nine dyadic components; each component is made of a scalar component F_{ij} and a dyad in the form of a pair of unit vectors $\hat{x}_i \hat{x}_j$ placed in that order.

Because a dyadic is formed by three vector functions and three unit vectors, the transform of a dyadic from its representation in one rectangular system (the unprimed) to another rectangular system (the primed) can most conveniently be executed by applying (1.61) to (1.83); thus,

$$\begin{aligned} \bar{\bar{F}} &= \sum_{i,j} F_{ij} \sum_m a_{mi} \hat{x}'_m \sum_n a_{nj} \hat{x}'_n \\ &= \sum_{i,j,m,n} a_{mi} a_{nj} F_{ij} \hat{x}'_m \hat{x}'_n. \end{aligned} \tag{1.84}$$

If we denote

$$F'_{mn} = \sum_{i,j} a_{mi} a_{nj} F_{ij}, \tag{1.85}$$

then

$$\begin{aligned} \bar{\bar{F}} &= \sum_{m,n} F'_{mn} \hat{x}'_m \hat{x}'_n \\ &= \sum_{i,j} F_{ij} \hat{x}_i \hat{x}_j. \end{aligned} \tag{1.86}$$

Equation (1.85) describes the rule of transform of the scalar components F_{ij} of a dyadic in two rectangular systems. By starting with the expression of $\bar{\bar{F}}'$ in the primed system, we find

$$F_{ij} = \sum_{m,n} a_{mi} a_{nj} F'_{mn}. \tag{1.87}$$

When the 3×3 scalar components F_{ij} are arranged in matrix form, denoted by $[F_{ij}]$, it is designated as a tensor or, more precisely, as a tensor of rank 2 or a tensor of valance 2. The exact form of $[F_{ij}]$ is

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}. \tag{1.88}$$

In tensor analysis, a vector is treated as a tensor of rank 1 and a scalar as a tensor of rank 0. In this book, tensor analysis is not one of our main topics. The subject has been covered by many excellent books such as Brand [1] and Borisenko and Tarapov [2]. However, many applications of tensor analysis can be treated equally well by dyadic analysis. In the previous section, we have already correlated a tensor of rank 2 in a Euclidean space with a dyadic. Tensor analysis is most useful in the theory of relativity, but one can formulate problems in the special theory of relativity using conventional vector analysis, as illustrated in Appendix E of this book.

1-6 Classification of Dyadics

When the scalar components of a dyadic are symmetrical, such that

$$D_{ij} = D_{ji}, \tag{1.89}$$

it is called a *symmetric dyadic* and the corresponding tensor, a *symmetric tensor.* When the components are antisymmetric, such that

$$D_{ij} = -D_{ji}, \tag{1.90}$$

such a dyadic is called an *antisymmetric dyadic* and the corresponding tensor, the *antisymmetric tensor.* For an antisymmetric dyadic, (1.90) implies $D_{ii} = 0$. An

antisymmetric tensor of this dyadic, therefore, has effectively only three distinct components, namely, D_{12}, D_{13}, and D_{23}. The other three components are $-D_{12}$, $-D_{13}$, and $-D_{23}$, which are not considered to be distinctly different.

Let us now introduce three terms with a single index, such that

$$D_1 = D_{23}, \qquad D_2 = D_{31}, \qquad D_3 = D_{12},$$

or

$$D_i = D_{jk}$$

with $i, j, k = 1, 2, 3$ in cyclic order; then the tensor of this antisymmetric dyadic has the form

$$[D_{ij}] = \begin{bmatrix} 0 & D_3 & -D_2 \\ -D_3 & 0 & D_1 \\ D_2 & -D_1 & 0 \end{bmatrix}. \tag{1.91}$$

The transform of these components to a primed system, according to (1.85), has the form

$$D'_{ij} = \sum_{m,n} a_{im} a_{jn} D_{mn}, \tag{1.92}$$

where we have interchanged the roles of the indices in (1.85). In terms of the single indexed components for D'_{ij},

$$D'_k = \sum_{m,n} a_{im} a_{jn} D_{mn}, \tag{1.93}$$

where (i, j, k) and $(l, m, n) = (1, 2, 3)$ in cyclic order. For example, the explicit expression of D'_1 is

$$\begin{aligned} D'_1 &= (a_{22}a_{33} - a_{23}a_{32}) D_{23} \\ &\quad + (a_{23}a_{31} - a_{21}a_{33}) D_{31} \\ &\quad + (a_{21}a_{32} - a_{22}a_{31}) D_{12}. \end{aligned} \tag{1.94}$$

The coefficients attached to D_{mn} are recognized as three cofactors of $[a_{ij}]$; they are, respectively, A_{11}, A_{12}, and A_{13}, which are equal to a_{11}, a_{12}, and a_{13}. By changing D_{23}, D_{31}, and D_{12} to D_1, D_2, and D_3, we obtain

$$\begin{aligned} D'_1 &= A_{11} D_1 + A_{12} D_2 + A_{13} D_3 \\ &= a_{11} D_1 + a_{12} D_2 + a_{13} D_3 \\ &= \sum_j a_{ij} D_j. \end{aligned} \tag{1.95}$$

Equation (1.95) describes, precisely, the transform of a polar vector in the two-coordinate system. By tracing back the derivations, we see that an antisymmetric

tensor is essentially an axial vector (defined in a right-hand system) and its components transform like a polar vector. The connection between an antisymmetric tensor, an axial vector, and a polar vector is well illustrated in this exercise. We now continue on to define some more quantities used in dyadic algebra.

When a symmetric dyadic is made of three dyads in the form

$$\bar{\bar{I}} = \hat{x}_1\hat{x}_1 + \hat{x}_2\hat{x}_2 + \hat{x}_3\hat{x}_3 = \sum_i \hat{x}_i\hat{x}_i, \tag{1.96}$$

it is called an *idemfactor*. Its significance and applications will be revealed shortly. When the positions of $\mathbf{F}_j$ and $\hat{x}_j$ are interchanged in (1.82), we form another dyadic that is called the *transpose* of $\bar{\bar{F}}$, and it is denoted by $[\bar{\bar{F}}]^{\mathrm{T}}$, that is,

$$[\bar{\bar{F}}]^{\mathrm{T}} = \sum_j \hat{x}_j\mathbf{F}_j = \sum_{j,i} F_{ij}\hat{x}_j\hat{x}_i = \sum_{i,j} F_{ji}\hat{x}_i\hat{x}_j. \tag{1.97}$$

The corresponding tensor will be denoted by

$$[F_{ij}]^{\mathrm{T}} = [F_{ji}] = \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}. \tag{1.98}$$

We therefore transpose the columns in $[F_{ij}]$ to form the rows in $[F_{ij}]^{\mathrm{T}}$. It is obvious that the transpose of $[F_{ij}]^{\mathrm{T}}$ goes back to $[\bar{\bar{F}}]$.

1-7 Products Between Vectors and Dyadics

There are two scalar products between a vector $\mathbf{A}$ and a dyadic $\bar{\bar{D}}$. The anterior scalar product is defined by

$$\begin{aligned} \mathbf{B} = \mathbf{A} \cdot \bar{\bar{D}} &= \mathbf{A} \cdot \sum_j \mathbf{D}_j\hat{x}_j \\ &= \mathbf{A} \cdot \sum_{i,j} D_{ij}\hat{x}_i\hat{x}_j \\ &= \sum_{i,j} A_i D_{ij}\hat{x}_j, \end{aligned} \tag{1.99}$$

which is a vector; hence we use the notation $\mathbf{B}$. Following the rules of the transform of a vector and a dyadic, we find that the same vector becomes

$$\mathbf{B} = \sum_{m,n} A'_m D'_{mn}\hat{x}'_n = \mathbf{A}' \cdot \bar{\bar{D}}'. \tag{1.100}$$

Hence

$$\mathbf{B} = \mathbf{B}'. \tag{1.101}$$

That means the product, like the scalar product $\mathbf{A} \cdot \mathbf{B}$, is independent of the coordinate system in which it is defined, or it is form-invariant.

The posterior vector between $\mathbf{A}$ and $\bar{\bar{D}}$ is defined by

$$\mathbf{C} = \bar{\bar{D}} \cdot \mathbf{A} = \sum_{i,j} D_{ij} \hat{x}_i \hat{x}_j \cdot \mathbf{A} = \sum_{i,j} D_{ij} A_j \hat{x}_i. \tag{1.102}$$

The scalar components of (1.102) can be cast as the product between the square matrix or tensor $[D_{ij}]$ and a column matrix $[A_j]$, that is,

$$[C_i] = [D_{ij}][A_j]. \tag{1.103}$$

A typical term of (1.103) reads

$$C_1 = D_{11} A_1 + D_{12} A_2 + D_{13} A_3. \tag{1.104}$$

Linear relations like (1.103) occur often in solid mechanics, crystal optics, and electromagnetic theory. Equation (1.102) is a more complete representation of these relations because the unit vectors are also included in the equation. We then speak of a scalar product between a vector and a dyadic instead of a product between a tensor and a column matrix.

By transforming A_j, D_{ij}, and $\hat{x}_i$ into the primed functions, we find

$$\mathbf{C} = \mathbf{C}' = \sum_{m,n} A'_n D'_{mn} \hat{x}'_m. \tag{1.105}$$

The anterior scalar product between $\mathbf{A}$ and $[\bar{\bar{D}}]^{\mathrm{T}}$, denoted by $\mathbf{T}$ for the time being, is given by

$$\mathbf{T} = \mathbf{A} \cdot [\bar{\bar{D}}]^{\mathrm{T}} = \mathbf{A} \cdot \sum_{i,j} D_{ji} \hat{x}_i \hat{x}_j \tag{1.106}$$

$$= \sum_{i,j} A_i D_{ji} \hat{x}_j = \sum_{i,j} A_j D_{ij} \hat{x}_i, \tag{1.107}$$

which is equal to $\mathbf{C}$ given by (1.102); hence we have a very useful identity:

$$\mathbf{A} \cdot [\bar{\bar{D}}]^{\mathrm{T}} = \bar{\bar{D}} \cdot \mathbf{A}. \tag{1.108}$$

Similarly, one finds

$$[\bar{\bar{D}}]^{\mathrm{T}} \cdot \mathbf{A} = \mathbf{A} \cdot \bar{\bar{D}}. \tag{1.109}$$

For a symmetric dyadic, denoted by $\bar{\bar{D}}_{\mathrm{s}}$,

$$[\bar{\bar{D}}_{\mathrm{s}}]^{\mathrm{T}} = \bar{\bar{D}}_{\mathrm{s}}. \tag{1.110}$$

Hence

$$\mathbf{A} \cdot \bar{\bar{D}}_{\mathrm{s}} = \bar{\bar{D}}_{\mathrm{s}} \cdot \mathbf{A}. \tag{1.111}$$

When $\bar{\bar{D}}_s$ is the idemfactor defined by (1.96), we have

$$\mathbf{A} \cdot \bar{\bar{I}} = \bar{\bar{I}} \cdot \mathbf{A} = \mathbf{A}. \tag{1.112}$$

The tensor of $\bar{\bar{I}}$ can be called a *unit tensor*, with the three diagonal terms equal to unity and the rest are null.

There are two vector products between $\mathbf{A}$ and $\bar{\bar{D}}$. These products are both dyadics. The anterior vector product is defined by

$$\begin{aligned} \bar{\bar{B}} &= \mathbf{A} \times \bar{\bar{D}} \\ &= \mathbf{A} \times \sum_{i,j} D_{ij} \hat{x}_i \hat{x}_j \\ &= \sum_{i,j,k} D_{ij} A_k (\hat{x}_k \times \hat{x}_i) \hat{x}_j, \end{aligned} \tag{1.113}$$

where $i, j, k = 1, 2, 3$ in cyclic order. The posterior vector product is defined by

$$\bar{\bar{C}} = \bar{\bar{D}} \times \mathbf{A} = \sum_{i,j,k} D_{ij} A_k \hat{x}_i (\hat{x}_j \times \hat{x}_k). \tag{1.114}$$

One important triple product involving three vectors is given by (1.23):

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}). \tag{1.115}$$

In dyadic analysis, we need a similar product, with one of the vectors changed to a dyadic. We can obtain such an identity by first changing (1.115) into the form

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}. \tag{1.116}$$

Now we let

$$\mathbf{C} = \bar{\bar{C}} \cdot \mathbf{F},$$

where $\mathbf{F}$ is an arbitrary vector function and $\bar{\bar{C}}$ is a dyadic. Then

$$\mathbf{A} \cdot (\mathbf{B} \times \bar{\bar{C}}) \cdot \mathbf{F} = -\mathbf{B} \cdot (\mathbf{A} \times \bar{\bar{C}}) \cdot \mathbf{F} = (\mathbf{A} \times \mathbf{B}) \cdot \bar{\bar{C}} \cdot \mathbf{F}. \tag{1.117}$$

Because this identity is valid for any arbitrary $\mathbf{F}$, we obtain

$$\mathbf{A} \cdot (\mathbf{B} \times \bar{\bar{C}}) = -\mathbf{B} \cdot (\mathbf{A} \times \bar{\bar{C}}) = (\mathbf{A} \times \mathbf{B}) \cdot \bar{\bar{C}}. \tag{1.118}$$

An alternative method of deriving (1.118) is to consider three sets of identities like (1.116) with three distinct $\mathbf{C}_j$, with $j = 1, 2, 3$. Then, by juxtaposing a unit vector $\hat{x}_j$ at the posterior position of each of these sets and summing the resultant equations, we again obtain (1.118) with

$$\bar{\bar{C}} = \sum_j \mathbf{C}_j \hat{x}_j.$$

Other dyadic identities can be derived in a similar manner. Many of them will be given in Chapter 7, which deals with dyadic analysis.

Finally, we want to introduce another class of dyadics in the form of

$$\begin{aligned}\bar{\bar{S}} &= \mathbf{MN} \\ &= \sum_i M_i \hat{x}_i \sum_j N_j \hat{x}_j \\ &= \sum_{i,j} M_i N_j \hat{x}_i \hat{x}_j;\end{aligned} \tag{1.119}$$

then,

$$S_{ij} = M_i N_j. \tag{1.120}$$

Because there are three components of **M** and three components of **N**, all together six functions, and they have generated nine dyadic components of $\bar{\bar{L}}$, three of the relations in (1.120) must be dependent. For example, we can write

$$\begin{aligned}S_{23} = M_2 N_3 &= (M_2 N_2) N_3 / N_2 \\ &= (M_2 N_2)(M_1 N_3)/(M_1 N_2) \\ &= S_{22} S_{13} / S_{12}.\end{aligned} \tag{1.121}$$

When the vectors in (1.120) are defined in two different coordinate systems unrelated to each other, we have a mixed dyadic. Let

$$\mathbf{M} = \mathbf{M}(x_1, x_2, x_3)$$

and

$$\mathbf{N}'' = \mathbf{N}''(x_1'', x_2'', x_3''),$$

where we use a double primed system to avoid a conflict of notation with (x_1', x_2', x_3'), which have been used to denote a system rotated with respect to (x_1, x_2, x_3). Here x_j'' with $j = 1, 2, 3$ are independent of x_i with $i = 1, 2, 3$. The mixed dyadic then has the form

$$\begin{aligned}\bar{\bar{T}} = \mathbf{MN}'' &= \sum_i M_i \hat{x}_i \sum_j N_j'' \hat{x}_j'' \\ &= \sum_{i,j} M_i N_j'' \hat{x}_i \hat{x}_j''.\end{aligned} \tag{1.122}$$

We can form an anterior scalar product of $\bar{\bar{T}}$ with a vector function **A** defined in the x_i system, but the posterior scalar product between **A** and $\bar{\bar{T}}$ is undefined or meaningless. Mixed dyadics can be defined in any two unrelated coordinate systems not necessarily rectangular, such as two spherical systems. These dyadics are frequently used in electromagnetic theory [3]. Many commonly used coordinate systems are introduced in the following chapter.

Chapter 2

Coordinate Systems

2-1 General Curvilinear System (GCS)

In the general curvilinear coordinate system (GCS), the coordinate variables will be denoted by ω^i with $i = 1, 2, 3$. The total differential of a position vector is defined by

$$d\mathbf{R}_p = \sum_i \frac{\partial \mathbf{R}_p}{\partial \omega^i} d\omega^i. \tag{2.1}$$

The geometrical interpretation of (2.1) is shown in Fig. 2-1. The vector coefficient $\partial \mathbf{R}_p / \partial \omega^i$ is a measure of the change of $\mathbf{R}_p$ due to a change of ω^i only; it will be denoted by

$$\mathbf{p}_i = \frac{\partial \mathbf{R}_p}{\partial \omega^i} . \tag{2.2}$$

The vectors $\mathbf{p}_i$ with $i = 1, 2, 3$ are designated as the primary vectors. They are, in general, not orthogonal to each other. If the system is orthogonal, we will specifically use the term *orthogonal curvilinear system.* Orthogonal linear system is the same as the rectangular system. In terms of the primary vectors,

$$d\mathbf{R}_p = \sum_i \mathbf{p}_i \, d\omega^i. \tag{2.3}$$

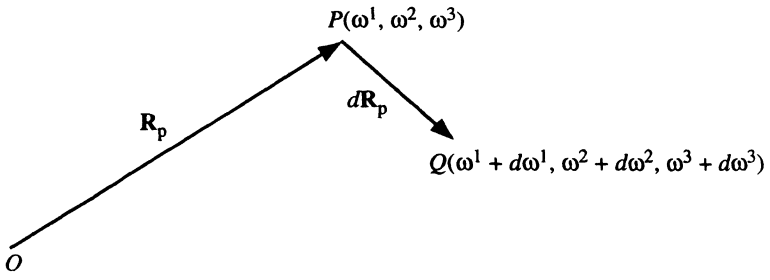


Figure 2-1 Position vector and its total differential.

The primary vectors are not necessarily of unit length, nor of the dimension of length. The three differential vectors $\mathbf{p}_i\, d\omega^i$ define a differential volume given by

$$dV = \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)\, d\omega^1\, d\omega^2\, d\omega^3 = \Lambda\, d\omega^1\, d\omega^2\, d\omega^3, \tag{2.4}$$

where

$$\Lambda = \mathbf{p}_i \cdot (\mathbf{p}_j \times \mathbf{p}_k) \tag{2.5}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order.

We now introduce three reciprocal vectors defined by

$$\mathbf{r}^j = \frac{1}{\Lambda}(\mathbf{p}_j \times \mathbf{p}_k) \tag{2.6}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. They are called reciprocal vectors because

$$\mathbf{p}_i \cdot \mathbf{r}^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.7}$$

The primary vectors can be expressed in terms of the reciprocal vectors in the form

$$\mathbf{p}_i = \Lambda \mathbf{r}^j \times \mathbf{r}^k, \tag{2.8}$$

which can be verified by means of (2.6). The reciprocal systems of vectors were originally introduced by Gibbs [4] without giving a nomenclature to the two systems of vectors. In Stratton's book [5], he designates our primary vectors as the unitary vectors and our reciprocal vectors as the reciprocal unitary vectors. His notations for our $\mathbf{p}_i$ and $\mathbf{r}^i$ are, respectively, $\mathbf{a}_i$ and $\mathbf{a}^i$. Superscript and subscript indices are used, following a tradition in tensor analysis. Incidentally, it should be mentioned that in Stratton's book, the relation described by (2.8) was inadvertently written, in our notation,

$$\mathbf{p}_i = \frac{\mathbf{r}^j \times \mathbf{r}^k}{\Lambda}.$$

A vector function $\mathbf{F}$ can be expressed either in terms of the primary vectors or the reciprocal vectors. We let

$$\mathbf{F} = \sum_i f_i \mathbf{r}^i = \sum_j g^j \mathbf{p}_j. \tag{2.9}$$

On account of the relations described by (2.7), one finds

$$f_i = \mathbf{p}_i \cdot \mathbf{F}, \tag{2.10}$$

$$g^j = \mathbf{r}^j \cdot \mathbf{F}. \tag{2.11}$$

f_i with $i = 1, 2, 3$ are designated as the primary components of $\mathbf{F}$, and g^j with $j = 1, 2, 3$ as the reciprocal components of $\mathbf{F}$. In tensor analysis, f_i are called the *covariant components* and g^j, the *contravariant components*. By substituting (2.10) and (2.11) into (2.9), we can write (2.9) in the form

$$\mathbf{F} = \sum_i \mathbf{F} \cdot \mathbf{p}_i \mathbf{r}^i = \sum_j \mathbf{F} \cdot \mathbf{r}^j \mathbf{p}_j$$

or

$$\mathbf{F} = \sum_i \mathbf{r}^i \mathbf{p}_i \cdot \mathbf{F} = \sum_j \mathbf{p}_j \mathbf{r}^j \cdot \mathbf{F}. \tag{2.12}$$

In the language of dyadic analysis, we denote

$$\sum_i \mathbf{r}^i \mathbf{p}_i = \sum_j \mathbf{p}_j \mathbf{r}^j = \bar{\bar{I}}, \tag{2.13}$$

where $\bar{\bar{I}}$ is an idemfactor, such that

$$\mathbf{F} = \mathbf{F} \cdot \bar{\bar{I}} = \bar{\bar{I}} \cdot \mathbf{F}. \tag{2.14}$$

This is an alternative representation of $\bar{\bar{I}}$ defined by (1.96).

The primary vectors $\mathbf{p}_i$ defined by (2.2), being used to represent $d\mathbf{R}_\mathrm{p}$ in (2.3), can be found if the scalar relations between the curvilinear coordinate variables ω^i, $i = 1, 2, 3$, and the rectangular variables x_j, $j = 1, 2, 3$, are known or given. In terms of the rectangular unit vectors $\hat{x}_j$ and the differentials dx_j, the differential position vector can be written as

$$d\mathbf{R}_\mathrm{p} = \sum_j \hat{x}_j \, dx_j. \tag{2.15}$$

By equating it to (2.3), we have

$$\sum_i \mathbf{p}_i \, d\omega^i = \sum_j \hat{x}_j \, dx_j. \tag{2.16}$$

Thus,

$$\mathbf{p}_i = \sum_j \hat{x}_j \frac{\partial x_j}{\partial \omega^i}, \qquad i = 1, 2, 3. \tag{2.17}$$

This is the explicit expression of $\mathbf{p}_i$ in terms of $\hat{x}_j$ and the derivative of x_j with respect to ω^i. Later, we will illustrate the application of (2.17) to determine $\mathbf{p}_i$ for many commonly used orthogonal curvilinear systems.

To determine the parameter Λ defined by (2.5), it is convenient to introduce the coefficients α_{ij} defined by

$$\mathbf{p}_i \cdot \mathbf{p}_j = \alpha_{ij}, \qquad i, j = 1, 2, 3. \tag{2.18}$$

It is obvious that $\alpha_{ij} = \alpha_{ji}$, thus we have only six distinct coefficients. By means of (2.17), we find

$$\alpha_{ij} = \sum_k \frac{\partial x_k}{\partial \omega^i} \frac{\partial x_k}{\partial \omega^j}. \tag{2.19}$$

When $i = j$,

$$\alpha_{ii} = \sum_k \left(\frac{\partial x_k}{\partial \omega^i} \right)^2. \tag{2.20}$$

These coefficients can now be used to determine the parameter Λ. By definition,

$$\Lambda = \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3). \tag{2.21}$$

The vector function $\mathbf{p}_2 \times \mathbf{p}_3$ in (2.21) can be expressed in terms of $\mathbf{r}^i$ with $i = 1, 2, 3$. According to (2.12) and (2.14),

$$\mathbf{p}_2 \times \mathbf{p}_3 = \sum_i [(\mathbf{p}_2 \times \mathbf{p}_3) \cdot \mathbf{r}^i] \mathbf{p}_i. \tag{2.22}$$

The reciprocal vectors $\mathbf{r}^i$ in (2.22) can be changed to

$$\mathbf{r}^i = \frac{1}{\Lambda} \mathbf{p}_j \times \mathbf{p}_k \tag{2.23}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. Thus, (2.22) becomes

$$\begin{aligned} \mathbf{p}_2 \times \mathbf{p}_3 = \frac{1}{\Lambda} \Big[& (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_2 \times \mathbf{p}_3) \mathbf{p}_1 \\ & + (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_3 \times \mathbf{p}_1) \mathbf{p}_2 \\ & + (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_1 \times \mathbf{p}_2) \mathbf{p}_3 \Big]. \end{aligned} \tag{2.24}$$

The scalar products in (2.24) can be simplified using

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Thus,

$$\begin{aligned} (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_2 \times \mathbf{p}_3) &= (\mathbf{p}_2 \cdot \mathbf{p}_2)(\mathbf{p}_3 \cdot \mathbf{p}_3) - (\mathbf{p}_2 \cdot \mathbf{p}_3)(\mathbf{p}_3 \cdot \mathbf{p}_2) \\ &= \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}. \end{aligned}$$

Similarly,

$$\begin{aligned} (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_3 \times \mathbf{p}_1) &= \alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}, \\ (\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_1 \times \mathbf{p}_2) &= \alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}. \end{aligned}$$

Hence by taking the scalar product of $\mathbf{p}_1$ with (2.24), we obtain

$$\begin{aligned}\Lambda &= \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3) \\ &= \frac{1}{\Lambda}\Big[\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) \\ &\quad + \alpha_{12}(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}) \\ &\quad + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31})\Big]\end{aligned}$$

or

$$\Lambda = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}^{1/2} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix}^{1/2}. \tag{2.25}$$

We take the positive square root of the determinant as the proper expression for Λ.

Before we leave this section, a theorem involving the sum of the derivative of the vectors $\Lambda \mathbf{r}^i$ should be presented. It is known in geometry that the total vectorial area of a closed surface vanishes, that is,

$$\oiint_S d\mathbf{s} = 0. \tag{2.26}$$

If we consider a small volume ΔV bounded by six coordinate surfaces located at $\omega^i \pm \Delta\omega^i/2$ with $i = 1, 2, 3$ in GCS, then

$$\Delta V = \Lambda\, \Delta\omega^1\, \Delta\omega^2\, \Delta\omega^3,$$

$$\Delta \mathbf{S}_i = \Lambda \mathbf{r}^i\, \Delta\omega^j\, \Delta\omega^k$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. The differential form of (2.26) can be obtained by taking the limit of the following identity:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S d\mathbf{s} = 0$$

or

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i=1}^{3} \left[(\Lambda \mathbf{r}^i)_{\omega^i + \Delta\omega^i/2} - (\Lambda \mathbf{r}^i)_{\omega^i - \Delta\omega^i/2}\right] \Delta\omega^j\, \Delta\omega^k = 0,$$

which yields

$$\sum_i \frac{\partial}{\partial \omega^i}(\Lambda \mathbf{r}^i) = 0. \tag{2.27}$$

Equation (2.27) is a very useful theorem; it will be designated as the *closed surface theorem.*

2-2 Orthogonal Curvilinear System (OCS)

The GCS degenerates to the orthogonal curvilinear system (OCS) when the primary vectors are mutually perpendicular to each other. In this case, we let

$$\mathbf{p}_i = \frac{\partial \mathbf{R}_p}{\partial \omega^i} = h_i \hat{u}_i, \qquad i = 1, 2, 3, \tag{2.28}$$

where $\hat{u}_i$ denote the unit vectors along the coordinates ω^i and h_i, the metric coefficients in the OCS. For a specific system, such as the spherical coordinate system with coordinate variables R, θ, ϕ, we use the notations $\hat{R}, \hat{\theta}$, and $\hat{\phi}$ to denote the unit vectors in that system. $\hat{u}_i$ are used when the orthogonal curvilinear system is arbitrary or unspecified. For the rectangular system, which is a special case of OCS, $h_i = 1$, and $\omega^i = x_i$, or more specifically, x, y, z. The reciprocal vectors in OCS now become

$$\mathbf{r}^j = \frac{\mathbf{p}_k \times \mathbf{p}_i}{\Lambda} = \frac{h_k h_i \hat{u}_j}{\Omega} = \frac{\hat{u}_j}{h_j}, \tag{2.29}$$

where $\Omega = h_i h_j h_k = h_1 h_2 h_3$. The metric coefficients h_i can be found if we know the relations between ω^i and x_j. In the rectangular system,

$$\frac{\partial \mathbf{R}_p}{\partial x_j} = \hat{x}_j. \tag{2.30}$$

By the chain rule of differentiation, we find

$$h_i \hat{u}_i = \frac{\partial \mathbf{R}_p}{\partial \omega^i} = \sum_j \frac{\partial \mathbf{R}_p}{\partial x_j} \frac{\partial x_j}{\partial \omega^i} = \sum_j \frac{\partial x_j}{\partial \omega^i} \hat{x}_j. \tag{2.31}$$

In (2.31), the summation with respect to j goes from $j = 1$ to $j = 3$; that labeling will be omitted henceforth. Thus,

$$h_i^2 = \sum_j \left(\frac{\partial x_j}{\partial \omega^i} \right)^2, \tag{2.32}$$

and the positive square root of (2.32) yields

$$h_i = \left[\sum_j \left(\frac{\partial x_j}{\partial \omega^i} \right)^2 \right]^{1/2}. \tag{2.33}$$

Equation (2.33) can be used to determine h_i when the relations between x_j, the dependent variables, and ω^i, the independent variables, are known. Another expression of h_i is sometimes useful when the roles of x_j and ω^i are interchanged with x_j as independent variables and ω^i as dependent variables. By definition,

$$d\mathbf{R}_p = \sum_j \hat{x}_j \, dx_j = \sum_i h_i \hat{u}_i \, d\omega^i; \tag{2.34}$$

hence

$$d\omega^i = \frac{1}{h_i} \sum_j \hat{u}_i \cdot \hat{x}_j \, dx_j; \tag{2.35}$$

thus,

$$\frac{\partial \omega^i}{\partial x_j} = \frac{1}{h_i} \hat{u}_i \cdot \hat{x}_j. \tag{2.36}$$

Let

$$\frac{\hat{u}_i}{h_i} = \sum_j c_{ij} \hat{x}_j. \tag{2.37}$$

Then

$$c_{ij} = \frac{1}{h_i} \hat{u}_i \cdot \hat{x}_j = \frac{\partial \omega^i}{\partial x_j}. \tag{2.38}$$

Equation (2.37) therefore becomes

$$\frac{\hat{u}_i}{h_i} = \sum_j \frac{\partial \omega^i}{\partial x_j} \hat{x}_j; \tag{2.39}$$

hence

$$\frac{1}{h_i^2} = \sum_j \left(\frac{\partial \omega^i}{\partial x_j} \right)^2. \tag{2.40}$$

We take the positive square root of (2.40) to be the expression for $1/h_i$:

$$\frac{1}{h_i} = \left[\sum_j \left(\frac{\partial \omega^i}{\partial x_j} \right)^2 \right]^{1/2}. \tag{2.41}$$

In contrast to (2.33), ω^i's are now the dependent variables and x_j, the independent variables. Unlike functions with one independent variable,

$$\frac{\partial \omega^i(x, y, z)}{\partial x} \neq 1 \Big/ \frac{\partial x(\omega^i, \omega^j, \omega^k)}{\partial \omega^i},$$

while

$$\frac{dy(x)}{dx} = 1 \Big/ \frac{dx(y)}{dy}.$$

The list that follows shows the metric coefficients of some commonly used OCS, and the relations between (v_1, v_2, v_3) and (x, y, z), based on which these metric coefficients are derived by means of (2.33).

Rectangular
Coordinate variables: (x, y, z)
Metric coefficients: $(1, 1, 1)$.

Cylindrical
Coordinate variables: (r, ϕ, z)
Metric coefficients: $(1, r, 1)$
Relations: $x = r\cos\phi,\ y = r\sin\phi,\ z = z.$

Spherical
Coordinate variables: (R, θ, ϕ)
Metric coefficients: $(1, R, R\sin\theta)$
Relations: $x = R\sin\theta\cos\phi,\ y = R\sin\theta\sin\phi,\ z = R\cos\theta.$

Elliptical Cylinder
Coordinate variables: (η, ξ, z)
Metric coefficients:

$$\left[c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2},\ c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2},\ 1\right]$$

Relations: $x = c\eta\xi,\ y = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2},\ z = z.$

Parabolic Cylinder
Coordinate variables: (η, ξ, z)
Metric coefficients:

$$\left[(\eta^2+\xi^2)^{1/2}, (\eta^2+\xi^2)^{1/2}, 1\right]$$

Relations: $x = \frac{1}{2}(\eta^2-\xi^2),\ y = \eta\xi,\ z = z.$

Prolate Spheroidal
Coordinate variables: (η, ξ, ϕ)
Metric coefficients:

$$\left[c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2},\ c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2},\ c(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}\right]$$

Relations:

$$x = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2}\cos\phi,$$
$$y = c\left[(1-\eta^2)(\xi^2-1)\right]^{1/2}\sin\phi,$$
$$z = c\eta\xi.$$

Oblate Spheroidal

Coordinate variables: (ξ, η, ϕ)

Metric coefficients:

$$\left[c\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2},\ c\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2},\ c\xi\eta\right]$$

Relations:

$$\begin{aligned} x &= c\xi\eta\cos\phi, \\ y &= c\xi\eta\sin\phi, \\ z &= c\left[(\xi^2-1)(1-\eta^2)\right]^{1/2}. \end{aligned}$$

Bipolar Cylinders

Coordinate variables: (η, ξ, z)

Metric coefficients:

$$\left[\frac{a}{\cosh\xi-\cos\eta},\ \frac{a}{\cosh\xi-\cos\eta},\ 1\right]$$

Relations:

$$\begin{aligned} x &= \frac{a\sinh\xi}{\cosh\xi-\cos\eta}, \\ y &= \frac{a\sin\eta}{\cosh\xi-\cos\eta}, \\ z &= z. \end{aligned}$$

In this list, for the case of the elliptical cylinder, the governing equations for the elliptical cylinder and the conformal hyperbolic cylinder are

$$\frac{x^2}{c^2\xi^2}+\frac{y^2}{c^2(\xi^2-1)}=1, \qquad \infty > \xi \geq 1, \tag{2.42}$$

$$\frac{x^2}{c^2\eta^2}-\frac{y^2}{c^2(1-\eta^2)}=1, \qquad 1 \geq \eta \geq -1, \tag{2.43}$$

where c denotes half of the focal distance between the foci of the ellipse. For the prolate spheroid, the governing equations are

$$\frac{z^2}{c^2\xi^2}+\frac{r^2}{c^2(\xi^2-1)}=1, \qquad 2\pi \geq \phi \geq 0, \quad \infty > \xi \geq 1, \tag{2.44}$$

$$\frac{z^2}{c^2\eta^2}-\frac{r^2}{c^2(1-\eta^2)}=1, \qquad 2\pi \geq \phi \geq 0, \quad 1 \geq \eta \geq -1, \tag{2.45}$$

where $r^2 = x^2 + y^2$. Equation (2.44) represents an ellipse of revolution revolving around the z axis, which is the major axis. The conformal hyperboloid is represented by (2.45). For the oblate spheroid, the governing equations are

$$\frac{r^2}{c^2\xi^2} + \frac{z^2}{c^2(\xi^2 - 1)} = 1, \qquad 2\pi \geq \phi \geq 0, \quad \infty > \xi \geq 1, \tag{2.46}$$

$$\frac{r^2}{c^2\eta^2} - \frac{z^2}{c^2(1 - \eta^2)} = 1, \qquad 2\pi \geq \phi \geq 0, \quad 1 \geq \eta \geq -1. \tag{2.47}$$

Equation (2.46) represents an oblate spheroid generated by revolving an ellipse around the z axis, which is the minor axis in this case, and (2.47) is the equation for the conformal hyperboloid.

For the bipolar cylinders, the governing equations are

$$(x - a \coth \xi)^2 + y^2 = a^2 \operatorname{csch}^2 \xi, \tag{2.48}$$

$$(y - a \cot \eta)^2 + x^2 = a^2 \csc^2 \eta, \tag{2.49}$$

with $\infty > \xi > -\infty$ and $2\pi > \eta > 0$, and where a denotes half of the distance between two pivoting points from which these circles are generated. In fact, (2.48) and (2.49) can be derived by considering a conformal transformation between the complex variables $x + jy$ and $\eta + j\xi$ in the form

$$\frac{(x + jy) - a}{(x + jy) + a} = e^{j(\eta + j\xi)}, \tag{2.50}$$

which is called a *bilinear transformation* in the theory of complex variables.

In the complex (x, y) plane, the numbers $c_1 = (x - a) + jy$ and $c_2 = (x + a) + jy$ are shown graphically in Fig. 2-2, where we assume a to be real. These numbers can also be written in the form

$$\begin{aligned} c_1 &= \left[(x - a)^2 + y^2\right]^{1/2} e^{j\alpha_1}, \qquad \alpha_1 = \tan^{-1} \frac{y}{x - a}, \\ c_2 &= \left[(x + a)^2 + y^2\right]^{1/2} e^{j\alpha_2}, \qquad \alpha_2 = \tan^{-1} \frac{y}{x + a}. \end{aligned} \tag{2.51}$$

Equation (2.50) is therefore equivalent to

$$\frac{c_1}{c_2} = \left[\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}\right]^{1/2} e^{j(\alpha_1 - \alpha_2)} = e^{-\xi} e^{j\eta}; \tag{2.52}$$

hence

$$\left[\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}\right]^{1/2} = e^{-\xi}, \tag{2.53}$$

$$\tan^{-1} \frac{y}{x - a} - \tan^{-1} \frac{y}{x + a} = \eta. \tag{2.54}$$

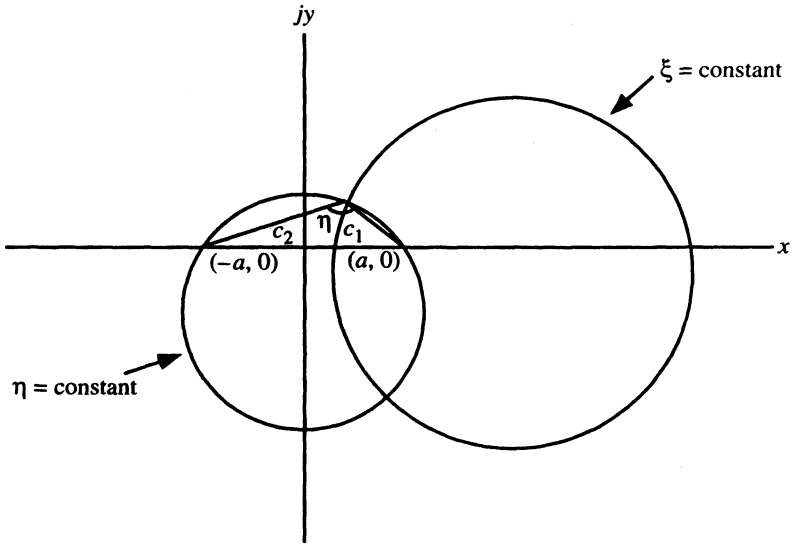


Figure 2-2 Locus of the complex number $x + jy$ resulting from a bilinear transformation.

It is not difficult to deduce (2.48) from (2.53), and (2.49) from (2.54). From this discussion, we see that the locus of constant ξ, which is a circle, corresponds to a constant ratio of the magnitude of c_1 and c_2, and the locus of constant η is also a circle conformal to the circle of constant ξ. The fact that they are conformal is because (2.50) is a conformal transformation in the theory of complex numbers.

2-3 Derivatives of Unit Vectors in OCS

Equation (2.28) states

$$\frac{\partial \mathbf{R}_p}{\partial \omega^i} = h_i \hat{u}_i, \qquad i = 1, 2, 3. \tag{2.55}$$

Let us change the variables ω^i to v_i for OCS; then,

$$\frac{\partial \mathbf{R}_p}{\partial v_j} = h_j \hat{u}_j, \qquad \frac{\partial \mathbf{R}_p}{\partial v_k} = h_k \hat{u}_k.$$

Hence

$$\frac{\partial (h_j \hat{u}_j)}{\partial v_k} = \frac{\partial (h_k \hat{u}_k)}{\partial v_j} \tag{2.56}$$

because both are equal to $\partial^2 \mathbf{R}_p / \partial v_j \, \partial v_k$. We assume that all of the first and second derivatives of $\mathbf{R}_p$ do exist. Equation (2.56) can be written in the form

$$h_j \frac{\partial \hat{u}_j}{\partial v_k} + \frac{\partial h_j}{\partial v_k} \hat{u}_j = h_k \frac{\partial \hat{u}_k}{\partial v_j} + \frac{\partial h_k}{\partial v_j} \hat{u}_k. \tag{2.57}$$

Because

$$\hat{u}_j \cdot \hat{u}_j = 1,$$

hence

$$\hat{u}_j \cdot \frac{\partial \hat{u}_j}{\partial v_k} = 0.$$

$\partial \hat{u}_j / \partial v_k$ is therefore perpendicular to $\hat{u}_j$. In the (v_j, v_k) plane, it is parallel to $\hat{u}_k$; similarly, $\partial \hat{u}_k / \partial v_j$ is parallel to $\hat{u}_j$. Thus, we can write

$$\frac{\partial \hat{u}_j}{\partial v_k} = \alpha \hat{u}_k, \qquad \frac{\partial \hat{u}_k}{\partial v_j} = \beta \hat{u}_j.$$

Equation (2.57) can now be put in the form

$$\left(\frac{\partial h_k}{\partial v_j} - \alpha h_j\right) \hat{u}_k = \left(\frac{\partial h_j}{\partial v_k} - \beta h_k\right) \hat{u}_j.$$

Because $\hat{u}_k$ and $\hat{u}_j$ are independent and orthogonal to each other, this equation can be satisfied only if

$$\frac{\partial h_k}{\partial v_j} = \alpha h_j, \qquad \frac{\partial h_j}{\partial v_k} = \beta h_k;$$

hence

$$\frac{\partial \hat{u}_j}{\partial v_k} = \frac{1}{h_j} \frac{\partial h_k}{\partial v_j} \hat{u}_k \tag{2.58}$$

and

$$\frac{\partial \hat{u}_k}{\partial v_j} = \frac{1}{h_k} \frac{\partial h_j}{\partial v_k} \hat{u}_j. \tag{2.59}$$

Equations (2.58) and (2.59) hold true for j and $k = 1, 2, 3$ with $j \neq k$.

The derivative $\partial \hat{u}_i / \partial v_i$ can be found by considering the relationship among the three orthogonal unit vectors $\hat{u}_i$, $\hat{u}_j$, and $\hat{u}_k$ in a right-hand system:

$$\hat{u}_i = \hat{u}_j \times \hat{u}_k, \qquad i, j, k = \begin{cases} 1, 2, 3 & \text{or} \\ 2, 3, 1 & \text{or} \\ 3, 1, 2. \end{cases} \tag{2.60}$$

The coordinate variables of the system are denoted by (v_1, v_2, v_3). The partial derivative of (2.60) with respect to v_i yields

$$\frac{\partial \hat{u}_i}{\partial v_i} = \hat{u}_j \times \frac{\partial \hat{u}_k}{\partial v_i} + \frac{\partial \hat{u}_j}{\partial v_i} \times \hat{u}_k.$$

In view of (2.58) or (2.59), this equation is equivalent to

$$\begin{aligned} \frac{\partial \hat{u}_i}{\partial v_i} &= \hat{u}_j \times \frac{1}{h_k} \frac{\partial h_i}{\partial v_k} \hat{u}_i + \frac{1}{h_j} \frac{\partial h_i}{\partial v_j} \hat{u}_i \times \hat{u}_k \\ &= -\left(\frac{1}{h_k} \frac{\partial h_i}{\partial v_k} \hat{u}_k + \frac{1}{h_j} \frac{\partial h_i}{\partial v_j} \hat{u}_j\right). \end{aligned} \tag{2.61}$$

Equations (2.58) and (2.59) are very important formulas that will be used frequently in subsequent sections. It can be easily verified that as a result of (2.61),

$$\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right) = 0, \tag{2.62}$$

where $\Omega = h_1 h_2 h_3$. Equation (2.62) can be derived readily from the general closed surface theorem stated in (2.27) by letting $\Lambda = \Omega$ and $\mathbf{r}^i = \hat{u}_i/h_i$. The derivation of this theorem for the OCS appears to be more complicated than for the GCS. However, the relations between the derivatives of the unit vectors give us a deeper understanding of the vector relations in OCS.

Another identity that can be proved with the aid of (2.58) and (2.62) is

$$\sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}\left(\frac{\hat{u}_j}{h_j}\right) = 0, \qquad j = 1, 2, 3. \tag{2.63}$$

Equations (2.61) and (2.63) will be used in the derivation of many important formulas. The interpretation of these two identities from the point of view of vector theorems will be discussed in Chapter 4.

2-4 Dupin Coordinate System

The Dupin coordinate system is an indispensable tool to treat vector analysis on a surface. In the general Dupin system, the coordinate variables will be denoted by (v_1, v_2, v_3) and the corresponding unit vectors by $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ with metric coefficients $(h_1, h_2, 1)$. The variables (v_1, v_2) are used to describe the coordinate lines on the surface, while v_3 denotes the normal distance measured linearly from the surface; hence $h_3 = 1$. For a right-hand system, the direction of $\hat{u}_3$ is determined by $\hat{u}_1 \times \hat{u}_2$. $\hat{u}_1$ and $\hat{u}_2$ are assumed to be orthogonal, and both are tangential to the surface. Figure 2-3 shows the disposition of these quantities. The total differential of the position vector measured from a point on the surface to a neighboring point in the space is then written as

$$d\mathbf{R}_p = h_1\, dv_1\, \hat{u}_1 + h_2\, dv_2\, \hat{u}_2 + dv_3\, \hat{u}_3. \tag{2.64}$$

When v_1 and v_2 have not yet been specified, we designate the system as the general Dupin system. The surface of a circular cylinder and that of a sphere are two of the simplest surfaces belonging to the Dupin system. The one-to-one correspondence of the variables, the unit vectors, and the metric coefficients is listed in Table 2-1.

Let us now consider a spheroidal surface described by

$$\frac{x^2 + y^2}{b^2} + \frac{z^2}{a^2} = 1$$

or

$$\frac{r^2}{b^2} + \frac{z^2}{a^2} = 1, \qquad 2\pi \geq \phi \geq 0, \tag{2.65}$$

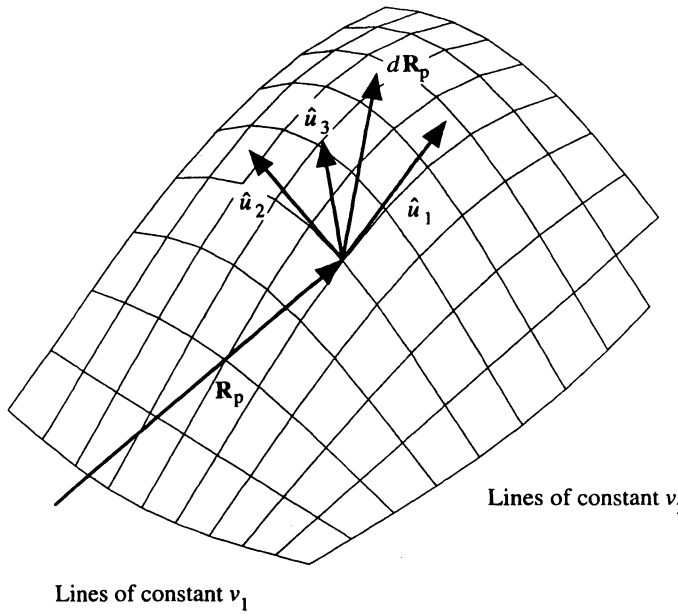


Figure 2-3 Dupin coordinate system, $d\mathbf{R}_p = h_1\, dv_1\, \hat{u}_1 + h_2\, dv_2\, \hat{u}_2 + dv_3\, \hat{u}_3$.

Table 2-1: Two Dupin coordinate systems

System	(v_1, v_2, v_3)	$(\hat{u}_1, \hat{u}_2, \hat{u}_3)$	(h_1, h_2, h_3)
Cylindrical surface	(ϕ, z, r)	$(\hat{\phi}, \hat{z}, \hat{r})$	$(r, 1, 1)$
Spherical surface	(θ, ϕ, R)	$(\hat{\theta}, \hat{\phi}, \hat{R})$	$(R, R\sin\theta, 1)$

where (ϕ, z, r) are the cylindrical variables. If we choose (ϕ, z) as (v_1, v_2), then

$$d\mathbf{R}_p = ds_1\, \hat{u}_1 + ds_2\, \hat{u}_2 + ds_3\, \hat{u}_3, \tag{2.66}$$

where

$$ds_1 = r\, d\phi = b\left[1 - \left(\frac{z}{a}\right)^2\right]^{1/2} d\phi = h_1\, dv_1,$$

$$ds_2 = [(dr)^2 + (dz)^2]^{1/2} = \left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2} dz = h_2\, dv_2,$$

$$ds_3 = dv_3.$$

Hence

$$h_1 = b\left[1 - \left(\frac{z}{a}\right)^2\right]^{1/2},$$

$$h_2 = \left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2} = \sec\,\alpha,$$

$$\frac{dr}{dz} = \tan\,\alpha = \text{slope of the tangent to the ellipse, (2.65), making an angle } \alpha \text{ with the } z \text{ axis.}$$

The corresponding unit vectors are

$$\hat{u}_1 = \hat{\phi}, \tag{2.67}$$

$$\hat{u}_2 = \sin\alpha\,\hat{r} + \cos\alpha\,\hat{z}, \tag{2.68}$$

$$\hat{u}_3 = \cos\alpha\,\hat{r} - \sin\alpha\,\hat{z}. \tag{2.69}$$

The choice of (v_1, v_2) in a Dupin system is not unique. In the previous example, we can use (ϕ, r) as (v_1, v_2); then,

$$ds_2 = \sqrt{1 + \left(\frac{dz}{dr}\right)^2}\,dr = \csc\alpha\,dr = h_2\,dv_2. \tag{2.70}$$

Hence $h_2 = \csc\alpha$, but h_1, $\hat{u}_1$, and $\hat{u}_2$ remain the same. In the case of a spherical surface, we can use (z, x) as (v_1, v_2). In certain problems dealing with integrations, such a choice sometimes is desirable, particularly from the point of view of numerical calculations.

