Gauge/Gravity Duality

Foundations and Applications

Solutions to Exercises

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References to equations involving Arabic chapter numbering refer to the equations given in the book. The equations in this set of solutions are labelled by a Roman number counting the chapters, followed by an arabic equation number (eg. (V.3)).

1 Elements of field theory

Exercise 1.1.1

The equation of motion for a free real scalar field reads $(\Box - m^2)\phi(x) = 0$ and is linear. It is therefore natural to expect that it is solved by a plane wave ansatz

$$\phi(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, a(k) e^{-ikx} \,, \tag{I.1}$$

where $k^\mu = (\omega, k)$, $x^\mu = (t, \vec{x})$ and $kx = k_\mu x^\mu = -\omega t + \vec{k}\vec{x}$. Inserting this ansatz into the equation of motion, we obtain

$$(\Box - m^2)\phi(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, (-k^2 - m^2) \, a(k) \, e^{-ikx} \,. \tag{I.2}$$

Thus the ansatz is only a solution if $k^2 + m^2 = 0$. Modifying the ansatz to implement this condition, we have

$$\phi(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, 2\pi \, \delta(k^2 + m^2) \Theta(k^0) \left(a(k) e^{-ikx} + \text{c.c.} \right) \,, \tag{I.3}$$

with Θ the step function. This ansatz satisfies the equation of motion. Note that $\phi(x)$ is real, which we achieved by adding the complex conjugated terms to the expression. We perform the k^0-integration using the identity

$$\delta(f(k^0)) = \sum_{i,\text{single zeros}} \frac{\delta(k^0 - k_i^0)}{f'(k_i^0)} \,, \tag{I.4}$$

where k_i^0 are the single zeros of the function. For $k^2 + m^2 = -(k^0)^2 + \vec{k}^2 + m^2$, the zeros are located at $k^0 = \pm\omega_k$ and we obtain

$$\phi(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^{d-1}\vec{k}}{2\omega_k} \left(a(k) e^{-ikx} + a(k)^* e^{ikx} \right) \Big|_{k^0 = \omega_k} \,. \tag{I.5}$$

Exercise 1.1.2

The Lagrangian reads

$$\mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 - \frac{g}{4!} \phi(x)^4 \,. \tag{I.6}$$

The associated Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right) = \frac{\partial \mathcal{L}}{\partial \phi(x)} \tag{I.7}$$

is

$$-\Box\phi(x) = -m^2\phi(x) - \frac{g}{3!}\phi(x)^3\,, \tag{I.8}$$

such that

$$(\Box - m^2)\phi(x) = \frac{g}{3!}\phi(x)^3\,. \tag{I.9}$$

Exercise 1.1.3

Using $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and $\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$, it is straightforward to rewrite

$$\mathcal{L}_{\text{free}} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - \frac{1}{2}m^2\phi_i^2\,, \tag{I.10}$$

with summation over i implied, as

$$\mathcal{L}_{\text{free}} = -\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi\,. \tag{I.11}$$

Moreover, we may introduce an interaction term $V_{\text{int}}[\phi^*\phi] = V_{\text{int}}[\frac{1}{2}(\phi_1^2 + \phi_2^2)]$ such that

$$\mathcal{L} = \mathcal{L}_{\text{free}} - V_{\text{int}}[\phi^*\phi]\,. \tag{I.12}$$

The Euler-Lagrange equations obtained by varying with respect to ϕ^* and ϕ respectively read

$$\begin{aligned}(\Box - m^2)\phi &= \frac{\partial V_{\text{int}}}{\partial(\phi^*\phi)}\phi\,,\\ (\Box - m^2)\phi^* &= \frac{\partial V_{\text{int}}}{\partial(\phi^*\phi)}\phi^*\,.\end{aligned} \tag{I.13}$$

Note that it is important to treat ϕ and ϕ^* as independent. This guarantees that when considering the Lagrangian in terms of the two real fields ϕ_1 and ϕ_2, we obtain a compatible set of equations of motion.

Exercise 1.2.1

The Noether charge Q is given by

$$Q = \int\limits_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, J^t \tag{I.14}$$

where the integration is performed for constant time slices $t = \text{const}$. The Noether charge is time-independent since

$$\frac{\mathrm{d}}{\mathrm{d}t}\,Q \;=\; \int\limits_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\,\partial_t J^t = -\int \mathrm{d}^{d-1}\vec{x}\,\partial_i J^i \;=\; 0\,, \tag{I.15}$$

where we used $\partial_\mu J^\mu = 0$. Moreover we assume that J^i decays fast enough at infinity such that the integral is zero.

Exercise 1.2.2

The Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = \Pi \partial_t \phi - \mathcal{L} \,. \tag{I.16}$$

For a free massive real scalar field, the Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 \,, \tag{I.17}$$

and hence the canonical momentum is given by

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi \,. \tag{I.18}$$

$\mathcal{H}$ may then be written as

$$\mathcal{H} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \,. \tag{I.19}$$

Exercise 1.2.3

$\Theta^{\mu\nu}$ and $N^{\mu\nu\rho}$ are conserved, i.e. $\partial_\mu\Theta^{\mu\nu} = \partial_\mu N^{\mu\nu\rho} = 0$. With $N^{\mu\nu\rho} = x^\nu\Theta^{\mu\rho} - x^\rho\Theta^{\mu\nu}$ we have

$$0 = \partial_\mu N^{\mu\nu\rho} = \delta^\nu_\mu\Theta^{\mu\rho} - \delta^\rho_\mu\Theta^{\mu\nu} = \Theta^{\nu\rho} - \Theta^{\rho\nu} \tag{I.20}$$

and hence $\Theta^{\nu\rho} = \Theta^{\rho\nu}$.

Exercise 1.2.4

The energy-momentum tensor given by

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\eta_{\mu\nu}(\partial_\rho\phi)(\partial^\rho\phi) - \frac{d-2}{4(d-1)}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2 \tag{I.21}$$

is symmetric, $T_{\mu\nu} = T_{\nu\mu}$, and conserved, $\partial^\mu T_{\mu\nu} = 0$. The latter follows from

$$\partial^\mu T_{\mu\nu} = (\partial^\mu\partial_\mu\phi)\partial_\nu\phi + (\partial_\mu\phi)(\partial^\mu\partial_\nu\phi) - (\partial_\nu\partial_\rho\phi)\partial^\rho\phi - \frac{d-2}{4(d-1)}(\Box\partial_\nu - \partial_\nu\Box)\phi^2 \,. \tag{I.22}$$

In this expression, the first term vanishes by the equation of motion $\Box\phi = 0$, while the second and third terms cancel each other. The last term vanishes by a mathematical identity. $T_{\mu\nu}$ is also traceless on-shell since

$$T^\mu{}_\mu = \frac{d-2}{2}\,\phi\,\partial_\mu\partial^\mu\phi \,, \tag{I.23}$$

which vanishes due to the equation of motion.

