Introduction to Lattice Field Theory[†]

Chuan Liu [◇]

Contents

1 Introduction **1**
- 1.1 What is this all about? . . . 1
- 1.2 Notations, units and conventions . . . 2

2 Path Integral Quantization **5**
- 2.1 Evolution operator of a harmonic oscillator . . . 5
- 2.2 Partition function of a harmonic oscillator . . . 8
- 2.3 Anharmonic oscillator and perturbative expansion . . . 9
  - 2.3.1 Generating functional and correlation function . . . 9
- 2.4 Euclidean versus Minkowski: Wick rotation . . . 11
- 2.5 Summaries . . . 12

3 Lattice Scalar Fields **13**
- 3.1 General discussions . . . 13
- 3.2 Schrödinger functional . . . 14
- 3.3 Coherent states path integral . . . 16

4 Euclidean Lattice Spinor Field Theory **19**
- 4.1 Naive fermions . . . 19
- 4.2 Grassman path integrals . . . 21
- 4.3 Fermion doubling problem . . . 24
- 4.4 Wilson fermions . . . 24

5 Lattice Gauge Fields **27**
- 5.1 Wilson plaquette action . . . 27
- 5.2 Wilson loops . . . 28
- 5.3 Strong coupling expansion . . . 29

6 Continuum Limit and Renormalization **31**
- 6.1 Continuum limit and critical phenomenon . . . 31
- 6.2 Block-spin: physical view of renormalization . . . 33
  - 6.2.1 Block-spin transformations . . . 33
  - 6.2.2 Fixed point solutions . . . 35
  - 6.2.3 Topology of renormalization group transformations . . . 37

6.3 Summary 38

7 Monte Carlo Simulations **41**
7.1 Importance sampling and Markov processes 41
7.2 Metropolis algorithm 43
7.3 Auto-correlation time and critical slowing down 45
7.4 Hybrid Monte Carlo algorithm 46
7.5 Triviality of scalar field theories 47
7.6 Simulation of fermions 49

8 Topics in Lattice QCD **51**
8.1 $Q\bar{Q}$ potential 51
8.2 Glueball spectrum 52
8.3 Quenched hadron spectrum 52

9 What's Left out **53**

Dedicated to my wife Dan and my son Mimi.

◇ Author's Email: liuchuan@mail.phy.pku.edu.cn

† Work supported by Project 19705001 of National Natural Science Foundation of China (NSFC), Pandeng fund, the startup fund from Educational Ministry of China and by the Natural Science Fund from Peking University.

Chapter 1

Introduction

1.1 What is this all about?

Lattice Field Theory, born in the 70's after the pioneer work by K. G. Wilson [35], has become an important branch in theoretical physics in recent years. With the fast development of computing resources, the growth of the field is likely to be accelerated in the 21 century. The theoretical and numerical methods used in lattice field theory have been applied to many fields of physics such as: particle physics, field theory, statistical physics, condensed matter physics and non-linear phenomenon. In this series of lectures, I would like to offer a brief introduction to lattice field theory to those who are now studying in theoretical particle physics and nuclear theory. Evidently, in such a short period of time, the lecture is not meant to be complete which means that several important topics are not included. However, I have tried to make it self-contained and pedagogical so that most of the audience can really benefit from these lectures. To fully benefit from these lectures, no knowledge of lattice field theory and path integral is required. However, some basic knowledge of canonical field quantization is helpful. Some standard textbooks for Quantum Field theory are listed at the end of this lecture notes: [14, 6, 32, 37, 1]. This list is far from complete, since the bibliography on this subject is just too vast to be included. For those topics that are not covered, please consult these books on lattice field theory:[8, 25].

Field Theory, or more precisely, Quantum Field Theory (QFT) is the quantum theory of a system with N degrees of freedom, where the default value for N is infinity although it could also be finite. According to the system it is used, QFT is further divided into two categories: Relativistic QFT (RQFT), which is the theory describing the fundamental interactions among elementary particles in Nature and Non-relativistic QFT (NRQFT), which is mainly used by condensed matter physicists to study low energy systems. Unfortunately, RQFT is sometimes called QFT while NRQFT is called The Theory of Greens Functions in theoretical condensed matter physics. According to the value of

N, QFT is categorized as Continuum QFT where N is "continuously infinite" and Lattice QFT where N is "countably infinite or finite". To be more precise, Lattice Quantum Field Theory is the quantum theory of a system whose degrees of freedom, called fields, are defined on a infinite or finite lattice.

Why do we need LQFT? Because it is the only non-perturbatively well-defined quantum field theory we have. This answer might sounds somewhat astonishing however it is indeed true. It turns out that, most of interacting Continuum QFT known so far have ultra-violet divergences. Therefore, without an ultra-violet regulator, the theory is not well-defined at all. Lattice is the only known non-perturbative regulator and therefore any ultra-violet divergent interacting theory is only well-defined with a lattice regulator. Continuum QFT is only defined via Lattice QFT in the continuum limit, which will be discussed in some detail in Chapter 4.

1.2 Notations, units and conventions

Throughout our discussion, we will use the "god-given" unit system in which $\hbar = c = k_B = 1$. With this unit convention, there is only one dimensional scale which could be taken as energy or length. Since we will be working constantly on a hyper-cubic lattice, it is natural to make a further choice that $a = 1$ where a is the lattice spacing. This is called the lattice units. Then all physical quantities are dimensionless numbers.

When working in the 4-dimensional continuum Minkowski space, we take the following convention for the free metric tensor: $\eta_{\mu\nu} = \eta^{\mu\nu} = (+,-,-,-)$. A covariant four vector is given as: $p^\mu = (p_0, \mathbf{p}) = (p_0, p^1, p^2, p^3)$, while the contra-variant vector is $p_\mu \equiv \eta_{\mu\nu} p^\nu = (p_0, -\mathbf{p})$. The Lorentz contraction of two 4 vectors is given by: $p \cdot k = p^\mu k_\mu = p_0 k_0 - \mathbf{p} \cdot \mathbf{k}$. The Dirac γ-matrices are denoted by $\Gamma^\mu = (\Gamma_0, \Gamma^1, \Gamma^2, \Gamma^3)$. They satisfy:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \quad , \tag{1.1}$$

where the matrix Γ_0 is hermitian while others are anti-hermitian. The chiral matrix Γ_5 is defined to be:

$$\Gamma_5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 \quad , \tag{1.2}$$

which is hermitian and anti-commutes with other γ-matrices.

In Euclidean space-time, the metric is just the unit matrix and there is no need to distinguish the superscripts and subscripts. We therefore write all vector indices as subscripts. Inner product of two 4 vectors is given by $p_\mu k_\mu$ where summation over μ is understood. The Dirac γ-matrices in Euclidean space are denoted by $\gamma_\mu = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$. All of them are hermitian and they satisfy:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad . \tag{1.3}$$

The chiral matrix γ_5 is defined to be:

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \quad , \tag{1.4}$$

which is hermitian and anti-commutes with other γ-matrices.

Chapter 2

Path Integral Quantization

2.1 Evolution operator of a harmonic oscillator

Consider a quantum one-dimensional harmonic oscillator whose Hamiltonian is given by:

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 \quad , \tag{2.1}$$

where ω is the frequency of the oscillator and x and p are the coordinate and conjugate momentum respectively. The creation and annihilation operators are introduced via:

$$a = \frac{1}{\sqrt{2\omega}}(\omega x + ip) \quad , \quad a^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x - ip) \quad . \tag{2.2}$$

They satisfy $[a, a^\dagger] = 1$. The Hamiltonian is then brought into the form: $H = (a^\dagger a + 1/2)\omega$ with eigenvalues of $(n + 1/2)\omega$.

We are interested in the evolution operator of the harmonic oscillator. The evolution operator is defined to be e^{-iHT} whose matrix element in the coordinate representation is $\langle x_F \mid e^{-iHT} \mid x_I \rangle$. To evaluate this expression, we chop the time interval T into N slices, each of size $a = T/N$. We then insert complete sets of states and write:

$$\begin{aligned} \langle x_F \mid e^{-iHT} \mid x_I \rangle \quad &= \quad \int \cdots \int \left(\prod_{t=1}^{N-1} dx_t \right) \\ &\underbrace{\langle x_N \mid e^{-iaH} \mid x_{N-1} \rangle \cdots \langle x_1 \mid e^{-iaH} \mid x_0 \rangle}_{N \text{factors}} \quad , \end{aligned} \tag{2.3}$$

with $x_N \equiv x_F$ and $x_0 \equiv x_I$. Keep in mind that $e^{-iaH} = \exp(-i(a/2)d^2/dx^2 - i(a/2)\omega^2 x^2)$, and use the operator identity:

$$e^{-i\frac{a}{2}\frac{d^2}{dx^2}} = \int d\xi e^{\frac{-i\xi^2}{2a}} e^{\xi\frac{d}{dx}} \quad , \tag{2.4}$$

we can bring the factors in eq. (2.3) into the following form:

$$\langle x_{t+1} \mid e^{-iaH} \mid x_t \rangle = \int d\xi_t \langle x_{t+1} \mid e^{\xi_t \frac{d}{dx}} \mid x_t \rangle e^{\frac{-i\xi_t^2}{2a}} e^{-i\frac{a}{2}\omega^2 x_t^2} \quad . \tag{2.5}$$

To obtain the above expression, we have used the fact that when a is small, the two factors arising from the kinetic energy and the potential energy nearly commute. It is easy to see that $\langle x_{t+1} \mid e^{\xi_t \frac{d}{dx}} \mid x_t \rangle = \delta(x_{t+1} + \xi_t - x_t)$, we therefore could carry out the integration over ξ's and obtain:

$$\langle x_{t+1} \mid e^{-iaH} \mid x_t \rangle = e^{-\frac{i}{2a}(x_{t+1}-x_t)^2} e^{-i\frac{a}{2}\omega^2 x_t^2} \quad . \tag{2.6}$$

Collecting all the terms, we finally obtain an (approximate) expression for the evolution operator:

$$\langle x_F \mid e^{-iTH} \mid x_I \rangle = \int \left(\prod_{t=1}^{N-1} dx_t \right) e^{ia \sum_{t=0}^{N-1} \frac{1}{2}\left[\frac{x_{t+1}-x_t}{a}\right]^2 - \frac{\omega^2}{2} x_t^2} \quad , \tag{2.7}$$

where $x_N = x_F$ and $x_0 = x_I$ are understood. The quantity in the exponential of the integrand in eq. (2.7) is nothing but $\exp(iS^{cl})$, with S^{cl} being a discretized version of the classical action of the harmonic oscillator. This constitutes the main idea of Feynman's path integral quantization [10]: in order to quantize a classical system, one only has to put the classical action (times i) in the exponential and integrate over *all* possible, not just the classical, paths that satisfy the boundary condition. In the limit of $a \to 0$, one formally writes:

$$\begin{aligned} \langle x_F \mid e^{-iTH} \mid x_I \rangle &= \int \mathcal{D}x \exp\left(i \int_0^T dt L(x, \dot{x}) \right) \quad , \\ L(x, \dot{x}) &= \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2} x^2 \quad , \quad x(0) = x_I \, , x(T) = x_F \quad . \end{aligned} \tag{2.8}$$

Note that, this is, and only is a formal expression. It is not meant to be exact in any mathematical sense. In fact, the functional measure $\mathcal{D}x$ in the above expression is not well-defined at all. Also, the major contribution to the path integral comes from paths that are not even once differentiable. If one would like to keep the mathematical rigor, one has to use the discrete version, i.e. eq. (2.7) as a definition. The formally continuous expression only helps to visualize what path integral resembles. We would also like to point out that this derivation easily generates to arbitrary potentials other than the harmonic oscillator. It can also be generalized to systems with many degrees of freedom which then becomes the path integral formulation of quantum field theory. For the moment, let us still focus on the simple one-dimensional harmonic oscillator example, since in this warm-up exercise, we can explicitly calculate everything we want.

Let us go back to eq. (2.7), the path integral representation for the evolution operator of the harmonic oscillator. It is a multi-dimensional Gaussian integral which can be evaluated exactly. To do this, we first find out the classical solution

$x_t^c, t = 1, 2 \cdots N-1$ which is the point that minimize the classical action. The classical equation of motion is:

$$x_{t+1}^c + x_{t-1}^c - 2x_t^c + \omega^2 a^2 x_t^c = 0 \ , \ \ t = 1, 2, \cdots N-1 \ . \tag{2.9}$$

Substitute in a special solution of the type $\exp(\pm i k_0 a t)$ we obtain:

$$2\sin(k_0 a/2) = \omega a \ . \tag{2.10}$$

Therefore, we get:

$$x_t^c = A\cos(k_0 a t) + B\sin(k_0 a t) \ , \tag{2.11}$$

where A and B are properly adjusted parameters such that $x_N^c = x_F$ and $x_0^c = x_I$ are fulfilled. We now make the change of variables: $x_t = x_t^c + X_t$. Then the boundary condition for X_t becomes $X_N = X_0 = 0$ and the path integral becomes:

$$\langle x_F \mid e^{-iTH} \mid x_I \rangle = e^{iS[x^c]} \int \left(\prod_{t=1}^{N-1} dX_t \right) e^{ia \sum_{t=0}^{N-1} \frac{1}{2} \left[\frac{X_{t+1} - X_t}{a} \right]^2 - \frac{\omega^2}{2} X_t^2} \ . \tag{2.12}$$

Now that X_t satisfy Dirichlet boundary condition in temporal direction, we can perform Fourier decomposition:

$$X_t = \sqrt{\frac{2}{N-1}} \sum_{n=1}^{N-1} \tilde{X}_n \sin(\frac{n\pi t}{N}) \ , \ \tilde{X}_n = \sqrt{\frac{2}{N-1}} \sum_{t=1}^{N-1} X_t \sin(\frac{n\pi t}{N}) \ , \tag{2.13}$$

where we have used the identity:

$$\frac{2}{N-1} \sum_{t=1}^{N-1} \sin(\frac{n\pi t}{N}) \sin(\frac{m\pi t}{N}) = \delta_{nm} \ . \tag{2.14}$$

EXCERCISE **2.1** *Prove eq. (2.13) and eq. (2.14).*

We therefore could rewrite the matrix element of the evolution operator into a diagonal form:

$$\langle x_F \mid e^{-iTH} \mid x_I \rangle = e^{iS[x^c]} \int \left(\prod_{n=1}^{N-1} d\tilde{X}_n \right) e^{\frac{i}{2a} \sum_{n=1}^{N-1} \tilde{X}_n^2 (2 - \omega^2 a^2 - 2\cos\frac{n\pi}{N} + i\epsilon)} \ . \tag{2.15}$$

Note that we have changed the integration variables to $\tilde{X}$. The Jacobian involved here is unity. Inspecting eq. (2.15) we find that, the integral is in fact not convergent. To remedy this technical problem, we have introduced the so-called Feynman's $+i\epsilon$ prescription.

EXCERCISE **2.2** *Prove that the absolute value of the Jacobian*

$$|\partial(X_1 \cdots X_{N-1}) / \partial(\tilde{X}_1 \cdots \tilde{X}_{N-1})| = 1 \ . \tag{2.16}$$

Hint: Using symmetry to show that the Jacobian J satisfies: $J = J^{-1}$.

Therefore, we finally obtain the Holy Grail expression for the evolution operator:

$$\langle x_F \mid e^{-iTH} \mid x_I \rangle = e^{iS[x^c]} \prod_{n=1}^{N-1} \left(\frac{2\pi i}{a(\frac{4}{a^2}\sin^2\frac{n\pi}{2N} - \omega^2 + i\epsilon)} \right)^{1/2} \quad , \tag{2.17}$$

where $S[x^c]$ is the discretized classical action.

