Cambridge Part III Maths

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Supersymmetry

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1 Introduction

Lecture 1
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1.1 What is SUSY?

Consider a quantum theory with Hilbert space $\mathcal{H}$, ground state $|\Omega\rangle \in \mathcal{H}$ and Hamiltonian H. Suppose that this theory contains fermions, i.e. that there are fermionic operators acting on $\mathcal{H}$. Then $\mathcal{H}$ necessarily decomposes as the tensor sum $\mathcal{H}_\mathrm{B} \oplus \mathcal{H}_\mathrm{F}$, where $\mathcal{H}_\mathrm{B}$ and $\mathcal{H}_\mathrm{F}$ contain states with an even and odd number of fermionic excitations respectively. What this means is that $|\Omega\rangle \in \mathcal{H}_\mathrm{B}$, and all fermionic operators map states in $\mathcal{H}_\mathrm{B}$ to states in $\mathcal{H}_\mathrm{F}$ and vice versa.

Such a theory is supersymmetric if it supports the following object:

Definition. A *supercharge* Q is a fermionic operator satisfying

$$\left\{Q, Q^\dagger\right\} = 2H, \qquad Q^2 = Q^{\dagger\,2} = 0.$$

Here $\{X, Y\} = XY + YX$ is the anticommutator of X and Y, and $Q^\dagger$ is the Hermitian conjugate of Q with respect to the inner product on $\mathcal{H}$. The factor of 2 in front of the Hamiltonian is a useful convention.

There are a couple of immediate consequences of the existence of a supercharge. First, the commutator of the Hamiltonian and a supercharge vanishes, since we have

$$\begin{aligned} 2[H, Q] &= \left[\left\{Q, Q^\dagger\right\}, Q\right] \\ &= (QQ^\dagger + Q^\dagger Q)Q - Q(QQ^\dagger + Q^\dagger Q) \\ &= QQ^\dagger Q - QQ^\dagger Q = 0, \end{aligned}$$

where the third line is reached by using $Q^2 = 0$. This means that Q is conserved, and that the transformations it generates are symmetries. The unitary operator implementing such symmetries is written $\exp(i\eta\, Q)$, where in order for the exponent to be bosonic, η must be a fermionic parameter. Whenever an object is parametrised by fermionic parameters, a common naming pattern in physics is to attach the prefix "super-". Therefore, we refer to the symmetries generated by Q as *supersymmetries*.

Second, for any state $|\psi\rangle \in \mathcal{H}$, the expectation value of the energy is non-negative:

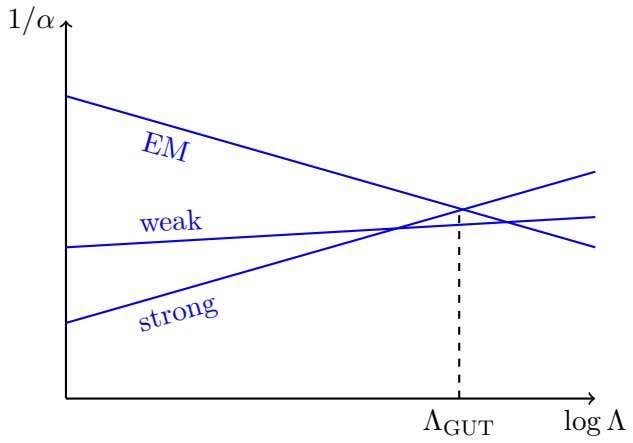
$$\begin{aligned} \langle H \rangle &= \langle \psi | H | \psi \rangle \\ &= \frac{1}{2} \langle \psi | (QQ^\dagger + Q^\dagger Q) | \psi \rangle \\ &= \frac{1}{2} \left\| Q^\dagger |\psi\rangle \right\|^2 + \frac{1}{2} \| Q |\psi\rangle \|^2 \geq 0. \end{aligned}$$

Furthermore, equality $\langle H \rangle = 0$ is attained if and only if $Q^\dagger |\psi\rangle = Q |\psi\rangle = 0$, i.e. if the state ψ is invariant under the supersymmetry transformations. We call such a state *supersymmetric*. Since the ground state must have zero energy, it must be supersymmetric.

1.2 Why is SUSY interesting?

Phenomenology

In QFT, the coupling constants α of interactions depend on the energy scale Λ at which we are making observations. One says that the coupling constants run with the energy scale, and understanding this running requires studying renormalisation. In the case of the Standard Model, three interesting coupling constants are the fine structure constants of the strong, weak and electromagnetic forces. One finds that these three constants come very close to being equal at a certain energy scale $\Lambda_{\text{GUT}} \approx 10^{16}\,\text{GeV}$.



The three constants do not appear to be exactly equal at this scale. But there is a reason that we might expect them to be. Matter in the standard model transforms in representations of the gauge group $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$. In fact, *all* of the matter in the standard model can be considered as a single representation of the larger group SO(10), which contains $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ as a subgroup. It is far from the case that such an extension can always be done, and therefore it seems natural that perhaps SO(10) is the 'true' unified gauge group of reality, which is broken down to $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ at some energy scale. But for this to be the case, the three fine structure constants should exactly agree at Λ_{GUT}.

So we might want to change the standard model such that this exact agreement does happen – the result of such a change is referred to as a GUT, or 'Grand Unified Theory'. One way of making such a change is to implement a supersymmetric extension to the Standard Model. There is a broad class of such supersymmetric extensions.

Another phenomenological reason to study SUSY is as follows. Various sources of evidence (e.g. galactic rotation curves) point to the existence of 'dark matter' which interacts very weakly with standard model particles. We don't know what dark matter is yet, but the particles necessary for a SUSY extension of the Standard Model provide candidates.

A further phenomenological reason (which has been called into question by observations at the LHC) is that SUSY may shed light on the Higgs mass. The Higgs boson is a scalar particle, and we expect such particles to recieve large quantum corrections to their masses, of order the Planck scale. However, according to observations the mass of the Higgs does not appear to recieve these corrections, so we are forced to ask what protects it from them. There are certain supersymmetric extensions of the Standard Model that could do this – but other observations at CERN have ruled

these extensions out. (Note that these observations do not rule out supersymmetry entirely. They just mean that it cannot explain the Higgs mass.)

Theory

The argument could be made that the above phenomological motivations are secondary to the following theoretical motivations.

QFT is hard, and SUSY makes it easier to understand. When learning quantum mechanics, one starts with the formalism, and then proceeds to analyse easy basic models such as the harmonic oscillator and the hydrogen atom. Then one finds out that these models are unrealistic, and uses perturbation theory to study deformations of them which are more realistic. The reason the basic models were easy is because they were idealised and contained a lot of symmetry. The perturbation theory enables us to study approximations to the consequences of breaking this symmetry.

In QFT, the *only* 'idealised' system one studies initially is free theory, with no interactions. Free theory is fine, but it is not obvious that it is a good approximation to reality. More importantly, free theory only allows us to study a small set of aspects of QFT.

SUSY allows us to do better. When we have supersymmetry we can compute some quantities exactly. This is almost always only true when studying toy models, with no pretense of realism, but this is a good starting poing. These quantities often reveal extremely mathematically interesting deep connections between QFT, geometry, and topology. Many modern breakthroughs in mathematics (e.g. mirror symmetry, Seiberg-Witten theory, Donaldson invariants...) are motivated by this point of view.

In this course, we will focus on these more theoretical and mathematical aspects, and not emphasise so much the phenomonology of supersymmetry.

Path integrals

Consider the following Euclidean signature path integral:

$$\mathcal{Z} = \int_{\mathscr{C}} e^{-S[X]/\hbar}\, \mathrm{D}X\,.$$

X is some field, $S[X]$ is the action, and $\mathscr{C}$ is the space of all possible field configurations. The configuration space $\mathscr{C}$ is an infinite-dimensional function space, which we need to be able to integrate over for the above expression to make sense. Such an integral is not often well-defined, and even if it were it is not always entirely obvious what it would mean.

The exact evaluation of a path integral is a formidable goal. As a very simple toy, suppose the whole universe $\mathcal{M}$ is just a single point. Then a field could be a map from that point to $\mathbb{R}$. The path integral would then just be a normal integral over the values taken by X:

$$\mathcal{Z} = \int_{\mathbb{R}} e^{-S[X]/\hbar}\, \mathrm{d}X\,.$$

Suppose the action takes the following very simple form:

$$S[X] = \frac{m}{2}X^2 + \frac{\lambda}{4!}X^4 + \frac{g}{6!}X^6.$$

m might be interpreted as the mass of X, and λ and g might be interpreted as higher order coupling constants. Even with such a simple action, the path integral is hard to evaluate explicitly. It is only easy if $\lambda = g = 0$, as then we have a Gaussian integral. But this would just be a free theory, which we are not particularly interested in.

One thing we can do is obtain an asymptotic series for $\mathcal{Z}$ in the semiclassical limit $\hbar \to 0$. If $S[X]$ has an isolated minimum at $X_0 \in \mathbb{R}$, one can approximate the path integral with a saddlepoint approximation as

$$\mathcal{Z} \overset{\hbar\to 0}{\approx} \underbrace{e^{-S[X_0]/\hbar}}_{\text{tree level}} \times \underbrace{\left(\frac{\partial^2 S}{\partial X^2}\right)^{-1/2}}_{\text{one loop}} \times \underbrace{(1+\mathcal{O}(\hbar))}_{\text{higher loops}}.$$

Each of the factors above come in at various loop orders in a Feynman diagram approach to the path integral. This series is at most asymptotic, as it can't possibly converge. If it did, it would have to converge for $\hbar$ in an open neighbourhood of 0 in the complex plane. But for negative $\hbar$, the path integral clearly blows up. So this semiclassical expression can be viewed at best as an approximation.

Supersymmetry allows us to do better than this approximation. This course will not be focussed only on classical supersymmetry, just adding new particles to a theory to make it supersymmetric, perhaps employing a superspace formalism. This can quickly become messy and complicated. We will try to understand why it is worthwhile, and in particular why doing it makes the path integral easier.

2 SUSY in zero dimensions

2.1 Grassman variables

Definition. A set of n *Grassman variables* are elements ψ^a, $a = 1, \dots, n$ which obey

$$\psi^a \psi^b = -\psi^b \psi^a.$$

In particular $(\psi^a)^2 = 0$. Grassman variables commute with bosonic variables x (e.g. real numbers):

$$\psi^a x = x \psi^a.$$

Grassman variables are vaguely reminiscent of the Dirac field $\left\{\psi^\alpha(\mathbf{x}), \psi^\beta(\mathbf{y})\right\} = 0$, and differential forms $\mathrm{d}x^a \wedge \mathrm{d}x^b = -\mathrm{d}x^b \wedge \mathrm{d}x^a$. This will turn out to not be a coincidence.

Any function $F(\psi)$ of Grassman variables has a terminating expansion:

$$F(\psi) = f + \rho_a \psi^a + \phi_{ab} \psi^a \psi^b + \dots + g \psi^1 \psi^2 \dots \psi^n.$$

Note that we can take the coefficients to be antisymmetric in their indices, e.g. $\phi_{ab} = -\phi_{ba}$. If $F(\psi)$ is bosonic (commuting), then the even coefficients $f, \phi_{ab}, \dots$ must also be bosonic, and the odd coefficients $\rho_a, \dots$ must be fermionic.

Definition. Derivatives with respect to Grassman variables are defined by

$$\frac{\partial}{\partial\psi^a}\left(\psi^b\ldots\right) = \delta^b_a(\ldots) - \psi^b\frac{\partial}{\partial\psi^a}(\ldots).$$

Lecture 2
23/01/19

This is reminiscent of the product rule, but with a minus sign instead of a plus sign, which reflects the fact that $\frac{\partial}{\partial\psi^a}$ is fermionic and so anticommutes with ψ^b.

We can also define integration with respect to Grassman variables. Note that since $\psi^2 = 0$, it is sufficient to define $\int 1\,\mathrm{d}\psi$ and $\int\psi\,\mathrm{d}\psi$. We want our integral to be 'translation invariant', i.e. $\int(\psi+\eta)\,\mathrm{d}\psi = \int\psi\,\mathrm{d}\psi$ for any η. Then we must have $\int\eta\,\mathrm{d}\psi = 0$, which implies that $\int 1\,\mathrm{d}\psi = 0$. Another way we could arrive at this condition is by insisting that the fundamental theorem of calculus holds, and supposing that the range of integration behaves as if it has no boundary. Then we would have $\int 1\,\mathrm{d}\psi = \int\frac{\partial}{\partial\psi}\psi\,\mathrm{d}\psi = 0$. It remains to define $\int\psi\,\mathrm{d}\psi$, and doing so is essentially just a normalisation convention. We will use the standard choice:

Definition. *Berezin integration* is defined by

$$\int 1\,\mathrm{d}\psi = 0, \qquad \int\psi\,\mathrm{d}\psi = 1.$$

Note that performing a Berezin integral actually just gives the same result as doing a derivative, because $\frac{\partial}{\partial\psi}1 = 0$ and $\frac{\partial}{\partial\psi}\psi = 1$.

Suppose we have n fermions $\psi^1,\ldots,\psi^n$. We have

$$\int\psi^1\psi^2\ldots\psi^n\underbrace{\mathrm{d}\psi^n\,\mathrm{d}\psi^{n-1}\ldots\mathrm{d}\psi^1}_{\mathrm{d}^n\psi} = 1.$$

Because the ψ^a anticommute, the order of the measures $\mathrm{d}\psi^a$ matters here – they may be taken to be anticommuting. In fact, we may write

$$\int\psi^{a_1}\psi^{a_2}\ldots\psi^{a_n}\,\mathrm{d}^n\psi = \epsilon^{a_1a_2\ldots a_n},$$

where $\epsilon^{a_1a_2\ldots a_n}$ is the totally antisymmetric symbol in n indices.

Now consider a change of variables $\psi'^a = N^a{}_b\,\psi^b$, where $N\in\mathrm{GL}(n)$ is some $n\times n$ matrix. We have

$$\begin{aligned}\int\psi'^{a_1}\psi'^{a_2}\ldots\psi'^{a_n}\,\mathrm{d}^n\psi &= N^{a_1}{}_{b_1}N^{a_2}{}_{b_2}\ldots N^{a_n}{}_{b_n}\int\psi^{b_1}\psi^{b_2}\ldots\psi^{b_n}\,\mathrm{d}^n\psi\\ &= N^{a_1}{}_{b_1}N^{a_2}{}_{b_2}\ldots N^{a_n}{}_{b_n}\epsilon^{b_1b_2\ldots b_n}\\ &= \det(N)\epsilon^{a_1a_2\ldots a_n}\\ &= \det(N)\int\psi'^{a_1}\psi'^{a_2}\ldots\psi'^{a_n}\,\mathrm{d}^n\psi'.\end{aligned}$$

This implies that under this change of variables the measure changes as $\mathrm{d}^n\psi' = \frac{1}{\det(N)}\,\mathrm{d}^n\psi$. Note that this is the opposite way round to what happens in bosonic integration, where we would have $\mathrm{d}^nx' = \det(N)\,\mathrm{d}^nx$ if $x'^a = N^a{}_b\,x^b$.

Example. Consider a single fermion ψ, and the change of variables $\chi = a\psi$. One has

$$\int \chi \, \mathrm{d}\chi = 1 = \int a\psi \, \mathrm{d}\chi \,,$$

which implies that $\mathrm{d}\chi = \mathrm{d}\psi \,/a$.

In QFT, we will often need to perform Gaussian integrals. Suppose ψ^1, ψ^2 are fermions, and let the 'action' be $S(\psi^a) = \frac{1}{2}\psi^1 M \psi^2$. One can think of this as a kinetic mass term (we are in zero dimensions, so there cannot be any derivatives). Then the path integral is

$$\int e^{-S(\psi^a)} \, \mathrm{d}\psi^1 \, \mathrm{d}\psi^2 = \int \left(1 - \frac{1}{2}\psi^1 M \psi^2\right) \mathrm{d}\psi^1 \, \mathrm{d}\psi^2 = \frac{1}{2} M,$$

where the series expansion for the exponential truncates at its first non-trivial term because $(\psi^1)^2 = (\psi^2)^2 = 0$. More generally, suppose we have $2m$ fermions ψ^a, and the action is $S(\psi^a) = \frac{1}{2}\psi^a M_{ab} \psi^b$, where M_{ab} is a $2m \times 2m$ which may be taken to be antisymmetric. Then

$$\int e^{-S(\psi^a)} \, \mathrm{d}^{2n}\psi = \int \sum^{\infty} \frac{(-1)^k}{2^k k!} (\psi^a M_{ab} \psi^b)^k \, \mathrm{d}^{2m}\psi$$

Note that the only contributing term is at $k = m$. This is because for $k < m$ there will be at least one factor of $\int 1 \, \mathrm{d}\psi^a = 0$, and for $k > m$ there will be at least one factor of $(\psi^a)^2 = 0$. So

$$\begin{aligned} \int e^{-S(\psi^a)} \, \mathrm{d}^{2n}\psi &= \frac{(-1)^m}{2^m m!} \int (\psi^a M_{ab} \psi^b)^m \, \mathrm{d}^{2m}\psi \\ &= \frac{(-1)^m}{2^m m!} \epsilon^{a_1 b_1 \ldots a_m b_m} M_{a_1 b_1} \ldots M_{a_m b_m} \\ &= \sqrt{\det(M)} = \mathrm{Pfaff}(M). \end{aligned}$$

So the path integral works out to the square root of the determinant of M, sometimes known as the Pfaffian. Note that this is again opposite to the bosonic case, where we would get

$$\int e^{-\frac{1}{2} x^a M_{ab} x^b} \, \mathrm{d}^{2m}x = \frac{(2\pi)^m}{\sqrt{\det(M)}} = \frac{(2\pi)^m}{\mathrm{Pfaff}(M)}.$$

2.2 Supersymmetric integrals and localisation

Consider a $d = 0$ theory of one bosonic variable x and two fermions ψ^1, ψ^2. The action must be bosonic and so generally takes the form

$$S(x, \psi^a) = V(x) - \psi^1 \psi^2 U(x),$$

where V, U are functions of x. If we imagine that V, U are polynomials, then the quadratic term in V is the kinetic term for x, and higher order terms give interactions. Even in $d = 0$, for generic U, V the path integral

$$\int e^{-S(x,\psi^a)} \, \mathrm{d}x \, \mathrm{d}\psi^1 \, \mathrm{d}\psi^2$$

is difficult.