2-5 Radii of Curvature

For a surface described in the general Dupin coordinate system, there are two radii of curvature of the surface associated with two contours in the (v_1, v_3) plane and the (v_2, v_3) plane. These are two normal planes containing $(\hat{u}_1, \hat{u}_3)$ and $(\hat{u}_2, \hat{u}_3)$ at $P\,(v_1, v_2, v_3)$ (see Fig. 2-3). These radii of curvature are closely related to the metric coefficients h_1, h_2 and the rate of change of these coefficients with respect to v_3. Figure 2-4 shows a section of the contour C in the neighborhood of P, resulting from the intersection of the $(\hat{u}_2, \hat{u}_3)$ plane with the surface.

Referring to Fig. 2-4, we denote

$$PQ = h_2\,dv_2,$$

$$OP = R_2 \text{ (second principal radius of the curvature),}$$

$$PS = QT = dv_3.$$

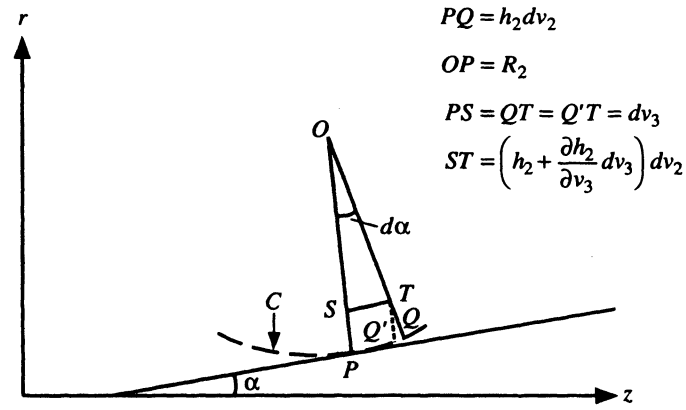


Figure 2-4 Radius of curvature of a surface in the plane containing $\hat{u}_2$ and $\hat{u}_3$ (r–z plane for the example illustrated in the text).

Then,

$$ST = \left(h_2 + \frac{\partial h_2}{\partial v_3} dv_3\right) dv_2,$$

$$Q'Q = PQ - ST = -\frac{\partial h_2}{\partial v_3} dv_3\, dv_2.$$

The triangles OPQ and $TQ'Q$ are similar; hence

$$\frac{PQ}{OP} = \frac{Q'Q}{QT}, \tag{2.71}$$

which yields

$$\frac{h_2}{R_2} = -\frac{\partial h_2}{\partial v_3} \tag{2.72}$$

or

$$\frac{1}{R_2} = -\frac{1}{h_2}\frac{\partial h_2}{\partial v_3}. \tag{2.73}$$

Equation (2.73) relates the second principal radius of curvature in the (v_2, v_3) plane at P to the metric coefficient h_2 and its derivative with respect to v_3 at that point. To find the expression of R_2 in terms of the shape of the curve, we have to know the governing equation for C. Let this equation be given in the form

$$r = f_1(z), \tag{2.74}$$

where z represents v_2; then,

$$ds_2 = \sqrt{(dr)^2 + (dz)^2} = \sqrt{1 + \left(\frac{dr}{dz}\right)^2}\, dz = h_2\, dv_2,$$

$$h_2 = \sqrt{1 + \left(\frac{dr}{dz}\right)^2} = \sqrt{1 + \tan^2\alpha} = \sec\alpha,$$

where α denotes the angle of inclination of the tangent at P made with the z axis. Now,

$$d\alpha = \frac{h_2\, dv_2}{R_2} = \frac{h_2\, dz}{R_2} ;$$

hence

$$\frac{1}{R_2} = \frac{1}{h_2}\frac{d\alpha}{dz} = \cos\alpha \frac{d\alpha}{dz} .$$

Because

$$\tan\alpha = \frac{dr}{dz} = r',$$

a differentiation of this equation with respect to z yields

$$\frac{1}{\cos^2\alpha}\frac{d\alpha}{dz} = \frac{dr'}{dz} = r''.$$

We have

$$\frac{1}{R_2} = (\cos^3\alpha)r'' = \frac{r''}{(1+r'^2)^{3/2}} . \tag{2.75}$$

The derivation of this formula is found in many books on calculus. It is repeated here to show its relationship with the derivative of the relevant metric coefficient. Equation (2.75) shows that for a concave surface, $r'' > 0$, so R_2 is positive, and for a convex surface, R_2 is negative. Similarly, one finds that in the $(\hat{u}_1, \hat{u}_3)$ plane,

$$\frac{1}{R_1} = \frac{-1}{h_1}\frac{\partial h_1}{\partial v_3}, \tag{2.76}$$

where R_1 denotes the first principal radius of curvature. A formula similar to (2.75) can be derived if the governing equation of the curve in the (v_1, v_3) plane is known. The reciprocals of the two radii of curvature are called Gaussian curvatures of the surface in the two orthogonal planes.

As an illustration of the application of these formulas, we consider the equation of a paraboloidal surface defined by

$$r^2 = 4fz, \qquad 2\pi \geq \phi \geq 0, \tag{2.77}$$

where f denotes the focal length of the paraboloid and (r, ϕ, z) denote the cylindrical variables. The coordinates in the Dupin system for the surface are identified as $(v_1, v_2, v_3) = (\phi, z, v_3)$ with

$$h_2 = \sqrt{1 + \left(\frac{dr}{dz}\right)^2} = \sqrt{1 + r'^2}, \qquad h_1 = r, \qquad h_3 = 1.$$

For the surface under consideration,

$$r' = \frac{dr}{dz} = \sqrt{\frac{f}{z}}, \qquad r'' = -\frac{1}{2}\sqrt{f}\, z^{-3/2}.$$

Upon substituting r' and r'' into (2.75), we find

$$R_2 = -2f\left(1 + \frac{z}{f}\right)^{3/2}. \tag{2.78}$$

At $z = 0$, $R_2 = -2f$, and at $z = f$, $R_2 = -4\sqrt{2}\, f$, and so on. To find R_1, it is simpler to use (2.76) instead of finding the equation of the cross section in the $(\hat{u}_2, \hat{u}_3)$ plane. Now,

$$\frac{1}{R_1} = \frac{-1}{h_1}\frac{\partial h_1}{\partial v_3} = \frac{-1}{r}\frac{\partial r}{\partial v_3} = -\frac{\sin\beta}{r}, \tag{2.79}$$

where β denotes the angle between the normal to the surface $\hat{u}_3$ and the z axis, and

$$\tan\beta = -\frac{1}{r'}, \qquad \sin\beta = \frac{1}{(1 + r'^2)^{1/2}}.$$

Thus,

$$R_1 = -r\,(1 + r'^2)^{1/2} = -2f\left(1 + \frac{z}{f}\right)^{1/2}.$$

It can be proved that $-R_1$ is the distance measured from the point $P(r, z)$ on the parabola along the inward normal to its intersect with the z axis. At $z = 0$, $R_1 = -2f$, and at $z = f$, $R_1 = -2\sqrt{2}f = \frac{1}{2}R_2$. The relationship between R_1 and R_2, in general, is $R_1^3 = 4f^2 R_2$.

As an exercise, the reader may be interested to verify that for a spheroidal surface defined by

$$\left(\frac{r}{b}\right)^2 + \left(\frac{z}{a}\right)^2 = 1, \qquad 2\pi \geq \phi \geq 0,$$

$$R_2 = -\frac{a^2}{b}\left[1 + \frac{z^2}{a^2}\left(\frac{b^2}{a^2} - 1\right)\right]^{3/2} \tag{2.80}$$

and

$$R_1 = -b\left[1 + \frac{z^2}{a^2}\left(\frac{b^2}{a^2} - 1\right)\right]^{1/2}. \tag{2.81}$$

The relationship between R_1 and R_2 is $R_1^3 = (b^4/a^2)R_2$.

The two Gaussian curvatures that are defined by (2.73) and (2.76) are related to the rate of change of an elementary area of the curved surface. For an orthogonal Dupin coordinate system under consideration,

$$\Delta S_3 = h_1 h_2\, \Delta v_1\, \Delta v_2; \tag{2.82}$$

hence

$$\lim_{\Delta S_3 \to 0} \frac{1}{\Delta S_3} \frac{\partial \Delta S_3}{\partial v_3} = \frac{1}{h_1 h_2} \frac{\partial (h_1 h_2)}{\partial v_3}$$
$$= \frac{1}{h_1} \frac{\partial h_1}{\partial v_3} + \frac{1}{h_2} \frac{\partial h_2}{\partial v_3} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{2.83}$$

The sum of the two principal Gaussian curvatures is therefore equal to the decrease of the normal derivative of an elementary area per unit area. We will denote the sum of the two principal curvatures by J:

$$J = \frac{1}{R_1} + \frac{1}{R_2}, \tag{2.84}$$

and it will be designated as the *surface curvature*. It is convenient to define a mean radius of curvature for a surface by R_m, such that

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_m}; \tag{2.85}$$

then,

$$J = \frac{1}{R_m}. \tag{2.86}$$

There is a simple graphical method to determine R_m for a given pair of R_1 and R_2 based on (2.85). We erect two vertical lines with lengths equal to R_1 and R_2 on a graded paper and draw two lines from the tips of these two vertical lines to the bases, as shown in Fig. 2-5. The intersecting point of the two inclined lines yields R_m. The validity of this method is based on the relation

$$\frac{R_m}{R_1} + \frac{R_m}{R_2} = \frac{d_2}{d_1 + d_2} + \frac{d_1}{d_1 + d_2} = 1.$$

The same method applies to the case when $R_1 > 0$ and $R_2 < 0$ (a saddle surface), as shown in Fig. 2-6. In this case,

$$\frac{R_m}{R_1} + \frac{R_m}{R_2} = \frac{-D_2}{D_1} + \frac{D_1 + D_2}{D_1} = 1.$$

This simple graphical method applies to problems involving the sum of two reciprocals such as two resistances or two reactances in parallel and two optical lenses with different focal lengths aligned in cascade. All these quantities could be of the same sign or opposite signs. The method is a visible aid to an algebraic identity.

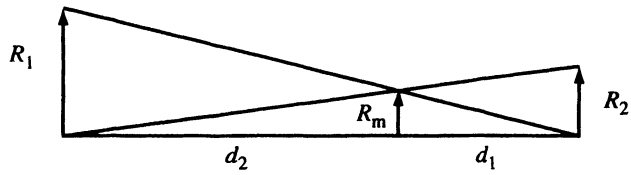


Figure 2-5 A graphical method to determine R_m for $R_1, R_2 > 0$.

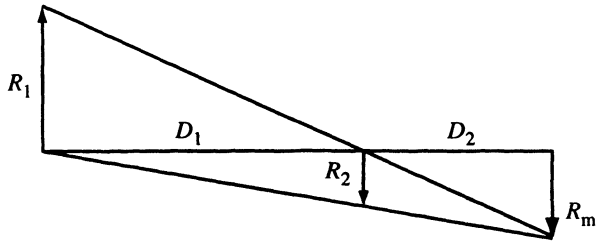


Figure 2-6 $R_1 > 0$, $R_2 < 0$.

Chapter 3

Line Integrals, Surface Integrals, and Volume Integrals

3-1 Differential Length, Area, and Volume

In this section, we shall give a brief review of the differential quantities to be used in vector analysis and, particularly, their notation. A differential length, in general, will be denoted by $d\ell$. It is the same as the *total differential* of the position vector $d\mathbf{R}_p$. In OCS,

$$d\ell = \sum_i h_i \, dv_i \, \hat{u}_i . \tag{3.1}$$

For a cell with its center located at (v_1, v_2, v_3) and bounded by six surfaces located at $v_i \pm dv_i/2$ with $i = 1, 2, 3$, the *vector differential area* of the three surfaces located at $v_i + dv_i/2$ is then given by

$$\begin{aligned} d\mathbf{S}_i &= h_j \, dv_j \, \hat{u}_j \times h_k \, dv_k \, \hat{u}_k \, |_{v_i + dv_i/2} \\ &= h_j \, h_k \, dv_j \, dv_k \, \hat{u}_i \, |_{v_i + dv_i/2}, \end{aligned} \tag{3.2}$$

where (i, j, k) follows the cyclic order of $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$. The vector differential areas of the other three surfaces are

$$d\mathbf{S}_i = -h_j \, h_k \, dv_j \, dv_k \, \hat{u}_i \, |_{v_i - dv_i/2} . \tag{3.3}$$

All of these vector areas are pointed away, or outward, from their surfaces. We should emphasize that the metric coefficients and the corresponding unit vectors

are evaluated at the sites of these surfaces $v_i \pm dv_i/2$, not at the center of the cell. The differential volume of the cell is given by

$$\begin{aligned} dV &= h_i\, dv_i\, \hat{u}_i \cdot (h_j\, dv_j\, \hat{u}_j \times h_k\, dv_k\, \hat{u}_k) \\ &= h_i\, h_j\, h_k\, dv_i\, dv_j\, dv_k \\ &= h_1\, h_2\, h_3\, dv_1\, dv_2\, dv_3. \end{aligned} \tag{3.4}$$

3-2 Classification of Line Integrals

If we continuously change the position vector of a point in space in a certain specified manner, the locus of the point will trace a curve in space (Fig. 3-1). Let a typical point on the curve be denoted by $P(x, y, z)$ in the Cartesian coordinate system. If (x, y, z) are functions of a single parameter t, then as t varies, $x(t)$, $y(t)$, and $z(t)$ will vary accordingly. We call such a description the *parametric representation* of a curve. We assume that there is a one-to-one correspondence between t and (x, y, z). The vector differential length of the curve can now be written as

$$d\ell = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} = \left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right) dt. \tag{3.5}$$

It should be pointed out here that we use dx, dy, dz, and dt to denote the total differential of these variables, but dx/dt, dy/dt, and dz/dt are the derivatives of x, y, z with respect to t. As an example, let

$$\left.\begin{aligned} x &= a\cos\frac{2\pi}{T}t \\ y &= a\sin\frac{2\pi}{T}t \\ z &= \frac{b}{T}t \end{aligned}\right\}, \tag{3.6}$$

where a, b, T are constants and t is the parameter. As t varies, the locus of P describes a right-hand spiral, advancing in the positive z direction as t is increased. When t changes from 0 to T, the spiral starts at $(a, 0, 0)$ and ends at $(a, 0, b)$; therefore, b denotes the height of the spiral after making one complete turn, and

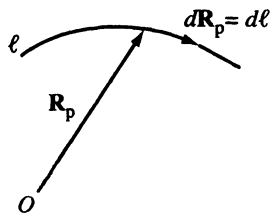


Figure 3-1 Curve in a three-dimensional space.

a denotes the radius of the circular projection of the spiral on the x–y plane. To calculate the length of the spiral, one starts with

$$(d\ell)^2 = d\ell \cdot d\ell = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt)^2;$$

hence

$$d\ell = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt. \tag{3.7}$$

The integral of (3.7) from $t = 0$ to T yields

$$L = \int_0^L d\ell = \frac{1}{T}\int_0^T \left[(2\pi a)^2 + b^2\right]^{1/2} dt$$

$$= \left[(2\pi a)^2 + b^2\right]^{1/2} = \left(c^2 + b^2\right)^{1/2}, \tag{3.8}$$

where c denotes the circumference of the projected circle. The pitch angle of the spiral is defined by

$$\alpha = \tan^{-1}\left(\frac{b}{c}\right). \tag{3.9}$$

Equation (3.8) represents the simplest form of a line integral. In general, we define the line integral of Type I as

$$I_1 = \int_c f(x, y, z)\, d\ell, \tag{3.10}$$

where c denotes the contour of the curve wherein the integration is executed.

As an example, let c be a parabolic curve described by

$$y^2 = 2x \quad \text{in the plane } z = 0, \tag{3.11}$$

and the contour extends from $x = 0$, $y = 0$ to $x = \frac{3}{2}$, $y = \sqrt{3}$, and the function in (3.10) is supposed to be $f(x, y) = xy$.

If we choose y as the parameter, then

$$d\ell = dx\,\hat{x} + dy\,\hat{y} = y\,dy\,\hat{x} + dy\,\hat{y},$$

$$d\ell = (y^2 + 1)^{1/2}\, dy,$$

$$f(x, y) = \frac{1}{2}y^3;$$

thus,

$$I_1 = \int_c f(x, y)\, d\ell = \int_0^{\sqrt{3}} \frac{1}{2}\, y^3 (y^2 + 1)^{1/2}\, dy = \frac{28}{15}.$$

We purposely choose $f(x, y)$ and c in such a manner that the integral can be evaluated in a closed form in order to clearly illustrate the steps.

The second type of line integral is defined by

$$\mathbf{I}_2 = \int_c \mathbf{f}(x, y, z)\, d\ell, \tag{3.12}$$

where $\mathbf{f}$ is a vector function.

If we write $\mathbf{f}$ in its component form in the Cartesian coordinate system,

$$\mathbf{f}(x, y, z) = f_x(x, y, z)\,\hat{x} + f_y(x, y, z)\,\hat{y} + f_z(x, y, z)\,\hat{z}, \tag{3.13}$$

and because $\hat{x}$, $\hat{y}$, $\hat{z}$ are constant vectors, we can change (3.12) into the form

$$\mathbf{I}_2 = \hat{x}\int_c f_x\, d\ell + \hat{y}\int_c f_y\, d\ell + \hat{z}\int f_z\, d\ell. \tag{3.14}$$

The three integrals contained in (3.14) are of Type I, which can be evaluated according to the method described previously. Line integrals of Types III, IV, and V are defined by

$$\mathbf{I}_3 = \int_c f(x, y, z)\, d\ell, \tag{3.15}$$

$$I_4 = \int_c \mathbf{f}(x, y, z) \cdot d\ell, \tag{3.16}$$

$$\mathbf{I}_5 = \int_c \mathbf{f}(x, y, z) \times d\ell. \tag{3.17}$$

Integrals of Type III can be resolved into three integrals, that is,

$$\mathbf{I}_3 = \hat{x}\int_c f\, dx + \hat{y}\int f\, dy + \hat{z}\int_c f\, dz. \tag{3.18}$$

The three scalar integrals in (3.18) can be evaluated by choosing a proper parameter for each integral. In fact, one can, for example, use x as a parameter for the first integral and express both y and z in terms of x. In the case of the spiral contour, if we let z be the parameter, then

$$x = a\cos\frac{2\pi}{b}z, \qquad y = a\sin\frac{2\pi}{b}z.$$

Integrals of Type IV can be converted to

$$I_4 = \int_c (f_x\, dx + f_y\, dy + f_z\, dz). \tag{3.19}$$

Again, each term in (3.19) can be evaluated by the parametric method. Integrals of Type V can be written in the form

$$\mathbf{I}_5 = \hat{x}\int_c (f_2\, dz - f_3\, dy) + \hat{y}\int (f_3\, dx - f_1\, dy) + \hat{z}\int_c (f_1\, dy - f_2\, dx). \tag{3.20}$$

All six terms in (3.20) can be evaluated in the same way. Thus, if the functions f and $\mathbf{f}$ and the differential lengths $d\ell$ and $\mathbf{d}\ell$ are expressed in a Cartesian system,

we have a systematic method to evaluate all different types of line integrals. In many cases, the curve under consideration may correspond to the intersection of two surfaces represented by

$$z = F_1(x, y), \tag{3.21}$$

$$z = F_2(x, y). \tag{3.22}$$

In that case, we can eliminate z between (3.21) and (3.22) so that

$$F_1(x, y) = F_2(x, y) \tag{3.23}$$

and then solve for x in terms of y to yield

$$x = F_3(y), \tag{3.24}$$

$$z = F_1[F_3(y), y]. \tag{3.25}$$

It is obvious that y can be used as the parameter for the curve. In many problems, it is sometimes rather difficult to find the explicit form of F_3 unless F_1 and F_2 are relatively simple functions.

If the integrals and the contour c are described in an orthogonal system other than a Cartesian system, then

$$d\ell = \left[\sum_{i=1}^{3}(h_i\, dv_i)^2\right]^{1/2} \tag{3.26}$$

and

$$d\boldsymbol{\ell} = \sum_{i=1}^{3} h_i\, dv_i\, \hat{u}_i. \tag{3.27}$$

A scalar function f is then assumed to be a function of (v_1, v_2, v_3), and a vector function $\mathbf{f}$ would be a function of both the v_i's and $\hat{u}_i$'s. Integrals of Types I and IV can be evaluated by expressing the v_i's and h_i's in terms of a single parameter, as was done previously. The integrands of integrals of Types II, III, and V contain $\hat{u}_i$'s that are, in general, not constant vectors, so they cannot be removed to the outside of these integrals. For these cases, we can transform the $\hat{u}_i$'s in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$ in the form

$$\hat{u}_i = \cos\alpha_i\, \hat{x} + \cos\beta_i\, \hat{y} + \cos\gamma_i\, \hat{z}, \qquad i = 1, 2, 3. \tag{3.28}$$

Then

$$\begin{aligned}\mathbf{f} = \sum_{i=1}^{3} f_i\, \hat{u}_i &= \sum_{i=1}^{3} f_i \left(\cos\alpha_i\, \hat{x} + \cos\beta_i\, \hat{y} + \cos\gamma_i\, \hat{z}\right) \\ &= f_x\, \hat{x} + f_y\, \hat{y} + f_z\, \hat{z},\end{aligned} \tag{3.29}$$

where

$$f_x = \sum_{i=1}^{3} f_i\, \cos\alpha_i, \tag{3.30}$$

$$f_y = \sum_{i=1}^{3} f_i \cos\beta_i, \tag{3.31}$$

$$f_z = \sum_{i=1}^{3} f_i \cos\gamma_i. \tag{3.32}$$

Afterwards, the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ can be removed to the outside of the integrals, and the remaining scalar integrals can be evaluated by the parametric method. In a later section, we will introduce a relatively simple method to find the transformation of the unit vectors from one system to another like (3.28).

3-3 Classification of Surface Integrals

A surface in a three-dimensional space, in general, is characterized by a governing equation

$$F(x, y, z) = 0, \tag{3.33}$$

in which we can select any two variables as independent and the remaining one will be a dependent variable. We assume that we can convert (3.33) into the explicit form

$$S: \qquad z = f(x, y). \tag{3.34}$$

For two neighboring points located on S, the total differential of the displacement vector between the two adjacent points can be written in the form

$$d\mathbf{R}_p = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}. \tag{3.35}$$

Only two of the Cartesian variables are independent because of the constraint stated by (3.34). If the same surface can be described by the coordinates (v_1, v_2, v_3) with unit vectors $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ and metric coefficients $(h_1, h_2, 1)$ in a Dupin system, then the total differential of a displacement on the surface can be written as (Fig. 3-2)

$$d\mathbf{R}_p = h_1\,dv_1\,\hat{u}_1 + h_2\,dv_2\,\hat{u}_2 \tag{3.36}$$

with $dv_3 = 0$.

The partial derivatives of (3.35) and (3.36) with respect to v_1 and v_2 are

$$\frac{\partial \mathbf{R}_p}{\partial v_1} = h_1\hat{u}_1 = \frac{\partial x}{\partial v_1}\hat{x} + \frac{\partial y}{\partial v_1}\hat{y} + \frac{\partial z}{\partial v_1}\hat{z}, \tag{3.37}$$

$$\frac{\partial \mathbf{R}_p}{\partial v_2} = h_2\hat{u}_2 = \frac{\partial x}{\partial v_2}\hat{x} + \frac{\partial y}{\partial v_2}\hat{y} + \frac{\partial z}{\partial v_2}\hat{z}. \tag{3.38}$$

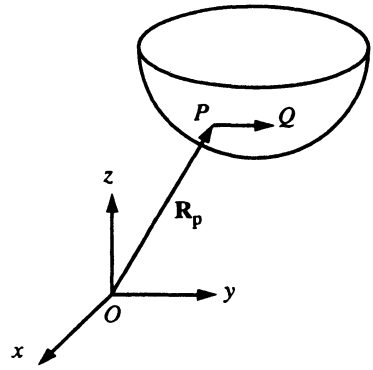


Figure 3-2 Total differential of the position vector on a surface $z = f(x, y)$ where $v_3 =$ constant.

Let the vector differential area of the surface be denoted by $d\mathbf{S}$; then

$$\begin{aligned} d\mathbf{S} &= h_1\, dv_1\, \hat{u}_1 \times h_2\, dv_2\, \hat{u}_2 \\ &= h_1 h_2\, dv_1\, dv_2\, \hat{u}_3 \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial x}{\partial v_1} & \dfrac{\partial y}{\partial v_1} & \dfrac{\partial z}{\partial v_1} \\ \dfrac{\partial x}{\partial v_2} & \dfrac{\partial y}{\partial v_2} & \dfrac{\partial z}{\partial v_2} \end{vmatrix} dv_1\, dv_2 = \mathbf{J}\, dv_1\, dv_2. \end{aligned} \tag{3.39}$$

The determinant in (3.39), denoted by $\mathbf{J}$, results from $h_1\, \hat{u}_1 \times h_2\, \hat{u}_2$. For convenience, we will call it the vector Jacobian of transformation between (x, y, z) and (v_1, v_2). If we write

$$\mathbf{J} = J_x\, \hat{x} + J_y\, \hat{y} + J_z\, \hat{z}, \tag{3.40}$$

then

$$J_x = \begin{vmatrix} \dfrac{\partial y}{\partial v_1} & \dfrac{\partial z}{\partial v_1} \\ \dfrac{\partial y}{\partial v_2} & \dfrac{\partial z}{\partial v_2} \end{vmatrix} \tag{3.41}$$

is the Jacobian of transformation between (y, z) and (v_1, v_2). Alternatively, it may be denoted by

$$J_x = \frac{\partial\,(y, z)}{\partial\,(v_1, v_2)}\,. \tag{3.42}$$

Similarly,

$$J_y = \frac{\partial (z, x)}{\partial (v_1, v_2)} \tag{3.43}$$

and

$$J_z = \frac{\partial (x, y)}{\partial (v_1, v_2)} . \tag{3.44}$$

Now, let us consider the case where the rectangular variables (x, y) are selected as (v_1, v_2); then

$$\mathbf{J}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & \dfrac{\partial z}{\partial x} \\ 0 & 1 & \dfrac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\hat{x} - \frac{\partial z}{\partial y}\hat{y} + \hat{z}. \tag{3.45}$$

The subscript 1 attached to $\mathbf{J}_1$ means that this is our first choice or first case. From (3.45), we can determine the unit normal vector $\hat{u}_3$, namely,

$$\hat{u}_3 = \frac{\mathbf{J}_1}{|\mathbf{J}_1|} = \frac{-\dfrac{\partial z}{\partial x}\hat{x} - \dfrac{\partial z}{\partial y}\hat{y} + \hat{z}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} . \tag{3.46}$$

The directional cosines of $\hat{u}_3$ are therefore given by

$$\cos \alpha_3 = \frac{-\dfrac{\partial z}{\partial x}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} , \tag{3.47}$$

$$\cos \beta_3 = \frac{-\dfrac{\partial z}{\partial y}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} , \tag{3.48}$$

$$\cos \gamma_3 = \frac{1}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} . \tag{3.49}$$

Based on (3.39) and (3.45), we find

$$\begin{aligned} dS = |d\mathbf{S}| &= |\mathbf{J}_1|\, dv_1\, dv_2 \\ &= \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} dx\, dy \\ &= \frac{1}{\cos\gamma_3}\, dx\, dy. \end{aligned} \tag{3.50}$$

Equation (3.50) can be used to find the area of a surface. As an example, let the surface be a portion of a parabola of revolution described by

$$S: \qquad z = \frac{1}{2}(x^2 + y^2), \qquad \frac{1}{2} \geq z \geq 0. \tag{3.51}$$

Then

$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2} = (x^2 + y^2 + 1)^{1/2}. \tag{3.52}$$

Hence

$$S = \iint_{S_1} dS = \iint_{S_1} (x^2 + y^2 + 1)^{1/2}\, dx\, dy, \tag{3.53}$$

where S_1 denotes the domain of integration with respect to (x, y), covering the projection of S in the x–y plane. For this particular example, it is convenient to convert (3.53) into an integral with respect to the cylindrical variables r and ϕ, that is,

$$S = \iint_{S_1} (1 + r^2)^{1/2}\, r\, dr\, d\phi = \int_0^{2\pi} \int_0^1 (1 + r^2)^{1/2}\, r\, dr\, d\phi = \frac{4\sqrt{2}\pi}{3}. \tag{3.54}$$

Here, we have used the transformation

$$dx\, dy = \frac{\partial(x, y)}{\partial(r, \phi)}\, dr\, d\phi, \tag{3.55}$$

where

$$\frac{\partial(x, y)}{\partial(r, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = r \tag{3.56}$$

with

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Equation (3.55) is a special case of (3.39) when it is applied to a plane surface corresponding to the x–y plane.

Returning now to the expression for $\mathbf{J}$ defined by (3.39), we can select either (y, z) or (z, x) as two alternative choices for (v_1, v_2); then

$$\mathbf{J}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial x}{\partial y} & 1 & 0 \\ \dfrac{\partial x}{\partial z} & 0 & 1 \end{vmatrix} = \hat{x} - \frac{\partial x}{\partial y}\hat{y} - \frac{\partial x}{\partial z}\hat{z}. \tag{3.57}$$

In this case, x is the dependent variable. The expression for the unit vector $\hat{u}_3$ is now given by

$$\hat{u}_3 = \frac{\mathbf{J}_2}{|\mathbf{J}_2|} = \frac{\hat{x} - \dfrac{\partial x}{\partial y}\hat{y} - \dfrac{\partial x}{\partial z}\hat{z}}{\left[1 + \left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2\right]^{1/2}} \tag{3.58}$$

and

$$d\mathbf{S} = \mathbf{J}_2\, dy\, dz = |\mathbf{J}_2|\, dy\, dz\, \hat{u}_3 = \frac{1}{\cos\alpha_3}\, dy\, dz\, \hat{u}_3.$$

Similarly, we have

$$\mathbf{J}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dfrac{\partial y}{\partial z} & 1 \\ 1 & \dfrac{\partial y}{\partial x} & 0 \end{vmatrix} = -\frac{\partial y}{\partial x}\hat{x} + \hat{y} - \frac{\partial y}{\partial z}\hat{z}, \tag{3.59}$$

$$\hat{u}_3 = \frac{\mathbf{J}_3}{|\mathbf{J}_3|} = \frac{-\dfrac{\partial y}{\partial x}\hat{x} + \hat{y} - \dfrac{\partial y}{\partial z}\hat{z}}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}, \tag{3.60}$$

$$d\mathbf{S} = \mathbf{J}_3\, dx\, dz = |\mathbf{J}_3|\, dx\, dz\, \hat{u}_3 = \frac{1}{\cos\beta_3}\, dx\, dz\, \hat{u}_3. \tag{3.61}$$

The directional cosines of $\hat{u}_3$, therefore, can be expressed in several different forms. Our three different choices of (v_1, v_2) yield

$$\cos\alpha_3 = \frac{-\dfrac{\partial z}{\partial x}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{1}{\left[\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1\right]^{1/2}}$$

$$= \frac{-\dfrac{\partial y}{\partial x}}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}, \tag{3.62}$$

$$\cos\beta_3 = \frac{-\dfrac{\partial z}{\partial y}}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{-\dfrac{\partial x}{\partial y}}{\left[\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1\right]^{1/2}}$$

$$= \frac{1}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}, \tag{3.63}$$

$$\cos\gamma_3 = \frac{1}{\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{-\dfrac{\partial x}{\partial z}}{\left[\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1\right]^{1/2}}$$

$$= \frac{-\dfrac{\partial y}{\partial z}}{\left[\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1\right]^{1/2}}. \tag{3.64}$$

We remind the student that

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} \tag{3.65}$$

for functions of single variables, such as $y = f(x)$, but

$$\frac{\partial y}{\partial x} \neq 1 \Big/ \frac{\partial x}{\partial y} \tag{3.66}$$

for functions of multiple variables. As an example, we consider the relations

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Then

$$r = (x^2 + y^2)^{1/2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right),$$

so that

$$\frac{\partial x}{\partial r} = \cos\phi$$

and

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2)^{1/2}} = \cos\phi.$$

That is,

$$\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x},$$

while

$$\frac{\partial x}{\partial \phi} = -r\sin\phi, \qquad \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r},$$

so that

$$\frac{\partial x}{\partial \phi} = r^2\frac{\partial \phi}{\partial x}.$$

One must therefore be very careful to distinguish between the dependent and independent variables.

Like the line integrals, there are five types of surface integrals. From now on, functions of space variables (x, y, z) or (v_1, v_2, v_3) will be denoted by $F(\mathbf{R}_p)$ or $\mathbf{F}(\mathbf{R}_p)$, where $\mathbf{R}_p$ denotes the position vector. The five types of surface integrals are as follows:

Type I:

$$I_1 = \iint_S F(\mathbf{R}_p)\, dS, \tag{3.67}$$

Type II:

$$\mathbf{I}_2 = \iint_S \mathbf{F}(\mathbf{R}_p)\, dS, \tag{3.68}$$

Type III:

$$\mathbf{I}_3 = \iint_S F(\mathbf{R}_p)\, d\mathbf{S}, \tag{3.69}$$

Type IV:

$$I_4 = \iint_S \mathbf{F}(\mathbf{R}_p) \cdot d\mathbf{S}, \tag{3.70}$$

Type V:

$$\mathbf{I}_5 = \iint_S \mathbf{F}(\mathbf{R}_p) \times d\mathbf{S}, \tag{3.71}$$

where $F(\mathbf{R}_p)$ is a scalar function of position and $\mathbf{F}(\mathbf{R}_p)$ denotes a vector function.

We assume that the surface S can be described by a governing equation of the form

$$z = f(x, y). \tag{3.72}$$

The same surface can always be considered as a normal surface ($v_3 = 0$) in a proper Dupin system with parameters (v_1, v_2), $(\hat{u}_1, \hat{u}_2)$ and metric coefficients (h_1, h_2). Treating v_1 and v_2 as two independent variables, we can write

$$x = f_1(v_1, v_2), \tag{3.73}$$

$$y = f_2(v_1, v_2), \tag{3.74}$$

$$z = f(x, y) = f[f_1(v_1, v_2), f_2(v_1, v_2)] = f_3(v_1, v_2). \tag{3.75}$$

The functions $F(\mathbf{r}_p)$ and $\mathbf{F}(\mathbf{r}_p)$ contained in (3.67)–(3.71), therefore, can be changed into functions of (v_1, v_2) for the scalar function, and of (v_1, v_2) as well as $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ for the vector function. An integral of Type I can be transformed into

$$I_1 = \iint_S F(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2, \tag{3.76}$$

which can be evaluated by the parametric method. Thus, if we let $(v_1, v_2) = (x, y)$, (3.76) becomes

$$I_1 = \iint_{S_3} F(x, y) \frac{1}{\cos \gamma_3} dx \, dy, \tag{3.77}$$

where S_3 denotes the domain of integration on the x–y plane covered by the projection of S on that plane. The execution to carry out the integration is very similar to the problem of finding the area of a curved surface, except that the integrand contains an additional function.

An integral of Type II is equivalent to

$$\begin{aligned} \mathbf{I}_2 = \hat{x} &\iint F_x(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2 \\ + \hat{y} &\iint F_y(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2 \\ + \hat{z} &\iint F_z(v_1, v_2) \, |\mathbf{J}| \, dv_1 \, dv_2. \end{aligned} \tag{3.78}$$

The three scalar integrals in (3.78) are of Type I. However, it is not necessary to use the same set of (v_1, v_2) for these integrals.

An integral of Type III, in view of (3.39) and (3.40), is equivalent to

$$\begin{aligned}\mathbf{I}_3 = \iint F(v_1, v_2)\, \mathbf{J}\, dv_1\, dv_2 = \hat{x} \iint J_x\, F(v_1, v_2)\, dv_1\, dv_2 \\ + \hat{y} \iint J_y\, F(v_1, v_2)\, dv_1\, dv_2 \\ + \hat{z} \iint J_z\, F(v_1, v_2)\, dv_1\, dv_2.\end{aligned} \tag{3.79}$$

The three scalar integrals in (3.79) are of Type I with different integrands.

An integral of Type IV is equivalent to

$$I_4 = \iint \left(J_x\, F_x + J_y\, F_y + J_z\, F_z\right) dS, \tag{3.80}$$

which belongs to Type I. Here, we have omitted the functional dependence of these functions and the Jacobians on (v_1, v_2).

Finally, an integral of Type V is equivalent to

$$\begin{aligned}\mathbf{I}_5 = \hat{x} \iint \left(J_z\, F_y - J_y\, F_z\right) dv_1\, dv_2 \\ + \hat{y} \iint \left(J_x\, F_z - J_z\, F_x\right) dv_1\, dv_2 \\ + \hat{z} \iint \left(J_y\, F_x - J_x\, F_y\right) dv_1\, dv_2.\end{aligned} \tag{3.81}$$

All of the scalar integrals in (3.81) are of Type I. In essence, an integral of Type I is the basic one; all other types of integrals can be reduced to that type. The choice of (v_1, v_2) depends greatly on the exact nature of the problem. Many integrals resulting from the formulation of physical problems may not be evaluated in a closed form. In these cases, we can seek the help of a numerical method.

3-4 Classification of Volume Integrals

There are only two types of volume integrals:

Type I:

$$I_1 = \iiint_V F(\mathbf{R}_p)\, dV, \tag{3.82}$$

Type II:

$$\mathbf{I}_2 = \iiint_V \mathbf{F}(\mathbf{R}_p)\, dV, \tag{3.83}$$

where V denotes the domain of integration, which can be either bounded or covering the entire space. We now have three independent variables. In an orthogonal system, they are (v_1, v_2, v_3). An integral of Type I, when expressed in that system, becomes

$$I_1 = \iiint_V F(v_1, v_2, v_3)\, h_1 h_2 h_3\, dv_1\, dv_2\, dv_3. \tag{3.84}$$

The choice of the proper coordinate system depends greatly on the shape of V. From the point of view of the numerical method, we can always use a rectangular coordinate system to partition the region of integration.

An integral of Type II is equivalent to

$$\mathbf{I}_2 = \hat{x} \iiint F_x\, dV + \hat{y} \iiint F_y\, dV + \hat{z} \iiint F_z\, dV. \tag{3.85}$$

The three scalar integrals in (3.85) are of Type I. We will not discuss the actual evaluation of (3.84), as the method is described in many standard books on calculus.

Chapter 4

Vector Analysis in Space

4-1 Symbolic Vector and Symbolic Vector Expressions

In this chapter, the most important one in the book, we introduce a new method in treating vector analysis called the *symbolic vector method.* The main advantages of this method are that (1) the differential expressions of the three key functions in vector analysis are derived based on one basic formula, (2) all of the integral theorems in vector analysis are deduced from one generalized theorem, (3) the commonly used vector identities are found by an algebraic method without performing any differentiation, and (4) two differential operators in the curvilinear coordinate system, called the *divergence operator* and the *curl operator* and distinct from the operator used for the gradient, are introduced. The technical meanings of the terms "divergence," "curl," and "gradient" will be explained shortly. Note that the nomenclature for some technical terms introduced in this chapter differs from the original one used in [6].

Because vector algebra is the germ of the method, we will review several essential topics covered in Chapter 1. In vector algebra, there are various products, such as

$$\begin{array}{llllll} a\mathbf{b}, & \mathbf{a}\cdot\mathbf{b}, & \mathbf{a}\times\mathbf{b}, & c(\mathbf{a}\cdot\mathbf{b}), & c(\mathbf{a}\times\mathbf{b}), \\ \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}), & \mathbf{c}\times(\mathbf{a}\times\mathbf{b}), & (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}). & & \end{array} \tag{4.1}$$

All of them have well-defined meanings in vector algebra. Here, we treat the scalar and vector quantities a, $\mathbf{a}$, b, $\mathbf{b}$, c, $\mathbf{c}$, $\mathbf{d}$ as functions of position, and they are

assumed to be distinct from each other. For purposes of identification, the functions listed in (4.1) will be referred to as *vector expressions*. A quantity like **ab** is not a vector expression, although it is a well-defined quantity in dyadic analysis, a subject already introduced in Chapter 1. For the time being, we are dealing with vector expressions only. In one case, a dyadic quantity will be involved, and its implication will be explained. All of the vector expressions listed in (4.1) are linear with respect to a single function, that is, the distributive law holds true. For example, if $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$, then

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}_1 + \mathbf{c}_2) \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{c}_1 \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{c}_2 \cdot (\mathbf{a} \times \mathbf{b}). \tag{4.2}$$

The important identities in vector algebra are listed here:

$$a\mathbf{b} = \mathbf{b}a, \tag{4.3}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}, \tag{4.4}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \tag{4.5}$$

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \tag{4.6}$$

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}. \tag{4.7}$$

The proofs of (4.6)–(4.7) are found in Section 1-2.

Now, if one of the vectors in (4.1) is replaced by a symbolic vector denoted by ∇, such as $a\nabla$, $\nabla \cdot \mathbf{b}$, $\mathbf{a} \cdot \nabla$, $\nabla \times \mathbf{b}$, $\mathbf{a} \times \nabla$, $c\nabla \cdot \mathbf{b}$, $c\mathbf{a} \cdot \nabla$, $\mathbf{c} \cdot (\nabla \times \mathbf{b})$, and so on, these expressions will be called *symbolic vector expressions* or *symbolic expressions* for short. The symbol ∇ is designated as the *symbolic vector*, or *S vector* for short. Besides ∇, a symbolic expression contains other functions, either scalar or/and vector. Thus, $c(\nabla \times \mathbf{b})$ contains one scalar, one vector, and the S vector. In general, a symbolic expression will be denoted by $T(\nabla)$. More specifically, if there is a need to identify the functions contained in $T(\nabla)$ besides ∇, we will use, for example, $T(\nabla, \tilde{a}, \tilde{b})$, which shows that there are two functions, $\tilde{a}$ and $\tilde{b}$, besides ∇. These functions may be both scalar, both vector, or one each. We use a tilde over these letters to indicate such options. The symbolic expression so created is *defined by*

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i)\, \Delta S_i}{\Delta V}, \tag{4.8}$$

where ΔS_i denotes a typical elementary area (scalar) of a surface enclosing the volume ΔV of a cell and $\hat{n}_i$ denotes the outward unit vector from ΔS_i. The running index i in (4.8) corresponds to the number of surfaces enclosing ΔV. For a cell bounded by six coordinate surfaces, i goes from 1 to 6. Because the definition of $T(\nabla)$ is independent of the choice of the coordinate system, (4.8) is *invariant to the coordinate system.* The expression on the right side of (4.8), from the analytical point of view, represents *the integral-differential transform of the symbolic expression* $T(\nabla)$, *or simply the functional transform of* $T(\nabla)$. By choosing the proper measures of ΔS_i and ΔV in a certain coordinate system, one can find the differential expression of $T(\nabla)$ based on (4.8).

There are several important characteristics of (4.8) that must be emphasized. In the first place, because all symbolic expressions are generated by well-defined vector expressions, they are, in general, involving simple multiplication, scalar product, and vector product. For example, if we replace the vector function $\mathbf{d}$ in the vector expression $a\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})$ by ∇, we would create a symbolic expression of the form

$$T(\nabla) = a\mathbf{b} \cdot (\mathbf{c} \times \nabla), \tag{4.9}$$

where a is a scalar function, $\mathbf{b}$ and $\mathbf{c}$ are vector functions; then, we have a multiplication between a and the rest of $T(\nabla)$; a scalar product between $\mathbf{b}$ and $(\mathbf{c} \times \nabla)$; and finally a vector product between $\mathbf{c}$ and ∇, or the S vector. The vector product $\mathbf{c} \times \nabla$ is not a true product because ∇ is a symbolic vector, or a dummy vector, not a function. However, the expression leads to a function $T(\hat{n}_j)$ in (4.8) in the form

$$T(\hat{n}_j) = a\mathbf{b} \cdot (\mathbf{c} \times \hat{n}_j). \tag{4.10}$$

This is a well-defined function in which $\mathbf{c} \times \hat{n}_j$ is truly a vector product. This function is used to find the differential expression of $T(\nabla)$ based on the right-side term of (4.8). This description consists of the most important concept in the method of symbolic vector: *the multiplication, the scalar product, and the vector product contained in* $\mathrm{T}(\nabla)$ *are executed in the function* $\mathrm{T}(\hat{\mathrm{n}}_\mathrm{j})$.

Another characteristic of (4.8) deals with the algebraic property of $T(\nabla)$. Because $T(\hat{n})$ is a well-defined function for a given $T(\nabla)$, the vector identities applicable to $T(n_i)$ are shared by $T(\nabla)$. In the previous example,

$$T(\hat{n}_i) = a\mathbf{b} \cdot (\mathbf{c} \times \hat{n}_i) = -\mathbf{b} \cdot (\hat{n}_i \times \mathbf{c})a = a\hat{n}_i \cdot (\mathbf{b} \times \mathbf{c}), \tag{4.11}$$

which implies

$$T(\nabla) = a\mathbf{b} \cdot (\mathbf{c} \times \nabla) = -\mathbf{b} \cdot (\nabla \times \mathbf{c})a = a\nabla \cdot (\mathbf{b} \times \mathbf{c}). \tag{4.12}$$

This property can be stated in a lemma:

Lemma 4.1. For any symbolic expression $T(\nabla)$ generated from a valid vector expression, we can treat the symbolic vector ∇ in that expression as a vector and all of the algebraic identities in vector algebra are applicable.

For example, we have the vector identities listed in (4.3) to (4.7); then, according to Lemma 4.1, the following relations hold true:

$$a\nabla = \nabla a, \tag{4.13}$$

$$\nabla \cdot \mathbf{a} = \mathbf{a} \cdot \nabla, \tag{4.14}$$

$$\nabla \times \mathbf{a} = -\mathbf{a} \times \nabla, \tag{4.15}$$

$$\mathbf{b} \cdot (\mathbf{a} \times \nabla) = \nabla \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\nabla \times \mathbf{b}), \tag{4.16}$$

$$\begin{aligned} \nabla \times (\mathbf{a} \times \mathbf{b}) &= (\mathbf{b} \times \mathbf{a}) \times \nabla \\ &= (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} \\ &= (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}. \end{aligned} \tag{4.17}$$

4-2 Differential Formulas of the Symbolic Expression in the Orthogonal Curvilinear Coordinate System for Gradient, Divergence, and Curl

Orthogonal coordinate systems are the most useful systems in formulating problems in physics and engineering. We will therefore derive the relevant differential expressions in OCS first based on the method of symbolic vector and treat the formulation for the general curvilinear system later. Actually, it is more efficient to do GCS first and consider OCS as special cases. For expository purposes, however, it is desirable to follow the proposed order because most readers are likely to have some acquaintance with vector analysis in orthogonal systems from a course in college physics or calculus. The mathematics in GCS would distract from the main feature of the new method at this stage.

To evaluate the integral-differential expression that defines $T(\nabla)$ in (4.8) in the OCS, we consider a volume bounded by six coordinate surfaces; then the running index in (4.8) goes from $i = 1$ to $i = 6$. Because $T(\hat{n}_i)$ is linear with respect to $\hat{n}_i$, we can combine $\hat{n}_i$ with ΔS_i to form a vector differential area. If that area corresponds to a segment $\hat{n}_i\,\Delta S$ on a coordinate surface with v_i being constant, then

$$\hat{n}_i\,\Delta S = h_j h_k\,\Delta v_j\,\Delta v_k\,\hat{u}_i = \frac{\Omega}{h_i}\,\Delta v_j\,\Delta v_k\,\hat{u}_i, \tag{4.18}$$

where $i, j, k = 1, 2, 3$ in cyclic order and $\Omega = h_1 h_2 h_3$. The h_i's are the metric coefficients of any, yet unspecified, orthogonal curvilinear system. For a cell bounded by six surfaces located at $v_i \pm \Delta v_i/2$, its volume, denoted by ΔV, is

$$\Delta V = \Omega\,\Delta v_i\,\Delta v_j\,\Delta v_k.$$

We can separate the sum in (4.8) into two groups:

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{3} \left[T\left(\frac{\Omega}{h_i}\,\hat{u}_i\right)_{v_i + \Delta v_i/2} - T\left(\frac{\Omega}{h_i}\,\hat{u}_i\right)_{v_i - \Delta v_i/2} \right] \Delta v_j\,\Delta v_k}{\Omega\,\Delta v_i\,\Delta v_j\,\Delta v_k}.$$

The limit yields

$$T(\nabla) = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i}\, T\left(\frac{\Omega}{h_i}\,\hat{u}_i\right). \tag{4.19}$$

Equation (4.19) is the differential expression for $T(\nabla)$ in any OCS. In (4.19), the summation goes from $i = 1$ to $i = 3$ and the metric coefficients and hence the parameter Ω are, in general, functions of v_i, the coordinate variables. Equation (4.19) is perhaps the most important formula in the method of symbolic vector when it is applied to an OCS.

We now consider the three simplest but also most basic forms of $T(\nabla)$, in which $T(\nabla)$ contains only one function besides ∇. There are only three

possibilities, namely,

$$T_1(\nabla) = \nabla f \quad \text{or} \quad f\nabla,$$
$$T_2(\nabla) = \nabla \cdot \mathbf{F} \quad \text{or} \quad \mathbf{F} \cdot \nabla,$$
$$T_3(\nabla) = \nabla \times \mathbf{F} \quad \text{or} \quad -\mathbf{F} \times \nabla,$$

where f denotes a scalar function and $\mathbf{F}$, a vector function. They are the same as the ones in (4.13), (4.14), and (4.15). The differential expressions of these functions can be found based on (4.19).