2.2 Partition function of a harmonic oscillator

Consider an ensemble of independent harmonic oscillators in contact with a heat bath whose inverse temperature is $\beta = 1/\mathcal{T}$.[1] The thermodynamic properties of the system is governed by the partition function of the system defined as: $Z = Tr\exp(-\beta H)$. For the harmonic oscillator, this can be written as:

$$Z = \int dx \langle x \mid \exp(-\beta H) \mid x \rangle \quad . \tag{2.18}$$

This expression looks so similar to the matrix element of the time evolution operator that we computed in the previous section, the only difference being: (a) the time variable becoming pure imaginary, i.e. $\beta = iT$; (b) the initial and final coordinates are the same and are integrated over to form the trace of the operator. We therefore obtain:

$$\begin{aligned} Z &= \int \left(\prod_{\tau=1}^{N} dX_\tau \right) e^{-a\sum_{\tau=0}^{N-1} \frac{1}{2}\left[\frac{X_{\tau+1}-X_\tau}{a}\right]^2 + \frac{\omega^2}{2}X_\tau^2} \quad , \\ &= \int \left(\prod_{\tau=1}^{N} dX_\tau \right) e^{-a\sum_{\tau=0}^{N-1} \frac{1}{2}X_\tau\left[-\hat{\partial}_0^2+\omega^2\right]X_\tau} \quad , \end{aligned} \tag{2.19}$$

where on the second line we have performed an integration by parts and introduced the lattice Laplacian in the temporal direction $\hat{\partial}_0^2$, defined by:

$$\hat{\partial}_0^2 X_\tau \equiv (X_{\tau+1} + X_{\tau-1} - 2X_\tau)/a^2 \quad . \tag{2.20}$$

The lattice spacing $a = \beta/N$ and periodic boundary condition is applied to X_τ. The kernel $(-\hat{\partial}_0^2 + \omega^2)$ is then diagonalized in Fourier space. Note that in the partition function there is no need to introduce the $+i\epsilon$ prescription since the integral is well-defined. The result is:

$$Z = \prod_k \left(\frac{2\pi}{a(\hat{k}^2 + \omega^2)} \right)^{1/2} = \prod_k \left(\frac{2\pi}{a(\frac{4}{a^2}\sin^2(ka/2) + \omega^2)} \right)^{1/2} \quad , \tag{2.21}$$

where $ka = 2\pi n/N$, $n = 0, 1, \cdots, N-1$ and $\hat{k}^2 \equiv (4/a^2)\sin^2(ka/2)$.

[1] We use $\mathcal{T}$ to denote the absolute temperature and the Boltzmann constant k_B is set to unity.

EXCERCISE **2.3** *In the limit $a \to 0$ but $Na = \beta$ held fixed, show that the above expression for the partition function goes over to the conventional expression:*

$$Z = e^{-\beta\omega/2} \sum_{n=0}^{\infty} e^{-n\beta\omega} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} \quad . \tag{2.22}$$

2.3 Anharmonic oscillator and perturbative expansion

So far we have only dealt with a trivial example–a harmonic oscillator, which is a complete solvable quantum mechanics model. Now we will add in some spice by considering a modle of a slight more complexity. We will utilize path integral formalism to analyse the quantum anharmonic oscillator whose Hamiltonian is given by:

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 + \frac{\lambda}{4!}x^4 \quad , \tag{2.23}$$

Since the extra interaction term $(\lambda/4!)x^4$ is completely diagonal in the coordinate space, the derivation of the path integral representation for the evolution operator remains valid except that in the final expression an extra term for the interaction has to be added in:

$$\langle x_F \mid e^{-iTH} \mid x_I \rangle = \int \left(\prod_{t=1}^{N-1} dx(t) \right) e^{ia \sum_{t=0}^{N-1} \frac{1}{2}\left[\frac{x(t+1)-x(t)}{a}\right]^2 - \frac{\omega^2}{2}x^2(t) - \frac{\lambda}{4!}x^4(t)} \quad , \tag{2.24}$$

with $x(Na) = x_F$ and $x(0) = x_I$. The partition function is also obtained:

$$Z = \int \left(\prod_{\tau=0}^{N-1} dx(\tau) \right) e^{-\sum_{\tau=0}^{N-1} \frac{1}{2}x(\tau)\left[-\hat{\partial}_0^2 + \omega^2\right]x(\tau) + \frac{\lambda}{4!}x^4(\tau)} \quad . \tag{2.25}$$

Further evaluation of this path integral analytically will rely on perturbation theory which is valid when λ is small. In this case, one expands the exponential of the interaction term into polynomials of fields $x(\tau)$:

$$\begin{aligned} Z &= \int \left(\prod_{\tau=0}^{N-1} dx(\tau) \right) \left(1 - \sum_{\tau=0}^{N-1} \frac{\lambda}{4!}x^4(\tau) + \frac{1}{2!}\left(\sum_{\tau=0}^{N-1} \frac{\lambda}{4!}X^4(\tau) \right)^2 + \cdots \right) \\ &\times \exp\left(-\sum_{\tau=0}^{N-1} \frac{1}{2}x(\tau)\left[-\hat{\partial}_0^2 + \omega^2\right]x(\tau) \right) \quad . \end{aligned} \tag{2.26}$$

2.3.1 Generating functional and correlation function

To simplify the path integral for the partition function, one defines the so-called (free) generating functional:

$$Z_0[J(\tau)] = \int \left(\prod_{\tau=0}^{N-1} dx(\tau) \right) e^{-\sum_{\tau=0}^{N-1} \frac{1}{2}x(\tau)\left[-\hat{\partial}_0^2 + \omega^2\right]x(\tau) - J(\tau)x(\tau)} \quad . \tag{2.27}$$

Once the functional form of the generating functional is obtained, the path integral with a function of local fields inserted in front of the exponential, like the one in eq. (2.26) can be evaluated by taking the derivatives with respect to corresponding sources and then set the external source J to zero. Therefore, the partition function for the anharmonic oscillator has the following formal perturbative expansion:

$$Z = \exp\left(-\sum_{\tau=0}^{N-1} \frac{\lambda}{4!}\frac{\partial^4}{\partial J^4(\tau)}\right) Z_0[J(\tau)]\Big|_{J=0} \quad . \tag{2.28}$$

The exact form of the free generating functional can be obtained by completing the square in the exponential with the result:

$$Z[J(\tau)] = Z[0]\exp\left(-\frac{1}{2}\sum_{\sigma,\tau=0}^{N-1} J(\tau)\left[\frac{1}{-\hat{\partial}_0^2+\omega^2}\right]_{\sigma,\tau} J(\sigma)\right) \quad . \tag{2.29}$$

EXCERCISE **2.4** *Prove eq. (2.29)*

We therefore have obtained a perturbative series for the partition function of the anharmonic oscillator. The perturbative expansion for the time evolution operator of the anharmonic oscillator can be obtained along similar reasoning. These perturbative series are usually asymptotic in nature. Each term of the series may be represented graphically as Feynman diagrams consisting of vertices and lines (propagators) with certain Feynman rules. For example, for the case of the anharmonic oscillator, each vertex introduces a factor of λ and each line corresponds to a free propagator, also called the free correlation function:[2]

$$< x(\tau_2)x(\tau_1) >_0 = \frac{1}{Z_0} Tr(x(\tau_2)x(\tau_1)e^{-\beta H_0}) \quad , \quad x(\tau) \equiv e^{\tau H_0} x e^{-\tau H_0} \quad . \tag{2.30}$$

EXCERCISE **2.5** *Using the cyclic property of the trace, show that the correlation function defined above is only a function of* $\tau_2 - \tau_1$.

$$\begin{aligned} < x(\tau_2)x(\tau_1) >_0 &= \frac{1}{Z_0}\frac{\partial^2 Z_0}{\partial J(\tau_2)\partial J(\tau_1)}\Big|_{J=0} = \left[\frac{1}{-\hat{\partial}_0^2+\omega^2}\right]_{\tau_2,\tau_1} \\ &= \frac{1}{N}\sum_p \frac{e^{ip(\tau_2-\tau_1)}}{\hat{p}^2+\omega^2} \quad , \end{aligned} \tag{2.31}$$

where $\hat{p}^2 = 4\sin^2(p/2)$ and $p = (2\pi)n/N$ with $n = 0, 1, \cdots, N-1$.

For the time evolution operator, Green's functions play similar role like the correlation functions in the partition function.

$$Z = \exp\left(-i\sum_{t=0}^{N-1} \frac{\lambda}{4!}(\frac{\partial}{i\partial J(t)})^4\right) Z_0[J(t)]\Big|_{J=0} \quad , \tag{2.32}$$

[2] It is named correlation function since it measures the thermal average of the correlation of the operator at two different temporal point. Recall that the thermal average of any operator $\mathcal{O}$ is given by $< \mathcal{O} >= (1/Z)Tr(\mathcal{O}e^{-\beta H})$ where Z is the partition function of the system.

where the free generating functional $Z_0[J(t)]$ is now defined by:

$$\begin{aligned} Z_0[J(\tau)] &\equiv \int \left(\prod_{t=0}^{N-1} dx(t) \right) e^{i \sum_{t=0}^{N-1} \frac{1}{2} x(t) \left[-\hat{\partial}_0^2 - \omega^2 \right] x(\tau) + J(t) x(t)} \\ &= Z[0] \exp \left(-\frac{i}{2} \sum_{s,t=0}^{N-1} J(s) \left[\frac{1}{-\hat{\partial}_0^2 - \omega^2 + i\epsilon} \right]_{s,t} J(t) \right) \quad . \end{aligned} \tag{2.33}$$

For the free two-point Green's function, or propagator, we have:

$$\langle 0 \mid T x_H(t_2) x_H(t_1) \mid 0 \rangle_0 = \frac{1}{N} \sum_p \frac{e^{ip(t_2 - t_1)}}{\hat{p}^2 - \omega^2 + i\epsilon} \quad , \tag{2.34}$$

Green's functions and the generating functional of Green's functions are so important since they are directly related to the matrix elements of the S-matrix of the theory by the famous LSZ redection formula. For a complete discussion of S-matrix theory and its relation to scattering cross section, Feynman rules and how to compute Feynman diagrams, please consult your textbook on Relativistic QFT.

2.4 Euclidean versus Minkowski: Wick rotation

The similarity between the partition function and the matrix element of the evolution operator that we have encountered in the quantum mechanics examples is a general feature that remains in quantum systems with many degrees of freedom, i.e. in QFT. Simple quantum-mechanical models like the harmonic oscillator can be viewed as a 0-dimensional quantum field theory if one desires. It is euqivalent to a $0+1$-dimensional classical statistical model when represented in terms of path integrals. The major difference is that one has to make the temporal separation pure-imaginary. This process is called a Wick rotation. In other words, after Wick rotation, the path integral representation of the evolution operator totally resembles the partition function of a classical statistical system with one extra dimension. More generally speaking, a $D+1$-dimensional quantum field theory in $D+1$-dimensional Minkowski space is equivalent to a $D+1$-dimensional classical statistical system in Euclidean space. With Wick rotation, the classical action in Minkowski space is transformed into the Hamiltonian of the classical statistical system in Euclidean space. The legitimacy of Wick rotation for a quantum field system is beyond the scope of this lecture. The main reason is that the Hamiltonian of the system pocesses no pole on the first sheet and therefore the $+i\epsilon$ prescription enables the Wick rotation into Euclidean. In the remaining lectures, we shall always assume that this is legitimate. Therefore, our study of QFT really deals with Euclidean QFT, which is easier both analytically and numerically. Finally, we point out that, Wick rotation only ensures that descent Minkowski Quantum Field Theories can be transformed into Euclidean Field Theories and not vice versa. That is

to say, constructing an arbitrary Euclidean QFT not necessarily corresponds to a descent QFT in Minkowski.

2.5 Summaries

In this chapter, using simple quantum mechanical systems as examples, we have illustrated the technique of path integral in the study of quantum systems. We have seen that, time evolution overators and Green's functions in real time can be related to the partition function and correlation functions in the imaginary time by Wick rotation; both bearing similar path integral representations. The quantum properties of the system can be studied by calculating the generating functional. For the case of harmonic oscillator, this generating functional can be obtained in a compact analytic form while for anharmonic oscillators, a perturbative expansion can be performed when the interaction is small. Each of the terms in the perturbative series corresponds to Feynman diagrams with definite Feynman rules. In the next two chapters, we will generalize the above discussion to field theories, i.e. systems with many degrees of freedom.

Further reading

Path integral was introduced into physcis by Feynman and can be found in [10]. It has now become a standard theoretic tool for quantizing field systems. Many books on advanced quantum mechanics and quantum field theory have excellent coverage on this topic. In [28], coherent states path integrals are introduced for both bosons and fermions. It is particularly useful for condensed matter physicists and neuclear theorists where one could see that path integral is really not just a patent to particle theorists. On the other hand, the book by Ramond [32] on Quantum Field Theory offers a particle physicst's view of path integrals in field theory. The whole book was based on path integral formalism without even touching the conventional canonical quantizations. Another useful book is Statistical Field Theory by Itzykson [13] which hopefully can make a bridge between these two view points.

Chapter 3

Lattice Scalar Fields

3.1 General discussions

Lattice scalar field system is a quantum system with many degrees of freedom. These degrees of freedom are characterized by a lattice field configuration $\{\phi_{\mathbf{x}} : \mathbf{x} \in \Lambda_3\}$ with Λ_3 being a non-empty set, which consists of sites of a 3-dimensional cubic lattice of lattice spacing a. In what follows, we usually assume that the lattice is finite with size L_i in direction i and periodic boundary conditions are applied to the fields $\phi_{\mathbf{x}}$. The total number of degrees of freedom is thus $L^3 \equiv L_1 L_2 L_3$. Sometimes, we are interested in the thermodynamic limit which means $L_i \to \infty$, in that case we call it an infinite lattice. These fields are the field theory counterparts of x in the harmonic oscillator.[1] The Lagrangian of the system is given by $L = L[\phi_{\mathbf{x}}, \dot{\phi}_{\mathbf{x}}]$. Of course, there are restrictions to the function $L[\phi_{\mathbf{x}}, \dot{\phi}_{\mathbf{x}}]$ in order to define a meaningful field theory. We first assume that the time derivative of fields in L is at most 1, which eliminates the possibilities of so-called higher derivative field theories. These theories usually results in a Hilbert space which is not positive definite. Another important restriction is that the system has to satisfy a property called locality. This means that, if we identify the base space Λ_3 with the sites of 3-dimensional cubic lattice, then field variables $\phi_{\mathbf{x}}$ at some lattice site $\mathbf{x}$ only couples to field variables $\phi_{\mathbf{y}}$ at sites $\mathbf{y}$ that are not infinitely away from $\mathbf{x}$. It turns out that the property of locality is necessary to guarantee a universal continuum limit of the theory. We assume that these general properties are satisfied.

Symmetry considerations play an essential role in quantum field theories. They are represented by group transformations which leave the physical content of the theory unchanged. The most important one is the symmetry of space-time. In the continuum limit, this symmetry is characterized by the Lorentz group, a Lie group with 6 generators 3 of which are the spatial rotations and

[1] One has to be very careful with the difference between a dynamical variable and a label here. In quantum mechanics, x is the dynamical variable, or a quantum mechanical operator, while in field theory the dynamical variables are $\phi_{\mathbf{x}}$, which are quantum mechanical operators. The subscripts $\mathbf{x}$ are only labels to count the degrees of freedom of the system.

the other 3 correspond to boosts along 3 directions. Lorentz invariance is a general requirement by the theory of special relativity. This means that all field theories must have a Lorentz invariant Lagrangian. On the 3-dimensional lattice, assuming time is still continuous, the symmetry is broken to a cubic symmetry characterized by the group O_h. Recall that when deriving the path integral representation, we will further discretized the temporal direction with spacing a. It is convenient to make a choice of units such that the lattice spacing in the temporal direction coincide with that in the spatial direction. This is said to be a symmetric lattice. The advantage of taking the symmetric lattice is the following. After Wick rotation to the Euclidean, the lattice model then possesses a hyper-cubic symmetry which makes the 4 directions symmetric. Only one scale, namely the lattice spacing a has to to taken to zero in order to restore the continuum limit Lorentz symmetry. Of course, asymmetric lattices are sometimes also useful, for example in the calculation of glueball spectrums, where one has to take both lattice spacings to zero to restore the continuum limit. In most of our discussions in this course, we will work with a symmetric lattice and usually take the lattice unit such that $a = 1$. Internal symmetries, for example isospins, are unchanged when going from the continuum to the lattice.