However, suppose we choose a polynomial $W(x)$, and take

$$S(x,\psi,\bar{\psi}) = \frac{1}{2}(\partial W)^2 - \bar{\psi}\psi\partial^2 W,$$

where $\psi = \psi_1 + i\psi_2$, $\bar{\psi} = \psi_1 - i\psi_2$, and $\partial = \frac{\partial}{\partial x}$. Then the functions $V = \frac{1}{2}(\partial W)^2$, $U = \partial^2 W$ are now related. We can view this action as more symmetric than the generic case, and in fact this extended symmetry is an example of supersymmetry, as we will now demonstrate. Consider the following variation:

$$\begin{aligned}\delta x &= \epsilon\psi - \bar{\epsilon}\bar{\psi},\\ \delta\psi &= \bar{\epsilon}\partial W,\\ \delta\bar{\psi} &= \epsilon\partial W,\end{aligned}$$

where $\epsilon, \bar{\epsilon}$ are some fermionic parameters. The action $S(x,\psi,\bar{\psi})$ is invariant under this transformation, since

$$\begin{aligned}\delta S &= \frac{1}{2}\delta((\partial W)^2) - \delta(\bar{\psi}\psi\partial^2 W)\\ &= \frac{1}{2}\delta x\partial((\partial W)^2) - \delta\bar{\psi}\psi\partial^2 W - \bar{\psi}\delta\psi\partial^2 W - \bar{\psi}\psi\delta x\partial^3 W\\ &= (\epsilon\psi - \bar{\epsilon}\bar{\psi})\partial W\partial^2 W - \epsilon\psi\partial W\partial^2 W + \bar{\epsilon}\bar{\psi}\partial W\partial^2 W - \bar{\psi}\psi(\epsilon\psi - \bar{\epsilon}\bar{\psi})\partial^3 W.\end{aligned}$$

The first three terms in the last line cancel each other, while the other vanishes because $\psi^2 = \bar{\psi}^2 = 0$. Hence $\delta S = 0$. We will write $\delta = \epsilon Q + \bar{\epsilon}\bar{Q}$, so that

$$\begin{aligned} Qx &= \psi, & \bar{Q}x &= -\bar{\psi},\\ Q\psi &= 0, & \bar{Q}\psi &= \partial W,\\ Q\bar{\psi} &= \partial W, & \bar{Q}\bar{\psi} &= 0.\end{aligned}$$

$Q, \bar{Q}$ are the generators of the above transformation, and may be written as vector fields

$$Q = \psi\frac{\partial}{\partial x} + \partial W\frac{\partial}{\partial\bar{\psi}}, \qquad \bar{Q} = -\bar{\psi}\frac{\partial}{\partial x} + \partial W\frac{\partial}{\partial\psi}.$$

These generators obey $\left\{Q,\bar{Q}\right\} = 0$. Furthermore, they obey $Q^2 = \bar{Q}^2 = 0$ when the equations $\psi\partial^2 W = \bar{\psi}\partial^2 W = 0$ hold, which are just the equations of motion for the $\psi, \bar{\psi}$ fields. We say that $Q^2 = \bar{Q}^2 = 0$ 'on-shell'. Later on, when we use the superspace formalism, we will see that this on-shell condition becomes unnecessary. Note that in 0 dimensions the Hamiltonian H must vanish, because there is no concept of time evolution. Therefore, we have (on-shell)

$$\left\{Q,\bar{Q}\right\} = 2H, \qquad Q^2 = \bar{Q}^2 = 0.$$

So $Q, \bar{Q}$ are supercharges, and hence they generate supersymmetries.

The presence of these supercharges makes the path integral much easier. Consider the supersymmetric path integral

$$\int e^{-S(x,\psi,\bar{\psi})}\,\mathrm{d}x\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,,$$

and suppose we rescale $W \to \lambda W$, for some $\lambda \in \mathbb{R}_+$. Then the action changes

$$S \to S_\lambda = \frac{1}{2}\lambda^2(\partial W)^2 - \bar{\psi}\psi\lambda\partial^2 W,$$

and so do the supercharges

$$Q \to Q_\lambda = \psi\frac{\partial}{\partial x} + \lambda\partial W\frac{\partial}{\partial\bar{\psi}}, \qquad \bar{Q} \to \bar{Q}_\lambda = -\bar{\psi}\frac{\partial}{\partial x} + \lambda\partial W\frac{\partial}{\partial\psi}.$$

Consider the path integral after this rescaling $I(\lambda) = \int e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi$. We have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\lambda} &= \int \frac{\partial}{\partial\lambda}e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi \\ &= -\int (\lambda(\partial W)^2 - \bar{\psi}\psi\partial^2 W)e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi \\ &= -\int \bar{Q}_\lambda(\partial W\psi)e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi \\ &= -\int \bar{Q}_\lambda(\partial W\psi e^{-S_\lambda})\,\mathrm{d}x\,\mathrm{d}^2\psi\end{aligned}$$

But since $\bar{Q}_\lambda = -\bar{\psi}\frac{\partial}{\partial x} + (\lambda\partial W)\frac{\partial}{\partial\psi}$, the first term is a total derivative in the x integral, while the second is a total derivative in the ψ integral, and so the whole expression vanishes. So $\frac{\mathrm{d}I}{\mathrm{d}\lambda} = 0$. The original path integral $I(1)$ is therefore equivalent to $I(\lambda)$, no matter the value of λ. In particular we can take

$$\int e^{-S(x,\psi,\bar{\psi})}\,\mathrm{d}x\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi} = \lim_{\lambda\to\infty} I(\lambda).$$

As $\lambda \to \infty$, the $e^{-\frac{\lambda^2}{2}(\partial W)^2}$ factor suppresses the integrand except near where $\partial W = 0$. So the path integral 'localises' to the critical points of $W(x)$.

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Let's consider a different, slightly more general perspective. Suppose there is a Lie group G which acts freely (which means there are no elements $g \in G$ other than the identity for which $g(x) = x$ for all x – so there are no fixed points) on the space of field configurations. Also, let us assume that both the action S and the integration measure on the space of fields are invariant under this G-action. In this case, one may decompose the configuration space $\mathscr{C} = G \times \mathscr{C}/G$, where $\mathscr{C}/G$ is the space of equivalence classes of configurations under the G-action. Then one may carry out the integral separately over each factor in this decomposition, and the integral over G just gives a factor of $\mathrm{vol}(G)$, i.e. the volume of the group G.

Now consider the case where G is a fermionic group, which means that the elements of G are parametrised by fermionic variables θ. Then we have $\mathrm{vol}(G) = 0$, because by definition

$$\mathrm{vol}(G) = \int_G 1\,\mathrm{d}^{\dim(G)}\theta = 0.$$

So the path integral, being proportional to $\mathrm{vol}(G)$, vanishes!

The situation changes when there are fixed points $x \in \mathscr{C}$, i.e. configurations such that there is a non-identity $g \in G$ satisfying $g(x) = x$. This is because the decomposition $\mathscr{C} = G \times \mathscr{C}/G$ is then no longer valid. However, if we exclude the fixed points, then the above construction still works. As a consequence, we can only get contributions to the path integral from a neighbourhood of the fixed points. So again the path integral 'localises' to a certain region in configuration space.

In our case, the supersymmetry transformation is $\delta\psi = \bar{\epsilon}\partial W$, $\delta\bar{\psi} = \epsilon\partial W$, and the only fixed points of this action are the critical points of $W(x)$. Away from such critical points, let us change coordinates in configuration space, defining $y, \chi, \bar{\chi}$ as follows:

$$y = x - \frac{\bar{\psi}\psi}{\partial W}, \quad \chi = \psi\sqrt{\partial W}, \bar{\chi} = \bar{\psi}.$$

The theory of *superdeterminants* (see the first example sheet) allows us to deduce $\mathrm{d}x\,\mathrm{d}^2\psi = \sqrt{\partial W(y)}\,\mathrm{d}y\,\mathrm{d}^2\chi$.

The first point of interest is that

$$\begin{aligned}\delta y &= \delta x - \frac{\delta\bar\psi\,\psi + \bar\psi\,\delta\psi}{\partial W} + \frac{\bar\psi\psi}{(\partial W)^2}\delta x\partial^2 W\\ &= \epsilon\psi - \epsilon\bar\psi - \frac{\epsilon\partial W\psi}{\partial W} - \frac{\bar\psi\bar\epsilon\partial W}{\partial W}\\ &= 0,\end{aligned}$$

where we used the fact that $\bar\epsilon\bar\psi = -\bar\psi\bar\epsilon$. So y is invariant under the supersymmetry transformations. The second point of interest is that

$$S(y,0,0) = \frac{1}{2}(\partial W(x))^2 - \partial^2 W(x)\bar\psi\psi = S(x,\psi,\bar\psi).$$

So in these coordinates, the action is independent of the fermionic variables! Thus, if we let $U \subset \mathscr{C}$ be an open neighbourhood of the critical points, and let U^c be its complement, then the path integral over U^c vanishes:

$$\int_{U^c} e^{-S(c,\psi,\bar\psi)}\,\mathrm{d}x\,\mathrm{d}^2\psi = \int_{U^c} e^{-S(y,0.0)}\sqrt{\partial W(y)}\,\mathrm{d}y\,\mathrm{d}^2\chi = 0.$$

So again, the path integral localises to the critical points.

Let us now evaluate the contributions from the critical points. We shall assume that all such critical points x_* are isolated, so that

$$W(x) = W(x_*) + \frac{c_*}{2}(x-x_*)^2 + \mathcal{O}\Big((x-x_*)^3\Big).$$

The action, to quadratic order in the fields, is then given by

$$S^{(2)}(x,\psi,\bar\psi) = \frac{c_*^2}{2}(x-x_*)^2 - \bar\psi\psi c_* + \dots$$

Because of the scaling symmetry $W \to \lambda W$ described above, one can always localise to an infinitesimal neighbourhood of x_*, and so we are free to ignore all of the higher order terms. Hence (including now a conventional factor of $\frac{1}{\sqrt{2\pi}}$ for all bosonic fields), the contribution to the path integral from this critical point is given by

$$\begin{aligned}I &= \int e^{-S(x,\psi,\bar\psi)}\frac{\mathrm{d}x}{\sqrt{2\pi}}\,\mathrm{d}^2\psi\\ &= \int e^{-\frac{c_*^2}{2}(x-x)}(-1+\bar\psi\psi c_*)\frac{\mathrm{d}x}{\sqrt{2\pi}}\,\mathrm{d}^2\psi\\ &= \frac{c_*}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{c_*^2}{2}(x-x)}\,\mathrm{d}x\\ &= \frac{c_*}{|c_*|} = \mathrm{sign}(c_*) = \pm 1.\end{aligned}$$

If W has several critical points, we thus have $I = \sum_{c_*}\mathrm{sign}(c_*)$. So the path integral reflects the topology of the polynomial W:

$$I = \begin{cases} 0 & \text{if } W \text{ has odd degree}\\ -1 & \text{if } W \text{ has even degree and } W \to -\infty \text{ as } |x| \to \infty\\ 1 & \text{if } W \text{ has even degree and } W \to +\infty \text{ as } |x| \to \infty\end{cases}$$

So there are two 'miracles'. First, we can actually do the path integral, and second, the answer is simple.

2.3 $d = 0$ Landau-Ginzburg theory

This is a complex generalisation of the theory described in the previous subsection. Our fields are now one complex number $z \in \mathbb{C}$ and two complex fermions ψ_1, ψ_2. We choose a holomorphic $W(z)$, and set the action to

$$S(z, \psi_i) = |\partial W|^2 + \partial^2 W \psi_1 \psi_2 - \overline{\partial^2 W} \bar{\psi}_1 \bar{\psi}_2,$$

where here $\partial = \partial_z$ is a holomorphic derivative.

This theory is supersymmetric. The supersymmetry transformations are

$$\begin{aligned} \delta z &= \epsilon_1 \psi_1 + \epsilon_2 \psi_2, & \bar{\delta} \bar{z} &= \bar{\epsilon}_1 \bar{\psi}_1 + \bar{\epsilon}_2 \bar{\psi}_2, \\ \delta \psi_1 &= \epsilon_2 \overline{\partial W}, & \bar{\delta} \bar{\psi}_1 &= \bar{\epsilon}_2 \partial W, \\ \delta \psi_2 &= -\epsilon_1 \overline{\partial W}, & \bar{\delta} \bar{\psi}_2 &= -\bar{\epsilon}_1 \partial W, \end{aligned}$$

with all other variations vanishing, $\bar{\delta} z = \bar{\delta} \psi_i = \delta \bar{z} = \delta \bar{\psi}_i = 0$. If we let $Q_i, \bar{Q}_i$ be the generators of the transformations with parameters $\epsilon_i, \bar{\epsilon}_i$, then one may verify that the supersymmetry relations

$$\{Q_i, Q_j, =\} \Big\{ \bar{Q}_i, \bar{Q}_j, = \Big\} \Big\{ Q_i, \bar{Q}_j, = \Big\} 0$$

hold (on-shell).

By again rescaling $W \to \lambda W$ for $\lambda \in \mathbb{R}_+$, one may see that the path integral localises to the critical points z_* of $W(z)$. Writing

$$W(z) = W(z_*) + \frac{\alpha_*}{2} (z - z_*)^2 + \mathcal{O}\Big((z - z_*)^3 \Big),$$

the quadratic order action takes the form

$$S^{(2)}(z, \psi_i) = |\alpha_*|^2 |z - z_*|^2 + \alpha_* \psi_1 \psi_2 - al\bar{p}ha_* \bar{\psi}_1 \bar{\psi}_2 + \dots$$

Therefore one may evaluate the path integral:

$$\begin{aligned} I &= \frac{1}{2\pi} \int e^{-S(z, \psi_i)} \, \mathrm{d}^2 z \, \mathrm{d}^4 \psi \\ &= \sum_{z_*} \frac{1}{2\pi} \int e^{-|\alpha_* (z - z_\star)|^2} |\alpha_*|^2 \, \mathrm{d}^2 z \\ &= \sum_{z_*} \frac{|\alpha_*|^2}{|\alpha_*|^2} = \sum_{z_*} 1. \end{aligned}$$

So this time the path integral counts the number of critical points of W.

More generally, let $f(z)$ be any holomorphic function. Then its expectation value[1]

$$\langle f(z) \rangle = \frac{1}{2\pi} \int e^{-S(z, \psi_i)} f(z) \, \mathrm{d}^2 z \, \mathrm{d}^4 \psi$$

[1] Note that we are defining this expectation value without a normalising factor of $\frac{1}{I}$.

is still invariant under the $\bar{\delta}$ transformations, because $\bar{\delta} f(z) = \bar{\delta} z \, \partial f(z) = 0$. So again this integral will localise to the critical points of $\bar{W}(\bar{z})$, and we get

$$\langle f(z) \rangle = \sum_{z_*} f(z_*).$$

This of course only works if $\bar{\delta} f = 0$. The holomorphic functions are however not the only ones satisfying this condition, and it is useful to think about how in general one might construct such a function. Since $\bar{\delta} = \bar{\epsilon}_1 \bar{Q}_1 + \bar{\epsilon}_2 \bar{Q}_2$, what we are really talking about are $\bar{Q}_i$-invariant functions. Since $\bar{Q}_i^2 = 0$, one way to construct such functions is simply to take $\bar{Q}_i \Lambda$, where $\Lambda = \Lambda(z, \bar{z}, \psi_j \bar{\psi}_k)$ is any smooth function. But these functions are actually not very interesting, because

$$\begin{aligned} \left\langle \bar{Q}\Lambda \right\rangle &= \int (\bar{Q}\Lambda) e^{-S} \frac{\mathrm{d}^2 z}{2\pi} \, \mathrm{d}^4 \psi \\ &= \int \bar{Q}(\Lambda e^{-S}) \frac{\mathrm{d}^2 z}{2\pi} \, \mathrm{d}^4 \psi = 0, \end{aligned}$$

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which vanishes because it is the integral of a total derivative.

A clear consequence is $\left\langle F + \bar{Q}\Lambda \right\rangle = \langle F \rangle$. So the 'interesting' functions are the ones which are $\bar{Q}$-invariant, but only up to addition of functions which are of the form $\bar{Q}\Lambda$ (which we say are $\bar{Q}$-exact). This is an example of cohomology. The interesting functions are elements of

$$H_{\bar{Q}} = \frac{\ker \bar{Q}}{\operatorname{im} \bar{Q}},$$

which is called the $\bar{Q}$-cohomology. $H_{\bar{Q}}$ is clearly closed under addition and subtraction, but note also that the product of a $\bar{Q}$-exact function with a $\bar{Q}$-invariant function is $\bar{Q}$-exact:

$$F \bar{Q} \Lambda = \bar{Q}(F\Lambda),$$

which implies that $H_{\bar{Q}}$ is also closed under multiplication. Thus $H_{\bar{Q}}$ is a ring. It is often called the chiral[2] ring in the context of supersymmetry.

In the case of $d = 0$ Landau-Ginzburg theory, for an example of a $\bar{Q}$-exact operator, we need only look at the transformation rule $\bar{\delta} \bar{\psi}_i = \bar{\epsilon}_i \partial W$, which implies that ∂W is $\bar{Q}$ of something. Hence if an operator $\mathcal{O}$ contains ∂W as a factor, then its correlator $\langle \mathcal{O} \rangle$ is trivial (it vanishes.) As a more specific example, suppose

$$W(z) = \frac{z^{n+1}}{n+1} - az, \qquad \partial W = z^n - a.$$

Then the non-trivial $\bar{Q}$-invariant operators are polynomials subject to the identification $z^n = a$, and these operators form a ring generated by $\{1, z, z^2, \ldots, z^{n-1}\}$, i.e.

$$H_{\bar{Q}} = \frac{\mathbb{C}[z]}{z^n - a}.$$

[2] 'Chiral' because we chose $\bar{Q}$ out of the two possible choices $Q, \bar{Q}$.

3 Supersymmetric Quantum Mechanics

We now move onto $d = 1$. We will try to understand this from two perspectives: first in a canonical framework, and then in a path integral framework. These two perspectives are fundamentally equivalent, so the results we get will be the same – but they will look quite different. Comparing the results in the two forms is a powerful tool.

