David Tong QFT solution manual

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Abstract

A solution to `http://www.damtp.cam.ac.uk/user/tong/qft.html`. There may exist some mistakes, please be careful

To my friend

Drake

1 Sheet 1

1.

$$
\begin{aligned}
L &= \int_0^a dx \left[\frac{\sigma}{2} \left(\sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n(t) \right)^2 - \frac{T}{2} \left(\sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \left(\frac{n\pi}{a} \cos\left(\frac{n\pi x}{a}\right) q_n(t) \right) \right)^2 \right] \\
&= \frac{\sigma}{a} \sum_{mn} \dot{q}_m(t) \dot{q}_n(t) \int_0^a dx \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) - \\
&\frac{T}{a} \sum_{mn} \frac{\pi^2 mn}{a^2} q_m(t) q_n(t) \int_0^a dx \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi x}{a}\right) \\
&= \frac{\sigma}{2} \sum_n \dot{q}_n^2(t) - \frac{T}{2} \sum_n \frac{n^2\pi^2}{a^2} q_n^2(t)
\end{aligned}
\tag{1}
$$

We can determine the equations of motion by the principle of least action. We vary *all* $q(t)$

$$
\begin{aligned}
0 \equiv \delta L &= \sigma \sum_n \dot{q}_n(t) \delta \dot{q}_n(t) - T \sum_n \frac{n^2\pi^2}{a^2} q_n(t) \delta q_n(t) \\
&= \sum_n \left[\sigma \ddot{q}_n(t) - T \frac{n^2\pi^2}{a^2} q_n(t) \right] \delta q_n(t) \\
\rightarrow 0 = \sigma \ddot{q}_n(t) - T \frac{n^2\pi^2}{a^2} q_n(t) &\rightarrow \ddot{q}_n(t) - \omega_n^2 q_n(t) = 0 \qquad \omega_n = \frac{n\pi}{a} \sqrt{\frac{T}{\sigma}}
\end{aligned}
\tag{2}
$$

Now, according to the Fourier's expansion, we transform the functional to a countable summation. In the third line of eq(2), the different $\delta q_n(t)$ are independent, so we can throw the summation away to get the harmonic equation for each q_n mode

2.

$$
\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0 \tag{3}
$$

If we impose the Lorentz transformation $x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$, we can get

$$
x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial \Lambda^\mu_{\ \nu} x^\nu} = \partial_\nu (\Lambda^{-1})^\nu_{\ \mu}
$$

$$
x_\mu \rightarrow x'_\mu = \Lambda_\mu^{\ \nu} x_\nu \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} \rightarrow \frac{\partial}{\partial x'_\mu} = \frac{\partial}{\partial \Lambda_\mu^{\ \nu} x_\nu} = (\Lambda^{-1})_\nu^{\ \mu} \partial^\nu
$$

We can check that

$$
x'^\mu x'_\mu = \Lambda^\mu_{\ \nu} \Lambda_\mu^{\ \sigma} x^\nu x_\sigma = (\Lambda^{-1})^\sigma_{\ \mu} \Lambda^\mu_{\ \nu} x^\nu x_\sigma = x^\nu x_\nu
$$

$$
\partial'_\mu \partial'^\mu = \partial_\nu (\Lambda^{-1})^\nu_{\ \mu} (\Lambda)^\mu_{\ \sigma} \partial^\sigma = \partial_\nu \partial^\nu
$$

We expect the field will be

$$
\phi'(x') = \phi(x) \rightarrow \phi'(\Lambda x) = \phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1} x)
$$

Now, to calculate $(\partial^2 + m^2)\phi(\Lambda^{-1}x) = 0$ We just need $\partial'^2 = \partial^2$. This is done in checking the Lorentz invariance above

3. For $\delta\psi$

$$
\begin{aligned}
0 \equiv \delta \mathcal{L} &= \partial_\mu \psi^* \partial^\mu \delta\psi - m^2 \psi^* \delta\psi - \lambda \psi^{*2} \psi \delta\psi \\
&= \left[-\partial^2 \psi^* - m^2 \psi^* - \lambda \psi^{*2} \psi \right] \delta\psi + \partial^\mu \left(\partial_\mu \psi^* \delta\psi \right) \\
&\rightarrow (\partial_\mu \partial^\mu + m^2)\psi^* + \lambda \psi^{*2} \psi = 0
\end{aligned}
\tag{4}
$$

For $\delta\psi^*$, procedures are similar.

$$(\partial_\mu\partial^\mu + m^2)\psi + \lambda\psi^2\psi^* = 0$$

If we do the infinitesimal transformation, we get

$$\delta\mathcal{L} = -i\alpha\partial_\mu\psi^*\partial^\mu\psi + i\alpha\partial_\mu\psi^*\partial^\mu\psi + i\alpha m^2\psi^*\psi - i\alpha m^2\psi^*\psi - i\alpha\lambda\psi^{*2}\psi^2 + i\alpha\lambda\psi^{*2}\psi^2 = 0$$

So $\delta\mathcal{L}$ is invariant under such a transform. To get the conservation current, we notice the surface term $\partial^\mu\left(\partial_\mu\psi^*\delta\psi\right)$

$$\delta\mathcal{L} = \partial^\mu\left(\partial_\mu\psi^*\delta\psi\right) + \partial^\mu\left(\partial_\mu\psi\delta\psi^*\right) = \alpha\partial_\mu i\left(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi\right)$$

$$j^\mu = i\left(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi\right)$$

To verify it explicitly, we contract it

$$\begin{aligned}\partial_\mu j^\mu &= i\left(\partial_\mu\psi\partial^\mu\psi^* + \psi\partial^2\psi^* - \partial_\mu\psi^*\partial^\mu\psi - \psi^*\partial^2\psi\right)\\ &= i(\psi\partial^2\psi^* - \psi^*\partial^2\psi) = i\left(-\psi\left(m^2\psi^* - \lambda\psi^{*2}\psi\right) + \psi^*\left(m^2\psi - \lambda\psi^2\psi^*\right)\right) = 0\end{aligned} \tag{5}$$

4. It's straightforward to show the field equation for each ϕ_a

$$\partial_\mu\partial^\mu\phi_a + m^2\phi_a = 0$$

$$\begin{aligned}\delta\mathcal{L} &= \frac{\theta}{2}\partial_\mu\epsilon_{abc}n_b\phi_c\partial^\mu\phi_a + \frac{\theta}{2}\partial_\mu\phi_a\partial^\mu\epsilon_{abc}n_b\phi_c - \theta m^2\phi_a\epsilon_{abc}n_b\phi_c\\ &= \theta\partial_\mu\phi_a\left(\epsilon_{abc}n_b\partial^\mu\phi_c\right) - \theta m^2\epsilon_{abc}n_b\phi_a\phi_c\end{aligned} \tag{6}$$

Because ϵ_{abc} is antisymmetric, the symmetric field components contracted with ϵ will vanish To get the Noether current j^μ, we use the surface term $\partial_\mu(\phi_a\epsilon_{abc}n_b\partial^\mu\phi_c)$

$$j^\mu = \phi_a\epsilon_{abc}n_b\partial^\mu\phi_c$$

The conserved charge is

$$Q = \int_{\mathbb{R}^3} d^3x j^0 = \int_{\mathbb{R}^3} d^3x\epsilon_{abc}n_b\phi_a\dot{\phi}_c = \int d^3x\epsilon_{bca}n_b\dot{\phi}_c\phi_a$$

Consider the unit vector takes a specific value, we can get

$$Q_a = \int d^3x\epsilon_{abc}\dot{\phi}_b\phi_c$$

To show this is conserved, we can take the time derivative.

$$\begin{aligned}\frac{dQ_a}{dt} &= \int d^3x\epsilon_{abc}\ddot{\phi}_b\phi_c + \int d^3x\epsilon_{abc}\dot{\phi}_b\dot{\phi}_c\\ &= \int d^3x\epsilon_{abc}(\vec{\nabla}^2\phi_b - m^2\phi_b)\phi_c\\ &= \int d^3x\epsilon_{abc}\phi_c\vec{\nabla}^2\phi_b\\ &= \int d\vec{S}\epsilon_{abc}\phi_c\vec{\nabla}\phi_b + \int d^3x\epsilon_{abc}\vec{\nabla}\phi_b\cdot\vec{\nabla}\phi_c\\ &= 0\end{aligned} \tag{7}$$

5.

$$\begin{aligned} \eta_{\mu\nu}x^\mu x^\nu &= \eta_{\mu\nu}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho x^\sigma x^\rho \\ \eta_{\mu\nu}x^\mu x^\nu &= \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu x^\mu x^\nu \\ \rightarrow \eta_{\mu\nu} &= \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu \end{aligned} \tag{8}$$