3.2 Schrödinger functional

Consider the Lagrangian of a scalar field theory in the continuum:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi(\mathbf{x})\partial_\mu\phi(\mathbf{x}) - \frac{m_0^2}{2}\phi^2(\mathbf{x}) - \frac{\lambda}{4!}\phi^4(\mathbf{x}) \quad . \tag{3.1}$$

The Lorentz invariance of this Lagrangian is obvious. The corresponding Hamiltonian density is:

$$\mathcal{H} = \frac{1}{2}\pi(\mathbf{x}) + \frac{1}{2}\partial_i\phi(\mathbf{x})\partial_i\phi(\mathbf{x}) + \frac{m_0^2}{2}\phi^2(\mathbf{x}) + \frac{\lambda}{4!}\phi^4(\mathbf{x}) \quad . \tag{3.2}$$

Using naive discretization of the derivatives, the Hamiltonian for the lattice model is:

$$H = \sum_{\mathbf{x}} \frac{1}{2}\pi_{\mathbf{x}}^2 + \frac{1}{2}\phi_{\mathbf{x}}\left[-\hat{\partial}_i\hat{\partial}_i^* + m_0^2\right]\phi_{\mathbf{x}} + \frac{\lambda}{4!}\phi_{\mathbf{x}}^4 \quad . \tag{3.3}$$

The field variables and their conjugate momenta satisfy the canonical commutation relation: $[\,\phi_{\mathbf{x}},\ \pi_{\mathbf{y}}\,] = \delta_{\mathbf{xy}}$. The matrix element of the time evolution operator is called the Schrödinger functional:[2]

$$\mathcal{Z}[\phi^F(\mathbf{x}), \phi^I(\mathbf{x}); T] = \langle\{\phi^F(\mathbf{x})\} \mid e^{-iHT} \mid \{\phi^I(\mathbf{x})\}\rangle \quad . \tag{3.4}$$

Performing the Wick rotation, we obtain for the Euclidean Schrödinger functional:

$$\mathcal{Z}_E[\phi^F(\mathbf{x}), \phi^I(\mathbf{x}); T] = \int \left(\prod_{\mathbf{x}}\prod_{\tau=1}^{N-1} d\phi_{\tau,\mathbf{x}}\right) \exp\left(-S_E[\phi_{\tau,\mathbf{x}}]\right) \quad ,$$

[2] It is a functional of given boundary fields $\{\phi^F(\mathbf{x})\}$ and $\{\phi^I(\mathbf{x})\}$.

$$S_E[\phi_{\tau,\mathbf{x}}] = \sum_{\mathbf{x},\tau=0}^{N-1} \frac{1}{2}[\hat{\partial}_0\phi_{\tau,\mathbf{x}}]^2 + \frac{1}{2}\phi_{\tau,\mathbf{x}}\left[-\hat{\partial}_i\hat{\partial}_i^* + m_0^2\right]\phi_{\tau,\mathbf{x}} + \frac{\lambda}{4!}\phi_{\tau,\mathbf{x}}^4 \quad (3.5)$$

The forward and backward lattice derivatives are defined as:

$$\hat{\partial}_\mu\phi_x = \phi_{x+\hat{\mu}} - \phi_x \ , \hat{\partial}_\mu^*\phi_x = \phi_x - \phi_{x-\hat{\mu}} \quad , \quad (3.6)$$

where we have introduced the 4 dimensional label $x = (\tau, \mathbf{x})$. The direction vector $\hat{\mu}$, $\mu = 0, 1, 2, 3$ indicates the 4 unit vectors along the four corresponding lattice directions. So if $x = (\tau, \mathbf{x}) = (x_0, x_1, x_2, x_3)$, then $x+\hat{0} = (x_0+1, x_1, x_2, x_3)$, $x + \hat{1} = (x_0, x_1 + 1, x_2, x_3)$ etc. The partition function of the system is obtained by taking periodic condition in temporal direction $\phi^F(\mathbf{x}) = \phi^I(\mathbf{x})$ and integrate over the boundary fields:

$$\begin{aligned} \mathcal{Z} &= \int \left(\prod_x d\phi_x\right) \exp\left(-S_E[\phi_x]\right) \quad , \\ S_E[\phi_x] &= \sum_x \frac{1}{2}\phi_x\left[-\hat{\partial}_\mu\hat{\partial}_\mu^* + m_0^2\right]\phi_x + \frac{\lambda_0}{4!}\phi_x^4 \quad . \end{aligned} \quad (3.7)$$

This path integral serves as the starting point of our study of the Euclidean lattice scalar field theory.

To calculate the partition function perturbatively, one introduces the free generating functional:

$$\mathcal{Z}_0[J(x)] = \int \left(\prod_x d\phi_x\right) \exp\left(-\sum_x \frac{1}{2}\phi_x\left[-\hat{\partial}_\mu\hat{\partial}_\mu^* + m_0^2\right]\phi_x + J(x)\phi(x)\right) \quad . \quad (3.8)$$

The partition function of the system then admits the following perturbative expansion:

$$\mathcal{Z} = \exp\left(-\sum_x \frac{\lambda_0}{4!}\left[\frac{\partial}{\partial\phi_x}\right]^4\right)\mathcal{Z}_0[J]\Big|_{J=0} \quad . \quad (3.9)$$

Again, each term in the perturbative expansion is represented by Feynman diagrams with vertices and propagators.

EXCERCISE **3.1** *Show that the free Euclidean propagator for the lattice scalar field is given by*

$$< \phi_x\phi_y >= \frac{1}{V}\sum_p \frac{e^{ip(x-y)}}{\hat{p}^2 + m_0^2} \quad , \quad (3.10)$$

where the lattice momentum squared $\hat{p}^2 = \sum_\mu 4\sin^2(p_\mu/2)$ *and* $V = \beta L^3$ *is size of the 4 volume. We have assumed periodic boundary conditions in all four directions. Therefore* $p_\mu = 2\pi n_\mu/L_\mu$*, where* L_μ *is the size of the lattice in* μ *direction and* n_μ *are integers ranging from* 0 *to* $L_\mu - 1$*.*

EXCERCISE **3.2** *For the free lattice scalar fields, the continuum limit is obtained by taking m_0 (measured in lattice units) to zero. Show that the free propagator in eq. (3.10) reduces to the continuum Euclidean space propagator, which in Fourier space is just $p^2 + m_0^2$, with p^2 the four momentum squared.*

To study the theory non-perturbatively, one would start with eq. (3.7) and use Monte Carlo methods to compute the path integral and relevant physical quantities numerically. We emphasize that this path integral is a well-defined multi-dimensional integral which is completely finite with no ultra-violet divergences in the formalism. It is also non-perturbative from the very beginning. In fact,eq. (3.7) serves as a non-perturbative definition of the scalar field theory.

3.3 Coherent states path integral

Consider a quantum system characterized by a set of bosonic creation and annihilation operators $a_j^\dagger$ and a_j, where $j = 1, 2, \cdots, M$ and $[a_i, a_j^\dagger] = \delta_{ij}$. Given a set of complex numbers ξ_j, the coherent ket state $\mid \xi \rangle$ is defined as:

$$\mid \xi \rangle = e^{\xi_j a_j^\dagger} \mid 0 \rangle \quad , \tag{3.11}$$

where $\mid 0 \rangle$ is the vacuum state annihilated by all a_i and summation over repeated indices are understood.

THEOREM **3.1** *The coherent state defined by eq. (3.11) satisfies the following:*

$$a_j \mid \xi \rangle = \xi_j \mid \xi \rangle \quad , \tag{3.12}$$

$$\langle \zeta \mid \xi \rangle = e^{\zeta_j^* \xi_j} \quad , \tag{3.13}$$

$$\int \prod_j \left(\frac{dRe\xi_j dIm\xi_j}{\pi} \right) e^{-\xi_j^* \xi_j} \mid \xi \rangle \langle \xi \mid = 1 \quad , \tag{3.14}$$

EXCERCISE **3.3** *Prove this theorem.*

Assuming that the Hamiltonian of the quantum system is known as a function of creation and annihilation operators $a_j^\dagger$ and a_j. We could use the over-complete relation of coherent states, i.e. eq. (3.14) to derive a path integral representation for the partition function of the system. Since:

$$Z = Tr(e^{-\beta H}) = \int \prod_j \left(\frac{dRe\xi_j dIm\xi_j}{\pi} \right) e^{-\xi_j^* \xi_j} \langle \xi \mid e^{-\beta H[a_j^\dagger, a_j]} \mid \xi \rangle \quad , \tag{3.15}$$

we further chop β into N slices each of separation a_t and insert a complete set of coherent states $\mid \xi_\tau \rangle \langle \xi_\tau \mid$ at each slice labeled by τ for $\tau = 0, 1, \cdots, N$; with ξ_N identified with ξ_0. Since a_t is small, the matrix element of $\exp(-a_t H[a_j^\dagger, a_j])$ between two coherent states may be reduced to the inner product of two coherent

states times a factor $\exp(-a_t H[\xi^*_\tau, \xi_{\tau+1}])$ with the corresponding operators $a^\dagger_j$ and a_j substituted by their eigenvalues. The final outcome of this is:

$$Z = \int \prod_{\tau=0}^{N-1} (D^2\xi_\tau) \exp\left(\sum_{\tau=0}^{N-1} \xi^*_\tau \cdot (\xi_{\tau+1} - \xi_\tau) - a_t H[\xi^*_\tau, \xi_{\tau+1}] \right) \quad , \tag{3.16}$$

where we have adopted the compact notation $\xi \cdot \zeta \equiv \xi_j \zeta_j$ and the integration measure $D^2\xi \equiv \prod_j (dRe\xi_j dIm\xi_j/\pi)$.

EXCERCISE **3.4** *Using Fourier transformation to show that the Hamiltonian for free lattice scalar field theory obtained by setting $\lambda = 0$ in eq. (3.3) is diagonalized into the form:*

$$H = \sum_{\mathbf{k}} (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + 1/2)\omega_{\mathbf{k}} \quad , \tag{3.17}$$

where $\mathbf{k} = (k_1, k_2, k_3) = (2\pi/L)(n_1, n_2, n_3)$, for $n_i = 0, 1, \cdots, L-1$. The energy $\omega_{\mathbf{k}}$ of a particle with 3-momenta $\mathbf{k}$ is given by $\omega^2_{\mathbf{k}} = \hat{\mathbf{k}}^2 + m_0^2$ with $\hat{\mathbf{k}}^2 = \sum_i 4\sin^2(k_i/2)$.

EXCERCISE **3.5** *Using the diagonal form of the free Hamiltonian (3.17) and derive a coherent state path integral representation for the partition function. Show that it is consistent with the result obtained using the coordinate representation.*

Further readings

Most of the quantum field theory textbooks covers the path integral for scalar fields.

Chapter 4

Euclidean Lattice Spinor Field Theory

4.1 Naive fermions

Fermions, the stuff that we are made up of, are more subtle objects to study quantum mechanically. The simplest fermions are so-called Weyl fermions which are divided into two categories: left-handed fermions and right-handed ones; both of which form a basic representation of the Lorentz group. Being only single-handed, left-handed Weyl fermions and right-handed Weyl fermions are handicapped in the sense they are not invariant under parity transformation, which transform the left-handed ones into the right-handed ones and vice versa. To preserve parity as a good symmetry, a direct sum of a left-handed fermion and a right-handed fermion has to be taken. The fermion constructed in this manner has 4 components and regards parity as a good symmetry. This type of fermion, first derived by Dirac, are called Dirac fermions and they are denoted by a spinor field $\psi_{\mathbf{x}}$.

The quantum mechanical nature of fermions are drastically different from those of scalars, or bosons. It turns out that, to quantize the theory in a consistent manner, one has to impose *anticommuation*, not commutation relations between the fields and their conjugate momenta: $\{\psi_{\mathbf{x}\alpha}, \pi_{\mathbf{y}\beta}\} = i\delta_{\alpha\beta}\delta_{\mathbf{xy}}$. The Lorentz invariant Lagrangian for free Dirac fermions in the continuum is given by:

$$\mathcal{L} = \bar{\psi}(i\Gamma^{\mu}\partial_{\mu} - m_0)\psi \quad , \tag{4.1}$$

where $\bar{\psi} = \psi^{\dagger}\Gamma_0$ and Γ_{μ} are conventional Dirac γ-matrices in Minkowski space. To get a lattice version of the Hamiltonian, we first introduce a two-component spinor field $\chi_{\mathbf{x}}$ on each lattice site. The reason for this will be explained later on [15]. Then in terms of this spinor field, the Hamiltonian for the naive fermion

is given by:

$$H = \sum_{\mathbf{x},\mathbf{n}} \chi^\dagger{}_{\mathbf{x}}(-i\sigma\cdot\mathbf{n})\chi_{\mathbf{x}+\mathbf{n}} + m_0 \sum_{\mathbf{x}} (-)^{\mathbf{x}} \chi^\dagger{}_{\mathbf{x}}\chi_{\mathbf{x}} \ , \tag{4.2}$$

where the unit vector $\mathbf{n}$ takes 6 possible values along 3 lattice axis, both positive and negative. The symbol $(-)^{\mathbf{x}} = (-)^{x_1+x_2+x_3}$ labels even (odd) lattice sites depending on being $+1$ (-1). The basic anti-commutation relation is imposed by:

$$\{\chi_{\alpha\mathbf{x}} \ , \ \chi^\dagger{}_{\beta\mathbf{y}}\} = \delta_{\alpha\beta}\delta_{\mathbf{xy}} \ . \tag{4.3}$$

The Dirac equation on 3-dimensional lattice becomes:

$$\left[i\partial_0 + i\frac{1}{2}(\hat{\partial}_i + \hat{\partial}_i^*)\sigma_i + m_0(-)^{\mathbf{x}}\right]\chi_{\mathbf{x}} = 0 \ . \tag{4.4}$$

EXCERCISE **4.1** *Verify this result.*

To keep the energy still finite as the lattice spacing is going to zero requires that the symmetrized lattice derivative $\chi_{\mathbf{x}+\mathbf{n}} - \chi_{\mathbf{x}-\mathbf{n}} \sim a$. However, notice that this does not impose any condition on $\chi_{\mathbf{x}+\mathbf{n}} - \chi_{\mathbf{x}}$. In other words, it can happen that the fields on even sites are drastically different from those on odd sites while the fields are smooth among themselves. To remedy this situation, we introduce two set of fields $\chi^{(e)}$ and $chi^{(o)}$, one on even sites and one on odd sites. They will be identified as the upper components and lower components of a 4 component spinor field ψ. The lattice Dirac equation now reads:

$$\begin{aligned} i\partial_0\chi^{(e)}_{\mathbf{x}} &= \left[-i\frac{1}{2}(\hat{\partial}_i + \hat{\partial}_i^*)\sigma_i\right]\chi^{(o)} + m_0\chi^{(e)}_{\mathbf{x}} = 0 \ . \\ i\partial_0\chi^{(o)}_{\mathbf{x}} &= \left[-i\frac{1}{2}(\hat{\partial}_i + \hat{\partial}_i^*)\sigma_i\right]\chi^{(e)} - m_0\chi^{(o)}_{\mathbf{x}} = 0 \ . \end{aligned} \tag{4.5}$$

Or, written in terms of 4-component spinor $\psi = (\chi^{(e)}, \chi^{(o)})$:

$$[i\Gamma^\mu\hat{\partial}_\mu - m_0]\psi = 0 \ , \tag{4.6}$$

where $\hat{\partial}_\mu = (\partial_0, (\hat{\partial}_i + \hat{\partial}_i^*)/2$.