Consider a worldline theory of a single bosonic field $x(t)$ and a single complex fermion $\psi(t)$ with action

$$S[x, \psi, \bar{\psi}] = \int \left[\frac{1}{2}\dot{x}^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2}(\partial h)^2 - \bar{\psi}\psi\partial^2 h\right] \mathrm{d}t\,,$$

where $h = h(x(t))$ is a function which plays the role of a potential. This theory is supersymmetric. We see this in the usual way: S is invariant under the following transformation:

$$\begin{aligned}
\delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi, \\
\delta\psi &= \epsilon(i\dot{x} + \partial h), \\
\delta\bar{\psi} &= \bar{\epsilon}(-i\dot{x} + \partial h).
\end{aligned}$$

If we allow the parameters $\epsilon, \bar{\epsilon}$ to be time-dependent, we find

$$\delta S = \int (\dot{\epsilon}Q + \dot{\bar{\epsilon}}\bar{Q})\,\mathrm{d}t\,,$$

so by the Noether procedure $Q, \bar{Q}$ are the generators of the transformations parametrised by $\epsilon, \bar{\epsilon}$ respectively. We have $Q^2 = \bar{Q}^2 = 0$. Also,

$$\begin{aligned}
\left\{Q, \bar{Q}\right\}x &= (Q\bar{Q} + \bar{Q}Q)x \\
&= -Q\psi + \bar{Q}\bar{\psi} \\
&= -(i\dot{x} + \partial h) + (-i\dot{x} + \partial h) = -2i\dot{x}, \\
\left\{Q, \bar{Q}\right\}\psi &= \bar{Q}(i\dot{x} + \partial h), \\
&= -i\dot{\psi} - \psi\partial^2 h, \\
&\simeq -2i\dot{\psi},
\end{aligned}$$

where $\simeq$ means an equality that holds on shell (this particular equality holds when the $\bar{\psi}$ equation of motion is satisfied). Thus, up to the fermionic equations of motion, $\left\{Q, \bar{Q}\right\}$ generates time evolution, with a factor of 2. The Hamiltonian generates time evolution by definition, so we have $\left\{Q, \bar{Q}\right\} = 2H$. So these are supercharges, and the theory is indeed supersymmetric.

To canonically quantise, first we need to move to a classical Hamiltonian mechanical point of view. The conjugate momenta are

$$\begin{aligned}
p &= \frac{\delta L}{\delta\dot{x}} = \dot{x}, \\
\pi &= \frac{\delta L}{\delta\dot{\psi}} = i\bar{\psi}.
\end{aligned}$$

The Hamiltonian is the Legendre transform of the Lagrangian:

$$\begin{aligned}
H &= p\dot{x} + \pi\dot{\psi} - L \\
&= \frac{1}{2}p^2 + \frac{1}{2}(\partial h)^2 + \frac{1}{2}\partial^2 h(\bar{\psi}\psi - \psi\bar{\psi}).
\end{aligned}$$

The reason we have written the final term in this way will become apparent.

Upon quantisation, we have (setting $\hbar = 1$)

$$[\hat{x}, \hat{p}] = i, \qquad \left\{\hat{\psi}, \hat{\bar{\psi}}\right\} = 1,$$

where a hat indicates that an object is an operator on a Hilbert space $\mathcal{H}$. For the Hilbert space factor describing x, we can as usual take $L^2(\mathbb{R}, \mathrm{d}x)$, the space of square integrable functions on the real line. Then $\hat{x}, \hat{p}$ are represented by $x, -i\frac{\partial}{\partial x}$ respectively. To deal with the fermions, we define a fermionic 'number operator' $\hat{F} = \hat{\bar{\psi}}\hat{\psi}$. We have

$$\left[\hat{F}, \hat{\psi}\right] = -\hat{\psi}, \qquad \left[\hat{F}, \hat{\bar{\psi}}\right] = +\hat{\bar{\psi}},$$

so $\hat{\bar{\psi}}$ adds 1 to the number of fermions, while $\hat{\psi}$ subtracts 1. We let the vacuum of the fermionic system be $|0\rangle$, defined by $\hat{\psi}\,|0\rangle = 0$. The first excited state is $|1\rangle = \hat{\bar{\psi}}\,|0\rangle$. Actually, since $\hat{\bar{\psi}}^2 = 0$, there are no further excited states. Therefore the full Hilbert space is

$$\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)\,|0\rangle \oplus L^2(\mathbb{R}, \mathrm{d}x)\,|1\rangle = L^2(\mathbb{R}, \mathrm{d}x) \otimes \mathbb{C}^2.$$

Now we need to convert the supercharges and the Hamiltonian to their quantum counterparts. For the supercharges, we can just write

$$\hat{Q} = \hat{\bar{\psi}}(i\hat{p} + \partial h),$$
$$\hat{\bar{Q}} = \hat{\psi}(-i\hat{p} + \partial h),$$

and there is no ordering ambiguity for the various operators involved. For the Hamiltonian, there is some ordering ambiguity. However, supersymmetry comes to the rescue here – if we want it to remain valid in the quantum theory, then we must have $2\hat{H} = \left\{\hat{Q}, \hat{\bar{Q}}\right\}$. Using this relation, we find

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}(\partial h)^2 + \frac{1}{2}\partial^2 h(\hat{\bar{\psi}}\hat{\psi} - \hat{\psi}\hat{\bar{\psi}}).$$

So the ordering is the same as the one we chose for the classical expression above.

From now on we will drop all hats on quantum operators.

3.1 Supersymmetric ground states

As before, we have $\langle\Psi|H|\Psi\rangle \geq 0$, with equality if and only if $Q\,|\Psi\rangle = \bar{Q}\,|\Psi\rangle = 0$. Thus a state of zero energy in SQM must be invariant under supersymmetry, and will be a ground state.

Let us represent the fermionic Hilbert space with $|0\rangle \to \binom{1}{0}, |1\rangle \to \binom{0}{1}$. Then the full Hilbert space consists of rank 2 vectors whose entries are square integrable functions on the real line. Suppose the state $|\Psi\rangle = \binom{f(x)}{g(x)}$ satisfies $Q\,|\Psi\rangle = \bar{Q}\,|\Psi\rangle = 0$. Then we may write this as

$$\begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial x} + \partial h & 0 \end{pmatrix}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + \partial h \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0$$

These equations are simple to solve, and we find that a zero energy ground state must be of the form

$$|\Psi\rangle = \begin{pmatrix} Ae^{-h(x)} \\ Be^{+h(x)} \end{pmatrix},$$

where A, B are some constants. For this to really be a state, it must be normalisable – so one of A, B must vanish. In fact:

- If $h(x) \to +\infty$ as $|x| \to \infty$, then the supersymmetric ground state is $\binom{Ae^{-h(x)}}{0}$.
- If $h(x) \to -\infty$ as $|x| \to \infty$, then the supersymmetric ground state is $\binom{0}{Be^{+h(x)}}$.
- If $h(x) \to \pm\infty$ as $x \to +\infty$ but $h(x) \to \mp\infty$ as $x \to -\infty$, then neither solution is normalisable, so there is no zero energy state! This is fine, since there is still a ground state, it just has non-zero energy, and is not supersymmetric. In this case we say that supersymmetry is spontaneously broken.

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In a supersymmetric theory, excited states (i.e. states with postive energy $E > 0$) come in pairs. To see this, let us decompose the Hilbert space into energy eigenspaces,

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n \quad \text{where} \quad H\,|\psi_n\rangle = E_n\,|\psi_n\rangle\,\forall\,|\psi_n\rangle\,\mathcal{H}_n.$$

We can further decompose each eigenspace $\mathcal{H}_n = \mathcal{H}_{n,B} \oplus \mathcal{H}_{n,F}$, where the bosonic part $\mathcal{H}_{n,B}$ contains states with an even number of fermions, and the fermionic part $\mathcal{H}_{n,F}$ contains states with an odd number of fermions. In particular, Q maps $\mathcal{H}_{n,F}$ to $\mathcal{H}_{n,B}$, since $[Q, H] = 0$. Also, Q annihilates $\mathcal{H}_{n,B}$. Thus, given a $|b\rangle \in \mathcal{H}_{n,B}$, we have

$$2E_n\,|b\rangle = (Q\bar{Q} + \bar{Q}Q)\,|b\rangle = Q(\bar{Q}\,|b\rangle).$$

So, for $E_n > 0$, $|b\rangle$ is Q of something in $\mathcal{H}_{n,F}$, namely $\frac{1}{2E_n}\bar{Q}\,|b\rangle$. Similarly, any state $|f\rangle$ in $\mathcal{H}_{n,F}$ with $E_n > 0$ may be written as $\bar{Q}$ of a state in $\mathcal{H}_{n,B}$. Thus the two spaces are isomorphic, $\mathcal{H}_{n,B} \simeq \mathcal{H}_{n,F}$, if $E_n > 0$. We say that each pair of excited states has a bosonic and fermionic partner.

This is not the case for states of zero energy, which makes the following a non-trivial object of study.

Definition. The *Witten index* of a supersymmetric theory is defined as the number of independent bosonic zero energy states minus the number of independent fermionic zero energy states:

$$I_W = \dim \mathcal{H}_{0,B} - \dim \mathcal{H}_{0,F}.$$

We may write the Witten index as

$$I_W = \mathrm{Tr}_{\mathcal{H}}(-1)^F,$$

the rationale being that each pair of excited states contribute a $+1$ and a -1 to this sum, which cancel cancel. One might be concerned that in the case of an infinite dimensional Hilbert space this sum is not well defined. A way to get around this is to instead write

$$I_W = \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right),$$

where now each pair of excited states of energy E contribute a $+e^{-\beta E}$ and $-e^{-\beta E}$ to this sum, which again cancel. This expression has much more reasonable convergence properties.

Definition. The *supertrace* of an operator $\mathcal{O}$ is defined as

$$\mathrm{STr}\,\mathcal{O} = \mathrm{Tr}\left((-1)^F \mathcal{O}\right).$$

So we may write

$$I_W = \mathrm{STr}(e^{-\beta H}).$$

This expression is completely independent of β. We included the $e^{-\beta H}$ factor to regularise the trace, but it also has another use – it allows us to make a connection to the path integral.

3.2 Path integrals in quantum mechanics

Consider a free particle travelling on $\mathbb{R}$. The time evolution operator e^{-iHt} becomes $e^{-H\tau}$ under a Wick rotation $t \to -i\tau$. By definition, if our particle is located at y_0 at $\tau = 0$, then the amplitude for it to be found at y_1 at $\tau = \beta$ is given by

$$\langle y_1|e^{-\beta H}|y_0\rangle = K_\beta(y_1, y_0) = \frac{1}{\sqrt{2\pi\beta}} \exp\left(-\frac{(y_0 - y_1)^2}{2\beta}\right).$$

This is sometimes called the heat kernel. We may break this evolution into N steps of length $\Delta\tau = \beta/N$, to obtain

$$\langle y_1|e^{-\beta H}|y_0\rangle = \int \langle y_1|e^{-\Delta\tau H}|x_{N-1}\rangle \, \langle x_{N-1}|e^{-\Delta\tau H}|x_{N-2}\rangle \dots \langle x_2|e^{-\Delta\tau H}|x_1\rangle \, \langle x_1|e^{-\Delta\tau H}|y_0\rangle \, \mathrm{d}^{N-1}x$$

$$= \int K_{\Delta\tau}(y_1, x_{N-1}) \dots K_{\Delta\tau}(x_1, y_0) \, \mathrm{d}^{N-1}x$$

$$= \int \exp\left[-\sum_{i=0}^{N} \frac{\Delta\tau}{2}\left(\frac{x_{i+1} - x_i}{\Delta\tau}\right)^2\right](2\pi\Delta\tau)^{-N/2} \prod_{i=1}^{N-1} \mathrm{d}x_i$$

where $x_N = y_1$ and $x_0 = y_0$. Taking the limit as $N \to \infty$ but keeping β fixed (so $\Delta\tau = \beta/N \to 0$), and defining

$$\exp\left(-\int_0^\beta \frac{1}{2}\dot{x}^2 \, \mathrm{d}\tau\right) \mathrm{D}x = \lim_{N\to\infty} \exp\left[-\sum_{i=0}^{N} \frac{\Delta\tau}{2}\left(\frac{x_{i+1} - x_i}{\Delta\tau}\right)^2\right](2\pi\Delta\tau)^{-N/2} \prod_{i=1}^{N-1} \mathrm{d}x_i \,,$$

then we have (at least heursitically)

$$\langle y_1|e^{-\beta H}|y_0\rangle = \int_{C[y_0,y_1]} \exp\left(-\int_0^\beta \frac{1}{2}\dot{x}^2 \, \mathrm{d}\tau\right) \mathrm{D}x \,,$$

where $C[y_0, y_1]$ is the space of all continuous maps $x : [0, \beta] \to \mathbb{R}$ such that $x(0) = y_0$ and $x(\beta) = y_1$.

A similar derivation works when the particle is in a potential, so $H = \frac{p^2}{2} + V(x)$. In that case, the action is $S = \int_0^\beta \left(\frac{1}{2}\dot{x}^2 + V(x)\right) \mathrm{d}\tau$.

The partition function $\mathcal{Z}(\beta)$ is closely related to the heat kernel. It is defined as

$$\mathcal{Z}(\beta) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) = \int_{\mathbb{R}} \langle x|e^{-\beta H}|x\rangle \, \mathrm{d}x \,.$$

Using the above expression, this may be written

$$\mathcal{Z}(\beta) = \int \left[\int_{\mathcal{C}[y,y]} e^{S[x]} \, \mathrm{D}x\right] \mathrm{d}y$$

$$= \int_{\mathcal{C}_{S^1}} e^{-S[x]} \, \mathrm{D}x \,,$$

where the range of integration in the last line is now the space of continuous maps from the circle S^1 to the real line.

3.3 Path integrals for fermions

Consider a Hilbert space for a fermionic theory. We can obtain a path integral in much the same way as with bosons. However, there are some subtleties of which one should be aware.

Definition. The fermionic *coherent states* are defined by

$$|\eta\rangle = e^{\bar{\psi}\eta}\,|0\rangle\,,$$

where $\bar{\psi}$ is the fermion creation operator, but η is just a c-number fermion.

These states obey the following:

$$\begin{aligned}
\psi\,|\eta\rangle &= \eta\,|\eta\rangle\,,\\
\langle\bar{\chi}|\eta\rangle &= e^{\bar{\chi}\eta},\\
I_{\mathcal{H}} &= \int e^{-\bar{\eta}\eta}\,|\bar{\eta}\rangle\,\langle\bar{\eta}|\,\mathrm{d}^2\eta\,,\\
\mathrm{Tr}(A) &= \int \langle -\bar{\eta}|A|\eta\rangle\,e^{-\bar{\eta}\eta}\,\mathrm{d}^2\eta\,,\\
\mathrm{STr}(A) &= \int \langle\bar{\eta}|A|\eta\rangle\,e^{-\bar{\eta}\eta}\,\mathrm{d}^2\eta\,.
\end{aligned}$$

Note the presence of the minus sign in the bra in the fourth line, and its absence in the fifth line.

Using these, and following the same procedure as for the bosons, we have

$$\langle\bar{\chi}'|e^{-\beta H}|\chi\rangle = \int \langle\bar{\chi}'|e^{-\Delta\tau H}|\eta_{N-1}\rangle\ldots\langle\bar{\eta}_1|e^{-\Delta\tau H}|\chi\rangle\prod_{i=1}^{N-1} e^{-\bar{\eta}_k\eta_k}\,\mathrm{d}^2\eta_k\,.$$

By using the commutation relations, we can order the operators that make up the Hamiltonian so that all ψs appear to the right hand side of all $\bar{\psi}$s. Then we have (to leading order in $\Delta\tau$)

$$\begin{aligned}
\langle\bar{\eta}_{k+1}|e^{-\Delta\tau H(\bar{\psi},\psi)}|\eta_k\rangle &= \langle\bar{\eta}_{k+1}|(1-\Delta\tau H(\bar{\psi},\psi))|\eta_k\rangle\\
&= \langle\bar{\eta}_{k+1}|(1-\Delta\tau H(\bar{\eta}_{k+1},\eta_k))|\eta_k\rangle\\
&= e^{-\Delta\tau H(\bar{\eta}_{k+1},\eta_k)}\,\langle\bar{\eta}_{k+1}|\eta_k\rangle\\
&= e^{-\Delta\tau H(\bar{\eta}_{k+1},\eta_k)}e^{\bar{\eta}_{k+1}\eta_k}.
\end{aligned}$$

Using this, we have

$$\begin{aligned}
\langle\bar{\chi}'|e^{-\beta H}|\chi\rangle &= \lim_{N\to\infty}\int\exp\left(\sum_{k=1}^{N}\bar{\eta}_k\eta_{k-1}-\Delta\tau H(\bar{\eta}_k,\eta_{k-1})\right)\prod_{k=1}^{N-1}e^{-\bar{\eta}_k\eta_k}\,\mathrm{d}^2\eta_k\\
&= \lim_{N\to\infty}\int\exp\left(-\sum_{k=1}^{N}\left[\bar{\eta}_k\frac{\eta_k-\eta_{k-1}}{\Delta\tau}+H(\bar{\eta}_k,\eta_{k-1})\right]\Delta\tau\right)e^{\bar{\eta}_N\eta_N}\prod_{k=1}^{N-1}\mathrm{d}^2\eta_k\,,
\end{aligned}$$

where $\eta_0=\chi, \eta_N=\chi'$. It is natural to write this as a path integral in the following way:

$$\langle\bar{\chi}'|e^{-\beta H}|\chi\rangle = \int e^{-S[\bar{\eta},\eta]}e^{\bar{\eta}(\beta)\eta(\beta)}\,\mathrm{D}\eta\,\mathrm{D}\bar{\eta}\,,$$

where the range of integration is over all fermions satisfying $\eta(0) = \chi, \eta(\beta) = \chi'$, and the action is

$$S[\bar\eta, \eta] = \int_0^\beta (\bar\eta\dot\eta - H(\bar\eta, \eta))\,\mathrm{d}\tau\,.$$

The partition function is

$$\begin{aligned}\mathcal{Z}(\beta) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) &= \int \langle -\bar\chi|e^{-\beta H}|\chi\rangle\, e^{-\bar\chi\chi}\,\mathrm{d}^2\chi \\ &= \int \exp\Big(-S[\bar\psi,\psi]\Big)\,\mathrm{D}\psi\,\mathrm{D}\bar\psi\,.\end{aligned}$$

Because of the minus sign, this integral must be done over all fermions satisfying $\psi(\tau+\beta) = -\psi(\tau)$. This is known as *antiperiodic* boundary conditions.

On the other hand, we have

$$\begin{aligned}\mathrm{STr}(e^{-\beta H}) &= \int \langle \bar\chi|e^{-\beta H}|\chi\rangle\, e^{-\bar\chi\chi}\,\mathrm{d}^2\chi \\ &= \int \exp\Big(-S[\bar\psi,\psi]\Big)\,\mathrm{D}\psi\,\mathrm{D}\bar\psi\,,\end{aligned}$$

where this time there is no minus sign, and so this integral is done over all configurations with *periodic* boundary conditions, i.e. $\psi(\tau+\beta) = \psi(\tau)$.