Exercise 1.2.5

The infinitesimal version of the transformation (1.35) reads

$$\begin{aligned}\phi(x) &\mapsto \phi'(x) = \phi(x) + i\alpha\phi(x) + \mathcal{O}(\alpha^2) = \phi(x) + \alpha\delta\phi(x)\,,\\ \phi^*(x) &\mapsto \phi'(x)^* = \phi(x)^* - i\alpha\phi(x) + \mathcal{O}(\alpha^2) = \phi(x)^* + \alpha\delta\phi(x)^*\,,\end{aligned} \tag{I.24}$$

where $\delta\phi(x) = i\phi(x)$ and $\delta\phi(x)^* = -i\phi(x)^*$. The conserved Noether current J^μ is given by

$$J^\mu = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\delta\phi^* = i\left(\phi(x)\overleftrightarrow{\partial}^\mu\phi^*(x)\right)\,, \tag{I.25}$$

where we have used the derivative $\overleftrightarrow{\partial}^\mu$, which is defined by

$$f(x)\overleftrightarrow{\partial}^\mu g(x) \equiv f(x)\partial^\mu g(x) - \left(\partial^\mu f(x)\right)g(x)\,. \tag{I.26}$$

The charge associated to the Noether current J^μ reads

$$Q = -i\int \mathrm{d}^{d-1}\vec{x}\,\phi(t,\vec{x})\overleftrightarrow{\partial}_t\,\phi^*(t,\vec{x})\,. \tag{I.27}$$

Exercise 1.2.6

The Lagrangian for N real free scalar fields ϕ_i with $i = 1,\ldots,N$ and mass $m_i = m$ reads

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - \frac{1}{2}m\phi_i^2\,, \tag{I.28}$$

where a sum over $i = 1,\ldots,N$ is implied. Introducing the vector

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \cdots\\ \phi_N\end{pmatrix} \tag{I.29}$$

we may rewrite the Lagrangian in the form

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi^T\partial_\nu\Phi - \frac{1}{2}m\Phi^T\Phi\,, \tag{I.30}$$

and hence it is obvious that the Lagrangian (I.30) is invariant under global $O(N)$ rotations R of the fields,

$$\Phi'(x) = R\Phi(x)\,, \qquad \text{or in components} \qquad {\phi'}^i(x) = R^i{}_j\,\phi^j(x) \tag{I.31}$$

since the matrix R satisfies $R^TR = \mathbb{1}$. For N complex free scalar fields ϕ_i with $i = 1,\ldots,N$ and with the same mass m, we may also introduce the vector Φ and hence the Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi^\dagger\partial_\nu\Phi - \frac{1}{2}m\Phi^\dagger\Phi \tag{I.32}$$

The only difference is that for complex scalar fields, we have to use † instead of transpose. Using the same logic as above, we conclude that the Lagrangian (I.32) is invariant under global $U(N)$ rotations R acting on the fields by (I.31). Note that $U(N)$ transformations R satisfy $R^\dagger R = \mathbb{1}$. However, this argument is not entirely correct. Rewriting each complex field as two real scalar fields with the same mass m, we may rewrite the Lagrangian for N complex free scalar fields in terms of $2N$ real free scalar fields and hence the Lagrangian of N complex free scalar fields is invariant under $O(2n)$ which contains $U(N)$ as a subgroup. Generic interactions break the $O(2N)$ invariance. This is the reason why usually only the global $U(N)$ invariance of the Lagriangian (I.32) is highlighted in the literature.

Exercise 1.3.2

Let us consider the following integral

$$\int \mathrm{d}^d k\, \delta^d(k^2+m^2)\Theta(k^0)f(k^2) = \int \mathrm{d}^{d-1}\vec{k} \int \mathrm{d}k^0\, \delta^d(k^2+m^2)\Theta(k^0)f(k^2) \quad \text{(I.33)}$$

In order to derive the identity asked for we perform the integration over k^0. Note that

$$k^2 + m^2 = -(k^0)^2 + \vec{k}^2 + m^2\,. \quad \text{(I.34)}$$

Hence k^2+m^2 vanishes for $k^0 = \pm\omega_k$ with $\omega_k = \sqrt{\vec{k}^2+m^2}$. Using the identity

$$\delta(f(k^0)) = \sum_i \frac{\delta(k^0 - k_i^0)}{|f'(k_i^0)|}\,, \quad \text{(I.35)}$$

where i label the simple zeros of the function f we obtain ofr $f = -(k^0)^2 + \vec{k}^2 + m^2$,

$$\delta(f(k^0)) = \frac{\delta(k^0-\omega_k)}{2\omega_k} + \frac{\delta(k^0+\omega_k)}{2\omega_k} \quad \text{(I.36)}$$

Due to the step function $\Theta(k^0)$ present in the integral only the first term contributes and therefore we obtain

$$\int \mathrm{d}^{d-1}\vec{k} \int \mathrm{d}k^0\, \delta^d(k^2+m^2)\Theta(k^0) = \int \frac{\mathrm{d}^{d-1}\vec{k}}{2\omega_k} \quad \text{(I.37)}$$

Exercise 1.3.6

Using the definiton of $\Gamma[\varphi]$, i.e. eq. (1.7.6) in the book, we obtain

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(y)} = \frac{\delta W[J]}{\delta\varphi(y)} - J(y) - \int \mathrm{d}^d x\, \frac{\delta J(x)}{\delta\varphi(y)}\,\varphi(x)\,. \quad \text{(I.38)}$$

Using the chain rule we may rewrite

$$\frac{\delta W[J]}{\delta\varphi(y)} = \int \mathrm{d}^d x \frac{\delta W[J]}{\delta J(x)}\,\frac{\delta J(x)}{\delta\varphi(y)} = \int \mathrm{d}^d x \varphi(x)\,\frac{\delta J(x)}{\delta\varphi(y)}\,, \quad \text{(I.39)}$$

where we have used the definition of $\varphi(x)$ in the last step. Combining the two equations we obtain

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(y)} = -J(y) \tag{I.40}$$

which is the correct equation of motion (note the typo in the book).

Exercise 1.3.7

The first part of the solution follows by applying the chain rule,

$$\frac{\delta}{\delta J(x)} = \int \mathrm{d}^d y \, \frac{\delta\varphi(y)}{\delta J(x)} \, \frac{\delta}{\varphi(y)} = \int \mathrm{d}^d y \, \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \, \frac{\delta}{\varphi(y)} \,, \tag{I.41}$$

where we have used the definition of $\varphi(x)$ again in the last step, i.e. equation (1.77) in the book. Applying the result (I.41) to $\varphi(x)$, we then obtain

$$\frac{\delta}{\delta J(y)}\varphi(x) = \int \mathrm{d}^d z \, \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)} \, \frac{\delta\varphi(x)}{\varphi(z)} = \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \,, \tag{I.42}$$

and hence

$$\Gamma^{(2)}(x,y) = \frac{\delta^2\,\Gamma[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)} = \frac{\delta J(y)}{\delta\varphi(x)} = -\left(\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\right)^{-1} \tag{I.43}$$

where we have used the correct result of exercise 1.3.6 (see above) and (I.42).

3 Symmetries in quantum field theory

Exercise 3.2.3

By consecutively inserting x into x', x' into x'' and x'' into x''', we obtain

$$x'''^{\mu} = \frac{x''^{\mu}}{x''^2} = \cdots = \frac{x^\mu/x^2 + b^\mu}{(x^\mu/x^2 + b^\mu)(x_\mu/x^2 + b_\mu)} \,. \tag{III.1}$$

Elementary algebra then gives (3.57).