1. Gradient of a scalar function

When $T(\nabla) = T_1(\nabla) = \nabla f = f\nabla$, (4.19) yields

$$\begin{aligned} \nabla f &= \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u}_i f \right) \\ &= \frac{1}{\Omega} \sum_i \left[\frac{\Omega}{h_i} \hat{u}_i \frac{\partial f}{\partial v_i} + f \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u}_i \right) \right]. \end{aligned} \tag{4.20}$$

The last sum vanishes because of the closed surface theorem stated by (2.62) in OCS, namely,

$$\sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u}_i \right) = 0. \tag{4.21}$$

Hence

$$\nabla f = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial v_i}. \tag{4.22}$$

The differential expression thus derived is called the *gradient* of the scalar function f. It is a vector function and it will be denoted by ∇f, where ∇ is a differential operator defined by

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial}{\partial v_i}. \tag{4.23}$$

It will be called a *gradient operator*; thus,

$$\nabla f = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial v_i}. \tag{4.24}$$

The linguistic notation used by some authors is grad f. The symbol ∇ is also called *del* or *nabla* or *Hamilton operator*.

2. Divergence of a vector function

When $T(\nabla) = T_2(\nabla) = \nabla \cdot \mathbf{F} = \mathbf{F} \cdot \nabla$, (4.19) yields

$$\nabla \cdot \mathbf{F} = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u} \cdot \mathbf{F} \right) = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} F_i \right). \tag{4.25}$$

The differential expression thus derived is called the *divergence* of the vector function **F**. There is another functional form of this function that can be obtained from the second term of (4.25). We split the derivative with respect to v_i into two terms, just as the splitting in (4.20), that is,

$$\frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}\cdot\mathbf{F}\right) = \frac{1}{\Omega}\sum_i \left[\frac{\Omega}{h_i}\hat{u}_i\cdot\frac{\partial \mathbf{F}}{\partial v_i} + \mathbf{F}\cdot\frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right)\right]. \tag{4.26}$$

The second term vanishes as a result of (4.21); hence

$$\nabla\cdot\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial v_i}. \tag{4.27}$$

Now, we introduced a divergence operator, denoted by ∇, and defined by

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial}{\partial v_i}. \tag{4.28}$$

Then,

$$\sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial v_i} = \nabla\mathbf{F}. \tag{4.29}$$

It can be verified that by evaluating the derivatives of **F** with respect to v_i, taking into due consideration that the unit vectors associated with **F** are functions of the coordinate variables and they can be expressed in terms of various $\hat{u}_i$'s according to (2.58) and (2.61), the function at the left of (4.29) reduces to (4.25) as it should be. The operand of a divergence operator must be a vector. Later on, we will show that it can also be applied to a dyadic. Comparing the divergence operator with the gradient operator defined by (4.23), we see that there is a dot, the scalar product symbol, between $\hat{u}_i/h_i$ and the partial derivative sign. It is this morphology that prompts us to use the symbol ∇ for the divergence operator. We must emphasize that there is no analytical relation between the gradient operator ∇ and the divergence operator ∇. They are distinct operators. In the history of vector analysis, there are two well-established notations for the divergence. One is the linguistic notation denoted by div **F**. The connotation of this notation is obvious. Another notation is $\nabla\cdot\mathbf{F}$, which was due to Gibbs, one of the founders of vector analysis. Unfortunately, some later authors treat $\nabla\cdot\mathbf{F}$ as the scalar product or "formal" scalar product between ∇ and **F** in the rectangular coordinate system, which is not a correct interpretation. The contradictions that resulted from the improper use of ∇ are discussed in the last chapter of this book. The evidence and the logic described therein strengthens our decision to adopt $\nabla\mathbf{F}$ as the new notation for the divergence. In the original edition of this book [7], we kept Gibbs's notations for the divergence and the curl, a function to be introduced

shortly. The use of Gibbs's notations does impede the understanding of the symbolic expressions $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$. We have therefore made a bold move in this edition by abandoning a long-established tradition. With this much discussion of the new operational notation for the divergence, we consider the last case of the triad.

3. **Curl of a vector function**

When $T(\nabla) = T_3(\nabla) = \nabla \times \mathbf{F} = -\mathbf{F} \times \nabla$, (4.19) yields

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u}_i \times \mathbf{F} \right) \\ &= \frac{1}{\Omega} \left[\frac{\partial}{\partial v_1} (h_2 h_3 \hat{u}_1 \times \mathbf{F}) + \frac{\partial}{\partial v_2} (h_1 h_3 \hat{u}_2 \times \mathbf{F}) \right. \\ &\quad \left. + \frac{\partial}{\partial v_3} (h_1 h_2 \hat{u}_3 \times \mathbf{F}) \right] \\ &= \frac{1}{\Omega} \left[\frac{\partial}{\partial v_1} h_2 h_3 (F_2 \hat{u}_3 - F_3 \hat{u}_2) + \frac{\partial}{\partial v_2} h_1 h_3 (-F_1 \hat{u}_3 + F_3 \hat{u}_1) \right. \\ &\quad \left. + \frac{\partial}{\partial v_3} h_1 h_2 (F_1 \hat{u}_2 - F_2 \hat{u}_1) \right].\end{aligned} \tag{4.30}$$

Each term in (4.30) can be split into two parts; for example,

$$\frac{1}{\Omega} \frac{\partial}{\partial v_1} (h_2 h_3 F_2 \hat{u}_3) = \frac{1}{\Omega} \left[h_2 F_2 \frac{\partial (h_3 \hat{u}_3)}{\partial v_1} + h_2 \hat{u}_3 \frac{\partial (h_2 F_2)}{\partial v_1} \right].$$

As a result of (2.56), one finds that all the terms containing the derivatives of $h_i \hat{u}_i$ cancel each other. The remaining terms are given by

$$\nabla \times \mathbf{F} = \frac{1}{\Omega} \sum_i h_i \hat{u}_i \left[\frac{\partial (h_k F_k)}{\partial v_j} - \frac{\partial (h_j F_j)}{\partial v_k} \right], \tag{4.31}$$

where $i, j, k = 1, 2, 3$ in cyclic order. This function is called the *curl* of $\mathbf{F}$. Like the divergence, we can find an operational form of this function. Using the first line of (4.30), we have

$$\frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u} \times \mathbf{F} \right) = \frac{1}{\Omega} \sum_i \left[\frac{\Omega}{h_i} \hat{u}_i \times \frac{\partial \mathbf{F}}{\partial v_i} - \mathbf{F} \times \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \hat{u}_i \right) \right].$$

The last term vanishes because of (4.21); hence

$$\nabla \times \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i}. \tag{4.32}$$

Now we introduce a curl operator, denoted by ∇, and defined by

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}. \tag{4.33}$$

The operand of this operator must be a vector. The curl of a vector function **F** therefore can be written in the form

$$\sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} = \nabla \mathbf{F}. \tag{4.34}$$

It can be verified that by evaluating the derivatives of **F** with respect to v_i, we can recover the differential expression of $\nabla \mathbf{F}$ given by (4.31). The linguistic notation for the curl is simply curl **F**, used mostly in English-speaking countries, and rot **F** in Germany. Gibbs's notation for this function is $\nabla \times \mathbf{F}$. As with the divergence, some authors treat $\nabla \times \mathbf{F}$ as a vector product or a "formal" vector product between ∇ and **F**, which is a misinterpretation. The new notation avoids this possibility. The location of the cross sign ($\times$) in the left side of (4.34) suggests to us to adopt the notation ∇ for this function. In summary, we have derived the differential expressions of three basic functions in vector analysis in the OCS based on a symbolic expression defined by (4.8); they are

$$\nabla f = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial v_i} \quad \text{(gradient)}, \tag{4.35}$$

$$\nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial v_i} = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} F_i \right) \quad \text{(divergence)}, \tag{4.36}$$

$$\begin{aligned} \nabla \mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} \\ &= \frac{1}{\Omega} \sum_i h_i \hat{u}_i \left[\frac{\partial (h_k F_k)}{\partial v_j} - \frac{\partial (h_j F_j)}{\partial v_k} \right] \quad \text{(curl)}, \end{aligned} \tag{4.37}$$

where $\Omega = h_1 h_2 h_3$. In the rectangular system, all the metric coefficients are equal to unity; hence

$$\nabla f = \sum_i \hat{x}_i \frac{\partial f}{\partial x_i}, \tag{4.38}$$

$$\nabla \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} = \sum_i \frac{\partial F_i}{\partial x_i}, \tag{4.39}$$

$$\nabla \mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i} = \sum_i \hat{x}_i \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} \right). \tag{4.40}$$

4-3 Invariance of the Differential Operators

From the definition of $T(\nabla)$ given by (4.8), the differential expressions evaluated from that formula should be independent of the coordinate system in which the differential expressions are derived, such as the gradient, the divergence, and the

curl. It is desirable, however, to show analytically that such an invariance is indeed true.

We consider the expressions for the divergence in two orthogonal curvilinear systems:

$$\nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial v_i} \tag{4.41}$$

and

$$\nabla' \mathbf{F}' = \sum_j \frac{\hat{u}'_j}{h'_j} \cdot \frac{\partial \mathbf{F}'}{\partial v'_j} . \tag{4.42}$$

All the primed functions and the operator ∇' are defined with respect to v'_j. Because $\mathbf{F}$ and $\mathbf{F}'$ are the same vector function expressed in two different systems, it is sufficient to show the invariance of the two divergence operators. The total differential of a position vector is given by

$$d\mathbf{R}_p = \sum_i h_i \hat{u}_i \, dv_i = \sum_j h'_j \hat{u}'_j \, dv'_j. \tag{4.43}$$

Thus,

$$h'_k \, dv'_k = \sum_i h_i (\hat{u}_i \cdot \hat{u}'_k) \, dv_i, \qquad k = 1, 2, 3$$

or

$$h'_j \, dv'_j = \sum_i h_i (\hat{u}_i \cdot \hat{u}'_j) \, dv_i, \qquad j = 1, 2, 3. \tag{4.44}$$

Hence

$$\frac{\partial v'_j}{\partial v_i} = \frac{h_i}{h'_j} (\hat{u}_i \cdot \hat{u}'_j). \tag{4.45}$$

Now,

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} = \sum_{i,j} \frac{\hat{u}_i}{h_i} \cdot \frac{\partial v'_j}{\partial v_i} \frac{\partial}{\partial v'_j} . \tag{4.46}$$

In view of (4.45), we obtain

$$\nabla = \sum_{i,j} \frac{\hat{u}_i}{h'_j} (\hat{u}_i \cdot \hat{u}'_j) \cdot \frac{\partial}{\partial v'_j} , \tag{4.47}$$

but

$$\sum_i \hat{u}_i (\hat{u}_i \cdot \hat{u}'_j) = \bar{\bar{I}} \cdot \hat{u}'_j = \hat{u}'_j, \tag{4.48}$$

where $\bar{\bar{I}}$ denotes the idemfactor introduced in Chapter 1, now in terms of the dyads $\hat{u}_i\hat{u}_i$. Another interpretation of (4.48) is to write

$$\sum_i \hat{u}_i(\hat{u}_i \cdot \hat{u}'_j) = \sum_i \hat{u}_i \cos\alpha_{ij} = \hat{u}'_j, \tag{4.49}$$

where $\cos\alpha_{ij}$ denotes the directional cosines between $\hat{u}_i$ and $\hat{u}'_j$. Substituting (4.48) into (4.47), we obtain

$$\nabla = \sum_j \frac{\hat{u}'_j}{h'_j} \cdot \frac{\partial}{\partial v'_i} = \nabla'. \tag{4.50}$$

The proofs of the invariance of ∇ and ∇ follow the same steps.

The invariance of the differential operators also ascertains the nature of these functions. To show their characteristics, let the primed and the unprimed coordinates represent two rectangular coordinate systems rotating with respect to each other as the ones formed in Section 1-3. The invariance of the gradient operator means

$$\sum_i \hat{x}_i \frac{\partial f}{\partial x_i} = \sum_j \hat{x}'_j \frac{\partial f}{\partial x'_j}.$$

By taking a scalar product of this equation with $\hat{x}'_k$, we find

$$\frac{\partial f}{\partial x'_k} = \sum_i a_{ki} \frac{\partial f}{\partial x_i}, \qquad k = 1, 2, 3.$$

These relations show that the components of the gradient obey the rules of transform of a polar vector.

The invariance of the divergence operator means

$$\sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} = \sum_j \hat{x}'_j \cdot \frac{\partial \mathbf{F}'}{\partial x'_j},$$

or

$$\sum_i \frac{\partial F_i}{\partial x_i} = \sum_j \frac{\partial F'_j}{\partial x'_j}.$$

The divergence of a vector function, therefore, is an invariant scalar.

The invariance of the curl operator has a more intricate implication. In the first place, the invariance of $\nabla \mathbf{F}$ means

$$\sum_i \hat{x}_i \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k}\right) = \sum_p \hat{x}'_p \left(\frac{\partial F'_r}{\partial x'_q} - \frac{\partial F'_q}{\partial x'_r}\right),$$

where $i, j, k = 1, 2, 3$ and $p, q, r = 1, 2, 3$ in cyclic order. By taking the scalar product of the previous equation with $\hat{x}'_\ell$, we obtain

$$\frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} = \sum_i a_{\ell i} \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k}\right), \qquad \ell = 1, 2, 3,$$

where ℓ, m, n as well as $i, j, k = 1, 2, 3$ in cyclic order. If we denote

$$\frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} = C'_\ell,$$

$$\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} = C_i,$$

then

$$C'_\ell = \sum_i a_{\ell i} C_i,$$

which shows C'_ℓ and C_i transform like a polar vector. On the other hand, if we denote

$$\frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} = C'_{mn},$$

$$\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} = C_{jk},$$

then

$$\begin{aligned} C'_{mn} &= \frac{\partial F'_n}{\partial x'_m} - \frac{\partial F'_m}{\partial x'_n} \\ &= \sum_j \frac{\partial F'_n}{\partial x_j}\frac{\partial x_j}{\partial x'_m} - \sum_j \frac{\partial F'_m}{\partial x_j}\frac{\partial x_j}{\partial x'_n} \\ &= \sum_j a_{mj}\frac{\partial}{\partial x_j}\sum_k a_{nk}F_k - \sum_j a_{nj}\frac{\partial}{\partial x_j}\sum_k a_{mk}F_k \\ &= \sum_{j,k}(a_{mj}a_{nk} - a_{nj}a_{mk})\frac{\partial F_k}{\partial x_j} \\ &= \sum_{j,k} a_{mj}a_{nk}\left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k}\right). \end{aligned}$$

Hence

$$C'_{mn} = \sum_{j,k} a_{mj}a_{nk}C_{jk}.$$

The preceding relation shows that C'_{mn} and C_{jk} transform according to the rule of an antisymmetric tensor in a three-dimensional space. The tensor is antisymmetric because

$$C'_{mn} = -C'_{nm}, \qquad C_{jk} = -C_{kj}$$

and

$$C'_{mm} = 0, \qquad C_{jj} = 0.$$

In summary, the curl of $\mathbf{F}$ is basically an antisymmetric tensor, but its three distinct vector components C_i and C'_ℓ also transform like a polar vector. Its property therefore resembles an axial vector. However, one must not treat $\nabla \mathbf{F}$ as the vector product between ∇ and $\mathbf{F}$, which is a misleading interpretation; it is fully explained in Chapter 8.

4-4 Differential Formulas of the Symbolic Expression in the General Curvilinear System

The integral-differential expression that defines $T(\nabla)$ in (4.8) will now be evaluated to obtain a differential expression for $T(\nabla)$ in GCS, which was introduced in Chapter 2. We consider a volume bounded by six coordinate surfaces; then the running index in (4.8) goes from $i = 1$ to $i = 6$. Because $T(\hat{n}_i)$ is linear with respect to $\hat{n}_i$, we can combine ΔS_i with $\hat{n}_i$ to form a vector differential area. If that area corresponds to a segment $\hat{n}_i \, \Delta S_i$ on a coordinate surface with v_i being constant in GCS, then

$$\hat{n}_i \, \Delta S_i = \Delta \mathbf{S}_i = \mathbf{p}_j \times \mathbf{p}_k \, \Delta\omega^j \, \Delta\omega^k = \Lambda \mathbf{r}^i \, \Delta\omega^j \, \Delta\omega^k. \tag{4.51}$$

For a cell bounded by six surfaces located at $\omega^i \pm (\Delta\omega^i/2)$ with $i = 1, 2, 3$, its volume would be equal to

$$\Delta V = \Lambda \, \Delta\omega^i \, \Delta\omega^j \, \Delta\omega^k.$$

We can separate the surface sum in (4.8) into two groups:

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{3} \left[T(\Lambda \mathbf{r}^i)_{\omega^i + \Delta\omega^i/2} - T(\Lambda \mathbf{r}^i)_{\omega^i - \Delta\omega^i/2} \right] \Delta\omega^j \, \Delta\omega^k}{\Lambda \, \Delta\omega^i \, \Delta\omega^j \, \Delta\omega^k}.$$

The limit yields

$$T(\nabla) = \frac{1}{\Lambda} \sum_{i=1}^{3} \frac{\partial}{\partial \omega^i} T(\Lambda \mathbf{r}^i). \tag{4.52}$$

Equation (4.52) is the differential expression for $T(\nabla)$ in GCS. From now on, it is understood that the summation index i goes from 1 to 3 unless specified otherwise. We now consider the three symbolic expressions ∇f, $\nabla \cdot \mathbf{F}$, and $\nabla \cdot \mathbf{F}$ in GCS.

Because $T(\Lambda \mathbf{r}^i)$ is linear with respect to $\Lambda \mathbf{r}^i$, there are also three possible combinations of $\Lambda \mathbf{r}^i$ with the remaining part of $T(\Lambda \mathbf{r}^i)$. In a rather compact notation, we can write $T(\Lambda \mathbf{r}^i)$ for the three cases in the form

$$T(\Lambda \mathbf{r}^i) = \Lambda \mathbf{r}^i * \tilde{f}, \tag{4.53}$$

where $*$ represents either a null (absent) when $\tilde{f}$ is a scalar function, or a dot (scalar product symbol), or a cross (vector product symbol) when $\tilde{f}$ is a vector function. We list these cases in Table 4-1. Substituting (4.53) into (4.52), we obtain

$$T(\nabla) = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda \mathbf{r}^i * \tilde{f}) = \frac{1}{\Lambda} \sum_i \left[\Lambda \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i} + \frac{\partial (\Lambda \mathbf{r}^i)}{\partial \omega^i} * \tilde{f} \right]. \tag{4.54}$$

Table 4-1: The Three Simplest Forms of $T(\Lambda \mathbf{r}^i)$

Case	$T(\nabla, \tilde{f})$	$T(\Lambda \mathbf{r}^i)$	$*$	$\tilde{f}$
1	$\nabla f = f\nabla$	$\Lambda \mathbf{r}^i f$	null	f, scalar
2	$\nabla \cdot \mathbf{F} = \mathbf{F} \cdot \nabla$	$\Lambda \mathbf{r}^i \cdot \mathbf{F}$	$\cdot$	$\mathbf{F}$, vector
3	$\nabla \times \mathbf{F} = -\mathbf{F} \times \nabla$	$\Lambda \mathbf{r}^i \times \mathbf{F}$	$\times$	$\mathbf{F}$, vector

The second term in (4.54) vanishes as a result of the closed surface theorem stated by (2.27), hence

$$T(\nabla) = \sum_i \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i}. \tag{4.55}$$

We now treat the cases listed in Table 4-1 individually.

1. **The gradient of a scalar function**

   In this case, we have

   $$T(\nabla) = \nabla f = f\nabla, \qquad \tilde{f} = f,$$

   and $*$ is null or absent in (4.55); then

   $$\nabla f = \sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i}. \tag{4.56}$$

   This function is the *gradient* of f in GCS. The gradient operator is now defined by

   $$\nabla = \sum_i \mathbf{r}^i \frac{\partial}{\partial \omega^i} \quad \text{(gradient operator)}. \tag{4.57}$$

   Thus,

   $$\sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i} = \nabla f. \tag{4.58}$$

   In summary, the gradient of a scalar function f in GCS is represented by

   $$\nabla f = \lim_{\Delta V \to 0} \frac{\sum_i (\hat{n}_i f)\, \Delta S_i}{\Delta V} = \sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i}. \tag{4.59}$$

2. **The divergence**

   In this case, we have

   $$T(\nabla) = \nabla \cdot \mathbf{F} = \mathbf{F} \cdot \nabla, \qquad T(\Lambda \mathbf{r}^i) = \Lambda \mathbf{r}^i \cdot \mathbf{F}, \qquad * = \cdot, \qquad \tilde{f} = \mathbf{F}.$$

   Substituting these quantities into (4.55), we obtain

   $$\nabla \cdot \mathbf{F} = \sum_i \mathbf{r}^i \cdot \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.60}$$

The divergence operator now has the form

$$\nabla = \sum_i \mathbf{r}^i \cdot \frac{\partial}{\partial \omega^i} \quad \text{(divergence operator)}. \tag{4.61}$$

Hence

$$\sum_i \mathbf{r}^i \cdot \frac{\partial \mathbf{F}}{\partial \omega^i} = \nabla \mathbf{F}. \tag{4.62}$$

In summary, the divergence of a vector function $\mathbf{F}$ in GCS is represented by

$$\nabla \mathbf{F} = \lim_{\Delta V \to 0} \frac{\sum_i (\hat{n}_i \cdot \mathbf{F})\, \Delta S_i}{\Delta V} = \sum_i \mathbf{r}^i \cdot \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.63}$$

It will be recalled that the operational form of this function was originally derived from the following expression:

$$\nabla \mathbf{F} = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda \mathbf{r}^i \cdot \mathbf{F}). \tag{4.64}$$

According to (2.11),

$$\mathbf{r}^i \cdot \mathbf{F} = g^i,$$

where g^i denotes the reciprocal components of $\mathbf{F}$ in GCS; hence the divergence of $\mathbf{F}$ can be written in the form

$$\nabla \mathbf{F} = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda g^i). \tag{4.65}$$

The differentiations are now applied to the scalar functions Λg^i; it is no longer applied to the full vector function $\mathbf{F}$ as in (4.62) or (4.63).

3. The curl

In this case, we have

$$T(\nabla) = \nabla \times \mathbf{F} = -\mathbf{F} \times \nabla, \qquad T(\Lambda \mathbf{r}^i) = \Lambda \mathbf{r}^i \times \mathbf{F}, \qquad * = \times, \qquad \tilde{f} = \mathbf{F}.$$

Then

$$\nabla \times \mathbf{F} = \sum_i \mathbf{r}^i \times \frac{\partial \mathbf{F}}{\partial \omega^i}.$$

The curl operator now has the form

$$\nabla = \sum_i \mathbf{r}^i \times \frac{\partial}{\partial \omega^i} \quad \text{(curl operator)}. \tag{4.66}$$

Hence

$$\sum_i \mathbf{r}^i \times \frac{\partial \mathbf{F}}{\partial \omega^i} = \nabla \mathbf{F}. \tag{4.67}$$

The curl of a vector function in GCS is therefore represented by

$$\nabla \mathbf{F} = \lim_{\Delta V \to 0} \frac{\sum_i (\hat{n}_i \times \mathbf{F})\, \Delta S_i}{\Delta V} = \sum_i \mathbf{r}^i \times \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.68}$$

The operational form of curl $\mathbf{F}$ was originally derived from the following expression:

$$\nabla \mathbf{F} = \frac{1}{\Lambda} \sum_i \frac{\partial}{\partial \omega^i} (\Lambda \mathbf{r}^i \times \mathbf{F}). \tag{4.69}$$

The vector product in (4.69) can be expressed in terms of the primary components of $\mathbf{F}$ in GCS. We start with

$$\mathbf{F} = \sum_j f_j \mathbf{r}^j, \tag{4.70}$$

where f_j denotes a primary component of $\mathbf{F}$ as stated by (2.10); then

$$\mathbf{r}^i \times \mathbf{F} = \sum_j f_j \mathbf{r}^i \times \mathbf{r}^j, \qquad i = 1, 2, 3. \tag{4.71}$$

More specifically,

$$\mathbf{r}^1 \times \mathbf{F} = f_2 \mathbf{r}^1 \times \mathbf{r}^2 + f_3 \mathbf{r}^1 \times \mathbf{r}^3 = \frac{1}{\Lambda}(f_2 \mathbf{p}_3 - f_3 \mathbf{p}_2),$$

$$\mathbf{r}^2 \times \mathbf{F} = f_1 \mathbf{r}^2 \times \mathbf{r}^1 + f_3 \mathbf{r}^2 \times \mathbf{r}^3 = \frac{1}{\Lambda}(-f_1 \mathbf{p}_3 + f_3 \mathbf{p}_1),$$

$$\mathbf{r}^3 \times \mathbf{F} = f_1 \mathbf{r}^3 \times \mathbf{r}^1 + f_2 \mathbf{r}^3 \times \mathbf{r}^2 = \frac{1}{\Lambda}(f_1 \mathbf{p}_2 - f_2 \mathbf{p}_1),$$

where $\mathbf{p}_i$ and $\mathbf{f}_i$ denote the primary vectors and the primary components, respectively, of $\mathbf{F}$. Substituting these expressions into (4.69), we have

$$\nabla \mathbf{F} = \frac{1}{\Lambda} \left[\frac{\partial}{\partial \omega^1}(f_2 \mathbf{p}_3 - f_3 \mathbf{p}_2) + \frac{\partial}{\partial \omega^2}(f_3 \mathbf{p}_1 - f_1 \mathbf{p}_3) + \frac{\partial}{\partial \omega^3}(f_1 \mathbf{p}_2 - f_2 \mathbf{p}_1) \right]. \tag{4.72}$$

The derivative of the first term in (4.72) consists of two parts:

$$\frac{\partial}{\partial \omega^1}(f_2 \mathbf{p}_3) = f_2 \frac{\partial \mathbf{p}_3}{\partial \omega^1} + \mathbf{p}_3 \frac{\partial f_2}{\partial \omega^1}. \tag{4.73}$$

The derivative of the last term in (4.72) gives

$$\frac{\partial}{\partial \omega^3}(-f_2 \mathbf{p}_1) = -f_2 \frac{\partial \mathbf{p}_1}{\partial \omega^3} - \mathbf{p}_1 \frac{\partial f_2}{\partial \omega^3}. \tag{4.74}$$

According to (2.2),

$$\mathbf{p}_3 = \frac{\partial \mathbf{R}_p}{\partial \omega^3} \quad \text{and} \quad \mathbf{p}_1 = \frac{\partial \mathbf{R}_p}{\partial \omega^1};$$

hence

$$\frac{\partial^2 \mathbf{R}_p}{\partial \omega^1 \partial \omega^3} = \frac{\partial \mathbf{p}_3}{\partial \omega^1} = \frac{\partial \mathbf{p}_1}{\partial \omega^3}.$$

The first two terms at the right sides of (4.73) and (4.74) are therefore equal and opposite in sign. Six terms in (4.69) involving the derivatives of the primary vectors cancel each other in this manner; the net result yields

$$\nabla \mathbf{F} = \frac{1}{\Lambda}\left[\mathbf{p}_1\left(\frac{\partial f_3}{\partial \omega^2} - \frac{\partial f_2}{\partial \omega^3}\right) + \mathbf{p}_2\left(\frac{\partial f_1}{\partial \omega^3} - \frac{\partial f_3}{\partial \omega^1}\right) + \mathbf{p}_3\left(\frac{\partial f_2}{\partial \omega^1} - \frac{\partial f_1}{\partial \omega^2}\right)\right],$$

or

$$\nabla \mathbf{F} = \frac{1}{\Lambda}\sum_i \mathbf{p}_i\left(\frac{\partial f_k}{\partial \omega^j} - \frac{\partial f_j}{\partial \omega^k}\right) \tag{4.75}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. The expressions for ∇f, $\nabla \mathbf{F}$, and $\nabla \mathbf{F}$ given by (4.60), (4.65), and (4.75) have previously been derived by Stratton [5, p. 44] [based on the variation of f (total differential of f) for ∇f], Gauss's theorem for $\nabla \mathbf{F}$, and Stokes's theorem for $\nabla \mathbf{F}$. We have not yet touched upon these theorems. Our derivation is based on only one formula, namely, the differential expression of the symbolic expression $T(\nabla)$ as stated by (4.8). In summary, the three differential operators in GCS have the forms

$$\nabla = \sum_i \mathbf{r}^i \frac{\partial}{\partial \omega^i} \quad \text{(gradient operator)},$$

$$\nabla = \sum_i \mathbf{r}^i \cdot \frac{\partial}{\partial \omega^i} \quad \text{(divergence operator)},$$

$$\nabla = \sum_i \mathbf{r}^i \times \frac{\partial}{\partial \omega^i} \quad \text{(curl operator)}.$$

The three operators can be condensed into one formula:

$$\nabla = \sum_i \mathbf{r}^i * \frac{\partial}{\partial \omega^i}, \tag{4.76}$$

where $*$ represents a null, a dot, or a cross. We would like to emphasize that these operators are independent of each other; in other words, they are distinctly different differential operators, and they are invariant with respect to the choice of the coordinate system. We leave the proof in GCS as an exercise for the reader. For the three functions, they can be written in a compact form

$$\nabla \tilde{f} = \sum_i \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i}, \tag{4.77}$$

where $\tilde{f}$ can be a scalar (for the gradient) or vector (for the divergence or curl). This completes our presentation of the differential expressions of the three key functions in their most general form.

The expressions of the three key functions in OCS given by (4.35) to (4.37) can now be treated as the special case of the formulas in GCS. We let

$$\mathbf{p}_i = h_i \hat{u}_i, \qquad i = 1, 2, 3 \tag{4.78}$$

according to (2.28), where h_i and $\hat{u}_i$ denote, respectively, the metric coefficients and the unit vectors in OCS. The unit vectors are orthogonal to each other; thus,

$$\mathbf{p}_i \cdot \mathbf{p}_j = \begin{cases} h_i^2, & i = j \\ 0, & i \neq j \end{cases}$$

$$\mathbf{p}_i \times \mathbf{p}_j = \begin{cases} h_i h_j \hat{u}_k, & i \neq j \neq k \\ 0, & i = j \end{cases}$$

for $(i, j, k) = (1, 2, 3)$ in cyclic order. The parameter Λ reduces to

$$\Lambda = \mathbf{p}_i \cdot (\mathbf{p}_j \times \mathbf{p}_k) = h_i h_j h_k = \Omega.$$

The reciprocal vectors $\mathbf{r}^i$ become

$$\mathbf{r}^i = \frac{1}{\Lambda} \mathbf{p}_j \times \mathbf{p}_k = \frac{h_j h_k}{\Omega} \hat{u}_i = \frac{\hat{u}_i}{h_i}. \tag{4.79}$$

The differential expression of the symbolic expression in OCS is then given by

$$T(\nabla) = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial \omega^i} T\left(\frac{\Omega}{h_i} \hat{u}_i\right). \tag{4.80}$$

We still use ω^i to denote the coordinate variables in OCS. The operational form of the three key functions previously described by

$$\nabla \tilde{f} = \sum_i \mathbf{r}^i * \frac{\partial \tilde{f}}{\partial \omega^i} \tag{4.81}$$

reduces to

$$\nabla \tilde{f} = \sum_i \frac{\hat{u}_i}{h_i} * \frac{\partial \tilde{f}}{\partial \omega^i}, \tag{4.82}$$

or more specifically,

$$\nabla f = \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial \omega^i}, \tag{4.83}$$

$$\nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial \omega^i}, \tag{4.84}$$

$$\nabla \mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial \omega^i}. \tag{4.85}$$

To find the component form of $\nabla \mathbf{F}$ and $\nabla \mathbf{F}$ in OCS, we let

$$\mathbf{F} = \sum_i F_i \hat{u}_i. \tag{4.86}$$

The primary and the reciprocal components of $\mathbf{F}$, in view of (2.10) and (2.11), are related to F_i by

$$f_i = h_i F_i, \qquad i = 1, 2, 3, \tag{4.87}$$

$$g_i = \frac{1}{h_i} F_i, \qquad i = 1, 2, 3. \tag{4.88}$$

The component forms of $\nabla \mathbf{F}$ and $\overline{\nabla}\, \mathbf{F}$ as given by (4.65) and (4.75) respectively become

$$\nabla \mathbf{F} = \frac{1}{\Omega} \sum_i \frac{\partial}{\partial \omega^i} \left(\frac{\Omega}{h_i} F_i \right), \tag{4.89}$$

$$\overline{\nabla}\, \mathbf{F} = \frac{1}{\Omega} \sum_i h_i \hat{u}_i \left[\frac{\partial (h_j f_j)}{\partial \omega^k} - \frac{\partial (h_k f_k)}{\partial \omega^j} \right]. \tag{4.90}$$

In the special case of an orthogonal linear system or the rectangular system $\hat{u}_i = \hat{x}_i$, $\omega^i = x_i$, $h_i = 1$ for $i = 1, 2, 3$, and $\Omega = 1$. Equations (4.83), (4.89), and (4.90) are the commonly used formulas for these functions in devising physical problems.

4-5 Alternative Definitions of Gradient and Curl

To distinguish these functions in a conceptual manner, we propose some names for the integral-differential expressions of these functions based on a "physical" model. Thus, the quantity $\sum_i \hat{n}_i f \Delta S_i$ in the second term of (4.59) will be identified as the total *directional radiance* of f from the volume cell ΔV, or *radiance* for short; the gradient is then a measure of radiance per unit volume. The quantity $\sum_i \hat{n}_i \cdot \mathbf{F} \Delta S_i$ in the second term of (4.63) has a well-established name used by many authors as the *total flux* of $\mathbf{F}$ from ΔV, or *flux* for short; the divergence is then a measure of flux per unit volume. For the vector quantity $\sum_i \hat{n}_i \times \mathbf{F} \Delta S_i$ in the second term of (4.68), we propose the name of the total *shear* of $\mathbf{F}$ around the enclosing volume ΔV, or *shear* for short; the curl is then a measure of shear per unit volume. From the mathematical point of view, there is no need to invoke this physical model. It is proposed here merely as an aid to distinguish these functions.

The expressions for the gradient and the curl that have been derived by the method of symbolic vector can be derived alternatively by two different approaches.

In the defining integral-differential expression for ∇f, let the shape of ΔV be a flat cell of uniform thickness Δs and area ΔA at the broad surfaces, as shown in Fig. 4-1. The outward normal unit vector is denoted by $\hat{s}$. By taking a scalar product of the third term of (4.59) with $\hat{s}$, we obtain

$$\hat{s} \cdot \nabla f = \lim_{\Delta V \to 0} \frac{\sum_i f(\hat{s} \cdot \hat{n}_i) \Delta S_i}{\Delta V} = \lim_{\substack{\Delta V \to 0 \\ \Delta s \to 0}} \frac{\sum_i f(\hat{s} \cdot \hat{n}_i) \Delta S_i}{\Delta A \, \Delta s}. \tag{4.91}$$

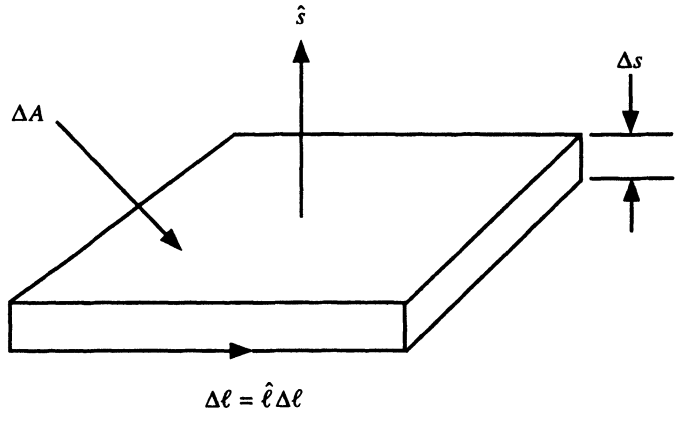


Figure 4-1 Thin flat volume of uniform thickness Δs and area ΔA.

The scalar product $\hat{s} \cdot \hat{n}_i$ vanishes for all the side surfaces because $\hat{s}$ is perpendicular to $\hat{n}_i$ therein. The only contributions result from the top and the bottom surfaces where $\hat{n}_i = \pm \hat{s}$ and $\Delta S_i = \Delta A$; thus, we obtain

$$\hat{s} \cdot \nabla f = \lim_{\Delta s \to 0} \frac{\left[f\left(s + \frac{\Delta s}{2}\right) - f\left(s + \frac{\Delta s}{2}\right) \right]}{\Delta s} = \frac{\partial f}{\partial s}, \tag{4.92}$$

where $s \pm \Delta s/2$ correspond to the locations of the broad surfaces along s, and the center of the flat cell is located at s. Equation (4.92) can be treated as an alternative definition of the component of the gradient in an arbitrary direction s.

By the rule of chain differentiation,

$$\frac{\partial f}{\partial s} = \sum_i \frac{\partial \omega^i}{\partial s} \frac{\partial f}{\partial \omega^i}, \tag{4.93}$$

where ω^i denotes one of the coordinate variables in GCS. Equation (4.93) can be interpreted as the scalar product between

$$\nabla f = \sum_i \mathbf{r}^i \frac{\partial f}{\partial \omega^i} \tag{4.94}$$

and

$$\hat{s} = \sum_j \mathbf{p}_j \frac{\partial \omega^j}{\partial s}, \tag{4.95}$$

where we have used the relation

$$\hat{s}\, ds = \sum \mathbf{p}_j\, d\omega^j \tag{4.96}$$

to obtain (4.95). $\mathbf{p}_j$ and $\mathbf{r}^i$ are, respectively, the primary and reciprocal vectors in GCS. Equation (4.93), therefore, is the same as

$$\frac{\partial f}{\partial s} = \hat{s} \cdot \nabla f,$$

previously derived by means of (4.91).

The same model can be used to find a typical component of the curl of $\mathbf{F}$ in GCS. Let us assume that $\hat{s}$ represents the unit normal in the direction of $\mathbf{r}^1$ in GCS, that is,

$$\hat{s} = \frac{\mathbf{r}^1}{(\mathbf{r}^1 \cdot \mathbf{r}^1)^{1/2}} = \hat{r}_1. \tag{4.97}$$

By taking the scalar product of the second term of (4.68) with $\hat{r}_1$, we obtain

$$\hat{r}_1 \cdot \nabla \mathbf{F} = \lim_{\Delta V \to 0} \frac{\sum_i \hat{r}_1 \cdot (\hat{n}_i \times \mathbf{F})\, \Delta S_i}{\Delta V} = \lim_{\substack{\Delta A \to 0 \\ \Delta s \to 0}} \frac{\sum_i (\hat{r}_1 \times \hat{n}_i) \cdot \mathbf{F}\, \Delta S_i}{\Delta A\, \Delta s}. \tag{4.98}$$

The area of the broad surface, ΔA, corresponds to

$$\Delta A = \left[(\mathbf{p}_2 \times \mathbf{p}_3) \cdot (\mathbf{p}_2 \times \mathbf{p}_3)\right]^{1/2} \Delta\omega^2\, \Delta\omega^3.$$

Because

$$\mathbf{r}^1 = \frac{1}{\Lambda}\, \mathbf{p}_2 \times \mathbf{p}_3,$$

it is evident that

$$\Delta A = \Lambda[\mathbf{r}^1 \cdot \mathbf{r}^1]^{1/2}\, \Delta\omega^2\, \Delta\omega^3. \tag{4.99}$$

In (4.98), the only contributions come from the side surfaces where

$$\hat{r}_1 \times \hat{n}_i = \hat{\ell}_j \quad \text{and} \quad \Delta S_i = \Delta s\, \Delta \ell_j.$$

$\hat{\ell}_j\, \Delta \ell_j$ represents a segment of the contour around the periphery of the broad surface. Equation (4.98) then reduces to

$$\hat{r}_1 \cdot \nabla \mathbf{F} = \lim_{\Delta A \to 0} \frac{\sum_j \mathbf{F} \cdot \Delta \boldsymbol{\ell}_j}{\Delta A}. \tag{4.100}$$

By considering a contour formed by $\mathbf{p}_2\, \Delta\omega^2$ and $\mathbf{p}_3\, \Delta\omega^3$ with center located at ω^2 and ω^3, we obtain

$$\begin{aligned} \hat{r}_1 \cdot \nabla \mathbf{F} = \lim_{\substack{\Delta\omega^2 \to 0 \\ \Delta\omega^3 \to 0}} & \left[\frac{(\mathbf{F} \cdot \mathbf{p}_2)_{\omega^3 - \Delta\omega^3/2} - (\mathbf{F} \cdot \mathbf{p}_2)_{\omega^3 + \Delta\omega^3/2}}{\Delta\omega^3} \right. \\ & \left. + \frac{(\mathbf{F} \cdot \mathbf{p}_3)_{\omega^2 + \Delta\omega^2/2} - (\mathbf{F} \cdot \mathbf{p}_3)_{\omega^2 - \Delta\omega^2/2}}{\Delta\omega^2} \right] \frac{1}{\Lambda(\mathbf{r}^1 \cdot \mathbf{r}^1)^{1/2}} \\ = & \frac{1}{\Lambda(\mathbf{r}^1 \cdot \mathbf{r}^1)^{1/2}} \left[\frac{\partial(\mathbf{p}_3 \cdot \mathbf{F})}{\partial\omega^2} - \frac{\partial(\mathbf{p}_2 \cdot \mathbf{F})}{\partial\omega^3} \right] \end{aligned}$$

or

$$\mathbf{r}^1 \cdot \nabla \mathbf{F} = \frac{1}{\Lambda}\left(\frac{\partial f_3}{\partial \omega^2} - \frac{\partial f_2}{\partial \omega^3}\right), \tag{4.101}$$

where f_2 and f_3 denote two of the primary components of $\mathbf{F}$ in GCS. In general, we find

$$\mathbf{r}^i \cdot \nabla \mathbf{F} = \frac{1}{\Lambda}\left(\frac{\partial f_k}{\partial \omega^j} - \frac{\partial f_j}{\partial \omega^k}\right), \tag{4.102}$$

where $(i, j, k) = (1, 2, 3)$ in cyclic order. Equation (4.102) represents one component of (4.75) because

$$\sum_i \mathbf{p}_i(\mathbf{r}^i \cdot \nabla \mathbf{F}) = \nabla \mathbf{F} = \sum_i \frac{\mathbf{p}_i}{\Lambda}\left(\frac{\partial f_k}{\partial \omega^j} - \frac{\partial f_j}{\partial \omega^k}\right).$$

4-6 The Method of Gradient

Once the differential expressions of certain functions are available in terms of different coordinate variables, we can derive many relations from them by taking advantage of the invariance property of these functions. The method of gradient is based on this principle. We will use an example to illustrate this method.

It is known that the relationships between (x, y), two of the rectangular variables, and (r, ϕ), two of the cylindrical variables, are

$$x = r\cos\phi, \tag{4.103}$$

$$y = r\sin\phi, \tag{4.104}$$

$$r = (x^2 + y^2)^{1/2}, \tag{4.105}$$

and

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{4.106}$$

By taking the gradient of (4.103) and (4.104) in the rectangular system for x and y on the left sides of these two equations, and in the cylindrical coordinate system on the right sides, we obtain

$$\hat{x} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}, \tag{4.107}$$

$$\hat{y} = \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}. \tag{4.108}$$

By doing the same for (4.105) and (4.106), but in reverse order, we obtain

$$\hat{r} = \frac{x}{(x^2 + y^2)^{1/2}}\hat{x} + \frac{y}{(x^2 + y^2)^{1/2}}\hat{y} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}, \tag{4.109}$$

$$\frac{\hat{\phi}}{r} = \frac{-y}{x^2 + y^2}\hat{x} + \frac{x}{x^2 + y^2}\hat{y},$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}. \tag{4.110}$$

These relations can be derived by a geometrical method, but the method of gradient is straightforward, particularly if the orthogonal system is a more complicated one compared to the cylindrical and spherical coordinate systems. Equations (4.107)–(4.110), together with the unit vector $\hat{z}$, can be tabulated in a matrix form, as shown in Table 4-2. The table can be used in both directions. Horizontally, it gives

$$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}, \tag{4.111}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}, \tag{4.112}$$

which are the same as (4.109) and (4.110). Vertically, it yields

$$\hat{x} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}, \tag{4.113}$$

$$\hat{y} = \sin\phi\,u_r + \cos\phi\,\hat{\phi}, \tag{4.114}$$

which can be derived algebraically by solving for $\hat{x}$ and $\hat{y}$ from (4.109) and (4.110) in terms of $\hat{r}$ and $\hat{\phi}$. Each coefficient in Table 4-2 corresponds to the scalar product of the two unit vectors in the intersecting column and row; thus, $\hat{r}\cdot\hat{x} = \cos\phi$, $\hat{\phi}\cdot\hat{x} = -\sin\phi$, and so on. For this reason, the same table is applicable to the transformation of the scalar components of a vector in the two systems. Because

$$\mathbf{f} = f_x\hat{x} + f_y\hat{y} + f_z\hat{z} = f_r\hat{r} + f_\phi\hat{\phi} + f_z\hat{z},$$

it follows that

$$f_x = \hat{x}\cdot\mathbf{f} = (\hat{x}\cdot\hat{r})f_r + (\hat{x}\cdot\hat{\phi})f_\phi = \cos\phi\,f_r - \sin\phi\,f_\phi \tag{4.115}$$

and

$$f_y = \hat{y}\cdot\mathbf{f} = \left(\hat{y}\cdot\hat{r}\right)f_r + \left(\hat{y}\cdot\hat{\phi}\right)f_\phi = \sin\phi\,f_r + \cos\phi\,f_\phi. \tag{4.116}$$

These relations are of the same form as (4.107) and (4.108). The transformations of the unit vectors of the orthogonal system reviewed in Section 2-1, and the unit vectors in the rectangular system, are listed in Appendix A, including the cylindrical system just described.

Table 4-2: Transportation of Unit Vectors

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{r}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

As another example, let us consider the problem of relating $(\hat{R}, \hat{\theta}, \hat{\phi})$ to $(\hat{R}, \hat{\alpha}, \hat{\beta})$ of another spherical system in which the polar angle α is measured from the x axis, and the azimuthal angle β is measured with respect to the x–y plane; thus,

$$x = R\sin\theta\cos\phi = R\cos\alpha, \tag{4.117}$$

$$y = R\sin\theta\sin\phi = R\sin\alpha\cos\beta, \tag{4.118}$$

$$z = R\cos\theta = R\sin\alpha\sin\beta. \tag{4.119}$$

We are seeking the relationships between $(\hat{\alpha}, \hat{\beta})$ and $(\hat{\theta}, \hat{\phi})$. The metric coefficients of the two systems are $(1, R, R\sin\theta)$ and $(1, R, R\sin\alpha)$. By taking the gradient of $\sin\theta\cos\phi = \cos\alpha$ in the two systems, we obtain

$$\frac{1}{R}\frac{\partial}{\partial\theta}(\sin\theta\cos\phi)\,\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}(\sin\theta\cos\phi)\,\hat{\phi} = \frac{1}{R}\frac{\partial}{\partial\alpha}(\cos\alpha)\,\hat{\alpha}. \tag{4.120}$$

Hence $-\sin\alpha\,\hat{\alpha} = \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$, or

$$\hat{\alpha} = \frac{-1}{(1-\sin^2\theta\cos^2\phi)^{1/2}}\left(\cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right). \tag{4.121}$$

By taking the gradient of

$$\cot\beta = \tan\theta\sin\phi = \left(\frac{y}{z}\right), \tag{4.122}$$

we obtain

$$\hat{\beta} = \frac{-1}{(1-\sin^2\theta\cos^2\phi)^{1/2}}\left(\sin\phi\,\hat{\theta} + \cos\theta\cos\phi\,\hat{\phi}\right). \tag{4.123}$$

From (4.121) and (4.123), we can solve for $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{\alpha}$ and $\hat{\beta}$. An alternative method is to use (4.119) and the relation

$$\tan\phi = \tan\alpha\cos\beta\left(=\frac{y}{x}\right), \tag{4.124}$$

and repeat the same operations; we then obtain

$$\hat{\theta} = \frac{-1}{(1-\sin^2\alpha\sin^2\beta)^{1/2}}\left(\cos\alpha\sin\beta\,\hat{\alpha} + \cos\beta\,\hat{\beta}\right) \tag{4.125}$$

and

$$\hat{\phi} = \frac{1}{(1-\sin^2\alpha\sin^2\beta)^{1/2}}\left(\cos\beta\,\hat{\alpha} - \cos\alpha\sin\beta\,\hat{\beta}\right). \tag{4.126}$$

The reader can verify these expressions by solving for $\hat{\theta}$ and $\hat{\phi}$ from (4.121) and (4.123) at the expense of a tedious calculation.

These relations are very useful in antenna theory when one is interested in finding the resultant field of two linear antennas placed at the origin, with one antenna pointed in the z direction and another one pointed in the x direction. In order to calculate the resultant distant field, the individual field must be expressed in a common coordinate system, say (R, θ, ϕ) in this case. Because the field of the x-directed antenna is proportional to $\hat{\alpha}$, (4.121) can be used to combine it with the field of the z-directed antenna, whose field is proportional to $\hat{\theta}$. In fact, it is this technical problem that motivated the author to formulate the method of gradient many years ago.

The method of gradient can also be used effectively to derive the expressions for the divergence operator and the curl operator in the orthogonal curvilinear system from their expressions in the rectangular system. In the rectangular system, the divergence operator and the curl operator are given respectively by

$$\nabla = \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i}, \tag{4.127}$$

$$\nabla = \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}. \tag{4.128}$$

Upon applying the method of gradient to the coordinate variables x_i with $i = (1, 2, 3)$, we obtain

$$\hat{x}_i = \nabla x_i = \sum_j \frac{\hat{u}_j}{h_j} \frac{\partial x_i}{\partial v_j}, \tag{4.129}$$

and by the chain rule of differentiation,

$$\frac{\partial}{\partial x_i} = \sum_k \frac{\partial v_k}{\partial x_i} \frac{\partial}{\partial v_k}. \tag{4.130}$$

We also have the relations

$$\frac{\partial v_l}{\partial v_j} = \sum_k \frac{\partial v_l}{\partial x_k} \frac{\partial x_k}{\partial v_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.131}$$

Upon substituting (4.129) and (4.130) into (4.127) and (4.128), and making use of (4.131), we find

$$\nabla = \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i}, \tag{4.132}$$

$$\nabla = \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i} = \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}. \tag{4.133}$$

This exercise shows again that the divergence operator and the curl operator, as with the del operator for the gradient, are invariant to the choice of the coordinate system, a property we have demonstrated before.