EXCERCISE **4.2** *Verify this result.*

Substitute a translational invariant solution of the form $w(\mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{x} - ip_0t)$ we obtain:

$$p_0^2 = \tilde{\mathbf{p}}^2 + m_0^2 \ , \tag{4.7}$$

with $\tilde{\mathbf{p}}^i = \sin(p^i)$ to be the mass-shell condition for nontrivial solutions. This means that p_0 has to be a definite function of $\mathbf{p}$ given above, which we now denote as $\omega_{\mathbf{p}} = +\sqrt{\tilde{\mathbf{p}}^2 + m_0^2} > 0$. Since the Dirac structure remains identical to that in the continuum, we can construct four linear independent solutions for a given $\mathbf{p}$, which we denote as $u_{s\mathbf{p}}$ and $v_{s\mathbf{p}}$ with $s = 1, 2$. They satisfy:

$$(\tilde{\not{p}} - m_0)u_{s\mathbf{p}} = 0 \ , \quad (\tilde{\not{p}} + m_0)v_{s\mathbf{p}} = 0 \ , \tag{4.8}$$

where $\tilde{\not{p}} \equiv \tilde{p}_\mu \Gamma^\mu$ and $p_0 \equiv \omega_{\mathbf{p}}$. This remains analogous to the continuum except that the continuum $\mathbf{p}$ is replaced by $\tilde{\mathbf{p}}$. These spinors satisfy certain orthonormal relations as their continuum counterparts:

$$\begin{aligned}
&\sum_s u_{s\mathbf{p}} \bar{u}_{s\mathbf{p}} = \Lambda^+_{\mathbf{p}} = \frac{\tilde{\not{p}} + m_0}{2m_0} \ , \ -\sum_s v_{s\mathbf{p}} \bar{v}_{s\mathbf{p}} = \Lambda^-_{\mathbf{p}} = \frac{-\tilde{\not{p}} + m_0}{2m_0} \ , \\
&\bar{u}_{s\mathbf{p}} u_{r\mathbf{p}} = \bar{v}_{s\mathbf{p}} v_{r\mathbf{p}} = \delta_{rs} \ , \ \bar{u}_{s\mathbf{p}} v_{r\mathbf{p}} = \bar{v}_{s\mathbf{p}} u_{r\mathbf{p}} = 0 \ , \\
&u^\dagger_{s\mathbf{p}} u_{r\mathbf{p}} = v^\dagger_{s\mathbf{p}} v_{r\mathbf{p}} = \frac{\omega_{\mathbf{p}}}{m_0} \delta_{rs} \ , \ u^\dagger_{s\mathbf{p}} v_{r,-\mathbf{p}} = v^\dagger_{s,-\mathbf{p}} u_{r\mathbf{p}} = 0 \ ,
\end{aligned} \tag{4.9}$$

We therefore could expand:

$$\begin{aligned}
\psi_{\mathbf{x}} &= \sum_{s\mathbf{p}} \sqrt{\frac{m_0}{\omega_{\mathbf{p}} V}} \left(b_{s\mathbf{p}} u_{s\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + d^\dagger_{s\mathbf{p}} v_{s\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \ , \\
\bar{\psi}_{\mathbf{x}} &= \sum_{s\mathbf{p}} \sqrt{\frac{m_0}{\omega_{\mathbf{p}} V}} \left(b^\dagger_{s\mathbf{p}} \bar{u}_{s\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + d_{s\mathbf{p}} \bar{v}_{s\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \right) \ ,
\end{aligned} \tag{4.10}$$

where the fermion creation and annihilation operators satisfy:

$$\{b_{s\mathbf{p}}, b^\dagger_{r\mathbf{k}}\} = \{d_{s\mathbf{p}}, d^\dagger_{r\mathbf{k}}\} = \delta_{sr}\delta_{\mathbf{kp}} \ . \tag{4.11}$$

EXCERCISE **4.3** *Using the basic anti-commutation relations (4.11) satisfied by creation and annihilation operators together with the Fourier expansion (4.10) of $\psi_{\mathbf{x}}$ and $\pi_{\mathbf{x}} = i\bar{\psi}_{\mathbf{x}}\Gamma_0$, show that the basic equal time anti-commutation relation for the fields and momenta: $\{\psi_{\mathbf{x}\alpha}, \pi_{\mathbf{y}\beta}\} = i\delta_{\alpha\beta}\delta_{\mathbf{xy}}$ is valid.*

Using the properties (4.9) satisfied by the spinors, it is easy to show that the free Hamiltonian is diagonalized:

$$H = \sum_{s\mathbf{p}} [b^\dagger_{s\mathbf{p}} b_{s\mathbf{p}} - d_{s\mathbf{p}} d^\dagger_{s\mathbf{p}}] \omega_{\mathbf{p}} \ . \tag{4.12}$$

EXCERCISE **4.4** *Verify this result.*

4.2 Grassman path integrals

Since the free lattice Hamiltonian is diagonalized using the creation and annihilation operators, it is easy to derive the partition function of the system using the coherent state technique just like in the bosonic case. Consider a fermionic system characterized by the set of creation and annihilation operators $c^\dagger_j$ and c_j for $j = 1, 2, \cdots M$. They satisfy $\{c_i, c^\dagger_j\} = \delta_{ij}$. Let ξ_j (and $\bar{\xi}_j$) be a set of Grassman variables. They satisfy $\xi_i \xi_j = -\xi_j \xi_i$ for all possible i and j. We define the fermionic coherent bra and ket states:

$$\begin{aligned}
| \, \xi \, \rangle &= \sum_{J=0}^{\infty} \frac{(-)^{J(J-1)}}{J!} (\xi \cdot c^\dagger)^J \, | \, 0 \, \rangle \\
\langle \, \bar{\xi} \, | &= \sum_{J=0}^{\infty} \langle 0 \, | \, \frac{(-)^{J(J-1)}}{J!} (\bar{\xi} \cdot c)^J \ ,
\end{aligned} \tag{4.13}$$

where $\xi \cdot c^\dagger = \xi_j c_j^\dagger$ and $\mid 0 \rangle$ is the vacuum annihilated by all c_j. Due to the anti-commuting nature of the Grassman variables, the range of the summation in the above definition is in fact finite. It is straightforward to show:

THEOREM **4.1** *The coherent state defined by eq. (4.13) satisfies the following:*

$$c_j \mid \xi \rangle = \mid \xi \rangle \xi_j \quad , \quad \langle \bar{\xi} \mid c_j^\dagger = \bar{\xi}_j \langle \bar{\xi} \mid \quad , \tag{4.14}$$

$$\langle \bar{\xi} \mid \xi \rangle = e^{\bar{\xi} \cdot \xi} \quad , \tag{4.15}$$

$$\int \prod_j (d\bar{\xi}_j d\xi_j) e^{-\bar{\xi} \cdot \xi} \mid \xi \rangle \langle \bar{\xi} \mid = 1 \quad , \tag{4.16}$$

The trace of any operator $\mathcal{O}$ is given by:

$$Tr\mathcal{O} = \int \prod_j (d\bar{\xi}_j d\xi_j) e^{-\bar{\xi} \cdot \xi} \langle \bar{\xi} \mid \mathcal{O} \mid -\xi \rangle \quad . \tag{4.17}$$

The matrix element of the operator $\exp(c_j^\dagger M_{jk} c_k)$ between two coherent states is given by:

$$\langle \bar{\xi} \mid \exp(c_j^\dagger M_{jk} c_k) \mid \xi \rangle = \exp\left(c_j^\dagger [e^M]_{jk} c_k \right) \quad . \tag{4.18}$$

A general Gaussian integral of Grassman integrals is given by:

$$\int \prod_j (d\bar{\xi}_j d\xi_j) \exp(-\bar{\xi}_j M_{jk} \xi_k) = \det(M) \quad . \tag{4.19}$$

EXCERCISE **4.5** *Prove this theorem.*

The integration and differentiation of Grassman variables are defined in the following way. The measure $d\xi_j$ and the partial derivatives $\partial / \partial \xi_j$ are themselves Grassman variables and satisfy:

$$\int d\xi_j \xi_j = \int d\bar{\xi}_j \bar{\xi}_j = 1 \quad , \quad \frac{\partial}{\partial \psi_i} \psi_i = \frac{\partial}{\partial \bar{\psi}_i} \bar{\psi}_i = 1 \quad . \tag{4.20}$$

With these properties, we can easily construct a path integral representation for the partition function of the free naive lattice fermion system in momentum space.

$$\begin{aligned} Z \quad = \quad & \int \prod_{s\mathbf{p},\tau=0}^{N-1} \left(d\bar{\xi}_{s\mathbf{p}\tau} d\xi_{s\mathbf{p}\tau} d\bar{\zeta}_{s\mathbf{p}\tau} d\zeta_{s\mathbf{p}\tau} \right) \\ & \times \exp\left(- \sum_{s\mathbf{p},\tau=1}^{N} \bar{\xi}_{s\mathbf{p},\tau-1} \cdot \xi_{s\mathbf{p}\tau} + \bar{\zeta}_{s\mathbf{p},\tau-1} \cdot \zeta_{s\mathbf{p}\tau} \right) \\ & \times \prod_{\tau=0}^{N-1} \langle \bar{\xi}_{s\mathbf{p}\tau} \bar{\zeta}_{s\mathbf{p}\tau} \mid e^{-a_t H} \mid \xi_{s\mathbf{p}\tau} \zeta_{s\mathbf{p}\tau} \rangle \quad , \end{aligned} \tag{4.21}$$

where $\bar{\xi}_{s\mathbf{p}\tau}$, $\xi_{s\mathbf{p}\tau}$, $\bar{\zeta}_{s\mathbf{p}\tau}$ and $\zeta_{s\mathbf{p}\tau}$ corresponds to the eigenvalues of $b^\dagger_{s\mathbf{p}}$, $b_{s\mathbf{p}}$, $d^\dagger_{s\mathbf{p}}$ and $d_{s\mathbf{p}}$ at time slice τ respectively. Anti-periodic boundary condition: $\xi_{s\mathbf{p},\tau=N} = -\xi_{s\mathbf{p},\tau=0}$ and $\zeta_{s\mathbf{p},\tau=N} = -\zeta_{s\mathbf{p},\tau=0}$ are employed in accordance with eq. (4.17). Using the property (4.18) and (4.15), we obtain for the free naive lattice fermion:

$$\begin{aligned} Z &= \int \prod_{s\mathbf{p},\tau=0}^{N-1} \left(d\bar{\xi}_{s\mathbf{p}\tau} d\xi_{s\mathbf{p}\tau} d\bar{\zeta}_{s\mathbf{p}\tau} d\zeta_{s\mathbf{p}\tau}\right) \\ &\quad \times \exp\left(-\sum_{s\mathbf{p},\tau=0}^{N-1} \bar{\xi}_{s\mathbf{p}\tau}\cdot(\xi_{s\mathbf{p},\tau+1}-\xi_{s\mathbf{p}\tau}) + \bar{\zeta}_{s\mathbf{p}\tau}\cdot(\zeta_{s\mathbf{p},\tau+1}-\zeta_{s\mathbf{p}\tau})\right) \\ &\quad \times \exp\left[-\sum_{s\mathbf{p},\tau=0}^{N-1} (\bar{\xi}_{s\mathbf{p}\tau}\xi_{s\mathbf{p}\tau} + \bar{\zeta}_{s\mathbf{p}\tau}\zeta_{s\mathbf{p}\tau})(1-e^{-a_t\omega_{\mathbf{p}}})\right] \quad , \end{aligned} \tag{4.22}$$

Taking the $d\tau = a_t \to 0$ limit, we could write:

$$\begin{aligned} Z &= \int \prod_{s\mathbf{p}\tau} \left(d\bar{\xi}_{s\mathbf{p}\tau} d\xi_{s\mathbf{p}\tau} d\bar{\zeta}_{s\mathbf{p}\tau} d\zeta_{s\mathbf{p}\tau}\right) \\ &\quad \times \exp\left(-\int_0^\beta d\tau \sum_{s\mathbf{p}} \bar{\xi}_{s\mathbf{p}\tau}\cdot\partial_0\xi_{s\mathbf{p}\tau} + \bar{\zeta}_{s\mathbf{p}\tau}\cdot\partial_0\zeta_{s\mathbf{p},\tau}\right) \\ &\quad \times \exp\left[-\int_0^\beta d\tau \sum_{s\mathbf{p}} (\bar{\xi}_{s\mathbf{p}\tau}\xi_{s\mathbf{p}\tau} + \bar{\zeta}_{s\mathbf{p}\tau}\zeta_{s\mathbf{p}\tau})\omega_{\mathbf{p}}\right] \quad , \end{aligned} \tag{4.23}$$

One could also make a change of variables, recall that the transformation from ψ to $b_{s\mathbf{p}}$ and $d_{s\mathbf{p}}$ is a unitary transformation, we could define a Grassman field:[1]

$$\begin{aligned} \psi_x &= \sum_{s\mathbf{p}} \sqrt{\frac{m_0}{\omega_{\mathbf{p}} V}} \left(\xi_{s\mathbf{p}\tau} u_{s\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + \bar{\zeta}_{s\mathbf{p}\tau} v_{s\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \quad , \\ \bar{\psi}_x &= \sum_{s\mathbf{p}} \sqrt{\frac{m_0}{\omega_{\mathbf{p}} V}} \left(\bar{\xi}_{s\mathbf{p}\tau} \bar{u}_{s\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \zeta_{s\mathbf{p}\tau} \bar{v}_{s\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \quad , \end{aligned} \tag{4.24}$$

where we have introduced the four-dimensional coordinate $x = (\tau, \mathbf{x})$. Changing the integration variables to ψ_x and $\bar{\psi}_x$, we finally obtain:

$$Z = \int \prod_x d\bar{\psi}_x d\psi_x \exp\left(-\int_0^\beta d\tau \sum_{\mathbf{x}} \bar{\psi}_x \left[\frac{1}{2}(\hat{\partial}_\mu + \hat{\partial}^*_\mu)\gamma_\mu + m_0\right]\psi_x\right) \quad , \tag{4.25}$$

where, we have used the symmetrized discrete version of ∂_0. The reason for this is the same as that for the symmetrized spatial derivatives. We have also

[1] This is not to be confused with the *field operator* in eq. (4.10) in the canonical quantization procedure. There, the field is an operator in Hilbert space. Here the field is just a Grassman variable taking values in the field of Grassman numbers. Therefore, the Grassman fields ψ_x and $\bar{\psi}_x$ are completely independent Grassman variables, have no relations with each other what so ever.

introduced the Euclidean γ-matrices:

$$\gamma_0 = \Gamma_0 \ , \ \ \gamma_i = i\Gamma^i \ . \tag{4.26}$$

EXCERCISE **4.6** *Verify this result.*

4.3 Fermion doubling problem

We have obtained the partition function as a path integral over Grassman fermion fields for the naive lattice fermion. On a 4-dimensional hyper-cubic lattice, the propagator of the naive fermion is easily obtained:

$$< \psi_x \bar{\psi}_y >= \frac{1}{V} \sum_p \frac{e^{ip\cdot(x-y)}}{i\tilde{p}_\mu \gamma_\mu + m_0} \ , \tag{4.27}$$

where $\tilde{p}_\mu = \sin p_\mu$. The pole of this propagator in momentum space thus represents real particle states of the system. We observe that, due to the periodic nature of the function $\sin(p_\mu)$, it develops poles not only for small p_μ but also at the conner of the Brillouin zone, namely at $p = (0, \pi, 0, 0)$ etc. In fact, in four dimensions, 16 such poles all correspond to real particle states of the lattice system. We are therefore led to the infamous doubling problem of lattice fermion. That is to say, a naive discretization of the continuum fermion action, which describe one spices of fermions in the continuum, leads to 2^D spices of fermions on the lattice and they all survive the continuum limit. Here D is the dimensionality of space-time. These extra degrees of freedom are called doublers.

The doubling problem we encountered here poses a serious challenge for lattice fermions. However, it is not just a lattice problem. In fact, this problem is intimately related to the chirality of fermions and the doubling problem is just a manifestation of the impossibility to define a fermion field theory of a single chirality non-perturbatively. In other words, the key issue is to formulate a non-perturbative definition of a chiral gauge theory. This question is now not fully solved yet, though interesting progress has been made recently. Here, we will only point out that, for vector-like theories, QCD for example, things are easier. One could adopt other fermion prescriptions suggested by Wilson or Kogut-Susskind to decouple the doubler fermions in the continuum limit.