Exercise 3.2.4

This is just the statement that we cannot Taylor expand x^μ/x^2 about $x^\mu = 0$. Moreover, aasume that the inversion I can be written as an infinitesimal transformation $x'^{\mu} = x^\mu + \epsilon^\mu(x)$. Then $\det I = 1 + \mathcal{O}(\epsilon)$, which contradicts the result of exercise 3.2.5.

Exercise 3.2.5

The determinant of the $d \times d$ symmetric matrix I is defined by

$$\det I \;=\; \epsilon^{\mu_1 \cdots \mu_d}\, I_{1\mu_1} \cdots I_{d\mu_d} \,, \tag{III.2}$$

Inserting the definition $I_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu/x^2$, we obtain

$$\det I \;=\; \epsilon^{1\ldots d} - 2({x_1}^2 + {x_2}^2 + \cdots {x_d}^2)/x^2 \;=\; -1 \,, \tag{III.3}$$

which is the desired result. It is straightforward to verify that all other possible contributions in (III.2) vanish due to the total antisymmetry of $\epsilon^{\mu_1 \cdots \mu_d}$.

Exercise 3.2.6

First, we recall that z and $\bar{z}$ are independent variables, i.e. $\partial_z \bar{z} = 0$. This implies the third commutation relation. For the first we have

$$\begin{aligned}
[l_m, l_n] &= -z^{m+1}\partial_z(-z^{n+1}\partial_z) + z^{n+1}\partial_z(-z^{m+1}\partial_z) \\
&= -z^{n+m+1}(m-n)\partial_z = (m-n)l_{n+m} \,.
\end{aligned} \tag{III.4}$$

The second relation follows from complex conjugation.

Exercise 3.2.7

From equation (1.238) we know that

$$[Q, \phi(x)] = i\delta\phi(x) \tag{III.5}$$

for any Noether charge Q. Inserting $\delta_\epsilon\phi(x)$ as given by (3.80) into this equation, and considering $Q = \{P_\mu, D, J_{\mu\nu}, K_\mu\}$ with the respective contribution to $\epsilon(x)$ given by (3.52), we obtain the commutation relations (3.79).

Exercise 3.2.8

The conformal algebra (3.53) is obtained by explicit evaluation of the commutators using the explicit expressions given in (3.79).

As an example, let us compute the commutator $[\mathcal{K}_\mu, \mathcal{P}_\nu]\phi$ explicitly:

$$\begin{aligned}[\mathcal{K}_\mu, \mathcal{P}_\nu]\phi &= [i(-x^2\partial_\mu + 2x_\mu x^\rho\partial_\rho + 2x_\mu\Delta) - 2x^\rho\mathcal{J}_{\mu\rho}, -i\partial_\nu] \\ &= -(-2x_\nu\partial_\mu + 2\delta_{\mu\nu}x^\rho\partial_\rho + 2x_\mu\partial_\nu + 2\eta_{\mu\nu}\Delta + 2i\mathcal{J}_{\mu\nu})\phi \\ &= -2i(\eta_{\mu\nu}\mathcal{D} - \tilde{\mathcal{J}}_{\mu\nu})\phi\,. \end{aligned} \tag{III.6}$$

This coincides with the appropriate expression in the algebra (3.53).

Exercise 3.2.9

We consider the action in the form $\mathcal{S}[g]$, i.e. we introduce a classical source $g^{\mu\nu}$ for the energy-momentum tensor. Under a symmetry transformation, we thus have

$$\delta\mathcal{S}[g] = \int \mathrm{d}^d x\, \delta g^{\mu\nu} \frac{\delta\mathcal{S}}{\delta g^{\mu\nu}}\,. \tag{III.7}$$

For a Weyl transformation under which $\delta_\sigma g^{\mu\nu}(x) = 2\sigma(x)\delta g^{\mu\nu}$, we find for a Weyl-invariant theory

$$0 \;=\; \delta_\sigma\mathcal{S} = \int \mathrm{d}^d x\, 2\sigma(x) g^{\mu\nu}\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}} = -\int \sqrt{-g}\sigma(x) g^{\mu\nu}T_{\mu\nu}\,. \tag{III.8}$$

Since $\sigma(x)$ is an arbitrary function, this implies that $T^\mu{}_\mu = 0$, i.e. the energy-momentum tensor is traceless.

Exercise 3.2.10

(i) Dilatations: The Ward identity is checked by inserting the explicit expression for the scalar two- and three-point functions, (3.91) and (3.92), into the dilatation Ward identity (3.88) and calculating the derivatives explicitly.

(ii) Special conformal transformations: From (3.87) with (3.80), the Ward identity for special conformal transformations reads

$$\sum_{i=1}^{n}\left((b^\mu x_i^2 - 2(b\cdot x_i)x_i^\mu)\frac{\partial}{\partial x_i^\mu} \;-\; 2\Delta_i\, b\cdot x_i\right)\langle\phi_1(x_1)\phi_2(x_2)\ldots\phi_i(x_i)\ldots\phi_n(x_n)\rangle \;=\; 0\,. \tag{III.9}$$

This Ward identity is satisfied by (3.91) and (3.92), as is checked by calculating the derivatives explicitly.

Exercise 3.2.11

(i) Using the definitions (3.62) as well as (3.65) and its cyclic permutations, the relations may be checked by explicit calculation, noting that

$$(z-x)^2 + (z-y)^2 = (x-y)^2 + 2(z-x)\cdot(z-y)\,. \tag{III.10}$$

(ii) These relations follow using the relations checked in (i) and the properties (3.101), (3.102) for the inversion, $R = I$.

(iii) This relation holds for three identical bosonic fields and is a consequence of the symmetry of the three-point function under interchanging any of the two operators. The first equality is obtained by considering the exchange of $\mathcal{O}^i$ and $\mathcal{O}^j$, while the second is obtained by interchanging $\mathcal{O}^i$ and $\mathcal{O}^k$.

Exercise 3.2.12

We start from (3.128),

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y) = -\frac{c}{48\pi^2}S^x{}_{\mu\nu}S^y{}_{\mu\nu}\ln(x-y)^2\,, \quad S_{\mu\nu} = \partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\,. \tag{III.11}$$

With $z = x_1 + ix_2$, $w = y_1 + iy_2$, as well as $T(z) = -2\pi T_{zz}(x)$ and $(\partial_z)^4\ln(z\bar{z}) = -6/z^4$ we find the expected result

$$\langle T(z)T(w)\rangle = \frac{c}{2}\frac{1}{(z-w)^4}\,. \tag{III.12}$$

Moreover, by explicit calculation it may be checked that (3.128) coincides with (3.96) for $C_T = c/(2\pi^2)$ in two dimensions, noting that

$$\partial_\mu\frac{1}{x^a} = -a\frac{x_\mu}{x^{a+2}}\,, \qquad x \equiv (x^\mu x_\mu)^{1/2}\,, \tag{III.13}$$

for any real number a. Taking the trace in (3.128) and using (3.130), we then obtain the desired result (3.131).

5 The AdS/CFT correspondence

Exercise 5.3.1

We insert $\phi \sim z^\Delta$ into (5.51) and perform the integration. Using $m^2L^2 = \Delta(\Delta - d)$ this gives

$$\begin{aligned}\mathcal{S} &\sim -\frac{CL^{d-1}}{2}\mathrm{Vol}\mathbb{R}^{d-1,1}\int_0^\epsilon \mathrm{d}z\frac{1}{z^{d+1}}\left(\Delta^2 + m^2L^2\right)z^{2\Delta} \\ &= -\frac{CL^{d-1}}{2}\mathrm{Vol}\mathbb{R}^{d-1,1}\Delta\epsilon^{2\Delta-d}\,. \end{aligned} \tag{V.1}$$

Factoring out the boundary volume, this is finite for $\epsilon \to 0$ provided that $\Delta \geq d/2$.