4-7 Symbolic Expressions with Two Functions and the Partial Symbolic Vectors

Symbolic expressions with two functions are represented by $T(\nabla, \widetilde{a}, \widetilde{b})$, where $\widetilde{a}$ and $\widetilde{b}$ both can be scalars, vectors, or one each. In this section, we will introduce a new method for finding the identities of these functions in terms of the individual functions $\widetilde{a}$ and $\widetilde{b}$ without using the otherwise tedious method in differential calculus.

Because of the invariance theorem, it is sufficient to use the differential expression of $T(\nabla, \tilde{a}, \tilde{b})$ in the rectangular system to describe this new method. In the rectangular system,

$$T(\nabla, \tilde{a}, \tilde{b}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \tilde{a}, \tilde{b}). \tag{4.134}$$

We now introduce two partial symbolic vectors, denoted by ∇_a and ∇_b, which are defined by the following two equations:

$$T(\nabla_a, \tilde{a}, \tilde{b}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \tilde{a}, \tilde{b})_{\tilde{b}=\text{constant}}, \tag{4.135}$$

$$T(\nabla_b, \tilde{a}, \tilde{b}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \tilde{a}, \tilde{b})_{\tilde{a}=\text{constant}}. \tag{4.136}$$

In (4.135), $\tilde{b}$ is considered to be constant, and in (4.136), $\tilde{a}$ is considered to be constant. The process is similar to the partial differentiation of a function of two independent variables, that is,

$$\frac{\partial f(x, y)}{\partial x} = \left[\frac{df(x, y)}{dx}\right]_{y=\text{constant}}. \tag{4.137}$$

The name *partial symbolic vector* was chosen because of this analogy. It is obvious that Lemma 4.1 is also applicable to symbolic expressions defined with a partial symbolic vector, because, in general,

$$T(\nabla_a, \tilde{a}, \tilde{b}) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i, \tilde{a}, \tilde{b})_{\tilde{b}=c} \Delta S_i}{\Delta V} \tag{4.138}$$

and

$$T(\nabla_b, \tilde{a}, \tilde{b}) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i, \tilde{a}, \tilde{b})_{\tilde{a}=c} \Delta S_i}{\Delta V}. \tag{4.139}$$

We now introduce the second lemma in the method of symbolic vector.

Lemma 4.2. For a symbolic expression containing two functions, the following relation holds true:

$$T(\nabla, \tilde{a}, \tilde{b}) = T(\nabla_a, \tilde{a}, \tilde{b}) + T(\nabla_b, \tilde{a}, \tilde{b}). \tag{4.140}$$

The proof of this lemma follows directly from the definition of the expressions (4.135) and (4.136) or (4.138) and (4.139). This lemma may also be called the *decomposition theorem.*

Let us now apply both Lemma 4.1 and Lemma 4.2 to derive various vector identities without actually performing any differentiation. Because the steps involved are algebraic, most of the time we merely write down the intermediate steps, omitting the explanation.

The symbolic expression with two functions besides the symbolic vector are listed here; there are only eight possibilities.

Two scalars:

$$\nabla ab = a\nabla b = b\nabla a. \tag{4.141}$$

One scalar and one vector:

$$\nabla \cdot a\mathbf{b} = a\nabla \cdot \mathbf{b} = \nabla a \cdot \mathbf{b}, \tag{4.142}$$

$$\nabla \times a\mathbf{b} = a\nabla \times \mathbf{b} = \nabla a \times \mathbf{b}. \tag{4.143}$$

Two vectors:

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\nabla = (\mathbf{b} \cdot \mathbf{a})\nabla, \tag{4.144}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \nabla) = \mathbf{b} \cdot (\nabla \times \mathbf{a}), \tag{4.145}$$

$$(\nabla \cdot \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \nabla)\mathbf{b} = \nabla \cdot \mathbf{ab}, \tag{4.146}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} = \nabla \cdot \mathbf{ba} - \nabla \cdot \mathbf{ab}, \tag{4.147}$$

$$(\nabla \times \mathbf{a}) \times \mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\nabla = \nabla \cdot \mathbf{ba} - \nabla(\mathbf{a} \cdot \mathbf{b}). \tag{4.148}$$

1.

$$\begin{aligned} \nabla ab &= \nabla_a(ab) + \nabla_b(ab) \quad \text{[Lemma 4.2]} \\ &= b\nabla_a a + a\nabla_b b; \quad \text{[Lemma 4.1]} \end{aligned}$$

hence

$$\nabla(ab) = b\nabla a + a\nabla b. \tag{4.149}$$

2.

$$\begin{aligned} \nabla \cdot (a\mathbf{b}) &= \nabla_a \cdot (a\mathbf{b}) + \nabla_b \cdot (a\mathbf{b}) \\ &= (\nabla_a a) \cdot \mathbf{b} + a\nabla_b \cdot \mathbf{b}; \end{aligned}$$

hence

$$\nabla(a\mathbf{b}) = \mathbf{b} \cdot \nabla a + a\nabla \mathbf{b}. \tag{4.150}$$

3.

$$\begin{aligned} \nabla \times (a\mathbf{b}) &= \nabla_a \times (a\mathbf{b}) + \nabla_b \times (a\mathbf{b}) \\ &= (\nabla_a a) \times \mathbf{b} + a\nabla_b \times \mathbf{b}; \end{aligned}$$

hence

$$\nabla(a\mathbf{b}) = -\mathbf{b} \times \nabla a + a\nabla \mathbf{b}. \tag{4.151}$$

4.

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \nabla_a(\mathbf{a} \cdot \mathbf{b}) + \nabla_b(\mathbf{a} \cdot \mathbf{b}).$$

In view of Lemma 4.1,

$$\nabla_b(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\times(\nabla_b\times\mathbf{b}) + (\mathbf{a}\cdot\nabla_b)\mathbf{b}$$
$$= \mathbf{a}\times\overline{\nabla}\,\mathbf{b} + \mathbf{a}\cdot\nabla\mathbf{b}.$$

By interchanging the role of $\mathbf{a}$ and $\mathbf{b}$, we obtain

$$\nabla_a(\mathbf{a}\cdot\mathbf{b}) = \mathbf{b}\times\overline{\nabla}\,\mathbf{a} + \mathbf{b}\cdot\nabla\mathbf{a};$$

hence

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{a} + \mathbf{a}\times\overline{\nabla}\,\mathbf{b} + \mathbf{b}\times\overline{\nabla}\,\mathbf{a}. \tag{4.152}$$

We would like to point out that the first two terms in (4.152) involve two new functions in the form of $\nabla\mathbf{b}$ and $\nabla\mathbf{a}$. They are two dyadic functions corresponding to the gradient of two vector functions. In the rectangular system, $\nabla\mathbf{b}$ is defined by

$$\nabla\mathbf{b} = \sum_i \hat{x}_i\frac{\partial\mathbf{b}}{\partial x_i}$$
$$= \sum_i\sum_j \hat{x}_i\frac{\partial}{x_i}(\hat{x}_j b_j) \tag{4.153}$$
$$= \sum_i\sum_j \hat{x}_i\hat{x}_j\frac{\partial b_j}{\partial x_i}.$$

Then,

$$\mathbf{a}\cdot\nabla\mathbf{b} = \sum_i\sum_j a_i\frac{\partial b_j}{\partial x_i}\hat{x}_j = \sum_i a_i\frac{\partial\mathbf{b}}{\partial x_i}. \tag{4.154}$$

In an orthogonal curvilinear system,

$$\mathbf{a}\cdot\nabla\mathbf{b} = \sum_i \frac{a_i}{h_i}\frac{\partial\mathbf{b}}{\partial v_i}$$
$$= \sum_i \frac{a_i}{h_i}\frac{\partial}{\partial v_i}\sum_j b_j\hat{u}_j \tag{4.155}$$
$$= \sum_i \frac{a_i}{h_i}\sum_j\left[b_j\frac{\partial\hat{u}_j}{\partial v_i} + \frac{\partial b_j}{\partial v_i}\hat{u}_j\right].$$

The derivatives of $\hat{u}_j$ with respect to v_i can be expressed in terms of the unit vectors themselves and the derivatives of the metric coefficients with the aid of (2.59) and (2.61). The result yields

$$\mathbf{a}\cdot\nabla\mathbf{b} = \sum_i\sum_j \frac{a_i}{h_i}\frac{\partial b_j}{\partial v_i}\hat{u}_j + \mathbf{A}\times\mathbf{b}, \tag{4.156}$$

where

$$\mathbf{A} = \frac{1}{\Omega} \sum_i \left(a_k \frac{\partial h_k}{\partial v_j} - a_j \frac{\partial h_j}{\partial v_k} \right) h_i \hat{u}_i$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order, and $\Omega = h_1 h_2 h_3$. To obtain (4.156) by means of (4.154) through coordinate transformation would be a very complicated exercise.

5.

$$\begin{aligned} \nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \nabla_a \cdot (\mathbf{a} \times \mathbf{b}) + \nabla_b \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{b} \cdot (\nabla_a \times \mathbf{a}) + \mathbf{a} \cdot (\mathbf{b} \times \nabla_b); \end{aligned}$$

hence

$$\nabla (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b}. \tag{4.157}$$

6.

$$(\nabla \cdot \mathbf{a})\mathbf{b} = (\nabla_a \cdot \mathbf{a})\mathbf{b} + (\nabla_b \cdot \mathbf{a})\mathbf{b};$$

hence

$$\nabla (\mathbf{ab}) = \mathbf{b} \nabla \mathbf{a} + \mathbf{a} \cdot \nabla \mathbf{b}. \tag{4.158}$$

It is seen that $(\nabla \cdot \mathbf{a})\mathbf{b}$ is not equal to $(\nabla \mathbf{a})\mathbf{b}$; rather, it is the divergence of a dyadic function $\mathbf{ab}$.

7.

$$\begin{aligned} \nabla \times (\mathbf{a} \times \mathbf{b}) &= \nabla_a \times (\mathbf{a} \times \mathbf{b}) + \nabla_b \times (\mathbf{a} \times \mathbf{b}) \\ &= (\nabla_a \cdot \mathbf{b})\mathbf{a} - (\nabla_a \cdot \mathbf{a})\mathbf{b} + (\nabla_b \cdot \mathbf{b})\mathbf{a} - (\nabla_b \cdot \mathbf{a})\mathbf{b}; \end{aligned}$$

hence

$$\nabla (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{b} \nabla \mathbf{a} + \mathbf{a} \nabla \mathbf{b} - \mathbf{a} \cdot \nabla \mathbf{b}. \tag{4.159}$$

8.

$$\begin{aligned} (\nabla \times \mathbf{a}) \times \mathbf{b} &= (\nabla_a \times \mathbf{a}) \times \mathbf{b} + (\nabla_b \times \mathbf{a}) \times \mathbf{b} \\ &= (\nabla_a \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_b \cdot \mathbf{b}) - \nabla_b (\mathbf{a} \cdot \mathbf{b}) \\ &= (\nabla_a \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_b \cdot \mathbf{b}) - \mathbf{a} \times (\nabla_b \times \mathbf{b}) - (\mathbf{a} \cdot \nabla_b)\mathbf{b}; \end{aligned}$$

hence

$$(\nabla \times \mathbf{a}) \times \mathbf{b} = -\mathbf{b} \times \nabla \mathbf{a} + \mathbf{a}(\nabla \mathbf{b}) - \mathbf{a} \times \nabla \mathbf{b} - \mathbf{a} \cdot \nabla \mathbf{b}. \tag{4.160}$$

Two vector identities can be conveniently derived by means of partial symbolic expressions.

9.

$$\mathbf{a} \times (\nabla_b \times \mathbf{b}) = \nabla_b (\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla_b)\mathbf{b}.$$

Hence

$$\mathbf{a} \times \nabla \mathbf{b} = (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b}. \tag{4.161}$$

10.

$$(\mathbf{a} \times \nabla_b) \times \mathbf{b} = \nabla_b(\mathbf{b} \cdot \mathbf{a}) - (\nabla_b \cdot \mathbf{b})\mathbf{a}.$$

Hence

$$\begin{aligned}(\mathbf{a} \times \nabla_b) \times \mathbf{b} &= (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a}\nabla \mathbf{b} \\ &= \mathbf{a} \cdot \nabla \mathbf{b} - \mathbf{a}\nabla \mathbf{b} + \mathbf{a} \times \nabla \mathbf{b}.\end{aligned} \tag{4.162}$$

The second line of (4.162) results from (4.161).

It is interesting to observe that the partial symbolic expression $(\mathbf{a} \times \nabla_b) \times \mathbf{b}$ involves the products of $\mathbf{a}$ with the gradient, divergence, and curl of $\mathbf{b}$. The convenience of using the method of symbolic vector in deriving the vector identities is evident.

4-8 Symbolic Expressions with Double Symbolic Vectors

When a symbolic expression is created with a vector expression containing two vector functions $\mathbf{g}_1$ and $\mathbf{g}_2$ and a third function $\tilde{f}$ that can be scalar or vector, we can generate a symbolic expression of the form $T(\nabla, \mathbf{g}_2, \tilde{f})$ after $\mathbf{g}_1$ is replaced by a symbolic vector ∇. In the rectangular system,

$$T(\nabla, \mathbf{g}_2, \tilde{f}) = \sum_i \frac{\partial}{\partial x_i} T(\hat{x}_i, \mathbf{g}_2, \tilde{f}). \tag{4.163}$$

Now, if $\mathbf{g}_2$ is replaced by another symbolic vector ∇' in (4.163), we would obtain a symbolic expression with double S vectors whose differential form in the rectangular system will be given by

$$T(\nabla, \nabla', \tilde{f}) = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} T(\hat{x}_i, \hat{x}_j, \tilde{f}). \tag{4.164}$$

It is obvious that Lemma 4.1 also applies to $T(\nabla, \nabla', \tilde{f})$. Several distinct cases will be considered.

1. Laplacian of a scalar function

$$T(\nabla, \nabla', \tilde{f}) = (\nabla \cdot \nabla')f,$$

where f is a scalar function; then

$$\nabla \cdot \nabla' f = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j}(\hat{x}_i \cdot \hat{x}_j) f = \sum_i \frac{\partial^2 f}{\partial x_i^2} = \nabla \nabla f. \tag{4.165}$$

Although we have arrived at this result using functions defined in the rectangular system, the result is applicable to any system because of the

invariance theorem of the differential operators. The function $\nabla\nabla f$ is called the *Laplacian of the scalar function* f, in honor of the French mathematical physicist Pierre Simon Laplace (1749–1827). In the past, many authors used the notation $\nabla^2 f$ for this function, which is a compact form of the original notation $\nabla \cdot \nabla f$ used by Gibbs. We have completely abandoned Gibbs's notations in this new edition so that the "formal" scalar and vector products, discussed in Chapter 8, will not appear and interfere with the operations involved in the method of symbolic vector.

2. Laplacian of a vector function

$$T(\nabla, \nabla', \widetilde{f}) = \nabla \cdot \nabla' \mathbf{F},$$

where $\mathbf{F}$ is a vector function; then,

$$\nabla \cdot \nabla' \mathbf{F} = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} (\hat{x}_i \cdot \hat{x}_j)\mathbf{F} = \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_i^2} = \nabla\nabla\mathbf{F}. \tag{4.166}$$

In this case, we encounter a dyadic function corresponding to the gradient of a vector function. The divergence of a dyadic is a vector function. In the rectangular coordinate system, we can write

$$\nabla\nabla\mathbf{F} = \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_1^2} = \sum_i \sum_j \frac{\partial^2}{\partial x_i^2} F_j \hat{x}_j = \sum_j (\nabla\nabla F_j)\, \hat{x}_j. \tag{4.167}$$

In the orthogonal curvilinear system,

$$\nabla\nabla\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \sum_j \frac{\hat{u}_j}{h_j} \frac{\partial \mathbf{F}}{\partial v_j}, \tag{4.168}$$

where the operational form of the divergence and the gradient have been used. The derivatives of the dyadic function $\nabla\mathbf{F}$ can be simplified as follows:

$$\frac{\partial}{\partial v_i} \nabla\mathbf{F} = \sum_j \left[\frac{\hat{u}_j}{h_j} \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} + \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right].$$

Hence

$$\begin{aligned} \nabla\nabla\mathbf{F} &= \sum_i \sum_j \frac{\hat{u}_i}{h_i} \cdot \left[\frac{\hat{u}_j}{h_j} \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} + \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right] \\ &= \sum_i \sum_j \left\{ \left(\frac{\hat{u}_i}{h_i} \cdot \frac{\hat{u}_j}{h_j} \right) \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} + \frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right\} \\ &= \sum_i \frac{1}{h_i^2} \frac{\partial^2 \mathbf{F}}{\partial v_i^2} + \sum_i \sum_j \left[\frac{\hat{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right]. \end{aligned} \tag{4.169}$$

The expression of $\nabla\nabla\mathbf{F}$ will be used later to demonstrate an identity involving this function.

3. The gradient of the divergence of a vector function

$$T(\nabla, \nabla', \tilde{f}) = \nabla(\nabla' \cdot \mathbf{F}).$$

Then,

$$\begin{aligned} \nabla(\nabla' \cdot \mathbf{F}) &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} \hat{x}_i (\hat{x}_j \cdot \mathbf{F}) \\ &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} \hat{x}_i F_j \\ &= \sum_i \hat{x}_i \frac{\partial}{\partial x_i} \sum_j \frac{\partial F_j}{\partial x_j} = \nabla\nabla\mathbf{F}. \end{aligned} \tag{4.170}$$

In OCS,

$$\begin{aligned} \nabla\nabla\mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i} \frac{\partial}{\partial v_i} \sum_j \frac{\hat{u}_j}{h_j} \cdot \frac{\partial \mathbf{F}}{\partial v_j} \\ &= \sum_i \sum_j \frac{\hat{u}_i}{h_i} \left[\frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \cdot \frac{\partial \mathbf{F}}{\partial v_j} + \frac{\hat{u}_j}{h_j} \cdot \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} \right]. \end{aligned} \tag{4.171}$$

We will leave it in this form for the time being.

4. The curl of the curl of a vector function

$$T(\nabla, \nabla', \tilde{f}) = \nabla \times (\nabla' \times \mathbf{F}).$$

Then,

$$\begin{aligned} \nabla \times (\nabla' \times \mathbf{F}) &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} \hat{x}_i \times (\hat{x}_j \times \mathbf{F}) \\ &= \sum_i \frac{\partial}{\partial x_i} \left[\hat{x}_i \times \sum_j \frac{\partial}{\partial x_j} (\hat{x}_j \times \mathbf{F}) \right] \\ &= \sum_i \frac{\partial}{\partial x_i} [\hat{x}_i \times \nabla\mathbf{F}] = \nabla\nabla\mathbf{F}. \end{aligned} \tag{4.172}$$

In OCS,

$$\begin{aligned} \nabla\nabla\mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i} \left(\sum_j \frac{\hat{u}_j}{h_j} \times \frac{\partial \mathbf{F}}{\partial v_j} \right) \\ &= \sum_i \sum_j \frac{\hat{u}_i}{h_i} \times \left[\frac{\partial}{\partial v_i} \left(\frac{\hat{u}_j}{h_j} \right) \times \frac{\partial \mathbf{F}}{\partial v_j} + \frac{\hat{u}_j}{h_j} \times \frac{\partial^2 \mathbf{F}}{\partial v_i \, \partial v_j} \right]. \end{aligned} \tag{4.173}$$

Because $T(\nabla, \nabla', \tilde{f})$ obeys Lemma 4.1, we can also change $\nabla \times (\nabla' \times \mathbf{F})$ to

$$\nabla \times (\nabla' \times \mathbf{F}) = (\nabla \cdot \mathbf{F})\nabla' - (\nabla \cdot \nabla')\mathbf{F}. \tag{4.174}$$

In view of (4.166), (4.170), and (4.172), we have

$$\nabla\nabla\mathbf{F} = \nabla\nabla\mathbf{F} - \nabla\nabla\mathbf{F}. \tag{4.175}$$

This identity has been derived using functions defined in the rectangular coordinate system. However, in view of the invariance theorem, it is valid in any coordinate system. The misinterpretation of $\nabla\nabla\mathbf{F}$ in (4.175) in OCS has troubled many authors in the past (see Chapter 8); it is therefore desirable to prove (4.175) analytically to confirm our assertion that (4.175) is an identity in any coordinate system. By taking the difference between (4.173) and (4.171) and rearranging the terms, we find

$$\nabla\nabla\mathbf{F} - \nabla\nabla\mathbf{F} = \nabla\nabla\mathbf{F} + \sum_i \sum_j \frac{\partial \mathbf{F}}{v_j} \times \left[\frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}\left(\frac{\hat{u}_j}{h_j}\right)\right], \tag{4.176}$$

where $\nabla\nabla\mathbf{F}$ corresponds to the function given by (4.169). The last term in (4.176) vanishes because

$$\sum_i \frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i}\left(\frac{\hat{u}_j}{h_j}\right) = \nabla\nabla v_j = 0.$$

The curl of the gradient of any differentiable scalar function vanishes by considering $\nabla\nabla f$ in the rectangular coordinate system that yields

$$\nabla\nabla f = \sum_i \hat{x}_i \left[\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)\right] = 0. \tag{4.177}$$

This theorem is treated later from the point of view of the method of symbolic vector. We have thus shown that (4.175) is indeed valid in OCS. The identity can be proved in GCS in a similar manner.

5. The curl of the gradient of a scalar or a vector function

$$T(\nabla, \nabla', \tilde{f}) = \nabla \times \nabla' \tilde{f}.$$

Then,

$$\nabla \times \nabla' \tilde{f} = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j}(\hat{x}_i \times \hat{x}_j \tilde{f})$$

$$= \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}\left[\sum_j \hat{x}_j \frac{\partial \tilde{f}}{\partial x_j}\right] = \nabla\nabla\tilde{f}.$$

But

$$\hat{x}_i \times \hat{x}_j = \begin{cases} 0, & i = j, \\ -\hat{x}_j \times \hat{x}_i, & i \neq j; \end{cases}$$

hence

$$\nabla\nabla\tilde{f} = 0, \tag{4.178}$$

where $\tilde{f}$ can be a scalar or a vector. When $\tilde{f}$ is a scalar, we have already proved it as stated by (4.177).

6. The divergence of the curl of a vector function

$$T(\nabla, \nabla', \widetilde{f}) = \nabla \cdot (\nabla' \times \mathbf{F}).$$

Then,

$$\begin{aligned}\nabla \cdot (\nabla' \times \mathbf{F}) &= \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} [\hat{x}_i \cdot (\hat{x}_j \times \mathbf{F})] \\ &= \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i} \left[\sum_j \hat{x}_j \times \frac{\partial \mathbf{F}}{\partial x_j} \right] \\ &= \nabla \nabla \mathbf{F}.\end{aligned}$$

But

$$\begin{aligned}\hat{x}_i \cdot (\hat{x}_j \times \mathbf{F}) &= \mathbf{F} \cdot (\hat{x}_i \times \hat{x}_j) \\ &= \begin{cases} 0, & i = j, \\ -\mathbf{F} \cdot (\hat{x}_j \times \hat{x}_i), & i \neq j. \end{cases}\end{aligned}$$

Hence

$$\nabla \nabla \mathbf{F} = 0. \tag{4.179}$$

When a symbolic expression consists of double S vectors and two functions, its definition in the rectangular system is

$$T(\nabla, \nabla', \widetilde{a}, \widetilde{b}) = \sum_i \sum_j \frac{\partial^2}{\partial x_i \, \partial x_j} T(\hat{x}_i, \hat{x}_j, \widetilde{a}, \widetilde{b}). \tag{4.180}$$

To simplify (4.180), we can apply Lemma 4.2 repeatedly to an expression with a single S vector, that is,

$$\begin{aligned}T(\nabla, \nabla', \widetilde{a}, \widetilde{b}) &= T(\nabla, \nabla'_a, \widetilde{a}, \widetilde{b}) + T(\nabla, \nabla'_b, \widetilde{a}, \widetilde{b}) \\ &= T(\nabla_a, \nabla'_a, \widetilde{a}, \widetilde{b}) + T(\nabla_b, \nabla'_a, \widetilde{a}, \widetilde{b}) \\ &\quad + T(\nabla_a, \nabla'_b, \widetilde{a}, \widetilde{b}) + T(\nabla_b, \nabla'_b, \widetilde{a}, \widetilde{b}).\end{aligned} \tag{4.181}$$

As an example, let

$$T(\nabla, \nabla', \widetilde{a}, \widetilde{b}) = (\nabla \cdot \nabla')ab. \tag{4.182}$$

According to (4.165), this is equal to $\nabla \nabla(ab)$. Equation (4.181) yields

$$\nabla \nabla(ab) = b \nabla \nabla a + 2(\nabla a)(\nabla b) + a \nabla \nabla b. \tag{4.183}$$

The same answer can be obtained by applying (4.149) and (4.150) to $\nabla \nabla(ab)$ in succession.

4-9 Generalized Gauss Theorem in Space

The principal integral theorem involving a symbolic expression can be formulated based on the very definition of $T(\nabla)$, namely,

$$T(\nabla) = \lim_{\Delta V \to 0} \frac{\sum_i T(\hat{n}_i)\, \Delta S_i}{\Delta V}. \tag{4.184}$$

Equation (4.184) can be considered as a limiting form of a parent equation

$$T(\nabla) = \frac{\sum_i T(\hat{n}_i)\, \Delta S_i}{\Delta V} + \varepsilon, \tag{4.185}$$

where $\varepsilon \to 0$ as $\Delta V \to 0$. If ΔV_j denotes a typical cell in a volume V with an enclosing surface S, then for that cell, we can write

$$T(\nabla)\, \Delta V_j = \sum_i T(\hat{n}_{ij})\, \Delta S_{ij} + \varepsilon_j \Delta V_j, \tag{4.186}$$

where ΔS_{ij} denotes an elementary area of ΔV_j, $\hat{n}_{ij}$ being an outward normal unit vector. By taking the Riemann sum of (4.186) with respect to j, we obtain

$$\sum_j T(\nabla)\, \Delta V_j = \sum_j \sum_i T(\hat{n}_{ij})\, \Delta S_{ij} + \sum_j \varepsilon_j\, \Delta V_j. \tag{4.187}$$

Then, as $\Delta V_j \to 0$, $\varepsilon_j \to 0$. By assuming $T(\nabla)$ to be continuous throughout V, we obtain

$$\iiint_V T(\nabla)\, dV = \oiint_S T(\hat{n})\, dS. \tag{4.188}$$

The sign around the double integral means that the surface is closed. It is observed that the contributions of $T(\hat{n}_{ij})\, \Delta S_{ij}$ from two contacting surfaces of adjacent cells cancel each other. The only contribution results from the exterior surface where there are no neighboring cells. In (4.188), $\hat{n}$ denotes the outward unit normal vector to S. The same formula can be obtained by integrating the symbolic expression in the rectangular system

$$\iiint_V T(\nabla)\, dV = \iiint_V \sum_i \frac{\partial T(\hat{x}_i)}{\partial x_i}\, dV. \tag{4.189}$$

The integral involving the partial derivative of $T(\hat{x}_i)$ with respect to x_i can be reduced to the surface integral found in (4.188). The linearity of $T(\hat{x}_i)$ with respect to $\hat{x}_i$ is a key link in that reduction. In a later section, we will give a detailed treatment of a two-dimensional version of a similar problem for a surface to demonstrate this approach. The formula that we have derived will be designated the *generalized Gauss theorem*, which converts a volume integral of $T(\nabla)$, continuous throughout V, to a surface integral evaluated at the enclosing surface S. Many of the classical theorems in vector analysis can be readily derived from this generalized theorem by a proper choice of the symbolic expression $T(\nabla)$.

1. *Divergence theorem or Gauss theorem.* Let $T(\nabla) = \nabla \cdot \mathbf{F} = \nabla\mathbf{F}$. Upon substituting these quantities into (4.188), we obtain the divergence theorem, or the standard Gauss theorem, named in honor of the great mathematician Karl Friedrich Gauss (1777–1855):

$$\iiint_V \nabla\mathbf{F}\, dV = \oiint_S (\hat{n} \cdot \mathbf{F})\, dS. \tag{4.190}$$

2. *Curl theorem.* Let $T(\nabla) = \nabla \times \mathbf{F} = \overline{\nabla}\,\mathbf{F}$; then $T(\hat{n}) = \hat{n} \times \mathbf{F}$. By means of (4.188), we obtain the curl theorem:

$$\iiint_V \overline{\nabla}\,\mathbf{F}\, dV = \oiint_S (\hat{n} \times \mathbf{F})\, dS. \tag{4.191}$$

3. *Gradient theorem.* This theorem is obtained by letting $T(\nabla) = \nabla f = \nabla f$; then $T(\hat{n}) = \hat{n} f$; hence

$$\iiint_V \nabla f\, dV = \oiint_S f\hat{n}\, dS = \oiint_S f\, d\mathbf{S}. \tag{4.192}$$

4. *Hallén's formula.* If we let $T(\nabla) = (\nabla\cdot\mathbf{a})\mathbf{b}$, then $T(\hat{n}) = (\mathbf{n}\cdot\mathbf{a})\mathbf{b}$. Because $T(\nabla)$ consists of two functions $\mathbf{a}$ and $\mathbf{b}$, we can apply Lemma 4.2 to obtain

$$(\nabla \cdot \mathbf{a})\mathbf{b} = (\nabla_a \cdot \mathbf{a})\mathbf{b} + (\nabla_b \cdot \mathbf{a})\mathbf{b} = \mathbf{b}(\nabla_a \cdot \mathbf{a}) + (\mathbf{a} \cdot \nabla_b)\mathbf{b}.$$

The rearrangement of the various terms follows Lemma 4.1; thus,

$$(\nabla \cdot \mathbf{a})\mathbf{b} = \mathbf{b}(\nabla\mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b}. \tag{4.193}$$

By substituting (4.193) into (4.188), we obtain

$$\iiint_V [\mathbf{b}\nabla\mathbf{a} + \mathbf{a} \cdot \nabla\mathbf{b}]\, dV = \oiint_S (\hat{n} \cdot \mathbf{a})\mathbf{b}\, dS. \tag{4.194}$$

Equation (4.194), with $\mathbf{b}$ equal to $\mathbf{c}/r$, where $r = \left(x^2 + y^2 + z^2\right)^{1/2}$ and $\mathbf{c}$ is a constant vector that can be deleted from the resultant equation, was derived by Hallén [8], based on differential calculus carried out in a rectangular system. We designate (4.194) as *Hallén's formula* for convenient identification.

The three theorems stated by (4.190)–(4.192) are closely related. In fact, it is possible to derive the divergence theorem and the curl theorem based on the gradient theorem. The derivation is given in Appendix D. The relationship between several surface theorems to be derived in Chapter 5 is also shown in that appendix.

With the vector theorems and identities at our disposal, it is of interest to give an interpretation of the closed surface theorem (2.62) based on the gradient theorem, and to identify (2.63) as a vector identity.

According to (4.19), when $T(\nabla) = \nabla f$ and $f =$ constant, we have

$$\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\right) = 0, \tag{4.195}$$

which is the same as (2.62), originally proved with the aid of the relationships between the derivatives of the unit vectors. From the point of view of the gradient theorem given by

$$\iiint \nabla f \, dV = \oiint f \, d\mathbf{S}$$

when f = constant, we obtain

$$\oiint d\mathbf{S} = 0, \tag{4.196}$$

which is the closed surface theorem in the integral form. Equation (4.195) can therefore be considered as the differential form of the integral theorem for a closed surface.

In view of the definition of the curl operator given by (4.85), (2.63) is recognized as

$$\nabla \left(\frac{\hat{u}_j}{h_j} \right) = 0, \qquad j = (1, 2, 3) . \tag{4.197}$$

Now,

$$\frac{\hat{u}_j}{h_j} = \nabla v_j;$$

hence (4.197) is equivalent to

$$\nabla \nabla v_j = 0, \tag{4.198}$$

which is a valid identity according to (4.178). By applying the curl theorem to the function $\mathbf{F} = \nabla v_j$, we obtain

$$\iiint \nabla \nabla v_j \, dV = \oiint \hat{n} \times \nabla v_j \, dS = \oiint \hat{n} \times \hat{n} \frac{\partial v_j}{\partial n} \, dS = 0. \tag{4.199}$$

Hence (4.198) may be considered as the differential form of the integral theorem stated by (4.199).

4-10 Scalar and Vector Green's Theorems

There are numerous theorems bearing the name of George Green (1793–1841). We first consider Green's theorem involving scalar functions. In the Gauss theorem, stated by (4.190), if we let

$$\mathbf{F} = a \nabla b, \tag{4.200}$$

where a and b are two scalar functions, then

$$\nabla \mathbf{F} = a \nabla \nabla b + (\nabla a) \cdot (\nabla b), \tag{4.201}$$

which is obtained by (4.150). Upon substituting (4.201) into (4.190) with $\hat{n} \cdot \mathbf{F} = a(\hat{n} \cdot \nabla b)$, we obtain

$$\iiint_V [a\nabla\nabla b + (\nabla a) \cdot (\nabla b)]\, dV = \oiint_S a(\hat{n} \cdot \nabla b)\, dS. \tag{4.202}$$

Because $\hat{n} \cdot \nabla b$ is the scalar component of ∇b in the direction of the unit vector $\hat{n}$, it is equal to $\partial b/\partial n$; (4.202) is often written in the form

$$\iiint_V [a\nabla\nabla b + (\nabla a) \cdot (\nabla b)]\, dV = \oiint_S a\frac{\partial b}{\partial n}\, dS. \tag{4.203}$$

For convenience, we will designate it as the *first scalar Green's theorem.*

If we let $\mathbf{F} = a\nabla b - b\nabla a$, then it is obvious that

$$\begin{aligned} \iiint_V (a\nabla\nabla b - b\nabla\nabla a)\, dV &= \oiint_S \left(a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right) dS \\ &= \oiint \hat{n} \cdot (a\nabla b - b\nabla a)\, dS. \end{aligned} \tag{4.204}$$

Equation (4.204) will be designated as the *second scalar Green's theorem.* Both (4.203) and (4.204) involve scalar functions only.

Two theorems involving one scalar and one vector can be constructed from (4.202) and (4.204). We consider three equations of the form (4.202) with three different scalar functions b_i $(i = 1, 2, 3)$. Then, by juxtaposing a unit vector $\hat{x}_i$ to each of these equations with $i = 1, 2, 3$ and summing the resultant equations, we obtain

$$\iiint [a\nabla\nabla\mathbf{b} + (\nabla a) \cdot (\nabla\mathbf{b})]\, dS = \oiint a(\hat{n} \cdot \nabla\mathbf{b})\, dS. \tag{4.205}$$

Similarly, by moving the function b to the posterior position in (4.204) and following the same procedure, we obtain

$$\iiint [a\nabla\nabla\mathbf{b} - (\nabla\nabla a)\mathbf{b}]\, dV = \oiint \hat{n} \cdot [a\nabla\mathbf{b} - (\nabla a)\mathbf{b}]\, dS. \tag{4.206}$$

Equation (4.205) is designated as the scalar–vector Green's theorem of the first kind and (4.206) as the second kind. Because of the invariance of the gradient and the divergence operators, these theorems, derived here using rectangular variables, are valid for functions defined in any coordinate system, including GCS. However, one must be careful to calculate $\nabla\mathbf{b}$, a dyadic, in a curvilinear system. In OCS,

$$\nabla\mathbf{b} = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial\mathbf{b}}{\partial v_i} = \sum_{i,j} \frac{\hat{u}_i}{h_i}\left[\frac{\partial b_j}{\partial v_i}\hat{u}_j + b_j\frac{\partial \hat{u}_j}{\partial v_i}\right], \tag{4.207}$$

which has appeared before in (4.155).

There are two vector Green's theorems, which are formulated, first, by letting

$$\mathbf{F} = \mathbf{a} \times \nabla\mathbf{b}. \tag{4.208}$$

In view of (4.157), we have

$$\nabla \mathbf{F} = \nabla \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \nabla \mathbf{b} \tag{4.209}$$

and

$$\hat{n} \cdot \mathbf{F} = \hat{n} \cdot (\mathbf{a} \times \nabla \mathbf{b}). \tag{4.210}$$

Upon substituting (4.209) and (4.210) into the Gauss theorem, we obtain

$$\iiint_V [\nabla \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \nabla \mathbf{b}]\, dV = \oiint_S \hat{n} \cdot (\mathbf{a} \times \nabla \mathbf{b})\, dS. \tag{4.211}$$

Equation (4.211) is designated as the *first vector Green's theorem*. By combining (4.211) with another equation of the same form as (4.211) with the roles of $\mathbf{a}$ and $\mathbf{b}$ interchanged, or by starting with $\mathbf{F} = \mathbf{a} \times \nabla \mathbf{b} - \mathbf{b} \times \nabla \mathbf{a}$, we obtain the *second vector Green's theorem*:

$$\iiint_V [\mathbf{b} \cdot \nabla \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \nabla \mathbf{b}]\, dV = \oiint_S \hat{n} \cdot (\mathbf{a} \times \nabla \mathbf{b} - \mathbf{b} \times \nabla \mathbf{a})\, dS. \tag{4.212}$$

The continuity of the function $\mathbf{F}$ imposed on the Gauss theorem is now carried over, for example, to the continuity of $\mathbf{a} \times \nabla \mathbf{b}$ in (4.212), and similarly for the other theorems.

4-11 Solenoidal Vector, Irrotational Vector, and Potential Functions

The main purpose of this book is to treat vector analysis based on a new symbolic method. The application of vector analysis to physical problems is not covered in this treatise. However, there are several topics in introductory courses on electromagnetics and hydrodynamics involving some technical terms in vector analysis that should be introduced in a book of this nature.

When the divergence of a vector function vanishes everywhere in the entire spatial domain, such a function is called a *solenoidal vector*, and it will be denoted by $\mathbf{F}_s$ in this section. If the curl of the same vector function also vanishes everywhere, it can be proved that the function under consideration must be a constant vector. Physically, when both the divergence and the curl of a vector vanish, it means that the field has no source. In general, a solenoidal field is characterized by

$$\nabla \cdot \mathbf{F}_s = 0, \tag{4.213}$$

$$\nabla \mathbf{F}_s = \mathbf{f}, \tag{4.214}$$

where we treat $\mathbf{f}$ as the source function responsible for producing the vector field.

When the curl of a vector vanishes but its divergence is nonvanishing, such a vector is called an *irrotational vector*, and it will be denoted by $\mathbf{F}_i$. Such a field vector is characterized by

$$\nabla \mathbf{F}_i = 0, \tag{4.215}$$

$$\nabla \mathbf{F}_i = f, \tag{4.216}$$

where the scalar function f is treated as the source function responsible for producing the field. In electromagnetics, $\mathbf{F}_s$ corresponds to the magnetic field in magnetostatics and $\mathbf{F}_i$ to the electric field in electrostatics. In hydrodynamics, $\mathbf{F}_i$ corresponds to the velocity field of a laminar flow, and $\mathbf{F}_s$ to that of a vortex.

In electrodynamics, the electric and magnetic fields are coupled, and they are both functions of space and time. Their relations are governed by Maxwell's equations. For example, in air, the system of equations is

$$\nabla \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{4.217}$$

$$\nabla \mathbf{H} = J + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{4.218}$$

$$\nabla (\varepsilon_0 \mathbf{E}) = \rho, \tag{4.219}$$

$$\nabla (\mu_0 \mathbf{H}) = 0, \tag{4.220}$$

$$\nabla \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{4.221}$$

where $\mathbf{J}$ and ρ denote, respectively, the current density and the charge density functions responsible for producing the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$, and μ_0 and ε_0 are two fundamental constants. It is seen that the magnetic field $\mathbf{H}$ is a solenoidal field, but the electric field is neither solenoidal nor irrotational, that is, $\nabla \mathbf{E} \neq 0$ and $\nabla \mathbf{E} \neq 0$.

The theoretical work in electrostatics and magnetostatics is to investigate the solutions of (4.213)–(4.214) and (4.215)–(4.216) under various boundary conditions of the physical problems. In electrodynamics, the theoretical work is to study the solutions of the differential equations such as (4.217)–(4.221) for various problems. In the case of electrostatics, in view of the vector identity (4.178), the electric field, now denoted by $\mathbf{E}$, can be expressed in terms of a scalar function V such that

$$\mathbf{E} = -\nabla V. \tag{4.222}$$

The negative sign in (4.222) is just a matter of tradition based on physical consideration; mathematically, it has no importance. The function V is called the *electrostatic potential function*. As a result of (4.219), we find that

$$\nabla \mathbf{E} = -\nabla \nabla V = \frac{\rho}{\varepsilon_0}, \tag{4.223}$$

where we have replaced the function f by ρ/ε_0, with ρ denoting the density function of a charge distribution and ε_0, a physical constant. The problem is now shifted to the study of the second-order partial differential equation

$$\nabla \nabla V = -\frac{\rho}{\varepsilon_0}, \tag{4.224}$$

which is called *Poisson's equation*. The operator $\nabla \nabla$, or div grad, is the Laplacian operator we introduced in Section 4.8.

In the case of magnetostatics, in view of identity (4.179), the magnetic field, now denoted by $\mathbf{H}$, replacing $\mathbf{F}_s$, can be expressed in terms of a vector function $\mathbf{A}$ such that

$$\mathbf{H} = \nabla \mathbf{A}. \tag{4.225}$$

$\mathbf{A}$ is called the *magnetostatic vector potential.* The function $\mathbf{f}$ in (4.214) corresponds to the density of a current distribution in magnetostatics, commonly denoted by $\mathbf{J}$. By taking the divergence of (4.214), we find that $\nabla \mathbf{f} = \nabla \mathbf{J} = 0$, which is true for a steady current. Upon substituting (4.225) into (4.214) with $\mathbf{F}_s$ and $\mathbf{f}$ replaced by $\mathbf{H}$ and $\mathbf{J}$, respectively, we obtain

$$\nabla \nabla \mathbf{A} = \mathbf{J}. \tag{4.226}$$

According to the Helmholtz theorem [9], in order to determine $\mathbf{A}$, one must impose a condition on the divergence of the vector function $\mathbf{A}$ in addition to (4.225). Because

$$\nabla \nabla \mathbf{A} = -\nabla \nabla \mathbf{A} + \nabla \nabla \mathbf{A}, \tag{4.227}$$

if we impose the condition

$$\nabla \mathbf{A} = 0, \tag{4.228}$$

then (4.226) becomes

$$\nabla \nabla \mathbf{A} = -\mathbf{J}. \tag{4.229}$$

The condition on the divergence of $\mathbf{A}$ so imposed upon is called the *gauge condition.* This condition must be compatible with the resultant differential equation for $\mathbf{A}$, (4.229). By taking the divergence of that equation, we observe that $\nabla \mathbf{A}$ must be equal to zero because $\nabla \mathbf{J} = 0$. Thus, the gauge condition so imposed is indeed compatible with (4.229). The analytical work in magnetostatics now rests on the study of the vector Poisson equation stated by (4.229) for various problems.

To solve the system of equations in electrodynamics such as the ones stated by (4.217)–(4.221), we let

$$\mu_0 \mathbf{H} = \nabla \mathbf{A}, \tag{4.230}$$

because $\mathbf{H}$ is a solenoidal vector. The function $\mathbf{A}$ is called the *dynamic vector potential.* Upon substituting (4.230) into (4.217), we obtain

$$\nabla \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{4.231}$$

Hence $\mathbf{E} + (\partial \mathbf{A}/\partial t)$ is irrotational, so we can express it in terms of a dynamic scalar potential ϕ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi. \tag{4.232}$$

Upon substituting the expressions for **H** and **E** given by (4.230) and (4.232) into (4.218), we obtain

$$\nabla\nabla\mathbf{A} = \mu_0\mathbf{J} - \frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla\frac{\partial\phi}{\partial t}\right), \tag{4.233}$$

where $c = (\mu_0\varepsilon_0)^{-1/2}$ is the velocity of light in free space.

In view of identity (4.227), we can impose a gauge condition on **A** such that

$$\nabla\mathbf{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}. \tag{4.234}$$

Then, (4.233) reduces to

$$\nabla\nabla\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}, \tag{4.235}$$

which is called the *vector Helmholtz wave equation.* By taking the divergence of (4.235) and making use of (4.221) and (4.234), we find that ϕ satisfies the following scalar Helmholtz wave equation:

$$\nabla\nabla\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}. \tag{4.236}$$

Once **A** and ϕ are known, the electromagnetic field vectors **E** and **H** can be found by using the following relations:

$$\mu_0\mathbf{H} = \nabla\mathbf{A}, \tag{4.237}$$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi. \tag{4.238}$$

The method of potentials in electrodynamics is a classical method. Another approach is to deal with the equations for **E** and **H** directly. Thus, by eliminating **E** or **H** between (4.217)–(4.218), we obtain

$$\nabla\nabla\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial\mathbf{J}}{\partial t}, \tag{4.239}$$

$$\nabla\nabla\mathbf{H} + \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla\times\mathbf{J}. \tag{4.240}$$

These are two basic equations that can be solved by the method of dyadic Green functions [3] or the vector Green's theorem [5].

Chapter 5

Vector Analysis on Surface

5-1 Surface Symbolic Vector and Symbolic Expression for a Surface

Vector analysis on a surface has previously been treated by Weatherburn [10]. His works are summarized by Van Bladel [11]. Most books on vector analysis do not cover this subject. The approach taken by Weatherburn is to define a two-dimensional surface operator similar to the del operator in space. Some key differential functions analogous to gradient, divergence, and curl are then introduced.

In this work, the treatment is different. We approach the analysis based on a symbolic vector method similar to the one found in Chapter 4 for vector analysis in space. A symbolic expression for a surface is defined in terms of a surface symbolic vector. Afterwards, several essential functions in vector analysis for a surface are introduced. They are different from the ones defined by Weatherburn. The relationships between the set introduced in this book and Weatherburn's set will be discussed later. Finally, it will be shown that there is an intimate relationship between the symbolic expression for a surface and the symbolic expression in space. In fact, the former can be deduced from the latter without an independent formulation. However, it is more natural to treat the vector analysis on a surface as an independent discipline first, and then point out its relationship to the vector analysis in space.

Following the symbolic method discussed in Chapter 4, we will introduce a symbolic surface vector, denoted by ∇_s, and the corresponding symbolic vector

expression $T(\nabla_s)$ for a surface that is defined by

$$T(\nabla_s) = \lim_{\Delta S \to 0} \frac{\sum_i T(\hat{m}_i)\, \Delta l_i}{\Delta S}, \tag{5.1}$$

where Δl_i denotes an elementary arc length of the contour enclosing ΔS, and $\hat{m}_i$ is the unit vector tangent to the surface and normal to its edge. The running index i covers the number of sides of ΔS. For a cell with four sides, i goes from 1 to 4. The symbolic expression is generated by replacing at least one vector in an algebraic vector expression with ∇_s. For example, $\nabla_s \times \mathbf{b}$ is created by replacing the vector $\mathbf{a}$ in $\mathbf{a} \times \mathbf{b}$ with ∇_s. The expression defined by (5.1) is invariant to, or independent of the choice of, the coordinates on the surface in the general Dupin system. It is recalled that the choice of (v_1, v_2) and the corresponding tangential unit vectors $(\hat{u}_1, \hat{u}_2)$ is quite arbitrary. To find the differential expression based on (5.1) in the general orthogonal Dupin system, let the sides of the surface cell be located at $v_1 \pm (\Delta v_1/2)$ and $v_2 \pm (\Delta v_2/2)$, with the corresponding unit normal vectors $\pm\hat{u}_1$ and $\pm\hat{u}_2$ located at these positions. The value of $\hat{u}_1$ evaluated at $v_1 + (\Delta v_1/2)$ is not equal to the value of the same unit vector evaluated at $v_1 - (\Delta v_1/2)$. The same is true for the metric coefficients h_1 and h_2. The area of the elementary surface ΔS is equal to $h_1 h_2\, \Delta v_1\, \Delta v_2$. Figure 5-1 shows the configuration of the cell. By substituting these quantities into (5.1) and taking the limit, one finds

$$T(\nabla_s) = \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial v_1} \left[h_2 T(\hat{u}_1) \right] + \frac{\partial}{\partial v_2} \left[h_1 T(\hat{u}_2) \right] \right\}. \tag{5.2}$$

For a plane surface located in the x–y plane in a rectangular system,

$$T(\nabla_s) = \frac{\partial}{\partial x} T(\hat{x}) + \frac{\partial}{\partial y} T(\hat{y}). \tag{5.3}$$

This is the only case where $T(\nabla_s)$ can be expressed conveniently in a rectangular system. In general, rectangular variables are not the proper ones to describe the

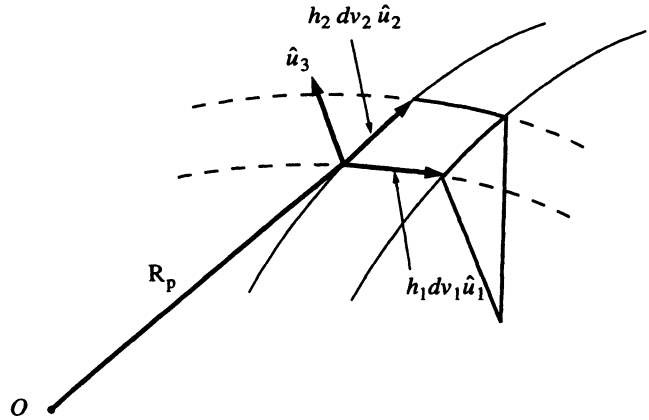


Figure 5-1 Cell on a surface in the general Dupin coordinate system.

function $T(\nabla_s)$ for a curved surface. From the definition of $T(\nabla_s)$ given by (5.1) and its differential form stated by (5.2), it is obvious that Lemma 4.1, introduced at the end of Chapter 4, Section 4-1, is also applicable to $T(\nabla_s)$ because $T(\hat{m}_i)$, $T(\hat{u}_1)$, and $T(\hat{u}_2)$ all have the proper form of a vector expression. By means of (5.2), it is now possible to derive the differential expression of some key functions in the vector analysis for a surface, analogous to the gradient, the divergence, and the curl in space.