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So the Witten index has the path integral representation

$$I_W = \mathrm{STr}(e^{-\beta H}) = \mathrm{Tr}\Big((-1)^F e^{-\beta H}\Big) = \int_{\text{periodic}} e^{-S_E[x,\psi,\bar\psi]}\,\mathrm{D}x\,\mathrm{D}\psi\,\mathrm{D}\bar\psi\,,$$

where the Euclidean action

$$S_E = \oint \left[\frac{1}{2}\dot x^2 + \bar\psi\dot\psi + \frac{1}{2}(\partial h)^2 + \partial^2 h\bar\psi\psi\right]\mathrm{d}\tau$$

is invariant under the supersymmetry transformations

$$\begin{aligned}\delta x &= \epsilon\bar\psi - \bar\epsilon\psi, \\ \delta\psi &= \epsilon(-\dot x + \partial h), \\ \delta\bar\psi &= \bar\epsilon(\dot x + \partial h).\end{aligned}$$

Note that these transformations make sense globally on S^1 because of the perioricity of $x, \psi, \bar\psi$.

Let's now compute the Witten index with this path integral. We expect to get the same answer as before. As in $d = 0$, we consider a rescaling $h \to \lambda h$ for $\lambda \in \mathbb{R}_+$, and consider how the Witten index changes under this rescaling. We have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} I_W(\lambda) = -\int \left[\oint \lambda(\partial h)^2 + \partial^2 h\bar\psi\psi\right] e^{-S_E[x,\psi,\bar\psi]}\,\mathrm{D}x\,\mathrm{D}\psi\,\mathrm{D}\bar\psi\,.$$

But

$$Q_\lambda\left(\oint \partial h\psi\,\mathrm{d}\tau\right) = \oint \left[\partial^2 h\bar\psi\psi + \lambda(\partial h)^2 - \partial h\frac{\mathrm{d}x}{\mathrm{d}\tau}\right]\mathrm{d}\tau\,.$$

Note that $\partial h\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}h}{\mathrm{d}\tau}$ is a total derivative, and $h(x(\tau))$ is periodic in τ, so the last term in the integrand vanishes. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} I_W(\lambda) = \langle Q_\lambda(\dots)\rangle = 0.$$

So I_λ is independent of the choice of λ. In particular as $\lambda \to \infty$, the term $\exp\left(-\frac{\lambda^2}{2}\oint(\partial h)^2\,\mathrm{d}\tau\right)$ suppresses all maps $x : S^1 \to \mathbb{R}$, except in a neighbourhood of constant maps to critical points of h. Near such critical points we can write $x(\tau) = x_* + \delta x(\tau)$, where δx is small. To quadratic order in the fields, we then have

$$S_E^{(2)} = \oint \left[\frac{1}{2}\delta x\left(-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + h''(x_*)^2\right)\delta x + \bar{\psi}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + h''(x_*)\right)\psi\right]\mathrm{d}\tau\,.$$

Since δx and the fermions must be periodic, they permit Fourier expansions

$$\delta x(\tau) = \sum_{n\in\mathbb{Z}} \delta x_n \exp\left(\frac{2\pi i n\tau}{\beta}\right),$$
$$\psi(\tau) = \sum_{n\in\mathbb{Z}} \psi_n \exp\left(\frac{2\pi i n\tau}{\beta}\right),$$

where $\delta x_{-n} = \overline{\delta x_n}$, since δx is real. We now find, near a critical point

$$\begin{aligned}\int e^{-S_E^{(2)}}\,\mathrm{D}\delta x\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi} &= \frac{\det(\partial_\tau + h''(x_*))}{\sqrt{\det(-\partial_\tau^2 + h''(x_*)^2)}} \\ &= \frac{\prod_{n\in\mathbb{Z}}\left(\frac{2\pi i n}{\beta} + h''(x_*)\right)}{\sqrt{\prod_{n\in\mathbb{Z}}\left(\left(\frac{2\pi n}{\beta}\right)^2 + h''(x_*)^2\right)}} \\ &= \frac{h''(x_*)}{|h''(x_*)|}.\end{aligned}$$

In the first line we used our formulae for Gaussian integrals we derived previously. In the second line, we used the fact that the Fourier modes $\exp\left(\frac{2\pi i n\tau}{\beta}\right)$ comprise an eigenbasis of each of the operators whose determinants we are computing. The third line resulted from noting that all factors with $n \neq 0$ cancel each other in the numerator and denominator – in particular, the combined contributions from $n = \pm k$ for $k \in \mathbb{Z}_+$ cancel to 1.

Therefore,

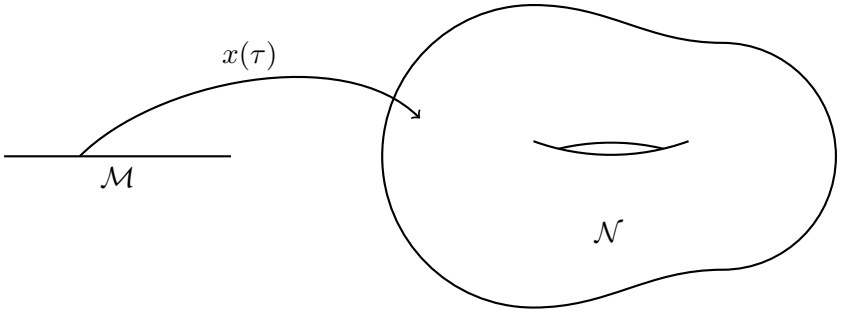
$$I_W = \sum_{x_*} \mathrm{sign}(h''(x_*)),$$

which is in fact the same answer as the canonical approach.

It is nice that we can study the same object from these two different points of view, but with this reasonably simple supersymmetric theory, the amount we can learn is limited. By investigating more involved theories, we can obtain more mathematically interesting results. We start to do this in the next section.

3.4 Non-linear sigma models

A non-linear sigma model is a theory involving a map x from a space $\mathcal{M}$ to a target space $\mathcal{N}$. To start with we will study the case where $\mathcal{M}$ is one-dimensional, and in particular given by $[0, \beta]$ or S^1, but one can also consider higher-dimensional versions – for example, string theory is a two-dimensional non-linear sigma model. In the one-dimensional case we call $\mathcal{M}$ the worldline, and we will assume that $\mathcal{N}$ is a compact Riemannian manifold with metric g.



Often we will let x^a be coordinates on some patch $U \subset \mathcal{N}$, and $x^a(\tau)$ the corresponding fields, where indices $a, b, \ldots$ take values in $1, \ldots, n = \dim \mathcal{N}$. We can choose the following (non-supersymmetric) action:

$$S[x] = \int \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b \, \mathrm{d}\tau \, .$$

This action gives the length of the image of x. Note that $g_{ab}(x)$ generically depends on x, so there are cubic and higher order in x terms appearing in the action – so this is an interacting worldline QFT.

If one varies x, the action changes by

$$\begin{aligned} \delta S &= \int_{\mathcal{M}} \left[g_{ab}(x) \dot{x}^a \frac{\mathrm{d}}{\mathrm{d}\tau} \delta x^b + \frac{1}{2} \partial_c g_{ab} \dot{x}^a \dot{x}^b \delta x^c \right] \mathrm{d}\tau \\ &= \int_{\mathcal{M}} \left[-\frac{\mathrm{d}}{\mathrm{d}\tau} (g_{ac} \dot{x}^a) + \frac{1}{2} \partial_c g_{ab} \dot{x}^a \dot{x}^b \right] \delta x^c \, \mathrm{d}\tau + \Big[g_{ab}(x) \dot{x}^a \delta x^b \Big]_{\partial \mathcal{M}}. \end{aligned}$$

Hence the classical equations of motion are just the geodesic equations

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0,$$

where Γ is the Levi-Civita connection on $(\mathcal{N}, g)$. So, classically, the particle travels on a geodesic.

For the quantum version, we have

$$p_q = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^b,$$

and the canonical quantisation relations are

$$[\hat{x}^a, \hat{p}_b] = i \delta^a_b.$$

We can choose the Hilbert space to be $\mathcal{H} = L^2(\mathcal{N}, \sqrt{g} \, \mathrm{d}^n x)$ (the space of square integrable functions on $\mathcal{N}$ with respect to the volume element induced from the metric g_{ab}). The canonical operators are then represented as usual by $\hat{x}^a = x^a$ and $\hat{p}^a = -i \frac{\partial}{\partial x^a}$.

However, there is no preferred choice of quantum Hamiltonian. Classically, as usual, the Hamiltonian is given by

$$H = p_a \dot{x}^a - L = \frac{1}{2} g^{ab}(x) p_a p_b,$$

but there's an ordering ambiguity in turning this into a quantum operator on $L^2(\mathcal{N}, \sqrt{g} \, \mathrm{d}^n x)$. We need some general principles which might fix this ordering ambiguity, and the following are reasonable choices:

- $\hat{H}$ should be generally covariant.
- $\hat{H}$ should reduce to $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$ in the case of flat space, $(\mathcal{N}, g) = (\mathbb{R}^n, \eta)$.
- $\hat{H}$ should contain no more than 2 derivatives acting on either the wavefunction or the metric.

Although we will not do so here, one can show that there is a 1-parameter family of such $\hat{H}$s, given by

$$\hat{H} = \frac{1}{2}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(g^{ab}\sqrt{g}\frac{\partial}{\partial x^b}\right) + \alpha R[g]\right).$$

The term on the left is just the Laplacian of the metric g, and $R[g]$ is its Ricci scalar, but α is an unfixed real number. There is no preferred choice of α, and different choices correspond to different ways to regularise the path integral.

It is possible to completely eliminate this ambiguity by adding supersymmetry.

3.5 Supersymmetric NLSM

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Let us now define a supersymmetric version of the NLSM introduced in the previous section. As before, the bosonic fields consist of a map $x : \mathcal{M} \to \mathcal{N}$. To these we add a set of fermionic fields $\psi^a, a = 1, \ldots, n$. Together these fields can be considered as a single fermionic function on $\mathcal{M}$ that takes values in the tangent space $T_x\mathcal{N}$. We choose the action

$$S[x, \psi] = \int_{\mathcal{M}} \left[\frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b + ig_{ab}\bar{\psi}^a(\nabla_t\psi)^b - \frac{1}{2}R_{abcd}\psi^a\bar{\psi}^b\psi^c\bar{\psi}^d\right]\mathrm{d}t\,,$$

where ∇ is the (pullback of) the Levi-Civita connection on $\mathcal{N}$, so that

$$\nabla_t\psi^a = \frac{\mathrm{d}\psi^a}{\mathrm{d}t} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}t}\psi^c,$$

and R_{abcd} is its Riemann curvature.

This action is invariant under the following supersymmetry transformations (again, the origin of these transformations will become clear when we have learnt about superfields and superspace):

$$\begin{aligned}
\delta x^a &= \epsilon\bar{\psi}^a - \bar{\epsilon}\psi^a,\\
\delta\psi^a &= \epsilon(i\dot{x}^a - \Gamma^a_{bc}\bar{\psi}^b\psi^c),\\
\delta\bar{\psi}^a &= \bar{\epsilon}(-i\dot{x}^a - \Gamma^a_{bc}\bar{\psi}^b\psi^c).
\end{aligned}$$

Using the Noether procedure, one may deduce that the transformations parametrised by $\epsilon, \bar{\epsilon}$ are generated by the supercharges $Q, \bar{Q}$ respectively, where

$$\begin{aligned}
Q &= i\bar{\psi}^a(g_{ab}\dot{x}^b + ig_{bc}\bar{\psi}^b\Gamma^c_{ad}\psi^d),\\
\bar{Q} &= -i\psi^a(g_{ab}\dot{x}^b + ig_{bc}\psi^b\Gamma^c_{ad}\bar{\psi}^d).
\end{aligned}$$

There is another more obvious symmetry of the action given by $\psi^a \to e^{i\alpha}\psi^a$, $\bar{\psi}^a \to e^{-i\alpha}\bar{\psi}^a$. It is generated by $F = g_{ab}\psi^a\bar{\psi}^b$, and the conservation of F in the quantum theory means that no fermionic excitations are created or destroyed by time evolution.

Now let's quantise. The conjugate momenta are

$$p_a = \frac{\delta L}{\delta\dot{x}^a} = g_{ab}\dot{x}^b + ig_{bc}\bar{\psi}^b\Gamma^c_{ad}\psi^d, \pi_a = \frac{\delta L}{\delta\dot{\psi}^a} \qquad = ig_{ab}\bar{\psi}^b,$$

which leads the canonical commuation and anticommutation relations

$$[\hat{x}^a, \hat{p}_b] = i\delta^a_b,$$
$$\left\{\hat{\psi}^a, \hat{\bar{\psi}}^b\right\} = g^{ab}(x),$$

with all others vanishing.

For the bosonic part of the Hilbert space, we can use $L^2(\mathcal{N}, \sqrt{g}\,\mathrm{d}^n x)$, as before. For the fermions, we proceed as previously, choosing $\hat{\bar{\psi}}^a$ to be a raising operator, and $\hat{\psi}^a$ to be a lowering operator. The vacuum is the unique state such that $\hat{\psi}^a\,|0\rangle = 0$ for all a. All other fermion states are generated by acting with $\hat{\bar{\psi}}$s on $|0\rangle$.

There is a remarkable geometric interpretation to the fermionic states. We interpret them as *forms* on $\mathcal{N}$. In particular, we make the following identifications (dropping now the hats):

$$|0\rangle \leftrightarrow 1,$$
$$\bar{\psi}^a\,|0\rangle \leftrightarrow \mathrm{d}x^a\,,$$
$$\bar{\psi}^a\bar{\psi}^b\,|0\rangle \leftrightarrow \mathrm{d}x^a \wedge \mathrm{d}x^b\,,$$

and so on. One can then identify the overall Hilbert space of this supersymmetric NLSM as

$$\mathcal{H} = \Omega^\bullet(\mathcal{N}) = \bigoplus_{p=0}^{n} \Omega^p(\mathcal{N}),$$

i.e. the space of all forms (technically with square-integrable coefficients) on $\mathcal{N}$. Normally, when considering forms, we do not consider addition of a p-form to a q-form if $p \neq q$. We will allow such an operation here, and the result is just a form of mixed degree. For example, a state $\Psi \in \mathcal{H}$ may be written

$$\Psi(x, \bar{\psi}) = f(x) + \alpha_a(x)\bar{\psi}^a + \beta_{ab}(x)\bar{\psi}^a\bar{\psi}^b + \ldots,$$

where f is a function on $\mathcal{N}$, $\alpha = \alpha_a\,\mathrm{d}x^a$ is a 1-form on $\mathcal{N}$, $\beta = \beta_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b$ is a 2-form on $\mathcal{N}$, and so on.

Acting on this space, the canonical operators $\hat{x}^a, \hat{p}_a, \hat{\bar{\psi}}^a, \hat{\psi}^a$ are represented in the following way:

$$\hat{x}^a \to x^a\times,$$
$$\hat{p}_a \to -i\frac{\partial}{\partial x^a},$$
$$\hat{\bar{\psi}}^a \to \mathrm{d}x^a \wedge,$$
$$\hat{\psi}^a \to \iota_{g^{ab}\frac{\partial}{\partial x^b}}.$$

Here, $\iota_V\omega$ is the form obtained by contracting the vector V into the form ω.

The inner product on $\mathcal{H}$ is just the Hodge inner product. In particular, if α, β are two p-forms, then

$$\langle\alpha|\beta\rangle = \int_{\mathcal{N}} \bar{\alpha} \wedge *\beta,$$

where $*$ is the Hodge star associated to the metric g_{ab}. In coordinates, if $\beta = \beta_{a_1\ldots a_p}\,\mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_p}$ then we have

$$*\beta = \frac{1}{(n-p)!}\sqrt{g}\epsilon^{a_1\ldots a_p}{}_{b_{p+1}\ldots b_n}\beta_{a_1\ldots a_p}\,\mathrm{d}x^{b_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{b_n}\,.$$

The Hodge inner product may be written

$$\langle\alpha|\beta\rangle = \int \bar{\alpha}^{a_1\ldots a_p}\beta_{a_1\ldots a_p}\sqrt{g}\,\mathrm{d}^n x\,.$$

If α, β have differing degree, then we define $\langle\alpha|\beta\rangle = 0$.

Furthermore, in the quantum theory, the supercharges are represented by the exterior derivative and its adjoint:

$$\begin{aligned} Q &= i\bar{\psi}^a p_a \to \mathrm{d}x^a \frac{\partial}{\partial x^a} = \mathrm{d}, \\ \bar{Q} &= -i\psi^a p_a \to \mathrm{d}^\dagger. \end{aligned}$$

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The adjoint of d is defined through $\left\langle\alpha\middle|\mathrm{d}^\dagger\beta\right\rangle = \langle\mathrm{d}\alpha|\beta\rangle$.

Recall that in the non-supsersymmetric theory there was an ordering ambiguity in the choice of the quantum Hamiltonian. Now that we have supersymmetry, we can fix this ambiguity by demanding that the quantum theory has the same supersymmetry as the classical theory. In particular, the supersymmetry algebra $\left\{\hat{Q}, \hat{\bar{Q}}\right\} = \hat{H}$ should still hold. In terms of the geometric interpretation, this fixes the quantum Hamiltonian to be $\hat{H} = \frac{1}{2}\Delta$, where

$$\Delta = \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger.$$

The operator Δ is known as the Laplace-de Rham operator. Since d raises the degree of a form by 1, and $\mathrm{d}^\dagger$ lowers the degree of a form by 1, Δ preserves the degree of the form it is acting on. Acting on functions, it may be shown that

$$\Delta f = \frac{1}{\sqrt{g}}\partial_b\Big(g^{ab}\sqrt{g}\partial_a f\Big),$$

which is just the Laplacian of f. Δ should be viewed as that natural generalisation of the usual Laplacian acting on functions (i.e. 0-forms), to an operator that acts on all forms.

Consider the expectation value of the energy of a state $|\omega\rangle$, where ω is a form. We have

$$\begin{aligned} \langle\omega|\hat{H}|\omega\rangle &= \frac{1}{2}\left\langle\omega\middle|\mathrm{d}\,\mathrm{d}^\dagger\omega\right\rangle + \frac{1}{2}\left\langle\omega\middle|\mathrm{d}^\dagger\,\mathrm{d}\omega\right\rangle \\ &= \frac{1}{2}\left\langle\mathrm{d}^\dagger\omega\middle|\mathrm{d}^\dagger\omega\right\rangle + \frac{1}{2}\langle\mathrm{d}\omega|\mathrm{d}\omega\rangle \\ &= \frac{1}{2}\left\|\mathrm{d}^\dagger\omega\right\|^2 + \frac{1}{2}\|\mathrm{d}\omega\|^2 \geq 0. \end{aligned}$$

So energy is non-negative. Also, iff $\hat{H}\omega = 0$, i.e. iff $|\omega\rangle$ is a supersymmetric ground state, then $\mathrm{d}\omega = \mathrm{d}^\dagger\omega = 0$.