Exercise 5.3.2

The calculation is analogous to exercise 5.3.1, inserting $\phi \sim z^\Delta$ into the action (5.54) and performing the integration using $m^2L^2 = \Delta(\Delta - d)$.

Exercise 5.4.1

To show that the bulk-to-boundary propagator $K(x, z; x')$ may be calculated from the bulk-to-bulk propagator $G(z, x; z', x')$ by

$$K(z'', x''; x') = (\Delta_+ - \Delta_-)\lim_{\epsilon\to 0}\frac{G(\epsilon, x'; z'', x'')}{\epsilon^{\Delta_+}}\,, \tag{V.2}$$

we use Green's second identity

$$\int_M \mathrm{d}^dx\mathrm{d}z\sqrt{g}\left[\phi(\Box_{x,z} - m^2)\psi - \psi(\Box_{x,z} - m^2)\phi\right] = \int_{\partial M}\mathrm{d}^dx\sqrt{\gamma}(\phi\partial_n\psi - \psi\partial_n\phi)\,, \tag{V.3}$$

where γ is the determinant of the induced metric on ∂M and ∂_n is the derivative normal to the boundary. Since ∂M is given by $z = \epsilon \ll 1$ (and since later the limit $\epsilon \to 0$ is taken), the derivative ∂_n is given by $\partial_n = n^\mu\partial_\mu$, where n has norm one, i.e. $g_{\mu\nu}n^\mu n^\nu = 1$. Since only n^z is non-vanishing, we obtain $n^z = z$ and therefore $\partial_n = z\partial_z$.

Let us consider the case $\phi(z,x) = K(z,x;x')$ and $\psi(z,x) = G(z,x;z'',x'')$, i.e.

$$\begin{aligned}\int_M \mathrm{d}^d x \mathrm{d}z \sqrt{g} \Big[& K(z,x;x')(\Box_{x,z} - m^2)G(z,x;z'',x'') \\ & -G(z,x;z'',x'')(\Box_{x,z} - m^2)K(z,x;x')\Big] \\ = \lim_{\epsilon\to 0}\int_{z=\epsilon} & \mathrm{d}^d x \sqrt{\gamma}(K(z,x;x') z\partial_z G(z,x;z'',x'') \\ & -G(z,x;z'',x'') z\partial_z K(z,x;x'))\,. \end{aligned} \tag{V.4}$$

Since $K(z,x;x')$ solves the equations of motion of the scalar field, i.e.

$$\left(\Box_{x,z} - m^2\right) K(z,x;x') = 0\,, \tag{V.5}$$

and $G(z,x;z'',x'')$ satisfies

$$\left(\Box_{x,z} - m^2\right) G(z,x;z'',x'') = \frac{1}{\sqrt{g}}\delta(x-x'')\delta(z-z'')\,, \tag{V.6}$$

we obtain for the left hand side of equation (V.4),

$$\int_M \mathrm{d}^d x \mathrm{d}z \sqrt{g} K(z,x;x')(\Box_{x,z} - m^2)G(z,x;z'',x'') = K(z'',x'';x')\,. \tag{V.7}$$

Evaluating the right hand side of equation (V.4) requires the behaviour of $K(z,x;x')$ and of $G(z,x;z'',x'')$ for small $z \to \epsilon \ll 1$, which is given by

$$\lim_{z\to\epsilon} K(z,x;x') = \epsilon^{\Delta_-}\delta(x-x')\,, \tag{V.8}$$

$$\lim_{z\to\epsilon} G(z,x;z'',x'') \sim \epsilon^{\Delta_+}\,. \tag{V.9}$$

With this asymptotic behaviour, the derivatives of K and G read

$$\partial_z K(z,x;x')\big|_{z\to\epsilon\ll 1} = \Delta_- \epsilon^{\Delta_- -1}\delta(x-x') \tag{V.10}$$

and

$$\partial_z G(z,x;z'',x'')\big|_{z\to\epsilon\ll 1} = \Delta_+ \epsilon^{-1} G(\epsilon,x;z'',x'')\,. \tag{V.11}$$

For obtaining equation (V.11), we use the asymptotics (V.9): Taking a derivative of G with respect to z (where $z \to \epsilon \ll 1$) in equation (V.9), we obtain $\partial_z G = \Delta_+ \epsilon^{\Delta_+ -1}\ldots$, where the dots represent the functional dependents on x, x'' and z''. Since the terms denoted by $\epsilon^{\Delta_+}\ldots$ can be written as $G(\epsilon,x;z'',x'')$ (see equation (V.9)), we obtain equation (V.11).

Moreover since $\sqrt{\gamma} = z^{-d}$, we may write the right hand side of equation (V.4) in the form

$$\begin{aligned}
&\int_{z=\epsilon} \mathrm{d}^d x \sqrt{\gamma}(K(z,x;x') z\partial_z G(z,x;z'',x'') - G(z,x;z'',x'') z\partial_z K(z,x;x')) \\
&= \int \mathrm{d}^d x\, \epsilon^{1-d} \left(\lim_{z\to\epsilon} K(z,x;x')\, \partial_z G(z,x;z'',x'')\big|_{z\to\epsilon} \right. \\
&\qquad\qquad \left. - \lim_{z\to\epsilon} G(z,x;z'',x'')\, \partial_z K(z,x;x')\big|_{z\to\epsilon} \right) \\
&= \int \mathrm{d}^d x\, \epsilon^{1-d} \left(\epsilon^{\Delta_-} \delta(x-x') \Delta_+ \epsilon^{-1} G(\epsilon,x;z'',x'') \right. \\
&\qquad\qquad \left. - G(\epsilon,x;z'',x'') \Delta_- \epsilon^{\Delta_- -1} \delta(x-x') \right) \\
&= \epsilon^{\Delta_- - d} (\Delta_+ - \Delta_-)\, G(\epsilon,x';z'',x'')\,.
\end{aligned} \tag{V.12}$$

Using $\Delta_+ + \Delta_- = d$ and equations (V.4), (V.7) and (V.12), we finally obtain

$$K(z'',x'';x') = (\Delta_+ - \Delta_-) \lim_{\epsilon\to 0} \frac{G(\epsilon,x';z'',x'')}{\epsilon^{\Delta_+}}\,. \tag{V.13}$$

Exercise 5.5.1

Inserting the expansion (5.92) into (5.94) and setting $\rho = 0$ gives

$$\Box_0 \phi_{(0)} + 2(d - 2\Delta + 2)\phi_{(2)}(x) = 0\,. \tag{V.14}$$

Rearranging the terms gives the desired result

$$\phi_{(2)}(x) = \frac{1}{2(2\Delta - d - 2)}\, \Box_0 \phi_{(0)}\,. \tag{V.15}$$