5-2 Surface Gradient, Surface Divergence, and Surface Curl

For convenience, we repeat here the differential expression for $T(\nabla_s)$ expressed in the general Dupin system:

$$T(\nabla_s) = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial v_1}\left[h_2\, T(\hat{u}_1)\right] + \frac{\partial}{\partial v_2}\left[h_1\, T(\hat{u}_2)\right]\right\}. \tag{5.4}$$

5-2-1 Surface Gradient

If we let $T(\nabla_s) = \nabla_s f$, where f is a scalar function of position, then $T(\hat{u}_1) = f\hat{u}_1$, $T(\hat{u}_2) = f\hat{u}_2$. Upon substituting these quantities into (5.4), we obtain

$$\begin{aligned} T(\nabla_s) &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial v_1}(h_2 f\hat{u}_1) + \frac{\partial}{\partial v_2}(h_1 f\hat{u}_2)\right] \\ &= \frac{1}{h_1 h_2}\left[h_2 f\frac{\partial \hat{u}_1}{\partial v_1} + \left(f\frac{\partial h_2}{\partial v_1} + h_2\frac{\partial f}{\partial v_1}\right)\hat{u}_1\right. \\ &\quad \left. + h_1 f\frac{\partial \hat{u}_2}{\partial v_2} + \left(f\frac{\partial h_1}{\partial v_2} + h_1\frac{\partial f}{\partial v_2}\right)\hat{u}_2\right]. \end{aligned}$$

By making use of (2.61), with $h_3 = 1$, we can express the derivatives of $\hat{u}_1$ and $\hat{u}_2$ in terms of $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$, which yields

$$\begin{aligned} T(\nabla_s) = \frac{1}{h_1 h_2}\left[-h_2 f\left(\frac{1}{h_2}\frac{\partial h_1}{\partial v_2}\hat{u}_2 + \frac{\partial h_1}{\partial v_3}\hat{u}_3\right) + \left(f\frac{\partial h_2}{\partial v_1} + h_2\frac{\partial f}{\partial v_1}\right)\hat{u}_1\right. \\ \left. - h_1 f\left(\frac{1}{h_1}\frac{\partial h_2}{\partial v_1}\hat{u}_1 + \frac{\partial h_2}{\partial v_3}\hat{u}_3\right) + \left(f\frac{\partial h_1}{\partial v_2} + h_1\frac{\partial f}{\partial v_2}\right)\hat{u}_2\right]. \end{aligned}$$

Some of the terms cancel each other. The net result is

$$\nabla_s f = \frac{1}{h_1}\frac{\partial f}{\partial v_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial f}{\partial v_2}\hat{u}_2 - \left(\frac{1}{h_1}\frac{\partial h_1}{\partial v_3} + \frac{1}{h_2}\frac{\partial h_2}{\partial v_3}\right) f\hat{u}_3. \tag{5.5}$$

The coefficient in front of $f\hat{u}_3$ can be written in several different forms. If we denote the product $h_1 h_2$ by H, which is equal to Ω $(= h_1 h_2 h_3)$ with $h_3 = 1$, then

$$\frac{1}{H}\frac{\partial H}{\partial v_3} = \frac{1}{h_1}\frac{\partial h_1}{\partial v_3} + \frac{1}{h_2}\frac{\partial h_2}{\partial v_3} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = -J, \tag{5.6}$$

where R_1 and R_2 denote the principal radii of curvature of the surface as stated by (2.73) and (2.76), and J is the surface curvature defined by (2.84). The differential function we have just derived is designated as the *surface gradient* of f, and it will be denoted by $\nabla_s f$, which can now be written in the form

$$\nabla_s f = \frac{\hat{u}_1}{h_1}\frac{\partial f}{\partial v_1} + \frac{\hat{u}_2}{h_2}\frac{\partial f}{\partial v_2} + J\hat{u}_3 f. \tag{5.7}$$

Equation (5.7) shows that the symbol ∇_s is indeed a differential-algebraic operator, defined by

$$\nabla_s = \frac{\hat{u}_1}{h_1}\frac{\partial}{\partial v_1} + \frac{\hat{u}_2}{h_2}\frac{\partial}{\partial v_2} + J\hat{u}_3. \tag{5.8}$$

The operation of $\hat{u}_3$ on f is a simple multiplication.

5-2-2 Surface Divergence

If we let $T(\nabla_s) = \nabla_s \cdot \mathbf{F}$, then $T(\hat{u}_i) = \hat{u}_i \cdot \mathbf{F}$; hence

$$\begin{aligned}\nabla_s \cdot \mathbf{F} &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial v_1}(h_2 F_1) + \frac{\partial}{\partial v_2}(h_1 F_2)\right] \\ &= \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}F_i\right).\end{aligned} \tag{5.9}$$

The function so obtained is designated as the *surface divergence* of $\mathbf{F}$, and it will be denoted by $\nabla_s \mathbf{F}$; thus,

$$\nabla_s \mathbf{F} = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}F_i\right). \tag{5.10}$$

Equation (5.10) can be converted to an operational form as follows:

$$\begin{aligned}\nabla_s \mathbf{F} &= \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i \cdot \mathbf{F}\right) \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial v_i} + \frac{1}{H}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i\right)\cdot\mathbf{F}\right] \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial v_i} + J\hat{u}_3 \cdot \mathbf{F}\right].\end{aligned} \tag{5.11}$$

This expression shows that ∇_s is another differential-algebraic operator, defined by

$$\nabla_s = \left[\sum_{i=1}^{2}\frac{\hat{u}_i}{h_i}\cdot\frac{\partial}{\partial v_i} + J\hat{u}_3\cdot\right]. \tag{5.12}$$

When this operator is applied to $\mathbf{F}$, it yields the surface divergence of $\mathbf{F}$.

5-2-3 Surface Curl

If we let $T(\nabla_s) = \nabla_s \times \mathbf{F}$, then $T(\hat{u}_i) = \hat{u}_i \times \mathbf{F}$; hence

$$\begin{aligned}\nabla_s \times \mathbf{F} &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial v_1}(h_2\hat{u}_1 \times \mathbf{F}) + \frac{\partial}{\partial v_2}(h_1\hat{u}_2 \times \mathbf{F})\right] \\ &= \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i \times \mathbf{F}\right) \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} + \frac{1}{H}\frac{\partial}{\partial v_i}\left(\frac{H}{h_i}\hat{u}_i\right) \times \mathbf{F}\right] \\ &= \sum_{i=1}^{2}\left[\frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i} + J\hat{u}_3 \times \mathbf{F}\right].\end{aligned} \tag{5.13}$$

The function so created is called the *surface curl* of $\mathbf{F}$ and it will be denoted by $\nabla_s\mathbf{F}$; thus,

$$\nabla_s\mathbf{F} = \left[\sum_{i=1}^{2}\frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i} + J\hat{u}_3\times\right]\mathbf{F}. \tag{5.14}$$

It is evident that ∇_s is another differential-algebraic operator, defined by

$$\nabla_s = \left(\sum_{i=1}^{2}\frac{\hat{u}_i}{h_i} \times \frac{\partial}{\partial v_i} + J\hat{u}_3\times\right). \tag{5.15}$$

When it is operated on $\mathbf{F}$, it yields the surface curl of $\mathbf{F}$. As with the vector analysis in space, we have three independent surface operators; they are partly differential and partly algebraic, a special feature of the surface operators. By evaluating the derivatives of $\mathbf{F}$ with respect to v_i in (5.14) and simplifying the result with the aid of (2.59) and (2.61), with $h_3 = 1$, one finds

$$\begin{aligned}\nabla_s\mathbf{F} = \frac{1}{H}\Bigg\{&\left[h_1\frac{\partial F_3}{\partial v_2} + h_1F_2\frac{\partial h_1}{\partial v_3}\right]\hat{u}_1 \\ &- \left[h_2\frac{\partial F_3}{\partial v_1} + h_1F_1\frac{\partial h_2}{\partial v_3}\right]\hat{u}_2 \\ &+ \left[\frac{\partial(h_2F_2)}{\partial v_1} - \frac{\partial(h_1F_1)}{\partial v_2}\right]\hat{u}_3\Bigg\}.\end{aligned} \tag{5.16}$$

The $\hat{u}_3$ component of $\nabla_s\mathbf{F}$ is the same as the corresponding component of the three-dimensional $\nabla\mathbf{F}$ in a Dupin coordinate system, but the two transversal components are different.

5-3 Relationship Between the Volume and Surface Symbolic Expressions

Although the surface symbolic vector ∇_s and the symbolic expression $T(\nabla_s)$ involving ∇_s as defined by (5.1) and its differential form by (5.2) appear to be independent of ∇ and $T(\nabla)$, actually they are intimately related. If we express $T(\nabla)$ in the general Dupin system ($h_3 = 1$), then (4.80) becomes

$$T(\nabla) = \frac{1}{H}\sum_{i=1}^{3}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i)\right], \tag{5.17}$$

where $H = h_1 h_2$. The differential expression of $T(\nabla_s)$ as given by (5.2) can be written in the form

$$T(\nabla_s) = \frac{1}{H}\sum_{i=1}^{2}\frac{\partial}{\partial v_i}\left[\frac{H}{h_i}T(\hat{u}_i)\right]. \tag{5.18}$$

It is obvious that the first two terms of (5.17) are exactly the same as $T(\nabla_s)$; hence

$$T(\nabla) = T(\nabla_s) + \frac{1}{H}\frac{\partial}{\partial v_3}\left[H\,T(\hat{u}_3)\right]. \tag{5.19}$$

Equation (5.19), therefore, can be used to find $T(\nabla_s)$ once $T(\nabla)$ is known or $T(\nabla_s)$ can be defined as the sum of the first two terms of $T(\nabla)$. From this point of view, $T(\nabla_s)$ is not an independent function, and ∇_s is not an independent symbolic vector.

The last term of (5.19) can be written in the form

$$\begin{aligned}\frac{1}{H}\frac{\partial}{\partial v_3}\left[HT(\hat{u}_3)\right] &= \frac{\partial T(\hat{u}_3)}{\partial v_3} + \frac{1}{H}\frac{\partial H}{\partial v_3}T(\hat{u}_3)\\ &= \frac{\partial T(\hat{u}_3)}{\partial v_3} - J\,T(\hat{u}_3).\end{aligned} \tag{5.20}$$

Equation (5.19) is therefore equivalent to

$$T(\nabla) = T(\nabla_s) + \frac{\partial T(\hat{u}_3)}{\partial v_3} - J\,T(\hat{u}_3). \tag{5.21}$$

By using the expression of ∇f, $\nabla\mathbf{F}$, and $\overline{\nabla}\mathbf{F}$ in the Dupin system and the expressions of $\nabla_s f$, $\nabla_s\mathbf{F}$, and $\overline{\nabla}_s\mathbf{F}$ given by (5.7), (5.10), (5.14), it can be easily verified that (5.21) is indeed satisfied when we let $T(\nabla)$ equal ∇f, $\nabla\mathbf{F}$, $\nabla \times \mathbf{F}$, respectively.

5-4 Relationship Between Weatherburn's Surface Functions and the Functions Defined in the Method of Symbolic Vector

In the classic work of Weatherburn [10], he defines the surface gradient, the surface divergence, and the surface curl by retaining the two transversal parts of the three-dimensional functions. Our notations for his functions are $\nabla_t f$, $\nabla_t\mathbf{F}$, and $\overline{\nabla}_t\mathbf{F}$.

Weatherburn originally used the same notations as the three-dimensional functions ∇f, $\nabla \cdot \mathbf{F}$, and $\nabla \times \mathbf{F}$. The surface functions defined by Brand [1] in an orthogonal Dupin system are the same as Weatherburn's. Van Bladel [11] uses three linguistic notations for these functions, namely, $\text{grad}_s f$, $\text{div}_s \mathbf{F}$, and $\text{curl}_s \mathbf{F}$. They are defined by

$$\text{grad}_s f = \nabla_t f = \sum_{i=1}^{2} \frac{\hat{u}_i}{h_i} \frac{\partial f}{\partial v_i}, \tag{5.22}$$

$$\text{div}_s \mathbf{F} = \nabla_t \mathbf{F} = \sum_{i=1}^{2} \frac{\hat{u}_i}{h_i} \cdot \frac{\partial \mathbf{F}}{\partial v_i}, \tag{5.23}$$

$$\text{curl}_s \mathbf{F} = \overline{\nabla}_t \mathbf{F} = \sum_{i=1}^{2} \frac{\hat{u}_i}{h_i} \times \frac{\partial \mathbf{F}}{\partial v_i}. \tag{5.24}$$

These three terms have appeared before in our surface functions defined by (5.7), (5.11), and (5.14). Thus, the relations between the two sets are

$$\nabla_s f = \nabla_t f + J\hat{u}_3 f, \tag{5.25}$$

$$\nabla_s \mathbf{F} = \nabla_t \mathbf{F} + J\hat{u}_3 \cdot \mathbf{F}, \tag{5.26}$$

$$\overline{\nabla}_s \mathbf{F} = \overline{\nabla}_t \mathbf{F} + J\hat{u}_3 \times \mathbf{F}. \tag{5.27}$$

We must emphasize that $\nabla_t \mathbf{F}$ and $\overline{\nabla}_t \mathbf{F}$ are not scalar and vector products between ∇_t and $\mathbf{F}$. As with our ∇_s, $\overline{\nabla}_s$, and $\overline{\nabla}_s$, they are three independent operators.

We have so far presented the three key surface functions in orthogonal Dupin coordinate systems. By following the same procedure in the three-dimensional GCS, it is not difficult to extend the formulation to nonorthogonal Dupin systems ($\hat{u}_1 \cdot \hat{u}_2 \neq 0$, but $\hat{u}_1 \cdot \hat{u}_3 = \hat{u}_2 \cdot \hat{u}_3 = 0$). By introducing the primary and reciprocal vectors on a curved surface, we can show the invariance of these surface functions in the nonorthogonal Dupin coordinate systems.

As far as the surface functions are concerned, once the relationships between the two sets are known, it is a matter of personal preference as to which set should be considered as the standard surface functions. In a subsequent section dealing with integral theorems, it will be evident that the set derived from the present method, that is, $\nabla_s f$, $\nabla_s \mathbf{F}$, and $\overline{\nabla}_s \mathbf{F}$, or in general, $T(\nabla_s)$, is much more convenient to formulate the generalized Gauss theorem for a surface. We may also inject a remark that in electromagnetic theory, the equation of continuity (the law of conservation of charge) relating the surface current density $\mathbf{J}_s$ and the time rate of change of the surface charge density ρ_s is described by

$$\nabla_s \mathbf{J}_s = -\frac{\partial \rho_s}{\partial t} \tag{5.28}$$

when the surrounding medium has no loss. Here, it is $\nabla_s \mathbf{J}$, not $\nabla_t \cdot \mathbf{J}$ or $\text{div}_s \mathbf{f}$, that enters the formulation. On the other hand, for the rate of change of a scalar function

on a surface in a direction tangent to the surface, both $\nabla_s f$ and $\nabla_t f$ produce the same result:

$$\frac{\partial f}{\partial s} = \hat{u}_s \cdot \nabla_s f = \hat{u}_s \cdot \nabla_t f. \tag{5.29}$$

The vector component $(\partial f/\partial v_3)\hat{u}_3$ in $\nabla_s f$ does not affect the value of $\partial f/\partial s$. For the curl function, one finds

$$\hat{u}_3 \cdot \nabla_s \mathbf{f} = \hat{u}_3 \cdot \nabla_t \times \mathbf{f} = \hat{u}_3 \cdot \nabla \mathbf{f}, \tag{5.30}$$

an identity to be used later.

5-5 Generalized Gauss Theorem for a Surface

By integrating the differential expression for $T(\nabla_s)$, (5.2), on an open surface S with contour L, we have

$$\iint_S T(\nabla_s)\, dS = \iint_S \sum_{i=1}^{2} \frac{\partial}{\partial v_i}\left[\frac{H}{h_i} T(\hat{u}_i)\right] dv_1\, dv_2, \tag{5.31}$$

where $dS = h_1 h_2\, dv_1\, dv_2 = H\, dv_1\, dv_2$. We assume that $T(\hat{u}_i)$ is continuous throughout S. The integrals in (5.31) can be carried out as follows:

$$\begin{aligned}\iint_S \frac{\partial}{\partial v_1}[h_2 T(\hat{u}_1)]\, dv_1\, dv_2 &= \int_{v_{2\min}}^{v_{2\max}} [h_2 T(\hat{u}_1)]_{P_1}^{P_2}\, dv_2 \\ &= \int_{L_2} h_2 T(\hat{u}_1)\, dv_2 - \int_{-L_1} h_2 T(\hat{u}_1)\, dv_2 \\ &= \oint_L h_2 T(\hat{u}_1)\, dv_2,\end{aligned} \tag{5.32}$$

where the locations (P_1, P_2), the segments L_1, L_2, and the two extreme values $v_{2\min}$, $v_{2\max}$ are shown in Fig. 5-2a. Similarly,

$$\begin{aligned}\iint_S \frac{\partial}{\partial v_2}[h_1 T(\hat{u}_2)]\, dv_1\, dv_2 &= \int_{v_{1\min}}^{v_{1\max}} [h_1 T(\hat{u}_2)]_{P_3}^{P_4}\, dv_1 \\ &= \int_{-L_4} h_1 T(\hat{u}_2)\, dv_1 - \int_{L_3} h_1 T(\hat{u}_2)\, dv_1 \\ &= -\oint_L h_1 T(\hat{u}_2)\, dv_1,\end{aligned} \tag{5.33}$$

where the locations P_3, P_4, the segments L_3, L_4, and the two extreme values $v_{1\min}$, $v_{1\max}$ are shown in Fig. 5-2b. Hence

$$\iint_S T(\nabla_s)\, dS = \oint_L \left[h_2 T(\hat{u}_1)\, dv_2 - h_1 T(\hat{u}_2)\, dv_1\right]. \tag{5.34}$$

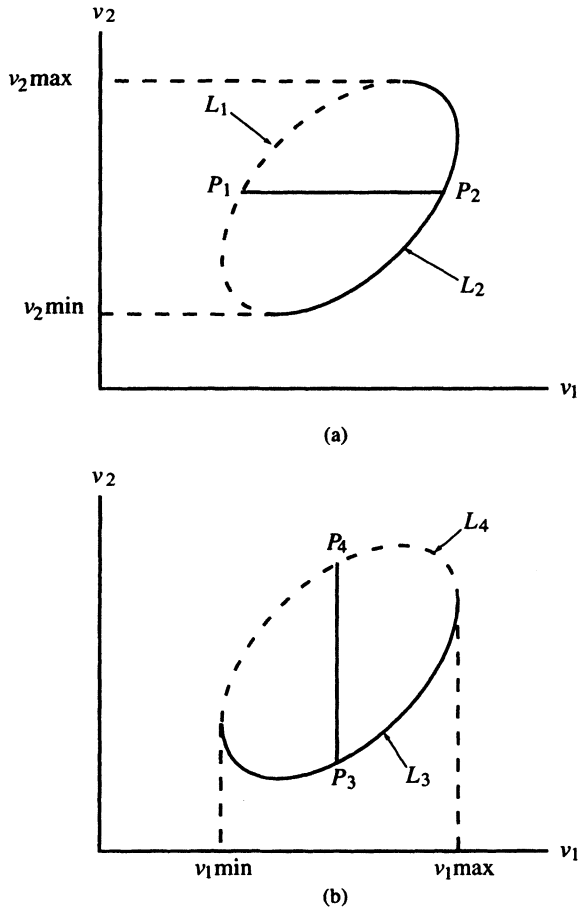


Figure 5-2 Domain of integration in the (v_1, v_2) plane of a simple region.

Because $T(\hat{u}_i)$ is linear with respect to $\hat{u}_i$ with $i = 1, 2$, the integrand in the line integral is proportional to

$$\hat{u}_1 h_2 \, dv_2 - \hat{u}_2 h_1 \, dv_1, \tag{5.35}$$

which can be simplified. Let us consider a segment of the contour L_1, which is the edge of S. In the tangential plane containing $\hat{u}_1$ and $\hat{u}_2$ at a typical point P, the four key unit vectors are shown in Fig. 5-3. All of them are tangential to the surface at P. A three-dimensional display of these vectors and the normal vector $\hat{u}_3$ is shown in Fig. 5-4. In these figures, $\hat{u}_\ell$ is tangential to the edge of the surface, and $\hat{u}_m$ is normal to the edge, but tangential to the surface. The relations between these unit vectors are

$$\hat{u}_1 \times \hat{u}_2 = \hat{u}_3 = \hat{u}_m \times \hat{u}_\ell. \tag{5.36}$$

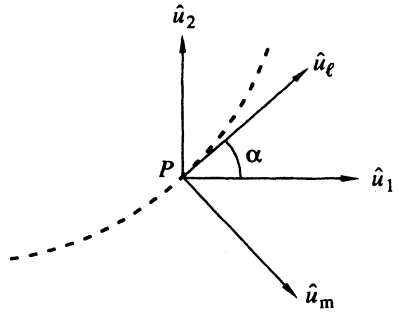


Figure 5-3 Four tangential vectors in the plane containing $\hat{u}_1$ and $\hat{u}_2$.

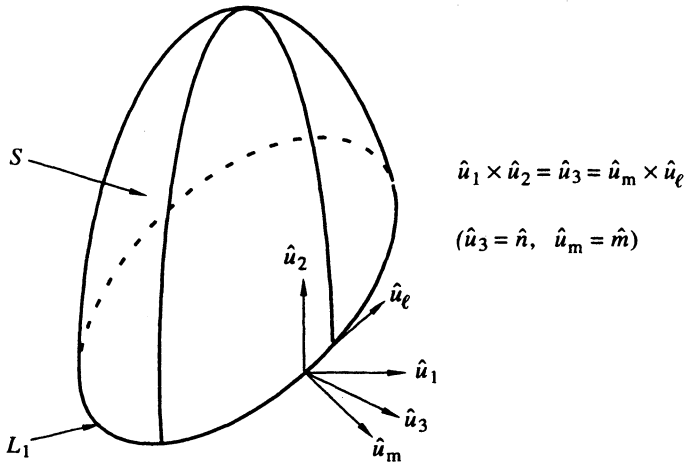


Figure 5-4 Three-dimensional view of the unit vectors at the edge of an open surface.

The algebraic relations between them are

$$\hat{u}_1 = \cos\alpha\,\hat{u}_\ell + \sin\alpha\,\hat{u}_m, \tag{5.37}$$

$$\hat{u}_2 = \sin\alpha\,\hat{u}_\ell - \cos\alpha\,\hat{u}_m, \tag{5.38}$$

where α is the angle between $\hat{u}_1$ and $\hat{u}_\ell$. If we denote the total differential arc length of the contour at P by dl, then

$$h_1\,dv_1 = \cos\alpha\,dl, \tag{5.39}$$

$$h_2\,dv_2 = \sin\alpha\,dl. \tag{5.40}$$

Upon substituting (5.39) and (5.40) into (5.35), we find

$$\hat{u}_1 h_2\,dv_2 - \hat{u}_2 h_1\,dv_1 = \hat{u}_m\,dl; \tag{5.41}$$

hence

$$T(\hat{u}_1)\,h_2\,dv_2 - T(\hat{u}_2)\,h_1\,dv_1 = T(\hat{u}_m)\,dl. \tag{5.42}$$

Equation (5.31) therefore reduces to

$$\iint_S T(\nabla_s)\, dS = \oint_L T(\hat{u}_m)\, dl. \tag{5.43}$$

The unit vector $\hat{u}_m$ is commonly denoted by $\hat{m}$, in contrast to the notation $\hat{n}$ used for $\hat{u}_3$; (5.43), therefore, will be written as

$$\iint_S T(\nabla_s)\, dS = \oint_L T(\hat{m})\, dl. \tag{5.44}$$

Equation (5.44) is designated as the *generalized Gauss theorem for a surface* or the *generalized surface Gauss theorem.* It converts an open surface integral into a closed line integral, and it has the same significance as the generalized Gauss theorem in space, which converts a volume integral into a closed surface integral. Various integral theorems can be derived by choosing the proper form for $T(\nabla_s)$.

1. Surface gradient theorem
   Let $T(\nabla_s) = \nabla_s f = \nabla_s f$; then $T(\hat{m}) = \hat{m} f$. Hence
   $$\iint_S \nabla_s f\, dS = \oint_L f \hat{m}\, dl. \tag{5.45}$$
2. Surface divergence theorem
   Let $T(\nabla_s) = \nabla_s \cdot \mathbf{F} = \nabla \cdot \mathbf{F}$; then $T(\hat{m}) = \hat{m} \cdot \mathbf{F}$. Hence
   $$\iint_S \nabla \cdot \mathbf{F}\, dS = \oint_L \hat{m} \cdot \mathbf{F}\, dl. \tag{5.46}$$
3. Surface curl theorem
   Let $T(\nabla_s) = \nabla_s \times \mathbf{F} = \nabla_s \mathbf{F}$; then $T(\hat{m}) = \hat{m} \times \mathbf{F}$. Hence
   $$\iint_S \nabla_s \mathbf{F}\, dS = \oint_L \hat{m} \times \mathbf{F}\, dl. \tag{5.47}$$

   In view of the relationship between $T(\nabla)$ and $T(\nabla_s)$ as described by (5.19), the generalized Gauss theorem for a surface can be written in the form

   $$\iint_S \left\{ T(\nabla) - \frac{1}{H} \frac{\partial}{\partial v_3} \left[H\, T(\hat{u}_3) \right] \right\} dS = \oint_L T(\hat{m})\, dl. \tag{5.48}$$

   If $T(\nabla)$ is proportional to $\hat{u}_3 \times \nabla$ or $\hat{n} \times \nabla$, then $T(\hat{u}_3) = 0$, and (5.48) becomes

   $$\iint_S T(\nabla)\, dS = \oint_L T(\hat{m})\, d\ell, \qquad T(\hat{n}) = 0. \tag{5.49}$$

   Three cases are considered now.

4. Cross-gradient theorem

Let $T(\nabla) = (\hat{n} \times \nabla) f$. As a result of Lemmas 4.1 and 4.2, we have

$$\begin{aligned}(\hat{n} \times \nabla) f &= (\hat{n} \times \nabla_n) f + (\hat{n} \times \nabla_f) f \\ &= -(\nabla \hat{n}) f + \hat{n} \times \nabla f = \hat{n} \times \nabla f,\end{aligned}$$

because $\nabla \hat{n} = 0$, and

$$T(\hat{m}) = (\hat{n} \times \hat{m}) f = \hat{u}_\ell f.$$

Hence

$$\iint_S \hat{n} \times \nabla f \, dS = \oint_L f \, d\ell. \tag{5.50}$$

For identification purposes, we designate it as the cross-gradient theorem.

5. Stokes's theorem

Let

$$\begin{aligned}T(\nabla) &= (\hat{n} \times \nabla) \cdot \mathbf{F} \\ &= (\hat{n} \times \nabla_n) \cdot \mathbf{F} + (\hat{n} \times \nabla_F) \cdot \mathbf{F} \\ &= -(\nabla \hat{n}) \cdot \mathbf{F} + \hat{n} \cdot (\nabla_F \times \mathbf{F}) = \hat{n} \cdot \nabla \mathbf{F}.\end{aligned}$$

Then

$$T(\hat{m}) = (\hat{n} \times \hat{m}) \cdot \mathbf{F} = \hat{u}_\ell \cdot \mathbf{F}.$$

Hence

$$\iint_S \hat{n} \cdot \nabla \mathbf{F} \, dS = \oint_L \mathbf{F} \cdot d\ell, \tag{5.51}$$

which is the famous theorem named after George Gabriel Stokes (1819–1903).

6. Cross–∇–cross theorem

Let $T(\nabla) = (\hat{n} \times \nabla) \times \mathbf{F}$. By means of Lemma 4.2, we have

$$\begin{aligned}(\hat{n} \times \nabla) \times \mathbf{F} &= (\hat{n} \times \nabla_n) \times \mathbf{F} + (\hat{n} \times \nabla_F) \times \mathbf{F} \\ &= -(\nabla \hat{n}) \times \mathbf{F} + (\hat{n} \times \nabla_F) \times \mathbf{F}.\end{aligned}$$

Now, $\nabla \hat{n} = 0$ because $\hat{n}$ is a linear vector, not curvilinear, and $T(\hat{m}) = (\hat{n} \times \hat{m}) \times \mathbf{F} = \hat{\ell} \times \mathbf{F}$; hence

$$\iint (\hat{n} \times \nabla_F) \times \mathbf{F} \, dS = \oint (\hat{\ell} \times \mathbf{F}) \, d\ell. \tag{5.52}$$

The function $(\hat{n} \times \nabla_F) \times \mathbf{F}$ is given by (4.162) with **a** and **b** therein replaced by $\hat{n}$ and **F**, respectively.

5-6 Surface Symbolic Expressions with a Single Symbolic Vector and Two Functions

A complete line of formulas and theorems can be derived covering these two topics. However, we will present only the essential formulation without actually going into detail. A third lemma dealing with symbolic expressions with a single-surface S vector and two functions is one of the main subjects to be covered. A scalar Green's theorem on a surface involving the surface Laplacian will also be presented.

For a symbolic expression with two functions, its differential form in the general Dupin system according to (5.2) is defined by

$$T(\nabla_s, \tilde{a}, \tilde{b}) = \frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left[\frac{H}{h_i} T(\hat{u}_i, \tilde{a}, \tilde{b}) \right], \tag{5.53}$$

where $H = h_1 h_2$. We now introduce two expressions with two *partial surface S vectors*, denoted by ∇_{sa} and ∇_{sb}, as follows:

$$T(\nabla_{sa}, \tilde{a}, \tilde{b}) = \frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left[\frac{H}{h_i} T(\hat{u}_i, \tilde{a}, \tilde{b}) \right]_{\tilde{b}=c}, \tag{5.54}$$

$$T(\nabla_{sb}, \tilde{a}, \tilde{b}) = \frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left[\frac{H}{h_i} T(\hat{u}_i, \tilde{a}, \tilde{b}) \right]_{\tilde{a}=c}. \tag{5.55}$$

It is obvious that Lemma 4.1 also applies to (5.54) and (5.55). Equation (5.53) can now be decomposed into three parts:

$$T(\nabla_s, \tilde{a}, \tilde{b}) = T(\nabla_{sa}, \tilde{a}, \tilde{b}) + T(\nabla_{sb}, \tilde{a}, \tilde{b}) - \frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left[\frac{H}{h_i} T(\hat{u}_i, \tilde{a}, \tilde{b}) \right]_{\tilde{a},\tilde{b}=c}. \tag{5.56}$$

Because $T(\hat{u}_i)$ is linear with respect to $\hat{u}_i$, we can combine $\hat{u}_i$ with H/h_i to form one function, and examine its derivatives. According to (2.62), with $h_3 = 1$,

$$\frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left(\frac{H}{h_i} \hat{u}_i \right) + \frac{1}{H} \frac{\partial}{\partial v_3} (H \hat{u}_3) = 0.$$

Thus,

$$\frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left(\frac{H}{h_i} \hat{u}_i \right) = -\frac{1}{H} \frac{\partial H}{\partial v_3} (H \hat{u}_3) = J \hat{u}_3. \tag{5.57}$$

The last term in (5.56), therefore, can be written as

$$-\frac{1}{H} \sum_{i=1}^{2} \frac{\partial}{\partial v_i} \left[\frac{H}{h_i} T(\hat{u}_i, \tilde{a}, \tilde{b}) \right]_{\tilde{a},\tilde{b}=c} = -J\, T(\hat{u}_3, \tilde{a}, \tilde{b}). \tag{5.58}$$

Lemma 5.1. For a symbolic expression defined with respect to a single-surface S vector and two functions, the following relation holds true:

$$T(\nabla_s, \tilde{a}, \tilde{b}) = T(\nabla_{sa}, f_1, f_2) + T(\nabla_{sb}, \tilde{a}, \tilde{b}) - J\,T(\hat{u}_3, \tilde{a}, \tilde{b}). \tag{5.59}$$

The proof of this lemma follows directly from (5.56) and (5.58). By means of this lemma, we can derive all the possible surface vector identities similar to the identities described by (4.140)–(4.151) and (4.156)–(4.160). We merely write down these relations without detailed explanation.

1.

$$\nabla_s(ab) = \nabla_{sa}(ab) + \nabla_{sb}(ab) - J\hat{u}_3 ab;$$

hence

$$\nabla_s(ab) = b\nabla_s a + a\nabla_s b - Jab\hat{u}_3. \tag{5.60}$$

2.

$$\nabla_s(a\mathbf{b}) = \nabla_{sa} \cdot (a\mathbf{b}) + \nabla_{sb} \cdot (a\mathbf{b}) - J\hat{u}_3 \cdot a\mathbf{b};$$

hence

$$\nabla_s(a\mathbf{b}) = a\nabla_s \mathbf{b} + \mathbf{b} \cdot \nabla_s a - Ja\mathbf{b} \cdot \hat{u}_3. \tag{5.61}$$

3.

$$\nabla_s \times (a\mathbf{b}) = \nabla_{sa} \times (a\mathbf{b}) + \nabla_{sb} \times (a\mathbf{b}) - J\hat{u}_3 \times a\mathbf{b};$$

hence

$$\nabla_s(a\mathbf{b}) = -b \times \nabla_s a + a\nabla_s \mathbf{b} + Ja\mathbf{b} \times \hat{u}_3. \tag{5.62}$$

4.

$$\begin{aligned}\nabla_s(\mathbf{a} \cdot \mathbf{b}) &= \nabla_{sa}(\mathbf{a} \cdot \mathbf{b}) + \nabla_{sb}(\mathbf{a} \cdot \mathbf{b}) - J\hat{u}_3(\mathbf{a} \cdot \mathbf{b}) \\ &= \mathbf{b} \times (\nabla_{sa} \times \mathbf{a}) + \mathbf{b} \cdot \nabla_{sa}\mathbf{a} \\ &\quad + \mathbf{a} \times (\nabla_{sb} \times \mathbf{b}) + \mathbf{a} \cdot \nabla_{sb}\mathbf{b} - J\hat{u}_3(\mathbf{a} \cdot \mathbf{b});\end{aligned}$$

hence

$$\nabla_s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{b} \times \nabla_s \mathbf{a} + \mathbf{b} \cdot \nabla_s \mathbf{a} + \mathbf{a} \times \nabla_s \mathbf{b} + \mathbf{a} \cdot \nabla_{sb}\mathbf{b} - J(\mathbf{a} \cdot \mathbf{b})\hat{u}_3. \tag{5.63}$$

We are making effective use of Lemma 4.1 in these exercises.

5.

$$\begin{aligned}\nabla_s \cdot (\mathbf{a} \times \mathbf{b}) &= \nabla_{sa} \cdot (\mathbf{a} \times \mathbf{b}) + \nabla_{sb} \cdot (\mathbf{a} \times \mathbf{b}) - J\hat{u}_3 \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{b} \cdot \nabla_{sa} \times \mathbf{a} + \mathbf{a} \cdot (\mathbf{b} \times \nabla_{sb}) - J\hat{u}_3 \cdot (\mathbf{a} \times \mathbf{b});\end{aligned}$$

hence

$$\nabla_s(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla_s \mathbf{a} - \mathbf{a} \cdot \nabla_s \mathbf{b} - J(\mathbf{a} \times \mathbf{b}) \cdot \hat{u}_3. \tag{5.64}$$

6.

$$(\nabla_s \cdot \mathbf{a})\mathbf{b} = (\nabla_{sa} \cdot \mathbf{a})\mathbf{b} + (\nabla_{sb} \cdot \mathbf{a})\mathbf{b} - J(\hat{u}_3 \cdot \mathbf{a})\mathbf{b};$$

hence

$$\nabla_s \mathbf{ab} = \mathbf{b}\nabla_s \mathbf{a} + \mathbf{a} \cdot \nabla_s \mathbf{b} - J\mathbf{ba} \cdot \hat{u}_3. \tag{5.65}$$

7.

$$\begin{aligned}\nabla_s \times (\mathbf{a} \times \mathbf{b}) &= \nabla_{sa} \times (\mathbf{a} \times \mathbf{b}) + \nabla_{sb} \times (\mathbf{a} \times \mathbf{b}) - J\hat{u}_3 \times (\mathbf{a} \times \mathbf{b}) \\ &= (\nabla_{sa} \cdot \mathbf{b})\mathbf{a} - (\nabla_{sa} \cdot \mathbf{a})\mathbf{b} + (\nabla_{sb} \cdot \mathbf{b})\mathbf{a} - (\nabla_{sb} \cdot \mathbf{a})\mathbf{b} \\ &\quad - j\hat{u}_3 \times (\mathbf{a} \times \mathbf{b});\end{aligned}$$

hence

$$\nabla_s(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla_s \mathbf{a} - \mathbf{b}\nabla_s \mathbf{a} + \mathbf{a}\nabla_s \mathbf{b} - \mathbf{a} \cdot \nabla_s \mathbf{b} + J(\mathbf{a} \times \mathbf{b}) \times \hat{u}_3. \tag{5.66}$$

8.

$$\begin{aligned}(\nabla_s \times \mathbf{a}) \times \mathbf{b} &= (\nabla_{sa} \times \mathbf{a}) \times \mathbf{b} + (\nabla_{sb} \times \mathbf{a}) \times \mathbf{b} - J(\hat{u}_3 \times \mathbf{a}) \times \mathbf{b} \\ &= (\nabla_{sa} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_{sb} \cdot \mathbf{b}) - \nabla_{sb}(\mathbf{a} \cdot \mathbf{b}) - J(\hat{u}_3 \times \mathbf{a}) \times \mathbf{b} \\ &= (\nabla_{sa} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a}(\nabla_{sb} \cdot \mathbf{b}) - \mathbf{a} \times (\nabla_{sb} \times \mathbf{b}) - (\mathbf{a} \cdot \nabla_{sb})\mathbf{b} \\ &\quad - J(\hat{u}_3 \times \mathbf{a}) \times \mathbf{b};\end{aligned}$$

hence

$$(\nabla_s \times \mathbf{a}) \times \mathbf{b} = -\mathbf{b} \times \nabla_s \mathbf{a} + \mathbf{a}\nabla_s \mathbf{b} - \mathbf{a} \times \nabla_s \mathbf{b} - \mathbf{a} \cdot \nabla_s \mathbf{b} - J\mathbf{b} \times (\mathbf{a} \times \hat{u}_3). \tag{5.67}$$

As with (4.159), the surface symbolic expression involves the vector product between $\mathbf{b}$ and $\nabla_s \mathbf{a}$ and the products of $\mathbf{a}$ with the surface gradient, the surface divergence, and the surface curl of $\mathbf{b}$.

5-7 Surface Symbolic Expressions with Two Surface Symbolic Vectors and a Single Function

In presenting the symbolic expressions with two symbolic expressions in a three-dimensional space, we define these functions in the rectangular system and then their relations. The rectangular system is not a convenient coordinate system for a curved surface. For this reason, the subject will be presented using the basic definition of the surface symbolic expression as stated by (5.1), namely,

$$T(\nabla_s) = \lim_{\Delta S \to 0} \frac{\sum_i T(\hat{m})\, \Delta \ell_i}{\Delta S}. \tag{5.68}$$

For an expression consisting of two surface symbolic vectors and a single function, we can apply the same formula twice, that is,

$$T(\nabla_s, \nabla_s', \tilde{f}) = \lim_{\Delta S' \to 0} \lim_{\Delta S \to 0} \frac{\sum_i \sum_j T(\hat{m}, \hat{m}', \tilde{f})\, \Delta \ell_\ell\, \Delta \ell'_j}{\Delta S\, \Delta S'}. \tag{5.69}$$

Several cases will be considered.

1. Surface Laplacian of a scalar function
Let $T(\nabla_s, \nabla_s', \tilde{f}) = \nabla_s \cdot \nabla_s' f$; then $T(\hat{m}, \hat{m}', \tilde{f}) = \hat{m} \cdot \hat{m}' f$. Thus,

$$\begin{aligned} \nabla_s \cdot \nabla_s' f &= \lim_{\Delta S \to 0} \lim_{\Delta S' \to 0} \frac{\sum_i \sum_j \hat{m} \cdot \hat{m}' f \, \Delta \ell \, \Delta \ell'}{\Delta S \, \Delta S'} \\ &= \lim_{\Delta S \to 0} \frac{\sum_i \hat{m} \cdot \nabla_s f \, \Delta \ell}{\Delta S} = \nabla_s \nabla_s f. \end{aligned} \tag{5.70}$$

The double operator $\nabla_s \nabla_s$ is designated as the surface Laplacian. No special notation will be introduced for this operator.

2. Surface Laplacian of a vector function
Let $T(\nabla_s, \nabla_s', \tilde{f}) = (\nabla_s \cdot \nabla_s')\mathbf{F}$; then $T(\hat{m}, \hat{m}', \tilde{f}) = (\hat{m} \cdot \hat{m}')\mathbf{F}$. Following the same procedure as the previous case, we obtain

$$\nabla_s \cdot \nabla_s' \mathbf{F} = \nabla_s \nabla_s \mathbf{F}. \tag{5.71}$$

3. The surface gradient of the surface divergence of a vector function
Let $T(\nabla_s, \nabla_s', \tilde{f}) = \nabla_s \nabla_s' \cdot \mathbf{F}$; then $T(\hat{m}, \hat{m}', \tilde{f}) = \hat{m}(\hat{m}' \cdot \mathbf{F})$. The double limit of this function yields

$$\nabla_s \cdot \nabla_s' \mathbf{F} = \nabla_s \nabla_s \mathbf{F}. \tag{5.72}$$

4. The double surface curl of a vector function
Let $T(\nabla_s, \nabla_s', \tilde{f}) = \nabla_s \times (\nabla_s' \times \mathbf{F})$; then $T(\hat{m}, \hat{m}', \mathbf{F}) = \hat{m} \times (\hat{m}' \times \mathbf{F})$. We find

$$\nabla_s \times (\nabla_s' \times \mathbf{F}) = \nabla_s \nabla_s \mathbf{F}. \tag{5.73}$$

As a result of Lemma 4.1,

$$\begin{aligned} \nabla_s \times (\nabla_s' \times \mathbf{F}) &= (\nabla_s \cdot \mathbf{F})\nabla_s' - (\nabla_s \cdot \nabla_s')\mathbf{F} \\ &= \nabla_s'(\nabla_s \cdot \mathbf{F}) - \nabla_s \cdot \nabla_s' \mathbf{F}. \end{aligned}$$

We obtain the identity

$$\begin{aligned} \nabla_s \nabla_s \mathbf{F} &= \nabla_s \nabla_s \mathbf{F} - \nabla_s \nabla_s \mathbf{F} \\ &= (\nabla_s \nabla_s - \nabla_s \nabla_s)\mathbf{F}. \end{aligned} \tag{5.74}$$

This is analogous to the three-dimensional identity

$$\begin{aligned} \nabla \nabla \mathbf{F} &= \nabla \nabla \mathbf{F} - \nabla \nabla \mathbf{F} \\ &= (\nabla \nabla - \nabla \nabla)\mathbf{F}. \end{aligned}$$

Finally, we would like to present a scalar Green's theorem for a curved surface. This theorem can be obtained by applying the generalized surface Gauss theorem, (5.44), with

$$T(\nabla_s) = \nabla_s \cdot (a \nabla_s b - b \nabla_s a). \tag{5.75}$$

As a result of Lemmas 4.1 and 5.1 or by means of (5.61), we find

$$T(\nabla_s) = a\nabla_s\nabla_s b - b\nabla_s\nabla_s a \tag{5.76}$$

and

$$\begin{aligned} T(\hat{m}) &= \hat{m} \cdot (a\nabla_s b - b\nabla_s a) \\ &= a\frac{\partial b}{\partial m} - b\frac{\partial a}{\partial m} . \end{aligned} \tag{5.77}$$

Substituting (5.76) and (5.77) into (5.44), we obtain the scalar surface Green's theorem of the second kind:

$$\iint (a\nabla_s\nabla_s b - b\nabla_s\nabla_s a)\, ds = \oint \left(a\frac{\partial b}{\partial m} - b\frac{\partial a}{\partial m} \right) d\ell. \tag{5.78}$$

Other theorems can be derived following similar procedures.

Chapter 6

Vector Analysis of Transport Theorems

6-1 Helmholtz Transport Theorem

Thus far, we have been dealing only with functions of position, that is, functions that are dependent on spatial variables only. In many engineering and physical problems, the quantities involved are functions of both space and time. Examples are the induced voltage in a moving coil of an electric generator and the transport of fluid in a channel. The mathematical formulation of these problems requires a knowledge of vector analysis involving a moving surface or a moving body. One of the fundamental theorems in this area is the Helmholtz transport theorem, named after the renowned German scientist Hermann Ludwig Ferdinand von Helmholtz (1821–1894). The theorem deals with the time rate of change of a surface integral of Type IV stated by (3.70), in which the domain of integration and the integrand are functions of both space and time. The quantity under consideration is defined by

$$\begin{aligned} I &= \frac{d}{dt}\iint_{S(\mathbf{R},t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\iint_{S_2(\mathbf{R},\ t+\Delta t)} \mathbf{F}(\mathbf{R},t+\Delta t)\cdot d\mathbf{S} - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S}\right], \end{aligned} \tag{6.1}$$

where $\mathbf{F}(\mathbf{R}, t)$ is an abbreviated notation for $\mathbf{F}(x_1, x_2, x_3, t)$. The x_i's denote the coordinate variables of the position vector where the function $\mathbf{F}$ is defined, and t is the time variable. The domain of integration changes from $S_1(\mathbf{R}, t)$ to $S_2(\mathbf{R},\ t + \Delta t)$ in a small time interval Δt, as shown in Fig. 6-1. To evaluate the

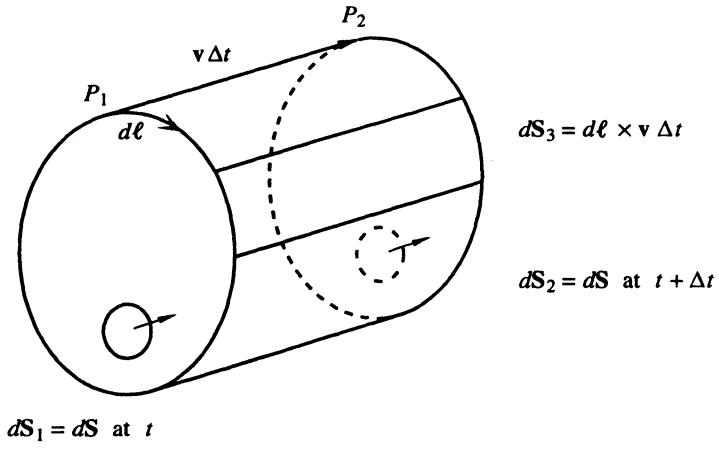


Figure 6-1 Moving surface at two different instants.

limiting value of the difference of the two surface integrals contained in (6.1), we first expand the integrand $\mathbf{F}(\mathbf{R}, t + \Delta t)$ in a Taylor series with respect to t:

$$\mathbf{F}(\mathbf{R}, t + \Delta t) = \mathbf{F}(\mathbf{R}, t) + \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \mathbf{F}(\mathbf{R}, t)}{\partial t^2}(\Delta t)^2 + \cdots . \tag{6.2}$$

Upon substituting (6.2) into (6.1), we have

$$\begin{aligned} I = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Bigg\{ & \iint_{S_2(\mathbf{R},\, t+\Delta t)} \left[\mathbf{F}(\mathbf{R}, t) + \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t}\Delta t + \cdots \right] \cdot d\mathbf{S} \\ & - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \Bigg\} \\ = \iint_{S(\mathbf{R},t)} & \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t} \cdot d\mathbf{S} + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\iint_{S_2(\mathbf{R},\, t+\Delta t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \right. \\ & \left. - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \right]. \end{aligned} \tag{6.3}$$

In (6.3), $S_1(\mathbf{R}, t)$ is the same as $S(\mathbf{R}, t)$; the subscripts 1 and 2 are used to identify the location of the surface at time t and at a later time $t + \Delta t$, respectively, as shown in Fig. 6-1. A point P_1 at the contour of S_1 is displaced to a point P_2 at the contour of S_2 during the time interval Δt. The displacement is equal to $\mathbf{v}\Delta t$, where $\mathbf{v}$ denotes the velocity of motion at that location, which may vary continuously from one location to another around the contour. For example, when a circular loop spins around its diagonal axis, the linear velocity varies around its circumference. The two surface integrals within the brackets of (6.3) can be

written in the form

$$\iint_{S_2(\mathbf{R},\, t+\Delta t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} = \oiint_{S_1+S_2+S_3} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} - \iint_{S_3} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S}_3, \tag{6.4}$$

where S_3 denotes the lateral surface swept by the displacement vector $\mathbf{v}\Delta t$ as S_1 moves to S_2. It is observed that $d\mathbf{S}_1$ is pointed into the volume bounded by S_1, S_2, and S_3, while $d\mathbf{S}_2$ is pointed outward. The closed surface integral in (6.4) can be transformed to a volume integral, that is,

$$\oiint_{\mathrm{S}} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} = \iiint_{\mathrm{V}} \nabla\, \mathbf{F}(\mathbf{R}, t)\, dV. \tag{6.5}$$

In (6.4) and (6.5),

$$d\mathbf{S}_3 = d\boldsymbol{\ell} \times (\mathbf{v}\, \Delta t)\,, \tag{6.6}$$

$$dV = (\mathbf{v}\, \Delta t) \cdot d\mathbf{S}. \tag{6.7}$$

By making use of the mean-value theorem in calculus, the surface integral in (6.4) evaluated on S_3 and the volume integral in (6.5) can be written in the following form:

$$\begin{aligned} -\iint_{S_3} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S}_3 &= \Delta t \oint_{\mathrm{L}} \mathbf{F}(\mathbf{R}, t) \cdot (\mathbf{v} \times d\boldsymbol{\ell}) \\ &= -\Delta t \oint_{\mathrm{L}} [\mathbf{v} \times \mathbf{F}(\mathbf{R}, t)] \cdot d\boldsymbol{\ell}, \end{aligned} \tag{6.8}$$

$$\iiint_{\mathrm{V}} \nabla\, \mathbf{F}(\mathbf{R}, t)\, dV = \Delta t \iint_{\mathrm{S}} [\mathbf{v}\nabla\, \mathbf{F}(\mathbf{R}, t)] \cdot d\mathbf{S}. \tag{6.9}$$

Equation (6.4) now becomes

$$\begin{aligned} &\iint_{S_2(\mathbf{R},\, t+\Delta t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} - \iint_{S_1(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} \\ &\quad = \Delta t \left\{ \iint_{\mathrm{S}} [\mathbf{v}\nabla\, \mathbf{F}(\mathbf{R}, t)] \cdot d\mathbf{S} - \oint_{\mathrm{L}} [\mathbf{v} \times \mathbf{F}(\mathbf{R}, t)] \cdot d\boldsymbol{\ell} \right\} \\ &\quad = \Delta t \left\{ \iint_{\mathrm{S}} \{\mathbf{v}\nabla\, \mathbf{F}(\mathbf{R}, t) - \nabla\, [\mathbf{v} \times \mathbf{F}(\mathbf{R}, t)]\} \cdot d\mathbf{S} \right\}. \end{aligned} \tag{6.10}$$

In (6.10), the line integral has been converted to a surface integral by means of the Stokes theorem. Equation (6.3), after taking the limit with respect to Δt, yields

$$\frac{d}{dt} \iint_{S(\mathbf{R},t)} \mathbf{F}(\mathbf{R}, t) \cdot d\mathbf{S} = \iint_{S(\mathbf{R},t)} \left\{ \frac{\partial \mathbf{F}(\mathbf{R}, t)}{\partial t} + \mathbf{v}\nabla\, \mathbf{F}(\mathbf{R}, t) - \nabla\, [\mathbf{v} \times \mathbf{F}(\mathbf{R}, t)] \right\} \cdot d\mathbf{S},$$

or simply,

$$\frac{d}{dt}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\nabla\ \mathbf{F} - \nabla\ (\mathbf{v}\times\mathbf{F})\right]\cdot d\mathbf{S}, \tag{6.11}$$

which represents the Helmholtz transport theorem [12] using the modern notation of vector analysis first formulated by Lorentz [13], and reiterated by Sommerfeld [14]. Equation (6.11) can be cast in a different form by making use of identity (4.159) for $\nabla\ (\mathbf{v}\times\mathbf{F})$, which yields

$$\frac{d}{dt}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{F} + \mathbf{F}\nabla\mathbf{v} - \mathbf{F}\cdot\nabla\mathbf{v}\right]\cdot d\mathbf{S}. \tag{6.12}$$

This version of the Helmholtz transport theorem is used by Candel and Poinsot [15] in formulating a problem in gas dynamics.