A form obeying $\Delta\omega = 0$ is said to be harmonic. Therefore, supersymmetric ground states are in one-to-one correspondence with harmonic forms of any degree on the target space $\mathcal{N}$. In notation, this space is denoted

$$\mathrm{Harm}^\bullet(\mathcal{N}) = \bigoplus_{p=0}^{n}\mathrm{Harm}^p(\mathcal{N}),$$

where $\mathrm{Harm}^p(\mathcal{N})$ is the space of harmonic p-forms on $\mathcal{N}$.

We are assuming that the metric g_{ab} on $\mathcal{N}$ is of Riemannian signature, i.e. that it is positive definite. In this case, the space of Harmonic forms is very closely related to certain topological properties of $\mathcal{N}$. The study of this connection is known as *Hodge theory*, and the main result is the following:

Theorem 1 (Hodge's theorem)**.** *The space of Harmonic p-forms is in one-to-one correspondence with the pth de Rham cohomology:*

$$\mathrm{Harm}^p(\mathcal{N}) \simeq H^p_{dR}(\mathcal{N}).$$

As a reminder:

Definition. The pth de Rham cohomology is the space of exact p-forms modulo closed p-forms:

$$H^p_{\mathrm{dR}}(\mathcal{N}) = \frac{\{\omega \in \Omega^p(\mathcal{N}) \,\mathrm{s.t.}\, \mathrm{d}\omega = 0\}}{\{\mathrm{d}\alpha \,\mathrm{s.t.}\, \alpha \in \Omega^{p-1}(\mathcal{N})\}}.$$

We may concisely write this as

$$H^p_{\mathrm{dR}}(\mathcal{N}) = \frac{\ker \mathrm{d}_p}{\mathrm{im}\, \mathrm{d}_{p-1}},$$

where $\mathrm{d}_q : \Omega^q(\mathcal{N}) \to \Omega^{q+1}(\mathcal{N})$ is the restriction of the exterior derivative to only acting on q-forms.

A harmonic form ω must be closed. In de Rham cohomology, two closed forms are considered equivalent if they differ by an exact form, so $\omega \sim \omega + \mathrm{d}\alpha$. Hodge's theorem works because the co-closure condition $\mathrm{d}^\dagger\omega$ works to select a unique representative of the resulting equivalence class. In particular, if $\mathrm{d}\omega = \mathrm{d}^\dagger\omega = 0$, then we can only rewrite $\omega \sim \omega + \mathrm{d}\alpha$ if $\mathrm{d}^\dagger\,\mathrm{d}\alpha = 0$. But one may show that the only solution to this equation is $\alpha = 0$.

So the space of supersymmetric ground states is equivalent to

$$H^\bullet_{\mathrm{dR}}(\mathcal{N}) = \bigoplus_{p=0}^{n} H^p_{\mathrm{dR}}(\mathcal{N}).$$

But how does de Rham cohomology tell us about the topology of $\mathcal{N}$? Consider the space of submanifolds of $\mathcal{N}$, and let the *boundary operator* δ map a submanifold to its boundary. ∂ maps p-dimensional submanifolds to $(p-1)$-submanifolds. Moreover, the boundary of any submanifold has no boundary, so $\partial^2 = 0$. So we can define:

Definition. The pth cohomology group of $\mathcal{N}$ is

$$H_p(\mathcal{N}) = \frac{\ker \partial_p}{\mathrm{im}\, \partial_{p+1}},$$

where ∂_q is the restriction of the boundary operator to acting on q-dimensional submanifolds.

In this cohomology, two submanifolds without boundary are considered equivalent if they differ by the boundary of a submanifold.

Recall:

Theorem 2 (Stokes' theorem)**.** *If U is a $(p+1)$-dimensional submanifold, and ω is a p-form, then*

$$\int_U \mathrm{d}\omega = \int_{\partial U} \omega.$$

Stokes' theorem indicates that there is a connection between the exterior derivative d and the boundary operator ∂. It is possible to exploit this connection to prove the following:

Theorem 3 (de Rham's theorem)**.** *The pth de Rham cohomology and the pth cohomology group are in one-to-one correspondence:*

$$H^p_{dR}(\mathcal{N}) \simeq H_p(\mathcal{N}).$$

So, in fact, the space of supersymmetric ground states is equivalent to

$$H_\bullet(\mathcal{N}) = \bigoplus_{p=0}^{n} H_p(\mathcal{N}).$$

The cohomology groups of $\mathcal{N}$ tells us many things about its topology. For example:

Definition. The kth *Betti number* b_k of $\mathcal{N}$ is the rank of its kth homology group.

One may think of b_k as the number of independent ways one may cut $\mathcal{N}$ along a k-dimensional surface, without increasing the number of connected components of $\mathcal{N}$.

Example. The Betti numbers of a Riemann surface Σ_g of genus g are

$$b_0 = 1, \quad b_1 = 2g, \quad b_2 = 1.$$

Definition. The *Euler characteristic* $\chi(\mathcal{N})$ of $\mathcal{N}$ is given by the alternating sum of its Betti numbers:

$$\chi(\mathcal{N}) = \sum_{k=0}^{n} (-1)^k b_k.$$

Example. The Euler characteristic of Σ_g is $2 - 2g$.

In our theory of a supersymmetric non-linear sigma model, the Betti numbers have a particular interpretation. $H_p(\mathcal{N})$ corresponds to the space of ground state p-forms, i.e. ground states with p fermionic excitations. The rank of $H_p(\mathcal{N})$ tells us how many independent such states there are. Thus, the kth Betti number tells us how many ground states there are with k fermions.

Let us now compute the Witten index. Using the above, we have

$$\begin{aligned} I_W &= \mathrm{Tr}\Big((-1)^F e^{-\beta H}\Big) \\ &= \#\text{ bosonic ground states} - \#\text{ fermionic ground states} \\ &= \sum_{k=0}^{n} (-1)^k b_k = \chi(\mathcal{N}). \end{aligned}$$

So the Witten index is none other than the Euler characteristic of $\mathcal{N}$!

As we have seen previously, one can compute the Witten index another way, using the periodic path integral. We thus may write

$$\chi(\mathcal{N}) = \int e^{-S[x,\psi,\bar{\psi}]}\,\mathrm{D}x\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi}\,,$$

where all fields are taken to be periodic with period β.

It is instructive to see how $I_W = \chi$ is obtained from the path integral perspective. Note that the action itself is supersymmetrically trivial, in that it is $\bar{Q}$-exact:

$$\begin{aligned} S &= \oint \left[\frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b + \frac{1}{2}g_{ab}\bar{\psi}^a\nabla_t\psi^b + \frac{1}{4}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\right]\mathrm{d}t \\ &= \bar{Q}\left[\oint \frac{1}{2}g^{ab}\bar{\psi}^a(ix^b + \Gamma^b_{cd}\bar{\psi}^c\psi^d)\,\mathrm{d}t\right]. \end{aligned}$$

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An infinitesimal change in β amounts to an insertion of the Lagrangian in the path integral. Consequently, the path integral is independent of β (of course, we already knew this had to be true by construction). In particular, if we expand

$$\begin{aligned} x^a(\tau) &= x^a_0 + \delta x^a(\tau), \\ \psi(\tau) &= \psi^a_0 + \delta\psi^a(\tau), \end{aligned}$$

where

$$\oint \delta x^a\,\mathrm{d}\tau = \oint \delta\psi^a\,\mathrm{d}\tau = 0,$$

then as $\beta \to 0$, all the contributions from the non-zero modes $\delta x^a, \delta\psi^a$ are highly suppressed. In fact, one can show that the contributions from $\delta x^a, \delta\psi^a$ precisely cancel each other, and one is left with just an integral over the zero-modes x^a_0, ψ^a_0. In other words, the path integral localises to constant maps. Since there is no preferred point in $\mathcal{N}$, we still need to integrate over $\mathcal{N}$, as well as the zero-modes of the fermions. Thus, the path integral becomes

$$\begin{aligned} \chi(\mathcal{N}) &= \int e^{-S[x_0,\psi_0]}\frac{\mathrm{d}^n x_0}{(2\pi)^{n/2}}\,\mathrm{d}^n\psi_0\,\mathrm{d}^n\bar{\psi}_0 \\ &= \int \exp\left[-\frac{1}{2}R_{abcd}(x_0)\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\right]\frac{\mathrm{d}^n x_0}{(2\pi)^{n/2}}\,\mathrm{d}^n\psi_0\,\mathrm{d}^n\bar{\psi}_0 \\ &= \begin{cases} 0 & \text{if } n \text{ odd}, \\ \frac{1}{(2\pi)^{n/2}}\int_{\mathcal{N}} \mathrm{tr}(R\wedge R\wedge\cdots\wedge R) & \text{if } n \text{ even}, \end{cases} \end{aligned}$$

where $R^a{}_b = R_{cd}{}^a{}_b\,\mathrm{d}x^c\wedge\mathrm{d}x^d$ is the curvature 2-form, and there are $n/2$ of these appearing in the last line. The last line is a well-known expression – it is the n-dimensional generalisation of the Gauss-Bonnet formula, and gives exactly the Euler characteristic of $\mathcal{N}$. So the path integral approach agrees with the canonical approach.

3.6 The Atiyah-Singer index theorem

Let us consider the case where $\mathcal{N}$ has even dimension $n = 2m$, and consider again the supersymmetric non-linear sigma model of the previous section, but now with the restriction that ψ^a is real. Then $\psi^a = \bar{\psi}^a$, and the action simplifies to

$$S[x,\psi] = \oint \left[\frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b + \frac{i}{2}g_{ab}\psi^a\nabla_\tau\psi^b\right]\mathrm{d}\tau\,,$$

where the term involving the Riemann tensor now vanishes due to the Bianchi identity $R_{a[bcd]} = 0$.

This action is still invariant under supersymmetry transformations which respect the restriction to real ψ^a. One can obtain such transformations by setting $\epsilon = -\bar{\epsilon}$. They take the form

$$\delta x^a = \epsilon\psi^a, \qquad \delta\psi^a = -\epsilon\dot{x}^a.$$

The momenta conjugate to x^a, ψ^a are

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab}\dot{x}^b + \frac{i}{2}\psi_c\Gamma^c_{ab}\psi^b,$$
$$\pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = \psi_a,$$

where $\psi_a = g_{ab}\psi^b$. The canonical commutators therefore take the form

$$[\hat{x}^a, \hat{p}_b] = i\delta^a_b,$$
$$\left\{\hat{\psi}^a, \hat{\psi}^b\right\} = 2g^{ab}.$$

This theory is sometimes referred to as a $\mathcal{N} = \frac{1}{2}$ theory, since there is half as much supersymmetry.

Before continuing with our analysis of this theory, it is useful to recall some basic facts about spinors in $n = 2m$ dimensions. We assume the existence of the following set of matrices:

Definition. The *Dirac γ matrices* are a set of n linear operators γ^i, $i \in 1, \ldots, n$ on the space of spinors obeying the anticommutation relations

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}.$$

In a curved space, one should choose an orthonormal frame, and the indices $i, j, \ldots$ should be understood as frame indices. We can construct m raising and lowering operators (over $\mathbb{C}$) from the γ matrices:

$$\gamma^I_\pm = \frac{1}{2}(\gamma^{2I} \pm i\gamma^{2I-1}), \quad I = 1, \ldots, m,$$

which obey

$$\left\{\gamma^I_+, \gamma^J_-\right\} = \delta^{IJ}, \qquad \left\{\gamma^I_\pm, \gamma^J_\pm\right\} = 0.$$

Starting from a spinor χ obeying $\gamma^I_-\chi = 0$ for all I, one can construct a basis of the space S of spinors by acting with any combination of the raising operators γ^I_+. Since ${\gamma^I_+}^2 = 0$, each γ^I_+ can act at most once, so $\dim(S) = 2^{n/2}$.

The group $\mathrm{Spin}(n)$, which is defined as the double cover of $SO(n)$, acts on these spinors via the generators

$$\Sigma^{ij} = -\frac{1}{4}\left[\gamma^i, \gamma^j\right],$$

which obey the $SO(n)$ algebra

$$\left[\Sigma^{ij}, \Sigma^{kl}\right] = i\Big(\delta^{ik}\Sigma^{jl} + \delta^{jl}\Sigma^{ik} - \delta^{jk}\Sigma^{il} - \delta^{il}\Sigma^{jk}\Big).$$

This is not an irreducible representation, and it is useful to decompose it.

Definition. The $n+1$-*th* γ *matrix* is

$$\gamma^{n+1} = i^{n/2}\gamma^1\gamma^2\dots\gamma^n.$$

We have

$$(\gamma^{n+1})^2 = 1, \qquad \left\{\gamma^{n+1}, \gamma^i\right\} = 0.$$

Moreover, γ^{n+1} transforms trivially under the above action of $\mathrm{Spin}(n)$, since

$$\left[\gamma^{n+1}, \Sigma^{ij}\right] = 0.$$

In other words, it is a Lorentz scalar. We may decompose $S = S^+ \oplus S^-$, where $S^\pm$ are the ± 1 eigenspaces of γ^{n+1}, and correspond to states constructed from an even/odd number of raising operators γ_+^I respectively acting on χ. A spinor in one of $S^\pm$ is called chiral. Spinors in S^+, S^- are called left-handed, right-handed respectively.

Definition. The *Dirac operator* is

$$i\not{\partial} = i\gamma^i\partial_i.$$

The Dirac operator anticommutes with γ^{n+1}, and thus decomposes as

$$i\not{\partial} = \begin{pmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{pmatrix},$$

where with this matrix notation the first row/column corresponds to S^-, while the second row/column corresponds to S^+. The operators $\partial^\pm$ map $S^\pm \to S^\mp$ respectively, and $\partial^\pm$ annihilates $S^\mp$, so $(\partial^\pm)^2 = 0$.

Definition. The *index* of the Dirac operator is defined as

$$\mathrm{index}(i\not{\partial}) = \dim(\ker(\partial^+)) - \dim(\ker(\partial^i)).$$

Let us now return to the $\mathcal{N} = \frac{1}{2}$ theory. The bosonic part of the wavefunction may again be taken to be in $L^2(\mathcal{N}, \sqrt{g}\,\mathrm{d}^n x)$. For the fermionic part, let e_a^i, $i = 1, \dots, n$ be a basis of c-number 1-forms on $\mathcal{N}$ obeying $e_a^i e_b^j g^{ab} = \delta^{ij}$ (such a basis is called an orthonormal frame). Then we have

$$\left\{e_a^i\hat{\psi}^a, e_b^j\hat{\psi}^b\right\} = 2\delta^{ij},$$

so the operators $\hat{\psi}^i = e_a^i\hat{\psi}^a$ obey the same anticommutation relations as the Dirac γ matrices, and all of the above constructions with the γ matrices also apply to these operators. In particular, we may construct raising and lowering operators $\hat{\psi}_\pm^I = \hat{\psi}^{2I} \pm i\hat{\psi}^{2I-1}$, and choose a fermionic vacuum state $|0\rangle$ which is annihilated by all lowering operators $\hat{\psi}_-^I$. The rest of the fermionic Hilbert space is generated by application of the raising operators $\hat{\psi}_+^I$ to the vacuum state.

By identifying the operators $\hat{\psi}^i$ with the γ matrices, and the vacuum state with the spinor χ, we can give fermionic states a geometric interpretation – they are just spinors on $\mathcal{N}$. Therefore, a

state in the full Hilbert space, including bosons and fermions, assigns an amplitude to each possible spinor at each possible $x \in \mathcal{M}$. In other words, the Hilbert space is

$$\mathcal{H} = L^2(S(\mathcal{N}), \sqrt{g}\,\mathrm{d}^n x),$$

the space of square integrable functions on the *spin bundle* over $\mathcal{M}$, which is just the bundle whose sections give a choice of a spinor at each point in $\mathcal{M}$. In this geometric interpretation, the supercharge

$$Q = \psi^a(i g_{ab}\dot{x}^b + \psi_c \Gamma^c_{ab}\psi^b)$$

can be shown to correspond to the covariant Dirac operator $i\not{\nabla} = i\gamma^i\nabla_i$, where ∇_i is the Levi-Civita covariant derivative associated with the frame e^i_a. As above, $i\not{\nabla}$ decomposes into two operators $i\nabla^{\pm} : S^{\pm}(\mathcal{N}) \to S^{\mp}(\mathcal{N})$, where $S(\mathcal{N}) = S^+(\mathcal{N}) \oplus S^-(\mathcal{N})$ is the chiral decomposition of the spin bundle. The Witten index of the theory is then just the index of the covariant Dirac operator:

$$\mathrm{Tr}\Big((-1)^F e^{-\beta H}\Big) = \dim(\ker(\nabla^+)) - \dim(\ker(\nabla^-)).$$

We can obtain another expression for this index by using a path integral approach. The path integral is again independence of β. As before, let us split separate the zero modes from the rest of the fields, writing

$$x^a(\tau) = x^a_0 + \delta x^a(\tau),$$
$$\psi^a(\tau) = \psi^a_0 + \delta\psi^a(\tau).$$

We will use Riemann normal coordinates near x_0 to write the metric as

$$g_{ab}(x) = \delta_{ab} - \frac{1}{3}R_{acbd}(x_0)\delta x^c \delta x^d + \mathcal{O}\Big(\delta x^3\Big).$$

In these coordinates, the Levi-Civita symbol may be written

$$\Gamma^a_{bc}(x) = \partial_d \Gamma^a_{bc}(x_0)\delta x^d = -\frac{1}{3}(R^a{}_{bcd}(x_0) + R^a{}_{cbd}(x_0))\delta x^d + \mathcal{O}\Big(\delta x^2\Big).$$

Using this, one may write the action to second order in the fluctuations as

$$S^{(2)}[x_0, \psi_0, \delta x, \delta\psi] = \oint \left[-\frac{1}{2}\delta x^b \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta x^a + \frac{i}{2}\delta_{ab}\delta\psi^b \frac{\mathrm{d}}{\mathrm{d}\tau}\delta\psi^a - \frac{i}{4}R_{abcd}\psi^a_0\psi^b_0\delta x^c \delta\dot{x}^d\right]\mathrm{d}\tau\,.$$