Exercise 5.5.2

Holographic renormalisation g The starting point is the Einstein equation

$$R_{MN} - \frac{1}{2} R\, G_{MN} + \Lambda G_{MN} = 0\,, \tag{V.16}$$

with cosmological constant $\Lambda = -d(d-1)/(2L^2)$, where d is the dimension of the boundary theory. Since AdS is an Einstein space, the equation of motion reduces to

$$R_{MN} = -\frac{d}{L^2}\, G_{MN}\,. \tag{V.17}$$

The next step is to insert the metric (5.117) into the Einstein equation (V.17), using the convention

$$R^M{}_{NKL} = \partial_K \Gamma^M{}_{NL} - \partial_L \Gamma^M{}_{KN} + \Gamma^M{}_{KP}\Gamma^P{}_{LN} - \Gamma^M{}_{LP}\Gamma^P{}_{KN}\,, \tag{V.18}$$

with $\Gamma^M{}_{NL}$ the Christoffel symbol. This is a tedious calculation for which we recommend the use of algebraic computing. For the (μ,ν) component, the result is

$$\rho\left(2g''_{\mu\nu}-2g'_{\mu\lambda}g^{\lambda\kappa}g'_{\kappa\nu}+g^{\kappa\lambda}g'_{\kappa\lambda}g'_{\mu\nu}\right)+R_{\mu\nu}[g]-(d-2)g'_{\mu\nu}-g^{\kappa\lambda}g'_{\kappa\lambda}g_{\mu\nu}=0\,, \tag{V.19}$$

where the prime denotes the derivative with respect to ρ and $g_{\mu\nu}$ is defined in (5.117). Setting ρ to zero, this gives, using the expansion (5.118) or (5.119),

$$R_{\mu\nu}[g^{(0)}]-(d-2)g^{(2)}_{\mu\nu}-g^{(0)\,\kappa\lambda}g^{(2)}_{\kappa\lambda}g^{(0)}_{\mu\nu}=0\,. \tag{V.20}$$

The trace of (V.20) reads

$$R[g^{(0)}]-2(d-1)g^{(0)\,\kappa\lambda}g^{(2)}_{\kappa\lambda}=0\,. \tag{V.21}$$

Inserting (V.21) into (V.20) gives the desired result (5.120).

6 Tests of the AdS/CFT correspondence

Exercise 6.2.1

The integral to be calculated is given by (6.62),

$$\langle J_\mu(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = \int \frac{\mathrm{d}^{d+1}w}{w_0^{d+1}} G_{m\mu}(w,\vec{z}) w_0^2 K_\Delta(w,\vec{x}) \frac{\overleftrightarrow{\partial}}{\partial w_m} K_\Delta(w,\vec{y}) \,. \tag{VI.1}$$

The gauge field propagator given by (6.58) may be written as

$$G_{m\mu}(w,\vec{z}) = C^d \left(\frac{w_0}{(w-\vec{z})^2}\right)^{d-2} \partial_m \left(\frac{(w_\mu - \vec{z}_\mu)}{(w-\vec{z})^2}\right) , \tag{VI.2}$$

where we have set $S_{m\mu} = 0$ in (6.58). We perform an inversion of the coordinates and set $\vec{z} = 0$. This gives

$$\begin{aligned}\langle J_\mu(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= \frac{C^d}{(\vec{x})^{2\Delta}(\vec{y})^{2\Delta}} \int \frac{\mathrm{d}^d w' \mathrm{d}w'_0}{{w'_0}^{d+1}} {w'_0}^d \delta_{m\mu} K_\Delta(w',\vec{x}\,') \frac{\overleftrightarrow{\partial}}{\partial w'_m} K_\Delta(w',\vec{y}\,') \\ &= -\frac{C^d}{(\vec{x})^{2\Delta}(\vec{y})^{2\Delta}} \left(\partial_{x'_\mu} - \partial_{y'_\mu}\right) \int \frac{\mathrm{d}^d w' \mathrm{d}w'_0}{w'_0} K_\Delta(w',\vec{x}\,') K_\Delta(w',\vec{x}\,') \,.\end{aligned} \tag{VI.3}$$

The remaining integral over scalar bulk-to-boundary propagators is evaluated using (6.32) with (6.35). The result is $\propto (\vec{x}\,' - \vec{y}\,')^{d-2\Delta}$. Reinstating the dependence on $\vec{z}$ by translating $\vec{x} \to \vec{x} - \vec{z}$, $\vec{y} \to \vec{y} - \vec{z}$, we obtain

$$\begin{aligned}&\langle J_\mu(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle \\ &= -\frac{(\Delta - d/2)\Gamma(d/2)\Gamma(\Delta)}{\pi^d \Gamma(\Delta - d/2)} \frac{1}{(\vec{x}-\vec{y})^{2\Delta-d++2}(\vec{x}-\vec{z})^{d-2}(\vec{y}-\vec{z})^{d-2}} Z_\mu .\end{aligned} \tag{VI.4}$$

This coincides with the desired result, with Z_μ as in (3.65) and ξ as in (6.65), using that

$$(\vec{x}\,' - \vec{y}\,')^2 = Z^2 = \frac{(x-y)^2}{(x-z)^2(y-z)^2} \,. \tag{VI.5}$$

Exercise 6.2.2

Using the explicit expression for the variable Z^μ as given by (3.65), the three-point function reads

$$\langle J^\mu(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = \frac{-\xi(d-2)}{(\vec{x}-\vec{y})^{2\Delta-d+2}(\vec{x}-\vec{z})^{d-2}(\vec{y}-\vec{z})^{d-2}}\left(\frac{\vec{x}^\mu-\vec{z}^\mu}{(\vec{x}-\vec{z})^2}-\frac{\vec{y}^\mu-\vec{z}^\mu}{(\vec{y}-\vec{z})^2}\right), \tag{VI.6}$$

where the arrow in e.g. $\vec{x}$ denotes the boundary field theory coordinates. Taking the derivative with respect to $\vec{z}^\mu$, we obtain

$$\begin{aligned}&\partial_{z_\mu}\langle J^\mu(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle\\ &= -\frac{\xi(d-2)}{(\vec{x}-\vec{y})^{(2\Delta-d+2)}}\left[\partial_{z_\mu}\left(\frac{\vec{x}^\mu-\vec{z}^\mu}{(\vec{x}-\vec{z})^d}\right)\frac{1}{(\vec{y}-\vec{z})^d}-\partial_{z_\mu}\left(\frac{\vec{y}^\mu-\vec{z}^\mu}{(\vec{y}-\vec{z})^d}\right)\frac{1}{(\vec{x}-\vec{z})^d}\right].\end{aligned} \tag{VI.7}$$

Using (3.129) we find that

$$\partial_\mu\frac{z^\mu}{z^d} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\delta^{(d)}(z)\,. \tag{VI.8}$$

Moreover, using (5.89) for the two-point function with $CL^{d-1}=1$, we have

$$\partial_{z_\mu}\langle J^\mu(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = \left(\delta^d(\vec{x}-\vec{z})-\delta^d(\vec{y}-\vec{z})\right)\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle\,. \tag{VI.9}$$