Because the material derivative of $\mathbf{F}$ or the total time derivative of $\mathbf{F}$ is defined by

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + \sum_{i=1}^{3}\frac{\partial \mathbf{F}}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{F}, \tag{6.13}$$

(6.12) can be written in the form

$$\frac{d}{dt}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \left[\frac{d\mathbf{F}}{dt} + \mathbf{F}\nabla\mathbf{v} - \mathbf{F}\cdot\nabla\mathbf{v}\right]\cdot d\mathbf{S}. \tag{6.14}$$

This form of the Helmholtz theorem is found in the treatment by Truesdell and Toupin [16].

6-2 Maxwell Theorem and Reynolds Transport Theorem

Two related theorems can now be derived from the Helmholtz transport theorem, although in the original works these two theorems were formulated independently of the Helmholtz theorem. In the Helmholtz theorem, if we let

$$\mathbf{F} = \nabla\mathbf{f} \tag{6.15}$$

and then convert the surface integral into a line integral, we find, noting that $\nabla\mathbf{F} = 0$ in view of (4.178),

$$\frac{d}{dt}\oint_S \mathbf{f}\cdot d\boldsymbol{\ell} = \oint_L \left(\frac{\partial \mathbf{f}}{\partial t} - \mathbf{v}\times\nabla\mathbf{f}\right)\cdot d\boldsymbol{\ell}. \tag{6.16}$$

This is the statement of the Maxwell theorem originally found in his great work on electromagnetic theory [17, 18].

In the Helmholtz theorem, if the surface is a closed one, we obtain

$$\frac{d}{dt}\oiint \mathbf{F}\cdot d\mathbf{S} = \oiint \left(\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\nabla\ \mathbf{F}\right)\cdot d\mathbf{S}. \tag{6.17}$$

The closed surface integral of ∇ $(\mathbf{v} \times \mathbf{F})$ vanishes because it is equal to a volume integral of $\nabla \nabla (\mathbf{v} \times \mathbf{F})$, which vanishes identically because of (4.179).

As a consequence of the Gauss theorem, (6.17) can be changed into the form

$$\frac{d}{dt} \iiint \nabla \mathbf{F} \, dV = \iiint \left[\frac{\partial}{\partial t} \nabla \mathbf{F} + \nabla (\mathbf{v} \nabla \mathbf{F}) \right] dV. \tag{6.18}$$

Now, if we let $\nabla \mathbf{F} = \rho$, a scalar function, then we obtain the Reynolds transport theorem [19], namely,

$$\frac{d}{dt} \iiint \rho \, dV = \iiint \left[\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) \right] dV, \tag{6.19}$$

where we identify $\mathbf{v}$ as the velocity of the fluid with density ρ. Because

$$\nabla (\rho \mathbf{v}) = \rho \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \rho, \tag{6.20}$$

and the total time derivative, or the material derivative, of ρ is given by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho,$$

(6.19) can be written in the form

$$\frac{d}{dt} \iiint \rho \, dV = \iiint \left[\frac{d\rho}{dt} + \rho \nabla \mathbf{v} \right] dV. \tag{6.21}$$

It should be mentioned that in the original work of Reynolds, (6.19) was derived by evaluating the total time derivative of $\iiint \rho \, dV$,

$$\frac{d}{dt} \iiint \rho \, dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\iiint_{V(t+\Delta t)} \rho(t + \Delta t) \, dV - \iiint_{V(t)} \rho(t) \, dV \right],$$

in a manner very similar to the derivation of the Helmholtz theorem.

From the preceding discussion, it is seen that the Helmholtz transport theorem can be considered as the principal transport theorem; both the Maxwell theorem and the Reynolds theorem can be treated as lemmas of that theorem.

Chapter 7

Dyadic Analysis

7-1 Divergence and Curl of Dyadic Functions and Gradient of Vector Functions

In Chapter 1, the definition of dyadic functions in a rectangular system and the algebra of these functions were introduced. In this chapter, the calculus of dyadics, or dyadic analysis, will be developed.

The divergence of a dyadic function $\bar{\bar{F}}$, expressed in a rectangular system by (1.82) in Chapter 1, will be denoted by $\nabla \bar{\bar{F}}$, and is defined by

$$\nabla \bar{\bar{F}} = \sum_j (\nabla \mathbf{F}_j)\hat{x}_j = \sum_i \sum_j \frac{\partial F_{ij}}{\partial x_i}\hat{x}_j, \tag{7.1}$$

which is a vector function.

The curl of a dyadic function of the form $\bar{\bar{F}}$, denoted by $\nabla \bar{\bar{F}}$, is defined by

$$\nabla \bar{\bar{F}} = \sum_j (\nabla \mathbf{F}_j)\hat{x}_j = \sum_i \sum_j (\nabla F_{ij} \times \hat{x}_i)\hat{x}_j, \tag{7.2}$$

which is also a dyadic function. Here, we have used the vector identity

$$\nabla \mathbf{F}_j = \nabla \sum_i F_{ij}\hat{x}_i = \sum_i (\nabla F_{ij} \times \hat{x}_i) \tag{7.3}$$

to convert the single sum in (7.2) to a double sum.

Sometimes we need the gradient of a vector function in dyadic analysis, denoted by $\nabla\mathbf{F}$. In a rectangular coordinate system, its expression is given by (4.153), that is,

$$\nabla\mathbf{F} = \sum_j (\nabla\mathbf{F}_j)\hat{x}_j = \sum_{i,j} \frac{\partial F_{ij}}{\partial x_i}\hat{x}_i\hat{x}_j.$$

In OCS, it is defined by

$$\begin{aligned}\nabla\mathbf{F} &= \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial \mathbf{F}}{\partial v_i} = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial}{\partial v_i}\sum_j F_j\hat{u}_j \\ &= \sum_{i,j}\frac{\hat{u}_i}{h_i}\left(\frac{\partial F_j}{\partial v_i}\hat{u}_j + F_j\frac{\partial \hat{u}_j}{\partial v_i}\right).\end{aligned} \tag{7.4}$$

The derivatives of the unit vector $\hat{u}_j$ in this equation can be expressed in terms of the other unit vectors $\hat{u}_i$, $\hat{u}_k$ with the aid of (2.59) and (2.61), which yields

$$\begin{aligned}\nabla\mathbf{F} = \sum_i \frac{\hat{u}_i}{h_i}\Bigg[&\left(\frac{\partial F_i}{\partial v_i} + \frac{F_j}{h_j}\frac{\partial h_i}{\partial v_j} + \frac{F_k}{h_k}\frac{\partial h_i}{\partial v_k}\right)\hat{u}_i \\ &+ \left(\frac{\partial F_j}{\partial v_i} - \frac{F_i}{h_j}\frac{\partial h_i}{\partial v_j}\right)\hat{u}_j + \left(\frac{\partial F_k}{\partial v_i} - \frac{F_i}{h_k}\frac{\partial h_i}{\partial v_k}\right)\hat{u}_k\Bigg]\end{aligned} \tag{7.5}$$

with $i, j, k = 1, 2, 3$ in cyclic order. Expression of $\nabla\mathbf{F}$ in GCS can be derived in a similar manner.

When a dyadic function is formed by a scalar function f with an idemfactor $\bar{\bar{I}}$ in the form of

$$f\bar{\bar{I}} = \sum_i f\hat{x}_i\hat{x}_i, \tag{7.6}$$

the divergence of this dyadic function is then a vector function, and it is given by

$$\nabla(f\bar{\bar{I}}) = \sum_i \frac{\partial f}{\partial x_i}\hat{x}_i = \nabla f. \tag{7.7}$$

In the OCS, $\bar{\bar{I}}$ is defined by

$$\bar{\bar{I}} = \sum_i \hat{u}_i\hat{u}_i, \tag{7.8}$$

and the divergence of $f\bar{\bar{I}}$ is defined by

$$\begin{aligned}\nabla(f\bar{\bar{I}}) &= \sum_i \frac{\hat{u}_i}{h_i}\cdot\frac{\partial}{\partial v_i}\sum_j f\hat{u}_j\hat{u}_j \\ &= \sum_i \frac{\hat{u}_i}{h_i}\cdot\sum_j\left[\frac{\partial f}{\partial v_i}\hat{u}_j\hat{u}_j + f\frac{\partial \hat{u}_j}{\partial v_i}\hat{u}_j + f\hat{u}_j\frac{\partial \hat{u}_j}{\partial v_i}\right].\end{aligned} \tag{7.9}$$

In (7.9), for $j = i$,

$$\hat{u}_i \cdot \frac{\partial \hat{u}_i}{\partial v_i} = 0, \tag{7.10}$$

which can be proved by taking the derivative of

$$\hat{u}_i \cdot \hat{u}_i = 1$$

with respect to v_i, or any variable for that matter. With the aid of (2.59) and (2.61), the last two terms in (7.9) cancel each other; hence

$$\nabla(f\bar{\bar{I}}) = \sum_i \frac{1}{h_i} \frac{\partial f}{\partial v_i} \hat{u}_i = \nabla f. \tag{7.11}$$

Equation (7.11), therefore, is invariant to the coordinate system. We demonstrate here once more the invariance of ∇. By following the same approach, we find

$$\nabla(f\bar{\bar{I}}) = \sum_j (\nabla f \hat{x}_j)\hat{x}_j = \sum_j (\nabla f \times \hat{x}_j)\hat{x}_j = \nabla f \times \bar{\bar{I}}, \tag{7.12}$$

which is a dyadic. This identity is valid in any coordinate system.

To find the divergence of a dyadic in an OCS, one can transform all the functions defined in a rectangular system to the functions in a specified OCS. We let

$$\mathbf{F} = \sum_i F_i \hat{x}_i = \sum_j F'_j \hat{u}_j = \mathbf{F}', \tag{7.13}$$

where F_i and F'_j (with $i, j = 1, 2, 3$) denote, respectively, the components of the function $\mathbf{F}$ or $\mathbf{F}'$ in the two systems; then

$$F'_j = \sum_i F_i \hat{x}_i \cdot \hat{u}_j = \sum_i c_{ji} F_i, \tag{7.14}$$

where c_{ji} denotes the directional cosines between $\hat{x}_i$ and $\hat{u}_j$. These coefficients can be found by the method of gradient in Section 4-5. The inverse transform is

$$F_i = \sum_j c_{ji} F'_j. \tag{7.15}$$

Equation (7.15) also applies to the transformation of the unit vectors. For example,

$$\hat{x}_i = \sum_j c_{ji} \hat{u}_j. \tag{7.16}$$

By definition,

$$\begin{aligned} \nabla \bar{\bar{F}} &= \sum_j (\nabla \mathbf{F}_j)\hat{x}_j = \sum_j (\nabla \mathbf{F}'_j)\hat{x}_j \\ &= \sum_j \sum_k \frac{1}{\Omega} \frac{\partial}{\partial v_k} \left(\frac{\Omega}{h_k} F'_{kj} \right) \sum_i c_{ij} \hat{u}_i \\ &= \sum_{i,j,k} \frac{c_{ij}}{\Omega} \frac{\partial}{\partial v_k} \left(\frac{\Omega}{h_k} F'_{kj} \right) \hat{u}_i, \end{aligned} \tag{7.17}$$

which is a vector function.

For mixed dyadics made of two independent vector functions of the form

$$\bar{\bar{F}} = \mathbf{M}(\mathbf{R})\,\mathbf{N}(\mathbf{R}'), \tag{7.18}$$

where $\mathbf{R}$ and $\mathbf{R}'$ represent two independent position vectors in some coordinate system, the divergence and the curl of $\bar{\bar{F}}$ with respect to the unprimed coordinates are defined as

$$\nabla \cdot \bar{\bar{F}} = [\nabla \cdot \mathbf{M}(\mathbf{R})]\,\mathbf{N}(\mathbf{R}'), \tag{7.19}$$

$$\nabla \times \bar{\bar{F}} = [\nabla \times \mathbf{M}(\mathbf{R})]\,\mathbf{N}(\mathbf{R}'). \tag{7.20}$$

The divergence and the curl of $\bar{\bar{F}}$ with respect to the primed coordinates are not defined, but

$$\nabla' \cdot [\bar{\bar{F}}]^{\mathrm{T}} = [\nabla' \cdot \mathbf{N}(\mathbf{R}')]\,\mathbf{M}(\mathbf{R}) \tag{7.21}$$

and

$$\nabla' \times [\bar{\bar{F}}]^{\mathrm{T}} = [\nabla' \times \mathbf{N}(\mathbf{R}')]\,\mathbf{M}(\mathbf{R}). \tag{7.22}$$

These functions are found in the application of dyadic analysis to electromagnetic theory [3].

A vector–dyadic identity to be quoted in the next chapter is derived here to show its origin. We have previously derived an identity, (4.158), showing that when a dyadic is formed by two vectors in the form of **ab**,

$$\nabla \cdot (\mathbf{ab}) = (\nabla \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \cdot \nabla \mathbf{b}. \tag{7.23}$$

Similarly,

$$\nabla \cdot (\mathbf{ba}) = (\nabla \cdot \mathbf{b})\mathbf{a} + \mathbf{b} \cdot \nabla \mathbf{a}. \tag{7.24}$$

The dyadic **ba** is the transpose of **ab**. By taking the difference between the last two equations, we obtain

$$\nabla \cdot (\mathbf{ba} - \mathbf{ab}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b}. \tag{7.25}$$

The right side of (7.25), according to (4.159), is equal to $\nabla \times (\mathbf{a} \times \mathbf{b})$; hence

$$\nabla \cdot (\mathbf{ba} - \mathbf{ab}) = \nabla \times (\mathbf{a} \times \mathbf{b}). \tag{7.26}$$

This identity was listed in Appendix B of reference [20] without derivation.

7-2 Dyadic Integral Theorems

There are several integral theorems in dyadic analysis that can be derived by changing the vector functions in the vector Green's theorems to dyadic functions.

1. First vector–dyadic Green's theorem

The first vector theorem stated by (4.211) will be written in the following form:

$$\iiint_V [(\nabla \mathbf{P}) \cdot (\nabla \mathbf{Q}) - \mathbf{P} \cdot \nabla \nabla \mathbf{Q}]\, dV = \oint_S \mathbf{n} \cdot (\mathbf{P} \times \nabla \mathbf{Q})\, dS. \tag{7.27}$$

We have purposely placed the function $\mathbf{Q}$ in the posterior position in (7.27), a practice that is used to change a vector to a dyadic. Consider now three distinct $\mathbf{Q}_j$ with $j = 1, 2, 3$ so that we have three identities of the same form as (7.27). By juxtaposing a unit vector $\hat{x}_j$ at the posterior position of each of the three equations and summing them, we obtain

$$\iiint_V [(\nabla \mathbf{P}) \cdot (\nabla \bar{\bar{Q}}) - \mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}}]\, dV = \oint_S \mathbf{n} \cdot (\mathbf{P} \times \nabla \bar{\bar{Q}})\, dS, \tag{7.28}$$

where, by definition,

$$\nabla \bar{\bar{Q}} = \sum_j (\nabla \mathbf{Q}_j)\hat{x}_j \tag{7.29}$$

and

$$\nabla \nabla \bar{\bar{Q}} = \sum_j (\nabla \nabla \mathbf{Q}_j)\hat{x}_j. \tag{7.30}$$

Equation (7.28) is designated as the first vector–dyadic Green's theorem of Type A because it involves a vector function $\mathbf{P}$ and a dyadic function $\bar{\bar{Q}}$. By interchanging $\mathbf{P}$ with $\mathbf{Q}$ in (7.27) and then raising the level of $\mathbf{Q}$ to a dyadic, we obtain

$$\iiint_V [(\nabla \mathbf{P}) \cdot (\nabla \bar{\bar{Q}}) - (\nabla \nabla \mathbf{P}) \cdot \bar{\bar{Q}}]\, dV = -\oint_S \mathbf{n} \cdot [(\nabla \mathbf{P}) \times \bar{\bar{Q}}]\, dS. \tag{7.31}$$

Equation (7.31) is designated as the first vector–dyadic Green's theorem of Type B. Except for the term $(\nabla \mathbf{P}) \cdot (\nabla \bar{\bar{Q}})$, which is common to (7.28) and (7.31), the rest are different.

2. Second vector–dyadic Green's theorem

By subtracting (7.28) from (7.31), we obtain the second vector–dyadic Green's theorem:

$$\iiint_V [\mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}} - (\nabla \nabla \mathbf{P}) \cdot \bar{\bar{Q}}]\, dV = -\oint_S \mathbf{n} \cdot [(\mathbf{P} \times \nabla \bar{\bar{Q}}) + (\nabla \mathbf{P}) \times \bar{\bar{Q}}]\, dS. \tag{7.32}$$

This theorem is probably the most useful formula in the application of dyadic analysis to electromagnetic theory [3].

3. First dyadic–dyadic Green's theorem

Equation (7.28) can be elevated to a higher level by moving $\mathbf{P}$ and $\nabla \mathbf{P}$ into the posterior position and transposing the dyadic terms into the anterior position so that

$$\iiint_V \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \mathbf{P}) - [\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \mathbf{P}\right\} dV = \oint_S [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \mathbf{P})\, dS. \tag{7.33}$$

The vector function $\mathbf{P}$ can now be elevated to a dyadic level that yields the first dyadic–dyadic Green's theorem:

$$\iiint_V \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \bar{\bar{P}}) - [\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \bar{\bar{P}}\right\} dV = \oint_S [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \bar{\bar{P}})\, dS. \tag{7.34}$$

By doing the same thing with (7.31), we obtain

$$\iiint_V \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \bar{\bar{P}}) - [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \nabla \bar{\bar{P}})\right\} dV = -\oint_S [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \nabla \bar{\bar{P}})\, dS. \tag{7.35}$$

4. Second dyadic–dyadic Green's theorem

By taking the difference between (7.34) and (7.35), we obtain the second dyadic–dyadic Green's theorem:

$$\iiint_V \left\{[\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \bar{\bar{P}} - [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\nabla \nabla \bar{\bar{P}})\right\} dV = -\oint_S \left\{[\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \bar{\bar{P}}) + [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\mathbf{n} \times \nabla \bar{\bar{P}})\right\} dS. \tag{7.36}$$

The two dyadic–dyadic Green's theorems involve two dyadics; hence we have the name. They can be used to prove the symmetrical property of the electric and magnetic dyadic Green's functions [3]. We have now assembled all of the important formulas in dyadic analysis, with the hope that they will be useful in digesting technical articles involving dyadic analysis, particularly in its application to electromagnetic theory.

Chapter 8

A Historical Study of Vector Analysis

8-1 Introduction[1]

In a book on the history of vector analysis [21], Michael J. Crowe made a thorough investigation of the decline of quaternion analysis and the evolution of vector analysis during the nineteenth century until the beginning of this century. The topics covered are mostly vector algebra and quaternion analysis. He had few comments to offer on the technical aspects of the subject from the point of view of a mathematician or a theoretical physicist. For example, the difference between the presentations of Gibbs and Heaviside, considered to be two founders of modern vector analysis, is not discussed in Crowe's book, and less attention is paid to the history of vector differentiation and integration, and to the role played by the del operator, ∇.

Brief descriptions of the history of vector analysis from the technical point of view are found in a few books. For example, in a book by Burali-Forti and Marcolongo [22] published in 1920, there are four historical notes in the appendix entitled, respectively, "On the definition of abstraction," "On vectors," "On vector and scalar (interior) products," and "On grad, rot, div." In a book by Moon and Spencer [23] published in 1965, there is a brief but critical review of the history

[1] This chapter is based on C. T. Tai, "A historical study of vector analysis," Technical Report RL 915, The Radiation Laboratory, Department of Electrical Engineering and Computer Science, The University of Michigan, May 1995.

of vector analysis from a technical perspective; this book will receive further discussion later. In their introduction, Moon and Spencer firmly state an important reason why they present vector analysis by way of tensor analysis [23, p. 9]:

> The present book differs from the customary textbook on vectors in stressing the idea of invariance under groups of transformations. In other words, elementary tensor technique is introduced, and in this way, the subject is placed on the firm, logical foundation which vector textbooks have previously lacked.

Further, in Appendix C [23, p. 323], Moon and Spencer write:

> In reading the foregoing book [referring to their book], one may wonder why nothing has been said about the operator ∇, which is usually considered such an important part of vector analysis. The truth is that ∇, though providing the subject with fluency, is an unreliable device because it often gives incorrect results. For this reason—and because it is not necessary—we have omitted it in the body of the book. Here, however, we shall indicate briefly the use of the operator ∇....

These two quotations are sufficient to indicate that after decades of application of vector analysis, there seems to be no systematic treatment of the subject that could be considered satisfactory according to these two authors. This observation is also supported by the fact that we have so far no standard notations in vector analysis. Many books on electromagnetics, for example, use the linguistic notations for the gradient, divergence, and curl—grad u, div $\mathbf{f}$, and curl $\mathbf{f}$—while many others prefer Gibbs's notations for these functions—∇u, $\nabla \cdot \mathbf{f}$, and $\nabla \times \mathbf{f}$. Can we offer a better explanation to students as to why we do not yet have a universally accepted standard notation than to say it is merely a matter of personal choice? In regard to Moon and Spencer's comments about the lack of a firm, logical foundation in previous books on vector analysis, there has been no elaboration. They do give an example of an incorrect result from using ∇ to form a scalar product with $\mathbf{f}$ to find the expression for divergence in an orthogonal curvilinear coordinate system, but they do not explain *why* the result is incorrect. In fact, the views expressed by these two authors are also found in many books treating vector analysis. These will be reviewed and commented upon later.

In this chapter, we assume that the reader has already read the previous chapters of this book, particularly the method of symbolic vector in Chapter 4. The aim of this chapter is to point out the inadequacy or illogic in the treatments of some basic topics in vector analysis during a period of approximately one hundred years. The mistreatments are then rectified by the proper tools introduced in this book.

8-2 Notations and Operators

8-2-1 Past and Present Notations in Vector Analysis

In a book on advanced vector analysis published in 1924, Weatherburn [24] compiled a table of notations in vector analysis that had been used up to that time. The authors represented in that table are Gibbs, Wilson, Heaviside, Abraham, Ignatowsky, Lorentz, Burali-Forti, and Marcolongo. A table of notations is also given in Moon and Spencer's 1965 book *Vectors* (from which the quotations in the previous section were taken). The authors represented in that table are Maxwell, Gibbs, Wilson, Heaviside, Gans, Lagally, Burali-Forti, Marcolongo, Phillips, Moon, and Spencer. Among these authors, Gibbs, Wilson, Phillips, Moon, and Spencer are American; Maxwell and Heaviside belong to the English schools; while Abraham, Ignatowsky, Gans, and Lagally belong to the German schools. Lorentz was a Dutch physicist, Burali-Forti and Marcolongo were Italian, and Ignatowsky was a native of Russia but was trained in Germany. For our study, we have prepared another list representing several contemporary authors and some additional notations; this is given in Table 8.1. The dyadic notation is added because we need it to characterize the gradient of a vector, which is a dyadic function. In perusing this table, the reader will recognize the linguistic notations grad u, div **A**, curl **A**, or rot **A** for the three key functions. The reader is probably familiar with Gibbs's notations ∇u, $\nabla \cdot \alpha$, and $\nabla \times \alpha$, except that here the period in $\nabla.\alpha$ has been replaced by a raised dot ($\cdot$) as in Wilson's notations, and his Greek letters for vectors are now commonly replaced by boldface Clarendon (or equivalent fonts), while the linguistic notations are used by many authors in Europe and a few in the United States. Gibbs's notations have been adopted in many books published in the United States. We will later quote from two books on electromagnetic theory, one by Stratton and another by Jackson. Their treatises are well known to many electrical engineers, as well as physicists.

Historically, vector analysis was developed a few years after Maxwell formulated his monumental work in electromagnetic theory. When he wrote his treatise on electricity and magnetism [25] in 1873, vector analysis was not yet available. Its forerunner, quaternion analysis, developed by Hamilton (1805–1865) in 1843, was then advocated by many of Hamilton's followers. It is probably for this reason that Maxwell wrote an article in his book (Article 618) entitled "Quaternion Expressions for the Electromagnetic Equations." Maxwell's notations in our list are based on this document. Actually, he made little use of these notations in his book and in his papers published elsewhere.

The notation used by Heaviside is unconventional from the present point of view. His notation for the scalar product and the divergence does not have a dot and his notation for the curl is of the quaternion form, as is Maxwell's. The notations used by Burali-Forti and Marcolongo are obsolete now. Occasionally, we still see

Table 8.1: List of Notations

Author(s)	Vectors	Scalar Product	Vector Product	Dyadic in 3-space	Tensor in 3-space	Gradient of a Scalar	Gradient of a Vector	Divergence of a Vector	Curl or Rot of a Vector	Laplacian of a Scalar	Laplacian of a Vector
Maxwell [25]	σ, ρ $\mathcal{A}, \mathcal{B}$	$S\sigma\rho$	$V\sigma\rho$	—	—	∇u	—	$-S\nabla\rho$	$V\nabla\rho$	$\nabla^2 u$	—
Gibbs [4]	α, β	α, β	$\alpha \times \beta$	$\alpha\beta$		∇u	$\nabla\alpha$	$\nabla \cdot \alpha$	$\nabla \times \alpha$	$\nabla \cdot \nabla u$	$\nabla \cdot \nabla\alpha$
Wilson [30]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	$\mathbf{AB}$	—	∇u	$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$	$\nabla \times \mathbf{A}$	$\nabla \cdot \nabla u$	$\nabla \cdot \nabla\mathbf{A}$
Heaviside [26]	$\mathbf{A}, \mathbf{B}$	$\mathbf{AB}$	$V\mathbf{AB}$	—	—	∇u	$\nabla \cdot \mathbf{A}$	$\nabla\mathbf{A}$; div $\mathbf{A}$	$V\nabla\mathbf{A}$; curl $\mathbf{A}$	$\nabla^2 u$	$\nabla^2\mathbf{A}$
Gans [37]	$\mathcal{A}, \mathcal{B}$	$(\mathcal{A}, \mathcal{B})$	$[\mathcal{A}, \mathcal{B}]$	—	—	∇u; grad u	—	$\nabla \cdot \mathbf{A}$; div $\mathbf{A}$	$\nabla \times \mathbf{A}$; rot $\mathbf{A}$	Δu	$\Delta\mathbf{A}$
Burali-Forti/ Marcolongo [22]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	$\mathbf{A} \wedge \mathbf{B}$	—	—	grad u	—	div $\mathbf{A}$	rot $\mathbf{A}$	$\Delta_2 u$	$\Delta_2'\mathbf{A}$
Stratton [5]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	—	T_{ij}	∇u	$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$	$\nabla \times \mathbf{A}$	$\nabla^2 u$	$\nabla^2\mathbf{A}$
Jackson [44]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	$\overleftrightarrow{T}$	T_{ij}	∇u	$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$	$\nabla \times \mathbf{A}$	$\nabla^2 u$	$\nabla^2\mathbf{A}$
Moon/ Spencer [23]	$\mathbf{A}, \mathbf{B}$	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \times \mathbf{B}$	—	T_{ij}	grad u	—	div$\mathbf{A}$	curl $\mathbf{A}$	$\nabla^2 u$	$\nabla\nabla\mathbf{A}$

Uppercase script symbols are used here in place of capital German letters originally used by Maxwell and Gans.

the notation $\mathbf{A} \wedge \mathbf{B}$ for the cross product in the works of European authors. On the whole, we now have basically two sets of notations in current use: the linguistic notation and Gibbs's notation. The names of Moon and Spencer are included in our list primarily because these two authors considered the use of ∇ to be unreliable and they frequently emphasize their view that the rigorous method of formulating vector analysis is to follow the route of tensor analysis. In addition, their new notation for the Laplacian of a vector function will receive detailed examination in the section on orthogonal curvilinear systems.

8-2-2 Quaternion Analysis

The rise of vector analysis as a distinct branch of applied mathematics has its origin in quaternion analysis. It is therefore necessary to review briefly the laws of quaternion analysis to show its influence on the development of vector analysis, and also explain the notations in the previous list. Quaternions are complex numbers of the form

$$q = w + ix + jy + kz, \tag{8.1}$$

where w, x, y, and z are real numbers, and i, j, and k are quaternion units, or quaternion unit vectors, associated with the x, y, and z axes, respectively. These units obey the following laws of multiplication:

$$\begin{gathered} ij = k, \qquad jk = i, \qquad ki = j, \\ ji = -k, \qquad kj = -i, \qquad ik = -j, \\ ii = jj = kk = -1. \end{gathered} \tag{8.2}$$

We must not at this stage associate these relations with our current laws of unit vectors in vector analysis. We consider the subject as a new algebra, which is indeed the case. The product of the multiplication of two quaternions σ and ρ in which the scalar parts w and w' are zero is obtained as follows:

We let

$$\begin{aligned} \sigma &= iD_1 + jD_2 + kD_3, \\ \rho &= iX + jY + kZ. \end{aligned}$$

Then,

$$\begin{aligned} \sigma\rho = &-(D_1X + D_2Y + D_3Z) \\ &+ i(D_2Z - D_3Y) + j(D_3X - D_1Z) + k(D_1Y - D_2X). \end{aligned} \tag{8.3}$$

The resultant quaternion, $\sigma\rho$, has two parts, one scalar and one vector. In Hamilton's original notation, they are

$$S.\sigma\rho = -(D_1X + D_2Y + D_3Z), \tag{8.4}$$

$$V.\sigma\rho = i(D_2Z - D_3Y) + j(D_3X - D_1Z) + k(D_1Y - D_2X). \tag{8.5}$$

The period between S or V and $\sigma\rho$ can be omitted without any resulting ambiguity. When one identifies σ as ∇, Hamilton's del operator defined with respect to the quaternion unit vectors, that is,

$$\sigma = \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}, \tag{8.6}$$

then,

$$S\nabla\rho = -\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right), \tag{8.7}$$

$$V\nabla\rho = i\left(\frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial z}\right) + j\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + k\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right). \tag{8.8}$$

Maxwell used the quaternion notation $S\nabla\rho$ for the negative of the divergence of ρ, which he termed the *convergence*. He used the quaternion notation $V\nabla\rho$ for the curl of ρ. The term *curl*, now standard, was coined by Maxwell. According to Crowe [21, p. 142] the term *divergence* was originally due to William Kingdom Clifford (1845–1879), who was also the first person to define the modern notations for the scalar and vector products. However, his original definition of the scalar product is the negative of the modern scalar product. In the list of notations, we notice that Heaviside used the quaternion notation for the curl even though he was opposed to quaternion analysis. In one of his writings [26, p. 35], he concurred with Gibbs's treatment of vector analysis but criticized Gibbs's notations without offering a reason; we discuss this comment of Heaviside's later in Section 8-5. Before we discuss the works of these various authors, a review of the meaning of the algebraic and differential operators is necessary.

8-2-3 Operators

For our convenience, we would like to discuss in sufficient detail the classification and the characteristics of a number of operators appearing in this study. We will focus on unary and binary operators and consider such operators in cascade or compound arrangements as the complexity of the case at hand requires.

A unary operator involves only one operand. A binary operator needs two operands, one anterior and another posterior. A cascade operator could be unary or binary. As an example, we consider the derivative symbol $\partial/\partial x$ to be a unary operator. When it operates on an operand P, it produces the derivative, $\partial P/\partial x$. In some writings, the operator $\partial/\partial x$ is denoted by D_x. The operand under consideration can be a scalar function of x and other independent variables or a vector function or a dyadic function; that is,

$$\frac{\partial P}{\partial x}; \quad \frac{\partial a}{\partial x}, \frac{\partial \mathbf{A}}{\partial x}, \frac{\partial \bar{\bar{F}}}{\partial x}$$

are all valid applications of the unary differential operator. The partial derivative

of a dyadic function in a rectangular system is defined by

$$\frac{\partial \bar{\bar{F}}}{\partial x} = \sum_j \frac{\partial \mathbf{F}_j}{\partial x}\hat{x}_j = \sum_i \sum_j \frac{\partial F_{ij}}{\partial x}\hat{x}_i\hat{x}_j. \tag{8.9}$$

We list in Table 8-2 several commonly used unary operators and their possible operands. The function a in the weighted differential operator $a(\partial/\partial x)$ is assumed to be a scalar function. A vector operator such as $\mathbf{A}(\partial/\partial x)$ can operate on a dyadic that would yield a "triadic"—a quantity that is not included in this study. The last operator in Table 8-2 is the del operator, or the gradient operator. It can be applied to an operand that is either a scalar or a vector.

Table 8-2: Valid Application of Some Unary Differential Operators

Operator	Type of Operand	Results
$\frac{\partial}{\partial x}$	b, $\mathbf{B}$, $\bar{\bar{B}}$	$\frac{\partial b}{\partial x}$, $\frac{\partial \mathbf{B}}{\partial x}$, $\frac{\partial \bar{\bar{B}}}{\partial x}$
$a\frac{\partial}{\partial x}$	b, $\mathbf{B}$, $\bar{\bar{B}}$	$a\frac{\partial b}{\partial x}$, $a\frac{\partial \mathbf{B}}{\partial x}$, $a\frac{\partial \bar{\bar{B}}}{\partial x}$
$\mathbf{A}\frac{\partial}{\partial x}$	b, $\mathbf{B}$	$\mathbf{A}\frac{\partial b}{\partial x}$, $\mathbf{A}\frac{\partial \mathbf{B}}{\partial x}$
$\nabla = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}$	b, $\mathbf{B}$	∇b, $\nabla \mathbf{B}$

A binary operator requires two operands. In arithmetic and algebra, we have four binary operators: + (addition), − (subtraction), × (multiplication), and ÷ (division). In these cases, we need two operands, one anterior and another posterior, as in $2 + 3$, $4 - 3$, 5×3, and $6 \div 3$. Note that the symbols + and − are also used to denote "plus" and "minus" signs. For example, $-a = |a|$ when a is negative. In this case, the minus sign is not considered to be a binary operator in our classification, but rather as a unary "sign change" operator. The two binary operators involved frequently in our work are the dot (·) and the cross (×). They appear in Gibbs's notations for the scalar and vector products, that is, $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. We consider the dot and the cross as two binary operators, and their operands, one anterior and one posterior, must be vectors; that is,

$$\mathbf{A} \cdot \mathbf{B} \quad \text{and} \quad \mathbf{A} \times \mathbf{B}.$$

The dot operator is not the same as the multiplication operator in arithmetic, nor is the cross operator the same as the multiplication operator, although we use the same symbol for both. According to the definitions of the scalar and vector products,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{B}| \cos\theta, \tag{8.10}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} = |\mathbf{A}||\mathbf{B}| \sin\theta \hat{c}, \tag{8.11}$$

where θ is the angle measured from **A** to **B** in the plane containing these two vectors, and $\hat{c}$ is the unit vector perpendicular to both **A** and **B** and is pointed in the right-screw advancing direction when **A** turns into **B**. The dot and the cross can also be applied to operands where one of them or both are dyadics. Thus, we have

$$\mathbf{A} \cdot \bar{\bar{B}}, \quad \bar{\bar{B}} \cdot \mathbf{A}, \quad \mathbf{A} \times \bar{\bar{B}}, \quad \bar{\bar{B}} \times \mathbf{A}, \quad \bar{\bar{A}} \cdot \bar{\bar{B}}, \quad \bar{\bar{B}} \cdot \bar{\bar{A}}. \tag{8.12}$$

The first two entities are vectors and the remaining four are dyadics.

The last group of operators are called *cascade* or *compound operators*. Of particular concern in this study is the proper treatment of a pair of operators of different types, which are applied sequentially. When one of the operators is a scalar differential unary operator, and the other is a vector binary operator, there arise a number of hazards in their application which, if not properly treated, could lead to invalid results. Several commonly used cascade operators are of the forms

$$\cdot\frac{\partial}{\partial y}, \qquad \cdot\nabla, \qquad \times\frac{\partial}{\partial y}, \qquad \times\nabla. \tag{8.13}$$

These operators also require two operands; the anterior operand must be a vector or a dyadic and the posterior operand must be compatible with the part in front. Thus, we can have

$$\begin{aligned} &\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial y}, && \mathbf{A} \cdot \frac{\partial \bar{\bar{B}}}{\partial y}; \\ &\mathbf{A} \cdot \nabla u, && \mathbf{A} \cdot \nabla \mathbf{B}; \\ &\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial y}, && \mathbf{A} \times \frac{\partial \bar{\bar{B}}}{\partial y}; \\ &\mathbf{A} \times \nabla u, && \mathbf{A} \times \nabla \mathbf{B}. \end{aligned} \tag{8.14}$$

In (8.13), the unary operators, $\partial/\partial y$ and ∇, and the binary operators, $\cdot$ and $\times$, are not commutative; hence the following combinations or assemblies are not valid cascade operators:

$$\frac{\partial}{\partial y}\cdot, \qquad \nabla\cdot, \qquad \frac{\partial}{\partial y}\times, \qquad \nabla\times. \tag{8.15}$$

These assemblies are formed by interchanging the positions of the symbols in (8.13). They are not operators in the sense that we cannot find an operand to form a meaningful entity. For example,

$$\begin{aligned} &\frac{\partial}{\partial y} \cdot \mathbf{A}, && \frac{\partial}{\partial y} \cdot \bar{\bar{B}}, && \nabla \cdot \mathbf{A}, && \nabla \cdot \bar{\bar{B}}, \\ &\frac{\partial}{\partial y} \times \mathbf{A}, && \frac{\partial}{\partial y} \times \bar{\bar{B}}, && \nabla \times \mathbf{A}, && \nabla \times \bar{\bar{B}}, \end{aligned} \tag{8.16}$$

do not have any meaningful interpretation. The reader has probably noticed that there are two assemblies, $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$, in (8.16) that correspond to *Gibbs's*

notation for the divergence and curl. This is true, but that does not mean that $\nabla \cdot \mathbf{A}$ is a scalar product between ∇ and $\mathbf{A}$, nor is $\nabla \times \mathbf{A}$ a vector product between ∇ and $\mathbf{A}$. In fact, this is a central issue in this study to be examined critically in the following sections. We now have the necessary tools to investigate many of the past presentations of vector analysis.

8-3 The Pioneer Works of J. Willard Gibbs (1839–1903)

8-3-1 Two Pamphlets Printed in 1881 and 1884

Gibbs's original works on vector analysis are found in two pamphlets entitled *Elements of Vector Analysis* [4], privately printed in New Haven. The first consists of 33 pages published in 1881 and the second of 40 pages published in 1884. These pamphlets were distributed to his students at Yale University and also to many scientists and mathematicians including Heaviside, Helmholtz, Kirchhoff, Lorentz, Lord Rayleigh, Stokes, Tait, and J. J. Thomson [27, Appendix IV]. The contents are divided into five chapters and a note on bivectors:

Chapter I.	Concerning the algebra of vectors
Chapter II.	Concerning the differential and integral calculus of vectors
Chapter III.	Concerning linear vector functions
Chapter IV.	Concerning the differential and integral calculus of vectors (Supplement to Chapter II)
Chapter V.	Concerning transcendental functions of dyadics
	A note on bivector analysis

The most important formulations for our immediate discussions are covered in Articles 50–54 and 68–71, which are reproduced here.

Functions of Positions in Space

50. Def.—If u is any scalar function of position in space (i.e., any scalar quantity having continuously varying values in space), ∇u is the vector function of position in space which has everywhere the direction of the most rapid increase of u, and a magnitude equal to the rate of that increase per unit of length. ∇u may be called the derivative of u, and u, the primitive of ∇u.

    We may also take any one of the Nos. 51, 52, 53 for the definition of ∇u.

51. If ρ is the vector defining the position of a point in space,

$$du = \nabla u \cdot d\rho.$$

52.

$$\nabla u = i\frac{du}{dx} + j\frac{du}{dy} + k\frac{du}{dz}. \tag{8.17}$$

53.

$$\frac{du}{dx} = i \cdot \nabla u, \qquad \frac{du}{dy} = j \cdot \nabla u, \qquad \frac{du}{dz} = k \cdot \nabla u.$$

54. Def.—If ω is a vector having continuously varying values in space,

$$\nabla \cdot \omega = i \cdot \frac{d\omega}{dx} + j \cdot \frac{d\omega}{dy} + k \cdot \frac{d\omega}{dz}, \tag{8.18}$$

$$\nabla \times \omega = i \times \frac{d\omega}{dx} + j \times \frac{d\omega}{dy} + k \times \frac{d\omega}{dz}. \tag{8.19}$$

$\nabla \cdot \omega$ is called the divergence of ω and $\nabla \times \omega$ its curl.

If we set

$$\omega = Xi + Yj + Zk,$$

we obtain by substitution the equation

$$\nabla \cdot \omega = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} \tag{8.20}$$

and

$$\nabla \times \omega = i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right), \tag{8.21}$$

which may also be regarded as defining $\nabla \cdot \omega$ and $\nabla \times \omega$.

Combinations of the Operators ∇, $\nabla\cdot$, and $\nabla\times$

68. If ω is any vector function of space, $\nabla \cdot \nabla \times \omega = 0$. This may be deduced directly from the definition of No. 54.

The converse of this proposition will be proved hereafter.

69. If u is any scalar function of position in space, we have by Nos. 52 and 54

$$\nabla \cdot \nabla u = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dx^2}\right) u. \tag{8.22}$$

70. Def.—If ω is any vector function of position in space, we may define $\nabla \cdot \nabla\omega$ by the equation

$$\nabla \cdot \nabla\omega = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) \omega, \tag{8.23}$$

the expression $\nabla\cdot\nabla$ being regarded, for the present at least, as a single operator when applied to a vector. (It will be remembered that no meaning has been attributed to ∇ before a vector.) Note that if

$$\omega = iX + jY + kZ,$$

then

$$\nabla\cdot\nabla\omega = i\nabla\cdot\nabla X + j\nabla\cdot\nabla Y + k\nabla\cdot\nabla Z, \tag{8.24}$$

that is, the operator $\nabla\cdot\nabla$ applied to a vector affects separately its scalar components.

71. From the above definition with those of Nos. 52 and 54, we may easily obtain

$$\nabla\cdot\nabla\omega = \nabla\nabla\cdot\omega - \nabla\times\nabla\times\omega. \tag{8.25}$$

The effect of the operator $\nabla\cdot\nabla$ is therefore independent of the direction of the axes used in its definition.

In quoting these sections, we have changed Gibbs's original notation for the divergence from $\nabla.\omega$ to $\nabla\cdot\omega$, that is, the period has been replaced by a dot. The equation numbers have been added for our reference later on.

After Gibbs revealed his new work on vector analysis, he was attacked fiercely by Tait, a chief advocate of the quaternion analysis, who stated [28, Preface]:

> Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on vector analysis; a sort of hermaphrodite monster, compounded by the notations of Hamilton and Grassman.

This infamous statement has been quoted by many authors in the past. Gibbs's gentlemanly but firm response to Tait's attack [29]:

> The merit or demerits of a pamphlet printed for private distribution a good many years ago do not constitute a subject of any great importance, but the assumption implied in the sentence quoted are suggestive of certain reflections and inquiries which are of broad interest; and seem not untimely at a period when the methods and results of the various forms of multiple algebra are attracting so much attention. It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain unchallenged, especially when supported by so high an authority. The criticism relates particularly to notations, but I believe that there is a deeper question of notions underlying that of notations. Indeed, if my offense had been solely in the matter of notation, it would have been less accurate to describe my production as a monstrosity, than to characterize its dress as uncouth.

Gibbs then went on to explain the advantage of his treatment of vector analysis compared with quaternion analysis. In the final part of that paper he stated:

> The particular form of signs we adopt is a matter of minor consequence. In order to keep within the resources of an ordinary printing office, I have used a dot and a cross, which are already associated with multiplication, which is best denoted by the simple juxtaposition of factors. I have no special predilection for these particular signs. The use of the dot is indeed liable to the objection that it interferes with its use as a separatrix, or instead of a parenthesis.

Although Gibbs considered his choice of the signs or notations a matter of minor importance, it was actually of great consequence, as will be shown in this study. Before we discuss his notations, a comment from Heaviside, generally considered by the scientific community as a cofounder with Gibbs of modern vector analysis, should be quoted. During the peak of the controversy between Tait and Gibbs, Heaviside made the following remark [26, p. 35]:

> Prof. W. Gibbs is well able to take care of himself. I may, however, remark that the modifications referred to are evidence of modifications felt to be needed, and that Prof. Gibbs' pamphlet (not published, New Haven, 1881–84, p. 83), is not a quaternionic treatise, but an able and in some respects original little treatise on vector analysis, though too condensed and also too advanced for learners' use, and that Prof. Gibbs, being no doubt a little touched by Prof. Tait's condemnation, has recently (in the pages of *Nature*) made a powerful defense of his position. He has by a long way the best of the argument, unless Prof. Tait's rejoinder has still to appear. Prof. Gibbs clearly separates the quaternionic question from the question of a suitable notation, and argues strongly against the quaternionic establishment of vector analysis. I am able (and am happy) to express a general concurrence of opinion with him about the quaternion and its comparative uselessness in practical vector analysis. As regards his notation, however, I do not like it. Mine is Tait's, but simplified, and made to harmonize with Cartesians.

There are two implications in Heaviside's remark that are of interest to us. When he considered Gibbs's pamphlet to be too condensed, it implies that some of the treatments may not have been obvious to him. Secondly, he stated his dislike for Gibbs's notations but without giving his reasons. The fact that Heaviside used some of Tait's quaternionic notations seems to indicate that he did not approve of Gibbs's notations at all. We now believe that many workers, including Heaviside, did not appreciate the most eloquent and complete theory of vector analysis formulated by Gibbs. For this reason, we would like to offer a digest of Gibbs's work so that we may have a clear understanding of his formulation.

8-3-2 Divergence and Curl Operators and Their New Notations

The basic definitions of the gradient, divergence, and the curl formulated by Gibbs are given by (8.18), (8.19), and (8.20). For convenience, we will make some changes in symbols to allow the convenience of using the summation sign. These changes are

$$x, y, z \quad \text{to} \quad x_1, x_2, x_3,$$
$$i, j, k \quad \text{to} \quad \hat{x}_1, \hat{x}_2, \hat{x}_3.$$

The old total derivative symbols will be replaced by partial derivatives and the Greek letters for vectors by boldface letters. Thus, Eqs. (8.17)–(8.19) become

$$\nabla u = \sum_i \hat{x}_i \frac{\partial u}{\partial x_i}, \tag{8.26}$$

$$\nabla \cdot \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i}, \tag{8.27}$$

$$\nabla \times \mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i}. \tag{8.28}$$

It is understood that the summation goes from $i = 1$ to $i = 3$.

The most important information passed to us by Gibbs concerns the nomenclature for the notations in these expressions. In the title preceding Article 68 quoted previously, he designated ∇, $\nabla\cdot$, and $\nabla\times$ as operators. If we examine the expressions given by (8.26), (8.27), and (8.28) it is obvious that the gradient operator, or the del operator, is unmistakably given by

$$\nabla = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}. \tag{8.29}$$

For the divergence, Gibbs used two symbols, a del followed by a dot, to denote his divergence operator. For the curl, he used a del followed by a cross to denote the curl operator. If we examine the expressions for the divergence and the curl defined by (8.27) and (8.28), it is clear that his two notations mean:

$$(\nabla\cdot)_{\mathrm{G}} \quad \rightarrow \quad \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i}, \tag{8.30}$$

$$(\nabla\times)_{\mathrm{G}} \quad \rightarrow \quad \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i}. \tag{8.31}$$

We emphasize this point by labeling his two notations with a subscript G, and we use an arrow instead of an equal sign to denote "a notation for."