4.4 Wilson fermions

Realizing the doubling problem for naive lattice fermions, Wilson suggested a slight modification of the lattice fermion action:

$$\begin{aligned} S_f &= \sum_x \bar{\psi}_x (D + m_0)\psi_x \ , \\ D &= \frac{1}{2}\left[(\partial_\mu + \partial^*_\mu)\gamma_\mu - \partial_\mu \partial^*_\mu)\right] \ . \end{aligned} \tag{4.28}$$

The free propagator of Wilson fermions is:

$$< \psi_x \bar{\psi}_y >= \frac{1}{V} \sum_p \frac{e^{ip\cdot(x-y)}}{i\tilde{p}_\mu \gamma_\mu + m_0 + \sum_\mu (1 - \cos p_\mu)} \quad . \tag{4.29}$$

Note that, an extra second derivative term is added to the naive fermion action so that the doubler at the corner of the Brillouin zone receive extra mass values which is proportional to $1/a$ where a is the lattice spacing. Therefore, in the continuum limit, only the fermion whose momenta are close to the origin of the Brillouin zone remains in the low energy spectrum while all other doublers are decoupled due to higher mass values. It can be shown that, for Wilson fermions coupled to the gauge fields in Euclidean, a well-defined Hamiltonian and Hilbert space can be constructed using the transfer matrix methods [18]. Therefore, Wilson fermion is a good starting point for a lattice QCD study. The disadvantage of Wilson fermion is that it breaks chiral symmetry of the action. Chiral symmetry is a symmetry for QCD with massless quarks.

Another formulation of lattice fermion exists which goes under the name of Kogut-Susskind fermions, or staggered fermions. It restores a residue chiral symmetry, though not the right symmetry that one prefers. Since the construction is too technical, we will not discuss it here.

Finally, we conclude by a famous theorem, also called no-go theorem due to Nielson-Ninomiya [29, 30, 31], which states:

THEOREM **4.2** *It is impossible to construct a lattice fermion action:* $S_f = \sum_x \bar{\psi}_x \mathcal{D}_{xy} \psi_y$ *which satisfy all of the following properties:*

1. *The fermion matrix* $\mathcal{D}$ *is local and translational invariant.*
2. *The correct continuum limit, i.e. the Fourier transform of the fermion matrix* $\tilde{\mathcal{D}}(p)$ *satisfies:* $\tilde{\mathcal{D}}(p) = i\gamma_\mu p_\mu$ *for small* p_μ.
3. *The fermion action contains no doublers. Therefore, the matrix* $\tilde{\mathcal{D}}(p)$ *is non-singular for* $p \neq 0$.
4. *The fermion action respects chiral symmetry:* $\gamma_5 D = -D\gamma_5$.

Chapter 5

Lattice Gauge Fields

5.1 Wilson plaquette action

Gauge fields and gauge symmetry plays an important role in modern quantum field theories. The Weinberg-Salam model of electro-weak interactions and QCD are all gauge theories. To study gauge fields on the lattice, one has to construct a lattice action which is explicitly gauge invariant. This is done by putting gauge fields on the link of the hyper-cubic lattice (not on the sites) and let them taking values in a gauge group. In other words, for a given oriented link pointing from x to $x + \hat{\mu}$, where x is a site on a 4-dimensional hyper-cubic lattice and $\hat{\mu}$ designates the 4 forward directions on the lattice, we associate a gauge field $U_{x,x+\hat{\mu}} \equiv U_\mu(x) \in G$ where G is some gauge group. By convention: $U_{x+\hat{\mu},x} = U^{-1}_{x+\hat{\mu},x}$. Under a gauge transformation, the gauge link variable $U_\mu(x)$ is transformed as:

$$U_\mu(x) \to g_x U_\mu(x) g^{-1}_{x+\hat{\mu}} \quad , \quad g_x \in G \quad . \tag{5.1}$$

In order to construct a gauge invariant action, we note that for a closed path C on the lattice, if we define:

$$U_C = U_{x_1,x_2} U_{x_2,x_3} \cdots U_{x_n,x_1} \quad , \tag{5.2}$$

where x_i are consecutive lattice sites along the path C. It is easy to see that under a gauge transformation:

$$U_C \to g_x U_C g_x^{-1} \quad , \quad g_x \in G \quad . \tag{5.3}$$

Therefore, if we take the character of U_C, or taking the trace of U_C is some representation of the group, it will be a gauge invariant quantity, due to the cyclic property of the trace. In what follows, we will consider the case where the gauge group is $SU(n)$ and we will assume that the gauge fields are in the fundamental representation of the gauge group. The simplest closed path on the

lattice is the so-called plaquette. We therefore arrive at the Wilson's plaquette action for the gauge fields:

$$S_g = \beta_g \sum_p \frac{1}{n} Tr U_p \quad . \tag{5.4}$$

where U_p is the multiplication of 4 gauge links along a plaquette p on the lattice and the summation extends over all distinct plaquettes. The naive continuum limit is obtained by substituting in

$$U_\mu(x) = \exp(a A_\mu(x + \hat{\mu}/2)) \quad . \tag{5.5}$$

Comparing with the continuum limit result of a gauge field action, one is led to the identification of:

$$\beta_g = 2n/g_0^2 \quad , \tag{5.6}$$

where g_0 is the bare gauge coupling in the continuum. The partition function of the pure gauge system is:

$$Z = \int \prod_{x,\mu} DU_\mu(x) e^{-S_g[U]} \quad , \tag{5.7}$$

where $DU_{\mu,x}$ is the invariant Haar measure for the gauge group.

5.2 Wilson loops

We have seen that the trace of the products of gauge fields along a closed path on a lattice is a gauge invariant quantity. A particular useful construction is called a Wilson loop, which is defined as:

$$\begin{aligned} W(R,T) \quad = \quad & Tr \left(\prod_{l=0}^{T-1} U_0(x + l\hat{0}) \prod_{k=0}^{R-1} U_i(x + T\hat{0} + k\hat{i}) \right. \\ & \left. \prod_{l=1}^{T} U_i^\dagger(x + (T-l)\hat{0} + R\hat{i}) \prod_{k=1}^{R} U_i^\dagger(x + (R-k)\hat{i}) \right) \quad . \end{aligned} \tag{5.8}$$

This object is of great importance, since it is related with the heavy quark-antiquark potential $V(R)$ by:

$$W(R,T) \propto e^{-V(R)T} \quad T \to \infty \quad , \tag{5.9}$$

where $V(R)$ is the heavy quark-antiquark potential. To see this, imagine that a quark-antiquark pair is created at some time with fixed distance R apart. The potential energy of the system is then $V(R)$. After some time separation T, the quark-antiquark pair is then annihilated. The probability amplitude for this process is then proportional to $\exp(-V(R)T)$. On the other hand, this amplitude is precisely the average of the Wilson loop operator.

Wilson loops could be evaluated in Monte Carlo simulations and in various approximation methods, e.g. strong coupling expansion, which will be discussed in the next section. Just by hyper-cubic symmetry, we see that,

$$V(R) \propto R \quad R \to \infty \ , \tag{5.10}$$

which says that the color potential between two color sources is a linear potential at large distances. This is a major feature of a confinement potential. We know that, the theory of strong interaction, namely QCD, is a theory with confinement. That is to say, a quark and anti-quark pair is always confined together to form a color singlet by the potential. Lattice QCD thus provides us an easy explanation for the linear potential between two color sources.

5.3 Strong coupling expansion

Strong coupling expansions refers to the limit where $\beta_g \to 0$. In this limit, one could expand the exponential of the Wilson action:

$$Z = \int DU \prod_p (1 - \beta_g \chi(U_p) + \cdots) \ , \tag{5.11}$$

where χ is the character function of the representation. To proceed, we recall the orthonormal properties of the character function on a compact group. Let us consider an irreducible unitary representation r of the group $SU(n)$ with dimension d_r. The representation matrices are denoted as $\mathcal{D}^r_{\alpha\beta}(U)$. We further normalize the Haar measure such that $\int DU = 1$. Then the representation satisfies the following:

$$\begin{aligned} &\int DU \mathcal{D}^r_{\alpha\beta}(U) \mathcal{D}^{s*}_{\gamma\delta}(U) = \delta_{rs}\delta_{\alpha\gamma}\delta_{\beta\delta}/d_r \ , \\ &\sum_{r\alpha\beta} d_r \mathcal{D}^r_{\alpha\beta}(U) \mathcal{D}^{r*}_{\alpha\beta}(V) = \delta(U,V) \ , \end{aligned} \tag{5.12}$$

where $\delta(U,V)$ is the δ-function on the group manifold. In terms of the character function,

$$\begin{aligned} &\int DU \chi_r(U) \chi^*_s(U) = \delta_{rs} \ , \ \sum_r d_r \chi_r(UV^{-1}) = \delta(U,V) \ , \\ &d_r \int DU \chi_r(U) \chi^*_s(U^{-1}V) = \delta_{rs}\chi_r(V) \ . \end{aligned} \tag{5.13}$$

These relations are then used to expand the exponential of the action. The key issue is to realize that:

$$\int DU f(U) = 0 \ , \tag{5.14}$$

unless the function $f(U)$ contains the trivial representation of the group. It is then clear that, in the strong coupling expansion of the partition function,

there is no term proportional to β_g. The lowest order contribution comes from β_g^2 where two contributing plaquettes has to be at the same location but the orientation is just opposite. In this case, the contribution is:

$$\begin{aligned} Z &\sim 1 - \frac{\beta_g^2}{2} \sum_p \int dU_p \chi(U_p)\chi(U_p^{-1}) \ , \\ &= 1 - \frac{\beta_g^2}{2} V D(D-1)/2 \ , \end{aligned} \tag{5.15}$$

where V is the total number of sites for the D-dimensional lattice.

Another interesting quantity is the average of the Wilson loop.

$$< W(R,T) >= \frac{1}{Z} \int DU W(R,T) e^{-S_g[U]} \ . \tag{5.16}$$

For the same reason as given above, the first nontrivial contribution comes from a term proportional to β_g^{RT} where $R \times T$ plaquettes exactly "tiles" the Wilson loop of size $R \times T$. It is therefore seen from strong coupling expansion that, the potential energy $V(R) \propto R$, offering a linear confining potential for two color sources.

The terms that enter the strong coupling expansion can be represented by diagrams with certain rules. Using diagrammatic techniques and integration rules for the character function, people have calculated the free energy and average of Wilson loops to rather high orders. Strong coupling expansions were extensively used in the early stage of lattice gauge theory when the capabilities of computers were still too limited. Along with the fast development of modern computers, Monte Carlo simulation have gradually become the major tool to study thermodynamic properties of the lattice gauge systems. In the regime where strong coupling expansion is valid, complete agreement between the two methods are found.

Chapter 6

Continuum Limit and Renormalization

In this chapter, we will discuss the issue of the continuum limit and renormalization, which in terms of statistical physics, the critical phenomenon. We will use lattice scalar field models, also called spin models, for our discussion although the main idea is certainly not restricted to spin models. Before going into the details of the discussion, I hope that the audience memorize the following point which I will try to convince you in this chapter:

- Renormalization is not a mathematical trick. It is completely physical.
- A renormalizable field theory is not a theory without a cutoff. Rather, it is a low-energy effective theory of a cutoff theory.

For a very good review, please refer to [36], where a large part of this chapter originates.

6.1 Continuum limit and critical phenomenon

The reason to study the continuum limit of a lattice model is quite obvious. We think that we are living in a continuous world, though this is not guaranteed to be true; at least, has not proved to be true. For a free lattice model, taking the continuum limit is trivial. It basically amounts to taking all the mass parameters in the theory to zero in lattice units, this is also called the naive continuum limit. For a free lattice theory, the continuum limit and the naive continuum limit coincide. However, the continuum limit of an interacting theory is not just an ordinary limit of certain parameters. It involves a mysterious procedure called renormalization. Imagine that an interacting lattice theory is parameterized by a set of bare parameters $\{c_i\}$. These parameters span a parameter space of certain dimension $\mathcal{D}$. What happens is that, when the bare parameters of the theory are approaching a lower dimensional subspace in the parameter space,

the interacting theory becomes singular, that is some correlation functions are diverging. For example, the basic mass scale in the theory is characterized by the correlation length ξ, which is defined to be the two-point correlation function at large distances, or equivalently, in Fourier space at small momentum:

$$< \tilde{\phi}(-p)\tilde{\phi}(p) > \sim (p^2 + (1/\xi)^2)^{-1} \quad . \tag{6.1}$$

The mass of the particle is related to the correlation length, or Compton wavelength, by $m = 1/\xi$. The continuum limit then is characterized by the limit that $\xi \to \infty$. In this limit, correlation functions in the interacting theory become divergent which is known as the ultra-violet divergence.

In the terminology of statistical physics, a statistical model is said to approach its critical regime where physical quantities like the specific heat, susceptibility are divergent. The subspace with lower dimension is called the critical surface of the theory. The way that physical quantities become divergent as the bare parameters of the theory approach the critical surface is characterized by the critical indices. All of this is called the critical phenomenon in statistical physics. Therefore, for a lattice model, continuum limit and the critical phenomenon is really the same process in which renormalization plays an essential role. For a particle physicist, he prefers to talk about continuum limit since he has a continuous world in mind and considers the lattice as an outside intruder which he likes to eliminate at last. For a statistical physicist, who is working with models on real lattices anyway, lattice is not a problem. Therefore he prefers to talk about critical phenomenon since it is an amusing and interesting regime for his model.

Intuitively, it is easy to visualize that, near the critical regime, the lattice effects indeed goes away and one is left with a continuum theory. We know that near the critical region the size (i.e. the Compton wavelength) of physical particles are large in lattice units. This means that each particle in the theory spans a space which is a large number of lattice spacing wide. Physical properties for such a "fat" particle becomes insensitive to the details of the lattice at the scale of one lattice spacing. Therefore all effects of the underlying lattice is going to zero. What is left is only the long-wavelength (or infrared, or low-energy) properties of the theory. This feature is called universality in the terminology of critical phenomenon. Universality guarantees that, in the critical region, the theory can be parametrized by only very few parameters, dictated by the symmetry and dimensionality of the model, no matter how the model might look like in the non-critical region. For particle physicists, universality explains why low-energy physics are described by the so-called renormalizable field theories, though at high energies the theory might be something that we do not know. We should therefore abandon the idea that quantum field theories are supposed to work till infinite energies. We accept that, these theories are only effective low-energy theories with a physical cutoff which we could take as the hypercubic lattice. Universality tells us that, some other valid regulators will also do the job and the low-energy physics does not depend on the details of the regulator.

6.2 Block-spin: physical view of renormalization

6.2.1 Block-spin transformations

We will consider a simple lattice Gaussian spin model on a D-dimensional infinite hyper-cubic lattice. We label the lattice sites by $x^{(0)}$, which is a D-dimensional lattice coordinate, each component of which being an integer ranging from $-\infty$ to $+\infty$. The original lattice fields, or spin fields in statistical physics language, are defined on these lattice sites and are labeled as: $\phi^{(0)}_{x^{(0)}}$. The Euclidean action of the model is given by:

$$S_E = \frac{1}{2} \sum_{x^{(0)} y^{(0)}} \phi^{(0)}_{x^{(0)}} K^{(0)}_{x^{(0)} y^{(0)}} \phi^{(0)}_{y^{(0)}} \quad , \tag{6.2}$$

where the kernel $K_{x^{(0)} y^{(0)}}$ is translational invariant such that:

$$K_{x^{(0)} y^{(0)}} = \int \frac{d^D p}{(2\pi)^D} e^{ip(x^{(0)} - y^{(0)})} \tilde{K}^{(0)}(p) \quad , \tag{6.3}$$

where the D-dimensional momentum p is restricted in the first Brillouin zone in D dimensions. We will also assume that the Fourier transformed kernel has an expansion in terms of the momentum when the momentum is small:

$$\tilde{K}^{(0)}(p) \sim c^{(0)}_0 + c^{(0)}_1 p^2 + c^{(0)}_2 p^4 + \cdots \quad . \tag{6.4}$$

In momentum space, the action of the model can be written as

$$S_E = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \tilde{K}^{(0)}_q |\tilde{\phi}^{(0)}_q|^2 \quad , \tag{6.5}$$

The model is thus parametrized by a set of bare parameters $\mathbf{c}^{(0)} \equiv (c^{(0)}_0, c^{(0)}_1, \cdots)$ which can be viewed as a point in an infinite dimensional parameter space spanned by the $c^{(0)}_i$'s. When these parameters are properly adjusted to approach the critical surface, the theory is driven to the critical regime where the correlation length becomes divergent. All relevant physical information is contained in the partition function of the system which is given by:

$$Z = \int \mathcal{D}\phi^{(0)} e^{-S_E[\phi^{(0)}]} = \int \mathcal{D}\tilde{\phi}^{(0)} e^{-S_E[\tilde{\phi}^{(0)}]} \quad , \tag{6.6}$$

where the measure $\mathcal{D}\phi^{(0)} = \prod_x d\phi^{(0)}_{x^{(0)}}$. Since we are interested in physics at the physical length scale ξ which is much larger than the lattice spacing, we could wash out the details of the fields at shorter length scales by defining an averaged field, or blocked field. In the mean time, we would like to keep "physics" unchanged, i.e. the partition function (6.6) of the system unchanged.