This expression is antisymmetric under the exchange of $\vec{x}$ and $\vec{y}$. For a symmetric expression, we have to consider a non-abelian $SU(2)$ global symmetry replacing the $U(1)$ considered above. Then, with $SU(2)$ representation T^{aIJ}, we have

$$\langle J^{\mu a}(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\rangle = -\xi(d-2)T^{aIJ}\frac{1}{(\vec{x}-\vec{y})^{2\Delta}}(Z^2)^{(d-2)/2}Z^\mu \tag{VI.10}$$

and

$$\begin{aligned}&\partial_{z_\mu}\langle J^{a\mu}(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\rangle\\ &\quad= \delta^d(\vec{x}-\vec{z})\,T^{aIK}\langle\mathcal{O}^K(\vec{x})\mathcal{O}^J(\vec{y})\rangle+\delta^d(\vec{y}-\vec{z})\,T^{aJK}\,\langle\mathcal{O}^I(\vec{x})\mathcal{O}^K(\vec{y})\rangle\,.\end{aligned} \tag{VI.11}$$

Exercise 6.3.1

$a^{(0)}$ and $a^{(2)}$ are given by (6.86). Under the transformation $\delta g^{(0)\,\mu\nu} = 2\sigma g^{(0)\,\mu\nu}$, $\delta\epsilon = -2\sigma\epsilon$ we have

$$\delta\sqrt{\det g^{(0)}} = -4\sigma\sqrt{\det g^{(0)}}\,,\qquad \delta\frac{1}{\epsilon^2} = 4\sigma\frac{1}{\epsilon^2}\,. \tag{VI.12}$$

This implies that

$$\delta\left(\sqrt{\det g^{(0)}}\epsilon^{-2}a_{(0)}\right) \;=\; 0 \tag{VI.13}$$

as expected. Moreover, we have

$$\delta\left(\sqrt{\det g^{(0)}}R\right) = -2\sigma\sqrt{\det g^{(0)}}R-2\sqrt{\det g^{(0)}}\nabla^2\sigma\,,\qquad \delta\frac{1}{\epsilon} = 2\sigma\frac{1}{\epsilon^2} \tag{VI.14}$$

such that for $\sigma(x)$ linear in x as required for a boundary conformal transformation, $\nabla^2\sigma = 0$ and

$$\delta\left(\sqrt{\det g^{(0)}}\epsilon^{-2}a_{(2)}\right) \;=\; 0\,. \tag{VI.15}$$

Exercise 6.3.2

$a_{(4)}$ in (6.86) may be written as

$$a_{(4)} = \frac{L^3}{8}(C^2 - E)\,, \tag{VI.16}$$

with C^2 the square of the Weyl tensor and E the Euler density. C^2 is a Weyl invariant, i.e. $\delta(\sqrt{\det g^{(0)}}C^2) = 0$. The transformation of the Euler density is given in (6.90): For a conformal boundary transformation for which $\nabla\nabla\sigma = 0$, $\sqrt{\det g^{(0)}}E$ is also an invariant. The only non-zero contribution in (6.89) thus comes from the derivative with respect to ϵ acting on $\ln\epsilon$ in (6.87). Since $\epsilon\partial_\epsilon\,\ln\epsilon = 1$, we obtain the result given in (6.89).

12 Linear response and hydrodynamics

Exercise 12.1.1

First we verify that $\mathcal{P}^L_{\mu\nu}$ and $\mathcal{P}^R_{\mu\nu}$ as given by (12.17), (12.18) are orthogonal,

$$\mathcal{P}^{T\,\mu\nu}\mathcal{P}^L_{\mu\nu} = \mathcal{P}^{T\,\mu\nu}(\mathcal{P}_{\mu\nu} - \mathcal{P}^T_{\mu\nu}) = \mathcal{P}^{T\,\mu\nu}(\mathcal{P}^T_{\mu\nu} - \mathcal{P}^T_{\mu\nu}) = 0\,. \tag{XII.1}$$

The orthogonality of $\mathcal{S}$ and $\mathcal{Q}$ of (12.29) follows similarly by direct computation.

Exercise 12.1.2

Using $\psi\left(z^*\right) = \psi^*\left(z\right)$, we have

$$\begin{aligned}\Pi\left(\omega\right) &= \frac{N^2T^2}{8}\left(i\omega + \omega^2\left[\psi\left(\frac{1-i}{2}\omega\right) + \psi\left(-\frac{1+i}{2}\omega\right)\right]\right) \\ &= \frac{N^2T^2}{8}\left(i\omega + \omega^2\left[\psi\left(\frac{1-i}{2}\omega\right) + \psi^*\left(-\frac{1-i}{2}\omega\right)\right]\right)\,.\end{aligned} \tag{XII.2}$$

Using the result $\mathrm{Im}[f\left(z\right) + f^*\left(-z\right)] = \mathrm{Im}[f\left(z\right) - f\left(-z\right)]$, where $z \in \mathbb{C}$, we find

$$\mathrm{Im}\Pi\left(\omega\right) = \frac{N^2T^2}{8}\left(\omega + \omega^2\mathrm{Im}\left[\psi\left(\frac{1-i}{2}\omega\right) - \psi\left(-\frac{1-i}{2}\omega\right)\right]\right)\,. \tag{XII.3}$$

Using $\psi\left(z\right) - \psi\left(-z\right) = -\pi\mathrm{cot}\pi z - 1/z$, we then find

$$\begin{aligned}\mathrm{Im}\Pi\left(\omega\right) &= -\frac{N^2T^2}{8}\omega^2\mathrm{Im}\left[\pi\mathrm{cot}\pi\left(\frac{1-i}{2}\omega\right)\right] \\ &= -\frac{N^2T^2}{8}\pi\omega^2\frac{\mathrm{sinh}\pi\omega}{\mathrm{cosh}\pi\omega - \mathrm{cos}\pi\omega},\end{aligned} \tag{XII.4}$$

where in the final line we use addition formulae for trigonometric and hyperbolic functions. That is,

$$\sin\left(\theta_1 \pm \theta_1\right) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2\,, \tag{XII.5}$$

$$\cos\left(\theta_1 \pm \theta_1\right) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2, \tag{XII.6}$$

and

$$\sinh\left(\theta_1 \pm \theta_1\right) = \sinh\theta_1\cosh\theta_2 \pm \cosh\theta_1\sinh\theta_2\,, \tag{XII.7}$$

$$\cosh\left(\theta_1 \pm \theta_1\right) = \cosh\theta_1\cosh\theta_2 \pm \sinh\theta_1\sinh\theta_2. \tag{XII.8}$$

The desired result follows from $\mathcal{R}\left(\omega\right) = -2\mathrm{Im}\Pi\left(\omega\right)$.

Exercise 12.2.1

We make use of the fact that $u^\mu u_\mu = -1$, which implies $u^\mu \nabla_\nu u_\mu = 0$. Moreover, $P^{\mu\nu} u_\nu = 0$ for $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. Conservation of the energy momentum tensor implies $\nabla^\mu T_{\mu\nu} = 0$. We multiply this equation with u^ν and obtain

$$\begin{aligned} 0 = u^\nu \nabla^\mu T_{\mu\nu} &= u^\nu \nabla^\mu \left(\epsilon u^\mu u^\nu + p P^{\mu\nu}\right) \\ &= -u_\mu \nabla^\mu \epsilon - (\epsilon + p) \nabla^\mu u_\mu \,. \end{aligned} \tag{XII.9}$$

Using the thermodynamic relations $\epsilon + p = sT + \mu_a \rho_a$ and $\mathrm{d}\epsilon = T\mathrm{d}s + \mu_a \mathrm{d}\rho_a$, we then have

$$0 = -T\nabla^\mu (s u_\mu) - \mu_a \nabla^\mu (\rho_a u_\mu) \,. \tag{XII.10}$$

Since T and μ_a are independent, this implies $\nabla^\mu s_\mu = \nabla^\mu (s u_\mu) = 0$ for the entropy current $s_\mu = s u_\mu$, which is hence conserved.