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For any fixed values of the zero modes x^a_0, ψ^a_0, this is a free action, so the path integral over fluctuations gives

$$\begin{aligned}\int e^{-S[x_0,\psi_0,\delta x,\delta\psi]}\,\mathrm{D}\delta x\,\mathrm{D}\delta\psi &= \frac{\sqrt{\det'(\partial_\tau \delta^b_a)}}{\sqrt{\det'(-\partial^2_\tau\delta^a_b - i\mathcal{R}^a{}_b(x_0,\psi_0)\partial_\tau)}} \\ &= \frac{\sqrt{\det'(\partial_\tau\delta^b_a)}}{\sqrt{\det'(\partial_\tau\delta^b_a)\det'(-\partial_\tau\delta^b_a - i\mathcal{R}^a{}_b(x_0,\psi_0))}},\end{aligned}$$

where $\mathcal{R}^a{}_b = R^a{}_{bcd}(x_0)\psi^c_0\psi^d_0$, and $\det'$ denotes a determinant without zero modes. The factors of $\sqrt{\det'(\partial_\tau\delta^b_a)}$ in the numerator and denominator clearly cancel. To deal with the remaining

expression, note that $\mathcal{R}^a{}_b$ is a real antisymmetric $2m$-dimensional matrix. This means we can decompose the tangent space $\mathrm{T}_{x_0}\mathcal{M}$ into m 2-dimensional subspaces on which $\mathcal{R}^a{}_b$, acts as

$$\mathcal{R}^a{}_b = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix},$$

where $i = 1, \ldots, m$, and $\pm i\omega_i$ are the eigenvalues of $\mathcal{R}^a{}_b$ with eigenvectors $v_i^{\pm}$. Let $-\mathcal{D}_i$ be the restriction of the operator $-\delta^a_b\partial_\tau - i\mathcal{R}^a{}_b$ to the ith subspace. On the basis $v_i^{\pm}e^{2\pi ik\tau}$, $k \neq 0$, $-\mathcal{D}_i$ acts as

$$-\mathcal{D}_i(v_i^{\pm}e^{2\pi ik\tau}) = -(2\pi ik \mp \omega_i)v_i^{\pm}e^{2\pi ik\tau}$$

Thus we have

$$\begin{aligned}\det{}'(-\mathcal{D}_i) &= \prod_{k\neq 0}(2\pi ik + \omega_i)(2\pi ik - \omega i) \\ &= \prod_{k\neq 0}(-4\pi^2k^2 - \omega_i^2) \\ &= \prod_{k=1}^{\infty}(2\pi k)^4 \prod_{k=1}^{\infty}\left(1 + \frac{\omega_i^2}{(2\pi k)^2}\right)^2.\end{aligned}$$

This product is clearly divergent, and so must be regularised in some way. One way to do this is with zeta-function regularisation, the result being the replacement $\prod_{k=1}^{\infty}(2\pi k)^4 \to 1$. What is left is the second factor in the last line, which, unlike the first factor, is dependent on ω_i. Recalling that

$$\sinh(z) = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{\pi^2k^2}\right),$$

we may therefore write

$$\sqrt{\det{}'(-\mathcal{D}_i)} = \frac{\sinh(\omega_i/2)}{\omega_i/2}.$$

Therefore,

$$\frac{1}{\sqrt{\det{}'(-\partial_\tau\delta^b_a - i\mathcal{R}^a{}_b(x_0,\psi_0))}} = \prod_{i=1}^{m}\frac{\omega_i/2}{\sinh(\omega_i/2)}.$$

To compute the Witten index, we still need to integrate over the zero modes. We have

$$\begin{aligned}I_W = \mathrm{index}(i\not{\nabla}) &= \int\prod_{i=1}^{m}\frac{\omega_i/2}{\sinh(\omega_i/2)}\frac{\mathrm{d}^nx_0}{(2\pi)^{n/2}}\,\mathrm{d}^n\psi_0 \\ &= \int\det\left(\frac{\mathcal{R}^a{}_b(x_0,\psi_0)/2}{\sinh(\mathcal{R}^a{}_b(x_0,\psi_0)/2)}\right)\frac{\mathrm{d}^nx_0}{(2\pi)^{n/2}}\,\mathrm{d}^n\psi_0 \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathcal{N}}\det\left(\frac{R/2}{\sinh(R/2)}\right),\end{aligned}$$

where $R^a{}_b = R^a{}_{bcd}\,\mathrm{d}x^c \wedge \mathrm{d}x^d$ is the curvature 2-form on $\mathcal{N}$.

This result, and the Gauss-Bonnet theorem in the previous section, are special cases of the *Atiyah-Singer index theorem*. This theorem allows us in a very general way to write the index of an elliptic operator (the Laplacian on forms, the Dirac operator, etc.) acting on sections of some bundle over $\mathcal{N}$ as the integral of certain quantities on $\mathcal{N}$.

4 Supersymmetric Quantum Field Theory

If we have a d-dimensional theory that is Lorentz invariant, we must compute the supersymmetry algebra $\left\{Q, Q^\dagger\right\} = 2H$. But the Hamiltonian is only one part of the d-momentum multiplet P_μ, which transforms non-trivially under Lorentz symmetry. Therefore it is necessary to introduce further supercharges which generate the rest of the d-momentum. Since P_μ is a vector, it is spin 1. The momentum is made from combining two supercharges with equal spin, and so the supercharges must have spin $\frac{1}{2}$ – in other words, they are spinors.

More specifically, the supersymmetry algebra in d dimensions is

$$\left\{Q_\alpha, Q^\dagger_\beta\right\} = 2\gamma^\mu_{\alpha\beta} P_\mu,$$

where $\alpha, \beta, \ldots$ are spinor indices, and γ^μ is a Dirac γ matrix.

We'll mostly be concerned with $d = 2$, whose Dirac spinors have $2^{d/2} = 2$ complex components. We will write $\psi = \binom{\psi_-}{\psi_+}$. Choosing coordinates $(t, s) \in \mathbb{R}^2$ in which the Minkowski metric takes the form $\eta = \mathrm{diag}(+1, -1)$, we can represent the Dirac matrices with

$$\gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which obey $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

The action for a free, massless Dirac spinor in $d = 2$ is

$$S[\psi] = \frac{1}{2\pi} \int_{\mathbb{R}} i\bar{\psi} \not{\partial} \psi \, \mathrm{d}t \, \mathrm{d}s \,,$$

where $\not{\partial} = \gamma^\mu \partial_\mu$, and $\bar{\psi} = \psi^\dagger \gamma^t$. We may expand this out to

$$S[\psi] = \frac{1}{2\pi} \int_{\mathbb{R}} \left[i\bar{\psi}_-(\partial_t + \partial_s)\psi_- + i\bar{\psi}_+(\partial_t - \partial_s)\psi_+ \right] \mathrm{d}t \, \mathrm{d}s$$

Classically, one finds that the equations of motion $(\partial_t + \partial_s)\psi_- = (\partial_t - \partial_s)\psi_+ = 0$ may be solved by a right-moving $\psi_-(t, s) = f(t - s)$, and a left-moving $\psi_+(t, s) = g(t + s)$.

Under a $SO(1, 1)$ transformation

$$\begin{pmatrix} t \\ s \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix},$$

the spinors transform as

$$\psi_\pm \mapsto e^{\pm\gamma/2}\psi_\pm, \qquad \bar{\psi}_\pm \mapsto e^{\pm\gamma/2}\bar{\psi}_\pm.$$

4.1 Superspace in $d = 2$

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Definition. A *superspace* is a space with both bosonic and fermionic directions.

Let $\mathbb{R}^{2|4}$ denote the superspace with bosonic coordinates $x^0, x^1 = t, s$ and fermionic coordinates $\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$. These coordinates are acted on by an $SO(1, 1)$ transformation in the way just

described, with $\theta^\pm, \bar{\theta}^\pm$ transforming like spinors of chirality ± 1 respectively. We can introduce the following fermionic derivatives on $\mathbb{R}^{2|4}$:

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm \frac{\partial}{\partial x^\pm},$$
$$\bar{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm \frac{\partial}{\partial x^\pm},$$

where $\partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$. These derivatives obey the anticommutation relations

$$\left\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\right\} = -2i\partial_\pm,$$

with others vanishing. Thus they represent the supersymmetry algebra on $\mathbb{R}^{2|4}$. SUSY transformations act geometrically on $\mathbb{R}^{2|4}$, being generated by

$$\delta = \epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \bar{\epsilon}_+\bar{\mathcal{Q}}_- + \bar{\epsilon}_-\bar{\mathcal{Q}}_+,$$

where $\epsilon_\pm, \bar{\epsilon}_\pm$ are some spinor parameters.

Definition. A *superfield* is a function on a superspace.

Since the $\theta^\pm, \bar{\theta}^\pm$ coordinates are anticommuting, a generic superfield $\mathcal{F}$ on $\mathbb{R}^{2|4}$ permits an expansion in the fermionic coordinates, with coefficients given by functions of the bosonic coordinates:

$$\mathcal{F}(x^\pm, \theta^\pm, \bar{\theta}^\pm) = f_0(x^\pm) + \theta^+ f_+(x^\pm) a + \theta^- f_-(x^\pm) + \bar{\theta}^+ g_+(x^\pm) + \bar{\theta}^- g_-(x^\pm) + \cdots + \theta^+\bar{\theta}^+\theta^-\bar{\theta}^- D(x^\pm).$$

There are $2^4 = 16$ such coefficient functions, sometimes collectively known as a supermultiplet. Note that under a supersymmetry transformation $\mathcal{F} \to \mathcal{F} + \delta F$, the highest component field $D(x^\pm)$ can only change at most by total bosonic derivatives.

It's often useful to have 'smaller' superfields that are constrained in some way. For this purpose, we introduce

$$\mathcal{D}_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm, \qquad \bar{\mathcal{D}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm,$$

which differ from $\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm$ by the signs of the second terms. One can think of $\mathcal{D}_\pm$ and $\bar{\mathcal{D}}_\pm$ as holomorphic antiholomorphic derivatives respectively. The anticommutators of these derivatives obey

$$\left\{\mathcal{D}_\pm, \bar{\mathcal{D}}_\pm\right\} = +2i\partial_\pm,$$

with others vanishing. Crucially, we also have

$$\{\mathcal{D}, \mathcal{Q}\} = \left\{\bar{\mathcal{D}}, \mathcal{Q}\right\} = \cdots = 0,$$

i.e. all anticommutators between these derivatives and the supercharges vanish. This makes the following definition a useful one.

Definition. A *chiral superfield* is a superfield Φ which obeys $\bar{\mathcal{D}}_\pm\Phi = 0$.

Chiral superfields can depend on $x^\pm, \theta^\pm, \bar\theta^\pm$ only through $y^\pm, \theta^\pm$, where $y^\pm = x^\pm - i\theta^\pm\bar\theta^\pm$, since $\bar{\mathcal{D}}_\pm y^\pm = \bar{\mathcal{D}}_\mp y^\pm = 0$. We can therefore expand a chiral superfield as

$$\Phi(y^\pm, \theta^\pm) = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm).$$

The product of any two chiral superfields is again chiral. A holomorphic function of any chiral superfield is chiral (holomorphic meaning no dependence on $\bar\Phi$). The conjugate of a chiral superfield obeys

$$\mathcal{D}_\pm\bar\Phi = 0,$$

and is called antichiral. Under a supersymmetry transformation $\Phi \to \Phi + \delta\Phi$, the chirality of Φ is preserved (because the supercharges anticommute with the $\mathcal{D}$s).

To work out how the supersymmetry transformations act on the component fields of a chiral superfield, it is useful to first write the supercharges in terms of $y^\pm$ instead of $x^\pm$. Using the chain rule, we have

$$\begin{aligned}
\mathcal{Q}_\pm &= \frac{\partial}{\partial\theta^\pm}\bigg|_{x,\bar\theta} + i\bar\theta^\pm \frac{\partial}{\partial x^\pm}\bigg|_{\theta,\bar\theta} \\
&= \frac{\partial}{\partial\theta^\pm}\bigg|_{y,\bar\theta} + \underbrace{\frac{\partial y^\pm}{\partial\theta^\pm}\bigg|_{x,\bar\theta}}_{-i\bar\theta^\pm} \frac{\partial}{\partial y^\pm}\bigg|_{\theta,\bar\theta} + i\bar\theta^\pm \frac{\partial}{\partial y^\pm}\bigg|_{\theta,\bar\theta} \\
&= \frac{\partial}{\partial\theta^\pm}\bigg|_{y,\bar\theta}.
\end{aligned}$$

One may similarly compute that

$$\bar{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar\theta^\pm}\bigg|_{y,\theta} - 2i\theta^\pm \frac{\partial}{\partial y^\pm}\bigg|_{\theta,\bar\theta}.$$

Using these, one finds that the components transform in the following way:

$$\begin{aligned}
\delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+, \\
\delta\psi_\pm &= \epsilon_\pm F \pm \bar\epsilon_\mp\partial_\pm\phi, \\
\delta F &= -2i\bar\epsilon_+\partial_-\psi_+ - 2i\bar\epsilon_-\partial_+\psi_-.
\end{aligned}$$

Again, note that the supersymmetry transformation of the highest component field $F(y^\pm)$ is at most a bosonic total derivative.

4.2 Supersymmetry invariant actions

The fact that the D term of a generic superfield and the F term of a chiral superfield vary only by total derivatives allows us to readily construct supersymmetry invariants.

In particular, let $K(\mathcal{F}_i, \Phi^a, \bar\Phi^a)$ be any real smooth function of real superfields $\mathcal{F}_i$ and chiral superfields Φ^a and their complex conjugates $\bar\Phi^a$ (K is sometimes called a Kähler potential). Then

$$\int_{\mathbb{R}^{2|4}} K(\mathcal{F}_i, \Phi^a, \bar\Phi^a)\,\mathrm{d}^2x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta$$

is a supersymmetry invariant, provided the component fields behave appropriately as $|x^\pm| \to \infty$. This is because after the fermionic integration only the $\theta^2\bar\theta^2$ coefficient in the expansion of K

remains, and this changes by a total derivative under a supersymmetry transformation, which vanishes upon doing the bosonic integration.

Similarly, suppose $W(\Phi^a)$ is a holomorphic function of Φ^a (in this context sometimes called a superpotential). Then W is chiral, and so

$$\int_{\mathbb{R}^{2|2}} W(\Phi^a)\,\mathrm{d}^2y\,\mathrm{d}^2\theta$$

is again a supersymmetry invariant, provided the coefficient fields are sufficiently well behaved as $|y^\pm| \to \infty$.

4.3 Wess-Zumino model in $d = 1 + 1$

Let's consider the simplest case of a single chiral superfield Φ and its conjugate $\bar{\Phi}$. We will take $K(\Phi, \bar{\Phi}) = |\Phi|^2 + \bar{\Phi}\Phi$, but leave the superpotential generic. Then we can use the real functional

$$S[\Phi, \bar{\Phi}] = \underbrace{\int_{\mathbb{R}^{2|4}} \bar{\Phi}\Phi\,\mathrm{d}^2x\,\mathrm{d}^4\theta}_{\text{kinetic terms}} + \underbrace{\int_{\mathbb{R}^{2|2}} W(\Phi)\,\mathrm{d}^2y\,\mathrm{d}^2\theta + \int_{\mathbb{R}^{2|2}} \bar{W}(\bar{\Phi})\,\mathrm{d}^2y\,\mathrm{d}^2\theta}_{\text{interactions}}$$

as the action for a field theory. For the reasons given in the previous subsection, this action is guaranteed to be supersymmetric, assuming the fields have the appropriate asymptotic behaviour. This theory is known as the Wess-Zumino model.

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Let's do the integration over the fermionic variables in the Wess-Zumino action. To do so, we need to find the θ^2 term in the superpotential, and the θ^4 term in the kinetic terms.

A Taylor series expansion gives that the coefficient of $\theta^+\theta^-$ in the superpotential is given by

$$W(\Phi)|_{\theta^+\theta^-} = F\partial W(\phi) - \psi_+\psi_-\frac{\partial^2 W}{\partial\phi^2}.$$

For the kinetic terms, we need to write the chiral superfield $\Phi(y^\pm, \theta^\pm)$ as a function of $x^\pm, \theta^\pm, \bar{\theta}^\pm$. Using $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$, we can expand the coefficients of the chiral superfield in a series in powers of $\theta^\pm\bar{\theta}^\pm$. Doing so, one finds

$$\begin{aligned}\Phi(x^\pm, \theta^\pm, \bar{\theta}^\pm) = \phi(x^\pm) - i\theta^+\bar{\theta}^+\partial_+\phi(x^\pm) - i\theta^-\bar{\theta}^-\partial_-\phi(x^\pm) - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_+\partial_-\phi(x^\pm)\\ + \theta^+\psi_+(x^\pm) + \theta^-\psi_-(x^\pm) - i\theta^+\theta^-\bar{\theta}^-\partial_-\psi_+(x^\pm) - i\theta^-\theta^+\bar{\theta}^+\partial_+\psi_-(x^\pm) + \theta^+\theta^-F(x^\pm).\end{aligned}$$

Similarly, for the conjugate $\bar{\Phi}$ we have

$$\begin{aligned}\bar{\Phi}(x^\pm, \theta^\pm, \bar{\theta}^\pm) = \bar{\phi}(x^\pm) + i\theta^+\bar{\theta}^+\partial_+\bar{\phi}(x^\pm) + i\theta^-\bar{\theta}^-\partial_-\bar{\phi}(x^\pm) - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_+\partial_-\bar{\phi}(x^\pm)\\ - \bar{\theta}^+\bar{\psi}_+(x^\pm) - \bar{\theta}^-\bar{\psi}_-(x^\pm) - i\bar{\theta}^+\theta^-\bar{\theta}^-\partial_-\bar{\psi}_+(x^\pm) - i\bar{\theta}^-\theta^+\bar{\theta}^+\partial_+\bar{\psi}_-(x^\pm) + \bar{\theta}^+\bar{\theta}^-\bar{F}(x^\pm).\end{aligned}$$

Using these, one can extract the coefficient of $\theta^2\bar{\theta}^2$ in $\bar{\Phi}\Phi$. One finds

$$\begin{aligned}\bar{\Phi}\Phi|_{\theta^4} = -\bar{\phi}\partial_+\partial_-\phi + \partial_+\bar{\phi}\partial_-\phi + \partial_-\bar{\phi}\partial_+\phi - \partial_+\partial_-\bar{\phi}\phi\\ + i\bar{\psi}_+\partial_-\psi_+ - i\partial_-\bar{\psi}_+\psi_+ + i\bar{\psi}_-\partial_+\psi_- - i\partial_+\bar{\psi}_-\psi_- + |F|^2.\end{aligned}$$

Combining all the pieces, we may now evaluate all of the fermionic integrals to get the Wess-Zumino action in terms of the component fields:

$$S[\phi, \psi, \bar{\psi}, F, \bar{F}] = \int_{\mathbb{R}^2} \Big[\partial^\mu \bar{\phi} \partial_\mu \phi + i\bar{\psi}_- \partial_+ \psi_- + i\bar{\psi}_+ \partial_- \psi_+ + |F|^2 + FW'(\phi) + \bar{F}\bar{W}'(\bar{\phi}) - \psi_+\psi_- W''(\phi) - \bar{\psi}_-\bar{\psi}_+\bar{W}''(\bar{\phi})\Big] \,\mathrm{d}^2x\,.$$

Note that the field F is auxiliary in this action, meaning that no derivatives of F appear, so the equations of motion for F are purely algebraic. This means that we can eliminate it using its equations of motion $F + \bar{W}'(\bar{\phi}) = 0$. Doing so yields

$$S[\phi, \psi, \bar{\psi}] = \int_{\mathbb{R}^2} \Big[\partial^\mu \bar{\phi} \partial_\mu \phi + i\bar{\psi}_- \partial_+ \psi_- + i\bar{\psi}_+ \partial_- \psi_+ - |W'(\phi)|^2 - \psi_+\psi_- W''(\phi) - \bar{\psi}_-\bar{\psi}_+\bar{W}''(\bar{\phi})\Big] \,\mathrm{d}^2x\,.$$

The first three terms are kinetic terms for the boson ϕ and fermions $\psi_\pm$. The other terms are interactions, the first one being a potential $V(\phi) = |W'(\phi)|^2$ for the scalar field.