We would now like to introduce a blocked field $\tilde{\phi}_q^{(1)}$ in Fourier space. To do this, we notice the following identity:[1]

$$\int \mathcal{D}\tilde{\phi}^{(1)} \exp\left[-\frac{a}{2}|\tilde{\phi}_q^{(1)} - b\tilde{\phi}_{q/\lambda}^{(0)}|^2\right] = 1 \quad , \tag{6.7}$$

where a and b are two free parameters to be adjusted later on. The scaling parameter λ is a positive real number that is greater than unity. The integration measure in the above formula is defined to be: $\mathcal{D}\phi^{(1)} = \prod_q d\tilde{\phi}_q^{(1)}$. Inserting eq. (6.7) into eq. (6.6) and interchanging the integral of $\tilde{\phi}^{(1)}$ and $\tilde{\phi}^{(0)}$, we obtain the following:

$$Z = \int \mathcal{D}\tilde{\phi}^{(1)} e^{-S_E^{(1)}[\tilde{\phi}^{(1)}]} \quad . \tag{6.8}$$

Therefore, we have expressed the same partition function as a path integral over the blocked fields $\tilde{\phi}_q^{(1)}$ with a modified action $S_E^{(1)}[\tilde{\phi}^{(1)}]$. Such a transformation is called a block-spin transformation, or renormalization group transformation. In this special case, our block-spin transformation is called a Gaussian block-spin transformation with parameters a, b and scale λ, denoted as $\mathcal{G}_\lambda(a,b)$. Other types of block-spin transformations could also be defined. Although the forms of the transformations might be different, the physical idea behind them are all the same. They all average out the high-momentum modes of the fields in some way, leaving physics, or the partition function, of the model unchanged. For our Gaussian model, the form of the transformed action $S_E^{(1)}[\phi^{(1)}]$ can be expressed in a compact form. It remains bilinear in the blocked fields:

$$\begin{aligned} S_E^{(1)} &= \frac{1}{2}\int \frac{d^D q}{(2\pi)^D} \tilde{K}_q^{(1)} |\tilde{\phi}_q^{(1)}|^2 \quad , \\ \tilde{K}^{(1)}(q) &= \frac{a\tilde{K}_{q/\lambda}^{(0)}}{ab^2\lambda^D + a\tilde{K}_{q/\lambda}^{(0)}} \quad . \end{aligned} \tag{6.9}$$

The new kernel $\tilde{K}_q^{(1)}$ now admits a similar low-momentum expansion as the original one except that the expansion coefficients are modified from $\mathbf{c}^{(0)}$ to $\mathbf{c}^{(1)}$.

Since the form of the partition function remains similar, we could carry out the block-spin transformation iteratively, each time replacing the original parameters by the new transformed ones. Such a sequence of block-spin transformation can be parametrized by a flow of points in the parameter space of the theory: $(\mathbf{c}^{(0)} \to \mathbf{c}^{(1)} \to \mathbf{c}^{(2)} \cdots)$. This flow is called a renormalization group flow of the theory and the trajectory of the flowing points is called a renormalization group trajectory. Note that, after each renormalization group transformation, the high-energy behavior of the theory is integrated out. The theory is then

[1] In fact, the result of the integral is not really unity but yields a constant which depends on a. However, since it is independent of the fields, it is irrelevant and can be viewed as unity.

driven to its infra-red limit as the renormalization group transformation is performed. The effect of this renormalization group transformation is merely a running of the parameters.

Let us now digress a bit to explain why the block-spin transformation is called renormalization group. In fact, it is easy to show that the transformation in eq. (6.7) forms a SIMD-group structure. That is to say,

$$\mathcal{G}(a_k, b_k) \cdot \mathcal{G}(a_l, b_l) = \mathcal{G}(a_{k+l}, b_{k+l}) \quad , \tag{6.10}$$

with the following identifications:

$$a_k = a(1 - b^2\lambda^D)/(1 - (b^2\lambda^D)^k) \quad , \quad b_k = b^k \quad . \tag{6.11}$$

EXCERCISE **6.1** *Verify this result.*

Using the semi-group property of the transformation, we obtain the Fourier transformed kernel $\tilde{K}^{(k)}(p)$ after k successive renormalization group transformations as:

$$\tilde{K}^{(k)}(p) = \frac{a_k \tilde{K}^{(0)}_{q/\lambda^k}}{a_k (b^2\lambda^D)^k + a_k \tilde{K}^{(0)}_{q/\lambda^k}} \quad , \tag{6.12}$$

EXCERCISE **6.2** *Verify this result.*

6.2.2 Fixed point solutions

It is now interesting to investigate the limit $k \to \infty$, i.e. the resulting action after infinite number of RG interactions. In this case, the renormalization group flow is driven towards the fixed points of the given renormalization group transformation. These are the special points in parameter space which are invariant under a renormalization group transformation. The corresponding kernel, denote as $\tilde{K}^*_q$ can be specified by the following equation:

$$\tilde{K}^*_q = \frac{a\tilde{K}^*_{q/\lambda}}{ab^2\lambda^D + a\tilde{K}^*_{q/\lambda}} \quad . \tag{6.13}$$

Assuming an expansion of the form:

$$\tilde{K}^*_q \sim c^*_0 + c^*_1 q^2 + c^*_2 q^4 + \cdots \quad , \tag{6.14}$$

we obtain the following relation for the subsequent expansion coefficients:

$$\begin{aligned} ac^*_0 &= c^*_0(ab^2\lambda^D + c^*_0) \quad , \\ ac^*_1/\lambda^2 &= c^*_1(ab^2\lambda^D + c^*_0) + c^*_0 c^*_1/\lambda^2 \quad , \\ ac^*_2/\lambda^4 &= c^*_2(ab^2\lambda^D + c^*_0) + c^*_0 c^*_2/\lambda^4 + (c^*_1)^2/\lambda^2 \quad , \\ &\cdots \end{aligned} \tag{6.15}$$

The first line of these equations solves c^*_0, which is then substituted into the second one to solve for c^*_1 etc.

- Case $c_0^* \neq 0$:

  In this case, we obtain: $c_0^* = a(1 - b^2\lambda^D)$. Substitute into the second one we get either $c_1^* = 0$ or $c_1^* = z$ being arbitrary but $b^2 = \lambda^{2-D}$. When c_1^* is non-zero, we obtain $c_2^* = z^2/(a(1-\lambda^2))$. When c_1^* is zero, we obtain $c_2^* = 0$ or $c_2^* \neq 0$ but $b^2 = \lambda^{4-D}$. To summarize, we obtained: $\mathbf{c}^* = (a(1-b^2\lambda^D), 0, 0, \cdots)$, $\mathbf{c}^* = (a(1-b^2\lambda^D), 0, c_2^*, \cdots)$ with $b^2 = \lambda^{4-D}$, $\mathbf{c}^* = (a(1-b^2\lambda^D), z, z^2/[a(1-\lambda^2)], \cdots)$ with $b^2 = \lambda^{2-D}$.

- Case $c_0^* = 0$:

  We get the following solutions: $\mathbf{c}^* = (0, 0, 0, \cdots)$, $\mathbf{c}^* = (0, 0, c_2^*, \cdots)$ with $b^2 = \lambda^{-D-4}$, $\mathbf{c}^* = (0, z, -z^2/[a(1-\lambda^{-2})], \cdots)$ with $b^2 = \lambda^{-D-2}$.

Not all of the above fixed points have physical significance on the continuum limit of the theory. The one that is of vital importance is given by: $\mathbf{c}^* = (0, z, -z^2/[a(1-\lambda^{-2})], \cdots)$ with $b^2 = \lambda^{-D-2}$. We now take the limit $a \to \infty$, which corresponds to the case of so-called δ-function blocking scheme. Therefore, $\mathbf{c}^* = (0, z, 0, \cdots)$ and the theory corresponds to a massless scalar field theory in D-dimensional Euclidean space. Consider a point that is close to this fixed point: $\mathbf{c} = \mathbf{c}^* + \delta\mathbf{c} = (\delta c_0, z + \delta c_1, \delta c_2, \cdots)$, with $\delta\mathbf{c}$ being small. Also, setting $\lambda = e^{\delta t}$, then the infinitesimal renormalization group transformation near the fixed point is written as:

$$\frac{d}{dt}\mathbf{c}(t) = L^* \cdot \mathbf{c}(t) \quad , \quad \mathbf{c}(0) = \mathbf{c}^* \quad , \tag{6.16}$$

where the matrix L^* in this case is a diagonal matrix i.e. $L^* = diag(2, 0, -2, \cdots)$. Equation (6.16) is in fact a linearized version of the renormalization group transformation near a fixed point. In general, the matrix L^* is not diagonal. However, we will assume that taking a suitable linear combination of the c_i's, it can be brought into a diagonal form with corresponding eigenvalues. The operators that corresponds to these eigen-coupling vectors are categorized according to the sign of their eigenvalues near the fixed point. An eigen operator is said to be *relevant* (*irrelevant*) if the corresponding eigenvalue is greater (less) than zero. If the eigenvalue vanishes, the eigen operator is said to be *marginal*. In this terminology, we see that, near the fixed point $\mathbf{c}^* = (0, z, 0, \cdots)$, the operator ϕ^2 is relevant, the operator $\phi\partial^2\phi$ is marginal[2] and the operator $\phi\partial^4\phi$ is irrelevant. For relevant operators, the coupling corresponding to it is growing exponentially with "time" t, when successive renormalization group transformation is carried out. Note that this parameter is nothing but proportional to the mass squared. Or equivalently in terms of correlation length, $\xi_t = e^{-t}\xi_0$. For irrelevant operators, the corresponding coupling is exponentially damped along a renormalization group trajectory. Therefore, no matter what finite values of coupling it started with, after infinite RG transformations, the effective coupling

[2] Note that, by redefining the field variables, one of the couplings in $\mathbf{c}$ can be absorbed and becomes redundant. For this case, we choose the convention such that $z = 1$. Therefore, the theory is really described by $(c_0, c_2, c_4, \cdots)$. So, the coupling corresponding to the operator $\phi\partial^2\phi$ is fixed.

associated with it is going to zero. We say that the fixed point $\mathbf{c} = (0, 1, 0, \cdots)$ has one relevant direction, which is associated with the relevant operator ϕ^2 and infinitely many irrelevant directions corresponding to the infinitely many operators: $\phi\partial^4\phi$, $\phi\partial^6\phi$ etc.

6.2.3 Topology of renormalization group transformations

In this part, we would like to elaborate some of the general topological features of renormalization group transformations. Let us consider a Euclidean field theory with short-ranged interactions and certain symmetry. The Euclidean action of the theory is parametrized by a set of bare parameters which we take to be a point $\mathbf{x}$ in the parameter space $\mathbf{S}$. The dimensionality of $\mathbf{S}$ could be infinite. We then perform a renormalization group transformation $\mathcal{G}_t$ upon all points in $\mathbf{S}$ and denote the resulting space $\mathbf{S}_t$ where t specifies the scaling factor of the RG transformation $\mathcal{G}_t$ with scale factor $\lambda = e^t$.. That is to say: $\mathbf{S}_t = \{\mathbf{x}_t \equiv \mathcal{G}_t \star \mathbf{x} : \ \mathbf{x} \in \mathbf{S}\}$. The flow $\mathbf{x}_t$ as a function of t defines a renormalization group trajectory in space $\mathbf{S}$, along which physics is unchanged and the dimensionless correlation length ξ_t is related to the original one by:

$$\xi_t = e^{-t}\xi_0 \quad , \tag{6.17}$$

as is seen from our previous examples. A critical surface $\mathbf{S}_c$ is defined to be the set of all points in $\mathbf{S}$ at which $\xi = \infty$. It is easy to see that, the sets $\mathbf{S}_t$ are nested, that is $\mathbf{S}_{t+t'} \subset \mathbf{S}_t$ for $t, t' > 0$. Sets $\mathbf{S}_t$ have the same dimensionality as the original $\mathbf{S}_0$. However, the limiting set $\mathbf{S}_\infty$, defined to the set $\mathbf{S}_t$ as $t \to \infty$, is expected to have a dimensionality that is a lot less than that of $\mathbf{S}$. A fixed point $\mathbf{x}^*$ in $\mathbf{S}$ is a point such that a renormalization group transformation leaves it invariant i.e. $\mathcal{G}_t \star \mathbf{x}^* = \mathbf{x}^*$. From eq. (6.17), it is clear that, at a fixed point the correlation length has to be either 0 or ∞. Obviously, renormalization group flow lines never intercects except at fixed points.

We will now consider the simplest case of topology in which there are only two fixed points in $\mathbf{S}$, one with correlation length 0, denoted by P_0; one with correlation length ∞ denoted by P_∞. For simplicity, we consider $\mathbf{S}$ to be a two-dimensional space and all renormalization group flows run into the two fixed points. It is clear that, any starting point in $\mathbf{S}$ with a finite correlation length will run into P_0 and points with infinite correlation length, i.e. points on the critical surface $\mathbf{S}_c$, which is now a one-dimensional line, will run into P_∞. It is then clear that the topology has to be the following. In $\mathbf{S}$, P_0 is the absolute stable fixed point and P_∞ is a saddle point. There is a line, denoted by G, running from P_∞ to P_0. This line segment G is of great physical importance and is called the renormalized trajectory. A little thought will then lead to the conclusion that G is nothing but the set $\mathbf{S}_\infty$ defined earlier. The importance of the renormalized trajectory G is that: any point on the renormalized trajectory defines a continuum theory with no lattice (cutoff) effects. In other words, any point $\mathbf{x} \in G$ defines a so-called perfect lattice action of the theory.

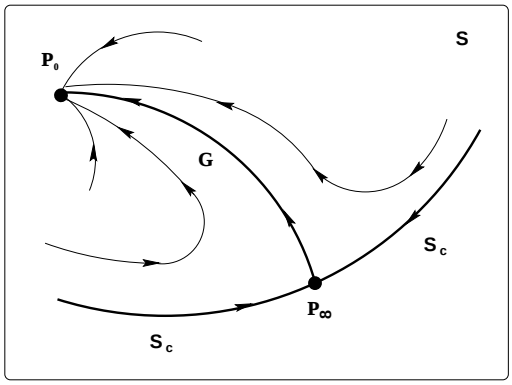

Figure 6.1: The topology of the renormalization group flow in parameter space **S** is shown schematically. Only two fixed points, one with infinite correlation length and one with zero correlation length, are displayed. All renormalization group flows run into these two fixed points.

6.3 Summary

Following Wilson, we have abandoned the idea that quantum field theories are working till zero distances or infinite energy. Quantum field theories are theories with a finite, though might be large, energy-momentum cutoff Λ, which could be taken as an underlying lattice, or something else like a sharp momentum cutoff. The Lagrangian of the theory could contain infinitely many operators that are consistent with the symmetry of the theory. One has no right, at priori, not to keep an operator that is consistent with the symmetry involved. Since the theory contains too many couplings, it obviously looses its predictive power at the energy scale close to Λ.