If $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, we can plug it in eq(8)

$$\begin{aligned} \eta_{\mu\nu} &= \eta_{\rho\sigma}(\delta^\rho_\mu + \omega^\rho{}_\mu)(\delta^\sigma_\nu + \omega^\sigma{}_\nu) \\ \eta_{\mu\nu} &= \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + o(\omega_{\sigma\mu}\omega^\sigma{}_\nu) \\ \rightarrow 0 &= \omega_{\mu\nu} + \omega_{\nu\mu} \\ \rightarrow \omega_{\mu\nu} &= -\omega_{\nu\mu} \end{aligned} \tag{9}$$

To write the matrix form for $\omega^\mu{}_\nu$, we make use of eq(1.25) given by the lecture

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

If the parameters are infinitesimal, then

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^\mu_\nu + \omega^\mu{}_\nu \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & -v & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \delta^\mu_\nu + \omega^\mu{}_\nu \tag{11}$$

We can get

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \omega^\mu{}_\nu = \begin{pmatrix} 0 & -v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

We may notice the ω is not antisymmetric for boost along x^1 axis. This is because the metric carries a sign difference when we deal with coordinate x^0 and x^1

6. Followed by

$$\phi'(x') = \phi(x)$$

$$\phi'(x') = \phi'((\mathbb{1}+\omega)x) = \phi(x) \rightarrow \phi'((\mathbb{1}-\omega^2)x) = \phi((\mathbb{1}-\omega)x) \rightarrow \phi'(x) \approx \phi((\mathbb{1}-\omega)x) \tag{13}$$

Use Taylor expansion, we get

$$\phi'(x) = \phi(x) - x^\nu\omega^\mu{}_\nu\partial_\mu\phi(x) \tag{14}$$

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu\delta\phi\partial^\mu\phi - m^2\phi\delta\phi \\ &= (-\partial_\mu\partial^\mu\phi - m^2\phi)\delta\phi + \partial_\mu\left(\delta\phi\partial^\mu\phi\right) \\ &= -\partial_\mu\left(x^\nu\omega^\sigma{}_\nu\partial_\sigma\phi\partial^\mu\phi\right) \\ &= -\omega^\sigma{}_\mu\partial_\sigma\phi\partial^\mu\phi - x^\nu\omega^\sigma{}_\nu\partial_\mu\left(\partial_\sigma\phi\partial^\mu\phi\right) \\ &= -x^\nu\omega^\sigma{}_\nu\partial_\mu\left(\partial_\sigma\phi\partial^\mu\phi\right) \\ &= -x^\nu\omega^\sigma{}_\nu\left(\partial_\sigma\partial_\mu\phi\partial^\mu\phi + \partial_\sigma\phi\partial_\mu\partial^\mu\phi\right) \\ &= -\omega^\mu{}_\nu x^\nu\left(\partial_\mu\partial^\rho\phi\partial_\rho\phi - m^2\phi\partial_\mu\phi\right) \end{aligned} \tag{15}$$

while

$$\begin{aligned} -\partial_\mu\left(\omega^\mu{}_\nu x^\nu \mathcal{L}\right) &= -\omega^\mu{}_\nu \delta^\nu_\mu \mathcal{L} - \omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} \\ &= -\omega^\mu{}_\nu x^\nu \partial_\mu \left(\frac{1}{2}\partial^\rho\phi\partial_\rho\phi - \frac{1}{2}m^2\phi^2\right) \\ &= -\omega^\mu{}_\nu x^\nu (\partial_\mu\partial^\rho\phi\partial_\rho\phi - m^2\phi\partial_\mu\phi) \end{aligned} \tag{16}$$

Using Noether theorem and $\delta\mathcal{L} = -\partial_\mu\left(\omega^\mu{}_\nu x^\nu \mathcal{L}\right)$, $\delta\mathcal{L} = -\partial_\mu\left(x^\nu \omega^\sigma{}_\nu \partial_\sigma\phi\partial^\mu\phi\right)$

$$0 = \partial_\mu\left(\omega^\mu{}_\nu x^\nu\mathcal{L} - x^\nu\omega^\sigma{}_\nu\partial_\sigma\phi\partial^\mu\phi\right) = \partial_\mu j^\mu \tag{17}$$

The energy-momentum tensor reads

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{18}$$

while the given conserved current reads

$$j^\mu = -x^\nu\omega^\rho{}_\nu(\partial^\mu\phi\partial_\rho\phi - \delta^\mu_\rho\mathcal{L}) \tag{19}$$

compare to eq(17) we get

$$\begin{aligned} j^\mu &= \omega^\mu{}_\nu x^\nu\mathcal{L} - x^\nu\omega^\sigma{}_\nu\partial_\sigma\phi\partial^\mu\phi \\ &= -x^\nu\omega^\rho{}_\nu\left(\partial_\rho\phi\partial^\mu\phi - \delta^\mu_\rho\mathcal{L}\right) \end{aligned} \tag{20}$$

The charges are given by

$$\int d^3x j^0 = -\int d^3x\omega_{\rho\nu}T^{0\rho}x^\nu \tag{21}$$

We should take $\rho,\nu \neq 0$ to get spatial components, so we use ijknotation

$$\begin{aligned} \int d^3x j^0 &= -\int d^3x\omega_{ij}T^{0i}x^j = -\int d^3x\epsilon_{ijk}\omega_k T^{0i}x^j \\ &= \frac{1}{2}\epsilon_{ijk}\omega_k\int d^3x\left(x^jT^{0i} - x^iT^{0j}\right) \\ &= \frac{1}{2}\epsilon_{ijk}\omega_k\int d^3x\left[\mathcal{J}^{0ij}\right] \end{aligned} \tag{22}$$

choose each ω_k, we can get the conserved current

If we consider the boost component, either $\rho = 0$ or $\nu = 0$, then

$$\int d^3x j^0 = -\int d^3x\omega_{0i}T^{00}x^i - \int d^3x\omega_{i0}T^{0i}x^0 \tag{23}$$

Then, take time derivative and notice the conservation of j^0

$$\begin{aligned} 0 = \frac{d}{dt}\int d^3x j^0 &= -\frac{d}{dt}\int d^3x\omega_{0i}T^{00}x^i - \frac{d}{dt}\int d^3x\omega_{0i}T^{0i}x^0 \\ \frac{d}{dt}\omega_{0i}\int d^3xT^{00}x^i &= -\omega_{0i}\int d^3xT^{0i} - \omega_{0i}t\int d^3x\frac{dT^{0i}}{dt} \\ &= -\omega_{0i}P^i - \omega_{0i}\frac{dP^i}{dt} \end{aligned} \tag{24}$$

But the P^i are conserved, s.t. $\frac{d}{dt}\int d^3xT^{00}x^i = P^i = constant$

If we regard T^{00} as energy, then the position weighted energy means the center of energy travels with momentum P^i

7.

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu\partial_\nu\xi - \partial_\nu\partial_\mu\xi \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \end{aligned} \tag{25}$$

So $\mathcal{L}$ is invariant under gauge transform

Let

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu \qquad A(x) \to A'_\mu(x) = A_\mu(x) + \epsilon^\nu \partial_\nu A_\mu(x)$$

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{2}\delta F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{2}F^{\mu\nu}\left(\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu\right) \\ &= -\frac{1}{2}F^{\mu\nu}\left(\epsilon^\sigma\partial_\mu\partial_\sigma A_\nu - \epsilon^\sigma\partial_\nu\partial_\sigma A_\mu\right) \\ &= \frac{\epsilon^\sigma}{2}(\partial_\mu F^{\mu\nu}\partial_\sigma A_\nu - \partial_\nu F^{\mu\nu}\partial_\sigma A_\mu) - \frac{\epsilon^\sigma}{2}\partial_\mu\left(F^{\mu\nu}\partial_\sigma A_\nu\right) + \frac{\epsilon^\sigma}{2}\partial_\nu\left(F^{\mu\nu}\partial_\sigma A_\mu\right) \\ &= -\frac{\epsilon^\sigma}{2}\partial_\mu\left(F^{\mu\nu}\partial_\sigma A_\nu - F^{\nu\mu}\partial_\sigma A_\nu\right) \\ &= -\epsilon^\sigma\partial_\mu\left(F^{\mu\nu}\partial_\sigma A_\nu\right) \end{aligned} \tag{26}$$