According to our classification of the operators in Section 8.2, Gibbs's $(\nabla\cdot)_{\mathrm{G}}$ and $(\nabla\times)_{\mathrm{G}}$ are not compound operators; they are assemblies used by Gibbs as the notations for the divergence and curl. On the other hand, the terms at the

right side of (8.30) and (8.31) are indeed compound operators, according to our classification. Because these operators are distinct from the gradient operator, we will introduce two notations for them. They are

$$\nabla = \sum_i \hat{x}_i \cdot \frac{\partial}{\partial x_i} , \tag{8.32}$$

$$\nabla = \sum_i \hat{x}_i \times \frac{\partial}{\partial x_i} . \tag{8.33}$$

They are called the *divergence operator* and the *curl operator*, respectively. Although these operators are so far defined in the rectangular coordinate system, we will demonstrate later that they are invariant to the choice of coordinate system. One important feature of ∇ and ∇ is that both these operators are independent of the gradient operator ∇. In other words, ∇ *is not a constituent of the divergence operator nor of the curl operator.* These two symbols are suggested by the appearance of the dot or the cross in between the unit vectors $\hat{x}_i$ and the partial derivatives $\partial/\partial x_i$ of the ∇ operator as defined by (8.29). In Gibbs's notations, $(\nabla\cdot)_G$ and $(\nabla\times)_G$, ∇ is a part of his notations for the divergence and the curl that leads to a serious misinterpretation by many later users and is a key issue in our study. With the introduction of these two new notations, Eqs. (8.18)–(8.26) become

$$\nabla u = \sum_i \hat{x}_i \frac{\partial u}{\partial x_i} , \tag{8.34}$$

$$\nabla \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} , \tag{8.35}$$

$$\nabla \mathbf{F} = \sum_i \frac{\partial F_i}{\partial x_i} , \tag{8.36}$$

$$\nabla \mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i} , \tag{8.37}$$

$$\nabla \mathbf{F} = \sum_i \hat{x}_i \left(\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} \right) \tag{8.38}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order,

$$\nabla \nabla u = \sum_i \frac{\partial^2 u}{\partial x_i^2} , \tag{8.39}$$

$$\nabla \nabla \mathbf{F} = \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_i^2} , \tag{8.40}$$

$$\nabla \nabla \mathbf{F} = \sum_i \hat{x}_i \nabla \nabla F_i , \tag{8.41}$$

$$\nabla \nabla \mathbf{F} = \nabla \nabla \mathbf{F} - \nabla \nabla \mathbf{F} . \tag{8.42}$$

In these formulas, the del operator only enters in the gradient of a scalar, (8.34), or of a vector, (8.40)–(8.42). Except for the notations for the divergence and the curl, we have not changed the content of Gibbs's work at all. These equations will be used later in our study of other people's presentations.

8-4 Book by Edwin Bidwell Wilson Founded upon the Lectures of J. Willard Gibbs

8-4-1 Gibbs's Lecture Notes

The first book on vector analysis by an American author was published in 1901. The author was Edwin Bidwell Wilson [30], then an instructor at Yale University. According to the general preface, the greater part of the material was taken from Prof. Gibbs's lectures on vector analysis delivered annually at Yale. A record of these lectures is preserved at Yale's Sterling Memorial Library: it is a clothbound book of notes, handwritten in ink on $8\frac{1}{2}$ in. $\times$ 11 in. ruled paper, about 290 pages long and consisting of 15 chapters [31]. The title page reads as follows:

Lectures Delivered

upon Vector Analysis

and its

Applications to Geometry and Physics

by

Professor J. Willard Gibbs 1899–90

reported by Mr. E. B. Wilson

The table of contents is given in Table 8-3.

8-4-2 Wilson's Book

Presumably, Wilson's book (436 pages) is based principally on these notes. The preface states, however, that some use has been made of the chapters on vector analysis in Heaviside's *Electromagnetic Theory* (1893) and in Föppl's lectures

Table 8-3: Table of Contents of E. B. Wilson's *Lectures Delivered upon Vector Analysis*

		page
Ch. 1	Fundamental Notions and Operators	1
Ch. 2	Geometrical Applications of Vector Analysis	11
Ch. 3	Products of Vectors	25
Ch. 4	Geometrical Applications of Products	50
Ch. 5	Crystallography	62
Ch. 6	Scalar Differentiation of Vectors	72
Ch. 7	Differentiating and Integrating Operations	83
Ch. 8	Potentials, Newtonians, Laplacians, Maxwellians	110
Ch. 9	Theory of Parabolic Orbits	125
Ch. 10	Linear Vector Functions	164
Ch. 11	Rotations and Strains	200
Ch. 12	Quadratic Surfaces	223
Ch. 13	Curvature of Curved Surfaces	234
Ch. 14	Dynamics of a Solid Body	261
Ch. 15	Hydrodynamics	276

on Maxwell's *Theory of Electricity* (1894). Apparently, Gibbs himself was not involved in the preparation of the body of the book, but he did contribute a preface, from which the following two paragraphs are taken:

> I was very glad to have one of the hearers of my course on Vector Analysis in the year 1899–1900 undertake the preparation of a text-book on the subject.
>
> I have not desired that Dr. Wilson should aim simply at the reproduction of my lectures, but rather that he should use his own judgment in all respects for the production of a text-book in which the subject should be so illustrated by an adequate number of examples as to meet the wants of students of geometry and physics.

In the general preface, Wilson stated:

> When I undertook to adapt the lectures of Professor Gibbs on Vector Analysis for publication in the Yale Bicentennial Series, Professor Gibbs himself was already so fully engaged in his work to appear in the same series, *Elementary Principles in Statistical Mechanics*, that it was understood no material assistance in the composition of this book could be expected from him. For this reason he wished me to feel entirely free to use my own discretion alike in the selection of the topics to be treated and in the mode

of treatment. It has been my endeavor to use the freedom thus granted only in so far as was necessary for presenting his method in text-book form.

The following passage from Wilson's preface is particularly significant for the present discussion:

It has been the aim here to give also an exposition of scalar and vector products of the operator ∇, of divergence and curl which have gained such universal recognition since the appearance of Maxwell's *Treatise on Electricity and Magnetism*, slope, potential, linear vector functions, etc. such as shall be adequate for the needs of students of physics at the present day and adapted to them.

We point out here that in Gibbs's pamphlets and in the lecture notes reported by Wilson, there is no mention of the scalar and vector products of the operator ∇. We believe this concept or interpretation was created by Wilson, and unfortunately it has had a detrimental effect upon the learning of vector analysis within the framework of Gibbs's original contributions.

In explaining the meaning of the divergence of a vector function, Wilson misinterpreted Gibbs's notation for this function, namely $\nabla \cdot \mathbf{F}$. After defining the ∇ operator for the gradient in a rectangular system as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}, \tag{8.43}$$

he stated in Section 70, p. 150, of Wilson's book [30]:

Although the operation $\nabla\mathbf{V}$ has not been defined and cannot be at present, two formal combinations of the vector operator ∇ and a vector function $\mathbf{V}$ may be treated. These are the (formal) scalar product and the (formal) vector product of ∇ into $\mathbf{V}$. They are:

$$\nabla \cdot \mathbf{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \mathbf{V}, \tag{8.44}$$

$$\nabla \times \mathbf{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \mathbf{V}. \tag{8.45}$$

The differentiations $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, being scalar operators, pass by the dot and the cross, that is

$$\nabla \cdot \mathbf{V} = \left(i \cdot \frac{\partial \mathbf{V}}{\partial x} + j \cdot \frac{\partial \mathbf{V}}{\partial y} + k \cdot \frac{\partial \mathbf{V}}{\partial z}\right), \tag{8.46}$$

$$\nabla \times \mathbf{V} = \left(i \times \frac{\partial \mathbf{V}}{\partial x} + j \times \frac{\partial \mathbf{V}}{\partial y} + k \times \frac{\partial \mathbf{V}}{\partial z}\right). \tag{8.47}$$

They may be expressed in terms of the components V_1, V_2, V_3 of $\mathbf{V}$.

We have identified the equations with our own numbers. In order to compare these expressions with Gibb's expressions now described by (8.34) to (8.38), we again will change the notations $\mathbf{V}, x, y, z, i, j, k$ to $\mathbf{F}, x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3$; and $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ to $\nabla\mathbf{F}$ and $\overline{\nabla}\mathbf{F}$. Equations (8.43) to (8.47) then become

$$\nabla = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}, \tag{8.48}$$

$$\nabla\mathbf{F} = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \mathbf{F}, \tag{8.49}$$

$$\overline{\nabla}\mathbf{F} = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \times \mathbf{F}, \tag{8.50}$$

$$\nabla\mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i}, \tag{8.51}$$

$$\overline{\nabla}\mathbf{F} = \sum_i \hat{x}_i \times \frac{\partial \mathbf{F}}{\partial x_i}. \tag{8.52}$$

Equations (8.48), (8.51), and (8.52) are identical to Gibbs's (8.29), (8.35), and (8.37). However, (8.49) and (8.50) are not found in Gibbs's works. Wilson obtained or derived (8.51) and (8.52) from (8.49) and (8.50). The derivation involves two crucial steps or assumptions. First, he considers Gibbs's notations $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ as "formal" scalar and vector products between ∇ and $\mathbf{F}$. In the following, we will refer to this model as the FSP (formal scalar product) and FVP (formal vector product). He did not explain the meaning of the word *formal*. Secondly, after he formed the FSP and FVP, he let the differentiation $\partial/\partial x_i$ "pass by" the dot and the cross with the argument that the differentiations $\partial/\partial x_i$ $(i = 1, 2, 3)$ are scalar operators. Wilson's statement appears to be quite firm, but the standard books on mathematical analysis contain no proofs of any such theorem. Later on [30, p. 152], Wilson attempts to modify his position by saying:

> From some standpoints objections may be brought forward against treating ∇ as a symbolic vector and introducing $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ as the symbolic scalar and vector products of ∇ into $\mathbf{V}$, respectively. These objections may be avoided by simply laying down the definition that the symbol $\nabla\cdot$ and $\nabla\times$, which may be looked upon as entirely new operators quite distinct from ∇, shall be
>
> $$\nabla \cdot \mathbf{V} = i \cdot \frac{\partial \mathbf{V}}{\partial x} + j \cdot \frac{\partial \mathbf{V}}{\partial y} + k \cdot \frac{\partial \mathbf{V}}{\partial z} \tag{8.53}$$

and

$$\nabla \times \mathbf{V} = i \times \frac{\partial \mathbf{V}}{\partial x} + j \times \frac{\partial \mathbf{V}}{\partial y} + k \times \frac{\partial \mathbf{V}}{\partial z} . \tag{8.54}$$

But for practical purposes and for remembering formulas, it seems by all means advisable to regard

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

as a symbolic vector differentiator. This symbol obeys the same laws as a vector just in so far as the differentiations $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ obey the same laws as ordinary scalar quantities.

The contradictions between Wilson's statement above and his assertion concerning FSP and FVP are evident. Equations (8.53) and (8.54), of course, are the same as Gibbs's (8.27) and (8.28) with $\mathbf{V}$ replaced by $\mathbf{F}$ and x, y, z, i, j, k replaced by $x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3$. The difference is that Gibbs never spoke of an FSP and FVP; Wilson introduced these concepts to derive the expressions for div $\mathbf{F}$ and curl $\mathbf{F}$ by imposing some nonvalid manipulations. What is the consequence? Many later authors followed his practice and encountered difficulties when the same treatment was applied to orthogonal curvilinear coordinate systems. Before we discuss this topic, we must review Heaviside's treatment of vector analysis, particularly his handling of ∇.

We have pointed out that Gibbs's pamphlets were communicated to Heaviside. On the other hand, Wilson also mentioned some use of Heaviside's treatment of vector analysis in the preface to his book, *Electromagnetic Theory* (1893). The exchange between Heaviside and Wilson was therefore reciprocal. However, Heaviside goes his own way in presenting the same topics. Before we turn to the next section, Wilson's FSP and FVP model will be analytically examined. If we start with one of Gibbs's definitions of divergence, without using his notation but rather by using the linguistic notation, that is,

$$\text{div } \mathbf{F} = \sum_i \frac{\partial F_i}{\partial x_i} , \tag{8.55}$$

then, by substituting $F_1 = \hat{x}_1 \cdot \mathbf{F}$ into (8.55), we find

$$\text{div } \mathbf{F} = \sum_i \frac{\partial(\hat{x}_i \cdot \mathbf{F})}{\partial x_i} = \sum_i \left[\hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} + \frac{\partial \hat{x}_i}{\partial x_i} \cdot \mathbf{F} \right]. \tag{8.56}$$

Because $\partial \hat{x}_i / \partial x_i = 0$, (8.56) reduces to

$$\text{div } \mathbf{F} = \sum_i \hat{x}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} , \tag{8.57}$$

which is obviously not equal to

$$\left(\sum \hat{x}_i \frac{\partial}{\partial x_i} \right) \cdot \mathbf{F},$$

or $\nabla \cdot \mathbf{F}$. This is a proof of the lack of validity of the FSP. A similar proof can be executed with respect to the FVP. Another demonstration of the fallacy of an FSP is to consider a "twisted" differential operator of the form

$$\nabla_t = \hat{x}_2 \frac{\partial}{\partial x_1} + \hat{x}_3 \frac{\partial}{\partial x_2} + \hat{x}_1 \frac{\partial}{\partial x_3} \tag{8.58}$$

and a "twisted" vector function defined by

$$\mathbf{F}_t = \hat{x}_2 F_1 + \hat{x}_3 F_2 + \hat{x}_1 F_3. \tag{8.59}$$

If the FSP were a valid product, then, by following Wilson's pass-by procedure, we obtain

$$\nabla_t \cdot \mathbf{F}_t = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}. \tag{8.60}$$

In other words, div $\mathbf{F}$ is now treated as the formal scalar product between ∇_t and $\mathbf{F}_t$. The result is the same as Wilson's FSP between ∇ and $\mathbf{F}$. Such a manipulation is not, of course, a valid mathematical procedure. We have now refuted Wilson's treatment of div $\mathbf{F}$ and curl $\mathbf{F}$ based on the FSP and FVP. The legitimate compound differential operators for the divergence and the curl are, respectively, $\overline{\nabla}$ and $\overline{\nabla}$, defined by (8.32) and (8.33). $(\nabla\cdot)_G$ and $(\nabla\times)_G$ are merely Gibbs's notations suggested for the divergence and the curl. They are not operators.

8-4-3 The Spread of the Formal Scalar Product (FSP) and Formal Vector Product (FVP)

Being the first book on vector analysis published in the United States, Wilson's book became quite popular. It received its eighth printing in 1943, and a paperback reprint by Dover Publications appeared in 1960. Many later authors freely adopted Wilson's presentation using the FSP and FVP to derive the expressions for divergence and curl in the Cartesian coordinate system. We have found over 50 books [32] containing such a treatment. We now quote a few examples to show Wilson's influence.

1. In the book by Weatherburn [24] published in 1924, we find the following statement:

   To justify the notation, we have only to expand the formal products according to the distributive law, then

   $$\nabla \cdot \mathbf{f} = \left[\sum_i \left(\hat{x}_i \frac{\partial}{\partial x_i}\right)\right] \cdot \mathbf{f} = \sum_i \frac{\partial f_i}{\partial x_i} = \text{div } \mathbf{f}.$$

   We remark here that any distributive law in mathematics must be proved. In this case, there is no distributive law to speak of because the author is dealing with an assembly of mathematical symbols and not a compound operator. Incidentally, Weatherburn's book appears to be

the first book published in England wherein Gibbs's notations, but not Heaviside's, have been used in addition to the linguistic notations, namely, grad u, div $\mathbf{f}$, and curl $\mathbf{f}$.

2. A book by Lagally [33] published in 1928 contains the following statement on p. 123 (the original text is in German):

   > The rotation (curl) of $\mathbf{f}$ is denoted by the vector product between ∇ with field function $\mathbf{f}$ ... and
   >
   > $$\text{div grad} f = \nabla \cdot \nabla f = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_i \hat{x}_j \frac{\partial f}{\partial x_j}\right) = \nabla^2 f.$$

   It is seen that a term like $\hat{x}_1(\partial/\partial x_1) \cdot \hat{x}_1(\partial/\partial x_1)$ is an assembly of symbols. It is not a compound operator.

3. In a book by Mason and Weaver [34, p. 336] we find the following statement:

   > The differential operator ∇ can be considered formally as a vector of components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, so that its scalar and vector products with another vector may be taken.

   In comparison with Wilson's treatment, Mason and Weaver have used the word *formally* to be associated with ∇ and then speak of scalar and vector products with vector functions.

4. In his book *Applied Mathematics*, Schelkunoff first derived the differential expression for the divergence based on the flux model [35, p. 126]; then he added:

   > In Section 6 the vector operator del was introduced. If we treat it as a vector and multiply it by a vector $\mathbf{F}$, we find
   >
   > $$\nabla \cdot \mathbf{F} = \left(\sum_i \hat{x}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_j \hat{x}_j F_j\right) = \sum_i \frac{\partial F_i}{\partial x_i} = \text{div } \mathbf{F}.$$
   >
   > For this reason, $\nabla\cdot$ may be used as an alternative for div; however, the notation is tied too specifically to Cartesian coordinates.

   There are two messages in this statement: the first one is his acceptance of the FSP as a valid entity. The second one is his implication that FSP only applies to the Cartesian system. Actually, the divergence operator, ∇, is invariant with respect to the choice of the coordinate system, a property shown in Chapter 4, but $\nabla\cdot$ is an assembly, not an operator. Only by means of an illegitimate manipulation does it yield the differential expression for the divergence in the Cartesian coordinate system.

5. In a well-known book by Feynman, Leighton, and Sands [36, pp. 2–7] we find the following statement:

Let us try the dot product between ∇ with a vector field that we know, say $\mathbf{f}$; we write

$$\nabla \cdot \mathbf{f} = \nabla_x f_x + \nabla_y f_y + \nabla_z f_z$$

or

$$\nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} .$$

The authors remarked on the same page before this statement:

> With operators we must always keep the sequence right, so that the operations make the proper sense ....

This remark is important. Our discussion and use of the operators in Section 8-2, particularly that related to the compound operators, closely adheres to this principle. In the case of Gibbs's notation, $\nabla \cdot \mathbf{f}$, we are faced with a dot symbol after ∇, so that the differentiation cannot be applied to $\mathbf{f}$; it is blocked by a dot in the assembly. Thus, the authors seem to have violated their own rule by trying to form a dot product or FSP.

6. In the English translation of a Russian book by Borisenko and Tarapov [2, p. 157], we find the following statement:

> The expression $\nabla = \sum i_k(\partial/\partial x_k)$ for the operator ∇ implies the following representation for the divergence of $\mathbf{A}$:
>
> $$\operatorname{div} \mathbf{A} = \frac{\partial A_k}{\partial x_k} = i_k \frac{\partial}{\partial x_k} \cdot \mathbf{A} = \nabla \cdot \mathbf{A}.$$
>
> A coordinate-free symbolic representation of the operator ∇ is
>
> $$\nabla(\ldots) = \lim_{V \to 0} \frac{1}{V} \oint_s \mathbf{N}(\ldots)\, dS \tag{8.61}$$
>
> where $(\ldots)$ is some expression (possibly preceded by a dot or a cross) on which the given operator acts. In fact, according to (4.31) and (4.29) [of their book],
>
> $$\operatorname{grad} \varphi = \lim_{V \to 0} \frac{1}{V} \oint_s \varphi \mathbf{N}\, dS, \tag{8.62}$$
>
> $$\operatorname{div} \mathbf{A} = \lim_{V \to 0} \frac{1}{V} \oint_s \mathbf{A} \cdot \mathbf{N}\, dS. \tag{8.63}$$

From this passage, we see that the two authors believe the FSP is valid. Equation (8.61) also implies that they consider ∇ to be a constituent of the divergence and the curl in addition to comprising the gradient operator. The formula described by (8.61) appeared earlier in the book by Gans [37,

p. 49], who used both Gibbs's notations and the linguistic notations in this edition.

There are several authors presenting ∇ as defined as $\sum(\partial/\partial x_i)\hat{x}_i$ instead of $\sum \hat{x}_i(\partial/\partial x_i)$; and the Laplacian, defined as div grad, is often treated as the scalar product between two nablas, presumably because Gibbs used $\nabla \cdot \nabla$ as the notation for this compound operator. These practices, including the use of an FSP and FVP, are not confined within the boundaries of the United States and continental Europe; some Chinese and Japanese books, for example, commit the same errors.

8-5 ∇ in the Hands of Oliver Heaviside (1850–1925)

Although we have traced the concept of the FSP and FVP as due to Wilson, the same practice is found in the works of Heaviside. In Volume I of his book *Electromagnetic Theory* [26, §127] published in 1893, Heaviside stated:

> When the operand of ∇ is a vector, say **D**, we have both the scalar product and the vector product to consider. Taking the formula along first, we have
>
> $$\text{div } \mathbf{D} = \nabla_1 D_1 + \nabla_2 D_2 + \nabla_3 D_3.$$
>
> This function of **D** is called the divergence and is an important function in physical mathematics.

He then considered the curl of a vector function as the vector product between ∇ and that vector. At the time of his writing, he was already aware of Gibbs's pamphlets on vector analysis but Wilson's book was not yet published. It seems, therefore, that Heaviside and Wilson independently introduced the misleading concept for the scalar and vector products between ∇ and a vector function. Both were, perhaps, induced by Gibbs's notations for the divergence and the curl. Heaviside did not even include the word *formal* in his description of the products. We should mention that Heaviside's notations for these two products and the gradient are not the same as Gibbs's (see the table of notations in Section 2.1). His notation for the divergence of **f** is ∇**f** and his notation for the curl of **f** is $V\nabla\mathbf{f}$ (a quaternion notation), while his notation for the gradient of a scalar function f is $\nabla.f$. Having treated $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$ (Gibbs's notations for the divergence and the curl) as two "products," Heaviside simply considered ∇ as a vector in deriving various differential identities. One of them was presented as follows [26, §132]:

> The examples relate principally to the modification introduced by the differentiating functions of ∇.
>
> (a) We have the parallelopiped property
>
> $$\mathbf{N}V\nabla\mathbf{E} = \nabla V\mathbf{E}\mathbf{N} = \mathbf{E}V\mathbf{N}\nabla \tag{176}$$
>
> where ∇ is a common vector. The equations remain true when ∇ is vex, provided we consistently employ the differentiating power in the three forms.

Thus, the first form, expressing **N** component of curl **E**, is not open to misconception. But in the second form, expressing the divergence of $V\mathbf{EN}$, since **N** follows ∇, we must understand that **N** is supposed to remain constant. In the third form, again, the operand **E** precedes the differentiator; we must either, then, assume that ∇ acts backwards, or else, which is preferable, change the third form to $V\mathbf{N}\nabla.\mathbf{E}$, the scalar product of $V\mathbf{N}\nabla$ and **E**, or $(V\mathbf{N}\nabla)\mathbf{E}$ if that is plainer.

(b) Suppose, however, that both vectors in the vector product are variable. Thus, required the divergence of $V\dot{E}\dot{H}$, expanded vectorially. We have,

$$\nabla V\mathbf{EH} = \mathbf{E}V\mathbf{H}\nabla = \mathbf{H}V\nabla\mathbf{E}, \tag{177}$$

where the first form alone is entirely unambiguous. But we may use either of the others, provided that the differentiating power of ∇ is made to act on both **E** and **H**. But if we keep to the plainer and more usual convention that the operand is to follow the operator, then the third term, in which **E** alone is differentiated, gives one part of the result, whilst the second form, or rather its equivalent, $-\mathbf{E}\nabla\mathbf{H}$, wherein **H** alone is differentiated, gives the rest. So we have, complete, and without ambiguity

$$\text{div } V\mathbf{EH} = \mathbf{H} \text{ curl } \mathbf{E} - \mathbf{E} \text{ curl } \mathbf{H}, \tag{178}$$

an important transformation.

First of all, in terms of Gibbs's notations, Heaviside's Eqs. (176), (177), and (178) would be written in the form

$$\mathbf{N} \cdot \nabla \times \mathbf{E} = \nabla \cdot (\mathbf{E} \times \mathbf{N}) = \mathbf{E} \cdot (\mathbf{N} \times \nabla), \tag{8.64}$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot (\mathbf{H} \times \nabla) = \mathbf{H} \cdot (\nabla \times \mathbf{E}), \tag{8.65}$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}. \tag{8.66}$$

According to the established mathematical rules, Heaviside's logic in arriving at his (178) or our (8.66) is entirely unacceptable; in particular, present-day students would never write an equation (177) or (8.65) with ∇ being the ∇ operator. The second term in (8.65) is a weighted operator, while the first and the third are functions and they are not equal to each other. His Eq. (178) or (8.66) in Gibbs's notation is a valid vector identity but his derivation of this identity is not based on established mathematical rules. It is obtained by a manipulation of mathematical symbols and selecting the desired forms. The most important message passed on to us is his practice of considering $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$ as two legitimate products, the same as Wilson's FSP and FVP. Heaviside's "equations" will be examined again in a later section and will be cast in proper form in terms of the symbolic vector and/or a partial symbolic vector.

Many authors in the past have considered Heaviside to be a cofounder with Gibbs of modern vector analysis. We do not share this view. In Heaviside's

treatment of vector analysis, he spoke freely of the scalar product and the vector product between ∇ and a vector function $\mathbf{F}$, and he used ∇ as a vector in deriving algebraic vector identities that incorporate differential entities. In view of these mathematically insupportable treatments, Heaviside's status as a pioneer in vector analysis is not of the same level as Gibbs's. In the historical introduction of a 1950 edition of Heaviside's book on *Electromagnetic Theory* [26], Ernst Weber stated:

> Chap. III of the *Electromagnetic Theory* dealing with 'The Elements of Vectorial Algebra and Analysis' is practically the model of modern treatises on vector analysis. Considerable moral assistance came from a pamphlet by J. W. Gibbs who independently developed vector analysis during 1881–84 in Heaviside sense—but using the less attractive notation of Tait; however, Gibbs deferred publication until 1901.

This statement unfortunately contains several misleading messages. In the first place, in view of our detailed study of Heaviside's works, his treatment would be a poor model if it were used to teach vector calculus. Secondly, if Heaviside truly received moral assistance from Gibbs's pamphlet, he would not have committed himself to the improper use of ∇, and would have restricted his use of it to the expression for the gradient. Most important of all, Gibbs did not develop his theory in the Heaviside sense. His development is completely different from that of Heaviside. Finally, the book published in 1901 was written by Wilson, not by Gibbs himself. Even though it was founded upon the lectures of Gibbs, it contained some of Wilson's own interpretations, which are not found in Gibbs's original pamphlets nor in his lecture notes reported by Wilson. The two prefaces, one by Gibbs and another by Wilson, which we quote in Section 8-4-2, are proofs of our assertion. We were reluctant to criticize a scientist of Heaviside's status and the opinion expressed by Prof. Weber. After all, Heaviside had contributed much to electromagnetic theory and had been recognized as a rare genius. However, in the field of vector analysis, we must set the record straight and call attention to the outstanding contribution of Gibbs, who stood above all his contemporaries in the last century. For the sake of future generations of students, we have the obligation to remove unsound arguments and arbitrary manipulations in an otherwise precise branch of mathematical science.

8-6 Shilov's Formulation of Vector Analysis

A book in Russian on vector analysis was written by Shilov [38] in 1954, who advocated a new formulation with the intent of providing a rather broader treatment of vector analysis. Shilov's work was adopted by Fang [39], who studied in the U.S.S.R. We were informed of Shilov's work through Fang. After a careful examination of the English translation of the two key chapters in that book, we found the contradictions as described below:

Shilov defined an "expression" for ∇ denoted by $T(\nabla)$ as

$$T(\nabla) = \frac{\partial}{\partial x}T(i) + \frac{\partial}{\partial y}T(j) + \frac{\partial}{\partial z}T(k), \tag{8.67}$$

where i, j, k denote the Cartesian unit vectors and ∇ (nabla) is identified as the Hamilton differential operator, that is,

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Equation (8.67) is the same as Shilov's Eq. (18) on p. 18 of [35]. We want to emphatically call attention to the fact that the only meaningful expression for $T(\nabla)$ involving ∇ are ∇f and $\nabla \mathbf{f}$, the gradient of f and $\mathbf{f}$. In the case of ∇f, (8.67) is an identity, because the right side of (8.67) yields

$$\frac{\partial}{\partial x}(if) + \frac{\partial}{\partial y}(jf) + \frac{\partial}{\partial z}(kf) = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z},$$

which is ∇f.

The most serious contradiction in Shilov's work is his derivation of the expression for the divergence and the curl by letting $T(\nabla)$ equal to $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$, respectively. We have pointed out before that these two products do not exist. Shilov is defining a meaningless assembly to make it meaningful. It is like defining $2 + \times 3$ to be equal to $2 \times +3$ $(= +6)$.

8-7 Formulations in Orthogonal Curvilinear Systems

After having revealed a number of "historical" confusions and contradictions in vector analysis so far presented in the rectangular system, we now examine several presentations in curvilinear coordinate systems. We will show even more clearly the sources of the various misrepresentations.

8-7-1 Two Examples from the Book by Moon and Spencer

In their book, Moon and Spencer write [23, p. 325]:

> Let me apply the definition, Eq. (1.4) [of ∇ in the orthogonal curvilinear system, our (7.20)], to divergence. By the usual definition of a scalar product,
>
> $$\nabla \cdot \mathbf{V} = \frac{1}{(g_{11})^{1/2}}\frac{\partial (V)_1}{\partial x^1} + \frac{1}{(g_{22})^{1/2}}\frac{\partial (V)_2}{\partial x^2} + \frac{1}{(g_{33})^{1/2}}\frac{\partial (V)_3}{\partial x^3}. \tag{8.68}$$
>
> But this is *not* divergence, which is found to be ....
>
> Similar inconsistencies are obtained with other applications of Eq. (1.4).

In (8.68), their $(g_{ii})^{1/2}$ correspond to our metric coefficients h_i and their x^i to our variables v_i.

In the first place, they have now applied the FSP to ∇ and $\mathbf{V}$ in an orthogonal curvilinear coordinate system without realizing that the FSP is not a valid entity in any coordinate system including the rectangular system. After obtaining a wrong formula for the divergence, (8.68), they did not offer an explanation of the reason for the failure.

In discussing the Laplacian of a vector function, Moon and Spencer state [23, p. 235]:

> Section (7.08) showed that there are three meaningful combinations of differential operators: div grad, grad div, and curl curl. Of these, the first is the scalar Laplacian, ∇^2. It is convenient to combine the other two operators to form the vector Laplacian, ✡:
>
> $$\text{✡} = \text{grad div} - \text{curl curl}. \tag{8.69}$$
>
> Evidently the vector Laplacian can operate only on a vector, so
>
> $$\text{✡}\,\mathbf{E} = \text{grad div}\,\mathbf{E} - \text{curl curl}\,\mathbf{E}. \tag{8.70}$$
>
> Since the quantities on the right are vectors, $\text{✡}\,\mathbf{E}$ transforms as a univalent tensor or vector.
>
> As noted in Table 1.01 [their table of notations on p. 10], the scalar and vector Laplacians are often represented by the same symbol. This is poor practice, however, since the two are basically quite different:
>
> $$\nabla^2 = \text{div grad}, \tag{8.71}$$
>
> $$\text{✡} \equiv \text{grad div} - \text{curl curl}. \tag{8.72}$$
>
> This difference is evident also when the expression for the vector Laplacian is expanded....

Analytically, we have proved that in any general curvilinear system,

$$\nabla\nabla\mathbf{f} = \nabla\nabla\mathbf{f} - \nabla\,\nabla\,\mathbf{f}, \tag{8.73}$$

where $\nabla\mathbf{f}$ denotes the gradient of a vector function that is a dyadic function. The divergence of a dyadic function is a vector function. The use of ∇^2 to denote the Laplacian is an old practice, but the use of $\nabla\,\nabla$ is preferred because it shows the structure of the Laplacian when it is applied to either a scalar function or a vector function. By treating (8.69) as the definition for the Laplacian applied to a vector function, the two authors have probably been influenced by a remark made by Stratton [5, p. 50]:

> The vector $\nabla\cdot\nabla\mathbf{F}$ may now be obtained by subtraction of (85) [an expansion of $\nabla\times\nabla\times\mathbf{F}$ in an orthogonal curvilinear system] from the expansion of $\nabla\nabla\cdot\mathbf{F}$, and the result differs from that which follows a direct application of the Laplacian to the curvilinear components of $\mathbf{F}$.

As shown in our proof, $\nabla\mathbf{F}$ is a dyadic, where the gradient operator must apply to the entire vector function containing both the components and the unit vectors. When this is done, we find that (8.73) is indeed an identity. In view of our analysis, it is clear that a special symbol for the Laplacian is not necessary when it is operating on a vector function. The same remark holds true for the two different notations for the Laplacian introduced by Burali-Forti and Marcolongo, as shown in Table 8.1.

These two examples also show why Moon and Spencer thought that ∇ is an unreliable device. The past history of vector analysis seems to have led them to such a conclusion. ∇ is a reliable device when it is used in the gradient of a scalar or vector function, but not in any other application. We emphasize once more that for the divergence and the curl, the divergence operator, $\nabla\cdot$, and the curl operator, $\nabla\times$, are the proper operators. They are distinctly different from ∇.

8-7-2 A Search for the Divergence Operator in Orthogonal Curvilinear Coordinate Systems

In a well-known book on the methods of theoretical physics [40, p. 44], the authors, Morse and Feshbach, try to find the differential operators for the three key functions in an orthogonal curvilinear coordinate system. They state:

The vector operator must have different forms for its different uses:

$$\nabla = \sum_i \frac{\hat{u}_i}{h_i}\frac{\partial}{\partial v_i} \quad \text{for the gradient}$$

$$= \frac{1}{\Omega}\sum_i \hat{u}_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\right) \quad \text{for the divergence}$$

and no form which can be written for the curl.

We have used Ω to represent $h_1h_2h_3$ and have changed their coordinate variables ξ_i to v_i and their symbols a_i to $\hat{u}_i$. It is obvious that the "operator" introduced by these two authors for the divergence can produce the correct expression for the divergence only if the operation is interpreted as

$$\left[\frac{1}{\Omega}\sum_i \hat{u}_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\right)\right]\cdot\mathbf{f} \quad \rightarrow \quad \frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\hat{u}_i\cdot\mathbf{f}\right). \tag{8.74}$$

Such an interpretation is quite arbitrary, and it does not follow the accepted rule of a differential operator because the first term within the bracket is a function, so the entire expression represents the scalar product of $[\cdots]$ and $\mathbf{f}$. One is not supposed to move the unit vector $\hat{u}_i$ to the right side of Ω/h and then combine $\hat{u}_i$ with $\cdot\mathbf{f}$, as shown in the right term of (8.74). It is a matter of creating a desired expression by arbitrarily rearranging the terms in a function and the position of the dot operator. A reader must recognize now that ∇ can never be a part of the divergence operator nor the curl operator. The proper operators for the divergence and the curl are $\nabla\cdot$

and ∇, respectively. We could have used any two symbols for that matter, such as **D** and **C**.

8-8 The Use of ∇ to Derive Vector Identities

There are many authors who have tried to apply identities in vector algebra to "derive" vector identities involving the differential functions ∇f, ∇f, and $\nabla \mathbf{f}$. We quote here two examples. The first example is from the book by Borisenko and Tarapov [2, p. 180], where a problem is posed and "solved":

Prob. 7. Find $\nabla(\mathbf{A} \cdot \mathbf{B})$.

Solution. Clearly

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla(\mathbf{A}_c \cdot \mathbf{B}) + \nabla(\mathbf{A} \cdot \mathbf{B}_c) \tag{8.75}$$

where the subscript 'c' has the same meaning as on p. 170 [the subscript 'c' denotes that the quantity to which it is attached is momentarily being held fixed]. According to formula (1.30)

$$\mathbf{c}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{c})\mathbf{B} - \mathbf{A} \times (\mathbf{B} \times \mathbf{c}). \tag{8.76}$$

Hence setting

$$\mathbf{A} = \mathbf{A}_c, \qquad \mathbf{B} = \mathbf{B}, \qquad \mathbf{c} = \nabla,$$

we have

$$\nabla(\mathbf{A}_c \cdot \mathbf{B}) = (\mathbf{A}_c \cdot \nabla)\mathbf{B} + \mathbf{A}_c \times (\nabla \times \mathbf{B}), \tag{8.77}$$

and similarly,

$$\nabla(\mathbf{A} \cdot \mathbf{B}_c) = \nabla(\mathbf{B}_c \cdot \mathbf{A}) = (\mathbf{B}_c \cdot \nabla)\mathbf{A} + \mathbf{B}_c \times (\nabla \times \mathbf{A}). \tag{8.78}$$

Thus, finally,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \operatorname{curl} \mathbf{B} + \mathbf{B} \times \operatorname{curl} \mathbf{A}. \tag{8.79}$$

As far as the final result, (8.79), is concerned, they have indeed obtained a correct answer. But there is no justification for applying (8.76) with **c** replaced by ∇.

The second example is found in the book by Panofsky and Phillips [41, p. 470]. They wrote:

$$\begin{aligned} \nabla \times (\mathbf{A} \times \mathbf{B}) &= (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} \\ &= (\nabla \cdot \mathbf{B}_c)\mathbf{A} - (\nabla \cdot \mathbf{B})\mathbf{A}_c \\ &\quad - (\nabla \cdot \mathbf{A}_c)\mathbf{B} - (\nabla \cdot \mathbf{A})\mathbf{B}_c \end{aligned} \tag{8.80}$$

where the subscript 'c' indicates that the function is constant and may be permuted with the vector operator, with due regard to sign changes if such changes are indicated by the ordinary vector relations.

It is seen that their $(\nabla \cdot \mathbf{B})\mathbf{A}$ in the first line is not (div $\mathbf{B}$)$\mathbf{A}$. Rather, it is equal to $(\nabla \cdot \mathbf{B}_c)\mathbf{A} + (\nabla \cdot \mathbf{B})\mathbf{A}_c$. Secondly, if $\mathbf{B}_c$ is constant, the established rule in differential calculus would consider their $\nabla \cdot \mathbf{B}_c$, (i.e., div $\mathbf{B}_c$) = 0. The use of algebraic identities to derive differential identities by replacing a vector by ∇ has no foundation—the first line of (8.80). For the exercise in consideration, one way to find the identity is to prove first that

$$\nabla(\mathbf{A} \times \mathbf{B}) = \nabla(\mathbf{BA} - \mathbf{AB}), \tag{8.81}$$

where $\mathbf{AB}$ is a dyadic and $\mathbf{BA}$ its transpose. Then, by means of dyadic analysis, one finds

$$\nabla(\mathbf{BA}) = (\nabla\mathbf{B})\mathbf{A} + \mathbf{B} \cdot \nabla\mathbf{A}, \tag{8.82}$$

$$\nabla(\mathbf{AB}) = (\nabla\mathbf{A})\mathbf{B} + \mathbf{A} \cdot \nabla\mathbf{B}. \tag{8.83}$$

Hence

$$\nabla(\mathbf{A} \times \mathbf{B}) = (\nabla\mathbf{B})\mathbf{A} + \mathbf{B} \cdot \nabla\mathbf{A} - (\nabla\mathbf{A})\mathbf{B} - \mathbf{A} \cdot \nabla\mathbf{B}, \tag{8.84}$$

where $\nabla\mathbf{A}$ and $\nabla\mathbf{B}$ are two dyadic functions. A simpler method of deriving (8.84) will be shown in a later section. It should be emphasized that one cannot legitimately write

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B}$$

as the two authors did and then change $(\nabla \cdot \mathbf{B})\mathbf{A}$ to $\nabla \cdot (\mathbf{BA})$, and similarly for $(\nabla \cdot \mathbf{A})\mathbf{B}$, in order to create a desired identity.

A general comment on the analogy and no analogy between algebraic vector identities and differential vector identities was made by Milne [42]. He states on p. 77:

> The above examples [referring to nine differential vector identities expressed in linguistic notations such as grad $(\hat{x} \cdot \mathbf{Y})$ = (grad $\hat{x}$) $\cdot$ $\mathbf{Y}$ + (grad $\mathbf{Y}$) $\cdot$ $\hat{x}$, etc.] whilst exhibiting the relations between the symbols in vector or tensor form, conceal the nature of the identities. A little gain of insight is obtained occasionally if the symbol is employed. E.g., Example (9) [curl curl $\hat{x}$ = grad div $\hat{x} - \nabla^2\hat{x}$] may be written
>
> $$\nabla \times (\nabla \times \hat{x}) = \nabla(\nabla \cdot \hat{x}) - \nabla^2\hat{x}, \tag{8.85}$$
>
> which bears an obvious analogy to
>
> $$\mathbf{Q} \times (\mathbf{Q} \times \hat{x}) = \mathbf{Q}(\mathbf{Q} \cdot \hat{x}) - \mathbf{Q}^2\hat{x} \tag{8.86}$$
>
> where $\mathbf{Q}$ denotes a vector function.
>
> On the other hand Example (5)
>
> $$\text{Curl } (\hat{x} \times \mathbf{Y}) = \mathbf{Y} \cdot \text{grad } \hat{x} - \hat{x} \cdot \text{grad } \mathbf{Y} + \hat{x} \text{ div } \mathbf{Y} - \mathbf{Y} \text{ div } \hat{x}$$
>
> may be written
>
> $$\nabla \times (\hat{x} \times \mathbf{Y}) = \mathbf{Y} \cdot \nabla\hat{x} - \hat{x} \cdot \nabla\mathbf{Y} + \hat{x}(\nabla \cdot \mathbf{Y}) - \mathbf{Y}(\nabla \cdot \hat{x}) \tag{8.87}$$

which bears no obvious analogy to

$$\mathbf{Q} \times (\hat{x} \times \mathbf{Y}) = \hat{x}(\mathbf{Q} \cdot \mathbf{Y}) - \mathbf{Y}(\mathbf{Q} \cdot \hat{x}) \tag{8.88}$$

To obtain a better analogy, one would have to write

$$\mathbf{Q} \times (\hat{x} \times \mathbf{Y}) = \mathbf{Q} \cdot (\mathbf{YX} - \mathbf{XY}) \tag{8.89}$$

and replace $\mathbf{Q}$ by ∇.

We do not understand why (8.89) is a better analogy than (8.88) because, as algebraic vector identities, they are equivalent. There is only one interpretation of (8.89), namely,

$$\mathbf{Q} \times (\hat{x} \times \mathbf{Y}) = (\mathbf{Q} \cdot \mathbf{Y})\hat{x} - (\mathbf{Q} \cdot \hat{x})\mathbf{Y}, \tag{8.90}$$

which is the same as (8.88).

By replacing $\mathbf{Q}$ by ∇ in (8.89), and treating the resultant expression as the divergence of the dyadic $\mathbf{XY} - \mathbf{YX}$, the manipulation is identical to the one used by Panofsky and Phillips. This short paragraph on the role played by del in an authoritative book on vectorial mechanics shows the consequence of treating Gibbs's notations for the divergence and the curl as two products, one scalar and one vector.

We have now shown the failures by several authors in trying to invoke ∇ as an operator, not only for the gradient but also for the divergence and the curl. The role is now filled in by the symbolic vector, to be discussed in the next section as introduced in this book. Many of the ambiguities that have occurred in the past presentations covered in this paper will be recast correctly and unambiguously by our new method utilizing the symbolic vector.

8-9 A Recasting of the Past Failures by the Method of Symbolic Vector

If we replace Heaviside's "equations" (8.64)–(8.66) with

$$\mathbf{N} \cdot \nabla \times \mathbf{E} = \nabla(\mathbf{E} \times \mathbf{N}) = \mathbf{E} \cdot (\mathbf{N} \times \nabla), \tag{8.91}$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot (\mathbf{H} \times \nabla) = \mathbf{H} \cdot (\nabla \times \mathbf{E}), \tag{8.92}$$

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \nabla_{\mathrm{E}} \cdot (\mathbf{E} \times \mathbf{H}) - \nabla_{\mathrm{H}} \cdot (\mathbf{H} \times \mathbf{E}) \\ &= \mathbf{H} \cdot (\nabla_{\mathrm{E}} \times \mathbf{E}) - \mathbf{E} \cdot (\nabla_{\mathrm{H}} \times \mathbf{H}), \end{aligned} \tag{8.93}$$

then (8.93) yields

$$\nabla(\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla\mathbf{E} - \mathbf{E} \cdot \nabla\mathbf{H}. \tag{8.94}$$

Although (8.91) and (8.92) have the same form as Heaviside's except that his ∇ has been replaced by the symbolic vector ∇, yet there is a vast difference in

meaning between the two sets. For example, his $\mathbf{H} \cdot \nabla \times \mathbf{E}$ in (8.65) is interpreted as $\mathbf{H} \cdot \text{curl } \mathbf{E}$, but our $\mathbf{H} \cdot \nabla \times \mathbf{E}$ is the same as $\nabla \cdot (\mathbf{E} \times \mathbf{H})$ because of Lemma 4.2 and it is equal to $\nabla (\mathbf{E} \times \mathbf{H})$.

Every term in (8.91) to (8.94) is well defined. Both Lemma 4.1 and Lemma 4.2 are used to obtain the vector identity stated by (8.94).

Returning now to the problems posed by Borisenko and Tarapov, we start with the symbolic expression $\nabla(\mathbf{A} \cdot \mathbf{B})$ for $\nabla(\mathbf{A} \cdot \mathbf{B})$; then, by applying Lemma 4.2, we have

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla_A(\mathbf{A} \cdot \mathbf{B}) + \nabla_B(\mathbf{A} \cdot \mathbf{B}). \tag{8.95}$$

Applying Lemma 4.1, we have

$$\nabla_A(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla_A)\mathbf{A} - \mathbf{B} \times (\mathbf{A} \times \nabla_A) \tag{8.96}$$

and

$$\nabla_B(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla_B)\mathbf{B} - \mathbf{A} \times (\mathbf{B} \times \nabla_B). \tag{8.97}$$

Hence

$$\nabla_A(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{A} + \mathbf{B} \times \nabla\mathbf{A} \tag{8.98}$$

and

$$\nabla_B(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{A} \times \nabla\mathbf{B}. \tag{8.99}$$

Thus,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\mathbf{A} + \mathbf{A} \times \nabla\mathbf{B} + \mathbf{B} \times \nabla\mathbf{A}. \tag{8.100}$$

Our derivation of (8.100) appears to be similar to the derivation by Borisenko and Tarapov in form, but the use of the FSP and FVP in their formulation and the treatment of (8.77) as an algebraic identity is entirely unacceptable, while each of our steps are supported by the basic principle in the method of symbolic vector, particularly the two lemmas therein.

The exercise posed by Panofsky and Phillips can be formulated correctly by our new method. The steps are as follows:

We start with $\nabla \times (\mathbf{A} \times \mathbf{B})$, which is the symbolic expression of $\nabla (\mathbf{A} \times \mathbf{B})$; then by means of Lemma 4.2,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \nabla_A \times (\mathbf{A} \times \mathbf{B}) + \nabla_B \times (\mathbf{A} \times \mathbf{B}). \tag{8.101}$$

By means of Lemma 4.1, we have

$$\begin{aligned} \nabla_A \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla_A)\mathbf{A} - (\nabla_A \cdot \mathbf{A})\mathbf{B} \\ &= \mathbf{B} \cdot \nabla\mathbf{A} - \mathbf{B}\nabla\mathbf{A}. \end{aligned}$$

Similarly,

$$\begin{aligned} \nabla_B \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla_B)\mathbf{A} - (\mathbf{A} \cdot \nabla_B)\mathbf{B} \\ &= \mathbf{A}\nabla\mathbf{B} - \mathbf{A} \cdot \nabla\mathbf{B}. \end{aligned}$$

Hence

$$\nabla (\mathbf{B} \times \mathbf{B}) = \mathbf{A}\nabla\mathbf{B} - \mathbf{A} \cdot \nabla\mathbf{B} - \mathbf{B}\nabla\mathbf{A} + \mathbf{B} \cdot \nabla\mathbf{A}, \tag{8.102}$$

which is the same as (8.84) obtained previously in Section 8-8 by a more complicated analysis. The convenience and the simplicity of the method of symbolic vector to derive vector identities has been clearly demonstrated in the last two examples. All commonly used vector identities have been derived in this way, as shown in Chapter 4.

8-9-1 In Retrospect

In this work, we have examined critically some practices of presenting vector analysis in several early works and in a few contemporary writings. We state with emphasis that the whole subject of vector analysis was formulated by the great American scientist J. Willard Gibbs in a precise and elegant fashion. Although his original works are confined to formulations in a Cartesian coordinate system, they can be extended to curvilinear systems as a result of the invariance of the differential operators, as reviewed in this book, without the necessity of resorting to the aid of tensor analysis.

In spite of the richness of Gibbs's theory of vector analysis, his notations for the divergence and the curl, in the opinion of this author, have induced several later workers, including one of his students, Wilson, to make some inappropriate interpretations. The adoption of these interpretations has been worldwide. We have selected a few examples from the works of several seasoned scientists and engineers to illustrate the prevalence of the improper use of ∇.

As a result of this study, we have justified our adoption of the new operational symbols as the notations for the divergence and the curl to replace Gibbs's old notations. It seems that our move is reasonable from the logistical point of view.

We have examined a history covering a period of over one hundred years. It represents a most interesting period in the development of the mathematical foundations of electromagnetic theory. However, in view of the long-entrenched and widespread misuse of the gradient operator ∇ as a constituent of the divergence and curl operators, the obligation of sharing the insight presented here with many of our colleagues in this field has been a labor fraught with frustration.