Symmetries of the WZ model

By construction, the Wess-Zumino model is invariant under supersymmetry transformations. The Noether currents for these transformations may be found to be

$$G^0_\pm = 2\partial_\pm \bar{\phi}\psi_\pm \pm i\bar{\psi}_\pm F,$$
$$G^1_\pm = \mp 2\partial_\pm \bar{\phi}\psi_\pm + i\bar{\psi}_\mp F.$$

$\bar{G}^\mu_\pm$ are just the conjugates of $\bar{G}^\mu_\pm$. As usual, we may obtain the Noether charges by integrating the Noether currents over a spatial surface, so

$$Q_\pm = \int_{\mathbb{R}} G^0_\pm \,\mathrm{d}x^1\,.$$

Notice that these currents $G^\mu_\pm$ are spin $\frac{3}{2}$. Integration reduces the spin by one, so the Noether charges are each spin $\frac{1}{2}$, as expected for supercharges.

The Wess-Zumino model has some more important symmetries. For example 'axial' transformations act on the chiral superfield as

$$\Phi(x^\pm, \theta^\pm, \bar{\theta}\pm) \to \Phi(x^\pm, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\bar{\theta}^\pm).$$

Under this transformation, $\theta^+\theta^-$ is invariant, and so $W(\Phi)|_{\theta^2}, \bar{\Phi}\Phi|_{\theta^4}$ are also invariant. This means that the action does not change, and so this is a symmetry. The group generated by axial symmetries is often denoted $U(1)_A$. In terms of the component fields, we can equivalently think of an axial transformation as

$$\phi \to \phi, \quad \psi_\pm \to e^{\mp i\alpha}\psi_\pm, \quad F \to F.$$

The corresponding Noether charge is

$$F_A = \int_{\mathbb{R}} (\bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_-)\,\mathrm{d}x^1\,.$$

Now consider the 'vector' transformation

$$\Phi(x^\pm, \theta^\pm, \bar{\theta}^\pm) \to e^{iq^\beta}\Phi(x^\pm, e^{-i\beta}\theta^\pm, e^{i\beta}\bar{\theta}^\pm).$$

The group of these transformations is often denoted $U(1)_V$. Under this transformation, we have $\theta^+\theta^- \to e^{-2i\beta}\theta^+\theta^-$, while $\theta^2\bar{\theta}^2$ is invariant. This means that the Kähler contribution $\bar{\Phi}\Phi|_{\theta^4}$ will be invariant in the action. However, the superpotential term will only be invariant if W has charge 2, meaning it transforms like $W \to e^{2i\beta}W$. In particular, for a monomial superpotential

$$W(\Phi) = c\Phi^k,$$

vector transformations are only symmetries if $kq = 2$. If this is the case, then vector transformations can be viewed as acting on the component fields in the following way:

$$\phi \to e^{\frac{2i}{k}\beta}\phi, \quad \psi_\pm \to e^{(\frac{2}{k}-1)i\beta}\psi_\pm, \quad F \to e^{(\frac{2}{k}-2)i\beta}F.$$

Vacuum moduli space

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So far we have only been considering the Wess-Zumino model classically. Similar classical field theories exist in higher dimensions. But we want to do *quantum* field theory. Generically, when quantising a classical field theory, quantum corrections from the microscopic theory become important. But a theory with supersymmetry is to some degree protected from these corrections. In particular, we will see that in a 2-dimensional Wess-Zumino model, the superpotential is not quantum corrected.

As noted above, the scalar potential of the WZ model is $V(\phi, \bar{\phi}) = |W'(\phi)|^2$. We only had one chiral superfield, but more generally we could have some number of them Φ^a, indexed by a and set the Kähler potential to $K(\Phi^a, \bar{\Phi}^a) = \bar{\Phi}^a\Phi^a$ (summing over a). The superpotential is allowed to depend on all of these fields. Carrying out the above analysis, we would find that the scalar potential is given by

$$V(\phi^a, \bar{\phi}^a) = \left|\frac{\partial W}{\partial \phi^a}\right|^2.$$

Recall that ground states $|\Omega\rangle$ are supersymmetric if $H|\Omega\rangle = 0$. In particular, since $V \geq 0$, a field configuration can only be a ground state if it sits in a minimum of $V(\phi)$ over all space. This ground state is supersymmetric if and only if $V = 0$, which is equivalent to requiring

$$\frac{\partial W}{\partial \phi^a} = 0 \text{ for all } \phi^a$$

when evaluated on the ground state. Let ϕ_0^a be the configuration of the scalar field on the ground state. In the quantum theory, this configuration is determined by an expectation value

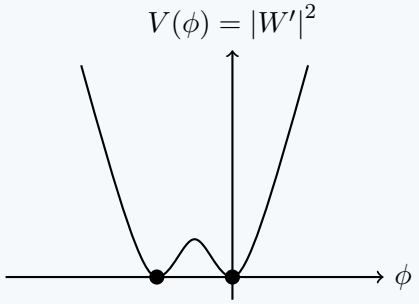
$$\phi_0^a = \langle\Omega|\phi^a|\Omega\rangle\,.$$

Typically, the holomorphic function $W(\phi^a)$ is a polynomial over $\mathbb{C}$, so the vacuum conditions $\frac{\partial W}{\partial \phi^a} = 0$ are a system of polynomials. The space of vacua, denoted $\mathscr{M}$, is then the zero set of these polynomials, i.e. the set satisfying

$$\frac{\partial W}{\partial \phi^1} = \frac{\partial W}{\partial \phi^2} = \cdots = 0.$$

This is an affine algebraic variety, which is a central object of study in algebraic geometry. $\mathscr{M}$ is sometimes called the *vacuum moduli space*.

Example. Suppose $W(\phi) = \frac{m\phi^2}{2} + \frac{\lambda\phi^3}{3}$. Then $\frac{\partial W}{\partial \phi} = m\phi + \lambda\phi^2 = 0$ at $\phi = 0, -\frac{m}{\lambda}$.



Thus there are two isolated supersymmetric minima, and these make up the vacuum space $\mathscr{M} = \{-\frac{m}{\lambda}, 0\}$.

Example. Suppose $W(\rho, h) = \frac{\lambda}{2}\rho h^2$. Then we have

$$\frac{\partial W}{\partial \rho} = \frac{\lambda}{2}h^2, \qquad \frac{\partial W}{\partial h} = \lambda\rho h.$$

The first of these implies that a supersymmetric vacuum must have $h = 0$. If this is true, then the second is automatically satified. In particular ρ can take any value in the complex plane. Thus the vacuum moduli space $\mathscr{M}$ is the complex ρ-plane.

For $\langle\rho\rangle \neq 0$, the field h is massive, with mass $|\lambda\langle\rho\rangle|$. The field ρ is always massless in the vacuum, because $\langle h\rangle = 0$, and so there is no term proportional to ρ^2 in the action.

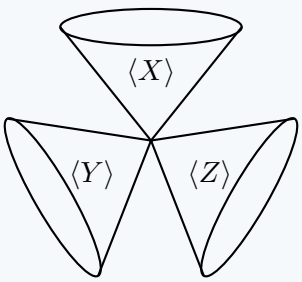
Example. Suppose we have three fields X, Y, Z, and the superpotential takes the form $W(X, Y, Z) = XYZ$. Then

$$\partial_X W = YZ, \quad \partial_Y W = ZX, \quad \partial_Z W = XY,$$

so

$$\mathscr{M} = \{X = Y = 0\} \cup \{Y = Z = 0\} \cup \{Z = X = 0\}.$$

This has the structure of three complex planes meeting at the origin:



In general the vacuum moduli space is the affine variety

$$\mathbb{C}[\phi^1, \dots]/\partial_a W.$$

If $\mathscr{M}$ is not just a set of isolated points, one says that the potential has flat directions. In this case, we can change some combination of the $\langle\phi^a\rangle$s continuously without leaving $V(\langle\phi^a\rangle) = 0$.

In a generic QFT, the structure of the classical potential is changed by quantum corrections. Couplings run with scale, and new couplings are generated (at least in an effective theory). In general, these corrections tend to lift flat directions, but supersymmetric theories are special.

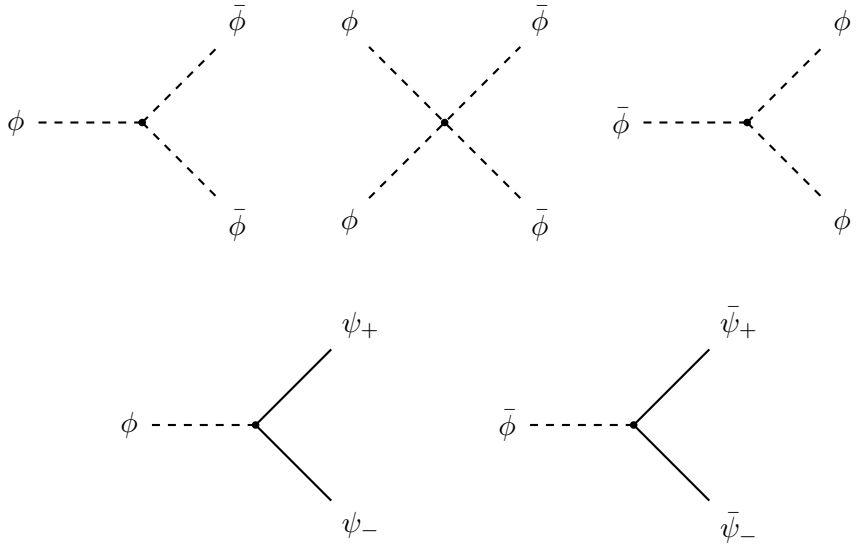
4.4 Seiberg non-renormalisation

The claim: in a supersymmetric theory, the effective superpotential $W_{\text{eff}}(\Phi)$ in the Wilsonian action is identical to the classical superpotential $W(\Phi)$.

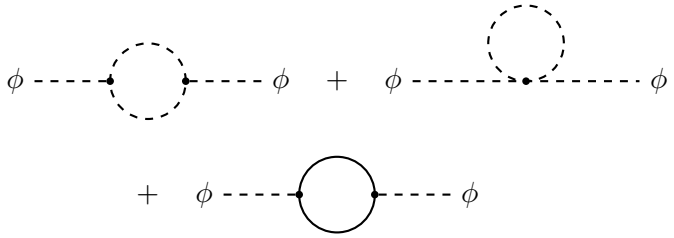
We won't prove this statement, but will instead try to understand the ideas behind it in a particular example. We set the superpotential to

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3.$$

This theory has interactions, and in particular has the following Feynman vertices:



As a consequence there are many non-trivial Feynman diagrams. For example, the 1-loop corrections to the mass m recieve contributions from:



However, it's possible to show that these diagrams cancel each other exactly. Moreover, the same cancellation holds to all orders in λ! Why has this happened?

Usually when a theory is protected from quantum corrections, it is due to a symmetry of that theory. Here, $W(\Phi)$ breaks the vector symmetry $U(1)_V$, so that can't possible help. Also, the axial symmetry $U(1)_A$ acts trivially on the scalars ϕ, so it can't constrain the form of the effective mass.

The right idea, due to Seiberg, is to promote the couplings m, λ themselves to chiral superfields M, Λ, such that m, λ are the vacuum expectation values of M, Λ. Note that M, Λ must be chiral, because they appear in the superpotential $W(\Phi, M, \Lambda)$. In promoting these couplings to fields, we give them kinetic terms, modifying the Kähler potential by

$$K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + \frac{1}{\epsilon}(\bar{M}M + \bar{\Lambda}\Lambda).$$

The factor of $\frac{1}{\epsilon}$ means that fluctuations in M, Λ will be highly suppressed as $\epsilon \to 0$.

The point is that if we give M, Λ the right charges, the superpotential $W(\Phi, M, \Lambda)$ now *does* preserve the $U(1)_V$ vector symmetry. There is also an additional symmetry $U(1)_{WZ}$ which acts trivially on $\theta^{\pm}, \bar{\theta}^{\pm}$. The correct charges are given by

	Φ	M	Λ
$U(1)_V$	1	0	-1
$U(1)_{WZ}$	1	-2	-3

Provided we choose a regularisation which is supersymmetric, and preserves these $U(1)$s, then the effective superpotential $W_{\text{eff}}(\Phi, M, \Lambda)$ in the Wilsonian action is constrained to:

- Be holomorphic in (Φ, M, Λ) (for supersymmetry),
- Have $U(1)_V$ charge +2 and $U(1)_{WZ}$ charge 0,
- Reduce to the classical $W(\Phi)$ in the limit M, Λ (i.e. weak coupling).

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The first two of these conditions fix the effective superpotnetial to be of the form

$$W_{\text{eff}}(\Phi, M, \Lambda) = M\Phi^2 f\left(\frac{\Phi\Lambda}{M}\right),$$

where $f(t)$ must be holomorphic in t. In particular, $f(t)$ must be regular as $t \to 0$, and $\frac{f(t)}{t}$ must be regular as $t \to \infty$, which means it must be of the form $f(t) = a + bt$. The final condition fixes $a = \frac{1}{2}$ and $b = \frac{1}{3}$. Hence

$$W_{\text{eff}}(\Phi, M, \Lambda) = \frac{M}{2}\Phi^2 + \frac{\Lambda}{3}\Phi^3 = W(\Phi, M, \Lambda).$$

Finally, we freeze the superfields M, Λ to their vacuum expectation values m, λ by sending $\epsilon \to 0$ in the kinetic term $\frac{1}{\epsilon}(\bar{M}M + \bar{\Lambda}\Lambda)$. The value of ϵ also can't effect W_{eff}, because we can always promote $\frac{1}{\epsilon}$ to a real superfield, and the superpotential can't depend on real superfields. (More generally, the quantum superpotential is always independent of couplings that appear only in the Kähler potential.)

Therefore:

$$W_{\text{eff}}(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 = W(\Phi).$$

In other words, there are no quantum corrections to W, and we have verified the claim at the beginning of this subsection (for this example).

However, the Kähler potential *does* generically get quantum corrections, and we will look at some examples of this.

4.5 Kähler geometry

Definition. A *Kähler manifold* $\mathcal{M}$ is a manifold with three compatible (in the sense described below) structures: a Riemannian metric g, a positive symplectic form ω, and a complex structure J.

Definition. A 2-form ω is *symplectic* if it is closed ($\mathrm{d}\omega = 0$) and non-degenerate (for any vector field X, $\omega(X, Y) = 0$ for all Y if and only if $X = 0$).

Definition. A rank $\binom{1}{1}$ tensor $J : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$ is an *almost complex structure* if $J^2 = -\mathrm{id}$, where id is the identity.

Example. On $\mathcal{M} = \mathbb{R}^2$, we have the tangent space basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$, and we could choose J such that

$$J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \qquad J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}.$$

In particular, such a J rotates the plane by $\frac{\pi}{2}$ in the anticlockwise direction. This should be compared with multiplication by i in the complex plane.

Note that since J is real, its eigenvalues must be $\pm i$. Complex vector fields are sections of $\mathrm{T}\mathcal{M} \otimes \mathbb{C}$. Thus we can define:

Definition. The holomorphic and antiholomorphic tangent bundles over $\mathcal{M}$ are

$$\mathrm{T}^{(1,0)}\mathcal{M} = \{X \in \mathrm{T}\mathcal{M} \otimes \mathbb{C} \text{ s.t. } \frac{1}{2}(1 - iJ)X = X\},$$
$$\mathrm{T}^{(0,1)}\mathcal{M} = \{X \in \mathrm{T}\mathcal{M} \otimes \mathbb{C} \text{ s.t. } \frac{1}{2}(1 + iJ)X = X\},$$

i.e. the $\pm i$ eigenspaces of J respectively.

An almost complex structure J is said to be integrable if for all $X, Y \in \mathrm{T}\mathcal{M} \otimes \mathbb{C}$, we have

$$\left(\frac{1 + iJ}{2}\right)\left[\left(\frac{1 - iJ}{2}\right)X, \left(\frac{1 - iJ}{2}\right)Y\right] = 0,$$

which is equivalent to the condition that the Lie bracket preserves (anti-)holomorphicity. Taking the real and imaginary parts of the above equation, it can be shown that it is equivalent to

$$N(X, Y) = -J^2([X, Y]) + J([JX, Y] + [X, JY]) - [JX, JY] = 0.$$

N is known as the Nijenhuis tensor.

Definition. A *complex structure* J is an integrable almost complex structure.

Theorem 4 (Newlander-Nirenberg)**.** *If J is an almost complex structure on a real manifold $\mathcal{M}$ whose Nijenhuis tensor vanishes, then there exist complex coordinates $x^i \to (z^a, \bar{z}^a)$ on any patch $U \subset \mathcal{M}$, and the transition functions on overlaps of patches are purely holomorphic.*

If J is a complex structure, we can split

$$\mathrm{T}\mathcal{M} \otimes \mathbb{C} = \mathrm{T}^{(1,0)}\mathcal{M} \oplus \mathrm{T}^{(0,1)}\mathcal{M}$$

globally. Similarly we may split the complexified cotangent bundle

$$\mathrm{T}^*\mathcal{M} \otimes \mathbb{C} = \mathrm{T}^{*(1,0)}\mathcal{M} \oplus \mathrm{T}^{*(0,1)}\mathcal{M},$$

where $\mathrm{T}_p^{*(1,0)}\mathcal{M}$ is the dual vector space to $\mathrm{T}_p^{(1,0)}\mathcal{M}$ for all $p \in \mathcal{M}$. Likewise, we can split

$$\Omega^k(\mathcal{M}, \mathbb{C}) = \bigoplus_{k=p+q} \Omega^{(p,q)}(\mathcal{M}),$$

where $\Omega^k(\mathcal{M}, \mathbb{C})$ is the space of complex valued k-forms, and if $\eta \in \Omega^{(p,q)}(\mathcal{M})$ then we may write

$$\eta(z, \bar{z}) = \eta_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q} \underbrace{\mathrm{d}z^{a_1} \wedge \dots \wedge \mathrm{d}z^{a_p}}_{p} \wedge \underbrace{\mathrm{d}\bar{z}^{\bar{b}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{b}_q}}_{q}.$$

The exterior derivative d maps k-forms to $(k+1)$-forms. On a complex manifold (i.e. a manifold with a complex structure), the exterior derivative itself splits as $d = \partial + \bar{\partial}$, where

$$\partial : \Omega^{(p,q)} \to \Omega^{(p+1,q)},$$
$$\bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}.$$

For example, on $\mathbb{R}^2$ we may write

$$\mathrm{d} = \mathrm{d}x \frac{\partial}{\partial x} + \mathrm{d}y \frac{\partial}{\partial y} = \underbrace{\mathrm{d}z \frac{\partial}{\partial z}}_{\partial} + \underbrace{\mathrm{d}\bar{z} \frac{\partial}{\partial \bar{z}}}_{\bar{\partial}},$$

where $z = x + iy$. These derivatives $\partial, \bar{\partial}$ are known as the Dolbeault operators. Since

$$0 = \mathrm{d}^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2,$$

we may write

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0,$$

because each of these operators take values in different vector spaces.