Remind

$$\begin{aligned} \delta\mathcal{L} &= \epsilon^\sigma\partial_\sigma\mathcal{L} \\ 0 &= -\epsilon^\sigma\partial_\mu\left(F^{\mu\nu}\partial_\sigma A_\nu + \delta^\mu_\sigma\mathcal{L}\right) \\ &\to j^\mu_\sigma = -F^{\mu\nu}\partial_\sigma A_\nu - \delta^\mu_\sigma\mathcal{L} \\ \to T^{\mu\sigma} &\equiv -F^{\mu\nu}\partial^\sigma A_\nu - \eta^{\mu\sigma}\mathcal{L} = -\eta^{\mu\rho}F_{\rho\nu}\partial^\sigma A^\nu - \eta^{\mu\sigma}\mathcal{L} \end{aligned} \tag{27}$$

Easy to see this is not symmetric and gauge invariant. If we consider a new tensor given by problem

$$\begin{aligned} \Theta^{\mu\nu} &= -\eta^{\mu\rho}F_{\rho\sigma}\partial^\nu A^\sigma - \eta^{\mu\nu}\mathcal{L} - F^{\rho\mu}\partial_\rho A^\nu \\ &= -\eta^{\mu\rho}F_{\rho\sigma}\partial^\nu A^\sigma - \eta^{\mu\nu}\mathcal{L} - \eta^{\mu\sigma}F_{\rho\sigma}\partial^\rho A^\nu \\ &= -\eta^{\mu\rho}F_{\rho\sigma}\left(F^{\nu\sigma} + \partial^\sigma A^\nu\right) - \eta^{\mu\nu}\mathcal{L} - \eta^{\mu\sigma}F_{\rho\sigma}\partial^\rho A^\nu \\ &= -\eta^{\mu\rho}F_{\rho\sigma}F^{\nu\sigma} - \eta^{\mu\nu}\mathcal{L} + \eta^{\mu\sigma}F_{\sigma\rho}\partial^\rho A^\nu - \eta^{\mu\sigma}F_{\rho\sigma}\partial^\rho A^\nu \\ &= -\eta^{\mu\rho}F_{\rho\sigma}F^{\nu\sigma} - \eta^{\mu\nu}\mathcal{L} \\ Symmetric \to \Theta^{\nu\mu} &= -\eta^{\nu\rho}F_{\rho\sigma}F^{\mu\sigma} - \eta^{\mu\nu}\mathcal{L} = -\eta_{\sigma\rho}F^{\nu\rho}F^{\mu\sigma} - \eta^{\mu\nu}\mathcal{L} \\ &= -\eta^{\mu\sigma}F_{\sigma\rho}F^{\nu\rho} - \eta^{\mu\nu}\mathcal{L} = \Theta^{\mu\nu} \\ Gauge \to \Theta^{\mu\nu} &\text{Gauge invariant because } F \text{ is invariant} \\ Traceless \to \Theta^\mu_\mu &= -F_{\nu\sigma}F^{\nu\sigma} - 4\mathcal{L} = -F_{\nu\sigma}F^{\nu\sigma} + F_{\mu\nu}F^{\mu\nu} = 0 \end{aligned} \tag{28}$$

The four conserved currents are easy to verify by using $\partial_\mu\Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} - \partial_\mu F^{\rho\mu}\partial_\rho A^\nu = 0$ For more details, consult J. D. Jackson's *Classical Electrodynamics* §12.10

8. Rua

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{2}F^{\mu\nu}\left(\partial_\mu\delta C_\nu - \partial_\nu\delta C_\mu\right) + m^2 C_\mu\delta C^\mu \\ &= \partial_\mu F^{\mu\nu}\delta C_\nu + m^2 C^\nu\delta C_\nu - \partial_\mu\left(F^{\mu\nu}\delta C_\nu\right) \end{aligned} \tag{29}$$

Equation of motion reads

$$\partial_\mu F^{\mu\nu} + m^2 C^\nu = 0$$

Then

$$\begin{aligned} &\partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu C^\nu = 0 \\ &\to \partial_\mu C^\mu = 0 \qquad \dot{C}^0 + \partial_i C^i = 0 \end{aligned} \tag{30}$$

We set $\nu = 0$, then

$$\begin{aligned} \ddot{C}^0 - \nabla^2 C^0 + m^2 C^0 = 0 \quad \text{use eq(30)} \\ -\partial_i \dot{C}^i + \partial_i \partial^i C^0 + m^2 C^0 = 0 \end{aligned} \tag{31}$$

We use the Lagrange density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu C_\nu)(\partial^\mu C^\nu) + \frac{1}{2}(\partial_\mu C^\mu)^2 + \frac{1}{2}m^2 C_\mu C^\mu$$

The canonical momentum conjugate to C_i would be

$$\begin{aligned} \Pi_i &= \frac{\partial \mathcal{L}}{\partial(\partial_0 C^i)} = \frac{\partial}{\partial(\partial_0 C^i)}\left(-\frac{1}{2}\partial_0 C_j \partial^0 C^j + \frac{1}{2}\left(\partial_0 C^0 + \partial_j C^j\right)\left(\partial_0 C^0 + \partial_j C^j\right)\right) \\ &= \frac{\partial}{\partial \partial^0 C^i}\left(-\frac{1}{2}\eta_{jk}\partial^0 C^j \partial^0 C^k + \eta_{jk}\partial^0 C^0 \partial^j C^k\right) \\ &= -\partial^0 C_i + \partial_i C^0 \\ \Pi_0 &= \frac{\partial \mathcal{L}}{\partial \dot{C}^0} = 0 \end{aligned} \tag{32}$$

The Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \Pi_i \dot{C}^i - \mathcal{L} = \partial_i C^0 \partial_0 C^i - \dot{C}_i \partial_0 C^i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 C^\mu C_\mu \\ &= \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{jk}F^{jk} + \partial_i C^0 \partial_0 C^i - \dot{C}_i \dot{C}^i \\ &= \frac{1}{2}\left(\dot{C}_i \dot{C}^i - 2\partial_i C_0 \partial^0 C^i + \partial_i C_0 \partial^i C^0\right) + \frac{1}{4}F_{jk}F^{jk} + \partial_i C^0 \partial_0 C^i - \dot{C}_i \dot{C}^i \\ &= \frac{1}{4}F_{jk}F^{jk} - \frac{1}{2}\dot{C}_i \dot{C}^i + \frac{1}{2}\partial_i C_0 \partial^i C^0 \\ &= \frac{1}{4}F_{jk}F^{jk} - \frac{1}{2}\Pi_i \Pi^i + \partial_i C_0 \partial^i C^0 - \partial_i C_0 \dot{C}^i \end{aligned} \tag{33}$$

9. Use the scaling transform

$$x^\mu \to x'^\mu = \lambda x^\mu \quad \phi'(x') = \Lambda(\lambda)\phi(x) \to \phi'(x) = \Lambda(\lambda)\phi(\lambda^{-1}x) \quad \Lambda(\lambda) \equiv \lambda^{-D} \tag{34}$$

For the action, we can get

$$\begin{aligned} &S \to S' = \int d^4x \lambda^4 \left(\frac{1}{2}\lambda^{-2D-2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\lambda^{-2D}\phi^2 - g\lambda^{-pD}\phi^p\right) \\ &\to D = 1 \quad \text{when derivative terms is invariant} \\ &\to p = 4 \quad m = -1 \end{aligned} \tag{35}$$

If the scalar field lives in an (n+1)-dimensional spacetime,

$$\begin{aligned} &S \to S' = \int d^dx \lambda^d \left(\frac{1}{2}\lambda^{-2D-2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\lambda^{-2D}\phi^2 - g\lambda^{-pD}\phi^p\right) \\ &\to D = \frac{d}{2} - 1 \quad \text{when derivative terms is invariant} \\ &\to p = \frac{d}{(\frac{d}{2} - 1)} \quad m = -1 \end{aligned} \tag{36}$$

To derive the conserved current, we do an infinitesimal scale transform

$$\lambda \to 1 + \alpha$$

$$x^\mu \to x'^\mu = \lambda x^\mu \quad \phi'(x') = \Lambda(\lambda)\phi(x) \to \phi'(x') = \Lambda(\lambda)\phi(x) \approx (\mathbb{1} - D\alpha)\phi(x) \quad \Lambda(\lambda) \equiv \lambda^{-D} \tag{37}$$

We use the current

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi(x) - T^{\mu\nu}\delta x_\nu \\ &= -\partial^\mu \phi D\phi - T^{\mu\nu}x_\nu \\ &= -\phi\partial^\mu\phi - T^{\mu\nu}x_\nu \end{aligned} \tag{38}$$