Exercise 12.2.2

For the mean free path, we have

$$l_{\mathrm{mfp}} = \frac{1}{n\sigma\nu} \,. \tag{XII.11}$$

Inserting the weak-coupling results as given by (12.92), as well as $\epsilon \sim T^4$ for the energy density, we obtain the result

$$l_{\mathrm{mfp}} \sim \frac{1}{g^2 T} \,, \qquad \eta \sim \epsilon l_{\mathrm{mfp}} \sim \frac{T^3}{g^2} \,, \tag{XII.12}$$

which implies that the mean free path is small at weak coupling, and $\eta/s \sim 1/g^2$ which is large.

Exercise 12.3.1

The choice of infalling boundary conditions gives $w_p \sim (1-u^2)^{\frac{-i\omega}{4\pi T}}$. To leading order in the frequency, this behaviour of w_p is also valid near the boundary given by $u \to 0$. For this choice for w_p, $\mathcal{F}$ as given by (12.123) reads

$$\mathcal{F} = \frac{\pi^2 N^2 T^4}{8} \times i \frac{\omega}{2\pi T} \frac{1}{1-u^2} \,. \tag{XII.13}$$

According to (12.121), the retarded Green's function then reads

$$G_{\mathrm{R}}(\omega) = -2 \lim_{u \to 0} \mathcal{F} = -\frac{\pi N^2 T^3}{8} i\omega \,, \tag{XII.14}$$

which is the desired result.

13 QCD and holography: confinement and chiral symmetry breaking

Exercise 13.1.1

From section 4.1.2 we know for an open string with Neumann boundary conditions on both endpoints in $D = 26$ dimensions that

$$\alpha' M^2 = N + \frac{2-D}{24} = N - 1\,, \tag{XIII.1}$$

where

$$N = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} n (a_n^i)^\dagger a_n^i\,. \tag{XIII.2}$$

In terms of a and $a^\dagger$, J^{23} reads

$$J^{23} = -i \sum_{n=1}^{\infty} \left((a_n^2)^\dagger a_n^3 - (a_n^3)^\dagger a_n^2 \right)\,. \tag{XIII.3}$$

By explicit calculation, we find that

$$\left[N, J^{23}\right] = 0\,, \tag{XIII.4}$$

which implies

$$\left[\alpha' M^2 + 1, J^{23}\right] = 0\,. \tag{XIII.5}$$

Hence we may use an eigenbasis of $\alpha' M^2 + 1$ for studying the action of J^{23}. We consider rotations in the $(2,3)$-plane only. We begin considering the case in which the eigenvalue of $\alpha' M^2 + 1$ vanishes and denote the corresponding eigenstate by $|0\rangle$,

$$(\alpha' M^2 + 1)|0\rangle = 0\,. \tag{XIII.6}$$

By explicit calculation using the definition of J^{23}, we find that also

$$J^{23}|0\rangle = 0\,, \tag{XIII.7}$$

so for the state $|0\rangle$, the eigenvalues are given by $J = 0 = \alpha' M^2 + 1$, and the Regge relation is satisfied. We now move on to the states with eigenvalue $\alpha' M^2 + 1 = 1$. The eigenvectors satisfying

$$(\alpha' M^2 + 1)|\psi\rangle = |\psi\rangle \tag{XIII.8}$$

are given by

$$|\psi_1\rangle = a_1^{3\dagger}|0\rangle\,, \qquad |\psi_2\rangle = a_1^{2\dagger}|0\rangle\,, \tag{XIII.9}$$

where we restrict to those eigenvectors relevant to rotations in the $(2,3)$ plane. By explicit calculation, we find

$$J^{23}\begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}. \tag{XIII.10}$$

The matrix on the right hand side has the eigenvalues $+1$ and -1. Thus $|J| \leq \alpha' M^2 + 1$ as required. We may now proceed by analysing the eigenstates for increasing eigenvalues to verify the relation $|J| \leq \alpha' M^2 + 1$ for each eigenvalue. For the next case, i.e. for states with eigenvalue $\alpha' M^2 + 1 = 2$, we note that the eigenvectors are given by

$$\begin{gathered} |\phi_1\rangle = a_2^{3\dagger}|0\rangle\,, \qquad |\phi_2\rangle = a_2^{2\dagger}|0\rangle,\,, \\ |\phi_3\rangle = a_1^{2\dagger}a_1^{3\dagger}|0\rangle\,, \qquad |\phi_4\rangle = a_1^{2\dagger}a_1^{2\dagger}|0\rangle\,, \\ |\phi_5\rangle = a_1^{3\dagger}a_1^{3\dagger}|0\rangle\,. \end{gathered} \tag{XIII.11}$$

The next step is to calculate the eigenvalues of the 5×5 matrix obtained by acting with J^{23} on the vectors given above.

15 Strongly coupled condensed matter systems

Exercise 15.2.1

The finite temperature case is obtained by repeating the steps described at the beginning of section 15.2.2 for the zero temperature case. In addition, a new scale z_0 is defined as given in (15.25). Then, expanding M as given by (15.25) in $\tilde{z}_0 \equiv z_h - z_0$, we obtain

$$f(z) = 1 - \tilde{M} z^d + \frac{d}{d-2}\left(\frac{z}{z_0}\right)^{2(d-1)} \tag{XV.1}$$

for $f(z)$ of (15.10), where to quadratic order in $\tilde{z}_0$,

$$\tilde{M} = \frac{2(d-1)}{d-2}\frac{1}{z_h^d} + \frac{d(d-1)}{z_h^d}\frac{\tilde{z}_h}{z_h^2}\,. \tag{XV.2}$$

Expanding also the last term in (XV.1) to second order in $\tilde{z}_0$, by writing $z_0 = z_h - \tilde{z}_0$, we obtain

$$f(z) = d(d-1)\frac{1}{z_h^2}\left(\tilde{z}^2 - \tilde{z}_0^2\right) = d(d-1)\frac{\tilde{z}^2}{z_h^2}\left(1 - \frac{\tilde{z}_0^2}{\tilde{z}^2}\right)\,. \tag{XV.3}$$

Then, inserting this expression for $f(z)$ into the metric (15.9) and performing the transformations of variables

$$\tilde{z} \mapsto \frac{\tilde{L}^2}{\zeta}\,, \qquad \tilde{z}_0 \mapsto \frac{\tilde{L}^2}{\zeta_h} \tag{XV.4}$$

with $\tilde{L}$ as in (15.22), we obtain the metric (15.27) of $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$.