We hope that this historical study has been sufficiently clear to enable the serious workers in this subject to understand the issues, and that future students will not have to ponder over contradictions and misrepresentations to learn this subject.

It may be proper to conclude this chapter and the book by quoting a remark made by E. B. Wilson 87 years ago [43]. In reviewing two Italian books on vector analysis by Burali-Forti and Marcolongo, Wilson concluded his article with the following remark:

> What the resulting residual system may be we will not venture to predict, but that there will be such a system fifty years hence we fully believe. And

whatever that system may be it should and probably will conform to two requirements:

1. correct ideas relative to vector fields;
2. analytical suggestions of notations.

We sincerely believe that the method of symbolic vector together with the new operational notations for the divergence and the curl have fulfilled Dr. Wilson's wish. Whether the new notations will be adopted by potential users we leave to future generations of students to decide. Our goal is to put a logical approach on record.

Appendix A

Transformation Between Unit Vectors

A-1 Cylindrical System

$$(v_1, v_2, v_3) = (r, \phi, z)$$
$$(h_1, h_2, h_3) = (1, r, 1)$$
$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{r}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

A-2 Spherical System

$$(v_1, v_2, v_3) = (R, \theta, \phi)$$
$$(h_1, h_2, h_3) = (1, R, R\sin\theta)$$
$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{R}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\hat{\theta}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0

A-3 Elliptical Cylinder

$$(v_1, v_2, v_3) = (\eta, \xi, z)$$

$$(h_1, h_2, h_3) = \left[c\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2},\ c\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2},\ 1 \right]$$

$$x = c\eta\xi, \quad y = c\left[(1 - \eta^2)(\xi^2 - 1)\right]^{1/2}, \quad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\eta}$	$\frac{c\xi}{h_1}$	$\frac{-c\eta}{h_2}$	0
$\hat{\xi}$	$\frac{c\eta}{h_2}$	$\frac{c\xi}{h_1}$	0
$\hat{z}$	0	0	1

A-4 Parabolic Cylinder

$$(v_1, v_2, v_3) = (\eta, \xi, z)$$

$$(h_1, h_2, h_3) = \left[(\eta^2 + \xi^2)^{1/2},\ (\eta^2 + \xi^2)^{1/2},\ 1 \right]$$

$$x = \frac{1}{2}(\eta^2 - \xi^2), \quad y = \eta\xi, \quad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\eta}$	$\frac{\eta}{h}$	$\frac{\xi}{h}$	0
$\hat{\xi}$	$\frac{-\xi}{h}$	$\frac{\eta}{h}$	0
$\hat{z}$	0	0	1

$$h = h_1 = h_2 = (\eta^2 + \xi^2)^{1/2}$$

A-5 Prolate Spheroid

$$(v_1, v_2, v_3) = (\eta, \xi, \phi)$$

$$(h_1, h_2, h_3) = \left[c\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2},\ c\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2},\ c\,(1 - \eta^2)^{1/2}\,(\xi^2 - 1)^{1/2} \right]$$

$$r = (x^2 + y^2)^{1/2} = c\left[(1 - \eta^2)\,(\xi^2 - 1)\right]^{1/2}, \qquad \phi = \phi, \qquad z = c\eta\xi$$

	$\hat{z}$	$\hat{r}$	$\hat{\phi}$
$\hat{\eta}$	$\dfrac{c\xi}{h_1}$	$\dfrac{-c\eta}{h_2}$	0
$\hat{\xi}$	$\dfrac{c\eta}{h_2}$	$\dfrac{c\xi}{h_1}$	0
$\hat{\phi}$	0	0	1

The unit vectors $\hat{r}$ and $\hat{\phi}$ can be expressed in terms of $\hat{x}$ and $\hat{y}$ covered in the cylindrical case; the same applies to the oblate spheroid.

A-6 Oblate Spheroid

$$(v_1, v_2, v_3) = (\eta, \xi, \phi)$$

$$(h_1, h_2, h_3) = \left[c\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2},\ c\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2},\ c\xi\eta \right]$$

$$r = (x^2 + y^2)^{1/2} = c\xi\eta, \qquad z = c\left[(\xi^2 - 1)\,(1 - \eta^2)\right]^{1/2}$$

	$\hat{z}$	$\hat{r}$	$\hat{\phi}$
$\hat{\xi}$	$\dfrac{c\xi}{h_2}$	$\dfrac{c\eta}{h_1}$	0
$\hat{\eta}$	$\dfrac{-c\eta}{h_1}$	$\dfrac{c\xi}{h_2}$	0
$\hat{\phi}$	0	0	1

A-7 Bipolar Cylinders

$$(v_1, v_2, v_3) = (\eta, \xi, z)$$

$$(h_1, h_2, h_3) = \left(\frac{a}{\cosh\xi - \cos\eta}, \frac{a}{\cosh\xi - \cos\eta}, 1\right)$$

$$x = \frac{a\sinh\xi}{\cosh\xi - \cos\eta}, \qquad y = \frac{a\sin\eta}{\cosh\xi - \cos\eta}, \qquad z = z$$

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\eta}$	$\frac{-h}{a}\sinh\xi\sin\eta$	$\frac{h}{a}(\cosh\xi\cos\eta - 1)$	0
$\hat{\xi}$	$\frac{-h}{a}(\cosh\xi\cos\eta - 1)$	$\frac{-h}{a}\sinh\xi\sin\eta$	0
$\hat{z}$	0	0	1

where

$$h = h_1 = h_2 = \frac{a}{\cosh\xi - \cos\eta}.$$

In all of these tables, the unit vectors are all arranged in the order of a right-handed system, that is, $\hat{x}_1 \times \hat{x}_2 = \hat{x}_3$.

Appendix B

Vector and Dyadic Identities

Vector Identities

1. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$
3. $\nabla (ab) = a\nabla b + b\nabla a$
4. $\nabla (a\mathbf{b}) = a\nabla \mathbf{b} + (\nabla a)\mathbf{b}$
5. $\nabla \cdot (a\mathbf{b}) = a\nabla \cdot \mathbf{b} + \mathbf{b} \cdot \nabla a$
6. $\nabla \times (a\mathbf{b}) = a\nabla \times \mathbf{b} - \mathbf{b} \times \nabla a$
7. $\nabla (\mathbf{a} \cdot \mathbf{b}) = (\nabla \mathbf{a}) \cdot \mathbf{b} + \mathbf{a} \cdot \nabla \mathbf{b} = \mathbf{a} \times \nabla \times \mathbf{b} + \mathbf{b} \times \nabla \times \mathbf{a} + \mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{a}$
8. $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$
9. $\nabla \cdot (\mathbf{ab}) = (\nabla \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \cdot \nabla \mathbf{b}$
10. $\mathbf{a} \times \nabla \times \mathbf{b} = (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b}$
11. $\nabla \times (\mathbf{a} \times \mathbf{b}) = \nabla (\mathbf{ba} - \mathbf{ab}) = \mathbf{a}\nabla \mathbf{b} - \mathbf{b}\nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{a}$
12. $\nabla \times \nabla \times \mathbf{a} = \nabla\nabla \cdot \mathbf{a} - \nabla \cdot \nabla \mathbf{a}$
13. $\nabla \cdot \nabla a = 0$
14. $\nabla \times \nabla \mathbf{a} = 0$

Dyadic Identities

15. $\mathbf{a} \cdot (\mathbf{b} \times \bar{\bar{c}}) = -\mathbf{b} \cdot (\mathbf{a} \times \bar{\bar{c}}) = (\mathbf{a} \times \mathbf{b}) \cdot \bar{\bar{c}}$
16. $\mathbf{a} \times (\mathbf{b} \times \bar{\bar{c}}) = \mathbf{b}(\mathbf{a} \cdot \bar{\bar{c}}) - (\mathbf{a} \cdot \mathbf{b})\bar{\bar{c}}$
17. $\nabla(a\bar{\bar{b}}) = a\nabla\bar{\bar{b}} + (\nabla a) \cdot \bar{\bar{b}}$
18. $\nabla(a\bar{\bar{b}}) = a\nabla\bar{\bar{b}} + (\nabla a) \times \bar{\bar{b}}$
19. $\nabla(\mathbf{a} \times \bar{\bar{b}}) = (\nabla \mathbf{a}) \cdot \bar{\bar{b}} - \mathbf{a} \cdot \nabla\bar{\bar{b}}$
20. $\nabla\nabla\bar{\bar{a}} = \nabla\nabla\bar{\bar{a}} - \nabla\nabla\bar{\bar{a}}$
21. $\nabla\nabla\bar{\bar{a}} = 0$
22. $\nabla\nabla\bar{\bar{a}} = 0$
23. $\mathbf{a} \cdot \bar{\bar{b}} = [\bar{\bar{b}}]^{\mathrm{T}} \cdot \mathbf{a}$
24. $\mathbf{a} \times \bar{\bar{b}} = -\left\{[\bar{\bar{b}}]^{\mathrm{T}} \times \mathbf{a}\right\}^{\mathrm{T}}$
25. $[\bar{\bar{c}}]^{\mathrm{T}} \cdot (\mathbf{a} \times \bar{\bar{b}}) = -[\mathbf{a} \times \bar{\bar{c}}]^{\mathrm{T}} \cdot \bar{\bar{b}}$

Appendix C

Integral Theorems

In this appendix, we use $\hat{n}$ to denote the normal unit vector to a surface that can be either open or closed. The two tangential vectors of an open surface will be denoted by $\hat{\ell}$ and $\hat{m}$; $\hat{\ell}$ is tangential to the edge of the contour and $\hat{m}$ is normal to the contour. Both are tangential to the surface. The triad forms an orthogonal relation $\hat{m} \times \hat{\ell} = \hat{n}$ and $d\mathbf{S} = \hat{n}\, ds$, $d\boldsymbol{\ell} = \hat{\ell}\, d\ell$.

1. Gauss theorem or divergence theorem:

$$\iiint \nabla \mathbf{F}\, dv = \oiint \hat{n} \cdot \mathbf{F}\, dS.$$

2. Curl theorem:

$$\iiint \nabla \mathbf{F}\, dV = \oiint \hat{n} \times \mathbf{F}\, dS.$$

3. Gradient theorem:

$$\iiint \nabla f\, dV = \oiint \hat{n} f\, dS.$$

4. Surface divergence theorem:

$$\iint \nabla_s \mathbf{F}\, dS = \oint \hat{m} \cdot \mathbf{F}\, d\ell.$$

5. Surface curl theorem:

$$\iint \nabla_s \mathbf{F}\, dS = \oint \hat{m} \times \mathbf{F}\, d\ell.$$

6. Surface gradient theorem:

$$\iint \nabla_s f \, dS = \oint \hat{m} f \, d\ell.$$

7. Cross-gradient theorem:

$$\iint \hat{n} \times \nabla f \, dS = \oint f \, d\ell.$$

8. Stokes's theorem:

$$\iint \hat{n} \cdot \nabla \mathbf{F} \, dS = \oint \mathbf{F} \cdot d\ell.$$

9. Cross-∇-cross theorem:

$$\iint (\hat{n} \times \nabla) \times \mathbf{F} \, dS = \iint [\hat{n} \times \nabla \mathbf{F} + \hat{n} \cdot \nabla \mathbf{F} - \hat{n} \nabla \mathbf{F}] \, dS = -\oint \mathbf{F} \times d\ell.$$

10. First scalar Green's theorem:

$$\iiint [a \nabla \nabla b + \nabla a \cdot \nabla b] \, dv = \oiint \hat{n} \cdot a \nabla b \, dS.$$

11. Second scalar Green's theorem:

$$\iiint (a \nabla \nabla b - b \nabla \nabla a) \, dV = \oiint \hat{n} \cdot (a \nabla b - b \nabla a) \, dS.$$

12. Second scalar surface Green's theorem:

$$\iint (a \nabla_s \nabla_s b - b \nabla_s \nabla_s a) \, dS = \oint \hat{m} \cdot (a \nabla_s b - b \nabla_s a) \, dS.$$

13. First scalar–vector Green's theorem:

Type 1: $$\iiint (f \nabla \nabla \mathbf{F} + \nabla f \cdot \nabla \mathbf{F}) \, dV = \oiint \hat{n} \cdot f \nabla \mathbf{F} \, dS,$$

Type 2: $$\iiint (\mathbf{F} \nabla \nabla f + \nabla f \cdot \nabla \mathbf{F}) \, dV = \oiint (\hat{n} \cdot \nabla f) \mathbf{F} \, dS.$$

14. Second scalar–vector Green's theorem:

$$\iiint (f \nabla \nabla \mathbf{F} - \mathbf{F} \nabla \nabla f) \, dV = \oiint \hat{n} \cdot [f \nabla \mathbf{F} - (\nabla f) \mathbf{F}] \, dS.$$

15. First vector Green's theorem:

$$\iiint [(\nabla \mathbf{P}) \cdot (\nabla \mathbf{Q}) - \mathbf{P} \cdot \nabla \nabla \mathbf{Q}] \, dV = \oiint \hat{n} \cdot (\mathbf{P} \times \nabla \mathbf{Q}) \, dS.$$

16. Second vector Green's theorem:

$$\iiint (\mathbf{Q} \cdot \nabla \nabla \mathbf{P} - \mathbf{P} \cdot \nabla \nabla \mathbf{Q}) \, dV = \oiint \hat{n} \cdot (\mathbf{P} \times \nabla \mathbf{Q} - \mathbf{Q} \times \nabla \mathbf{P}) \, dS.$$

17. First vector–dyadic Green's theorem:

$$\iiint [(\nabla \mathbf{P}) \cdot \nabla \bar{\bar{Q}} - \mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}}]\, dV = \oiint \hat{n} \cdot (\mathbf{P} \times \nabla \bar{\bar{Q}})\, dS.$$

18. Second vector–dyadic Green's theorem:

$$\iiint [(\nabla \nabla \mathbf{P}) \cdot \bar{\bar{Q}} - \mathbf{P} \cdot \nabla \nabla \bar{\bar{Q}}]\, dV = \oiint \hat{n} \cdot [\mathbf{P} \times \nabla \bar{\bar{Q}} + (\nabla \mathbf{P}) \times \bar{\bar{Q}}]\, dS.$$

19. First dyadic–dyadic Green's theorem:

$$\iiint \left\{ [\bar{\bar{Q}}]^{\mathrm{T}} \cdot \nabla \nabla \bar{\bar{P}} - [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \nabla \bar{\bar{P}} \right\} dV = \oiint [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\hat{n} \times \nabla \bar{\bar{P}})\, dS.$$

20. Second dyadic–dyadic Green's theorem:

$$\iiint \left\{ [\bar{\bar{Q}}]^{\mathrm{T}} \cdot \nabla \nabla \bar{\bar{P}} - [\nabla \nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot \bar{\bar{P}} \right\} dV$$

$$= \oiint \left\{ [\bar{\bar{Q}}]^{\mathrm{T}} \cdot (\hat{n} \times \nabla \bar{\bar{P}}) + [\nabla \bar{\bar{Q}}]^{\mathrm{T}} \cdot (\hat{n} \times \bar{\bar{P}}) \right\} dS.$$

21. Helmholtz transport theorem:

$$\frac{d}{dt} \iint_{S(t)} \mathbf{F} \cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \nabla \mathbf{F} - \nabla (\mathbf{v} \times \mathbf{F}) \right] \cdot d\mathbf{S}.$$

22. Maxwell's theorem:

$$\frac{d}{dt} \oint_{L(t)} \mathbf{f} \cdot d\boldsymbol{\ell} = \int_{L(t)} \left(\frac{\partial \mathbf{f}}{\partial t} - \mathbf{v} \times \nabla \mathbf{f} \right) \cdot d\boldsymbol{\ell}.$$

23. Reynolds's transport theorem:

$$\frac{d}{dt} \iiint_{V(t)} \rho\, dV = \iiint_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) \right] dV$$

$$= \iiint_{V(t)} \left(\frac{d\rho}{dt} + \rho \nabla \mathbf{v} \right) dV.$$

Appendix D

Relationships Between Integral Theorems

The integral theorems stated by (1)–(3) and (7)–(8) in Appendix C are closely related. By means of the gradient theorem, we can derive both the divergence theorem and the curl theorem. From this point of view, we must first prove the gradient theorem, leaving aside its derivation by the symbolic method. The theorem states that

$$\iiint \nabla f \, dV = \oiint_S f \, d\mathbf{S}. \tag{D.1}$$

In a rectangular system with coordinate variables (x_1, x_2, x_3), the x_1 component of (D.1) corresponds to

$$\iiint \frac{\partial f}{\partial x_1} \, dx_1 \, dx_2 \, dx_3 = \iint_{S_2} f \, dx_2 \, dx_3 - \iint_{S_1} f \, dx_2 \, dx_3, \tag{D.2}$$

where S_1 and S_2 denote the two sides of an enclosed surface S viewed in the x_1 direction. The negative sign associated with the surface integral evaluated on S_1 is due to the fact that the vector component of $d\mathbf{S}_1$ is equal to $-dx_2 \, dx_3 \, \hat{x}_1$. Equation (D.2) is a valid identity because the volume integral is given by

$$\begin{aligned} \iiint \frac{\partial f}{\partial x_1} \, dx_1 \, dx_2 \, dx_3 &= \iint [f(P_2) - f(P_1)] \, dx_2 \, dx_3 \\ &= \iint_{S_2} f \, dx_2 \, dx_3 - \iint_{S_1} f \, dx_2 \, dx_3, \end{aligned} \tag{D.3}$$

where P_2 and P_1 denote two stations located at opposite sides of the surface along the x_1 direction. The same procedure can be used to prove the remaining two components of (D.1). Having proved the validity of (D.1), we can use it to deduce the divergence theorem (Gauss theorem) and the curl theorem.

We now consider three distinct sets of (D.1) in the form

$$\iiint \nabla F_i \, dV = \oiint_S F_i \, d\mathbf{S}, \qquad i = 1, 2, 3. \tag{D.4}$$

By taking the scalar product of (D.4) with $\hat{x}_i$ and summing the resultant equations, we obtain

$$\sum_i \hat{x}_i \cdot \iiint \nabla F_i \, dV = \sum_i \hat{x}_i \cdot \oiint_S F_i \, d\mathbf{S}. \tag{D.5}$$

Let

$$\mathbf{F} = \sum F_i \hat{x}_i, \tag{D.6}$$

and because

$$\hat{x}_i \cdot \nabla F_i = \nabla \cdot (F_i \hat{x}_i), \tag{D.7}$$

we obtain

$$\iiint \nabla \mathbf{F} \, dV = \oiint_S \mathbf{F} \cdot d\mathbf{S}, \tag{D.8}$$

which is the divergence theorem. Similarly, by taking the cross product of $\hat{x}_i$ with (D.4), we obtain

$$\sum_i \hat{x}_i \times \iiint \nabla F_i \, dV = \sum_i \hat{x}_i \times \oiint F_i \, d\mathbf{S}. \tag{D.9}$$

Because

$$\hat{x}_i \times \nabla F_i = -\nabla (F_i \hat{x}_i), \tag{D.10}$$

(D.9) is equivalent to

$$\iiint \nabla \mathbf{F} \, dV = -\oiint \mathbf{F} \times d\mathbf{S}, \tag{D.11}$$

which is the curl theorem. The approach that we took can be applied to the other two theorems listed as (7)–(8) in Appendix C. In this case, we consider the cross-gradient theorem as the key theorem that must be proved first. The theorem states that

$$\iint \hat{n} \times \nabla f \, dS = \oint f \, d\ell. \tag{D.12}$$

In a rectangular system, we can write

$$\hat{n} = \sum_i n_i \hat{x}_i,$$

$$d\ell = \sum_i dx_i \, \hat{x}_i.$$

Then the x_1 component of (D.12) reads

$$\iint \left(n_2 \frac{\partial f}{\partial x_3} - n_3 \frac{\partial f}{\partial x_2} \right) dS = \oint f \, dx_1. \tag{D.13}$$

The surface integral in (D.13) can be written in the form

$$\begin{aligned} \iint \left(\frac{\partial f}{\partial x_3} dx_1 \, dx_3 + \frac{\partial f}{\partial x_2} dx_1 \, dx_2 \right) &= \iint \left(\frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \right) dx_1 \\ &= \iint df \, dx_1 \\ &= \int [f(p_2) - f(p_1)] \, dx_1 \\ &= \int_{c_2} f(p_2) \, dx_1 - \int_{-c_1} f(p_1) \, dx_1 \\ &= \oint_c f \, dx_1, \end{aligned}$$

where p_1 and p_2 denote two stations on the closed contour, which consists of two segments $c_1 + c_2$. We have thus proved the validity of (D.13). The same procedure applies to the x_2 and x_3 components of (D.12). Once we have proved the cross-gradient theorem, it can be used to deduce the Stokes theorem.

We consider three distinct sets of (D.12) in the form

$$\iint \hat{n} \times \nabla F_i \, dS = \oint F_i \, d\ell, \qquad i = 1, 2, 3. \tag{D.14}$$

By taking the scalar product of (D.14) with $\hat{x}_i$ and summing the resultant equations, we obtain

$$\sum_i \hat{x}_i \cdot \iint (\hat{n} \times \nabla F_i) \, dS = \sum_i \hat{x}_i \cdot \oint F_i \, d\ell. \tag{D.15}$$

Because

$$\hat{x}_i \cdot (\hat{n} \times \nabla F_i) = -\hat{n} \cdot (\hat{x}_i \times \nabla F_i) = \hat{n} \cdot \nabla (F_i \hat{x}_i), \tag{D.16}$$

and we let

$$\mathbf{F} = \sum_i F_i \hat{x}_i,$$

(D.15) can be written in the form

$$\iint \hat{n} \cdot \nabla \mathbf{F}\, dS = \oint \mathbf{F} \cdot d\ell, \tag{D.17}$$

which is the Stokes theorem. It is seen that in this analysis, the gradient theorem is considered the key theorem based on which the other three theorems can be readily derived. The approach taken here has its own merit without considering the derivation of these theorems, independently, by the symbolic method.

The relationships among the gradient theorem, the divergence theorem, and the curl theorem have previously been pointed out by Van Bladel [11, Appendix I]. Alternatively, we can use the divergence theorem and the Stokes theorem as the key theorems to derive the other three theorems. The manipulations, however, are more complicated.

Appendix E

Vector Analysis in the Special Theory of Relativity

To study the theory of relativity, the most efficient mathematical tool is multidimensional tensor analysis with a dimension greater than three. Within the realm of the special theory of relativity, the subject can be treated by ordinary vector analysis in three dimensions. In fact, this is what Einstein did in his original work published in 1905. In this appendix, we shall first follow this approach and then show how the same result is obtained by a four-dimensional analysis.

Based on the experimental evidence that the velocity of light is independent of the status of source, moving or stationary, that emits the light signal, Einstein postulated the doctrine in his special theory of relativity that for two coordinate systems in relative motion with a constant velocity of separation $\mathbf{v} = v\hat{z}$, the space and the time variables in the two systems must obey the Lorentz transform stated by the following relations:

$$x = x', \tag{E.1}$$

$$y = y', \tag{E.2}$$

$$z = \gamma(z' + vt'), \tag{E.3}$$

$$t = \gamma\left(t' + \frac{v}{c^2}z'\right); \tag{E.4}$$

the reverse transforms of (E.3) and (E.4) are

$$z' = \gamma(z - vt) \tag{E.5}$$

and

$$t' = \gamma\left(t - \frac{v}{c^2}z\right), \tag{E.6}$$

where v is the velocity of separation in the z or z' direction between the two systems, $\gamma = 1/(1-\beta^2)^{1/2}$, $\beta = v/c$, and c is the velocity of light in free space. Another principle contained in Einstein's theory is the invariance of Maxwell's equations in the two coordinate systems, that is,

$$\nabla \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{E.7}$$

$$\nabla \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{E.8}$$

$$\nabla \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{E.9}$$

$$\nabla \mathbf{D} = \rho, \tag{E.10}$$

$$\nabla \mathbf{B} = 0, \tag{E.11}$$

and

$$\nabla' \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}, \tag{E.12}$$

$$\nabla' \mathbf{H}' = \mathbf{J}' + \frac{\partial \mathbf{D}'}{\partial t'}, \tag{E.13}$$

$$\nabla' \mathbf{J}' = -\frac{\partial \rho'}{\partial t'}, \tag{E.14}$$

$$\nabla' \mathbf{D}' = \rho', \tag{E.15}$$

$$\nabla' \mathbf{B}' = 0, \tag{E.16}$$

where the unprimed operators and the unprimed functions are defined with respect to $(x, y, z; t)$ and the primed ones with respect to $(x', y', z'; t')$. The Lorentz transform assures us the relation that when

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \tag{E.17}$$

then

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \tag{E.18}$$

(E.17) or (E.18) corresponds to "equation of motion" of the propagation of the light signal. In fact, these are two of the equations used to derive (E.3) and (E.4). With this background we can find the transform between the field vectors. Let us consider first the x-component of (E.7) pertaining to Faraday's law in the rectangular systems, that is,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}. \tag{E.19}$$

The derivatives with respect to y, z, and t can be converted to the derivatives with respect to y', z', and t' with the aid of (E.1) to (E.6). Thus, (E.19) can be written as

$$\frac{\partial E_z}{\partial y'} - \left(\frac{\partial E_y}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial E_y}{\partial t'}\frac{\partial t'}{\partial z}\right) = -\left(\frac{\partial B_x}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial B_x}{\partial z'}\frac{\partial z'}{\partial t}\right) \tag{E.20}$$

or

$$\frac{\partial E_z}{\partial y'} - \gamma\frac{\partial E_y}{\partial z'} + \frac{\gamma v}{c^2}\frac{\partial E_y}{\partial t'} = -\gamma\frac{\partial B_x}{\partial t'} + \gamma v\frac{\partial B_x}{\partial z'};$$

hence

$$\frac{\partial E_z}{\partial y'} - \frac{\partial}{\partial z'}\left[\gamma\left(E_y + vB_z\right)\right] = -\frac{\partial}{\partial t'}\left[\gamma\left(B_x + \frac{v}{c^2}E_y\right)\right]. \tag{E.21}$$

The x-component of (E.12) reads

$$\frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} = -\frac{\partial B'_x}{\partial t'}. \tag{E.22}$$

The matching of (E.21) and (E.22) yields

$$E'_z = E_z, \tag{E.23}$$

$$E'_y = \gamma\left(E_y + vB_x\right), \tag{E.24}$$

$$B'_x = \gamma\left(B_x + \frac{v}{c^2}E_y\right). \tag{E.25}$$

By working, similarly, on the other equations and combining the resultant equations, we find

$$\mathbf{E}' = \bar{\bar{\gamma}}\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \tag{E.26}$$

$$\mathbf{B}' = \bar{\bar{\gamma}}\cdot\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right), \tag{E.27}$$

$$\mathbf{H}' = \bar{\bar{\gamma}}\cdot(\mathbf{H} - \mathbf{v}\times\mathbf{D}), \tag{E.28}$$

$$\mathbf{D}' = \bar{\bar{\gamma}}\cdot\left(\mathbf{D} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H}\right), \tag{E.29}$$

$$\mathbf{J}' = \gamma\bar{\bar{\gamma}}^{-1}(\mathbf{J} - \rho\mathbf{v}), \tag{E.30}$$

$$\rho' = \gamma\left(\rho - \frac{1}{c^2}\mathbf{v}\cdot\mathbf{J}\right), \tag{E.31}$$

where $\mathbf{v} = v\hat{z}$ and the dyadics $\bar{\bar{\gamma}}$ and $\bar{\bar{\gamma}}^{-1}$ are defined by

$$\bar{\bar{\gamma}} = \gamma\left(\hat{x}\hat{x} + \hat{y}\hat{y}\right) + \hat{z}\hat{z},$$

$$\bar{\bar{\gamma}}^{-1} = \frac{1}{\gamma}\left(\hat{x}\hat{x} + \hat{y}\hat{y}\right) + \hat{z}\hat{z};$$

so

$$\gamma\bar{\bar{\gamma}}^{-1} = \hat{x}\hat{x} + \hat{y}\hat{y} + \gamma\hat{z}\hat{z}$$

and

$$\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}}^{-1} = \bar{\bar{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} .$$

(E.26) to (E.29) are the transforms of the field vectors defined in the two coordinate systems. These are the equations based on which the problems involving moving media can be formulated. When $v^2/c^2 \ll 1$,

$$\gamma \doteq 1,$$

the following relations hold true

$$\mathbf{E}'^* = \mathbf{E} + \mathbf{v} \times \mathbf{B} , \tag{E.32}$$

$$\mathbf{B}'^* = \mathbf{B} , \tag{E.33}$$

$$\mathbf{H}'^* = \mathbf{H} - \mathbf{v} \times \mathbf{D} , \tag{E.34}$$

$$\mathbf{D}'^* = \mathbf{D} , \tag{E.35}$$

$$\mathbf{J}'^* = \mathbf{J} - \rho\mathbf{v} , \tag{E.36}$$

$$\rho'^* = \rho . \tag{E.37}$$

The symbol * means that these expressions are approximate under the condition $v^2/c^2 \ll 1$.

These transforms have been derived with a rather tedious procedure involving altogether nine scalar differential equations. Many of the details are not shown here. A more elegant method is to recast the Lorentz transform into a pseudo-real orthogonal transform, a method due to Sommerfeld [45]. According to that method, we let

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}} = \cos\eta , \tag{E.38}$$

then

$$\sin\eta = (1 - \cos^2\eta)^{1/2} = \frac{i\beta}{(1-\beta^2)^{1/2}} . \tag{E.39}$$

Because

$$\beta = \frac{v}{c} < 1,$$

η must be an imaginary angle, hence the term "pseudo-real." Now, we introduce the four-dimensional coordinate variables (x_1, x_2, x_3, x_4) defined by

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict,$$

and similarly for x'_j with $j = 1, 2, 3, 4$; then (E.1) to (E.6) can be written as

$$\begin{aligned} x_1 &= x'_1 , \\ x_2 &= x'_2 , \\ x_3 &= x'_3 \cos\eta - x'_4 \sin\eta = a_{33}x'_3 + a_{43}x'_4 , \\ x_4 &= x'_4 \cos\eta + x'_3 \sin\eta = a_{34}x'_3 + a_{44}x'_4 , \end{aligned} \tag{E.40}$$

where

$$\begin{aligned} a_{33} &= \cos\eta, & a_{34} &= \sin\eta, \\ a_{43} &= -\sin\eta, & a_{44} &= \cos\eta. \end{aligned} \tag{E.41}$$

The "directional cosines" between the two sets of axes can be tabulated as follows:

	x_1	x_2	x_3	x_4
x'_1	1	0	0	0
x'_2	0	1	0	0
x'_3	0	0	a_{33}	a_{34}
x'_4	0	0	a_{43}	a_{44}

(E.42)

As with the three-dimensional coefficients, they satisfy the orthogonal relations

$$\sum_{j=1}^{4} a_{ij}a_{kj} = \delta_{ik} \tag{E.43}$$

and

$$|a_{ij}| = 1.$$

The field vectors in Maxwell's equations can now be formulated as vectors or tensors in a four-dimensional manifold. We first define a four-potential vector, denoted by **P**, as

$$\mathbf{P} = \sum_{i=1}^{4} P_i \hat{x}_i$$

with $P_1 = A_1$, $P_2 = A_2$, $P_3 = A_3$, $P_4 = i\phi/c$, where A_i with $i = 1, 2, 3$ are the components of the vector potential **A** and ϕ is the dynamic scalar potential. The two functions have been introduced previously in Section 4-10. We define the components of the curl of **P** as

$$\nabla_{mn}\mathbf{P} = \frac{\partial P_n}{\partial x_m} - \frac{\partial P_m}{\partial x_n}, \qquad m, n = 1, 2, 3, 4. \tag{E.44}$$

They are functions with two indices and there are 12 of them, but because

$$\nabla_{mn}\mathbf{P} = -\nabla_{nm}\mathbf{P} \tag{E.45}$$

and

$$\nabla_{mm}\mathbf{P} = 0,$$

there are actually only six distinct components. These components or functions will be denoted by F_{mn} and are designated as the *six-vectors*. The components of the field vectors **B** and **E** can now be treated as six vectors. For example,

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = \frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_3} = \nabla_{23}\mathbf{P} = F_{23}\,, \tag{E.46}$$

$$E_1 = -\frac{\partial \phi}{\partial x_1} - \frac{\partial A_1}{\partial t} = ic\left(\frac{\partial P_4}{\partial x_1} - \frac{\partial P_1}{\partial x_4}\right) = ic\nabla_{14}\mathbf{P} = icF_{14},$$

or

$$\frac{-iE_1}{c} = \nabla_{14}\mathbf{P} = F_{14}\,. \tag{E.47}$$

The six functions $(B_i, -iE_i/c)$ with $i = 1, 2, 3$ form the components of a 4×4 antisymmetric tensor, which is shown here:

$$[F_{ij}] = \begin{bmatrix} 0 & B_3 & -B_2 & \frac{-i}{c}E_1 \\ -B_3 & 0 & B_1 & \frac{-i}{c}E_2 \\ B_2 & B_1 & 0 & \frac{-i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix}. \tag{E.48}$$

The 3×3 minor in the upper left corner, or the adjoint of F_{44}, is recognized as the antisymmetric tensor of the axial vector $\mathbf{B} = \nabla \mathbf{A}$.
Another six-vector is defined by

$$\mathbf{G} = (\mathbf{H}, -ic\mathbf{D})\,. \tag{E.49}$$

In free space,

$$\mathbf{G} = \frac{1}{\mu_0}\mathbf{F} = \frac{1}{\mu_0}\nabla \mathbf{P}\,. \tag{E.50}$$

The 4×4 antisymmetric tensor of **G** is shown here:

$$[G_{ij}] = \begin{bmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{bmatrix}. \tag{E.51}$$

It is possible to construct two four-dimensional dyadics using these tensors, but they are not necessary in this presentation.

We can now extend the rules of the transformation of 3×3 tensors in Chapter 1 for two orthogonal rectangular systems to the 4×4 antisymmetric tensors under consideration, that is,

$$f'_{ij} = \sum_{m=1}^{4} \sum_{n=1}^{4} a_{im} a_{jn} f_{mn}. \tag{E.52}$$

Applying (E.52) to **F** and **G** with a_{ij} given by (E.42), we find

$$\mathbf{E}' = \bar{\bar{\gamma}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{E.53}$$

$$\mathbf{B}' = \bar{\bar{\gamma}} \cdot \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right), \tag{E.54}$$

$$\mathbf{H}' = \bar{\bar{\gamma}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}), \tag{E.55}$$

$$\mathbf{D}' = \bar{\bar{\gamma}} \cdot \left(\mathbf{D} + \frac{\mathbf{v}}{c^2} \times \mathbf{H}\right). \tag{E.56}$$

They are the same as (E.26) to (E.29) obtained before by the classical method. The transform of the current density function **J** and the charge density function ρ can be found by applying the four-dimensional rule to the components of a four-current vector defined by

$$K_i = (J_1, J_2, J_3, ic\rho), \qquad i = 1, 2, 3, 4. \tag{E.57}$$

The transform is

$$K'_i = \sum_{j=1}^{4} a_{ij} K_j, \qquad i = 1, 2, 3, 4, \tag{E.58}$$

which yields

$$K'_1 = K_1; \qquad \text{hence } J'_1 = J_1, \tag{E.59}$$

$$K'_2 = K_2; \qquad \text{hence } J'_2 = J_2, \tag{E.60}$$

$$K'_3 = a_{33} K_3 + a_{34} K_4;$$

hence

$$\begin{aligned} J'_3 &= \gamma (J_3 - v\rho), \\ K'_4 &= a_{43} K_3 + a_{44} K_4 \end{aligned} \tag{E.61}$$

or

$$ic\rho' = -\frac{i\gamma v}{c} J_3 + \gamma (ic\rho);$$

hence

$$\rho' = \gamma \left(\rho - \frac{v}{c^2} J_3\right). \tag{E.62}$$

(E.53) to (E.62) are identical to (E.26) to (E.31). It is seen that the four-dimensional analysis to derive the transform of the field vectors is very elegant as long as we get used to the rather novel concept of pseudo-real representation of Lorentz transform in a four-dimensional space or manifold.

Appendix F

Comparison of the Nomenclatures and Notations of the Quantities Used in This Book and in the Book by Stratton [5]

Present Book (1)	Quantities	Stratton's Book (2)
$\mathbf{a} \cdot \mathbf{b}$	scalar product	$\mathbf{a} \cdot \mathbf{b}$
$\mathbf{a} \times \mathbf{b}$	vector product	$\mathbf{a} \times \mathbf{b}$
$\mathbf{p}_i$	(1) primary vectors (2) unitary vectors	$\mathbf{a}_i$
$\mathbf{r}^i$	(1) reciprocal vectors (2) reciprocal unitary vectors	$\mathbf{a}^i$
$\mathbf{p}_i \cdot \mathbf{p}_j = \alpha_{ij}$	(1) scalar products of $\mathbf{p}_i$ (2) scalar products of $\mathbf{a}_i$	$\mathbf{a}_i \cdot \mathbf{a}_j = g_{ij}$
—	(2) scalar products of $\mathbf{a}^i$	$\mathbf{a}^i \cdot \mathbf{a}^j = g^{ij}$

Present Book (1)	Quantities	Stratton's Book (2)
$\Lambda = \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)$	a volume parameter in GCS	$g^{1/2} = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)$
$\mathbf{F} = \sum_i f_i \mathbf{r}^i$ $= \sum_j g^j \mathbf{p}_j$	vector function in component form in GCS	$\mathbf{F} = \sum_i f_i \mathbf{a}^i$ $= \sum_j f^j \mathbf{a}_j$
$f_i = \mathbf{p}_i \cdot \mathbf{F}$	(1) primary component (2) covariant component	$f_i = \mathbf{a}_i \cdot \mathbf{F}$
$g^j = \mathbf{r}^j \cdot \mathbf{F}$	(1) reciprocal component (2) contravariant component	$f^j = \mathbf{a}^j \cdot \mathbf{F}$
v_i	(1) coordinate variables (2) coordinate variables along $\mathbf{a}_i$	u^i
—	(2) coordinate variables along $\mathbf{a}^i$	u_i
$\hat{u}_i$	unit vectors in OCS	$\mathbf{i}_i$
h_i	metric coefficients in OCS	h_i
$\Omega = h_1 h_2 h_3$	product of metric coefficients	—
$x, y, z; x_i$	rectangular variables	x, y, z
$\hat{x}, \hat{y}, \hat{z}; \hat{x}_i$	unit vectors in rectangular system	$\mathbf{i}_i$
r, θ, z	cylindrical variables	r, θ, z
$\hat{r}, \hat{\theta}, \hat{z}$	unit vectors in cylindrical coordinate system	$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$
R, θ, ϕ	spherical variables	r, θ, ϕ
$\hat{R}, \hat{\theta}, \hat{\phi}$	unit vectors in spherical coordinate system	$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$
$d\mathbf{R}_p$	differential of position vector	$d\mathbf{r}$
∇	symbolic vector	—
∇f	gradient of a scalar function	∇f
$\nabla \mathbf{F}$	gradient of a vector function	$\nabla \mathbf{F}$
$\nabla \cdot \mathbf{F}$	divergence of a vector function	$\nabla \cdot \mathbf{F}$
$\nabla \times \mathbf{F}$	curl of a vector function	$\nabla \times \mathbf{F}$
$\nabla \cdot \nabla f$	Laplacian of a scalar function	$\nabla \cdot \nabla f; \nabla^2 f$
$\nabla \cdot \nabla \mathbf{F}$	Laplacian of a vector function	$\nabla \cdot \nabla \mathbf{F}; \nabla^2 \mathbf{F}$
$\nabla \times \nabla \times \mathbf{F}$	curl curl $\mathbf{F}$	$\nabla \times \nabla \times \mathbf{F}$
$\nabla_s f$	(1) surface gradient of a scalar function	—
$\nabla_s \cdot \mathbf{F}$	(1) surface divergence of a vector function	—
$\nabla_s \times \mathbf{F}$	(1) surface curl of a vector function	—
$[T_{ij}]$	tensor of rank 2	$^2\mathbf{T}$
T_{ij}	(2) component of a tensor of rank 2	T_{ij}
—	(2) divergence of a tensor of rank 2	$\text{div}\,^2\mathbf{T}$
$\bar{\bar{F}}$	(1) dyadic function	—
$\nabla \cdot \bar{\bar{F}}$	(1) divergence of a dyadic	—
$\nabla \times \bar{\bar{F}}$	(1) curl of a dyadic	—

F-1 Typesetting the New Notations for Divergence and Curl

To facilitate the new notations for curl and divergence, it helps to know how to typeset them. Dr. Leland Pierce of the University of Michigan has created TEX symbols (usable also in LATEX) for the S-vector ∇, divergence operator $\nabla\cdot$, and curl operator $\nabla\times$. The following LATEX document:

```
\documentstyle[12pt]{article}
\begin{document}
\font\tttt=cmsy5
\def\taisvec{\nabla\!\!\!\!\raise0.31ex\hbox{- -}\,}
\def\taidivg{\nabla\!\!\kern-2.5pt\raise0.5ex\hbox to6pt{$\cdot$}}
\def\taicurl{\nabla\!\!\!\kern-1.5pt\raise0.8ex\hbox to7pt{\tttt\char'002}}


\begin{eqnarray*}
\taicurl\bf{E} &=& -\frac{\partial\bf{B}}{\partial t}
\\
\taicurl\bf{H} &=& \bf{J} + \frac{\partial \bf{D}}{\partial t}
\\
\taidivg\bf{D} &=& \rho
\\
\taidivg\bf{B} &=& 0
\end{eqnarray*}
\end{document}
```

will produce

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

$$\nabla\cdot\mathbf{D} = \rho$$

$$\nabla\cdot\mathbf{B} = 0$$

For Macintosh© or PC users of Microsoft Word 6.0© or higher, the following steps can be used to typeset the new divergence and curl operators.

1. Pull down the "Insert" menu and select "Field."
2. Select "Equation" to insert a blank equation field in the document.
3. For Macintosh users, hold down the control key and click on the blank equation. For PC users, right click the blank equation field. Select "Change" to show the equation field.

4. Edit the equation field as follows:
   (a) For $\vec{\nabla}$ vector: {EQ\O(∇,\S\UP5(−))}.
   (b) For $\vec{\nabla}$ operator: {EQ\O(∇,\S\UP5(.))}.
   (c) For $\vec{\nabla}$ operator: {EQ\O(∇,\S\UP5(×))}.

   Note that "∇," "." and "×" are characters of Symbol Font and entered by using "Insert Symbol" command. The size of "∇" is 12 point. The size of "." and "×" is 4 point. Note that "." is the big dot character of Symbol Font, not period, and "×" is the cross character of Symbol Font, not "x."
5. To save typing time, the above can be saved and recalled by using the "Autotext" command. For instance, select the operators $\vec{\nabla}$, pull down the "Edit" menu, and select "Autotext" to store the operator, say, by the name "dot." For subsequent use of the operator, type the name "dot" followed by the function key F3, then the $\vec{\nabla}$ operator will appear automatically.

For more information, see the help menu of Microsoft Word 6.0© on the following topics: Insert Field, Insert Symbol, EQ, and Autotext.

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Index

A
Associative rule, 2
B
Bilinear transformation, 32
Bivector,135
Bladel, J. Van, 99, 105
Borisenko, A. J., 17, 148,155
Brand, L., 17, 105
Burali-Forti, C., 130
C
Candel, S. M., 119
Clifford, W. K., 132
Closed surface theorem, 27, 92
  integral form of, 93
Cofactors, 11
Conformal transformation, 33
Convergence, 132
Coordinate system
  bipolar cylinders, 31, 164
  Cartesian or rectangular, 3
  cylindrical, 30, 161
  Dupin, 35, 100
  elliptical cylinder, 30, 162
  general curvilinear system (GCS), 23
  oblate spherical, 31, 163
  orthogonal curvilinear system (OCS), 28
  parabolic cylinder, 30, 162
  prolate spheroidal, 30, 163
  spherical, 30, 161
Cross-∇-cross theorem, 110
Cross-gradient theorem, 110
Cross product or vector product, 5, 131
Crowe, M. J., 127
Curl
  alternative definition of, 77
  of a dyadic function, 121
  in Cartesian system, 65, 132
  in general curvilinear system, 71
  Gibbs's notation for, 65, 135

Curl (*cont.*)
linguistic notation of, 128
in orthogonal curvilinear system, 64
Curl theorem, 92
surface, 109
Curvature
Gaussian, 39
radius of, 37
surface, 41
D
Del operator, 62
Derivatives of unit vectors, 33
Differential-algebraic operators, 102, 103
Differential area, 43
Differential length, 43
Differential volume, 44
Directional cosines, 3, 10
Directional radiance, 75
Distributive law
for scalar products, 5
for vector products, 6
Divergence
of a dyadic function, 123
in Cartesian system, 65
in general curvilinear system, 71
Gibbs's notation for, 63, 134
linguistic notation of, 128
in orthogonal curvilinear system, 62
Divergence theorem, 92
surface, 109
Dot product, 4
Dyadic algebra, 16
Dyadic function, 16
antisymmetric, 17
Cartesian, 16
classification of, 17
scalar product of, 19
symmetric, 17
transpose of, 19
vector product of, 21
Dyadic Green function, 98
Dyadic identities, 124, 166
Dyadic integral theorems, 124
E
Einstein, A., 174
notation, 10
F
Fang, N. H., 58, 151
Feshback, H., 154
Feynman, R. P., 148
Flux, 75
Four-vector, 15, 178
G
Gans, R., 149
Gauge condition, 98
Gauss, theorem or divergence theorem, 92
generalized, 91
generalized surface, 106
Gibbs, J. W., 15, 24, 63, 129, 135
Gradient
alternative definition of, 75
in Cartesian system, 65
in general curvilinear system, 70
method of, 78
in orthogonal curvilinear system, 62
of a vector, 87, 122
Gradient theorem, 92
surface, 109
Green's theorem
first dyadic-dyadic, 126
first scalar, 94
first vector, 95
first vector-dyadic, 125
scale-vector, 168
second dyadic-dyadic, 126
second scalar, 94
second vector, 95
second vector-dyadic, 125
surface, 114

H
Hallén's formula, 92
Hamilton, W. R., 129
Haus, H., 124
Heaviside, O., 129, 141, 149
Helmholtz, H. L. F. von, 116
Helmholtz theorem, 97
  transport theorem, 116
Helmholtz wave equation
  scalar, 98
  vector, 98
I
Idemfactor, 19, 25, 122
Integral theorems, 167
  relationship between, 170
J
Jackson, J. D., 130
Jacobian
  scalar, 49
  vector, 49
K
Kronecker δ function, 12
L
Lagally, M., 147
Laplace, P. S., 87
Laplacian
  of a scalar, 86
  surface, 114
  of a vector, 87, 153
Leighton, R. B., 148
Lemma 4.1, 60
Lemma 4.2, 82
Lemma 5.1, 112
Line integrals, classification of, 44
Linguistic notation, 105, 128
Lorentz, H. A., 119
  transform, 174
M
Marcolongo, R., 130
Mason, M., 147
Material derivative, 119
Maxwell's equations, 96, 175
Maxwell's theorem, 119
Metric coefficients, 28
Milne, E. A., 156
Moon, P., 127, 130, 152
Morse, P. M., 154
N
Nomenclature and notations, 181
O
Operand, 132
Operator(s), 132
  binary, 132
  cascade, 132
  cross, 133
  curl, 58, 64, 140
  del, 62, 127
  differential-algebraic surface, 102, 103
  divergence, 58, 63, 140
  dot, 133
  gradient, 58, 62, 139
  Hamilton, 62
  invariance of, 65
  nabla, 62
  unary, 132
P
Panofsky, W. K. H., 155
Penfield, P., 124
Phillips, H. B., 97
Phillips, M., 155
Poinsot, T. J., 119
Poisson's equation
  scalar, 96
  vector, 97
Position vector, 23
Potential function
  dynamic scalar, 97
  dynamic vector, 97
  electrostatic, 96
  magnetostatic vector, 97
Q
Quarternion, 129, 131

R
Radiance, 75
Reynolds transport theorem, 120
Rotation of Cartesian coordinate system, 8
S
S-vector, 59
  typesetting of, 183, 184
Sands, M., 148
Scalar product, 4
Schelkunoff, S. A., 147
Shear, 75
Shilov, G. E., 151
Six-vector, 15, 179
Sommerfeld, A., 177
Special theory of relativity, 174
Spencer, D. E., 127, 130, 152
Stokes, G. G., 110
Stokes theorem, 110
Stratton, J. A., 73, 130, 153
Surface curl, 103
  theorem, 109
  of Weatherburn, 105
Surface gradient, 101
  theorem, 109
  of Weatherburn, 105
Surface integrals, classification of, 48
Surface symbolic expression, 100
  with two functions, 110
  with two surface *S*-vectors, 113
Surface symbolic vector, 99
  partial, 111
T
Tai, C. T., 21, 58, 63, 119, 127, 146
Tarapov, I. E., 17, 148, 155
Tensor, 17, 179
Toupin, R., 119
Transform of field vectors, 176
Triple product
  of dyadics, 21
  of vectors, 6
Truesdell, C., 119
Twisted differential operator, 146
Twisted vector function, 146
Typesetting of *S*-vector, divergence and curl operator, 183, 184
U
Unit vectors, 3
  derivatives of, 33
  transformation of, 79, 161
V
Vector or vector function, 1
  axial, 15
  components of, 3
  controvariant components of, 25
  covariant components of, 25
  irrotational, 95
  orthogonal transformation of, 8
  polar, 14
  position, 23
  primary, 23
  reciprocal, 24
  reciprocal unitary, 24
  screw, 15
  solenoidal, 95
  unitary, 24
Vector identities, 82, 165
Vector Laplacian, 153
Vector product, 5
  formal (FVP), 143
Volume integrals, classification of, 56
W
Weatherburn, C. E., 99, 104
Weaver, W., 147
Weber, E., 151
Wilson, E. B., 129, 141, 160