Reference: Voja Radovanovic Problem Book in QFT §5 Problem 19

2 Sheet 2

1.

$$\begin{aligned} &[a_m, a_n] = \frac{i}{2}[q_m, p_n] + \frac{i}{2}[p_m, q_n] = 0 \\ &[a_m^\dagger, a_n^\dagger] = ... = 0 \\ &[a_m, a_n^\dagger] = -\frac{i}{2}[q_m, p_n] + \frac{i}{2}[p_m, q_n] = \delta_{mn} \end{aligned} \tag{39}$$

To write the Hamiltonian, we note that

$$q_n = \frac{1}{\sqrt{2\omega_n}}(a_n + a_n^\dagger) \quad p_n = \frac{1}{i}\sqrt{\frac{\omega_n}{2}}(a_n - a_n^\dagger)$$

$$\begin{aligned} H &= \sum_n \left(\frac{1}{2}p_n^2 + \frac{1}{2}\omega_n^2 q_n^2\right) \\ &= \sum_n \left(-\frac{\omega_n}{4}(a_n - a_n^\dagger)^2 + \frac{1}{4}\omega_n(a_n + a_n^\dagger)^2\right) \\ &= \frac{1}{2}\sum_n \omega_n \left(a_n a_n^\dagger + a_n^\dagger a_n\right) \\ &= \sum_n \omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right) \end{aligned} \tag{40}$$

If we remove the vacuum energy, which means

$$H|0\rangle = \frac{1}{2}\sum_n \omega_n |0\rangle$$

should be removed. Then we get

$$H = \sum_n \omega_n a_n^\dagger a_n$$

$$\begin{aligned} [H, a_m^\dagger] &= [\omega_m a_m^\dagger a_m, a_m^\dagger] = \omega_m a_m^\dagger \\ H|l_1, l_2 ... l_N\rangle &= [H, \prod_i a_i^{\dagger l_i}]|0\rangle \\ &= \sum_i \prod_i a_i^{\dagger l_i}[H, a_{i+1}^{l_{i+1}}] \prod_k a_k^{l_k} \\ &= \sum_i l_i \omega_i \prod_j a_j^{\dagger l_j} = \sum_i l_i \omega_i |l_1, l_2 ... l_N\rangle \end{aligned} \tag{41}$$

2.

$$\begin{aligned}
\int d^3x\phi(\vec{x})e^{-i\vec{p'}\cdot\vec{x}} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p\int d^3xe^{i(\vec{p}-\vec{p'})\cdot\vec{x}} + a_p^\dagger\int d^3xe^{-i(\vec{p}+\vec{p'})\cdot\vec{x}}) \\
&= \frac{1}{\sqrt{2E_{p'}}}(a_{p'} + a_{-p'}^\dagger) \\
\int d^3x\pi(x)e^{-i\vec{p'}\cdot\vec{x}} &= -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}(a_p\int d^3xe^{i(\vec{p}-\vec{p'})\cdot\vec{x}} - a_p^\dagger\int d^3xe^{-i(\vec{p}+\vec{p'})\cdot\vec{x}}) \\
&= -i\sqrt{\frac{E_p'}{2}}\left(a_{p'} - a_{-p'}^\dagger\right) \\
\rightarrow a_p &= \int d^3x\left(\sqrt{\frac{E_p}{2}}\phi(\vec{x}) + i\frac{1}{\sqrt{2E_p}}\pi(\vec{x})\right)e^{-i\vec{p}\cdot\vec{x}} \\
a_p^\dagger &= \int d^3x\left(\sqrt{\frac{E_p}{2}}\phi(\vec{x}) - i\frac{1}{\sqrt{2E_p}}\pi(\vec{x})\right)e^{+i\vec{p}\cdot\vec{x}}
\end{aligned} \tag{42}$$

Just plug in the commutational relation, we get the answer

3. Remember

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\begin{aligned}
P^\mu &= \int d^3xT^{0\mu} = \dot{\phi}\partial^\mu\phi - \eta^{0\mu}\mathcal{L} \\
P^0 &= \int d^3xH = \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}a_{\vec{p}}^\dagger a_{\vec{p}} = \int \frac{d^3p}{(2\pi)^3}p^0a_{\vec{p}}^\dagger a_{\vec{p}} \\
P^i &= \int d^3x\dot{\phi}\partial^i\phi = \int d^3x\pi\partial^i\phi \\
&= \frac{-i}{2}\int d^3x\int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{E_p}{E_{p'}}}(a_pe^{i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{-i\vec{p}\cdot\vec{x}})ip'^i(a_pe^{i\vec{p'}\cdot\vec{x}} - a_{p'}^\dagger e^{-i\vec{p'}\cdot\vec{x}}) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{E_{\pm p}}}(-p^i)(-a_pa_{-p} - a_pa_p^\dagger - a_p^\dagger a_p - a_p^\dagger a_{-p}^\dagger) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a_pa_p^\dagger + a_p^\dagger a_p\right) \\
&= \int \frac{d^3p}{(2\pi)^3}a_p^\dagger a_p + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\delta(0)
\end{aligned} \tag{43}$$

If

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_pe^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x})$$

$$\begin{aligned}
[P^\mu, \phi(x)] &= \int \frac{d^3p'd^3p''}{(2\pi)^6}\frac{1}{\sqrt{2E_{p''}}}p'^\mu([a_{p'}^\dagger a_{p'}, a_{p''}]e^{-ip''\cdot x} + [a_{p'}^\dagger a_{p'}, a_{p''}^\dagger]e^{ip''\cdot x}) \\
&= \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}p'^\mu\left(a_{p'}^\dagger e^{ip'\cdot x} - a_{p'}e^{-ip'\cdot x}\right) \\
-i\partial^\mu\phi(x) &= -i\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\partial^\mu(a_{p'}e^{-ip'\cdot x} + a_{p'}^\dagger e^{ip'\cdot x}) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}p'^\mu\left(a_{p'}^\dagger e^{ip'\cdot x} - a_{p'}e^{-ip'\cdot x}\right)
\end{aligned} \tag{44}$$

4.

$$\dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{-iE_p}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} - a_p^\dagger e^{ip\cdot x} \right) = \pi(x)$$

$i[H, \phi(x)]$ Followed by the commutator relation

$$\begin{aligned} \dot{\pi}(x) &= -\int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} E_p \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} ((ip)^2 - m^2)(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}) \\ &= (\nabla^2 - m^2)\phi(x) \end{aligned} \tag{45}$$

Then

$$\ddot{\phi}(x) = \nabla^2\phi(x) - m^2\phi(x) \to (\partial_\mu \partial^\mu + m^2)\phi(x) = 0 \tag{46}$$

5.

$$\begin{aligned} \langle 0|\phi(x)|p\rangle &= \int \frac{d^3p'}{(2\pi)^3} \sqrt{\frac{E_p}{E_{p'}}} e^{-ip\cdot x} \langle 0|a_{p'} a_p^\dagger|0\rangle \\ &= e^{-ip\cdot x} \end{aligned} \tag{47}$$

6.

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\begin{aligned} Q_i &= \epsilon_{ijk} \int d^3x (x^j \dot{\phi}\partial^k\phi - x^k\dot{\phi}\partial^j\phi) \\ &= \frac{1}{2}\epsilon_{ijk} \int d^3x \int \frac{d^3p'd^3p''}{(2\pi)^6} \left[x^j \sqrt{\frac{E_{p'}}{E_{p''}}} \left(a_{p'} e^{-ip'x} + a_{p'}^\dagger e^{ip'x} \right) p''^k \left(a_{p''} e^{-ip''x} - a_{p''}^\dagger e^{ip''x} \right) \right] - (j \leftrightarrow k) \\ &= \frac{1}{2}\epsilon_{ijk} \int d^3x \int \frac{d^3p'd^3p''}{(2\pi)^6} \sqrt{\frac{E_{p'}}{E_{p''}}} p''^k \left[-a_{p'} a_{p''}^\dagger x^j e^{i(\vec{p'}-\vec{p''})\cdot\vec{x}} + a_{p'}^\dagger a_{p''} x^j e^{-i(\vec{p'}-\vec{p''})\cdot\vec{x}} \right] \\ &= \frac{1}{2}\epsilon_{ijk} \int d^3x \int \frac{d^3p'd^3p''}{(2\pi)^6} \sqrt{\frac{E_{p'}}{E_{p''}}} p''^k \left[a_{p'} a_{p''}^\dagger i \frac{\partial}{\partial p'_j} e^{i(\vec{p'}-\vec{p''})\cdot\vec{x}} + a_{p'}^\dagger a_{p''} i \frac{\partial}{\partial p'_j} e^{-i(\vec{p'}-\vec{p''})\cdot\vec{x}} \right] \\ &= -i\frac{1}{2}\epsilon_{ijk} \int d^3x \int \frac{d^3p'd^3p''}{(2\pi)^6} \frac{p''^k}{\sqrt{E_{p''}}} \frac{\partial}{\partial p'_j} \left[\sqrt{E_{p'}} \left(a_{p'} a_{p''}^\dagger e^{i(\vec{p'}-\vec{p''})\vec{x}} + a_{p'}^\dagger a_{p''} e^{-i(p'-p'')x} \right) \right] \\ &= -i\epsilon_{ijk} \int \frac{d^3p'}{(2\pi)^3} p'^k \left(\frac{\partial}{\partial p'_j} a_{p'}^\dagger a_{p'} \right) + i\epsilon_{ijk} \int \frac{d^3p'}{(2\pi)^3} p'^j \left(\frac{\partial}{\partial p'_k} a_{p'}^\dagger a_{p'} \right) \\ &= -i\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger (p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j}) a_{\vec{p}} \end{aligned} \tag{48}$$