Exercise 15.3.1

Consider the case of an uncharged scalar with mass close to the Breitenlohner-Freedman bound $m_{\mathrm{BF}}^2 = -d^2/(4L^2)$ in $d > 2$, for instance the example given by (15.96), where $m^2 = -2/L^2$ with $d = 3$. For an uncharged scalar, there is no coupling to the gauge field and the term $g^{tt}A_t^2$ is absent, such that $m_{\mathrm{eff}} = m$. For the Reissner-Nordström metric, in the IR the Breitenlohner-Freedman bound for an AdS_2 space with radius $\tilde{L}^2 = L^2/6$ is relevant, which is given by

$$m^2_{\mathrm{BF},AdS_2} = -\frac{1}{4\tilde{L}^2} = -\frac{3}{2L^2}\,. \tag{XV.5}$$

Thus a scalar field with mass $m^2 = -2/L^2$ violates the Breitenlohner-Freedman bound for AdS_2 relevant in the IR, although it satisfies the bound for AdS_4 in the UV.

Exercise 15.3.2

The equations of motion for the $SU(2)$ gauge field in the Einstein-Yang-Mills action (15.106) read

$$\nabla^\mu F^a{}_{\mu\nu} = -\epsilon^{abc} A^{b\mu} F^c_{\mu\nu}\,. \tag{XV.6}$$

With the ansatz

$$A = A^t(z)\tau^3 \mathrm{d}t + w(z)\tau^1 \mathrm{d}x\,, \tag{XV.7}$$

where τ^3, τ^1 are Pauli matrices, the right hand side of (XV.6) gives rise to a mass term for the fluctuation $w(z)$ of the form

$$m^2_{\text{eff}} = g^{tt}(z) A_t(z) A_t(z)\,. \tag{XV.8}$$

This may be below the Breitenlohner-Freedman bound for particular values of z.

Exercise 15.5.1

We consider flat Minkowski space $\mathbb{R}^{n,1}$. First, we take $n = 3$. In momentum space, the density of states for fermions (whose degeneracy is two) is $g(p) = 2/h^3$. With $\hbar = h/(2\pi)$ and setting $\hbar = 1$ this is $2/(2\pi)^3$. Using $E^2 = \vec{k}^2 + m^2$ to change variables to energy, we have, in radial coordinates in momentum space,

$$\rho(k)\mathrm{d}k = \frac{2}{(2\pi^3)} 4\pi k^2 \mathrm{d}k = \frac{1}{\pi^2}\sqrt{E^2 - m^2} E \mathrm{d}E = g(E)\mathrm{d}E\,. \tag{XV.9}$$

In general dimensions, we have

$$g(k)\mathrm{d}k = 2\frac{n\pi^{n/2}}{\Gamma(n/2+1)} k^{d-1}\mathrm{d}k = 2\frac{n\pi^{n/2}}{\Gamma(n/2+1)} (E^2 - m^2)^{(d-2)/2} E \mathrm{d}E\,. \tag{XV.10}$$

Exercise 15.5.2

For an isotropic system, the grand canonical potential is given by $G = -pV$ and $G = -p/V$. With $E = U - TS$ and $\epsilon = E/V$, $\rho = N/V$ with the chemical potential kept fixed, we have $-p = \epsilon - \mu\rho$.

Exercise 15.5.3

By performing the integrals with $g(E)$ as given by (15.161), we find

$$\rho = \frac{1}{\pi^2}\int_m^\mu \sqrt{E^2-m^2}E\mathrm{d}E = \frac{1}{3\pi^2}(\mu^2-m^2)^{3/2}\,, \tag{XV.11}$$

$$\begin{aligned}\epsilon = \frac{1}{\pi^2}\int_m^\mu \sqrt{E^2-m^2}E^2\mathrm{d}E = \frac{1}{8\pi^2}\Big(\mu\sqrt{\mu^2-m^2}(2\mu^2-m^2)\\ + m^4\left(\ln m\, - \,\ln\left[\mu+\sqrt{\mu^2-m^2}\right]\right)\Big)\,.\end{aligned} \tag{XV.12}$$

p is given by inserting these results into (15.163).

Exercise 15.5.4

We derive the equations of motion for an asymptotically AdS_{d+1} space. For the dilaton, the equation of motion reads

$$\frac{1}{\kappa^2}\sqrt{-g}^{-1}\partial_m\left(\sqrt{-g}g^{mn}\partial_n\Phi\right) - \frac{1}{4\kappa^2L^2}V'(\Phi) - \frac{1}{8g^2}F^{mn}F_{mn}Z'(\Phi) = 0\,, \tag{XV.13}$$

which when inserting the explicit expression for the metric becomes

$$\Phi''(r) - \frac{d-1}{r}\Phi'(r) + \frac{1}{2}\left(\frac{f'(r)}{f(r)} - \frac{g'(r)}{g(r)}\right)\Phi'(r) - \frac{g(r)}{4}V'(\Phi) + \frac{h'(r)^2}{4f(r)}Z'(\Phi) = 0\,. \tag{XV.14}$$

For the $U(1)$ gauge field, the equation of motion reads

$$\frac{1}{4g^2}\sqrt{-g}^{-1}\partial_m\left(\sqrt{-g}g^{mk}g^{nl}F_{kl}\right) = 0\,, \tag{XV.15}$$

which becomes

$$h''(r) - \frac{d-1}{r}h'(r) - \frac{1}{2}\left(\frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)}\right)h'(r) = 0\,. \tag{XV.16}$$

Finally, for the metric we have the Einstein equation

$$R_{mn} - \frac{1}{2}Rg_{mn} - \kappa^2T_{mn} = 0\,, \tag{XV.17}$$

with

$$\begin{aligned}\kappa^2T_{mn} =& 2\partial_m\Phi\partial_n\Phi + \frac{\kappa^2}{g^2}g^{kl}F_{mk}F_{nl}\\ &- \left(\partial^k\Phi\partial_k\Phi + \frac{1}{2L^2}V(\Phi) + \frac{\kappa^2}{4g^2}Z(\Phi)F^{kl}F_{kl}\right)g_{mn}\,.\end{aligned} \tag{XV.18}$$

The energy-momentum tensor takes the explicit form

$$\begin{aligned}\kappa^2 T_{mn} =& 2\Phi'(r)^2\delta_{mr}\delta_{nr} + \left(\frac{1}{g(r)}\delta_{mt}\delta_{nt} - \frac{1}{f(r)}\delta_{mr}\delta_{nr}\right) Z(\Phi)h'(r)^2 \\ &- \left(\frac{1}{L^2 g(r)}\Phi'(r) + \frac{1}{2L^2}V(\Phi) - \frac{2}{L^2 f(r)g(r)}Z(\Phi)h'(r)^2\right) g_{mn}\,. \end{aligned} \tag{XV.19}$$

For the tt-component, the equation of motion reads

$$-\frac{(d-1)f(r)\left((d+2)g(r)+rg'(r)\right)}{2r^2 g(r)^2} - \kappa^2 T_{tt} = 0\,, \tag{XV.20}$$

while for the xx-component, we have

$$\begin{aligned}&\frac{1}{4r^2 f(r)g(r)}\left(f'(r)\left(\frac{f'(r)}{f(r)}+\frac{g'(r)}{g(r)}\right) - 2f''(r)\right) \\ &+ \frac{d-2}{2r^3 g(r)}\left(\frac{-d}{r} + \frac{f'(r)}{f(r)} - \frac{g'(r)}{g(r)}\right) - \kappa^2 T_{xx} = 0\,, \end{aligned} \tag{XV.21}$$

and for the rr-component

$$\frac{(d-1)((d-2)f(r) - rf'(r))}{2r^2 f(r)} - \kappa^2 T_{rr} = 0\,. \tag{XV.22}$$