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A Kähler manifold has a symplectic form that is compatible with the complex structure in the sense that

$$\omega(JX, JY) = \omega(X, Y) \text{ for all vectors } X, Y.$$

This implies that $\omega \in \Omega^{(1,1)}(\mathcal{M})$, i.e. it has one holomorphic and one antiholomorphic component, so we may write

$$\omega = \omega_{a\bar{b}}(z, \bar{z})\, \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}}.$$

Given any such ω, we get a Hermitian metric for free, by writing $g(X,Y) = \omega(X, JY)$. We can check that this obeys all the properties we expect of a Hermitian metric. We have

$$g(Y,X) = \omega(Y,JX) = -\omega(J^2Y, JX) = \omega(JX, J^2Y) = \omega(X,JY) = g(X,Y),$$

so it is symmetric. Also,

$$g(JX,JY) = \omega(JX, J^2Y) = -\omega(JX,Y) = \omega(Y,JX) = g(Y,X) = g(X,Y),$$

so it is Hermitian. Additionally g is positive if and only if ω is positive (i.e. $\omega(x,JX) > 0$ for non-zero X). g is non-degenerate because ω, J are non-degenerate.

Since $\mathrm{d}\omega = 0$, on a complex manifold we have $\partial\omega + \bar\partial\omega = 0$, and since these two forms live in different vector spaces, we must have $\partial\omega = 0, \bar\partial\omega = 0$. The complex form of the Poincaré lemma works just like the real form, but there is one result for each of ∂ and $\bar\partial$. In this case we can apply it to locally write

$$\omega = i\partial\bar\partial K,$$

where K is a real function on $\mathcal{M}$. Thus the metric may be written

$$g_{a\bar b} = \partial_a \partial_{\bar b} K.$$

(In this notation, $\partial_{\bar b} = \frac{\partial}{\partial \bar z^{\bar b}}$ is an antiholomorphic derivative.) K is called the *Kähler potential*, and is defined up to

$$K(z,\bar z) \to K(z,\bar z) + f(z) + \bar f(\bar z),$$

which leaves ω, g unchanged. This is called a Kähler transformation.

Example. $\mathbb{C}^n$ is Kähler, with

$$\begin{aligned} K &= \sum_{a=1}^n |z^a|^2, \\ g &= \sum_{a=1}^n \mathrm{d}z^a \, \mathrm{d}\bar z^{\bar a}, \\ \omega &= i\sum_{a=1}^n \mathrm{d}z^a \wedge \mathrm{d}\bar z^{\bar a}. \end{aligned}$$

Example. $\mathbb{CP}^n$ is Kähler, with

$$K = \ln\left(1 + \sum_{a=1}^n |z^a|^2\right),$$

where $z^a, \bar z^{\bar a}$ are defined on a $\mathbb{C}^n$ coordinate patch. The associated metric is called the Fubini-Study metric.

On a Kähler manifold, the only non-vanishing Christoffel symbols are those for which the indices are either all holomorphic or all antiholomorphic. In particular we have

$$\begin{aligned}\Gamma^a_{bc} &= \frac{1}{2}g^{ai}(\partial_b g_{ic} + \partial_c g_{ib} - \partial_i g_{bc}) \\ &= \frac{1}{2}g^{a\bar{d}}(\partial_b g_{\bar{d}c} + \partial_c g_{\bar{d}b}) \\ &= \frac{1}{2}g^{a\bar{d}}(\partial_b \partial_{\bar{d}} \partial_c K + \partial_c \partial_{\bar{d}} \partial_b K) \\ &= g^{a\bar{d}}\partial_b g_{c\bar{d}},\end{aligned}$$

where i is a real index. Similarly

$$\Gamma^{\bar{a}}_{\bar{b}\bar{c}} = g^{\bar{a}d}\partial_{\bar{b}} g_{\bar{c}d}.$$

All others vanish.

4.6 Kähler manifolds and SUSY

The general kinetic term in a supersymmetric nonlinear sigma model on $\mathbb{R}^{2|4}$ is

$$\int_{\mathbb{R}^{2|4}} K(\Phi^a, \bar{\Phi}^a)\, \mathrm{d}^2x\, \mathrm{d}^4\theta\,.$$

This is defined up to $K \to K + F(\Phi) + \bar{F}(\bar{\Phi})$ as, up to total derivatives, the F terms to not survive the $\int \mathrm{d}^4\theta$ integration.

Performing the fermionic integrals gives

$$\begin{aligned}S_{\text{kin}}[\Phi^a, \bar{\Phi}^b] = \int_{\mathbb{R}^2} \Big[&- g_{a\bar{b}}\partial^\mu \phi^a \partial_\mu \bar{\phi}^{\bar{b}} + i g_{a\bar{b}} \bar{\psi}^{\bar{b}}_+ \nabla_- \psi^a_+ + i g_{a\bar{b}} \bar{\psi}^{\bar{b}}_- \nabla_+ \psi^a_- \\ &+ R_{a\bar{b}c\bar{d}} \psi^a_+ \bar{\psi}^{\bar{b}}_+ \psi^c_- \bar{\psi}^{\bar{d}}_- + g_{a\bar{b}}(F^a - \Gamma^a_{cd}\psi^c_+\psi^d_-)(\bar{F}^{\bar{b}} - \Gamma^{\bar{b}}_{\bar{e}\bar{f}}\bar{\psi}^{\bar{e}}_-\bar{\psi}^{\bar{f}}_+)\Big]\, \mathrm{d}^2x\,,\end{aligned}$$

where $g_{a\bar{b}}(\phi, \bar{\phi}) = \partial_a \partial_{\bar{b}} K(\phi, \bar{\phi})$ is the Kähler metric, and

$$\nabla_\mu \psi^a_+ = \partial_\mu \psi^a_+ + \Gamma^a_{bc}\partial_\mu \phi^b \psi^c_+, \text{ etc.}$$

It is not trivial that this is true, but it is clearly plausible.

Eliminating the auxiliary fields by their equations of motion $F^a = \Gamma^a_{bc}\psi^b_+\psi^c_-$, the remaining action is invariant under:

- Complex coordinate transformations of the target space.
- Kähler transformations.
- Supersymmetry transformations on the worldsheet $\mathbb{R}^2$.
- $U(1)_V$ transformations $\psi_\pm \to e^{i\beta}\psi_\pm, \bar{\psi}_\pm \to e^{-i\beta}\bar{\psi}_\pm$.
- $U(1)_A$ transformations $\psi_\pm \to e^{\pm i\alpha}\psi_\pm, \bar{\psi}_\pm \to e^{\mp i\alpha}\bar{\psi}_\pm$.
- Dilatations (scale transformations) on the worldsheet.

In fact it is also invariant under conformal transformations, which are positition dependent generalisations of dilatations. One can see this by noting that the action has no dimensionful couplings. The metric g contains a very complicated hidden dependence on ϕ, which generates many interactions – but all of the couplings are dimensionless.

Thus at the classical level, this defines a supersymmetric conformal field theory. Turning on a superpotential

$$\int_{\mathbb{R}^{2|2}} W(\Phi)\,\mathrm{d}^2\theta\,\mathrm{d}^2x = \int_{\mathbb{R}^2}\left[F^a\partial_a W - \frac{1}{2}\partial_a\partial_b W\psi_+^a\psi_-^b\right]\mathrm{d}^2x$$

typically breaks conformal invariance, because the couplings in $W(\Phi)$ can be dimensionful.

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These symmetries may or may not survive at the quantum level.

To better understand the dilatation invariance, let us explicitly include the worldsheet metric $h_{\mu\nu}$ and gamma matrices γ^μ in the action. We'll also allow the worldsheet Σ to be more general than $\mathbb{R}^2$. Then, after eliminating $F^a, \bar{F}^{\bar{a}}$, the action may be written

$$S[\phi,\psi] = \int_\Sigma \left[g_{a\bar{b}}h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\bar{\phi}^{\bar{b}} + ig_{a\bar{b}}\bar{\psi}^{\bar{b}}\gamma^\mu\nabla_\mu\psi^a + R_{a\bar{b}c\bar{d}}\psi_+^a\psi_-^c\bar{\psi}_-^{\bar{b}}\bar{\psi}_+^{\bar{d}}\right]\sqrt{h}\,\mathrm{d}^2x\,.$$

Classically, this is invariant under the scale transformation

$$h_{\mu\nu}\to\lambda^2 h_{\mu\nu},\quad \gamma^\mu\to\lambda^{-1}\gamma^\mu,\quad \psi\to\lambda^{-\frac{1}{2}}\psi,\quad \lambda\in\mathbb{R}_{>0}.$$

The quantum mechanical violation of scale invariance is encoded in the β-function, and in this case there can be a non-zero β-function for the target space metric $g_{a\bar{b}}(\phi)$. To understand this, let us first consider a purely bosonic NLSM with Riemannian target. The action is

$$S[\phi] = \frac{1}{2}\int_\Sigma g_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j\,\mathrm{d}^2x\,.$$

Let us write $\phi^i = \phi_0^i + \xi^i$, where ϕ_0^i is constant, and ξ^i is assumed to be some small vector. Then we have in Riemannian normal coordinates

$$g_{ij}(\phi) = \delta_{ij} - \frac{1}{3}R_{ikjl}\xi^k\xi^l + \mathcal{O}\Big(\xi^3\Big),$$

and the action becomes

$$S[\xi] = \frac{1}{2}\int_\Sigma\left[\delta_{ij}\partial^\mu\xi^i\partial_\mu\xi^j - \frac{1}{3}R_{ikjl}(\phi_0)\xi^k\xi^l\partial_\mu\xi^i\partial^\mu\xi^j + \mathcal{O}\Big(\xi^5\Big)\right]\mathrm{d}^2x\,.$$

The first term gives a propagator and the second term gives a four-point vertex for the ξ field. We may use these to compute a 1-loop correction to the propagator. We have

$$\Big\langle\xi^i(x)\xi^j(y)\Big\rangle^{\text{1-loop}} = \quad \xi^i(x) \xrightarrow[k]{} \xi^j(y) \quad + \quad \xi^i(x) \xrightarrow[k]{} \bullet \xrightarrow[k]{} \xi^j(y) \quad (\text{loop } p)$$

$$= \int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{e^{ik\cdot(x-y)}}{k^2}\left[\delta^{ij} + \frac{1}{3}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{p^2}R^{ij}(\phi_0)\right].$$

To do the p integral, we need to introduce a UV cutoff Λ, and an IR cutoff μ. Then we have

$$\int_{\mu\le|p|\le\Lambda}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{p^2} = \frac{1}{2\pi}\int_\mu^\Lambda\frac{\mathrm{d}p}{p} = \frac{1}{2\pi}\ln\Big(\frac{\Lambda}{\mu}\Big).$$

By comparing with the classical propagator

$$\left\langle \xi^i(x)\xi^j(y)\right\rangle^{\text{0-loop}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{e^{ik\cdot(x-y)}}{k^2}\delta^{ij},$$

we may conclude that the metric at ϕ_0 recieves the following correction:

$$g_{ij}(\mu) = \delta_{ij} + \frac{1}{6\pi}R_{ij}\ln\left(\frac{\Lambda}{\mu}\right).$$

This is a renormalisation with β-function

$$\beta_{ij} = \frac{1}{6\pi}R_{ij}$$

which is proportional to the Ricci curvature. There are then three possibilities:

- If $R_{ij} > 0$, then $\beta_{ij} > 0$, so the model is asymptotically free. This means that the curvature of the target becomes less important at short distances on the worldsheet, so the theory makes sense in the UV.
- If $R_{ij} < 0$, then the theory only makes sense as an effective theory, but becomes trivial in the IR.
- If $R_{ij} = 0$ (so the target is Ricci-flat, i.e. it solves the vacuum Einstein equations), then the theory is conformally invariant to (at least) 1-loop order accuracy. Higher order corrections may still change this.

Exactly the same calculations can be done in supersymmetric models. In this case there is a non-zero running of the Kähler metric, which changes by

$$g_{a\bar{b}} \to g_{a\bar{b}} + cR_{a\bar{b}},$$

where c is some constant. Again, conformal invariance is respected to 1-loop if the target is Ricci-flat. Ricci-flat Kähler spaces are called *Calabi-Yau*. It used to be the belief that this continues to hold to all loop orders in the supersymmetric theory. Indeed, the 2-loop and 3-loop corrections also vanish on a Calabi-Yau target space. However, there actually are further corrections, which start being non-trivial at 4 loops. This is apparently a problem for string theory, which we want to remain conformally invariant.

However, there are ways to compute exact constraints for conformal invariance. In particular, this may be done without going to higher and higher loops in a Feynman diagram expansion. As an example, consider the following correlation function:

$$f(h,g) = \left\langle (\psi_-)^k(\bar{\psi}_+)^k \right\rangle,$$

where k is some positive integer. $f = f(h,g)$ depends on the worldsheet metric $h_{\mu\nu}$ and target metric $g_{a\bar{b}}$. This correlation function usually vanishes, because there are no Feynman diagrams that one can draw which will connect ψ_- and $\bar{\psi}_+$. However, the Feynman expansion only works perturbatively, and there are non-perturbative effects which can contribute to the correlator. In particular, if there are exactly k zero-modes of ψ_- and $\bar{\psi}_+$, $f(h,g)$ does not have to vanish. A zero mode of ψ_- is one that satisfies $\nabla_+\psi_- = 0$. This means it does not contribute to the quadratic part of the action, and is said to be non-propagating. A zero mode ψ_- is a member of

$$H^0(\Sigma, \phi^*\mathrm{T}^{(1,0)}\mathcal{N} \otimes S_-).$$

In other words, it is holomorphic on the worldsheet Σ, carries a holomorphic target space index, and is a spinor.

Let us focus on the case where the worldsheet is a 2-torus $\Sigma = T^2$. Then it is possible to apply the index theorem (and a vanishing theorem) to show that the number of zero modes is equal to

$$\frac{i}{2\pi}\int_{T^2} R_{a\bar{b}}\,\mathrm{d}\phi^a \wedge \mathrm{d}\bar{\phi}^{\bar{b}}\,,$$

which is sometimes called the 'first Chern class' of the image of the worldsheet in the target space, and is a positive integer. Since $\bar{\psi}_+$ is related to ψ_- by complex conjugation, it has the same number of zero modes.

Let's set k to be equal to the number of zero modes. We have

$$f(h,g) = f(\lambda^2 h, g)\lambda^k.$$

This is because a rescaling $h \to \lambda^2 h$ of the worldsheet metric can be compensated by a rescaling $\psi \to \lambda^{-\frac{1}{2}}\psi$ of the fermions. Also, since the fermions $\psi_-, \bar{\psi}_+$ are invariant under $\bar{Q}_+, Q_-$ respectively, we expect some kind of localisation to apply. In fact, one can use this localisation to show that

$$f(h,g) = n_h e^{-\operatorname{Area}(T^2,\phi^* g)},$$

where n_h is some number that depends on h, and

$$\operatorname{Area}(\Sigma,\phi^* g) = \int_\Sigma h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\bar{\phi}^{\bar{b}} g_{a\bar{b}}\sqrt{h}\,\mathrm{d}^2x$$

is the area of the worldsheet according to the pullback of the target metric.

Combining the above, we see that

$$f(h,g) = n_{\lambda^2 h} e^{-(\operatorname{Area} - k\ln\lambda)} = g(\lambda^2 h, g'),$$

where g' is a target space metric which satisfies

$$\operatorname{Area}(T^2,\phi^* g') = \operatorname{Area}(T^2,\phi^* g) - k\ln\lambda.$$

However

$$\begin{aligned}\operatorname{Area}(T^2,\phi^* g) &= i\int_{T^2} g_{a\bar{b}}(\partial_z\phi^a\partial_z\bar{\phi}^{\bar{b}} + \partial_{\bar{z}}\phi^a\partial_z\bar{\phi}^{\bar{b}})\,\mathrm{d}^2z \\ &= 2i\int_{T^2} g_{a\bar{b}}\partial_{\bar{z}}\phi^a\partial_z\bar{\phi}^{\bar{b}}\,\mathrm{d}^2z + i\int_{T^2}(\partial_z\phi^a\partial_z\bar{\phi}^{\bar{b}} - \partial_{\bar{z}}\phi^a\partial_z\bar{\phi}^{\bar{b}})\,\mathrm{d}^2z\,.\end{aligned}$$

In the course of the localisation procedure referred to above, one finds that the correlator localises to holomophic ϕ, so the first term in the last line vanishes. Also, the second term is just equal to the integral of $\phi^*\omega$ (the pullback of the target symplectic form to the worldsheet) over T^2. Thus

$$\operatorname{Area}(T^2,\phi^* g) = \int_{T^2}\phi^*\omega.$$

Thus, the effect of rescaling $h \to \lambda^2 h$ means that

$$\int\phi^*\omega \to \int\phi^*\omega - \frac{i\ln\lambda}{2\pi}\int R_{a\bar{b}}\,\mathrm{d}\phi^a\wedge\mathrm{d}\bar{\phi}^{\bar{b}}\,,$$

where we used the index theorem result for k. In other words

$$[\omega] \to [\omega] - \frac{\ln \lambda}{2\pi}[R],$$

where the notation $[\alpha]$ refers to the equivalence class of the form α up to addition of exact forms. Thus, the class of the symplectic form changes by the first Chern class. For the theory to be scale-invariant, we therefore require that the first Chern class of the target space vanishes. This result is *exact* in a supersymmetric theory.

Fin