If the state has momentum $\vec{p} = 0$, then the commutator gives $\delta^3(\vec{p})$ which makes the $p^j = 0$, the angular momentum vanishes

7. The variation becomes

$$\begin{aligned} \delta\mathcal{L} &= i\delta\phi^*\partial_0\phi + i\phi^*\partial_0\delta\phi - \frac{1}{2m}(\nabla\delta\phi^*\nabla\phi + \nabla\phi^*\nabla\delta\phi) \\ &= \delta\phi^*(i\partial_0\phi + \frac{1}{2m}\nabla^2\phi) + (-i\partial_0\phi^* + \frac{1}{2m}\nabla^2\phi)\delta\phi \\ &+ i\left[\partial_0(\phi^*\delta\phi) - \frac{1}{2m}\nabla(\delta\phi^*\nabla\phi + \nabla\phi^*\delta\phi) \right] \end{aligned} \tag{49}$$

Equation of motion reads

$$i\partial_0\phi + \frac{1}{2m}\nabla^2\phi = 0 \tag{50}$$

To derive the energy-momentum, we use the equation

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

$$\begin{aligned} T^0{}_0 &= \frac{1}{2m}\nabla\phi^*\nabla\phi \\ T^0{}_i &= i\phi^*\nabla_i\phi \\ T^i_j &= -\frac{1}{2m}\nabla_i\phi^*\nabla_j\phi - \delta^i_j\mathcal{L} \end{aligned} \tag{51}$$

To derive the conserved current, we notice the infinitesimal transform is

$$\phi \to \phi + i\alpha\phi$$

Then, we use the surface term

$$\begin{aligned} &i\alpha\partial_0(\phi^*\phi) - \frac{-i\alpha}{2m}\nabla(\phi^*\nabla\phi - \phi\nabla\phi^*) \\ &= i\alpha\partial_\mu j^\mu \quad j^\mu = (\phi^*\phi, -\frac{i}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*)) \end{aligned} \tag{52}$$

This is eq(2.107)

The momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = i\phi^*$$

The rest discussion is finished in the lecture note after eq(2.109)

8.

$$\begin{aligned} T(\phi(x_2)\phi(x_1)) &= \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1) + \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) = T(\phi(x_1)\phi(x_2)) \\ :\phi(x_1)\phi(x_2): &\propto a_p a_{p'} e^{-ipx_1 - ip'x_2} + a_p^\dagger a_{p'}^\dagger e^{ipx_1 + ip'x_2} + (2\pi)^3 e^{-ip(x_1-x_2)}\delta^3(p-p') \\ &\quad + a_p^\dagger a_{p'} e^{ipx_1 - ip'x_2} + a_{p'}^\dagger a_p e^{-ipx_1 + ip'x_2} \xrightarrow{p\to -p, p'\to -p'} :\phi(x_2)\phi(x_1): \end{aligned} \tag{53}$$

$$\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(y)\phi(x)|0\rangle = \Delta_F(y-x) \tag{54}$$

9. Brute Expansion

$$\begin{aligned} T\phi_1\phi_2\phi_3 &= (\phi_1^+ + \phi_1^-)(:\phi_2\phi_3: + \Delta_F(x_2 - x_3)) \\ &=: \phi_1^-\phi_2\phi_3: + \phi_1\Delta_F(x_2 - x_3) + :\phi_1^+\phi_2\phi_3: + \phi_3\Delta_F(x_1 - x_2) + \phi_2\Delta_F(x_1 - x_3) \\ &=: \phi_1\phi_2\phi_3: + + \phi_1\Delta_F(x_2 - x_3) + \phi_3\Delta_F(x_1 - x_2) + \phi_2\Delta_F(x_1 - x_3) \end{aligned} \tag{55}$$

PS:for more detailed solution
`https://janosh.io/berges-midterm-solution-93dcdeba8ac622750661ee488e7cbea2.pdf`

10. The first order of g is

$$|i\rangle = \sqrt{4E_{\vec{q}_1}R_{\vec{q}_2}}b^\dagger_{\vec{q}_1}c^\dagger_{\vec{q}_2}|0\rangle$$

$$|f\rangle = \sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|0\rangle$$

$$\begin{aligned}
\langle f|S|i\rangle &= -ig\langle f|\int d^4x\phi(x)\psi(x)\psi^\dagger(x)|i\rangle \\
&= -ig\langle 0|\int d^4x\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{k}}}}a_{\vec{p}}a^\dagger_{\vec{k}}e^{ik\cdot x}\psi(x)\psi^\dagger(x)|i\rangle \\
&= -ig\langle 0|\int d^4x\psi^\dagger(x)\psi(x)e^{ip\cdot x}|i\rangle \\
&= -ig\langle 0|\int\frac{d^4xd^3k_1d^3k_2}{(2\pi)^6}\sqrt{\frac{4E_{\vec{q}_q}E_{\vec{q}_2}}{4E_{\vec{k}_1}E_{\vec{k}_2}}}b_{\vec{k}_1}c_{\vec{k}_2}e^{-i(k_1+k_2)\cdot x+ip\cdot x}b^\dagger_{\vec{q}_1}c^\dagger_{\vec{q}_2}|0\rangle \\
&= -ig\int d^4xe^{i(p-(q_1+q_2))x} = -ig(2\pi)^4\delta(q_1+q_2-p)
\end{aligned} \tag{56}$$

To calculate the Nucleon-Meson scattering $\phi+\psi\to\phi+\psi$ at order g^2

$$|i\rangle = \sqrt{4E_{\vec{p}_1}E_{\vec{q}_1}}a^\dagger_{p_1}b^\dagger_{\vec{q}_1}|0\rangle$$

$$|f\rangle = \sqrt{4E_{\vec{p}_2}E_{\vec{q}_2}}a^\dagger_{p_2}b^\dagger_{\vec{q}_2}|0\rangle$$

$$\begin{aligned}
\langle f|S|i\rangle &= -\frac{(ig)^2}{2}\langle f|\int d^4xd^4yT\left[\phi(x)\psi(x)\psi^\dagger(x)\phi(y)\psi(y)\psi^\dagger(y)\right]|i\rangle \\
&= -\frac{(ig)^2}{2}\Bigg\{\langle f|\int d^4xd^4y\left[:\phi(x)\psi(x)\phi(y)\psi(y)^\dagger:\right]\{\psi^\dagger(x)\psi(y)\}|i\rangle \\
&+\langle f|\int d^4xd^4y\left[:\phi(x)\psi^\dagger(x)\phi(y)\psi(y):\right]\{\psi(x)\psi^\dagger(y)\}|i\rangle\Bigg\} \\
&= -(ig)^2\int d^4xd^4y\Delta_F(x-y)\langle f|:\phi(x)\psi^\dagger(x)\phi(y)\psi(y):|i\rangle \\
&= -(ig)^2\int d^4xd^4y\Delta_F(x-y)\langle f|\int\frac{d^4k_1d^4k_2d^4k_3d^4k_4}{(2\pi)^{12}}(a^\dagger_{k_1}b^\dagger_{k_2}b_{k_3}a_{k_4}+a^\dagger_{k_4}...)|i\rangle \\
&= -(ig)^2\int d^4xd^4yd^4kd^4k_1d^4k_2d^4k_3d^4k_4\Bigg(e^{ikx-iky+ik_1x-ik_4y+ik_2x-ik_3y} \\
&\times\delta(k_1-p_2)\delta(k_2-q_2)\delta(p_1-k_4)\delta(k_3-q_1)+e^{ikx-iky-ik_1x+ik_4y+ik_2x-ik_3y}\times \\
&\delta(k_4-p_2)\delta(k_2-q_2)\delta(p_1-k_1)\delta(k_3-q_1)\Bigg)\frac{i}{(2\pi)^4(k^2-m^2+i\epsilon)} \\
&= -(ig)^2\int d^4xd^4yd^4k\left(e^{ix(k+p_2+q_2)+iy(-k-p_1-q_1)}+e^{ix(k-p_1+q_2)+iy(-k+p_2-q_1)}\right)\frac{i}{(2\pi)^4(k^2-m^2+i\epsilon)} \\
&= -i(ig)^2(2\pi)^4\int d^4k\frac{1}{k^2-m^2+i\epsilon}\left(\delta(k+p_2+q_2\delta(k+p_1+q_1)+\delta(k-p_1+q_2)\delta(k-p_2+q_1)\right)) \\
&= (-ig)^2(2\pi)^4\Bigg(\delta^4(p_1+q_1-p_2-q_2)\frac{i}{(p_1+q_1)^2-m^2+i\epsilon} \\
&+\delta^4(p_1-q_2+q_1-p_2)\frac{i}{(p_1-q_2)^2-m^2+i\epsilon}\Bigg)
\end{aligned} \tag{57}$$