However, suppose we are not interested in physics at a scale close to Λ, but at some other scale $\Lambda_R << \Lambda$. The low-energy behavior of the theory is obtained by integrating out the modes with high-momentum. This procedure has to be done in a way that physics remain unchanged. Wilson's renormalization group idea tells us that, upon doing this, the original coulpings of the theory are changed to another set of coulpings as one is moving down to lower and lower energies. In other words, these couplings forms a flow in the parameter space of the theory as one progressively integrating out the high-momentum modes. What is remarkable about renormalization is that, after performing the RG transformation over over again, one finds that the all the renormalization group trajectories flows to a subspace which is low-dimensional, say $\mathcal{D}$-dimensional, not like the original parameter space which is usually infinite-dimensional. Therefore, no matter where your starting point in the parameter space is, you are driven down to a $\mathcal{D}$-dimensional space as far as the low-energy physics is concerned. One ends up with a theory that could be parameterized by $\mathcal{D}$ couplings and the theory indeed has predictive power. The other couplings which corre-

spond to irrelevant operators have died out in the process of renormalization. That is to say, the couplings of these irrelevant operators have no effects on the low-energy physics of the theory. What are the operators that irrelevant? The answer is simple, all those that are power-counting non-renormalizable operators. We therefore see that, the low-energy physics is bound to be described by a renormalizable field theory with power-counting renormalizable couplings. These renormalizable couplings are determined totally by the properties of the fixed points of the RG transformation and universality holds in the low-energy regime.

Further reading

Physical view of renormalization was first suggested by K. G. Wilson and a good review can be found in [36]. Several field theory textbooks also contains a systematic description of the material [1, 37]. They also contain discussions on subjects that we have not even touched like: renormalization group equations, calculation of critical exponents, finite size scaling and many others. Should the reader be interested in these topics, please consult these relevant textbooks.

Chapter 7

Monte Carlo Simulations

7.1 Importance sampling and Markov processes

In the previous chapters, we have understood that a lattice quantum field system is fully determined by the partition function of the system. For scalar field system or gauge field systems, we write the partition function as:[1]

$$\begin{aligned} Z &= \int \mathcal{D}\phi e^{-S[\phi]} \ , \\ < \mathcal{O}[\phi] > &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}[\phi] e^{-S[\phi]} \ , \end{aligned} \tag{7.1}$$

where we have symbolically denoted the fields by ϕ. The continuum limit of the theory is obtained as the low-energy limit of lattice theory, or equivalently by driving the system to criticality. In this way, a field theory is completely defined non-perturbatively. Of course, one could perform perturbative expansions of theory. However, many aspects of the theory have to be studied using pure non-perturbative methods. Since the partition function is nothing but a multi-dimensional integral, one could imagine that we just compute this integral numerically using computers. However, as I will show you immediately, this multi-dimensional integral usually cannot be evaluated using conventional numerical methods.

Imagine a Ising spin system in 4 dimensions whose action is

$$S[\{\sigma_x\}] = -J \sum_{<xy>} \sigma_x \sigma_y \ , \tag{7.2}$$

where the summation is over all nearest neighbor pairs $< xy >$ on the lattice and the Ising field σ_x takes discrete values of $+1$ and -1. The size of this

[1] Fermion systems involves integration over Grassman variables and can not be evaluated numerically directly. We will deal with this in section 3 of this chapter.

4-dimensional lattice is $V = L^4$. The partition function of the system is:

$$Z = \sum_{\{\sigma_x\}} e^{-S[\{\sigma_x\}]} \quad . \tag{7.3}$$

Performing the summation numerically requires a sum of 2^V terms. Let us make a conservative estimate of this number. Suppose that we are not too greedy and only works on a lattice with $L = 4$. Then, the number of terms to be summed over is $2^{256} \sim 10^{80}$. Assuming the fastest computer in the world can perform 10^{12} summations in a second. Then this numerical summation will cost about 10^{68} seconds, to be compared with the age of the universe 10^{17} seconds! It is therefore hopeless to evaluate the partition function using a conventional numerical summation.

The way out of this is to use a method called *importance sampling*. We know that, exactly summing up all terms in the partition function is hopeless. However, note that a lot of the terms in the summation really contributes little to the partition function. What really contribute most part of the partition function are those configurations which are near the minimum of the action. The configurations which are far away from the minimum configuration will be exponentially suppressed relative to the ones close to the minimum configuration. Therefore, to numerically evaluate the partition function, one will not sample the whole configuration space. One only tries to sample the part of the configuration space that makes the most important contributions to the partition function. This is the idea of importance sampling.

One good way to implement the idea of importance sampling is to use Monte Carlo methods. Note that the average of any operator can be viewed as an average with a probability distribution $\exp(-S[\phi])$. Once the probability distribution is correctly generated, we can just take samples from this probability distribution and average over different samples. The Boltzmann distribution can be generated by a Markov process. Suppose one starts with an arbitrary probability distribution $P^{(0)}[\{\phi_x^{(0)}\}]$. A probability transition amplitude $W[\{\phi_x^{(0)}\} \to \{\phi^{(1)}\}]$ is defined and the system is updated. The resulting probability distribution becomes:

$$P^{(1)}[\{\phi_x^{(1)}\}] = \int \mathcal{D}\phi^{(0)} W[\{\phi_x^{(0)}\} \to \{\phi^{(1)}\}] P^{(0)}[\{\phi_x^{(0)}\}] \quad . \tag{7.4}$$

This process is called a Markov process. It can be iterated and the desired probability distribution, namely a Boltzmann distribution proportional to $\exp(-S)$, has to satisfy the stationary condition:

$$\exp(-S[\{\phi_x^{(1)}\}]) = \int \mathcal{D}\phi^{(0)} W[\{\phi_x^{(0)}\} \to \{\phi^{(1)}\}] \exp(-S[\{\phi_x^{(0)}\}]) \quad . \tag{7.5}$$

Therefore, one has to choose the function $W[\{\phi_x^{(0)}\} \to \{\phi^{(1)}\}]$ such that the stability condition (7.5) is satisfied. We further assume that, when iterated long enough, the Markov chain converges to the desired distribution:

$$\lim_{\tau \to \infty} P^{(\tau)}[\{\phi_x\}] = \exp(-S[\{\phi_x\}]) \quad . \tag{7.6}$$

Any probability transition function $W[\{\phi_x^{(0)}\} \to \{\phi^{(1)}\}]$ satisfying the stability condition (7.5) and eq (7.6) is called an *exact algorithm*. Monte Carlo simulations are based on these exact algorithms which generates the desired Boltzmann probability distribution. Another related condition is the so-called detailed balance condition given by:

$$W[\{\phi_x^{(1)}\} \to \{\phi^{(0)}\}] \exp(-S[\{\phi_x^{(1)}\}]) = W[\{\phi_x^{(0)}\} \to \{\phi^{(1)}\}] \exp(-S[\{\phi_x^{(0)}\}]) \quad . \tag{7.7}$$

Integration over $\phi^{(1)}$ on both sides yields the stability condition (7.5). Therefore, detailed balance condition is sufficient for an exact algorithm, though not necessary.

Numerically, one starts with some arbitrary configuration $\{\phi_x\}$ in the beginning. Then the field configuration is *updated* according to the probability transition function which constitutes an exact algorithm. The field configuration is then updated over and over again. After some Monte Carlo time, the Boltzmann distribution is established. That is to say, further updates of the field configurations can be viewed as samples taken from the desired distribution. This process is called thermalization process. How fast, or how slow, the thermalization process will take depends on the initial conditions and, most importantly, on the concrete algorithm that one uses. Good algorithms usually takes much less Monte Carlo time to finish thermalization process.

Assuming the desired distribution has been established, then the sample average serves as a unbiased estimator for the average value of the operator:

$$\begin{aligned} \bar{\mathcal{O}} &\sim < O >= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O} e^{-S[\phi]} \quad , \\ \bar{\mathcal{O}} &= \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}_i \quad , \end{aligned} \tag{7.8}$$

where N is the sample size and $\mathcal{O}_i$ is the measured value of the operator $\mathcal{O}$ on sample i. The sample variance is defined to be:

$$(\delta \mathcal{O})^2 = \frac{1}{N(N-1)} \sum_{i=1}^{N} (\mathcal{O}_i - \bar{\mathcal{O}})^2 \quad , \tag{7.9}$$

and serves as an unbiased estimator for the error of the quantity $< \mathcal{O} >$.

7.2 Metropolis algorithm

We now illustrate some of the algorithms by a specific example: the lattice ϕ^4 theory with one-component scalar field. The Euclidean lattice action of the theory is given by:

$$S_E[\phi_x] = \sum_x \frac{1}{2} \phi_x \left[-\hat{\partial}_\mu \hat{\partial}_\mu^* + m_0^2 \right] \phi_x + \frac{\lambda_0}{4!} \phi_x^4 \quad . \tag{7.10}$$

The theory is parametrized by two parameters: the bare mass m_0^2 and the bare coupling λ_0. Sometimes, another parametrization of the same theory is used which amounts to a redefinition of the fields and the parameters. The lattice action is given by:

$$S_E[\phi_x] = -2\kappa \sum_{x\mu} \phi_x \phi_{x+\hat{\mu}} + \sum_x \phi_x^2 + \lambda(\phi_x^2 - 1)^2 \quad , \tag{7.11}$$

which is parametrized by two parameters: hopping parameter κ and the coupling λ. Note that, if the parameter λ is set to infinity, one recovers the Ising model. That is to say, Ising model is just a special case of one component ϕ^4 theory.

To simulate this theory, we consider the ancient Metropolis algorithm which amounts to the following probability transition function:

Algorithm 7.1 *Given a field configuration $\{\phi_x\}$, each field is updated by: $\phi_x \to \phi'_x = \phi_x + \delta_x$, where δ_x is a uniformly distributed random variable within the interval $(-\Delta, \Delta)$. The new configuration is accepted/rejected according to the following rule. If the new configuration yields a smaller action, it is accepted. If the new configuration yields a larger action, it is accepted with the probability $\exp(-\Delta S_E)$, where ΔS_E is the change of the action $\Delta S_E = S_E[\phi'] - S_E[\phi]$.*

How does one realize the "accept with a probability $\exp(-\Delta S_E)$"? First, one generates a uniformly distributed random number r in the interval $(0, 1)$. Then, $\exp(-\Delta S_E)$ is compared with r. If $\exp(-\Delta S_E) > r$ the new configuration is accepted otherwise rejected.

EXCERCISE **7.1** *Show that the Metropolis algorithm satisfies the detailed balance condition and therefore offers an exact algorithm.*

Since Metropolis algorithm is an exact algorithm, one could now use it to simulate lattice scalar ϕ^4 theory. So now you need a computer, a stupid equipment that can only do what you told it to do. Computers have only finite memory. Therefore, we can only work on finite lattices and try to take the infinite volume limit later on. This is done by a clever creature such as you, not by the stupid computer which cannot handle infinite stuff. So we will work on a hyper-cubic lattice with size, say 8^4. You will create an array in the memory of the computer to store these fields. For this example, an array of real numbers, or floats in the language of computer, with size 8^4 will be enough, assuming that your computer has enough memory for this. Now you have to implement what you learned from the Metropolis algorithm into the computer. Depending on your capability of writing computer codes and the ability to debug them, this could take as little as a few hours or as long as months. After some painful experiences, let us assume you finally get the right code working in some language you prefer. Executing the program and wait for some time, you get a Markov chain of configurations popping out of the computer regularly. These configurations, assuming that the thermalization process has finished, can now be used to measure physical observables of interests.

7.3 Auto-correlation time and critical slowing down

Metropolis algorithm is an exact algorithm. However, it is not guaranteed that it is a efficient algorithm. This has to do with the fact that, the configurations generated from any Markov process are not really decorrelated. There is a built-in parameter, called the auto-correlation time, which characterizes the number of steps that a Markov chain gets decorrelated. Why is this parameter relevant for our Monte Carlo simulation? The answer is quite simple. We would like to measure physical observables as accurate as possible so as to make some theoretical prediction of the theory. Being a sample mean drawn from a probability distribution, the true error of this measurement is determined by the effective size of the sample. The larger the sample is the more accurate result one gets. The effective size of the sample is given by the decorrelated, or independent number of measurements in the sample, not merely by the size of the sample. Suppose that the size of the sample is given by N. However, these samples are not all decorrelated or independent. Only samples separated by τ_a in the Markov chain is decorrelated. This means that about after every τ_a steps in the Markov chain, one gets an independent, or decorrelated configuration. Therefore, the effective size of the sample is not N but only N/τ_a. The true error for the measurements will vanish roughly according to $1/\sqrt{N/\tau_a}$. Note that, there is no guarantee that τ_a is of order 1. It could be as large as thousands or millions. Efficient algorithms will be an algorithm such that it can generate an independent configuration with a minimum number of computer operations. An poor algorithm will do the opposite. The value of τ_a also depends on the values of other bare parameters of the theory. In other words, it crucially depends on where you are in the parameter space of the theory. Typically, close to the critical region, that is where the continuum limit takes place, auto-correlation time τ_a will grow and, in most cases, it will diverge as the correlation length ξ is approaching infinity. The behavior is given by:

$$\tau_a \sim \xi^{-z} \quad , \tag{7.12}$$

where z is called the dynamical critical index for the algorithm. Usually, the value of z is greater than zero which means that, close to the critical region, the algorithm is slowing down. This is the so-called critical slowing down of the algorithm. It is found that, for Metropolis algorithm, the value of z is about 2. This is a bad news for us, since we are particularly interested in the physics close to the critical line where continuum physics dominates.

How do we cope with this situation? We try to find better algorithms. Good algorithms will give a value of z with is less than that of the Metropolis algorithm. For example, there is an algorithm, called Hybrid Monte Carlo algorithm, which will give a value of $z \sim 1$. Of course, Hybrid Monte Carlo algorithm would probably need more computer operations (and therefore more computer time) to generate one configuration than the Metropolis does. However, since in the critical region Hybrid Monte Carlo has less auto-correlation

time, it will eventually win over the Metropolis algorithm. For the scalar lattice field theory, there exists a state of the art algorithm, called cluster algorithm, which gives a value for the dynamical critical index $z \sim 0$. This is almost the best algorithm that one can get. For the sake of simplicity, cluster algorithm will not be discussed here. Interested readers should consult the relevant references. We will introduce the Hybrid Monte Carlo algorithm, which turns out to be a broadly applicable algorithm even for gauge fields and fermions.

7.4 Hybrid Monte Carlo algorithm

Our aim is to generate a probability distribution which is of the form $\exp(-S[\phi]$. Hybrid Monte Carlo algorithm realizes this by mimicking the molecular dynamics, treating the fields ϕ as coordinates of gas molecules. Therefore, imagine that we introduce a momentum π_x which is conjugate to each field variable ϕ_x. We also introduce the "Hamiltonian" H,[2]

$$H[\pi_x, \phi_x] = \sum_x \pi_x^2 + S[\phi_x] \quad . \tag{7.13}$$

The equation of motion for this classical Hamilton system is:

$$\frac{d}{d\tau}\phi_x = \phi_x \quad , \quad \frac{d}{d\tau}\pi_x = -\frac{\partial S}{\partial \phi_x} \quad . \tag{7.14}$$

Hybrid Monte Carlo algorithm then consists of the following steps:

Algorithm 7.2 *Hybrid Monte Carlo Algorithm:*

1. *For each field variable ϕ_x, generate a momentum variable π_x from the Gaussian distribution $\exp(-\phi_x^2/2)$. These values of the fields and momenta are then taken as initial values for time evolution.*

2. *Let the classical system described by the Hamiltonian (7.13) evolve in Monte Carlo time τ according to the equations (7.14). This is achieved by a numerical integration with the Monte Carlo time discretized. This forms a Hybrid Monte Carlo trajectory. At the end of the trajectory, both momenta and fields are updated to ϕ_x^{new} and π_x^{new}.*

3. *At the end of the trajectory, the new Hamiltonian $H^{new} = H[\pi_x^{new}, \phi_x^{new}]$ is evaluated and the difference with respect to the old Hamiltonian is formed: $\Delta H = H^{new} - H[\pi_x, \phi_x]$. The new configuration is then accepted with a probability $\min(1, \exp(-\Delta H))$.*

[2] This Hamiltonian is not to be confused with the Hamiltonian of the system in the Hilbert space. Also, the momenta π_x is not to be confused with the conjugate momenta in the canonical quantization procedure. Here the Hamiltonian is purely classical and it governs the the classical system with "coordinates" ϕ_x. The classical system is evolving in Monte Carlo time τ, which is not the real time, according to the Hamilton's equation of motion.