3 Sheet 3

In Dirac equations, I used conventions in $ar\chi iv$:1605.00146

$$\sigma_\mu = \bar{\sigma}^\mu = (\sigma_0, \vec{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \quad \bar{\sigma}_\mu = \sigma^\mu = (\mathbb{1}, -\bar{\sigma})$$

$$\frac{1}{2}tr(\sigma_i\sigma_j) = \delta_{ij} \quad \frac{1}{2}tr(\sigma_\mu\bar{\sigma}_\nu) = \eta_{\mu\nu}$$

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

1. Direct calculation, for example

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \tag{58}$$

The unitary matrix reads

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix} \tag{59}$$

Then

$$\gamma'^0 = U\gamma U^\dagger = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \gamma'^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{60}$$

2. Use the commutational identity

$$[AB, C] = A[B, C] + [A, C]B = A\{B, C\} - \{A, C\}B$$

$$[A, BC] = [A, B]C + B[A, C] = \{A, B\}C - B\{A, C\}$$

$$\begin{aligned} [\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] &= [\gamma^\kappa, \gamma^\mu\gamma^\nu]\gamma^\lambda + \gamma^\kappa[\gamma^\lambda, \gamma^\mu\gamma^\nu] \\ &= 2\eta^{\kappa\mu}\gamma^\nu\gamma^\lambda - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda + 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\lambda\nu}\gamma^\kappa\gamma^\mu \\ &= [\gamma^\kappa\gamma^\lambda, \gamma^\mu]\gamma^\nu + \gamma^\mu[\gamma^\kappa\gamma^\lambda, \gamma^\nu] \\ &= 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\lambda^k - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda \end{aligned} \tag{61}$$

$$S^{\mu\nu} = \frac{1}{4}(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu - 2\eta^{\mu\nu}) = \frac{1}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}) \tag{62}$$

$$\begin{aligned} [S^{\kappa\lambda}, S^{\mu\nu}] &= \frac{1}{4}([\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu]) = \frac{1}{4}(2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\lambda^k - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda) \\ &= \eta^{\lambda\mu}S^{\kappa\nu} + \frac{1}{2}\eta^{\lambda\mu}\eta^{\kappa\nu} - \eta^{\kappa\mu}S^{\lambda\nu} - \frac{1}{2}\eta^{\kappa\mu}\eta^{\lambda\nu} + \frac{1}{2}\eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - \frac{1}{2}\eta^{\kappa\nu}\gamma^\nu\gamma^\lambda \\ &= \eta^{\lambda\mu}S^{\kappa\nu} - \eta^{\kappa\mu}S^{\lambda\nu} + \eta^{\kappa\nu}S^{\lambda\mu} - \eta^{\lambda\nu}S^{\kappa\mu} \end{aligned} \tag{63}$$

3. See *Reference: Voja Radovanovic Problem Book in QFT §3*

(1) $Tr[\gamma^\mu] = 0$

$$\begin{aligned} \gamma^\mu\gamma^\mu &= g^{\mu\mu} \\ \frac{\gamma^\mu\gamma^\mu}{g^{\mu\mu}} &= I \\ Tr[\gamma^\mu] &= Tr[\gamma^\mu\gamma^\nu\gamma^\nu]/g^{\nu\nu} \\ &= -Tr[\gamma^\nu\gamma^\mu\gamma^\nu]/g^{\nu\nu} + 2Tr[\gamma^\nu]/g^{\nu\nu}g^{\nu\mu} \\ &= -Tr[\gamma^\mu] \end{aligned} \tag{64}$$

(2) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0$ odd number of gamma matrices are zero

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] &= \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\sigma]/g^{\sigma\sigma} \\ &= -\mathrm{Tr}[\gamma^\sigma\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho]/g^{\sigma\sigma}\end{aligned} \tag{65}$$

for more, we choose γ^5 which is $\{\gamma^5, \gamma^\mu\} = 0$

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho...] &= \mathrm{Tr}[\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho...] \\ &= -\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho...\gamma^5]\end{aligned} \tag{66}$$

(3) $Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

$$\begin{aligned}Tr[\gamma^\mu\gamma^\nu] =& Tr[\{\gamma^\mu\gamma^\nu\}] - Tr[\gamma^\nu\gamma^\mu] \\ 2Tr[\gamma^\mu\gamma^\nu] =& 2g^{\mu\nu}Tr[1_{4\times4}]\end{aligned} \tag{67}$$

(4) $Tr[\gamma^5] = 0$

$$\begin{aligned}iTr[\gamma^0\gamma^1\gamma^2\gamma^3] &= -iTr[\gamma^1\gamma^0\gamma^2\gamma^3] \\ &= -iTr[\gamma^0\gamma^2\gamma^3\gamma^1] \\ &= -iTr[\gamma^0\gamma^1\gamma^2\gamma^3]\end{aligned} \tag{68}$$

(5) $(\gamma^5)^2 = 1$

$$\begin{aligned}&-1(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3) \\ &= +(\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3\gamma^1\gamma^2\gamma^3) \\ &= +(\gamma^0\gamma^1\gamma^0\gamma^2\gamma^1\gamma^3\gamma^2\gamma^3) \\ &= -(\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3) \\ &= -(1\times(-1)^3)\end{aligned} \tag{69}$$

(6) $\not{p}\not{q} = 2p\dot{q} - \not{q}\not{p} = p\dot{q} - 2iS^{\mu\nu}p_\mu q_\nu$
easy to prove

(7) $\mathrm{Tr}[\not{p}\not{q}] = 4p\dot{q}$
easy to prove

(8) $\mathrm{Tr}[\not{p}_1\not{p}_2...\not{p}_n] = 0$ if n is odd
easy to prove

(9) $\mathrm{Tr}[\not{p}_1\not{p}_2\not{p}_3\not{p}_4] = 4[(p_1\dot{p}_2)(p_3\dot{p}_4) + (p_1\dot{p}_4)(p_2\dot{p}_3) - (p_1\dot{p}_3)(p_2\dot{p}_4)]$
easy to prove

(10) $\mathrm{Tr}[\gamma^5\not{p}_1\not{p}_2] = 0$
easy to prove, notice $\gamma_\mu\gamma^\mu = \gamma_0\gamma^0 - \gamma_i\gamma^i = 4$

(11) $\gamma_\mu\not{p}_1\not{p}_2\gamma^\mu = 4p_1\dot{p}_2$ notice $\gamma_\mu\gamma^\nu = (2\delta^\nu_\mu - \gamma^\nu\gamma_\mu)$, $\gamma^\mu\gamma^\nu = (2\iota^\nu_\mu - \gamma^\nu\gamma^\mu)$

(12) $\gamma_\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1$
use the anti-commutation relation

(13) $\mathrm{Tr}[\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4] = 4i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$
first, if two indices are the same, then they are vanishes by the anti-commutation trick and No.10's result
then if they are different, we can use the result of No.5 and trace an identity matrix

4. *In Dirac equations, I used conventions in $ar\chi iv$:1605.00146*

$$\sigma_\mu = \bar{\sigma}^\mu = (\sigma_0, \vec{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \quad \bar{\sigma}_\mu = \sigma^\mu = (\mathbb{1}, -\bar{\sigma})$$

$$\frac{1}{2} tr(\sigma_i \sigma_j) = \delta_{ij} \quad \frac{1}{2} tr(\sigma_\mu \bar{\sigma}_\nu) = \eta_{\mu\nu}$$

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$\begin{aligned} \xi^{r\dagger} p \cdot \sigma \cdot \xi^s + \xi^{r\dagger} p \cdot \bar{\sigma} \cdot \xi^s &= 2p^0 \xi^{r\dagger} \xi^s = 2p^0 \delta^{rs} \\ \xi^{r\dagger} \sqrt{(p \cdot \bar{\sigma})(p \cdot \sigma)} \xi^s + \xi^{r\dagger} \sqrt{(p \cdot \sigma)(p \cdot \bar{\sigma})} \xi^s &= 2m \xi^{r\dagger} \xi^s = 2m \delta^{rs} \end{aligned} \tag{70}$$

For the other formulas, see the notes

5. See the notes

6.

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \sum_{r,s} \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}} E_{\vec{q}}}} \Big(\{b, b\} ... + \{c, c\} ... + \{b, c^\dagger\} ... + c^\dagger, b \Big) \\ &= 0 \\ \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} &= 0 \\ \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} &= \sum_{r,s} \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}} E_{\vec{q}}}} \big(\{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} u^r_\alpha(\vec{p}) u^{s\dagger}_\beta(\vec{q}) e^{i(\vec{x}\cdot\vec{p} - \vec{y}\cdot\vec{q})} \\ &\quad + \{c^{r\dagger}_{\vec{p}}, c^s_{\vec{q}}\} v^r_\alpha(\vec{p}) v^{s\dagger}_\beta(\vec{q}) e^{-i(\vec{x}\cdot\vec{p} - \vec{y}\cdot\vec{q})} \big) \\ &= \delta_{\alpha\beta} \sum_r \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (u^r(\vec{p}) \bar{u}^r(\vec{p}) \gamma^0 e^{i\vec{p}(\vec{x} - \vec{y})} + v^r(\vec{p}) \bar{v}^r(\vec{p}) \gamma^0 e^{-i\vec{p}\cdot(\vec{x} - \vec{y})}) \\ &= \delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} ((\not{p} + m) \gamma^0 e^{i\vec{p}(\vec{x} - \vec{y})} + (\not{p} - m) \gamma^0 e^{-i\vec{p}(\vec{x} - \vec{y})}) \\ &= \delta_{\alpha\beta} \delta(\vec{x} - \vec{y}) \end{aligned} \tag{71}$$

7.

$$
\begin{aligned}
H &= \int d^3x \bar{\psi}(-i\gamma^i\partial_i + m)\psi \\
&= \left\{\int d^3x \sum_{rs}\int \frac{d^3\vec{p}d^3\vec{q}}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\right\}\left[b^{r\dagger}_{\vec{p}}\bar{u}^r(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^r_{\vec{p}}\bar{v}^r(\vec{p})e^{i\vec{p}\cdot\vec{x}}\right] \\
&(-i\gamma^i\partial_i + m)\left[b^s_{\vec{q}}u^s(\vec{q})e^{i\vec{q}\cdot\vec{x}} + c^{s\dagger}_{\vec{q}}v^s(\vec{q})e^{-i\vec{q}\cdot\vec{x}}\right] \\
&= \{\}\left[b^{r\dagger}_{\vec{p}}\bar{u}^r(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^r_{\vec{p}}\bar{v}^r(\vec{p})e^{i\vec{p}\cdot\vec{x}}\right]\left[(-\gamma^i q_i + m)b^s_{\vec{q}}u^s(\vec{q})e^{i\vec{q}\cdot\vec{x}} + (\gamma^i q_i + m)c^{s\dagger}_{\vec{q}}v^s(\vec{q})e^{-i\vec{q}\cdot\vec{x}}\right] \quad \text{See 5.9} \\
&= \{\}\left[b^{r\dagger}_{\vec{p}}\bar{u}^r(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^r_{\vec{p}}\bar{v}^r(\vec{p})e^{i\vec{p}\cdot\vec{x}}\right]\gamma^0 E_{\vec{q}}\left[b^s_{\vec{q}}u^s(\vec{q})e^{i\vec{q}\cdot\vec{x}} - c^{s\dagger}_{\vec{q}}v^s(\vec{q})e^{-i\vec{q}\cdot\vec{x}}\right] \\
&= \left\{\sum_{rs}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2}\right\}\left[b^{r\dagger}_{\vec{p}}b^s_{\vec{p}}u^{r\dagger}(\vec{p})u^s(\vec{p}) - b^{r\dagger}_{\vec{p}}c^{s\dagger}_{-\vec{p}}u^{r\dagger}(\vec{p})v^s(-\vec{p})\right. \\
&\quad \left. - c^r_{\vec{p}}c^{s\dagger}_{\vec{p}}v^{r\dagger}(\vec{p})v^s(\vec{p}) + c^{r\dagger}_{\vec{p}}b^s_{-\vec{p}}v^{r\dagger}(\vec{p})u^s(-\vec{p})\right] \\
&= \sum_r\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}(b^{r\dagger}_{\vec{p}}b^r_{\vec{p}} - c^r_{\vec{p}}c^{r\dagger}_{\vec{p}}) \\
&= \sum_r\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left[b^{r\dagger}_{\vec{p}}b^r_{\vec{p}} + c^{r\dagger}_{\vec{p}}c^r_{\vec{p}} - (2\pi)^3\delta^3(0)\right]
\end{aligned}
\tag{72}
$$

8. The first question has been addressed in the note eq(5.7), and the sick hamiltonian is eq(5.12)

4 Sheet 4

1.

$$
\begin{aligned}
|i\rangle &= \sqrt{8E_{\vec{p}_1}E_{\vec{p}_2}E_{\vec{p}_3}}a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}a^\dagger_{\vec{p}_3} \\
|f\rangle &= \sqrt{8E_{\vec{p'}_1}E_{\vec{p'}_2}E_{\vec{p'}_3}}a^\dagger_{\vec{p'}_1}a^\dagger_{\vec{p'}_2}a^\dagger_{\vec{p'}_3} \\
&\frac{(-i\lambda)^2}{2(4!)^2}\langle f|\int d^4xd^4yT[\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)]|i\rangle \\
&= (-i\lambda)^2\frac{4^2}{2(4!)^2}\langle f|\int d^4xd^4y[:\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y):]\{\phi(x)\phi(y)\}|i\rangle \\
&= (-i\lambda)^2\frac{4^2}{2(4!)^2}\langle f|\int d^4xd^4y[:\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y):]\{\phi(x)\phi(y)\}|i\rangle \\
&= (-i\lambda)^2\frac{4^2}{2(4!)^2}\langle f|\int d^4xd^4y\Delta_F(x-y)\langle f|\int\frac{d^3k_1d^3k_2d^3k_3d^3k_4d^3k_5d^3k_6}{(2\pi)^{18}}... \\
&\quad (a^\dagger_{k_1}a^\dagger_{k_2}a^\dagger_{k_3}a_{k_4}a_{k_5}a_{k_6} + ...) \\
&= ... \\
&= (-i\lambda)^2\frac{i}{(p_1+p_2+p_3)^2 - m^2 + i\epsilon}
\end{aligned}
\tag{73}
$$

For the other diagrams contribute to this process, there might be some bubble on the propagator

2. I read the solution

https://janosh.io/berges-midterm-solution-93dcdeba8ac622750661ee488e7cbea2.pdf

$$
\begin{aligned}
&\langle 0|T\left[\exp\left(-i\lambda\int d^4x\frac{\phi^4}{4!}\right)\right]|0\rangle = 1+\frac{-i\lambda}{4!}\int d^4x\langle 0|\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle \\
&\quad+\frac{(-i\lambda)^2}{2(4!)^2}\int d^4xd^4y\langle 0|T\left[\phi(x)^4\phi(y)^4\right]|0\rangle \\
&=1+\langle 0|:\phi(x)^4:|0\rangle+\frac{-i3\lambda}{4!}\Delta_F(0)^2+\langle 0|:\phi(x)^4\phi(y)^4:|0\rangle \\
&\quad+\frac{(-i\lambda)^2}{2(4!)^2}\left(3^2\Delta_{F_x}(0)^2\Delta_{F_y}(0)^2+2\times 3!^2\Delta_{F_x}(0)\Delta_{F_y}(0)\Delta_F(x-y)^2+4!\Delta_F(x-y)^4\right) \\
&=1+\frac{-i\lambda}{8}Fig_1+\frac{(-i\lambda)^2}{2}(\frac{1}{8}Fig_1)^2+\frac{(-i\lambda)^2}{2}\frac{1}{8}Fig_2+\frac{(-i\lambda)^2}{2}\frac{1}{4!}Fig_4
\end{aligned}
\tag{74}
$$