4. Go back to Step 1 and regenerate the momenta and iterate the process.

It can be shown that Hybrid Monte Carlo algorithm is an exact algorithm. The third step in the algorithm is called a Metropolis test of new configurations. This step is necessary because, upon numerical integration of the time evolution, the Monte Carlo time is discretized. Therefore, the classical Hamiltonian is not conserved anymore. Without the Metropolis test, the algorithm would not be exact. Hybrid Monte Carlo algorithms is also applicable to gauge fields and fermions. In fact, it is more or less the only valid algorithm for dynamical fermion simulations.

7.5 Triviality of scalar field theories

Scalar field theories have been extensively studied using various methods, both analytically and numerically. These studies basically investigate two important physical quantities.

One important physical observable is the so-called averaged magnetization, or vacuum expectation value (VEV), $<\bar{\phi}>$, defined as:

$$\begin{aligned} <\bar{\phi}> &= \frac{1}{Z}\int \mathcal{D}\phi\bar{\phi}e^{-S[\phi]} \\ \bar{\phi} &= \frac{1}{V}\sum_x \phi_x \ , \\ <\bar{\phi}> &\sim \frac{1}{N}\sum_i \bar{\phi}_i \ , \end{aligned} \tag{7.15}$$

where $\bar{\phi}_i$ is the value of $\bar{\phi}$ measured from the configuration i and N is the total number of configurations you have. Suppose you measured this quantity for several values of κ and a fixed value of λ. You would discover that, the averaged magnetization is very small for κ values that are less than a certain critical value $\kappa_c(\lambda)$, which is a function of λ. Above κ_c the averaged magnetization will rise significantly. Close to the neighborhood of κ_c the lattice system undergoes a phase transition which is related to a symmetry breaking. The symmetry that is broken is the $\phi \to -\phi$ symmetry, or a Z_2 symmetry. If this symmetry were not broken, then it is easy to show that the average magnetization should always vanish. However, for a infinite lattice system, this symmetry could be broken spontaneously if κ is tuned to be larger than κ_c. In the parameter space of the theory, which is a two-dimensional plane with the horizontal and vertical axis labeled by λ and κ respectively, a critical line $\kappa_c(\lambda)$ divides the plane into two parts. Points that are below the critical line $\kappa_c(\lambda)$ correspond to a scalar field theory with a Z_2 symmetry. Points that are above the critical line corresponds to a scalar field theory with a broken symmetry. A picture like this is called a phase diagram of the system. For our example, the phase diagram consists of two phases, one called symmetric phase and the other called broken phase, separated by a second order phase transition line $\kappa_c(\lambda)$. On the critical line

$\kappa_c(\lambda)$, the system undergoes a second order phase transition. The region close to the critical line is the critical regime where the correlation length of the system is diverging and the continuum limit of the theory is obtained.

Another important physical quantity is the physical correlation length itself, which is intimately related to the renormalized mass of the particle.[3] This can be measured by measuring a correlation function of the fields $< \bar{\phi}_t \bar{\phi}_0 >$, where $\bar{\phi}(t) = (\sum_{\mathbf{x}} \phi_{\mathbf{x},t})/V_3$ is the average magnetization on time slice t and V_3 is the 3-volume. In the transfer matrix formalism, this operator creates one particle from the vacuum at Euclidean time 0 and annihilates it at a later time t. Therefore, we know that the correlation function has the following asymptotic behavior:

$$< \bar{\phi}_t \bar{\phi}_0 > \overset{t\to\infty}{\sim} e^{-mt} \ , \tag{7.16}$$

where m is the lowest energy gap from vacuum. Since the operator carries zero momentum, m can thus be identified as the mass of the particle. Higher excited states are more exponentially damped and therefore could be neglected in the large t limit. The goal of the Monte Carlo simulation is then directly measuring this quantity, fitting it to the exponential form above and extracting the renormalized mass of the particle directly. The above analyses could be done in both symmetric phase and broken phase. In particular, in the broken phase where the vacuum expectation value is not vanishing, the ratio of the mass and the vacuum expectation value is directly related to the renormalized coupling constant λ_R.

To understand the continuum limit of the lattice ϕ^4 theory requires a renormalization group study near the critical region. The result in 4-dimensions is quite striking. The theory turns out to have only one relevant direction and two marginal directions in the parameter space. There is only one relevant operator, namely the operator ϕ_x^2, which serves as the mass term of the theory. There are two marginal operators, one is the kinetic term $\phi\partial^2\phi$ and the other one is ϕ_x^4 which is the interaction term. However, the topology of the parameter space is such that it contains no extra fixed points than the free field theory. The result of this is a feature called triviality [24, 7, 9, 16, 17, 12, 19, 22, 21, 20, 23]. That is to say, the renormalization property of the theory is completely governed by the trivial Gaussian fixed point. Therefore, in the continuum limit, the coupling λ_t as a function of the renormalization scale t is going to zero. This means that, the so-called renormalized coupling constant is zero in the low-energy limit of the theory. In fact, if the cutoff of the theory is Λ, then the renormalized coupling at a physical scale μ_R much below Λ is of the form $\lambda(\mu_R) \sim 1/\log(\Lambda/\mu_R)$. If we use the lattice units, then $\lambda(\mu_R) \sim 1/\log(a\mu_R)$. In order to keep the low-energy theory still interacting, a finite cutoff has to remain in the theory. This is the lesson that triviality teaches us. Using this fact, an upper bound of the Higgs mass has been established with the result 700MeV.

[3] The mass of the particle and the correlation length is related via $m = 1/\xi$ in lattice units. Since the Higgs sector of the standard model has exactly the same Lagrangian, the mass of the particle is sometimes also called Higgs mass, with an abuse of language.

7.6 Simulation of fermions

Fermions are more difficult to simulate since our computer can not manipulate Grassman variables directly. Fermions has to be integrated out first, yielding the fermion determinant. Then the fermion determinant is evaluated numerically. Obviously, numerical calculation of the determination of a huge matrix is a much more difficult problem than just computing local functions, which is the case in scalar and pure gauge theories. In practice, simulation of fermions is performed by introducing the so-called pseudo-fermions, which are really bosonic fields. For lattice QCD, the partition function is:

$$\begin{aligned} Z &= \int \mathcal{D}U e^{-S_g[U]} \det \mathcal{D}[U] \\ &= \int \mathcal{D}U \mathcal{D}\phi^\dagger \mathcal{D}\phi e^{-S_g[U] - \phi^\dagger \mathcal{D}^{-1}[U]\phi} \quad . \end{aligned} \tag{7.17}$$

where $\mathcal{D}[U]$ is the fermion matrix of Wilson lattice QCD and $S_g[U]$ is the pure lattice gauge action. We have assumed that the fermion matrix involved here is positive definite so that the introduction of pseudo-fermion fields are legitimate. This system can be simulated using the Hybrid Monte Carlo algorithm. The difference between the fermionic simulation is that, in the process of the HMC update, one has to find the solution vector X for the linear system:

$$\mathcal{D} \cdot X = \phi \quad , \tag{7.18}$$

at each step of the HMC trajectory. This is done by iterative solvers like the conjugate gradient algorithm. It is exactly this step which makes the simulation of fermions at least two to three orders of magnitude slower than systems without fermions.

Further readings

Discussions on algorithm issues have appeared in various books on lattice field theory [8, 33, 25]. For a discussion on the triviality of scalar field theory in four dimensions and its consequences on the upper bound of the Higgs mass, please consult [24, 7, 9, 16, 17, 12, 19, 22, 21, 20, 23].

Chapter 8

Topics in Lattice QCD

8.1 $Q\bar{Q}$ potential

As was pointed out in Chap. (5), the static potential between two infinitely heavy color sources, which are called heavy quark and anti-quark, can be evaluated non-perturbatively using Monte Carlo simulations. To do this, one conceives an exact algorithm for the pure gauge theory. By now, there are two major types, one is called heat-bath with over-relaxation and the other is the Hybrid Monte Carlo algorithm, properly modified for the group manifold. Then, using these algorithms, a Markov chain of gauge field configurations are generated. Rectangle Wilson loops of size $R \times T$, with R and T being the spatial and temporal extent of the loop respectively, can then be measured using these gauge field configurations. These measured loops are then fitted for large T values to offer us the static potential $V(R)$. Typically, the potential can be fitted according to a Coulomb term plus a linear potential:

$$V(R) = V_0 + e_c/R + \sigma R \quad , \tag{8.1}$$

where σ is called the string tension, the physical interpretation of which is the energy per unit length if the color fields are viewed as being confined into a string connecting the two quarks. Obviously, at large distances, the potential is dominated by the linear term and this term is called the confinement term since it prevents the two quarks to be separated apart. This is only the result for the pure gauge theory. In real QCD with quarks, strings will break at roughly the energy scale of $2m_q$ where a $q\bar{q}$ pair is popped up from the vacuum and one ends up with two mesons.

The quark anti-quark potential has been measured in both $SU(2)$ and $SU(3)$ pure gauge theory very accurately. The measurement was so accurate that people can even measure the spin-dependent potential.

8.2 Glueball spectrum

If QCD were the correct theory of strong interactions, it would provide us with a special type of particle, a particle made up of purely gluonic content. This is called a glueball. Of course, in reality, glueballs mix with hadrons that have the same quantum numbers. However, since the degrees of freedom for the gluons are present, there has to be particle excitations associated with it. To predict the mass of the glueball requires an intimate knowledge of the QCD vacuum. Therefore, this can only be done non-perturbatively. Lattice QCD provides us with the most promising methods to deal with this issue.

To measure the mass of the glueball in physical units, one has to do two things. One first has to measure the Wilson loop and the static potential. From this measurement, one can convert the lattice spacing to a physical unit, using either the string tension or some other scale in the potential [34, 11]. Secondly, one constructs the appropriate glueball operators which can create and annihilate glueballs. Then a correlation function is measured and fitted according to the exponential decaying form. The mass of the glueball is then extracted in lattice units. Using now the conversion between the lattice units and physical units obtained from the Wilson loop, one finally obtains the mass of the glueball in physical units, say in MeV. The strategy of the calculation is quite simple. To really carry out this calculation requires a lot of special techniques. For details of this calculation, please consult [27, 26, 2, 5, 3, 4].

8.3 Quenched hadron spectrum

Hadron spectrum is a mystery of non-perturbative QCD. Lattice calculations have offered the most credible result for this theoretically. Hadron spectrums have been calculated in Lattice QCD using the so-called quenched approximation, or valence quark approximation. This approximation amounts to neglecting internal quark loop effects and treat quarks only as external particles. This might sounds like a terrible approximation. However, it bears some of the physical significance. Since part of the effects due to internal quark loops is only a renormalization of bare parameters. This is basic idea of quenched approximation. Despite of this terrible, you might call it, approximation, the outcome of the hadron spectrum is spectacularly good when compared with the experimental result. The mass of the mesons and baryons are within 6% and the decay constants are within 10-20%.

Chapter 9

What's Left out

Obviously, in a short course such as this one, it is impossible to cover all the areas of Lattice Field theory, which now has become a major branch of theoretical physics, covering areas ranging from particle physics, field theory to condensed matter physics and probably many others. The intention of these lectures is not to stuff the audience with all the lattice theories, applications and algorithms. Rather, I do hope to give you a flavor of the subject and hopefully bring up some interest in the field among young physicists, which is much more important than just memorizing the formulae in this lecture. Whenever necessary, the interested reader is directed to relevant references for further reading and thinking of the problem. I am fully aware that, even the partial list below cannot exhaust the merits of Lattice field theory.

In this lecture, I have left out a discussion of improved latticed actions, which has become so popular recently in simulations of various field theories. I have not discussed the problem of chiral fermions in the light of the newly proposed Ginsparg-Wilson fermions. Lattice perturbation theory for gauge fields are completely omitted. The whole glory of renormalization group is only briefly glimpsed through the surface. Things like renormalization group equations, calculations of critical indices, finite size scaling are all left out. In the discussion of Monte Carlo simulations, a comprehensive discussion about stochastic process is omitted. Fokker-Planck equation is not introduced and only every simple algorithms are discussed. In the examples, I have not discussed the running coupling calculations, hadronic matrix element calculations, random surfaces plus a lot more.

Bibliography

[1] Amit, D. J. *Field Theory: the Renormalization Group and Critical Phenomena*. World Scientific, 1984.

[2] Bali, G., et al. *Phys. Lett. B* **309** (1993), 378.

[3] Berg, B., and Billoire, A. *Nucl. Phys. B* **221** (1983), 109.

[4] Berg, B., and Billoire, A. *Nucl. Phys. B* **226** (1983), 405.

[5] Berg, B., and Billoire, A. *Nucl. Phys. B* **314** (1989), 347.

[6] Bjorken, J. D., and Drell, S. D. *Relativistic Quantum Fields*. Mc-Graw Hill, 1965.

[7] Cabibbo, N., Maiani, L., Parisi, G., and Petronzio, R. *Nucl. Phys. B* **158** (1979), 295.

[8] Creutz, M. *Quarks, Gluons and Lattices*. Cambridge University Press, 1983.

[9] Dashen, R., and Neuberger, H. *Phys. Rev. Lett.* **50** (1983), 1897.

[10] Feynman, R. P., and Hibbs, A. R. *Quantum Mechanics and Path Integrals*. Mc-Graw Hill, 1965.

[11] Guagnelli, M., Sommer, R., and Wittig, H. *Nucl. Phys. B* **535** (1998), 389.

[12] Hasenfratz, A., et al. *Nucl. Phys. B* **317** (1989), 81.

[13] Itzykson, C., and Drouffe, J. M. *Statistical Field Theory*. Cambridge University Press, 1989.

[14] Itzykson, C., and Zuber, J. B. *Quantum Field Theory*. McGraw-Hill, 1980.

[15] Kogut, J., and Susskind, L. *Phys. Rev. D* **11** (1975), 395.

[16] Kuti, J., Lin, L., and Shen, Y. *Nucl. Phys. (Proc. Suppl.) B* **4** (1988), 397.

[17] Kuti, J., Lin, L., and Shen, Y. *Phys. Rev. Lett.* **61** (1988), 678.

[18] Lüscher, M. *Commun. Math. Phys.* **54** (1977), 283.

[19] Lüscher, M., and Weisz, P. *Nucl. Phys. B* **290** (1987), 25.

[20] Lüscher, M., and Weisz, P. *Nucl. Phys. B* **300** (1988), 325.

[21] Lüscher, M., and Weisz, P. *Phys. Lett. B* **212** (1988), 472.

[22] Lüscher, M., and Weisz, P. *Nucl. Phys. B* **295** (1988), 65.

[23] Lüscher, M., and Weisz, P. *Nucl. Phys. B* **318** (1989), 705.

[24] Maiani, L., Parisi, G., and Petronzio, R. *Nucl. Phys. B* **136** (1978), 115.

[25] Montvay, I., and Munster, G. *Quantum Fields on a Lattice.* Cambridge University Press, 1994.

[26] Morningstar, C., and Peardon, M. *Phys. Rev. D* **56** (1997), 4043.

[27] Morningstar, C., and Peardon, M. *Phys. Rev. D* **60** (1999), 034509.

[28] Negele, J. W., and Orland, H. *Quantum Many-Particle Systems.* Addison-Wesley, 1988.

[29] Nielsen, H. B., and Ninomiya, M. *Nucl. Phys. B* **185** (1981), 20.

[30] Nielsen, H. B., and Ninomiya, M. *Nucl. Phys. B* **193** (1981), 173.

[31] Nielsen, H. B., and Ninomiya, M. *Phys. Lett. B* **105** (1981), 219.

[32] Ramond, P. *Field Theory: a modern primer.* Addison-Wesley, 1989.

[33] Rebbi, C. *Lattice Gauge Theories and Monte-Carlo Simulations.* World Scientific, Singapore, 1983.

[34] Sommer, R. *Nucl. Phys. B* **411** (1994), 839.

[35] Wilson, K. G. *Phys. Rev. D* **3** (1974), 2445.

[36] Wilson, K. G., and Kogut, J. *Phys. Rep.* **C12** (1974), 75.

[37] Zinn-Justin, J. *Quantum Field Theory and Critical Phenomena.* Clarendon Press, 1996.