3.

$$
\begin{aligned}
\langle 0|T(\phi_i(x)\phi_j(y))|0\rangle &= \theta(x^0-y^0)\langle 0|\phi(x)_i\phi(y)_j|0\rangle+\theta(y^0-x^0)\langle 0|\phi(y)_j\phi(x)_i|0\rangle \\
&=\int\frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\big(\theta(x^0-y^0)\langle 0|a_{i\vec{p}}a^\dagger_{j\vec{q}}|0\rangle e^{-ip\cdot x+iq\cdot y} \\
&\quad+\theta(y^0-x^0)\langle 0|a_{j\vec{q}}a^\dagger_{i\vec{p}}|0\rangle e^{-iq\cdot y+ip\cdot x}\big) \\
&=\delta_{ij}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{+ip\cdot(x-y)}\right) \\
&=\delta_{ij}D_F(x-y)
\end{aligned}
\tag{75}
$$

The feynmann rules

- Add a momentum k^i to each internal line, where the superscript means the field is ϕ_i
- To each vertex, write down a factor of
$$(-i\lambda)$$
- For each internal line, write down the propagator

The first order term should be

$$
\begin{aligned}
&-\frac{i\lambda}{8}\langle k,l|T\left(\sum_{i=1}^3\phi_i^2\right)^2|i,j\rangle \\
&=\frac{(-i\lambda)}{8}\big[4!(\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}+\delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}+\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}) \\
&+2(\delta_{1i}\delta_{1j}\delta_{2k}\delta_{2l}+(1,2\to 3)+(1\to 2,2\to 3)+4(\delta_{1i}\delta_{2j}\delta_{1k}\delta_{2l}+(...)))\big]
\end{aligned}
\tag{76}
$$

4.

$$
\begin{aligned}
&\frac{(-i\lambda)^2}{2}\int d^4xd^4y\langle f|T\left[\phi(x)\bar\psi(x)\psi(x)\phi(y)\bar\psi(y)\psi(y)\right]|i\rangle \\
&=\frac{(-i\lambda)^2}{2}\int d^4xd^4y\langle f|\left[:\bar\psi(x)\psi(x)\bar\psi(y)\psi(y):\right]\{\phi(x)\phi(y)\}|i\rangle \\
&=... \quad \text{See note after (5.41)}
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
&\frac{(-i\lambda)^2}{2}\int d^4xd^4y\langle f|T[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]|i\rangle \\
&=\frac{(-i\lambda)^2}{2}\int d^4xd^4y\langle f|\left[:\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y):\right]\{\phi(x)\phi(y)\}|i\rangle \\
&=\frac{(-i\lambda)^2}{2}\int d^4xd^4y\Delta_F(x-y)\langle 0|c^{r'}_{\vec{q}'}b^{s'}_{\vec{p}'}:\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y):b^{s\dagger}_{\vec{p}}c^{r\dagger}_{\vec{q}}|0\rangle \\
&=\frac{(-i\lambda)^2}{2}\int d^4xd^4y\Delta_F(x-y) \\
&\quad\int\prod\frac{d^3k_i}{(2\pi)^3}\langle 0|c^{r'}_{\vec{q}'}b^{s'}_{\vec{p}'}(b^\dagger_x c^\dagger_x c_y b_y e^{ik_1x+ik_2x-ik_3y-ik_4y}+...)b^{s\dagger}_{\vec{p}}c^{r\dagger}_{\vec{q}}|0\rangle \\
&=\frac{(-i\lambda)^2}{2}\int d^4xd^4y\Delta_F(x-y) \\
&(+[\bar{u}^{s'}(p')\cdot v^{r'}(q)][\bar{v}^r(q)\cdot u^s(p)]e^{ip'x+iq'x-iqy-ipy} \\
&-[\bar{u}^{s'}(p')\cdot u^s(p)][\bar{v}^r(q)\cdot v^{r'}(q')]e^{ip'x-ipx-iqy+iq'y} \\
&+[\bar{u}^{s'}(p')\cdot v^{r'}(q)][\bar{v}^r(q)\cdot u^s(p)]e^{ip'y+iq'y-ipx-iqx} \\
&-[\bar{u}^{s'}(p')\cdot u^s(p)][\bar{v}^r(q)\cdot v^{r'}(q')]e^{iq'x+ip'y-ipy-iqx}) \\
&=\frac{(-i\lambda)^2}{2}\int d^4k\frac{i}{k^2-m^2+i\epsilon}(\delta...\delta...) \\
&=-(-i\lambda)^2\left(\frac{[\bar{u}^{s'}(p')\cdot u^s(p)][\bar{v}^r(q)\cdot v^{r'}(q')]}{(p-p')^2-m^2}-\frac{[\bar{u}^{s'}(p')\cdot v^{r'}(q)][\bar{v}^r(q)\cdot u^s(p)]}{(p+q)^2-m^2}\right)
\end{aligned}
\tag{78}
$$

5. Analogous to previous question

6.

$$
\begin{aligned}
\nabla\times\mathbf{f}&=-\nabla^2\mathbf{g} \\
\mathbf{g}&=\frac{1}{4\pi}\int d^3y\frac{\nabla\times\mathbf{f}(\vec{y})}{|\vec{x}-\vec{y}|}
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
\nabla\cdot\mathbf{f}&=\nabla^2 h \\
h&=-\frac{1}{4\pi}\int d^3y\frac{\nabla\cdot\mathbf{f}(\vec{y})}{|\vec{x}-\vec{y}|}
\end{aligned}
\tag{80}
$$

$$
\mathbf{f}^T=\mathbf{f}-\mathbf{f}^L=\mathbf{f} \tag{81}
$$

To show the formal solution, we notice the $\mathbf{f}$ consists of two part $\mathbf{f}^T$ and $\mathbf{f}^L$, we have

$$
\begin{aligned}
\mathbf{f}^T&=\mathbf{f}-\mathbf{f}^L \\
\nabla\cdot\mathbf{f}^L&=\nabla\cdot\mathbf{f}=\nabla^2 h \\
h&=(\nabla^2)^{-1}\nabla\cdot\mathbf{f}^L \\
\mathbf{f}^L&=\nabla h=\nabla(\nabla^2)^{-1}\nabla\cdot\mathbf{f} \\
\mathbf{f}^T&=\mathbf{f}-\mathbf{f}^L=\mathbf{f}-\nabla(\nabla^2)^{-1}\nabla\cdot\mathbf{f}
\end{aligned}
\tag{82}
$$

In electromangetic field, the vector potential is zero divergence, so $\vec{A}=\vec{f}^T$. The operator

$$
\delta_{ij}-\frac{\nabla_i\nabla_j}{(\nabla^2)^{-1}}
$$

project any vector valued function into its transverse component, it's a projection operator and imposed the constraint $\nabla \cdot \vec{A} = 0$ into the Poisson bracket, so we get the modified commutation relation by imposing the projection operator:

$$\begin{aligned}
&[A_i(\vec{x}), E^j(\vec{y})] = i\delta_i^j \delta^3(\vec{x} - \vec{y}) \\
&\rightarrow (\delta_{ki} - \frac{\nabla_k \nabla_i}{(\nabla^2)^{-1}})[A_i(\vec{x}), E^j(\vec{y})] = i(\delta_{ki} - \frac{\nabla_k \nabla_i}{(\nabla^2)^{-1}})\delta_i^j \delta^3(\vec{x} - \vec{y}) \\
&\rightarrow [A_k^T(\vec{x}), E^j(\vec{y})] = i(\delta_k^j - \frac{\nabla_k \nabla^j}{(\nabla^2)^{-1}})\delta^3(\vec{x} - \vec{y})
\end{aligned} \tag{83}$$

7. Just read this diagram using the rules given in §6.5, the second diagram is given above (6.94) The reason why the total amplitude vanishes is given below (6.94)

8. two vortex $(-ie\gamma)^2$, photon propagator $-\frac{i\eta_{\mu\nu}}{p^2 + i\epsilon}$ and four fermions

$$(-ie)^2 \frac{[\bar{v}_e^r(\vec{q})\gamma^\mu u_e^s(\vec{p})](-i)\eta_{\mu\nu}[\bar{u}_{\mu^-}^{s'}(\vec{p}')\gamma^\nu v_{\mu^+}^{r'}(\vec{q}')]}{(p+q)^2 + i\epsilon} \tag{84}$$

9. Rua?Rua

10. Rua!